SOLUTIONS TO THREE LAMINAR VISCIOUS
FLOW PROBLEMS BY AN IMPLICIT
FINITE-DIFFERENCE METHOD

by

Wei Jao Chyu, B. S., M. S.

Thesis submitted to the Graduate Faculty of the
Virginia Polytechnic Institute
in candidacy for the degree of

DOCTOR OF PHILOSOPHY

in

ENGINEERING MECHANICS

May, 1965
Blacksburg, Virginia
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>1</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>iv</td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. REVIEW OF LITERATURE</td>
<td>6</td>
</tr>
<tr>
<td>III. IMPLICIT FINITE-DIFFERENCE METHOD OF SOLUTION</td>
<td>8</td>
</tr>
<tr>
<td>A. Implicit Finite-Difference Relations</td>
<td>8</td>
</tr>
<tr>
<td>B. Relations used for Numerical Integration and the Computation of Shear</td>
<td>10</td>
</tr>
<tr>
<td>C. Illustrative Example—The Solution of the Boundary-Layer Equations</td>
<td>11</td>
</tr>
<tr>
<td>(1) Method A</td>
<td>13</td>
</tr>
<tr>
<td>(2) Method B</td>
<td>18</td>
</tr>
<tr>
<td>(3) Discussion of Results</td>
<td>25</td>
</tr>
<tr>
<td>(4) Conclusions</td>
<td>30</td>
</tr>
<tr>
<td>IV. LAMINAR INCOMPRESSIBLE VISCOUS FLOW PAST A FINITE FLAT PLATE (SECOND-ORDER SOLUTIONS)</td>
<td>34</td>
</tr>
<tr>
<td>Part I: LAMINAR FLOW IN THE WAKE OF A FINITE FLAT PLATE</td>
<td>34</td>
</tr>
<tr>
<td>A. Formulation of the Problem</td>
<td>34</td>
</tr>
<tr>
<td>(1) Coordinate System</td>
<td>34</td>
</tr>
<tr>
<td>(2) Dimensionless Quantities</td>
<td>34</td>
</tr>
<tr>
<td>(3) Starting Flow Quantities</td>
<td>36</td>
</tr>
<tr>
<td>B. Solution to the Problem</td>
<td>41</td>
</tr>
<tr>
<td>Chapter</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>------</td>
</tr>
<tr>
<td>C. Discussion of the Results</td>
<td>43</td>
</tr>
<tr>
<td>D. Conclusions</td>
<td>48</td>
</tr>
<tr>
<td>Part 2: SECOND-ORDER BOUNDARY-LAYER SOLUTIONS</td>
<td>48</td>
</tr>
<tr>
<td>A. Formulation of the Problem</td>
<td>48</td>
</tr>
<tr>
<td>(1) Coordinate System</td>
<td>48</td>
</tr>
<tr>
<td>(2) Dimensionless Quantities</td>
<td>48</td>
</tr>
<tr>
<td>(3) Governing Equations</td>
<td>49</td>
</tr>
<tr>
<td>(4) Starting Quantities for Computations and a Second-Order Coefficient of Friction</td>
<td>52</td>
</tr>
<tr>
<td>B. Solutions to the Problem</td>
<td>54</td>
</tr>
<tr>
<td>C. Discussion of the Results</td>
<td>59</td>
</tr>
<tr>
<td>D. Conclusions</td>
<td>67</td>
</tr>
<tr>
<td>V. LAMINAR VISCOUS FLOW PAST A SPHERE AT HIGH MACH NUMBER</td>
<td>71</td>
</tr>
<tr>
<td>A. Formulation of the Problem</td>
<td>71</td>
</tr>
<tr>
<td>(1) Coordinate system</td>
<td>71</td>
</tr>
<tr>
<td>(2) Dimensionless Quantities</td>
<td>73</td>
</tr>
<tr>
<td>(3) Governing Equations (General)</td>
<td>74</td>
</tr>
<tr>
<td>B. Truncated Series Method of Solutions</td>
<td>75</td>
</tr>
<tr>
<td>(1) Governing Equations</td>
<td>75</td>
</tr>
<tr>
<td>(2) Boundary Conditions</td>
<td>79</td>
</tr>
<tr>
<td>C. Implicit Finite-Difference Method of Solutions</td>
<td>83</td>
</tr>
<tr>
<td>(1) Governing Equations</td>
<td>83</td>
</tr>
<tr>
<td>(2) Solution of Velocity ( u )</td>
<td>84</td>
</tr>
<tr>
<td>(3) Radial Distance to the Shock from the Center</td>
<td></td>
</tr>
<tr>
<td>Chapter</td>
<td>Page</td>
</tr>
<tr>
<td>--------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>of a Sphere (unit radius)</td>
<td>85</td>
</tr>
<tr>
<td>(4) Solutions of Velocity v and Pressure P</td>
<td>87</td>
</tr>
<tr>
<td>(5) Shear Stress</td>
<td>87</td>
</tr>
<tr>
<td>(6) Solutions of a Flow with $Re_s = \infty$</td>
<td>87</td>
</tr>
<tr>
<td>D. Discussion of the Results</td>
<td>90</td>
</tr>
<tr>
<td>E. Conclusions</td>
<td>116</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENT</td>
<td>118</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>119</td>
</tr>
<tr>
<td>VITA</td>
<td>122</td>
</tr>
<tr>
<td>APPENDIX A. DIFFERENCE EQUATION FOR THE MASS CONSERVATION</td>
<td>123</td>
</tr>
</tbody>
</table>
## LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 a-b</td>
<td>Coordinate system</td>
<td>9</td>
</tr>
<tr>
<td>3.2</td>
<td>Velocity Distribution in the Boundary-Layer on a Circular Cylinder</td>
<td>27</td>
</tr>
<tr>
<td>3.3</td>
<td>Velocity Distribution in the Boundary-Layer on a Sphere</td>
<td>28</td>
</tr>
<tr>
<td>3.4</td>
<td>Distribution of the Velocity Component Normal to the Sphere Surface in the Boundary-Layer</td>
<td>29</td>
</tr>
<tr>
<td>3.5</td>
<td>Variation of Shearing Stress at the Body Surface</td>
<td>31</td>
</tr>
<tr>
<td>3.6</td>
<td>Displacement-Thickness</td>
<td>32</td>
</tr>
<tr>
<td>4.1</td>
<td>Coordinate System</td>
<td>35</td>
</tr>
<tr>
<td>4.2 a-b</td>
<td>Distribution of Wake Velocity Behind a Finite Flat Plate</td>
<td>44-45</td>
</tr>
<tr>
<td>4.3</td>
<td>Variation of Displacement-Thickness in the Wake Behind a Finite Flat Plate</td>
<td>46</td>
</tr>
<tr>
<td>4.4</td>
<td>Variation of Velocity Along the Axis, N = 0 in the Wake Behind a Finite Flat Plate</td>
<td>47</td>
</tr>
<tr>
<td>4.5</td>
<td>Variation of Displacement-Thickness (δ)</td>
<td>57</td>
</tr>
<tr>
<td>4.6</td>
<td>Distribution of Second-Order Velocity on a Finite Flat Plate (Based on the constant δ)</td>
<td>60</td>
</tr>
<tr>
<td>4.7 a-b</td>
<td>Distribution of the Second-order Velocity Component Normal to the Finite Flat Plate</td>
<td>61-62</td>
</tr>
<tr>
<td>4.8</td>
<td>Variation of Second-Order Coefficient of Skin-Friction Over the Finite Flat Plate</td>
<td>63</td>
</tr>
<tr>
<td>4.9</td>
<td>Variation of $u_2$ as $N \to \infty$ (Based on the Constant δ and the Calculated δ)</td>
<td>65</td>
</tr>
</tbody>
</table>
## LIST OF ILLUSTRATIONS (Continued)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.10</td>
<td>Variation of $C_f$$_S$ Over the Finite Flat Plate (Based on the Constant $\delta$ and the Calculated $\delta$)</td>
<td>66</td>
</tr>
<tr>
<td>4.11 a-b</td>
<td>Distribution of the $u_0$ on a Finite Flat Plate (Based on the Constant $\delta$ and the Calculated $\delta$)</td>
<td>68-69</td>
</tr>
<tr>
<td>5.1</td>
<td>Coordinate System</td>
<td>72</td>
</tr>
<tr>
<td>5.2</td>
<td>Radial Distance to the Shock at Various Reynolds Numbers</td>
<td>92</td>
</tr>
<tr>
<td>5.3</td>
<td>Distribution of $u_1$ at Various Reynolds Numbers (Truncated Series Method)</td>
<td>93</td>
</tr>
<tr>
<td>5.4</td>
<td>Distribution of $P_1$ and $P_2$ at Various Reynolds Numbers (Truncated Series Method)</td>
<td>94</td>
</tr>
<tr>
<td>5.5a</td>
<td>Velocity Distribution in the Shock Layer for $Re_s = 49$</td>
<td>95</td>
</tr>
<tr>
<td>5.5b</td>
<td>Velocity Distribution in the Shock Layer for $Re_s = 100$</td>
<td>96</td>
</tr>
<tr>
<td>5.5c</td>
<td>Velocity Distribution in the Shock Layer for $Re_s = 900$</td>
<td>97</td>
</tr>
<tr>
<td>5.6a</td>
<td>Distribution of Velocity Component $v$ at Various Reynolds Number for $s = 0$ (Truncated Series Method)</td>
<td>98</td>
</tr>
<tr>
<td>5.6b</td>
<td>Distribution of Velocity Component $v$ at Various Reynolds Numbers for $s = .40$ (Finite-Difference Method)</td>
<td>99</td>
</tr>
<tr>
<td>5.7a</td>
<td>Distribution of Velocity Component $v$ at $Re_s = 49$</td>
<td>100</td>
</tr>
<tr>
<td>5.7b</td>
<td>Distribution of Velocity Component $v$ at $Re_s = 100$</td>
<td>101</td>
</tr>
<tr>
<td>5.7c</td>
<td>Distribution of Velocity Component $v$ at $Re_s = 900$</td>
<td>102</td>
</tr>
<tr>
<td>5.8a</td>
<td>Pressure Distribution for the Flow with $Re_s = 49$</td>
<td>103</td>
</tr>
</tbody>
</table>
LIST OF ILLUSTRATIONS (Continued)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.8b</td>
<td>Pressure Distribution for the Flow with $Re_z = 100$.</td>
<td>104</td>
</tr>
<tr>
<td>5.8c</td>
<td>Pressure Distribution for the Flow with $Re_z = 900$.</td>
<td>105</td>
</tr>
<tr>
<td>5.9a</td>
<td>Variation of Skin Friction Along the Body Surface for $Re_z = 49$.</td>
<td>107</td>
</tr>
<tr>
<td>5.9b</td>
<td>Variation of Skin Friction for $Re_z = 100$.</td>
<td>108</td>
</tr>
<tr>
<td>5.9c</td>
<td>Variation of Skin Friction for $Re_z = 900$.</td>
<td>109</td>
</tr>
<tr>
<td>5.9d</td>
<td>Variation of Skin Friction at Various Reynolds Numbers.</td>
<td>110</td>
</tr>
<tr>
<td>5.10a</td>
<td>Variation of Surface Pressure Along the Body Surface for $Re_z = 49$.</td>
<td>112</td>
</tr>
<tr>
<td>5.10b</td>
<td>Variation of Surface Pressure Along the Body Surface for $Re_z = 100$.</td>
<td>113</td>
</tr>
<tr>
<td>5.10c</td>
<td>Variation of Surface Pressure Along the Body Surface for $Re_z = 900$.</td>
<td>114</td>
</tr>
<tr>
<td>5.10d</td>
<td>Variation of Surface Pressure Along the Body Surface for Various Reynolds Numbers</td>
<td>115</td>
</tr>
</tbody>
</table>
NOMENCLATURE

\( a = 1 \) ...................... Radius of a cylinder or a sphere (see Figure 5.1)

\( A_n, B_n, C_n, F_n \) ................ Coefficients in the difference equations (3.6a) or (5.26a) and defined by equations (3.6b-e), (4.14a-d), (4.36a-d) and (5.26b-e)

\( D_n, E_n \) ......................... Coefficients in equation (3.12) and defined by equations (3.14a-b)

c ................................. Shock radius

c_p ................................. Specific heat at constant pressure

c_v ................................. Specific heat at constant volume

\( C_f \) ................................. Coefficient of friction defined by equation (4.20a)

\( C_{f_1}, C_{f_2} \) ..................... First- and second-order coefficients of friction defined by equations (4.33a-c)

\( C_{f_3} \) ................................. Second-order coefficients of friction due to a concentrated force at the leading edge of the plate

f ................................. Functions defined by equations (3.20), (3.29) and (4.6)

\( f_0, f_3, f_6 \) ......................... Functions defined by equation (4.6)
Function defined by Blasius equation (4.25)

Function defined by equation (4.26b)

Typical quantity in the boundary-layer

Constant which equals 0 for plane flow and 1 for axisymmetric flow

Coefficients defined by equations (4.16a-d)

Density ratio at the shock defined by equation (5.1b)

Coefficients defined by equation (4.16a)

Dimensionless length of the finite flat plate (see Figure 4.1)

Free stream Mach number

Dimensionless coordinate normal to the body surface defined by equations (4.20b) or (5.1c)

Coordinate normal to body surface defined by equation (3.5d)

Coordinate normal to body surface defined by equation (5.7b)

Dimensionless pressure defined by equation (3.5e)

Dimensionless surface pressure
\[ P_1, P_2; \overline{P}_1, \overline{P}_2 \] Coefficients of truncated series for pressure defined by equations (5.3a) and (5.8a)

\[ Q, \overline{Q} \] Derivative of \( u \) with respect to \( n \) defined by equations (5.6c) and (5.9c)

\[ r \] Dimensionless distance from axis of symmetry to body surface shown in Figure 5.1

\[ R \] Radial distance from the center of a sphere defined by equation (5.1e)

\[ \text{Re} \] Reynolds number defined by equations (3.5f) and (4.20d)

\[ \text{Re}_s \] Shock Reynolds number defined by equation (5.1f)

\[ \text{Rs} \] Radial distance from the center of a sphere (unit radius) to the shock

\[ s \] Dimensionless coordinate along the body surface defined by equation (3.5a)

\[ u \] Velocity component in the \( s \)-direction defined by equation (3.5b)

\[ u_1, u_2 \] First- and second-order velocity component in the \( s \)-direction expressed in equations (4.21a-b)

\[ u_1, \overline{u}_1 \] Coefficients of truncated series for velocity component \( u \) defined by equa-
tions (5.3b) and (5.7c)

Potential velocity defined by equation (3.5g)

Second-order velocity component in the s-direction in the outer flow

Velocity component in the n-direction defined by equation (3.5c)

Coefficients of truncated series for velocity component v defined by equations (5.3c) and (5.7d)

Constant expressed in equation (4.21e)

Ratio of specific heats $c_p/c_v$

Displacement-thickness defined by equation (3.5i)

Arbitrarily small quantity

Nondimensional distance between the gridpoints in the n-direction

Nondimensional distance between the gridpoints in the N-direction

Nondimensional distance between the gridpoints in the s-direction

A function defined by equation (4.13b)

"Backward" difference defined by equations (3.32a-c)

Dimensionless coordinate defined by
equations (4.4b) and (5.1j)

\( \mu \) ........................................ Viscosity coefficient defined by equation (5.1i)

\( \xi \) ........................................ Dimensionless coordinate defined by equation (4.4a)

\( \xi_s \) ........................................ Coordinate measured from the leading edge of the finite flat plate along the plate as shown in Figure 4.5

\( \rho \) ........................................ Dimensionless density defined by equation (5.1h)

\( \rho_s \) ........................................ Density evaluated at the shock

\( \nu_s \) ........................................ Kinematic viscosity evaluated at the shock defined by equation (5.1k)

\( \epsilon \) ........................................ A parameter defined by equations (4.29b) and (5.7g)

\( \tau_f \) ........................................ Shear stress defined by equations (3.5h) and (4.20g)

\( \psi \) ........................................ A function defined by equations (4.4c) and (5.17b)

\( \psi_1 \) ........................................ First-order stream function defined by equation (4.20e)

\( \psi_2 \) ........................................ Second-order stream function near to the leading edge of the plate defined by equation (4.20f)

\( \psi_0, \psi_1, \ldots, \psi_8 \) ........................ Functions defined by equations (4.12a-i)
Subscripts:
- \( e \) Condition at the edge of boundary-layer
- \( H \) Homogeneous solution
- \( P \) Particular solution
- \( b \) Condition at the body surface
- \( m \) Designation of the gridpoint in the \( s \)-direction, see Figures 3.1 a-b
- \( n \) Designation of the gridpoints in the \( N \)-direction, see Figures 3.1 a-b
- \( s \) Condition at the shock
- \( os \) Condition ahead of the shock
- \( ls \) Condition after the shock
- \( \infty \) Condition at free stream

Superscripts:
- \( * \) Dimensional quantity
- \( ' \) Differentiation with respect to the given argument
- \( - \) Dimensionless quantity defined by equation (5.7a-f)

Other Notations:
A coordinate used as a subscript means a partial differentiation with respect to the coordinate.
In recent years the problem of finding exact solutions to the laminar boundary-layer equations by finite-difference methods has become the subject of intensive research. This is due to the non-linearity of the boundary-layer equations and the difficulty of obtaining analytical solutions except in certain special cases such as those where self-similar solutions can be found. Since the boundary-layer equations are of the parabolic type, solutions can be obtained by finite-difference methods. These methods consist of step-by-step integration starting from the stagnation-point and proceeding downstream along the body surface. For partial differential equations of the elliptic type, solutions must be found for the total flow region at once and therefore step-by-step computation is not applicable since it would result in an unstable numerical scheme. In applying finite-difference methods to partial differential equations of the parabolic type, we replace the derivatives (derivative with respect to the coordinates parallel and normal to the body) by difference quotients, and then solve the resulting difference equations step-by-step along the body surface.

The boundary-layer equations can be converted to difference equations of either explicit or implicit type depending on whether

---

*A solution is regarded as exact if it is accurate either from the analytical or the numerical point of view.*
forward or backward differences are taken in the coordinate along
the body surface. If forward difference quotients are taken,
we obtain difference equations of the explicit type. If backward
difference quotients are used we obtain difference equations
of the implicit type. The explicit method of solution to the boundary-
layer equations has been studied by Baxter and Flügge-Lotz\textsuperscript{(1)},
Flügge-Lotz and Yu\textsuperscript{(5)}, and others. It was found from their studies
that the method has limited applications due to the severe require-
ments for stability placed on the difference equations\textsuperscript{(3)}. The
implicit finite-difference method of solution to the boundary-layer
equations has been investigated by Flügge-Lotz and Blottner\textsuperscript{(4)},
Smith and Clutter\textsuperscript{(18,19)}, Davis and Flügge-Lotz\textsuperscript{(3)}, and others, and
has met with considerable success. It has been found that in an
implicit finite-difference method, a smaller step-size in the computation
gives more accurate solutions. On the other hand, in an explicit
finite-difference method the step-size requirements are usually so
severe that it is very difficult to obtain both stable and accurate
solutions at the same time. In this paper the implicit finite-
difference method due to Davis and Flügge-Lotz\textsuperscript{(3)} is used as it is
found to be both stable and accurate.

In applying the implicit finite-difference method, the
boundary-layer is divided into a rectangular grid which consists of
lines parallel and normal to the body surface (see Figure 3.1).
The differential equations are then integrated for each gridpoint
starting from the stagnation-point and proceeding downstream. In order to obtain a practical computation, the differential equations are linearized to form a number of algebraic equations which can be solved simultaneously. The solution to these algebraic equations are repeated at each downstream step. The computation can be carried out easily on an electronic digital computer as it is most suited for the repeated computation.

In order to start the computation by the implicit finite-difference method, it is necessary to obtain starting flow quantities at the stagnation-point. This can be obtained by representing the flow properties in series form such as a Blasius series or a truncated series.

This paper includes three problems, (1) laminar incompressible viscous flow past a cylinder and a sphere, (2) laminar incompressible viscous flow past a finite flat plate (second-order solutions), and (3) laminar viscous flow past a sphere at a high Mach number. The flow in the boundary-layer can be laminar, transitional and turbulent. We restrict the problems to laminar flow as it can be solved analytically without resorting to experimental results. Furthermore, the study of laminar flow is important because laminar flow results in a lower skin-friction and better thermal insulation to the body than the corresponding turbulent flow does (1).

The first problem (flow past a sphere and cylinder) involves
the classical boundary-layer equations. These equations are the first approximation to the Navier-Stokes equations in a region near to the body surface for high Reynolds number. The solutions to these equations are obtained by an implicit finite-difference method. Another approach to the problem was attempted by linearizing the boundary-layer equations and then writing only the derivatives along the body surface in difference form. We then attempted to solve the resulting difference-differential equation numerically. The solution encountered difficulties due to the fact that very accurate first derivatives of the velocity at the wall are necessary in order to obtain accurate solutions. This requires a time-consuming technique of trial and error and therefore the method proved less desirable than the finite-difference method.

The second problem (second-order flow past a finite flat plate) involves the second-order boundary-layer equations which introduce only the effect of the displacement-thickness in the case of flow past a flat plate. An assumption is made that the displacement-thickness is constant in the wake behind the flat plate. The adequacy of this assumption is checked from solutions based on the calculated displacement-thickness in the wake. The wake behind the finite flat plate is assumed laminar, and its displacement-thickness is computed downstream by using the implicit finite-difference method.
In the third problem (high Mach number flow past a sphere), constant density is assumed in the shock layer. This is nearly true in the stagnation-point region especially if the flow is hypersonic and the temperature of the sphere is nearly the same as the stagnation-temperature. It is also assumed that the shock is nearly spherical, even though it is not spherical as it is in the inviscid case. The numerical results will show that the assumption of a spherical shock will, however, nearly be true. This problem involves the solution of the complete Navier-Stokes equations. These equations are solved for various Reynolds numbers by two methods; namely the truncated series method and the implicit finite-difference method. Results for flow at Reynolds number 900 are compared with the solutions by Lighthill\(^\text{12}\) for constant-density inviscid flow past a sphere.

The solutions by the implicit finite-difference method are in excellent accord with those obtained by the series solutions in the stagnation-point region. As the computation by the finite-difference method proceeds downstream, the deviation of the finite-difference solution from the series truncation solution increases. This is due to the fact that the series is valid only around the stagnation-point, and is thus expected to give inaccurate solutions downstream. The finite-difference method has no such restrictions, however, and gives accurate results in the whole flow field.
Implicit finite-difference methods of solution to the boundary-layer equations have been studied by Flügge-Lotz and Blottner (4), Smith and Clutter (18,19), Davis and Flügge-Lotz (3), and others. The method due to Davis and Flügge-Lotz involves the use of three-point differences in the s-direction to produce truncation errors of order $\Delta s^2$ rather than of order $\Delta s$ as contained in the Flügge-Lotz and Blottner (4) method. Advantages (3) of this three-point scheme over the two-point scheme are that truncation errors in the s and n directions are of the same order, and that the method produces a solution which is highly accurate with a fairly large step size. Solutions by the implicit finite-difference method have proven not only to be accurate but also to be stable in all examples computed. The summary of the method is presented in Chapter III.

The explicit finite-difference method of solution has been studied by Baxter and Flügge-Lotz (1), Flügge-Lotz and Yu (5) and others. This method encountered difficulties in pursuing stability of the finite-difference equations, and thus has limited applications. Hence, this method is not used in this paper.

In order to use the finite-difference method in a computation, starting data such as the initial velocities at the stagnation-
region are necessary. These are expressed in a series such as the Blasius series or the truncated series. Tifford\(^{20}\) has solved for the coefficients of the first six terms of the Blasius series for plane flow from which the starting data for the calculation of the flow past a cylinder is obtained. Similarly, Frössling\(^{(6)}\) has given several coefficients of the Blasius series for axisymmetric flow from which the boundary-layer flow past a sphere can be calculated. The starting data for the calculation of the wake velocity was obtained from a power series solution given by Goldstein\(^{(7)}\). This series is obtained by expanding in the distance measured from the trailing edge of the plate. The initial velocity distribution for the flow past a sphere at a high Mach number is assumed expressible by a truncated series. In obtaining this series, one is guided by the constant-density solutions of Lighthill\(^{(12)}\) and the solution of the compressible problem by Kao\(^{(10)}\), who also used the truncated series method.
CHAPTER III

IMPLICIT FINITE-DIFFERENCE METHOD OF SOLUTION

This chapter presents a summary of an implicit finite-difference scheme due to Davis and Flügge-Lotz\(^{(3)}\), and its application to the problem of the flow of an incompressible viscous fluid past a cylinder and a sphere.

A. Implicit Finite-Difference Relations

Only a brief summary of the implicit finite-difference method is presented in this section since details of the method have been presented by Davis and Flügge-Lotz\(^{(3)}\).

We construct a rectangular grid in the flow field as shown in Figure 3.1 a-b. The orthogonal curvilinear coordinates \((s, n)\) are parallel and normal to the body surface, respectively. The point of intersection of the grid lines \(m\) and \(n\) is identified by the subscripts \(m, n\). We assume that all necessary data are given at gridpoints along the lines \(m - 1\) and \(m\), and that the quantities at the section \(m + 1\) are to be computed. Then the three-point implicit difference scheme gives the following difference quotients for the evaluation of derivatives of a typical quantity \(F\) at point \(m + 1, n\).

\[
\frac{\partial F}{\partial s} = \frac{3 F_{m+1, n} - 4 F_{m, n} + F_{m-1, n} + \frac{1}{3} (\Delta s)^2 F_{nnn} + ...}{2\Delta s} \tag{3.1 a}
\]

\[
\frac{\partial F}{\partial n} = \frac{F_{m+1, n+1} - F_{m+1, n-1} - \frac{1}{6} (\Delta n)^2 F_{nnn} + ...}{2\Delta n} \tag{3.1 b}
\]
Figure 3.1 a-b  Coordinate System
\[ \frac{\partial^2 F}{\partial n^2} = \frac{F_{m+1, n+1} - 2F_{m+1, n} + F_{m+1, n-1}}{(\Delta n)^2} - \frac{1}{12}(\Delta n)^2 F_{nnn} + \ldots \]  

(3.1 c)

\[ F_{m+1, n} = 2F_m, n - F_{m-1, n} + (\Delta s)^2 F_{ss} + \ldots \]  

(3.1 d)

The terms with order higher than \((\Delta s)^2\) or \((\Delta n)^2\) are negligibly small if the step size is small enough and therefore are not taken into account when the above relations are substituted in the differential equations. In order to linearize a quantity such as \(u \frac{\partial u}{\partial s}\) in the differential equations the following relation is used:

\[ \left( u \frac{\partial u}{\partial s} \right)_{m+1, n} = \left( 2u_m, n - u_{m-1, n} \right) \left( \frac{3u_{m+1, n} - 4u_m, n + u_{m-1, n}}{2\Delta s} \right) + 0(\Delta s^2) \]  

(3.1 e)

B. Relations used for Numerical Integration and the Computation of Shear

The relation necessary for the integration of the continuity equation and the \(s\) - momentum equation are derived as follows. We assume that an integrand \(F\) can be approximated by a second order polynomial

\[ F = \alpha_1 + \alpha_2 n + \alpha_3 n^2 \]  

(3.2 a)

and denote the integration by

\[ I = \int_0^n Fdn \]  

(3.2 b)

Here \(\alpha_1, \alpha_2\) and \(\alpha_3\) can be evaluated by solving three algebraic equations which satisfy equation (3.2 a) at various points, say \(n, n + \Delta n,\) and \(n + 2\Delta n.\) Then, the integration from \(n\) to \(n + \Delta n\) gives:
\[ I(n + \Delta n) = I(n) + \frac{\Delta n}{12} \left[ 5 F(n) + 8 F(n + \Delta n) - F(n + 2\Delta n) \right] \]

(3.2 c)

The following difference equation for \( \left( \frac{\partial u}{\partial n} \right) \) is used for the evaluation of the shear at the body surface,

\[ \left( \frac{\partial u}{\partial n} \right)_{n=0} = \frac{1}{6\Delta n} \left[ -11 u(0) + 18 u(\Delta n) - 9 u(2\Delta n) + 2 u(3\Delta n) \right] \]

(3.2 d)

C. Illustrative Example - The Solution of the Laminar Boundary-Layer Equations

The implicit finite-difference methods of solution suitable for solving parabolic type partial differential equations are best illustrated by solving the boundary-layer equations (17):

\[ u \frac{\partial u}{\partial s} + v \frac{\partial u}{\partial N} = - \frac{dp}{ds} + 2 \frac{\partial^2 u}{\partial N^2} \]

(3.4 a)

\[ \frac{\partial}{\partial s} (r^j u) + \frac{\partial}{\partial N} (r^j v) = 0 \]

(3.4 b)

with the boundary conditions

\[ u, v = 0 \text{ at } N = 0 \]

(3.4 c)

\[ u = U \text{ as } N \rightarrow \infty \]

(3.4 d)

The curvilinear coordinate system \((s^*, n^*)\) is adopted in the above equations. The coordinate \(s^*\) is the distance measured along the body surface from the stagnation point, and \(n^*\) is the normal distance measured from the body surface (see Figure 5.1).

In this section dimensionless quantities are denoted as follows. Quantities without a superscript * are dimensionless, while
those with a superscript $*$ are dimensional. The subscript $\infty$ denotes a free stream quantity.

$$s = \frac{s^*}{a^*} \quad \text{coordinate along body surface} \quad (3.5 \text{ a})$$

$$u = \frac{u^*}{U^*_\infty} \quad \text{velocity component parallel to body surface} \quad (3.5 \text{ b})$$

$$v = \text{Re} \frac{1}{2} \frac{v^*}{U^*_\infty} \quad \text{velocity component normal to body surface} \quad (3.5 \text{ c})$$

$$N = \text{Re} \frac{1}{2} \frac{N^*}{a^*} \quad \text{coordinate normal to body surface} \quad (3.5 \text{ d})$$

$$P = \frac{p^*}{\rho^* U^*_\infty^2} \quad \text{pressure} \quad (3.5 \text{ e})$$

$$\text{Re} = \frac{2U^*_\infty a^*}{v^*} \quad \text{Reynolds number} \quad (3.5 \text{ f})$$

$$U = \frac{U^*}{U^*_\infty} \quad \text{potential velocity parallel to the wall} \quad (3.5 \text{ g})$$

$$\tau_f = \sqrt{2} \text{Re} \frac{1}{\rho^* U^*_\infty^2} \frac{\tau^*}{v^*} \quad \text{shear stress} \quad (3.5 \text{ h})$$

$$\delta = \text{Re} \frac{1}{2} \frac{\delta^*}{a^*} \quad \text{displacement-thickness} \quad (3.5 \text{ i})$$
Method A

Replacing terms \( \frac{\partial u}{\partial s}, \frac{\partial u}{\partial N}, \frac{\partial^2 u}{\partial N^2} \), and \( v \) in equation (3.4 a) by the relations (3.1 a-d) respectively, we obtain the following recurrence formula in terms of the velocity \( u_{m+1} \):

\[
A_n u_{m+1, n-1} + B_n u_{m+1, n} + C_n u_{m+1, n+1} + D_n = F_n
\]

where

\[
A_n = -\frac{\Delta s}{2\Delta N} (2 v_{m,n} - v_{m-1,n}) - 2 \frac{\Delta s}{\Delta N^2} \tag{3.6 b}
\]

\[
B_n = \frac{3}{2} (2 u_{m,n} - u_{m-1,n}) + 4 \frac{\Delta s}{\Delta N^2} \tag{3.6 c}
\]

\[
C_n = -A_n - 4 \frac{\Delta s}{\Delta N^3} \tag{3.6 d}
\]

\[
F_n = \frac{1}{2} (2 u_{m,n} - u_{m-1,n}) (4 u_{m,n} - u_{m-1,n}) - \Delta s \left( \frac{dp}{ds} \right)_{m+1,n} \tag{3.6 e}
\]

It is noted that the coefficients \( A_n, B_n, C_n, \) and \( F_n \) are functions of the quantities at sections \( m \) and \( m - 1 \), and therefore they can be computed for each grid point located at station \( m + 1 \). The term \( \left( \frac{dp}{ds} \right)_{m+1, n} \) in the relation (3.6 e) can be evaluated as follows.

In the potential flow, \( u = U, v = 0 \). Thus equation (3.4 a) reduces to

\[
\frac{dp}{ds} = -U \frac{du}{ds} \tag{3.7}
\]

The flow past a cylinder gives

\[
U = \sin s \tag{3.8}
\]

and therefore
\[
\frac{dp}{ds} = -2 \sin(2s) \tag{3.9}
\]

The flow past a sphere gives
\[
U = \frac{3}{2} \sin s \tag{3.10}
\]
and therefore
\[
\frac{dp}{ds} = -\frac{9}{8} \sin(2s) \tag{3.11}
\]

We assume that a linear relation exists between \(u_{m+1, n+1}\) and \(u_{m+1, n}\) as follows,
\[
u_{m+1, n} = E_n u_{m+1, n+1} + D_n \tag{3.12}
\]
Introducing the above relation into equation (3.6a), we obtain
\[
u_{m+1, n} = \frac{-C_n}{B_n + A_n E_n - 1} u_{m+1, n+1} + \frac{F_n - A_n D_n - 1}{B_n + A_n E_n - 1} \tag{3.13}
\]

It is found from equations (3.12) and (3.13) that
\[
E_n = \frac{-C_n}{B_n + A_n E_n - 1} \tag{3.14a}
\]
and
\[
D_n = \frac{F_n - A_n D_n - 1}{B_n + A_n E_n - 1} \tag{3.14b}
\]

The no-slip boundary condition \(u = 0\) at the wall applied to equation (3.12) leads to
\[
E_0 = D_0 = 0 \tag{3.15}
\]
where the subscript \(o\) denotes the quantities evaluated at the wall.
Therefore, the quantities of $E_n$ and $D_n$ at each gridpoint located at station $m + 1$ can be obtained if we repeat the computation of (3.14) starting from the wall and proceeding toward the potential velocity. Introducing the quantities $E_n$ and $D_n$ into (3.12), we can compute $u$ at station $m + 1$ in a similar way as above except that we start from the outer edge of the boundary-layer and proceed toward the wall.

The velocity component $v$ can now be integrated from equation (3.4 b) as follows,

$$v = -\int_0^N \frac{\partial u}{\partial s} \, dN - \frac{\partial r}{\partial s} \int_0^N u \, dN \quad (3.16 \, a)$$

For a flow past a cylinder ($j = 0$)

$$v = -\int_0^N \frac{\partial u}{\partial s} \, dN \quad (3.16 \, b)$$

For a flow past a sphere ($j = 1$)

$$v = -\int_0^N \frac{\partial u}{\partial s} \, dN - \tan s \int_0^N u \, dN \quad (3.16 \, c)$$

where the relation $r = \sin s$ is used. The above integration is carried out numerically by using formula (3.2 c). The integrand $\frac{\partial u}{\partial s}$ is computed by using (3.1 a) for each gridpoint located at the station $m + 1$.

The normal velocity component $v$ for large $N$ can be
derived from equation (3.16 a) as follows. Since U is not a function of N, equation (3.16 a) reduces to

$$v_{m+1, N + \Delta N} = v_{m+1, N} - \left( \frac{dU}{ds} \right) \Delta N - \frac{\partial r_j}{\partial s} U \Delta N$$  \hspace{1cm} (3.17 a)

For a flow past a cylinder ($U = 2 \sin s$)

$$v_{m+1, N + \Delta N} = v_{m+1, N} - 2 \Delta N \cos s$$  \hspace{1cm} (3.17 b)

For a flow past a sphere ($U = \frac{3}{2} \sin s$)

$$v_{m+1, N + \Delta N} = v_{m+1, N} - 3 \Delta N \cos s$$  \hspace{1cm} (3.17 c)

In the numerical computation, we assume that the edge of the boundary-layer exists where the following condition is satisfied,

$$\left| u_{m+1, n_e - 10} - u_{m+1, n_e - 11} \right| < \delta'$$  \hspace{1cm} (3.18)

Here $m + 1, n_e$ denotes the gridpoint at the outer edge of the boundary-layer, and $\delta'$ denotes an arbitrarily small quantity. ($\delta' = 0.00005$ is used in this problem.) If the above condition is not satisfied, we increase the thickness of the boundary-layer until it is satisfied. This is carried out as follows. At stations $m$ and $m - 1$, the velocities at the outer edge of the boundary-layer are computed. (Say, five steps beyond the point at which the edge condition above is satisfied.) In this way, the velocities at the outer edge of the boundary-layer at stations $m$ and $m - 1$
are available for the computation at station \( m + 1 \) where the condition (3.18) may not be satisfied and the boundary-layer thickness must be increased.

The starting quantities for \( u \) and \( v \) for the computation are obtained from the Blasius series\(^{(17)}\). This can be written as follows for the flow near to the stagnation-point.

\[
\begin{align*}
\text{u}^* &= u_1^* s^* f'(N) \quad (3.19 \ a) \\
\text{v}^* &= -\sqrt{\frac{\text{u}^*}{u_1^*}} u_1^* f(N) \quad (3.19 \ b)
\end{align*}
\]

where

\[
\begin{align*}
u_1^* &= 2 \frac{U_\infty^*}{a^*} \quad (3.19 \ c)
\end{align*}
\]

The functions \( f(N) \) and \( f'(N) \) are the solutions of the following differential equations,

for a flow past a cylinder

\[
f^3 - f f'' = 1 + f''' \quad (3.20)
\]

for a flow past a sphere

\[
f''' = -f f'' + \frac{1}{2} (f')^2 - 1 \quad (3.21)
\]

Equations (3.20-21) have been evaluated by Tifford\(^{(20)}\) and Frossling,\(^{(6)}\) respectively. In dimensionless form, equations (3.19 a-b) are expressed as

\[
\begin{align*}
u &= 2 s f'(N) \quad (3.22 \ a) \\
v &= -2 f(N) \quad (3.22 \ b)
\end{align*}
\]
The displacement-thickness and the skin friction are obtained from the following equations,

Displacement-thickness:

$$\delta = \int_0^\infty \left( 1 - \frac{u}{U} \right) dN$$  \hspace{1cm} (3.23)

Skin friction:

$$\begin{bmatrix} r_f \end{bmatrix}_b = 2 \sqrt{2} \left( \frac{\partial u}{\partial N} \right)_{N=0}$$  \hspace{1cm} (3.24)

Equations (3.23-24) can be computed by using (3.2 c) and 3.1 b), respectively.

(2) Method B

Replacing the term $\frac{\partial u}{\partial s}$ in equation (3.4 a) by three-point s-differences (3.1 a) gives

$$\left( \frac{d^2 u}{dN^2} \right)_{m+1, n} - \frac{1}{2} v_{m+1, n} \left( \frac{d u}{dN} \right)_{m+1, n} - \frac{3}{4\Delta s} u_{m+1, n} + \frac{4 u_{m,n} - u_{m-1,n}}{4\Delta s} u_{m+1, n} = \frac{1}{2} \left( \frac{dP}{ds} \right)_{m+1, n}$$

where the N-derivatives are taken at constant $s$. The above equation is nonlinear and thus superposition of homogeneous and particular solutions as in the case of linear differential equation is not
The above equation can be solved by using Smith and Clutter's method (18, 19). The method involves a two-point boundary value problem in which the boundary conditions at the wall (i.e. \( u = v = 0 \) at \( n = 0 \)), and at infinity (i.e. \( u = U \) as \( n \to \infty \)) must be satisfied. The computation to satisfy the latter boundary condition involves a trial and error technique. In order to avoid this time-consuming computation, we attempt to linearize equation (3.4 a) so that superposition of the homogeneous and particular solutions can be applied. Replacing the nonlinear term \( (u \frac{\partial u}{\partial s})_{m+1,n} \) in equation (3.4 a) by the difference relation (3.1 e), equation (3.4 a) reduces to the following linear difference-differential equation,

\[
\left( \frac{d^2 u}{dn^2} \right)_{n+1, m} - \frac{1}{2} (2 v_{m,n} - v_{m-1, n}) \left( \frac{du}{dn} \right)_{m+1, n} - \frac{3}{4\Delta s} \left( \frac{2 u_{m,n} - u_{m-1, n}}{4\Delta s} \right) (4 u_{m,n} - u_{m-1, n}) - \frac{1}{2} \left( U \frac{du}{ds} \right)_{m+1, n} = 0
\]

(3.25)

In this section we attempt to solve the above equation for its particular \( (u_p) \) and homogeneous \( (u_H) \) solution. The solution for \( u \) then is given by

\[
u = u_p + Au_H
\]

(3.26)

Here \( A \) is a constant which can be obtained from the boundary condition at infinity, i.e.

\[
u_p(\infty) + Au_H(\infty) = U
\]

(3.27 a)

as \( n \) approaches infinity. Thus,

\[
A = \frac{U - \nu_p(\infty)}{u_H(\infty)}
\]

(3.27 b)
The no-slip boundary condition at the wall gives
\begin{align}
  u_p(0) &= 0 \quad (3.28\, a) \\
  u_H(0) &= 0 \quad (3.28\, b)
\end{align}

Following Smith and Clutter's\cite{18,19} predictor and corrector technique, the velocity $u_{m+1,\, n+1}$ and its derivatives $u'_{m+1,\, n+1}$ are extrapolated (predicted) and then interpolated (corrected) by applying Falkner's\cite{2} formulas. For an equation of type

$$u^m = f(N, u(N), u'(N)) \quad (3.29)$$

Falkner's formula gives the following relations,

**Extrapolation:**

\begin{align}
  u_{m+1,\, n+1} &= u_{m+1,\, n} + \Delta N \left( \frac{du}{dN} \right)_{m+1,\, n} + (\Delta N)^2 \left( \frac{1}{2} f_n + \frac{1}{6} \nabla f_n + \frac{1}{8} \nabla^2 f_n + \frac{19}{180} \nabla^3 f_n + \ldots \right) \\
  \left( \frac{du}{dN} \right)_{m+1,\, n+1} &= \left( \frac{du}{dN} \right)_{m+1,\, n} + \Delta N \left[ f_n + \frac{1}{2} \nabla f_n + \frac{5}{12} \nabla^2 f_n + \frac{3}{8} \nabla^3 f_n - \ldots \right] \quad (3.30\, a)
\end{align}

**Interpolation:**

\begin{align}
  u_{m+1,\, n+1} &= u_{m+1,\, n} + \Delta N \left( \frac{du}{dN} \right)_{m+1,\, n} + (\Delta N)^2 \left[ \frac{1}{2} f_n + 1 - \frac{1}{3} \nabla f_n + 1 - \frac{1}{24} \nabla^2 f_n + 1 - \frac{7}{360} \nabla^3 f_n + 1 - \ldots \right] \\
  \left( \frac{du}{dN} \right)_{m+1,\, n+1} &= \left( \frac{du}{dN} \right)_{m+1,\, n} + \Delta N \left[ f_n + 1 - \frac{1}{3} \nabla f_n + 1 - \frac{1}{24} \nabla^2 f_n + 1 - \frac{7}{360} \nabla^3 f_n + 1 - \ldots \right] \quad (3.31\, a)
\end{align}
\[
\left( \frac{du}{dN} \right)_{m+1, n+1} = \left( \frac{du}{dN} \right)_{m+1, n} + \Delta N \left[ f_n + 1 - \frac{1}{2} \nabla f_n + 1 - \frac{1}{12} \nabla^2 f_n + 1 - \cdots \right]
\]

where

\[
\nabla f_n = f_n - f_n - 1 \quad (3.32a)
\]

\[
\nabla^2 f_n = f_n - 2f_n - 1 + f_n - 2 \quad (3.32b)
\]

\[
\nabla^3 f_n = f_n - 3f_n - 1 + 3f_n - 2 - f_n - 3 \quad (3.32c)
\]

Comparing equation (3.29) with (3.25), we obtain the following relations for the particular and homogeneous solutions. To obtain the particular solution of (3.25), we let

\[
f_n = \frac{1}{2} \left( 2v_{m,n} - v_m - 1,n \right) \left( \frac{du}{dN} \right)_{m+1, n} + \frac{3}{4\Delta s} \left( 2u_{m,n} - u_{m-1,n} \right) u_{m+1,n} - \left( \frac{2u_{m,n} - u_{m-1,n}}{4\Delta s} \right) (4u_{m,n} - u_{m-1,n}) - \frac{1}{2} \left( U \frac{dU}{ds} \right)_{m+1, n} \quad (3.33a)
\]

and to obtain the homogeneous solution,

\[
f_n = \frac{1}{2} \left( 2v_{m,n} - v_m - 1,n \right) \left( \frac{du}{dN} \right)_{m+1, n} + \frac{3}{4\Delta s} \left( 2u_{m,n} - u_{m-1,n} \right) u_{m+1,n} \quad (3.33b)
\]

We assume that the velocity components \( u \) and \( v \) at stations \( m \) and \( m - 1 \) are known quantities which may be the starting data for the
computation or may have been computed at the previous stations, m and m - 1. We also assume that the quantities \( \left( \frac{du}{dN} \right)_{m+1,n} \) and \( u_{m+1,n} \) have been computed.

The quantities \( \nabla f_n, \nabla^2 f_n \) and \( \nabla^3 f_n \) of (3.32 a-c) can be obtained by using (3.33 a) and (3.33 b). Substituting the above quantities into (3.30 a-b) gives the extrapolated quantities of \( u_{m+1, n+1} \) and \( \left( \frac{du}{dN} \right)_{m+1, n+1} \). Based on these predicted quantities we calculate \( \nabla f_{n+1}, \nabla^2 f_{n+1} \) and \( \nabla^3 f_{n+1} \). We then obtain the corrected quantities of \( u_{m+1, n+1} \) and \( \left( \frac{du}{dN} \right)_{m+1, n+1} \), \( n+1 \) from the interpolation formula (3.31 a-b). The computations following the above procedure are repeated independently for the particular and homogeneous solutions. In each solution, the computation starts from the body surface and proceeds outward to the outer edge of the boundary-layer. We can then compute the constant \( A \) by substituting the velocity components at the outer edge of the boundary-layer, i.e. \( u_p(\infty) \) and \( u_H(\infty) \) into (3.27 b). Thus, the velocity component \( u \) can be obtained from equation (3.26).

The starting data at sections \( m - 1 \) and \( m \) for the computations are known quantities which may be initial values from (3.22 a-b) or the quantities obtained in the previous steps of computation. In addition to the above, we need additional starting data at station \( m + 1 \) in the flow field adjacent to the body surface.
We assume that a no-slip boundary condition exists at the body surface. We impose a condition for \( \frac{du}{dN} \) at the wall as follows. For the particular solution:

\[
    u_p = 0 \quad (3.34 \text{ a})
\]

and

\[
    \left( \frac{du_p}{dN} \right)_{m+1,o} = 0 \text{ at the body surface.} \quad (3.34 \text{ b})
\]

For the homogeneous solution:

\[
    u_H = 0 \quad (3.34 \text{ c})
\]

and

\[
    \left( \frac{du_H}{dN} \right)_{m+1,o} = 1 \text{ at the body surface.} \quad (3.34 \text{ d})
\]

The quantities \( u \) and \( \frac{du}{dN} \) at gridpoint \((m+1, l)\) can now be approximated by the following Maclaurin series which retains terms through \( \frac{d^4u}{dN^4} \),

\[
    u_{m+1, l} = u_{m+1,o} + \frac{\Delta N}{1!} \left( \frac{du}{dN} \right)_{m+1, o} + \frac{(\Delta N)^2}{2!} \left( \frac{d^2u}{dN^2} \right)_{m+1, o} + \ldots
\]

\[
    \left( \frac{d^3u}{dN^3} \right)_{m+1, o} + \ldots
\]

\[
    \left( \frac{du}{dN} \right)_{m+1, l} = \left( \frac{du}{dN} \right)_{m+1,o} + \frac{\Delta N}{1!} \left( \frac{d^2u}{dN^2} \right)_{m+1, o} + \frac{(\Delta N)^2}{2!} \left( \frac{d^3u}{dN^3} \right)_{m+1, o} + \ldots
\]

\[
    \left( \frac{d^3u}{dN^3} \right)_{m+1, o} + \ldots
\]

\[
    \left( \frac{d^4u}{dN^4} \right)_{m+1, o} + \ldots
\]
Here the derivatives \( \frac{d^3u}{dN^3} \) and \( \frac{d^4u}{dN^4} \) can be obtained by taking higher derivatives of equation (3.29) in which the function \( f \) represents (3.33 a-b).

Starting from gridpoint \((m + 1, 1)\) and proceeding toward the edge of the boundary-layer, we apply Falkner's formula (3.30 a-b) and (3.31 a-b) for the computation of \( u \) and \( \frac{du}{dN} \). We retain terms through \( \nabla f_n \) for the computation at gridpoint \((m + 1, 2)\) and for the gridpoint \((m + 1, 3)\) we retain terms through \( \nabla^2 f_n \). Beyond the gridpoint \((m + 1, 3)\), we use (3.30 a-b) and (3.31 a-b) which retains terms through \( \nabla^3 f_n \).

The computation based on the above procedure leads to numerical difficulties. This is brought about because small changes in \( \frac{du}{dN} \) at the body surface leads to large changes in \( u \) at the outer edge. This means that to satisfy the outer edge condition to a few significant digits requires that \( \frac{du}{dN} \) at the body surface must be satisfied to a large number of significant places. We can see the cause of these numerical difficulties from equation (3.27 b). Unless an accurate value of \( \frac{du}{dN} \) at the body surface is used in the computation, \( u_H(\infty) \) and \( u_p(\infty) \) are large in comparison with the potential velocity \( U \). Thus \( A \) is approximated by \( A \approx u_p(\infty)/u_H(\infty) \), and equation (3.26) becomes

\[
\frac{u(N)}{u_H(\infty)} \approx u_p(N) - \frac{u_p(\infty)}{u_H(\infty)} u_H(N) \quad (3.36)
\]
As the computation proceeds toward the outer edge, \( u \) does not approach the potential velocity \( U \). This can be observed from equation (3.36) in which the quantity \( U \) is essentially neglected in the calculation of \( A \).

In order to cope with the above numerical difficulties we predict \( \left( \frac{du}{dN} \right) \) by using equation (3.1 d) which can be written in the following form,

\[
\left( \frac{du_p}{dN} \right)_{m+1,0} = 2 \left( \frac{du}{dN} \right)_{m,0} - \left( \frac{du}{dN} \right)_{m-1,0} \tag{3.37}
\]

However, the quantity \( \frac{du}{dN} \) at the wall computed from the above equation is not accurate enough to satisfy the outer edge condition. We can predict \( \frac{du}{dN} \) at the body surface by using Newton technique\(^{(14)}\), and solve equation (3.25) without superposition. This is essentially the method of Smith and Clutter\(^{(18,19)}\) which is not what we intend to use in this problem.

(3) Discussion of Results

The computations for the velocity components, the skin friction, and the displacement-thickness by Method A were carried out on the IBM 7040 electronic digital computer. These computations took approximately one second per step in \( \Delta s \) for approximately seventy steps in \( \Delta N \). The step sizes in \( \Delta s \) and \( \Delta N \) were .01 and .1, respectively. The computation started with 58 steps in \( \Delta N \) at the stagnation-point and ended with 100 steps in \( \Delta N \) at the separation point with a total of 183 steps in \( \Delta s \) along the body surface. The
results are shown in Figures 3.2 to 3.6. The distribution of the velocity component $u$ is plotted in Figure 3.2 and 3.3 for the flow past a cylinder and a sphere, respectively. We see from the above figures that the boundary-layer thickness increases as the distance $s$ increases downstream. The solutions by the finite-difference method are compared with those given in Schlichting's book which are the solutions obtained by using the Blasius series. As the distance $s$ and the normal distance $N$ increase the deviation of the curves between the two results increases. This is due to the fact that the solutions by the Blasius series may not be accurate at large distances from the stagnation point if the higher terms in the series are neglected in the computation. On the other hand the finite-difference method should give accurate solutions since the boundary-layer equations and the boundary conditions are satisfied at each station on the body during the computation.

The point of separation where $\frac{\partial u}{\partial N} = 0$, occurs at $105^\circ$ and $105.37^\circ$ measured from the stagnation point for the cylinder and the sphere, respectively. Schlichting gives $108.8^\circ$ for the cylinder and $109.6^\circ$ for the sphere.

The velocity components $v$ at various positions $s$ are shown in Figure 3.4. The value of $\frac{\partial v}{\partial N}$ vanishes at the wall ($N = 0$) as can be observed from equation (3.16 c) if we substitute $\frac{\partial u}{\partial s} = u = 0$ at $N = 0$.

The shearing stress at the body surface is plotted in
Figure 3.2 Velocity Distribution in the Boundary-Layer on a Circular Cylinder.
Figure 3.3 Velocity Distribution in the Boundary-Layer on a Sphere.
Figure 3.4 Distribution of the Velocity Component Normal to the Sphere Surface in the Boundary-Layer.
(4) Conclusions

The problems of the laminar incompressible flow past a circular cylinder and a sphere were solved by the implicit finite-difference method. The computational results were obtained for the distribution of velocity components in the boundary-layer, and the variation of skin friction and displacement-thickness along the body. Another attempt was made to approach the same problem by solving the second-order linear difference-differential equation which was reduced from the boundary-layer equations. The superposition of solutions, and Smith and Clutter's predictor and corrector technique were used in the computation. The computation encountered numerical difficulties. This is brought about because small changes in \( \frac{du}{dN} \) at the body surface lead to large changes in \( u \) at the outer edge of the boundary-layer. This means that to satisfy the outer edge condition to a few significant digits requires that \( \frac{du}{dN} \) at the body surface must be satisfied to a large number of significant places. Thus, we have to predict \( \frac{du}{dN} \) at the body surface by using the time-consuming Newton technique. The governing equation then must be solved without superposition. This is essentially the method of Smith and Clutter which is not what we intended to use in the second method of solution. Thus the attempt at the second method was terminated.
Figure 3.5 Variation of Shearing Stress at the Body Surface.
Figure 3.6 Displacement-Thickness.
In conclusion method A, (the finite-difference method) is found to be the most desirable method. It has been shown to be highly accurate and to be free from computational difficulties. In the remainder of the problems to be solved we will use method A or a modification of it.
CHAPTER IV

LAMINAR INCOMPRESSIBLE VISCOUS FLOW PAST A
FINITE FLAT PLATE (SECOND-ORDER SOLUTIONS)

This chapter deals with viscous flow past a finite flat plate. It is assumed that the flow is laminar, viscous and incompressible. Further, in the computation of the second-order velocities in the second part of this chapter, the assumption is made that the displacement-thickness is constant in the wake. The adequacy of this assumption is checked with solutions based on the calculated displacement-thickness in the wake.

PART 1: LAMINAR FLOW IN THE WAKE OF A FINITE FLAT PLATE

A. Formulation of the Problem

(1) Coordinate System

The coordinates used in this chapter are an orthogonal Cartesian system in which the s* and N* coordinates are along and normal to the flat plate respectively. In the first part of the chapter the coordinate s* is measured from the leading edge of the plate, and in the second part it is measured from the trailing edge of the plate (see Figure 4.1). The velocity components (u*, v*) are parallel to the coordinates lines (s*, n*) respectively.

(2) Dimensionless Quantities

The dimensionless quantities used in this chapter are
Figure 4.1. Coordinate System.
the same as those in Chapter III, Method A.

(3) Starting Flow Quantities

Goldstein\(^{(7)}\) has solved the problem of flow in the wake behind a finite flat plate by assuming that the velocity can be expanded in a power series at the trailing edge of the plate. In the first part of this Chapter, the same problem is approached by the implicit finite-difference method. In order to obtain accurate starting flow quantities for the calculation we recalculate the equations given by Goldstein\(^{(7)}\) by using Runge-Kutta's method. This computation gives the starting wake velocities behind the trailing edge of the plate.

In Goldstein's paper\(^{(7)}\), \(4 \, L^*\) is chosen for the characteristic length. Thus, for instance Reynolds number is defined as

\[ Re = \frac{\infty}{\nu^*} \cdot \frac{4 \, U^* \, L^*}{\nu^*}. \]

It is assumed that the flow outside the boundary-layer is uniform and that the pressure gradient is zero. Thus, the boundary-layer equations reduce to

\[
\frac{\partial u}{\partial s} + v \frac{\partial u}{\partial n} = \frac{\partial^2 u}{\partial n^2} \quad (4.1 \, a)
\]

\[
\frac{\partial u}{\partial s} + \frac{\partial v}{\partial n} = 0 \quad (4.1 \, b)
\]

The boundary conditions are:

\[ \frac{\partial u}{\partial n} = 0 \, \text{and} \, v = 0 \, \text{at} \, N = 0 \quad (4.2 \, a) \]
The continuity equation (4.1 b) is satisfied if a function $\psi$ is defined such that

$$u = \frac{\partial \psi}{\partial N}, \quad v = -\frac{\partial \psi}{\partial s}$$

(4.3)

We now define new coordinates $(\xi, \eta)$ and a function $f$ as follows,

\[ \xi = \frac{1}{3} \]
\[ \eta = N s - \frac{1}{3} \]

(4.4 a, b)

\[ \psi = \xi^2 f(\xi, \eta) \]

(4.4 c)

Then the velocity components can be written as

\[ u = \frac{\partial \psi}{\partial N} = \frac{1}{3} \xi f_\eta \]
\[ v = -\left(2 f + \xi f_\xi - \eta f_\eta\right)/3 \xi \]

(4.5 a, b)

For the flow in the neighborhood of the plate the function $f$ is defined as follows,

\[ f = f_o(\eta) + \xi^3 f_3(\eta) + \xi^6 f_6(\eta) + \ldots \]

(4.6)

The velocity component $u$ becomes

\[ u = \frac{1}{3} \xi \left[ f_o'(\eta) + \xi^3 f_3'(\eta) + \xi^6 f_6'(\eta) + \ldots \right] \]

(4.7)
\[ f'' + 2 f f' - f^2 = 0 \] 
\[ (4.8 \text{a}) \]

\[ f_3'''' + 2 f_0 f_3'' - 5 f_0 f_3' + 5 f_0^2 f_3 = 0 \] 
\[ (4.8 \text{b}) \]

\[ f_6'''' + 2 f_0 f_6'' - 8 f_0 f_6' + 8 f_0^2 f_6 = 4 f_3'''' - 5 f_3 f_3'' \] 
\[ (4.8 \text{c}) \]

Here dashes denote differentiation with respect to \( \eta \). The above equations (4.8 a-c) are solved simultaneously by using Runge-Kutta's method with the boundary conditions (4.2 a). These conditions leads to the initial values at \( N = 0 \) for the \( f \)'s which are given by Goldstein (7) as:

\[ f_r' (0) = f_r''' (0) = 0 \] 
\[ (4.9 \text{a}) \]

where \( r = 0, 3, 6 \ldots \)

\[ f_0' (0) = 3.67869 \] 
\[ (4.9 \text{b}) \]

\[ f_3' (0) = -3.543 \] 
\[ (4.9 \text{c}) \]

\[ f_6' (0) = 8.119 \] 
\[ (4.9 \text{d}) \]

For large values of \( \eta \), Goldstein expanded \( \psi \) in the form
\[ \psi = \psi_0 + \xi \psi_1 + \xi^2 \frac{\psi_2}{2!} + \xi^3 \frac{\psi_3}{3!} + \xi^4 \frac{\psi_4}{4!} + \ldots \] (4.10)

where \( \psi_0, \psi_1, \) etcetera are functions of \( N. \) Thus

\[ u = \psi_0' + \xi \psi_1' + \xi^2 \frac{\psi_2'}{2!} + \xi^3 \frac{\psi_3'}{3!} + \ldots \] (4.11)

The derivatives of \( \psi_0, \psi_1, \) etcetera are given as follows,

\[ \frac{\psi_0'}{2!} = \frac{1}{2} \xi' \] (4.12 a)

\[ \frac{\psi_1'}{2!} = \frac{1}{2} A \xi'' \] (4.12 b)

\[ \frac{\psi_2'}{2!} = \frac{1}{2} \frac{A^2}{2!} \xi''' \] (4.12 c)

\[ \frac{\psi_3'}{3!} = \frac{1}{2} \frac{A^3}{3!} \xi^{(4)} - N \xi'' \] (4.12 d)

\[ \frac{\psi_4'}{4!} = \frac{1}{2} \frac{A^4}{4!} \xi^{(5)} - AN\xi''' - B\xi'' \] (4.12 e)

\[ \frac{\psi_5'}{5!} = \frac{1}{2} \frac{A^5}{5!} \xi^{(6)} - \frac{A^3}{2!} N\xi^{(4)} - AB\xi''' \] (4.12 f)

\[ \frac{\psi_6'}{6!} = \frac{1}{2} \frac{A^6}{6!} \xi^{(7)} - \frac{A^3}{3!} N\xi^{(6)} - \frac{A^3 B}{2!} \xi^{(4)} + N^2\xi''' + 3 N\xi'' \] (4.12 g)

\[ \frac{\psi_7'}{7!} = \frac{1}{2} \frac{A^7}{7!} \xi^{(8)} - \frac{A^4}{4!} N\xi^{(6)} - \frac{A^3 B}{3!} \xi^{(5)} + AN^2 \xi^{(4)} + (3A + 2B) \]

\[ N\xi''' + C\xi'' \] (4.12 h)
\[
\frac{V}{8} = \frac{1}{2} \frac{A^8}{8!} \zeta^{(8)} - \frac{A^5}{5!} N^e(7) - \frac{A^4 B}{4!} \zeta^{(6)} + \frac{A^2 c}{2!} N^e \zeta^{(5)} + \\
\left( \frac{3A^2}{2!} + 2 AB \right) N^e(4) + (AC - B^2) \zeta^{(')}
\]

(4.12 i)

where

\[ A = 1.0224 \]
\[ B = .4491 \]
\[ C = .9865 \]

\[ \zeta, \zeta', \zeta''', \ldots \zeta^{(9)} \] are computed by using Runge-Kutta's method from the equation

\[
\zeta^{(0)} + \zeta \zeta^{(9)} + 7 \zeta' \zeta^{(8)} + 22 \zeta''' \zeta^{(7)} + 42 \zeta'''' \zeta^{(6)} + \\
56 \zeta^{(4)} \zeta^{(5)} = 0
\]

(4.13 a)

which is obtained from the repeated differentiation of the following equation,

\[
\frac{d^3 \zeta}{dN^3} + \zeta \frac{d^3 \zeta}{dN^3} = 0
\]

(4.13 b)

Thus, the component of velocity \( u \) can be computed from equations (4.7) and (4.11). Integrating the continuity equation (4.1 b) gives

\[
v = \int_0^N \frac{dN}{dN} \frac{\partial u}{\partial N} \, dN
\]

which is integrated numerically by using (3.2 c).
B. Solutions to the Problem

The boundary-layer equations (4.1 a-b) are solved by using the implicit finite-difference method as illustrated in Chapter III, Method A. As a result we obtain the recurrence formula (3.6 a) in which the coefficients are expressed as follows,

\[
A_n = -\frac{1}{2} \frac{\Delta x}{\Delta N} (2 v_{m,n} - v_{m - 1,n}) - \frac{\Delta y}{\Delta N^2} \quad (4.14 \text{ a})
\]

\[
B_n = \frac{3}{2} (2 u_{m,n} - u_{m - 1,n}) + 2 \frac{\Delta s}{\Delta N^2} \quad (4.14 \text{ b})
\]

\[
C_n = -A_n - 2 \frac{\Delta s}{\Delta N^2} \quad (4.14 \text{ c})
\]

\[
F_n = \frac{1}{2} (u_{m,n} - u_{m - 1,n}) (4 u_{m,n} - u_{m - 1,n}) \quad (4.14 \text{ d})
\]

Using the above coefficients, the velocity component \( u \) can be computed from equation (3.12). In order to obtain the starting quantities \( E_0 \) and \( D_0 \) at the wall we express the boundary condition (4.2 a) in differential form (3.2 d), and then write the recurrence formula (3.6 a) for gridpoints \((m + 1, 1)\) and \((m + 1, 2)\) as follows,

\[
11 u_{m + 1, 0} - 18 u_{m + 1, 1} + 9 u_{m + 1, 2} - 2 u_{m + 1, 3} = 0 \quad (4.15 \text{ a})
\]

\[
A_1 u_{m + 1, 0} + B_1 u_{m + 1, 1} + C_1 u_{m + 1, 2} = F_1 \quad (4.15 \text{ b})
\]

\[
A_2 u_{m + 1, 1} + B_2 u_{m + 1, 2} + C_2 u_{m + 1, 3} = F_2 \quad (4.15 \text{ c})
\]
Eliminating the term $u_{m+1,3}$ gives

$$u_{m+1,0} = \frac{K_0}{K_1} u_{m+1,1} + \frac{K_3}{K_1}$$

where

$$K_1 = 11 - 9 \frac{A_1}{C_1} - 2 \frac{A_1 B_2}{C_1 C_2}$$

(4.16 b)

$$K_2 = 18 + 9 \frac{B_1}{C_1} - 2 \frac{A_2}{C_2} + 2 \frac{B_1 B_2}{C_1 C_2}$$

(4.16 c)

$$K_3 = -9 \frac{F_1}{C_1} + 2 \frac{F_3}{C_2} - 2 \frac{B_2 F_1}{C_1 C_2}$$

(4.16 d)

At gridpoint $(m+1,0)$, equation (3.12) can be written as

$$u_{m+1,0} = E_0 u_{m+1,1} + D_0$$

(4.17)

From equations (4.16 a) and (4.17), it is found that

$$E_0 = \frac{K_2}{K_1} \text{ and } D_0 = \frac{K_3}{K_1}$$

(4.18)

The procedure of the computation is described as follows.

First we compute the coefficients $A_n$, $B_n$, $C_n$, and $F_n$ for each gridpoint at station $m+1$. Based on these coefficients, we can compute $K_1$, $K_2$, $K_3$, $E_0$, and $D_0$. Then $E_n$ and $D_n$ are computed starting from the wall and proceeding toward the edge of boundary-layer.

With the known quantities of $E_n$ and $D_n$ at each gridpoint on station $m+1$, the velocity component $u$ is computed from equation (3.12).

This is carried out starting from the edge of the boundary-layer where $u = 1$, and proceeding toward the plane of the plate. The
widening of the boundary-layer is treated by using the method described in Method A, Chapter III. The displacement-thickness is obtained from integrating the equation

$$\delta = \int_{1}^{\infty} (1 - u) \, dN$$

Equations (4.19) can be integrated numerically by using the relation (3.2 c). The above computation is carried out at each station in m starting from the trailing edge of the plate and proceeding downstream.

C. Discussion of Results

The variation of the wake velocity distribution behind a finite flat plate is plotted in Figures 4.2 a-b. We see that as the boundary-layer broadens downstream, the velocity distribution approaches the free stream velocity profile. The variation of the displacement-thickness through $s = 10$ is plotted in Figure 4.3. The variation of the velocity at the axis $N = 0$ is shown in Figure 4.4.

The computations were carried out on the IBM 7040 electronic digital computer starting from $s = 0$ to $s = 10$. These computations took approximately 0.6 second per step in $\Delta s$ for approximately 130 steps in $\Delta N$. The step sizes in $\Delta s$ and $\Delta N$ are .01 and .1, respectively. The computation started with 76 steps in $\Delta N$ at the stagnation-point and ended with 180 steps in $\Delta N$ at
Figure 4.2a Distribution of Wake Velocity Behind a Finite Flat Plate.
Figure 4.2b Distribution of Wake Velocity Behind a Finite Flat Plate.
Figure 4.3 Variation of Displacement-Thickness in the Wake Behind a Finite Flat Plate.
Figure 4.4 Variation of Velocity Along the Axis, $N = 0$ in the Wake Behind a Finite Flat Plate.
s = 10 with a total of 1000 steps in $\Delta s$ along the axis, $N = 0$.

D. Conclusions

The distributions of the wake velocity behind a finite flat plate were obtained by using the implicit finite-difference method. The displacement-thickness and the variation of velocity along the axis, $N = 0$ in the range $s = 0$ to $s = 10$, are illustrated. The agreement with Goldstein's(7) series expansion near the trailing edge is excellent in the region where his expansion is valid, i.e. near the trailing edge. Goldstein(8) has also found an asymptotic solution for large distances from the trailing edge. The agreement there is also excellent. The results obtained here provide for the first time a solution which is valid in the whole wake, i.e. for small, intermediate, and large distances from the trailing edge.

PART 2: SECOND-ORDER BOUNDARY-LAYER SOLUTIONS

A. Formulation of the problem

(1) Coordinate system

The same coordinate system as was used in the first part of this chapter is used here except that the coordinate $s^*$ is measured from the leading edge of the plate.

(2) Dimensionless Quantities

The dimensionless quantities not listed in the following can be found in the first part of this chapter. One notes that the
Reynolds number $Re$ is defined as $\frac{UL}{\nu}$ here. The characteristic length $L^*$ (i.e., the length of the flat plate) must replace $a^*$ in Chapter III for the quantities $s$, $N$ and $\delta$. In addition we define the following quantities,

$$C_f = \frac{(\tau_f)^*_b}{\frac{1}{2} \rho^* U^*_\infty^2} \quad \text{coefficient of friction} \quad (4.20 \ a)$$

$$n = \frac{N}{\sqrt{s}} \quad \text{coordinate normal to the flat plate} \quad (4.20 \ b)$$

$$L = 1 \quad \text{length of the plate} \quad (4.20 \ c)$$

$$Re = \frac{\infty^* L^*}{\nu^*} \quad \text{Reynolds number} \quad (4.20 \ d)$$

$$\psi_1 = \sqrt{s} f_1(n) \quad \text{first-order stream function} \quad (4.20 \ e)$$

$$\psi_2 = \frac{1}{\pi} \frac{1}{s^2} f_2(n) \quad \text{second-order stream function valid near the leading-edge region of the plate} \quad (4.20 \ f)$$

$$\tau_f = \frac{\tau_f^*}{\rho^* U^*_\infty^2} \quad \text{shear stress} \quad (4.20 \ g)$$

Subscript 1 and 2 denote first and second order quantities respectively.

(3) **Governing Equations**

The governing equations given by Van Dyke$^{(21)}$ for the
second-order boundary-layer flow can be reduced to the following set of equations for the steady laminar incompressible flow past a finite flat plate.

\[
\frac{\partial u_2}{\partial s} + \frac{\partial v_2}{\partial N} = 0 \quad (4.21\ a)
\]

\[
 u_1 \frac{\partial u_2}{\partial s} + u_2 \frac{\partial u_2}{\partial s} + v_1 \frac{\partial u_2}{\partial N} + v_2 \frac{\partial u_2}{\partial N} + \frac{\partial p_2}{\partial s} - \frac{\partial^2 u_2}{\partial N^2} = 0 \quad (4.21\ b)
\]

\[
\frac{\partial p_2}{\partial N} = 0 \quad (4.21\ c)
\]

From equation \((4.22\ c)\) and matching, it can be shown that

\[
p_2 (s, N) = - U_2 (s, o) \quad (4.21\ d)
\]

In the above, Kuo\(^{(11)}\) gives

\[
U_2 (s, o) = \frac{\beta}{2\pi} \frac{1}{\sqrt{s}} \ln \frac{1 + \sqrt{s}}{1 - \sqrt{s}} \quad (4.21\ e)
\]

where \(\beta = 1.7208\).

The assumption made in deriving equation \((4.21\ e)\) is that the displacement-thickness in the wake can be approximated as being constant (see Figure 4.5). Figure 4.3 shows this to be incorrect. The accuracy of the approximation of constant displacement-thickness will be checked from the solutions based on the calculated displacement-thickness. Since we know the first order solution for flow past a flat plate we define a stream function
where \( f_1 \) is the familiar Blasius function. From this the following velocity components are obtained,

\[
u_1 = \frac{\partial \psi_1}{\partial n} = f_1' \quad (n)
\]

\[
u_1 = -\frac{\partial \psi_1}{\partial s} = \frac{1}{2\sqrt{s}} \left[ -nf_1' \quad (n) - f_1 \quad (n) \right] \quad (4.23 \quad b)
\]

Substituting the above into equations (4.21 a-b) and expressing the result by the new dimensionless coordinates \((s, n)\), we obtain the following set of linear partial-differential equations,

\[
sf_1' \quad (n) \frac{\partial u_2}{\partial s} - \frac{n}{2} f_1'' \quad (n) u_2 - \frac{f_1' \quad (n)}{2} \frac{\partial u_2}{\partial n} - s \frac{\partial P_2}{\partial s} = -\sqrt{s} f_1''' \quad (n) v_2 \quad (4.24 \quad a)
\]

\[
\frac{\partial v_2}{\partial n} = \frac{n}{2\sqrt{s}} \frac{\partial u_2}{\partial n} - \sqrt{s} \frac{\partial u_2}{\partial s} \quad (4.24 \quad b)
\]

where

\[
s \frac{\partial P_2}{\partial s} = \frac{\beta}{4\pi} \left[ \frac{1}{\sqrt{s}} \ln \frac{1 + \sqrt{s}}{1 - \sqrt{s}} - \frac{2}{1 - s} \right] \quad (4.24 \quad c)
\]

if the displacement-thickness is assumed to be constant in the wake. We see that equations (4.24 a-b) have two unknowns, \( u_2 \) and \( v_2 \). \( f_1, f_1' \) and \( f_1'' \) are known and are solutions to the Blasius equation

\[
2 f_1''' + f_1 f_1'' = 0 \quad (4.25)
\]

They can be found in Schlichting's (17) book.
(4) Starting Quantities for the Computations and a Second-Order Coefficient of Friction.

The second order stream function given by Kuo\(^{(11)}\) can be reduced as follows for the flow in the neighborhood of the leading edge of the plate,

\[
\psi_2 = \frac{1}{\pi} \frac{1}{s^2} f_2(n) \tag{4.26 a}
\]

where \(f_2 = \frac{1}{2} (f_1 + n f_1')\) \( \tag{4.26 b}\)

From the above, we can derive the following equations for the second order velocities which provide the starting data for the computation,

\[
u_2 = \frac{\partial \psi_2}{\partial n^*} = \frac{\beta}{2\pi} \left[ 2 f_1' (n) + n f_1'' (n) \right] \tag{4.27 a}
\]

\[
v_2 = \frac{\partial \psi_2}{\partial s^*} = -\frac{\beta}{4\pi} \frac{1}{s^2} \left[ f_1 (n) - n f_1' (n) - n^2 f_1'' (n) \right] \tag{4.27 b}
\]

The no-slip boundary condition at the body surface gives

\[
(u_2)_b = (v_2)_b = 0 \tag{4.28 a}
\]

The boundary condition at the outer edge of the boundary-layer is given by Van Dyke\(^{(21)}\) as follows,

\[
u_2 (s,N) = U_2 (s,0) \text{ as } N \rightarrow \infty \tag{4.28 b}
\]

The shear stress which is valid through the second approximation can be written as follows,
\[ \tau_f^* = \mu^* \frac{\partial u^*}{\partial n^*} = \mu^* \left( \frac{\partial u_1}{\partial n} + \varepsilon \frac{\partial u_2}{\partial n} \right) \]  
where 
\[ \varepsilon = \frac{1}{\sqrt{Re}} \]

Expressing the differentiation in equation (4.29 a) by the new coordinate \( n \) yields 
\[ \tau_f^* = \frac{\rho^* U^*}{\sqrt{Re}} \left[ \frac{1}{2} \left( \frac{\partial u_1}{\partial n} + \varepsilon \frac{\partial u_2}{\partial n} \right) \right] \]

Substituting (4.23 a) into the above equation (4.30) gives 
\[ \tau_f^* = \frac{\rho^* U^*}{\sqrt{Re}} \left[ s^2 \left[ f_1''(n) + \varepsilon \frac{\partial u_2}{\partial n} \right] \right] \]

The coefficient of friction is defined as 
\[ C_f = \frac{1}{\frac{1}{2} \rho^* U^* L^*} \int_0^L (\tau_f)^*_b \, ds^* \] 
where \((\tau_f)_b\) is evaluated at the body surface. After substituting from (4.31) into (4.32), we obtain the following coefficient of friction consisting of \( C_{f_1} \) and \( C_{f_2} \); namely the first and second order coefficients of friction respectively, 
\[ C_f = \frac{1}{\sqrt{Re}} C_{f_1} + \frac{1}{Re} C_{f_2} \]

where 
\[ C_{f_1} = 4 f_1''(0) \]
\[ C_{f_2} = 2 \int_{0}^{1} s \frac{1}{2} \frac{\partial u_2}{\partial n} (s, 0) \, ds \]  \hspace{1cm} (4.33 \, c)

For the flow in the neighborhood of the leading edge of the plate, (4.27 \, a) gives

\[ \frac{\partial u_2}{\partial n} (0) = \frac{3}{2\pi} \left[ 3 f_1'' (0) \right] \]  \hspace{1cm} (4.34)

Thus equation (4.33 \, c) reduces to

\[ C_{f_2} = \frac{6\beta}{\pi} s \frac{1}{2} f_1'' (0) \]  \hspace{1cm} (4.35)

B. Solutions to the Problem

The governing equations (4.24 \, a-b) are solved step-by-step starting from the leading edge of the plate and proceeding downstream by using the implicit finite-difference method. This method has been discussed in Chapter III as a general technique in numerical computation. This section deals with the application to the flow past a finite plate. Using the same approach as in Chapter III, Method A, we can reduce equation (4.24 \, a) to the recurrence formula (3.6 \, a) in which the coefficients are expressed as follows,

\[ A_n = \frac{f_1}{4N} - \frac{1}{N^3} \]  \hspace{1cm} (4.36 \, a)

\[ B_n = \frac{3}{2} \frac{sf_1'}{\Delta s} - \frac{n}{2} f_1'' + \frac{2}{\Delta n^3} \]  \hspace{1cm} (4.36 \, b)

\[ C_n = -A_n - \frac{2}{\Delta n^3} \]  \hspace{1cm} (4.36 \, c)

\[ F_n = \frac{sf_1'}{2\Delta s} (4 u_2 m, n - u_2 m, n) - s p \phi_2 s - \sqrt{s} f_1'' \left( 2 v_2 m, n - v_2 m, 1, n \right) \]  \hspace{1cm} (4.36 \, d)
The velocity \( u_2 \) may be calculated as in Chapter III.

Integrating equation (4.24 b), we obtain

\[
v_2 = \int_0^n \left( \frac{n}{2s} \frac{\partial u_2}{\partial n} - \sqrt{s} \frac{\partial u_2}{\partial s} \right) \, dn
\]  

(4.37)

Applying the difference relation (3.2 c) to the above, the velocity \( v_2 \) can be integrated numerically along the station \( m + 1 \). The boundary condition (4.28 a) gives a starting quantity at the plate surface. The terms \( \frac{\partial u_2}{\partial s} \) and \( \frac{\partial u_2}{\partial n} \) in (4.37) are reduced to difference forms for computation by using the relations (3.1 a-b), respectively.

In the computation of \( C_{f2} \), equation (4.35) is used for the flow near to the leading edge, and equation (4.33 c) for the flow on the rest of the plate. If \( \Delta s \) denotes a small distance from the leading edge, \( C_{f2} \) can be written as follows,

\[
C_{f2} = \frac{6\beta}{\pi} (\Delta s)^{1/2} f'(0) + 2 \int_{\Delta s}^{1} s^{-1/2} \frac{\partial u_2}{\partial n} (s,0) \, ds
\]

(4.38)

The above integration is carried out by using the relation (3.2 c).

Equation (4.21 e) gives the second-order velocity \( U_2 (s,0) \) as \( N \) approaches infinity. The above equation is based on the assumption that the displacement-thickness is constant in the wake as shown in Figure 4.3. For the solutions based on the calculated displacement-thickness, \( U_2 (s,0) \) can be obtained from the following equation (see figure 4.5), which can be recognized as the thin
airfoil approximation for flow past a plane body,

\[
U_2 (s,o) = \frac{1}{\pi s} \int_0^\infty \frac{\delta'_{ab}}{s - \xi_o} \, d \xi_o = \frac{1}{\pi s} \int_0^\infty \frac{\delta'_{bd} - \delta'_{bc}}{s - \xi_o} \, d \xi_o \quad (4.39)
\]

Here dashes denote the differentiation with respect to $\xi_o$. $\delta'_{bd}$ represents the calculated displacement-thickness in the wake of a flow past a finite flat plate, and $\delta'_{bc}$ represents the displacement-thickness on the semi-infinite flat plate for the portion $s \geq 1$ (see Figures 4.3 and 4.5). We denote

\[
I_{bc} = - \frac{1}{\pi s} \int_0^\infty \frac{\delta'_{bc}}{s - \xi_o} \, d \xi_o \quad (4.40 a)
\]

\[
I_{bd} = \frac{1}{\pi s} \int_0^\infty \frac{\delta'_{bd}}{s - \xi_o} \, d \xi_o \quad (4.40 b)
\]

then

\[
U_2 (s,o) = I_{bc} + I_{bd} \quad (4.41)
\]

Schlichting\(^{(17)}\) gives

\[
\delta'_{bc} = 1.72077 \sqrt{\xi_o} \quad (4.42)
\]

which is obtained from the solution to Blasius problem. Hence, $I_{bc}$ is found as follows.

\[
I_{bc} = \frac{0.860385}{\pi} \frac{1}{\sqrt{s}} \ln \left[ \frac{1 + \sqrt{s}}{1 - \sqrt{s}} \right] \quad (4.43)
\]

We note that equation (4.43) is same as equation (4.21 e) which is given by Kuo.
Figure 4.5 Variation of Displacement-Thickness ($\delta$).
In the calculation of $I_{bd}$, we let

$$
I_{bd} = \frac{1}{\pi} \int_{1}^{4} \frac{\delta_{bd}'}{s - \xi} \, d\xi + \frac{1}{\pi} \int_{4}^{\infty} \frac{\delta_{bd}'}{s - \xi} \, d\xi \tag{4.44}
$$

For the integration between $\xi = 1$ to $\xi = 4$, the displacement-thickness calculated in the first part of this chapter is used (see Figure 4.3). Equations (3.2 c) is used for the integration between $\xi = 1$ to $4$ in which $\delta_{bd}'$ is obtained by using equation (3.1 a). For the integration between $\xi = 4$ to infinity, we use

$$
\delta_{bd}' = -\frac{15095}{(\xi - .48)^{3/2}} \tag{4.45}
$$

The above equation is derived from an asymptotic solution for large distances from the trailing edge given by Goldstein\textsuperscript{(8)}. The quantity .15095 is approximated from the displacement-thickness between $\xi = 4$ to 10 calculated in the first part of this chapter. We denote the second integration in equation (4.44) by $\overline{I}_{bd}$, then

$$
\overline{I}_{bd} = \frac{15095}{\pi} \frac{1}{3} \frac{1}{(\xi - .48)^{2}} \frac{1}{(\xi - s)} \, d\xi \tag{4.46}
$$

The above integration can be carried out analytically. At $s = .48$, $\overline{I}_{bd} = .004850$. Thus the second order velocity $U_2(s,o)$ can be calculated from equation (4.41). After calculating $U_2(s,o)$ on
the plate we may obtain \( \frac{\partial \hat{P}_2}{\partial s} \) from the following equation,

\[
\frac{\partial \hat{P}_2}{\partial s} = -\frac{d U_2(s, o)}{d s}
\]  

(4.47)

\( \frac{d U_2(s, o)}{d s} \) in the above equation now can be evaluated numerically by using equation \((3.1 \, a)\). Equation \((4.47)\) must replace equation \((4.24 \, c)\) if the solution based on the calculated displacement-thickness is to be obtained.

C. Discussion of the Results

The results of the solutions based on the constant displacement-thickness are discussed first. Figure 4.6 shows the distribution of the second order velocity \( u_2 \) on a finite flat plate. As \( N \) approaches infinity (the region \( n > 3.5 \) in Figure 4.6) \( u_2 \) approaches a value which can be represented by equation \((4.21 \, e)\). Due to a singularity at \( s = 1.0 \), the computation fails to give a correct value for \( u_2 \) at the trailing edge of the plate.

Figures 4.7 a-b show the distributions of the second-order velocity component \( v_2 \). A logarithmic scale is used in Figure 4.7 b for the \( v_2 \) coordinate.

The second-order coefficient of friction is plotted in Figure 4.8. At \( s = .999 \), the computation gives \( C_{f_2} = 3.61881 \). Kuo\((11)\) estimated the term \( C_{f_2} \) from the sum of a series, and gave
Figure 4.6 Distribution of Second-Order Velocity on a Finite Flat Plate (Based on the constant $\delta$)
Figure 4.7 a Distribution of the Second-Order Velocity Component Normal to the Finite Flat Plate.
Figure 4.7b Distribution of the Second-Order Velocity Component Normal to the Finite Flat Plate.
\[ C_f = \frac{1}{\sqrt{Re}} C_{f1} + \frac{1}{Re} (C_{f2} + C_{f2}') \]

Figure 4.8 Variation of Second-Order Coefficient of Skin-Friction Over the Finite Flat Plate.
\( C_{f_2} = 4.12 \) at \( s = 1.0 \). The term \( C'_{f_2} \) (Figure 4.8), which arises due to a concentrated force at the leading edge of the plate, was estimated by Imai\(^9\) as \( C'_{f_2} = 2.326 \). With this term included (this term was overlooked by Kuo) \( C_{f_2} \) becomes 5.3\(^{22}\). Taking \( C'_{f_2} \) into account, the second-order coefficient of friction \( C_{f_2} \) will be larger than 5.9448 which is obtained from the computation based on constant displacement-thickness.

The solutions based on the calculated displacement-thickness are shown in Figures 4.9-11. The second-order velocities \( U_2(s,o) \) are shown in Figure 4.9 for the solutions based on the constant and the calculated displacement-thickness. A rapid increase in the deviation of \( U_2(s,o) \) is noted in the region of large \( s \). The deviation is due to the term \( I_{bd} \) in equation (4.41). The term \( I_{bc} \) is due to the constant displacement-thickness only. The computations of equation (4.44) show that the integrals from \( \xi = 4 \) to \( \xi = \infty \) give negligible values in comparison with the integral from \( \xi = 1 \) to \( \xi = 4 \). Thus the deviations in \( U_2(s,o) \) between the solutions based on the constant and the calculated displacement-thickness are mainly due to the sudden decrease of the displacement-thickness between \( \xi = 1 \) to \( \xi = 4 \). Large deviations in the second-order coefficient of friction are seen in Figure 4.10. At \( s = .999 \) the computation shows that \( C_{f_2} = 57.1838 \) for the solutions based on the calculated displacement-thickness, whereas the solutions based on the constant
Figure 4.9 Variation of $u_2$ as $N \to \infty$ (Based on the Constant $\delta$ and the Calculated $\delta$).
Figure 4.10 Variation of $C_{f_2}$ over the Finite Flat Plate (Based on the $C_{f_2}$ Constant $\delta$ and Calculated $\delta$).
displacement-thickness gives $C_{x_2} = 3.61881$.

The distribution of $u_2$ on a finite flat plate is shown in Figures 4.11 a-b. The deviations of $u_2$ between the solutions based on the constant and the calculated displacement-thickness increase as $s$ increases.

The computations were carried out on the IBM 7040 electronic digital computer. It took approximately 0.5 seconds per step in $\Delta s$ for 89 steps in $\Delta n$. The step sizes in $\Delta s$ and $\Delta n$ are .001 and .1, respectively.

E. Conclusions

The second-order solutions to the laminar incompressible viscous flow past a finite flat plate were approached by using the implicit finite-difference method. The second-order boundary-layer equations, derived by Van Dyke (22), and Kuo's (11) second-order velocity were used for the solutions to the problem. An assumption was made that the displacement-thickness in the wake can be approximated as being constant. The accuracy of this approximation was checked from the solutions based on the calculated displacement-thickness. The deviations of solutions based on the constant and the calculated displacement-thickness ($\delta$) increase as $s$ increases. This is due to the sudden decrease of the displacement-thickness near to the trailing edge of the plate which increases $U_2 (s, o)$ in the similar way as sinks do if they were distributed in the wake.
Figure 4.11a Distribution of $u_2$ on a Finite Plate
(Based on the Constant $\delta$ and the Calculated $\tilde{\delta}$).
Figure 4.11 b. Distribution of $u_2$ on a Finite Plate (Based on the Constant $\delta$ and the Calculated $\delta$)

(SEE FIG. 4.3)
Due to the sudden decrease in \( \delta \) near to the trailing edge of the plate, sinks with strength greater than in the case of constant \( \delta \) are produced, and thus \( U_2 (s, o) \) increases rapidly especially at large \( s \). The assumption of constant displacement-thickness gives approximate solutions on the plate near to the leading edge.

The solutions to the second-order velocities and the coefficients of frictions are shown for two cases which are based on the constant and the calculated displacement-thickness.
CHAPTER V
LAMINAR VISCOSOUS FLOW PAST A SPHERE AT HIGH MACH NUMBER

This chapter deals with the laminar flow past a sphere at a high Mach number. The flow after the shock is assumed to have constant density and viscosity. This is a good approximation in the region near to the stagnation-point especially if the flow is hypersonic and the temperature of the sphere is near to the stagnation-temperature. The solution to the above problem is approached by two methods: the truncated series method and the implicit finite-difference method. The truncated series method gives accurate solutions for the flow in the neighborhood of the stagnation-point, whereas the implicit finite-difference method carries the step-by-step computation along the body surface and thus gives accurate solutions away from the stagnation-point. The method of solution to the above problem will also apply to the flow of a compressible fluid where the density and viscosity are variable in the shock layer.

A Formulation of the Problem

(1) Coordinate system

The governing equations to be treated in this chapter are expressed in an orthogonal curvilinear coordinate system \((s^*, n^*)\)
Figure 5.1  Coordinate System
as described by Figure 5.1.

\( n^* \) is the normal distance measured from the surface of the body, and \( s^* \) the distance along the body surface from the stagnation point. The components of velocity, \( u^* \) and \( v^* \) are parallel to the coordinate lines \( s^* \) and \( n^* \) respectively.

(2) **Dimensionless Quantities**

The dimensionless quantities not listed in the following may be found in Chapter III, Method A,

\[
\begin{align*}
\text{a} &= 1 & \text{radius of sphere} & (5.1 \text{ a}) \\
K &= \frac{\rho^*_{ls}}{\rho^*_{os}} & \text{density ratio at the shock} & (5.1 \text{ b}) \\
n &= \frac{n^*}{a^*} & \text{coordinate normal to body surface} & (5.1 \text{ c}) \\
r &= \frac{r^*}{a^*} & \text{distance to body surface from axis of symmetry} & (5.1 \text{ d}) \\
R &= \frac{R^*}{a^*} & \text{radial distance from the center of a sphere} & (5.1 \text{ e}) \\
Re_s &= \frac{U^*_\infty a^*}{v^*_s} & \text{shock Reynolds number} & (5.1 \text{ f}) \\
v &= \frac{v^*}{U^*_\infty} & \text{velocity component normal to body surface} & (5.1 \text{ g})
\end{align*}
\]
### Governing Equations (General)

The compressible Navier-Stokes equations have been expressed in a curvilinear coordinates system by various authors. In particular these equations can be obtained from Van Dyke and Maslen. If constant density and viscosity are assumed in the shock layer, the Navier-Stokes equations can be reduced to the following form if one retains only terms to second order in Reynolds number in both the boundary-layer and inviscid flow region,

**Continuity equation**

\[
\left\{ \left[ (1 + n) \sin s \right] u \right\}_s + \left\{ (1 + n) \left[ (1 + n) \sin s \right] v \right\}_n = 0
\]

\[ (5.2 \text{ a}) \]
s-momentum equation

\[ \rho \left( \frac{u u_s}{1 + n} + v u_n + \frac{1}{1 + n} u v \right) + \frac{p_s}{1 + n} = \frac{1}{Re_s} \left( u_{nn} + 2 \mu u_n \right) \]  

(5.2b)

n-momentum equation

\[ \rho \left( \frac{u v_s}{1 + n} + v v_n - \frac{u^2}{1 + n} \right) + p_n = 0 \]  

(5.2c)

The above three equations involve three unknowns u, v and P. The possibility of solving such a set of equations for the more general case of a compressible fluid has been discussed by Davis and Flugge-Lotz\(^3\). These equations represent a simplification to their equations, however the numerical approach is the same.

In the following sections, the solution to the governing equations (5.2 a-c) are discussed for two different methods of solution, namely the truncated series method and the implicit finite-difference method.

B Truncated Series Method of Solutions

(1) Governing Equations

We assume the following truncated series for the flow variables P, u and v. In obtaining this series, one is guided by
the constant density solution of Lighthill (12) and the solution of the compressible problem by Kao (10) by the same method,

\[ P(n, s) = P_1(n) + P_2(n) \sin^2 s + \ldots \]  

(5.3 a)

\[ u(n, s) = u_1(n) \sin s + \ldots \]  

(5.3 b)

\[ v(n, s) = -v_1(n) \cos s + \ldots \]  

(5.3 c)

Substituting the above series into the governing equations (5.2 a-c) and equating like powers of \( \sin s \), we obtain the following equations,

\[ 2(1 + n)u_1 - \left[ (1 + n)^2 v_1 \right]_n = 0 \]  

(5.4 a)

\[ \rho \left( \frac{u_1^2}{1 + n} - v_1 u_1 n - \frac{u_1 v_1}{1 + n} \right) + \frac{2 P_2}{1 + n} = \frac{1}{Re_s} \left( u_{1n} + 2 u_{1n} \right) \]  

(5.4 b)

\[ \rho v_1 v_{1n} + p_{1n} = 0 \]  

(5.4 c)

\[ \rho \left( \frac{u_1 v_1}{1 + n} - v_1 v_{1n} - \frac{u_1^2}{1 + n} \right) + p_{2n} = 0 \]  

(5.4 d)

Denoting \( u_{1n} = 0 \)

the above equations reduce to the following set of first order non-linear ordinary differential equations,

\[ v_{1n} = \frac{2}{1 + n} (u_1 - v_1) \]  

(5.6 a)

\[ p_{2n} = \frac{\rho}{1 + n} (u_1^2 - 2v_1^2 + u_1 v_1) \]  

(5.6 b)
We define new coordinates as follows. These coordinates are order unity in the boundary-layer region,

\[ s = \bar{s} \]  

\[ n = \epsilon \bar{N} \]  

\[ u = \bar{u} \]  

\[ v = \epsilon \bar{v} \]  

\[ Q = \frac{1}{\epsilon} \bar{Q} \]  

\[ P = \bar{P} \]  

where \( \epsilon = \frac{1}{\text{Re}_{\bar{S}}} \)  

Introducing the above into the truncated series (5.3 a-c) and equations (5.6 a-e) we obtain the following relations,

\[ \bar{P} = \bar{P}_1 + \bar{P}_2 \sin^2 s + ... \]  

\[ \bar{u} = \bar{u}_1 \sin s + ... \]  

\[ \bar{v} = -\bar{v}_1 \cos s + ... \]
Based on conservation of mass between the body and the shock, we can derive an equation as follows,

\[ U_\infty \frac{R_s^* \sin s^*}{\rho \int_0^{R_s^*} u^* \, dR^*} = \frac{\rho_0 s^* \int_{R_s^*}^{R_s^*} R_s^* u^* \, dR^*}{1 + \epsilon N} \] 

(5.10 a)

where \( R_s^* \) denotes the radial distance from the center of a sphere to the shock. Introducing the relation

\[ R = \frac{R_s^*}{a^*} = 1 + n = 1 + \epsilon \frac{N}{N} \]

Along with equation (5.8 b), equation (5.10 a) reduces to

\[ \frac{(1 + \epsilon \frac{N}{N})^2 \sin s}{2 \frac{K}{\epsilon}} = \int_1^{N_s} \frac{\bar{u}}{N_s} (1 + \epsilon \frac{N}{N}) \, dN \] 

(5.10 b)

The above equation is used for finding the shock radius.
The Rankine-Hugoniot relations for a spherical shock are given as follows,

\[
\begin{align*}
    p_s &= \left[ \frac{2}{\gamma + 1} - \frac{\gamma - 1}{\gamma (\gamma + 1) M_\infty^2} \right] - \frac{2}{\gamma + 1} \sin^2 \theta \\
    \bar{u}_s &= \sin \theta \\
    \bar{v}_s &= -\frac{2}{\gamma + 1} \left( \gamma - 1 + \frac{2}{M_\infty^2} \right) \cos \theta
\end{align*}
\]

where the subscript \( s \) denotes the quantity evaluated at the shock. Even though the shock is not spherical as it is in the inviscid case we will assume that it is close to the spherical case. The numerical results will show this to be true. Comparing the coefficients of the above relation with equation (5.8 a-c), we obtain the boundary condition at the shock for Mach number infinity as follows,

\[
\begin{align*}
    \bar{\rho}_s &= \frac{2}{\gamma + 1} \\
    \bar{P}_s &= -\frac{2}{\gamma + 1} \\
    \bar{u}_s &= 1 \\
    \bar{v}_s &= \left( \frac{\gamma - 1}{\gamma + 1} \right) \frac{1}{\epsilon}
\end{align*}
\]
The no-slip boundary conditions at the body surface are

\[ \overline{u_b} = \overline{v_b} = 0 \quad (5.13) \]

The integration of equations (5.9 a-e) must be carried out numerically. The solution of these equations is complicated by the fact that the boundary conditions are given at the two ends of the integration, i.e. it is a two-point boundary value problem. Due to the nonlinearity of the equations, the integration must be carried out by guessing values for \( \overline{u_n} \) and \( \overline{v_b} \) at the body surface. These may be estimated on the basis of boundary-layer theory with the use of the first term of the Blasius series expansion (see Frössling\(^6\)) and the constant-density pressure distribution of Lighthill\(^{12}\). The flow velocity near to the stagnation-point can be written as the first term of a Blasius series as follows,

\[ u^* = U_1^* s^* f_1' (\eta) + ... \quad (5.14) \]

The coefficient \( U_1^* \) is to be determined from Lighthill's\(^{12}\) constant-density solution. As \( n^* \) approaches infinity (i.e. at the edge of the boundary-layer), \( f_1' = 1 \). Equation (5.14) therefore leads to the relation

\[ u^* = U_1^* s^* \quad (5.15) \]

Differentiating (5.14) with respect to \( n^* \), we obtain the following relation,
where Li Shilling's constant-density solution gives

\[
\sin s u^* = \frac{v_1}{v_\infty} \sqrt{\frac{2}{s}} u_1^* f_1''(\eta)
\]

\[
u_1 = \frac{a_*}{u_\infty} u_1^* s^* f_1''(\eta) \sqrt{\frac{2}{s}} u_1^*
\]

(5.16)

Lighthill's constant-density solution gives

\[
u_1^* = \psi'(R^*) \frac{\sin s}{R^*}
\]

(5.17 a)

where

\[
\psi(R^*) = \frac{U_\infty c}{30 K} \left\{ 3 (K - 1)^2 \left( \frac{R^*}{c} \right)^4 - 5 K (K - 4) \left( \frac{R^*}{c} \right)^2 + 2 (K - 1) (K - 6) \left( \frac{R^*}{c} \right)^{-1} \right\}
\]

(5.17 b)

If we let \( c = 1 \) (i.e. unit shock radius) and if we use \( K = 6 \), the proper value of \( K \) for Mach number infinity, in equation (5.17 a-b), we obtain the following equation,

\[
u_1^* = U_\infty \sin s \left( \frac{5}{3} R^*^2 - \frac{2}{3} \right)
\]

(5.18)

At the body surface in the neighborhood of the stagnation-point \( \sin s \approx s \) and \( R_b^* = \sqrt{\frac{4}{5}} \) (obtained by setting \( \psi(R^*) = 0 \)). With this substitution equation (5.18) leads to

\[
u_{b*}^* = \frac{2}{3} U_\infty \frac{s^*}{a^*}
\]

(5.19)

Equations (5.19) and (5.15) must match, therefore we obtain the quantity for \( u_1^* \) as follows,
Equation (5.16) becomes, after substituting the relation (5.20)

\[ u_n = \frac{4}{3} \sqrt{2} f_{1}'' \sqrt{Re_s} \sin s \quad (5.21 \text{a}) \]

From (5.7 b) and (5.8), we see that the above equation leads to

\[ u_{N} = \frac{4}{3} \sqrt{2} f_{1}'' \sin s \quad (5.21 \text{b}) \]

and

\[ u_{N} = \frac{4}{3} \sqrt{2} f_{1}'' \quad (5.21 \text{c}) \]

At \( N = 0 \), Frössling's(6) results give \( f_{1}'' = .9277 \) and thus

(5.21 c) becomes

\[ u_{N} = 1.75 \text{ at } N = 0 \quad (5.21 \text{d}) \]

This gives approximate starting values of \( u_{N} \) at the body surface for the numerical solution of equation (5.9 c).

Lighthill's constant-density solution gives for the surface pressure

\[ \bar{p} = \frac{11}{12} - \frac{25}{18} \sin^2 s \quad (5.22 \text{a}) \]

The starting conditions for pressure are, therefore

\[ \bar{P}_1 = \frac{11}{12} \quad (5.22 \text{b}) \]

\[ \bar{P}_2 = -\frac{25}{18} \quad (5.22 \text{c}) \]

The set of five first order non-linear differential equations...
(5.9 a-e) is numerically solved by Runge-Kutta's method. The starting conditions for the computation are summarized as follows,

\[ u = 0 \]  \hspace{1cm} (5.23 a)
\[ v = 0 \]  \hspace{1cm} (5.23 b)
\[ F_1 = 0.91662 \]  \hspace{1cm} (5.23 c)
\[ F_2 = -1.389 \]  \hspace{1cm} (5.23 d)
\[ Q = 1.75 \]  \hspace{1cm} (5.23 e)

The starting values for \( F_2 \) and \( Q \) are adjusted by using Newton technique (14) until the boundary conditions at the shock (5.12 a-d) are satisfied.

C. Implicit Finite-Difference Method of Solutions

(1) Governing Equations

In the case of flow past a sphere of unit radius, equation (5.2 a-c) can be written as follows,

\[
\left\{ \begin{array}{l}
(1 + \varepsilon N) \sin \bar{s} \bar{u} + (1 + \varepsilon N) \sin \bar{s} \bar{v} = 0 \\
\rho \left( \frac{\bar{u} \bar{s}}{1 + \varepsilon N} + \bar{v} \bar{u} \right) + \frac{F_s}{1 + \varepsilon N} = \frac{u_{NN}}{u_N} + 2 \varepsilon \frac{\bar{u}}{\bar{u_N}} \\
\rho \left( \frac{\bar{u} \varepsilon \bar{v} \bar{s}}{1 + \varepsilon N} + \varepsilon \bar{v} \bar{u} \bar{v} - \frac{u^2}{1 + \varepsilon N} \right) + \frac{\bar{p}}{\varepsilon} = 0
\end{array} \right. 
\]  \hspace{1cm} (5.24 a, b, c)
After rearranging equation (5.24 b) and integrating equation (5.24 a) and (5.24 c), they reduce to:

\[
\ddot{u}_{NN} + (2 \varepsilon - \rho \ddot{V}) \ddot{u}_N - \frac{\rho \ddot{u}}{1 + \varepsilon N} \ddot{u}_S - \frac{\varepsilon \ddot{V}}{1 + \varepsilon N} \ddot{u} = \frac{P_S}{1 + \varepsilon N}
\]

(5.25 a)

\[
\ddot{V} = -\frac{1}{(1 + \varepsilon N)^2 \sin \vec{s}} \int_0^{\vec{N}} (1 + \varepsilon N) (\ddot{u} \sin \vec{s}) \, d\vec{N}
\]

(5.25 b)

\[
\ddot{P} = \ddot{P}_0 + \varepsilon \rho \int_0^{\vec{N}} \left[ \frac{\ddot{u}^2}{1 + \varepsilon N} - \frac{\varepsilon \ddot{u} \ddot{V} \ddot{S}}{1 + \varepsilon N} - \frac{\varepsilon \ddot{V} \ddot{V}_N}{1 + \varepsilon N} \right] \, d\vec{N}
\]

(5.25 c)

(2) **Solution of Velocity** \(u\)

Using the same approach as in Chapter III, Method A, we can reduce equation (5.25 a) to the following recurrence formula for \(\ddot{u}\),

\[
A_n \ddot{u}_m + 1, n - 1 + B_n \ddot{u}_m + 1, n + C_n \ddot{u}_m + 1, n + 1 = F_n
\]

(5.26 a)

where

\[
A_n = \frac{1}{\Delta N^2} - \frac{1}{2\Delta N} \left[ 2 \varepsilon - \rho (2 \ddot{V}_m, n - \ddot{V}_m - 1, n) \right]
\]

(5.26 b)

\[
B_n = -\frac{2}{\Delta N^2} - \left( \frac{3}{2\Delta N} \right) \left( \frac{\rho}{1 + \varepsilon N} \right) (2 \ddot{u}_m, n - \ddot{u}_m - 1, n)
\]

(5.26 c)

\[
C_n = -A_n + \frac{2}{\Delta N^2}
\]

(5.26 d)
\[
F_n = \left( \frac{1}{2\Delta s} \right) \left( \frac{\rho}{1 + \epsilon \bar{N}} \right) (2 \bar{u}_{m,n} - \bar{u}_{m-1,n}) (\bar{u}_{m-1,n} - 4 \bar{u}_{m,n}) + \\
\left( \frac{\epsilon}{1 + \epsilon \bar{N}} \right) (2 \bar{v}_{m,n} - \bar{v}_{m-1,n}) (2 \bar{u}_{m,n} - \bar{u}_{m-1,n}) + \\
\left( \frac{1}{1 + \epsilon \bar{N}} \right) \left[ 2 (\bar{P}_s)_{m,n} - (\bar{P}_s)_{m-1,n} \right]
\]

(5.26 e)

Applying the same technique as in Chapter III, the velocity component \( u \) can be computed at station \( m + 1 \) starting from the shock and proceeding toward the body surface. The evaluation for the radial distance to the shock is discussed in the next section. The value of \( \frac{d\bar{P}}{ds} \) at station \( m + 1 \) are extrapolated by using equation (3.1 d), and then interpolated by equation (3.1 a) after the pressure at station \( m + 1 \) is integrated from equation (5.25 c).

(3) **Radial Distance to the Shock from the Center of a Sphere**

(unit radius)

The position of the shock is necessary in order to evaluate the flow quantities. The technique of evaluating the radial distance is that if a correct value of \( \bar{N}_s \) is substituted into equation (5.10 b), it must be satisfied. The correct values of \( \bar{N}_s \) is evaluated numerically as follows.

Denoting

\[
I_n = \int_0^{\bar{N}} \bar{u} (1 + \epsilon \bar{N}) \, d\bar{N}
\]

(5.27)
we can write $I_n$ in the difference form as follows (see Appendix A),

$$I_n = \frac{\Delta N}{12} \left[ (L_n + M_n) \bar{u}_n + P_n + Q_n \right]$$  \hspace{1cm} (5.28 a)

where

$$L_n = (L_n - 1 + M_n - 1) E_n - 1$$  \hspace{1cm} (5.28 b)

$$P_n = (L_n - 1 + M_n - 1) D_n - 1 + P_n - 1 + Q_n - 1$$  \hspace{1cm} (5.28 c)

$$M_n = 5 (1 + \varepsilon \bar{N})_{n - 1} E_n - 1 + 8 (1 + \varepsilon \bar{N})_n - \frac{D}{E_n} (1 + \varepsilon \bar{N})_n + 1$$  \hspace{1cm} (5.28 d)

$$Q_n = 5 (1 + \varepsilon \bar{N})_{n - 1} D_n - 1 + \frac{D_n}{E_n} (1 + \varepsilon \bar{N})_n + 1$$  \hspace{1cm} (5.28 e)

and $I_0 = M_0 = 0$ at the body surface. We note that $M_n$ and $Q_n$ are known functions of $E_n$ and $D_n$ which are given by (3.14 a-b). The relations (5.28 b-c) are recurrence formulas and thus $L_n$ and $P_n$ can be evaluated for each station $n$ starting from the body surface and proceeding toward the shock. Based on the known quantity $L_n'$, $M_n'$, $P_n'$ and $Q_n'$, the difference equation (5.28 a) gives the integrated values $I_n$ for each station $n$. If we compute the quantity

$$\frac{(1 + \varepsilon \bar{N})^2 \sin \bar{s}}{2 K \varepsilon} - I_n$$

for each station $n$ from the body to the shock, the above quantity will
change its numerical sign when the section $n_s$ is reached which satisfies the shock condition on $u$. Thus we see that the shock exists between the sections $n_s$ and $n_s - 1$, and the corresponding $N_s$ can be calculated by interpolating the above quantities between sections $n_s$ and $n_s - 1$.

(4) **Solutions of Velocity $v$ and Pressure $P$**

The velocity $v$ and pressure $P$ can be integrated from equations (5.25 b-c) by using the integral formula (3.2 c).

(5) **Shear Stress**

The dimensionless shear stress is defined as follows,

\[ \tau_f = \frac{\partial \bar{u}}{\partial N} \]  

(5.29)

The shear stress at the wall is obtained by using equation (3.2 d).

(6) **Solutions of a Flow with $Re_s = \infty$**

In the preceding articles, the shock Reynolds number is regarded as a parameter and represented in terms of $\varepsilon$. The governing equations (5.9 a-e) or (5.24 a-c) are formulated for the problem with various shock Reynolds numbers. In the case $Re_s = \infty$ (or $\varepsilon = 0$), equations (5.24 a-c) reduce to the following boundary-layer equations,

\[ (\bar{u} \sin \bar{s})_s + (\bar{v} \sin \bar{s})_N = 0 \]  

(5.30 a)

\[ \bar{u} \bar{u}_s + \bar{v} \bar{u}_N = -\frac{1}{\rho} \bar{F}_s + \frac{1}{\rho} \bar{u}_{NN} = 0 \]  

(5.30 b)

\[ \bar{F}_N = 0 \]  

(5.30 c)
The above equations can be solved by a method similar to the one discussed in Chapter III, C. Method A. Thus equation (5.30 b) reduces to equation (5.26 a) in which the coefficients are expressed as follows.

\[ A_n = - \frac{\Delta \bar{s}}{2 \Delta N} (2 \bar{V}_m, n - \bar{V}_m - 1, n) - \frac{1}{6} \frac{\Delta \bar{s}}{(\Delta N)^2} \]  \hspace{1cm} (5.31 a)

\[ B_n = \frac{3}{2} (2 \bar{U}_m, n - \bar{U}_m - 1, n) + \frac{1}{3} \frac{\Delta \bar{s}}{(\Delta N)^2} \]  \hspace{1cm} (5.31 b)

\[ C_n = - A_n - \frac{1}{3} \frac{\Delta \bar{s}}{(\Delta N)^2} \]  \hspace{1cm} (5.31 c)

\[ F_n = \frac{1}{2} (2 \bar{U}_m, n - \bar{U}_m - 1, n) (4 \bar{U}_m, n - \bar{U}_m - 1, n) - \frac{1}{6} \left( \frac{d \bar{P}}{d \bar{s}} \right)_m + 1, n \]  \hspace{1cm} (5.31 d)

Differentiating equation (5.22 a) gives

\[ \frac{d \bar{P}}{d \bar{s}} = - \frac{25}{18} \sin 2 \bar{s} \]  \hspace{1cm} (5.32)

which is to be substituted into (5.31 d).

Starting values of the velocity components \( u \) and \( v \) for the computation are obtained from the Blasius series \(^{(17)}\). In the neighborhood of the stagnation-point, the Blasius series can be written as follows,

\[ u^* = U_1^* s^* f' (N) \]  \hspace{1cm} (5.31 a)

\[ v^* = - \sqrt{\frac{2}{\rho}} U_1^* f (N) \]  \hspace{1cm} (5.31 b)

where \( U_1^* \) is defined by equation (5.20) and \( f \) is the solution to equation (3.21).
In dimensionless form, the above equations become

\[ \bar{u} = \frac{2}{3} \bar{s} f (\bar{N}) \]  
\[ \bar{v} = -\frac{1}{\sqrt{3}} f (\bar{N}) \]  

(5.32 a)  
(5.32 b)

Upon substituting \( R^* = \sqrt{\frac{4}{5}} \) (obtained by setting \( \psi (R^*) = 0 \) in equation (5.17 b)) into equation (5.18), we obtain the following inviscid velocity component,

\[ U = \frac{2}{3} \sin \bar{s} \text{ as } N \text{ approaches infinity.} \]  

(5.33)

We assume that a linear relation exists between \( u_{m+1,n+1} \) and \( u_{m+1,n} \) as expressed by equation (3.12). The no-slip boundary condition at the wall leads to \( E_o = D_o = 0 \). Then we can compute \( u_n \) at station \( m+1 \) in a way similar to that used in Chapter III.

Integrating equation (5.30 a) leads to the following equations for the velocity component \( v \),

\[ \bar{v} = -\int_{0}^{\bar{N}} \frac{\partial \bar{u}}{\partial \bar{s}} \ d\bar{N} - \tan \bar{s} \int_{0}^{\bar{N}} \bar{u} \ d\bar{N} \]  

(5.33)

The above integral is carried out numerically by using (3.2 c).

The inviscid velocity component adjacent to the body surface can be approximated from equation (5.33) as
\[ \nabla m + 1, n + \Delta n = \nabla m + 1, n - \frac{4}{3} (\cos \theta) \Delta N \]  \hspace{1cm} (5.34)

D. Discussion of the Results

The solutions by the truncated series method and the implicit finite-difference method are presented in Figure 5.2 to 5.10 for the flow with \( \text{Re}_s = 49, 100 \) and 900. The solutions obtained from Lighthill's constant-density approximation for which \( \text{Re}_s = \infty \) are also plotted in the figures for comparison.

Figure 5.2 shows that the shock is approximately spherical even at low Reynolds number. Thus the approximation of a spherical shock is good for the range \( s = 0 \) to \( s = 0.8 \).

The coefficients \( u_1, P_1, \) and \( P_3 \) in the truncated series (5.3 a-b) are plotted in Figures 5.3 and 5.4. Based on these coefficients the velocity \( u \) and the pressure \( P \) are computed. These results are plotted in Figures 5.5 a-c and 5.8 a-c. The pressure distributions evaluated by the above method are compared with the results evaluated by the finite-difference method in Figures 5.8 a-c and 5.10 a-d.

In the region near the stagnation point the velocity distribution, evaluated by the truncated series method, is in close accord with the results obtained by the finite-difference method (see Figures 5.5 a-c).
However, the above agreement is not extended to the region away from the stagnation-point.

In Figures 5.5 b-c, the approximate positions of the outer edge of the boundary-layer are observed for the flow with $Re_s = 100$ and 900. At low Reynolds numbers, as in the case $Re_s = 49$, the boundary-layer extends to the shock (see Figure 5.5 a).

The velocity distributions based on Lighthill's constant-density approximation is compared with the case $Re_s = 900$ in Figure 5.5 c. The curves obtained by Lighthill's solutions and those by the finite-difference method show a similar trend. (See the shock layer between the shock and the outer edge of the boundary-layer, Figure 5.5 c.)

The distribution of the velocity component $v$ evaluated by the finite-difference method is shown in Figures 5.6 a-b and 5.7 a-c.

Figures 5.9 a-d show the variation of skin-friction along the body surface. The deviations in the results by the truncated series method and the finite-difference method increase as the distance $s$ increases. In Figure 5.9 d the skin-frictions, obtained by the finite-difference method, are compared with those based on the boundary-layer theory for which $Re_s = \infty$. 
Figure 5.2 Radial Distance to the Shock at Various Reynolds Numbers
Figure 5.3 Distribution of $u_1$ at Various Reynolds Numbers (Truncated Series Method).
Figure 5.4 Distribution of $R_1$ and $R_2$ at Various Reynolds Numbers (Truncated Series Method).
Figure 5.5a  Velocity Distribution in the Shock Layer for $Re_s = 49$
Figure 5.5b Velocity Distribution in the Shock Layer for $Re_s = 100$. 

$Re_s = 100$

---

**FINITE-DIFFERENCE METHOD**

---

**TRUNCATED SERIES METHOD**

---

$R =$ Radial Distance from the Center of a Sphere (Unit Radius)

---

$u =$
Figure 5.5 c: Velocity Distribution in the Shock Layer for $Re_s = 900$. 

Legend:
- **Solid Line**: Finite-Difference Method
- **Dash Line**: Truncated Series Method
- **Dashed-Dot Line**: Lieghtill's Constant-Density Approximation
Figure 5.6a Distribution of Velocity Component $v$ at Various Reynolds Numbers for $s = 0$ (Truncated Series Method).
Figure 5.6 b Distribution of Velocity Component v at Various Reynolds Numbers for $s = .40$ (Finite-Difference Method).
Figure 5.7  a Distribution of Velocity Component $v$ at $Re_s = 49$. 

$R = \text{Radial Distance from the Center of a Sphere (Unit Radius)}$

$Re_s = 49$

Shock $s = 0.8, 0.6, 0.4, 0.2$

BY FINITE-DIFFERENCE METHOD EXCEPT $s = 0$ FROM TRUNCATED SERIES METHOD
Figure 5.7 b Distribution of Velocity Component v at $Re_s = 100$. 

BY FINITE-DIFFERENCE METHOD EXCEPT $S = 0$ FROM TRUNCATED SERIES METHOD
Figure 5.7c Distribution of Velocity Component $v$ at $Re_s = 900$. 

METHOD EXCEPT FROM TRUNCATED RIDGE METHOD
Figure 5.8 a Pressure Distribution for the Flow with $Re_s = 49$. 

$Re = 49$

$s = 0.8$

$S = 0.6$

$S = 0.4$

$S = 0.2$

$P$

$R = \text{radial distance from the center of a sphere (unit radius)}$
Figure 5.8b Pressure Distribution for the Flow with $Re = 100$. 

- **SHOCK**
- **FINITE-DIFFERENCE METHOD**
- **TRUNCATED SERIES METHOD**
Figure 5.8c  Pressure Distribution for the Flow with Re = 900.
INTERPRETATION OF CURVES IN FIGURES 5.9 a-d

Methods and Equations used in the Computations

Curve (1)  Truncated Series Method

\[ (\tau_f)_{N=0} = \left[ \frac{d u_1 (N)}{d N} \right]_{N=0} \sin (s) \]

Curve (2)  Finite-Difference Method

\[ \frac{dP}{ds} = P_2 \sin (2s) \]

Curve (3)  Finite-Difference Method

Predicted Value

\[ \left( \frac{dP}{ds} \right)_{m+1} = 2 \left( \frac{dP}{ds} \right)_m - \left( \frac{dP}{ds} \right)_{m-1} \]

Corrected Value

\[ \left( \frac{dP}{ds} \right)_{m+1} = \left( 3 P_m + 1 - 4 P_m + P_{m-1} \right) / (2 \Delta s) \]

Curve (4)  Boundary-Layer Theory with \( \frac{dP}{ds} = -\frac{25}{18} \sin (2s) \)

From Lighthill's Constant-Density Solution
Figure 5.9a Variation of Skin Friction for $Re_s = 49$. 
Figure 5.9 b  Variation of Skin Friction for $Re_s = 100$. 
Figure 5.9c Variation of Skin Friction for Reₙ = 900.
Figure 5.9 d  Variation of Skin Friction at Various Reynolds Numbers.
INTERPRETATION OF CURVES IN FIGURES 5.10 a-d

Methods and Equations used in the Computations

Curve (1) Finite-Difference Method

\[ P = P_0 + \varepsilon \rho \int_0^N \left[ \frac{u^2}{1 + \varepsilon N} - \frac{\varepsilon u v_s}{1 + \varepsilon N} + \varepsilon v v_N \right] dN \]

Curve (2) Truncated Series Method

\[ P = P_1 + P_2 \sin^2 s \]

<table>
<thead>
<tr>
<th>Re</th>
<th>( P_2 )</th>
<th>( P_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>49</td>
<td>1.292836</td>
<td>-</td>
</tr>
<tr>
<td>100</td>
<td>1.303816</td>
<td>.916620</td>
</tr>
<tr>
<td>900</td>
<td>1.322773</td>
<td>-</td>
</tr>
</tbody>
</table>

Curve (3) Lighthill's Constant-Density Solution

\[ P = .9166 - 1.389 \sin^2 s \]

Curve (4) Finite-Difference Method

\[ P = P_0 + \varepsilon \rho \int_0^N \frac{u^3}{1 + \varepsilon N} dN \]
Figure 5.10 a Variation of Surface Pressure Along the Body Surface for $Re_s = 49$
Figure 5.10 b  Variation of Surface Pressure Along the Body Surface for $Re_s = 100$. 
Figure 5.10 c  Variation of Surface Pressure Along the Body Surface for $Re_s = 900$. 

$P_0 = \text{SURFACE PRESSURE}$ 

$S$ 

$Re_s = 900$
Figure 5.10d  Variation of Surface Pressure Along the
Body Surface for Various Reynolds Numbers.
The above computations were carried out on the IBM 7040 electronic digital computer. The computation, based on the finite-difference method, took approximately two seconds per step in \( \Delta s \) for approximately 90 steps in \( \Delta N \). The size of the steps in \( \Delta s \) and \( \Delta N \) are .005 and .02, respectively.

E. Conclusions

The problem of the laminar flow past a sphere at a high Mach number is solved with the assumptions that (1) the flow after the shock has constant density and viscosity, and (2) the shape of the shock is nearly spherical. The first assumption is true in the stagnation region especially if the flow is hypersonic and the temperature of the sphere is near to the stagnation-temperature. The second assumption is justified to be appropriate for the region \( s = 0 \) to \( s = .8 \).

The Navier-Stokes equations due to Van Dyke\(^{(21)}\) and Maslen\(^{(13)}\) are used to solve the problem. These equations are solved by the implicit finite-difference method and the truncated series method. The computations are made for the distribution of velocities, pressure, and skin-friction in the case of \( Re_s = 49, 100 \) and 900. The solutions by both methods are in good accord in the stagnation-point region. The deviations of both solutions increase in downstream as would be expected since the truncated
A series solution is not accurate away from the stagnation-point. The velocity distributions based on Lighthill's constant-density solution are compared with the case Re_s = 900, which are also solved by both of the above mentioned methods. The curves for the velocity distributions show similar trends. For the flow with Re_s = \infty, the Navier-Stokes equations reduce to the boundary-layer equations. These equations were solved by the implicit finite-difference method. In this case the skin-friction was compared with the flow where Re_s = 49, 100, and 900.
ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation to his advisor, Dr. R. T. Davis, for suggesting the topic and the invaluable guidance rendered throughout this investigation.

The author also extends his gratitude to the members of his graduate committee, Professors J. B. Eades, Jr., D. Frederick, S. T. Gormsen, and H. L. Wood for their criticisms of this work.

The author is indebted to his wife, Nancy, for her patience and encouragement during the years of his graduate study.
REFERENCES


7. Goldstein S., 1929, "Concerning some solutions of the Boundary-
Layer Equations in Hydrodynamics" Proceedings of the Cambridge
Philosophical Society, Volume xxvi.
8. Goldstein S., 1933, "On the Two-Dimensional Steady flow of a
Viscous Fluid Behind a Solid Body—I and II" Proceedings of the
9. Imai, I. (1957) "Second Approximation to the Laminar Boundary-
Layer Flow over a Flat Plate" Jr. of Aeronaut. Sci. 24,155-156.
Streamline of a Blunt Body: I. A Test of Local Similarity
11. Third-Order Boundary-Layer Theory and Comparison with Other
Past a Flat Plate at Moderate Reynolds numbers. Jr. Math. and
Phys. 32.
12. Lighthill M. J., Jan. 1957, "Dynamics of a Dissociating Gas,
Boundary-Layer over a Yawed Infinite Cylinder with Heat Transfer
15. Richtmyer, R. D., 1957, "Difference Methods for Initial-Value


The vita has been removed from the scanned document
Based on conservation of mass between the body and the shock, we can derive an equation as follows,

\[
\frac{(1 + \varepsilon \bar{N}_s)^2 \sin s}{2 \kappa \varepsilon} = \int_0^{\bar{N}} \bar{u} (1 + \varepsilon \bar{N}) \, d\bar{N} \quad \text{(A. 1 a)}
\]

We denote

\[
I_n = \int_0^{\bar{N}} \bar{u} (1 + \varepsilon \bar{N}) \, d\bar{N} \quad \text{(A. 2 a)}
\]

This integration, after using the relation (3.2c) becomes

\[
I_{m+1, n+1} = I_{m+1, n} + \frac{\Delta \bar{N}}{12} \left[ 5 \bar{u}_m + 1, n (1 + \varepsilon \bar{N}) + 1, n + 8 \bar{u}_m + 1, n + 1 (1 + \varepsilon \bar{N}) + 1, n + 1 - \bar{u}_m + 1, n + 2 (1 + \varepsilon \bar{N}) + 1, n + 2 \right] \quad \text{(A. 3 a)}
\]

Hereafter the subscript \( m+1 \) is omitted. Expressing \( \bar{u}_n \) and \( \bar{u}_n + 2 \) in terms of \( \bar{u}_n + 1 \) by the use of (3.12), equation (A. 3 a) becomes

\[
I_{n+1} = I_n + \frac{\Delta \bar{N}}{12} (M_n + 1 \bar{u}_n + 1 + Q_n + 1) \quad \text{(A. 4 a)}
\]

where \( M_n + 1 = 5 (1 + \varepsilon \bar{N})_n E_n + 8 (1 + \varepsilon \bar{N})_n + 1 - \frac{1}{E_n + 1} (1 + \varepsilon \bar{N})_{n+2} \quad \text{(A. 4 b)} \)
\[ Q_n + 1 = 5 (1 + e N) D_n + \frac{D_n + 1}{E_n + 1} (1 + e N) n + 2 \]  
\hspace{1cm} (A. 4 c)

The equation (A. 4 a) can also be written in the following form:

\[ I_n + 1 = \frac{\Delta N}{12} (L_n + 1 \bar{u}_n + 1 + P_n + 1) + \frac{\Delta N}{12} (M_n + 1 \bar{u}_n + 1 + Q_n + 1) \]  
\hspace{1cm} (A. 5 a)

Similarly,

\[ I_n = \frac{\Delta N}{12} (L_n \bar{u}_n + P_n) + \frac{\Delta N}{12} (M_n \bar{u}_n + Q_n) \]  
\hspace{1cm} (A. 5 b)

Substituting (A. 5 b) into (A. 4 a), and expressing \( \bar{u}_n \) in terms of \( \bar{u}_n + 1 \) by the use of (3.12) we obtain

\[ I_n + 1 = \frac{\Delta N}{12} \left[ (L_n E_n + M_n E_n) \bar{u}_n + 1 + L_n D_n + M_n D_n + P_n + Q_n \right] + \frac{\Delta N}{12} (M_n + 1 \bar{u}_n + 1 + Q_n + 1) \]  
\hspace{1cm} (A. 6 a)

Comparing the above equation with (A. 5 a), it is found that

\[ L_n + 1 = (L_n + M_n) E_n \]  
\hspace{1cm} (A. 7 a)

\[ P_n + 1 = (L_n + M_n) D_n + P_n + Q_n \]  
\hspace{1cm} (A. 7 b)

Thus, equation (A. 2 a) can be represented in the difference form as follows,

\[ I_n = \frac{\Delta N}{12} \left[ (L_n + M_n) \bar{u}_n + P_n + Q_n \right] \]  
\hspace{1cm} (A. 8 a)
where

\[ L_n = (L_n - 1 + M_n - 1) E_n - 1 \quad (A. 8 \, b) \]

\[ P_n = (L_n - 1 + M_n - 1) D_n - 1 + P_n - 1 + Q_n - 1 \quad (A. 8 \, c) \]

\[ M_n = 5 (1 + \varepsilon \overline{N})_n - 1 E_n - 1 + 8 (1 + \varepsilon \overline{N})_n - \]

\[ \frac{1}{E_n} (1 + \varepsilon \overline{N})_n + 1 \quad (A. 8 \, d) \]

\[ Q_n = 5 (1 + \varepsilon \overline{N})_n - 1 D_n - 1 + \frac{D_n}{E_n} (1 + \varepsilon \overline{N})_n + 1 \quad (A. 8 \, e) \]
ABSTRACT

This paper deals with three problems, (1) laminar incompressible viscous flow past a cylinder and a sphere, (2) laminar incompressible viscous flow past a finite flat plate (second-order solutions), and (3) laminar viscous flow past a sphere at a high Mach number. These problems are solved by using an implicit finite-difference method.

The first problem (flow past a sphere and a cylinder) involves the classical boundary-layer equations which are the first approximation to the Navier-Stokes equations in a region near to the body surface for high Reynolds number. The computational results were obtained for the distribution of velocity components in the boundary-layer, and the variation of skin-friction and displacement-thickness along the body.

The second problem (second-order flow past a finite flat plate) involves the second-order boundary-layer equations which introduce only the effect of the displacement-thickness in the case of flow past a flat plate. An assumption is made that the displacement-thickness is constant in the wake behind the flat plate. The adequacy of this assumption is checked from solutions based on the calculated displacement-thickness in the wake. The wake behind the finite flat plate is assumed laminar, and its displacement-thickness and the velocity distribution are computed
downstream, by using the implicit finite-difference method.

In the third problem (high Mach number flow past a sphere), constant density is assumed in the shock layer. This is nearly true in the stagnation-point region, especially if the flow is hypersonic and the temperature of the sphere is nearly the same as the stagnation-temperature. It is also assumed that the shock is nearly spherical, even though it is not spherical as it is in the inviscid case. The numerical results will show that the assumption of a spherical shock will, however, nearly be true. This problem involves the solution of the complete Navier-Stokes equations. These equations are solved for various Reynolds numbers by two methods; namely the truncated series method and the implicit finite-difference method.

The solutions by the implicit finite-difference method are in excellent accord with those obtained by the series solutions in the stagnation-point region. As the computation by the finite-difference method proceeds downstream, the deviation of the finite-difference solution from the series solution increases. This is due to the fact that the series is valid only around the stagnation-point, and is thus expected to give inaccurate solutions downstream. The finite-difference method has no such restrictions, however, and gives accurate results in the whole flow field. In conclusion, solutions by the implicit finite-difference method have proven not only to be accurate but also to be stable in all examples computed.