Three Essays on Networks

Bowen Shi

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Sudipta Sarangi, Chair
Eric A. Bahel
Sheryl B. Ball
Christophe Bravard
Hans H. Haller
Kwok Ping ‘Byron’ Tsang

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(ABSTRACT)

This dissertation consists of three essays studying human behavior and contagion phenomenon in networks. The analysis especially focuses on information sharing, trade relationship and pest spread in networks. The first chapter outlines the dissertation by briefly discussing the motivations, methods, and main findings in each of the following chapters.

Chapter two examines the information sharing in networks. We develop a heterogeneous agents model in which connections between players act as a channel to exchange information. We focus on specialized equilibria, which is based on Nash tâtonnement. It is shown that players utilize the signals in the linear form and only specialized equilibria can be stable. We also compare the sequential equilibria and stable equilibria, and it is shown that stable equilibria form a proper subset of the sequential ones, which gives a sharper prediction. The stable equilibria demonstrate star-like graphs, which is similar to the phenomenon “the law of the few” in the literature.

Chapter three investigates the trade relationship among players where trade between two players can bring benefits as well as conflict. And if conflict happens, the players coordinate based on received information. We show that the optimal structure of trade networks ranges from complete market to Autarky. Also, we study the optimal timing for trade relationship establishment and the optimal size of organizations when facing scarce members. It is shown that when potential neighbors become more scarce, people care more about the future, or new technology breakthroughs occur more frequently, it is optimal to have more neighbors to back up for the potential technological breakthrough.

The last chapter studies the pest spread in the networks. We use a directional and weighted network to study the spread of Tuta absoluta. A robust network-based approach is proposed to model seasonal flow of agricultural produce and examine its role in pest spread. Furthermore, the long-term establishment potential of the pest and its economic impact on the country are assessed. Preliminary analyses indicate that Tuta absoluta will invade most major tomato production regions within a year of introduction and the economic impact of invasion could range from $17-25 million.
Networks play an important role in the society today, for example, the Internet, Facebook and other social media, cell phones, and communication networks. Networks have undeniable effects on human activity in modern society and this dissertation will focus on three topics: information sharing in networks, trade networks and diffusion in the networks.

Information sharing in networks: In the society, individuals have different ability to refine signals, so their strategy about information will be different. In this paper, we assume that there are noisy signals about the true state of the world and individuals have a tradeoff between refining their signals by themselves and connecting to others to receive information. It is shown that players utilize the signals in the linear form and only specialized equilibria can be stable. Furthermore, the stable equilibria demonstrate star-like graphs, which is similar to the phenomenon “the law of the few” in the literature.

The structure of trade networks: Trade is ubiquitous in modern society and has undeniable effects on human activity. The trade relationship among players can bring benefits as well as conflict. Facing the advantages and disadvantages of trade, players need to make decisions about who to trade with. We show that the optimal structure of trade networks ranges from complete market to Autarky as economy environment changes. Also, we study the optimal timing for trade relationship establishment and the optimal size of organizations when facing scarce members. It is shown that when potential neighbors become more scarce, people care more about the future, or new technology breakthroughs occur more frequently, it is optimal to have more neighbors to back up for the potential technological breakthrough.

Diffusion in networks: Diffusions like information, diseases, rumor are usually through the network in the society. We study the pest spread Nepal in the networks. Based on road networks and trade patterns, we model the flow networks of the pest spread. Based on the flow network, the long-term establishment potential of the pest and its economic impact on the country are assessed. Preliminary analyses indicate that Tuta absoluta will invade most major tomato production regions within a year of introduction and the economic impact of invasion could range from $17-25 million.
Life is like a journey on a train and the station at Department of Economics, Virginia Tech is the most important part of my whole journey. Five years at Virginia Tech is not a short period of time, however, my unforgettable, enjoyable and fruitful Ph.D. life at Virginia Tech is just like the blink of an eye, the same as it happens to a lot of wonderful things. Through this journey, I harvest knowledge, friendship, wisdom from the professors, classmates and so on.

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Networks are ubiquitous in modern society and have undeniable effects on human activity. Traditional economics assume the interactions among individuals as centralized and anonymous. However, in the real world, many, if not most, activities in the economy do not function as centralized and anonymous ones. Usually, individuals in the society interact with each other through social links, to name a few, such as job searching through social contacts (Gagnon and Goyal, 2017; Granovetter, 1995), informal insurance mechanism through social networks (Ambrus et al., 2014), diffusion of innovations (Coleman et al., 1966; Robertson et al., 1996; Udry and Conley, 2005), criminal behavior (Glaeser et al., 1996), learning in network (Acemoglu et al., 2011a; Golub and Jackson, 2010), etc. Furthermore, new applications have arisen for which networks are of clear importance, including, the Internet, Facebook and other social media, cell phones, and communication networks. Additionally, increased access to relevant data from both new and more traditional applications, along with the stronger computing power, leads to a growing body of empirical work showed phenomena not observed in traditional economic analysis. Kirman (2016) argues that a focus on the structure of interactions between agents should become the new benchmark in economic modeling.

Over the past two decades, economists have realized the importance of network structures in the economic analysis and have started to recognize and systematically study the impact of networks on economic and social interactions. The adoption of the network framework in economic analysis spread to almost all fields in Economics, including the study of exchange market, the financial crisis, international trade, migration, development, the labor market, and so on (Bramoullé et al., 2016). As (Jackson, 2008) said, social networks play a central role in the transmission of information, the trade of many goods and services, determining how diseases spread, etc.
This dissertation covers three topics: information sharing, trade networks and the spread of pests. One approach in the literatures on network economics assumes the structure of links as fixed and exogenous, and most empirical works are based on this framework, as well as learning in networks, etc. As said in Durieu et al. (2011), “To network or not to network, that is the question.” So, another approach in network economics assumes that network formation is endogenously determined. Chapter 2 and Chapter 3 adopts the second approach to study players’ linking decisions in the environment of information sharing and trade relationship, while Chapter 4 mainly focuses on exogenous network.

Information plays a very important role in economic analysis. For example, the price mechanism is the principle device of coordination among individual actions, especially the theory of general equilibrium and the theory of oligopoly. In traditional economics, the information of price is usually assumed as centralized and anonymous. However, in the real world, individuals usually rely on their friends and social contacts to gather information. In Chapter 2, we study the information problem in the framework of network where players can only receive information from their neighbors and they have a tradeoff between the cost of linking to others and the benefit from more information from their neighbors. We find that according to the criterion of stability, the resulting network structure will be star-like, which is similar as the phenomenon “the law of the few” (Galeotti and Goyal, 2010), where only a small fraction of players contribute and others free ride. And only highest type players are potential to refine their information and others will link to them and free ride for the information.

Trade is perhaps the most important activity in modern human society, prevailing almost at every corner of the modern world. Chapter 3 studies the trade relationship among players. However, free trade was not a widespread idea among economists prior to Adam Smith. At that time, it was widely believed that an appropriate use of government trade restrictions was likely a better economic policy than free trade. In his illuminating intellectual history book of free trade, Against the Tide (Irwin, 1996), Douglas Irwin traces views on the virtues and vices of foreign trade back to early Greek and Roman writers, demonstrating the evolution of people towards trade along the history of humankind. In Chapter 3, we introduce the idea from Dessein et al. (2016) to assume that the trade among trade partners can bring benefit as well as conflict. So we study the optimal and equilibrium trade networks. We find that as the gains from trade, the communication quality or the importance of the fundamental value increases, or the coordination difficulty or the risk of trade opportunity decreases, the optimal market structure tends toward a complete market where all members trade with each other. On the opposite side, the optimal market structure tends toward Autarky. From this perspectives, we give an explanation about the change of views of public towards the international trade.

Finally, Chapter 4 studies another phenomenon - pest spread, where network is the
desirable framework to start with. In this paper, we use a directional and weighted network to study the contagion network. As an application, the Tuta absoluta in Nepal is studied, of which the contagion network is built based on the trade and the transport patterns. A robust network-based approach is proposed to model seasonal flow of agricultural produce and examine its role in pest spread. Furthermore, the long-term establishment potential of the pest and its economic impact on the country are assessed. Preliminary analyses indicate that Tuta absoluta will invade most major tomato production regions within a year of introduction and the economic impact of invasion could range from $17-25 million. The proposed approach is generic and particularly suited for data-poor scenarios.
We develop a heterogenous agents model in which connections between players act as a channel to exchange information. We show that: (i) In all such games, Nash equilibria and specialized equilibria exist, (ii) Only specialized equilibrium can be stable, (iii) In endogenous network games, according to the stability criterion, only star-like graphs can exist. Only highest type players can potentially refine their information and others will link to them and free ride for the information. And also, we compare the stable equilibria and the sequential equilibria of the game. We find that the set of stable equilibria is a proper subset of the set of sequential equilibria.

2.1 Introduction

In this paper, we build a network model of information exchange, where information exchange only occurs when two players are neighbors. It is assumed that there exists an uncertain world, with the true state unknown to the players. However, players can observe some noisy signals to gain information about the true state, where the accuracy of the signals is assumed to depend on the players’ effort. It is further assumed that different players have different abilities to refine the signals. As for the decisions, there are three dimensions for each player: linking decision; effort to refine the signal and prediction about the true state of the world. Players’ utilities depend on the deviation of prediction from the true state, effort making and linking cost. We find that based on the information available, each player’s optimal prediction about true state is a linear combination of signals available for her, observed by herself and her neighbors. Furthermore, in equilibrium, only a
specialized strategy profile is stable. When the network is endogenously formed, only a star-like equilibrium network is stable. In such a stable equilibrium, only highest type players can potentially refine their information and others will link to them and free ride for the information.

We will firstly analyze the case where the network is exogenously given. Given a graph, players will maximize their utility by choosing how much effort to contribute and how to use the information available for them. Examples include the information about the introduction of new technology. When players obtain some information about new technology, they can share this information with their friends. The previous literature demonstrates such a phenomenon. In medicine and other technical fields, professional networks shape research patterns (Coleman et al., 1957; Valente, 1996). In agriculture, a farmer’s experience with a new crop can benefit others (Foster and Rosenzweig, 1995). Here, Bramoullé and Kranton (2007) is the closest one to our model. In their model, they studied public goods in networks with homogeneous agents. We include the strategy for dealing with information in our analysis. Further, the analysis is based on a heterogeneous agents model.

However, in many cases, the network is not exogenously given. players will establish their network intentionally to better their welfare. In the previous literature, the micro analysis of economic networks relies on game theory, which aims at identifying Nash equilibria. The game-theoretic literature has highlighted the crucial role of incentives in the endogenous and induced behavior of agents in socioeconomic networks (Bala and Goyal, 2000; Jackson and Wolinsky, 1996). In agriculture, a farmer’s experience with a new crop can benefit others (Foster and Rosenzweig, 1995). Here, Bramoullé and Kranton (2007) is the closest one to our model. In their model, they studied public goods in networks with homogeneous agents. We include the strategy for dealing with information in our analysis. Further, the analysis is based on a heterogeneous agents model.

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Galeotti and Goyal (2010) demonstrate an interesting phenomenon “Law of the Few”. However, they do not consider stability of equilibria. In our heterogeneous agents model, by considering stability, we find similar network structure as “Law of the Few”: star-like networks. There are three potential types of players in the society. One of them is the group of highest type players, who are potential contributors; the second type is the group of middle type players, who just contribute to themselves, with no connection to others and third type is the group of low type players, who connect to highest type players and free ride for the information.

The rest of the paper is organized as follows. Firstly, we introduce one scenario for the information problem for players in society. The model is set up in Section 2.2. Then, the exogenous network model is studied in section 2.3, where we study the
optimal strategy of player for prediction of the true state of the world and equilibrium existence and stability, and we will study some specific graphs about stars and complete graphs. In section 2.4, we will continue to study endogenous network formation problem, in equilibrium, we have the star-like stable graph, which illustrates the law of the few. And in the conclusion section, we present some extensions for further research.

2.2 Model Setup

Let \( \mathcal{N} = \{1, 2, ..., n\} \), \( n \geq 3 \), be the set of agents with types \( a_1, \cdots, a_n \) and let \( i \) and \( j \) be typical members of this set. Here, the types of players represent the ability to refine the signals. The players face an uncertain world - whose state can be represented by the random variable \( \hat{\theta} \), with mean \( \mathbb{E}(\hat{\theta}) = 0 \). The timing of moves in the game proceeds as follows: In the first stage, each player chooses whom to link with at a constant marginal cost. Then, using the network formed, each player chooses her effort to refine her own information regarding \( \hat{\theta} \). Finally, in the third stage, based on the information available, each agent \( i \) will make the prediction about the true state of the world and obtain her payoffs. Here, if two players are neighbors, their respective signals are observable by each other. It is assumed that the types of players, the structure of signals and the payoff structure are common knowledge among all players.

Our notational setup draws heavily on the network literature (Jackson, 2008). The network of relations is represented as a graph \( g \), and we denote \( \mathcal{G} \) as the set of all possible graphs on \( n \) vertices. In this paper, \( g \) is assumed to be a simple directed and unweighted network, that is, \( g_{ij} \in \{0, 1\}, \forall i, j \in \mathcal{N} \); and \( g_{ij} = 1 \) means that player \( i \) connects with player \( j \). Without specification, \( g \) is assumed to be directed in the following. There is a cost per link, denoted by \( \eta \). Also, we denote \( N_i(g) = \{j \in \mathcal{N} : g_{ij} = 1\} \) as the set of agents who are neighbors of \( i \) and \( n_i(g) \) as the cardinality of the set \( N_i(g) \). Nodes \( i \) and \( j \) are connected if and only if there exist \( i_1, \cdots, i_l \) such that \( i_1 = i, i_l = j \) and \( g_{i_ki_{k+1}} = 1, \forall k = 1, \cdots, l - 1 \). A component \( C \) of the graph \( g \) containing node \( i \) is a subset of \( \mathcal{N} \) such that \( \forall k \in C, i \) and \( k \) are connected and \( \forall j \in \mathcal{N}, \) if \( i \) and \( j \) are connected, \( j \in C \). Some special graphs discussed in this paper include stars, \( g^s \); complete graphs, \( g^c \) and regular graphs, \( g^r \). Graph \( g^s \) is said to be star if and only if there exists \( i \in \mathcal{N} \) such that \( g_{ij} = 1, \forall j \neq i \) and \( g_{jk} = 0, \forall j, k \neq i \). In this network, player \( i \) is said to be center of the graph. Graph \( g^c \) is said to be complete if and only if \( g_{ij} = 1, \forall i, j \in \mathcal{N} \). Graph \( g^k \) is said to be regular of degree \( k \) if \( N_i(g) = k, \forall i \in \mathcal{N} \).

Next we introduce the concept of maximal independent set. Given an undirected graph \( g \), a set \( M \) is said to be maximally independent with respect to \( g \) if and only if \( \forall i, j \in M, g_{ij} = 0 \) and \( \forall k \notin M \), there exists \( i \in M \), such that \( g_{ik} = 1 \). Furthermore,
a set \( M \) is a maximal independent set of order \( r \) with respect to \( g \) if and only if \( \forall i, j \in M, g_{ij} = 0 \) and there exist \( i_1, \ldots, i_r \in M \), such that \( g_{i_k k} = 1, \ldots, g_{i_k k} = 1 \), \( \forall k \notin M \). As shown in Jackson (2008), maximal independent sets always exist for any graph. The following example to illustrates this concept.

**Example 2.1. Maximal Independent Set with Respect to Star and Complete Graph.**

Let \( n = 6 \). The star and complete graph with 6 players are depicted in Figure 2.1. In the complete graph, maximal independent sets are: \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, which means that each player is a maximal independent set. Any other set will not be maximal independent set, since if the \( g_{ij} = 1 \), \( \forall i \neq j \in \{1, 2, 3, 4, 5, 6\} \). As for the star, there are two maximal independent sets: \{1\} and \{2, 3, 4, 5, 6\}.

![Figure 2.1: Maximal Independent Set with Respect to Star and Complete Graph](image)

The structure of information is modeled as follows. The true state of the world is represented by a random variable \( \hat{\theta} \), where \( \hat{\theta} \sim N(0, 1) \). Each player \( i \) observes a signal \( s_i \), conveying information about the true state of the world with Gaussian noise \( \varepsilon_i \sim N(0, 1) \). Here, \( \hat{\theta}, \varepsilon_1, \ldots, \varepsilon_n \) are assumed to be i.i.d. However, agents can make this signal more accurate by investing costly effort \( x_i \in [0, \infty) \). We assume that with effort \( x_i \), the signal player \( i \) will receive is \( s_i = \hat{\theta} + \varepsilon_i / \sqrt{x_i} \). Besides her own signal, player \( i \) can also observe the signals of her neighbors. Now, for any given specific network \( g \in G \), the information available for player \( i \) is \( \{s_i, \{s_j\}_{j \in N_i(g)}\} \). Based on the information obtained \( I_i \), player \( i \) updates her information about \( \hat{\theta} \) following Bayesian rule.

Given common knowledge about the structure of signals, the types of players, each player \( i \) chooses a set of links with others to access their information. This is represented as a vector \( g_i = (g_{i1}, \ldots, g_{ii-1}, g_{ii+1}, \ldots, g_{in}) \), where \( g_{ij} \in \{0, 1\}, \forall j \in \mathcal{N}/\{i\} \). Based on the network formed by \( g \), player \( i \) chooses a level of personal information acquisition \( x_i \in X \). Then, we have the strategy space for players in the society as
\( A_i = G_i \otimes X \otimes \Theta \), where \( g_i \in G_i = \{0, 1\}^{n-1} \); \( X : \mathcal{G} \to [0, \infty) \); \( \Theta : \mathcal{I} \to \mathbb{R}_+ \). More specifically, for player \( i \), \( \alpha_i = (g_i, x_i, \theta_i) \) represents her strategy in each stage: in stage one, she choose her neighbors \( \{j : g_{ij} = 1\} \); in stage two, given the realized network \( g \), she chooses her effort level described by \( x_i \); in stage three, facing information set \( \mathcal{I} \), she makes a prediction \( \theta_i \) about the true state of the world. If \( g_{ij} = 1 \), it means that player \( i \) offers to form a link with \( j \), and then both players can share information with each other. Here, we also define \( \mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n \) as the set of strategies of all players. An action profile \( \alpha = (g, x, \theta) \in \mathcal{A} \) specifies the network of relations \( g = (g_1, \cdots, g_n) \), the respective efforts chosen, \( x = (x_1, \cdots, x_n) \) and the predictions about the true state \( \theta = (\theta_1, \cdots, \theta_n) \). Note that \( \theta_i \) should depend on the information available \( \mathcal{I}_i \), which means that \( \theta_i \) is a measurable function of \( \mathcal{I}_i \). Now, given an action profile \( \alpha \) and realized signal \( \{s_i\}_{i \in N} \), the payoff for player \( i \) is:

\[
U_i(\alpha) = - (\theta_i - \hat{\theta})^2 - x_i/(1 + a_i)^2 - n_i(g) \eta
\]

And the corresponding expected payoff for player \( i \) is:

\[
E_{\mathcal{I}_i}(U_i) = -E_{\mathcal{I}_i}[(\theta_i - \hat{\theta})^2] - x_i/(1 + a_i)^2 - n_i(g) \eta
\]

here, \( \mathcal{I}_i = \{s_i\} \cup \{s_j : g_{ij} = 1 \text{ or } g_{ji} = 1\} \). The first term is the expected loss because the player’s prediction \( \theta_i \) differs from true state of the world; the second one is the utility loss from the effort involved in refining information and the last term is the linking cost. Furthermore, we assume that if \( i \) initiates a link to \( j \), there is a constant cost \( \eta \) independent of all other things. Here, \( a_i \geq 0 \) measures the ability of a player to refine information. The higher \( a_i \) is, the more efficiently player \( i \) refines information. As we can see from the form of utility function, information is non-rival in this framework. So, for any player \( i \), she has no incentive to misreport her information or prevent the neighbors from accessing her information even when her neighbors do not contribute to refining their information. In the following sections, we will firstly study some results in exogenous networks and then we come to the analysis of endogenous networks.

### 2.3 Results for Exogenous Network

We begin by studying the exogenous network situation. Because the information flow is two-way, in this section, it is assumed that \( g \) is exogenously given and undirected. \footnote{Here, there is another similar framework using the concept from information theory, which replace with the first part with the corresponding log function, standing for entropy of Normal distribution. Here, we use \( (1 + a_i)^2 \) just for the convenience of notation.}
Then player $i$’s strategy has two parts: $x_i$, the effort for refining information and $\theta_i$, the predication about true state. Here, even though players have no choice about their neighbors, without confusion, we still use the same notations as in Section 2.2 to denote the players’ strategies. The strategy space is identical for all players: $A_i = X \otimes \Theta$, where $X: G \to [0, \infty)$ and $\Theta: \mathcal{I} \to \mathbb{R}$. Also, we define $A = A_1 \otimes \cdots \otimes A_n$ as the joint strategy space. In this framework, given network $g$ and action profile $\alpha = (x_i,\theta_i)_{i \in \mathcal{N}}$, the payoff for player $i$ turns out to be:

$$U_i(\alpha; g, s_1, \cdots, s_n) = - (\theta_i - \hat{\theta})^2 - \frac{x_i}{(1 + a_i)^2}$$

Note that there is no cost for link formation since the network is exogenously given.

In this paper, the game is two-stage simultaneous game and we define the equilibrium concept as follows.

**Definition 2.1.** Given network structure $g$, strategy profile $\alpha^*$ is a Subgame Perfect Nash equilibrium (SPNE) if and only if

1. In stage 2, for any given effort level $x$, for each player $i$, $\theta^*_i$ is measurable to information set $\mathcal{I}_i$ and $-\mathbb{E}[\theta^*_i + \hat{\theta})^2 - x_i/(1 + a_i)^2 | \mathcal{I}_i] \geq -\mathbb{E}[(\theta_i - \hat{\theta})^2 + x_i/(1 + a_i)^2 | \mathcal{I}_i], \forall \mathcal{I}_i$-measurable function, $\theta_i$;
2. In stage 1, $-\mathbb{E}_{\mathcal{I}_i}[(\theta^*_i + \hat{\theta})^2 - x^*_i/(1 + a_i)^2] \geq -\mathbb{E}_{\mathcal{I}_i}[(\theta^*_i - \hat{\theta})^2 + x_i/(1 + a_i)^2] \forall x_i \in [0, \infty)$. Here, $\mathcal{I}_i = \{s_i, s_j : j \in \mathcal{N}_i(g)\}$.

We will begin by studying the final stage of the game about players’ predictions of the true state. In stage 2, consider a specific player $i$, who has $k$ neighbors denoted by $i_1, \ldots, i_k$, with each neighbor choosing effort $x_{ij}$ in stage 1, respectively. Then, player $i$’s relevant information set about the true state is $\mathcal{I}_i = \{s_i, s_{i_1}, \ldots, s_{i_k}\}$. We now have the following lemma to describe the optimal behavior.

**Lemma 2.1.** For player $i$, given her information $\mathcal{I}_i$, the best response function is:

If $\sum_{j=1}^k x_{ij} \geq a_i$, then $x^*_i = 0$; otherwise, $x^*_i = a_i - \sum_{j=1}^k x_{ij}$. Moreover, the optimal $\theta_i$ is given by $\theta^*_i = \mathbb{E}(\hat{\theta}|\mathcal{I}_i) = \frac{x_{ij}}{1 + x_i + x_{i_1} + \cdots + x_{i_k}} s_i + \sum_{j=1}^k \frac{x_{ij}}{1 + x_{i_1} + x_{i_2} + \cdots + x_{i_k}} s_{ij}$.

**Proof:** The proof is in the Appendix.

The proof of the Lemma 1 is mainly based on the theory of optimal Bayesian filtering (Vives, 2010). The linear form comes from the assumption of Normal noisy. And the weights are based on the quality of the signals. Intuitively, the better quality the received signal has, the more weight will assigned to it. As we can see from Lemma 1, if a player has no neighbors, she will choose $x_i = a_i$. If a player $i$ is isolated and
chooses effort \( x_i = 0 \), then the signal observed by her has infinite variance. However, player \( i \) still can use some information to guide her prediction. The optimal strategy is just to ignore the signal, and use prior belief about \( \hat{\theta} \) to set \( \theta_i = E(\hat{\theta}) = 0 \).

Remarks: By lemma above, we notice that i) Given other things fixed, the optimal effort \( x_i^* \) for refining information is non-decreasing with her ability \( a_i \); ii) The more effort \( j \) makes to refine information, the more weight her neighbors will assign to her signal for the predictions.

The two results in the above remark are consistent with intuition: The higher the ability of player \( i \), the less costly is effort, and therefore the more effort she will exert on refining information; the more player \( j \) contributes to refine her information, the better is the quality of her information, and the more weight will be assigned to her signal by player \( i \).

Now, based on players’ optimal strategy in stage 2, player \( i \)’s expected payoff following players’ effort levels \( x \) in stage 1 is given by 

\[-\frac{1}{1+x_i+\sum_{j \in N_i(g)} x_j} - \frac{x_i}{(1+a_i)^2} \].

As for the effort levels, an observation from Lemma 1 is that if player is isolated, she will choose effort \( x_i = a_i \). For those players choosing \( x_i = a_i \), they are denoted as specialists. We will define the specialized strategy profiles as follows.

Definition 2.2. A strategy profile \( \alpha \) is said to be specialized if and only if for any agent \( i \), either she chooses \( x_i = a_i \) or \( x_i = 0 \).

By this definition, a specialized strategy profile means that for any player, either she acts as if she has no neighbor or she totally free rides. Next, we come to study the optimal choice of \( x \). In stage 1, the players will simultaneously choose \( x \) with the expected payoff as 

\[-\frac{1}{1+x_i+\sum_{j \in N_i(g)} x_j} - \frac{x_i}{(1+a_i)^2}, \forall i \in N. \]

Now, we come to the theorem about the equilibria in stage 1.

Theorem 2.1. For any SPNE \( \alpha^* \), in stage 1, players’ strategies must satisfy 

\[x_i^* + \sum_{j \in N_i(g)} x_j^* \geq a_i, (x_i^* + \sum_{j \in N_i(g)} x_j^* - a_i)x_i^* = 0, \forall i \in N. \]

Furthermore, there always exists specialized SPNE.

Theorem 2.1 tells us that when a player’s neighbors contribute more than her desired one, she will just contribute nothing, otherwise, she will contribute until the quality of the aggregated signals reaches her desired level. Since the system of equations 

\[x_i^* + \sum_{j \in N_i(g)} x_j^* \geq a_i, (x_i^* + \sum_{j \in N_i(g)} x_j^* - a_i)x_i^* = 0, \forall i \in N \]

may have multiple solutions, multiple SPNEs may exist for the exogenous network case. For example, for the complete graph structure \( g^c \), \( \alpha^* \) is SPNE if and only if \( \sum_{i \in N} x_i^* = a \) in the homogeneous agents model.
2.3.1 Stability of Equilibria

As we have already mentioned in Section 2.3, multiple SPNEs may exist. In this section, we introduce stability to refine the SPNEs and we will see that the specialized equilibria are the only candidates for stable equilibria. In the literature of network economics and game theory, there are alternative criteria of stability. One criterion for stability is related to the non-credible threat, which takes the possibility of off-the-equilibrium play into account. According to this criterion, the concepts of sequential equilibrium and trembling-hand perfect equilibrium were introduced by game theorists for equilibrium refinements, see Fudenberg and Tirole (1991), Kan-dori et al. (1993) and Young (1996), among others. The idea is that if an equilibrium $e$ is stable for a game $\Pi$, then, the sequence of disturbed games $\Pi_\varepsilon$ around game $\Pi$ should have a sequence of equilibria $e_\varepsilon$ converge to $e$. Following this idea, Bala and Goyal (2000) and Feri (2007) introduce the concept of stochastic stability to refine the converging network structures in the framework of dynamic network formation games with decay. Durieu et al. (2011) also uses stochastic stability with logit rule for trembling-hand to study the relationship between stochastically stable states and the set of potential maximizers.

In this paper, we consider a classic definition of stability by Nash tâtonnement (Bramoullé and Kranton, 2016; Fisher, 1961). For stability of a SPNE $\alpha$, we ask whether players’ strategies lead back to $\alpha$ when there is an initial small deviation from $\alpha$ and players update their beliefs about other players’ actions adaptively. The main difference from the stochastic stability is: stochastic stability assumes there is a sequence of perturbed games and here stability assumes there is an initial perturbation of strategies. We define $f_i(x)$ as the best-response of player $i$ to a strategy profile $x$ and denote by $f$ the collection of these player best-response $f = \{f_1, ..., f_n\}$. Then, we have a stability criterion that can be defined as follows.

**Definition 2.3.** An equilibrium $x^*$ is said to have the property of stability if and only if there exists a positive number $\rho > 0$ such that for any vector $\epsilon$ satisfying $\forall i, |\epsilon_i| \leq \rho$ and $x^*_i + \epsilon_i \geq 0$, the sequence $x^{(n)}$ defined by $x^{(0)} = x^* + \epsilon$ and $x^{(n+1)} = f(x^{(n)})$ converges to $x^*$.

Before we go to the general results about the stability of Nash equilibrium in our model, we consider two examples to illustrate this.

**Example 2.2.** Stability of Nash Equilibria. We assume that there are 5 players, the core player 1 has competitiveness $a_1 = 5$; and periphery players have competitiveness as $a_2 = \cdots = a_5 = 2$, as shown in Figure 2.2. As we can see in the Figure 2.2(a), if the core player decreases her effort by a small amount $\epsilon$, the process will diverge, oscillating between Figure 2.2(b) and Figure 2.2(c); and Figure 2.2(b) and Figure 2.2(c) are stable.
In the example above, it seems that all specialized equilibria are stable, however, it is not always the case as the following example shows.

**Example 2.3. Stability of Specialized Equilibria.** We assume the network of neighbors are shown in Figure 2.3, and the types for all players are $a_1 = 4, a_2 = 1, a_3 = 1, a_4 = 1, a_5 = 1$, respectively. Then, there are only two Nash equilibria, with both being specialized equilibria. However, one of them Figure 2.3(b) is stable and the other one Figure 2.3(a) is unstable.
above two examples gives us some hints that to ensure the stability of equilibrium, the
free riders should have their neighbors contribute more than their optimal demand
for effort to refine her signal. We have the following theorem about the stability of
Nash equilibrium.

**Theorem 2.2.** For any given network structure, an equilibrium \( \mathbf{x} \) is stable if and only
if it is specialized and every non-specialist has their neighbors contributing strictly
more than \( a_i \), which means that \( \sum_{j \in N_i(\mathbf{g})} x_j > a_i, \forall i \in \{k: x_k = 0\} \).

**Proof:** The details of the proof are shown in the Appendix.

The intuition behind Theorem 2.2 is that for any non-specialist \( i \), if her neighbors
contribute more than \( a_i \), even though she contributes some effort \( \epsilon_i \) in some stage,
her neighbors will change their effort by \( \epsilon_i \), with \( \epsilon_i \) small enough, in the next stage,
her neighbors will still contribute more than \( a_i \), player \( i \) will come back to the original
state. However, it does not hold when her neighbors contribute no more than \( a_i \).

As we can see, our result about stability is a little different from that of Bramoullé
and Kranton (2007). Actually, the stability of equilibrium depends more on the
information structure than on the network structure. In their paper, a specialized
equilibrium is stable if and only if specialists forms a maximal independent set of
order 2. However, in our model, in Example 2.3(a), even the only non-specialist has
more than 2 neighbors, the equilibrium is not stable; also, in Example 2.3(b), even
though the specialists form a maximal independent set of order 1, however, it is still
a stable specialized equilibrium. If players are homogeneous, then, we have the same
conclusion as Bramoullé and Kranton (2007), as the following corollary shows.

**Corollary 2.1.** Bramoullé and Kranton (2007) In the homogeneous agents model
with each player having the type \( a_i = a \), one specialized equilibrium is stable if and
only if specialists forms a maximal independent set of order 2.

**Proof:** For a specialized equilibrium \( \mathbf{x} \), where the set of specialists is denoted by
\( \mathcal{S} \). Then, it must be that \( x_i = a, \forall i \in \mathcal{S} \) and \( x_i = 0, \forall i \notin \mathcal{S} \). Now, by Theorem
2.2, \( \sum_{j \in N_i(\mathbf{g})} x_j > a_i, \forall i \in \{k: x_k = 0\} \), which is equal to \( \sum_{j \in N_i(\mathbf{g}) \cap \mathcal{S}} a > a_i \),
\( \forall i \in \{k: x_k = 0\} \). Then, we have \( \#(N_i(\mathbf{g}) \cap \mathcal{S}) \geq 2, \forall i \in \{k: x_k = 0\} \). The proof is
concluded.

Now, we want to show that the stable equilibria in our paper satisfy the requirement
of sequential equilibrium. First of all, for each player \( i \), her strategy for \( x_i \) should
lie in the interval \([0, a_i]\), so, for the disturbed game, we assume that the strategy \( \sigma_i \)
must satisfy \( \sigma_i(x_i) > 0, \forall i \in \mathcal{N}, x_i \in [0, a_i] \). Based on this disturbed game, we define
the sequential equilibrium as follows (Fudenberg and Tirole, 1991):
Definition 2.4. An equilibrium \( x^* \) is said to be an sequential equilibrium if and only if:

i) \( \mathbb{E}(U_i(x_{i}^*, x_{-i}^*)) \geq \mathbb{E}(U_i(x_i, x_{-i}^*)), \forall x_i \in X_i, \forall i \in N. \)

ii) There exists a sequence of positive numbers \( \lambda_k \), and a sequence of strategies and belief \( \{\sigma_k^i, \mu_k\}_{k=1}^{\infty} \) satisfying \( \mathbb{E}(U_i(\sigma^k)|\mu^k) \geq \mathbb{E}(U_i(\sigma_{i}^k', \sigma_{-i}^k)|\mu^k) \), with the constraint \( \forall x_i \in [0, c_i], \sigma^k(x_i) \geq \lambda_k \). And also, \( \lim_{k \to \infty} \sigma^k(\{x^*\}) = 1 \) and \( \lim_{k \to \infty} \mu^k = \mu^* \), where \( \mu^* \) denotes the beliefs derived from strategy profile \( \{\sigma^k\}_{k=1}^{\infty} \) using Bayes rule.\(^2\)

Then, we will show that the stable equilibrium in Theorem 2.2 is also a sequential equilibrium as the following proposition demonstrates.

Proposition 2.1. If an equilibrium \( x^* \) is specialized and for every non-specialist \( i \), \( \sum_{j \in N_i} x_{i}^* > a_i \), then \( x^* \) is sequential equilibrium.

Proof: The details of proof are shown in Appendix. \( \blacksquare \)

Here, one thing to be noticed is that there exists sequential equilibria which are not stable. In other words, stable equilibria may be a proper subset of sequential equilibria, so stable equilibria can give us sharper prediction about the equilibrium refinement. This can be shown in the following example.

Example 2.4. Non-stable sequential equilibrium. We assume the network structure is shown in Figure 2.4, and the types for all players are \( a_1 = 2, a_2 = 1, a_3 = 1 \), respectively. Then, we can verify the equilibrium shown in Figure 2.4 is a sequential equilibrium, but is not stable.

Example 2.4: Non-stable Sequential Equilibrium

The analysis of why such an equilibrium is sequential equilibrium but not stable equilibrium is left in the Appendix.

\(^2\)Here, since the state space is continuous, the probability density is different from the probability of each state. We denote \( \sigma(\{x\}) \) as the probability of state \( x \), and \( \sigma(x) \) as the probability density of state \( x \). The same applies to \( \mu \).
2.4 Endogenous Network Formation

Galeotti and Goyal (2010) studied the network formation problem in the framework where players have two choice variables, one about information acquisition and the other about linking decisions, which is similar as our framework. They conclude that this leads to the law of the few phenomenon which means that only a small number of players will contribute to the local public goods, which will be enjoyed by others who connect to the suppliers. In this section, we will refine SPNEs by the criterion of stability in our heterogeneous agents model. As we can see, when the linking cost is relatively large, the criterion of stability can help us obtain a sharper prediction of the equilibrium network - star-like networks; however when linking cost is not large enough and players with highest types are not unique, the criterion of stability fails to do so.

In this part of the paper we assume that the network \( g \) is directed. Strategy space for player \( i \) is \( A_i = G_i \otimes X \otimes \Theta \), where \( g_i \in G_i = \{0, 1\}^{n-1} \), \( X : G \rightarrow [0, \infty) \) and \( \Theta : \mathcal{I}_i \rightarrow \mathbb{R} \). If \( g_{ij} = 1 \), it means that player \( i \) requests a link to \( j \), and if the link is formed, both players can share information with each other. Here, we also define \( A = A_1 \otimes \cdots \otimes A_n \) as the joint strategy set.

**Definition 2.5.** Strategy profile \( \alpha^* \) is a Subgame Perfect Nash equilibrium (SPNE) if and only if

i) In stage 3, for any given network \( g \) and effort level \( x \), for each player \( i \), \( \theta^*_i \) is measurable to information set \( \mathcal{I}_i \) and \( -\mathbb{E}[(\theta^*_i + \hat{\theta})^2 - x_i/(1 + a_i)^2 | \mathcal{I}_i] \geq -\mathbb{E}[(\theta_i - \hat{\theta})^2 + x_i/(1 + a_i)^2 | \mathcal{I}_i], \forall \mathcal{I}_i \)-measurable function, \( \theta_i \). Here, \( \mathcal{I}_i = \{s_i, s_j : j \in N_i(g)\}; \)

ii) In stage 2, for any given network \( g \), \( -\mathbb{E}_{\mathcal{I}_i}[(\theta^*_i + \hat{\theta})^2 - x^*_i/(1 + a_i)^2 | g] \geq -\mathbb{E}_{\mathcal{I}_i}[(\theta^*_i - \hat{\theta})^2 + x_i/(1 + a_i)^2 | g], \forall x_i \in [0, \infty); \)

iii) In stage 1, \( -\mathbb{E}_{\mathcal{I}_i}[(\theta^*_i + \hat{\theta})^2 - x^*_i/(1 + a_i)^2 | g^*] \geq -\mathbb{E}_{\mathcal{I}_i}[(\theta^*_i - \hat{\theta})^2 + x_i/(1 + a_i)^2 | g_i \cup g^*], \forall g_i \in G_i. \)

By Lemma 1, we have the optimal strategy for \( \theta \) in stage 3. In section 2.3, we have already studied the subgame Nash equilibrium in stage 2. Now, we come to the analysis of players’ strategies in stage 1. Because we will use the criterion introduced in section 2.3 to refine the SPNEs, we will study the stages 1 and 2 together even though we have already studied the subgame Nash equilibrium in stage 2. In stage 2, given a strategy profile \( \alpha \), the expected payoff for player \( i \) is:

\[
\mathbb{E}_{\mathcal{I}_i}(U_i) = -(1 + x_i + \sum_{j \in N_i(g)} x_j)^{-1} - x_i/(1 + a_i)^2 - n_i(g)\eta
\]
where, \( \mathcal{I}_i = \{s_i\} \cup \{s_j : g_{ij} = 1 \ or \ g_{ji} = 1\} \). The first term is the expected loss from the deviation of prediction from the true state of the world; the second one is the utility loss from contribution to refine information and the last term is the fixed linking cost.

First, we consider the group of players with the highest ability, and denote their type as \( \bar{a} \). By Theorem 2.2, stable equilibria must be specialized equilibria, so, by the form of their utility function, we have the following lemma about stable equilibria of the endogenous network formation game.

**Lemma 2.2.** Consider the types of the players are \( a_1, a_2, \cdots, a_n \), respectively and denote \( \bar{a} = \max_{i=1,2,\cdots,n} \{a_1, a_2, \cdots, a_n\} \), then if \( \#\{i : i = 1, 2, \cdots, n; a_i = \bar{a}\} = 1 \) or \( \eta > \frac{\bar{a}}{(1+\bar{a})^2} \), stable equilibrium will always exist; otherwise stable equilibria will not exist.

**Proof:** Firstly, we notice that for a specified player \( i \), if she established no relation with others, then her optimal choice gives her expected payoff as \( -(1 + a_i)^{-1} - a_i/(1 + a_i)^2 \); and if she established a link with others, the best she can get is \( -(1 + \bar{a})^{-1} - \eta \).

If \( \#\{i : i = 1, 2, \cdots, n; a_i = \bar{a}\} = 1 \), it means that the player with highest type is unique. Now, the SPNE such that \( g_{ij} = 1 \) only if \( a_j = \bar{a} \) and \( \eta \leq (1 + a_i)^{-1} + a_i/(1 + a_i)^2 - (1 + \bar{a})^{-1} \) will be stable.

If \( \eta > \frac{\bar{a}}{(1+\bar{a})^2} \), now for the players with the highest type, \( -(1 + a_i)^{-1} - a_i/(1 + a_i)^2 > -(1 + \bar{a})^{-1} - \eta \), they have no incentive to link with anyone else. Now, the SPNE such that \( g_{ij} = 1 \) only if \( a_j = \bar{a} \) and \( \eta \leq (1 + a_i)^{-1} + a_i/(1 + a_i)^2 - (1 + \bar{a})^{-1} \) will be stable.

If \( \eta \leq \frac{\bar{a}}{(1+\bar{a})^2} \), by Theorem 2.2, only stable equilibrium is specialized one. Assuming there is a stable equilibrium, then there must be one player contributes \( x_i = \bar{a} \). Now, the graph must be connected, for the reason \( \eta \leq \frac{\bar{a}}{(1+\bar{a})^2} \), which means each isolated player can profit by linking to the specialists. And if there are two specialists, then one specialist has the incentive by deviating from contributing \( x_i = \bar{a} \) to linking to the other specialist and reducing her contribution to 0. So, it must be a connected graph with only one specialist. However, by Corollary about homogeneous population, it can’t be stable. We come to a contradiction, then the lemma is concluded.

By Lemma 2.2, we find that if the cost of linking is relatively high, then the stable equilibrium will exist. Furthermore, if the linking cost is high enough, the stable equilibriums will be an empty network. In this case, the criterion of stability can help us refine the SPNEs. However, when the linking cost is small enough and there are more than one player has the highest type, it fails to refine the SPNEs. Now, we consider that there is more than one type of abilities, from the observation in the
exogenous network framework, stable equilibrium can be the case that players with highest ability are free riders, as Example 2.2(c) shows. However, the same thing cannot happen in the framework of endogenous network, as shown in the following lemma.

**Lemma 2.3.** Consider the types of the players are \(a_1, a_2, \ldots, a_n\), respectively. For any stable equilibrium \(\alpha\), assuming \(C\) is a component of \(g\), and denote \(a_C\) as \(a_C = \max\{a_i : i \in C\}\), then anyone with ability \(a_C\) should be a specialist.

**Proof:** Assume not, then there exists a player \(i\) such that \(a_i = a_C\), who is nonspecialist. By the fact that it’s stable equilibrium, then we have \(x_i = 0\).

According to Theorem 2.2, for player \(i\), \(\sum_{j \in N_i} x_j > a_C\) must hold. Along with the definition of \(a_C\) and the fact that \(x_j \leq a_j, \forall j \in N, i\) must have at least two neighbors. We rank all her neighbors in decreasing order according to their abilities, and denote the first two as \(i_1, i_2\).

Because \(x_i = 0\), \(i_1\) and \(i_2\) has no incentive to link with \(i\), then \(g_{i_1} = 1, g_{i_2} = 1\) should hold. Now, considering player \(i\)'s strategy, based on that she has already linked to \(i_1\), whether she will link to \(i_2\) decides on the marginal benefit, which is \(-\frac{1}{1+x_{i_1}+x_{i_2}} + \frac{1}{1+x_{i_1}} - \eta\). From the fact that she has \(i_1, i_2\) as her neighbors, \(-\frac{1}{1+x_{i_1}+x_{i_2}} + \frac{1}{1+x_{i_1}} - \eta > 0\) must hold.

Now, for player \(i_1\), there are two cases:

Case 1: All her neighbors are free riders, then she will find it beneficial to link to \(i_2\), for the reason that the marginal benefit \(-\frac{1}{1+x_{i_1}+x_{i_2}} + \frac{1}{1+x_{i_1}} - \eta > 0\). But then, it’s not stable, even not an equilibrium, contradiction.

Case 2: She has at least one neighbor who is not free rider. Then, this also can’t happen in equilibrium, contradiction.

This concludes the proof.

By Lemma 2.3, for any component of stable equilibrium, the highest ability players in that component will choose to be specialists. If we consider the strategy of players among different components, then players with highest ability among all players in the society should be specialists. Then we have the following theorem.

**Theorem 2.3.** Suppose that the types of the players are \(a_1, a_2, \ldots, a_n\), respectively and denote \(\bar{a} = \max_{i=1,2,\ldots,n}\{a_1, a_2, \ldots, a_n\}\), if \(#\{i : i = 1, 2, \ldots, n; a_i = \bar{a}\} = 1\) or \(\eta > \frac{\bar{a}^{3}}{(1+\bar{a})^2}\), there is a potential threshold \(\tilde{a}\) satisfying \(\frac{1}{1+\tilde{a}} + \frac{\tilde{a}}{(1+\tilde{a})^2} = \eta + \frac{1}{1+\tilde{a}}\).

If \(a_i = \bar{a}\), player \(i\)'s linking decision is \(g_{ij} = 0, \forall j \in N\); her effort is \(x_i = \bar{a}\);
If \( a_i \in (\hat{a}, \bar{a}) \), player \( i \)'s linking decision is \( g_{ij} = 0 \), \( \forall j \in \mathcal{N} \); her effort is \( x_i = a_i \);

If \( a_i < \hat{a} \), player \( i \)'s linking decision is \( g_{ij} = 1 \), \( \exists j \in \mathcal{N} \) and \( g_{ik} = 0 \), \( \forall k \in \mathcal{N}/j \); her effort is \( x_i = 0 \).

**Proof:** If \#\{\( i : i = 1, 2, \cdots, n; a_i = \bar{a} \)\} = 1 or \( \eta > \frac{\bar{a}}{(1+\bar{a})^2} \), according to Lemma 2.2, the stable SPNE exists. Also by Lemma 2.2, \( g_{ij} = 0 \) if \( a_i = a_j = \bar{a} \), \( \forall i, j \in \mathcal{N} \), which means the players with highest ability must be separated with each other. They will choose effort level \( x_i = \bar{a} \), these players are specialists, only potential linked by others.

For a specific player \( i \), if \(-\frac{1}{1+a_i} - \frac{a_i}{(1+a_i)^2} > -\eta - \frac{1}{1+\bar{a}} \), which means that her expected utility without linking to anyone else is greater than the one with linking to a player with highest type, she will choose not linking to any other such that \( g_{ij} = 0 \), \( \forall j \in \mathcal{N}/\{i\} \) and choose her own effort as \( x_i = a_i \). And these players are also specialists, however being isolated from all others.

For a specific player \( i \), if \(-\frac{1}{1+a_i} - \frac{a_i}{(1+a_i)^2} < -\eta - \frac{1}{1+\bar{a}} \), which means that her expected utility without linking to anyone else is less than the one with linking to a player with highest type, she will choose to link to one of the players with highest ability such that \( \exists j, a_j = \bar{a}, g_{ij} = 1, g_{ik} = 0, \forall k \in \mathcal{N}/\{i,j\} \) and choose \( x_i = 0 \). These players are free riders and link to specialists with highest ability.

As we can see from Theorem 2.3, for any stable equilibria, there are at most three types of players: highest type players - specialists connected by others; medium type players - isolated specialists; and low type players - free riders connecting to specialists. The intuition behind the theorem is that the players with the highest type have advantages to refine signals, so they will contribute; and for other individuals, if their ability is low enough, they will give up refining signal by themselves and connect to others to gather information. One example of such a network structure is about production structure for agriculture in ancient world: landowners rent their lands to tenant farmers; farmers with land will not just produce on their own lands and tenant farmers just connect to landowners. We notice that in stable equilibria, for any connected component, the law of the few happens, only the one with highest ability contribute, and all others will free ride. If the linking cost is so high, then only empty graph will survive and it’s stable; if the linking cost is so low, then no stable equilibria will exist. And if linking cost is on medium level, three types of players will appear, highest, medium and low type players as described in Theorem 2.3. We will use an example to illustrate this.

**Example 2.5. Stable Equilibria with Endogenous Network Formation.** We assume there is one player with \( a = 4 \); 8 players with \( a = 2 \) and 8 players with \( a = 1 \). If \( \eta \) is low enough, then all will connect to the one with \( a = 4 \), as shown in Figure 2.5(a); when \( \eta \) is medium high, then only the ones with \( a = 1 \) will connect, as shown in
Figure 2.5(b); and if $\eta$ is high enough, all players are isolated, as shown in Figure 2.5(c). As we can see, as $\eta$ grows, the graph will be sparser.

$\eta = 0.2$, $\hat{a} \approx 3.44$

$\eta = 0.4$, $\hat{a} \approx 1.72$

$\eta = 0.6$, $\hat{a} \approx 0.81$

Figure 2.5: Stability of Specialized Equilibria

In reality, there are more than one player who contribute for the following reasons: 1) the homophily effect will incur different linking costs among different types, in this way, $\eta$ will depend on the similarities between players; 2) even though, one player contributes no effort, she may still transmit some information, in this sense, if the signal is $s_i = \hat{\theta} + \varepsilon_i/(1 + x_i)$, then players in the society have the incentive to have more neighbors, even though their neighbors make no effort to refine their information. All of these are further studies we will focus on. Also, we will study the dynamic model of network formation and information exchange, which will lead us to a deeper understandings about network formation phenomena.

2.5 Conclusion

One criterion about good prediction of the real world is stability. In this paper, we studied the stable equilibria in both exogenous networks and endogenous networks. We found that only specialized equilibria can be stable. For specialized equilibria, first of all, they are the only potential candidates to be stable; and at the same time, specialized equilibria are finite, which can help us reduce the number of Nash equilibria significantly. Also, we found the phenomenon of the law of the few, where only highest type players contribute to the quality of the information and others either are isolated from the society or link to highest type players and free ride for the information.

One extension of our model is about changing the information structure. In our model, if one player doesn’t make effort to refine her information, then others have
no incentive to link to her. Especially, for the highest type players, they have no incentive to attract other players to link to them. However, in some case, even though one player doesn’t make effort to increase her information’s quality, her information is still helpful for other players. In this sense, highest type players may have incentives to offer better information than her type to attract other players linking to them, so that they can benefit from the players who link to them.

Another extension is about changing the network formation structure. We can introduce some random effect into our model, for example, we can assume that there is some positive probability that two players can form a link exogenously without any cost, or there is some positive probability that the existing link in the network will break down. So, in this sense, we introduce dynamic network formation framework into our model, which potentially will give us more understanding about the network in the real world.
2.6 Appendix

Proof of Lemma 2.1:

Proof: Firstly, we will calculate player $i$’s updated distribution about $\hat{\theta}$ based on her information $\mathcal{I}_i = \{s_i, s_{i_1}, \ldots, s_{i_k}\}$. Because $\theta, s_i, s_{i_1}, \ldots, s_{i_k}$ are all Normal distributed, the updated $\hat{\theta}$ has the same distribution as $\gamma + \gamma_0 s_i + \gamma_1 s_{i_1} + \cdots + \gamma_k s_{i_k}$. Now, we need to determine values for $\gamma, \gamma_0, \gamma_1, \cdots, \gamma_k$.

We observe that $\mathbb{E}(\hat{\theta} | \mathcal{I}_i) = \gamma + \gamma_0 s_i + \gamma_1 s_{i_1} + \cdots + \gamma_k s_{i_k}$, then by tower property of conditional expectation, we have the following system of equations:

$$
\mathbb{E}(\hat{\theta}) = \mathbb{E}[\mathbb{E}(\hat{\theta} | \mathcal{I}_i)] = \mathbb{E}(\gamma + \gamma_0 s_i + \gamma_1 s_{i_1} + \cdots + \gamma_k s_{i_k}) \\
\mathbb{E}(\hat{\theta} s_i) = \mathbb{E}[s_i \mathbb{E}(\hat{\theta} | \mathcal{I}_i)] = \mathbb{E}(\gamma s_i + \gamma_0 s_i s_i + \gamma_1 s_{i_1} s_i + \cdots + \gamma_k s_{i_k} s_i) \\
\cdots \\
\mathbb{E}(\hat{\theta} s_{i_k}) = \mathbb{E}[s_{i_k} \mathbb{E}(\hat{\theta} | \mathcal{I}_i)] = \mathbb{E}(\gamma s_{i_k} + \gamma_0 s_i s_{i_k} + \gamma_1 s_{i_1} s_{i_k} + \cdots + \gamma_k s_{i_k} s_{i_k})
$$

which is equal to:

$$
0 = \gamma + 0\gamma_0 + 0\gamma_1 + \cdots + 0\gamma_k \\
1 = 0\gamma + (1 + \frac{1}{x_i})\gamma_0 + 1\gamma_1 + \cdots + 1\gamma_k \\
\cdots \\
1 = 0\gamma + 1\gamma_0 + 1\gamma_1 + \cdots + (1 + \frac{1}{x_{i_k}})\gamma_k
$$

We have solution for this system of equations as: $\gamma = 0, \gamma_0 = \frac{x_i}{1+x_i+x_{i_1}+\cdots+x_{i_k}}$ and $\forall j = 1, \cdots, k, \gamma_j = \frac{x_{i_j}}{1+x_i+x_{i_1}+\cdots+x_{i_k}}$.

Now, player $i$’s expected utility is:

$$
\mathbb{E}(\mathcal{U}_i | \mathcal{I}_i) = -\mathbb{E}[(\theta_i - \mathbb{E}(\hat{\theta} | \mathcal{I}_i) + \mathbb{E}(\hat{\theta} | \mathcal{I}_i) - \hat{\theta})^2 | \mathcal{I}_i] - x_i/(1 + a_i)^2 \\
= -\mathbb{E}[(\theta_i - \mathbb{E}(\hat{\theta} | \mathcal{I}_i))^2 | \mathcal{I}_i] - 2\mathbb{E}[(\theta_i - \mathbb{E}(\hat{\theta} | \mathcal{I}_i)) \cdot (\mathbb{E}(\hat{\theta} | \mathcal{I}_i) - \hat{\theta}) | \mathcal{I}_i] \\
= -\mathbb{E}[(\theta_i - \mathbb{E}(\hat{\theta} | \mathcal{I}_i))^2 | \mathcal{I}_i] - \mathbb{E}[(\mathbb{E}(\hat{\theta} | \mathcal{I}_i) - \hat{\theta})^2 | \mathcal{I}_i] - x_i/(1 + a_i)^2 \\
= -\mathbb{E}[(\theta_i - \mathbb{E}(\hat{\theta} | \mathcal{I}_i))^2 | \mathcal{I}_i] - \frac{1}{1+x_i+x_{i_1}+\cdots+x_{i_k}} - x_i/(1 + a_i)^2
$$

The third equality holds for the reason that $\theta_i$ is $\mathcal{I}_i$-measurable, so, $\mathbb{E}[(\theta_i - \mathbb{E}(\hat{\theta} | \mathcal{I}_i)) \cdot (\mathbb{E}(\hat{\theta} | \mathcal{I}_i) - \hat{\theta}) | \mathcal{I}_i] = (\theta_i - \mathbb{E}(\hat{\theta} | \mathcal{I}_i)) \cdot \mathbb{E}[(\mathbb{E}(\hat{\theta} | \mathcal{I}_i) - \hat{\theta}) | \mathcal{I}_i] = 0$. Now, we have that the best strategy for $\theta_i$ should be $\theta_i^* = \mathbb{E}(\hat{\theta} | \mathcal{I}_i)$, which is equal to $\frac{x_i}{1+x_i+x_{i_1}+\cdots+x_{i_k}}s_i + \sum_{j=1,\ldots,k} \frac{x_{i_j}}{1+x_i+x_{i_1}+\cdots+x_{i_k}}s_{i_j}$.
Now, after some simple calculation, we have the expected utility of player \( i \) as 
\[
- \frac{1}{1+x_i+x_{i1}+\ldots+x_{ik}} - \frac{x_i}{(1+a_i)^2}.
\]
Also, by first order conditions, we can easily have the solution as:

If \( \sum_{j=1}^{k} x_{ij} \geq a_i \), then \( x_i^* = 0 \); otherwise, \( x_i^* = a_i - \sum_{j=1}^{k} x_{ij} \).

Proof of Theorem 2.1:

Proof: For the first part, by F.O.Cs, player \( i \)'s best response function with respect to \( x \in \otimes_{i \in \mathcal{N}} [0, a_i] \), i.e. \( b_i(x) = a_i - \sum_{j \in \mathcal{M}_i(x)} x_{ij} \) if \( a_i - \sum_{j \in \mathcal{M}_i(x)} x_{ij} \leq 0 \); otherwise, \( b_i(x) = 0 \). So, for SPNE, it must satisfy \( x_i^* + \sum_{j \in \mathcal{M}_i(x)} x_{ij}^* \geq a_i \), \( (x_i^* + \sum_{j \in \mathcal{M}_i(x)} x_{ij}^* - a_i)x_i^* = 0 \), \( \forall i \in \mathcal{N} \).

For the second part, for a fixed graph \( g \), we use maximal independent set successively to come to the results. The procedure is as follows.

Step 1, we consider the group of players with the highest ability, denoted as \( \mathcal{V}_1 = \{i \in \mathcal{N} : a_i \geq a_j, \forall j \in \mathcal{N}\} \), now we have the corresponding graph constrained on \( \mathcal{V}_1 \), denoted as \( g|_{\mathcal{V}_1} \). Then for graph \( g|_{\mathcal{V}_1} \), we can find one maximal independent set with respect to \( g|_{\mathcal{V}_1} \), denote as \( \mathcal{S}_1 \), meaning the specialized players at step 1.

Step 2, we remove all players in \( \mathcal{S}_1 \) and their neighbors, for the rest players \( \mathcal{V}_1 = \mathcal{N} - (\mathcal{S}_1 \cup \mathcal{N}_{\mathcal{S}_1}(g)) \), we consider the group of players with the highest ability among the remaining players, denoted as \( \mathcal{V}_2 = \{i \in \mathcal{N}_1 : a_i \geq a_j, \forall j \in \mathcal{N}_1\} \). Then we can define the similar graph as the first step \( g|_{\mathcal{V}_2} \), then find one maximal independent set with respect to \( g|_{\mathcal{V}_2} \), denoted as \( \mathcal{S}_2 \), meaning the specialized players at step 2.

Continuing this procedure, as \( \mathcal{N} \) is finite, the procedure will finally terminate after finite steps. We denote the union of \( \mathcal{S}_k \) at each step as \( \mathcal{S} \). Then the specialized equilibrium will be that \( \forall i \in \mathcal{S}, x_i = c_i \); otherwise, \( x_i = 0 \). It's easy to check such a strategy profile is Nash equilibrium. The second part is concluded.

Proof of Theorem 2.2:

Proof: We denote the transition function as \( f(x^{(n)}) \), where \( f_i(x^{(n)}) = \max\{a_i - \sum_{j \in \mathcal{M}_i(x)} x_{ij}^{(n)}, 0\} \). Firstly, we will show that if \( x' \geq x \), then \( f \circ f(x') \geq f \circ f(x) \). This is because that if \( x' \geq x \), then \( \max\{a_i - \sum_{j \in \mathcal{M}_i(x)} x_{ij}', 0\} \leq \max\{a_i - \sum_{j \in \mathcal{M}_i(x)} x_{ij}, 0\} \), that is, \( f(x') \leq f(x) \). Now, following the same argument, we have \( f \circ f(x') \geq f \circ f(x) \).

Consider first an equilibrium that is not specialized and denote by \( J = \{j : 0 < x_j < a_j\} \). Let \( \rho \) be a small number such that \( 0 < \rho < \min\{a_j - x_j, \frac{x_j}{n-1} : j \in J\} \). Define a perturbation \( \epsilon \) as: \( \forall j \in J, \epsilon_j = \rho \) and \( \forall j \notin J, \epsilon_j = 0 \).
If $i$ is such that $x_i = 0$, then $\epsilon_i = 0$, now $f_i(x + \epsilon) = 0$ and $f_i(f(x + \epsilon)) \geq 0 = x_i + \epsilon_i$.

If $i$ is such that $x_i = a_i$, then $\epsilon_i = 0$. Now, $\forall j \in N_i(g)$, $x_j = \epsilon_j = 0$. So, $f_i(x + \epsilon) = f_i(f(x + \epsilon)) = a_i = x_i + \epsilon_i$.

If $i$ is such that $0 < x_j < a_j$, by definition of $\rho$, we have $\forall j \in J$, $\sum_{k \in N_i(g)} \epsilon_k < x_j$, then $f_i(x + \epsilon) = x_i - \sum_{k \in N_i(g)} \epsilon_k$. So, $f_i(f(x + \epsilon)) = x_i + \sum_{j \in J \cap N_i(g)} \sum_{k \in N_j(g)} \epsilon_k$.

Since $i$ has at least one neighbor in $J$, hence $\sum_{j \in J \cap N_i(g)} \sum_{k \in N_j(g)} \epsilon_k \geq \rho$ and then $f_i(f(x + \epsilon)) \geq x_i + \epsilon_i$.

Summing all above, we have that $f \circ f(x + \epsilon) \geq x + \epsilon$. And by the statement that: if $x' \geq x$, then $f \circ f(x') \geq f \circ f(x)$, we have $f^{(2k)}(x + \epsilon) \geq x + \epsilon$ for any $k$. Then we have that $x$ can’t be stable equilibrium.

Now, consider a specialized equilibrium $x$ such that $i$ is a non-specialist who has neighbors such that $\sum_{j \in N_i(g)} x_j = a_i$, and denote specialists as $S$. Then if we let $\rho$ be a small number such that $0 < \rho < \min \{a_j, \frac{x_j}{n-1} : j \in S \}$. Then, we define one perturbation $\epsilon$ such that $\epsilon_i = \rho$ and $\epsilon_l = 0$, $\forall l \neq i$. Then, $x_i^{(1)} = x_i$, $\forall l \notin \{i\} \cup (N_i(g) \cap S)$ and $x_i^{(1)} = x_i - \rho$, $\forall l \in N_i(g) \cap S$. And then, $x_i^{(2)} = n_i(g) \rho \geq \rho$.

Also, if $l \in N_i(g)$ and $x_l = 0$, then it’s sure that $x_l^{(2)} \geq x_l$; if $l \notin N_i(g) \cap S$, then it must be that $x_l^{(2)} = x_l$, for the reason that she faces only non-specialists and all non-specialists will choose 0 in the first stage; if $l \notin \{i\} \cup (N_i(g) \cap S)$, then whatever neighbors they face, their neighbors don’t increase their effort in the first stage, then $x_l^{(2)} \geq x_l$. Then we have $f \circ f(x + \epsilon) \geq x + \epsilon$, following the same argument as non-specialists case, we have equilibrium $x$ is not stable.

Now, we will prove that for specialized equilibria $x$ such that all non-specialists with $\sum_{j \in N_i(g)} x_j > a_i$ is stable. Defining $\rho = \min \{\sum_{j \in N_i(g)} x_j - a_i : j \in S \}$, now, we consider any perturbation $\epsilon$ such that $\forall i, |\epsilon_i| < \rho/(n-1)^2$ and $\epsilon_i + x_i \geq 0$.

For $x^{(1)}$, if $i \notin S$, then $\sum_{j \in N_i(g)} x_j^{(0)} = \sum_{j \in N_i(g)} (x_j + \epsilon_j) = \sum_{j \in N_i(g)} x_j + \sum_{j \in N_i(g)} \epsilon_j$.

Now, $\sum_{j \in N_i(g)} x_j^{(0)} \geq a_i + \sum_{j \in N_i(g)} x_j - a_i - \rho$, which is greater than $a_i$ by definition of $\rho$. So, $x_i^{(1)} = 0$. If $i \in S$, because they only face non-specialized neighbors, then $x_i^{(1)} = x_i - \sum_{j \in N_i(g)} \epsilon_j$.

As for $x^{(2)}$, if $i \notin S$, then $\sum_{j \in N_i(g)} x_j^{(1)} = \sum_{j \in N_i(g)} (x_j - \sum_{j \in N_i(g)} \epsilon_j)$, which is greater than $\sum_{j \in N_i(g)} x_j - (n-1)^2 \cdot \max \{\epsilon_i : i \in N \} \geq \sum_{j \in N_i(g)} x_j - \rho$, which implies that $\sum_{j \in N_i(g)} x_j^{(1)} \geq a_i$. So, $x_i^{(2)} = 0 = x_i$. If $i \in S$, because specialists only face non-specialists, and non-specialists don’t contribute in the first stage, so $x_i^{(2)} = a_i = x_i$. So, $f \circ f(x + \epsilon) = x$, hence $f^{(k)}(x + \epsilon) = x$, $\forall k \geq 2$. The theorem is concluded.

Proof of Proposition 2.1:
Proof: Condition i) of sequential equilibrium is obvious. For condition ii), we assume that for any given $\lambda_k$,
non-specialist $i$’s strategy is of the form: $\sigma^k_i(\{x^s_i\}) = 1 - a_i \lambda_k\sigma^k_i(x_i) = \lambda_k, \forall x_i \in [0, a_i]/\{x^s_i\};$ denote $x^{(k)}_i = x^s_i$, here.
Believes for non-specialist $i$: $\mu^k_i(\{x^s_i\}) = 1 - a_i \lambda_k, \mu^k_i(x_i) = \lambda_k, \forall x_i \in [0, a_i]/\{x^s_i\}$.
Specialist $j$’s strategy if of the form: $\sigma^k_j(\{x^{(k)}_j\}) = 1 - a_j \lambda_k; \sigma^k_j(x_j) = \lambda_k, \forall x_j \in [0, a_j]/\{x^{(k)}_j\}$;
Believes for specialist $j$: $\mu^k_j(\{x^{(k)}_j\}) = 1 - a_j \lambda_k, \mu^k_j(x_j) = \lambda_k, \forall x_j \in [0, a_j]/\{x^{(k)}_j\}$.
Here, $x^{(k)}_j$ will be defined in the analysis.

We will show that for appropriate choice of $\lambda_k$, such $\{\sigma^k, \mu^k\}_{k=1}^\infty$ defines a sequential equilibrium.

Firstly, for a set $S$, we define $E_S$ as the expectation operator on the probability space $\{\otimes j \in S [0, a_i], \mathcal{L}(\otimes j \in S [0, a_i]), U\}$, which is uniform distribution on state space $\otimes j \in S [0, a_i]$ with measurable sets as class of Lebesgue-measurable sets. Also, we denote $\xi = \min_{i: x^*_i = 0} \{a_i - \sum_{j \in N_i(g)} x_j\}$.

Now, for given $\lambda_k > 0$, we have the optimal problem for player $i$ as:

$$\max_{\sigma^k_i} \left[ -\left( \prod_{j \in N_i(g)} (1 - \lambda_k a_j) \right) \frac{1}{1 + x_i + \sum_{j \in N_i(g)} x^{(k)}_j} - \sum_{m=1}^{n_i(g)} \frac{1}{c^k_{n_i(g)}} \sum_{i_1, \ldots, i_m \in N_i(g)} \left( \prod_{l=1, \ldots, m} (1 - \lambda_k a_{i_l}) \right) \lambda^{n_i(g) - m}_{k} \times \left( \prod_{l=m+1}^{n_i(g)} 1 \right) \frac{1}{1 + x_i + \sum_{j \in N_i(g)} x^{(k)}_j} \frac{1}{1 + x_i + \sum_{j \in N_i(g)} x^{(k)}_j} \frac{1}{1 + x_i + \sum_{j \in N_i(g)} x^{(k)}_j} \right] - \lambda^{n_i(g)}_{k} \frac{1}{1 + x_i + \sum_{j \in N_i(g)} x^{(k)}_j}$$

$$\text{s.t. } \sigma^k_i(x_i) \geq \lambda_k \quad \forall x_i \in [0, a_i]$$

If $i$ is a specialist, by ignoring the constraint, we observe that for any $x_j, \forall j \in N_i(g)$, marginal benefit for $-\frac{1}{1 + x_i + \sum_{j \in N_i(g)} x^{(k)}_j}$ will be no less than than $\frac{1}{1 + \sum_{j \in N_i(g)} x^{(k)}_j}$. Then, the expected marginal benefit at $x^{(k)}_i$ should be no less than $(\prod_{j \in N_i(g)} (1 - \lambda_k a_j)) \frac{1}{1 + x_i} + (1 - (\prod_{j \in N_i(g)} (1 - \lambda_k a_j))) \frac{1}{1 + \sum_{j \in N_i(g)} a_j} - \frac{1}{1 + a_i}$, furthermore, no less than $(\prod_{j \in N_i(g)} (1 - \lambda_k a_j)) \frac{1}{1 + x_i} - \frac{1}{1 + a_i}$, and also, we also know that if for some $x \in [0, a_i]$, the corresponding expected marginal benefit is greater than 0, then optimal solution should lie in $[x, a_i]$. Now, we can find $\lambda^{(i)} > 0$ such that $\forall \lambda_k < \lambda^{(i)}$, the corresponding expected marginal benefit at $a_i - \xi/n$ is greater than 0, that is, $(\prod_{j \in N_i(g)} (1 - \lambda_k a_j)) \frac{1}{1 + a_i - \xi/n} - \frac{1}{1 + a_i} > 0$. So, optimal solution should lie in $[a_i - \xi/n, a_i]$. Also, the optimal solution for $x_i$, denoted as $x^{(k)}_i$ will converge to $a_i$ as
\( \lambda_k \to 0 \), for the reason that the solution to \( \left( \prod_{j \in N_i(g)} (1 - \lambda_k a_j) \right) \frac{1}{1 + x_i} - \frac{1}{1 + a_i} = 0 \) approaches to \( a_i \) as \( \lambda_k \to 0 \). If we add the constraint to the analysis, we have \( \sigma^k\{x^{(k)}\} = 1 - a_j \lambda_k; \sigma^k(x_i) = \lambda_k, \forall x_i \in [0, a_i]/\{x_i^{(k)}\} \).

As for non-specialist \( i \), if we ignore the constraint, firstly, we observe that for any \( x_j \geq 0, \forall j \in N_i(g) \), marginal benefit for \( \frac{1}{1 + x_i + \sum_{j \in N_i(g)} x_j} \) will be no greater than 1. Then, for the second and third part, expected marginal benefit will be no greater than \( \left( \prod_{j \in N_i(g)} (1 - \lambda_k a_j) \right) \frac{1}{1 + \sum_{j \in N_i(g)} x_j^{(k)}} + \left( 1 - \left( \prod_{j \in N_i(g)} (1 - \lambda_k a_j) \right) \right) - \frac{1}{1 + a_i} \), but we have \( \sum_{j \in N_i(g)} x_j^{(k)} > a_i \) by assumption. Along with \( x_i^{(k)} \) for specialists constructed before, there exists \( \lambda^{(i)} > 0 \) such that \( \forall \lambda_k < \min\{\lambda^{(i)}, \lambda^{(j)} : x_j^{*} = a_j\} \), marginal benefit will be less than 0, then optimal strategy for player \( i \) should be \( x_i = 0 \). If we add the constraint into the analysis, we have \( \sigma^k\{x_i^{*}\} = 1 - a_i \lambda_k; \sigma^k(x_i) = \lambda_k, \forall x_i \in [0, a_i]/\{x_i^{*}\} \).

More rigorous analysis will use functional Lagrangian methods by Fréchet derivatives, which is not shown here. Now, denote \( \lambda = \min\{\lambda^{(i)} : i \in N\} \), we can define \( \lambda_k = \lambda/k \). We can check that the sequence of \( \lambda_k \) and corresponding \( \{\sigma^k, \mu^k\}_{k=1}^\infty \) satisfy the condition ii) of sequential equilibrium, this concludes Proposition 2.1.

Proof of *Equilibrium in Example 2.4 is sequential but not stable*:

*Proof*: By Theorem 2.2, it is obviously not stable.

As for the sequential rationality, we will find a sequence \( \lambda_k \), and a sequence of equilibria such that the sequence converges to the equilibrium in Example 2.4. The analysis is similar with proof of Proposition 2.1 but a little more difficult.

Firstly, condition i) of sequential equilibrium is obvious. For condition ii), we assume that for any given \( \lambda_k \), Player 1’s strategy is of the form: \( \sigma^k\{x_1^{(k)}\} = 1 - 2\lambda_k; \sigma^k(x_1) = \lambda_k, \forall x_1 \in [0, 2]/\{x_1^{(k)}\} \); here, \( x_1^{(k)} \) is the atom of the mixed strategy of equilibria of perturbed game \( k \), which will be defined during the analysis.

Believes about player 1: \( \mu^k\{x_1^{(k)}\} = 1 - 2\lambda_k, \mu^k(x_1) = \lambda_k, \forall x_1 \in [0, 2]/\{x_1^{(k)}\} \).

Player \( j = 2, 3 \)’s strategy if of the form: \( \sigma^k\{x_j^{(k)}\} = 1 - \lambda_k; \sigma^k(x_j) = \lambda_k, \forall x_j \in [0, 1]/\{x_j^{(k)}\} \).

Believes about players 2, 3: \( \mu^k\{x_j^{(k)}\} = 1 - \lambda_k, \mu^k(x_j) = \lambda_k, \forall x_j \in [0, 1]/\{x_j^{(k)}\} \).

We will show that for appropriate choice of \( \lambda_k \), such \( \{\sigma^k, \mu^k\}_{k=1}^\infty \) defines a sequential equilibrium. Actually, we will verify that for each small enough \( \lambda_k > 0 \), there exists a \( \varepsilon_k \) such that \( x_1^{(k)} \in [0, \varepsilon_k]; x_2^{(k)}, x_3^{(k)} \in [1 - \varepsilon_k, 1] \), and furthermore, \( \varepsilon_k \to 0 \) as \( \lambda_k \to 0 \).
Now, for given \( \lambda_k > 0 \), we have the optimal problem for player 1 as:

\[
\begin{align*}
\max_{\sigma_1} & \quad \left[ - (1 - \lambda_k)^2 \sum_{j \in N_i(x)} \frac{1}{1 + x_1 + \sum_{j \in N_i(x)} x_j} - (1 - \lambda_k) \int_0^1 \frac{1}{1 + x_1 + x_3^{(k)} + x_2} - \lambda_k dx_2 \\
& \quad - (1 - \lambda_k) \int_0^1 \frac{1}{1 + x_1 + x_3^{(k)} + x_3} \lambda_k dx_3 \right] \\
\text{s.t.} & \quad \sigma_k(x_1) \geq \lambda_k \quad \forall x_1 \in [0, 2]
\end{align*}
\]

Now, for the first term, we observe that if \( x_2^{(k)}, x_3^{(k)} \in [1 - \varepsilon_k, 1] \), marginal benefit for

\[
\int_0^1 \frac{1}{1 + x_1 + x_3^{(k)} + x_2} - \lambda_k dx_2
\]

will be no greater than \( \frac{1}{(1 + x_1 + 2(1 - \varepsilon_k))^2} \). As for the 2nd, 3rd and 4th term, the total marginal benefits will be no greater than \( (1 - \lambda_k)^2 \). So the total marginal effect at \( x_1 \) will be no greater than \( \frac{1}{(1 + x_1 + 2(1 - \varepsilon_k))^2} \). Now, there exists \( \varepsilon_k = \max \{ \varepsilon > 0 : \frac{1}{(1 + x_1 + 2(1 - \varepsilon_k))^2} (1 - \lambda_k)^2 + (1 - (1 - \lambda_k)^2) - 1/9 \geq 0 \} \) such that it is optimal for player 1 to weight some action \( x_1(k) \in [0, \varepsilon_k] \) to have the highest utility.

As for player 2 or 3, for example player 2, her optimal problem comes to be:

\[
\begin{align*}
\max_{\sigma_2} & \quad \left[ - (1 - 2\lambda_k) \frac{1}{1 + x_2 + x_1^{(k)}} - 2\lambda_k \int_0^1 \frac{1}{1 + x_1 + x_2} dx_1 \\
& \quad - x_2/(1 + 1)^2 \right] \\
\text{s.t.} & \quad \sigma_k(x_2) \geq \lambda_k \quad \forall x_2 \in [0, 1]
\end{align*}
\]

Now, if she believes that \( x_1^{(k)} \in [0, \varepsilon_k] \), so for the first term, marginal benefit will be no less than \( (1 - 2\lambda_k) \frac{1}{1 + x_2 + x_1^{(k)}} \) and for the second term, the expected marginal benefit will be no less than \( 2\lambda_k \frac{1}{1 + x_1 + a_2} \). Then the total expected marginal benefit at \( x_2 \) will be no less than \( (1 - 2\lambda_k) \frac{1}{1 + x_2 + x_1^{(k)}} + 2\lambda_k \frac{1}{1 + a_1 + a_2} - \frac{1}{(1 + a_2)^2} \). Now, \( \varepsilon_k = \max \{ \varepsilon > 0 : (1 - 2\lambda_k) \frac{1}{1 + x_2 + x_1^{(k)}} + 2\lambda_k \frac{1}{1 + a_1 + a_2} \geq 0 \} \) such that it is optimal for player 2 to put as much weight as possible on some action \( x_2(k) \in [0, 1 - \varepsilon_k] \) to have the highest utility.

Now, if we define \( \varepsilon_k = \min \{ \varepsilon_1, \varepsilon_2 : (1 - \varepsilon_1)^2 + (1 - (1 - \lambda_k)^2) - 1/9 = 0; (1 - 2\lambda_k) \frac{1}{1 + x_2 + x_1^{(k)}} + 2\lambda_k \frac{1}{1 + a_1 + a_2} - \frac{1}{(1 + a_2)^2} = 0 \} \), we can see that \( \varepsilon_k \to 0 \) as \( \lambda_k \to 0 \).

And also, \( x_1^{(k)} \in [0, \varepsilon_k]; x_2^{(k)}, x_3^{(k)} \in [1 - \varepsilon_k, 1] \), which verify the sequential rationality of the equilibrium shown in Example 2.4.
CHAPTER 3

THE STRUCTURE OF TRADE NETWORKS

The doctrine of free trade, however widely rejected in the world of politics, holds its own in the sphere of the intellect.

— Frank Taussig (1905, 65)

This paper examines the structure of trade relationships with coordination failures. The static model shows that as the gains from trade, the communication quality or the importance of the fundamental value increases, or the coordination difficulty or the risk of trade opportunity decreases, the optimal market structure tends toward a complete market where all members trade with each other. On the opposite side, the optimal market structure tends toward Autarky. Furthermore, the dynamic model demonstrates that when potential neighbors become more scarce, people care more about the future, or new technology breakthroughs occur more frequently, it is optimal to have more neighbors to back up for the potential technological breakthrough. An additional dynamic model shields insights into why the intellectual history documents an evolution towards the increasing support for free trade.

3.1 Introduction

Trade is perhaps the most important activity in modern human society, prevailing almost at every corner of the modern world. Many organizations (like European Economic Area and World Trade Organization, etc.) emerge, facilitating nations building trade relationship with each other. It is widely believed that “...freedom of
trade is on the whole economically more beneficial than protection is one of the most fundamental propositions economic theory has to offer for the guidance of economic policy”, as shown in Johnson (1971). The belief about free trade has survived along the long repeated debates among economists ever since Adam Smith foresaw the essence of trade in his book, *An Inquiry into the Nature and Causes of the Wealth of Nations* and continues to receive overwhelming support from professional economists today. In the paper Krugman (1988), Paul Krugman wrote:

*One of the things that almost all economists agree on is the desirability of free trade. Adam Smith said it; David Ricardo, the great English economist of the early nineteenth century, provided a mathematical justification; and ever since, an understanding of the reasons why international trade is a good idea and free trade is best has been a key part of the professional training of every economist. Probably no other idea so well defines what an economist is: every economist understands Ricardos theory, and almost nobody else does. Or as the Nobel laureate Paul Samuelson puts it, the concept of comparative advantage that underlies the economists case for free trade is one of the few ideas in economics that is both important and right without being obvious.*

However, free trade was not a widespread idea among economists prior to Adam Smith. At that time, it was widely believed that an appropriate use of government trade restrictions was likely a better economic policy than free trade. In his illuminating intellectual history book of free trade, *Against the Tide* (Irwin, 1996), Douglas Irwin traces views on the virtues and vices of foreign trade back to early Greek and Roman writers, demonstrating the evolution of people towards trade along the history of humankind. There are various reasons why people in the early human history tended to think negatively about trade. All of these reasons are related to different kinds of conflicts associated with trade activities.

In the paper, we take into considerations both advantages and disadvantages of trade, where advantages bring cash flow for a member from each of her trade partners and disadvantages bring negative shocks, whose occurrences are modeled as a Poisson process. We firstly study three different kinds of trade networks: ideal trade networks, optimal trade networks and equilibrium trade networks. It is shown that ideal trade networks do not always exist. And for the optimal trade networks, as cash flow and communication quality increase or explicit value of private information, difficulty of coordination, frequency of negative shocks and uncertainty about the state of negative shocks decrease, the optimal trade networks will tend to change from Autarky to complete trade networks where all members trade with each other. We also fully describe all equilibrium trade networks by the system of equations, showing that in the limit, the equilibrium trade networks approach the optimal trade networks by the criterion of efficiency.
We also propose two dynamic models, one of them analyzes the optimal time to build trade relationships between countries; and the other one studies the optimal size of trade organizations when there is scarcity of potential members. We find that the optimal time to build a trade relationship depends on the quality of communication, the explicit value of private information, the difficulty of coordination, the frequency of negative shocks and the uncertainty about the state of negative shocks. These models offer insight into why people’s attitudes towards trade have become more positive (Irwin, 1996). The second model demonstrates that as the frequency of technology breakthroughs increases, the frequency of members arriving decreases or the discount rate decreases, trade organizations will tend to increase their size to take full advantage of potential technology breakthroughs.

Our paper belongs to the literature in organizational economics, most prominently (Bolton et al., 2012; Dessein et al., 2016; Dessein and Matouschek, 2008; Dessein and Santos, 2006), where it is assumed that the number of the things of interest is exogenous and fixed. In our paper, the number is endogenously determined, and the optimal size and the structure of trade networks will be studied. This paper is also part of literature in trade networks, Goyal and Joshi (2006); Furusawa and Konishi (2007); Kinateder and Merlino (2017); Lake (2017) investigate the trade relationship based on network formation game. This paper differs from them by studying the behavior about choice of trade partners without specific network formation game involved.

3.2 Benchmark Model

In this paper, it is assumed that there are $n$ members in the economy, $i \in \mathcal{N} = \{1, 2, ..., n\}$, which can be seen as families, cities, countries, organizations, etc. It is assumed to be a continuous-time model. At each time, members make decisions about their trade partners, as well as adaptive and coordinative actions when necessary. Establishment of trade relationship between two members needs permission from both sides, which is a reasonable assumption in the framework of trade relationship. Here, there is no explicit cost to establish trade relationship, however, there does exist implicit cost from trade relationship between trade partners, as we can see soon.

Trade Networks — To represent trade relationship among members, we follow the conventions in the literature of networks (see Jackson (2008)). In this paper, the trade relationship among members is represented as a graph $\mathbf{g}$, and we denote $\mathcal{G}$ as the set of all possible graphs on $n$ vertices. Here, $\mathbf{g}$ is assumed to be undirected and unweighted network, that is to say, $g_{ij} \in \{0, 1\}, \forall i, j$, where $g_{ij} = 1$ means that member $i$ connects with member $j$. Also, we denote $\mathcal{N}_i(\mathbf{g}) = \{j \in \mathcal{N} : g_{ij} = 1\}$ as the set of agents who are partners of $i$ and $n_i(\mathbf{g})$ as the cardinal number of the set
Special graphs like empty graphs, $g^e$; complete graphs, $g^c$ and regular graphs, $g^r$ will be discussed in this paper. Graph $g^e$ is said to be empty if and only if $g_{ij} = 0$, $\forall j \neq i$; graph $g^c$ is said to be complete if and only if $g_{ij} = 1$, $\forall i, j \in N$; and graph $g^r$ is said to be regular of degree $k$ if $N_i(g) = k$, $\forall i \in N$. Empty graphs and complete graphs are special cases of regular graphs. And $g - ij$ stands for a new graph where link $ij$ is deleted from $g$ if $ij \in g$; $g + ij$ stands for a new graph where link $ij$ is added to $g$. The set of components of a network $(N, g)$ is denoted as $C(N, g)$ where $\forall(N', g') \in C(N, g)$, $(N', g')$ is connected and if $i \in N'$, $ij \in g$, then $j \in N'$, $ij \in g'$.

Pros and Cons of Trade — As discussed in the introduction, both positive and negative things can happen to the members with trade relationship. On one hand, the positive part of trade relationship between two members is assumed to bring return $P_i$ for both of them at each time $t$, where $\{P_t: t \in [0, \infty)\}$ are independent random variables with mean $p$. So, at time $t$, with trade network as $g_t$, the cash flow for member $i$ is $n_i(g_t)P_t$. Here, the positive things can be assumed to be the welfare gain from exchanges among trade partners. On the other hand, the negative part of trade relationship brings negative shocks to trade participants from their trade partners. Here, the negative shocks can be seen as the conflicts between members and their trade partners. For member $i$, it is assumed that the arrival processes of negative shocks from each of her trade partners follow independent a Poisson processes with Poisson rate $\lambda$. So, the more trade partners member $i$ has, it will bring more cash flow to her, and at the same time, she will face more negative shocks from her trade partners. More specifically, the negative shocks for member $i$ follows a Poisson process with Poisson rate $n_i(\tilde{g})\lambda$, where $\tilde{g} \equiv \{g_t, t \in [0, \infty)\}$ stands for the trade network process. We denote the attack arrival process for member $i$ as $\{A_{i\mid g}(t), t \in [0, \infty)\}$, where $A_{i\mid g}(t) = \#\{s \leq t: \text{attack happens to } i \text{ at time } s\}$. Here and in the following parts of the paper, $\#S$ stands for the number of elements in the set $S$.

Adaption and Coordination — At each time, when an attack happens to member $i$, the state of the attack is represented as a random variable $\theta_{i,t}$, following a normal distribution with variance $\sigma^2_\theta$ and mean $\hat{\theta}_t$, where $\hat{\theta}_t$ can be seen as fundamental value about the attack. Member $i$ owns private information about the attack, so her strategy will depend on privately perfectly observed signal $\theta_{i,t}$ and fundamental value $\hat{\theta}_i$. Her strategy will take into considerations of her trade partners’ actions, when she tries to minimize the miscoordination cost. Introducing similar idea of Dessein and Santos (2006) and Dessein et al. (2016), the implicit cost from negative shocks comes from two parts: maladaption cost and miscoordination cost. More specifically, at time $t$, when an attack arrives for $i$, given a particular

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1 As we can see later from Lemma 3.2, without receiving information successfully, members optimally choose action $\hat{\theta}_i$. In this sense, $\hat{\theta}_i$ is treated by members as fundamental value to react the attack. With information available, members will update their actions optimally based on fundamental value.
realization \( \theta_t = (\theta_{1,t}, \theta_{2,t}, ..., \theta_{n,t}) \); a choice of actions, \( q_t = (q_{1,t}, q_{2,t}, ..., q_{n,t}) \), with \( q_{i,t} = \{q_{i1,t}, q_{i2,t}, ..., q_{inn,t}\} \) and trade network \( g \), the loss is assumed to be as follows,

\[
L_{ij|g}(q_t|\theta_t) = \phi(q_{ii,t} - \theta_{i,t})^2 + \beta \sum_{j \in N_i(g)} (q_{ji,t} - q_{ii,t})^2
\]  

(3.1)

here the loss depends on (i) how well member \( i \) adapts her action \( q_{ii,t} \) to utilize her private information \( \theta_{i,t} \), and (ii) how well her trade partners’ coordinating their actions \( q_{ji,t} \) to \( q_{ii,t} \). The parameter \( \phi \) measures the relative importance of member \( i \)'s private information or explicit value of private information and \( \beta \) measures relative importance of coordination or difficulties of coordination among trade partners. Another perspective of \( \phi \) shows that its inverse \( \frac{1}{\phi} \) measures the relative importance of fundamental value.

Now, it is assumed that the discount rate is \( \gamma \) for the members, and then given stochastic process \( \vec{\theta} = \{\theta_t, t \in [0, \infty)\} \); trade network process \( \vec{g} \); and actions \( \vec{q} = \{q_t, t \in [0, \infty)\} \) of members when needed, the expected net gain from trade for member \( i \) can be represented as:

\[
\mathbb{E}[\pi_i(\vec{q}|\vec{g}, \vec{\theta})] = \mathbb{E} \int_0^\infty \gamma e^{-\gamma t} \left[ n_i(g_t)P dt - L_i(q_t|g_t, \theta_t)dA_{ij|g}(t) \right]
\]  

(3.2)

here, given \( \vec{g}, \vec{\theta} \) and \( A_{ij|g} \) are assumed to be independent. The first part of the equation is the cash flow from trade activities and the second part is the potential cost when an attack happens to member \( i \), where \( A_{ij|g} \) has a jump at time \( t \). According to the assumption about attack process, the net gain can be simplified as:

\[
\mathbb{E}[\pi_i(\vec{q}|\vec{g}, \vec{\theta})] = \int_0^\infty \gamma e^{-\gamma t} n_i(g_t) \left[ p - \lambda \mathbb{E} \left( L_i(q_t|g_t, \theta_t) \right) \right] dt
\]  

(3.3)

Communication Frictions. — Similar as in Dessein et al. (2016), the coordination of trade partners is based on communication channel which is assumed to be noisy. A member’s communication quality \( r \) stands for the probability that her partners learn her message. So, if member \( i \) has communication quality \( r \), she can send message successfully to her trade partners with probability \( r \) when an attack happens to her. Otherwise, with the probability \( 1 - r \), communication is uninformative, which will incur some miscoordination cost.

Timing. — At each time \( t \), the members determine their trade partners based on information they obtain. At the same time, if member \( i \) is under an attack at time \( t \), the timeline of events will go as follows:
i. $\theta_{i,t}$ is realized and observed by member $i$ who suffers an attack.

ii. Communication state: Member $i$ communicates the realization of $\theta_{i,t}$ to her trade partners, and with probability $r$, the information is successfully transmitted. Here, whether the information is successfully transmitted is unknown to $i$.\footnote{Here, in our framework, it does not matter whether the realization of $\theta_{i,t}$ is sent or the action is sent by member $i$, as can be seen in section 3.2.1.}

iii. Action stage: Member $i$ makes her decision about adaptive action $q_{ii,t}$ based on her available information, and her trade partners will make coordinating actions $\{q_{ji,t}, j \in N_i(g_t)\}$ based information transmitted to them.

3.2.1 Optimal Adaptive Actions and Coordinative Actions

As mentioned before, when member $i$ encounters an attack, to deal with such an attack, she needs to negotiate with all her trade partners to react, which may bring maladaptation and miscoordination costs. It is assumed that members are rational, so in equilibrium, the optimal strategy form $s_{i|g_t}(\theta_{i,t})$ is common knowledge.\footnote{We point out this fact here because of the assumption that the information of $\theta_{i,t}$ is transmitted. Actually, it is not necessary to have this by just assuming that they transmit signals about actions. But the timeline is a little different, it should be the member who encounters an attack acts firstly and her trade partners act according to her action. Here, with $\theta_{i,t}$ transmitted, member who encounters an attack and her trade partners can be assumed to act simultaneously.}

The coordinative actions of trade partners are based on whether the information is transmitted successfully or not. For each of member $i$’s trade partners $j$, when $\theta_{i,t}$ is successfully received with probability $r$, he will optimally set $q_{ji,t} = s_{i|g_t}(\theta_{i,t}) = q_{ii,t}$. Otherwise, with probability $1 - r$, trade partner $j$ will choose coordinative action based on his information and knowledge, which are $\hat{\theta}_i$ and $s_{i|g_t} : \theta_{i,t} \to \mathbb{R}$. When trade partner $j$ of member $i$ did not receive the signal successfully, even though he does not know the actual action of member $i$, he does know member $i$’s strategy form, so his coordinative strategy comes to be $q_{ji,t} = s_{ji,t}^u(s_{i|g_t})$ (here, superscript $u$ stands for when he did not receive the signal). So, the expected loss, $\mathbb{E}[L_{i|g_t}(\theta_{i,t})] \equiv \max_{q_t} \mathbb{E}[L_i(q_t|g_t, \theta_{i,t})]$, with realization $\theta_{i,t}$, is equal to:

$$\phi(s_{i|g_t}(\theta_{i,t}) - \theta_{i,t})^2 + \beta n_i(g_t)(1 - r)(s_{i|g_t}(\theta_{i,t}) - s_{ji,t}^u(s_{i|g_t}))^2$$ (3.4)

Firstly, we notice that member $i$’s trade partners’ objective is to minimize the expected loss of member $i$, which is:

$$\mathbb{E}_{\theta_{i,t}}[\phi(s_{i|g_t}(\theta_{i,t}) - \theta_{i,t})^2 + \beta n_i(g_t)(1 - r)(s_{i|g_t}(\theta_{i,t}) - s_{ji,t}^u(s_{i|g_t}))^2]$$ (3.5)

which leads to Lemma 3.1 about the optimal coordinative strategies of members as follows.
Lemma 3.1. For any member $i$’s adaptive strategy $s_{i|g} : \Theta_i \to \mathbb{R}$, her trade partners’ optimal strategy when he does not receive the signal successfully is: $s_{j|s_{i|g}}^*(\theta_i) = \mathbb{E}_{\theta_i}(s_{i|g}(\theta_i))$.

Lemma 3.1 shows that when trade partner $j$ of member $i$ receives no signal of $i$’s private information, he will choose his optimal coordinative strategy based on expectation of $i$’s adaptive behavior. To study the optimal adaptive behavior of members, we have the definition as follows.

Definition 3.1. Given trade network $g$, member $i$’s optimal adaptive strategy $s_{i|g} : \Theta_i \to \mathbb{R}$ satisfies that $\forall \theta_i \in \Theta_i$, 

$$s_{i|g}(\theta_i) \in \arg \min_{s_i \in \mathbb{R}} \left[ \phi(s_i - \theta_i)^2 + \beta n_i(g)(1 - r)(s_i - \mathbb{E}_{\theta_i}(s_{i|g}(\theta_i)))^2 \right].$$

So, member $i$’s optimal adaptive strategy $s_{i|g}(\theta_i)$ is a fixed point of $\phi(s_i - \theta_i)^2 + \beta n_i(g)(1 - r)(s_i - \mathbb{E}_{\theta_i}(s_{i|g}(\theta_i)))^2$. As for optimal adaptive strategy of member $i$, she takes into consideration both loss from deviation about true state and loss of miscoordination from her trade partners. Even though she has information about true state of the attack, she optimally chooses not to fully adhere to that information, which may reduce miscoordination loss. The optimal adaptive strategies of members are demonstrated in the following lemma.

Lemma 3.2. At time $t$, given trade networks $g_t$, when member $i$ encounters an attack, her optimal adaptive strategy, $s_{i|g_t}(\cdot) : \Theta_{i,t} \to \mathbb{R}$, is linear in her private information $\theta_{i,t}$:

$$s_{i|g_t}(\theta_{i,t}) = \hat{\theta}_{i,t} + \alpha_{i,t}(g_t)(\theta_{i,t} - \hat{\theta}_{i,t})$$

where $\alpha_{i,t}(g_t)$ measures members’ adaptiveness to her private information, which can be written as follows:

$$\alpha_{i,t}(g_t) = \frac{\phi}{\phi + \beta n_i(g_t)(1 - r)}$$

which increases with $\phi$, $r$ and decreases with $\beta$ and her degrees in the trade network $g_t$.

Lemma 3.2 tells us that for each member, to deal with an attack, her optimal strategy depends linearly on her private information. Here, $\alpha_{i,t}(g_t)$ measures the implicit value of her private information, which gives weight of her private information on her optimal decision. When $\alpha_{i,t}(g_t)$ becomes greater, member $i$ will optimally utilize more of her private information when negotiating with her trade partners. On the other hand, when $\alpha_{i,t}(g_t)$ becomes smaller, it is optimal to refer more to fundamental value to react to such an attack. Furthermore, we observe that $\alpha_{i,t}(g_t)$ decreases with
β, n_i(g_t) and increases with φ, r. It tells us that members will sacrifice their private information to alleviate miscoordination loss when complexity n_i(g_t) or difficulties β grows. At the same time, when explicit value of private information φ or communication quality r increases, members will utilize more of her private information.

Now, we will look closer at trade partners’ coordination behavior. As we have already seen, when he observes his trade partner’s adaptive behavior successfully, he will imitate her behavior; otherwise, he will follow the expectation about his trade partner’s adaptive behavior. And the expectation of member i’s adaptive behavior is actually the fundamental value \hat{\theta}_i. So, members’ coordinative behaviors follow simple rule: either imitate their trade partners’ adaptive behavior whenever it is possible, or just use the fundamental value to coordinate his trade partner.

Now, given trade network g_t at time t, when an attack happens to member i, by substituting members optimal adaptive strategy and coordinative strategy in to loss function, we have the expected loss for member i as:

$$\mathbb{E}[L_i(g_t)] = \frac{\phi \sigma^2 \beta n_i(g_t)(1-r)}{\phi + \beta n_i(g_t)(1-r)}$$

which depends only on g_t. Here, when member i encounters an shock, the expected loss from each of her trade partner will increase with \phi, β, n_i(g_t) and decrease with r. This will simplify the functional form of net gain from trade of member i as:

$$\mathbb{E}[\pi_i(g_t)] = \int_0^\infty \gamma e^{-\gamma t} n_i(g_t) \left[ p - \lambda \phi \sigma^2 \frac{\beta n_i(g_t)(1-r)}{\phi + \beta n_i(g_t)(1-r)} \right] dt$$

3.2.2 Ideal Number of Trade Partners

In section 3.2.1, we have studied the optimal adaptive behaviors of members when they encounter an attack and the optimal coordinative behaviors of members when their trade partners encounter an attack. We found that when an attack happens to member i at time t, she will choose partially utilizing her private information, which depends on explicit value of information φ, communication quality r, coordination difficulties β and coordination complexity n_i(g_t). The minimized expected loss when member i encounters an attack at time t depends only on trade networks g_t and other exogenous parameters. In this section, we will study the ideal number of trade partners of the members, with exogenous parameters p, λ, φ, β, r, σ^2_θ, γ.

When exogenous parameters p, λ, φ, β, r, σ^2_θ, γ are given at time t, so the ideal number of trade partners of each member is solution of the problem \min_{n \in \mathbb{N}} n[p - \lambda \phi \sigma^2 \frac{\beta n(1-r)}{\phi + \beta n(1-r)}].
As we can see, for member $i$, $p - \lambda \phi \sigma^2_\theta \frac{\beta_n (1-r)}{\phi + \beta n (1-r)}$ is decreasing with the number of her trade partners, which means that the expected instantaneous net gain from each trade partner is less when a member has more trade partners. The reason for this is that the coordination problem will be more complicated when members have more trade partners. However, on the other hand, if a member has more trade partners, the cash flow is also more for her, which will potentially increase her total utility. So, when trying to decide their trade partners, the members face tradeoff between the extra coordination complexity and the extra cash flow coming from more trade partners. Based on this tradeoff, we have the following lemma about the ideal number of trade partners for members.

**Lemma 3.3.** Define $\bar{p}_k = \int_{k-1}^{k} \lambda \phi \sigma^2_\theta (1 - \frac{\phi^2}{(\phi + \beta x (1-r))^{2/3}}) dx, k \in \mathbb{N}$, then for member $i \in \mathcal{N}$, her ideal number of trade partners as a function of $p$ is,

$$n^*_i(p) = \begin{cases} 0 & \text{if } p < \bar{p}_1 \\ \{k-1, k\} & \text{if } p = \bar{p}_k \\ k & \text{if } p \in (\bar{p}_k, \bar{p}_{k+1}) \\ \infty & \text{if } p > \phi \sigma^2_\theta. \end{cases}$$

Here, $\bar{p}_k = \int_{k-1}^{k} \lambda \phi \sigma^2_\theta (1 - \frac{\phi^2}{(\phi + \beta x (1-r))^{2/3}}) dx$ is the extra cost for each member when her trade partners increase from $k-1$ to $k$. In the framework of all members are identical, each member has the same function about ideal number of trade partners, we denote them as $n^*(p)$ for this situation and denote $n^*$ without $p$ as the ideal number of trade partners when $p$ is given and fixed. It has two parts, the first part comes from adaptive cost with one extra trade partner; the second part comes from complexity increase from coordination with an additional trade partner. However, it is not guaranteed that the ideal number of trade partners for all members can be achieved according to the well-known theorem in graph theory (see Chartrand et al. (2015)). The theorem says that $k$-degree regular graph of order $n$ exists if and only if $kn$ is even and $k \leq n - 1$. We have the following lemma about ideal trade networks.

**Lemma 3.4.** Ideal trade network where each member achieves her ideal number of trade partners exists if and only if i) $p < \bar{p}_1$; or ii) $p = \bar{p}_k$, and $n \geq k$; or iii) $p \in (\bar{p}_k, \bar{p}_{k+1})$, $n \geq k + 1$ and $nk$ is even.

Otherwise, for any trade networks, at least one member can not achieve her ideal number of trade partners.

As we can see from Lemma 3.4, ideal trade networks may not exist for two reasons. On reason is that the total population in the economy is not large enough, and the other reason is that the total population $n$ and ideal number of trade partners are
not both even. With taking both of these into considerations, in the next sections, we will introduce the concept of optimal and equilibrium networks and gives a full description of the network structure.

3.3 Static Model: Optimal and Equilibrium Trade Networks

As we have already mentioned before, the ideal number of trade partners for every member can not be guaranteed. In this section, we introduce two concepts about trade networks: optimal and equilibrium trade networks and study the trade network structure under these two criterions. When exogenous parameters \( p, \lambda, \phi, \beta, r, \sigma^2, \gamma \) do not evolve with time, so the optimal and equilibrium trade network problems come to maximize the expected instantaneous net gain

\[
\pi_i(g_t) = p - \lambda \phi \sigma^2_0 \beta n_i(g_t)(1-r) + \bar{\beta} n_i(g_t)(1-r)
\]

at time \( t \), which is the same for all \( t \in [0, \infty) \). Then, it comes to be a static model. Firstly, we come to the analysis of optimal trade networks.

Definition 3.2. Trade network \( g \) is an optimal trade network if and only if \( \forall g' \in \mathcal{G} \),

\[
\sum_{i \in \mathbb{N}} \mathbb{E}[\pi_i(g)] \geq \sum_{i \in \mathbb{N}} \mathbb{E}[\pi_i(g')].
\]

In the following part about optimal trade networks, we study the case where \( n \) is even.\(^5\) We will study the relationship between the exogenous parameters and the structure of optimal trade networks as shown in the proposition below.

Proposition 3.1. As for optimal trade network, given parameters \( p, r, \lambda, \phi, \beta, \sigma^2_0 \) and total population \( n \) which is even:

i. If \( p \geq \int_{k-1}^{k} \lambda \phi \sigma^2_0 (1 - \frac{\phi^2}{\phi+\beta x(1-r)\pi}) dx \), or in other words, \( p \geq \bar{p}_{n-1} \), the optimal trade network will be complete graph, which is a perfect market economy where all members trade with each other.

ii. If \( \lambda \phi \sigma^2_0 \beta(1-r) \leq p < \int_{k-1}^{k} \lambda \phi \sigma^2_0 (1 - \frac{\phi^2}{\phi+\beta x(1-r)\pi}) dx \), or in other words, \( \bar{p}_1 \leq p < \bar{p}_{n-1} \), the optimal trade network consists of components of regular graphs with degree \( n^*(p) \), which are local market economies where members will only trade within their specific local market.

\(^4\)Here and in the following parts of the paper, to simplify notations, in the framework with constant parameters, with a little abuse of notations, we denote \( g = g = \{g_t, t \in [0, \infty)\} \).

\(^5\)If \( n \) is not even and ideal number of trade partners for each member \( n^* \) is also not even, then optimal trade network still exists by choosing one member with degree \( n^*-1 \) and all other members with degree \( n^* \). The existence of such a graph is proved in the proof of Proposition 3.2. The statements will remain almost the same in this scenario.
iii. If $p < \lambda \sigma_\theta^2 \frac{\beta(1-r)}{\phi + \beta(1-r)}$, or in other words, $p < \bar{p}_1$, the optimal trade network will be empty graph, which is an Autarky where members refuse to trade with each other.

Furthermore, for the optimal trade networks, the degree is increasing with $p$, $r$ and decreasing with $\phi$, $\beta$, $\sigma_\theta^2$.

Proposition 3.1 shows that, in the optimal trade network, on one hand, when the constant returns for trade partners are higher or the communication quality is better, members tend to have more trade partners; on the other hand, when the cost of deviation from true state, cost of miscoordination, or uncertainty of the state becomes more, members tend to decrease their trade members. Despite the direction of these factors’ influence on choices of trade partners for the members, these factors also play different roles about limiting behavior of members’ choices of trade partners.

As for the limiting behavior, expected returns from the trade, $p$; and uncertainty of shocks, $\sigma_\theta^2$ demonstrate similar behaviors about members’ choices of trade partners in optimal trade networks. When $p$ is large enough or $\sigma_\theta^2$ is small enough, the optimal trade network will be complete graph, which is perfect market where all members trade with each other. On the other hand, when $p$ is small enough or $\sigma_\theta^2$ is large enough, the optimal trade network will be empty graph, which is an Autarky. So, as $p$ or $\sigma_\theta^2$, the optimal trade network ranges from Autarky to perfect market economy. However, $\phi$, $\beta$ will behave differently, range of optimal network when varying only $\phi$ or $\beta$ is not always from Autarky to perfect market economy. For example, we consider the case for explicit value of private information $\phi$: even though the degree of optimal trade network decreases with $\phi$, when $\phi$ approaches infinity, the optimal network will not converge to Autarky when $\beta$ is small enough, more specifically, when $p < \sigma_\theta^2 \beta (1-r)$. The reason for this is that when explicit value of her private information $\phi$ is very high, she can choose fully utilize her private information, and marginal loss from each extra trade partner is bounded by $\sigma_\theta^2 \beta (1-r)$. Similar argument applies to $\beta$, when $\beta$ is high enough, members will totally abandoned their private information, use the fundamental value to coordinate with their trade partners.

As we have already seen that $1/\phi$ can be seen as the relative importance of fundamental value. Now, Proposition 3.1 tells us that if for one society, the importance of fundamental value is more for its members, it will more likely to be totally market economy where members trade with each other. On the other hand, if the importance of private information is more for its members, the optimal way for them is to negotiate with each other, where total market economy is not the optimal trade network.

We have studied the optimal trade networks as above, however, the resulting networks may be not optimal one. Optimal trade networks usually evolve when there
is a social planner to maximize the total utility of the whole society. But, if there is no such a social planner, the resulting trade networks may be different from optimal trade networks. In the following part of this section, we will study the trade networks formed when members in the economy form trade relationship among them individually and rationally. We will study all possible equilibrium trade networks and gives a full description of them. For the definition of equilibrium trade network, we use the criterion of pairwise stability introduced by Jackson and Wolinsky (1996), and the equilibrium is defined as follows.

**Definition 3.3.** Trade network $g$ is an equilibrium trade network if and only if:

1. for all $ij \in g$, $E[\pi_i(g)] \geq E[\pi_i(g - ij)]$ and $E[\pi_j(g)] \geq E[\pi_j(g - ij)]$, and

2. for all $ij \notin g$, if $E[\pi_i(g + ij)] > E[\pi_i(g)]$ and $E[\pi_j(g + ij)] < E[\pi_j(g)]$.

By definition of equilibrium trade network, $g$ is an equilibrium trade network means that anyone in the network has no incentive to destroy the trade relationship with her trade partners unilaterally and also has no incentive to form any other trade relationship with other members who are willing to do so. To fully describe the equilibrium trade networks, we notice that it is not always that any $n$ integers can be degrees of a graph, so, in order to study equilibrium trade networks, we introduce Erdős-Gallai criterion (see Graham and Nesetril (2012)) as follows.

**Definition 3.4.** The set of integers $\{d_1, \cdots, d_n\}$ is said to satisfy Erdős-Gallai criterion if and only if the permutation $\{d^1, \cdots, d^n\}$ of $\{d_1, \cdots, d_n\}$ where $d^1 \geq \cdots \geq d^n$ satisfy

$$\sum_{i=1}^{k} d^i \leq k(k - 1) + \sum_{k=i+1}^{n} \min(d^i, k), \forall 1 \leq k \leq n.$$  

Now, we have the following proposition about equilibrium trade networks.

**Proposition 3.2.** For the model with homogenous members, equilibrium trade network $g$ with $m$ components such that $n_i(g) = d_i, i \in \mathcal{N}$ exists if and only if there exist $m$ groups and function $h : \mathcal{N} \to \{1, \cdots, m\}$ which assigns members to different groups such that:

1. $\forall i \in \mathcal{N}$, we have $d_i \leq n^*$;

2. If for $k \in \{1, \cdots, m\}$, there exists $i \in \mathcal{N}$ such that $d_i < n^*$ and $h(i) = k$, then either $d_i = \#(\{i : h(i) = k\}) - 1$ or $\{i \in \mathcal{N} : d_i < n^*\}$ is a singleton;
iii. The set \( \{ k : 1 \leq k \leq m, \exists i \in \mathcal{N} \text{ such that } h(i) = k \text{ and } d_i < n^* \} \) is a singleton.

iv. \( \forall k \in \{1, \cdots, m\}, \{ d_i : h(i) = k \} \) satisfies Erdős-Gallai criterion; \( d_i \geq 1; \sum_{h(i)=k} d_i \) is even and \( \sum_{h(i)=k} d_i \geq 2(\#(\{i : h(i) = k\}) - 1) \);

And equilibrium trade network always exists.

Now, we can have all equilibrium trade networks by finding all \( n \) integers satisfying i), ii), iii) and iv) in Proposition 3.2. For each equilibrium trade network there exists \( n \) integers satisfying i), ii), iii) and iv) in Proposition 3.2 and for each \( n \) integers satisfying i), ii), iii) and iv) in Proposition 3.2, there exists at least one equilibrium trade network, maybe not unique.

Proposition 3.2 also tells us that the components where there exist members who do not achieve their ideal number of trade partners are at most one. Furthermore, for such a component, either it is a complete graph or there is just one member who does not achieve her ideal number of trade partners. So, in this sense, the members who do not achieve in equilibrium trade networks are bounded above by \( n^* - 1 \). So, as the population of the whole society increases, the equilibrium trade networks approach optimal trade networks in the sense that the fraction of members who do not achieve their ideal number of trade partners will decrease to 0 in the limit. So, we have the following proposition about comparisons between optimal trade networks and equilibrium trade networks.

**Proposition 3.3.** The equilibrium trade networks \( g^e \) will approach to optimal trade networks \( g^o \) as total population increases. Mathematically, \( \lim_{n \to \infty} \frac{\#(\{d_i(g^e) < n^*\})}{n} = \lim_{n \to \infty} \frac{\#(\{d_i(g^o) < n^*\})}{n} \), which means that the fraction of members who do achieve their ideal number of trade partners in equilibrium trade networks will approach to that in optimal trade networks as total population increases.

For the more general case where \( p, r, \lambda, \phi, \beta, \sigma^2 \) may be different for different members, we still have the following proposition about equilibrium trade network.

**Proposition 3.4.** For the model with homogenous members, equilibrium trade network \( g \) with \( m \) components such that \( n_i(g) = d_i, i \in \mathcal{N} \) exists if and only if there exist \( m \) groups and a function \( h : \mathcal{N} \to \{1, \cdots, m\} \) which assigns members to different groups such that:

i. \( \forall i \in \mathcal{N}, \text{ we have } d_i \leq n_i^*; \)

ii. If for \( k \in \{1, \cdots, m\}, \text{ there exists } i \in \mathcal{N} \text{ such that } d_i < n_i^* \text{ and } h(i) = k, \text{ then either } d_i = \#(\{i : h(i) = k\}) - 1 \text{ or } \{i \in \mathcal{N} : d_i < n_i^*\} \text{ is a singleton}; \)
iii. The set \( \{k : 1 \leq k \leq m, \exists i \in \mathcal{N} \text{ such that } h(i) = k \text{ and } d_i < n_i^* \} \) is a singleton.

iv. \( \forall k \in \{1, \cdots, m\}, \{d_i : h(i) = k\} \text{ satisfies Erdős-Gallai criterion; } d_i \geq 1; \sum_{h(i)=k} d_i \) is even and \( \sum_{h(i)=k} d_i \geq 2(\#(\{i : h(i) = k\}) - 1) \);

And equilibrium trade network always exists.

## 3.4 Dynamic Models

We have already studied the optimal and the equilibrium trade networks in the framework where the exogenous parameters remain the same along the time. In the real world, the exogenous parameters usually evolve, which may lead to different outcomes. In this section, we will study two dynamic models to see how members behave in such environments. In section 3.4.1, by assuming specific processes about the growth of cash flow, we study evolution of trade relationship within and between countries where each of them have fixed number of population. In section 3.4.2, we study trade organizations’ behavior in the situation where there is scarcity about potential members when they want to increase their size.

### 3.4.1 Optimal Timing for Trade Relationship

In this section, to simplify the analysis, it is assumed that there are two countries \( \mathcal{N}_1, \mathcal{N}_2 \), with population \( n_1, n_2 \), which are even, respectively. Here, again the assumption about \( n_1, n_2 \) is made to guarantee the existence of regular graph with any degree less than its order. For the general case, the results in the section still hold. In each country, there is no explicit cost to establish trade relationship among members in it. However, there is a one-time cost \( c \) (and the average cost is \( \bar{c} = c/(n_1 + n_2) \)) for the two countries to establish trade relationship with each other. The cost can be the resources to build the shipping channel, or the friction of negotiating trade rules between two countries, etc.

The evolution of economy environment is represented by an increasing process \( \{p_t : t \in [0, \infty)\} \), where \( p_{t''} \geq p_t, \forall t'' > t \), which means that the benefit from trade opportunity will not decrease. Facing the one-time cost, each country is optimal to firstly fully utilize local market until the complete market where every member trades with each other. Later, as economic grows, the countries will make tradeoff between the one-time cost and extra utility from foreign trade. We study the optimal timing problem for the two countries about trade relationship building. To simplify the analysis, it is further assumed that \( n_1 = n_2 = \hat{n} \). First of all, it is obvious that
if at time $t$, $n^*(p_t) \leq \hat{n}$, it is optimal not to establish trade relationship between two countries. If the two countries establish trade relationship at time $\bar{t}$ such that $n^*(p_t) > \hat{n}$, then, with trade network process $g$, we will have the total utility for the whole economy as:

$$
\mathbb{E}[\pi(\hat{g})] = \sum_{i \in N_1 \cup N_2} \left\{ \int_0^\ell \gamma e^{-\gamma t} n_i(\hat{g}_t) \left[ p_t - \lambda \phi \sigma^2 \frac{\beta n_i(\hat{g}_t)(1-r)}{\phi + \beta n_i(\hat{g}_t)(1-r)} \right] dt + \int_\ell^\infty \gamma e^{-\gamma t} n_i(\hat{g}_t) \left[ p_t - \lambda \phi \sigma^2 \frac{\beta n_i(\hat{g}_t)(1-r)}{\phi + \beta n_i(\hat{g}_t)(1-r)} \right] dt - e^{-\gamma \bar{t}} \right\}
$$

By noticing that $e^{-\gamma \bar{t}} \hat{c} = \int_\ell^\infty \gamma e^{-\gamma s} \hat{c} ds$, we can rewrite the total utility as:

$$
\mathbb{E}[\pi(\hat{g})] = \sum_{i \in N_1 \cup N_2} \left\{ \int_0^\ell \gamma e^{-\gamma t} n_i(\hat{g}_t) \left[ p_t - \lambda \phi \sigma^2 \frac{\beta n_i(\hat{g}_t)(1-r)}{\phi + \beta n_i(\hat{g}_t)(1-r)} \right] dt + \int_\ell^\infty \gamma e^{-\gamma t} \left( n_i(\hat{g}_t) \left[ p_t - \lambda \phi \sigma^2 \frac{\beta n_i(\hat{g}_t)(1-r)}{\phi + \beta n_i(\hat{g}_t)(1-r)} \right] - \hat{c} \right) dt \right\}
$$

Now, according to Proposition 3.1, the optimal trade network is regular graph with degree $\min\{n^*(p_t), \hat{n} - 1\}$ when $t < \bar{t}$ and with degree $\min\{n^*(p_t), 2\hat{n} - 1\}$ when $t \geq \bar{t}$. So, the total utility for the whole economy can be further simplified as:

$$
2\hat{n} \left\{ \int_0^\ell \gamma e^{-\gamma t} \min\{n^*(p_t), \hat{n} - 1\} \left[ p_t - \lambda \phi \sigma^2 \frac{\beta \min\{n^*(p_t), \hat{n} - 1\}(1-r)}{\phi + \beta \min\{n^*(p_t), \hat{n} - 1\}(1-r)} \right] dt + \int_\ell^\infty \gamma e^{-\gamma t} \left( \min\{n^*(p_t), 2\hat{n} - 1\} \left[ p_t - \lambda \phi \sigma^2 \frac{\beta \min\{n^*(p_t), 2\hat{n} - 1\}(1-r)}{\phi + \beta \min\{n^*(p_t), 2\hat{n} - 1\}(1-r)} \right] - \hat{c} \right) dt \right\}
$$

In the following proposition, we give the optimal time that the two counties establish trade relationship.

**Proposition 3.5.** The optimal time $t^*$ to establish the trade relationship between two countries with process $\{p_t : t \in [0, \infty)\}$ is the interval $[t^*, \bar{t}^*]$, with $t^* = \sup \{ t : u(p_t, 2\hat{n} - 1) - u(p_t, \hat{n} - 1) < \hat{c} \}$ and $\bar{t}^* = \inf \{ t : u(p_t, \hat{n}) - u(p_t, \hat{n} - 1) > \hat{c} \}$, respectively, where $u(p_t, \hat{n}) = \min\{n^*(p_t), \hat{n}\} \left[ p_t - \lambda \phi \sigma^2 \frac{\beta \min\{n^*(p_t), \hat{n}\}(1-r)}{\phi + \beta \min\{n^*(p_t), \hat{n}\}(1-r)} \right]$. And when $\{p_t : t \in [0, \infty)\}$ is strictly increasing, $[t^*, \bar{t}^*]$ is a singleton. Furthermore, $t^*, \bar{t}^*$ are decreasing with $r$ and increasing with $\hat{c}, \phi, \beta, \lambda, \sigma^2$. According to Proposition 3.5, as for the trade relationship between two countries, it shows that the agreement of such a trade relationship will arrive earlier when the one-time cost $\hat{c}$ is smaller, communication quality $r$ is better, the explicit value of private information $\phi$ is smaller, coordination complexity $\beta$ is smaller, the shock happens less frequently or the variance of trade opportunity $\sigma^2$ is smaller. In the journey of
human history, at the beginning, there do not exist common customs with respect to trade, so, trade conflicts happened frequently. At the same time, production level at that time is low and variety of products is small, which leads low rewards from trade. All of these leads people to think negatively about trade and trade happens only among people in different groups of small size. Later, as economy developed, more and more customs emerged and productions level also improved, people came to think more and more positively about trade.

It is consistent with the observation about attitudes of economists, scholars, philosophers, officers and so on, towards trade. As pointed out in Irwin (1996), at the very beginning, people even thought trade or commerce within border of a country generally brings harm to the human society. Later, people’s attitudes towards trade evolve, they accepted the existence of commercial activities, but still thought negatively with respect to foreign trade. Starting from Adam Smith, more and more people began to support free trade, and nowadays, almost all economists agree on the desirability of free trade.

In the analysis above, we did not consider the problem how to raise the fund for the one-time cost. The trade channel between two countries is a public good, if a government has no enough power, then it is difficult to build such a trade channel to benefit its citizens. Based on the successful paths to industrialization taken by Britain, the United States, and Japan, their governments all play an important role to assist their international trade.

3.4.2 Trade Organizations’ Behavior with Scarce Members

In this section, we consider a sequential arrival model. In the model, it is assumed that there is a trade organization, consisting of members who trade with each other within the trade organization. Initially the technology for the whole economy is $p$, however there is a potential breakthrough about technology, which increase $p$ to $\bar{p}$. Here, the difference between $p$ and $\bar{p}$ are assumed to large enough such that the ideal number of trade partners will be different for $p$ and $\bar{p}$. The breakthrough is assumed to follow a Poisson process with Poisson rate $\zeta$. If the organization places a request for new members, the new members will arrive according to a Poisson process with Poisson rate $\eta$, and if a member is accepted by the trade organization, she can neither escape nor be kicked out.\footnote{In our model, it is not necessary to have such an assumption, but it will make the analysis simpler.}

If the trade organization is myopic, the analysis is simple. According to Lemma 3.3, when the state of $p$ is $\bar{p}$ and the members in the trade organization is already $n^*(\bar{p}) + 1$, then it will not place any request for new members. When $p$ comes to be
the trade organization will place request for new members until the total number of members comes to be $n^*(\bar{p}) + 1$. We state this in the following claim.

**Claim.** As for the myopic trade organization, at each time $t$, if its population is less than $n^* (p_t) + 1$, it will place a request for new members. Otherwise, it will not.

However, if the trade organization is far-sighted, the outcome may be different. For the reason that if the new arrivers do not arrive frequently enough after the trade organization places a request for new members, when the breakthrough comes, it can not take full advantage of the breakthrough immediately. So, there is a tradeoff for the trade organization, the loss from having more members when the state is $p$ and the gains by having enough members to take full advantage of technology breakthrough when technology breakthrough arrives. The strategy of trade organization is $R_t : \mathbb{N}^{[0,t]} \otimes \{p, \bar{p}\} \to \{0, 1\}$, which determines whether to place an order or not based on the information available. Because the trade organization can not kick out its members, so the strategy only depends on the information at $t$, which simplify the strategy space as $R_t : \mathbb{N} \otimes \{p, \bar{p}\} \to \{0, 1\}$. Now we come to the analysis with far-sighted trade organization.

We define the expected current value function for member $i$ in the trade organization as $V_i(k, p)$ when the state is $p$ and the total number of members in it is $k + 1$:

$$V_{i,t}(k, p) = \max_{R_{i,s}, s \geq t} \mathbb{E} \left[ e^{rt} \int_t^\infty \gamma e^{-\gamma s} n_{i,s} \left( p - \lambda \phi \sigma^2 \frac{\beta n_{i,s}(1-r)}{\phi + \beta n_{i,s}(1-r)} \right) ds \right]$$

(3.8)

where, the expectation is on $n_{i,s}$, which is an increasing random process such that $n_{i,t} = k$, $n_{i,s'} \geq n_{i,s''}, \forall s' \geq s''$ and also $n_{i,s}$ changes at time $s$ only when there is new arriver in the trade organization. Here, the assumption $n_{i,s}$ is an increasing is not necessary, in equilibrium, $n_{i,s}$ will not decrease. To simplify the analysis, we assume that it holds anyway. These mean that at time $t$, members in the organization face the condition with the state as $p$ and $k$ trade partners. Facing such a condition, the organization needs to make decisions about whether to place a request for new members.

For the reason that new members arrives according to Poisson process after the organization places a request for new members and the technology breakthrough arrives also according to a Poisson process, the value function $V_{i,t}(k, p)$ is independent with time $t$, so we can denote is as $V_i(k, p)$:

$$V_i(k, p) = \max_{R_{i,s}, s \geq t} \mathbb{E} \left[ \int_0^\infty \gamma e^{-\gamma s} n_{i,s} \left( p - \lambda \phi \sigma^2 \frac{\beta n_{i,s}(1-r)}{\phi + \beta n_{i,s}(1-r)} \right) ds \right]$$

(3.9)

where $n_{i,0} = k$. 
Firstly, we study the value function \(V_i(k,p)\) when \(p = \bar{p}\). We know that whenever, \(k < n^*(\bar{p})\), the trade organization is optimal to place a request for new members. And when \(k \geq n^*(\bar{p})\), it is optimal for the organization not to place a request. Now, we have the following proposition about \(V_i(k,p)\). To simplify notations, we denote 
\[
h(k,p) = k \left[ p - \lambda \phi \sigma^2_{\theta} \frac{\beta k(1-r)}{\theta + \beta k(1-r)} \right].
\]

**Proposition 3.6.** The value function \(V_i(k,\bar{p})\) satisfies the following recursive equation:

\[
V_i(k,\bar{p}) = \begin{cases} 
\gamma \gamma + \eta h(k,\bar{p}) + \frac{\eta}{\gamma + \eta} V_i(k+1,\bar{p}), & \text{if } k < n^*(\bar{p}); \\
h(k,\bar{p}), & \text{if } k \geq n^*(\bar{p}). 
\end{cases}
\]

By solving the recursive equation, we have the value function \(V_i(k,\bar{p})\) as:

\[
V_i(k,\bar{p}) = \sum_{m=k}^{n^*(\bar{p})} \gamma \left( \frac{\eta}{\gamma + \eta} \right)^{m-k} h(m,\bar{p}) + \left( \frac{\eta}{\gamma + \eta} \right)^{n^*(\bar{p})-k+1} h(n^*(\bar{p}),\bar{p}), \quad \text{if } k < n^*(\bar{p});
\]

\[
V_i(k,\bar{p}) = h(k,\bar{p}), \quad \text{if } k \geq n^*(\bar{p}).
\]

In Proposition 3.6, we obtain value function for \(V_i(k,\bar{p})\) where the state is \(\bar{p}\). Now, based on this, we come to the analysis about value function \(V_i(k,p)\). As we have discussed before, the threshold of number of trade partners for optimal strategy may not be \(n^*(\bar{p})\) for the reason that members should consider the situation when technology breakthrough actually happens, there are no enough members to take full advantage of this breakthrough.

Suppose that when the organization’ member faces the situation where there are \(k+1\) members and the state is \(\bar{p}\), the organization chooses to place a request for new members. Now, by the same method as in the proof of Proposition 3.6, we have the following recursive function:

\[
V_i(k,p) = \frac{\gamma}{\gamma + \eta + \zeta} h(k,p) + \frac{\eta}{\gamma + \eta + \zeta} V_i(k+1,p) + \frac{\zeta}{\gamma + \eta + \zeta} V_i(k,\bar{p}) \tag{3.10}
\]

And, when the organization chooses not to place a request for new members in such a situation, the value function will satisfy the following recursive function:

\[
V_i(k,p) = \frac{\gamma}{\gamma + \zeta} h(k,p) + \frac{\zeta}{\gamma + \zeta} V_i(k,\bar{p}) \tag{3.11}
\]

So, by equations 3.10 and 3.11, the threshold for the trade organization to choose placing a request should be the one that solves the problem \(\max_{k \in \mathbb{N}} \frac{\gamma}{\gamma + \zeta} h(k,p) + \frac{\zeta}{\gamma + \zeta} V_i(k,\bar{p})\), we will state this in the following proposition.
Proposition 3.7. There exists a threshold \( \hat{k} \in [n^*(\bar{p}), n^*(\bar{\bar{p}})] \cap \mathbb{N} \) which solves the problem \( \max_{k \in \mathbb{N}} \frac{\gamma}{\gamma + \xi} h(k, p) + \frac{\zeta}{\gamma + \xi} V_i(k, \bar{p}) \) such that the value function \( V_i(k, p) \) follows the following recursive function:

\[
V_i(k, p) = \begin{cases} 
\frac{\gamma}{\gamma + \eta + \xi} h(k, p) + \frac{\eta}{\gamma + \eta + \xi} V_i(k + 1, p) + \frac{\zeta}{\gamma + \eta + \xi} V_i(k, \bar{p}), & \text{if } k < \hat{k}; \\
\frac{\eta}{\gamma + \xi} h(k, p) + \frac{\zeta}{\gamma + \xi} V_i(k, \bar{p}), & \text{if } k \geq \hat{k}.
\end{cases}
\]

Furthermore, \( \hat{k} \) is increasing with \( \zeta \) and decreasing with \( \gamma, \eta \).

As we can see from Proposition 3.7, it is optimal for trade organization to have more members than its myopic optimal number of members when the technology has not arrived yet. So, when technology breakthrough has not arrived yet, it is optimal to have extra members as backup to take full advantage of the forthcoming technology breakthrough. When \( \zeta \) is higher, which means that it is highly possible that the technology breakthrough arrives earlier, the organization will choose to have more extra members. At the same time, when \( \eta \) is higher, the organization will choose to have less extra members, which is intuitive. If the request for new members is placed, with higher \( \eta \), it is easier to find new members, there is no need to have more extra human capital. Furthermore, the discount factor also influences organization’ behavior about optimal number of members, if \( \gamma \) is larger, which means the organization’s members care less about future, so the breakthrough is less important for them, they have no incentive to have more member as backup.

### 3.5 Conclusion

In this paper, we studied the trade behavior when people face coordination cost. In the benchmark model, we study the optimal and equilibrium networks, we find that three types of markets will exist for optimal trade networks. The three markets are complete market, local market and Autarky market, respective. And for the local market, the average trade volume is increasing with expected benefit and communication quality; decreasing with explicit value of private information, coordination difficulty and risk of trade opportunity. Also, we fully describe the equilibrium trade networks by solving system of inequalities. Furthermore, we find that equilibrium trade networks approach to optimal trade networks in the sense that the whole welfare from these two kind of networks will converge in the limit.

And we also studied two dynamic models. In the first model with one-time cost to build trade relationship between two countries, we find that when the countries with similar culture and customs or closer geographical distance, they are more likely to form trade relationship with each other. In the model with sequential arrivers,
it is shown that when trade organizations are farsighted, they will optimally have extra members as backup to take advantage of technology breakthrough. When potential members are scarcer, or the organization care more about the future, or the technology breakthrough is highly possible to arrive earlier, trade organizations have more incentive to have more extra members as backup.
Appendix

PROOF OF LEMMA 3.1:

According to trade partners’ object function 3.5, we can rewrite it as:

\[
\mathbb{E}_\theta \left[ \phi(s_{i|g}(\theta_i) - \theta_i)^2 + \beta n_i(g)(1 - r)(s_{i|g}(\theta_i) - s_{ji}^u(s_{i|g}))^2 \right] = \\
\mathbb{E}_\theta \left[ \phi(s_{i|g}(\theta_i) - \theta_i)^2 ight] + \beta n_i(g)(1 - r)(s_{i|g}(\theta_i) - \mathbb{E}_\theta(s_{i|g}(\theta_i)))^2 \\
\mathbb{E}_\theta \left[ \phi(s_{i|g}(\theta_i) - \theta_i)^2 \right] + \beta n_i(g)(1 - r)(s_{i|g}(\theta_i) - \mathbb{E}_\theta(s_{i|g}(\theta_i)))^2 \\
\mathbb{E}_\theta \left[ \phi(s_{i|g}(\theta_i) - \theta_i)^2 \right] + \beta n_i(g)(1 - r)(s_{i|g}(\theta_i) - \mathbb{E}_\theta(s_{i|g}(\theta_i)))^2
\]

So, we have the optimal coordinative strategy as \( s_{ji}^u(s_{i|g}) = \mathbb{E}_\theta(s_{i|g}(\theta_i)) \) when trade partner \( j \) does not receive the signal successfully.

PROOF OF LEMMA 3.2:

The proof is similar as Dessein and Santos (2006), as we can see from definition 3.1, member \( i \)’s optimal strategy is to minimize the loss, which leads to the optimal strategy \( s_{i|g}(\theta) \) as the solution of the following problem:

\[
\min_{s_i \in \mathbb{R}} \left[ \phi(s_i - \theta_i)^2 + \beta n_i(g)(1 - r)(s_i - \mathbb{E}_\theta(s_{i|g}(\theta_i,t)))^2 \right]
\]

When member \( i \) tries to minimize the loss when realization is \( \theta_i,t \), her strategy can not change \( \mathbb{E}_\theta(s_{i|g}(\theta_i,t)) \) because integration remain the same when the value of function just changes on the domain with measure 0. So, \( \mathbb{E}_\theta(s_{i|g}(\theta_i,t)) \) seems to be exogenously given for her. Now, we can arrive optimal strategy \( s_{i|g}(\theta) \) by first order condition of the minimization problem above, which gives:

\[
s_{i|g}(\theta_i,t) = \hat{\theta}_i,t + \alpha_{i,t}(g_t)(\theta_i,t - \hat{\theta}_i,t)
\]

where \( \alpha_{i,t}(g_t) \) measures member’s adaptiveness to her private information, which has the form as follows:

\[
\alpha_{i,t}(g_t) = \frac{\phi}{\phi + \beta n_i(g_t)(1 - r)}
\]

PROOF OF LEMMA 3.3:

Firstly, we ignore the fact that \( n_i \) is an integer and study the function \( f(x) = x[p - \lambda \phi \sigma_\theta^2 \beta x(1 - r)] \). Taking derivative of \( f(x) \) with respect to \( x \), we come to the derivative as \( f'(x) = p - \lambda \phi \sigma_\theta^2 (1 - \frac{\phi^2}{[\phi + \beta x(1 - r)]^2}) \). It is obvious that \( f'(x) \) is decreasing
Now, we have the equation \( f(k) - f(k-1) = \int_{k-1}^{k} \left[ p - \lambda \phi \sigma^2 \left( 1 - \frac{\phi^2}{(\phi + \beta x(1-r))^2} \right) \right] dx = p - \int_{k-1}^{k} \lambda \phi \sigma^2 \left( 1 - \frac{\phi^2}{(\phi + \beta x(1-r))^2} \right) dx. \) It is clear that \( \bar{p}_k \equiv \int_{k-1}^{k} \lambda \phi \sigma^2 \left( 1 - \frac{\phi^2}{(\phi + \beta x(1-r))^2} \right) dx \) is increasing with \( k \), so if \( k_1 > k_2 \) and \( f(k_2) - f(k_2 - 1) \leq 0 \), then \( f(k_2) - f(k_2 - 1) < 0 \); and if \( k_1 < k_2 \) and \( f(k_2) - f(k_2 - 1) \geq 0 \), then \( f(k_2) - f(k_2 - 1) > 0 \). So, for any given \( p \), the ideal number of trade partners is \( n_i^* \) if and only if \( f(n_i^*) - f(n_i^* - 1) \geq 0 \) and \( f(n_i^* + 1) - f(n_i^*) \leq 0 \).

So, we have the following three cases about ideal number of trade partners of member \( i \):

i. \( n_i^* = 0 \) if and only if \( f(1) - f(0) < 0 \), which is equal to \( p < \bar{p}_1 \).

ii. \( n_i^* = k \) if and only if \( f(k) - f(k - 1) > 0 \) and \( f(k + 1) - f(k) < 0 \), which is equal to \( \int_{k-1}^{k} \lambda \phi \sigma^2 \left( 1 - \frac{\phi^2}{(\phi + \beta x(1-r))^2} \right) dx < p < \int_{k-1}^{k+1} \lambda \phi \sigma^2 \left( 1 - \frac{\phi^2}{(\phi + \beta x(1-r))^2} \right) dx \), or \( \bar{p}_k < p < \bar{p}_{k+1} \).

iii. \( n_i^* = k - 1 \) if and only if \( f(k) - f(k - 2) \geq 0 \) and \( f(k) - f(k - 1) \leq 0 \); and \( f(k) - f(k - 1) \geq 0 \) and \( f(k + 1) - f(k) \leq 0 \). It holds only when \( f(k) - f(k - 1) = 0 \), which is equal to \( p = \bar{p}_{k-1} \).

iv. \( n_i^* = \infty \) if and only if \( f(k) - f(k - 1) \geq 0, \forall k \in \mathcal{N} \). It is equal to \( p > \lim_{k \to \infty} \int_{k-1}^{k} \lambda \phi \sigma^2 \left( 1 - \frac{\phi^2}{(\phi + \beta x(1-r))^2} \right) dx = \lambda \phi \sigma^2 \).

**Proof of Lemma 3.4:**

The lemma is proved by the well-known theorem in graph theory (see Chartrand et al. (2015)) that \( n^* \)-degree regular graph of order \( n \) exists if and only if \( n^*n \) is even and \( n^* \leq n - 1 \).

According to Lemma 3.3, we have the following cases about ideal trade networks:

i. When \( p < \bar{p}_1 \), then \( n^* = 0 \), so empty graph makes each member achieves her ideal number of trade networks;

ii. When \( p = \bar{p}_k \), we have \( n^* = \{k - 1, k\} \). If \( n = k \), then \( n(k - 1) = k(k - 1) \) must be even, so \((k - 1)\)-degree regular graph with order \( k \) exists where each member can achieve her ideal number of trade partners; if \( n \geq k + 1 \), then either \( k - 1 \) or \( k \) is even, so either \((k - 1)\)-degree regular graph with order \( k \) exists or \( k\)-degree regular graph with order \( k \) exists, where both of them can guarantee each member achieving her ideal number of trade partners.
iii. When \( p \in (\bar{p}_k, \bar{p}_{k+1}) \), we have \( n^* = k \); then \( k \)-degree regular graph with order \( n \) exists if and only if \( n \geq k + 1 \) and \( nk \) is even.

This completes the proof of Lemma 3.4.

PROOF OF PROPOSITION 3.1:

By Lemma 3.3, we have the following cases about optimal trade networks:

Now, if \( p \geq \bar{p}_{n-1} \), then for each member \( i \), at least one of her ideal number of trade partners is larger than \( n - 1 \). So, each member will form trade relationship with all other members to gain more advantage from trade, which results in complete trade network.

If \( \bar{p}_1 \leq p < \bar{p}_{n-1} \), then for each member \( i \), her ideal number of trade partners is less than \( n - 1 \). And the regular graph of order \( n \) with degree \( n^*(p) \) makes each member achieve her ideal number of trade partners. And the existence of such a graph is guaranteed by the assumption that \( n \) is even. In this case, each member will try to form trade relationship with parts of the other members but not all of them.

If \( p < \bar{p}_1 \), then for each member \( i \), her ideal number of trade partners is 0. So, optimal trade network will be empty graph where every member will not trade with any other members. This concludes the first part of Proposition 3.1.

For the second part of Proposition 3.1, firstly, it is obvious that when \( p \) is larger, the \( k \) such that \( p \in (\bar{p}_k, \bar{p}_{k+1}) \) is larger. So, the degree of optimal trade network is increasing with \( p \). For other parameters, it is sufficient to prove that for \( \bar{p}_k \), \( k \in \mathbb{N} \), it is increasing with \( \beta, \lambda, \sigma^2, \phi \) and decreasing with \( r \). Furthermore, this can be further simplified by studying function \( f(p, r, \lambda, \phi, \beta, \sigma^2) = \lambda \phi \sigma^2 (1 - \frac{\phi^2}{[\phi + \beta x(1-r)]^2}) \) is increasing with \( \beta, \lambda, \sigma^2, \phi \) and decreasing with \( r \). The monotonicity of \( f \) with respect to \( \beta, \lambda, \sigma^2, \phi, r \) is obvious. We come to prove \( f \) is increasing with \( \phi \) now. By taking derivative of \( f \) with respect to \( \phi \), we have:

\[
\begin{align*}
  f_\phi &= \lambda \sigma^2 \left[ 1 - \frac{\phi^3 + 3\phi^2 \beta x(1-r)}{[\phi + \beta x(1-r)]^3} \right] \\
  &= \lambda \sigma^2 \left[ \frac{[\phi + \beta x(1-r)]^3 - \phi^3 - 3\phi^2 \beta x(1-r)}{[\phi + \beta x(1-r)]^3} \right] \\
  &= \lambda \sigma^2 \left[ \frac{\beta x(1-r)^3 - 3\phi \beta x(1-r)}{[\phi + \beta x(1-r)]^3} \right] \\
  &\geq 0
\end{align*}
\]

Summing all above, we conclude Proposition 3.1.
Proof of Proposition 3.2:

First of all, we prove the following statement about sufficient and necessary conditions of that a specific graph is an equilibrium trade network.

Trade network \( g \) is an equilibrium if and only if:

i. \( \forall i \in N ', \) we have \( n_i(\ g) \leq n^*; \)

ii. If component \( (N', g') \in (N, g) \) has one member \( i \in N ' \) such that \( n_i(\ g) < n^*; \) then component \( (N', g') \) is either complete or the member of \( \{ i \in N ': n_i(\ g) < n^* \} \) is unique;

iii. The number of components \( (N', g') \in (N, g) \) such that \( \exists i \in N ', n_i(\ g) < n^* \) is at most one.

"If" parts:

By definition of \( n^*; \) by i), it is obvious that \( \forall ij \in g, \ E[\pi_i(g)] \geq E[\pi_i(g - ij)] \) and \( E[\pi_j(g)] \geq E[\pi_j(g - ij)] \).

Now, if \( ij \notin g, \) by ii) and iii), at most one of them is in the component \( (N', g') \in (N, g) \) where \( \exists i \in N ' \) such that \( n_i(\ g) < n^* \). Now, at least one of them, for example \( i \notin N ', \) then \( E[\pi_i(g + ij)] < E[\pi_i(g)] \). So, "if" part is completed.

"Only if" parts:

i) \( n_i(\ g) \leq n^* \) is obvious by definition of \( n^* \).

ii) If for component \( (N', g') \in (N, g), \exists i \in N ' \) such that \( n_i(\ g) < n^*; \) Suppose \( (N', g') \) is not complete and \( \{ i \in N ': n_i(\ g) < n^* \} \) is not unique, then for \( i, j \in \{ i \in N ': n_i(\ g) < n^* \}; \) both \( E[\pi_i(g + ij)] > E[\pi_i(g)] \) and \( E[\pi_j(g + ij)] > E[\pi_j(g)] \) will hold, which violates the definition of equilibrium trade network. So ii) holds.

iii) If there are two components \( (N_1, g_1) \in (N, g) \) and \( (N_2, g_2) \in (N, g) \) where \( \exists i_1 \in N_1 \) and \( \exists i_2 \in N_2 \) such that \( n_{i_1}(\ g) < n^* \) and \( n_{i_2}(\ g) < n^*; \) then for \( i_1, i_2, \)

\( E[\pi_{i_1}(g + i_1i_2)] > E[\pi_{i_1}(g)] \) and \( E[\pi_{i_2}(g + i_1i_2)] > E[\pi_{i_2}(g)] \) will hold, which violates the definition of equilibrium trade network. So iii) holds.

Now, by modified version of Erdös-Gallai theorem which gives a sufficient and necessary condition of existence of connected graph with a specific degree sequence (see Berge and Minieka (1973)) and substituting \( n_i(\ g) \) with \( d_i; \) it is easy to prove the first part of the proposition.

For the part of the existence of equilibrium trade network, we have three cases as follows.
• When \( n \leq n^* + 1 \), it is easy to show that complete graph satisfy i), ii) and iii);

• When \( n > n^* + 1 \), and either \( n \) or \( n^* \) is even, then it is easy to show that \( n^* \)-degree regular graph with order \( n \) satisfy i), ii) and iii);

• When \( n > n^* + 1 \), and both \( n \) and \( n^* \) are not even, then graph where one member has degree \( n^* - 1 \), while all other members have degree \( n^* \) satisfy i), ii) and iii).

The existence of graph in case 3 can be proved as follows. Step 1, we choose \( n - 1 \) members and form regular graph with degree \( n^* \) and order \( n - 1 \); step 2, choose \( (n^* - 1)/2 \) edges formed in step 1, and break these edges; step 3, add \( n^* - 1 \) edges between the \( n \)-th member and the members whose edges have been broken in step 2.

This concludes Proposition 3.2.

PROOF OF PROPOSITION 3.5:

According to the analysis in the main part of the paper, if \( n^*(p_\infty) \leq \hat{n} - 1 \), where \( p_\infty = \lim_{t \to \infty} p_t \), the two countries will never trade with each other. If \( n^*(p_\infty) > \hat{n} - 1 \) and the two countries rationally choose to establish trade relationship at time \( \bar{t} \), then \( \bar{t} \) must satisfy \( n^*(p_t) > \hat{n} - 1 \). Now, if we deviate from the optimal time \( \bar{t} \) by postponing the time by \( \Delta t \) to \( \bar{t} + \Delta t \), the change of total utility will be close to \( 2\hat{n}[u(p_t, 2\hat{n} - 1) - u(p_t, \hat{n} - 1) - \delta]\Delta t \) when \( \Delta t \) is small enough, where

\[
\begin{align*}
&u(p_t, \hat{n}) = \min\{n^*(p_t), \hat{n}\} \left[ p_t - \lambda \phi \sigma_2^{\beta \min\{n^*(p_t), \hat{n}\}(1-r)} \right] \\
&\min\{n^*(p_t), \hat{n}\} \left[ p_t - \lambda \phi \sigma_2^{\beta \min\{n^*(p_t), \hat{n}\}(1-r)} \right]
\end{align*}
\]

First of all, we notice that \( \min\{n^*(p), \hat{n} - 1\} \left[ p - \lambda \phi \sigma_2^{\beta \min\{n^*(p), \hat{n} - 1\}(1-r)} \right] \) is strictly increasing with respect to \( p \). To see this, assume that \( p' > p'' \), we have

\[
\begin{align*}
&\min\{n^*(p'), \hat{n} - 1\} \left[ p' - \lambda \phi \sigma_2^{\beta \min\{n^*(p'), \hat{n} - 1\}(1-r)} \right] > \min\{n^*(p''), \hat{n} - 1\} \left[ p'' - \lambda \phi \sigma_2^{\beta \min\{n^*(p''), \hat{n} - 1\}(1-r)} \right].
\end{align*}
\]
And also, we notice that from Lemma 3.3, \( \min\{n^*(p'), \hat{n} - 1\} \) is the optimal solution of the following problem

\[
\max_{x \in \mathbb{R}, 0 \leq x \leq \hat{n} - 1} x \left[ p - \frac{\lambda \phi \sigma^2}{\phi + \beta x(1-r)} \right]
\]

Then, we have

\[
\min\{n^*(p'), \hat{n} - 1\} \left[ p' - \frac{\lambda \phi \sigma^2}{\phi + \beta \min\{n^*(p'), \hat{n} - 1\}(1-r)} \right] \geq \min\{n^*(p'), \hat{n} - 1\} \left[ p - \frac{\lambda \phi \sigma^2}{\phi + \beta \min\{n^*(p), \hat{n} - 1\}(1-r)} \right]
\]

which proves strict monotonicity of \( \min\{n^*(p), \hat{n} - 1\} \) with respect to \( p \).

By the fact that \( \min\{n^*(p), \hat{n} - 1\} \left[ p - \frac{\lambda \phi \sigma^2}{\phi + \beta \min\{n^*(p), \hat{n} - 1\}(1-r)} \right] \) is strictly increasing with respect to \( p \) and \( \{p_t : t \in [0, \infty)\} \) is an increasing process, we have \( \{t : u(p_t, 2\hat{n} - 1) - u(p_t, \hat{n} - 1) < c\} \) is an interval with left endpoint 0 and \( \{t : u(p_t, 2\hat{n} - 1) - u(p_t, \hat{n} - 1) > c\} \) is an interval with right endpoint \( \infty \).

Now, denote \( \bar{t}^* = \inf \{t : u(p_t, 2\hat{n} - 1) - u(p_t, \hat{n} - 1) > c\} \), for any \( t > \bar{t}^* \), for the reason \( \{t : u(p_t, 2\hat{n} - 1) - u(p_t, \hat{n} - 1) > c\} \) is an interval, there must exist an interval \( (t - \varepsilon, t) \) where \((t - \varepsilon, t) \subseteq \{t : u(p_t, 2\hat{n} - 1) - u(p_t, \hat{n} - 1) > c\} \). Now,

\[
\int_{t-\varepsilon}^{t} \gamma e^{-\gamma s} \lambda u(p_s, 2\hat{n} - 1) - u(p_s, \hat{n} - 1) - c \left[ e^{-\gamma(t-\varepsilon)} - e^{-\gamma t} \right] ds
\]

It means that \( t \) can not be the optimal time for the two countries to establish trade relationship. Otherwise, the two countries can benefit by establishing trade relationship \( \varepsilon \) time earlier. So, we must that \( t^* \leq \bar{t}^* \).

Similarly, we can prove that \( t^* \geq \bar{t}^* \). And for any point \( t \in [\bar{t}^*, \bar{t}^*] \), it is easy to prove that any deviation from \( t \) can not increase the total utility, so the first part of the proposition finished.

As for the second part, firstly, we notice that:

\[
u(p_t, 2\hat{n} - 1) - u(p_t, \hat{n} - 1) - c
\]

\[
= \min\{n^*(p_t), 2\hat{n} - 1\} \left[ p_t - \frac{\lambda \phi \sigma^2}{\phi + \beta \min\{n^*(p_t), 2\hat{n} - 1\}(1-r)} \right]
\]

\[
- (\hat{n} - 1) \left[ p_t - \frac{\lambda \phi \sigma^2}{\phi + \beta (\hat{n} - 1)(1-r)} \right] - c
\]

\[
= \left[ \min\{n^*(p_t), 2\hat{n} - 1\} - \hat{n} + 1 \right] p_t
\]

\[
- \int_{\hat{n} - 1}^{\min\{n^*(p_t), 2\hat{n} - 1\}} \lambda \phi \sigma^2 \left( 1 - \frac{\phi^2}{[\phi + \beta x(1-r)]^2} \right) dx - c
\]

As we can see from proof of Proposition 3.1, \( \lambda \phi \sigma^2 \left( 1 - \frac{\phi^2}{[\phi + \beta x(1-r)]^2} \right) \) is increasing with \( \beta, \lambda, \sigma^2, \phi \) and decreasing with \( r \). Denote \( u(p_t, \hat{n}, \phi) = \min\{n^*(p_t, \phi), \hat{n}\} [p_t - \frac{\lambda \phi \sigma^2}{\phi + \beta \min\{n^*(p_t, \phi), \hat{n}\}(1-r)}] \), then if \( \phi_1 > \phi_2 \), we have
where the first inequality come from the fact that \( \min\{n^*(p_t, \phi_2), 2\hat{n} - 1\} \) is the optimal solution of the following problem

\[
\max_{x \in \mathbb{N}, 0 \leq x \leq 2\hat{n} - 1} x \left[ p - \lambda \phi_2 \sigma^2_\theta \frac{\beta x}{\phi_2 + \beta x(1 - r)} \right]
\]

and the second inequality comes from that \( \lambda \phi_2 \sigma^2_\theta \left(1 - \frac{\phi^2}{|\phi + \beta x(1 - r)|^2}\right) \) is increasing with \( \beta \).

So, \( \{t : u(p_t, 2\hat{n} - 1, \phi_2) - u(p_t, \hat{n} - 1, \phi_2) > c\} \supseteq \{t : u(p_t, 2\hat{n} - 1, \phi_1) - u(p_t, \hat{n} - 1, \phi_1) > c\} \), then \( \inf\{t : u(p_t, 2\hat{n} - 1, \phi_2) - u(p_t, \hat{n} - 1, \phi_2) > c\} \leq \inf\{t : u(p_t, 2\hat{n} - 1, \phi_1) - u(p_t, \hat{n} - 1, \phi_1) > c\} \), which implies \( \hat{t}^*(\phi_2) \geq \hat{t}^*(\phi_2) \).

And \( \{t : u(p_t, 2\hat{n} - 1, \phi_2) - u(p_t, \hat{n} - 1, \phi_2) < c\} \subseteq \{t : u(p_t, 2\hat{n} - 1, \phi_1) - u(p_t, \hat{n} - 1, \phi_1) < c\} \), then \( \sup\{t : u(p_t, 2\hat{n} - 1, \phi_2) - u(p_t, \hat{n} - 1, \phi_2) < c\} \leq \sup\{t : u(p_t, 2\hat{n} - 1, \phi_1) - u(p_t, \hat{n} - 1, \phi_1) < c\} \), which implies \( \hat{t}^*(\phi_1) \geq \hat{t}^*(\phi_2) \).

The analysis also holds for \( \beta, r, \lambda, \sigma^2_\theta \), which concludes the proposition.

PROOF OF PROPOSITION 3.6:

Here, we use Bellman equation to derive the value function (see Stokey et al. (1989)), as for the value function \( V_i(k, \bar{p}) \):

\[
V_i(k, \bar{p}) = \max_{R_{i,s}, n_{i,s}} \mathbb{E} \left[ \int_0^\infty \gamma e^{-\gamma s} n_{i,s} \left[ \bar{p} - \lambda \phi \sigma^2_\theta \frac{\beta n_{i,s}(1 - r)}{\phi + \beta n_{i,s}(1 - r)} \right] ds \right] \quad (3.12)
\]

where \( n_{i,0} = k \).

It is obvious that when \( k \geq n^*(\bar{p}) \), \( R_{i,s} = 0, \forall s \geq 0 \). So then \( V_i(k, \bar{p}) = \lambda(k, \bar{p}) \). Now, if \( k < n^*(\bar{p}) \), it is optimal to choose \( R_{i,s} = 1 \) until at least time \( t \) where new member
arrives, now with $\Delta t$ small enough,

\[
V_i(k, \bar{p}) = \max_{R_{i,s} \geq t} \mathbb{E} \left[ \int_0^\infty \gamma e^{-\gamma s} h(n_{i,s}, \bar{p}) ds \right] \\
= \max_{R_{i,s} \geq t} \mathbb{E} \left[ \int_0^{\Delta t} \gamma e^{-\gamma s} h(n_{i,s}, \bar{p}) ds + \int_{\Delta t}^\infty \gamma e^{-\gamma s} h(n_{i,s}, \bar{p}) ds \right] \\
= \max_{R_{i,s} \geq t} \mathbb{E} \left[ \int_0^{\Delta t} \gamma e^{-\gamma s} h(n_{i,s}, \bar{p}) ds + e^{-\gamma \Delta t} \int_{\Delta t}^\infty \gamma e^{-\gamma(s-\Delta t)} h(n_{i,s}, \bar{p}) ds \right] \\
= \mathbb{E} \left[ \int_0^{\Delta t} \gamma e^{-\gamma s} h(n_{i,s}, \bar{p}) ds \right] + \max_{R_{i,s} \geq t} \mathbb{E} \left[ e^{-\gamma \Delta t} \int_{\Delta t}^\infty \gamma e^{-\gamma(s-\Delta t)} h(n_{i,s}, \bar{p}) ds \right] \\
\approx h(k, \bar{p}) \gamma \Delta t + (1 - \gamma \Delta t) \eta \Delta t V_i(k + 1, \bar{p}) + (1 - \eta \Delta t) V_i(k, \bar{p})
\]

Rearranging the equation $V_i(k, \bar{p}) = h(k, \bar{p}) \gamma \Delta t + (1 - \gamma \Delta t) [\eta \Delta t V_i(k + 1, \bar{p}) + (1 - \eta \Delta t) V_i(k, \bar{p})]$ we can obtain the equation $V_i(k, \bar{p}) = \frac{\eta}{\gamma+\eta} h(k, \bar{p}) + \frac{\eta}{\gamma+\eta} V_i(k + 1, \bar{p})$ when $k < n^*(\bar{p})$. The remain part of the proposition is easy to obtain by backward substituting recursively.

PROOF OF PROPOSITION 3.7:

Firstly, we come to derive the value function $V_i(k, \bar{p})$ when the state is $\bar{p}$.

As for the value function $V_i(k, \bar{p})$:

\[
V_i(k, \bar{p}) = \max_{R_{i,s} \geq t} \mathbb{E} \left[ \int_0^\infty \gamma e^{-\gamma s} n_{i,s} \left( \bar{p} - \lambda \phi \sigma^2 \frac{\beta n_{i,s}(1-r)}{\phi + \beta n_{i,s}(1-r)} \right) ds \right] \\
= \max_{R_{i,s} \geq t} \mathbb{E} \left[ \int_0^{\Delta t} \gamma e^{-\gamma s} h(n_{i,s}, \bar{p}) ds + \int_{\Delta t}^\infty \gamma e^{-\gamma s} h(n_{i,s}, \bar{p}) ds \right]
\]

where $n_{i,0} = k$.

Suppose in the situation where there are $k + 1$ members and the state is $\bar{p}$, the trade organization optimally chooses to place a request for new members, then with $\Delta t$ small enough,

\[
V_i(k, \bar{p}) = \max_{R_{i,s} \geq t} \mathbb{E} \left[ \int_0^\infty \gamma e^{-\gamma s} h(n_{i,s}, \bar{p}) ds \right] \\
= \max_{R_{i,s} \geq t} \mathbb{E} \left[ \int_0^{\Delta t} \gamma e^{-\gamma s} h(n_{i,s}, \bar{p}) ds + \int_{\Delta t}^\infty \gamma e^{-\gamma s} h(n_{i,s}, \bar{p}) ds \right] \\
= \max_{R_{i,s} \geq t} \mathbb{E} \left[ \int_0^{\Delta t} \gamma e^{-\gamma s} h(n_{i,s}, \bar{p}) ds + e^{-\gamma \Delta t} \int_{\Delta t}^\infty \gamma e^{-\gamma(s-\Delta t)} h(n_{i,s}, \bar{p}) ds \right] \\
= \mathbb{E} \left[ \int_0^{\Delta t} \gamma e^{-\gamma s} h(n_{i,s}, \bar{p}) ds \right] + \max_{R_{i,s} \geq t} \mathbb{E} \left[ e^{-\gamma \Delta t} \int_{\Delta t}^\infty \gamma e^{-\gamma(s-\Delta t)} h(n_{i,s}, \bar{p}) ds \right] \\
\approx h(k, \bar{p}) \gamma \Delta t + (1 - \gamma \Delta t) [\eta \Delta t V_i(k + 1, \bar{p}) + \eta \Delta t \zeta \Delta t V_i(k, \bar{p})] + (1 - \eta \Delta t) V_i(k, \bar{p})
\]

Rearranging the equation, we can obtain the value function $\frac{\gamma}{\gamma+\eta+\zeta} h(k, \bar{p}) + \frac{\eta}{\gamma+\eta+\zeta} V_i(k+1, \bar{p}) + \frac{\zeta}{\gamma+\eta+\zeta} V_i(k, \bar{p})$ when the organization optimally chooses placing a request
in such a situation. Similarly, we have the value function \( V_i(k, p) = \frac{\gamma}{\gamma + \zeta} h(k, p) + \frac{\zeta}{\gamma + \zeta} V_i(k, \bar{p}) \) when islands optimally choose not placing a request in such a situation.

Now, we come to the analysis about the optimal threshold of placing a request. To study this, we firstly come to prove that when we ignore the fact \( x \in \mathbb{N} \), \( \frac{\gamma}{\gamma + \zeta} h(x, p) + \frac{\zeta}{\gamma + \zeta} V_i(x, \bar{p}) \) is strictly concave with respect to \( x \). Firstly, \( h(x, p) \) is strictly concave with respect to \( x \). Secondly, if \( x \geq n^* \bar{p} \), \( V_i(x, \bar{p}) \) is strictly concave with respect to \( x \) for the reason that \( V_i(x, \bar{p}) = h(x, p) \); if \( n^* \bar{p} - 1 \leq x < n^* \bar{p} \), by the equation of \( V_i(x, \bar{p}) \), it still is strictly concave with respect to \( x \) for the reason that it is the positive sum of two strictly concave functions. Going on this way, we can prove that \( V_i(x, \bar{p}) \) is strictly concave with respect to \( x \) for all \( x \). This leads to the strict concavity of \( \frac{\gamma}{\gamma + \zeta} h(x, p) + \frac{\zeta}{\gamma + \zeta} V_i(x, \bar{p}) \). Now, it is easy to prove that the solution \( \hat{k} \) of problem \( \max_{k \in \mathbb{N}} \frac{\gamma}{\gamma + \zeta} h(k, p) + \frac{\zeta}{\gamma + \zeta} V_i(k, \bar{p}) \) exists and satisfies \( \hat{k} \in [n^*(p), n^*(\bar{p})] \cap \mathbb{N} \).

Next, we prove that \( \hat{k} \) is the optimal threshold. For any \( k \geq \hat{k} \), by the strict concavity of function \( \frac{\gamma}{\gamma + \zeta} h(x, p) + \frac{\zeta}{\gamma + \zeta} V_i(x, \bar{p}) \) with respect to \( x \), and \( \hat{k} \) is the optimal solution of \( \max_{k \in \mathbb{N}} \frac{\gamma}{\gamma + \zeta} h(k, p) + \frac{\zeta}{\gamma + \zeta} V_i(k, \bar{p}) \), if islands choose to place a request, the utility will decrease, so the optimal decision is not to place a request.

And for \( k < \hat{k} \), if islands choose not to place a request, the utility is \( \frac{\gamma}{\gamma + \zeta} h(k, p) + \frac{\zeta}{\gamma + \zeta} V_i(k, \bar{p}) \), and if they choose to place a request, the utility is \( \frac{\gamma}{\gamma + \zeta} \left[ \frac{\gamma}{\gamma + \zeta} h(k, p) + \frac{\zeta}{\gamma + \zeta} V_i(k, \bar{p}) \right] + \frac{n}{\gamma + \eta + \zeta} V_i(k + 1, p) \). Here, by the fact that \( \hat{k} \) solves \( \max_{k \in \mathbb{N}} \frac{\gamma}{\gamma + \zeta} h(k, p) + \frac{\zeta}{\gamma + \zeta} V_i(k, \bar{p}) \) and function \( \frac{\gamma}{\gamma + \zeta} h(x, p) + \frac{\zeta}{\gamma + \zeta} V_i(x, \bar{p}) \) is strictly concave with respect to \( x \), we have \( \frac{\gamma}{\gamma + \zeta} h(k, p) + \frac{\zeta}{\gamma + \zeta} V_i(k, \bar{p}) < \left[ \frac{\gamma}{\gamma + \zeta} h(\hat{k}, p) + \frac{\zeta}{\gamma + \zeta} V_i(\hat{k}, \bar{p}) \right] + \frac{n}{\gamma + \eta + \zeta} V_i(\hat{k} + 1, p) \), so it is optimal to choose placing a request.

By finding such a threshold \( \hat{k} \), the recursive function of value function follows directly from the analysis above.

As for the monotonicity of \( \hat{k} \), if \( \zeta_1 \geq \zeta_2 \), here we notice that \( V_i(k, \bar{p}) \) remains the same, now the solution of \( \max_{k \in \mathbb{N}} \frac{\gamma}{\gamma + \zeta} h(x, p) + \frac{\zeta}{\gamma + \zeta} V_i(x, \bar{p}) \) will be \( x(\zeta_1) \geq x(\zeta_2) \). The reason is that the derivative of \( \frac{\gamma}{\gamma + \zeta} h(x, p) + \frac{\zeta}{\gamma + \zeta} V_i(x, \bar{p}) \) with respect to \( x \) at \( x(\zeta_1) \) is \( 0 \). And \( \frac{\gamma}{\gamma + \zeta_1} \leq \frac{\gamma}{\gamma + \zeta_2} \), \( \frac{\zeta_1}{\gamma + \zeta_1} \geq \frac{\zeta_2}{\gamma + \zeta_2} \), also \( h(x(\zeta_1), p) \leq 0 \) and \( V_i(x(\zeta_1), \bar{p}) \geq 0 \), we have the derivative of \( \frac{\gamma}{\gamma + \zeta} h(x, p) + \frac{\zeta}{\gamma + \zeta} V_i(x, \bar{p}) \) with respect to \( x \) at \( x(\zeta_1) \) is less than \( 0 \). So, by concavity of function \( \frac{\gamma}{\gamma + \zeta_1} h(x, p) + \frac{\zeta_1}{\gamma + \zeta_1} V_i(x, \bar{p}) \) with respect to \( x \), we have \( x(\zeta_1) \geq x(\zeta_2) \). As for \( \hat{k}(\zeta_1) \geq \hat{k}(\zeta_2) \), the logic is the same as in the proof of Proposition 3.1. And the monotonicity of \( \hat{k} \) with respect to \( \gamma, \eta \) can be proved similarly, which is ignored here.
Contagion is an commonly observed phenomenon in everyday life. In this paper, we use a directional and weighted network to study the contagion network. As an application, the Tuta absoluta in Nepal is studied, of which the contagion network is built based on the trade and transport patterns. A robust network-based approach is proposed to model seasonal flow of agricultural produce and examine its role in pest spread. Furthermore, the long-term establishment potential of the pest and its economic impact on the country are assessed. Preliminary analyses indicate that Tuta absoluta will invade most major tomato production regions within a year of introduction and the economic impact of invasion could range from $17-25 million. The proposed approach is generic and particularly suited for data-poor scenarios.

4.1 Introduction

Food security is an increasingly important societal problem. Increased globalization, climate change, population growth, scarce per capita resources, international travel and invasive species are important factors contributing to the issue of global food

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This paper is coauthored with Srinivasan Venkatramanan, et al. during the work at Biocomplexity Institute of Virginia Tech, which has been published in “2017 IEEE International Conference on Big Data (Big Data 2017), December 11-14, 2017, Boston, MA, USA”.
security. This paper examines how commodity flows affect food security. Production and consumption of agricultural produce is no longer localized - agro products travel thousands of miles over global supply chain networks. While economically attractive in the short term, global trade increases the risk of rapid spread of invasive species and possibly even bioterrorism. The situation is quite similar to spread of infectious diseases in human and animal populations. In this paper, we study the seasonal flow of agricultural commodities, focusing in particular on their role in factors: a spread of invasive species. The spread of pests and pathogens is driven by various understanding of the biology and climatic conditions is essential to assess establishment risk and devise sustainable management strategies and has been the focus of ecologists for a long time. In contrast, not much is understood regarding to the role of human mediated pathways (including trade and travel) to prevent introduction and mitigate immediate impact (Banks et al., 2015; Carrasco et al., 2010; Cunniffe et al., 2015; Hulme, 2009; Robinet et al., 2012). See Colunga-Garcia and Haack (2015); Early et al. (2016); Ercsey-Ravasz et al. (2012); Nopsa et al. (2015); Tatem and Hay (2007) for further discussion on this important subject.

In this paper, we develop an integrated methodology that combines data science, algorithmic, machine learning, ecological modeling and economic welfare analysis that allow us to address important factors that affect the human mediated pathways contributing to invasive species spread and its consequence of economic welfare. i) We develop an integrated data-driven methodology for synthesizing realistic spatio-temporal networks of seasonal agro products between major markets. ii) We illustrate the methodology by developing a spatio-temporal domestic tomato trade network in Nepal and investigate its role in the spread of *T. absoluta*, a devastating pest of the tomato crop (Campos et al., 2017) and an emerging pest in Nepal (Bajracharya et al., 2016). iii) We analyze the spatio-temporal properties of the flow networks. Further, through dynamical analysis of the networks and applying a novel rank-based inference approach to compare the simulation outputs to the time series of pest distribution. iv) We conduct an in-depth sensitivity analysis and this analysis is used in validating our synthesized networks; furthermore the analysis provides improved understanding of the pest dynamics. v) Based on the predictions about *T. absoluta* spread, we study the welfare effect of *T. absoluta* in Nepal among different interventions.

One thing to be noticed here is validating the network representations. While international trade data is available at the commodity level, domestic data is hard to come by. Even in data-rich regions such as the US, the available sample data is aggregated at the commodity level. The role of these networks in the study of invasive species requires one to understand the ecological contagion processes. Monitoring is a resource intensive task: the placement of traps is largely determined by accessibility and availability of trained personnel. Also, pest might not be detected during off season due to host unavailability. In the absence of monitoring, the pest’s presence will become apparent only during the growing season. But its reporting
might be delayed by farmers due to lack of awareness or fear of quarantining. Given these constraints, there might be several months of delay in reporting pest presence.

**Related literature:** Our paper is part of a larger literature on diffusions in network (Acemoglu et al., 2011b; Jackson and Yariv, 2005, 2007), network securities (Haller, 2016; Miller and Kiss, 2014). This paper In recent years, there has been a lot of interest in studying the role of international trade and travel in invasive species spread. Ercsey-Ravasz et al. (2012) analyzes the International Agro-Food Trade Network. Using network measures, they identify countries of importance in the trade network. Early et al. (2016) study the terrestrial threat from invasive species and evaluate national capacities to prevent and manage invasions. Tatem and Hay (2007) showed that the world-wide airline network increases the risks of establishment by providing busy transport links between spatially distant, but climatically similar regions of the world. There has been some work on domestic commodity flow and its role in pest spread. Nopsa et al. (2015) evaluated the structure of rail networks in the US and Australia for pest and mycotoxin dispersal. Colunga-Garcia and Haack (2015) use the regional freight transport information to characterize risk of urban and periurban areas to exotic forest insect pests in the US. Robinet et al. (2012) provides a survey of recent modeling efforts.

**T. absoluta.** There is general consensus that vegetable and seedling trade is a primary driver of *T. absoluta* spread (Campos et al., 2017). However, previous modeling efforts have only focused on establishment potential (Tonnang et al., 2015) and spatial dispersion (Guimapi et al., 2016). This is the first work that analyzes human-mediated pathways in the context of *T. absoluta*. Nepal’s vegetable production and trade has been extensively studied from a socio-economic perspective. But, to the best of our knowledge, there is no such work in the context of invasive species spread with focus on this region.

### 4.2 Network Structures

Let $\mathcal{N} = \{1, 2, ..., n\}$, $n \geq 3$, be the set of nodes, which stands for different markets. In each market $i$, we denote by $O_i$ and $I_i$ the respective volumes of outflows and inflows of the commodity. Following the conventions in network economics literature (Jackson, 2008), we denote the neighbors of $i$ in the network $\mathbf{g}$ as $N_i(\mathbf{g}) = \{g_{ij} > 0\}$. There are several different networks in existence between the markets, and all networks are weighted in this paper. The first type of network is the road network $\mathbf{g}^R$, where $g_{ij}^R \geq 0$ stands for the road connection between market $i$ and market $j$.

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1See “Value Chain/Market Analysis of the vegetable Sub-Sector in Nepal (August)” 2011, USAID/Nepal, for example.
if \( g_{ij}^R = 0 \), there is no road directly connecting the two markets; if \( g_{ij}^R > 0 \), it is the distance from \( i \) to \( j \).

Next we define the flow network (FN): \( g^F \): an end-to-end flux for each pair of markets. Here, the flow network of commodity among markets depends on: the total outflow from a market - the amount of commodity that enters it; and the total inflow towards a market - a function based on the population it caters to and the corresponding per capita income. The main assumptions here are: (i) imports and exports are not significant enough to influence domestic trade, (ii) fresh tomatoes are mainly traded for consumption, and (iii) the higher the per capita income, the greater the consumption. For the study of Nepal, these are fair assumptions. The flows are estimated using a doubly constrained gravity model (Anderson, 2011; Kaluza et al., 2010). In its general form, the flow \( g^F_{ij} \) from market \( i \) to market \( j \) is given by

\[
g^F_{ij} = a_i b_j O_i I_j f(g_{ij}^R),
\]

where, \( O_i \) is the total outflow of the commodity from \( i \), \( I_j \) is the total inflow to \( j \), \( g_{ij}^R \) is the distance to travel from \( i \) to \( j \), and \( f(\cdot) \) is the distance deterrence function. Here, \( a_i, b_j \) are parameters that are calibrated to coincide the model and the data. Following Kaluza et al. (2010), we use a general deterrence function: \( (g_{ij}^R)^{-\beta} \exp(-g_{ij}^R/\kappa) \), combining power-law and exponential decay with \( g_{ij}^R \) which can be controlled by the tunable parameters \( \beta \), the power-law exponent, and \( \kappa \), the cutoff time.

### 4.2.1 Spread Dynamics

In this section, we develop a discrete-time SI (Susceptible-Infected) epidemic model on directed weighted networks (Pastor-Satorras et al., 2015) to model pest dispersal. Each node is either susceptible (free from pest) or infected (pest is present). Henceforth, we use the term “infected” for a node or a region frequently to imply pest infestation at that location. A node \( i \) in state \( I \) infects each of its out-neighbors \( j \) in the network with probability proportional to the flow \( g^F_{ij} \) at each time step \( t \). The infection probabilities are obtained by normalizing flows globally:

\[
\lambda_{i,j} = \frac{g^F_{ij}}{\max_{i,j} g^F_{ij}}.
\]

The model is based on two assumptions: (i) an infected node remains infected and continues to infect its neighbors and (ii) the chance of infection is directly proportional to the volume traded. Considering the fact that Nepal was ill-prepared for this invasion and the lack of effective intervention methods, (i) is a fair assumption. Historically, pest has spread rapidly in regions where tomato trade has been the highest (parts of Europe and Middle-East for example) thus motivating assumption (ii).

Here, we interpret each node as a value from \([0, 1]\), standing for the probability that the market is uninfected, or the fraction of the market is uninfected. We introduce an efficient methods proposed in Lokhov et al. (2014) to derive the probability. The susceptibility of a node to infestation depends on the initial conditions and time of
observation. We denote initial condition as $f_0$, which assigns probability of infection at time $t = 0$ to each node. Based on initial condition and pest spread dynamics, we denote $\mathcal{P}_S(i, t, f_0)$ as the probability that node $i$ remains uninfected (i.e., susceptible) by time $t$ given initial condition as $f_0$. Here, we adopt the dynamic message passing algorithm introduced in Lokhov et al. (2014) as follows.

The pest dispersal process can be interpreted as the propagation of infection signals from infected to susceptible nodes. At time $t$, each node $i$ can be either susceptible or infected, which will be denoted as $q_i(t) = S, q_i(t) = I$, respectively. And we use indication function $\delta_{q_i(t), I} = 1$ if at time $t$, node $i$ is infected, and $\delta_{q_i(t), I} = 1$ otherwise. The infection signal $d^{i\rightarrow j}(t)$ is defined as a random variable which is equal to one with probability $\delta_{q_i(t-1), I} \lambda_{ij}(t)$ and each to zero otherwise. Consider an auxiliary dynamic $D_j$ where node $j$ receives infection signals, but ignores them and thus is fixed to the susceptible status at all times. In order to have a closed system of message-passing equations, we have the concepts of two cavity messages as follows.

Equation 4.1 defines the message $\theta^{i\rightarrow j}(t)$ as the probability that the infection signal has not been passed from node $i$ to node $j$ up to time $t$ in the dynamics $D_j$. While equation 4.2 defines the probability that the infection signal has not been passed from node $i$ to node $j$ up to time $t$ in the dynamic $D_j$ and that node $i$ is in the state $I$ at time $t$. And also, we denote $\mathcal{P}_S^{i\rightarrow j}(t)$ as the probability that node $i$ is in the state $S$ at time $t$ in the dynamics $D_j$ as follows.

$$\mathcal{P}_S^{i\rightarrow j}(t) = \text{Prob}^{D_j}[q_i(t) = S] \quad (4.3)$$

Following Lokhov et al. (2014), we have the following proposition about the transition of random variables about the pest dispersal.

**Proposition 4.1.** (Lokhov et al., 2014) The evolution of variables $\mathcal{P}_S^{i\rightarrow j}(t), \theta^{i\rightarrow j}(t), \phi^{i\rightarrow j}(t)$ about the pest dispersal process follows the following dynamics.

$$\mathcal{P}_S^{i\rightarrow j}(t + 1) = \mathcal{P}_S(i, 0, f_0) \prod_{k \in N_i(g^p)} \theta^{k\rightarrow i}(t + 1) \quad (4.4)$$

$$\theta^{k\rightarrow i}(t + 1) - \theta^{k\rightarrow i}(t) = -\lambda_{ki}(t) \phi^{k\rightarrow i}(t) \quad (4.5)$$
\[ \phi^{k \rightarrow i}(t) = (1 - \lambda_{ki}(t - 1)) \phi^{k \rightarrow i}(t - 1) - [\mathcal{P}_S^{j \rightarrow i}(t) - \mathcal{P}_S^{j \rightarrow i}(t - 1)] \] (4.6)

where \( \lambda_{ki}(t) \) is the infection probability across edge \((k, i)\) at time \(t\).

In the analysis of \( T. \) absoluta in Nepal, we choose initial configuration \( f_0 \) to mimic a spatially dispersed seeding scenario. We first select a central seed node, and then use a Gaussian kernel with parameter \( \sigma \) around the seed node to assign initial infection probabilities for neighboring markets. A market at a geodesic distance \( d \) from the seed, is assigned the infection probability \( e^{-\frac{d^2}{2\sigma^2}} \). The kernel accounts for factors such as uncertainty in determining the pest location, the possibility of spread of the pest through natural means as well as interactions between these markets.

### 4.2.2 Economic Impact

For the Economic impact analysis, we use simulate the spread dynamics firstly and then based on the simulation results, we analyze both the direct and indirect economic outcome in this scenario. Here, from observations about tomato production in Nepal, we can see the tomato production in Nepal as a small open economy, and also, Nepal just imports the tomatoes, consisting of around 6-7\% of the total consumption, barely exports, consisting of less than 1\% of total consumption. So, in this circumstance, it is assumed that the world price \( (P_W) \) is exogenously given; at the same time, imported tomatoes and domestic tomatoes are substitutes but not perfect substitutes, because the qualities or the types of these tomatoes are different. Hence, we assume that the import function is given by \( Im = v(P_w)^{-\omega} \cdot P^d \); here, \( v > 0 \) is a scale factor, and \( \omega, d \) are the elasticities of import to world price and domestic price, respectively. The assumption takes the substitute effect between the local tomatoes and world tomatoes into consideration.

We assume that the demand for tomatoes in Nepal is given by \( D = \chi P^{-\eta} \), where \( \eta \) is the demand elasticity at the domestic price. Domestic supply part consists of two parts: non-affected supply, \( S_N = \beta P^\theta (1 - z) \); affected supply: \( S_A = (1 - h)\beta(v P)^\theta z \), where, \( \beta \) represents the factor related to productivity. Here, \( \theta \) is the supply elasticity to domestic price, \( z \) stands for the proportion of the affected area, \( h \) represents the production loss due to the disease, and \( v \) counts for the increased cost of production with respect to extra work on control or sanitation of the producers.

In the economic impact analysis, we assume that tomatoes are freely distributed to the whole market, so the domestic price is assumed to be the same. For each polygon \( i \), we have the prediction data about the disease, so we have \( z_i \), so for each polygon \( i \), we can calculate both non-affected supply, affected supply as \( \beta_i P^\theta l_i (1 - z_i), (1 - h)\beta_i (v P)^\theta l_i z_i \), where \( l_i \) is the areas of land to produce tomatoes for polygon \( i \). In this
sense, we have the total domestic supply as \( \sum_i \beta_i P^\theta l_i (1 - z_i) + \sum_i (1 - h) \beta_i (vP)^\theta l_i z_i \).

In equilibrium, the market price \( P \) must satisfy

\[
\chi P^{-\eta} = \sum_i \beta_i P^\theta l_i (1 - z_i) + \sum_i (1 - h) \beta_i (vP)^\theta l_i z_i + v(P_W)^{-\omega} \cdot P^d
\] (4.7)

where, the left-hand side is the demand and the right-hand side is the sum of non-affected supply, affected supply form domestic market, and also the supply from the rest of the world.

Based on simulation results of pest spread in Nepal and equilibrium price, the direct impact assessment is calculated by:

\[
\text{Direct Impact} = P \sum_i h y_i l_i z_i
\] (4.8)

here, \( h \) is proportion loss if affected as described above, \( P \) is the market price and \( y_i \) is the yield per amount of land before being affected. This assessment doesn’t consider the influence of market price on producers’ choice on how much to plant tomatoes, for the reason that for the same land, if the profit from tomatoes is very low, then the producers may have incentives to plant other things. Also, this doesn’t consider the whole utility change of the society. In the following, we calculate the total economic impact.

For each intervention (this includes the case where there is no disease and also, no interventions), we have the prediction data about the disease. When we try to compare two interventions \( \text{Intervention}_1, \text{Intervention}_2 \), firstly, we use this prediction data to calculate market prices \( P_1, P_2 \), respectively, by using the market clearing condition 4.8. Secondly, we calculate the social welfare changes from intervention 1 to intervention 2, this comes from three parts: consumer surplus change, producer surplus change, and influence from import change.

For consumer surplus change, we have

\[
\Delta CS = \int_{P_1}^{P_2} \chi P^{-\eta} dP = \frac{\chi}{1 - \eta} P_1^{1-\eta} \bigg|_{P_1}^{P_2}
\] (4.9)

As for producer surplus, we firstly calculate the producer surplus for each polygon \( i \) as
\[ \Delta P S_i = \frac{\beta_i}{1 + \theta} l_i z_i,2 P_2^{1+\theta} - \frac{\beta_i}{1 + \theta} l_i z_i,1 P_1^{1+\theta} + \left( \frac{\beta_i}{1 + \theta} l_i z_i,2 (1 - h) v^\theta P_2^{1+\theta} \right) \]

\[ - \frac{\beta_i}{1 + \theta} l_i z_i,1 (1 - h) v^\theta P_1^{1+\theta} \]  

(4.10)

And for surplus change due to import, we have

\[ \Delta I S = \frac{v}{1 - \omega} (P_W)_{1-\omega} (P_d^d - P_1^d) \]  

(4.11)

Based on these three effect about social surplus, we will have the total economic impact as \( \Delta C S + \sum_i \Delta P S_i + \Delta I S \) between two interventions.

4.3 Analyses and Results

In this section, we will analyze the spread of *T. absoluta* in Nepal. Firstly, we will describe data structure in this paper; secondly, using the data collected, we will build the flow network; thirdly, based on flow network and data collected, we will analyze the spread of *T. absoluta*, and finally we will analyze the economic impact of the spread of *T. absoluta* in Nepal.

4.3.1 Data Structure

Agro-trade networks for moving agricultural products is a complex system problem. The networks depend on numerous factors, including seasonal production, population distribution, cultural factors, economic activity, storage and transport infrastructure. Furthermore, data needed to develop agro-trade networks is often sparse, noisy and is not openly available. For instance, even standard information such as region-level production is unavailable for many countries. Even if available, these datasets vary in format, they are misaligned in reporting time and vary in spatial and temporal resolution. Apart from quantitative datasets, there is also need for qualitative information pertaining to the study region such as cultural practices, seasonal production cycles, etc. In this section, Figure 4.1 outlines the different components that constitute the framework and Table I lists the datasets used in our framework. As we can see the attributes of each market will consist of information about market locations, tomato production, production seasonality, population and income per capita. By combining road network and the attributes of market nodes, we can build the seasonal flow
network in Nepal. Based on flow network, by dynamic message passing algorithm, we come to the spread dynamics of *T. absoluta*, which will lead to the probability prediction of *T. absoluta* in Nepal. Finally, based on the probability prediction and the market condition, we can estimate the economic impact of *T. absoluta* in Nepal.

![Modeling framework and some databases](image)

**Figure 4.1: Modeling framework and some databases**

<table>
<thead>
<tr>
<th>Description</th>
<th>Source</th>
<th>Resolution</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market distances</td>
<td>Google Maps, Distance matrix API</td>
<td>Market</td>
<td>2017</td>
</tr>
<tr>
<td>Tomato consumption</td>
<td>MOAD, FAOSTAT, MOAD</td>
<td>Country, Annual</td>
<td>2013</td>
</tr>
<tr>
<td>USAID IPM Innovation Lab, iDE Nepal</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 4.1: Datasets**

**Estimate of Flow Network**

As already mentioned, the flow network will be built on doubly constrained gravity model Anderson (2011); Kaluza et al. (2010), where \( g_{ij}^R = a_i b_j O_i I_j f(g_{ij}^R) \). So, we need to gather data for \( O_i, I_j, g_{ij}^R \) for the tomato markets in Nepal. The nodes of the
flow network are the major markets, 69 in all, after merging markets that belong to the same town.

**Outflow, $O_i$:** Recall that the amount of production is specified at the district level. In order to obtain the production estimates at market level, we defined market scope as follows: The country’s map was overlaid by a grid cell of size 5km-5km and we constructed a Voronoi partition of these cells using market locations as centroids. This was motivated by the fact that tomato sellers and buyers will seek out the nearest market. We assumed uniform spatial distribution of production within each district. Each grid cell was assigned a value of production in a particular season proportional to the fraction of the area of the district covered by the cell. The total outflow from the market is the sum of production of the grid cells assigned to it for a particular season.

**Inflow, $I_j$:** We modeled the total inflow $I_j$ into a market as a product of the population catered to by the market and a function of the average per capita income associated with the market $\eta_i$, $\eta_i^\gamma$, where $\gamma$ is a tunable parameter. The population catered to by the market, was derived from district level population data and the market scope as defined for production redistribution.

**Road network, $g^R$:** Owing to the diverse landscape of Nepal and varying road conditions we used travel time by road instead of the geodesic or road distance between the markets. We begin with list of major vegetable markets in Nepal (see Table 4.1), and
geolocate them using Google Maps. We then manually embedded the market locations onto Nepal road network, and constructed a planar network by connecting the markets which have a direct route (without going through other markets) between them. We also removed markets which were completely inaccessible by road. We used Google Distance Matrix API to compute travel times by road along the edges of this planar network. This in turn, yields a road network among the markets, where the edges are weighted by their travel time. Distance between any two markets is then obtained as the shortest travel time on the road network.

Now, based on data about $O_i, I_j, g^R$, after finishing the estimate of $a_i, b_j$, we come to the flow trade of the *T. absoluta* in Nepal. Here, to estimate $a_i, b_j$, we notice that $O_i = \sum_{j \in N} g_{ij}^F$ and $I_j = \sum_{i \in N} g_{ij}^F$, which means that the total outflow of each market is equal to the sum of the outflows to all other markets and the total inflow of each market is equal to the sum of the inflows from all other markets. So we must have the following equations:

$$a_i = \frac{1}{\sum_{j \in N} b_j I_j f(g_{ij}^R)}$$  \hspace{1cm} (4.12)

$$b_j = \frac{1}{\sum_{i \in N} a_i O_i f(g_{ij}^R)}$$  \hspace{1cm} (4.13)

There are total $2n$ unknowns and $2n$ equations. Kaluza et al. (2010) show that the iterative process converges to fixed values of $a_i$ and $b_j$. There is a tolerance which enables faster convergence at the cost of accuracy of these parameters, and in turn the flow. We set the tolerance factor to 0.01.

*Seasonal flow network:* Here, we should notice that there are two flow networks for *T. absoluta* in Nepal according to the trade pattern changes in different seasons. Based on the physiography, districts of Nepal are partitioned into three regions, namely Terai, Mid Hills and High Hills (see Figure 4.4(e)). Due to altitude and temperature variations, the tomato production season varies among these regions (see Figure 4.2(b)). Production in the Mid Hills and High Hills is largely restricted to the summer months of June to November (referred to as season S1), while Terai region produces during the winter months of December to May (referred to as season S2). As a result, we have two distinct flow networks, one for each season. We partitioned the districts into two groups: Mid Hills and High Hills belong to group 1, while the Terai districts belong to group 2. All districts belonging to group $i$ were assigned their respective annual production for season S2 and zero for the other season.
4.3.2 Results

Flow validation: The unavailability of sample data on seasonal trade of tomato crop makes it challenging to calibrate and validate the flow network model. In fact, to the best of our knowledge, even information on annual flow of vegetables between markets is not available. However, for the largest wholesale market of Nepal, Kalimati (located in Kathmandu), yearly data on volume of tomato arriving from each district is available (Table I). In Figures 4.3(d)-4.3(f), we compare this data with the network flows. Given a set of network parameters ($\beta, \kappa, \gamma$), we obtained the inflow from a particular district to Kathmandu as follows: We combined the weights of all edges of the corresponding network with destination node “Kathmandu” and source nodes belonging to that district.

As seen in Figure 4.3(d), for $\gamma$ values between 0.5 and 1, the flows from the networks are comparable to the Kalimati data except for two districts: Dhading (the top contributor) and Sarlahi (third highest). Upon further investigation we find that Dhading, which is a major producer west of Kathmandu, serves the Mid Hills and Terai regions of the Central Development Region in the flow networks (Figure 4.3(e)). While the gravity model predicts that these flows will be directly delivered to these regions, in reality, it is possible that Dhading’s produce is routed through Kalimati market as there are several traders from Dhading registered in the Kalimati market\(^2\).

As for Sarlahi, even though there is little inflow to Kalimati market in the flow networks, other markets in the Kathmandu valley (belonging to Bhaktapur and

\(^2\)Refer to “Value chain development plan for tomato”, 2012, Nepal Ministry of Agriculture Development.
The first row shows the flow from east to west between development regions of Nepal. The second row depicts the flow from north to south between regions of different altitudes. While the second and third columns correspond to seasonal flow, the last column corresponds to the flows generated from annual data.

Figure 4.4: The Spatio-temporal structure of the flow network.

Lalitpur districts) receive significant flows from Sarlahi (Figure 4.3(f)), which could, as in the previous case be routed through Kalinmati market. These issues highlight some of the limitations of the gravity model, which do not account for real-world trader dynamics.

Structural properties

For each set of network parameters \( (\beta, \kappa, \gamma) \), there are two networks, one for each season. Both networks have 69 nodes. The cumulative distribution of flows with respect to travel time are plotted in Figures 4.3(a) - 4.3(c) for different values of network parameters for season S1 (the network corresponding to season S2 have similar properties). Except for \( \beta = 1 \), the flow for \( t > 500 \) minutes, which corresponds to \( \approx 8 \) hours of travel time.

For further analysis of the flow network we describe the different regions within Nepal. Nepal has significant altitude variations along the North-South axis, and is divided into three major physiographic regions namely: Terai, Mid-hills and High hills (Figure 4.4(e)). For administrative reasons, Nepal has been divided along the East-West axis (Figure 4.4(a)) into five major development regions. Kathmandu, for instance,
belongs to Mid-hills and Central Development Region. It is useful to remember that the Central Development Region is by far the most economically prosperous, while the population density is high along the Terai region and Kathmandu valley (Table I).

The general trends of tomato trade between markets is depicted in Figure 4.4 (generated for $\beta = 2, \kappa = 500$ and $\gamma = 1.0$). We recall that our model accounts for the fact that the Hills/Mid Hills and the Terai are the primary sources of tomato during seasons S1 and S2 respectively. This is clearly reflected in the net flow diagram between geographic regions: north (Hills/Mid Hills) to south (Terai) in S1 and south to north in S2. However, an interesting pattern to be noted is the significant flow from east to west during S1 as observed in the net flow diagram between the Development Regions. These could be due to the variability in vegetable production, and the presence of an arterial East-West highway that almost covers the entire breadth of the country.

**Comparison with the annual flow network**

To evaluate the importance of seasons, we constructed the annual flow network by using the gravity model with annual production for each district. The resulting flows are shown in Figures 4.4(d) and 4.4(h). Compared to the seasonal flows we see that annual flows are of shorter distance and thus there is not much flow between regions (either between east and west or south and north).

**Sensitivity analysis of the flow network**

Figures 4.3(a) - 4.3(c) show the sensitivity of edge weight distribution of season S1 network to $\beta, \kappa, \gamma$. We find that the for $\beta \geq 2$ and $\kappa \geq 500$ the weight distribution is relatively stable. A similar behavior was observed for the season S2 flow network with respect to $\beta$ and $\kappa$. Increasing $\gamma$ tends to redistribute flows towards high income regions (in this case, regions around Kathmandu in the Mid Hills, Central Development Region, see Figure 4.3(a)), and leads to higher maximum flows in the network in season S1, and lower maximum flows in season S2 (not shown here). However, changing $\gamma$ had minimal effect on most of the low weight edges in the network.

### 4.3.3 Role of trade network in pest spread

We applied the network diffusion model described in Section 4.2.1 to study the role of the flow networks in the spread of *T. absoluta* in Nepal. To interpret the spread
model’s output, in terms of incidence reports, we need to translate the SI model to a real-world equivalent temporal unit (e.g., month). Validating this requires pest reports at a high spatiotemporal resolution. Since this is absent in the case of *T. absoluta*, to circumvent this problem, we make use of SI model’s monotonic property: For any $t' > t$, $P_S(i, t', f_0) \leq P_S(i, t, f_0)$, and thus the ranking of relative vulnerabilities of market nodes could inform how the process unfolds. We also observed that the rank list is stable (or changes slowly) with respect to $t$ with other parameters fixed (see Table 4.2).

<table>
<thead>
<tr>
<th>Parm.</th>
<th>Levels</th>
<th>t-ratio</th>
<th>F-value</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>[0, 1, 2]</td>
<td>-5.16</td>
<td>26.6059</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>[5, 10, 15, 20]</td>
<td>-3.29</td>
<td>10.8424</td>
<td>&lt;0.0001</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>[100, 500, 1000]</td>
<td>-0.42</td>
<td>0.1758</td>
<td>0.6753</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>[0, 0.5, 1.0]</td>
<td>0.89</td>
<td>0.7970</td>
<td>0.3727</td>
</tr>
<tr>
<td>$T$</td>
<td>[5, 10, 20]</td>
<td>1.14</td>
<td>1.2976</td>
<td>0.2556</td>
</tr>
</tbody>
</table>

Table 4.2: Analyzing sensitivity to model parameters using ANOVA

The experiment was setup under the following premise: *T. absoluta* was first introduced to the Kathmandu valley. Ground experts have high confidence in this assumption since the pest was not discovered in the previous growing season in other parts of Nepal. Given the pest reports till December 2016 (Figure 4.1(d)), we evaluate our model based on the following backward inference problem: for an observation of node states at time $t$, what is the most likely origin of invasion? (also known as the source detection problem (Shah and Zaman, 2011)). We examine the likelihood of markets or regions being the source nodes, and in particular, we compare this with the likelihood of the region around Kathmandu being the source (see Figure 4.5). Suppose $\mathcal{O}$ is the observation criteria; it consists of pairs $(v, X)$ where $v$ is a node and $X \in \{S, I\}$ is a state. For each candidate initial condition $f_0$, we estimate the joint probability of $\mathcal{O}$ at a time step $t$, as a product of the marginal probability estimates from the message passing algorithm and define an energy function for each tuple $(f_0, t)$ as

$$
\phi(\mathcal{O}|f_0, t) = -\log \left( \prod_{(v, X) \in \mathcal{O}} \mathcal{P}_X(v, t, f_0) \right)
$$

The lower the value of $\phi$, the higher the likelihood of $f_0$ being the initial condition. Secondly, recalling the uncertainty in interpreting time step $t$, we examined the relative likelihoods of each $f_0$ and the stability of the ranking across a range of model parameters.

We consider the spread during June-November (season S1) for model evaluation. Using the S1 flow network, our objective was to rank various starting configurations.
$f_0$ based on $\phi(O|f_0, t)$ given $O, t$. For a given $\sigma$, we evaluated the likelihood of each node being the central node. We considered two criteria based on which the likelihood of each $f_0$ as the starting configuration was computed: (i) $O_G$: this is the set of all pairs $(v, I)$ where $v$ is a market node that belongs to a district that reported pest presence by December 2016. (ii) $O_B$: this is the set of $(v, I)$ for all nodes $v$. This is the baseline which assumes no observational data.

![Graph](image)

(a) The average rank of each market based on the likelihood for the criterion $O_G$ for a range of model parameters (see Table 4.2). (b) Same as (a), but for criterion $O_B$. (c) Spread in S2. The parameters used were $\beta = 2$, $\kappa = 500$, $\sigma = 15$, $\gamma = 1$, $T_1 = T_2 = 10$ with Kathmandu as the seed node. The blue dots correspond to markets whose districts reported $T.\textit{absoluta}$ presence before December 2016 (season S1), while the red dots correspond to markets which reported later.

Figure 4.5: Evaluating the spread model using epidemic source inference framework.

The results are shown in Figure 4.5. Firstly, we observed that for both criteria $O_G$ and $O_B$, the top few ranks are relatively robust to varying network and model parameters. Also, for both criteria, markets from the Central Development Region (CDR) that belong to Kathmandu and its adjacent districts are among the top ranked nodes. Interestingly, for the criterion $O_G$, Dhankuta (EDR), with the highest assigned production has a very low rank (Figure 4.5(a)) and a low $\phi$ value compared to the top market in $O_G$. However, for $O_B$, it is ranked second (Figure 4.5(b)). This clearly shows that while Dhankuta has the potential to infect a large number of areas, given what has been observed, it is very unlikely that it was the source of infection. Dhankuta reported presence of the pest only towards the end of 2016 (see Figure 4.1(d)).
Spread in season S2

To study the spread from November 2016 to May 2017, we considered the dynamics on season S2 network. To set the initial conditions, we used the results of our inference study, and chose Kathmandu with $\sigma = 10$ as the seed distribution. For this initial condition, we obtained the probability of infection for all nodes in S1 for T1 time steps. This distribution is used as initial condition for the S2 network spread. Figure 4.5(c) shows the infection probabilities for a particular combination of $(T1, T2)$. As seen in Figure 4.5(c), our model suggests that most Terai and Mid Hills regions of CDR, WDR would be affected by the end of May 2017, and subsequent seasons are only going to see increasing incidence of the pest throughout the country. From Figure 4.2(c), we see that regions belonging to Terai in CDR and Mid Hills of WDR and MWDR have already reported pest presence (marked in Figure 4.5(c)). While the intended usage of the origin inference formulation is to determine the source of infection, we have adapted it to compare expected spread in the model with observed data. Our results demonstrate that this framework is in general very useful in finding the likely pathways of introduction of the pest.

Sensitivity analyses

A full factorial design was performed with levels for the parameters of interest as given in Table II, and analysis of variance (ANOVA) was used to evaluate single parameter effect. It is worth noting that assessment of parameter sensitivity depends on the choice of quantity of interest. Since the outcome of origin inference is a ranking on markets, we used Spearman’s rho to test its stability across the parameter space. The experiment was set up within the GENEUS framework (Wu et al., 2017), a general computational environment for experimental design, uncertainty quantification and sensitivity analysis.

We studied the sensitivity of individual market ranks as well as rank lists to network parameters ($\beta, \kappa, \gamma$), and diffusion model parameters ($\sigma, t$). We found that the market ranks are more sensitive to spatial seeding parameter $\sigma$ and distance exponent $\beta$ than other parameters. In particular, we observed that the sensitivity was highest when $\sigma = 0$ was included in the analysis. In this case (and in general for very low values of $\sigma$), substantial spread occurs only when the seed node is a source. Even if a node is in close proximity to several sources (such as Kathmandu), there is hardly any spread. This is unrealistic in the context of pest and pathogen dispersal. Hence, we restricted $\sigma$ to be greater than 0 in our analysis. Also, we observe that the variance in rank is small for higher ranked nodes. This can be seen in Figure 4, and is more pronounced in the single parameter analyses. This property gives higher confidence in interpreting the results on top markets.
We used Spearman’s rank correlation coefficient to analyze the rank stability. Here we use the rank list that results from configuration \((\beta = 2, \kappa = 500, \gamma = 0, \sigma = 5, T = 10)\) as the reference and calculate the Spearman’s rho value with respect to it for rank lists induced by other parameter settings. Table II gives the Analysis of Variance (ANOVA) results. Under 95% confidence level, \(p\)-value < 0.05 means that the particular parameter has a significant effect. Therefore, we see that \(\beta\) and \(\sigma\) have significant effects, while others do not. Here, we note that this is despite not considering \(\sigma = 0\) in the analysis.

### 4.3.4 Economic Impact Analysis

In this section, we will analyze the economic impact of spread of *T. absoluta* in Nepal. We evaluate the economic impact of *T. absoluta* in Nepal as we project its spread from the initial location in Kathmandu. Nepal’s tomato production is less than 0.2% of the global tomato production, most of which is used to meet the domestic demand for tomatoes, and it exports only 1% of its production, so we treat it as a small closed economy\(^3\). To compute the impact, we use two different measures: (i) the direct impact and (ii) the total impact, following the methodology in Soliman et al. (2012). The direct impact measures the direct revenue loss from the tomato crop as the sum of loss encountered by each district, which in turn depends on proportional loss in the affected area, yield per unit of land in the district before being affected, tomato production area in the district, and the proportion of area affected by the pest which is informed by the spread model.

A 25% crop loss (Bajracharya et al., 2016) in cultivated areas infected by *T. absoluta* results in a direct economic impact of $19.7M. The direct impact, however, does not account for the change in the market price of tomatoes due to loss in the crop or the impact of price change on consumers’ and producers’ welfare.

To calculate a more comprehensive economic impact, we use the partial equilibrium approach (Alston et al., 1995; Soliman et al., 2012). This method focuses on the dynamics of the tomato market and assumes that the price for substitute and complementary goods are given, i.e., no changes occur in substitute and complementary goods market due to changes in tomato prices. The comprehensive economic impact analysis shows a social welfare loss of $22.4M and a price increase of 32%. The analysis assumes ‘demand elasticity to price’ to be -0.7, ‘supply elasticity to price’ to be 0.5, base price at $400 per ton\(^4\), crop loss due to pest invasion to be 25% (Bajracharya et al., 2016) and reduced net price due to increased cost of production from pest con-


trol to be 80% (Khidr et al., 2013). The pest risk probabilities are obtained after running the spread model for 10 time steps.

As mentioned in the model evaluation, the time duration of each time step \( t \) can be arbitrary, so we perform a sensitivity analysis of the economic impact for two other time steps; one where the spread model is run for only 5 time steps and the other where it is run for 20 time steps. The longer run forecasts result in higher pest risk probabilities and hence higher economic impact, and vice versa. Pest risk probabilities based on 5 time steps, ceteris paribus, result in a direct economic impact of $16M and a comprehensive economic impact of $17.5M, whereas after 20 time steps these numbers increase to $21.5M and $24.7M respectively.

### 4.4 Conclusion

Although there is general consensus that vegetable and seedling trade is a primary driver of *T. absoluta* spread Campos et al. (2017); Desneux et al. (2010); Karadjova et al. (2013), previous modeling efforts have only focused on establishment potential Desneux et al. (2010); Tonnang et al. (2015) and spatial dispersion Guimapi et al. (2016). This is the first work that analyzes human-mediated pathways in the context of *T. absoluta*. In recent years however, the role of human-mediated dispersal is being increasingly accounted for in the modeling community. Robinet et al. Robinet et al. (2009, 2017) develop a long-distance dispersal model to study the spread of pests that accounts for heterogeneous human population densities in the study region. International Early et al. (2016); Ercsey-Ravasz et al. (2012); Kaluza et al. (2010) and domestic Colunga-Garcia and Haack (2015); Magarey et al. (2011) trade datasets have been analyzed to assess the susceptibility of countries to invasive alien species and contaminants. Nopsa et al. Nopsa et al. (2015) study the structure of stored grain network induced by storage facilities and an underlying rail network. Nepal’s vegetable production and trade has been extensively studied from a socio-economic perspective Adhikari et al. (2012); Ali (2000), But, to the best of our knowledge, there is no such work in the context of invasive species spread with focus on this region.

Even though Nepal is mainly an agrarian society, its agriculture has been predominantly characterized by subsistence farming, poor marketing infrastructure and dependency on imports from neighboring India. However, in the past decade, there has been a surge of efforts to improve this situation. Increasingly, farmers have been adapting protected cultivation methods such as tunnel farming to increase yield, and

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5For example, “Value chain development plan for tomato”, 2012, Nepal Ministry of Agriculture Development; and “Value Chain/Market Analysis of the vegetable Sub-Sector in Nepal (August)” 2011, USAID/Nepal.
reverse the trend of importing. Also, given its unique geography, Nepal is among the most susceptible regions to climate change (Kindlmann, 2011, Chapter 2). Under such conditions, major invasion events such as *T. absoluta* are detrimental to its biodiversity, economy and societal well-being in general. Therefore, there is a desperate need for concerted efforts to understand the complex food system of this country and increase its reactive capacity and resilience to attacks from invasive species. We believe that our work has taken the first steps in this direction.

Since our study is one of the first to consider commodity flow analysis in the context of *T. absoluta*, we believe there are several avenues for improvement. While some of the limitations arise from lack of refined data, others are due to the limited understanding of the underlying complexity of pest invasions. The former may be the norm for emerging contagions in a data-poor region, whereas the latter will need several iterations of model development and validation by the scientific community.

While gravity models have been applied to study a diverse set of phenomena that concern interaction between entities Bossenbroek et al. (2001); Erlander and Stewart (1990); Jongejans et al. (2015); Kaluza et al. (2010); Krings et al. (2009); Simini et al. (2012); Thiemann et al. (2010) they do have known shortcomings Rothlisberger and Lodge (2011); Simini et al. (2012). The illustrative example and the analysis of the flow network with respect to data on flows in Kalimati markets brought some of them to the fore. Further, the model could be refined by accounting for price variations and commodity varieties. The assignment of demand and supply attributes was done assuming homogeneous distribution of district level production and population information. This can be improved with (a) population and production estimates at a higher spatial resolution, (b) ground survey data on each market’s scope. The same holds for temporal resolution in assigning seasonal production. Further, a better understanding of production cycles, underlying infrastructure for commodity transport will greatly improve the flow estimates.

Our model predominantly focuses on the commodity flow, and does not explicitly account for natural spread or ecological parameters. A more comprehensive model will need to integrate ecological suitability directly in the diffusion process, and also account for spatial diffusion. Further studies will be needed to understand the pest’s flying capacity, influence of wind direction, etc. to realistically capture natural diffusion. In this regard, more general SEIR models can be considered to account for time taken for pest establishment and interventions if available.

*T. absoluta* is also known to survive on alternative hosts such as eggplant, potato and pepper, at varying degrees of suitability Bawin et al. (2016). Transportation of infected seedlings from nurseries to production sites and dispersal through packaging material are other possible causes of long-distance spread, thus hinting at multiple pathways that need to be accounted for within the umbrella term of “commodity flow”.
Despite these limitations, having such a bare bones framework means that it can be quickly extended to other vegetables, pests and regions with minimal effort. We also believe that our approach provides a modular framework for integration of other models that can be refined with increased availability of data and sophisticated methods.
4.5 Appendix

Information about Tuta absoluta

The Tuta absoluta is a pest of great economic importance in several countries in Latin America and the Mediterranean basin. Its primary host is tomato, although potato, aubergine, common bean, physalis and various wild solanaceous plants are also suitable hosts. In tomato, it can attack any plant part at any crop stage and can cause up to 100% crop destruction, although the larvae prefer apical buds, tender new leaflets, flowers, and green fruits.

Infestation of tomato plants occurs throughout the entire crop cycle. Feeding damage is caused by all larval instars and throughout the whole plant. On leaves, the larvae feed on the mesophyll tissue, forming irregular leaf mines which may later become necrotic. Larvae can form extensive galleries in the stems which affect the development of the plants. Fruit are also attacked by the larvae, and the entry-ways are used by secondary pathogens, leading to fruit rot. The extent of infestation is partly dependent on the variety. Potential yield loss in tomatoes (quantity and quality) is significant and can reach up to 100% if the pest is not managed.

The total life cycle is completed in an average of 24-38 days, with the exception of winter months, when the cycle could be extended to more than 60 days. Laval development time depends on temperatures: 14 °C - 76 Days, 20 °C - 40 Days and 27 °C - 24 Days, respectively. Most favorable temperature: 15 °C to 29 °C. The minimum temperature for activity is 9 °C and above 35 °C the growth and reproductions is arrested. 10-12 generations can be produced each year. Tuta absoluta can overwinter as eggs, pupae or adults depending on environmental conditions. Under open-field conditions Tuta absoluta is usually found up till 1000 m above sea level.

The basis for effective and sustainable management of Tuta absoluta is the integration of cultural, behavioral, biological and chemical control. In the analysis of the paper, we study the economic impact by interventions with different combination of such managements.

Dynamic Message-passing Algorithm and Energy Function

Dynamic message-passing algorithm in this paper is an algorithm for the estimation of the origin of an SI epidemic from the knowledge of the network and the snapshot

\footnote{For further details about Tuta absoluta, refer to http://www.irac-online.org/documents/tuta-absoluta-irm-booklet/.}
of some nodes at a certain time. For every possible origin of the epidemic, dynamic message-passing method provides a fast way to estimate the probability that a given node in the network was in the observed state.

Dynamic message-passing algorithm belongs to the class of message-passing algorithms that includes the standard belief propagation (BP) method, also known in different fields as cavity method or sum-product equations (Mezard and Montanari, 2009). The BP equations are derived from the Boltzmann-Gibbs distribution under the assumption that the marginals defined on an auxiliary cavity graph (a graph with a removed node) are uncorrelated, which is exact if the underlying network is a tree (for a general discussion, see Mezard and Montanari (2009)).

Dynamic message-passing algorithm calculates marginals of a graphical model by operating on ‘messages’ associated with edges of the disjoint factor graphs with respect to a specific node. Then through a ‘dynamic programming’ procedure, it calculates the probability associated with each factor graphs. Then based on probabilities of different factor graphs with respect to the node, it comes to the marginal of that node.

For the energy function, it introduces the concept of energy in Physics. Intuitively, it is similar with maximal likelihood estimate. It operates as follows: for each node \( i \) (which is assumed as possible source of infection), it calculates the likelihood of the observed data \( \phi(O|i,t) = -\log \left( \prod_{(v,X) \in O} P_X(v,t,i) \right) \), and the node \( i^* \in \arg \max_{i \in N} \phi(O|i,t) \), which results in the largest likelihood is the one that has the highest probability is set to be the source of infection.


