Stochastic Programming Approaches to Multi-product Inventory Management Problems with Substitution

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(ABSTRACT)

The presence of substitution among multiple similar products plays an important role in inventory management. It has been observed in the literature that incorporating the impact of substitution among products can substantially improve the profit and reduce the understock or overstock risk. This thesis focuses on exploring and exploiting the impact of substitution on inventory management problems by theoretically analyzing mathematical models and developing efficient solution approaches.

To that end, we address four problems. In the first problem, we study different pricing strategies and the role of substitution for new and remanufactured products. Our work presents a two-stage model for an original equipment manufacturer (OEM) in this regard. A closed-form one-to-one mapping of product designs onto the optimal product strategies is developed, which provides useful information for the retailer.

Our second problem is a multi-product newsvendor problem with customer-driven demand substitution. We completely characterize the optimal order policy when the demand is known and reformulate this nonconvex problem as a binary quadratic program. When the demand is stochastic, we formulate the problem as a two-stage stochastic program with mixed integer recourse, derive several necessary optimality conditions, prove the submodularity of the profit function, develop polynomial-time approximation algorithms, and show their performance guarantees. Our numerical investigation demonstrates the effectiveness of the proposed algorithms and, furthermore, reveals several useful findings and managerial insights.

In the third problem, we study a robust multi-product newsvendor model with substitution (R-MNMS), where both demand and substitution rates are uncertain and are subject to cardinality-constrained uncertainty set. We show that for given order quantities, computing the worst-case total profit, in general, is NP-hard, and therefore, address three special cases for which we provide closed-form solutions.

In practice, placing an order might incur a fixed cost. Motivated by this fact, our fourth problem extends the R-MNMS by incorporating fixed cost (denoted as R-MNMSF) and develop efficient approaches for its solution. In particular, we propose an exact branch-and-cut algorithm to solve small- or medium-sized problem instances of the R-MNMSF, and for large-scale problem instances, we develop an approximation algorithm. We further study effects of the fixed cost and show how to tune the parameters of the uncertainty set.

Stochastic Programming Approaches to Multi-product Inventory Management Problems with Substitution

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(GENERAL AUDIENCE ABSTRACT)

In a multi-product supply chain, the substitution of products arises if a customer's first-choice product is out-of-stock, and she/he have to turn to buy another similar product. It has been shown in the literature that the presence of product substitution reduces the assortment size, and thus, brings in more profit. However, how to quantitatively study and analyze substitution effects has not been addressed in the literature. This thesis fills this gap by developing and analyzing the profit model, and therefore, providing judicious decisions for the retailer to make in order to maximize their profit.

In our first problem, we consider substitution between new products and remanufactured products. We provide closed-form solutions, and a mapping that can help the retailer in choosing optimal prices and end-of-life options given a certain product design.

In our second problem, we study multi-product newsvendor model with substitution. We first show that, when the probability distribution of customers' demand is known, we can tightly approximate the proposed model as a stochastic integer program under discrete support. Next, we provide effective solution approaches to solve the multi-product newsvendor model with substitution.

In practice, typically, there is a limited information available on the customers' demand or substitution rates, and therefore, for our third problem, we study a robust model with a cardinality uncertainty set to account for these stochastic demand and substitution rates. We give closed-form solutions for the following three special cases: (1) there are only two

products, (2) there is no substitution among different products, and (3) the budget of uncertainty is equal to the number of products.

Finally, similar to many inventory management problems, we include a fixed cost in the robust model and develop efficient approaches for its solution. The numerical study demonstrates the effectiveness of the proposed methods and the robustness of our model. We further illustrate the effects of the fixed cost and how to tune the parameters of the uncertainty set.

Dedication

To my mother Chunlan and father Zhinan.

Acknowledgments

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Chapter 1

Introduction

1.1 Background and Motivation

As one of the classic problems in inventory management, the newsvendor model has been successfully applied to many areas including airline, health care, retail industry, among others. In the newsvendor model, the retailer optimizes the inventory policy to maximize its profit, where the customers' demand is assumed to be stochastic, salvage value is incurred when the order quantity exceeds the demand, and a penalty cost is encouraged when the demand is unsatisfied. Although useful in many areas, the newsvendor model has only been applied to a single product, while multi-product inventory management problems are commonly encountered in practice. An important feature that arises for these problems is the mutual substitution of products because of their similar function, color, style, size, or price (cf. [91]).

In fact, many e-commerce supply chains typically involve products with many alternatives. As it has also been observed by others (cf. [26]), often, such products can substitute each other's demand proportionally, in particular, when some of them are out-of-stock. For instance, when shopping at Amazon.com, a customer might turn to the green luggage if the orange one is out of stock. This phenomenon is also known as "*customer-driven demand substitution*".

As observed by many researchers (cf, [45]), there also exists a substitution between new products and remanufactured products. For example, Marshall and Archibald [45] built a semi-Markov decision process to derive the production, remanufacturing, and two-way substitution decisions. Inderfurth [36] studied the expected profit model with downward substitution by incorporating stochastic returns of used products and stochastic demands of serviceable products. Their work highlighted the importance of coordinating the manufacturing and remanufacturing systems in order to maximize the expected profit. However, they did not consider product design in their analysis, which can significantly impact the customers' demand, end-of-life options, and salvage value. The fact that the design for remanufacturing can result in significant profits has been recognized by companies like Kodak, BMW, IBM, SMEA, DEC, and Xerox (cf. [29, 44]). There has been very limited work reported in the literature that jointly considers substitution, design, prices, and end-of-life options, all of which are important to help the retailer make better decisions.

For the general multi-product supply chain, it is important for the retailer to determine and maintain a proper inventory level in the presence of substitution. As far as we know, only a limited amount of work has been reported that explores approaches to solve the multi-product newsvendor model with customer-driven substitution. Besides, due to the stochasticity and unpredictability of customers' demand or substitution rates, it is challenging for the retailer to make judicious production planning-related decisions as well as effectively manage resources for the supply chain companies of these products. Moreover, the information of historical data for the customers' demand and substitution rates may not be sufficient to make an accurate prediction of their true probability distributions, and a misleading distribution may cause inferior decisions. Therefore, modeling and analyzing these difficulties, as well as developing efficient algorithms for their implementation in multi-product supply chain environments, are essential for the retailer to make effective decisions in order to maximize its profit.

1.2 Research Objectives and Questions

As alluded to earlier, it adds to modeling challenges when there is only a limited data available for customers' demand and substitution rates. Also, the non-convexity and non-smoothness of the models as a result of incorporating the features of substitution and stochasticity further complicate the profit model. This thesis develops mathematical tools to address these challenges. In particular, we address the following main research questions in this thesis:

(i) In the face of substitution between new and remanufactured products for primary customers and green customers in a closed-loop supply chain, how to formulate the profit model that also incorporates selection of product design? What is the best decision for the retailer?

(ii) If the demand distribution is known, what should be the optimal order quantities in multi-product newsvendor model with customer-driven substitution? Are there efficient algorithms to solve it?

(iii) If the data is not enough to predict the demand and substitution rates, how can we formulate the model to maximize total profit, how can we solve this model, and can we provide useful insights for the retailer?

(iv) Can we extend the analysis and develop an algorithm to the case where a fixed cost is non-negligible? What insights can be obtained to help the retailer make judicious decisions?

1.3 Dissertation Organization

This thesis is organized as follows.

In Chapter 2, we present a two-stage model for the reverse supply chain of an original equipment manufacturer (OEM) to study the impact of product design and the end-of-life options. In Chapter 3, we study a multi-product newsvendor problem with customer-driven demand substitution, where each product, once run out of stock, can be proportionally substituted by the others. In Chapter 4, we investigate robust multi-product newsvendor model with Substitution (R-MNMS), where the demand is stochastic and is subject to cardinality-constrained uncertainty set. In Chapter 5, we study multi-product newsvendor model with substitution rates and fixed cost (R-MNMSF) and develop algorithms to solve it.

Chapter 2

Design, Pricing, and End-of-life Option of a Product

2.1 Introduction

In this chapter, we address a problem faced by an OEM to price a product and select its end-of-life (EOL) option. The EOL options that we consider are remanufacturing, salvage, and disposal. Remanufacturing of a product is the process of making it like a new product. Salvaging a product refers to selling it as is, while a product that cannot be remanufactured or salvaged is disposed of. The EOL options for a product can be affected by its characteristics, design, and pricing (cf. [78]). Research reported in the reverse supply chain area has been devoted to designing and planning of the supply chain, product pricing and coordination, production and planning, inventory management, and vehicle routing problem (cf. [28, 77]). However, product design is an important consideration for an OEM because of its impact on overall profit, EOL option, and the environment. The design for remanufacturing can be categorized into design for disassembly, design for the environment, and design for the EOL option (cf. [32]). Debo et al. [19, 20], and Robotis et al. [63] have referred to product design as "remanufacturability level". A value of zero means that the product cannot be remanufactured. Ramoni and Zhang [60] have developed a metric to quantify quality level, which is incorporated into a new product design. Sundin [81] has analyzed interchangeable design in remanufacturing. Product design is a complex issue, and it pertains to determining characteristics of a product, choice of its quality/reliability, and choice of joining techniques (cf. [80]). Omwando et al. [52] have presented a decision support system based on a bi-level fuzzy computing approach to determine the remanufacturability of a product, while Farahani et al. [24] study the remanufacturing process and have specified a return quality threshold. Arredondo-Soto et al. [3] introduced the tools, methods, and techniques for application in remanufacturing. Steeneck and Sarin [79] explore how the trend of extended producer responsibility affects product design when the OEM leases their products and then remanufacture them at the end of a lease period. Haziri and Sundin [33] present a framework that supports design for remanufacturing, which aims at outlining and implementing information feedback from remanufacturing to product design. In this chapter, we define product design as its yield, which can take a value between 0 and 1. A value of zero indicates that a collected product cannot be used for remanufacturing or for salvage, while a value of 1 implies that it can be used like new. Our objective is to study the impact of a product design on an OEM's decisions at the end of its useful life. *We explore relationships between product design, product pricing, and product's EOL options, and define a combination of product pricing and EOL option as a "product strategy".* Although Atasu et al. [5] have presented the concept of high pricing and low pricing, and Wu [86] has studied product design and its pricing strategies, the utility function that they consider is independent of product design, and the product design only affects its production cost. Moreover, there are just a few papers that consider the impact of product design on the EOL options for an OEM (cf. [4, 29]). However, we provide a more comprehensive study by considering the impact of product design on the utility, demand, and salvage value functions, which would enable determination of product design, product price, and EOL option simultaneously in order to maximize profit. In particular, we address the following questions in this chapter:

(i) How should an OEM price their remanufactured and new products, and what should be the best EOL option for a product?

(ii) What is the relationship between product design, product price, and EOL options? More specifically, given a certain product design, which optimal product strategy should the OEM select?

2.2 Preliminaries

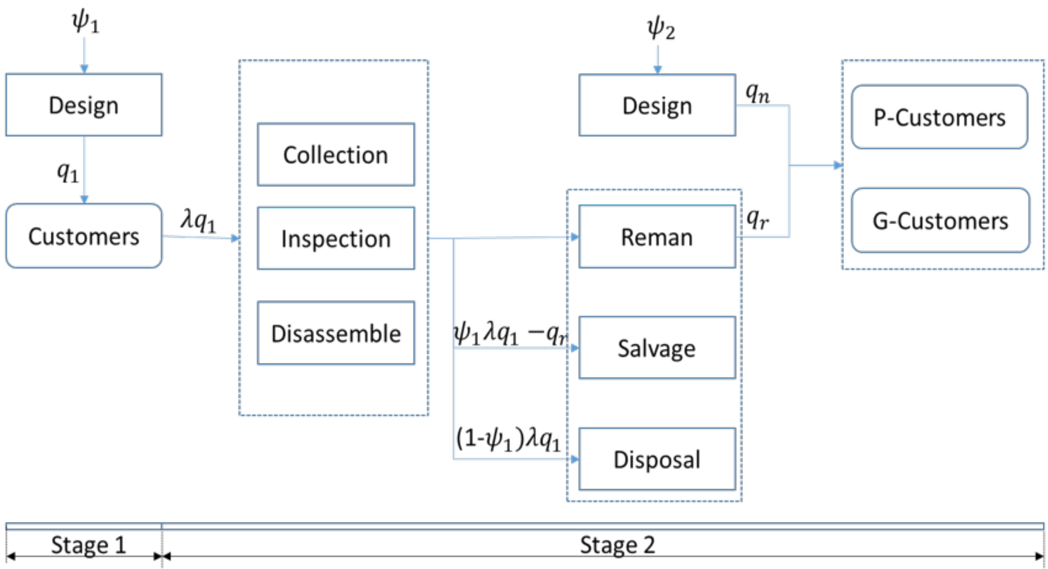


Figure 2.1: System description.

For our analysis, we model the problem at hand as a two-stage process. An OEM designs and sells its new products at the first stage. At the second stage, the OEM collects, inspects, and dissembles those products at the end of their useful lives. Not only that the recovered products undergo the EOL options of remanufacturing, salvage, or disposal, but also, the remanufactured products can compete with new products. Therefore, we assume existence of primary customers' demands and green customers' demands for both new and remanufactured products at the second stage. For the sake of convenience, the features present at both the stages are presented in Figure 2.1.

The notation that we use is presented in Table 2.1. For the sake of completeness, we consider three cases. Case 1: The product design at Stage 2 is different from that at Stage 1; however, the collected products (produced at Stage 1) are upgraded (while remanufacturing) to include features of Stage 2 design Case 2: The product design at Stage 2 is identical to that of Stage 1. Case 3: The product design at Stage 2 is different from that at Stage 1; however, the recovered product is remanufactured retaining its Stage 1 design. In the sequel, we analyze all of these cases. Case 1 is the most general and is presented in detail.

2.2.1 Production Cost Function

Production cost is typically an increasing function of product design because of the use of more complex production processes (cf. [20]). Production cost has been used as a linear function of product design in [58], as a quadratic function in [4, 5, 32, 48, 53] and as a convex function in [19]. Here, we assume production cost of new and remanufactured products to be a quadratic function of product design (ψ_2). We also discuss the remanufactured products are increased to ψ_1 in Section 2.5.

2.2.2 Salvage Value Function

Salvage value reflects asset depreciation and it is an increasing function of quality level (cf. [25]). Steeneck and Sarin [78] have introduced an increasing "knee-in-curve" function for the quality level. Guide et al. [30] have assumed the salvage value to be equal to the remanufacturing cost. Bakal and Akcali [7] assume salvage value of the returned products

Table 2.1: Notation

θ	Each customer's willingness to pay for a product except for primary customers' willingness to pay for remanufactured products, which is denoted as $\kappa\theta$, where $0 < \kappa < 1$.
ψ_i	Product design at the *ith* stage.
U_n^p	Utility of primary customers for new products.
U_n^g	Utility of green customers for new products.
U_r^p	Utility of primary customers for remanufactured products.
U_r^g	Utility of green customers for remanufactured products.
q_n^p	Primary customers' demand for new products.
q_n^g	Green customers' demand for new products.
q_r^p	Primary customers' demand for remanufactured products.
q_r^g	Green customers' demand for remanufactured products.
p_1	Unit price of new products at Stage 1.
q_1	Quantity of new products produced at Stage 1.
α	Impact factor of product design on a new products production cost.
β	Impact factor of product design on a remanufactured product's production cost.
$\gamma\psi_1$	Unit salvage value at Stage 2.
q_n	Demand of new products at Stage 2 ($= q_n^p + q_n^g$).
q_r	Demand of remanufactured products at Stage 2 ($= q_r^p + q_r^g$).
q_s	Quantity of the collected products used for salvage.
p_n	Unit price of new products at Stage 2.
p_r	Unit price of remanufactured products at Stage 2.
λ	Collection rate.
C_c	Unit collection cost (includes the acquisition price and transportation cost).
C_d	Unit disassembling cost.
C_{disp}	Unit disposal cost.
Π	Profit function.

to be the same regardless of its quality level and it is proportional to the recyclable material content. Guide et al. [30] have studied salvage value under different scenarios, and they point out that when the demand is high, the salvage value is related to quality level; and otherwise, it is independent of quality level. We assume salvage value at Stage 2 to be a linear function of its product design (ψ_1).

2.3 Discussion of Case 1

2.3.1 Market Definition and Demand Function

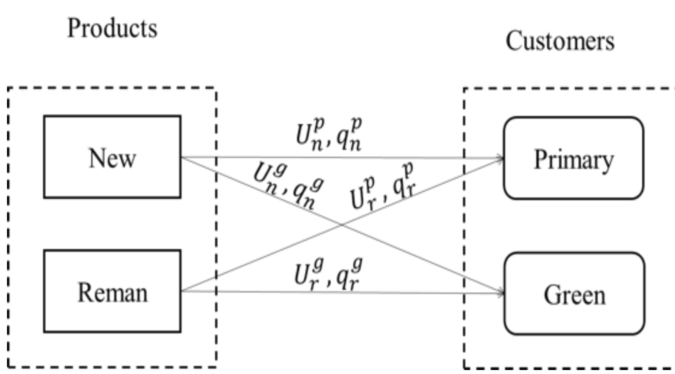


Figure 2.2: Utility and demand for products under segmented market.

The market size at Stage 1 is designed by A, and in the second period, the market size is designed by ΔA, which depends on the products' position in its life cycle and Δ is a scalar parameter. We assume the market at Stage 2 to consist of two types of customers: primary customers and green customers. The green customers' proportion is $\rho < 1$, that is, the green customers' size is $\rho\Delta A$, and the primary customers' size is $(1-\rho)\Delta A$. When the product is in its growth (decline) phase of its life cycle, the market size of the second period expands (shrinks), so that $\Delta > 1$ ($\Delta < 1$). The market consists of two types of customers: primary customers and green customers. The customers' willingness to pay (WTP) for a product is denoted as θ, where $\theta \sim U[0,1]$. In the following analysis, we assume $\theta = \frac{1}{2}, A = 1, \Delta = 1$ to keep the market size unbiased for both product types. Both the customer types are not only sensitive to product price, but also, they are sensitive to product design. At the first stage, only new products are in the market, and therefore, there is no competition. We assume the utility functions for both the primary and green customers to be dependent on the prices of new and remanufactured products as well as on product design. Moreover, the primary customers have a lower valuation for the remanufactured products than that for new products. We define this as a discounted value for remanufactured products, which we denote as $\kappa(0 < \kappa < 1)$ of the WTP for new products. Accordingly, the utility functions of the primary customers for new and remanufactured products are as follows: $U_n^p = \psi_2\theta - p_n$, $U_r^p = \kappa\psi_2\theta - p_r$, $U_n^g = \psi_2\theta - p_n$, and $U_r^g = \psi_2\theta - p_r$. The number of primary customers to buy new and remanufactured products are χ_n^p and χ_r^p, respectively, where $\chi_n^p = \{\theta, U_n^p \geq \max\{U_r^p, 0\}\}$, $\chi_n^p = \{\theta, U_r^p \geq \max\{U_n^p, 0\}\}$. Similarly, the sets of green customers to buy new products and remanufactured products are: χ_n^g and χ_r^g, where $\chi_n^g = \{\theta, U_r^g \geq \max\{U_r^g, 0\}\}$, $\chi_n^g = \{\theta, U_r^g \geq \max\{U_n^g, 0\}\}$.

Thus, we have the following proposition.

Proposition 2.1. *The demand functions for new and remanufactured products are as follows.*

(i) If $\kappa p_n < p_r < p_n$, the demand functions of new and remanufactured products are:

$$q_n = q_n^p + q_n^g = \frac{1}{2}(1 - \frac{p_r}{\kappa\psi_2}) \tag{2.1a}$$

$$q_r = q_r^p + q_r^g = \frac{1}{2}(1 - \frac{p_r}{\psi_2}) \tag{2.1b}$$

(ii) If $p_n - (1-\kappa)\psi_2 < p_r \leq \kappa p_n$, we have

$$q_n = q_n^p + q_n^g = \frac{1}{2}(1 - \frac{p_n - p_r}{(1-\kappa)\psi_2}) \tag{2.2a}$$

$$q_r = q_r^p + q_r^g = \frac{1}{2}(\frac{p_n - p_r}{(1-\kappa)\psi_2} - \frac{p_n}{\psi_2}) + \frac{1}{2}(1 - \frac{p_r}{\psi_2}) \tag{2.2b}$$

Proof. For primary customers:

(i) $\chi_n^p = \{\theta \geq \frac{p_r}{\psi_2 \kappa}, \theta \geq \frac{p_n - p_r}{(1-\kappa)\psi_2)}\}, U_n^p > U_r^p \geq 0$

(a) $\frac{p_r}{\psi_2 \kappa} > \frac{p_n - p_r}{(1-\kappa)\psi_2}$ that is, $p_r > \kappa p_n$, $\frac{p_n}{p_r} < \kappa \leq 1$, $\theta \geq \frac{p_r}{\kappa \psi_2}$. Therefore, the demand of primary customers for the new products,

$$q_n^p = \frac{1}{2}\int_{\frac{p_r}{\kappa\psi_2}}^1 d\theta = \frac{1}{2}\left(1 - \frac{p_r}{\kappa\psi_2}\right)$$

(b) $\frac{p_r}{\kappa\psi_2} \leq \frac{p_n - p_r}{(1-\kappa)\psi_2}$ that is, $p_r \leq \kappa p_n$, $\theta \geq \frac{p_n - p_r}{(1-\kappa)\psi_2}$. Therefore, to determine the demand of primary customers for new products, we need to consider whether $\frac{p_n - p_r}{(1-\kappa)\psi_2}$ is greater than one or not.

If $\frac{p_n - p_r}{(1-\kappa)\psi_2} \leq 1$, then $\kappa \leq 1 - \frac{p_n - p_r}{\psi_2}$, $q_n^p = \frac{1}{2}\int_{\frac{p_n - p_r}{(1-\kappa)\psi_2}}^1 d\theta = \frac{1}{2}\left(1 - \frac{p_n - p_r}{(1-\kappa)\psi_2}\right)$.

If $\frac{p_n - p_r}{(1-\kappa)\psi_2} > 1$, then $\kappa > 1 - \frac{p_n - p_r}{\psi_2}$, $q_n^p = 0$.

(ii) $\chi_r^p = \{\theta \geq \frac{p_n}{\psi_2}, \theta \leq \frac{p_n - p_r}{(1-\kappa)\psi_2}\}, 0 \leq U_n^p \leq U_r^p$.

(a) $\frac{p_n}{\psi_2} \leq \frac{p_n - p_r}{(1-\kappa)\psi_2}$, $p_r \leq \kappa p_n$ or $\kappa \geq \frac{p_n}{p_r}$.

If $\frac{p_n - p_r}{(1-\kappa)\psi_2} \leq 1$, then $q_r^p = \frac{1}{2}\int_{\frac{p_n}{\psi_2}}^{\frac{p_n - p_r}{(1-\kappa)\psi_2}} d\theta = \frac{1}{2}(\frac{p_n - p_r}{(1-\kappa)\psi_2} - \frac{p_n}{\psi_2})$.

If $\frac{p_n - p_r}{(1-\kappa)\psi_2} > 1$, that is, $\kappa > 1 - \frac{p_n - p_r}{\psi_2}$, then $q_r^p = \frac{1}{2}\int_{\frac{p_n}{\psi_2}}^1 d\theta = \frac{1}{2}(1 - \frac{p_n}{\psi_2})$.

(b) $\frac{p_n}{\psi_2} > \frac{p_n - p_r}{(1-\kappa)\psi_2}$, $p_r > \kappa p_n$ or $\kappa < \frac{p_r}{p_n}$, then $q_r^p = 0$.

Thus, for primary customers, the demand functions are:

$$(q_n^p, q_r^p) = \begin{cases} \left(\frac{1}{2}(1 - \frac{p_r}{\kappa\psi_2}), 0\right), & \text{when } p_r > \kappa p_n \\ \left(\frac{1}{2}(1 - \frac{p_n - p_r}{(1-\kappa)\psi_2}), \frac{1}{2}(\frac{p_n - p_r}{(1-\kappa)\psi_2} - \frac{p_n}{\psi_2})\right) & \text{when } p_n - (1-\kappa)\psi_2 < p_r \leq \kappa p_n \\ \left(0, \frac{1}{2}\left(1 - \frac{p_n}{\psi_2}\right)\right), & \text{when } p_r \leq p_n - (1-\kappa)\psi_2 \end{cases}$$

In the above equation, $p_r \leq p_n - (1-\kappa)\psi_2$ is not possible, since we assume that the OEM will always produce new products.

For green customers, $\chi_n^g = \{\theta, U_n^g \geq \max\{U_r^g, 0\}\}$, $\chi_r^g = \{\theta, U_r^g \geq \max\{U_n^g, 0\}\}$, $U_n^g = \psi_2\theta - p_n$, $U_r^g = \psi_2\theta - p_r$. Since $p_r < p_n$, $U_n^g < U_r^g$, they will always buy remanufactured products if their utility is positive. Therefore, $q_r^g = \frac{1}{2}\left(1 - \frac{p_r}{\psi_2}\right)$, thus, $(q_n^g, q_r^g) = \left(0, \frac{1}{2}(1 - \frac{p_r}{\psi_2})\right)$, when $p_r < p_n$. From above, the demand function can be written as:

$$q_n = q_n^p + q_n^g = \begin{cases} \frac{1}{2}(1 - \frac{p_r}{\kappa\psi_2}) & \text{when } \kappa p_n < p_r < p_n \\ \frac{1}{2}(1 - \frac{p_n - p_r}{(1-\kappa)\psi_2}) & \text{when } p_n - (1-\kappa)\psi_2 < p_r \leq \kappa p_n \end{cases}$$

$$q_r = q_r^p + q_r^g = \begin{cases} \frac{1}{2}(1 - \frac{p_r}{\psi_2}) & \text{when } \kappa p_n < p_r < p_n \\ \frac{1}{2}(\frac{p_n - p_r}{(1-\kappa)\psi_2} - \frac{p_n}{\psi_2}) + \frac{1}{2}(1 - \frac{p_r}{\psi_2}) & \text{when } p_n - (1-\kappa)\psi_2 < p_r \leq \kappa p_n \end{cases}$$

□

When $\kappa p_n < p_r < p_n$, we call it as a high pricing strategy for remanufactured products; and when $p_n - (1-\kappa)\psi_2 < p_r \leq \kappa p_n$, we call it as a low pricing strategy for remanufactured products.

2.3.2 Model Formulation

As depicted in Figure 2.1, both the product design and product price are set by the OEM. The customers' demand is determined by the demand function, which consists of product design (ψ_2) and prices (p_n, p_r) for the products at the second stage. The quantity of each product that the OEM produces is equal to the demand for each product at the second stage. The quantity of the product salvaged can be affected by customers' demand. Because the quantity of the collected product is limited, if the customers' demand is high, then the remanent of collected products available for salvaging will be low.

Since the utility and demand functions are non-negative, we can infer that $p_n \leq \psi_2$. If p_n is equal to ψ_2, then q_n will be equal to zero under the high pricing strategy for remanufactured products. In reality, we consider the price of new products to be less than the product design at the second stage (i.e., $p_n < \psi_2$) to guarantee that the OEM produces new products at the second stage. And, when the collected products attain a certain level, they can be used for remanufacturing and savaging. The demand of remanufactured products should be less than or equal to the quantity of good quality collected products (i.e., $q_r \leq \psi_1 \lambda q_1$).

Proposition 2.2. *If the OEM chooses low pricing strategy for its remanufactured products, then there exists a lower bound for its new products' price, that is,*

$$p_n \geq \frac{\psi_2}{\kappa}(1 - 2\psi_1 \lambda q_1). \tag{2.3}$$

Proof. Since $q_r \leq \psi_1 \lambda q_1$, we have, by (2.2b), $p_r \geq \frac{\kappa p_n + (1-2\psi_1\lambda q_1)(1-\kappa)\psi_2}{2-\kappa}$. In view of the condition $p_n - (1-\kappa)\psi_2 < p_r \leq \kappa p_n$, we have $\frac{\kappa p_n + (1-2\psi_1\lambda q_1)(1-\kappa)\psi_2}{2-\kappa} \leq \kappa p_n$. Then, $p_n \geq \frac{\psi_2}{\kappa}(1 - 2\psi_1 \lambda q_1)$. Moreover, by (2.2b), $\frac{\partial q_r}{\partial p_r} = -\frac{1}{2\psi_2} - \frac{1}{2(1-\kappa)\kappa} < 0$. Therefore, the remanufactured products could have a large market size when its price is low. Consequently, the customers'

Table 2.2: Notations and formulas of pricing strategies and end-of-life options.

i	L	$p_n - (1-\kappa)\psi_2 < p_r \leq \kappa p_n$
	H	$\kappa p_n < p_r < p_n$
j	L	$p_n < \frac{\psi_2}{\kappa}(1 - 2\psi_1 \lambda q_1)$
	H	$\frac{\psi_2}{\kappa}(1 - 2\psi_1 \lambda q_1) \leq p_n < \psi_2$
m	S	$\psi_1 \lambda q_1 > q_r$
	N	$\psi_1 \lambda q_1 = q_r$

demand for new products is small, which dictates a relatively high price for new products. $\square$

When $p_n \geq \frac{\psi_2}{\kappa}(1 - 2\psi_1 \lambda q_1)$, we call it as a high pricing strategy for new products, and consequently, when $p_n < \frac{\psi_2}{\kappa}(1 - 2\psi_1 \lambda q_1)$, we call it a low pricing strategy for new products. By Proposition 1, if the OEM chooses a low pricing strategy for its remanufactured products, then they are compelled to choose a high pricing strategy for their new products. In case, the OEM chooses a high pricing strategy for remanufactured products, we assume that the OEM will then choose a low pricing strategy for its new products, to attract customers and gain more market share, that is, it will not set the prices of both the products to be very high.

The OEM chooses an optimal pricing strategy and end-of-life option to maximize its second stage profit. We denote by i and j the pricing strategies used for the remanufactured and new products, and by κ the salvage option. We have, $i, j \in \{H, L\}$, $m \in \{S, N\}$, where H and L represent high pricing and low pricing strategies respectively; and S and N represent the salvage option and no-salvage option, respectively. If $\psi_1 \lambda q_1 = q_r$, then all the collected products are used for remanufacturing, and the OEM will not salvage any products; and if $\psi_1 \lambda q_1 > q_r$, the OEM will use both salvage and remanufacturing options. The pricing strategy selected by an OEM affects the EOL option used. The formulas to express different strategies are shown in Table 2.2.

When $p_r = \kappa p_n$, we call the prices of both the products to be binding. We call it as "binding pricing strategy", and denote it as $\{\underline{LH}\}$. The product strategy space $\{ijm\}$ is as follows: $\{\underline{LH}N\}$, $\{\underline{LH}S\}$, $\{LHN\}$, $\{LHS\}$, $\{HLN\}$, and $\{HLS\}$. For each product strategy, there will be three constraints corresponding to product strategies $\{i, j, m\}$ from those listed in Table 2.2, and an additional constraint, $p_r = \kappa p_n$, for the case of "binding pricing strategy". For each product strategy, we solve the following problem:

$$\max \Pi^{ijm} = (p_n - \alpha\psi_2^2)q_n + (p_r - \beta\alpha\psi_2^2)q_r + \gamma\psi_1(\psi_1\lambda q_1 - q_r) - (C_c + C_d + C_{disp}(1 - \psi_1))\lambda q_1 \tag{2.4a}$$

$$\text{s.t. } q_r \leq q_1\lambda\psi_1, \tag{2.4b}$$

$$\text{Constraints corresponding to product strategy } ijm, \tag{2.4c}$$

$$p_n, p_r \geq 0. \tag{2.4d}$$

Inequality (2.4b) is due to the fact that the number of collected products that can be used for remanufacturing should be greater or equal to the demand of remanufactured products. For the model for each strategy, we will add the corresponding constraints from Table 2.2 to Model (2.4). Note that q_n and q_r are expressed in terms of p_n and p_r (See Equations (2.1a) to (2.2b)). If we fix ψ_1 and ψ_2, then the unknown variables in Π^{ijm} are basically p_n and p_r. The constraints are dictated by the product strategy under consideration. The objective function Π^{ijm} consists of four terms. The first term represents the net profit gained by selling new products. The second term captures the profit resulting from selling the remanufactured products, and the third term depicts the profit obtained by salvaging products. The last term captures the total cost of collection, disassembly, and disposal. A summary of the features of the model is as follows.

(i) The problem is a two-stage decision problem for an OEM, we allow a different product design for each stage, i.e., ψ_1 and ψ_2 represent the product designs at the first stage and the second stage respectively.

(ii) The salvage value at the second stage is a linear function of the product design at the first stage. The production and remanufacturing costs are quadratic functions of product design, which refers to part yield.

(iii) The primary and green customers have a different willingness to pay for new and remanufactured products.

(iv) Market segmentation: the market is segmented as primary customers and the green customers after the first stage.

(v) The price of new products is always greater than that of the remanufactured products at the second stage.

(vi) For primary customers, their willingness to pay for remanufactured products is a fraction of that for new products.

2.3.3 Determination of Optimal Decisions for the OEM

Note that the decisions made by an OEM at the second stage depend on those made at the first stage (p_1, q_1, ψ_1), given ψ_2 and parameters value. We solve the above model by using KKT conditions.

Proposition 2.3. *For each pricing strategies $\{i, j, m\}$, there exists a unique solution of (p_n^*, p_r^*) that maximizes the profit function Π^{ijm}.*

Proof. At the second stage, the profit function is jointly concave on (p_n, p_r) as follows. For a low pricing strategy for remanufactured products, the Hessian matrix of the profit function, $H_1 = \begin{pmatrix} -\frac{1}{\psi_2(1-\kappa)} & \frac{1}{\psi_2(1-\kappa)} \\ \frac{1}{\psi_2(1-\kappa)} & -\frac{1}{\psi_2} - \frac{1}{\psi_2(1-\kappa)} \end{pmatrix}$, is negative definite, since its leading coefficient is negative and the determinant, $(1+\kappa)/(\psi_2^2(\kappa - \kappa^2)) > 0$. For a high pricing strategy for remanufactured products, the Hessian matrix of the profit function, $H_2 = \begin{pmatrix} -\frac{1}{\psi_2} & 0 \\ 0 & -\frac{1}{\psi_2} \end{pmatrix}$, which is clearly negative definite. $\square$

The optimal prices for each product strategy are shown in Table 2.3, while the optimal quantity and profit are depicted in Table 2.4 and Table 2.5, repeatedly. The optimal conditions for each product strategy are given in Table 2.6. The quantity for end-of-life option can be determined by comparing q_r^* and $\psi_1 \lambda q_1$.

Table 2.3: Optimal prices for each strategy

Strategies	Optimal price of new product and remanufactured product
$\underline{LH}N$	$p_r^{\underline{LH}N} = (1-2\lambda q_1\psi_1)\psi_2, p_n^{\underline{LH}N} = \frac{\psi_2}{\kappa}(1-2\lambda q_1\psi_1)$
$\underline{LH}S$	$p_r^{\underline{LH}S} = \frac{\gamma\kappa^2\psi_1}{2+2\kappa^2} + \frac{\kappa\psi_2(1+\alpha\beta\kappa\psi_2-\alpha\kappa\psi_2^2)}{2(1+\kappa^2)}, p_n^{\underline{LH}S} = \frac{\gamma\kappa\psi_1}{2+2\kappa^2} + \frac{\psi_2+\alpha\beta\kappa\psi_2^2-\alpha\kappa\psi_2^3}{2+2\kappa^2}$
LHN	$p_r^{LHN} = \frac{2(1-\kappa)\kappa\lambda q_1\psi_1\psi_2}{-1-\kappa+\kappa^2} + \frac{\kappa\psi_2(3+2\kappa-3\kappa^2+\alpha(1+\kappa)\psi_2)}{2(1+2\kappa-\kappa^3)}$, $p_n^{LHN} = \frac{(1+2\kappa-\kappa^2)\psi_2}{2+2\kappa} + \frac{\alpha\psi_2^2}{2}$
LHS	$p_r^{LHS} = \frac{\gamma\psi_1}{2} + \frac{\kappa\psi_2}{1+\kappa} + \frac{\alpha\beta\psi_2^2}{2}$, $p_n^{LHN} = \frac{(1+2\kappa-\kappa^2)\psi_2}{2+2\kappa} + \frac{\alpha\psi_2^2}{2}$
HLN	$p_r^{HLN} = (1-2\lambda q_1\psi_1)\psi_2$, $p_n^{HLN} = \frac{\psi_2(1+\alpha\psi_2)}{2}$
HLS	$p_r^{HLS} = \frac{\gamma\psi_1+\psi_2+\alpha\beta\psi_2^2}{2}$, $p_n^{HLS} = \frac{\psi_2(1+\alpha\psi_2)}{2}$

Table 2.4: Optimal quantities for each strategy

Strategies	Optimal quanitties
$\underline{LH}N$	$q_r^{\underline{LH}N} = \lambda q_1\psi_1, q_n^{\underline{LH}N} = \frac{-1+\kappa+2\lambda q_1\psi_1}{2\kappa}, q_s^{\underline{LH}N} = 0$
$\underline{LH}S$	$q_r^{\underline{LH}S} = -\frac{\gamma\kappa^2\psi_1}{(4+4\kappa^2)\psi_2} - \frac{\alpha\beta\kappa^2\psi_2}{4+4\kappa^2} - \frac{\alpha\kappa\psi_2}{4+4\kappa^2} + \frac{2-\kappa+\kappa^2}{4+4\kappa^2}, q_n^{\underline{LH}S} = -\frac{\gamma\kappa\psi_1}{(4+4\kappa^2)\psi_2} - \frac{(\alpha+\alpha\beta\kappa)\psi_2}{4+4\kappa^2} - \frac{-1+\kappa-2\kappa^2}{4+4\kappa^2}$, $q_s^{\underline{LH}S} = \frac{\gamma\kappa^2\psi_1}{(4+4\kappa^2)\psi_2} + \frac{\alpha\beta\kappa^2\psi_2}{4+4\kappa^2} + \frac{\alpha\kappa\psi_2}{4+4\kappa^2} - \frac{2-\kappa+\kappa^2}{4+4\kappa^2} + \lambda q_1\psi_1$
LHN	$q_r^{LHN} = \lambda q_1\psi_1$, $q_n^{LHN} = \frac{1+2\kappa-\kappa^2-4\kappa\lambda q_1\psi_1}{4+4\kappa-4\kappa^2} + \frac{\alpha(1+\kappa)\psi_2}{4(-1-\kappa+\kappa^2)}$, $q_s^{LHN} = 0$
LHS	$q_r^{LHS} = \frac{1}{4} + \frac{\gamma(1+\kappa-\kappa^2)\psi_1}{4(-1+\kappa)\kappa\psi_2} - \frac{\alpha(\kappa+\beta(-1-\kappa+\kappa^2))\psi_2}{4(-1+\kappa)\kappa}$, $q_n^{LHN} = \frac{1}{4} + \frac{\gamma\psi_1}{(4-4\kappa)\psi_2} + \frac{(\alpha-\alpha\beta)\psi_2}{4(-1+\kappa)}$, $q_s^{LHS} = -\frac{1}{4} + \lambda q_1\psi_1 + \frac{\gamma(1+\kappa-\kappa^2)\psi_1}{4(-1+\kappa)\kappa\psi_2} + \frac{\alpha(\kappa+\beta(-1-\kappa+\kappa^2))\psi_2}{4(-1+\kappa)\kappa}$
HLN	$q_r^{HLN} = \lambda q_1\psi_1$, $q_n^{HLN} = \frac{1-\alpha\psi_1}{4}$, $q_s^{HLN} = 0$
HLS	$q_r^{HLS} = \frac{1}{4} - \frac{\gamma\psi_1}{4\psi_2} - \frac{\alpha\beta\psi_2}{4}$, $q_n^{HLS} = \frac{1-\alpha\psi_2)}{4}$, $q_s^{HLS} = \psi_1\lambda q_1 - \frac{1}{4} + \frac{\gamma\psi_1}{4\psi_2} + \frac{\alpha\beta\psi_2}{4}$

Table 2.5: Optimal profit for each strategy

Strategies	Profits
$\underline{LH}N$	$\Pi^{\underline{LH}N} = -\lambda(C_c + C_d + C_{disp})q_1 + \frac{(-1+\kappa)\psi_2}{2\kappa^2} - \frac{2(1+\kappa^2)\lambda^2 q_1^2 \psi_1^2 \psi_2}{\kappa^2} - \frac{\alpha(1+\kappa)\psi_2^2}{2\kappa} + \psi(\lambda C_{disp} q_1 + \frac{(2-\kappa+\kappa^2)\lambda q_1 \psi_2}{\kappa^2}) - \frac{\alpha(1+\beta\kappa)\lambda q_1 \psi_2^2}{\kappa}$
$\underline{LH}S$	$\Pi^{\underline{LH}S} = -\lambda(C_c + C_d + C_{disp})q_1 + \psi_1^2(\gamma\lambda q_1 + \frac{\gamma^2\kappa^2}{(8+8\kappa^2)\psi_2}) + \frac{(1+\kappa)^2\psi_2}{8(1+\kappa^2)} + \frac{(\alpha+\alpha\beta\kappa)^2\psi_2^3}{8(1+\kappa^2)} + \psi_1\left(\frac{\gamma(-2+\kappa-\kappa^2)+4(1+4\kappa^2)\lambda C_{disp} q_1}{4(1+\kappa^2)} + \frac{\alpha\gamma\kappa(1+\beta\kappa)\psi_2}{4(1+\kappa^2)}\right) - \frac{\alpha\left(1-\kappa+2\kappa^2+\beta(2-\kappa+\kappa^2)\right)\psi_2^2}{4(1+\kappa^2)}$
LHN	$\Pi^{LHN} = -\lambda(C_c + C_d + C_{disp})q_1 - \frac{(-1-2\kappa+\kappa^2)\psi_2}{8(-1-2\kappa+\kappa^3)} - \frac{2(1+\kappa)\kappa\lambda^2 q_1^2\psi_1^2\psi_2}{-1-\kappa+\kappa^2} - \frac{\alpha(-1-2\kappa+\kappa^2)\psi_2^2}{4(-1-\kappa+\kappa^2)} + \frac{\alpha^2(1+\kappa)\psi_2^3}{8+8\kappa-8\kappa^2} + \psi_1\left(\lambda C_{disp} q_1 + \frac{(-1+\kappa)\kappa\lambda q_1\psi_2}{-1-\kappa+\kappa^2} - \frac{\alpha(\kappa+\beta(-1-\kappa+\kappa^2))\lambda q_1\psi_2^2}{-1-\kappa+\kappa^2}\right)$
LHS	$\Pi^{LHS} = -\lambda(C_c + C_d + C_{disp})q_1 + \psi_1^2(\gamma\lambda q_1 + \frac{\gamma^2(-1-\kappa+\kappa^2)}{8(-1+\kappa)\kappa\psi_2}) + \frac{(1+4\kappa-\kappa^2)\psi_2}{8+8\kappa} - \frac{1}{4}\alpha(1+\beta)\psi_2^2 + \frac{\alpha^2(-\kappa+2\beta\kappa+\beta^2(-1-\kappa+\kappa^2))\psi_2^3}{8(-1+\kappa)\kappa} + \psi_1\left(-\frac{\gamma}{4} + \lambda C_{disp} q_1 + \frac{\alpha\gamma(\kappa+\beta(-1-\kappa+\kappa^2))\psi_2}{4(-1+\kappa)\kappa}\right)$
HLN	$\Pi^{LHS} = q_1\lambda\left(-C_c - C_d - C_{disp} + \psi_1(C_{disp} + 2\psi_2 - \alpha(1+\beta))\psi_2^2 - 4\psi_2 q_1^2\psi_1^2\lambda^2\right)$
HLS	$\Pi^{HLS} = \frac{\gamma^2\psi_1^2+2\gamma\psi_1\psi_2(-1+\alpha\beta\psi_2)+\psi_2^2\left(2-2\alpha(1+\beta)\psi_2+\alpha^2(1+\beta^2)\psi_2^2\right)}{8\psi_2} - \lambda q_1(C_c + C_d + C_{disp} - C_{disp}\psi_1 - \gamma\psi_1^2)$

Table 2.6: Conditions for each strategy

Strategies	Conditions
$\{\underline{LH}N\}$	$2 + \kappa - 2\kappa^2 + \kappa^3 - 4(1+\kappa)\lambda q_1\psi_1 - \alpha\kappa(1+\kappa)\psi_2 \geq 0$, $-1 + 2\lambda q_1\psi_1 + \frac{\kappa(\gamma\kappa\psi_1+\psi_2(1+\kappa+(\alpha+\alpha\beta\kappa)\psi_2))}{2(1+\kappa^2)\psi_2} \leq 0$, $-1+\kappa+2\lambda q_1\psi_1 > 0$
$\{\underline{LH}S\}$	$\gamma(1+\kappa)\psi_1 + \psi_2((-1+\kappa)^2\kappa + \alpha(\beta-\kappa)(1+\kappa)\psi_2) \geq 0$, $-1 + 2\lambda q_1\psi_1 + \frac{\kappa(\gamma\kappa\psi_1+\psi_2(1+\kappa+(\alpha+\alpha\beta\kappa)\psi_2))}{2(1+\kappa^2)\psi_2} > 0$, $\gamma\kappa\psi_1 + \psi_2(-1+\kappa-2\kappa^2+(\alpha+\alpha\beta\kappa)\psi_2) < 0$
$\{LHN\}$	$2 + \kappa - 2\kappa^2 + \kappa^3 - 4(1+\kappa)\lambda q_1\psi_1 - \alpha\kappa(1+\kappa)\psi_2 < 0$, $q_1\lambda\psi_1 - \frac{\gamma(1+\kappa-\kappa^2)\psi_1+\psi_2((-1+\kappa)\kappa+\alpha(\beta-\kappa+\beta\kappa-\beta\kappa^2)\psi_2)}{4(-1+\kappa)\kappa\psi_2} \leq 0, 1-4\kappa\lambda q_1\psi_1-\alpha(1+\kappa)\psi_2+2\kappa-\kappa^2 > 0$
$\{LHS\}$	$\gamma(1+\kappa)\psi_1 + \psi_2((-1+\kappa)^2\kappa + \alpha(\beta-\kappa)(1+\kappa)\psi_2) < 0$, $q_1\lambda\psi_1 - \frac{\gamma(1+\kappa-\kappa^2)\psi_1+\psi_2((-1+\kappa)\kappa+\alpha(\beta-\kappa+\beta\kappa-\beta\kappa^2)\psi_2)}{4(-1+\kappa)\kappa\psi_2} > 0$, $\gamma\psi_1+\psi_2(1-\kappa+\alpha(-1+\beta)\psi_2) > 0$
$\{HLN\}$	$\psi_1\gamma - (\psi_2 - \psi_2^2\alpha\beta - 4\psi_2\lambda q_1\psi_1) \leq 0$, $\lambda q_1\psi_1 - \frac{1}{4}(1-\psi_2\alpha) > 0$, $\lambda q_1\psi_1 - \frac{1}{4}(2-\kappa-\psi_2\alpha\kappa) < 0$
$\{HLS\}$	$\psi_1\gamma - (\psi_2 - \psi_2^2\alpha\beta - 4\psi_2\lambda q_1\psi_1) \leq 0$, $\psi_1\gamma - (\psi_2^2(1-\alpha\beta)) < 0$, $\lambda q_1\psi_1 - \frac{1}{4}(2-\kappa-\psi_2\alpha\kappa) < 0$

2.3.4 Impact of Product Design on the OEM's Product Strategies

Our analysis of the results presented in Section 2.3.3, has revealed the following insights on the impact of product design on the OEM's product strategies. Referring to Table 2.6, and in accordance with the constraints for the three pricing strategies, i.e., $\{\underline{LH}\}, \{LH\}, \{HL\}$, regardless of the salvage option, we have the following:

(i) There exists a lower bound on product design, ψ_1, for the OEM to select the salvage option under various pricing strategies; otherwise, the salvage option is not as profitable. They are as follows:

   (a) For pricing strategy $\{\underline{LH}\}$, $\psi_1 > -\frac{\psi_2(-2+\kappa-\kappa^2+\alpha\kappa(1+\beta\kappa\psi_2)}{\gamma\kappa^2+4(1+\kappa^2)\lambda q_1\psi_2}$.

   (b) For pricing strategy $\{LH\}$, $\psi_1 > \frac{\psi_2^2\alpha(-\kappa+\beta(1+\kappa-\kappa^2))}{(-1-\kappa+\kappa^2)\gamma} - \frac{\psi_2(-1+\kappa)\kappa(-1+4\psi_1 q_1\lambda)}{(-1-\kappa+\kappa^2)\gamma}$.

   (c) For pricing strategy $\{HL\}$, $\psi_1 > \frac{\psi_2-\psi_2^2\alpha\beta-4\psi_2\psi_1 q_1\lambda}{\gamma}$.

   A high value of ψ_1 implies a high part yield and a high salvage value at the second stage. Since ψ_1 does not affect the demand of new and remanufactured products at the second stage, a higher value of ψ_1 leads to salvage option for each pricing strategy, if other parameters are kept fixed. For different pricing strategies, there exist conditions to guide an OEM to choose or not to choose the salvage option as mentioned above.

(ii) There exist an upper bound and a lower bound on product design ψ_1, respectively, to select a binding pricing strategy if the OEM does not or does use salvage option. They are as follows:

   (a) If the OEM does not choose salvage option, it is optimal for the OEM to use a binding pricing strategy when $\psi_1 \leq \frac{2+\kappa-2\kappa^2+\kappa^3}{4\lambda q_1+4\kappa\lambda q_1} - \frac{\alpha\kappa\psi_2}{4\lambda q_1}$ or when the impact factor of product design on new products' production cost (α) is small, that is, $\alpha \leq \alpha' = \frac{2+\kappa-2\kappa^2+\kappa^3}{(\kappa+\kappa^2)\psi_2} - \frac{4\lambda q_1\psi_1}{\kappa\psi_2}$.

(b) If the OEM chooses salvage option, it is optimal for the OEM to use binding pricing strategy when $\psi_1 \geq -\frac{(-1+\kappa)^2\kappa\psi_2}{\gamma(1+\kappa)} + \frac{\alpha(-\beta+\kappa)\psi_2^2}{\gamma}$ or when the impact factor of product design on remanufactured products' production cost (β) is significantly large, that is, $\beta \geq \beta' = -\frac{\gamma\psi_1}{\alpha\psi_2^2} + \frac{\kappa(-(-1+\kappa)^2+\alpha(1+\kappa)\psi_2)}{\alpha(1+\kappa)\psi_2}$.

(iii) Based on our analysis, a mapping can be generated of product designs at the two stages onto the optimal regions of product strategies. We illustrate this in Figure 2.3. Such a mapping can aid the decision maker in choosing an optimal product strategy for a given product design. For this illustration, the parameters value are fixed as follows: $q_1 = 0.9, \lambda = 0.6, \gamma = 0.4, \beta = 0.4, \kappa = 0.9, \alpha = 0.95$). Note that, in Figure 2.3(a), the salvage option dominates for large value of ψ_1. Furthermore, for binding pricing strategy $\{\underline{LH}\}$, the OEM chooses salvage option if $\psi_1 > 0.27$ (point B); while for no binding pricing strategy $\{LH\}$, the OEM chooses salvage option if $\psi_1 > 0.3849$ (point C). Thus, the conditions for the OEM to choose strategies $\{LHN\}$, $\{LHN\}$, $\{LHS\}$, and $\{LHS\}$ are, respectively, $\psi_1 \leq 0.27$, $\psi_1 \leq 0.3849$, $\psi_1 > 0.27$, and $\psi_1 > 0.3849$. From (ii) above, $\alpha = 0.95 \leq \alpha'$, which implies that $\psi_1 \leq 0.252$. Also, $\beta = 0.4 < \beta'$ implies $\psi_1 < 0.42$. Thus, the condition for the OEM to choose product strategies $\{LHN\}$, $\{LHN\}$, $\{LHS\}$, and $\{LHS\}$ are, respectively, $\psi_1 \leq 0.252, \psi_1 > 0.252, \psi_1 > 0.42$, and $\psi_1 \leq 0.42$. Moreover, to meet the conditions specified in Table 2.6 for the OEM to choose product strategies $\{LHN\}, \{LHS\}, \{LHN\}$, and $\{LHS\}$ are, respectively, $\psi_1 > 0.093, \psi_1 < 1.57467, \psi_1 < 0.466564$, and $\psi_1 > 0.363$.

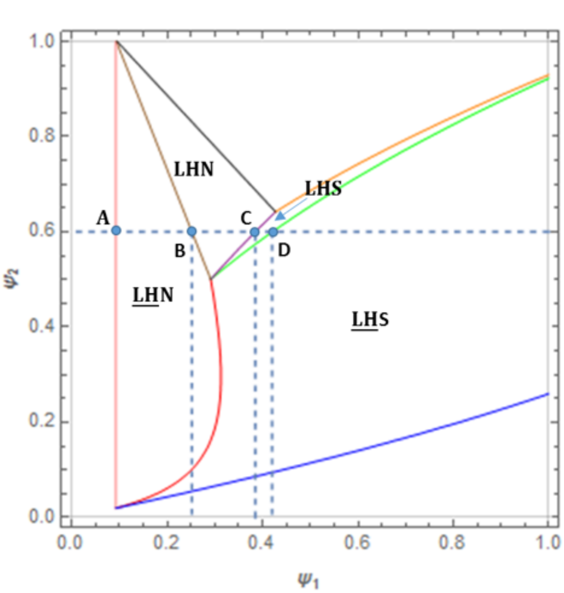
Thus, the feasible regions for product strategies $\{\underline{LH}N\}$, $\{\underline{LH}S\}$, $\{LHN\}$, and $\{LHS\}$ are, respectively, $0.093 < \psi_1 \leq 0.252, 0.42 < \psi_1 \leq 1$, $0.252 < \psi_1 \leq 0.3849$, and $0.3849 < \psi_1 \leq 0.42$. These are denoted in Figure 2.3(b), and this can be obtained by fixing $\psi_2 = 0.6$ in Figure 2.3(a). The coordinates of A, B, C, D are, respectively, $(0.093, 0.6)$, $(0.252, 0.6)$, $(0.3849, 0.6)$, and $(0.42, 0.6)$.

In Figure 2.3(a) and Figure 2.3(c), if ψ_2 is large ($\psi_2 \to 1$), ψ_1 should be set very small and the OEM should not choose salvage option. This shows that if the OEM wants to choose a high-level product design at Stage 2, they should choose a low-level product design at Stage 1 and should not choose salvage option.

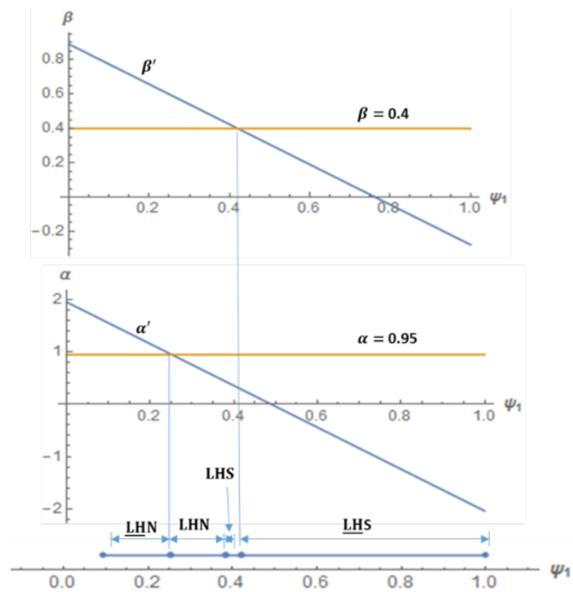
(iv) At the second stage, when the OEM selects a low product design ($\psi_2 \to 0$), then the product strategy $\{LHS\}$ dominates the other product strategies.

If the aim of the OEM is to choose a very low product design at the second stage, then they should choose product strategy $\{LHS\}$. The OEM can forcast customers' demand by using the demand function for given values of prices (p_n, p_r), and design (ψ_2). In Equations (2.1a), (2.1b), (2.2a), and (2.2b), ψ_2 appears in the denominator. Therefore if $\psi_2 \to 0$, the demand function will not become negative only if the corresponding nominator approximates to 0, i.e. $p_r \to p_n \to 0$. In reality, the price of any low quality product should be set very low, because otherwise, the customers will not buy it. At the same time, $q_r \to 0, q_n \to 1$, and the profit for the OEM at the second stage is mainly generated by salvage value. Apparently, the salvage option dominates the no salvage option. For the product strategy $\{LHS\}$, the salvage value $\psi_1 \gamma q_S^{ijk}$ is maximum, since $q_S^{LHS} > \{q_S^{\underline{LHS}}, q_S^{HLS}\}$.

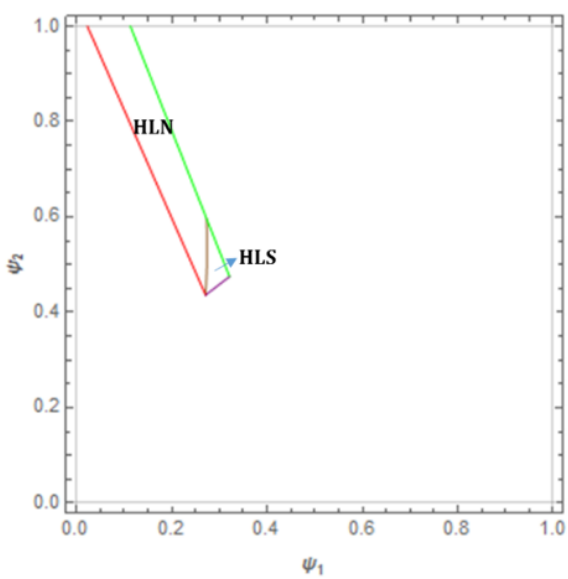
(v) In view of (iii) above, once a mapping of product designs onto end-of-life options is obtained for a given set of parameters, it can be used to obtain optimal values of ψ_1 and ψ_2 for that set of parameters. To that end, we search over ψ_1 and ψ_2. For every pair of ψ_1 and ψ_2, we can obtain the objective function value, Π^{ijm}, corresponding to strategy ijm from the mapping. Then, the best design corresponds to a pair ψ_1 and ψ_2 that maximize Π^{ijm}. Consequently, we have the best solution for product design, its end-of-life option, product prices, and production quantities for new and remanufactured products, and total profit.



(a)



(b)



(c)

Figure 2.3: (a) Optimal regions of product strategies over ψ_1 and ψ_2 for pricing strategies $\{LH\}$ and $\{LH\}$. (b) Optimal regions of product strategies over ψ_1, given $\psi_2 = 0.6$ for pricing strategies $\{LH\}$ and $\{LH\}$. (c) Optimal regions of product strategies over, ψ_1 and ψ for pricing strategy $\{HL\}$.

2.4 Discussion of Case 2

In this section, we discuss the special case when $\psi_1 = \psi_2 = \psi$. Under this settling, Model 2.4 is revised accordingly and is used to obtain optimal prices for each product strategy (shown in Table 2.7), while the optimal quantity and profit are depicted in Table 2.8 and Table 2.9, repeatedly. The optimal conditions for each product strategy are given in Table 2.10.

Table 2.7: Optimal prices for each strategy

Strategies	Optimal price of a new product and a remanufactured product
$\underline{LH}N$	$p_r^{\underline{LH}N} = (1-2\lambda q_1\psi)\psi, p_n^{\underline{LH}N} = \frac{\psi}{\kappa}(1-2\lambda q_1\psi)$
$\underline{LH}S$	$p_r^{\underline{LH}S} = \frac{\gamma\kappa^2\psi}{2+2\kappa^2} + \frac{\kappa\psi(1+\alpha\beta\kappa\psi-\alpha\kappa\psi^2)}{2(1+\kappa^2)}, p_n^{\underline{LH}S} = \frac{\gamma\kappa\psi}{2+2\kappa^2} + \frac{\psi+\alpha\beta\kappa\psi^2-\alpha\kappa\psi^3}{2+2\kappa^2}$
LHN	$p_r^{LHN} = \frac{2(1-\kappa)\kappa\lambda q_1\psi^2}{-1-\kappa+\kappa^2} + \frac{\kappa\psi(3+2\kappa-3\kappa^2+\alpha(1+\kappa)\psi)}{2(1+2\kappa-\kappa^3)}$, $p_n^{LHN} = \frac{(1+2\kappa-\kappa^2)\psi}{2+2\kappa} + \frac{\alpha\psi^2}{2}$
LHS	$p_r^{LHS} = \frac{\gamma\psi}{2} + \frac{\kappa\psi}{1+\kappa} + \frac{\alpha\beta\psi^2}{2}$, $p_n^{LHN} = \frac{(1+2\kappa-\kappa^2)\psi}{2+2\kappa} + \frac{\alpha\psi^2}{2}$
HLN	$p_r^{HLN} = (1-2\lambda q_1\psi)\psi$, $p_n^{HLN} = \frac{\psi(1+\alpha\psi)}{2}$
HLS	$p_r^{HLS} = \frac{(\gamma+1+\alpha\beta\psi)\psi}{2}$, $p_n^{HLS} = \frac{\psi(1+\alpha\psi)}{2}$

Table 2.8: Optimal quantities for each strategy

Strategies	Optimal quanitties
$\underline{LH}N$	$q_r^{\underline{LH}N} = \lambda q_1\psi, q_n^{\underline{LH}N} = \frac{-1+\kappa+2\lambda q_1\psi}{2\kappa}, q_s^{\underline{LH}N} = 0$
$\underline{LH}S$	$q_r^{\underline{LH}S} = -\frac{\gamma\kappa^2}{(4+4\kappa^2)} - \frac{\alpha\beta\kappa^2\psi}{4+4\kappa^2} - \frac{\alpha\kappa\psi}{4+4\kappa^2} + \frac{2-\kappa+\kappa^2}{4+4\kappa^2}, q_n^{\underline{LH}S} = -\frac{\gamma\kappa}{(4+4\kappa^2)} - \frac{(\alpha+\alpha\beta\kappa)\psi}{4+4\kappa^2} - \frac{-1+\kappa-2\kappa^2}{4+4\kappa^2}$, $q_s^{\underline{LH}S} = \frac{\gamma\kappa^2}{(4+4\kappa^2)} + \frac{\alpha\beta\kappa^2\psi}{4+4\kappa^2} + \frac{\alpha\kappa\psi}{4+4\kappa^2} - \frac{2-\kappa+\kappa^2}{4+4\kappa^2} + \lambda q_1\psi$
LHN	$q_r^{LHN} = \lambda q_1\psi$, $q_n^{LHN} = \frac{1+2\kappa-\kappa^2-4\kappa\lambda q_1\psi}{4+4\kappa-4\kappa^2} + \frac{\alpha(1+\kappa)\psi}{4(-1-\kappa+\kappa^2)}$, $q_s^{LHN} = 0$
LHS	$q_r^{LHS} = \frac{1}{4} + \frac{\gamma(1+\kappa-\kappa^2)}{4(-1+\kappa)\kappa} - \frac{\alpha(\kappa+\beta(-1-\kappa+\kappa^2))\psi}{4(-1+\kappa)\kappa}$, $q_n^{LHN} = \frac{1}{4} + \frac{\gamma}{(4-4\kappa)} + \frac{(\alpha-\alpha\beta)\psi}{4(-1+\kappa)}$, $q_s^{LHS} = -\frac{1}{4} + \lambda q_1\psi + \frac{\gamma(1+\kappa-\kappa^2)}{4(-1+\kappa)\kappa} + \frac{\alpha(\kappa+\beta(-1-\kappa+\kappa^2))\psi}{4(-1+\kappa)\kappa}$
HLN	$q_r^{HLN} = \lambda q_1\psi$, $q_n^{HLN} = \frac{1-\alpha\psi}{4}$, $q_s^{HLN} = 0$
HLS	$q_r^{HLS} = \frac{1}{4} - \frac{\gamma}{4} - \frac{\alpha\beta\psi}{4}$, $q_n^{HLS} = \frac{1-\alpha\psi}{4}$, $q_s^{HLS} = \psi\lambda q_1 - \frac{1}{4} + \frac{\gamma}{4} + \frac{\alpha\beta\psi}{4}$

We can make the following observations. According to Table 2.10, regardless of the salvage option, according to the constraints for the three pricing strategies, i.e., $\{\underline{LH}\}, \{LH\}, \{HL\}$, we have the following results.

Table 2.9: Optimal profit for each strategy

Strategies	Profits
$\underline{LH}N$	$\Pi^{\underline{LH}N} = -\lambda(C_c+C_d+C_{disp})q_1 + \frac{(-1+\kappa)\psi}{2\kappa^2} - \frac{2(1+\kappa^2)\lambda^2 q_1^2\psi^3}{\kappa^2} - \frac{\alpha(1+\kappa)\psi^2}{2\kappa} + \psi(\lambda C_{disp}q_1 + \frac{(2-\kappa+\kappa^2)\lambda q_1\psi}{\kappa^2}) - \frac{\alpha(1+\beta\kappa)\lambda q_1\psi^2}{\kappa}$
$\underline{LH}S$	$\Pi^{\underline{LH}S} = -\lambda(C_c + C_d + C_{disp})q_1 + \psi^2(\gamma\lambda q_1 + \frac{\gamma^2\kappa^2}{(8+8\kappa^2)\psi}) + \frac{(1+\kappa)^2\psi}{8(1+\kappa^2)} + \frac{(\alpha+\alpha\beta\kappa)^2\psi^3}{8(1+\kappa^2)} + \psi\left(\frac{\gamma(-2+\kappa-\kappa^2)+4(1+4\kappa^2)\lambda C_{disp}q_1}{4(1+\kappa^2)} + \frac{\alpha\gamma\kappa(1+\beta\kappa)\psi}{4(1+\kappa^2)}\right) - \frac{\alpha\left(1-\kappa+2\kappa^2+\beta(2-\kappa+\kappa^2)\right)\psi^2}{4(1+\kappa^2)}$
LHN	$\Pi^{LHN} = -\lambda(C_c + C_d + C_{disp})q_1 - \frac{(-1-2\kappa+\kappa^2)\psi}{8(-1-2\kappa+\kappa^3)} - \frac{2(1+\kappa)\kappa\lambda^2 q_1^2\psi^3}{-1-\kappa+\kappa^2} - \frac{\alpha(-1-2\kappa+\kappa^2)\psi^2}{4(-1-\kappa+\kappa^2)} + \frac{\alpha^2(1+\kappa)\psi^3}{8+8\kappa-8\kappa^2} + \psi\left(\lambda C_{disp}q_1 + \frac{(-1+\kappa)\kappa\lambda q_1\psi}{-1-\kappa+\kappa^2} - \frac{\alpha(\kappa+\beta(-1-\kappa+\kappa^2))\lambda q_1\psi^2}{-1-\kappa+\kappa^2}\right)$
LHS	$\Pi^{LHS} = -\lambda(C_c + C_d + C_{disp})q_1 + \psi^2(\gamma\lambda q_1 + \frac{\gamma^2(-1-\kappa+\kappa^2)}{8(-1+\kappa)\kappa\psi}) + \frac{(1+4\kappa-\kappa^2)\psi}{8+8\kappa} - \frac{1}{4}\alpha(1+\beta)\psi^2 + \frac{\alpha^2(-\kappa+2\beta\kappa+\beta^2(-1-\kappa+\kappa^2))\psi^3}{8(-1+\kappa)\kappa} + \psi\left(-\frac{\gamma}{4} + \lambda C_{disp}q_1 + \frac{\alpha\gamma(\kappa+\beta(-1-\kappa+\kappa^2))\psi}{4(-1+\kappa)\kappa}\right)$
HLN	$\Pi^{HLN} = q_1\lambda\left(-C_c - C_d - C_{disp} + \psi(C_{disp} + 2\psi - \alpha(1+\beta))\psi^2 - 4q_1^2\psi^3\lambda^2\right)$
HLS	$\Pi^{HLS} = \frac{\gamma^2\psi+2\gamma\psi(-1+\alpha\beta\psi)+\psi\left(2-2\alpha(1+\beta)\psi+\alpha^2(1+\beta^2)\psi^2\right)}{8} - \lambda q_1(C_c + C_d + C_{disp} - C_{disp}\psi - \gamma\psi^2)$

Table 2.10: Conditions for each strategy

Strategies	Conditions
$\{\underline{LH}N\}$	$2+\kappa-2\kappa^2+\kappa^3-4(1+\kappa)\lambda q_1\psi-\alpha\kappa(1+\kappa)\psi \geq 0$, $-1+2\lambda q_1\psi+\frac{\kappa(\gamma\kappa+(1+\kappa+(\alpha+\alpha\beta\kappa)\psi))}{2(1+\kappa^2)} \leq 0$, $-1+\kappa+2\lambda q_1\psi > 0$
$\{\underline{LH}S\}$	$\gamma(1+\kappa)+((-1+\kappa)^2\kappa+\alpha(\beta-\kappa)(1+\kappa)\psi) \geq 0$, $-1+2\lambda q_1\psi+\frac{\kappa(\gamma\kappa+(1+\kappa+(\alpha+\alpha\beta\kappa)\psi))}{2(1+\kappa^2)} > 0$, $\gamma\kappa+-1+\kappa-2\kappa^2+(\alpha+\alpha\beta\kappa)\psi < 0$
$\{LHN\}$	$2+\kappa-2\kappa^2+\kappa^3-4(1+\kappa)\lambda q_1\psi-\alpha\kappa(1+\kappa)\psi < 0$, $q_1\lambda\psi-\frac{\gamma(1+\kappa-\kappa^2)\psi+\psi((-1+\kappa)\kappa+\alpha(\beta-\kappa+\beta\kappa-\beta\kappa^2)\psi)}{4(-1+\kappa)\kappa\psi} \leq 0, 1-4\kappa\lambda q_1\psi-\alpha(1+\kappa)\psi+2\kappa-\kappa^2 > 0$
$\{LHS\}$	$\gamma(1+\kappa)\psi+\psi((-1+\kappa)^2\kappa+\alpha(\beta-\kappa)(1+\kappa)\psi) < 0$, $q_1\lambda\psi-\frac{\gamma(1+\kappa-\kappa^2)+((-1+\kappa)\kappa+\alpha(\beta-\kappa+\beta\kappa-\beta\kappa^2)\psi)}{4(-1+\kappa)\kappa} > 0$, $\gamma+(1-\kappa+\alpha(-1+\beta)\psi) > 0$
$\{HLN\}$	$\gamma-(1-\psi\alpha\beta-4\psi\lambda q_1) \leq 0$, $\lambda q_1\psi-\frac{1}{4}(1-\psi\alpha) > 0$, $\lambda q_1\psi-\frac{1}{4}(2-\kappa-\psi\alpha\kappa) < 0$
$\{HLS\}$	$\gamma-(1-\psi\alpha\beta-4\psi\lambda q_1) \leq 0$, $\gamma-(\psi(1-\alpha\beta)) < 0$, $\lambda q_1\psi-\frac{1}{4}(2-\kappa-\psi\alpha\kappa) < 0$

(i) There exists a lower bound on collection rate λ, for the OEM to select the salvage option under various pricing strategies; otherwise, the salvage option is not as profitable. They are as follows:

(a) For pricing strategy $\{\underline{LH}\}$, $\lambda > \frac{2-\kappa+(1-\gamma)\kappa^2-\alpha\kappa(1+\beta\kappa\psi)}{4(1+\kappa^2)q_1\psi}$.

(b) For pricing strategy $\{LH\}$, $\lambda > \frac{(1+\kappa-\kappa^2)(\gamma+\psi\alpha\beta)-\psi\alpha\beta-(1-\kappa)\kappa}{4\psi q_1(\kappa-1)\kappa}$.

(c) For pricing strategy $\{HL\}$, $\lambda > \frac{1-\gamma-\psi\alpha\beta}{4\psi q_1}$.

A high value of λ implies a high salvage value at the second stage. For each pricing strategy, if λ increases, it is more profitable to choose the salvage option.

(ii) (a) If the OEM does not choose salvage option, it is optimal for the OEM to use a binding pricing strategy when $\psi \leq \frac{2+\kappa-2\kappa^2+\kappa^3}{(1+\kappa)(4\lambda q_1+\alpha\kappa)}$ or when the impact factor of product design on new products' production cost (α) is small, that is, $\alpha \leq \alpha' = \frac{2+\kappa-2\kappa^2+\kappa^3}{(\kappa+\kappa^2)\psi} - \frac{4\lambda q_1}{\kappa}$.

(b) If the OEM chooses salvage option, it is optimal for the OEM to use a binding pricing strategy when $\psi \leq \frac{(1+\kappa)+(1-\kappa)^2\kappa}{(1+\kappa)(\alpha(-\beta+\kappa))}$, if $\kappa > \beta$, and $\psi \geq \frac{(1+\kappa)+(1-\kappa)^2\kappa}{(1+\kappa)(\alpha(-\beta+\kappa))}$, if $\kappa < \beta$; or when the impact factor of product design on remanufactured products' production cost (β) is significantly large, that is, $\beta \geq \beta' = -\frac{\gamma}{\alpha\psi} + \frac{\kappa(-(-1+\kappa)^2+\alpha(1+\kappa)\psi)}{\alpha(1+\kappa)\psi}$.

Apparently, the profit functions even for the case $\psi_1 = \psi_2$ are not convex in ψ. However, it is instructive to observe how their magnitude vary with ψ. We have demonstrated one instance in this regard in Figure 2.4 by setting $q_1 = 0.6, \gamma = 0.8, \beta = 0.4, \kappa = 0.8, \alpha = 0.02, \lambda = 0.6, C_c = C_d = 0.1$, and $C_{disp} = 0.3$. Note that, at this setting, the product strategy $\{\underline{LH}S\}$ dominates the other product strategies and there is a lower bound on product design ψ for the OEM to choose this product strategy, otherwise, this strategy is not profitable. In Figure 2.4(b), with increment in product design, it is more profitable for the OEM to choose $\{HLS\}$ at first and then choose $\{LHS\}$.

If we increase α to 0.2 (keeping all other parameters the same), then Π^{LHN}, $\Pi^{\underline{LH}N}$, and Π^{HLN} decrease as depicted in Figure 2.5 as expected since for these strategies, the OEM only produces new and remanufactured products. and α appears as a coefficient in the new and remanufactured products' cost. However, the relative order of their magnitudes remains the same.

If we decrease β to 0.0002 (keeping all other parameters the same), then Π^{LHN}, $\Pi^{\underline{LH}N}$, and Π^{HLN} decrease as depicted in Figure 2.6. If we increase γ to 0.001 (keeping all other parameters the same), then Π^{LHN}, $\Pi^{\underline{LH}N}$, and Π^{HLN} decrease as depicted in Figure 2.7. Note that, even for different values of β and γ, the product strategy $\underline{LH}N$ always dominates other product strategies, and HLS dominates at a low product design ψ, otherwise HLN is dominant.

(iii) The impacts of per unit change in the value of α, β, and γ (the factors that determine the production costs of the new and remanufactured products and salvage value, respectively) on profits are shown in Table 2.11, 2.12, and 2.13. In Table 2.11 and 2.12,, for product strategies $\underline{LH}N$, LHN, and HLN, the derivatives w.r.t α and β are negative, and thus, an increment in α and β will result in decrement of profit. In Table 2.13, for product strategies $\underline{LH}N$, LHN, and HLN, the derivative w.r.t γ are negative, and thus, an increment of γ will result in decrement in profit.

(iv) As mentioned in (iii) for Case 1, a mapping of ψ_1 can be generated onto product strategies, and correspondingly, Like (iv) of Case 1, we can obtain optimal ψ_1, its end-of-life option, product prices, optimal quantities, and total profit, for a given set of parameters.

Table 2.11: Impact of α on profits.

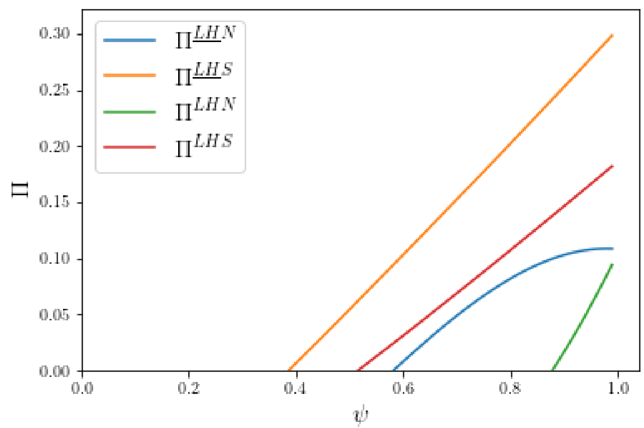
$\underline{LH}N$	$\frac{\partial \Pi^{\underline{LH}N}}{\partial \alpha} = -\frac{(1+\kappa)\psi^2}{2\kappa} - \frac{(1+\beta\kappa)\lambda q_1 \psi^2}{\kappa}$
$\underline{LH}S$	$\frac{\partial \Pi^{\underline{LH}S}}{\partial \alpha} = \frac{(1+\beta\kappa)^2\psi^3 - 2\left(1-\kappa+2\kappa^2+\beta(2-\kappa+\kappa^2)-\alpha\gamma\kappa(1+\beta\kappa)\right)\psi^2}{8(1+\kappa^2)}$
LHN	$\frac{\partial \Pi^{LHN}}{\partial \alpha} = \frac{1+2\kappa-\kappa\psi^2-\alpha(1+\kappa)\psi^2-4\lambda\kappa q_1\psi^2}{4(-1-\kappa+\kappa^2)} - \alpha\beta\lambda q_1\psi^3$
LHS	$\frac{\partial \Pi^{LHS}}{\partial \alpha} = -\frac{(1+\beta)\psi^2}{4} + \frac{\alpha\kappa\psi^2((-1+2\beta)\psi+\gamma)+\alpha\psi^2(\kappa+\beta(-1-\kappa+\kappa^2))}{4(-1+\kappa)\kappa}$
HLN	$\frac{\partial \Pi^{HLN}}{\partial \alpha} = -\alpha\lambda q_1\psi^3$
HLS	$\frac{\partial \Pi^{HLS}}{\partial \alpha} = \frac{2\gamma\beta\psi^2-2\psi^2+2\alpha(1+\beta^2)\psi^3}{8}$

Table 2.12: Impact of β on profits.

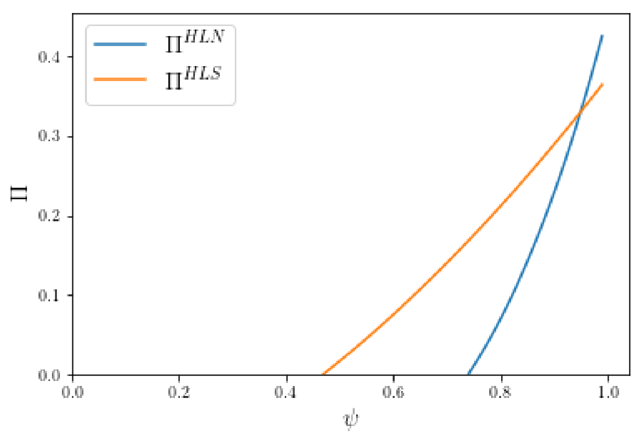
$\underline{LH}N$	$\frac{\partial \Pi^{\underline{LH}N}}{\partial \beta} = -\alpha\lambda q_1\psi^2$
$\underline{LH}S$	$\frac{\partial \Pi^{\underline{LH}S}}{\partial \beta} = \frac{\beta(\alpha\kappa)^2\psi^3-\alpha(2-\kappa+(1-\gamma)\kappa^2)\psi^2}{4(1+\kappa^2)}$
LHN	$\frac{\partial \Pi^{LHN}}{\partial \beta} = -\alpha\lambda q_1\psi^3$
LHS	$\frac{\partial \Pi^{LHS}}{\partial \beta} = -\frac{\alpha\psi^2}{4} + \frac{\alpha^2\kappa\psi^3+\alpha\psi^2(2\beta\psi+\gamma)(-1-k+\kappa^2)}{4(-1+\kappa)\kappa}$
HLN	$\frac{\partial \Pi^{HLN}}{\partial \beta} = -\alpha\lambda q_1\psi^3$
HLS	$\frac{\partial \Pi^{HLS}}{\partial \beta} = \frac{2\gamma\alpha\psi^2-2\alpha\psi^2+2\alpha^2\beta\psi^3}{8}$

Table 2.13: Impact of γ on profits.

$\underline{LH}N$	$\frac{\partial \Pi^{\underline{LH}N}}{\partial \gamma} = 0$
$\underline{LH}S$	$\frac{\partial \Pi^{\underline{LH}S}}{\partial \gamma} = \frac{\gamma(-2+\kappa-\kappa^2)+4(1+4\kappa^2)\lambda C_{disp} q_1}{4(1+\kappa^2)} + \frac{\alpha\gamma\kappa(1+\beta\kappa)\psi}{4(1+\kappa^2)} + \frac{\alpha\gamma\kappa(1+\beta\kappa)\psi}{4(1+\kappa^2)}$
LHN	$\frac{\partial \Pi^{LHN}}{\partial \gamma} = 0$
LHS	$\frac{\partial \Pi^{LHS}}{\partial \gamma} = \psi^2 \lambda q_1 - \frac{\psi}{4} + \frac{\gamma\psi[\alpha\psi + (1+\beta\psi)(-1-\kappa+\kappa^2)]}{4(-1+\kappa)\kappa}$
HLN	$\frac{\partial \Pi^{HLN}}{\partial \gamma} = 0$
HLS	$\frac{\partial \Pi^{HLS}}{\partial \gamma} = \frac{2\gamma\psi + 2\psi(-1+\alpha\beta\psi)}{8} + \lambda q_1 \psi^2$

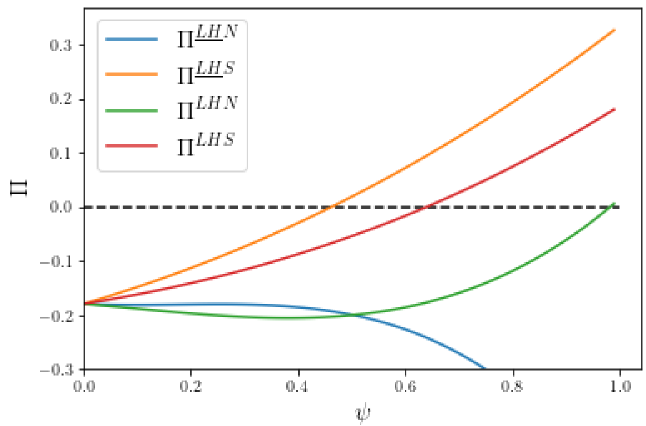


(a) The profit function for pricing strategies $\{LH\}$ and $\{\underline{LH}\}$.

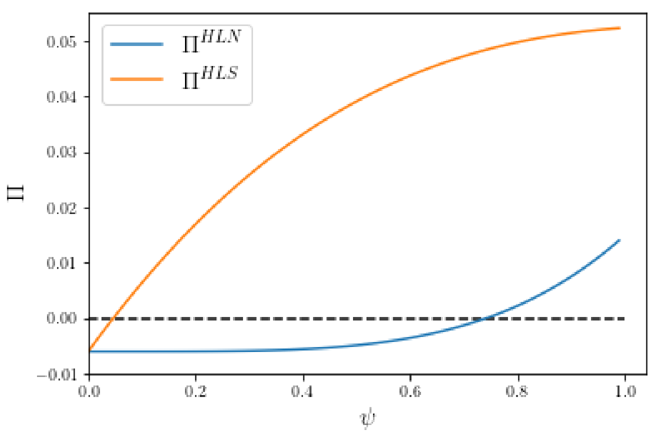


(b) The profit function for pricing strategies $\{HL\}$.

Figure 2.4: The profit function for different product strategies.

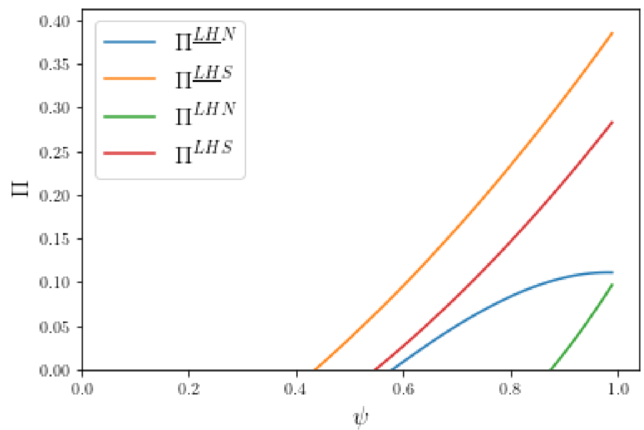


(a) The profit function for pricing strategies $\{LH\}$ and $\{\underline{LH}\}$ ($\alpha = 0.2$).

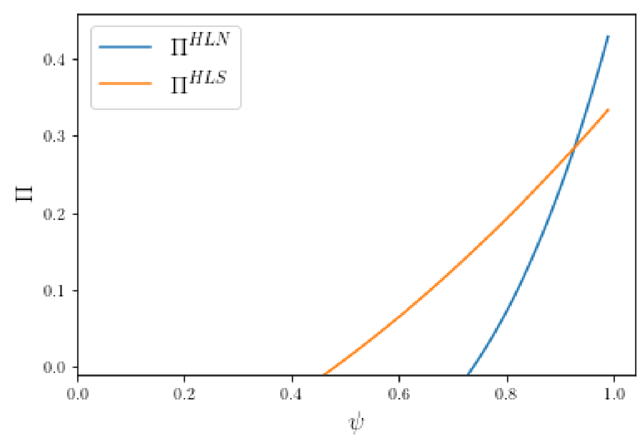


(b) The profit function for pricing strategy $\{HL\}$ ($\alpha = 0.2$).

Figure 2.5: The profit functions for different product strategies ($\alpha = 0.2$).

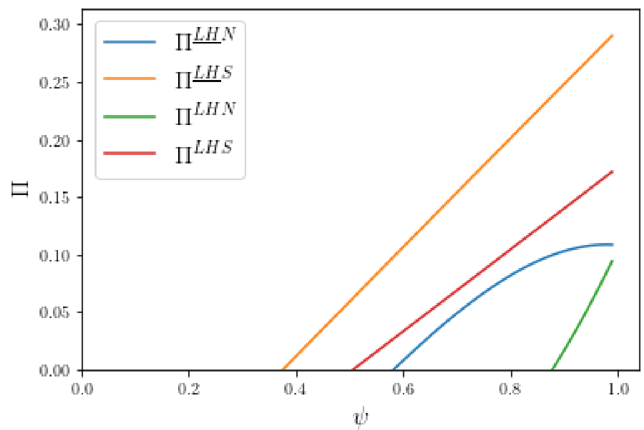


(a) The profit function for pricing strategies $\{LH\}$ and $\{\underline{LH}\}$ ($\beta = 0.0002$).

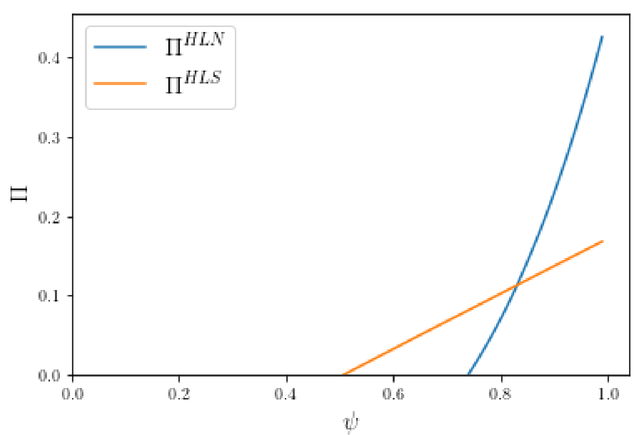


(b) The profit function for pricing strategies $\{HL\}$ ($\beta = 0.0002$).

Figure 2.6: The profit function for different product strategies ($\beta = 0.0002$).



(a) The profit function for pricing strategies $\{LH\}$ and $\{\underline{LH}\}$ ($\gamma = 0.001$).



(b) The profit function for pricing strategies $\{HL\}$ ($\gamma = 0.001$).

Figure 2.7: The profit function for different product strategies ($\gamma = 0.001$).

2.5 Discussion of Case 3

In this case, $U_n^p = \psi_2\theta - p_n$, $U_r^p = \kappa\psi_1\theta - p_r$, $U_n^g = \psi_2\theta - p_n$, and $U_r^g = \psi_1\theta - p_r$.

Proposition 2.4. *The demand functions for new and remanufactured products are as follows.*

(i) If $\frac{\kappa p_n\psi_1}{\psi_2} < p_r \le \frac{p_n\psi_1}{\psi_2}$ *, the demand functions of new and remanufactured products are:*

$$q_n = q_n^p + q_n^g = \frac{1}{2}(1 - \frac{p_r}{\kappa\psi_2}) \tag{2.5a}$$

$$q_r = q_r^p + q_r^g = 0 \tag{2.5b}$$

(ii) If $p_n - (\psi_2 - \psi_1) < p_r \le \frac{\kappa p_n\psi_1}{\psi_2}$*, we have*

$$q_n = q_n^p + q_n^g = \frac{1}{2}(1 - \frac{p_n - p_r}{\psi_2 - \psi_1}) + \frac{1}{2}(1 - \frac{p_n - p_r}{\psi_2 - \kappa\psi_1}) \tag{2.6a}$$

$$q_r = q_r^p + q_r^g = \frac{1}{2}(\frac{p_n - p_r}{\psi_2 - \kappa\psi_1} - \frac{p_n}{\psi_2}) + \frac{1}{2}(\frac{p_n - p_r}{\psi_2 - \psi_1} - \frac{p_n}{\psi_2}) \tag{2.6b}$$

Proof. For primary customers:

(i) $\chi_n^p = \{\theta \ge \frac{p_r}{\psi_2 1\kappa}, \theta \ge \frac{p_n - p_r}{\psi_2 - \kappa\psi_1}\}, U_n^p > U_r^p \ge 0$

(a) $\frac{p_r}{\psi_1\kappa} > \frac{p_n - p_r}{\psi_2 - \kappa\psi_1}$ that is, $p_r\psi_2 > \kappa p_n\psi_1$. Therefore, the demand of primary customers for the new products,

$$q_n^p = \frac{1}{2}\int_{\frac{p_r}{\kappa\psi_1}}^{1} d\theta = \frac{1}{2}\left(1 - \frac{p_r}{\kappa\psi_1}\right)$$

(b) $\frac{p_r}{\kappa\psi_1} \leq \frac{p_n-p_r}{\psi_2-\kappa\psi_1}$ that is, $p_r\psi_2 \leq \kappa p_n\psi_1$. Therefore, to determine the demand of primary customers for new products, we need to consider whether $\frac{p_n-p_r}{\psi_2-\kappa\psi_1}$ is greater than one or not.

If $\frac{p_n-p_r}{\psi_2-\kappa\psi_1} \leq 1$, then $q_n^p = \frac{1}{2}\int_{\frac{p_n-p_r}{(1-\kappa)\psi_2}}^{1} d\theta = \frac{1}{2}\left(1 - \frac{p_n-p_r}{(1-\kappa)\psi_2}\right)$.

If $\frac{p_n-p_r}{\psi_2-\kappa\psi_1} > 1$, then $q_n^p = 0$.

(ii) $\chi_r^p = \{\theta \geq \frac{p_n}{\psi_2}, \theta \leq \frac{p_n-p_r}{\psi_2-\kappa\psi_1}\}, 0 \leq U_n^p \leq U_r^p$.

(a) $\frac{p_n}{\psi_2} \leq \frac{p_n-p_r}{\psi_2-\kappa\psi_1}$, that is, $p_r\psi_2 \leq \kappa p_n\psi_1$.

If $\frac{p_n-p_r}{\psi_2-\kappa\psi_1} \leq 1$, then $q_r^p = \frac{1}{2}\int_{\frac{p_n}{\psi_2}}^{\frac{p_n-p_r}{\psi_2-\kappa\psi_1}} d\theta = \frac{1}{2}(\frac{p_n-p_r}{\psi_2-\kappa\psi_1} - \frac{p_n}{\psi_2})$.

If $\frac{p_n-p_r}{\psi_2-\kappa\psi_1} > 1$, that is, $p_r < p_n - (\psi_2 - \kappa\psi_1)$, then $q_r^p = \frac{1}{2}\int_{\frac{p_n}{\psi_2}}^{1} d\theta = \frac{1}{2}(1 - \frac{p_n}{\psi_2})$.

(b) $\frac{p_n}{\psi_2} > \frac{p_n-p_r}{\psi_2-\kappa\psi_1}$, then $q_r^p = 0$.

Thus, for primary customers, the demand functions are:

$$(q_n^p, q_r^p) = \begin{cases} \left(\frac{1}{2}(1 - \frac{p_r}{\kappa\psi_1}), 0\right), & \text{when } p_r > \frac{\kappa p_n\psi_1}{\psi_2} \\ \left(\frac{1}{2}(1 - \frac{p_n-p_r}{\psi_2-\kappa\psi_1}), \frac{1}{2}(\frac{p_n-p_r}{\psi_2-\kappa\psi_1} - \frac{p_n}{\psi_2})\right) & \text{when } p_n - (\psi_2 - \kappa\psi_1) < p_r \leq \frac{\kappa p_n\psi_1}{\psi_2} \\ \left(0, \frac{1}{2}\left(1 - \frac{p_n}{\psi_2}\right)\right), & \text{when } p_r \leq p_n - (\psi_2 - \kappa\psi_1) \end{cases}$$

In the above equation, $p_r \leq p_n - (\psi_2 - \kappa\psi_1)$ is not possible, since we assume that the OEM will always produce new products.

For green customers, $\chi_n^g = \{\theta, U_n^g \geq \max\{U_r^g, 0\}\}$, $\chi_r^g = \{\theta, U_r^g \geq \max\{U_n^g, 0\}\}$. Similarly, we can obtain the quantity as

$$(q_n^g, q_r^g) = \begin{cases} \left(\frac{1}{2}(1-\frac{p_r}{\psi_1}), 0\right), & \text{when } p_r > \frac{p_n\psi_1}{\psi_2} \\ \left(\frac{1}{2}(1-\frac{p_n-p_r}{\psi_2-\psi_1}), \frac{1}{2}(\frac{p_n-p_r}{\psi_2-\psi_1}-\frac{p_n}{\psi_2})\right) & \text{when } p_n-(\psi_2-\psi_1) < p_r \leq \frac{p_n\psi_1}{\psi_2} \\ \left(0, \frac{1}{2}\left(1-\frac{p_n}{\psi_2}\right)\right), & \text{when } p_r \leq p_n-(\psi_2-\psi_1) \end{cases}$$

In the above equation, $p_r \leq p_n - (\psi_2 - \psi_1)$ is not possible, since we assume that the OEM will always produce new products.

From above, the demand function can be written as:

$$q_n = q_n^p + q_n^g = \begin{cases} \frac{1}{2}(1-\frac{p_r}{\kappa\psi_2}), & \text{when } \frac{\kappa p_n\psi_1}{\psi_2} < p_r \leq \frac{p_n\psi_1}{\psi_2} \\ \frac{1}{2}(1-\frac{p_n-p_r}{\psi_2-\psi_1}) + \frac{1}{2}(1-\frac{p_n-p_r}{\psi_2-\kappa\psi_1}), & \text{when } p_n-(\psi_2-\psi_1) < p_r \leq \frac{\kappa p_n\psi_1}{\psi_2} \end{cases}$$

$$q_r = q_r^p + q_r^g = \begin{cases} 0, & \text{when } \frac{\kappa p_n\psi_1}{\psi_2} < p_r \leq \frac{p_n\psi_1}{\psi_2} \\ \frac{1}{2}(\frac{p_n-p_r}{\psi_2-\kappa\psi_1}-\frac{p_n}{\psi_2}) + \frac{1}{2}(\frac{p_n-p_r}{\psi_2-\psi_1}-\frac{p_n}{\psi_2}), & \text{when } p_n-(\psi_2-\psi_1) < p_r \leq \frac{\kappa p_n\psi_1}{\psi_2} \end{cases}$$

$\square$

The product design (ψ_1 and ψ_2) and their end-of-life options, production quantities, optimal price, and total profit can be derived in the same manner as it is presented in Section (2.3.4).

Chapter 3

Multi-product Newsvendor Model with Customer-driven Substitution: A Stochastic Program

3.1 Introduction

The substitution problem in a multi-product supply chain contains the following three unique aspects. First, the customers' demand is, typically, highly uncertain. However, even if the demand were known, because of substitution, it would be challenging to distribute across different products. Second, due to substitution effect, the order quantities of some products can be reduced while those of the others might be increased, which causes the underlying supply chain to be significantly different from a single product supply chain that is typically addressed in the literature. Third, the left-over products at the end of the planning period have to be salvaged at a relatively low price, which requires a sophisticated predetermined optimal ordering policy. These aspects result in severe modeling and algorithmic challenges that require sophisticated methodologies to address them. In this chapter, we plan to directly address these features.

The multi-product substitution in the inventory management problem has been extensively studied in the literature. In [16, 59, 76], the authors revealed that the substitution is very effective in hedging against risks from demand uncertainty and increasing the sales for retailers. In the work of [17], the authors studied and characterized the substitution as the use of one product to satisfy the demand of a different product within a specific product category. Zhang et al. [91] suggested that the products with similar functionality, color, style, size, or price can be substituted with each other. In [39], the authors studied different decision scenarios for a basic supply chain with one manufacturer and one retailer. There are many notions of substitution and interested readers can refer to [71] for a comprehensive review. Other extensions of the inventory problem with substitution can be found in [8], [39], [12], [49], [34], [73], [87], and [89], etc. In this thesis, we will focus on customer-driven demand substitution, where the substitution is driven by customers rather than by companies and there is no substitution restriction among different products.

There are many studies on analyzing the multi-product newsvendor problem with *customer-driven demand substitutions* (e.g., [13, 35, 51, 56, 59]).

An earlier work can be found in [56], where the authors proved the concavity of the objective function for a two-product newsvendor problem under a mild condition. This thesis also provided necessary conditions to quantify the optimal solution. When the products are decentralized among different retailers, Cachon and Netessine [13] formulated the problem as a decentralized game and investigated properties of their underlying model through game theory. In their model, there are two retailers who sell the same products, where the customers of an out-of-stock retailer can turn to the products of the other retailer. Later on, Huang et al. [35] characterized the conditions for existence and uniqueness of Nash equilibrium of a decentralized model and provided an iterative algorithm to obtain it. In this thesis, our focus is on modeling a centralized multi-product newsvendor problem. This model is highly nonconvex, and for which only very few solution algorithms have been attempted.

In [59], the authors developed a service rate heuristic algorithm to solve a model developed only for two products. In addition, they proposed an upper bound by solving a Lagrangian relaxation problem in order to evaluate the performance of the proposed heuristic approach. Netessine and Rudi [51] showed some analytical properties of the centralized multi-product newsvendor problem, where they demonstrated that the deterministic objective function can be quasiconcave or bimodal. However, their results are not sufficient to completely characterize the model properties or develop any efficient solution algorithms. In this thesis, we develop several useful insights and efficient solution algorithms with performance guarantees for both deterministic and stochastic demand.

Summary of Main Contributions: The objective of this study, motivated by e-commerce, is to determine optimal order quantities of a multi-product supply chain under customer-driven demand substitution, which maximizes the expected profit including sales profit and salvage value. The main contributions of this work are summarized as below:

(i) When the demand is deterministic, we show a complete characterization of the optimal order quantity for each product, i.e., the optimal order quantity of each product will be either 0 or equal to its effective demand. This characterization allows us to reformulate the entire problem as a binary quadratic program (BQP), which admits a tight semidefinite program (SDP) relaxation bound.

(ii) When the demand is stochastic, we first apply sampling average approximation (SAA) to approximate the model, i.e., we formulate the model as a two-stage stochastic program with finite support. We derive first order necessary conditions of the optimal order quantities, and based on these conditions, we give tight lower and upper bounds of optimal order quantities. We then prove that the profit function is continuous submodular in the order quantities, i.e., the marginal benefit of increasing one product's order quantity decreases as another product's order quantity increases.

(iii) The model properties in Part (ii) further motivate us to derive efficient solution approaches. First of all, the observation that optimal order quantity of each product is bounded, allows us to derive two mixed-integer linear program (MILP) formulations. We show that one MILP formulation is stronger than the other, which can be solved to optimality by the off-the-shelf solvers if the number of products is not very large.

(iv) We also investigate approximation algorithms for the stochastic model, which are still efficiently computable, in particular when MILP formulations fail. Our first approximation algorithm is based on relaxing nonanticipativity constraints of the two-stage stochastic program model, which enables the decomposition of the stochastic model into a series of deterministic ones.

(v) Finally, we conduct numerical experiments to test the performance of proposed algorithms as well as to illustrate useful managerial insights. We show that the MILP models work well for small- or medium-size instances; however, for larger instances, the approximation algorithms consistently outperform the MILP models. In addition, we show that the substitution among multi-products can reduce the risks from demand uncertainty significantly as well as increase the expected profit.

The remainder of the chapter is organized as follows. Section 4.2 introduces the problem setting and our model formulation. Section 3.3 presents the properties and main results of the model for the special case of known demand. In Sections 3.4 and 3.5, we study model properties for the case of stochastic demand, and develop two mixed integer programming formulations along with several efficient solution approaches. Section 3.6 presents results of our numerical investigation of proposed solution algorithms and also discusses managerial insights gained from this investigation.

Notation: The following notation is used throughout the chapter. We use bold-letters (e.g., $\boldsymbol{x}, \boldsymbol{A}$) to denote vectors and matrices, and use corresponding non-bold letters to denote their components. We let $\boldsymbol{e}$ be the vector of all ones, and let $\boldsymbol{e}_i$ be the ith standard basis vector. Given an integer n, we let $[n] := \{1, 2, \ldots, n\}$, and use $\mathbb{R}^n_+ := \{\boldsymbol{x} \in \mathbb{R}^n : x_i \geq 0, \forall i \in [n]\}$ and $\mathbb{R}^n_{++} := \{\boldsymbol{x} \in \mathbb{R}^n : x_i > 0, \forall i \in [n]\}$. Given a real number t, we let $(t)_+ := \max\{t, 0\}$, $\lceil t \rceil$ be its round-up value and $\lfloor t \rfloor$ be its round-down value. Given a finite set I, we let $|I|$ denote its cardinality. We let $\tilde{\boldsymbol{\xi}}$ denote a random vector with support Ξ and denote its realizations by $\boldsymbol{\xi}$. Given a vector $\boldsymbol{x} \in \mathbb{R}^n$, let $\mathrm{supp}(\boldsymbol{x})$ be its support, i.e., $\mathrm{supp}(\boldsymbol{x}) := \{i \in [n] : x_i \neq 0\}$, and let $(\boldsymbol{x}|x_j \leftarrow a)$ be a vector obtained by replacing jth entry of $\boldsymbol{x}$ with a. Additional notation will be introduced as needed.

3.2 Model Formulation

In this section, we present a model formulation for our Multi-product Newsvendor Problem with Customer-driven Demand Substitution (MPNP-CDS).

In the MPNP-CDS, we suppose that there are n products for sale which are indexed by $[n] := \{1, \cdots, n\}$. These products are similar to each other, and each product $i \in [n]$, bears a random demand $\tilde{D}_i$. Similar to many newsvendor problems, we assume that each product $i \in [n]$ has cost c_i, price p_i, and salvage value s_i at the end of planning horizon with $p_i \geq c_i \geq s_i$. Due to substitution effects, when a product $j \in [n]$ runs out of stock, its demand can be often proportionally substituted by other products. Let α_{ji}[1] be the substitution rate

[1]Note that we allow the units of products to be different from each other, thus, α_{ji} might be larger than 1. For example, one unit of product 1 is a dozen of apples, and one unit of product 2 is only 1 apple. Suppose that 50% customers of product 1 will choose to buy product 2 if product 1 is unavailable, then $\alpha_{12} = 50\% \times 12 = 6$.

of the unmet demand of product $j \in [n]$ by another product $i \in [n]$, i.e., there are α_{ji} units of product i that can substitute one unit of the unmet demand of product j. By convention, we let $\alpha_{ii} = 0$. In this model, the decision variable is the order quantity of each product $i \in [n]$, denoted as Q_i.

Note that for each product $i \in [n]$, its effective demand function, denoted as $\tilde{D}_i^s(\boldsymbol{Q})$, constitutes of two parts, i.e., primary demand $\tilde{D}_i$ and substitutable demand $\sum_{j\in[n]} \alpha_{ji}(\tilde{D}_j - Q_j)_+$, i.e.,

$$\tilde{D}_i^s(\boldsymbol{Q}) = \tilde{D}_i + \sum_{j\in[n]} \alpha_{ji}(\tilde{D}_j - Q_j)_+. \tag{3.1}$$

In view of the notation introduced above, the MPNP-CDS can be formulated as follows:

$$v^* = \max_{\boldsymbol{Q}\in\mathbb{R}_+^n} \left\{ \Pi(\boldsymbol{Q}) := \mathbb{E}\left[\sum_{i\in[n]} \left(p_i \min\left(Q_i, \tilde{D}_i^s(\boldsymbol{Q})\right) - c_i Q_i + s_i \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q})\right)_+ \right) \right] \right\}, \tag{3.2}$$

where $(x)_+ = \max\{x, 0\}$ denotes the nonnegative part of number x and $\Pi(\cdot)$ denotes the expected profit function. In the above Model (3.2), the objective is to maximize the expected profit, where the first term is the expected revenue from sales, the second term is the cost incurred, and the last term is the expected salvage value. Let $\bar{P}_i = p_i - c_i$ and $\bar{S}_i = p_i - s_i$ for each product $i \in [n]$. Also, because of the identity that $\min(Q_i, \tilde{D}_i^s(\boldsymbol{Q})) = Q_i - (Q_i - \tilde{D}_i^s(\boldsymbol{Q}))_+$, the above Model (3.2) is equivalent to

$$v^* = \max_{\boldsymbol{Q}\in\mathbb{R}_+^n} \left\{ \Pi(\boldsymbol{Q}) = \sum_{i\in[n]} \bar{P}_i Q_i - \mathbb{E}\left[\sum_{i\in[n]} \bar{S}_i \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q})\right)_+ \right] \right\}, \tag{3.3}$$

Note that if there is no substitution effect (i.e., $\alpha_{ji} = 0$ for each $i, j \in [n]$), then Model (3.3) reduces to n classical newsvendor problems, one for each product. On the other hand, with the customer-driven demand substitution, the profit function $\Pi(\boldsymbol{Q})$ is usually neither convex nor concave even if there are only two products ($n = 2$) (cf. [51]).

Finally, we remark that, for the MPNP-CDS, the substitution rate matrix $\boldsymbol{\alpha}$ plays an important role, therefore various methods have been developed in the literature to estimate matrix $\boldsymbol{\alpha}$. For example, [84] have suggested to treat the substitution rates as probabilities and assumed independent choices among different alternatives for a given product. [21] have presented three different approaches, i.e., multi-logit model, locational choice model, and exogenous demand model, to construct the substitution rate matrix. [75] have proposed to build the substitution rate matrix by using the proportional substitution matrix, given that the market share of each product and the loss probability for each product are known. [38] have studied a random substitution model, where the substitution rates are estimated by regression.

3.3 Deterministic Demand

In this section, we study a special case of MPNP-CDS (denoted as MPNP-CDS(D)), for which the demand is known. We will first show a complete characterization of optimal order quantities, reformulate the MPNP-CDS(D) as a discrete submodular maximization problem as well as a binary quadratic program (BQP), and also show the complexity of the MPNP-CDS(D). The formulation and model properties developed in this section also serve as a foundation for subsequent sections.

To begin with, we formally state our assumption on the deterministic demand.

Assumption 1. *The demand for each product $i \in [n]$ is known, i.e., $\tilde{D}_i = D_i$, where D_i is a positive constant.*

Under this assumption, the MPNP-CDS (3.3) reduces to MPNP-CDS(D), which can be formulated as below:

$$v_D^* = \max_{\boldsymbol{Q}\in\mathbb{R}_+^n} \left\{ \Pi(\boldsymbol{Q}) = \sum_{i\in[n]} \bar{P}_i Q_i - \sum_{i\in[n]} \bar{S}_i \left(Q_i - D_i^s(\boldsymbol{Q})\right)_+ \right\}, \tag{3.4}$$

where the ith effective demand function is $D_i^s(\boldsymbol{Q}) = D_i + \sum_{j\in[n]} \alpha_{ji}(D_j - Q_j)_+$ for each $i \in [n]$.

3.3.1 Characterization of Optimal Order Quantities and Model Reformulation of MPNP-CDS(D)

For notational convenience, we let $\boldsymbol{Q}^*$ denote the optimal order quantities in (3.4). In this subsection, we first show a characterization of optimal order quantities $\boldsymbol{Q}^*$ and thus, reformulate the MPNP-CDS(D) (3.4) as a submodular optimization problem.

In the next theorem, we show a characteristics of the optimal order quantities $\boldsymbol{Q}^*$ of MPNP-CDS(D) (3.4). To derive this result, we first compare Q_i^* with D_i and $D_i^s(\boldsymbol{Q}^*)$, analyze the optimality condition of $\boldsymbol{Q}^*$, and finally reformulate MPNP-CDS(D) (3.4) as a combinatorial optimization problem.

Theorem 3.1. *Let $\boldsymbol{Q}^*$ be optimal order quantities for the MPNP-CDS(D) (3.4). Then,*

(i)

$$Q_j^* = \begin{cases} 0, & \text{if } \bar{P}_j - \sum_{i\in[n]\setminus\Gamma^*} \alpha_{ji}\bar{P}_i < 0 \\ D_j^s(\boldsymbol{Q}^*) = D_j + \sum_{i\in\Gamma^*} \alpha_{ij} D_i, & \text{otherwise.} \end{cases} \tag{3.5}$$

for each $j \in [n]$; and

(ii)

$$v_D^* = \max_{\Gamma \subseteq [n]} \left\{ f(\Gamma) := \sum_{j \in \Gamma} \sum_{i \in [n] \setminus \Gamma} \alpha_{ji} \bar{P}_i D_j + \sum_{i \in [n] \setminus \Gamma} \bar{P}_i D_i \right\} := f(\Gamma^*), \tag{3.6}$$

where $[n] \setminus \mathrm{supp}(\boldsymbol{Q}^*) = \Gamma^*$, *i.e.,* $\Gamma^* = \{i \in [n] : Q_i^* = 0\}$.

Proof. See Appendix A.1. □

By Proposition 4.9, we can conclude that, when demand is known, the optimal order quantity of product $i \in [n]$, denoted as Q_i^*, is either zero or equal to its effective demand $D_i^s(\boldsymbol{Q}^*)$, depending on the difference between its own maginal profit $\bar{P}_i$ and its substituted marginal profit $\sum_{i \in [n] \setminus \Gamma^*} \alpha_{ji} \bar{P}_i$. In fact, $\bar{P}_j - \sum_{i \in [n] \setminus \Gamma^*} \alpha_{ji} \bar{P}_i < 0$ implies that directly ordering a product j is less profitable than substituting it. Thus, the optimal order quantity of product j should be 0; otherwise, ordering a product j is more profitable by other products. Therefore, the decision maker should order up to its effective demand, that is, the sum of its primary demand and substitution part. In objective function of (4.18), the first term corresponds to the profit generated by using each product $i \in [n] \setminus \Gamma$ to satisfy the demand of its substitutable product $j \in \Gamma$, while the second term consists of the profit generated by using each product $i \in [n] \setminus \Gamma$ to satisfy its own demand.

Another interesting observation is that, under some special conditions, the optimal order quantity of a product is equal to its demand, i.e., substitution does not take effect. One particular condition is given as below.

Corollary 3.2. *Suppose (1)* $\bar{P}_i = \bar{P}_j, \forall i, j \in [n]$ *and (2) for each product* $j \in [n]$, $\sum_{i \in [n]} \alpha_{ji} \leq 1$. *Then* $Q_j^* = D_j$ *for all* $j \in [n]$.

Proof. Note that in this case, the optimal subset $\Gamma^* = \emptyset$ according to Proposition 4.9. Therefore, $Q_j^* = D_j^s(\boldsymbol{Q}^*) = D_j$ for all $j \in [n]$. □

Next, we show that the set function $f(\Gamma)$ in (4.18) is submodular, i.e., it has diminished marginal benefit when the size of set Γ grows. Below, we briefly introduce the definition of submodularity and interested readers are referred to [22, 43, 72] for more details.

Definition 3.3. (Discrete Submodularity) Given a finite set Θ, let 2^{Θ} denote its power set. Then a set function $g : 2^{\Theta} \to \mathbb{R}$ is "submodular" if and only if it satisfies the following condition:

- for every $X, Y \subseteq \Theta$ with $X \subseteq Y$ and every $x \in \Theta \backslash Y$, we must have $g(X \cup \{x\}) - g(X) \geq g(Y \cup \{x\}) - g(Y)$.

By directly checking the definition, we can show that

Proposition 3.4. *The set function $f(\Gamma)$, defined in (4.18), is submodular.*

Proof. See Appendix A.2. $\square$

The submodular property in Proposition 3.4 implies that the increment in expected profit becomes smaller as the subset Γ (denoting a subset of products with zero order quantities) grows. Therefore, inferring from this property, we anticipate that the optimal set Γ^* might not be large. In the next subsection, we derive a BQP reformulation of (4.18), which has a tight semidefinite program (SDP) relaxation bound.

3.3.2 Complexity of the MPNP-CDS and Alternative BQP Formulation

In this subsection, we first show that the MPNP-CDS(D) is strongly NP-hard. Since Model (4.18) is a special case of the MPNP-CDS (i.e., with known demand), therefore, solving the MPNP-CDS in general is also strongly NP-hard. Then, in light of Model (4.18), we present an alternative BQP for MPNP-CDS(D) and its SDP relaxation. As we will show in subsequent sections, this result is also useful in designing a tight bound for Model (3.3) with stochastic demand. To prove the complexity result, we show that the well-known strongly NP-hard problem - weighted max-cut problem is polynomially reducible to Model (4.18) of MPNP-CDS(D).

Theorem 3.5. *The MPNP-CDS(D) is strongly NP-hard, so is the MPNP-CDS (3.3).*

Proof. See Appendix A.3. ☐

To reformulate Model (4.18) as an equivalent BQP, we follow the idea from [27] on reformulating a Maximum Directed Cut Problem (MAX DICUT) as a BQP. To do so, we first introduce a binary variable $y_i \in \{-1, 1\}$ for each product $i \in [n]$. We also introduce an additional variable $y_{n+1} \in \{-1, 1\}$ to differentiate between sets Γ and $[n] \setminus \Gamma$, that is, set $\Gamma = \{j \in [n] : y_j = y_{n+1}\}$. For notational convenience, let us denote $w_{ij} = \alpha_{ji}\bar{P}_i D_j, w_{i(n+1)} = \bar{P}_i D_i, w_{ii} = w_{(n+1)i} = 0$ for all $i, j \in [n]$.

With the notation above, we are ready to show that

Proposition 3.6. *Model (4.18) is equivalent to*

$$v_D^* = \max_{\boldsymbol{y}} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} w_{ij} \left(1 - y_i y_{n+1} + y_j y_{n+1} - y_i y_j\right) : y_i \in \{-1, 1\}, \forall i \in [n+1] \right\}. \tag{3.7}$$

Proof. Let $\widehat{v}_D^*$ be the optimal value of Model (3.7). We need to show $\widehat{v}_D^* = v_D^*$.

($\widehat{v}_D^* \leq v_D^*$) Given an optimal solution $\boldsymbol{y}^*$ of Model (3.7), we define a set $\widehat{\Gamma} = \{j \in [n] : y_j^* =$

$y_{n+1}^*\}$. Clearly, $\widehat{\Gamma}$ is a feasible solution of Model (4.18) and

$$\begin{aligned}
\widehat{v}_D^* &= \sum_{i\in[n]}\sum_{j\in[n+1]} \frac{1}{4} w_{ij}\left(1 - y_i^* y_{n+1}^* + y_j^* y_{n+1}^* - y_i^* y_j^*\right), \\
&= \sum_{j\in[n]}\sum_{i\in[n]} \frac{1}{4} w_{ij}\left(1 - y_i^* y_{n+1}^* + y_j^* y_{n+1}^* - y_i^* y_j^*\right) + \sum_{i\in[n]} \frac{1}{2} w_{i(n+1)}\left(1 - y_i^* y_{n+1}^*\right), \\
&= \sum_{j\in\widehat{\Gamma}}\sum_{i\in[n]} \frac{1}{4} w_{ij}\left(2 - y_i^* y_{n+1}^* - y_i^* y_j^*\right) + \sum_{j\in[n]\setminus\widehat{\Gamma}}\sum_{i\in[n]} \frac{1}{4} w_{ij}\left(-y_i^* y_{n+1}^* - y_i^* y_j^*\right) + \\
&\quad \sum_{i\in[n]\setminus\widehat{\Gamma}} w_{i(n+1)}, \qquad (3.8)\\
&= \sum_{j\in\widehat{\Gamma}}\sum_{i\in[n]} \frac{1}{4} w_{ij}\left(2 - 2y_i^* y_{n+1}^*\right) + \sum_{i\in[n]\setminus\widehat{\Gamma}} w_{i(n+1)}, \\
&= \sum_{j\in\widehat{\Gamma}}\sum_{i\in[n]\setminus\widehat{\Gamma}} w_{ij} + \sum_{i\in[n]\setminus\widehat{\Gamma}} w_{i(n+1)}, \\
&= \sum_{j\in\widehat{\Gamma}}\sum_{i\in[n]\setminus\widehat{\Gamma}} \alpha_{ji}\bar{P}_i D_j + \sum_{i\in[n]\setminus\widehat{\Gamma}} \bar{P}_i D_i,
\end{aligned}$$

where the third equality is because $y_j^* y_{n+1}^* = 1$ for all $j \in \widehat{\Gamma}$, and -1, otherwise; the fourth equality is due to the definition of set $\widehat{\Gamma}$, we have $y_i^* y_{n+1}^* + y_i^* y_j^* = 2y_i^* y_{n+1}^*$ if $j \in \widehat{\Gamma}$, and 0, otherwise; the fifth equality is due to that $y_i^* y_{n+1}^* = 1$ for all $i \in \widehat{\Gamma}$, -1, otherwise; and the last equality is due to the definition of w_{ij}. Thus, $\widehat{v}_D^* \leq v_D^*$.

($\widehat{v}_D^* \geq v_D^*$) Given an optimal solution Γ^* of Model (4.18), let us construct vector $\widehat{\boldsymbol{y}} \in \{-1,1\}^{n+1}$ as follows: If $j \in \Gamma^*$, then $\widehat{y}_j = \widehat{y}_{n+1} = 1$, otherwise, $\widehat{y}_j = -1 \neq \widehat{y}_{n+1}$. Clearly, $\widehat{\boldsymbol{y}}$ is feasible to Model (3.7) and following the same derivation as (3.8),we have

$$v_D^* = \sum_{i\in[n]}\sum_{j\in[n+1]} \frac{1}{4} w_{ij}\left(1 - \widehat{y}_i\widehat{y}_{n+1} + \widehat{y}_j\widehat{y}_{n+1} - \widehat{y}_i\widehat{y}_j\right).$$

Thus, $\widehat{v}_D^* \geq v_D^*$.

□

In Model (3.7), let us define $\boldsymbol{Y} = \boldsymbol{y}\boldsymbol{y}^T$. Then, Model (3.7) is equivalent to:

$$v_D^* = \max_{\boldsymbol{Y}} \left\{ \sum_{i\in[n]} \sum_{j\in[n+1]} \frac{1}{4} w_{ij} \left(1 - Y_{i(n+1)} + Y_{j(n+1)} - Y_{ij}\right) : Y_{jj} = 1, \forall j \in [n+1], \boldsymbol{Y} \succeq 0, \operatorname{rank}(\boldsymbol{Y}) = 1 \right\} \tag{3.9}$$

If we relax the constraint $\operatorname{rank}(\boldsymbol{Y}) = 1$ in Model (3.9), v_D^* is upper bounded by the optimal value of the following SDP:

$$\bar{v}_D = \max_{\boldsymbol{Y}} \left\{ \sum_{i\in[n]} \sum_{j\in[n+1]} \frac{1}{4} w_{ij} \left(1 - Y_{i(n+1)} + Y_{j(n+1)} - Y_{ij}\right) : Y_{jj} = 1, \forall j \in [n+1], \boldsymbol{Y} \succeq 0 \right\}. \tag{3.10}$$

It follows from [27] that BQP Model (3.7) can be viewed a special case of MAX DICUT, which implies the following bound comparison result.

Corollary 3.7. $0.79607\bar{v}_D \leq v_D^* \leq \bar{v}_D$.

Proposition 3.7 provides the strength of SDP relaxation Model (3.10), and this result will be useful for the analysis of Lagrangian relaxation approach in section 3.5.2.

3.4 Stochastic Demand: Model Properties and MILP Reformulations

In this section, we study the general MPNP-CDS (3.3) when the demand is stochastic. We derive first-order necessary conditions, and show that the profit function is continuous submodular and that the optimal order quantities are bounded. Consequently, we formulate the MPNP-CDS (3.3) as a two-stage stochastic MILP and further improve this formulation by exploring the model properties.

To begin with, we make the following assumption.

Assumption 2. *The random demand $\tilde{\boldsymbol{D}}$ has a finite support $\{\boldsymbol{D}^k\}_{k\in[N]}$, where each $k \in [N]$ is referred to as a scenario. For each scenario $k \in [N]$, m_k denotes its associated probability mass, i.e.,* $\mathbb{P}\left\{\tilde{\boldsymbol{D}} = \boldsymbol{D}^k\right\} = m_k$.

Under this assumption, the expectation in Model (3.3) is equivalent to a finite summation, thus MPNP-CDS (3.3) can be reformulated as the following scenario-based model

$$\begin{aligned} v^* &= \max_{\boldsymbol{Q}\in\mathbb{R}^n_+} \left\{ \Pi(\boldsymbol{Q}) = \sum_{i\in[n]} \bar{P}_i Q_i - \mathbb{E}\left[\sum_{i\in[n]} \bar{S}_i \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q}) \right)_+ \right] \right\} \\ &= \max_{\boldsymbol{Q}\in\mathbb{R}^n_+} \left\{ \Pi(\boldsymbol{Q}) = \sum_{i\in[n]} \bar{P}_i Q_i - \sum_{k\in[N]} m_k \left[\sum_{i\in[n]} \bar{S}_i \left(Q_i - D_i^{sk}(\boldsymbol{Q}) \right)_+ \right] \right\}, \end{aligned} \tag{3.11}$$

where for each product $i \in [n]$ and scenario $k \in [N]$, its effective demand function is

$$D_i^{sk}(\boldsymbol{Q}) = D_i^k + \sum_{j\in[n]} \alpha_{ji}(D_j^k - Q_j)_+.$$

We remark that if the random demand $\tilde{\boldsymbol{D}}$ is not finitely supported, then one might generate N i.i.d samples, $\{\boldsymbol{D}^k\}_{k\in[N]}$. By the sampling average approximation (SAA) method (cf. [70]), Model (3.3) can be approximated to an arbitrary accuracy by the scenario Model (3.11) if N is large enough (polynomial both in n and accuracy).

3.4.1 Model Properties

In this subsection, we derive the first-order necessary conditions for the scenario Model (3.11) of MPNP-CDS and show that the profit function is continuous submodular.

Note that the objective function $\Pi(\boldsymbol{Q})$ is a nonsmooth function. Therefore, the main proof idea in this subsection is based upon the perturbation method, i.e., suppose the vector of the optimal order quantities $\boldsymbol{Q}^*$ is known, then we analyze the inequality $\Pi(\boldsymbol{Q}^* + \boldsymbol{\epsilon}) \leq \Pi(\boldsymbol{Q}^*)$ for a sufficiently small vector $\boldsymbol{\epsilon} \in \mathbb{R}^n$. Our first result specifies the range of $\boldsymbol{Q}^*$, which further implies that $\boldsymbol{Q}^*$ is a bounded vector. A similar result has been developed in [51] with continuous demand, however, the discrete demand has different necessary conditions and also requires a very different proof.

Theorem 3.8. *Let $\boldsymbol{Q}^*$ be the vector of optimal quantities of Model (3.11). Then,*

$$\mathbb{P}\left(Q_i^* \geq \tilde{D}_i^s(\boldsymbol{Q}^*)\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ji}\mathbb{P}\left(Q_j^* \geq \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* < \tilde{D}_i\right) \geq \frac{\bar{P}_i}{\bar{S}_i}, \forall i \in [n], \tag{3.12a}$$

$$\mathbb{P}\left(Q_i^* > \tilde{D}_i^s(\boldsymbol{Q}^*)\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ji}\mathbb{P}\left(Q_j^* > \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* \leq \tilde{D}_i\right) \leq \frac{\bar{P}_i}{\bar{S}_i}, \forall i \in [n]. \tag{3.12b}$$

Proof. See Appendix A.4. □

We remark that: (i) in (3.12a), if there is no substitution effect (i.e., $\alpha_{ij} = 0$ for all $i, j \in [n]$), then the second term is zero since $\tilde{\boldsymbol{D}}^{\boldsymbol{s}}(\boldsymbol{Q}^*) = \tilde{\boldsymbol{D}}$. Thus, (3.12a) reduces to the necessary optimal condition for the classical newsvendor problem with discrete demand; (ii) in (3.12), as $\boldsymbol{Q}^*$ exists in the effective demand function $\tilde{\boldsymbol{D}}^s(\boldsymbol{Q}^*)$, it can be very difficult to obtain a closed-form expression of the optimal order quantity for each product.

Next, we use the result of Proposition 3.8 to derive upper and lower bounds on the optimal order quantities. The main idea in proving this result is to relax the inequalities in (3.12) until arriving at the desired results.

Proposition 3.9. *Let $\boldsymbol{Q}^*$ be the vector of optimal quantities of Model (3.11). Then, $\boldsymbol{Q}^*$ is upper and lower bounded by $\overline{\boldsymbol{Q}}$ and $\underline{\boldsymbol{Q}}$, respectively, i.e., for each product $i \in [n]$, $\overline{Q}_i \geq Q_i^* \geq$*

$\underline{Q}_i$ *with*

$$\underline{Q}_i = \begin{cases} F^{-1}_{\tilde{D}_i}\left(\frac{\bar{P}_i - \sum_{j\in[n]} \alpha_{ij}\bar{S}_j}{\bar{S}_i - \sum_{j\in[n]} \alpha_{ij}\bar{S}_j}\right), & \textit{if} \quad \bar{S}_i > \sum_{j\in[n]} \alpha_{ij}\bar{S}_j \\ 0, & \textit{otherwise} \end{cases}, \tag{3.13a}$$

$$\overline{Q}_i = \bar{F}^{-1}_{\tilde{D}_i + \sum_{j\in[n]} \alpha_{ji}\tilde{D}_j}\left(\frac{\bar{P}_i}{\bar{S}_i}\right), \tag{3.13b}$$

where $F^{-1}_{\tilde{X}}, \bar{F}^{-1}_{\tilde{X}}$ *denote the lower and upper inverse distribution function of random variable* $\tilde{X}$*, respectively, i.e.,* $F^{-1}_{\tilde{X}}(t) = \inf\left\{\kappa : \mathbb{P}\left(\tilde{X} \le \kappa\right) \ge t\right\}$ *and* $\bar{F}^{-1}_{\tilde{X}}(t) = \inf\left\{\kappa : \mathbb{P}\left(\tilde{X} < \kappa\right) \ge t\right\}$.

Proof. See Appendix A.5. □

Note that to compute the lower bound in (3.13a), we can simply sort $\{D_i^k\}_{k\in[N]}$ in an ascending order such that $D_i^{(1)} \le \ldots \le D_i^{(N)}$, where $\{(1), \ldots, (N)\}$ is a permutation of $[N]$, and choose the smallest index $k_{\min}$ such that $\sum_{i\in[k_{\min}]} m_{(i)} \ge \frac{\bar{P}_i - \sum_{j\in[n]} \alpha_{ij}\bar{S}_j}{\bar{S}_i - \sum_{j\in[n]} \alpha_{ij}\bar{S}_j}$, then $\underline{Q}_i = D_i^{(k_{\min})}$. Similarly, we can compute the upper bound $\overline{Q}_i$ in (3.13b) efficiently.

Next, we show that profit function $\Pi(\boldsymbol{Q})$ defined in (3.11) is a continuous submodular function, which is defined as below.

Definition 3.10. (Continuous Submodularity) Given a closed set Θ, let $g : \Theta \mapsto \mathbb{R}$ be a continuous function. Then $g(\cdot)$ is continuous submodular if and only if for every $\boldsymbol{x}, \boldsymbol{y} \in \Theta$, we must have $g(\boldsymbol{x}) + g(\boldsymbol{y}) \ge g(\max\{\boldsymbol{x}, \boldsymbol{y}\}) + g(\min\{\boldsymbol{x}, \boldsymbol{y}\})$.

Other equivalent definitions of continuous submodularity can be found in [6, 9].

Next we will show that $\Pi(\boldsymbol{Q})$ is continuous submodular. Note that in [51], the authors proved that the profit function is continuous submodular in demand when $N = 1$, i.e., demand is determinitic.

Proposition 3.11. *The profit function* $\Pi(\boldsymbol{Q})$ *defined in* (3.11) *is continuous submodular.*

Proof. See Appendix A.6. □

The continuous submodular property in Proposition 3.11 implies that if the ith product's order quantity becomes larger, then an increment in the expected profit becomes smaller when another product j's order quantity increases, i.e., the marginal benefit diminishes if we increase order quantities of both the products i, j. We will show in the next section that by exploring the submodularity of profit function $\Pi(\boldsymbol{Q})$, there exists an efficient double greedy algorithm that can solve Model (3.11) to near optimality with an approximation ratio 1/3.

3.4.2 MILP Formulations

Note that Model (3.11) is in general nonconvex and nonsmooth. In this subsection, we introduce two different MILP formulations by linearizing nonconvex functions in the form $g(x, y) = \max\{x, y\}$ with additional binary variables. We also show that the second model, which is slightly less intuitive, is stronger than the first model.

We first reformulate Model (3.11) as a two-stage stochastic program as follows:

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}_+^n} \left\{ \sum_{i \in [n]} \bar{P}_i Q_i + \sum_{k \in [N]} m_k H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right) \right\}, \tag{3.14a}$$

where the second-stage function

$$H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right) = -\sum_{i \in [n]} \bar{S}_i \left(Q_i - D_i^k - \sum_{j \in [n]} \alpha_{ji} (D_j^k - Q_j)_+ \right)_+, \tag{3.14b}$$

for each $k \in [N]$. Note that the nonconvexity of function $H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right)$ arises from unmet demand terms $\{(D_j^k - Q_j)_+\}_{j \in [n]}$. Our main idea is to linearize them by introducing additional variables.

Before introducing our two MILP models, we first start with the following two properties of Model (3.14). From Proposition 3.9, we know that the optimal order quantities $\boldsymbol{Q}^*$ can be bounded from above, which is formally stated as follows:

Property 1. *In* (3.14)*, let* $Q_i \leq M_i$ *for each* $i \in [n]$*.*

Note that one might simply choose $\boldsymbol{M} := \overline{\boldsymbol{Q}}$ in Proposition 3.9 as a valid upper bound, or derive another tighter one.

For the second property, as there is only a finite number of realizations $\{\boldsymbol{D}^k\}_{k\in[N]}$ of the random demand $\tilde{\boldsymbol{D}}$, therefore, for each product, we can sort the realizations of its demand in an ascending order, i.e.,

Property 2. *for each* $i \in [n]$*, let* $D_i^{(1)} \leq \ldots \leq D_i^{(N)}$*, where* $\{(1), \ldots, (N)\}$ *is a permutation of* $[N]$*.*

Please note that the demand realizations of different products can be sorted differently, i.e., different products might not share the same permutations to sort their demand realizations.

MILP Model 1. In this model, we introduce nonnegative variables y_i^k and u_i^k to represent salvaged units of product $\left(Q_i - D_i^k - \sum_{j\in[n]} \alpha_{ji}(D_j^k - Q_j)_+\right)_+$ and unmet demand $(D_i^k - Q_i)_+$ for each product $i \in [n]$, respectively. In addition, we introduce a binary variable $z_i^k = 1$ if $Q_i \geq D_i^k$ (i.e., $u_i^k = 0$) and $z_i^k = 0$ if $Q_i < D_i^k$ (i.e., $u_i^k = D_i^k - Q_i$). In view of the notation above, the second-stage function $H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right)$ can be equivalently represented by the following MILP:

$$H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right) = \max_{\boldsymbol{u}^k, \boldsymbol{y}^k, \boldsymbol{z}^k} \quad -\sum_{i\in[n]} \bar{S}_i y_i^k \tag{3.15a}$$

$$\text{s.t.} \quad y_i^k \geq Q_i - D_i^k - \sum_{j\in[n]} \alpha_{ji} u_j^k, \forall i \in [n], \tag{3.15b}$$

$$D_i^k - Q_i + M_j z_i^k \geq u_i^k \geq D_i^k - Q_i - M_j z_i^k, \forall i \in [n], \tag{3.15c}$$

$$0 \leq u_i^k \leq D_i^k \left(1 - z_i^k\right), \forall i \in [n], \tag{3.15d}$$

$$y_i^k \geq 0, \forall i \in [n], \tag{3.15e}$$

$$z_i^k \in \{0, 1\}, \forall i \in [n]. \tag{3.15f}$$

Note that for each product $i \in [n]$, the objective function (3.15a), constraints (3.15b) and nonnegativity constraints (3.15e) together enforce that $y_i^k = Q_i - D_i^k - \sum_{j\in[n]} \alpha_{ji} u_j^k$. Constraints (3.15d), (3.15e), and (3.15f) along with Property 1 imply that

$$u_i^k = (D_i^k - Q_i)_+ = \begin{cases} 0, & \text{if } Q_i \geq D_i^k \\ D_i^k - Q_i, & \text{otherwise} \end{cases}.$$

We conclude the validity of MILP Model 1 in the following proposition.

Proposition 3.12. *The second-stage function $H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right)$ is equivalent to the MILP formulation (3.15) for each scenario $k \in [N]$, i.e., and MILP Model 1 is*

$$v^* = \max_{\boldsymbol{Q}\in\mathbb{R}_+^n} \left\{ \sum_{i\in[n]} \bar{P}_i Q_i + \sum_{k\in[N]} m_k H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right) \right\}$$

where $H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right)$ is defined in (3.15).

MILP Model 2. In the formulation (3.15), note that, the expressions $\left\{(D_i^k - Q_i)_+\right\}_{i\in[n]}$ share the same monotonicity as $\left\{D_i^k\right\}_{i\in[n]}$. This property motivates us to derive a stronger formulation to represent expected second-stage functions, i.e., $\sum_{k\in[N]} m_k H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right)$.

In MILP Model 2, we use the same variables $\{\boldsymbol{u}^k\}_{k\in[N]}, \{\boldsymbol{y}^k\}$ as in MILP Model 1, i.e., nonnegative variables y_i^k and u_i^k represent salvaged units of product $\left(Q_i - D_i^k - \sum_{j\in[n]} \alpha_{ji}(D_j^k - Q_j)_+\right)_+$ and unmet demand $(D_i^k - Q_i)_+$ for each product $i \in [n]$, respectively. From Property 2, we know that the demand realizations for each product are sorted as

$$D_i^{(1)} \leq \ldots \leq D_i^{(N)}.$$

Let $\widehat{D}_i^{(k)} = \min\left\{D_i^{(k)}, M_i\right\}$ for each $k \in [N]$ since M_i might be smaller than some of $\{D_i^{(k)}\}_{k\in[N]}$. We note that, for each product $i \in [n]$, its order quantity Q_i must belong to one of the following $N+1$ intervals:

$$\left[\widehat{D}_i^{(0)}, \widehat{D}_i^{(1)}\right], \left[\widehat{D}_i^{(1)}, \widehat{D}_i^{(2)}\right], \ldots, \left[\widehat{D}_i^{(N)}, \widehat{D}_i^{(N+1)}\right].$$

where $\widehat{D}_i^{(0)} = 0, \widehat{D}_i^{(N+1)} = M_i$. Therefore, we introduce one binary variable for each interval to indicate whether Q_i is in this interval or not, i.e., we let $\chi_i^{(k)} = 1$ if $Q_i \in [\widehat{D}_i^{(k-1)}, \widehat{D}_i^{(k)}]$, and 0, otherwise. Also, we let $\sum_{k\in[N+1]} \chi_i^{(k)} = 1$ to enforce that Q_i indeed belong to only one interval (we break the boundary points arbitrarily).

In order to formulate the model as a mathematical program, for each product $i \in [n]$ and $k \in [N+1]$, we introduce another variable $w_i^{(k)}$ to be equal to Q_i if $Q_i \in [\widehat{D}_i^{(k-1)}, \widehat{D}_i^{(k)}]$; and 0, otherwise. That is, $\widehat{D}_i^{(k-1)}\chi_i^{(k)} \leq w_i^{(k)} \leq \widehat{D}_i^{(k)}\chi_i^{(k)}$, and $\sum_{k\in[N+1]} w_i^{(k)} = Q_i$.

Now, we can represent $u_i^{(k)}$ with variables $\{\chi_i^{(\tau)}\}_{\tau\in[N]}$ and $\{w_i^{(\tau)}\}_{\tau\in[N]}$ for each product $i \in [n]$ and $k \in [N]$ as follows:

$$u_i^{(k)} = D_i^{(k)} \sum_{\tau\in[k]} \chi_i^{(\tau)} - \sum_{\tau\in[k]} w_i^{(\tau)}$$

which is equal to 0 if $Q_i > D_i^{(k)}$, and otherwise, it is equal to $D_i^{(k)} - Q_i$.

In view of the above development, we can represent $\sum_{k\in[N]} m_k H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right)$ as the following mathematical program:

$$\sum_{k\in[N]} m_k H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right) = \max_{\boldsymbol{u},\boldsymbol{w},\boldsymbol{\chi},\boldsymbol{y}} \quad -\sum_{i\in[n]}\sum_{k\in[N]} \bar{S}_i m_i^{(k)} y_i^{(k)} \tag{3.16a}$$

$$\text{s.t.} \quad y_i^{(k)} \geq Q_i - D_i^{(k)} - \sum_{j\in[n]} \alpha_{ji} u_j^{(k)}, \forall i \in [n], \forall k \in [N], \tag{3.16b}$$

$$\widehat{D}_i^{(k-1)} \chi_i^{(k)} \leq w_i^{(k)} \leq \widehat{D}_i^{(k)} \chi_i^{(k)}, \forall i \in [n], \forall k \in [N], \tag{3.16c}$$

$$u_i^{(k)} = D_i^{(k)} \sum_{\tau\in[k]} \chi_i^{(\tau)} - \sum_{\tau\in[k]} w_i^{(\tau)}, \forall i \in [n], \forall k \in [N], \tag{3.16d}$$

$$\sum_{k\in[N+1]} w_i^{(k)} = Q_i, \forall i \in [n], \forall k \in [N], \tag{3.16e}$$

$$\sum_{k\in[N+1]} \chi_i^{(k)} = 1, \forall j \in [n], \forall k \in [N], \tag{3.16f}$$

$$y_i^{(k)} \geq 0, u_i^{(k)} \geq 0, \forall i \in [n], w_i^{(k)} \geq 0, \forall i \in [n], \forall k \in [N], \tag{3.16g}$$

$$\chi_i^{(k)} \in \{0, 1\}, \forall i \in [n], \forall k \in [N], \tag{3.16h}$$

where for notational convenience, we let $\left\{m_i^{(k)}\right\}_{k\in[N]}$ be the permutation of $\{m_k\}_{k\in[N]}$ in the same order as $\left\{D_i^{(k)}\right\}_{k\in[N]}$.

From the discussion above, we conclude that Model (5.6) is a valid representation of $\sum_{k\in[N]} m_k H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right)$. Therefore, we can formally state the validity of MILP Model 2 as follows.

Proposition 3.13. *The expected second-stage function* $\sum_{k\in[N]} m_k H\left(\boldsymbol{Q}, \boldsymbol{D}^k\right)$ *is equivalent to the MILP formulation* (5.6)*, and MILP Model 2 is equivalent to*

$$v^* = \max_{\boldsymbol{Q},\boldsymbol{\chi},\boldsymbol{u},\boldsymbol{w},\boldsymbol{y}} \left\{ \sum_{i\in[n]} \bar{P}_i Q_i - \sum_{i\in[n]}\sum_{k\in[N]} \bar{S}_i m_i^{(k)} y_i^{(k)} : (5.6b) - (3.16h) \right\}.$$

Next, we show that the MILP Model 2 (from Proposition 3.13) is stronger than the MILP

Model 1 (from Proposition 3.12). To this end, the main idea is to show that $\bar{v}_M^2$ is no smaller than $\bar{v}_M^1$, where $\bar{v}_M^1, \bar{v}_M^2$ denote the continuous relaxation values of these models, respectively, i.e.,

$$\bar{v}_M^1 = \max_{\boldsymbol{Q},\boldsymbol{z},\boldsymbol{u},\boldsymbol{y}} \left\{ \sum_{i\in[n]} \bar{P}_i Q_i - \sum_{k\in[N]} \sum_{i\in[n]} \bar{S}_i m_k y_i^k : (3.15\text{b}) - (3.15\text{e}), \boldsymbol{Q} \in \mathbb{R}_+^n, \boldsymbol{z} \in [0,1]^{n\times N} \right\}, \tag{3.17a}$$

$$\bar{v}_M^2 = \max_{\boldsymbol{Q},\boldsymbol{\chi},\boldsymbol{u},\boldsymbol{w},\boldsymbol{y}} \left\{ \sum_{i\in[n]} \bar{P}_i Q_i - \sum_{i\in[n]} \sum_{k\in[N]} \bar{S}_i m_i^{(k)} y_i^{(k)} : (5.6\text{b}) - (3.16\text{h}), \boldsymbol{Q} \in \mathbb{R}_+^n, \boldsymbol{\chi} \in [0,1]^{n\times(N+1)} \right\}. \tag{3.17b}$$

Theorem 3.14. *The MILP Model 2 is stronger than MILP Model 1, i.e., their continuous relaxation values satisfy $\bar{v}_M^1 \leq \bar{v}_M^2$, where $\bar{v}_M^1, \bar{v}_M^2$ are defined in (3.17a), (3.17b), respectively.*

Proof. See Appendix A.7. ☐

3.5 Stochastic Demand: Approximation Algorithms

Although the MILP Model 2 is proven to be stronger, there are $O(nN)$ binary variables in the both formulations. These additional binary variables may cause difficulties in solving these models, especially for large-sized instances, where the number of products or number of scenarios is large. Therefore, in this section, we present efficient approximation algorithms to solve Model (3.11) with provable performance guarantees. We will adopt the same notation and assumption as the previous section.

3.5.1 Double Greedy Algorithm

From Proposition 3.11, we know that the profit function (3.11) is continuous submodular. Recent development in continuous submodular optimization has shown that a double greedy algorithm can solve a nonnegative continuous submodular function maximization with bounded feasible region efficiently, leading to an approximation ratio of 1/3 (cf. [9]), i.e., suppose $\boldsymbol{Q}$ denotes the output of double greedy algorithm, then $\Pi(\boldsymbol{Q}) \geq v^*/3$.

The detailed implementation of double greedy can be found in Algorithm 1. The algorithm requires n iterations and proceeds as follows. Let $\underline{\boldsymbol{Q}}, \overline{\boldsymbol{Q}}$ be the lower and upper bounds of optimal order quantities $\boldsymbol{Q}$ according to Proposition 3.9. We first initiate two vectors $\boldsymbol{x}^0, \boldsymbol{y}^0$ to be $\underline{\boldsymbol{Q}}, \overline{\boldsymbol{Q}}$, respectively. During ith iteration, we solve the following two univariate optimization problems

$$\max_{Q_a \in \left[\underline{Q}_i, \overline{Q}_i\right]} \Pi\left(\boldsymbol{x}^{i-1} | x_i^{i-1} \leftarrow Q_a\right) \tag{3.18a}$$

$$\max_{Q_b \in \left[\underline{Q}_i, \overline{Q}_i\right]} \Pi\left(\boldsymbol{y}^{i-1} | y_i^{i-1} \leftarrow Q_b\right) \tag{3.18b}$$

where for a vector $\boldsymbol{x}$, $(\boldsymbol{x}|x_j \leftarrow a)$ denotes a copy of vector $\boldsymbol{x}$ except the jth coordinate is replaced by a. Let $\widehat{Q}_a, \widehat{Q}_b$ be the optimal solutions to optimization Models (3.18a) and (3.18b), respectively. Next we check the improvements of the new solutions, $\delta_a = \Pi\left(\boldsymbol{x}^{i-1}|x_i^{i-1} \leftarrow Q_a\right) - \Pi\left(\boldsymbol{x}^{i-1}\right)$ and $\delta_b = \Pi\left(\boldsymbol{y}^{i-1}|y_i^{i-1} \leftarrow Q_b\right) - \Pi\left(\boldsymbol{y}^{i-1}\right)$. If $\delta_a \geq \delta_b$, then let $\boldsymbol{x}^i, \boldsymbol{y}^i$ be $\left(\boldsymbol{x}^{i-1}|x_i^{i-1} \leftarrow \widehat{Q}_a\right), \left(\boldsymbol{y}^{i-1}|y_i^{i-1} \leftarrow \widehat{Q}_a\right)$, respectively; otherwise, let $\boldsymbol{x}^i, \boldsymbol{y}^i$ be $\left(\boldsymbol{x}^{i-1}|x_i^{i-1} \leftarrow \widehat{Q}_b\right), \left(\boldsymbol{y}^{i-1}|y_i^{i-1} \leftarrow \widehat{Q}_b\right)$.

Next we show that both the univariate optimization models in (3.18) are efficiently solvable.

Proposition 3.15. *Suppose that $\boldsymbol{Q} \in \mathbb{R}_+^n$ is known. Then,* *(i) the following optimization model is efficiently solvable,*

$$\max_{q\in\left[\underline{Q}_i,\overline{Q}_i\right]} \Pi\left(\boldsymbol{Q}|Q_i \leftarrow q\right) \tag{3.19}$$

for each $i \in [n]$*; and*

(ii) an optimal solution to Model (3.19) *belongs to set* $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$*, where*

$$\mathcal{R}_1 = \left\{D_i^k : D_i^k \in \left[\underline{Q}_i, \overline{Q}_i\right], \forall k \in [N+1]\right\}, \tag{3.20a}$$

$$\mathcal{R}_2 = \left\{D_i^{sk} : D_i^{sk} \in \left[\underline{Q}_i, \overline{Q}_i\right], \forall k \in [N]\right\}, \tag{3.20b}$$

$$\mathcal{R}_3 = \left\{D_i^k - \frac{Q_j - D_{j,-i}^{sk}}{\alpha_{ij}} : D_i^k - \frac{Q_j - D_{j,-i}^{sk}}{\alpha_{ij}} \in \left[\underline{Q}_i, D_i^k\right], \forall j \in [n], k \in [N]\right\}. \tag{3.20c}$$

Proof. See Appendix A.8. □

By Proposition 3.15, we also note that to solve Model (3.19), we might simply check the objective value for each $q \in \mathcal{R}$ and the one with the largest objective value must be an optimal solution. There are at most $2N+nN$ points in set $\mathcal{R}$, thus, Model (3.19) is efficiently solvable.

From the discussions above, we can conclude that

Corollary 3.16. *Algorithm 1 is a polynomial-time approximation algorithm with an approximation ratio of* $1/3$*, i.e., suppose* $\boldsymbol{Q}$ *denotes the output of Algorithm 1, then* $\Pi(\boldsymbol{Q}) \geq v^*/3$.

3.5.2 Lagrangian Relaxation Approach

In this subsection, we derive an efficiently computable upper bound of Model (3.11) based on the nonanticipativity Lagrangian dual of stochastic program (cf. [70]) as well as the results of the deterministic MPNP-CDS (i.e., MPNP-CDS(D)) in Section 3.3. We show that

Algorithm 1 Double greedy algorithm to solve Model (3.11) ([9])

$\boldsymbol{x}^0 \leftarrow \underline{\boldsymbol{Q}}$, $\boldsymbol{y}^0 \leftarrow \overline{\boldsymbol{Q}}$
for $i = 1$ to n **do**
  Find $\widehat{Q}_a \in \arg\max_{Q_a \in [\underline{Q}_i, \overline{Q}_i]} \Pi\left(\boldsymbol{x}^{i-1} | x_i^{i-1} \leftarrow Q_a\right)$, $\widehat{Q}_b \in \arg\max_{Q_b \in [\underline{Q}_i, \overline{Q}_i]} \Pi\left(\boldsymbol{y}^{i-1} | y_i^{i-1} \leftarrow Q_b\right)$
  Let $\delta_a \leftarrow \Pi\left(\boldsymbol{x}^{i-1} | x_i^{i-1} \leftarrow \widehat{Q}_a\right) - \Pi\left(\boldsymbol{x}^{i-1}\right)$ and $\delta_b \leftarrow \Pi\left(\boldsymbol{y}^{i-1} | y_i^{i-1} \leftarrow \widehat{Q}_b\right) - \Pi\left(\boldsymbol{y}^{i-1}\right)$
  if $\delta_a \geq \delta_b$ **then**
    $\boldsymbol{x}^i \leftarrow \left(\boldsymbol{x}^{i-1} | x_i^{i-1} \leftarrow \widehat{Q}_a\right)$, $\boldsymbol{y}^i \leftarrow \left(\boldsymbol{y}^{i-1} | y_i^{i-1} \leftarrow \widehat{Q}_a\right)$;
  else
    $\boldsymbol{y}^i \leftarrow \left(\boldsymbol{y}^{i-1} | y_i^{i-1} \leftarrow \widehat{Q}_b\right)$, $\boldsymbol{x}^i \leftarrow \left(\boldsymbol{x}^{i-1} | x_i^{i-1} \leftarrow \widehat{Q}_b\right)$
  end if
end for
Output $\boldsymbol{Q} = \boldsymbol{x}^n$ (or $\boldsymbol{y}^n$);

this upper bound can be equivalently computed via a SDP, which is more effective than computing the Lagrangian dual. We prove that this upper bound is only a constant-factor away from the true optimal value v^* given that the random demand is mainly concentrated on the mean. Finally, we derive a heuristic algorithm simply by letting order quantities $\boldsymbol{Q}$ equal to one vector of this linear program, which has a similar economic interpretation as order quantities.

The nonanticipativity Lagrangian dual method has been demonstrated as one of the effective approaches to solve the two-stage stochastic (integer) program (cf. [1, 10, 11, 64, 67]). It decomposes a large-scale stochastic program model into scenario-based subproblems. As a special case of a two-stage stochastic program, we apply this approach to Model (3.11). First of all, we create N copies of vector $\boldsymbol{Q}$, one for each scenario, denoted as $\{\boldsymbol{Q}^k\}_{k\in[N]}$, and enforce them to be equal, i.e.,

$$\boldsymbol{Q}^k = \boldsymbol{Q}, \forall k \in [N], \tag{3.21}$$

where constraints (3.21) are known as "*nonanticipativity constraints*".

Then, Model (3.11) can be equivalently reformulated as follows:

$$v^* = \max_{\boldsymbol{Q}^k \in \mathbb{R}^n_+, \forall k \in [N], \boldsymbol{Q}} \left\{ \sum_{k\in[N]} m_k \sum_{i\in[n]} \bar{P}_i Q_i^k - \sum_{k\in[N]} m_k \left[\sum_{i\in[n]} \bar{S}_i \left(Q_i^k - D_i^{sk}(\boldsymbol{Q}) \right)_+ \right] : (3.21) \right\}. \tag{3.22}$$

The Lagrangian dual problem is to relax the nonanticipativity constraints with Lagrangian multipliers $\boldsymbol{\lambda} = \{\boldsymbol{\lambda}^k\}_{k\in[N]} \in \mathbb{R}^{n\times N}$, which can be written as

$$v^{LD} = \inf_{\boldsymbol{\lambda}} \mathcal{L}(\lambda) \tag{3.23a}$$

where

$$\begin{aligned} \mathcal{L}(\lambda) := \max_{\boldsymbol{Q}^k \in \mathbb{R}^n_+, \forall k \in [N], \boldsymbol{Q}} & \left\{ \sum_{k\in[N]} m_k \sum_{i\in[n]} \left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) Q_i^k - \sum_{k\in[N]} m_k \left[\sum_{i\in[n]} \bar{S}_i \left(Q_i^k - D_i^{sk}(\boldsymbol{Q}) \right)_+ \right] \right. \\ & \left. - \sum_{k\in[N]} \sum_{i\in[n]} \lambda_i^k Q_i \right\}. \end{aligned} \tag{3.23b}$$

We note that: (i) since $\boldsymbol{Q}$ is a free vector in (3.23b), we must have $\sum_{k\in[N]} \lambda_i^k = 0$, otherwise, $\mathcal{L}(\lambda) = +\infty$; and (ii) since Q_i^k can be positive infinity, we also have $\bar{P}_i + \frac{\lambda_i^k}{m_k} \le \bar{S}_i$ for all $i \in [n]$ and $k \in [N]$, otherwise, if we suppose that there exists a pair (i, k) such that $\bar{P}_i + \frac{\lambda_i^k}{m_k} > \bar{S}_i$, then we can have $\left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) Q_i^k - \bar{S}_i \left(Q_i^k - D_i^{sk}(\boldsymbol{Q}) \right)_+ \to \infty$ if $Q_i^k \to +\infty$, i.e., $\mathcal{L}(\lambda) \to +\infty$; and (iii) we also notice that for any given Lagrangian multipliers $\boldsymbol{\lambda}$, the maximization of the dual problem (3.23b) can be decomposed into N subproblems, one for each scenario. Therefore, the Lagrangian dual problem can be rewritten as

$$v^{LD} = \inf_{\boldsymbol{\lambda}\in\Omega} \sum_{k\in[N]} m_k \max_{\boldsymbol{Q}^k \in \mathbb{R}^n_+} \left\{ \sum_{i\in[n]} \left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) Q_i^k - \sum_{i\in[n]} \bar{S}_i \left(Q_i^k - D_i^{sk}(\boldsymbol{Q}) \right)_+ \right\}, \tag{3.24}$$

where

$$\Omega = \left\{ \boldsymbol{\lambda} : \sum_{\tau \in [N]} \lambda_i^\tau = 0, \bar{P}_i + \frac{\lambda_i^k}{m_k} \le \bar{S}_i. \ \forall k \in [N], \ \forall i \in [n] \right\}.$$

For each $k \in [N]$, the inner maximization in (3.24) is a special case of the deterministic MPNP-CDS (3.4) with $\bar{P}_i \leftarrow \bar{P}_i + \frac{\lambda_i^k}{m_k}$ and $D_i \leftarrow D_i^k$ for all $i \in [n]$. Therefore, by Proposition 3.6, the inner maximization in (3.24) is equivalent to Model (3.9), so Lagrangian dual (3.24) is equivalent to

$$v^{LD} = \inf_{\boldsymbol{\lambda} \in \Omega} \sum_{k \in [N]} m_k \max_{\boldsymbol{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k \right) \right\}, \tag{3.25}$$

where $C = \{\boldsymbol{Y} : Y_{jj} = 1, \forall j \in [n+1], \boldsymbol{Y} \succeq 0, \operatorname{rank}(\boldsymbol{Y}) = 1\}$, $w_{ij}^k = \alpha_{ji} D_j^k, w_{i(n+1)}^k = D_i^k, w_{j,i}^k = w_{(n+1)i}^k = 0$ for all $i, j \in [n]$. If we relax the rank-one constraint, then we obtain an upper bound of v^{LD}, denoted as v_R^{LD}, as follows:

$$v_R^{LD} = \inf_{\boldsymbol{\lambda} \in \Omega} \sum_{k \in [N]} m_k \max_{\boldsymbol{Y}^k \in C_R} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k \right) \right\}, \tag{3.26}$$

where $C_R = \{\boldsymbol{Y} : Y_{jj} = 1, \forall j \in [n+1], \boldsymbol{Y} \succeq 0\}$.

The following theorem summarizes the above model comparison results, i.e., $v^* \le v^{LD} \le v_R^{LD}$ and shows an equivalent SDP to obtain v_R^{LD}.

Theorem 3.17. *Let v^*, v^{LD}, v_R^{LD} denote the optimal values obtained for Models (3.11), (3.23b), and (3.26), respectively. Then,*

(i) $v^* \le v^{LD} \le v_R^{LD}$; and

(ii)

$$
\begin{aligned}
v_R^{LD} = & \max_{\boldsymbol{\pi},\boldsymbol{\beta}} \quad \sum_{i\in[n]} \bar{P}_i\beta_i - \sum_{k\in[N]} m_k \sum_{i\in[n]} \bar{S}_i \pi_i^k, && \text{(3.27a)}\\
s.t. \quad & \beta_i - \pi_i^k = \sum_{j=1}^{n+1} \frac{1}{4}\bar{P}_i w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right), \forall i\in[n], \forall k\in[N], && \text{(3.27b)}\\
& \pi_i^k \geq 0, \forall i \in [n], \forall k\in[N], && \text{(3.27c)}\\
& \beta_i \geq 0, \forall i\in[n] && \text{(3.27d)}\\
& Y_{jj}^k = 1, \forall j \in [n+1], \forall k \in [N] && \text{(3.27e)}\\
& \boldsymbol{Y}^k \succeq 0, \forall k \in [N] && \text{(3.27f)}
\end{aligned}
$$

Proof. (i) Clearly, by the discussion above, we have $v^* \leq v^{LD} \leq v_R^{LD}$.

(ii) Notice that in Model (3.26), the inner maximization problem are seperable, thus, we can swap summation and max operators as below,

$$
v_R^{LD} = \inf_{\boldsymbol{\lambda}\in\Omega} \max_{\boldsymbol{Y}\in C_R^N} \sum_{k\in[N]} m_k \left\{ \sum_{i\in[n]} \sum_{j\in[n+1]} \frac{1}{4}\left(\bar{P}_i + \frac{\lambda_i^k}{m_k}\right) w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right) \right\},
$$

where C_R^N denotes n-fold Cartesian product of set C_R and $\boldsymbol{Y} = \{\boldsymbol{Y}^k\}_{k\in[N]}$. Note that C_R is a bounded convex set and Ω is a nonempty polyhedral set, The above function is bilinear in $\boldsymbol{\lambda}$ and $\boldsymbol{Y}$. According to the well-known Sion's minimax theorem (cf. [74]), we can switch the inf and max operators, i.e., Model (3.26) is equivalent to

$$
v_R^{LD} = \max_{\boldsymbol{Y}\in C_R^N} \inf_{\boldsymbol{\lambda}\in\Omega} \sum_{k\in[N]} m_k \left\{ \sum_{i\in[n]} \sum_{j\in[n+1]} \frac{1}{4}\left(\bar{P}_i + \frac{\lambda_i^k}{m_k}\right) w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right) \right\}.
$$

By the strong duality of linear program, we can reformulate the inner infimum in the above formulation as an equivalent maximization problem with dual variables $\boldsymbol{\beta} =$

$\{\beta_i\}_{i\in[n]}, \boldsymbol{\pi} = \{\pi_i^k\}_{i\in[n],k\in[N]}$ corresponding to the constraints in Ω, i.e., Model (3.26) is equivalent to

$$
\begin{aligned}
v_R^{LD} =& \max_{\boldsymbol{\pi},\boldsymbol{\beta}} \quad \sum_{i\in[n]} \bar{P}_i\beta_i - \sum_{k\in[N]}\sum_{i\in[n]} \bar{S}_i\pi_i^k,\\
\text{s.t.} \quad & \beta_i m_k - \pi_i^k = \sum_{j\in[n+1]} \frac{1}{4} m_k \bar{P}_i w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k\right), \forall i \in [n], \forall k \in [N],\\
& \pi_i^k \geq 0, \forall i \in [n], \forall k \in [N],\\
& \beta_i \geq 0, \forall i \in [n]\\
& Y_{jj}^k = 1, \forall j \in [n+1], \forall k \in [N]\\
& Y^k \succeq 0, \forall k \in [N]
\end{aligned}
$$

Redefining $\pi_i^k := \frac{\pi_i^k}{m_k}$ for each $i \in [n], k \in [N]$, we arrive at Model (3.27).

□

Suppose that $(\boldsymbol{\pi}^*, \boldsymbol{\beta}^*)$ is an optimal solution to Model (3.27). From constraints (3.27d), it is clear that $\boldsymbol{\beta}^* \in \mathbb{R}_+^n$. We also note that from the proof of Proposition 3.17, variable β_i^* in (3.27) is the shadow price of Lagrangian dual constraint $\sum_{\tau\in[N]} \lambda_i^\tau = 0$ for each product $i \in [n]$, while from (3.23b) and the subsequent derivations, we can also see that, its order quantity Q_i can be also viewed as the shadow price of Lagrangian dual constraint $\sum_{\tau\in[N]} \lambda_i^\tau = 0$ for each $i \in [n]$. This motivates us to construct a lower bound of Model (3.11) simply by letting $\boldsymbol{Q} = \boldsymbol{\beta}^*$.

Remark 3.18. Let $(\boldsymbol{\pi}^*, \boldsymbol{\beta}^*)$ be an optimal solution to Model (3.27). Then, $\boldsymbol{\beta}^*$ is feasible to Model (3.11), and $\Pi(\boldsymbol{\beta}^*)$ is a lower bound to the optimal value v^*.

Proposition 3.17 provides an efficiently computable upper bound v_R^{LD} for MPNP-CDS (3.11). Next, we investigate the quality of the proposed bound compared with the optimal value

v^* of Model (3.11). Before introducing our main result, we first assume that the random demand $\tilde{\boldsymbol{D}}$ mainly concentrates on its mean or a particular constant vector $\boldsymbol{D}$.

Assumption 3. *Given an n-dimensional nonnegative vector $\boldsymbol{D} \in \mathbb{R}_+^n$ and two constants $\underline{\delta} \in [0,1), \overline{\delta} \in [0,+\infty)$, let the random demand $\tilde{\boldsymbol{D}}$ satisfy*

$$\mathbb{P}\left\{(1-\underline{\delta})D_i \leq \tilde{D}_i \leq (1+\overline{\delta})D_i\right\} = 1.$$

i.e., $(1-\underline{\delta})D_i \leq D_i^k \leq (1+\overline{\delta})D_i$ for all $k \in [N]$.

Under this assumption, we are able to show that v_R^{LD} and v^{LD} are only constant-factor away from the true optimal v^*.

Theorem 3.19. *Let v^*, v^{LD}, v_R^{LD} denote the optimal value of Models (3.11), (3.23b), and (3.26), respectively. Then,*

(i) $v_R^{LD} \leq \frac{v^{LD}}{0.79607}$; and

(ii) if Assumption 3 holds, then

$$v_R^{LD} \leq \frac{v^{LD}}{0.79607} \leq \frac{(1+\overline{\delta})}{0.79607(1-\underline{\delta})} v^*$$

Proof. See Appendix A.9. □

3.6 Numerical Investigation

We conduct two sets of numerical experiments on Model (3.11). For the first set of numerical experiments, we test the performances of solving Model (3.11) by MILP Model 1 (from Proposition 3.12); MILP Model 2 (from Proposition 3.13); double greedy Algorithm 1, relaxed Lagrangian dual bound v_R^{LD} in (3.27) and its related heuristic algorithm from Proposition 3.18. For the second set of numerical experiments, we investigate model properties and conduct sensitivity analyses that reveal useful managerial insights.

3.6.1 Performances of Different Approaches

To study the computational effectiveness of the two MILP formulations and the approximation algorithms, 20 numerical instances of varying sizes were generated. We consider two values of n, namely, $n = 10$ and $n = 20$ products. For each product, its demand was assumed to be uniformly distributed between 5 and 100, its unit price to range from 85 to 95, unit cost to vary from 40 to 50, and the salvage value between 22 and 30. All the products are sold or purchased in the same unit of measurement, and thus their substitution rates are between 0 and 1. [2] These substitution rates were chosen uniformly between 0 and 1, and then were normalized to satisfy $\sum_{j\in[n]} \alpha_{ij} = 0.8$ and $\alpha_{ii} = 0$ for each $i \in [n]$. To study the model performances across different scenarios, we generated 10 different types of samples with sample size, $N = 100, 200, \ldots, 1000$. The results for each pair value of (N, n) are the average values by performing 5 replications. All the algorithms were coded in Python 2.7 with calls to Gurobi 7.5 on a personal computer with 2.3 GHz Intel Core i5 processor and 8G of memory. The CPU time limit of Gurobi was set to be 3600 seconds.

Table 5.1 and Table 5.2 display the computational results of MILP Model 1 in Proposition 3.12 and MILP Model 2 in Proposition 3.13 with $n = 10$ and $n = 20$, respectively. In particular, LB and UB denote best lower and upper bounds, RB represents the root bound (i.e., continuous relaxation value), while Gap is the optimality gap, computed as

[2] This assumption is due to the demonstration convenience. If there are different units of measurement among the products, then we can stick to one particular product's unit of measurement and convert all the other products' values including their unit costs, unit prices, salvage values, and demands into this unit of measurement.

(UB-LB)/LB. Note that, when $n = 10$, MILP Model 2 can be solved to optimality or near-optimality, while MILP Model 1 takes a longer time to solve or it ends with larger optimality gaps. When the number of products increases to 20, both models cannot be solved to optimality; however, MILP Model 2 yields much smaller optimality gaps. The root bounds of MILP Model 2 are also much smaller than those for MILP Model 1, which is consistent with the model comparison results presented in Proposition 3.17. Therefore, MILP Model 2 outperforms MILP Model 1. We also notice that the computational time and optimality gaps for both the models increase as the number of scenarios N grows. Thus, a good choice of N will be crucial for these models, in particular for MILP Model 2.

The results of the approximation algorithms presented in Section 3.4 are shown in Table 3.3 and Table 3.4. In particular, we let LB denote the best objective value obtained using the procedure specified in Proposition 3.18 or Algorithm 1, Gap is computed as (v_R^{LD}-LB)/LB, where v_R^{LD} denotes relaxed Lagrangian dual bound.

Note that when there are only $n = 10$ products, all of these approximation algorithms can find good-quality feasible solutions. The relaxed Lagrangian dual bound v_R^{LD} is also quite close to the true optimal value. When $n = 20$, Lagrangian relaxation can find good-quality feasible solutions, while the solutions obtained by the double greedy algorithm are slightly worse. On the other hand, the running times of double greedy algorithm grow significantly as the number of scenarios N increases, while the relaxed Lagrangian dual bound, v_R^{LD}, can be obtained within 4 seconds even when $N = 1000$. In comparison with the results of the MILP models in Table 5.1 and Table 5.2, the relaxed Lagrangian dual bound can be tighter than the best upper bound generated by the MILP Model 2, especially when $n = 20, N \geq 800$. These observations suggest that for large-scale instances, the relaxed Lagrangian dual bound and the feasible solution constructed by the method specified in Proposition 3.18 are preferable.

We further compare the theoretical approximation ratio with that obtained computation-

Table 3.1: Computational results of MILP Model 1, and MILP Model 2 with $n = 10$

	MILP Model 1					MILP Model 2				
N	time	LB	UB	RB	Gap	time	LB	UB	RB	Gap
100	26	20111	20113	28354	0.01%	29	20111	20113	21798	0.01%
200	153	19244	19246	27513	0.01%	253	19244	19246	20965	0.01%
300	774	19437	19439	27741	0.01%	593	19437	19439	21162	0.01%
400	3600	19017	19022	27288	0.03%	1122	19017	19019	20704	0.01%
500	3600	19384	19389	27702	0.02%	2788	19384	19387	21126	0.02%
600	3600	19547	19568	27866	0.11%	3600	19547	19551	21276	0.02%
700	3600	19429	19448	27728	0.10%	3600	19429	19435	21133	0.03%
800	3600	19318	19350	27591	0.16%	3600	19318	19323	21013	0.02%
900	3600	19434	19515	27744	0.42%	3600	19434	19437	21167	0.02%
1000	3600	19545	19631	27852	0.44%	3600	19545	19548	21266	0.02%

Table 3.2: Computational results of MILP Model 1, and MILP Model 2 with $n = 20$

	MILP Model 1					MILP Model 2				
N	time	LB	UB	RB	Gap	time	LB	UB	RB	Gap
100	3600	39879	42703	80394	7.08%	3600	39907	40656	48553	1.88%
200	3600	39342	45561	79994	15.81%	3600	39326	40489	47970	2.96%
300	3600	38970	45563	79704	16.92%	3600	38911	40089	47575	3.03%
400	3600	39448	46603	80250	18.14%	3600	39442	40550	48181	2.81%
500	3600	39339	46808	80124	18.99%	3600	38994	40680	48132	4.32%
600	3600	39014	46738	79778	19.90%	3600	38596	40715	47738	5.49%
700	3600	39593	47365	80546	19.63%	3600	39277	42285	48489	7.66%
800	3600	38088	47565	80181	24.88%	3600	38552	43510	48219	12.86%
900	3600	39110	46951	80003	20.05%	3600	38861	43998	47914	13.22%
1000	3600	37473	47156	79910	25.84%	3600	38478	45470	47810	18.17%

Table 3.3: Computational results of approximation algorithms when $n = 10$

	Lagrangian Relaxation				Double Greedy		
N	time	LB	v_R^{LD}	Gap	time	LB	Gap
100	0.19	20105	20263	0.79%	6	19538	3.71%
200	0.33	19240	19409	0.88%	20	18768	3.42%
300	0.47	19432	19598	0.85%	47	18935	3.50%
400	0.62	19011	19182	0.90%	81	18559	3.35%
500	0.85	19378	19545	0.86%	127	18924	3.28%
600	0.90	19541	19709	0.86%	190	19070	3.35%
700	1.08	19422	19596	0.89%	259	18951	3.40%
800	1.15	19311	19484	0.89%	348	18845	3.39%
900	1.31	19430	19601	0.88%	430	18964	3.36%
1000	1.42	19538	19709	0.87%	520	19051	3.45%

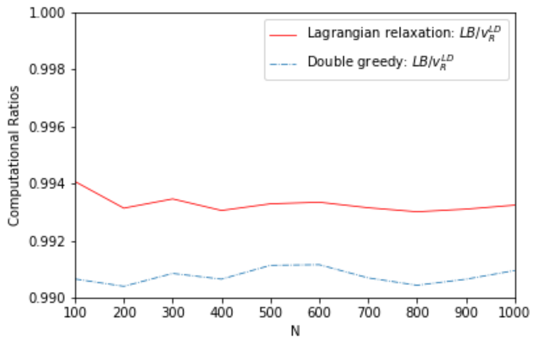
Table 3.4: Computational results of approximation algorithms when $n = 20$

	Lagrangian Relaxation				Double Greedy		
N	time	LB	v_R^{LD}	Gap	time	LB	Gap
100	0.62	39809	41801	5.00%	22	35572	17.51%
200	1.09	39341	41307	5.00%	83	35283	17.07%
300	1.34	39029	41028	5.12%	193	35098	16.90%
400	2.14	39459	41445	5.03%	336	35335	17.29%
500	2.40	39387	41409	5.14%	506	35278	17.38%
600	2.67	39082	41077	5.11%	738	35069	17.13%
700	3.24	39718	41713	5.02%	1040	35603	17.16%
800	3.57	39467	41463	5.06%	1403	35423	17.05%
900	3.79	39221	41179	4.99%	1708	35207	16.96%
1000	4.36	39157	41159	5.11%	2071	35085	17.31%

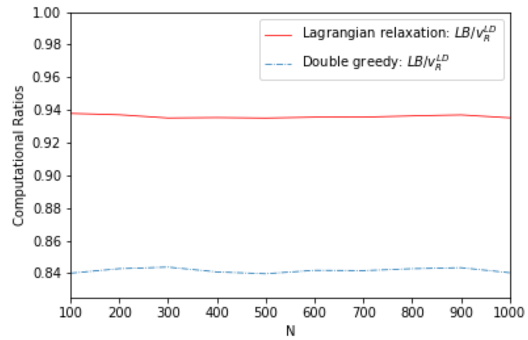
ally, termed *"computational ratio"*, of double greedy algorithm and Lagrangian relaxation approach. The computational ratios are illustrated in Figures 3.1(a) and 3.1(b). Note that, for double greedy algorithm, the computational approximation ratios (i.e., best lower bound divided by v_R^{LD}) for all the instances are larger than the theoretical ratio 1/3. For the Lagrangian relaxation approach, according to Theorem 3.19, the theoretical approximation ratio is $1/19 * 0.79607 \approx 0.042$, however, the computational ratios (lower bound divided by v_R^{LD}) for all the instances are much larger than the theoretical ratio. We also notice that in both figures, the computational ratios of the Lagrangian relaxation approach are even better than that of the double greedy algorithm. This suggests a potential to improve the approximation result in Proposition 3.19.

3.6.2 Model Properties and Sensitivity Analyses

In this subsection, we illustrate model properties, and also conduct sensitivity analyses for the MPNP-CDS by using the case of two products, i.e., $n = 2$. For all the different instances below, the optimal solutions and values are obtained by solving the MILP Model 2 to optimality.



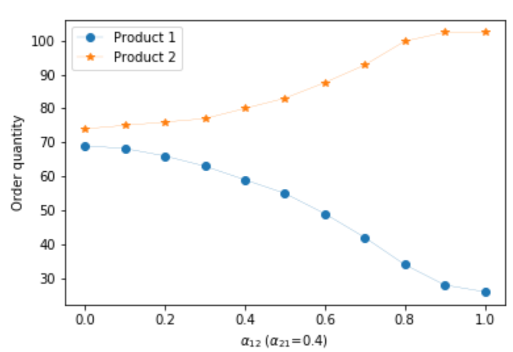
(a) Computational ratios when $n = 10$



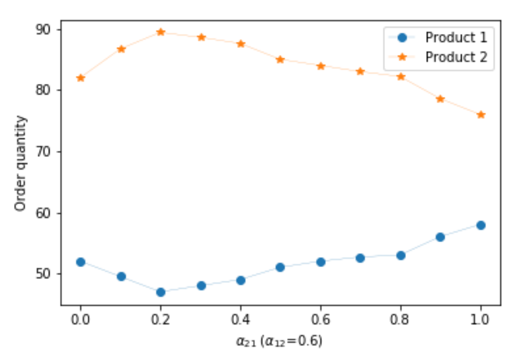
(b) Computational ratios when $n = 20$

Figure 3.1: Computational ratios of Lagrangian relaxation approach and double greedy algorithm

Effects of Substitution Rates. To study this effect, the demand was assumed to follow a uniform distribution between 5 and 100, and we generated $N = 1000$ i.i.d. samples. Also, we let $p_1 = 91, p_2 = 92, c_1 = 45, c_2 = 41, s_1 = 23, s_2 = 28$. Figure 3.2 illustrates how order quantities vary with different substitution rates, where in Figure 3.2(a), we set $\alpha_{21} = 0.4$ and let α_{12} vary from 0 to 1, while in Figure 3.2(b), we set $\alpha_{12} = 0.6$ and let α_{21} vary from 0 to 1.



(a) Order quantity versus α_{12}



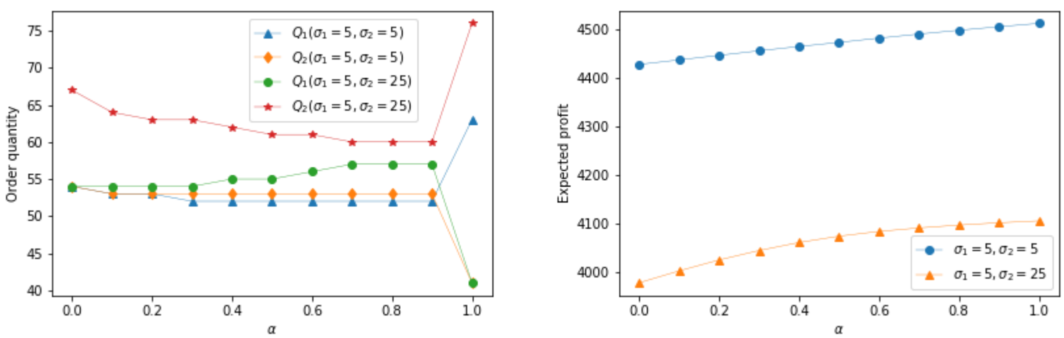
(b) Order quantity versus α_{21}

Figure 3.2: Optimal order quantities versus substitution rates

In Figure 3.2(a), we observe that as the value of α_{12} increases, the optimal order quantity of Product 2 increases as well, while the optimal order quantity of Product 1 decreases. In Figure 3.2(b), as α_{21} increases, the optimal order quantity of Product 1 decreases in the beginning and then increases. On the contrary, the optimal order quantity of Product 2 increases at first and then decreases. Both figures imply that the optimal order quantity does not necessarily monotonically increase or decrease with increment in substitution rate. These observations are consistent with the result in Proposition 3.8. Therefore, we recommend that the retailer should increase the orders of products which have higher substitution rates and order less on the other products which will be substituted.

Effects of Demand Variation. To study the effect of demand variation, we assumed that the demand for each product to be uniformly distributed with mean $\mu = 50$, and the standard deviation is $\sigma_1 = 5$ for product 1. The standard deviation of product 2 (i.e., σ_2) varied from 5 to 25. For each configuration (σ_1, σ_2) of random demand, we generated $N = 500$ samples and solved their corresponding MILP Model 2. For the rest of parameters, we let $p_i = 92, s_i = 24, c_i = 45$ for each product $i = 1, 2$, and also, let $\alpha_{12} = \alpha_{21} = \alpha$, which varies from 0 to 1.

Figure 3.3 illustrates the results obtained. In Figure 3.3(a), we see that, when the product 2's standard deviation increases, both products' order quantities will increase. This suggests that the retailer should increase the orders of both products to hedge against the risks from demand variation. When the substitution rate is close to 1, we observe that the difference between these two products' order quantities becomes large. This might be because the customers are insensitive to either product and thus, the retailer probably can remove one product to reduce the products' assortment on the shelf. In Figure 3.3(b), we observe that as the demand standard deviation increases, the expected profit decreases. This is mainly because a higher standard deviation implies a higher risk of understock or overstock. On the contrary, increasing substitution rate α can mitigate the risks caused by demand variability, thus contributes to a higher expected profit, which demonstrates the important role of substitution rates. As a result, we recommend that the retailer should focus on a subset of products, among which there are substantial substitution effects.



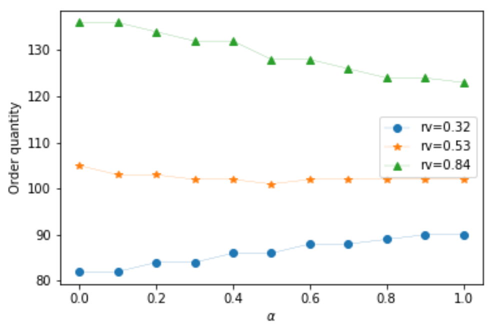
(a) Order quantitiy versus α

(b) Expected profit versus α

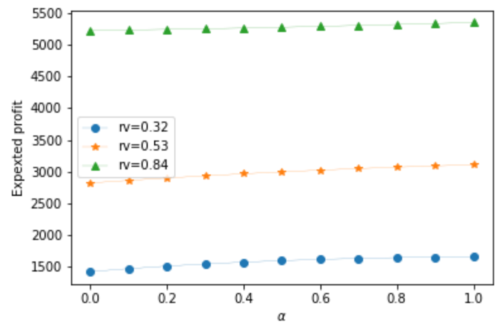
Figure 3.3: Optimal order quantities and the optimal expected profit versus substitution rate under different standard deviations

Effects of Ratio Value. To study this effect, we assume the demand for each product to be uniformly distributed between 5 and 100. We generate $N = 1000$ samples. For the rest of parameters, we let $p_i = 92, s_i = 24$ while $c_i = c$, which takes three values, namely, 35, 45 and 70, for each product $i = 1, 2$, and we also let $\alpha_{12} = \alpha_{21} = \alpha$, which varies from 0 to 1. Therefore, we define the "*ratio value*" as $rv = \frac{\bar{P}}{\bar{S}}$, where $\bar{P}_i = \bar{P}$ and $\bar{S}_i = \bar{S}$ for each product $i = 1, 2$. Clearly, rv takes three values-0.84, 0.53, 0.32.

Figure 5.1 illustrates how the sum of optimal order quantities and the expected profit change with different substitution rates and ratio values. Note that both the order quantities and expected profit increase as ratio value increases, which is because the higher ratio value is, the more profitable products are. In Figure 3.4(a), the sum of total order quantities can increase or decrease with an increment in substitution rate, which depends on their current ratio values. On the contrary, as depicted in Figure 3.4(b), the expected profit increases with an increment in substitution rate. Thus, a higher ratio value will help increase the retailer's expected profit, but does not necessarily imply that the retailer should always order more products.



(a) Sum of order quantities versus α



(b) Expected profit versus α

Figure 3.4: Sum of optimal order quantities and expected profit versus substitution rates under different ratio values.

Chapter 4

Robust Multi-product Newsvendor Model with Substitution under Cardinality-constrained Uncertainty Set

4.1 Introduction

This chapter studies Multi-product Newsvendor Model with Substitution (MNMS) under demand and substitution rate uncertainty, in which a retailer determines the optimal order quantity for each product to maximize its total profit. Due to similarity among different products and their occasional unavailability, the phenomenon of substitution among different products is quite common and has been observed in many studies (cf. [8, 16, 17, 59, 73, 76, 89]). For instance, when shopping at Amazon.com, a customer might turn to a blue hat if his or her first-choice of green hat were currently unavailable. The existence of substitution somehow increases the profit of the retailer (cf. [59]), however, on the other hand, it significantly complicates the problem and makes the problem very challenging to handle. Besides, due to the stochasticity of customers' demand and substitution rates, it might be hard to forecast the demand and substitution rates accurately. Therefore, many works (cf. [23, 59, 61, 68, 90]) proposed stochastic programming models to tackle the demand uncertainty by assuming that the probability distribution of the demand is known. However, in many cases, a good estimation of probability distribution might be very challenging. Nowadays, technology companies and original equipment manufacturers frequently release their new products. For example, every year, Apple Inc. releases its new-generation iPhones and MacBooks. Without enough historic sales data, it is almost impossible to have an accurate prediction of these new products' demand and substitution rates and inaccurate estimations can cause misleading decisions (cf. [88]). Therefore, to foster a more reliable decision, instead we study the "Robust" Multi-product Newsvendor Model with Substitution (R-MNMS) subject to cardinality-constrained uncertainty set.

A R-MNMS encounters the following technical features. First of all, due to the substitution effect, it has been shown in [90], even when the demand is deterministic, MNMS can be NP-hard. Second, most of the existing works assumed that the customers' demand follows a given probability distribution, which, however, might result in a loss of sales due to inaccurate demand forecasting. Third, although many existing works illustrated interesting properties of MNMS, the closed-form optimal solutions are rarely known, therefore very limited managerial insights have been discovered so far. In this chapter, we will show that under some conditions, all of these features can be appropriately addressed.

Many works on MNMS assume that the probability distribution of the demand is known, for example, [35, 51, 90]. Huang et al. [35] analyzed the decentralized MNMS, gave the conditions in which the Nash equilibrium exists and presented an iterative algorithm to solve the model. However, its centralized counterpart becomes highly non-convex, and will be studied in this thesis. Netessine and Rudi [51] demonstrated that the profit function could be quasi-concave or bi-modal when the demand is deterministic. Recently, Zhang et al. [90] formulated the stochastic MNMS as a mixed integer linear program and developed polynomial-time approximation algorithms with performance guarantee to solve it. As also noted above, different from these works, this thesis will study centralized R-MNMS under cardinality-constrained uncertainty set.

In practice, it might not be easy to learn distribution of the random demand completely, in particular, when the random demand is not stationary, i.e., the probability distribution of the random demand is subject to change from time to time. In addition, an inaccurate probability distribution might result in unreliable or misleading decisions. Under these circumstances, alternatively, one can choose a robust approach to formulate the model with partial information of the demand, which can be easily characterized or will stay the same for a relatively long period of time (i.e., mean, variance, or support). Therefore, some works have applied robust optimization to the newsvendor problem [2, 14, 15, 31, 42, 54, 62, 65, 85]. In this pioneering work, Scarf [65] introduced the idea of robustness to analyze a single-product newsvendor problem with known mean and variance of the demand. Vairaktarakis [85] studied several minimax regret formulations for robust multi-item newsvendor models with a budget constraint when the support of demand is known. They developed efficient algorithms to solve the proposed robust models. Similarly, when the demand is constrained over given interval, Lin and Ng [42] determined an optimal order quantity as well as addressed market selection for a minimax regret multi-market newsvendor model. They further developed an approximation algorithm for solving large-sized problem instances. With known first and second moments and shape of the demand distribution, Perakis and Roels [57] derived an optimal order policy by minimizing the maximum regret of the newsvendor problem. Ardestani-Jaafari and Delage [2] studied robust optimization with sum of piecewise linear functions and polyhedral uncertainty set, which can be applied to solve the robust multi-product newsvendor problem under budget uncertainty set. All of these works have either studied robust single-product newsvendor problem or multi-product newsvendor problem without substitution. In this paper, we study robust multi-product newsvendor problem with substitution.

There have been very limited works reported in the literature on the R-MNMS. For decentralized R-MNMS, Jiang et al. [37] used the absolute regret criterion to obtain a unique Nash equilibrium. In their work, only the support of the demand is known, and they also showed that the robust model tends to be more tractable than its stochastic counterpart. In their recent work, [41] studied a robust two-product newsvendor model with substitution, when the first two moments of demand are known. However, the authors were only able to provide the optimal solution for the following two extreme cases: (1) when there exists no substitution, or (2) when there is a perfect substitution between products. However, in this thesis we study centralized R-MNMS, and also, it is not restricted to two-product cases.

Summary of Main Contributions: The objective of this chapter is to help a retailer determine optimal order quantities of a single-period multi-product newsvendor model with substitution, which optimizes the worst-case total profit under the cardinality-constrained uncertainty set. The main contributions of this chapter are summarized as below[1]:

(i) We develop an equivalent reformulation of R-MNMS and prove that computing the worst-case total profit in general is NP-hard for given order quantities.

(ii) We derive closed-form solutions for the following three special cases: (1) when there are only two products; (2) when there is no substitution among different products; or (3) when the budget of demand uncertainty is equal to the number of products.

The remainder of the chapter is organized as follows. Section 4.2 introduces the problem

[1] Note that this chapter is quite different with our previous work [90]. In this chapter, we assume the probability distribution of the demand is unknown and derive closed-form optimal solutions for three special cases with managerial insights. However, in [90], the probability distribution of the demand were assume to be known, we were unable to derive closed-form solutions but instead, we developed exact and approximation algorithms to solve the model.

setting and the model. Section 4.3 presents the properties of the model and proves the complexity of computing the worst-case total profit. In Section 4.4, we derive the optimal order quantities for three special cases of the model. Section 5.2 reformulates the R-MNMS as an MILP, and develops a branch-and-cut algorithm and a conservative approximation to solve it. Section 5.3 presents the results of our numerical investigation on the proposed algorithms.

Notation: The following notation is used throughout the chapter. We use bold-letters (e.g., $\boldsymbol{x}, \boldsymbol{A}$) to denote vectors and matrices, and use corresponding non-bold letters to denote their components. Given a vector or matrix $\boldsymbol{x}$, its zero norm $\|\boldsymbol{x}\|_0$ denotes the number of its nonzero elements. We let $\boldsymbol{e}$ be the vector or matrix of all ones, and let $\boldsymbol{e}_i$ be the ith standard basis vector. Given an integer n, we let $[n] := \{1, 2, \ldots, n\}$, and use $\mathbb{R}^n_+ := \{\boldsymbol{x} \in \mathbb{R}^n : x_i \geq 0, \forall i \in [n]\}$. Given a real number t, we let $(t)_+ := \max\{t, 0\}$. Given a finite set I, we let $|I|$ denote its cardinality. We let $\tilde{\boldsymbol{\xi}}$ denote a random vector and denote its realizations by $\boldsymbol{\xi}$. Additional notation will be introduced as needed.

4.2 Model Formulation

In this section, we present the model formulation for R-MNMS.

To begin with, suppose that there is a retailer selling n similar products in the market indexed by $[n] := \{1, \cdots, n\}$ at a given time period. For each product $i \in [n]$, its cost is c_i, price is p_i, and salvage value is s_i, where by convention, we assume that $p_i \geq c_i \geq s_i$. Each product also bears with a random demand $\tilde{D}_i$ for each $i \in [n]$. Ideally, the retailer would like to determine the optimal order quantity for each product $i \in [n]$, denoted as Q_i. Due to the substitution effect, the effective demand of each product will be affected by its realized demand, its order quantity as well as other products' conditions (i.e., whether out-of-stock or not). To formulate this effect, we suppose that the demand of product $j \in [n]$ can be proportionally substituted by another product $i \in [n]$ and $i \neq j$, once the part of the demand of product j cannot be satisfied by its order quantity Q_j. In particular, we let $\tilde{\alpha}_{ji}$ be the substitution rate, which is the proportion of the unmet demand of product j substituted by product i. In this chapter, we assume that all the products have the same unit, therefore, substitution rate satisfies $\tilde{\alpha}_{ji} \in [0, 1]$ for each pair of products $i, j \in [n]$. Also, by default, we let $\tilde{\alpha}_{ii} = 0$ for each product $i \in [n]$. We let $\tilde{D}_i^s(\boldsymbol{Q})$ denote the effective demand function of product $i \in [n]$ as below:

$$\tilde{D}_i^s(\boldsymbol{Q}) = \tilde{D}_i + \sum_{j \in [n]} \tilde{\alpha}_{ji}(\tilde{D}_j - Q_j)_+, \forall i \in [n], \tag{4.1}$$

where the second term in the sum is due to its substitution to the unavailable products.

As shown in [90], the retailer's total profit for given order quantities $\boldsymbol{Q}$, substitution rates $\tilde{\boldsymbol{\alpha}}$, and demand $\tilde{\boldsymbol{D}}$ can be formulated as:

$$\widehat{\Pi}(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \tilde{\boldsymbol{\alpha}}) := \sum_{i \in [n]} \left(p_i \min\left(Q_i, \tilde{D}_i^s(\boldsymbol{Q})\right) - c_i Q_i + s_i \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q})\right)_+ \right). \tag{4.2}$$

4.2.1 Constructing Uncertainty Sets of Demand and Substitution Rates

Oftentimes, the substitution rates ($\tilde{\boldsymbol{\alpha}}$) and the demand ($\tilde{\boldsymbol{D}}$) of products in (4.1) are stochastic and their probability distributions are difficult to characterize. To well address the uncertainties of substitution rates and the demand, we will use robust optimization. In particular, we will study R-MNMS under cardinality-constrained uncertainty sets.

First of all, in the demand uncertainty set, suppose that the demand of the n products (i.e.,

$\tilde{\boldsymbol{D}}$) is within a box, e.g., $\tilde{\boldsymbol{D}} \in [\boldsymbol{D} - \boldsymbol{l}, \boldsymbol{D} + \boldsymbol{u}]$, where $\boldsymbol{D}$ denotes the nominal demand, $\boldsymbol{l}, \boldsymbol{u}$ denote the lower and upper deviations of the demand respectively satisfying $\boldsymbol{l} \in [\boldsymbol{0}, \boldsymbol{D}]$ and $\boldsymbol{u} \geq \boldsymbol{0}$. We also assume that at most $k \in [n] \cup \{0\}$ products are allowed to deviate from their nominal demand $\boldsymbol{D}$. We will discuss the impact of the budget of uncertainty k on optimal order quantities. Therefore, the uncertainty set of the demand can be written as

$$\mathcal{U}_0 = \left\{ \tilde{\boldsymbol{D}} : \tilde{D}_i = D_i + \Delta_i, -l_i \leq \Delta_i \leq u_i, \forall i \in [n], \|\Delta\|_0 \leq k \right\}, \tag{4.3}$$

Similarly, let us denote the uncertainty set of substitution rate as below

$$\mathcal{U}_\alpha = \left\{ \tilde{\boldsymbol{\alpha}} : \tilde{\alpha}_{ji} = \alpha_{ji} + \Delta^{\alpha}_{ji}, -l^{\alpha}_{ji} \leq \Delta^{\alpha}_{ji} \leq u^{\alpha}_{ji}, \forall\ i, j \in [n], \|\boldsymbol{\Delta}^{\alpha}\|_0 \leq k^{\alpha} \right\}, \tag{4.4}$$

where $\|\cdot\|_0$ denotes the zero-norm, and k^{α} is the budget of uncertainty. We suppose that the substitution rates are within a box, e.g., $\tilde{\boldsymbol{\alpha}} \in [\boldsymbol{\alpha} - \boldsymbol{l}^{\boldsymbol{\alpha}}, \boldsymbol{\alpha} + \boldsymbol{u}^{\boldsymbol{\alpha}}]$, where $\boldsymbol{\alpha}$ denotes the nominal substitution rates, $\boldsymbol{l}^{\boldsymbol{\alpha}}, \boldsymbol{u}^{\boldsymbol{\alpha}}$ denote the lower and upper deviations of the substitution rates respectively satisfying $\boldsymbol{l}^{\boldsymbol{\alpha}} \in [\boldsymbol{0}, \boldsymbol{\alpha}]$ and $\boldsymbol{u}^{\boldsymbol{\alpha}} \in [0, \boldsymbol{e} - \boldsymbol{\alpha}]$. For notational convenience, we let $\alpha^{\alpha}_{ii} = l^{\alpha}_{ii} = u^{\alpha}_{ii} = 0$ for each $i \in [n]$.

With the notation introduced above, R-MNMS can be formulated as:

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}^n_+} \min_{\tilde{\boldsymbol{D}} \in \mathcal{U}_0, \tilde{\boldsymbol{\alpha}} \in \mathcal{U}_\alpha} \left\{ \widehat{\Pi}(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \tilde{\boldsymbol{\alpha}}) := \sum_{i \in [n]} \left(p_i \min\left(Q_i, \tilde{D}^s_i(\boldsymbol{Q})\right) - c_i Q_i + s_i \left(Q_i - \tilde{D}^s_i(\boldsymbol{Q})\right)_+ \right) \right\}. \tag{4.5}$$

In Model (4.5), the objective is to find optimal order quantities to maximize the worst-case total profit over the uncertainty sets $\mathcal{U}_0, \mathcal{U}_\alpha$. For each product $i \in [n]$, we let $\overline{P}_i = p_i - c_i \geq 0$ and $\overline{S}_i = p_i - s_i \geq 0$. Note that $\overline{P}_i$ can be interpreted as the marginal profit or underage cost of product $i \in [n]$, while $\overline{S}_i$ is the sum of the underage cost $(p_i - c_i)$ and overage cost

$(c_i - s_i)$ of product $i \in [n]$, where their ratio $\frac{\overline{P}_i}{\overline{S}_i}$ is known as the critical ratio of newsvendor model (c.f., [50]). Since $\min(Q_i, \tilde{D}_i^s(\boldsymbol{Q})) = Q_i - (Q_i - \tilde{D}_i^s(\boldsymbol{Q}))_+$ for each $i \in [n]$, the above Model (4.5) is equivalent to

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}_+^n} \min_{\tilde{\boldsymbol{D}} \in \mathcal{U}_0, \tilde{\boldsymbol{\alpha}} \in \mathcal{U}_\alpha} \left\{ \widehat{\Pi}(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \tilde{\boldsymbol{\alpha}}) := \sum_{i \in [n]} \overline{P}_i Q_i - \sum_{i \in [n]} \overline{S}_i \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q}) \right)_+ \right\}. \tag{4.6}$$

For notational convenience, throughout this chapter, we will let $\boldsymbol{Q}^*$ denote an optimal solution to R-MNMS (4.6).

4.2.2 Discussion about How to Estimate the Uncertainty Sets

The budgets of uncertainty (i.e., k, k^α) in Model (4.6) plays an important role, and a good choice of these values can achieve both least-conservatism and robustness. The following steps show how to find the optimal budgets of uncertainty $k^*, k^{\alpha*}$ using possibly limited historical data:

Step 0: We split the historical data into two groups Υ_i, $i \in [2]$, and select a candidate set $\mathcal{K} \subseteq \{0\} \cup [n] \times \{0\} \cup [n^2 - n]$ to choose the best $(k^*, k^{\alpha*})$.

Step 1.1: **Determine nominal demand $\widehat{\boldsymbol{\mu}}$, the lower deviation $\boldsymbol{l}$, and the upper deviation $\boldsymbol{u}$.** To do so, we compute the sample mean $\widehat{\mu}_i$ and standard deviation $\widehat{\sigma}_i$ of the first group of historical demand data Υ_1 for each product $i \in [n]$. Then we set the nominal demand $D_i = \widehat{\mu}_i$, and $u_i = l_i = 1.96\widehat{\sigma}_i$ for each product $i \in [n]$.

Step 1.2: **Determine nominal substitution rate $\widehat{\boldsymbol{\mu}}^\alpha$, the lower deviation $\boldsymbol{l}^\alpha$, and the upper deviation $\boldsymbol{u}^\alpha$.** Similarly, we compute the sample mean $\widehat{\mu}_{ji}^\alpha$ and standard deviation $\widehat{\sigma}_{ji}^\alpha$ of the first group of historical substitution rate data Υ_1 for each pair of products $i, j \in [n]$. Then we set the nominal substitution rate $\alpha_{ji} = \widehat{\mu}_{ji}^\alpha$, and $u_{ji}^\alpha = l_{ji}^\alpha = 1.96\widehat{\sigma}_{ji}^\alpha$ for each pair of products $i, j \in [n]$.

Step 2: **Calculate the optimal order quantities $\boldsymbol{Q}^*(k, k^\alpha)$ and objective value $v^*(k, k^\alpha)$ for each $(k, k^\alpha) \in \mathcal{K}$ by solving Model (4.6).**

Step 3: **Compute the objective value $\widehat{\Pi}(\boldsymbol{Q}^*(k, k^\alpha), \boldsymbol{D}, \boldsymbol{\alpha})$ of Model (4.2) for each $(k, k^\alpha) \in \mathcal{K}$ and each pair of demand and substitution rates $(\boldsymbol{D}, \boldsymbol{\alpha})$ in the second group of historical data Υ_2.**

Step 4: **Determine the optimal $k^*, k^{\alpha*}$.** For each $(k, k^\alpha) \in \mathcal{K}$, we compute the qth percentile of $\{\widehat{\Pi}(\boldsymbol{Q}^*(k, k^\alpha), \boldsymbol{D}, \boldsymbol{\alpha})\}_{(\boldsymbol{D},\boldsymbol{\alpha})\in\Upsilon_2}$, and denote it as $\widehat{\Pi}^{q\%}(k, k^\alpha)$. Given two nonnegative weights $w_1, w_2 \in \mathbb{R}_+$, we choose the optimal budgets of uncertainty $k^*, k^{\alpha*}$ which achieve the smallest weighted value $w_1 k + w_2 k^\alpha$ such that $v^*(k, k^\alpha) \leq \widehat{\Pi}^{q\%}(k, k^\alpha)$.

4.3 Equivalent Reformulation and Model Properties

In this section, we study R-MNMS under cardinality-constrained uncertainty set and derive its equivalent reformulation. We also provide upper bounds of optimal order quantities and show that computing the worst-case total profit for given order quantities in general is NP-hard.

Throughout the rest of the chapter, we will make the following assumption.

Assumption 4. *Suppose that $k^\alpha = n^2 - n$ in the substitution uncertainty set $\mathcal{U}_\alpha$.*

Assumption 4 implies that the substitution uncertainty set $\mathcal{U}_\alpha$ is purely a box. We make this assumption for the following reasons: (i) first of all, it is often more difficult to substitution rates $\tilde{\boldsymbol{\alpha}}$ than the demand; (ii) second, under this assumption, we can derive some interesting analytical results; and (iii) third, our exact branch-and-cut algorithm in Section 5.2 can be applied to the general k^α, and it follows directly from the derivation in Section 5.2.

4.3.1 Equivalent Reformulation

In this subsection, we provide an alternative formulation for Model (4.6).

First, we make the following observation.

Lemma 4.1. *For any $\boldsymbol{Q}, \tilde{\boldsymbol{D}} \in \mathbb{R}^n_+$, the profit function $\widehat{\Pi}\left(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \cdot\right)$ is monotone nondecreasing in $\tilde{\boldsymbol{\alpha}}$; and for any $\boldsymbol{Q}, \tilde{\boldsymbol{\alpha}} \in \mathbb{R}^n_+$, the profit function $\widehat{\Pi}\left(\boldsymbol{Q}, \cdot, \tilde{\boldsymbol{\alpha}}\right)$ is monotone nondecreasing in $\tilde{\boldsymbol{D}}$.*

Proof. According to Model (4.6), the profit function $\widehat{\Pi}(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \cdot)$ is nondecreasing in $\tilde{D}^s_i(\boldsymbol{Q})$ and from (4.1), the effective demand $\tilde{D}^s_i(\boldsymbol{Q})$ is also nondecreasing in $\tilde{\alpha}_{ji}$ for each product $i, j \in [n]$. Therefore, the profit function $\widehat{\Pi}(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \cdot)$ is nondecreasing in $\tilde{\boldsymbol{\alpha}}$. Similarly, from (4.1), $\tilde{D}^s_i(\boldsymbol{Q})$ is also nondecreasing in $\tilde{D}_i$ for each product $i \in [n]$. Therefore, the profit function $\widehat{\Pi}(\boldsymbol{Q}, \cdot, \tilde{\boldsymbol{\alpha}})$ is nondecreasing in the demand $\tilde{\boldsymbol{D}}$. ☐

According to Lemma 4.1 and Assumption 4, $\min_{\tilde{\alpha}\in\mathcal{U}_\alpha} \widehat{\Pi}(\boldsymbol{Q}, \cdot, \tilde{\boldsymbol{\alpha}})$ is achieved by $\tilde{\alpha}_{ji} = \alpha_{ji} - l^\alpha_{ji} := \underline{\alpha}_{ji}$ for all products $i \neq j$ and $i, j \in [n]$. In this case, Model (4.6) becomes

$$v^* = \max_{\boldsymbol{Q}\in\mathbb{R}^n_+} \min_{\tilde{\boldsymbol{D}}\in\mathcal{U}_0} \left\{ \Pi(\boldsymbol{Q}, \tilde{\boldsymbol{D}}) := \sum_{i\in[n]} \overline{P}_i Q_i - \sum_{i\in[n]} \overline{S}_i \left(Q_i - \tilde{D}^s_i(\boldsymbol{Q}) \right)_+ \right\}, \tag{4.7}$$

where we let $\Pi(\boldsymbol{Q}, \tilde{\boldsymbol{D}}) = \widehat{\Pi}(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \underline{\boldsymbol{\alpha}})$.

Now we are ready to show our equivalent reformulation. The main idea of the derivation is to show that in the worst-case, the uncertainty set $\mathcal{U}_0$ can be restricted to the following mixed integer set:

$$\mathcal{U} = \left\{ \tilde{\boldsymbol{D}} : \sum_{i \in [n]} z_i \leq k, \tilde{D}_i = D_i - l_i z_i, z_i \in \{0,1\}, \forall i \in [n] \right\}. \tag{4.8}$$

Clearly, set $\mathcal{U} \subseteq \mathcal{U}_0$, since for any feasible point $(\tilde{\boldsymbol{D}}, \boldsymbol{z})$ satisfying constraints in (4.8), let us define $\Delta_i = -l_i z_i$ for each $i \in [n]$, then $(\tilde{\boldsymbol{D}}, \boldsymbol{\Delta})$ satisfies the constraints in (4.3). Indeed, we can show that

Proposition 4.2. *R-MNMS is equivalent to*

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}^n_+} \min_{\tilde{\boldsymbol{D}} \in \mathcal{U}} \Pi\left(\boldsymbol{Q}, \tilde{\boldsymbol{D}}\right), \tag{4.9}$$

where $\mathcal{U}$ is defined in (4.8)*.*

Proof. Let v_1 denote the optimal value of Model (4.9), then we only need to show $v_1 = v^*$.

(i) $v_1 \geq v^*$. For any $\tilde{\boldsymbol{D}} \in \mathcal{U}_0$, there exists $\boldsymbol{\Delta}$ such that $\|\boldsymbol{\Delta}\|_0 \leq k, \tilde{D}_i = D_i + \Delta_i, -l_i \leq \Delta_i \leq u_i$. Let us define binary variable $z_i = \begin{cases} 0, & \text{if } \Delta_i = 0 \\ 1, & \text{if } \Delta_i \neq 0 \end{cases}$ for each $i \in [n]$. Since $\|\boldsymbol{\Delta}\|_0 \leq k$, thus we must have $\sum_{i \in [n]} z_i \leq k$. Let us define $\tilde{D}_i^* = D_i - l_i z_i$ for each $i \in [n]$. Clearly, we have $\tilde{\boldsymbol{D}}^* \in \mathcal{U}$ and $\tilde{\boldsymbol{D}}^* \leq \tilde{\boldsymbol{D}}$. For any fixed $\boldsymbol{Q} \in \mathbb{R}^n_+$, by Proposition 4.1, we know that the profit function $\Pi\left(\boldsymbol{Q}, \tilde{\boldsymbol{D}}\right)$ is nondecreasing in the demand $\tilde{\boldsymbol{D}}$. Thus, $\Pi\left(\boldsymbol{Q}, \tilde{\boldsymbol{D}}\right) \geq \Pi\left(\boldsymbol{Q}, \tilde{\boldsymbol{D}}^*\right)$, which implies $\min_{\tilde{\boldsymbol{D}} \in \mathcal{U}_0} \Pi\left(\boldsymbol{Q}, \tilde{\boldsymbol{D}}\right) \geq \min_{\tilde{\boldsymbol{D}} \in \mathcal{U}} \Pi\left(\boldsymbol{Q}, \tilde{\boldsymbol{D}}\right)$ for any $\boldsymbol{Q} \in \mathbb{R}^n_+$. This proves $v_1 \geq v^*$.

(ii) $v_1 \leq v^*$. Since $\mathcal{U}_0 \supseteq \mathcal{U}$, thus for any $\boldsymbol{Q} \in \mathbb{R}^n_+$, $\min_{\tilde{\boldsymbol{D}} \in \mathcal{U}_0} \Pi\left(\boldsymbol{Q}, \tilde{\boldsymbol{D}}\right) \leq \min_{\tilde{\boldsymbol{D}} \in \mathcal{U}} \Pi\left(\boldsymbol{Q}, \tilde{\boldsymbol{D}}\right)$, thus, $v_1 \leq v^*$.

$\square$

From Proposition 4.2, by substituting $\tilde{D}_i = D_i - l_i z_i$ in (4.6) and defining the following cardinality set

$$X = \left\{ \boldsymbol{z} : \sum_{i \in [n]} z_i \leq k, z_i \in \{0,1\} \right\}, \tag{4.10}$$

then we can have the following equivalent formulation of R-MNMS:

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}_+^n} \left\{ f(\boldsymbol{Q}) := \sum_{i \in [n]} \overline{P}_i Q_i - R(\boldsymbol{Q}) \right\}, \tag{4.11a}$$

where

$$R(\boldsymbol{Q}) := \max_{\boldsymbol{z} \in X} \sum_{i \in [n]} \overline{S}_i \left(Q_i - D_i + l_i z_i - \sum_{j \in [n]} \underline{\alpha}_{ji} \left(D_j - l_j z_j - Q_j \right)_+ \right)_+ . \tag{4.11b}$$

This new equivalent formulation (4.11) allows us to compute the worst-case profit function via an integer program rather than a nonconvex program, which can be further reduced to a mixed integer linear program (MILP) in Section 5.2.

One direct benefit of formulation (4.11) is that we can easily derive upper bounds of optimal order quantities. The result can be proved by contradiction.

Proposition 4.3. *There exists an optimal solution $\boldsymbol{Q}^*$ to R-MNMS such that for each product $i \in [n]$, $Q_i^* \leq M_i$, where $M_i = D_i + \sum_{j \in [n]} \underline{\alpha}_{ji} D_j$.*

Proof. See Appendix B.1. □

This result is very useful to derive an equivalent MILP formulation of R-MNMS in Section 5.2.

4.3.2 Complexity of the Inner Maximization Problem (4.11b)

In this subsection, we show that the inner maximization problem (4.11b) of R-MNMS is NP-hard.

First, observe that

$$(D_j - l_j z_j - Q_j)_+ = \begin{cases} (D_j - l_j - Q_j)_+ , & \text{if } z_j = 1 \\ (D_j - Q_j)_+ , & \text{if } z_j = 0 \end{cases}$$

for each $j \in [n]$. This observation allows us to linearize the nonlinear expressions $\{(D_j - l_j z_j - Q_j)_+\}_{j\in[n]}$ and to rewrite (4.11b) as

$$R(\boldsymbol{Q}) = \max_{\boldsymbol{z}\in X} \left\{ \sum_{i\in[n]} \overline{S}_i \left[Q_i - D_i + l_i z_i - \sum_{j\in[n]} \underline{\alpha}_{ji} \left((D_j - l_j - Q_j)_+ z_j + (D_j - Q_j)_+ (1 - z_j) \right) \right]_+ \right\}. \tag{4.12}$$

Next, we show that the inner maximization problem (4.12) is NP-hard via a reduction to the well known clique problem.

Theorem 4.4. *The inner maximization problem (4.12) in general is NP-hard.*

Proof. See Appendix B.2. □

Proposition 4.4 shows that unlike many robust optimization problems, it might be difficult to derive a tractable form for the general inner maximization problem (4.12). Thus, instead, in Section 4.4, we propose three special cases such that both inner maximization (4.12) and R-MNMS are tractable. For general R-MNMS, we propose an equivalent MILP reformulation and develop exact and approximate algorithms to solve it, which will be presented in Section 5.2.

4.4 Three Special Cases: Closed-form Optimal Solutions

In this section, we will study three different special cases of R-MNMS (4.11) and derive their closed-form optimal solutions.

4.4.1 Special Case I: $n = 2, k = 1$

In this section, we study R-MNMS with only two products (i.e., $n = 2$) with the budget of uncertainty is 1 (i.e., $k = 1$ in set X defined in (4.10)). Note that if $k = 0$ or 2, it reduces to Special Case III, which will be discussed in Section 4.4.3. Under this setting, R-MNMS (4.11) becomes:

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}_+^2} \left\{ \sum_{i \in [2]} \overline{P}_i Q_i - \max_{\boldsymbol{z} \in X} \sum_{i \in [2]} \overline{S}_i \left(Q_i - D_i + l_i z_i - \sum_{j \in [2]} \underline{\alpha}_{ji} \left(D_j - l_j z_j - Q_j \right)_+ \right)_+ \right\}, \tag{4.13}$$

and $X = \{\boldsymbol{z} : z_1 + z_2 \leq 1, z_i \in \{0, 1\}, \forall i \in [2]\}$. To simplify our closed-form solutions, we further make the following assumption.

Assumption 5. $D_2 \underline{\alpha}_{21} \geq l_1 \geq \underline{\alpha}_{21} l_2, D_1 \underline{\alpha}_{12} \geq l_2 \geq \underline{\alpha}_{12} l_1$.

Assumption 5 postulates that the demand deviation of one product cannot be smaller than the substitution part of the other product's demand deviation and cannot be larger than the substitution part of the other product's nominal demand. Please note that our analysis is general and can be also applied to the other parametric settings without satisfying Assumption 5. However, for the brevity of this thesis, we will stick to this assumption.

The next theorem presents our main findings of the optimal order quantities for this special case under Assumption 5. The key ideas to these results are: (1) to divide the feasible regions into 9 subregions by comparing Q_i with $D_i - l_i$ and D_i for each $i \in [2]$; (2) for each subregion, R-MNMS (4.13) becomes a concave maximization problem with a piecewise linear objective function, thus one of its optimal solutions can be achieved by an extreme point; and (3) for each subregion, there are not too many potential optimal solutions, thus, we enumerate all the candidate solutions and find the one which achieves the highest total profit across all the 9 subregions.

Theorem 4.5. *Suppose $n = 2$, $k = 1$, and Assumption 5 holds, then the optimal order quantities $\boldsymbol{Q}^* = (Q_1^*, Q_2^*)$ are characterized by the following three cases:*

Case 1: If $\overline{P}_1 \le \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \ge \overline{P}_1 \underline{\alpha}_{21}$, then $(Q_1^*, Q_2^*) = (0, D_2 - l_2 + \underline{\alpha}_{12} D_1)$.

Case 2: If $\overline{P}_2 \le \overline{P}_1 \underline{\alpha}_{21}$ and $\overline{P}_1 \ge \overline{P}_2 \underline{\alpha}_{12}$, then $(Q_1^*, Q_2^*) = (D_1 - l_1 + \underline{\alpha}_{21} D_2, 0)$.

Case 3: If $\overline{P}_1 \ge \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \ge \overline{P}_1 \underline{\alpha}_{21}$, then we have the following two sub-cases:

Sub-case 3.1: If $\overline{S}_1 l_1 \ge \overline{S}_2 l_2$, then $(Q_1^*, Q_2^*) = \left(D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}, D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}\right)$ or $(Q_1^*, Q_2^*) = \left(D_1, D_2 - \frac{\overline{S}_2 l_2 - \overline{S}_1 l_1}{\overline{S}_2 - \overline{S}_1 \underline{\alpha}_{21}}\right)$.

Sub-case 3.2: If $\overline{S}_1 l_1 \le \overline{S}_2 l_2$, then $(Q_1^*, Q_2^*) = \left(D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}, D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}\right)$ or $(Q_1^*, Q_2^*) = \left(D_1 - \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}}, D_2\right)$.

Proof. See Appendix B.3. □

Proposition 4.5 provides a complete characterization of optimal order quantities of the two-product case, which highly depend on the comparison between the marginal profit of product i and the profit generated by using product j to substitute product i. In particular, we make the following remarks.

Remark 4.6. (i) Suppose that the marginal profit of product 1 is lower than the profit generated by using product 2 to substitute product 1, but the marginal profit of product

2 is higher than the profit generated by using product 1 to substitute product 2, i.e., product 2 is much more profitable than product 1. Thus, in this case, the decision makers should only order product 2 to satisfy their customers' demand as well as to satisfy part of the customers' demand of product 1 by substitution. In this case, the worst-case demand of product 2 is $D_2 - l_2$ while the worst-case demand of product 1 is equal to the nominal demand D_1. The interpretation for case 2 is similar with case 1 since they are symmetric in both products.

(ii) If the marginal profit of one product is higher than the profit generated by using the other product to substitute this product (i.e., both products are similarly profitable), then the optimal order quantities depend on the relationship between $\overline{S}_1 l_1$ and $\overline{S}_2 l_2$. One special case is that when $s_i = c_i$ for each product $i \in [2]$, i.e., the salvage value of each product is equal to its unit production cost, the optimal order quantity of product 1 is $Q_1^* = D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}$ and the optimal order quantity of product 2 is $Q_2^* = D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}$, while the worst-case demand of products 1 and 2 can be $(D_1, D_2 - l_2)$ or $(D_1 - l_1, D_2)$, respectively.

(iii) It is impossible to have the case that $\overline{P}_1 < \overline{P}_2 \underline{\alpha}_{12}$, $\overline{P}_2 < \overline{P}_1 \underline{\alpha}_{21}$, which implies $1 < \underline{\alpha}_{12}\underline{\alpha}_{21}$, contradicting the assumption that all the substitution rates are between 0 and 1.

4.4.2 Special Case II: $\underline{\boldsymbol{\alpha}} = \mathbf{0}$

In this subsection, we analyze robust multi-product newsvendor problem without substitution, i.e., $\underline{\boldsymbol{\alpha}} = \mathbf{0}$. In this setting, the effective demand becomes $\tilde{D}_i^s(\boldsymbol{Q}) = \tilde{D}_i = D_i - l_i z_i$.

Thus, R-MNMS (4.11) reduces to:

$$v^* = \max_{\boldsymbol{Q}\in\mathbb{R}_+^n} \left\{ f(\boldsymbol{Q}) := \sum_{i\in[n]} \overline{P}_i Q_i - \max_{\boldsymbol{z}\in X} \sum_{i\in[n]} \overline{S}_i \left(Q_i - D_i + l_i z_i\right)_+ \right\}, \tag{4.14}$$

where set X is defined in (4.10). We first make the following observation.

Lemma 4.7. *There exists an optimal solution $\boldsymbol{Q}^*$ of Model (4.14) such that $D_i - l_i \le Q_i^* \le D_i$ for all $i \in [n]$.*

Proof. For notational convenience, let us define $\boldsymbol{Q}_{-i} = [Q_1, \cdots, Q_{i-1}, Q_{i+1}, \cdots, Q_n]^\top$ to be the vector of the remaining elements of $\boldsymbol{Q}$. It is sufficient to show that for any fixed $\boldsymbol{Q}_{-i} \in \mathbb{R}_+^{n-1}$, the objective function of Model (4.14), $f(Q_i, \boldsymbol{Q}_{-i})$, is monotone nondecreasing in Q_i when $Q_i \in [0, D_i - l_i]$ and monotone nonincreasing in Q_i when $Q_i \in [D_i, +\infty)$. Indeed, we note that

$$\begin{aligned}
& f(Q_i, \boldsymbol{Q}_{-i}) \\
=& \sum_{\tau\in[n]\setminus\{i\}} \overline{P}_\tau Q_\tau + \overline{P}_i Q_i - \max_{\boldsymbol{z}\in X} \left(\sum_{\tau\in[n]\setminus\{i\}} \overline{S}_\tau \left(Q_\tau - D_\tau + l_\tau z_\tau\right)_+ + \overline{S}_i \left(Q_i - D_i + l_i z_i\right)_+ \right) \\
=& \begin{cases} \sum\limits_{\tau\in[n]\setminus\{i\}} \overline{P}_\tau Q_\tau - \max\limits_{\boldsymbol{z}\in X} \sum\limits_{\tau\in[n]\setminus\{i\}} \overline{S}_\tau \left(Q_\tau - D_\tau + l_\tau z_\tau\right)_+ + \overline{P}_i Q_i, & \text{if } Q_i \in [0, D_i - l_i], \\ \sum\limits_{\tau\in[n]\setminus\{i\}} \overline{P}_\tau Q_\tau - \max\limits_{\boldsymbol{z}\in X} \left(\sum\limits_{\tau\in[n]\setminus\{i\}} \overline{S}_\tau \left(Q_\tau - D_\tau + l_\tau z_\tau\right)_+ - D_i + l_i z_i \right) + \left(\overline{P}_i - \overline{S}_i\right) Q_i, & \text{if } Q_i \in [D_i, +\infty). \end{cases}
\end{aligned}$$

Clearly, from the above equation, we know that if $Q_i \in [0, D_i - l_i]$, the coefficient of Q_i is $\overline{P}_i$, which is nonnegative, while if $Q_i \in [D_i, +\infty)$, the coefficient of Q_i is $\overline{P}_i - \overline{S}_i$, which is nonpositive. Thus, $f(Q_i, \boldsymbol{Q}_{-i})$ is nondecreasing on Q_i when $Q_i \in [0, D_i - l_i]$ and nonincreasing on Q_i when $Q_i \in [D_i, +\infty)$. This completes the proof. □

According to Proposition 4.7, without loss of generality, we can assume in Model (4.14), $\boldsymbol{Q} \in [\boldsymbol{D} - \boldsymbol{l}, \boldsymbol{D}]$. Thus, for each $i \in [n]$, $(Q_i - D_i + l_i z_i)_+ = \begin{cases} 0, & \text{if } z_i = 0 \\ Q_i - D_i + l_i, & \text{if } z_i = 1 \end{cases} =$

$(Q_i - D_i + l_i)\, z_i$. Therefore, Model (4.14) is equivalent to

$$v^* = \max_{\boldsymbol{Q} \in [\boldsymbol{D}-\boldsymbol{l}, \boldsymbol{D}]} \left(\sum_{i \in [n]} \overline{P}_i Q_i - \max_{\boldsymbol{z} \in X} \sum_{i \in [n]} \overline{S}_i \left(Q_i - D_i + l_i\right) z_i \right), \tag{4.15}$$

where X is defined in (4.10).

Suppose that $\{(1), (2), \cdots, (n)\}$ is a permutation of $[n]$ such that $\overline{S}_{(1)} l_{(1)} \geq \overline{S}_{(2)} l_{(2)} \geq \cdots \geq \overline{S}_{(n)} l_{(n)}$. We can obtain a closed-form optimal solution to Model (4.15) as follows.

Theorem 4.8. *When $\underline{\boldsymbol{\alpha}} = 0$, the optimal solutions $\boldsymbol{Q}^*$ of Model (4.15) are characterized as follows:*

(i) If $\sum_{i \in [n]} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, then $Q_i^* = D_i - l_i$, and $v^* = \sum_{i \in [n]} \overline{P}_i \left(D_i - l_i\right)$.

(ii) If $\sum_{i \in [n]} \frac{\overline{P}_i}{\overline{S}_i} > k$,

$$Q_i^* = \begin{cases} D_i - l_i + \frac{\overline{S}_{(t+1)} l_{(t+1)}}{\overline{S}_i}, & \text{if } i \in T \\ D_i, & \text{if } i \in [n] \setminus T \end{cases},$$

and

$$v^* = \sum_{i \in [n] \setminus T} \overline{P}_i l_i - \overline{S}_{(t+1)} l_{(t+1)} k + \sum_{i \in T} \frac{\overline{P}_i}{\overline{S}_i} \overline{S}_{(t+1)} l_{(t+1)} + \sum_{i \in [n]} \overline{P}_i \left(D_i - l_i\right),$$

where set $T := \{(1), (2), \cdots, (t)\}$ satisfying $\sum_{i \in T} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, $\sum_{i \in T \cup \{(t+1)\}} \frac{\overline{P}_i}{\overline{S}_i} > k$.

Proof. See Appendix B.5.

□

Proposition 4.8 reveals the impact of the budget of uncertainty on the optimal order quantities. Indeed, if $\sum_{i \in [n]} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, i.e., the budget of uncertainty k is no smaller than the sum of the critical ratios of all the products, then in this case, the optimal order quantity for each product is equal to the lower bound of the demand, i.e., $Q_i^* = D_i - l_i$ for each $i \in [n]$. Hence, this implies that when the products are not very profitable or the accuracy of demand forecasting is relatively low, then the decision of the retailer should be conservative to hedge against unnecessary loss from demand forecasting. Suppose that $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} > k$, i.e, the budget of uncertainty is smaller than the sum of critical ratios of all the products, or equivalently, relatively a small number of demand can be allowed to deviate from the nominal demand $\boldsymbol{D}$. Also, note that for each product $i \in [n]$, the value of $\overline{S}_i l_i$ can be interpreted as the risk of lost sales for product i when its order quantity is D_i with the worst-case demand $D_i - l_i$ (i.e, the sum of underage cost and overage cost multiplies the demand deviation). In this case, for each product $i \in T$ whose risk of lost sales is larger than a threshold $\overline{S}_{(t+1)} l_{(t+1)}$, its order quantity should be equal to $D_i - l_i + \frac{\overline{S}_{(t+1)} l_{(t+1)}}{\overline{S}_i}$; otherwise, it should be D_i. The threshold $\overline{S}_{(t+1)} l_{(t+1)}$ can be determined by searching for the product such that the sum of the critical ratios of the products, whose risk is higher than product $(t+1)$, is no larger than the budget of uncertainty k, but including the critical ratio of this product into the sum will make it above k. This result implies that the products with lower risk of lost sales should be ordered up to the nominal demand, while those with higher risk should be ordered less than the nominal demand.

4.4.3 Special Case III: $k = n$

When the budget of uncertainty is equal to n, i.e., $k = n$, the uncertainty set $\mathcal{U}$ becomes

$$\mathcal{U} = \left\{ \tilde{\boldsymbol{D}} : \tilde{D}_i = D_i - l_i z_i, \sum_{i\in[n]} z_i \leq n, z_i \in \{0,1\}, \forall i \in [n] \right\}.$$

From Proposition 4.1, we know that the profit function $\Pi(\boldsymbol{Q}, \tilde{\boldsymbol{D}})$ is nonincreasing in $\tilde{\boldsymbol{D}}$, thus at the optimality, we must have $z_i = 1$ for all $i \in [n]$ in the inner maximization problem (4.11b), i.e., the worst-case demand in this special case will always be equal to $\boldsymbol{D} - \boldsymbol{l}$. Thus, Model (4.11) becomes

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}^n_+} \sum_{i \in [n]} \overline{P}_i Q_i - \sum_{i \in [n]} \overline{S}_i \left[Q_i - D_i + l_i - \sum_{j \in [n]} \underline{\alpha}_{ji} \left(D_j - l_j - Q_j \right)_+ \right]_+ . \tag{4.16}$$

Note that Model (4.16) is a multi-product newsvendor model with substitution when the demand is deterministic and is equal to $\boldsymbol{D} - \boldsymbol{l}$. According to the recent work in [90], the optimal order quantities of Model (4.16) can be completely characterized as follows (For more details, please refer to [90]).

Theorem 4.9. *(Theorem 1, [90]) When $k = n$, the optimal order quantities $\boldsymbol{Q}^*$ and the optimal total profit v^* are characterized as follows:*

(i)

$$Q_j^* = \begin{cases} D_j^s(\boldsymbol{Q}^*) = D_j - l_j + \sum\limits_{i \in \Gamma^*} \underline{\alpha}_{ij}(D_i - l_i), & \text{if } \overline{P}_j - \sum\limits_{i \in [n] \setminus \Gamma^*} \underline{\alpha}_{ji} \overline{P}_i \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{4.17}$$

for each $j \in [n]$, where $[n] \setminus \operatorname{supp}(\boldsymbol{Q}^*) = \Gamma^*$, i.e., $\Gamma^* = \{i \in [n] : Q_i^* = 0\}$; and

(ii)

$$v^* = \max_{\Gamma \subseteq [n]} \left\{ f(\Gamma) := \sum_{j \in \Gamma} \sum_{i \in [n] \setminus \Gamma} \underline{\alpha}_{ji} \overline{P}_i (D_j - l_j) + \sum_{i \in [n] \setminus \Gamma} \overline{P}_i (D_i - l_i) \right\} := f(\Gamma^*), \tag{4.18}$$

In Proposition 4.9, if the budget of uncertainty is equal to the number of products, then for each product $j \in [n]$, its optimal order quantity $\boldsymbol{Q}_j^*$ is equal to its effective demand if its marginal profit $\overline{P}_j$ is larger than or equal to the sum of the profits generated by using other products to substitute it, and 0, otherwise. This suggests that the retailer does not need to order a product if its marginal profit is relatively low and should order up to its effective demand, otherwise. Also, in (4.18), the first term is the sum of the total profit for selling product $i \in [n] \setminus \Gamma$ to meet the demand of its substitutable products $j \in \Gamma$ and the second term is the profit of selling product $i \in [n] \setminus \Gamma$ to meet its own demand. Finally, please note that although we completely characterize the optimal order quantities for all the products, obtaining these value is in general NP-hard (cf. [90]).

Another interesting observation from Proposition 4.9 is that the optimal order quantity for each product can be equal to their worst-case demand, i.e., $Q_j^* = D_j - l_j$ for each product $j \in [n]$, under the following assumptions.

Corollary 4.10. *Suppose (1)* $\overline{P}_i = \overline{P}_j, \forall i, j \in [n]$ *and (2) for each product* $j \in [n]$, $\sum\limits_{i \in [n]} \underline{\alpha}_{ji} < 1$. *Then* $Q_j^* = D_j - l_j$ *for all* $j \in [n]$.

Proof. Note that from Proposition 4.9, the optimal subset $\Gamma^* = \emptyset$. Therefore, $Q_j^* = D_j^s(\boldsymbol{Q}^*) = D_j - l_j$ for all $j \in [n]$. □

Proposition 4.10 shows that if all the products share the same underage cost and cannot be completely substituted by the others, then the optimal order quantities are equal to the worst-case demand, i.e., $Q_j^* = D_j - l_j$ for each product $j \in [n]$.

Finally, we remark that if $k = 0$, then the results in Proposition 4.9 will also hold simply by replacing $l_i = 0$ for each $i \in [n]$.

Chapter 5

Robust Multi-product Newsvendor Model with Uncertain Substitution Rates, Uncertain Demand, and Fixed Cost

5.1 Introduction

In practice, it is more realistic to consider fixed cost to the objective function when determining a product's optimal order quantity. For example, a retailer needs to create more space on the shelf for a new product to display when the assortment size increases. There are several existing works on addressing the fixed cost issue in the multi-product newsvendor problem. For example, [47] applied dynamic programming procedures and heuristic algorithms to study the multi-product newsvendor problem with fixed ordering cost subject to a budget constraint. By developing heuristic algorithms based on dynamic programming, simulated-based optimization, and network flow, [61] solved the multi-product inventory model with setup cost for production. To the best of our knowledge, there is no existing work on multi-product newsvendor model with customer-driven substitution and fixed-cost. In this chapter, we plan to explore the model properties and develop efficient solution algorithms for this model.

Let us denote C_i^F to be the fixed cost of product $i \in [n]$. Then following the notation in Chapter 4, R-MNMS with fixed cost (R-MNMSF) can be formulated as

$$v_K^* = \max_{\boldsymbol{Q} \in \mathbb{R}_+^n} \min_{\tilde{\boldsymbol{D}} \in \mathcal{U}_0} \left\{ \Pi_K(\boldsymbol{Q}, \tilde{\boldsymbol{D}}) := \sum_{i \in [n]} \bar{P}_i Q_i - \sum_{i \in [n]} \overline{S}_i \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q}) \right)_+ - \sum_{i \in n} C_i^F \mathbb{1} \left(Q_i > 0 \right) \right\}, \tag{5.1}$$

where indicator function $\mathbb{1}(Q_i > 0) = 1$ if $Q_i > 0$, and 0, otherwise.

Note that the inner maximization Model (4.11b) is a nonconvex and nonsmooth optimization problem, the developed methods in Chapter 4 cannot apply. According to Section 4.3.2, we can conclude that solving Model (5.1) is NP-hard in general. Thus, in this paper, we will resort to mixed integer programming techniques and algorithms to solve Model (4.3.2).

Summary of Main Contributions: The objective of this chapter is to help a retailer determine optimal order quantities of a single-period multi-product newsvendor model with substitution rates and fixed cost, which optimizes the worst-case total profit under the cardinality-constrained uncertainty set. However, this chapter will provide ways to solve Model (5.1). The main contributions of this chapter are summarized as follows:

(i) We reformulate R-MNMSF as a mixed-integer linear program (MILP) with an exponential number of constraints, and develop branch-and-cut algorithm to solve it.

(ii) We provide a conservative approximation of R-MNMSF, which can be solved more efficiently, and also prove that under certain conditions, the proposed conservative approximation is equivalent to R-MNMSF.

(iii) Computational studies are conducted to prove the effectiveness of the proposed algorithms, determine the optimal budget of uncertainty and demonstrate the robustness of our robust model.

The remainder of the chapter is organized as follows. Section 5.2 introduces the solution approaches, i,e., mixed integer programming reformulation, branch-and-cut algorithm, and conservative approximation algorithm to solve Model (5.1). Section 5.3 provides numerical experiments to prove the performance efficiency of the proposed algorithms and demonstrated the robustness of the robust model versus the risk neutral model.

5.2 Solution Approaches

In this section, we introduce equivalent MILP formulations for R-MNMS (4.11) and its inner maximization Model (4.11b) by linearizing the nonconvex terms in the profit function. These equivalent formulations allow us to develop an effective branch-and-cut algorithm and an alternative conservative approximation to solve R-MNMSF.

5.2.1 An Equivalent MILP Formulation of the Inner Maximization Problem

In this subsection, we present an MILP formulation, which is equivalent to the inner maximization problem (4.11b)[1]. To begin with, in (4.11b), let us define two new variables

$$u_j = (D_j - l_j - Q_j)_+, \quad \psi_j = (D_j - Q_j)_+$$

[1]For the general k^α, we can derive the similar MILP formulation, which can be found in Appendix C.3.

for each $j \in [n]$. Clearly, we have $\psi_j \geq u_j$ for each $j \in [n]$. For simplicity, we still use the function $R(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi})$ to denote the optimal value of inner maximization problem (4.11b) for any given $\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}$, i.e., the inner maximization problem becomes

$$R(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}) = \max_{\boldsymbol{z}} \sum_{i \in [n]} \overline{S}_i \left[Q_i - D_i + l_i z_i - \sum_{j \in [n]} \underline{\alpha}_{ji} \left(u_j z_j + \psi_j (1 - z_j) \right) \right]_+, \tag{5.2a}$$

$$\text{s.t.} \sum_{i \in [n]} z_i \leq k, \tag{5.2b}$$

$$z_i \in \{0, 1\}, \forall i \in [n]. \tag{5.2c}$$

Note that Model (5.2) is a convex integer maximization problem. Thus, we will further linearize the objective function into a linear form. To do so, for each $i \in [n]$, let us define a binary variable $x_i = 1$, if $Q_i - D_i + l_i z_i - \sum_j \underline{\alpha}_{ji} \left(u_j z_j + \psi_j (1 - z_j) \right) \geq 0$, and 0, otherwise. Thus, Model (5.2) is equivalent to

$$R(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}) = \max_{\boldsymbol{x}, \boldsymbol{z}} \sum_{i \in [n]} \overline{S}_i \left[Q_i - D_i + l_i z_i - \sum_{j \in [n]} \underline{\alpha}_{ji} \left(u_j z_j + \psi_j (1 - z_j) \right) \right] x_i, \tag{5.3a}$$

$$\text{s.t.} \sum_{i \in [n]} z_i \leq k, \tag{5.3b}$$

$$x_i, z_i \in \{0, 1\}, \forall i \in [n]. \tag{5.3c}$$

The above Model (5.3) now becomes a binary bilinear program, which can be further linearized by introducing new variables representing the bilinear terms. The final reformulation result is shown below.

Proposition 5.1. *The inner maximization problem* (5.2) *is equivalent to*

$$R(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}) = \max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \sum_{i\in[n]} \overline{S}_i \left[(Q_i - D_i)x_i + l_i y_{ii} - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(u_j y_{ji} + \psi_j (x_i - y_{ji}) \right) \right] \tag{5.4a}$$

$$s.t. \sum_{i\in[n]} z_i \leq k. \tag{5.4b}$$

$$y_{ji} \leq x_i, \forall i, j \in [n], \tag{5.4c}$$

$$y_{ji} \leq z_j, \forall i, j \in [n], \tag{5.4d}$$

$$z_i, x_i \in \{0, 1\}, y_{ji} \geq 0, \forall i, j \in [n]. \tag{5.4e}$$

Proof. See Appendix C.1. □

5.2.2 Reformulation of R-MNMSF and branch-and-cut algorithm

Next we are going to investigate an MILP reformulation for R-MNMSF (4.11), which is amenable for a branch-and-cut algorithm. First, from Proposition 4.3, without loss of generality, we can assume that the order quantities $\boldsymbol{Q}$ can be upper bounded by $\boldsymbol{M}$. Thus, for each product $i \in [n]$, its order quantity Q_i must belong to one of the following three intervals: $[0, D_i - l_i], [D_i - l_i, D_i], [D_i, M_i]$ (we break the boundary points arbitrarily). For notational convenience, let us denote $D_i^{(0)} = 0$, $D_i^{(1)} = D_i - l_i$, $D_i^{(2)} = D_i$, and $D_i^{(3)} = M_i$. Next, we introduce one binary variable for each interval to indicate whether Q_i is in this interval or not, i.e., we let $\chi_i^{(e)} = 1$ if $Q_i \in [D_i^{(e-1)}, D_i^{(e)}]$ for each $e \in [3]$; and 0, otherwise. And we let

$$\sum_{e\in[3]} \chi_i^{(e)} = 1, \tag{5.5a}$$

to enforce that Q_i indeed belongs to only one interval. Moreover, for each product $i \in [n]$, we use $\chi_i^{(0)} = 1$ if $Q_i = 0$, and 0, otherwise. Correspondingly, for each product $i \in [n]$ and

$e \in [3]$, we further introduce another variable $w_i^{(e)}$ to be equal to Q_i if $Q_i \in [D_i^{(e-1)}, D_i^{(e)}]$, and 0, otherwise. That is,

$$D_i^{(e-1)}\chi_i^{(e)} \le w_i^{(e)} \le D_i^{(e)}\chi_i^{(e)}, \forall i \in [n], e \in [3], \tag{5.5b}$$

$$\sum_{e\in[3]} w_i^{(e)} = Q_i, \forall i \in [n]. \tag{5.5c}$$

Next, we can express u_i and ψ_i (recall that $u_i = (D_i - l_i - Q_i)_+$ and $\psi_i = (D_i - Q_i)_+$) as linear functions of variables $\{\chi_i^{(e)}\}_{e\in[2]}$ and $\{w_i^{(e)}\}_{e\in[2]}$ for each product $i \in [n]$, i.e.,

$$u_i = (D_i - l_i)\chi_i^{(1)} - w_i^{(1)}, \forall i \in [n], \tag{5.5d}$$

$$\psi_i = D_i \sum_{e\in[2]} \chi_i^{(e)} - \sum_{e\in[2]} w_i^{(e)}, \forall i \in [n], \tag{5.5e}$$

Clearly, in (5.5d), if $Q_i > D_i - l_i$, then u_i is equal to 0 since both $\chi_i^{(0)} = \chi_i^{(1)} = 0, w_i^{(1)} = 0$ and otherwise, it is equal to $D_i - l_i - Q_i$. And in (5.5e), if $Q_i > D_i$, then ψ_i is equal to 0 since $\chi_i^{(0)} = \chi_i^{(1)} = \chi_i^{(2)} = 0, w_i^{(1)} = w_i^{(2)} = 0$, and otherwise, it is equal to $D_i - Q_i$. For the inner maximization problem (5.4), let us also define function $g(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ to be its objective function, i.e.,

$$g(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \sum_{i\in[n]} \overline{S}_i \left[(Q_i - D_i)\, x_i + l_i y_{ii} - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(u_j y_{ji} + \psi_j \left(x_i - y_{ji} \right) \right) \right],$$

and set Ξ to be its feasible region, i.e.,

$$\Xi = \left\{ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) : (5.4\text{b}) - (5.4\text{e}) \right\}.$$

In view of the above development, we have the following equivalent MILP formulation of

R-MNMSF (5.1):

$$
\begin{aligned}
v^* = \max_{\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{\chi},\boldsymbol{w},\eta} \quad & \sum_{i\in[n]} \overline{P}_i Q_i - \sum_{i\in n} C_i^F \left(1-\chi_i^{(0)}\right) - \eta, && \text{(5.6a)}\\
\text{s.t.} \quad & \eta \geq g\left(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\right), \forall (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in \Xi, && \text{(5.6b)}\\
& Q_i \leq M_i(1-\chi_i^{(0)}), \forall i \in [n], && \text{(5.6c)}\\
& \chi_i^{(0)} \leq \chi_i^{(1)}, \forall i \in [n], && \text{(5.6d)}\\
& w_i^{(e)}, u_i, \psi_i \geq 0, \chi_i^{(e)} \in \{0,1\}, \forall i \in [n], e \in [3]. && \text{(5.6e)}\\
& (5.5a) - (5.5e).
\end{aligned}
$$

Note that in (5.6b), there can be exponentially many constraints. Therefore, we propose a branch-and-cut algorithm to solve Model (5.6). To begin with, suppose we are given a subset $\widehat{\Xi} \subseteq \Xi$, which can be empty, then the master problem is formulated as below:

$$
\begin{aligned}
& \max_{\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{\chi},\boldsymbol{w},\eta} \left\{ \sum_{i\in[n]} \overline{P}_i Q_i - \sum_{i\in n} C_i^F \left(1-\chi_i^{(0)}\right) - \eta \right\}\\
& \text{s.t. } \eta \geq g\left(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\right), \forall (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in \widehat{\Xi},\\
& (5.5a) - (5.5e), (5.6c) - (5.6e).
\end{aligned} \tag{5.7}
$$

Clearly, Model (5.7) is a relaxation of Model (5.6), since $\widehat{\Xi} \subseteq \Xi$. Given an optimal solution $\left(\widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\psi}}, \widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{w}}, \widehat{\eta}\right)$ to the master problem (5.7) to check whether this solution is optimal to original Model (5.6) or not, it is sufficient to check whether it satisfies constraints (5.6b), i.e., solve the inner maximization problem (5.4) by letting $(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})= (\widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\psi}})$ as below:

$$
R(\widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\psi}}) = \max_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in\Xi} \left\{ g\left(\widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\psi}}, \boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\right)\right\}, \tag{5.8}
$$

and check if $\widehat{\eta} \geq R(\widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\psi}})$ or not. If $\widehat{\eta} \geq R(\widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\psi}})$, then $\left(\widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\psi}}, \widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{w}}, \widehat{\eta}\right)$ is optimal to Model (5.6). Otherwise, let $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}})$ be an optimal solution to Model (5.8). Then add a new constraint

$$\eta \geq g(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}, \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}})$$

into the master problem (5.7) and continue. Note that this solution procedure can be integrated with branch-and-bound, which is known as “branch and cut” (cf. [55, 69]).

Below, we summarize the proposed branch-and-cut algorithm to solve Model (5.6), i.e., at each branch-and-bound node, we proceed the following solution procedure.

Step 0: Initialize set $\widehat{\Xi} = \emptyset$.

Step 1: Solve the proposed master problem (5.7) with an optimal solution $\left(\widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\psi}}, \widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{w}}, \widehat{\eta}\right)$.

Step 2: Solve Model (5.8), denote its optimal solution by $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}})$ and optimal value $R(\widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\psi}})$.

Step 3: There are two cases:

Csse 1: If $\widehat{\eta} \geq R(\widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\psi}})$, set $\boldsymbol{Q}^* \leftarrow \widehat{\boldsymbol{Q}}, \boldsymbol{u}^* \leftarrow \widehat{\boldsymbol{u}}, \boldsymbol{\psi}^* \leftarrow \widehat{\boldsymbol{\psi}}, \boldsymbol{\chi}^* \leftarrow \widehat{\boldsymbol{\chi}}, \boldsymbol{w}^* \leftarrow \widehat{\boldsymbol{w}}$, $\eta^* \leftarrow \widehat{\eta}$, stop and output the optimal solution $(\boldsymbol{Q}^*, \boldsymbol{u}^*, \boldsymbol{\psi}^*, \boldsymbol{\chi}^*, \boldsymbol{w}^*, \eta^*)$.

Csse 2: If $\widehat{\eta} < R(\widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\psi}})$, then augment set $\widehat{\Xi} = \widehat{\Xi} \cup (\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}})$, and go to Step 1.

Note that although this branch-and-cut algorithm will terminate in a finite number of steps since there are only a finite number of points in set Ξ, as well as finite number of binary variables in the master problem. However, to generate a new constraint at Step 2 might be very time-consuming since it involves solving an MILP (5.8), i.e., the inner maximization problem (5.4). In the remaining part of this section, we will replace this MILP (5.8) by its continuous relaxation and derive a conservative approximation for R-MNMSF.

5.2.3 Conservative Approximation

In practice, branch-and-cut algorithm might not be efficient to solve very large-scale problem instances. In this section, we propose a simple but very effective conservative approximation to solve R-MNMSF (5.6), i.e., the optimal solution from conservative approximation is a feasible solution to R-MNMSF (5.6). We also provide some sufficient conditions under which this conservative approximation yields an exact optimal solution to R-MNMSF (5.6).

To derive the conservative approximation, we simply relax variables $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ in set Ξ to be continuous in R-MNMSF (5.6), then we can obtain the following lower bound, i.e., a conservative approximation to Model (5.6):

$$
\begin{aligned}
v^{CA} = \max_{\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{\chi},\boldsymbol{w},\eta} & \left\{ \sum_{i\in[n]} \overline{P}_i Q_i - \sum_{i\in n} C_i^F \left(1-\chi_i^{(0)}\right) - \eta \right\} \\
\text{s.t. } & \eta \geq g\left(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\right), \forall (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in \Xi_C, \\
& (5.5\text{a}) - (5.5\text{e}), (5.6\text{c}) - (5.6\text{e}),
\end{aligned}
\tag{5.9}
$$

where Ξ_C denotes the continuous relaxation of set X.

Note that the constraints $\eta \geq g(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}), \forall (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in \Xi_C$ is equivalent to

$$
\eta \geq \max_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in \Xi_C} \left\{ g\left(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\right)\right\},
$$

where the right-hand side is a linear program with nonempty and bounded feasible region for any given $(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$. Therefore, according to the strong duality of linear program, we can replace the max operator by its dual, i.e., an equivalent min operator, and further change the min operator with the existence one. Let $\varpi, \boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{\zeta}, \boldsymbol{\xi}$ be the dual variables associated with constraints (5.4b),(5.4c), (5.4d), $\boldsymbol{z} \leq \boldsymbol{e}$ and $\boldsymbol{x} \leq \boldsymbol{e}$, respectively. Then the conservative approximation (5.9) is equivalent to the following MILP:

$$v^{CA} = \max_{\boldsymbol{Q},\boldsymbol{u},\boldsymbol{v},\eta,\varpi,\boldsymbol{\sigma},\boldsymbol{\rho},\boldsymbol{\zeta},\boldsymbol{\xi}} \sum_{i\in[n]} \overline{P}_i Q_i - \sum_{i\in n} C_i^F \left(1-\chi_i^{(0)}\right) - \eta, \tag{5.10a}$$

$$\text{s.t. } \eta \geq k\varpi + \sum_{i\in[n]} (\zeta_i + \xi_i), \tag{5.10b}$$

$$\varpi + \zeta_j - \sum_{i\in[n]} \rho_{ji} \geq 0, \forall j \in [n], \tag{5.10c}$$

$$\sigma_{ji} + \rho_{ji} \geq -\overline{S}_i \underline{\alpha}_{ji}(u_j - \psi_j), \forall i,j \in [n], i \neq j, \tag{5.10d}$$

$$\sigma_{jj} + \rho_{jj} \geq \overline{S}_j l_j, \forall j \in [n], \tag{5.10e}$$

$$\xi_i - \sum_{j\in[n]} \sigma_{ji} \geq \overline{S}_i(Q_i - D_i) - \overline{S}_i \sum_{j\in[n]} \underline{\alpha}_{ji}\psi_j, \forall i \in [n], \tag{5.10f}$$

$$\zeta_i, \xi_i, \sigma_{ij}, \rho_{ij} \geq 0, \forall i,j \in [n], \tag{5.10g}$$

$$(5.5a) - (5.5e), (5.6c) - (5.6e).$$

The following result summarizes the above development of the conservative approximation and also shows that under some sufficient conditions, this approximation can be exact, i.e., $v^{CA} = v^*$.

Theorem 5.2. *Let v^{CA} denote the optimal value of Model (5.10). Then*

(i) $v^{CA} \leq v^*$; and

(ii) $v^{CA} = v^*$, if one of the following conditions holds: (1) $\underline{\boldsymbol{\alpha}} = \mathbf{0}$, or (2) $n = k$.

Proof. See Appendix C.2. □

From Proposition 5.2, we see that the conservative approximation (5.10) provides a feasible solution to the R-MNMSF (5.6). In addition, the conservative approximation can find a result as very good-quality solution, which can even be optimal to the R-MNMSF (5.6). We illustrate these facts in Section 5.3.

5.3 Computational Study

In this section, we test the performances of branch-and-cut algorithm and conservative approximation to solve the R-MNMSF (5.6). Also, we test the reliability of the R-MNMSF (5.6) compared with the stochastic model.

5.3.1 Effectiveness of Algorithms

In our experiments, we considered instances with $n = 10$ and $n = 20$ products. For each $n \in \{10, 20\}$, we generated 10 random instances, where for each product $i \in [n]$, the nominal demand D_i is between 5 and 100, the unit price, p_i, from 85 to 95, unit cost, c_i, from 40 to 50, the salvage value, s_i, between 22 and 30 and the fixed cost $C_i^F = 200$. All the products were assumed to be similar. We assume $k^\alpha = n^2 - n$, and thus, the substitution rates were generated uniformly between 0 and 1, satisfying $\sum_{i\in[n]} \underline{\alpha}_{ji} = 0.8$ and $\underline{\alpha}_{jj} = 0$ for each $j \in [n]$. The lower bound of the demand was set to be proportional to the nominal demand, i.e., $\boldsymbol{l} = \theta \boldsymbol{D}$, where $\theta \in (0, 1)$ is called "deviation ratio". We tested these instances with the deviation ratio $\theta \in \{0.2, 0.4\}$ and the budget of uncertainty $k \in \{5, 10\}$. Both approaches were coded in Python 2.7 with calls to Gurobi 7.5 on a personal computer with 2.3 GHz Intel Core i5 processor and 8G of memory. The CPU time limit of Gurobi was set to be 3600 seconds.

Table 5.1 and Table 5.2 display computational results of the proposed branch-and-cut algorithm and conservative approximation method for $n = 10$ and $n = 20$, respectively. For the branch-and-cut algorithm, Opt.val denotes the optimal value if available, LB and UB denote the best lower and upper bounds, Gap denotes the optimality gap, computed as (UB-LB)/UB; for conservative approximation, C.val denotes its output objective value and A-Gap represents its optimality gap, computed as (UB-C.val)/UB (note that UB is equal to the Opt.val if available).

From Table 5.1, we see that when $n = 10$, both approaches find good-quality feasible solutions. As explained in Section 5.2.3, the solutions obtained from conservative approximation method are equal to the solutions calculated from branch-and-cut algorithm when $k = n = 10$, i.e., A-Gap=0. In Table 5.2, we note that when the number of products increases to 20, the conservative approximation method can find good-quality feasible solutions within the time limit specified. Oftentimes, the conservative approximation solution can be even better than that obtained by the branch-and-cut algorithm. Also from Table 5.2, we see that if the budget of uncertainty k increases and $k \leq \frac{n}{2}$, i.e., the number of possible realizations of products' demand grows, then the computational time tends to be longer. Also, since a larger k implies that a larger number of products whose worst-case demand can be equal to their lower bounds, therefore we can anticipate that the total profit becomes smaller. Since θ denotes how much the worst-case demand can deviate from the nominal demand, the increase of θ implies that the variance of random demand grows, which means a chance of being understock or overstock to become larger, and it leads to a smaller total profit.

5.3.2 Effects of Fixed Cost

To study the effect of the fixed cost, we consider two products and their substitution rates are equal to each other, $\underline{\alpha}_{12} = \underline{\alpha}_{21} = \underline{\alpha} \in \{0, 0.4, 0.6, 1\}$. For each product $i \in [2]$, the fixed cost $C_1^F = C_2^F = C^F \in \{0, 50, 500, 1000, 1500, 2000, 2500\}$. We set the budget of uncertainty $k = 1$, the deviation ratio $\theta = 0.4$, unite price $p_1 = p_2 = 86$, unit salvage value $s_1 = s_2 = 25$,

Table 5.1: Computational results of branch-and-cut algorithm and conservative approximation method with $n = 10$

k	θ	Instances	Branch and Cut		Conservative Approximation		
			Time	Opt.val	Time	C.val	A-Gap (%)
5	0.2	1	14.64	27366.4	0.24	27246.3	0.44
		2	19.98	28903.2	0.23	28738.8	0.57
		3	12.87	25826.4	0.45	25742.2	0.33
		4	14.24	25509.3	0.26	25431.1	0.31
		5	13.20	31482.8	0.22	31362.9	0.38
		6	7.22	28826.7	0.27	28730.1	0.34
		7	6.84	28653.4	0.34	28614.1	0.14
		8	8.79	27820.5	0.20	27651.1	0.61
		9	4.95	29935.1	0.26	29876.4	0.20
		10	6.26	31229.0	0.30	31163.8	0.21
Average			10.90	28555.3	0.28	28455.7	0.35
5	0.4	1	13.90	22225.8	0.30	21985.6	1.08
		2	20.33	23434.4	0.20	23105.6	1.40
		3	12.69	20961.9	0.26	20793.5	0.80
		4	11.87	20841.6	0.19	20685.1	0.75
		5	13.20	25626.2	0.21	25386.8	0.93
		6	6.58	23459.5	0.26	23266.2	0.82
		7	7.57	23482.7	0.29	23404.2	0.33
		8	8.78	22566.1	0.25	22227.1	1.50
		9	5.47	24448.9	0.25	22227.1	9.09
		10	6.48	25464.0	0.27	25333.6	0.51
Average			10.69	23251.1	0.25	22841.5	1.72
10	0.2	1	3.84	25605.6	0.14	25605.6	0
		2	0.12	27097.6	0.12	27097.6	0
		3	3.67	24152.8	0.15	24152.8	0
		4	4.33	23741.6	0.13	23741.6	0
		5	3.78	29471.2	0.19	29471.2	0
		6	1.45	26955.2	0.18	26955.2	0
		7	1.87	26659.2	0.15	26659.2	0
		8	1.65	26060.0	0.16	26060.0	0
		9	1.45	27938.4	0.18	27938.4	0
		10	1.66	29195.2	0.22	29195.2	0
Average			2.38	26687.7	0.16	26687.7	0
10	0.4	1	7.42	18704.2	0.18	18704.2	0
		2	5.51	19823.2	0.178	19823.2	0
		3	4.78	17614.6	0.16	17614.6	0
		4	5.12	17306.2	0.15	17306.2	0
		5	3.11	21603.4	0.16	21603.4	0
		6	1.98	19716.4	0.18	19716.4	0
		7	2.44	19494.4	0.15	19494.4	0
		8	1.71	19045.0	0.19	19045.0	0
		9	1.74	20453.8	0.18	20453.8	0
		10	1.78	21396.4	0.18	21396.4	0
Average			3.56	19515.8	0.17	19515.8	0

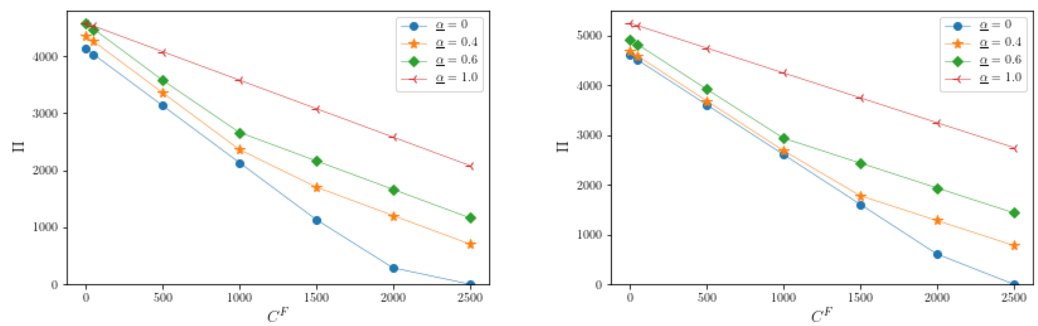
Table 5.2: Computational results of branch-and-cut algorithm and conservative approximation method with $n = 20$

k	θ	Instances	Branch and Cut				Conservative Approximation		
			Time	Opt.val	UB	Gap(%)	Time	C.val	A-Gap (%)
5	0.2	1	3600	58812.3	60941.6	3.49	3600	59997.9	1.55
		2	3600	53937.1	56148.8	3.94	3600	55076.9	1.91
		3	3600	63557.9	65977.3	3.67	3600	64671.3	1.98
		4	3600	59756.0	62592.4	4.53	3600	60689.4	3.04
		5	3600	64192.1	67435.0	4.81	3600	65370.1	3.06
		6	3600	59303.4	61772.4	4.00	3600	60558.1	1.97
		7	3600	53211.0	56000.9	4.98	3600	54273.8	3.08
		8	3600	55788.2	59677.1	6.52	3600	56882.0	4.68
		9	3600	59063.8	62064.2	4.83	3600	60415.2	2.66
		10	3600	59695.7	63675.8	6.25	3600	60840.0	4.45
Average			3600	58731.8	61628.5	4.70	3600	59877.5	2.84
5	0.4	1	3600	52957.7	56016.0	5.46	3600	54005.3	3.59
		2	3600	48566.5	51319.9	5.37	3600	49702.7	3.15
		3	3600	57336.3	60980.3	5.98	3600	58435.1	4.17
		4	3600	54037.5	56894.9	5.02	3600	54891.9	3.52
		5	3600	58116.7	61963.3	6.21	3600	59187.7	4.48
		6	3600	53573.8	57188.7	6.32	3600	54816.5	4.15
		7	3600	47931.5	51935.4	7.71	3600	48908.8	5.83
		8	3600	50216.4	53901.2	6.84	3600	51081.1	5.23
		9	3600	53362.2	57599.6	7.36	3600	54504.5	5.37
		10	3600	53864.9	59522.8	9.51	3600	54843.9	7.86
Average			3600	52996.4	56732.2	6.58	3600	54037.8	4.74
10	0.2	1	3600	54374.2	59500.2	8.62	3600	55160.4	7.29
		2	3600	49793.9	52853.2	5.79	3600	50708.9	4.06
		3	3600	58769.5	65572.3	10.37	3600	59388.7	9.43
		4	3600	54677.0	62196.7	12.09	3600	55823.0	10.25
		5	3600	58627.1	67370.1	12.98	3600	60206.7	10.63
		6	3600	54645.0	61036.6	10.47	3600	55717.5	8.71
		7	3600	48513.5	57023.5	14.92	3600	50031.6	12.26
		8	3600	51468.7	58932.2	12.66	3600	52543.3	10.84
		9	3600	54145.4	62775.3	13.75	3600	55720.9	11.24
		10	3600	54947.5	62572.2	12.19	3600	56077.8	10.38
Average			3600	53996.2	60983.3	11.38	3600	55137.9	9.51
10	0.4	1	3600	44742.6	47414.3	5.63	3600	44490.8	6.17
		2	3600	40880.5	43543.2	6.11	3600	41069.3	5.68
		3	3600	48395.0	51770.6	6.52	3600	48012.7	7.26
		4	3600	44748.2	49526.2	9.65	3600	45158.0	8.82
		5	3600	48737.5	51881.3	6.06	3600	48952.8	5.64
		6	3600	44888.1	48318.8	7.10	3600	45265.4	6.32
		7	3600	38653.0	47586.5	18.77	3600	40515.2	14.86
		8	3600	42418.5	47482.8	10.67	3600	42382.7	10.74
		9	3600	44756.4	51500.9	13.10	3600	45261.3	12.12
		10	3600	44004.0	52751.0	16.58	3600	45356.5	14.02
Average			3600	44222.4	49177.6	10.02	3600	44646.5	9.16

and unit cost $c_1 = c_2 = 45$.

Figure 5.1(a) illustrates how the total profit change with different fixed cost and different substitution rates. In Figure 5.1(a), Π denotes the total profit. The total profit increases with the substitution rates and decreases with the fixed cost for both cases. The value of the profit are in Table 5.3. Table 5.3 describes the change of the optimal order quantities and the total profit with the fixed cost and substitution rates for different nominal demand settings.

In Table 5.3, there are two cases with different nominal demand: Case 1: $D_1 = 56, D_2 = 93$ and Case 2: $D_1 = 80, D_2 = 80$. We see that for both cases, when the fixed cost increases, the retailer might prefer to order one product instead of two products to avoid the fixed cost. When the substitution rate increases, the retailer inclines to decrease the assortment size even the fixed cost is not large. For example, when the substitution rate increases to 1 and $C^F > 0$, the retailer will only order one product and use this product to substitute the other product; when the fixed cost $C^F = 0$ and substitution rate $\underline{\alpha} = 1$, the total order quantities of these two products will be 112 for the first case and 128 for the second case, and there are many different optimal solutions for each product since customers are completely insensitive between these two products. In Case 1, when the fixed cost C^F increases from 2000 to 2500, the retailer only orders the second product and Q_2 will be equal to $93 - 93 \times 0.4 \approx 56$, which is the lower bound of its demand since the profit function is nondecreasing in demand as described in Lemma 4.1 and the first product is no longer considered; in Case 2, when the fixed cost C^F increases from 2000 to 2500, the retailer will not order any product. In Case 2, when $\alpha = 0$ and all the parameters of product 1 and product 2 are equal to each other, we always have $Q_1 = Q_2$ for different fixed costs.



(a) $D_1 \neq D_2$ (b) $D_1 = D_2$

Figure 5.1: Total profit under different fixed cost and different substitution rates

5.3.3 Robustness of Model (5.6)

In this subsection, we illustrate how to find the optimal budget of uncertainty and also test robustness of Model (5.6). Suppose that there are 10 products. The values of $\boldsymbol{p}, \boldsymbol{c}$, $\boldsymbol{s}$, $\underline{\boldsymbol{\alpha}}$, and k^α are the same as those in Section 5.3.1. We also assumed that there are 200 historical data and we split them into two groups, Υ_1, Υ_2, with equal size. These historical demand were generated by sampling from independent uniformly random variables between 20 and 80. We choose the candidate set $\mathcal{K}$ of budget of uncertainty k as $\{0, 1, \cdots, 10\}$. According to Section 4.2.2 with percentile $q = 10$, we found the optimal budget of uncertainty $k^* = 6$, which is the smallest $k \in \mathcal{K}$ such that $v^*(k, k^\alpha) \leq \widehat{\Pi}^{10\%}(k, k^\alpha)$ as shown in Figure 5.2.

We also tested the reliability of the solution from robust Model (5.6) by comparing with the risk neutral solution presented in [90], which has the following form:

$$v^{rn} = \max_{\boldsymbol{Q} \in \mathbb{R}^n_+} \left\{ \sum_{i \in [n]} \overline{P}_i Q_i - \mathbb{E}_{\mathbb{P}} \left[\sum_{i \in [n]} \overline{S}_i \left(Q_i - \tilde{D}^s_i(\boldsymbol{Q}) \right)_+ \right] \right\}, \tag{5.11}$$

where $\mathbb{P}$ denotes a particular probability distribution.

Table 5.3: Optimal order quantities and total profit under different fixed cost and different substitution rates.
$(\underline{\alpha}_{12} = \underline{\alpha}_{21} = \underline{\alpha}, C_1^F = C_2^F = C^F, k = 1, \theta = 0.4, p_1 = p_2 = 86, s_1 = s_2 = 25, c_1 = c_2 = 45)$

Case	D_1	D_2	C^F	$\alpha = 0$			$\alpha = 0.4$			$\alpha = 0.6$			$\alpha = 1$		
				Q_1	Q_2	Π	Q_1	Q_2	Π	Q_1	Q_2	Π	Q_1	Q_2	Π
			0	56	78	4135.8	47	59	4363.6	56	56	4581.8	56	56	4583.8
			50	56	78	4035.8	47	59	4263.6	56	56	4481.8	0	112	4533.8
			500	56	78	3135.8	47	59	3363.6	56	56	3581.8	0	112	4083.8
1	56	93	1000	56	78	2135.8	47	59	2363.6	0	89	2665.4	0	112	3583.8
			1500	56	78	1135.8	0	78	1706.2	0	89	2165.4	0	112	3083.8
			2000	0	56	287.8	0	78	1206.2	0	89	1665.4	0	112	2583.8
			2500	0	0	0	0	78	0	0	89	1165.4	0	112	2083.8
			0	80	80	4608	57	57	4685.7	60	60	4920	64	64	5248
			50	80	80	4508	57	57	4585.7	60	60	4820	0	128	5198
			500	80	80	3608	57	57	3685.7	60	60	3920	0	128	4748
2	80	80	1000	80	80	2608	57	57	2685.7	0	96	2936	0	128	4248
			1500	80	80	1608	0	80	1780	0	96	2436	0	128	3748
			2000	80	80	608	0	80	1280	0	96	1936	0	128	3248
			2500	0	0	0	0	80	780	0	96	1436	0	128	2748

We first used the demand data in Υ_2 to obtain the optimal order quantities of robust Model (5.6) with $k^* = 6$. Also, we obtained the optimal order quantities from the risk neutral Model (5.11) by solving the sample average approximation (SAA) with the demand realizations from set Υ_2. To compare the quality of both solutions, we assumed that the underlying true probability distribution of each product's demand is independent Gaussian $\mathcal{N}(\mu, \sigma^2)$ truncated at the interval $[20, 80]$. We selected different parametric pairs (μ, σ^2) of Gaussian random vectors, and for each pair (μ, σ^2), we generated 10^5 i.i.d. samples to evaluate the solutions from robust Model (5.6) and risk neutral Model (5.11) and also to compute their statistical confidence intervals. The computational results are presented in Table 5.4.

From Table 5.4, we see that if the data are very limited and unable to predict the underlying true probability distribution or if the underlying true probability distribution is not the same as the one we stick to, then the solution from robust Model (5.6) is more reliable than that

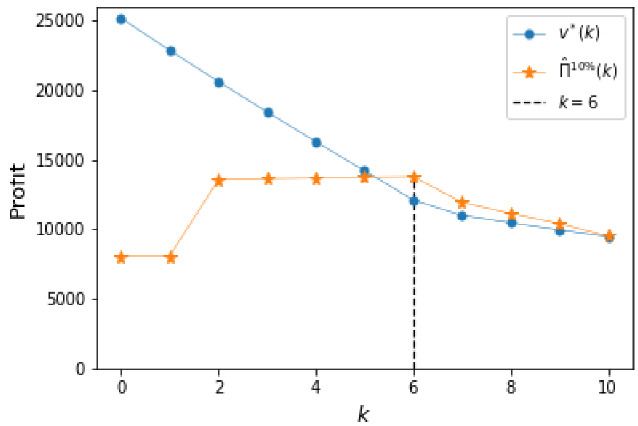


Figure 5.2: 10th percentile of profits for Model (4.2) by plugging in the optimal order quantities of robust model (5.6) for different k.

from risk neutral Model (5.11). On the other hand, if the underlying true distribution is close to the one we predict using the historical data (e.g., in the cases of Gaussian$(40, 40^2)$ and Gaussian$(40, 50^2)$), then risk neutral Model (5.11) can be more accurate. In practice, if there are limited data or the demand is changing rapidly, then we recommend the robust Model (5.6), and if there are plenty of historical data and products' demand is quite stable, then risk neutral Model (5.11) is more desirable.

Table 5.4: 95% confidence interval (CI) objective value of robust model (5.6) and risk neutral model (5.11) under different distributions.

Distribution	95% CI of Robust Model	95% CI of Risk Neutral Model
Gaussian$(20, 10^2)$	6416.22 ± 6.29	5066.66 ± 6.33
Gaussian$(20, 40^2)$	11559.16 ± 15.64	10951.76 ± 17.74
Gaussian$(20, 50^2)$	12316.00 ± 16.38	11929.89 ± 18.96
Gaussian$(30, 10^2)$	10875.82 ± 9.24	9554.59 ± 9.30
Gaussian$(30, 50^2)$	13643.80 ± 16.16	13387.78 ± 18.70
Gaussian$(30, 80^2)$	14013.59 ± 16.63	13959.36 ± 19.53
Gaussian$(40, 10^2)$	16618.50 ± 9.61	15553.12 ± 10.27
Gaussian$(40, 40^2)$	15780.94 ± 15.98	15964.15 ± 18.90
Gaussian$(40, 50^2)$	15734.04 ± 16.32	16037.57 ± 19.47

Chapter 6

Concluding Remarks

In this thesis, we have studied the impact of product substitution in the reverse supply chain and multi-product inventory environment. Background and motivation for this work is presented in Chapter 1.

In Chapter 2, we study the impact of product design on its pricing and end-of-life (EOL) options. The product design is explicitly included in the profit functions for new and remanufactured products and salvage value. These functions are quadratic in nature. We model the problem as a two-stage process in which new products are produced at Stage 1 and they undergo EOL options after their collection at Stage 2. At the second stage, remanufactured products also compete with new products. We provide a mapping to show how a product design impacts the OEM's choice of product strategies, and present lower bounds for product design at the first stage that leads to the selection of salvage option at the second stage under different pricing strategies. Moreover, we provide an upper bound on the product design at the second stage for the OEM to select a binding pricing strategy. Finally, we show that the product strategy of low pricing for remanufactured products, high pricing for new products, and salvage option dominates other product strategies if the OEM chooses a low product design at the second stage.

In Chapter 3, we investigated a multi-product newsvendor problem with customer-driven demand substitution (MPNP-CDS) for both the cases of deterministic and stochastic demands. When demand is known, we show that each product is either not ordered or ordered up to the effective demand. This fact allows us to reformulate the MPNP-CDS as an equivalent binary quadratic program, and to prove that the MPNP-CDS is NP-hard. When the demand is stochastic, we derive first-oder necessary conditions for the MPNP-CDS, and show that the profit function is continuous submodular. This fact enables us to develop two different mixed integer linear program (MILP) models and to compare their strengths. Inspired by the model properties, we develop several approximation algorithms and prove their performance guarantees. Our numerical investigation on the performance of the proposed solution algorithms shows that the stronger MILP model works well for small- or medium- sized problem instances, while the approximation algorithms consistently provide high-quality solutions. We further conducted sensitivity analyses to reveal how the model performs when the values of parameters change. A potential idea for future research includes investigation of distributionally robust MPNP-CDS, where the probability distribution of the random demand is not fully specified but instead, some empirical data is available. Also, it will be interesting to develop exact and efficient algorithms to solve large-scale instance of MPNP-CDS instances when demand is stochastic.

In Chapter 4, we have focused on the robust multi-product newsvendor problem with substitution (R-MNMS) under cardinality-constrained uncertainty set. We first prove that evaluating the worst-case total profit for given order quantities in general is NP-hard. Next, we identify three solvable special cases of R-MNMS and derive their closed-form optimal solutions. One possible future direction is to incorporate pricing decision into R-MNMS, i.e., to study joint inventory and pricing optimization in R-MNMS.

In Chapter 5, we studied the robust multi-product newsvendor problem with substitution under cardinality-constrained uncertainty sets with fixed cost (R-MNMSF). For a general R-MNMSF with fixed cost, we propose a mixed integer linear program formulation which can be solved by a branch-and-cut algorithm. We also develop a conservative approximation method to solve the R-MNMSF, and show that under certain conditions, it yields optimal solution to the R-MNMSF. We conduct numerical studies to illustrate the effectiveness and solution quality of the proposed algorithms, and also, test the reliability of robust model. Also, we study the effects of the fixed cost and choose the optimal budget of uncertainty for the R-MNMSF. One possible future direction is to study the robust model by incorporating other uncertainty sets, for example, ellipsoidal uncertainty set, polyhedral uncertainty set, and norm uncertainty set.

Appendices

Appendix A

Proofs in Chapter 2

Appendix A.1. Proof of Proposition 4.9

Theorem A.1. *(Theorem 1, [90]) When $k = n$, the optimal order quantities $\boldsymbol{Q}^*$ and the optimal total profit v^* are characterized as follows:*

(i)

$$Q_j^* = \begin{cases} D_j^s(\boldsymbol{Q}^*) = D_j - l_j + \sum_{i \in \Gamma^*} \underline{\alpha}_{ij}(D_i - l_i), & \text{if } \overline{P}_j - \sum_{i \in [n] \setminus \Gamma^*} \underline{\alpha}_{ji} \overline{P}_i \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{4.17}$$

for each $j \in [n]$, where $[n] \setminus \operatorname{supp}(\boldsymbol{Q}^*) = \Gamma^*$, i.e., $\Gamma^* = \{i \in [n] : Q_i^* = 0\}$; and

(ii)

$$v^* = \max_{\Gamma \subseteq [n]} \left\{ f(\Gamma) := \sum_{j \in \Gamma} \sum_{i \in [n] \setminus \Gamma} \underline{\alpha}_{ji} \overline{P}_i (D_j - l_j) + \sum_{i \in [n] \setminus \Gamma} \overline{P}_i (D_i - l_i) \right\} := f(\Gamma^*), \tag{4.18}$$

Proof. We prove the result by using the following three arguments.

(1) Let $x_i = Q_i - D_i$ denotes the unsold units of ith product for each $i \in [n]$. Then, Model

(3.4) is equivalent to

$$v_D^* = \max_{\boldsymbol{x} \geq -\boldsymbol{D}} \left\{ g(\boldsymbol{x}) := \sum_{i \in [n]} \bar{P}_i x_i - \sum_{i \in [n]} \bar{S}_i \left(x_i - \sum_{j \in [n]} \alpha_{ji} (-x_j)_+ \right)_+ + \sum_{i \in [n]} \bar{P}_i D_i \right\}, \quad \text{(A.1)}$$

To simplify function $g(\boldsymbol{x})$, we classify n products into the following three sets according to the value of $\boldsymbol{x}$, i.e.,

$$I_+ = \{i : x_i \geq 0\}, I_- = \{i : x_i \leq 0\}, I_{++} = \left\{ i \in I_+ : x_i + \sum_{j \in I_-} \alpha_{ji} x_j > 0 \right\}.$$

Consequently, we can remove $(\cdot)_+$ from (A.1), and we have

$$g(\boldsymbol{x}) = \sum_{i \in I_+ \setminus I_{++}} \bar{P}_i x_i + \sum_{i \in I_{++}} \left(\bar{P}_i - \bar{S}_i \right) x_i + \sum_{j \in I_-} \left(\bar{P}_j - \sum_{i \in I_{++}} \alpha_{ji} \bar{S}_i \right) x_j + \sum_{i \in [n]} \bar{P}_i D_i.$$

Note that in the above function, for each $i \in I_+ \setminus I_{++}$, the coefficient of x_i is positive as $\bar{P}_i = p_i - c_i > 0$, and by definition, $x_i \leq -\sum_{j \in I_-} \alpha_{ji} x_j$. Therefore, by letting $x_i = -\sum_{j \in I_-} \alpha_{ji} x_j$ for each $i \in I_+ \setminus I_{++}$, function $g(\boldsymbol{x})$ is upper bounded by

$$\begin{aligned} g(\boldsymbol{x}) &\leq \sum_{i \in I_+ \setminus I_{++}} \bar{P}_i \left(-\sum_{j \in I_-} \alpha_{ji} x_j \right) + \sum_{i \in I_{++}} \left(\bar{P}_i - \bar{S}_i \right) x_i + \sum_{j \in I_-} \left(\bar{P}_j - \sum_{i \in I_{++}} \alpha_{ji} \bar{S}_i \right) x_j + \sum_{i \in [n]} \bar{P}_i D_i \\ &= \sum_{i \in I_{++}} \left(\bar{P}_i - \bar{S}_i \right) \left(x_i + \sum_{j \in I_-} \alpha_{ji} x_j \right) + \sum_{j \in I_-} \left(\bar{P}_j - \sum_{i \in I_+} \alpha_{ji} \bar{P}_i \right) x_j + \sum_{i \in [n]} \bar{P}_i D_i. \end{aligned}$$

For each $i \in I_{++}$, we have $\bar{P}_i - \bar{S}_i = s_i - c_i < 0$ by definition, and $x_i + \sum_{j \in I_-} \alpha_{ji} x_j > 0$ by definition of set I_{++}. Thus, by letting $I_{++} = \emptyset$, function $g(\boldsymbol{x})$ is further upper bounded by

$$g(\boldsymbol{x}) \leq \sum_{j \in I_-} \left(\bar{P}_j - \sum_{i \in I_+} \alpha_{ji} \bar{P}_i \right) x_j + \sum_{i \in [n]} \bar{P}_i D_i.$$

Note that for each $j \in I_-$, we note that $x_j \in [-D_j, 0]$. Hence, for each $j \in I_-$, let $x_j = 0$ if $\bar{P}_j - \sum_{i \in I_+} \alpha_{ji} \bar{P}_i \geq 0$, and $-D_j$, otherwise. Then $g(\boldsymbol{x})$ is further upper bounded by

$$g(\boldsymbol{x}) \leq \sum_{j \in I_-} \left(\sum_{i \in I_+} \alpha_{ji} \bar{P}_i - \bar{P}_j \right)_+ D_j + \sum_{i \in [n]} \bar{P}_i D_i,$$

where the equality is achieved when $I_{++} = \emptyset$ and for each $j \in [n]$,

$$x_j = \begin{cases} 0, & \text{if } \sum_{i \in I_+} \alpha_{ji} \bar{P}_i - \bar{P}_j \leq 0, j \in I_- \\ -D_j, & \text{if } \sum_{i \in I_+} \alpha_{ji} \bar{P}_i - \bar{P}_j > 0, j \in I_- \\ -\sum_{i \in I_-} \alpha_{ij} x_i, & \text{otherwise} \end{cases}. \tag{A.2}$$

Note that $I_+ = [n] \setminus I_-$. Therefore, Model (A.1) is further equivalent to the following combinatorial optimization problem

$$v_D^* = \max_{I_- \subseteq [n]} \left\{ \widehat{g}(I_-) := \sum_{j \in I_-} \left(\sum_{i \in [n] \setminus I_-} \alpha_{ji} \bar{P}_i - \bar{P}_j \right)_+ D_j + \sum_{i \in [n]} \bar{P}_i D_i \right\}. \tag{A.3}$$

(2) Next, we prove the following property of Model (A.3).

Claim 1. In the Model (A.3), for any subset $I_- \subseteq [n]$, let $J_0 = \left\{ j \in I_- : \sum_{i \in [n] \setminus I_-} \alpha_{ji} \bar{P}_i \leq \bar{P}_j \right\}$, then $\widehat{g}(I_-) \leq \widehat{g}(I_- \setminus J_0)$.

Proof. Let us define $\widehat{I}_- = I_- \setminus J_0$. By definitions of sets I_-, J_0 and $\widehat{I}_-$, we have

$$\begin{aligned}
\widehat{g}(I_-) &= \sum_{j\in I_-}\left(\sum_{i\in[n]\setminus I_-}\alpha_{ji}\bar{P}_i - \bar{P}_j\right)_+ D_j + \sum_{i\in[n]}\bar{P}_i D_i \\
&= \sum_{j\in I_-\setminus J_0}\left(\sum_{i\in[n]\setminus I_-}\alpha_{ji}\bar{P}_i - \bar{P}_j\right) D_j + \sum_{i\in[n]}\bar{P}_i D_i \\
&= \sum_{j\in I_-\setminus J_0}\left(\sum_{i\in[n]\setminus(I_-\setminus J_0)}\alpha_{ji}\bar{P}_i - \bar{P}_j\right) D_j + \sum_{i\in[n]}\bar{P}_i D_i - \sum_{j\in I_-\setminus J_0}\sum_{i\in J_0}\alpha_{ji}\bar{P}_i D_j \\
&= \widehat{g}(\widehat{I}_-) - \sum_{j\in I_-\setminus J_0}\sum_{i\in J_0}\alpha_{ji}\bar{P}_i D_j \\
&\leq \widehat{g}(\widehat{I}_-)
\end{aligned}$$

where the inequality holds due to $\sum_{j\in I_-\setminus J_0}\sum_{i\in J_0}\alpha_{ji}\bar{P}_i D_j \geq 0$.

$\diamond$

By Claim 1 and equation (A.2), we note that there exists an optimal solution to Model (A.1) $\boldsymbol{x}^*$ with subset $I_-^* := \left\{ j : \sum_{i\in[n]\setminus I_-^*}\alpha_{ji}\bar{P}_i > \bar{P}_j \right\}$ such that

$$x_j^* = \begin{cases} -D_j, & \text{if } j \in I_-^* \\ -\sum_{i\in I_-}\alpha_{ij}x_i^*, & \text{otherwise.} \end{cases}.$$

Let $\boldsymbol{Q}^* = \boldsymbol{x}^* + \boldsymbol{D}$, and $\Gamma^* = I_-^*$. Clearly, $\boldsymbol{Q}^*$ satisfies (4.17) and is an optimal solution to Model (3.4) and $v_D^* = f(\Gamma^*)$.

Finally, by Claim 1 and letting $\Gamma = I_-$, Model (A.3) reduces to

$$v_D^* = \max_{\Gamma\subseteq[n]}\left\{ f(\Gamma) := \sum_{j\in\Gamma}\left(\sum_{i\in[n]\setminus S}\alpha_{ji}\bar{P}_i - \bar{P}_j\right) D_j + \sum_{i\in[n]}\bar{P}_i D_i \right\},$$

which is equivalent to (4.18).

∎

Appendix A.2. Proof of Proposition 3.4

Proposition 3.4. *The set function $f(\Gamma)$, defined in (4.18), is submodular.*

Proof. Let $A \subseteq B \subseteq [I], k \in [n] \setminus B$. By Proposition 3.3, we only need to show that

$$f(A \cup \{k\}) - f(A) \geq f(B \cup \{k\}) - f(B).$$

Note that

$$\begin{aligned} f(B \cup \{k\}) - f(B) &= \sum_{j \in B \cup \{k\}} \sum_{i \in [n] \setminus (B \cup \{k\})} \alpha_{ji} \bar{P}_i D_j - \sum_{j \in B} \sum_{i \in [n] \setminus B} \alpha_{ji} \bar{P}_i D_j - \bar{P}_k D_k \\ &= \sum_{j \in B} \sum_{i \in [n] \setminus (B \cup \{k\})} \alpha_{ji} \bar{P}_i D_j - \sum_{j \in B} \sum_{i \in [n] \setminus B} \alpha_{ji} \bar{P}_i D_j + \sum_{i \in [n] \setminus (B \cup \{k\})} \alpha_{ki} \bar{P}_i D_k - \bar{P}_k D_k \\ &= -\sum_{j \in B} \alpha_{jk} \bar{P}_k D_j + \sum_{i \in [n] \setminus (B \cup \{k\})} \alpha_{ki} \bar{P}_i D_k - \bar{P}_k D_k \end{aligned}$$

Similarly,

$$f(A \cup \{k\}) - f(A) = -\sum_{j \in A} \alpha_{jk} \bar{P}_k D_j + \sum_{i \in [n] \setminus (A \cup \{k\})} \alpha_{ki} \bar{P}_i D_k - \bar{P}_k D_k.$$

Hence,

$$(f(B \cup \{k\}) - f(B)) - (f(A \cup \{k\}) - f(A))$$

$$
\begin{aligned}
&= -\sum_{j\in B}\alpha_{jk}\bar{P}_k D_j + \sum_{i\in[n]\setminus(B\cup\{k\})}\alpha_{ki}\bar{P}_i D_k - \left(-\sum_{j\in A}\alpha_{jk}\bar{P}_k D_j + \sum_{i\in[n]\setminus(A\cup\{k\})}\alpha_{ki}\bar{P}_i D_k\right)\\
&= -\sum_{j\in B\setminus A}\alpha_{jk}\bar{P}_k D_j - \sum_{i\in B\setminus A}\alpha_{ki}\bar{P}_i D_k \leq 0
\end{aligned}
$$

where the inequality follows because $\sum\limits_{j\in B\setminus A}\alpha_{jk}\bar{P}_k D_j \geq 0$ and $\sum\limits_{i\in B\setminus A}\alpha_{ki}\bar{P}_i D_k \geq 0$. Thus, $f(\Gamma)$ is submodular. □

Appendix A.3. Proof of Proposition 3.5

Proposition 4.2. *R-MNMS is equivalent to*

$$
v^* = \max_{\boldsymbol{Q}\in\mathbb{R}_+^n}\min_{\tilde{\boldsymbol{D}}\in\mathcal{U}}\Pi\left(\boldsymbol{Q},\tilde{\boldsymbol{D}}\right), \tag{4.9}
$$

where $\mathcal{U}$ is defined in (4.8).

Proof. We prove this result by showing that the weighted max-cut problem (WMCP) is a special case of the MPNP-CDS(D).

> **(Weighted Max-Cut Problem)** Given an undirected graph $G = (V, E)$ with $|V| = n$, and a nonnegative integer weight w_{ij} associated with each edge $(i, j) \in E$ in the graph, (and $w_{ij} = 0$ if there is no edge between nodes i, j.) find a subset $\Lambda \subseteq V$ which maximizes the total weights of edges between subsets Λ and $[n]\setminus\Lambda$.

Clearly, this problem can be formulated as:

$$
v_w = \max_{\Lambda\subseteq[n]}\left\{\sum_{j\in\Lambda}\sum_{i\in[n]\setminus\Lambda} w_{ji}\right\}. \tag{A.4}
$$

Without loss of generality, we assume that there is at lease one edge $(i, j) \in E$ such that $w_{ij} > 0$, otherwise, the weighted max-cut problem is trivial.

Consider a special instance of MPNP-CDS(D), where $\alpha_{ji} = \alpha_{ij} = (n+1)w_{ji}$ and $\bar{P}_i = D_i = 1$ for all $i, j \in [n]$. Under this setting, Model (4.18) reduces to

$$v_{DW} = \max_{\Lambda \subseteq [n]} \left\{ (n+1) \sum_{j \in \Lambda} \sum_{i \in [n] \setminus \Lambda} w_{ji} + n - |\Lambda| \right\}. \tag{A.5}$$

Let $\lfloor x \rfloor$ denote the floor function of number x. It remains to show that

Claim 2. $\lfloor \frac{v_{DW}}{n+1} \rfloor = v_w$.

Proof. We separate the proof into two steps.

$v_w \leq \lfloor \frac{v_{DW}}{n+1} \rfloor$ Let Λ^* be an optimal solution to (A.4). Clearly, Λ^* is feasible to (A.5), thus

$$(n+1)v_w \leq (n+1) \sum_{j \in \Lambda^*} \sum_{i \in [n] \setminus \Lambda^*} w_{ji} + n - |\Lambda^*| \leq v_{DW}.$$

Due to our assumption that all the weights are integral, we have $v_w \leq \lfloor \frac{v_{DW}}{n+1} \rfloor$. Next we show that

$v_w \geq \lfloor \frac{v_{DW}}{n+1} \rfloor$ Suppose that $v_w < \lfloor \frac{v_{DW}}{n+1} \rfloor$, i.e., $v_w \leq \lfloor \frac{v_{DW}}{n+1} \rfloor - 1$, which implies that

$$(n+1)v_w \leq v_{DW} - (n+1).$$

Let $\widehat{\Lambda}$ be an optimal solution to (A.5). We have

$$(n+1)v_w \leq v_{DW} - (n+1) = (n+1) \sum_{j \in \widehat{\Lambda}} \sum_{i \in [n] \setminus \widehat{\Lambda}} w_{ji} + n - |\widehat{\Lambda}| - (n+1)$$

which implies that

$$\sum_{j\in\widehat{\Lambda}}\sum_{i\in[n]\setminus\widehat{\Lambda}} w_{ji} \geq v_w + \frac{1+|\widehat{\Lambda}|}{n+1} > v_w$$

a contradiction that v_w is the optimal value to (A.4).

$\diamond$

Hence, it follows that we can solve the MPNP-CDS(D) efficiently, only if we can solve the weighted max-cut problem (A.4) efficiently. However, the weighted max-cut problem is strongly NP-hard. Therefore, the MPNP-CDS is also NP-hard, and consequently, so is the MPNP-CDS. $\square$

Appendix A.4. Proof of Proposition 3.8

Theorem A.2. *Let $\boldsymbol{Q}^*$ be the vector of optimal quantities of Model (3.11). Then,*

$$\mathbb{P}\left(Q_i^* \geq \tilde{D}_i^s(\boldsymbol{Q}^*)\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ji}\mathbb{P}\left(Q_j^* \geq \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* < \tilde{D}_i\right) \geq \frac{\bar{P}_i}{\bar{S}_i}, \forall i \in [n], \tag{3.12a}$$

$$\mathbb{P}\left(Q_i^* > \tilde{D}_i^s(\boldsymbol{Q}^*)\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ji}\mathbb{P}\left(Q_j^* > \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* \leq \tilde{D}_i\right) \leq \frac{\bar{P}_i}{\bar{S}_i}, \forall i \in [n]. \tag{3.12b}$$

Proof. For notational convenience, given a vector $\boldsymbol{Q}$, let $(\boldsymbol{Q}|Q_i \leftarrow q)$ denote a new vector that is the same as $\boldsymbol{Q}$ except that the ith entry is q. Note that $\boldsymbol{Q}^*$ is optimal to (3.11), i.e.,

$$\boldsymbol{Q}^* \in \arg\max_{\boldsymbol{Q}\in\mathbb{R}_n^+}\left\{\Pi(\boldsymbol{Q}) = \sum_{i\in[n]}\bar{P}_iQ_i - \mathbb{E}\left[\sum_{i\in[n]}\bar{S}_i\left(Q_i - \tilde{D}_i^s(\boldsymbol{Q})\right)_+\right]\right\}.$$

We note that in the above optimization model, since $0 < \bar{P}_i < \bar{S}_i$ for each $i \in [n]$ and demand $\tilde{\boldsymbol{D}}$ is nonnegative, thus, $\Pi(\boldsymbol{Q}) < \Pi((\boldsymbol{Q})_+)$ if $\boldsymbol{Q} \notin \mathbb{R}_n^+$ is not a nonnegative vector. Therefore, we can relax the domain of $\boldsymbol{Q}$ to be $\mathbb{R}^n$ as below:

$$\boldsymbol{Q}^* \in \arg\max_{\boldsymbol{Q}} \left\{ \Pi(\boldsymbol{Q}) = \sum_{i\in[n]} \bar{P}_i Q_i - \mathbb{E}\left[\sum_{i\in[n]} \bar{S}_i \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q}) \right)_+ \right] \right\}.$$

By the optimality of $\boldsymbol{Q}^*$, for each $i \in [n]$, Q_i^* is also optimal to the following mathematical program

$$\begin{aligned} Q_i^* \in \operatorname{argmax} \Bigg\{ & G(Q_i) := \bar{P}_i Q_i - \bar{S}_i \mathbb{E} \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q}^* | Q_i^* \leftarrow Q_i) \right)_+ \\ & - \sum_{j\in[n], j\neq i} \mathbb{E} \left(\bar{S}_j \left(Q_j^* - \tilde{D}_j^s \left(\boldsymbol{Q}^* | Q_i^* \leftarrow Q_i \right) \right)_+ \right) \Bigg\}. \end{aligned}$$

Let $Q_i^1 := Q_i^* + \varepsilon, Q_i^2 := Q_i^* - \varepsilon$, where $\varepsilon > 0$ is a sufficiently small positive constant. Simple calculation yields

$$\begin{aligned} & G\left(Q_i^1\right) - G\left(Q_i^*\right) \\ & = \bar{P}_i \varepsilon - \mathbb{P}\left(Q_i^* \geq \tilde{D}_i^s(\boldsymbol{Q}^*) \right) \bar{S}_i \varepsilon - \sum_{j\in[n], j\neq i} \mathbb{P}\left(Q_j^* \geq \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* < \tilde{D}_i \right) \bar{S}_j \alpha_{ji} \varepsilon \leq 0, \\ & G\left(Q_i^2\right) - G\left(Q_i^*\right) \\ & = -\bar{P}_i \varepsilon + \mathbb{P}\left(Q_i^* > \tilde{D}_i^s(\boldsymbol{Q}^*) \right) \bar{S}_i \varepsilon + \sum_{j\in[n], j\neq i} \mathbb{P}\left(Q_j^* > \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* \leq \tilde{D}_i \right) \bar{S}_j \alpha_{ji} \varepsilon \leq 0, \end{aligned}$$

where $G\left(Q_i^1\right) - G\left(Q_i^*\right) \leq 0$, and $G\left(Q_i^2\right) - G\left(Q_i^*\right) \leq 0$ are due to the optimality of Q_i^*. Thus, we arrive at (3.12). $\square$

Appendix A.5. Proof of Proposition 3.9

Proposition 3.9. *Let $\boldsymbol{Q}^*$ be the vector of optimal quantities of Model (3.11). Then, $\boldsymbol{Q}^*$ is upper and lower bounded by $\overline{\boldsymbol{Q}}$ and $\underline{\boldsymbol{Q}}$, respectively, i.e., for each product $i \in [n]$, $\overline{Q}_i \geq Q_i^* \geq \underline{Q}_i$ with*

$$\underline{Q}_i = \begin{cases} F_{\tilde{D}_i}^{-1}\left(\frac{\bar{P}_i - \sum_{j\in[n]} \alpha_{ij}\bar{S}_j}{\bar{S}_i - \sum_{j\in[n]} \alpha_{ij}\bar{S}_j}\right), & \text{if} \quad \bar{S}_i > \sum_{j\in[n]} \alpha_{ij}\bar{S}_j \\ 0, & \text{otherwise} \end{cases}, \tag{3.13a}$$

$$\overline{Q}_i = \bar{F}^{-1}_{\tilde{D}_i + \sum_{j\in[n]} \alpha_{ji}\tilde{D}_j}\left(\frac{\bar{P}_i}{\bar{S}_i}\right), \tag{3.13b}$$

where $F_{\tilde{X}}^{-1}, \bar{F}_{\tilde{X}}^{-1}$ denote the lower and upper inverse distribution function of random variable $\tilde{X}$, respectively, i.e., $F_{\tilde{X}}^{-1}(t) = \inf\left\{\kappa : \mathbb{P}\left(\tilde{X} \leq \kappa\right) \geq t\right\}$ and $\bar{F}_{\tilde{X}}^{-1}(t) = \inf\left\{\kappa : \mathbb{P}\left(\tilde{X} < \kappa\right) \geq t\right\}$.

Proof. Note that by definition, for each product $i \in [n]$, we have

$$\tilde{D}_i \leq \tilde{D}_i^s(\boldsymbol{Q}^*) = \tilde{D}_i + \sum_{j\in[n]} \alpha_{ji}\left(\tilde{D}_j - Q_j^*\right)_+ \leq \tilde{D}_i + \sum_{j\in[n]} \alpha_{ji}\tilde{D}_j,$$

where the second inequality holds because $\tilde{D}_j, Q_j^*$ are nonnegative for all $j \in [n]$. Therefore,

$$\begin{aligned} &\mathbb{P}\left(\tilde{D}_i \leq Q_i^*\right) \geq \mathbb{P}\left(\tilde{D}_i^s \leq Q_i^*\right) \geq \mathbb{P}\left(\tilde{D}_i + \sum_{j\in[n]} \alpha_{ji}\tilde{D}_j \leq Q_i^*\right), \\ &\mathbb{P}\left(\tilde{D}_i < Q_i^*\right) \geq \mathbb{P}\left(\tilde{D}_i^s < Q_i^*\right) \geq \mathbb{P}\left(\tilde{D}_i + \sum_{j\in[n]} \alpha_{ji}\tilde{D}_j < Q_i^*\right), \end{aligned} \tag{A.6}$$

Now, we separate the rest of the proof into two parts.

(1) Clearly, by (3.11), $\boldsymbol{Q}_+^* \in \mathbb{R}_+^n$. According to (3.12a) in Proposition 3.8, we have

$$\begin{aligned}
&\mathbb{P}\left(Q_i^* \geq \tilde{D}_i\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ij}\mathbb{P}\left(Q_i^* < \tilde{D}_i\right) \\
&\geq \mathbb{P}\left(Q_i^* \geq \tilde{D}_i^s(\boldsymbol{Q}^*)\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ij}\mathbb{P}\left(Q_j \geq \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* < \tilde{D}_i\right) \\
&\geq \frac{\bar{P}_i}{\bar{S}_i}
\end{aligned}$$

where the first inequality follows because (A.6) and $\mathbb{P}\left(Q_j \geq \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* < \tilde{D}_i\right) \leq \mathbb{P}\left(Q_i^* < \tilde{D}_i\right)$. The above inequalities imply that

$$\mathbb{P}\left(Q_i^* \geq \tilde{D}_i(Q)^*\right) \geq \frac{\bar{P}_i - \sum_{j\in[n]}\alpha_{ij}\bar{S}_j}{\bar{S}_i - \sum_{j\in[n]}\alpha_{ij}\bar{S}_j}$$

given that $\bar{S}_i > \sum_{j\in[n]}\alpha_{ij}\bar{S}_j$, i.e., we arrive at (3.13a).

(2) According to (3.12b) in Proposition 3.8, we have

$$\mathbb{P}\left(Q_i^* > \tilde{D}_i + \sum_{j\in[n]}\alpha_{ji}\tilde{D}_j\right) \leq \mathbb{P}\left(Q_i^* > \tilde{D}_i^s(\boldsymbol{Q}^*)\right) + \sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ij}\mathbb{P}\left(Q_j > \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* \leq \tilde{D}_i\right) \leq \frac{\bar{P}_i}{\bar{S}_i}$$

where the first inequality is due to (A.6) and $\sum_{j\in[n]} \frac{\bar{S}_j}{\bar{S}_i}\alpha_{ij}\mathbb{P}\left(Q_j > \tilde{D}_j^s(\boldsymbol{Q}^*), Q_i^* \leq \tilde{D}_i\right) \geq 0$. Thus, we arrive at (3.13b).

□

Appendix A.6. Proof of Proposition 3.11

Proposition 3.11. *The profit function* $\Pi(\boldsymbol{Q})$ *defined in* (3.11) *is continuous submodular.*

Proof.

$$(3.11) = \sum_{i\in[n]} \bar{P}_i Q_i - \sum_{k\in[N]} m_k \left[\sum_{i\in[n]} \bar{S}_i \left(Q_i - D_i^{sk}(\boldsymbol{Q})\right)_+ \right] \tag{A.7a}$$

$$= \sum_{i\in[n]} \bar{P}_i Q_i + \sum_{k\in[N]} m_k \left[\sum_{i\in[n]} \bar{S}_i \min\left(Q_i - D_i^{sk}(\boldsymbol{Q}), 0\right) \right] \tag{A.7b}$$

In (A.7b), $D_i^{sk}(\boldsymbol{Q}) = D_i^k + \sum_{i\neq j} \alpha_{ij} \left(D_j^k - Q_j\right)_+ = D_i^k + \sum_{i\neq j} \alpha_{ij} \min\left(D_j^k - Q_j, 0\right) = D_i^k - \sum_{i\neq j} \alpha_{ij} Q_j + \sum_{i\neq j} \alpha_{ij} \min\left(D_i^k, Q_j\right)$. As proved in [83], $D_i^{sk}(\boldsymbol{Q})$ is submodular and supermodular on $\boldsymbol{D}$. Thus, $D_i^s(\boldsymbol{Q})$ is submodular and supermodular on $\boldsymbol{Q}$, since $\boldsymbol{Q}$ and $\boldsymbol{D}$ are symmetric in the function $\min(D_i, Q_i)$, and also the summation of linear funciton are still submodular or supermodular. So $Q_i - D_i^{sk}(\boldsymbol{Q})$ is submodular on $\boldsymbol{Q}$. Since $\min\{t, 0\}$ is non-decreasing and concave on t, according to [82], $\min\{f(\boldsymbol{Q}), 0\}$ is submodular if $f(\boldsymbol{Q})$ is submodular. Therefore, $\min\left(Q_i - D_i^{sk}(\boldsymbol{Q}), 0\right)$ is submodualr on $\boldsymbol{Q}$. The first term $\sum_{i\in[n]} \bar{P}_i Q_i$ in (A.7b) is linear function and $m_k, \bar{P}_i \geq 0,$ for all $k \in [N]$ and $i \in [n]$. Thus, (3.11) is submodular on $\boldsymbol{Q}$. $\square$

Appendix A.7. Proof of Proposition 3.14

Theorem A.3. *The MILP Model 2 is stronger than MILP Model 1, i.e., their continuous relaxation values satisfy $\bar{v}_M^1 \leq \bar{v}_M^2$, where $\bar{v}_M^1, \bar{v}_M^2$ are defined in (3.17a), (3.17b), respectively.*

Proof. Let $(\boldsymbol{Q}^*, \boldsymbol{\chi}^*, \boldsymbol{u}^*, \boldsymbol{w}^*, \boldsymbol{y}^*)$ be an optimal solution to relaxed Model (3.17b). For each $i \in [n], k \in [N]$, define

$$z_i^{(k)*} = 1 - \sum_{\tau\in[k]} \chi_i^{(\tau)*}.$$

Clearly, $\boldsymbol{z}^* \in [0,1]^{n\times N}$. We need to show that $(\boldsymbol{Q}^*, \boldsymbol{z}^*, \boldsymbol{u}^*, \boldsymbol{y}^*)$ is feasible to relaxed Model

(3.17a). Note that $(\boldsymbol{Q}^*, \boldsymbol{z}^*, \boldsymbol{u}^*, \boldsymbol{y}^*)$ satisfies constraints (3.15b) and (3.15e).

According to (3.16d), for each $i \in [n]$ and $k \in [N]$, we have

$$\begin{aligned} u_i^{(k)*} + Q_i^* - D_i^{(k)} &= D_i^{(k)} \left[\sum_{\tau \in [k]} \chi_i^{(\tau)^*} - 1 \right] + Q_i^* - \sum_{\tau \in [k]} w_i^{(\tau)*} \\ &\geq D_i^{(k)} \left[\sum_{\tau \in [k]} \chi_i^{(\tau)^*} - 1 \right] \geq -M_i z_i^{(k)*} \end{aligned}$$

where the first inequality is due to $Q_i^* \geq \sum_{\tau \in [k]} w_i^{(\tau)*}$ and the second inequality is due to $z_i^{(k)*} = 1 - \sum_{\tau \in [k]} \chi_i^{(\tau)*} = 0$ if $D_i^{(k)} > M_i$, and $z_i^{(k)*} = 1 - \sum_{\tau \in [k]} \chi_i^{(\tau)*} \in [0, 1]$, otherwise. On the other hand,

$$\begin{aligned} u_i^{(k)*} + Q_i^* - D_i^{(k)} &= D_i^{(k)} \left[\sum_{\tau \in [k]} \chi_i^{(\tau)^*} - 1 \right] + Q_i^* - \sum_{\tau \in [k]} w_i^{(\tau)*} \\ &= D_i^{(k)} \left[\sum_{\tau \in [k]} \chi_i^{(\tau)^*} - 1 \right] + \sum_{\tau \in [N+1] \setminus [k]} w_i^{(\tau)*} \\ &\leq M_i \sum_{\tau \in [N+1] \setminus [k]} \chi_i^{(\tau)*} := M_i z_i^{(k)*} \end{aligned}$$

where the second equality follows because $Q_i^* = \sum_{\tau \in [N+1]} w_i^{(\tau)*}$, the first inequality is due to $D_i^{(k)} \left[\sum_{\tau \in [k]} \chi_i^{(\tau)^*} - 1 \right] \leq 0$ and $w_i^{(\tau)*} \leq \widehat{D}_i^{(\tau)} \chi_i^{(\tau)*} \leq M_i \chi_i^{(\tau)*}$ for each $\tau \in [N+1] \setminus [k]$. Therefore, $(\boldsymbol{Q}^*, \boldsymbol{z}^*, \boldsymbol{u}^*, \boldsymbol{y}^*)$ satisfies constraints (3.15c).

Finally,we note that $u_i^{(k)*} \geq 0$ for each $i \in [n]$ and $k \in [N]$. In addition, by (3.16d), we have

$$u_i^{(k)*} = D_i^{(k)} \sum_{\tau \in [k]} \chi_i^{(\tau)^*} - \sum_{\tau \in [k]} w_i^{(\tau)*} \leq D_i^k \sum_{\tau \in [k]} \chi_i^{(\tau)*} := D_i^k (1 - z_i^{(k)*})$$

where the inequality because $\sum_{\tau \in [k]} w_i^{(\tau)*} \geq 0$. Thus, $(\boldsymbol{Q}^*, \boldsymbol{z}^*, \boldsymbol{u}^*, \boldsymbol{y}^*)$ satisfies constraints

(3.15d). $\square$

Appendix A.8. Proof of Proposition 3.15

Proposition 3.15. *Suppose that $\boldsymbol{Q} \in \mathbb{R}_+^n$ is known. Then,*

(i) the following optimization model is efficiently solvable,

$$\max_{q \in \left[\underline{Q}_i, \overline{Q}_i\right]} \Pi\left(\boldsymbol{Q} | Q_i \leftarrow q\right) \tag{3.19}$$

for each $i \in [n]$; and

(ii) an optimal solution to Model (3.19) belongs to set $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, where

$$\mathcal{R}_1 = \left\{ D_i^k : D_i^k \in \left[\underline{Q}_i, \overline{Q}_i\right], \forall k \in [N+1] \right\}, \tag{3.20a}$$

$$\mathcal{R}_2 = \left\{ D_i^{sk} : D_i^{sk} \in \left[\underline{Q}_i, \overline{Q}_i\right], \forall k \in [N] \right\}, \tag{3.20b}$$

$$\mathcal{R}_3 = \left\{ D_i^k - \frac{Q_j - D_{j,-i}^{sk}}{\alpha_{ij}} : D_i^k - \frac{Q_j - D_{j,-i}^{sk}}{\alpha_{ij}} \in \left[\underline{Q}_i, D_i^k\right], \forall j \in [n], k \in [N] \right\}. \tag{3.20c}$$

Proof. First of all, we can simplify Model (3.19) to an equivalent form by eliminating all of the constant terms, i.e., the following optimization problem has the same optimal solutions as Model (3.19):

$$\max_{q \in \left[\underline{Q}_i, \overline{Q}_i\right]} \bar{P}_i q - \sum_{k \in [N]} m_k \bar{S}_i \left(q - D_i^{sk}\left(\boldsymbol{Q} | Q_i \leftarrow q\right)\right)_+ - \sum_{k \in [N]} m_k \sum_{j \in [n], j \neq i} \bar{S}_j \left(Q_j - D_j^{sk}\left(\boldsymbol{Q} | Q_i \leftarrow q\right)\right)_+,$$

which is further equivalent to

$$\max_{q\in\left[\underline{Q}_i,\overline{Q}_i\right]} \bar{P}_i q - \sum_{k\in[N]} m_k \bar{S}_i \left(q - D_i^{sk}\left(\boldsymbol{Q}\right)\right)_+ - \sum_{k\in[N]} m_k \sum_{j\in[n],j\neq i} \bar{S}_j \left(Q_j - D_j^{sk}\left(\boldsymbol{Q}|Q_i \leftarrow q\right)\right)_+, \tag{A.8}$$

since $\alpha_{ii} = 0$ and $D_i^{sk}\left(\boldsymbol{Q}|Q_i \leftarrow q\right) = D_i^{sk}(\boldsymbol{Q}) = D_i^k + \sum_{j\in[n]} \alpha_{ji}(D_j^k - Q_j)_+$ is a constant.

Notice that

$$D_j^{sk}\left(\boldsymbol{Q}|Q_i \leftarrow q\right) = D_j^k + \sum_{\tau\in[n],\tau\neq i} \alpha_{\tau j}\left(D_\tau^k - Q_\tau\right)_+ + \alpha_{ij}\left(D_i^k - q\right)_+ := D_{j,-i}^{sk}(\boldsymbol{Q}) + \alpha_{ij}\left(D_i^k - q\right)_+$$

where $D_{j,-i}^{sk}(\boldsymbol{Q}) = D_j^k + \sum_{\tau\in[n],\tau\neq i} \alpha_{\tau j}\left(D_\tau^k - Q_\tau\right)_+$.

From Property 2, we know that the demand of product i is sorted as

$$D_i^{(1)} \leq \ldots \leq D_i^{(N)}.$$

Now let $\widehat{D}_i^{(k)} = \max\left\{\min\left\{D_i^{(k)}, \overline{Q}_i\right\}, \underline{Q}_i\right\}$. Hence, the optimal order quantity q^* of Model (A.8) must belong to one of the following $N+1$ intervals:

$$\left[\widehat{D}_i^{(0)}, \widehat{D}_i^{(1)}\right], \left[\widehat{D}_i^{(1)}, \widehat{D}_i^{(2)}\right], \ldots, \left[\widehat{D}_i^{(N)}, \widehat{D}_i^{(N+1)}\right].$$

where $\widehat{D}_i^{(0)} = \underline{Q}_i, \widehat{D}_i^{(N+1)} = \overline{Q}_i$. Let us set $I_r = \left\{\tau : D_i^\tau \geq \widehat{D}_i^{(r)}\right\}$ for each $r \in [N]$. By removing constant terms, Model (A.8) further reduces to

$$\begin{aligned}\max_{r\in[N+1]} \max_{q\in[\widehat{D}_i^{(r-1)},\widehat{D}_i^{(r)}]} \bar{P}_i q &- \sum_{k\in[N]} m_k \bar{S}_i \left(q - D_i^{sk}\left(\boldsymbol{Q}\right)\right)_+ \\ &- \sum_{k\in I_r} m_k \sum_{\substack{j\in[n]\\ j\neq i}} \bar{S}_j \left(\alpha_{ij} q - D_{j,-i}^{sk}(\boldsymbol{Q}) - \alpha_{ij} D_i^k + Q_j\right)_+.\end{aligned}$$

Note that in the above optimization model, the inner optimization is to maximize a piecewise linear concave function with optimal value achieved by one of its extreme points, which are included in the set of all the breaking points of the piecewise linear concave function. Therefore, one of the optimal solution to the above maximization model is contained in a set $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are defined in (3.20). There are at most $2N + nN$ points in set $\mathcal{R}$, thus, Model (3.19) is efficiently solvable. $\square$

Appendix A.9. Proof of Proposition 3.19

Theorem A.4. *Let v^*, v^{LD}, v_R^{LD} denote the optimal value of Models (3.11), (3.23b), and (3.26), respectively. Then,*

(i) $v_R^{LD} \leq \frac{v^{LD}}{0.79607}$*; and*

(ii) if Assumption 3 holds, then

$$v_R^{LD} \leq \frac{v^{LD}}{0.79607} \leq \frac{(1+\overline{\delta})}{0.79607(1-\underline{\delta})} v^*$$

Proof. (i) We first prove $v_R^{LD} \leq \frac{v^{LD}}{0.79607}$. From (3.26), we have

$$\begin{aligned}
v_R^{LD} &= \inf_{\boldsymbol{\lambda}\in\Omega} \sum_{k\in[N]} m_k \max_{\boldsymbol{Y}^k \in C_R} \left\{ \sum_{i\in[n]} \sum_{j\in[n+1]} \frac{1}{4} \left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k \right) \right\} \\
&\leq \frac{1}{0.79607} \inf_{\boldsymbol{\lambda}\in\Omega} \sum_{k\in[N]} m_k \max_{\boldsymbol{Y}^k \in C} \left\{ \sum_{i\in[n]} \sum_{j\in[n+1]} \frac{1}{4} \left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k \right) \right\} \\
&= \frac{1}{0.79607} v^{LD}
\end{aligned}$$

where the inequality follows by the result of Proposition 3.7.

(ii) It remains to show that $v^{LD} \le \frac{1+\bar{\delta}}{1-\underline{\delta}} v^*$ under Assumption 3. By (3.25), we have

$$\begin{aligned}
v^{LD} &= \inf_{\boldsymbol{\lambda} \in \Omega} \sum_{k \in [N]} m_k \max_{\boldsymbol{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \left(\bar{P}_i + \frac{\lambda_i^k}{m_k} \right) w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k \right) \right\} \\
&\le \sum_{k \in [N]} m_k \max_{\boldsymbol{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \bar{P}_i w_{ij}^k \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k \right) \right\} \\
&\le (1+\bar{\delta}) \sum_{k \in [N]} m_k \max_{\boldsymbol{Y}^k \in C} \left\{ \sum_{i \in [n]} \sum_{j \in [n+1]} \frac{1}{4} \bar{P}_i w_{ij} \left(1 - Y_{i(n+1)}^k + Y_{j(n+1)}^k - Y_{ij}^k \right) \right\} \\
&= \left(1+\bar{\delta}\right) \sum_{k \in [N]} m_k v_D^* \qquad \text{(A.9)} \\
&= \left(1+\bar{\delta}\right) v_D^*, \qquad \text{(A.10)}
\end{aligned}$$

where the first inequality follows because we let $\boldsymbol{\lambda} = \mathbf{0}$, the second inequality holds because $D_i^k \le (1+\bar{\delta}) D_i$ for all $k \in [N]$, the second equality follows by the definition of v_D^* in (3.9), and the third equality is due to $\sum_{k \in [N]} m_k = 1$. On the other hand, note that for any fixed $\boldsymbol{Q} \in \mathbb{R}_+^n$, $D_i^{sk}(\boldsymbol{Q}) = D_i^k + \sum_{j \in [n]} \alpha_{ji} (D_j^k - Q_j)_+$ is nondecreasing in $\boldsymbol{D}^k$. Since $(1-\underline{\delta}) D_i \le D_i^k$ for all $k \in [N]$, we have

$$D_i^{sk}(\boldsymbol{Q}) \ge (1-\underline{\delta}) D_i^s \left(\frac{\boldsymbol{Q}}{1-\underline{\delta}} \right) := (1-\underline{\delta}) \left[D_i + \sum_{j \in [n]} \alpha_{ji} \left(D_j - \frac{Q_j}{1-\underline{\delta}} \right)_+ \right].$$

Thus, by (3.11), we have

$$\begin{aligned}
v^* &= \max_{\boldsymbol{Q} \in \mathbb{R}_+^n} \left\{ \sum_{i \in [n]} \bar{P}_i Q_i - \sum_{k \in [N]} m_k \left[\sum_{i \in [n]} \bar{S}_i \left(Q_i - D_i^{sk}(\boldsymbol{Q}) \right)_+ \right] \right\} \\
&\ge \max_{\boldsymbol{Q} \in \mathbb{R}_+^n} \left\{ \sum_{i \in [n]} \bar{P}_i Q_i - \sum_{k \in [N]} m_k \left[\sum_{i \in [n]} \bar{S}_i \left(Q_i - (1-\underline{\delta}) D_i^s \left(\frac{\boldsymbol{Q}}{1-\underline{\delta}} \right) \right)_+ \right] \right\} \\
&= (1-\underline{\delta}) \max_{\boldsymbol{Q} \in \mathbb{R}_+^n} \left\{ \sum_{i \in [n]} \bar{P}_i \frac{Q_i}{1-\underline{\delta}} - \sum_{k \in [N]} m_k \left[\sum_{i \in [n]} \bar{S}_i \left(\frac{Q_i}{1-\underline{\delta}} - D_i^s \left(\frac{\boldsymbol{Q}}{1-\underline{\delta}} \right) \right)_+ \right] \right\} \\
&= (1-\underline{\delta})\, v^*(\boldsymbol{D}) \qquad \text{(A.11)}
\end{aligned}$$

where the first inequality follows because $D_i^{sk}(\boldsymbol{Q}) \geq (1-\underline{\delta})D_i^s\left(\frac{\boldsymbol{Q}}{1-\underline{\delta}}\right)$ for each $k \in [N]$, and the third equality obtained by letting $Q_i := \frac{Q_i}{1-\underline{\delta}}$ for each $i \in [n]$.

Combining (A.10) and (A.11), we have

$$v^* \geq \frac{1-\underline{\delta}}{1+\overline{\delta}} v^{LD}.$$

$\square$

Appendix B

Proofs in Chapter 4

Appendix B.1. Proof of Proposition 4.3

Proposition 4.3. *There exists an optimal solution* $\boldsymbol{Q}^*$ *to R-MNMS such that for each product* $i \in [n]$, $Q_i^* \leq M_i$, *where* $M_i = D_i + \sum_{j \in [n]} \underline{\alpha}_{ji} D_j$.

Proof. We prove the result by contradiction. Suppose for any optimal solution $\boldsymbol{Q}^*$, there exists a product $i \in [n]$ such that $Q_i^* > M_i$. Let set $\mathbb{B} := \{i \in [n] : Q_i^* > M_i\}$. Hence, $\mathbb{B} \neq \emptyset$. Let us define a new solution $\widehat{\boldsymbol{Q}}$ such that $\widehat{Q}_i = \begin{cases} M_i, & \text{if } i \in \mathbb{B} \\ Q_i^*, & \text{otherwise} \end{cases}$ for each product $i \in [n]$. Clearly, $\widehat{Q}_i \leq M_i$ for each $i \in [n]$. Then the objective value $f(\widehat{\boldsymbol{Q}})$ is equal to

$$
\begin{aligned}
f(\widehat{\boldsymbol{Q}}) =& \sum_{i \in \mathbb{B}} \overline{P}_i M_i + \sum_{i \in [n] \setminus \mathbb{B}} \overline{P}_i Q_i^* \\
& - \max_{\boldsymbol{z} \in X} \left\{ \sum_{i \in \mathbb{B}} \overline{S}_i \left(M_i - D_i + l_i z_i - \sum_{j \in \mathbb{B}} \underline{\alpha}_{ji} \left(D_j - l_j z_j - M_j\right)_+ - \sum_{j \in [n] \setminus \mathbb{B}} \underline{\alpha}_{ji} \left(D_j - l_j z_j - Q_j^*\right)_+ \right)_+ \right. \\
& \left. + \sum_{i \in [n] \setminus \mathbb{B}} \overline{S}_i \left(Q_i^* - D_i + l_i z_i - \sum_{j \in \mathbb{B}} \underline{\alpha}_{ji} \left(D_j - l_j z_j - M_j\right)_+ - \sum_{j \in [n] \setminus \mathbb{B}} \underline{\alpha}_{ji} \left(D_j - l_j z_j - Q_j^*\right)_+ \right)_+ \right\} \\
=& \sum_{i \in \mathbb{B}} \overline{P}_i M_i + \sum_{i \in [n] \setminus \mathbb{B}} \overline{P}_i Q_i^* + \sum_{i \in \mathbb{B}} \overline{S}_i \left(Q_i^* - M_i\right) \\
& - \max_{\boldsymbol{z} \in X} \left\{ \sum_{i \in \mathbb{B}} \overline{S}_i \left(Q_i^* - D_i + l_i z_i - \sum_{j \in [n] \setminus \mathbb{B}} \underline{\alpha}_{ji} \left(D_j - l_j z_j - Q_j^*\right)_+ \right) \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i\in[n]\setminus\mathbb{B}}\overline{S}_i\left(Q_i^*-D_i+l_iz_i-\sum_{j\in[n]\setminus\mathbb{B}}\underline{\alpha}_{ji}\left(D_j-l_jz_j-Q_j^*\right)_+\right)_+\Bigg\}\\
=&\sum_{i\in[n]}\overline{P}_iQ_i^*-\max_{\boldsymbol{z}\in X}\left\{\sum_{i\in[n]}\overline{S}_i\left(Q_i^*-D_i+l_iz_i-\sum_{j\in[n]}\underline{\alpha}_{ji}\left(D_j-l_jz_j-Q_j^*\right)_+\right)_+\right\}\\
&+\sum_{i\in\mathbb{B}}\left(\overline{S}_i-\overline{P}_i\right)\left(Q_i^*-M_i\right)\\
=&v^*+\sum_{i\in\mathbb{B}}\left(\overline{S}_i-\overline{P}_i\right)\left(Q_i^*-M_i\right)\\
\geq& v^*,
\end{aligned}
$$

where the second equality is because of $M_j=D_j+\sum_{i\in[n]}\underline{\alpha}_{ji}D_i$ for all $j\in\mathbb{B}$, the third equality is because for each $j\in\mathbb{B}$, we have $Q_j^*>M_j=D_j+\sum_{i\in[n]}\underline{\alpha}_{ij}D_i$, the forth equality is due to the optimality of $\boldsymbol{Q}^*$, and the last inequality holds because $\overline{S}_i\geq\overline{P}_i$ and $Q_i^*>M_i$ for each $i\in[n]$. This implies $\widehat{\boldsymbol{Q}}$ is also an optimal solution, a contradiction. □

Appendix B.2. Proof of Proposition 4.4

Theorem B.1. *The inner maximization problem (4.12) in general is NP-hard.*

Proof. We prove this result from a reduction of clique problem to be a special case of Model (4.12).

> **(Clique Problem)** Given an undirected graph $G(V,E)$, does it have a size-τ clique?

Let us consider a special instance of the inner maximization problem (4.12): suppose that there are $n=|V|+|E|$ products, and for each product $i\in E$, we let $\overline{S}_i=1,Q_i=D_i,l_i=1$, while for each product $j\in V$, we let $\overline{S}_j=1,Q_j=D_j-l_j,l_j=1$. Additionally, the substitution rate matrix $\underline{\boldsymbol{\alpha}}$ is defined as

$$
\underline{\alpha}_{ji}=\begin{cases}1, & \text{if edge } i\in E \text{ contains node } j\in V\\ 0, & \text{otherwise}\end{cases}
$$

for all $i, j \in V \cup E$. Let the budget of uncertainty $k = \frac{\tau(\tau+1)}{2}$. Under this setting, the inner maximization problem (4.12) reduces to

$$R(\boldsymbol{Q}) = \max_{\boldsymbol{z}} \sum_{i \in E} \left(z_i^{(E)} - \sum_{j \in V} \underline{\alpha}_{ji} \left(1 - z_j^{(V)}\right) \right)_+ + \sum_{j \in V} z_j^{(V)}, \tag{B.1a}$$

$$\text{s.t.} \sum_{j \in V} z_j^{(V)} + \sum_{i \in E} z_i^{(E)} \leq \frac{\tau(\tau+1)}{2}, \tag{B.1b}$$

$$z_j^{(V)}, z_i^{(E)} \in \{0, 1\}. \tag{B.1c}$$

It is sufficient to show that the Clique Problem is equivalent to Model (B.1), i.e., we only need to show the following claim.

Claim 3. There is a clique with τ nodes in the undirected graph $G(V, E)$ if and only if $R(\boldsymbol{Q}) = \frac{\tau(\tau+1)}{2}$.

Proof. Before we prove the result, let us denote $\boldsymbol{z}^*$ as an optimal solution of Model (B.1), and also define the following two sets: $V^* = \{j \in V : (z_j^{(V)})^* = 1\}$, $E^* = \{i \in E : (z_i^{(E)})^* = 1\}$, i.e., $\widehat{G}(V^*, E^*)$ is a substructure of $G(V, E)$. Note that $\widehat{G}(V^*, E^*)$ might not be a graph since we might not choose enough nodes to cover all the edges, i.e., there might exist an edge in set E^* but not both of its two nodes are selected in set V^*. Thus, $R(\boldsymbol{Q})$ is equal to

$$R(\boldsymbol{Q}) = \sum_{i \in E^*} \left(1 - \sum_{j \in V} \underline{\alpha}_{ji} \left(1 - (z_j^{(V)})^*\right) \right)_+ + \sum_{j \in V^*} (z_j^{(V)})^* = \sum_{i \in E^*} \left(1 - \sum_{j \in V \setminus V^*} \underline{\alpha}_{ji} \right)_+ + |V^*|. \tag{B.2}$$

From (B.2), we have the following inequality:

$$R(\boldsymbol{Q}) = \sum_{i \in E^*} \left(1 - \sum_{j \in V \setminus V^*} \underline{\alpha}_{ji} \right)_+ + |V^*| \leq \sum_{i \in E} \left(1 - \sum_{j \in V \setminus V^*} \underline{\alpha}_{ji} \right)_+ + |V^*| \leq \binom{|V^*|}{2} + |V^*|, \tag{B.3}$$

where the first inequality is due to $E^* \subseteq E$, and the second inequality is because of $\left(1 - \sum_{j \in V \setminus V^*} \underline{\alpha}_{ji}\right)_+ = 0$ if at least one of the two nodes from edge i is not covered by set V^*.

Now we are ready to prove the main results.

"only if". Suppose that there exists a size-τ clique (V_τ, E_τ) in the graph $G(V, E)$. Let us denote a binary vector $\widehat{\boldsymbol{z}}$ as

$$\widehat{z}_j^{(V)} = \begin{cases} 1, & j \in V_\tau \\ 0, & \text{otherwise} \end{cases}, \forall i \in V, \quad \widehat{z}_i^{(E)} = \begin{cases} 1, & i \in E_\tau \\ 0, & \text{otherwise} \end{cases}, \forall j \in E.$$

Clearly, $\widehat{\boldsymbol{z}}$ is a feasible solution to Model (B.1), with an objective value equal to $\frac{\tau(\tau+1)}{2}$. Thus, $R(\boldsymbol{Q}) \geq \frac{\tau(\tau+1)}{2}$.

Now suppose that $R(\boldsymbol{Q}) > \frac{\tau(\tau+1)}{2}$. According to the objective function (B.1a), we must have

$$|E^*| + |V^*| = \sum_{i \in E} (z_i^{(E)})^* + \sum_{j \in V} (z_j^{(V)})^* \geq R(\boldsymbol{Q}) > \frac{\tau(\tau+1)}{2}.$$

Also, the constraint (B.1b) implies that

$$|E^*| + |V^*| \leq \frac{\tau(\tau+1)}{2},$$

a contradiction.

"if". Suppose that $R(\boldsymbol{Q}) = \frac{\tau(\tau+1)}{2}$. According to (B.2), we must have

$$\frac{\tau(\tau+1)}{2} = R(\boldsymbol{Q}) = \sum_{i \in E^*} \left(1 - \sum_{j \in V \setminus V^*} \underline{\alpha}_{ji}\right)_+ + |V^*| \leq |E^*| + |V^*|$$

and by (B.3), we also have

$$\frac{\tau(\tau+1)}{2} = R(\boldsymbol{Q}) \leq \binom{|V^*|}{2} + |V^*|,$$

On the other hand, the constraint (B.1b) implies that $|V^*| + |E^*| \leq \frac{\tau(\tau+1)}{2}$. Thus, we must have $|V^*| + |E^*| = \frac{\tau(\tau+1)}{2}$. Suppose that the substructure $\widehat{G}(V^*, E^*)$ is not a clique, then there exists $i_0 = (u_0, v_0) \in E^*$ such that at least one of its nodes is not chosen, i.e., $\left(1 - \sum_{j \in V \setminus V^*} \underline{\alpha}_{ji_0}\right)_+ = 0$. Thus, by (B.2), we have

$$\frac{\tau(\tau+1)}{2} = R(\boldsymbol{Q}) = \sum_{i \in E^* \setminus \{i_0\}} \left(1 - \sum_{j \in V \setminus V^*} \underline{\alpha}_{ji}\right)_+ + |V^*| \leq |E^*| - 1 + |V^*| < \frac{\tau(\tau+1)}{2},$$

a contradiction.

□

Appendix B.3. Proof of Proposition 4.5

Theorem B.2. *Suppose $n = 2$, $k = 1$, and Assumption 5 holds, then the optimal order quantities $\boldsymbol{Q}^* = (Q_1^*, Q_2^*)$ are characterized by the following three cases:*

Case 1: If $\overline{P}_1 \leq \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \geq \overline{P}_1 \underline{\alpha}_{21}$, then $(Q_1^*, Q_2^*) = (0, D_2 - l_2 + \underline{\alpha}_{12} D_1)$.

Case 2: If $\overline{P}_2 \leq \overline{P}_1 \underline{\alpha}_{21}$ and $\overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12}$, then $(Q_1^*, Q_2^*) = (D_1 - l_1 + \underline{\alpha}_{21} D_2, 0)$.

Case 3: If $\overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \geq \overline{P}_1 \underline{\alpha}_{21}$, then we have the following two sub-cases:

Sub-case 3.1: If $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, then $(Q_1^*, Q_2^*) = \left(D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}, D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}\right)$ or $(Q_1^*, Q_2^*) = \left(D_1, D_2 - \frac{\overline{S}_2 l_2 - \overline{S}_1 l_1}{\overline{S}_2 - \overline{S}_1 \underline{\alpha}_{21}}\right)$.

Sub-case 3.2: If $\overline{S}_1 l_1 \leq \overline{S}_2 l_2$, then $(Q_1^*, Q_2^*) = \left(D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}, D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}\right)$ or $(Q_1^*, Q_2^*) = \left(D_1 - \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}}, D_2\right)$.

Proof. According to Model (4.13), we have

$$v^* = \max_{Q_1, Q_2 \geq 0} \left\{ f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 - R(\boldsymbol{Q}) \right\}, \tag{B.4}$$

and

$$
\begin{aligned}
R(\boldsymbol{Q}) &= \max_{\boldsymbol{z}\in X} \sum_{i\in[2]} \overline{S}_i \left[Q_i - D_i + l_i z_i - \sum_{j\in[2]} \underline{\alpha}_{ji} \left((D_j - l_j - Q_j)_+ z_j + (D_j - Q_j)_+ (1 - z_j) \right) \right]_+ \\
&= \max \Big\{ \overline{S}_1 \left(Q_1 - D_1 + l_1 - \underline{\alpha}_{21} (D_2 - Q_2)_+ \right)_+ + \overline{S}_2 \left(Q_2 - D_2 - \underline{\alpha}_{12} (D_1 - l_1 - Q_1)_+ \right)_+ , \\
&\quad \overline{S}_1 \left(Q_1 - D_1 - \underline{\alpha}_{21} (D_2 - l_2 - Q_2)_+ \right)_+ + \overline{S}_2 \left(Q_2 - D_2 + l_2 - \underline{\alpha}_{12} (D_1 - Q_1)_+ \right)_+ , \\
&\quad \overline{S}_1 \left(Q_1 - D_1 - \underline{\alpha}_{21} (D_2 - l_2 - Q_2)_+ \right)_+ + \overline{S}_2 \left(Q_2 - D_2 - \underline{\alpha}_{12} (D_1 - Q_1)_+ \right)_+ \Big\} \\
&= \max \Big\{ \overline{S}_1 \left(Q_1 - D_1 + l_1 - \underline{\alpha}_{21} (D_2 - Q_2)_+ \right)_+ + \overline{S}_2 \left(Q_2 - D_2 - \underline{\alpha}_{12} (D_1 - l_1 - Q_1)_+ \right)_+ , \\
&\quad \overline{S}_1 \left(Q_1 - D_1 - \underline{\alpha}_{21} (D_2 - l_2 - Q_2)_+ \right)_+ + \overline{S}_2 \left(Q_2 - D_2 + l_2 - \underline{\alpha}_{12} (D_1 - Q_1)_+ \right)_+ \Big\},
\end{aligned}
$$

where the second equality is due to $X = \left\{ \boldsymbol{z} : \sum_{i\in[2]} z_i \leq 1, z_i \in \{0,1\}, \forall i \in [2] \right\} = \{(0,1),(1,0),(0,0)\}$ and the third equality is because $\overline{S}_1 \left(Q_1 - D_1 - \underline{\alpha}_{21} (D_2 - l_2 - Q_2)_+ \right)_+ + \overline{S}_2 \left(Q_2 - D_2 - \underline{\alpha}_{12} (D_1 - Q_1)_+ \right)_+$ is dominated by the other two.

Note that for each $i \in [2]$, the optimal order quantity Q_i^* must belong to one of the three intervals $[0, D_i - l_i]$, $[D_i - l_i, D_i]$, or $[D_i, +\infty)$. Thus, we can divide the feasible region into 9 subregions (see Section B.3 for an illustration), where under each subregion, function $R(\boldsymbol{Q})$ becomes piecewise convex, thus Model (B.4) is solvable. Therefore, we can optimize Model (B.4) over each subregion, and the solution with the largest objective value corresponds to an optimal solution to the original problem (B.4). Therefore, we need to discuss the 9 cases, corresponding to 9 subregions.

Before we derive the main results, we observe a characterization of an optimal solution of maximizing a piecewise concave function over a box.

Observation 1. *Given an integer number τ, consider the following piecewise concave optimization program:*

$$\max_{\boldsymbol{x}} \left\{ \min_{i\in[\tau]} (\boldsymbol{c}^i)^\top \boldsymbol{x} : \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{U} \right\}.$$

Then an optimal solution of the above optimization problem can be one of the following points:

(i) the extreme points of the box $[\boldsymbol{0}, \boldsymbol{U}]$*; and*

(ii) the intersection point of any affine system $(\boldsymbol{c}^i)^\top \boldsymbol{x} = (\boldsymbol{c}^j)^\top \boldsymbol{x}$ *for all* $i, j \in \mathbb{B} \subseteq [\tau]$ *with* $2 \le |\mathbb{B}| \le n$ *and the boundary of the box* $[\boldsymbol{0}, \boldsymbol{U}]$*.*

(iii) the unique solution of the affine system $(\boldsymbol{c}^i)^\top \boldsymbol{x} = (\boldsymbol{c}^j)^\top \boldsymbol{x}$ *for all* $i, j \in \mathbb{B}$ *with* $|\mathbb{B}| = n+1$*, which is in the box* $[\boldsymbol{0}, \boldsymbol{U}]$*.*

Proof. Note that the piecewise concave optimization program can be written as the following linear program:

$$\max_{\boldsymbol{x}} \left\{ w : w \le (\boldsymbol{c}^i)^\top \boldsymbol{x}, \forall i \in [\tau], \boldsymbol{0} \le \boldsymbol{x} \le \boldsymbol{U} \right\}.$$

The conclusion follows by the fact that one optimal solution of the above linear program must be an extreme point, and condition (i), (ii), and (iii) exactly characterize all the extreme points. ◇

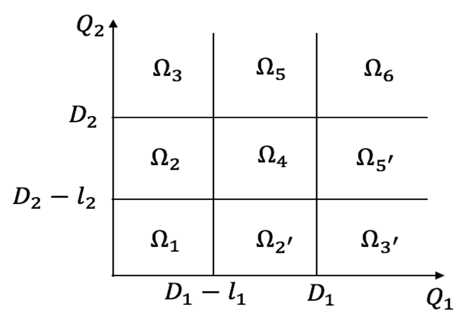


Figure B.1: Decomposition of feasible solution regions into 9 sub-regions

.

In Figure B.1, $\Omega_1 = \{(Q_1, Q_2) : 0 \le Q_1 \le D_1 - l_1, 0 \le Q_2 \le D_2 - l_2\}, \Omega_2 = \{(Q_1, Q_2) : 0 \le Q_1 \le D_1 - l_1, D_2 - l_2 \le Q_2 \le D_2\}, \Omega_3 = \{(Q_1, Q_2) : 0 \le Q_1 \le D_1 - l_1, D_2 \le Q_2\}, \Omega_4 = \{(Q_1, Q_2) : D_1 - l_1 \le Q_1 \le D_1, D_2 - l_2 \le Q_2 \le D_2\}, \Omega_5 = \{(Q_1, Q_2) : D_1 - l_1 \le Q_1 \le D_1, D_2 \le Q_2\}, \Omega_6 = \{(Q_1, Q_2) : D_1 \le Q_1, D_2 \le Q_2\}, \Omega_{2'} = \{(Q_1, Q_2) : D_1 - l_1 \le Q_1 \le D_1, 0 \le Q_2 \le D_2 - l_2\}, \Omega_{3'} = \{(Q_1, Q_2) : D_1 \le Q_1, 0 \le Q_2 \le D_2 - l_2\}, \Omega_{5'} = \{(Q_1, Q_2) : D_1 \le Q_1, D_2 - l_2 \le Q_2 \le D_2\}$

.

Now we are ready to discuss the following 9 cases.

Case 1. Suppose $(Q_1, Q_2) \in \Omega_1$**, i.e.,** $0 \le Q_1 \le D_1 - l_1$**,** $0 \le Q_2 \le D_2 - l_2$**.**

In this case, $R(\boldsymbol{Q}) = 0$ and Model (B.4) becomes:

$$\max_{\boldsymbol{Q}} \left\{ f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 : 0 \le Q_1 \le D_1 - l_1, 0 \le Q_2 \le D_2 - l_2 \right\}.$$

Clearly, the optimal solution of the above linear program is $\boldsymbol{t}_1^1 = (D_1 - l_1, D_2 - l_2)$, and its optimal total profit is

$$f(\boldsymbol{t}_1^1) = \overline{P}_1(D_1 - l_1) + \overline{P}_2(D_2 - l_2).$$

Case 2. Suppose $(Q_1, Q_2) \in \Omega_2$**, i.e.,** $0 \le Q_1 \le D_1 - l_1, D_2 - l_2 \le Q_2 \le D_2$**.**

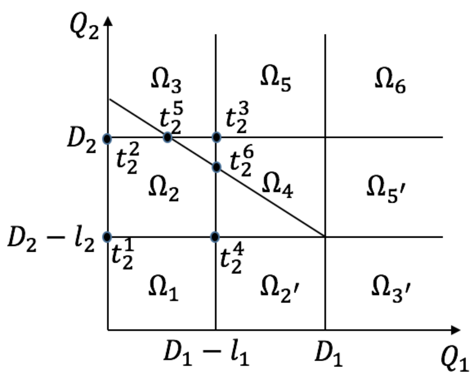


Figure B.2: Illustration of possible solutions of Case 2.

In this case, we have $R(\boldsymbol{Q}) = \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))_+$ and Model (B.4) becomes:

$$\max_{\boldsymbol{Q}} \{ f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 - \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))_+ : 0 \le Q_1 \le D_1 - l_1, D_2 - l_2 \le Q_2 \le D_2 \}.$$

According to Observation 1, the optimal solution of above optimization problem can be one of the following points: (1) extreme points of Ω_2 i.e., $\boldsymbol{t}_2^1, \boldsymbol{t}_2^2, \boldsymbol{t}_2^3, \boldsymbol{t}_2^4$; and (2) the intersection points of linear equation $Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1) = 0$ with feasible regions, i.e., $\boldsymbol{t}_2^5, \boldsymbol{t}_2^6$ (see Figure B.2 for an illustration). These potential optimal solutions and their corresponding total profits are listed in Table B.1.

Table B.1: The possible solutions and their total profits in Case 2

Possible solutions	Total profit
$\boldsymbol{t}_2^1 = (0, D_2 - l_2)$	$f(\boldsymbol{t}_2^1) = \overline{P}_2(D_2 - l_2)$
$\boldsymbol{t}_2^2 = (0, D_2)$	$f(\boldsymbol{t}_2^2) = \overline{P}_2 D_2$
$\boldsymbol{t}_2^3 = (D_1 - l_1, D_2)$	$f(\boldsymbol{t}_2^3) = \overline{P}_1(D_1 - l_1) + \overline{P}_2 D_2 - \overline{S}_2(l_2 - \underline{\alpha}_{12} l_1)$
$\boldsymbol{t}_1^1 = \boldsymbol{t}_2^4 = (D_1 - l_1, D_2 - l_2)$	$f(\boldsymbol{t}_2^4) = f(\boldsymbol{t}_1^1) = \overline{P}_1(D_1 - l_1) + \overline{P}_2(D_2 - l_2)$
$\boldsymbol{t}_2^5 = (D_1 - l_2/\underline{\alpha}_{12}, D_2)$	$f(\boldsymbol{t}_2^5) = \overline{P}_1(D_1 - l_2/\underline{\alpha}_{12}) + \overline{P}_2 D_2$
$\boldsymbol{t}_2^6 = (D_1 - l_1, D_2 - l_2 + \underline{\alpha}_{12} l_1)$	$f(\boldsymbol{t}_2^6) = \overline{P}_1(D_1 - l_1) + \overline{P}_2(D_2 - l_2 + \underline{\alpha}_{12} l_1)$

It remains to compare these solutions. Clearly, we have

- $f(\boldsymbol{t}_2^2) \geq f(\boldsymbol{t}_2^1)$, since $\overline{P}_2, l_2 \geq 0$,
- $f(\boldsymbol{t}_2^5) - f(\boldsymbol{t}_2^2) = \overline{P}_1(D_1 - l_2/\underline{\alpha}_{12}) \geq 0$ due to Assumption 5 that $\underline{\alpha}_{12} D_1 - l_2 \geq 0$,
- $f(\boldsymbol{t}_2^6) - f(\boldsymbol{t}_2^3) = \overline{P}_2(-l_2 + \underline{\alpha}_{12} l_1) + \overline{S}_2(l_2 - \alpha_{12} l_1) = (-\overline{P}_2 + \overline{S}_2)(l_2 - \underline{\alpha}_{12} l_1) \geq 0$ due to $\overline{S}_2 \geq \overline{P}_2$ and Assumption 5 that $l_2 - \underline{\alpha}_{12} l_1 \geq 0$,
- $f(\boldsymbol{t}_2^6) - f(\boldsymbol{t}_2^4) = \overline{P}_2 \underline{\alpha}_{12} l_1 \geq 0$, and
- $f(\boldsymbol{t}_2^6) - f(\boldsymbol{t}_2^5) = -\overline{P}_1 l_1 + \overline{P}_2(-l_2 + \underline{\alpha}_{12} l_1) + \overline{P}_1 \frac{l_2}{\underline{\alpha}_{12}} = \frac{1}{\underline{\alpha}_{12}}(\overline{P}_1 - \overline{P}_2 \underline{\alpha}_{12})(l_2 - \underline{\alpha}_{12} l_1) \begin{cases} \geq 0, & \text{if } \overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12} \\ \leq 0, & \text{if } \overline{P}_1 \leq \overline{P}_2 \underline{\alpha}_{12} \end{cases}$,

  due to Assumption 5 that $l_2 - \underline{\alpha}_{12} l_1 \geq 0$.

From the above comparison, we can draw the following conclusions on the best solution in the subregions Ω_1 and Ω_2:

(i) If $\overline{P}_1 - \overline{P}_2\underline{\alpha}_{12} \leq 0$, then the point $\boldsymbol{t}_2^5$ dominates the other points in the subregions Ω_1 and Ω_2, since $f(\boldsymbol{t}_2^5) \geq f(\boldsymbol{t}_2^2) \geq f(\boldsymbol{t}_2^1), f(\boldsymbol{t}_2^5) \geq f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_2^4)$, and $f(\boldsymbol{t}_2^5) \geq f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_2^3)$.

(ii) If $\overline{P}_1 - \overline{P}_2\underline{\alpha}_{12} \geq 0$, then the point $\boldsymbol{t}_2^6$ dominates the other points in the subregions Ω_1 and Ω_2, since $f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_2^5) \geq f(\boldsymbol{t}_2^2) \geq f(\boldsymbol{t}_2^1), f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_2^4)$, and $f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_2^3)$.

Case 3. Suppose $(Q_1, Q_2) \in \Omega_3$**, i.e.,** $0 \leq Q_1 \leq D_1 - l_1, D_2 \leq Q_2$**.**

In this case, $R(\boldsymbol{Q}) = \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))_+$ and Model (B.4) becomes:

$$\max_{\boldsymbol{Q}} \left\{ f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 - \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))_+ : 0 \leq Q_1 \leq D_1 - l_1, D_2 \leq Q_2 \right\}.$$

According to Observation 1, the optimal solution can be one of the following points: (1) the extreme points in Ω_3, i.e., $\boldsymbol{t}_3^1, \boldsymbol{t}_3^2$; and (2) the intersection points of linear $Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1) = 0$ and the boundary of Ω_3, i.e., $\boldsymbol{t}_3^3, \boldsymbol{t}_3^4$ (see Figure B.3 for an illustration). These solutions and their corresponding total profits are listed in Table B.2.

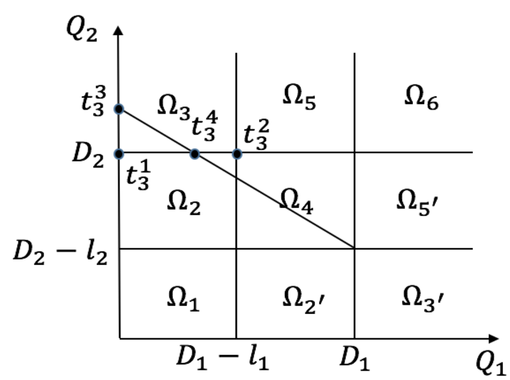


Figure B.3: Possible solutions in Case 3

Table B.2: Possible solutions and their total profits in Case 3

Possible solutions	Total profit
$\boldsymbol{t}_3^1 = \boldsymbol{t}_2^2 = (0, D_2)$	$f(\boldsymbol{t}_3^1) = f(\boldsymbol{t}_2^2) = \overline{P}_2 D_2$
$\boldsymbol{t}_3^2 = \boldsymbol{t}_2^3 = (D_1 - l_1, D_2)$	$f(\boldsymbol{t}_3^2) = f(\boldsymbol{t}_2^3) = \overline{P}_1(D_1 - l_1) + \overline{P}_2 D_2 - \overline{S}_2(l_2 - \underline{\alpha}_{12} l_1)$
$\boldsymbol{t}_3^3 = (0, D_2 - l_2 + \underline{\alpha}_{12} D_1)$	$f(\boldsymbol{t}_3^3) = \overline{P}_2(D_2 - l_2 + \underline{\alpha}_{12} D_1)$
$\boldsymbol{t}_3^4 = \boldsymbol{t}_2^5 = (D_1 - l_2/\underline{\alpha}_{12}, D_2)$	$f(\boldsymbol{t}_3^4) = f(\boldsymbol{t}_2^5) = \overline{P}_1(D_1 - l_2/\underline{\alpha}_{12}) + \overline{P}_2 D_2$

In view of the results in Case 1 and Case 2, it remains to compare $f(\boldsymbol{t}_2^5), f(\boldsymbol{t}_3^3)$ and also $f(\boldsymbol{t}_2^6), f(\boldsymbol{t}_3^3)$. Clearly, we have

- $$\begin{aligned} f(\boldsymbol{t}_2^5) - f(\boldsymbol{t}_3^3) &= \overline{P}_1(D_1 - l_2/\underline{\alpha}_{12}) - \overline{P}_2(-l_2 + \underline{\alpha}_{12} D_1) \\ &= \frac{1}{\underline{\alpha}_{12}}(\overline{P}_1 - \overline{P}_2\underline{\alpha}_{12})(\underline{\alpha}_{12} D_1 - l_2) \begin{cases} \geq 0, & \text{if } \overline{P}_1 \geq \overline{P}_2\underline{\alpha}_{12} \\ \leq 0, & \text{if } \overline{P}_1 \leq \overline{P}_2\underline{\alpha}_{12} \end{cases}, \end{aligned}$$

  due to Assumption 5 that $\underline{\alpha}_{12} D_1 \geq l_2$,

- $f(\boldsymbol{t}_2^6) - f(\boldsymbol{t}_3^3) = \overline{P}_1(D_1 - l_1) + \overline{P}_2\underline{\alpha}_{12}(l_1 - D_1) = (\overline{P}_1 - \overline{P}_2\underline{\alpha}_{12})(D_1 - l_1) \begin{cases} \geq 0, & \text{if } \overline{P}_1 \geq \overline{P}_2\underline{\alpha}_{12} \\ \leq 0, & \text{if } \overline{P}_1 \leq \overline{P}_2\underline{\alpha}_{12} \end{cases}$

  since $D_l \geq l_1$,

From the above comparison as well as the results of Case 1 and Case 2, we can draw the following conclusion on the best solution in subregions Ω_1, Ω_2 and Ω_3:

(i) If $\overline{P}_1 \geq \overline{P}_2\underline{\alpha}_{12}$, then $\boldsymbol{t}_2^6$ dominates all the other points in Ω_1, Ω_2, and Ω_3 since $f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_2^5) \geq f(\boldsymbol{t}_3^3)$.

(ii) If $\overline{P}_1 \leq \overline{P}_2 \underline{\alpha}_{12}$, then $\boldsymbol{t}_3^3$ dominates all the other points in Ω_1, Ω_2, and Ω_3 since $f(\boldsymbol{t}_2^6) \leq f(\boldsymbol{t}_2^5) \leq f(\boldsymbol{t}_3^3)$.

Case 4. Suppose $(Q_1, Q_2) \in \Omega_4$**, i.e.,** $D_1 - l_1 \leq Q_1 \leq D_1, D_2 - l_2 \leq Q_2 \leq D_2$**.**

In this case, $R(\boldsymbol{Q}) = \max\left\{\overline{S}_1(Q_1 - D_1 + l_1 - \underline{\alpha}_{21}(D_2 - Q_2))_+, \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))_+\right\}$ and Model (B.4) becomes:

$$\max_{\boldsymbol{Q}} \left\{f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 - R(\boldsymbol{Q}) : D_1 - l_1 \leq Q_1 \leq D_1, D_2 - l_2 \leq Q_2 \leq D_2\right\}.$$

According to Observation 1, the optimal solution can be one of the following points: (1) the extreme points of Ω_4, i.e., $\boldsymbol{t}_4^1, \boldsymbol{t}_4^2, \boldsymbol{t}_4^3, \boldsymbol{t}_4^4$; (2) the intersection points of the line $Q_1 - D_1 + l_1 - \underline{\alpha}_{11}(D_2 - Q_2) = 0$ with the boundary of Ω_4, i.e., $\boldsymbol{t}_4^2, \boldsymbol{t}_4^5$; (3) the intersection points of the line $Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1) = 0$ with the boundary of Ω_4, i.e., $\boldsymbol{t}_4^4, \boldsymbol{t}_4^6$; and (4) the intersection point of two lines $\overline{S}_1(Q_1 - D_1 + l_1 - \underline{\alpha}_{21}(D_2 - Q_2))_+ = 0$ and $\overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))_+ = 0$, i.e., $\boldsymbol{t}_4^7$ (see Figure B.4 for an illustration). These solutions and their total profits are listed in Table B.3.

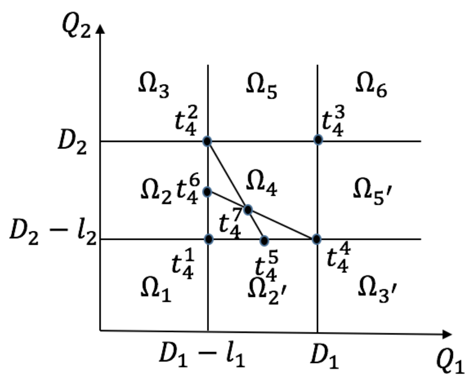


Figure B.4: Possible solutions in Case 4

Table B.3: Possible solutions and their total profits in Case 4

Possible solutions	Total profit
$\boldsymbol{t}_4^1 = \boldsymbol{t}_2^4 = \boldsymbol{t}_1^1 = (D_1 - l_1, D_2 - l_2)$	$f(\boldsymbol{t}_4^1) = f(\boldsymbol{t}_2^4) = f(\boldsymbol{t}_1^1) = \overline{P}_1(D_1 - l_1) + \overline{P}_2(D_2 - l_2)$
$\boldsymbol{t}_4^2 = \boldsymbol{t}_3^2 = \boldsymbol{t}_2^3 = (D_1 - l_1, D_2)$	$f(\boldsymbol{t}_4^2) = f(\boldsymbol{t}_3^2) = f(\boldsymbol{t}_2^3)$ $= \overline{P}_1(D_1 - l_1) + \overline{P}_2 D_2 - \overline{S}_2(l_2 - \underline{\alpha}_{12} l_1)$
$\boldsymbol{t}_4^3 = (D_1, D_2)$	$f(\boldsymbol{t}_4^3) = \overline{P}_1 D_1 + \overline{P}_2 D_2 - \max\left\{\overline{S}_1 l_1, \overline{S}_2 l_2\right\}$
$\boldsymbol{t}_4^4 = (D_1, D_2 - l_2)$	$f(\boldsymbol{t}_4^4) = \overline{P}_1 D_1 + \overline{P}_2(D_2 - l_2) - \overline{S}_1(l_1 - \underline{\alpha}_{21} l_2)$
$\boldsymbol{t}_4^5 = (D_1 - l_1 + \underline{\alpha}_{21} l_2, D_2 - l_2)$	$f(\boldsymbol{t}_4^5) = \overline{P}_1(D_1 - l_1 + \underline{\alpha}_{21} l_2) + \overline{P}_2(D_2 - l_2)$
$\boldsymbol{t}_4^6 = \boldsymbol{t}_2^6 = (D_1 - l_1, D_2 - l_2 + \underline{\alpha}_{12} l_1)$	$f(\boldsymbol{t}_4^6) = f(\boldsymbol{t}_2^6) = \overline{P}_1(D_1 - l_1) + \overline{P}_2\left(D_2 - l_2 + \underline{\alpha}_{12} l_1\right)$
$\boldsymbol{t}_4^7 = \left(D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}, D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}\right)$	$f(\boldsymbol{t}_4^7) = \overline{P}_1\left(D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}\right) + \overline{P}_2\left(D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}\right)$

In view of the results in Case 1- Case 3, we know that point $\boldsymbol{t}_2^6$ or $\boldsymbol{t}_3^3$ dominates all the other points of Ω_1, Ω_2, and Ω_3, so we only need to compare $f(\boldsymbol{t}_2^6)$, $f(\boldsymbol{t}_3^3)$, $f(\boldsymbol{t}_4^3)$, $f(\boldsymbol{t}_4^4)$, $f(\boldsymbol{t}_4^5)$, $f(\boldsymbol{t}_4^7)$.

- $f(\boldsymbol{t}_4^4) - f(\boldsymbol{t}_4^5) = (\overline{P}_1 - \overline{S}_1)(l_1 - \underline{\alpha}_{21} l_2) \leq 0$ due to Assumption 5 that $l_1 - \underline{\alpha}_{21} l_2 \geq 0$ and the fact that $\overline{P}_1 \leq \overline{S}_1$,

- If $\overline{P}_1 \leq \overline{P}_2 \underline{\alpha}_{12}$, then

$$
\begin{aligned}
f(\boldsymbol{t}_4^3) - f(\boldsymbol{t}_3^3) &= \overline{P}_1 D_1 - \max\{\overline{S}_1 l_1, \overline{S}_2 l_2\} + \overline{P}_2 l_2 - \overline{P}_2 \underline{\alpha}_{12} D_1 \\
&= (\overline{P}_1 - \overline{P}_2 \underline{\alpha}_{12}) D_1 + (\overline{P}_2 l_2 - \max\{\overline{S}_1 l_1, \overline{S}_2 l_2\}) \leq 0,
\end{aligned}
$$

  where the inequality is because of $D_1 \geq 0$ and $\overline{P}_2 l_2 \leq \overline{S}_2 l_2 \leq \max\{\overline{S}_1 l_1, \overline{S}_2 l_2\}$. Otherwise, $f(\boldsymbol{t}_4^3)$ and $f(\boldsymbol{t}_3^3)$ are incomparable.

- Compare $f(\boldsymbol{t}_2^6)$ with $f(\boldsymbol{t}_4^7)$

$$
\begin{aligned}
f(\boldsymbol{t}_2^6) - f(\boldsymbol{t}_4^7) &= \overline{P}_2(-l_2 + \underline{\alpha}_{12} l_1) + \overline{P}_1(-l_1) + \overline{P}_1 \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} + \overline{P}_2 \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} \\
&= -\underline{\alpha}_{21} \frac{(\overline{P}_1 - \overline{P}_2\underline{\alpha}_{12})(l_2 - l_1\underline{\alpha}_{12})}{1 - \underline{\alpha}_{21}\underline{\alpha}_{21}} \begin{cases} \leq 0, & \text{if } \overline{P}_1 \geq \overline{P}_2\underline{\alpha}_{12} \\ \geq 0, & \text{if } \overline{P}_1 \leq \overline{P}_2\underline{\alpha}_{12} \end{cases},
\end{aligned}
$$

where the inequalities are due to Assumption 5 that $l_2 \geq \underline{\alpha}_{12} l_1$ and the fact that $0 \leq \underline{\alpha}_{21} \leq 1, 0 \leq \underline{\alpha}_{12} \leq 1$.

- If $\overline{P}_1 \leq \overline{P}_2\underline{\alpha}_{12}$, then

$$
f(\boldsymbol{t}_4^5) - f(\boldsymbol{t}_3^3) = \overline{P}_1(D_1 - l_1 + \underline{\alpha}_{21} l_2) - \overline{P}_2\underline{\alpha}_{12} D_1 = (\overline{P}_1 - \overline{P}_2\underline{\alpha}_{12}) D_1 - \overline{P}_1(l_1 - \underline{\alpha}_{21} l_2) \leq 0,
$$

where the inequality is due to Assumption 5 that $l_1 \geq \underline{\alpha}_{21} l_2$ and the fact that $\overline{P}_1 \geq 0, D_1 \geq 0$.

- Compare $f(\boldsymbol{t}_2^6)$ with $f(\boldsymbol{t}_3^3)$

$$
\begin{aligned}
f(\boldsymbol{t}_2^6) - f(\boldsymbol{t}_3^3) &= \overline{P}_1(D_1 - l_1) + \overline{P}_2\underline{\alpha}_{12} l_1 - \overline{P}_2\underline{\alpha}_{12} D_1 = (\overline{P}_1 - \overline{P}_2\underline{\alpha}_{12})(D_1 - l_1) \\
&\begin{cases} \geq 0, & \text{if } \overline{P}_1 \geq \overline{P}_2\underline{\alpha}_{12} \\ \leq 0, & \text{if } \overline{P}_1 \leq \overline{P}_2\underline{\alpha}_{12} \end{cases},
\end{aligned}
$$

where the inequalities are due to the fact that $D_1 \geq l_1$.

- Compare $f(\boldsymbol{t}_4^5)$ with $f(\boldsymbol{t}_4^7)$

$$
\begin{aligned}
f(\boldsymbol{t}_4^5) - f(\boldsymbol{t}_4^7) &= \overline{P}_1(-l_1 + \underline{\alpha}_{21} l_2) + \overline{P}_2(-l_2) + \overline{P}_1 \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} + \overline{P}_2 \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} \\
&= -\underline{\alpha}_{12} \frac{(\overline{P}_2 - \overline{P}_1\underline{\alpha}_{21})(l_1 - l_2\underline{\alpha}_{21})}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} \begin{cases} \leq 0, & \text{if } \overline{P}_2 \geq \overline{P}_1\underline{\alpha}_{21} \\ \geq 0, & \text{if } \overline{P}_2 \leq \overline{P}_1\underline{\alpha}_{21} \end{cases},
\end{aligned}
$$

where the inequalities are due to Assumption 5 that $l_1 \geq \underline{\alpha}_{21} l_2$ and the fact that $0 \leq \underline{\alpha}_{21} \leq 1, 0 \leq \underline{\alpha}_{12} \leq 1$.

- Compare $f(\boldsymbol{t}_2^5)$ with $f(\boldsymbol{t}_4^7)$

$$f(\boldsymbol{t}_2^5) - f(\boldsymbol{t}_4^7) = -\overline{P}_1 \frac{l_2}{\underline{\alpha}_{12}} + \overline{P}_1 \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} + \overline{P}_2 \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} = -\frac{(\overline{P}_1 - \overline{P}_2 \underline{\alpha}_{12})(l_2 - l_1 \underline{\alpha}_{12})}{\underline{\alpha}_{12}(1 - \underline{\alpha}_{21}\underline{\alpha}_{21})}$$
$$\begin{cases} \leq 0, & \text{if } \overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12} \\ \geq 0, & \text{if } \overline{P}_1 \leq \overline{P}_2 \underline{\alpha}_{12} \end{cases},$$

where the inequalities are due to Assumption 5 that $l_2 \geq \underline{\alpha}_{12} l_1$ and the fact that $0 \leq \underline{\alpha}_{21} \leq 1, 0 \leq \underline{\alpha}_{12} \leq 1$.

From the above comparison as well as the results of Case 1-Case 3, we can draw the following conclusion on the best solution in subregions Ω_1, Ω_2, Ω_3, and Ω_4:

(i) If $\overline{P}_1 \leq \overline{P}_2 \underline{\alpha}_{12}$, then we must have $\overline{P}_2 \geq \overline{P}_1 \underline{\alpha}_{21}$ since $\underline{\alpha}_{12}, \underline{\alpha}_{21} \in [0, 1]$, and $\boldsymbol{t}_3^3$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, and Ω_4, since $f(\boldsymbol{t}_3^3) \geq f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_4^7)$, $f(\boldsymbol{t}_4^5) \leq f(\boldsymbol{t}_3^3)$, and $f(\boldsymbol{t}_4^3) \leq f(\boldsymbol{t}_3^3)$.

(ii) If $\overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \leq \overline{P}_1 \underline{\alpha}_{21}$, then $\boldsymbol{t}_4^3$ or $\boldsymbol{t}_4^5$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, and Ω_4, since $f(\boldsymbol{t}_4^5) \geq f(\boldsymbol{t}_4^7) \geq f(\boldsymbol{t}_2^5)$, and $f(\boldsymbol{t}_4^7) \geq f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_3^3)$.

(iii) If $\overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \geq \overline{P}_1 \underline{\alpha}_{21}$, then $\boldsymbol{t}_4^3$ or $\boldsymbol{t}_4^7$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, and Ω_4, since $f(\boldsymbol{t}_4^7) \geq f(\boldsymbol{t}_3^3)$, $f(\boldsymbol{t}_4^7) \geq f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_3^3)$, and $f(\boldsymbol{t}_4^7) \geq f(\boldsymbol{t}_4^5)$.

Case 5. Suppose $(Q_1, Q_2) \in \Omega_5$**, i.e.,** $D_1 - l_1 \leq Q_1 \leq D_1, D_2 \leq Q_2$**.**

In this case, $R(\boldsymbol{Q}) = \max\left\{\overline{S}_1(Q_1 - D_1 + l_1) + \overline{S}_2(Q_2 - D_2), \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))\right\}$ and Model (B.4) becomes:

$$\max_{\boldsymbol{Q}} \left\{f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 - R(\boldsymbol{Q}) : D_1 - l_1 \leq Q_1 \leq D_1, D_2 \leq Q_2\right\}.$$

According to Observation 1, the optimal solution can be one of the following points: (1) the extreme points of Ω_5, i.e, $\boldsymbol{t}_5^1, \boldsymbol{t}_5^2$; (2) the intersection point of line $\overline{S}_1(Q_1 - D_1 + l_1) + \overline{S}_2(Q_2 - D_2) = \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))$ and the boundary of Ω_5, i.e., $\boldsymbol{t}_5^3$ (See Figure B.5 for an illustration). Note that $\boldsymbol{t}_5^3 \in \Omega_5$ if $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, otherwise, $\boldsymbol{t}_5^3 \notin \Omega_5$. These possible solutions are listed in Table B.4.

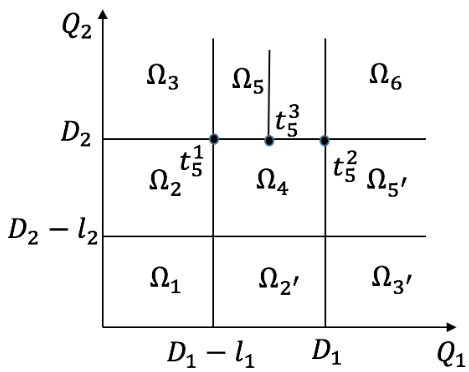


Figure B.5: Possible solutions in Case 5

Table B.4: Possible solutions and their total profits in Case 5

Possible solutions	Total profit
$\boldsymbol{t}_5^1 = \boldsymbol{t}_4^2 = \boldsymbol{t}_3^2 = \boldsymbol{t}_2^3 = (D_1 - l_1, D_2)$	$f(\boldsymbol{t}_5^1) = f(\boldsymbol{t}_4^2) = f(\boldsymbol{t}_3^2) = f(\boldsymbol{t}_2^3)$ $= \overline{P}_1(D_1 - l_1) + \overline{P}_2 D_2 - \overline{S}_2(l_2 - \underline{\alpha}_{12} l_1)$
$\boldsymbol{t}_5^2 = \boldsymbol{t}_4^3 = (D_1, D_2)$	$f(\boldsymbol{t}_5^2) = f(\boldsymbol{t}_4^3) = \overline{P}_1 D_1 + \overline{P}_2 D_2 - \max\left\{\overline{S}_1 l_1, \overline{S}_2 l_2\right\}$
$\boldsymbol{t}_5^3 = (D_1 - \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}}, D_2)$	$f(\boldsymbol{t}_5^3) = \overline{P}_1 D_1 + \overline{P}_2 D_2 - \overline{P}_1 \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} - \overline{S}_1 \overline{S}_2 \frac{l_2 - l_1 \underline{\alpha}_{12}}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}}$

In view of the results in Case 1- Case 4, the only new point is $\boldsymbol{t}_5^3$, which is in subregion 5 if $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$. Thus, suppose that $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, we will compare $f(\boldsymbol{t}_5^3)$ with $f(\boldsymbol{t}_3^3)$, $f(\boldsymbol{t}_4^3)$.

- Compare $f(\boldsymbol{t}_4^3)$ with $f(\boldsymbol{t}_5^3)$:

$$f(\boldsymbol{t}_4^3) - f(\boldsymbol{t}_5^3) = \overline{P}_1 D_1 + \overline{P}_2 D_2 - \overline{S}_1 l_1 - \overline{P}_1 D_1 - \overline{P}_2 D_2 + \overline{P}_1 \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} + \overline{S}_1 \overline{S}_2 \frac{l_2 - l_1 \underline{\alpha}_{12}}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}}$$

$$= -\frac{(\overline{S}_1 - \overline{P}_1)(\overline{S}_1 l_1 - \overline{S}_2 l_2)}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} \leq 0,$$

where the inequality is due to $\overline{S}_1 \geq \overline{P}_1$ and $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$.

- Compare $f(\boldsymbol{t}_3^3)$ with $f(\boldsymbol{t}_5^3)$

$$\begin{aligned}
f(\boldsymbol{t}_3^3) - f(\boldsymbol{t}_5^3) &= \overline{P}_2\left(D_2 - l_2 + \underline{\alpha}_{12} D_1\right) - \overline{P}_1 D_1 - \overline{P}_2 D_2 + \overline{P}_1 \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} + \overline{S}_1 \overline{S}_2 \frac{l_2 - l_1 \underline{\alpha}_{12}}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} \\
&\geq \overline{P}_2\left(-l_2 + \underline{\alpha}_{12} D_1\right) - \overline{P}_1 D_1 + \overline{P}_1 \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} + \overline{S}_1 \overline{P}_2 \frac{l_2 - l_1 \underline{\alpha}_{12}}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} \\
&= \left(\overline{P}_2 \underline{\alpha}_{12} - \overline{P}_1\right)\left(D_1 - \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}}\right).
\end{aligned}$$

Since $D_1 - \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} \geq D_1 - l_1 \geq 0$ and $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, thus, $f(\boldsymbol{t}_3^3) \geq f(\boldsymbol{t}_5^3)$, if $\overline{P}_1 \leq \overline{P}_2 \underline{\alpha}_{12}$.

From the above comparison results as well as the results of Case 1-Case 4, we can draw the following conclusion on the best solution in subregions Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5:

(i) If $\overline{P}_1 \leq \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \geq \overline{P}_1 \underline{\alpha}_{21}$, then $\boldsymbol{t}_3^3$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5, since $f(\boldsymbol{t}_3^3) \geq f(\boldsymbol{t}_5^3)$.

(ii) If $\overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \leq \overline{P}_1 \underline{\alpha}_{21}$, then

  (a) if $\overline{S}_1 l_1 \leq \overline{S}_2 l_2$, then $\boldsymbol{t}_4^3$ or $\boldsymbol{t}_4^5$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5, and

  (b) if $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, then $\boldsymbol{t}_5^3$ or $\boldsymbol{t}_4^5$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5.

(iii) If $\overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \geq \overline{P}_1 \underline{\alpha}_{21}$, then

  (a) if $\overline{S}_1 l_1 \leq \overline{S}_2 l_2$, then $\boldsymbol{t}_4^3$ or $\boldsymbol{t}_4^7$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5, and

(b) if $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, then $\boldsymbol{t}_5^3$ or $\boldsymbol{t}_4^7$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5.

Case 6. Suppose $(Q_1, Q_2) \in \Omega_6$**, i.e.,** $D_1 \leq Q_1, D_2 \leq Q_2$**.**

In this case, $R(\boldsymbol{Q}) = \max\{\overline{S}_1(Q_1 - D_1 + l_1) + \overline{S}_2(Q_2 - D_2), \overline{S}_1(Q_1 - D_1) + \overline{S}_2(Q_2 - D_2 + l_2)\}$ and Model (B.4) becomes:

$$\max_{\boldsymbol{Q}} \left\{ f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 - R(\boldsymbol{Q}) : D_1 \leq Q_1, D_2 \leq Q_2 \right\}.$$

According to Observation 1, the optimal solution can only be $\boldsymbol{t}_6^1$, which is listed in Table B.5.

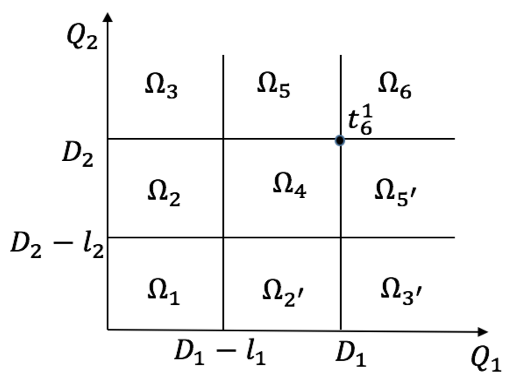


Figure B.6: Possible solutions in Case 6

Table B.5: Possible solutions and their total profits in Case 6

Possible solutions	Total profit
$t_6^1 = t_5^2 = t_4^3 = (D_1, D_2)$	$f(\boldsymbol{t}_6^1) = f(\boldsymbol{t}_5^2) = f(\boldsymbol{t}_4^3) = \overline{P}_1 D_1 + \overline{P}_2 D_2 - \max\left\{\overline{S}_1 l_1, \overline{S}_2 l_2\right\}$

Note that there is no new optimal solution generated in the case, thus the conclusion in Case 5 still follows.

Next, for the Cases $2'$, $3'$, $5'$, since they are symmetric to Cases 2, 3, 5, thus we will directly write down the possible solutions.

Case $2'$. Suppose $(Q_1, Q_2) \in \Omega_{2'}$, i.e., $D_1 \leq Q_1, D_2 \leq Q_2$.

Case $2'$ is symmetric to Case 2, and its possible solutions are listed in Table B.6.

Table B.6: Possible solutions and their total profits in Case 2'

Possible solutions	Total profit
$t_{2'}^1 = (D_1 - l_1, 0)$	$f(\boldsymbol{t}_{2'}^1) = \overline{P}_1(D_1 - l_1)$
$t_{2'}^2 = (D_1, 0)$	$f(\boldsymbol{t}_{2'}^2) = \overline{P}_1 D_1$
$t_{2'}^3 = (D_1, D_2 - l_2)$	$f(\boldsymbol{t}_{2'}^3) = \overline{P}_2(D_2 - l_2) + \overline{P}_1 D_1 - \overline{S}_1(l_1 - \underline{\alpha}_{21} l_2)$
$t_{2'}^4 = t_4^1 = t_2^4 = t_1^1 = (D_1 - l_1, D_2 - l_2)$	$f(\boldsymbol{t}_{2'}^4) = f(\boldsymbol{t}_4^1) = f(\boldsymbol{t}_2^4)$ $= f(\boldsymbol{t}_1^1) = \overline{P}_2(D_2 - l_2) + \overline{P}_1(D_1 - l_1)$
$t_{2'}^5 = (D_1, D_2 - l_1/\underline{\alpha}_{21})$	$f(\boldsymbol{t}_{2'}^5) = \overline{P}_2(D_2 - l_1/\underline{\alpha}_{21}) + \overline{P}_1 D_1$
$t_{2'}^6 = (D_1 - l_1 + \underline{\alpha}_{21} l_2, D_2 - l_2)$	$f(\boldsymbol{t}_{2'}^6) = \overline{P}_2(D_2 - l_2) + \overline{P}_1(D_1 - l_1 + \underline{\alpha}_{21} l_2)$

Case $3'$. Suppose $(Q_1, Q_2) \in \Omega_{3'}$, i.e., $D_1 \leq Q_1, 0 \leq Q_2 \leq D_2 - l_2$.

Case $3'$ is symmetric to Case 3 and its possible solutions are listed in Table B.7.

Table B.7: Possible solutions and their total profits in Case 3'

Possible solutions	Total profit
$t_{3'}^1 = t_{2'}^2 = (D_1, 0)$	$f(\boldsymbol{t}_{3'}^1) = f(\boldsymbol{t}_{2'}^2) = \overline{P}_1 D_1$
$t_{3'}^2 = t_{2'}^3 = (D_1, D_2 - l_2)$	$f(\boldsymbol{t}_{3'}^2) = f(\boldsymbol{t}_{2'}^3) = \overline{P}_2(D_2 - l_2) + \overline{P}_1 D_1 - \overline{S}_1(l_1 - \underline{\alpha}_{21} l_2)$
$t_{3'}^3 = (D_1 - l_1 + \underline{\alpha}_{21} D_2, 0)$	$f(\boldsymbol{t}_{3'}^3) = \overline{P}_1(D_1 - l_1 + \underline{\alpha}_{21} D_2)$
$t_{3'}^4 = t_{2'}^5 = (D_1, D_2 - l_1/\underline{\alpha}_{21})$	$f(\boldsymbol{t}_{3'}^4) = f(\boldsymbol{t}_{2'}^5) = \overline{P}_2(D_2 - l_1/\underline{\alpha}_{21}) + \overline{P}_1 D_1$

Case $5'$. Suppose $(Q_1, Q_2) \in \Omega_{5'}$, i.e., $D_1 \le Q_1, D_2 - l_2 \le Q_2 \le D_2$.

Case $5'$ is symmetric to Case 5 and its possible solutions are listed in Table B.8.

Table B.8: Possible solutions and their total profits for Case 5'

Possible solutions	Total profit
$t^1_{5'} = t^2_4 = t^2_{3'} = t^3_{2'} = (D_1, D_2 - l_2)$	$f(\boldsymbol{t}^1_{5'}) = f(\boldsymbol{t}^2_4) = f(\boldsymbol{t}^2_{3'})$ $= f(\boldsymbol{t}^3_{2'}) = \overline{P}_2(D_2 - l_2) + \overline{P}_1 D_1 - \overline{S}_1(l_1 - \underline{\alpha}_{21} l_2)$
$t^2_{5'} = t^2_5 = t^1_4 = (D_1, D_2)$	$f(\boldsymbol{t}^2_{5'}) = f(\boldsymbol{t}^2_5) = f(\boldsymbol{t}^1_4)$ $= \overline{P}_2 D_2 + \overline{P}_1 D_1 - \max\left\{\overline{S}_2 l_2, \overline{S}_1 l_1\right\}$
$t^3_{5'} = (D_1, D_2 - \frac{\overline{S}_2 l_2 - \overline{S}_1 l_1}{\overline{S}_2 - \overline{S}_1 \underline{\alpha}_{21}})$	$f(\boldsymbol{t}^3_{5'}) = \overline{P}_1 D_1 + \overline{P}_2 D_2 - \overline{P}_2 \frac{\overline{S}_2 l_2 - \overline{S}_1 l_1}{\overline{S}_2 - \overline{S}_1 \underline{\alpha}_{21}} - \overline{S}_1 \overline{S}_2 \frac{l_1 - l_2 \underline{\alpha}_{21}}{\overline{S}_2 - \overline{S}_1 \underline{\alpha}_{21}}$

Based on the results in Case 1-Case 6, thus symmetricly, we can also draw the following conclusions in the subregions Ω_1, $\Omega_{2'}$, $\Omega_{3'}$, Ω_4, $\Omega_{5'}$ and Ω_6:

(i) If $\overline{P}_1 \le \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \ge \overline{P}_1 \underline{\alpha}_{21}$, then

  (a) if $\overline{S}_1 l_1 \ge \overline{S}_2 l_2$, then $\boldsymbol{t}^3_4$ or $\boldsymbol{t}^5_4$ dominates all the other points in subregions Ω_1, $\Omega_{2'}$, $\Omega_{3'}$, Ω_4, and $\Omega_{5'}$.

  (b) if $\overline{S}_1 l_1 \le \overline{S}_2 l_2$, then $\boldsymbol{t}^3_{5'}$ or $\boldsymbol{t}^5_4$ dominates all the other points in subregions Ω_1, $\Omega_{2'}$, $\Omega_{3'}$, Ω_4, and $\Omega_{5'}$, since $f(\boldsymbol{t}^3_4) \le f(\boldsymbol{t}^3_{5'})$.

(ii) If $\overline{P}_1 \ge \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \le \overline{P}_1 \underline{\alpha}_{21}$, then $\boldsymbol{t}^3_{3'}$ dominates all the other points in subregions Ω_1, $\Omega_{2'}$, $\Omega_{3'}$, Ω_4, and $\Omega_{5'}$.

(iii) If $\overline{P}_1 \ge \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \ge \overline{P}_1 \underline{\alpha}_{21}$, then

  (a) if $\overline{S}_1 l_1 \ge \overline{S}_2 l_2$, then $\boldsymbol{t}^3_4$ or $\boldsymbol{t}^7_4$ dominates all the other points in subregions Ω_1, $\Omega_{2'}$, $\Omega_{3'}$, Ω_4, and $\Omega_{5'}$, and

(b) if $\overline{S}_1 l_1 \le \overline{S}_2 l_2$, then $\boldsymbol{t}^3_{5'}$ or $\boldsymbol{t}^7_4$ dominates all the other points in subregions Ω_1, $\Omega_{2'}$, $\Omega_{3'}$, Ω_4, and $\Omega_{5'}$, since $f(\boldsymbol{t}^3_4) \le f(\boldsymbol{t}^3_{5'})$.

Thus, combining all the comparison results, we can conclude that

Case 1: If $\overline{P}_1 \le \overline{P}_2\underline{\alpha}_{12}$ and $\overline{P}_2 \ge \overline{P}_1\underline{\alpha}_{21}$, then $\boldsymbol{t}^3_3$ dominates all the other points of the 9 subregions, since $f(\boldsymbol{t}^5_4) \le f(\boldsymbol{t}^3_3)$, $f(\boldsymbol{t}^3_4) \le f(\boldsymbol{t}^3_3)$ from the results of Case 4, and when $\overline{S}_1 l_1 \le \overline{S}_2 l_2$, we have

$$\begin{aligned} f(\boldsymbol{t}^3_{5'}) - f(\boldsymbol{t}^3_3) &= \overline{P}_1 D_1 + \overline{P}_2 D_2 - \overline{P}_2 \frac{\overline{S}_2 l_2 - \overline{S}_1 l_1}{\overline{S}_2 - \overline{S}_1\underline{\alpha}_{21}} - \overline{S}_1\overline{S}_2 \frac{l_1 - l_2\underline{\alpha}_{21}}{\overline{S}_2 - \overline{S}_1\underline{\alpha}_{21}} - \left(\overline{P}_2(D_2 - l_2 + \underline{\alpha}_{12} D_1)\right) \\ &= \left(\overline{P}_1 - \overline{P}_2\underline{\alpha}_{12}\right) D_1 + \overline{S}_1 \left(\overline{S}_2 - \overline{P}_2\right) \frac{l_2\underline{\alpha}_{21} - l_1}{\overline{S}_2 - \overline{S}_1\underline{\alpha}_{21}} \le 0, \end{aligned}$$

where the inequality is due to $\overline{P}_1 \le \overline{P}_2\underline{\alpha}_{12}$, $\overline{S}_1 \ge 0, \overline{S}_2 - \overline{P}_2 \ge 0$, $l_2\underline{\alpha}_{21} \le l_1$ and $\overline{S}_2 \ge \overline{S}_1\underline{\alpha}_{21}$ (due to $\overline{S}_1 l_1 \le \overline{S}_2 l_2$ and $l_2\underline{\alpha}_{21} \le l_1$).

Case 2: If $\overline{P}_2 \le \overline{P}_1\underline{\alpha}_{21}$ and $\overline{P}_1 \ge \overline{P}_2\underline{\alpha}_{12}$, then $\boldsymbol{t}^3_{3'}$ dominates all the other points of the 9 subregions, since $f(\boldsymbol{t}^5_4) \le f(\boldsymbol{t}^3_{3'})$, $f(\boldsymbol{t}^3_4) \le f(\boldsymbol{t}^3_{3'})$ which is due to the symmetric results from Case 4 that $f(\boldsymbol{t}^5_4) \le f(\boldsymbol{t}^3_3)$, $f(\boldsymbol{t}^3_4) \le f(\boldsymbol{t}^3_3)$, and when $\overline{S}_1 l_1 \ge \overline{S}_2 l_2$, by the results that $f(\boldsymbol{t}^3_{5'}) \le f(\boldsymbol{t}^3_3)$, we must have $f(\boldsymbol{t}^3_5) \le f(\boldsymbol{t}^3_{3'})$.

Case 3: If $\overline{P}_1 \ge \overline{P}_2\underline{\alpha}_{12}$, $\overline{P}_2 \ge \overline{P}_1\underline{\alpha}_{21}$, there we can separate the results into two sub-cases:

Sub-case 3.1: If $\overline{S}_1 l_1 \ge \overline{S}_2 l_2$, $\boldsymbol{t}^3_5$ or $\boldsymbol{t}^7_4$ dominates all the other points, since $f(\boldsymbol{t}^3_5) \ge f(\boldsymbol{t}^3_4)$ from Case 5;

Sub-case 3.2: If $\overline{S}_1 l_1 \le \overline{S}_2 l_2$, $\boldsymbol{t}^3_{5'}$ or $\boldsymbol{t}^7_4$ dominates all the other points, since $f(\boldsymbol{t}^3_{5'}) \ge f(\boldsymbol{t}^3_4)$ by symmetry.

This completes the proof.

$\square$

Appendix B.4. Proof of Proposition 4.8

Theorem B.3. *When $\underline{\boldsymbol{\alpha}} = 0$, the optimal solutions $\boldsymbol{Q}^*$ of Model (4.15) are characterized as follows:*

(i) If $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, then $Q_i^* = D_i - l_i$, and $v^* = \sum_{i\in[n]} \overline{P}_i (D_i - l_i)$.

(ii) If $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} > k$,

$$Q_i^* = \begin{cases} D_i - l_i + \frac{\overline{S}_{(t+1)} l_{(t+1)}}{\overline{S}_i}, & \text{if } i \in T \\ D_i, & \text{if } i \in [n] \setminus T \end{cases},$$

and

$$v^* = \sum_{i\in[n]\setminus T} \overline{P}_i l_i - \overline{S}_{(t+1)} l_{(t+1)} k + \sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \overline{S}_{(t+1)} l_{(t+1)} + \sum_{i\in[n]} \overline{P}_i (D_i - l_i),$$

where set $T := \{(1), (2), \cdots, (t)\}$ satisfying $\sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, $\sum_{i\in T\cup\{(t+1)\}} \frac{\overline{P}_i}{\overline{S}_i} > k$.

Proof. Let $\widehat{X} = \{\boldsymbol{z} : \sum_{i\in[n]} z_i \leq k, z_i \in [0,1], \forall i \in [n]\}$, which is a well-known integral polytope. Thus, $\text{conv}(X) = \widehat{X}$ and the inner maximization problem of (4.14) is equivalent to maximize a linear function of set $\widehat{X}$. Thus, we have

$$v^* = \max_{\boldsymbol{Q}\in[\boldsymbol{D}-\boldsymbol{l},\boldsymbol{D}]} \left(\sum_{i\in[n]} \overline{P}_i Q_i - \max_{\boldsymbol{z}\in\widehat{X}} \sum_{i\in[n]} \overline{S}_i (Q_i - D_i + l_i) z_i \right). \tag{B.5}$$

Let $q_i = Q_i - D_i + l_i$ for each $i \in [n]$, then Model (B.12) is equivalent to

$$v^* = \max_{\boldsymbol{q}\in[\boldsymbol{0},\boldsymbol{l}]} \min_{\boldsymbol{z}\in\widehat{X}} \left(\sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right) q_i + \sum_{i\in[n]} \overline{P}_i (D_i - l_i) \right). \tag{B.6}$$

Let λ be the dual variable associated with constraint $\sum_{i\in[n]} z_i \leq k$ and β_i be the dual variable associated with constraint $z_i \leq 1$ for each $i \in [n]$. Then by reformulating the inner maximization into its dual form, Model (B.13) is equivalent to

$$v^* = \max_{\boldsymbol{q},\lambda,\boldsymbol{\beta}} k\lambda + \sum_{i\in[n]} \beta_i + \sum_{i\in[n]} \overline{P}_i q_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \tag{B.7a}$$

$$\text{s.t. } \lambda + \beta_i \leq -\overline{S}_i q_i, i \in [n], \tag{B.7b}$$

$$0 \leq q_i \leq l_i, i \in [n], \tag{B.7c}$$

$$\lambda, \beta_i \leq 0, i \in [n]. \tag{B.7d}$$

In Model (B.13), since the objective function is concave in $\boldsymbol{q}$ and convex in $\boldsymbol{z}$, and set $\widehat{X}$ is convex compact set, thus according to Sion's minimax theorem (cf. [74]), we can equivalently reformulate Model (B.13) by switching the min with max operators as follows:

$$v^* = \min_{\boldsymbol{z}\in\widehat{X}} \max_{\boldsymbol{q}\in[\boldsymbol{0},\boldsymbol{l}]} \left\{ \sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right) q_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \right\}.$$

Note that $\max_{q_i\in[0,l_i]} \left(\sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right) q_i\right) = \sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right)_+ l_i$ for each $i \in [n]$. Thus, Model (B.13) is equivalent to

$$v^* = \min_{\boldsymbol{z}\in\widehat{X}} \left\{ \sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right)_+ l_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \right\}. \tag{B.8}$$

we also observe that

Claim 4. Model (B.15) is equivalent to

$$v^* = \min_{\boldsymbol{z}\in\widehat{X}_1} \left\{ \sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right) l_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \right\}, \tag{B.9}$$

where $\widehat{X}_1 = \left\{ \boldsymbol{z} : \sum_{i\in[n]} z_i \leq k, 0 \leq z_i \leq \frac{\overline{P}_i}{\overline{S}_i} \right\}$.

Proof. Let v_1^* denote the optimal value of Model (B.16). To prove Model (B.15) is equivalent to

Model (B.16), we only need to show $v^* = v_1^*$.

$v^* \geq v_1^*$. Given an optimal solution $\boldsymbol{z}^*$ of Model (B.15), let us define set $\mathcal{J}_1 = \{i \in [n] : 0 \leq z_i^* \leq \frac{\bar{P}_i}{\bar{S}_i}\}$ and $\mathcal{J}_2 = \{i \in [n] : \frac{\bar{P}_i}{\bar{S}_i} < z_i^* \leq 1\}$. Clearly, $\mathcal{J}_1 \cup \mathcal{J}_2 = [n]$ and $\mathcal{J}_1 \cap \mathcal{J}_2 = \emptyset$. Next, we define a new solution $\widehat{\boldsymbol{z}}$ such that

$$\widehat{z}_i = \begin{cases} z_i^*, & \text{if } i \in \mathcal{J}_1 \\ \frac{\bar{P}_i}{\bar{S}_i}, & \text{otherwise} \end{cases},$$

for each $i \in [n]$. Clearly, $\widehat{\boldsymbol{z}} \in \widehat{X}_1$. We also have

$$\begin{aligned} v^* &= \sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i^*\right)_+ l_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \\ &= \sum_{i\in\mathcal{J}_1} \left(\overline{P}_i - \overline{S}_i z_i^*\right) l_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \\ &= \sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i \widehat{z}_i\right) l_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right), \end{aligned}$$

where the third equality is due to the definition of $\widehat{\boldsymbol{z}}$. Therefore, $\widehat{\boldsymbol{z}}$ is feasible to Model (B.16) with the same objective value v^*. Thus, we have $v^* \geq v_1^*$.

$v^* \leq v_1^*$. Since $\widehat{X}_1 \subseteq \widehat{X}$, thus, $v^* \leq v_1^*$.

$\diamond$

Note that Model (B.16) is a continuous knapsack minimization problem and can be solved by greedy procedure (c.f., [18, 40]). Let $\boldsymbol{z}^*$ denote an optimal solution to Model (B.16). To obtain $\boldsymbol{z}^*$, we first sort $\{\overline{S}_i l_i\}_{i\in[n]}$ in the descending order $\overline{S}_{(1)} l_{(1)} \geq \overline{S}_{(2)} l_{(2)} \geq \cdots \geq \overline{S}_{(n)} l_{(n)}$. Next, we discuss two cases:

Case 1. If $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, then we have

$$z_i^* = \frac{\overline{P}_i}{\overline{S}_i}, \forall i \in [n],$$

and $v^* = \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right)$. On the other hand, in (B.14), let us consider the following feasible solution $\lambda^* = 0, \beta_i^* = 0, q_i^* = 0$ for all $i \in [n]$ with objective value equal to $\sum_{i\in[n]} \overline{P}_i (D_i - l_i)$. Therefore, $(\boldsymbol{Q}^*, \lambda^*, \boldsymbol{\beta}^*)$ is optimal to (B.14). Hence, the optimal order quantity for each product $i \in [n]$ is

$$Q_i^* = q_i^* + D_i - l_i = D_i - l_i.$$

Case 2. If $\sum_i \frac{\overline{P}_i}{\overline{S}_i} > k$, then let us define set $T := \{(1), (2), \ldots, (t)\}$ such that $\sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, $\sum_{i\in T\cup(t+1)} \frac{\overline{P}_i}{\overline{S}_i} > k$. Then we have

$$z_i^* = \begin{cases} \frac{\overline{P}_i}{\overline{S}_i} & \text{if } i \in T \\ k - \sum_{\tau\in T} \frac{\overline{P}_\tau}{\overline{S}_\tau} & \text{if } i = (t+1) \\ 0 & \text{otherwise} \end{cases},$$

for each $i \in [n]$, and

$$\begin{aligned} v^* &= \sum_{i\in[n]\setminus T} \left(\overline{P}_i - \overline{S}_i z_i^*\right) l_i + \sum_{i\in[n]} \overline{P}_i (D_i - l_i) \\ &= \overline{P}_{(t+1)} l_{(t+1)} - \overline{S}_{(t+1)} \left(k - \sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i}\right) l_{(t+1)} + \sum_{i\in\{[n]\setminus T\cup(t+1)\}} \overline{P}_i l_i + \sum_{i\in[n]} \overline{P}_i (D_i - l_i) \\ &= \sum_{i\in[n]\setminus T} \overline{P}_i l_i - \overline{S}_{(t+1)} l_{(t+1)} k + \sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \overline{S}_{(t+1)} l_{(t+1)} + \sum_{i\in[n]} \overline{P}_i (D_i - l_i) \end{aligned}$$

Next $\lambda^* = -\overline{S}_{t+1} l_{t+1}, \beta_i^* = 0$ and

$$q_i^* = \begin{cases} \frac{\overline{S}_{t+1} l_{t+1}}{\overline{S}_i}, & \text{if } i \in T \\ l_i, & \text{if } i \in [n] \setminus T \end{cases}$$

for each product $i \in [n]$. Clearly, $(\boldsymbol{Q}^*, \lambda^*, \boldsymbol{\beta}^*)$ is feasible to (B.14) with objective value equal to v^*. Therefore, $(\boldsymbol{Q}^*, \lambda^*, \boldsymbol{\beta}^*)$ is optimal to (B.14). Hence, the optimal order quantity for each product $i \in [n]$ is

$$Q_i^* = q_i^* + D_i - l_i = \begin{cases} D_i - l_i + \frac{\overline{S}_{t+1} l_{t+1}}{\overline{S}_i}, & \text{if } i \in T \\ D_i, & \text{if } i \in [n] \setminus T \end{cases}.$$

$\square$

Appendix B.5. Proof of Proposition 4.8

Theorem B.4. *When $\underline{\boldsymbol{\alpha}} = 0$, the optimal solutions $\boldsymbol{Q^*}$ of Model (4.15) are characterized as follows:*

(i) If $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, then $Q_i^* = D_i - l_i$, and $v^* = \sum_{i\in[n]} \overline{P}_i (D_i - l_i)$.

(ii) If $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} > k$,

$$Q_i^* = \begin{cases} D_i - l_i + \frac{\overline{S}_{(t+1)} l_{(t+1)}}{\overline{S}_i}, & \text{if } i \in T \\ D_i, & \text{if } i \in [n] \setminus T \end{cases},$$

and

$$v^* = \sum_{i\in[n]\setminus T} \overline{P}_i l_i - \overline{S}_{(t+1)} l_{(t+1)} k + \sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \overline{S}_{(t+1)} l_{(t+1)} + \sum_{i\in[n]} \overline{P}_i (D_i - l_i),$$

where set $T := \{(1), (2), \cdots, (t)\}$ satisfying $\sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, $\sum_{i\in T\cup\{(t+1)\}} \frac{\overline{P}_i}{\overline{S}_i} > k$.

Proof. Let $\widehat{X} = \{\boldsymbol{z} : \sum_{i\in[n]} z_i \leq k, z_i \in [0, 1], \forall i \in [n]\}$, which is a well-known integral polytope. Thus, $\text{conv}(X) = \widehat{X}$ and the inner maximization problem of (4.14) is equivalent to maximize a linear function of set $\widehat{X}$. Thus, we have

$$v^* = \max_{\boldsymbol{Q}\in[\boldsymbol{D}-\boldsymbol{l},\boldsymbol{D}]} \left(\sum_{i\in[n]} \overline{P}_i Q_i - \max_{\boldsymbol{z}\in\widehat{X}} \sum_{i\in[n]} \overline{S}_i (Q_i - D_i + l_i) z_i \right). \tag{B.12}$$

Let $q_i = Q_i - D_i + l_i$ for each $i \in [n]$, then Model (B.12) is equivalent to

$$v^* = \max_{\boldsymbol{q}\in[\boldsymbol{0},\boldsymbol{l}]} \min_{\boldsymbol{z}\in\widehat{X}} \left(\sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right) q_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \right). \tag{B.13}$$

Let λ be the dual variable associated with constraint $\sum_{i\in[n]} z_i \leq k$ and β_i be the dual variable associated with constraint $z_i \leq 1$ for each $i \in [n]$. Then by reformulating the inner maximization into its dual form, Model (B.13) is equivalent to

$$v^* = \max_{\boldsymbol{q},\lambda,\boldsymbol{\beta}} k\lambda + \sum_{i\in[n]} \beta_i + \sum_{i\in[n]} \overline{P}_i q_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \tag{B.14a}$$

$$\text{s.t. } \lambda + \beta_i \leq -\overline{S}_i q_i, i \in [n], \tag{B.14b}$$

$$0 \leq q_i \leq l_i, i \in [n], \tag{B.14c}$$

$$\lambda, \beta_i \leq 0, i \in [n]. \tag{B.14d}$$

In Model (B.13), since the objective function is concave in $\boldsymbol{q}$ and convex in $\boldsymbol{z}$, and set $\widehat{X}$ is convex compact set, thus according to Sion's minimax theorem (cf. [74]), we can equivalently reformulate Model (B.13) by switching the min with max operators as follows:

$$v^* = \min_{\boldsymbol{z}\in\widehat{X}} \max_{\boldsymbol{q}\in[\boldsymbol{0},\boldsymbol{l}]} \left\{ \sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right) q_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \right\}.$$

Note that $\max_{q_i\in[0,l_i]} \left(\sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right) q_i\right) = \sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right)_+ l_i$ for each $i \in [n]$. Thus, Model (B.13) is equivalent to

$$v^* = \min_{\boldsymbol{z}\in\widehat{X}} \left\{ \sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right)_+ l_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \right\}. \tag{B.15}$$

we also observe that

Claim 5. Model (B.15) is equivalent to

$$v^* = \min_{\boldsymbol{z} \in \widehat{X}_1} \left\{ \sum_{i \in [n]} \left(\overline{P}_i - \overline{S}_i z_i \right) l_i + \sum_{i \in [n]} \overline{P}_i \left(D_i - l_i \right) \right\}, \tag{B.16}$$

where $\widehat{X}_1 = \left\{ \boldsymbol{z} : \sum_{i \in [n]} z_i \le k, 0 \le z_i \le \frac{\overline{P}_i}{\overline{S}_i} \right\}$.

Proof. Let v_1^* denote the optimal value of Model (B.16). To prove Model (B.15) is equivalent to Model (B.16), we only need to show $v^* = v_1^*$.

$v^* \ge v_1^*$. Given an optimal solution $\boldsymbol{z}^*$ of Model (B.15), let us define set $\mathcal{J}_1 = \{i \in [n] : 0 \le z_i^* \le \frac{\bar{P}_i}{\bar{S}_i}\}$ and $\mathcal{J}_2 = \{i \in [n] : \frac{\bar{P}_i}{\bar{S}_i} < z_i^* \le 1\}$. Clearly, $\mathcal{J}_1 \cup \mathcal{J}_2 = [n]$ and $\mathcal{J}_1 \cap \mathcal{J}_2 = \emptyset$. Next, we define a new solution $\widehat{\boldsymbol{z}}$ such that

$$\widehat{z}_i = \begin{cases} z_i^*, & \text{if } i \in \mathcal{J}_1 \\ \frac{\bar{P}_i}{\bar{S}_i}, & \text{otherwise} \end{cases},$$

for each $i \in [n]$. Clearly, $\widehat{\boldsymbol{z}} \in \widehat{X}_1$. We also have

$$\begin{aligned} v^* &= \sum_{i \in [n]} \left(\overline{P}_i - \overline{S}_i z_i^* \right)_+ l_i + \sum_{i \in [n]} \overline{P}_i \left(D_i - l_i \right) \\ &= \sum_{i \in \mathcal{J}_1} \left(\overline{P}_i - \overline{S}_i z_i^* \right) l_i + \sum_{i \in [n]} \overline{P}_i \left(D_i - l_i \right) \\ &= \sum_{i \in [n]} \left(\overline{P}_i - \overline{S}_i \widehat{z}_i \right) l_i + \sum_{i \in [n]} \overline{P}_i \left(D_i - l_i \right), \end{aligned}$$

where the third equality is due to the definition of $\widehat{\boldsymbol{z}}$. Therefore, $\widehat{\boldsymbol{z}}$ is feasible to Model (B.16) with the same objective value v^*. Thus, we have $v^* \ge v_1^*$.

$v^* \le v_1^*$. Since $\widehat{X}_1 \subseteq \widehat{X}$, thus, $v^* \le v_1^*$.

$\diamond$

Note that Model (B.16) is a continuous knapsack minimization problem and can be solved by greedy procedure (c.f., [18, 40]). Let $\boldsymbol{z}^*$ denote an optimal solution to Model (B.16). To obtain $\boldsymbol{z}^*$, we first sort $\{\overline{S}_i l_i\}_{i\in[n]}$ in the descending order $\overline{S}_{(1)}l_{(1)} \geq \overline{S}_{(2)}l_{(2)} \geq \cdots \geq \overline{S}_{(n)}l_{(n)}$. Next, we discuss two cases:

Case 1. If $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, then we have

$$z_i^* = \frac{\overline{P}_i}{\overline{S}_i}, \forall i \in [n],$$

and $v^* = \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right)$. On the other hand, in (B.14), let us consider the following feasible solution $\lambda^* = 0, \beta_i^* = 0, q_i^* = 0$ for all $i \in [n]$ with objective value equal to $\sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right)$. Therefore, $(\boldsymbol{Q}^*, \lambda^*, \boldsymbol{\beta}^*)$ is optimal to (B.14). Hence, the optimal order quantity for each product $i \in [n]$ is

$$Q_i^* = q_i^* + D_i - l_i = D_i - l_i.$$

Case 2. If $\sum_i \frac{\overline{P}_i}{\overline{S}_i} > k$, then let us define set $T := \{(1), (2), \ldots, (t)\}$ such that $\sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, $\sum_{i\in T\cup(t+1)} \frac{\overline{P}_i}{\overline{S}_i} > k$. Then we have

$$z_i^* = \begin{cases} \frac{\overline{P}_i}{\overline{S}_i} & \text{if } i \in T \\ k - \sum_{\tau\in T} \frac{\overline{P}_\tau}{\overline{S}_\tau} & \text{if } i = (t+1) \\ 0 & \text{otherwise} \end{cases},$$

for each $i \in [n]$, and

$$\begin{aligned} v^* &= \sum_{i\in[n]\setminus T} \left(\overline{P}_i - \overline{S}_i z_i^*\right) l_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \\ &= \overline{P}_{(t+1)} l_{(t+1)} - \overline{S}_{(t+1)} \left(k - \sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i}\right) l_{(t+1)} + \sum_{i\in\{[n]\setminus T\cup(t+1)\}} \overline{P}_i l_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \\ &= \sum_{i\in[n]\setminus T} \overline{P}_i l_i - \overline{S}_{(t+1)} l_{(t+1)} k + \sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \overline{S}_{(t+1)} l_{(t+1)} + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \end{aligned}$$

Next $\lambda^* = -\overline{S}_{t+1} l_{t+1}, \beta_i^* = 0$ and

$$q_i^* = \begin{cases} \frac{\overline{S}_{t+1} l_{t+1}}{\overline{S}_i}, & \text{if } i \in T \\ l_i, & \text{if } i \in [n] \setminus T \end{cases}$$

for each product $i \in [n]$. Clearly, $(\boldsymbol{Q}^*, \lambda^*, \boldsymbol{\beta}^*)$ is feasible to (B.14) with objective value equal to v^*. Therefore, $(\boldsymbol{Q}^*, \lambda^*, \boldsymbol{\beta}^*)$ is optimal to (B.14). Hence, the optimal order quantity for each product $i \in [n]$ is

$$Q_i^* = q_i^* + D_i - l_i = \begin{cases} D_i - l_i + \frac{\overline{S}_{t+1} l_{t+1}}{\overline{S}_i}, & \text{if } i \in T \\ D_i, & \text{if } i \in [n] \setminus T \end{cases}.$$

$\square$

Appendix C

Proofs in Chapter 5

Appendix C.1. Proof of Proposition 5.1

Proposition 5.1. *The inner maximization problem (5.2) is equivalent to*

$$R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) = \max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \sum_{i\in[n]} \overline{S}_i \left[(Q_i - D_i)x_i + l_i y_{ii} - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(u_j y_{ji} + \psi_j (x_i - y_{ji}) \right) \right] \tag{5.4a}$$

$$s.t. \sum_{i\in[n]} z_i \le k. \tag{5.4b}$$

$$y_{ji} \le x_i, \forall i,j \in [n], \tag{5.4c}$$

$$y_{ji} \le z_j, \forall i,j \in [n], \tag{5.4d}$$

$$z_i, x_i \in \{0,1\}, y_{ji} \ge 0, \forall i,j \in [n]. \tag{5.4e}$$

Proof. Let $\widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$ denote the optimal value of Model (5.4). It is sufficient to show that $R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) = \widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$ for any feasible $(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) \in \mathbb{R}^{3n}_+$.

$R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) \le \widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$. Suppose $(\boldsymbol{x}^*,\boldsymbol{y}^*) \in \{0,1\}^{2n}$ is an optimal solution of Model (5.3). Define $y^*_{ji} = z^*_j x^*_i$ for each $i,j \in [n]$. Clearly, $(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{z}^*)$ is feasible to Model (5.4) and

$$\begin{aligned} R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) &= \sum_{i\in[n]} \overline{S}_i \left[Q_i - D_i + l_i z^*_i - \sum_{j} \underline{\alpha}_{ji} \left(u_j z^*_j + v_j (1 - z^*_j) \right) \right] x^*_i \\ &= \sum_{i\in[n]} \overline{S}_i \left(Q_i - D_i)x^*_i + l_i y^*_{ii} - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(u_j y^*_{ji} + \psi_j (x^*_i - y^*_{ji}) \right) \right], \end{aligned}$$

i.e., it yields the same objective value as $R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$. Thus, $R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) \leq \widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$.

$R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) \geq \widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$. Suppose $(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{z}^*)$ is an optimal solution to Model (5.4). Since both $\boldsymbol{x}^*$ and $\boldsymbol{z}^*$ are binary, thus according to constraints (5.4c) and (5.4d), we must have $y^*_{ji} \leq z^*_j x^*_i = \min\{z^*_j, x^*_i\}$ for each $i,j \in [n]$. Therefore,

$$\begin{aligned}\widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) &= \sum_{i\in[n]} \overline{S}_i \left[(Q_i - D_i)x^*_i + l_i y^*_{ii} - \sum_{j\in[n]} \underline{\alpha}_{ji}\left(u_j y^*_{ji} + \psi_j(x^*_i - y^*_{ji})\right)\right] \\ &\leq \sum_{i\in[n]} \overline{S}_i \left[Q_i - D_i + l_i z^*_i - \sum_{j} \underline{\alpha}_{ji}\left(u_j z^*_j + v_j(1 - z^*_j)\right)\right] x^*_i\end{aligned}$$

where the first inequality is due to the coefficients of $\{y^*_{ji}\}_{j,i\in[n]}$ are all nonnegative, i.e., $\overline{S}_i l_i \geq 0$ and $\psi_j \geq u_j$ for all $i,j \in [n]$. Hence, $(\boldsymbol{x}^*,\boldsymbol{z}^*)$ is feasible to Model (5.3) and yields an objective value at least as large as $\widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$, which implies that $R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) \geq \widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$.

□

Appendix C.2. Proof of Proposition 5.2

Theorem C.1. *Let v^{CA} denote the optimal value of Model (5.10). Then*

(i) $v^{CA} \leq v^*$; and

(ii) $v^{CA} = v^*$, if one of the following conditions holds: (1) $\underline{\boldsymbol{\alpha}} = \boldsymbol{0}$, or (2) $n = k$.

Proof. $v^* \geq v^{CA}$ holds since in (5.10), we replace set Ξ to be its continuous relaxation Ξ_C. It remains to show that $v^* = v^{CA}$ if $\underline{\boldsymbol{\alpha}} = \boldsymbol{0}$ or $n = k$. To prove this result, it is sufficient to show that for any given $(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$ satisfying constraints (5.5a) − (5.5e), (5.6b) − (5.6e), the following linear program

$$\max_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in\Xi_C} g\left(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\right)$$

has an integral optimal solution, i.e., the continuous relaxation of Model (5.4) has an integral optimal solution.

$\underline{\boldsymbol{\alpha}} = \mathbf{0}$. In this case, Model (5.4) is equivalent to

$$R(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}) = \max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \sum_{i\in[n]} \overline{S}_i \left[(Q_i - D_i)x_i + l_i y_{ii}\right] \tag{C.1a}$$

$$\text{s.t.} \sum_{i\in[n]} z_i \leq k. \tag{C.1b}$$

$$y_{ii} \leq x_i, \forall i \in [n], \tag{C.1c}$$

$$y_{ii} \leq z_i, \forall i \in [n], \tag{C.1d}$$

$$z_i, x_i \in \{0,1\}, y_{ii} \geq 0, \forall i \in [n]. \tag{C.1e}$$

We let $\widehat{\Xi}_C$ denote the continuous relaxation of the feasible region of Model (C.1), where we relax $\boldsymbol{x}, \boldsymbol{z}$ to be continuous. Then, it is sufficient to show that $\widehat{\Xi}_C$ is an integral polytope.

First of all, let us write the constraints (C.1b) – (C.1e) in the matrix form as below:

$$\begin{bmatrix} \boldsymbol{e}^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{y} \\ \boldsymbol{x} \end{bmatrix} \leq \begin{bmatrix} k \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{C.2}$$

where $\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$, $\boldsymbol{y} = \begin{bmatrix} y_{11} \\ y_{22} \\ \vdots \\ y_{nn} \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. By Theorem 19.3 on Page 269 of [66], to prove $\mathcal{P}$ is a integral polytope, it is sufficient to prove $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{e}^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} & \mathbf{0} \end{bmatrix}$ is totally unimodular

(TU). Indeed, if $\boldsymbol{A}$ is TU, and since $\begin{bmatrix} k \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$ is integral, thus, $\mathcal{P}$ is an integral polytope. Hence, the continuous relaxation of Model (C.1), which is a linear program, has an integral optimal solution. According to Theorem 19.3 on Page 269 of [66], to prove $\boldsymbol{A}$ is a totally unimodular matrix, it is sufficient to prove that for any $S \subseteq [3n]$, there exist S_1 and S_2 such that $S_1 \cap S_2 = \emptyset, S_1 \cup S_2 = S, \sum_{i \in S_1} \boldsymbol{A}_{.i} - \sum_{i \in S_2} \boldsymbol{A}_{.i} \in \{-1, 0, 1\}^{2n+1}$.

We let $\widehat{S}_1 = S \cap \{1, 2, \cdots, n\} = \{i_1, \cdots, i_{|\overline{S}_1|}\}$, $T_z^1 = \{i_\tau\}_{\tau \leq |\overline{S}_1|, \tau \text{ is odd}}$ and $T_z^2 = \{i_\tau\}_{\tau \leq |\overline{S}_1|, \tau \text{ is even}}$. Also, we let $\widehat{S}_2 = S \cap \{n+1, \cdots, 2n\}$, $\widehat{S}_3 = S \cap \{2n+1, \cdots, 3n\}$, $T_y^1 = \left\{j \in \widehat{S}_2 : j - n \in T_z^1\right\}$, $T_y^2 = \widehat{S}_2 \setminus T_y^1$, $T_x^1 = \left\{j \in \widehat{S}_3 : j - n \in T_y^1\right\}$, $T_x^2 = \overline{S}_3 \setminus T_x^1$. Clearly, we have $S_1 = T_z^1 \cup T_y^1 \cup T_x^1$, $S_2 = S \setminus S_1$. For such S_1 and S_2, we have $\sum_{i \in S_1} \boldsymbol{A}_{.i} - \sum_{i \in S_2} \boldsymbol{A}_{.i} \in \{-1, 0, 1\}^{2n+1}$.

$k = n$. From the discussion in Section 4.4.3, we already know at the optimality, we must have $z_i^* = 1$ for all $i \in [n]$ when $k = n$. In (5.4a), the coefficient of y_{ji} is $\sum_{j \in [n]} \underline{\alpha}_{ji}(\psi_j - u_j) \geq 0$, since $\psi_j \geq u_j$ for each $j, i \in [n]$ and $j \neq i$. Also, the coefficient of y_{ii} is l_i, which is nonnegative, for each $i \in [n]$. Thus, at the optimality of the continuous relaxation of Model (5.4), we must have $y_{ji} = \min(x_i, z_j) = \min(x_i, 1) = x_i$ for all $i, j \in [n]$. Then, the continuous relaxation of Model (5.4) is equivalent to

$$\begin{aligned}
&\max_{\boldsymbol{x} \in [0,1]^n} \sum_{i \in [n]} \overline{S}_i \left((Q_i - D_i)x_i + l_i y_{ii} - \sum_{j \in [n]} \underline{\alpha}_{ji} \left(u_j y_{ji} + \psi_j (x_i - y_{ji}) \right) \right) \\
= &\max_{\boldsymbol{x} \in [0,1]^n} \sum_{i \in [n]} \overline{S}_i \left((Q_i - D_i + l_i) + \sum_{j \in [n]} \underline{\alpha}_{ji} (Q_j - D_j + l_j)_+ \right) x_i,
\end{aligned}$$

which is a linear program over a unit box. Thus, there exists an optimal solution $\boldsymbol{x}^*$ of the above linear program, which corresponds to an extreme point of the box $[0, 1]^n$, i.e., $x_i^* \in \{0, 1\}$ for all $i \in [n]$. Thus, $y_{ji}^* = x_i^* \in \{0, 1\}$ for all $i, j \in [n]$. Therefore, the continuous relaxation of Model (5.4) has an integral optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$.

□

Appendix C.3. MILP Reformulation of Model (5.2) for General k^α

Suppose k^α is general, i.e., Assumption 4 does not hold. Then the formulation (5.2) becomes

$$R(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}) = \max_{\boldsymbol{z}} \sum_{i\in[n]} \overline{S}_i \left[Q_i - D_i + l_i z_i - \sum_{j\in[n]} (\alpha_{ji} - l_{ji}^\alpha z_{ji}^\alpha)\left(u_j z_j + \psi_j(1-z_j)\right) \right]_+, \tag{C.3a}$$

$$\text{s.t.} \sum_{i\in[n]} z_i \le k, \tag{C.3b}$$

$$\sum_{j\in[n]} \sum_{i\in[n]} z_{ji}^\alpha \le k^\alpha, \tag{C.3c}$$

$$z_i \in \{0,1\}, \forall i \in [n], \tag{C.3d}$$

$$z_{ji}^\alpha \in \{0,1\}, \forall j, i \in [n]. \tag{C.3e}$$

Similar to Proposition 5.1, We can reformulate (C.3) as an MILP.

Proposition C.2. *The inner maximization problem (C.3) is equivalent to*

$$\begin{aligned}
R(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}) = \max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \sum_{i\in[n]} \overline{S}_i \Bigg\{ & (Q_i - D_i)\, x_i + l_i y_{ii} - \sum_{j\in[n]} \alpha_{ji}\left[u_j y_{ji} + \psi_j\left(x_i - y_{ji}\right)\right] \\
& + \sum_{j\in[n]} l_{ji}^\alpha \left[u_j T_{ji}^\alpha + \psi_j\left(B_{ji}^\alpha - T_{ji}^\alpha\right)\right] \Bigg\} \\
s.t. \quad & \sum_{i\in[n]} z_i \le k. \\
& \sum_{j\in[n]} \sum_{i\in[n]} z_{ji}^\alpha \le k^\alpha, \\
& y_{ji} \le x_i, \quad y_{ji} \le z_j, \forall i, j \in [n], \\
& T_{ji}^\alpha \le z_{ji}^\alpha, T_{ji}^\alpha \le y_{ji}, \forall i, j \in [n], \\
& B_{ji}^\alpha \le z_{ji}^\alpha, B_{ji}^\alpha \le x_i, B_{ji}^\alpha \ge z_{ji}^\alpha + x_i - 1, \forall i, j \in [n] \\
& z_{ji}^\alpha, z_i, x_i \in \{0,1\}, y_{ji}, T_{ji}^\alpha \ge 0, \forall i, j \in [n].
\end{aligned}$$

Proof. The proof is similar to that of Proposition 5.1, i.e., we eliminate the bilinear terms by introducing variables $y_{ji} = x_i z_j, T^{\alpha}_{ji} = z^{\alpha}_{ji} y_{ji}, B^{\alpha}_{ji} = z^{\alpha}_{ji} x_i$ for each $i, j \in [n]$, and then applying McCormick inequalities [46]. $\square$

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