Number Sequences as Explanatory Models for Middle-Grades Students' Algebraic Reasoning

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ABSTRACT

Early algebraic reasoning can be viewed as developing a bridge between arithmetic and algebra. Accordingly, this research examines how middle-grades students' arithmetic reasoning, classified by their number sequences, can be used to model their algebraic reasoning as it pertains to generalizing, writing, and solving linear equations and systems of equations. In the quantitative phase of research, 326 students in grades six through nine completed a survey to assess their number sequence construction. In the qualitative phase, 18 students participated in clinical interviews, the purpose of which was to elicit their algebraic reasoning. Results show that the numbers of students who had constructed the two least sophisticated number sequences did not change significantly across grades six through nine. In contrast, the numbers of students who had constructed the three most sophisticated number sequences did change significantly from grades six and seven to grades eight and nine. Furthermore, students did not consistently reason algebraically unless they had constructed at least the fourth number sequence. Thus, it is concluded that students with the two least sophisticated number sequences are no more prepared to reason algebraically in ninth grade than they were in sixth.

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GENERAL AUDIENCE ABSTRACT

Early algebraic reasoning can be viewed as developing a bridge between arithmetic and algebra. This study examines how students in grades six through nine reason about numbers, and whether their reasoning about numbers can be used to explain how they reason on algebra tasks. Particularly, the students were asked to extend numerical patterns by writing algebraic expressions, and were asked to read contextualized word problems and write algebraic equations and systems of equations to represent the problems. In the first phase of research, 326 students completed a survey to assess their understanding of numbers and their ability to reason about numbers. In the second phase, 18 students participated in interviews, the purpose of which was to elicit their algebraic reasoning. Results show that the numbers of students who had constructed a more sophisticated understanding of number did not change significantly across grades six through nine. In contrast, the numbers of students who had constructed a less sophisticated understanding of number did change significantly from grades six and seven to grades eight and nine. Furthermore, students were not consistently successful on algebraic tasks unless they had constructed a more sophisticated understanding of number. Thus, it is concluded that students with an unsophisticated understanding of number are no more prepared to reason algebraically in ninth grade than they were in sixth.

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LIST OF ABBREVIATIONS

aTNS: Advanced Tacitly Nested Number Sequence

ENS: Explicitly Nested Number Sequence

GNS: Generalized Number Sequence

IFS: Iterative Fraction Scheme

INS: Initial Number Sequence

MC1: The first multiplicative concept

MC2: The second multiplicative concept

MC3: The third multiplicative concept

PFS: Partitive Fraction Scheme

PUFS: Partitive Unit Fraction Scheme

RPFS: Reversible Partitive Fraction Scheme

TNS: Tacitly Nested Number Sequence

UFCS: Unit Fraction Composition Scheme

Chapter 1: Introduction

The next-day phenomenon (Tzur & Simon, 2004) is likely a familiar phenomenon to all mathematics teachers: A student, when engaged in a mathematical activity, demonstrates what appears to the teacher to be a complete understanding of the concept under instruction. Much to the teacher's dismay, during the next day's lesson the same student cannot reason in a manner that is consistent with the reasoning they demonstrated the previous day. However, when the previous day's activities are brought to the student's recollection, their understanding "miraculously" returns. In my own teaching experience, I encountered this phenomenon on a daily basis and imagined explanations that ranged from insufficient studying on the student's part to poor instruction on my part. (Incidentally, I also praised my teaching efforts when a brief reminder of the previous day's lesson helped students recall what I had "taught" them. Little did I know the underlying structures at play.) Tzur and Simon (2004) have explained how the next-day phenomenon is the result of students constructing only participatory conceptions of the mathematics under consideration.

Within the frame of radical constructivism, students' mathematical conceptions are modeled by three-part schemes, which include an assimilatory structure, mental activities, and an expected outcome (Glasersfeld, 1995). Ideally, instruction leads students to construct *anticipatory* conceptions of mathematics. An anticipatory scheme implies that the three parts of the scheme are constructed and mentally linked by the student (most likely subconsciously) (Tzur & Simon, 2004). The implication of having constructed an anticipatory conception is that when a mathematical task is given, the student perceives similarities between the task at hand and tasks that she has previously encountered. As a result, she is able to assimilate the task into an appropriate scheme, and the scheme's actions will be engaged.

On the other hand, a *participatory* conception implies that the student has linked the activity and effect of a scheme, but has not yet connected the activity and effect to an appropriate assimilatory structure (Tzur & Simon, 2004). A participatory conception can be crippling to students. Without having connected the assimilatory structure of a scheme to the appropriate activity and effect the student will attempt to assimilate tasks into a scheme that is, for them, anticipatory (Tzur & Simon, 2004), and to an observer such as their teacher, inappropriate. While the scheme under construction is active, students with a participatory structure are able to apply the appropriate mental activity and to interpret the results of the activity as either expected or unexpected; that is, they apply the scheme's activity and effect. To the teacher, it appears that the student has learned the mathematical concept under instruction. However, when the scheme under construction has not been activated by some activity or prompt, students with a participatory scheme will attempt to assimilate a task using a scheme that is, to them, anticipatory. To the teacher, this comes across as a lack of recall because the same student who was successfully solving tasks while in activity the previous day is no longer able to do so.

Tzur and Simon's (2004) next-day phenomenon highlights the importance of distinguishing between participatory and anticipatory schemes for students, and has been researched in the context of students' understanding of unit fractions. The application of these stages of understanding, however, has the potential to be applied in a multitude of other concepts in mathematics and in particular, to students' construction of number.

Construction of Number

Students construction of the concept of number, when considered within the frame of radical constructivism, develops through a series of four number sequences (Steffe & Cobb, 1988). The number sequence a student is able to construct is directly influenced by the student's stage of units coordination (Ulrich, 2015, 2016a), and a student's number sequence further influences her ability to reason multiplicatively (Hackenberg & Tillema, 2009), and construct fraction schemes and operations (Steffe, 2010b). Steffe and Cobb's (1988) hierarchy of number sequences include the initial number sequence (INS), tacitly nested number sequence (TNS), explicitly nested number sequence (ENS), and generalized number sequence (GNS). In addition to these, Ulrich (2016b) defined the advanced tacitly nested number sequence (aTNS), which falls within the TNS stage of Steffe and Cobb's (1988) original progression, and constitutes a substage. As students construct the ability to assimilate tasks with greater levels of units, more sophisticated number sequences fall within their zone of potential construction[1] (ZPC; Norton & D'Ambrosio, 2008), and consequently, the construction of each number sequence allows students access to increasingly sophisticated mathematical reasoning (Ulrich, 2015, 2016a).

The level of units students are able to coordinate directly relates to the number sequence they are able to construct, and is furthermore an indicator for the student's ability to reason multiplicatively. According to Hackenberg and Tillema (2009) students who can assimilate[2] tasks with one level of units and coordinate two levels of units in activity are said to have constructed the first multiplicative concept (MC1). Assimilating tasks with two levels of units and coordinating a third level of units in activity is synonymous with the second multiplicative concept (MC2), and assimilating tasks with three levels of units aligns with the third

[1] The zone of potential construction (cf. ZPD; Vygotsky, 1986) indicates "a hypothetical reorganization of a students' present ways of operating" (Norton & D'Ambrosio, 2008, p. 236). Thus, to say that the INS, for example, lies within the ZPC of students who can construct one level of units in activity means that a student who can construct one level of units in activity can possibly be perturbed into reorganizing their units coordinating scheme into the INS.

[2] In the context of this introduction, I use the term assimilate to indicate a student's ability to "operate on the results of their units coordinating activity" (Hackenberg & Lee, 2009, p. 3). In the literature review, I will define the term assimilation more thoroughly and so as to include situations that extend beyond units coordinating activity.

multiplicative concept (MC3). The implications of the multiplicative concepts students have constructed have been studied as they relate to their understanding of fractions and algebra (Hackenberg & Tillema, 2009; Hackenberg, 2013; Hackenberg & Lee, 2015; Hackenberg, Jones, Eker, & Creager, 2017).

While the implications of having constructed one of Steffe and Cobb's (1988) original four number sequences is clearly delineated in the extant research, the full implications for having constructed only an aTNS remain open to examination. Thus far, Ulrich (2016b) has outlined the implications for aTNS students' units coordination and for their understanding of fraction operations. This leaves the opportunity to further examine the ability of aTNS students to reason algebraically.

Rationale for the Study

With each increasingly sophisticated level of units coordination, students can construct corresponding number sequences, each of which have implications for their multiplicative reasoning. Hackenberg (2013) and Hackenberg and Lee (2015) have studied how middle grades students operating with an MC1, MC2, and MC3 are able to reason algebraically. In particular, she found that MC1 students were at a significant disadvantage in terms of algebraic reasoning and generalizing algebraic relationships because they had not yet constructed the disembedding operation. Furthermore, MC1 students were heavily dependent upon their iterating operation to complete algebraic tasks (Hackenberg, 2013). While MC2 students are likely to have constructed the disembedding operation, and therefore reason algebraically and make generalizations more readily than do MC1 students, MC2 students are still limited. Hackenberg and Lee (2015) found that MC2 students failed to use fractional, and sometimes integer, coefficients in equation writing (Hackenberg & Lee, 2015). This limitation "demonstrated a lack of reversibility in equation writing as well as a lack of reciprocal reasoning (Hackenberg & Lee, 2015, p. 214). Furthermore, MC2 students tended to build equations involving multiplicative relationships using numerical examples, and demonstrated non-standard conceptions of variable following operations (Hackenberg et al., 2017).

Steffe and Cobb's (1988) number sequences can generally be understood to align with Hackenberg and Tillema's (2009) multiplicative concepts—a TNS is within the ZPC of students who have constructed the first multiplicative concept, and ENS is within the ZPC of students who have constructed the second multiplicative concept. This alignment implies that the algebraic limitations of an MC1 student would be similar, if not identical, to those of a TNS student, and that the algebraic limitations of an MC2 student would be similar to those of an ENS student. However, the limitations of an aTNS student to reason algebraically remain hypothetical. aTNS students, like TNS and MC1 students, have not yet constructed the disembedding operation; it follows, theoretically, that an aTNS student's algebraic limitations would be similar to those of a TNS student. However, aTNS students assimilate tasks with a composite unit, which is more sophisticated than TNS students who can only construct composite units in activity. Does this more advanced assimilatory structure advantage aTNS students' algebraic reasoning over TNS students in some way? These questions require further empirical consideration, which provide, in part, the rationale for this research study.

Understanding the limitations of aTNS students with regard to algebra has implications for instruction and curricula. The impact of these constraints in algebraic reasoning can only be fully understood by also developing a picture of the population of students in middle and high school, as it relates to the number sequences. Steffe (2007) approximates that by fifth grade, between 30 and 50% of students are still operating with at most a TNS. Furthermore, Ulrich and

Wilkins (2017) found approximately 42% of 93 sixth grade students surveyed to be operating with an aTNS. It follows that many middle grades students continue to operate with limited ability to coordinate units, and are therefore constrained in their ability to reason algebraically. This understanding is critical to instructional decision making. It follows that constructing a broader picture of the population of students across grades six through nine will only further benefit these efforts to understand the mental structures that undergird students' difficulty to reason algebraically. Furthermore, an examination of the limitations of aTNS students' algebraic reasoning coupled with a snapshot of the number of aTNS students across grades six through nine has the potential to provide insight into why some students struggle to develop more advanced number sequences.

Purpose and Research Questions

The purpose of this study is to construct a more complete understanding of the algebraic reasoning of students who have constructed only an aTNS in grades six through nine. In particular, the current study will examine middle and high school students' construction of number sequences, with specific attention paid to the relationship between their number sequence and grade level, and the algebraic reasoning of students who have constructed each number sequence. This purpose will be addressed through an examination of the following questions:

1. How do aTNS students reason algebraically?
2. How is the algebraic reasoning of aTNS students constrained compared to that of ENS students, or advantaged over that of TNS students?
3. How does the number of aTNS students compare across grades six through nine?

Chapter 2: Literature Review

"The crucial realization for us is that the children cannot construct our knowledge, because our knowledge is essentially inaccessible to them. The best they can do is to modify their own knowledge as a result of interacting with us and with each other" (Steffe & Tzur, 1994, p. 102). This quote brings attention to one of the basic tenets of radical constructivism (Glasersfeld, 1995)—that children's knowledge is necessarily different than our own. Accordingly, research aimed at understanding how children learn mathematics from this point of view focuses on building models of student's mathematics (Steffe & Thompson, 2000; Steffe & Ulrich, 2013), so as to understand the students' ways of interacting with mathematics, instead of trying to impart upon the children a mathematics that matches that of the researchers.

Steffe (2010f) argues that these models should be the driving force behind a child's mathematics education, however today's accepted school mathematics is driven by prescribed knowledge. The curricula and standards documents that *a priori* determine what mathematics students should learn do not take into account the schemes (Glasersfeld, 1995) that students have constructed. As a result, mathematics instruction is at times far too advanced for students' schemes and operations (e.g., Hackenberg, 2013; Hackenberg & Lee, 2015; Norton & Wilkins, 2012). In this chapter, I will delineate several of the key schemes and operations identified in the current research for modeling students' mathematics as they construct what Steffe (2010f) would encourage schools to use to inform the new "school mathematics." These include units coordination (Steffe, 1992), number sequences (Steffe & Cobb, 1988), multiplicative concepts (Hackenberg & Tillema, 2009), and fraction schemes and operations (Steffe & Olive, 2010). After summarizing these schemes, I will synthesize the extant research examining the relationships among the schemes.

Learning

Radical constructivism (Glasersfeld, 1995) is a theory of knowing based on several Piagetian ideas, including cognitive adaptation, assimilation, accommodation, and schemes (Glasersfeld, 1995). The first of these, cognitive adaptation, has its roots in biological adaptation, but rather than survival of the fittest, refers to a conceptual state of equilibrium and a cycle of equilibration (Glasersfeld, 1995). As imagined by Piaget, conceptual equilibration is a cycle involving the assimilation of sensory information into an action scheme through a recognition template, the execution of a mental or physical action, and a result, which may be expected or unexpected. When the result is expected, the cycle of equilibration is complete. That is to say, the mind is pleased with the result of its actions and there is never a sense of uneasiness or dissatisfaction; it is not perturbed (Glasersfeld, 1995, p. 65). On the other hand, when a situation is assimilated into a scheme, the actions of the scheme can also lead to an unexpected or unsatisfactory result—a perturbation. Perturbation may lead to an accommodation to the existing scheme. Glasersfeld (1995) describes an accommodation as “an act of learning” (p. 66) because making accommodations to an existing scheme or constructing new schemes based on a perturbation will allow future assimilations of this stimuli to result an expected, and hopefully pleasant result. It is the ability to maintain equilibrium that marks cognitive adaptation.

If learning is to be described as accommodations to one’s schemes, then it becomes important to highlight the manner by which these accommodations are made. Piaget (1977/2001) explains these to be the result of reflective abstractions, which can occur on three levels: pseudo-empirical, reflecting, and reflected. Each of the levels of reflective abstraction rely on one’s abstraction of generalized properties from actions imposed on objects (pseudo-empirical), thoughts (reflecting), or previous reflections (reflected). It is these forms of abstraction that allow an object to be re-presented and understood in the absence of physical material (Piaget, 1977/2001).

While accommodation describes learning, accommodations to schemes are impossible without assimilation. Glasersfeld (1995) cautions that Piagetian assimilation does not refer to bringing environmental characteristics into the organism. Rather, it "must instead be understood as treating new material *as an instance of something known*" (p. 66, emphasis in original). In other words, the cognizing subject perceives similarities between an object, task, or experience, and objects, tasks, or experiences that they have previously acted upon. Perceived similarities may or may not be the same similarities perceived by others. Regardless, to the cognizing subject, it is the perceived similarities that determine the scheme into which a situation is assimilated. Furthermore, what one perceives and does not perceive will also vary by individual based on their existing experiences and schemes. "The cognitive organism perceives (assimilates) only what it can fit into the structures it already has. … it remains unaware of, or disregards, whatever does not fit into the conceptual structures it possesses" (Glasersfeld, 1995, p. 63).

Consider, for example, a typical struggle for algebra students—learning to solve quadratic equations. At this point in a beginning high school algebra course, it is assumed that the students have learned to solve linear equations. The process for solving linear equations is to isolate the variable by performing inverse operations. This is a practiced skills and when given the directions to "solve," students are presumably triggered to begin performing inverse operations until the variable in question stands alone on one side of the equation. The introduction of quadratic equations, then, are treated by the cognizing subject *"as an instance of* *something known*" (Glasersfeld, 1995, p. 66, emphasis in original), namely, linear equations. A common mistake is that students attempt to isolate the variable in a quadratic equation; although this is generally unproductive, it is consistent with the actions applied to linear equations. Thus, if the student is unaware of the difference between linear and quadratic equations then this behavior makes perfect sense.

This is an example of the potential observed behaviors of a high school algebra student who assimilates a quadratic equation into a scheme for solving linear equations because they perceive similarities between the two situations. Figure 2.1 shows Glasersfeld's (1995) three-part scheme, in which the cognizing subject (1) assimilates stimuli based on perceived similarities, (2) uses those perceptions of similarity to determine an appropriate action, (3) resulting in an expected outcome. This model, of course, assumes that the student's result is expected which, as previously discussed, may or may not be the case. Relying on the example of learning to solve quadratic equations, students are likely to initially perceive the situation of a quadratic to be equivalent to that of a linear equation, thereby assimilating the task into a scheme for linear equations. Hopefully, the solution that results from the activity of a linear equations scheme perturbs the student, causing them to make accommodations to their scheme or to construct new schemes to reason about quadratic equations.

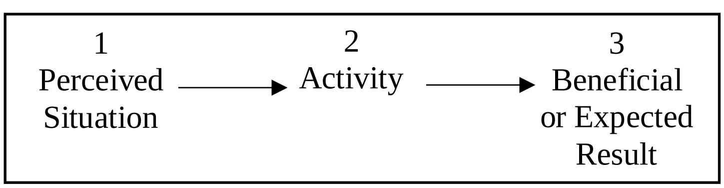

Figure 2.1. Glasersfeld's (1995) three-part model of a scheme (p. 65).

The purpose of this example is to demonstrate one way in which students might assimilate mathematical tasks into existing schemes. Schemes themselves, however, can also become assimilatory when the three parts can be taken as material for further operations. Steffe

(2010c) describes the difference between two students as being "in" versus "outside of the re-presentation" (p. 64). In this characterization, students who are "in" the scheme, must mentally carry out the activity of the scheme in order to determine its result. The result is not anticipated. Furthermore, the result of the scheme cannot be taken as material for further operating. When a student can stand "outside of" a scheme, they "can set it at a distance and look at it" (Steffe, 2010c, p. 64). The student who has constructed an assimilatory scheme can reflect and operate on the activity and result of the scheme.

Units Coordination

At the root of students' development of number is the concept of units coordination (Steffe, 1992), which describes the manner by which children construct and operate with different levels of units (Hackenberg & Tillema, 2009; Ulrich, 2015, 2016a). Units coordination is a construct that explains, to a certain extent, students' mathematical understanding spanning from a very young age through adulthood. In this section, I will begin by defining the unitizing operation, as it is envisioned by Glasersfeld (1981), and then how students' units coordination advances from the construction of one level of units in activity through a much more powerful coordination of four and even five levels of units in activity.

What is a unit? In its simplest form, *unitizing* is the ability to focus one's attention on a single object, recognize its singularity, and distinguish it from other background objects (Glasersfeld, 1981, p. 87). Interpreting any sensory-motor object requires that we focus our attention on it, as a means of identifying its qualitatively distinct characteristics. In unitizing the object, we go beyond simply recognizing the attributes of the object that distinguish it from other types of objects, and additionally bound the attentional pulses that identify that object from other background objects (Glasersfeld, 1981, p. 88). For example, when a young child encounters a dog for the first time, they must first determine what qualities of the object make it a dog (e.g., four legs, fur, wagging tail, etc.). Although each of these qualities are distinct, and therefore represent different attentional pulses, the process of unitizing these qualities into a singular object—a dog—is achieved by the unfocused attentional pulses occurring before and after the pulses indicating qualities of the dog. Moreover, an empirical abstraction[3] (Piaget, 1977/2001) will eventually lead the child to recognize the dog through a single bounded, focused pulse, rather than a series of distinct bounded, focused pulses; this describes the process through which we create units (Glasersfeld, 1981, p. 89).

While the construction of the unitizing operation seems rather elementary, and is one that is likely taken for granted in day to day activities, its importance cannot be overstated as an instigator to a child's construction of the concept of number. Ulrich (2015) summarizes this importance:

> characterizing the amount of quantitative complexity students can assimilate is useful for understanding how they experience the world mathematically. In the stages of unit construction and coordination, quantitative complexity is characterized in terms of the types of units students work with and the relationships students construct between the different types of units. (p. 3)

The unitizing operation forms a foundation for these constructions and coordinations, and provides teachers and researchers with a means of understanding and interpreting children's mathematical actions.

Constructing one level of units in activity. The construction of one level of units is possible through the construction of an *arithmetic unit* (Ulrich, 2015), which is the outcome of a

[3] An empirical abstraction is the process by which subjects notice properties of physical objects, and generalize those properties into sensorimotor or conceptual schemes (Piaget, 1977/2001).

reflective abstraction on the child's act of counting physical objects (Glasersfeld, 1981, p. 91). As Glasersfeld (1981) specifies in the instance of an arithmetic unit, a reflective abstraction involves

> the focusing of attention not on sensory-motor signals but on the results or products of prior attentional operations. Something that has been constructed by means of an attentional pattern is now reprocessed and used as raw material for a new sequence of focused and unfocused pulses, (pp. 91–92)

and further summarizes this abstraction with the words of Piaget – arithmetic units are "elements … stripped of their qualities" (Piaget, 1970, p. 37). Thus, a reflective abstraction on previously unitized objects results in the construction of a new type of unit, an arithmetic unit; this unit no longer requires sensory-motor material to be meaningful, and facilitates a child's understanding of the cardinality of a set. The implication of this construction is that the child has reprocessed the act of counting sensory-motor items, although only in activity (Ulrich, 2015, pp. 4–5).

Having constructed an arithmetic unit facilitates the construction of one level of units. The ability of a child to construct one level of units indicates that the child is capable of creating arithmetic units while they are in the act of counting, however, those arithmetic units are not anticipated prior to the counting act, nor are they available to the child for reflection after completing the counting act (Ulrich, 2015). A limitation of operating with one level of units is that while the arithmetic unit is available to the student for counting activity by ones, the result of their counting activity is not available to them as a unit on which they can operate further; in other words, that unit decays (Ulrich, 2015).

Consider, for example, Tyrone—a child who can construct arithmetic units in activity (Steffe & Cobb, 1988)—is asked to find the total number of blocks in two screened groups, consisting of eight and three blocks, respectively. As he is unable to anticipate the result of counting eight blocks, he must begin his counting at one, and is likely to re-present the hidden three blocks using a figurative pattern. This could include a finger pattern, a spatio-motor pattern, an auditory pattern, or some other type. Tyrone, for example, made a linear three pattern by tapping his finger over the second cloth (Steffe & Cobb, 1988). Regardless of what type of figurative material children create, what is important is that the child's assimilation of the situation will likely involve creating stand-in objects to count in the absence of the actual blocks. Furthermore, he can continue to count, beginning at nine, and extend his counting to include the second group of three blocks, for a result of eleven blocks total. While this counting behavior could be indicative of a child constructing one level of units or of a child counting figurative unit items, the distinction is in Tyrone's ability to abstract the records of his counting; this is marked by an awareness of the counting acts while in the process of counting (Steffe & Cobb, 1988; see also Ulrich, 2015).

Being able to monitor and reflect upon the counting while in activity is powerful in the child's conception of number, however, their counting activity is still extremely limited because the arithmetic unit they construct in activity decays (Ulrich, 2015). In the previous example Tyrone found the total number of blocks in two groups of eight and three, and he successfully determined that there are eleven blocks in all. After completing the activity, however it is likely that he would lose track of the fact that the original task was to join groups of eight and three blocks; the one level of units that were constructed to facilitate the completion of the task is no longer available to the child because it was constructed in activity. It is the ability to coordinate with two levels of units in activity that allows the student to reflect upon the first level of units after completing the activity. Therefore, this is the next important step in their units coordinating activity.

Coordinating two levels of units in activity. Advancing beyond the construction of one level of units is the student's ability to assimilate tasks with one level of units and coordinate a second level of units in activity. This development is made possible by the construction of a *composite unit* (Steffe, 2010a; Steffe & Cobb, 1988; Ulrich, 2015). If an arithmetic unit is understood to be the construction of five as five individual units of one, then a composite unit can be understood as the construction of five as one group of five individual units. This advance allows the result of their counting activity, the composite unit, to be available for further operating (Ulrich, 2015).

One distinction between constructing composite and arithmetic units can be understood in the reflections hypothesized to engender their constructions. Arithmetic units may be formed through the child's reflection on and reprocessing of their counting acts, allowing them to abstract their records of counting. Constructing composite units, on the other hand, may be instigated by the child's reflection on subsequences of counting (Ulrich, 2015). A child who can construct a composite unit understands that five represents a single group of the counting sequence, and furthermore, that it contains five individual units. With this understanding, five may represent the subsequence from one through five, or it may represent the subsequence sixteen through twenty (Steffe, 2010b; Ulrich, 2015). The benefit of constructing a composite unit is that the child can anticipate the result of their counting acts. The child plans to accomplish the task of enumerating a set of five, for example, and is aware before they begin counting that they are able to monitor their counting acts. It is this anticipation and monitoring that allows her to embed the subsequence one through five within a larger sequence beginning with a numeral other than one.

A limitation of operating with only a composite unit is students' understanding of the relationship between the unit of one and the composite unit. While a composite unit of five is understood to be a single unit made up of five individual units, it is not understood that five is five times larger than one (Steffe, 2010b; Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins, 2015); there is no multiplicative relationship between the unit of one and the composite unit. This is an additional limitation for students only able to coordinate two levels of units in activity that is not resolved until they are able to coordinate three levels of units (Norton et al., 2015; Ulrich, 2016a).

Coordinating three levels of units in activity. The construction of an iterable unit of one is the next advancement of students' quantitative reasoning, as it facilitates the ability of students to coordinate three levels of units in activity. To iterate in this sense means "repeatedly instantiating an amount in order to produce another amount" (Hackenberg, 2007, p. 28). The most notable development of students operating with an iterable unit of one lies in their understanding of composite units (Ulrich, 2016a). Although students operating with composite units understand five to be a single group made of five units, they do not necessarily interpret the five individual units as identical. Rather, at first, they understand the five units to be five counting acts which result in the composite unit of five. On the other hand, operating with an iterable unit of one implies an understanding that the five individual units within a composite unit of five are identical units any one of which can be iterated to create five (Steffe, 2010b; Ulrich, 2016a). Thus, "the construction of iterable units … decreases the information students must attend to when operating with composite units" (Ulrich, 2016a, p. 34) because they need not consider the entire sequence of numbers from one through five, for example, but can think of five as five copies of one unit; the iterable unit of one also facilitates an awareness of the multiplicative relationship between the single unit and the composite unit of five.

The iterable unit of one makes a difference in students' reasoning with the task previously described in which the student is asked to find the number of blocks in two groups of six and five. Whereas a child coordinating two levels of units in activity (and constructing a composite unit) can determine the sum to be 11, her understanding is limited. She understands the problem as counting out six blocks, counting out five more blocks, and ending the counting activity at 11. A student operating with iterable units, on the other hand, understands that the sum, 11, can be decomposed into two distinct subsequences, six and five (Steffe, 2010b; Ulrich, 2016a).

The coordination of three levels of units in activity, implying that the student assimilates with two levels of units and coordinates the third level in activity, is available to students operating with an iterable unit of one (Ulrich, 2016a). To illustrate this point, consider a student's understanding of the number 18. With the construction of three levels of units in activity, a student can create 18 by iterating a composite unit of six, three times. The three levels of units at play for this student are the individual unit, which is iterated six times to construct the second level of units, the composite unit of six, which in turn, can be iterated three times to form 18. The iterable unit of one facilitates an understanding that a composite unit of six is the repetition of six identical units, thus freeing the student from constructing six as a sequence of counting six units (Ulrich, 2016a), that enables her to coordinate the third level of units (iterating the composite unit of six) in activity; the student has constructed 18 as three composite units of six, or a composite unit of composite units.

The limitation of this understanding is that the student must build the third level of units in activity. However, it is an improvement of those students coordinating only two levels of units because without the availability of an iterable unit of one, the student would be unable to interpret the resulting 18 as 18 units of one while maintaining that it is three units of six; one of the levels of units will decay, which implies the result is not available for further operating. For students coordinating three levels of units in activity, the iteration of a composite unit is done in activity (Ulrich, 2016a). It is a reflection on and reprocessing of the activity of iterating composite units that engenders the construction of an iterable composite unit as an assimilatory structure (Olive, 2001).

Assimilating with three levels of units. Once students construct iterable composite units, constructing three levels of units as an assimilatory structure becomes possible (Ulrich, 2016a). Assimilating tasks with an iterable composite unit involves an understanding that composite units of, say, six are identical to one another which allows students to assimilate mathematical tasks with three levels of units and construct four or even five levels of units in activity (Ulrich, 2016a; Wilkins, Norton, & Ulrich, 2017). In considering the same example as above ($3 \times 6 = 18$), the result to a student assimilating with three levels of units is a collection of three units, each of which contains an identical unit of six. This structure can be expanded and interpreted as 18 individual units or, because the composite units are interpreted as identical, can be understood in its more compact form.

Understanding 18 as three iterations of six in this compact form is available to students operating with an iterable composite unit because the iterable composite unit implies an understanding that the three composite units of six are identical – not just equal. There are several advantages to students interpreting this situation having constructed the assimilatory structure for three levels of units. First, students can keep track of each of the three levels of units, and can flexibly switch between them (Steffe, 2010b; Ulrich, 2016a)—18 can be interpreted as three groups of six or six groups of three. Additionally, students assimilating with three levels of units can take this compact structure of 18 for further operating (Steffe, 2010b; Ulrich, 2016a). For example, if they were asked to determine the total when adding four more groups of six, assimilating with three levels of units enables the extension of 18 as three units of six to become 42 as seven units of six.

Summary. Students' advancing constructions of units and their corresponding abilities to coordinate and assimilate with increasingly complex levels of units provides a foundation for researchers and teachers to understand students' quantitative reasoning. Interpreting students' quantitative reasoning in terms of the types of units they are constructing and coordinating provides insight into some of their understandings of mathematical tasks, and the levels of units coordination can be used to further understand the number sequence, multiplicative concepts, and fraction schemes and operations they have constructed.

Number Sequences

Students construction of number sequences relies directly on the types of units they are able to construct and the level of units with which they can assimilate mathematical tasks. Specifically, number sequences are a "recognition template of a numerical counting scheme; that is, its assimilating structure. … At all stages of construction, children use their number sequences to provide meaning for number words" (Steffe, 2010b, p. 27). Steffe's number sequences outline the trajectory through which students construct an increasingly complex understanding for their number words, which is based in their construction of units and facilitates their assimilation of tasks with increasing levels of complexity. The number sequences align closely to the levels of units coordination, and include: the initial number sequence (INS), the tacitly nested number sequence (TNS), the explicitly nested number sequence (ENS), and the generalized number sequence (GNS; Steffe & Cobb, 1988; Steffe, 2010b). In addition to these four number sequences, Ulrich's (2016b) advanced tacitly nested number sequence (aTNS), falls in between the tacitly and explicitly nested number sequences. In this section, I will outline each number sequence as it relates to units coordination, and will additionally outline the hypothesized reorganization of each subordinate number sequence into the superseding sequence, and some of the distinctions between students' reasoning and mathematical behaviors when operating with each sequence.

Initial number sequence. The behavioral marker for an initial number sequence (Steffe, 2010b; Steffe & Cobb, 1988), is the onset of the activity of counting on (Ulrich, 2015). Prior to counting on, children will *count all*. In counting all, when joining two groups, the child will always begin at one and then count all to determine the total sum. For example, if asked how many blocks there are all together in two groups of eight and three, a child who is counting all would first count out the two individual sets: "one, two, three, four, five, six, seven, eight and one, two, three," and then would find the total by counting the two sets together: "one, two, three, … ten, eleven," enumerating each and every item. Counting all is a behavioral indication that the student has yet to internalize her counting acts (Steffe, 2010b; Ulrich, 2015); it is the internalization of the counting acts that allows the child to make meaning of the cardinality of eight, for example, without actually counting from one to eight.

Advancing beyond counting all is the behavior of *counting on*. Given the same task of finding the total number of blocks, a child who is counting on can begin with eight, and then extend her counting an additional three units by saying, "eight…nine, ten, eleven." To the child who is counting on, saying the number word "eight" represents the cardinality of the subsequence from one through eight (Steffe, 2010b; Ulrich, 2015; Wilkins & Ulrich, 2017), making it unnecessary for the child to carry out the initial counting act. The child's ability to use the final number in a counting sequence to stand in for the entire sequence, and to count on from there, is one reason this sequence is termed the initial number sequence (Olive, 2001; Steffe, 2010b).

Counting on, the behavioral indicator used as the hallmark characteristic of children who have constructed the INS, can be understood in terms of the types of units children with an INS can construct. Because the child has internalized her counting acts from one to six, say, the final number in the counting sequence can replace the act of counting out that sequence; this is the result of having constructed an arithmetic unit (Steffe, 2010b). When joining the next sequence of numbers (eight through eleven in the example above), an INS student is likely to rely on figurative material in completing the task (Ulrich, 2015) because they are unable to coordinate a second level of units and monitor their counting on activity. Counting figurative material implies that the child creates something to count, such as finger patterns or visual re-presentations, and takes each created item as something to which she can apply her counting sequence (Steffe, 2010b). Thus, constructing the first level of units allows an INS student to count on with the aid of figurative material to monitor their counting of the second number sequence; for children without the ability to construct one level of units in activity, counting on holds no meaning, and they will instead resort to counting all.

Tacitly nested number sequence. While INS students can construct arithmetic units in activity, the tacitly nested number sequence (Steffe, 2010b; Steffe & Cobb, 1988) enters within the ZPC of students who can construct composite units in activity (Steffe, 2010b; Ulrich, 2015).

According to Olive (2001), the TNS is the result of reinteriorizing the INS. This reinteriorization is instigated by the monitoring of counting activity. A TNS indicates that the student now understands that nine, for example, need not be the sequence from one through nine, but can also be the sequence from 24 through 32. It is important to note that at this point the student's awareness of this is tacit, rather than explicit, hence, the tacitly nested number sequence (Olive, 2001). This is made possible by the construction of a composite unit because the composite unit allows students to consider the number nine as one group containing nine units. Initially, this may simply allow TNS students to count on without the aid of figurative materials (Ulrich, 2015), but will lead to other, more sophisticated counting techniques, such as double counting.

One of the ways TNS students can engage a composite unit in double counting is by making sense of questions such as, how many sixes are in 18? For an INS student, this question does not make sense because six stands for the sequence from one through six, and has not yet been united into a composite unit that can be inserted as a subsequence beginning with a number other than one. TNS students, on the other hand, can monitor their counting from one through 18 and can further divide it into subgroups of six (Olive, 2001). For example, they may count one through six and denote that as the first group by holding up one finger, then count from seven through 12 and denote that as the second group by holding up a second finger, and finally count from 13 through 18 and denote that as the third group by holding up a third finger. The TNS student recognizes that the task of finding how many sixes are in 18 will require them to count in groups of six until they reach 18, carries out the activity of counting out composite units of six and keeping track of them, and appropriately interprets the response—there are three sixes in 18.

Another way in which operating with a composite unit allows TNS students to act in a more sophisticated way than INS students is in their ability to solve missing addend tasks. Steffe and Cobb (1988) describe several teaching episodes in which students solve missing addend tasks, such as 8+___=12. In one specific instance of this task, they indicate that one student, Scenetra's, solution is evidence of her formation of composite units. Scenetra began counting at eight, and then counted 9, 10, 11, 12 while touching her lips four times with her fingers. Then, she counted the four fingers she had touched to her lips, which is an indication "that she formed the goal of specifying their numerosity" (p. 80). Ulrich (2015) discusses students' possible solutions to problems similar to those solved by Scenetra. An INS student is unlikely to solve these problems independently. A TNS student, on the other hand, can form the goal of starting at nine and keeping track of how many more times they must count to arrive at 12. As Ulrich (2016a) further explains, while TNS students are capable of solving this type of problem using an additive comparison to "transform" eight into 12, the student does not think about the number 12 as being comprised of the subsequences eight and four; this relates to the tacit nature of their understanding. For TNS students, four is not an explicit subsequence embedded within 12. Four "is instead like an adjective describing the action of counting on" (Ulrich, 2016a, p. 36). The result is that this missing addend problem is possible for a TNS student, but because they have not yet constructed an iterable unit of one, they are limited in their understanding of the task (Steffe, 2010b).

Explicitly nested number sequence. The next advancement in students' number sequences is the reorganization of the student's TNS into an explicitly nested number sequence (Steffe, 2010b; Steffe & Cobb, 1988). According to Steffe (2010b), the characteristics of an ENS include an iterable unit of one, the disembedding operation, and strategic reasoning (cf. Ulrich, 2016b). The first of these, an iterable unit of one, indicates the student's understanding that a composite unit, such as five, is comprised not of a sequence of numbers from one to five, but of five identical units, any of which could be iterated to fill in the composite unit (Steffe, 2010b; Ulrich, 2016a). Because students with an ENS can assimilate tasks with a composite unit, they do not need to carry out the activity of creating the sequence from one to five, as a TNS student would; they can anticipate the filling in of the composite unit. In this sense, the iterable unit of one "provide[s] great economy in the child's reasoning … [because it] opens the possibility for a child to 'collapse' a composite unit into a unit structure containing a singleton unit, which can be iterated many times" (Steffe, 2010b, p. 42); this "collapse" is what enables ENS students to coordinate three levels of units in activity—the student assimilates with a composite unit that can be iterated in activity.

The next defining characteristic of an ENS is the ability to disembed composite units (Steffe, 2010b). *Disembedding* denotes the student's ability to imagine the embedded composite unit being removed from the larger composite unit without destroying either of the composite units. This allows the smaller composite unit to be taken as material for further operations (Steffe, 2010b). Consider the missing addend task previously discussed ($8 + ? = 12$; Steffe & Cobb, 1988[4]). Because an ENS student assimilates this task with a composite unit, it can be interpreted as a composite unit of twelve, which is comprised of two smaller, embedded composite units—eight and an unknown—either of which can be disembedded; the disembedding of a smaller composite unit from the whole becomes possible for ENS students because they understand the iterable units of one to be identical (Ulrich, 2016a). The reason this number sequence is termed explicit is that students are becoming more explicitly aware of "the nested nature" (Ulrich, 2016a, p. 34) of these units. Furthermore, the explicit understanding of

[4] A similar task, 27 + __ = 36 was presented to a student, Jason, who had constructed an ENS (Steffe, 1992, 2010a). Steffe's (1992, 2010a) discussion of Jason's solution and the manner by which his ENS supported his solution is similar to Ulrich's (2015).

the nesting of composite units within composite units demonstrates an ENS student's ability to construct a composite unit of composite units in activity (Ulrich, 2016a).

To disembed a composite unit from the whole and operate on that composite unit is usually involved in strategic reasoning, the third characteristic of an ENS (Steffe, 2010b). A student who engages in strategic reasoning in the missing addend task, $8 + ? = 12$ (Steffe & Cobb, 1988), might explain that because ten plus two is twelve, and eight is two less than ten, eight plus four will equal 12[5]. A possible explanation for this student's thinking is that they assimilated the task using a composite unit of 12, which indicates that they understand 12 to be a composite unit that could be filled in with identical units of one. The student mentally subdivides 12 into two smaller, embedded composite units, ten and two, perhaps because this is a number fact that they are familiar with. Because these smaller composite units can be taken as material for further operating, the movement of two units from the ten (leaving a composite unit of eight) to the two (increasing it to a composite unit of four) is characteristic of strategic reasoning. These mental operations are not possible without having constructed an iterable unit of one and the disembedding operation (Steffe, 2010b).

Generalized number sequence. The culminating number sequence, a reorganization of the ENS, is the generalized number sequence (Steffe & Cobb, 1988). A main characteristic of the GNS is an iterable composite unit, constructed when students repeatedly reflect upon the results of iterating a composite unit in activity within their ENS (Olive, 2001). To illustrate the advantages and meaning of assimilating tasks with an iterable composite unit, consider the example of evaluating seven times four (Ulrich, 2015, 2016a).

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[5] Jason, an ENS student, estimated a solution to the parallel task, 27 + __ = 36, to be 7 and then adjusted his estimate. Jason's adjustment is a different manner by which students might reason strategically (see Steffe, 1992, 2010a for a complete discussion of Jason's solution).

Whereas students with a TNS can likely solve this task by keeping track of counting by four seven times using some sort of figurative material, such as their fingers (1, 2, 3, 4 – that's one, 5, 6, 7, 8 – that's two, etc.), once they arrive at the product of 28, the composite units of four decay because they were constructed in activity. For ENS students, assimilating the task with a composite unit allows them to construct a composite unit (of seven) of composite units (of four) in activity. Thus, the result of 28 as seven groups of four does not decay for ENS students; these students can simultaneously interpret 28 as seven groups of four and as 28 individual units. The distinction for a GNS student is that they can assimilate tasks with an iterable composite unit. Much like an iterable unit of one, this affords the understanding that composite units of four are not only equal, but identical. With this understanding, the structure of a composite unit of seven comprised of composite units of four becomes assimilatory, and can be taken for further operating to determine, for example, how many more groups of four are in 44.

Advanced tacitly nested number sequence. In Steffe and Cobb's (1988) traditional number sequences, outlined for children through grade five, students reorganize their TNS into an ENS. However, in her research with middle grade students, Ulrich (2016b) has outlined the advanced tacitly nested number sequence, as a possible bridge between the TNS and the ENS. She describes the aTNS as a "natural outgrowth" (Ulrich, 2016b, p. 1) of the TNS that results from operating extensively with the TNS over an extended period of time, as well as high levels of endurance in problem solving.

While TNS students construct composite units in activity, aTNS students assimilate with composite units; although assimilating with composite units was previously thought to be possible only for ENS students (Steffe, 2010b), Ulrich (2016b) identified a student who could assimilate with composite units but was unable to construct the iterable unit of one and the disembedding operation, which are characteristic of the ENS (Steffe, 2010b). First, it is important to note that assimilating with composite units is a demonstration of aTNS students' extreme fluency in operating with a TNS, and second, assimilating with composite units allows aTNS students to engage in strategic reasoning, although in a manner that is distinct from ENS students because aTNS students are not yet disembedding (Ulrich, 2016b). In their strategic reasoning, both aTNS and ENS students will assimilate the task with a composite unit, however, whereas ENS students would then disembed and operate on the smaller composite units, aTNS students are more likely to operate on the embedded smaller composite units in activity; their cognizance of the subsequences being embedded within the larger composite unit is tacit, aligning their reasoning more closely with a TNS than an ENS (Ulrich, 2016b). Therefore, aTNS students' abilities to reason strategically because they can assimilate with composite units makes them more advanced than TNS students, but their inability to construct the iterable unit of one and the disembedding operation disqualifies them as ENS.

Summary. The number sequences explained here describe the assimilatory structures students use to make sense of whole numbers. When students are initially numerical (INS), they can understand a number word as standing for the counting sequence from one to that number. With a TNS, students can unite the results of their counting acts, and accordingly, a number word can stand in for a sequence of numbers beginning with one or any other number. An ENS allows students to think about a number as a set of identical iterations that no longer require the activity of or imagined activity of carrying out the counting sequence, which opens the door to more quantitative flexibility. The hierarchy culminates with the construction of a GNS, with which students can interpret composite units as identical. This allows them to collapse composite units into units of one and coordinate with increasing levels of units (Steffe, 2010b). As students' assimilatory structure, and corresponding interpretation of numerosity becomes more sophisticated, so does their ability to operate on other types of numbers, including fractions.

Multiplicative Concepts

Just as students' number sequences are closely linked to their units coordination, so are the multiplicative concepts with which they operate; in particular, "How students generate and coordinate composite units is the foundation of how we understand students' multiplicative concepts" (Hackenberg & Lee, 2015, p. 205). To coordinate two levels of units involves distributing the units of one composite unit across the units of another (Steffe, 1992).

The first multiplicative concept (MC1; Hackenberg & Tillema, 2009) is characterized by students who can coordinate two levels of units in activity. In finding three groups of six, for example, a student who has constructed an MC1 will count by ones from 1 through 18 and is likely to use their fingers, or some other figurative material, to keep track of how many times they have counted by six (Hackenberg & Tillema, 2009). While in the act of counting, an MC1 student can monitor their counting to successfully complete the task, but the constraint of this level of operating is that they "insert" the six units across the three units with their activity, which indicates that they can neither anticipate, nor reflect upon the activity (Hackenberg & Lee, 2015).

Students assimilating with two levels of units and coordinating a third level in activity are considered to have constructed the second multiplicative concept (MC2; Hackenberg & Tillema, 2009). Consider again the multiplicative task of finding three groups of six. Assimilating with a composite unit, MC2 students understand three as one group containing three units (these are the two levels with which they assimilate the task), and they can distribute a composite unit of six across each of the individual units of three units within the composite unit; the distribution of a composite unit of six across the individual units within the composite unit of three is done in activity, and represents the third level of units being coordinated. For MC2 students, the resulting 18 is interpreted as a composite unit containing 18 individual units. Although an MC2 student completes this problem in a more sophisticated way than does an MC1 student, the resulting three-level structure of units cannot yet be taken as material for further operating (Hackenberg & Tillema, 2009). Thus, if asked to determine how many more were in five groups of six, compared to three groups of six, the student would have to separately calculate five groups of six to be 30, and then find the difference between 30 and 18 to determine there to be 12 more. MC2 students necessarily treat these two problems as separate because the third level of units is constructed in activity, and thus decays, leaving the student with 18 individual units and 30 individual units, rather than 18 as a structure of three groups of six, and 30 as a structure of five groups of six.

The most powerful, and third multiplicative concept (MC3; Hackenberg & Tillema, 2009) requires the interiorization of three levels of units. The interiorization of the third level of units indicates that students assimilate tasks with a structure involving three levels and take it as material for further operating. So, when asked to find three groups of six, the task is assimilated as a structure of a composite unit of three, each unit of which contains a composite unit of six. The difference for this student is that the composite unit of six does not need to be distributed across each unit in activity, as it is for MC2 students. The advantage for MC3 students' thinking is that the result, 18, is still understood as three composite units of six; the resulting three level structure does not decay, as it does for MC2 students. Thus, if asked the same extension task as above (how many more are in five groups of six), MC3 students can extend the three-level units structure contained by 18 to include five more groups of six, resulting in a composite unit of eight groups of six, or 48 total. MC3 students can disembed composite units of three groups of six and five groups of six, and can operate on them (Hackenberg & Tillema, 2009). The tasks of finding three groups of six and five groups of six are no longer separate tasks, as they were for MC2 students, and the resulting units structure within 48 does not decay.

Fraction Schemes and Operations

Steffe (2002) hypothesizes that “children’s fractional schemes can emerge as accommodations in their numerical counting schemes” (p. 267). The implication of this hypothesis is that when children begin to encounter situations that their numerical schemes for whole numbers (the assimilatory structures of which are the number sequences) cannot handle, the students’ perturbations lead to the construction of new operations for fractions. These operations, when used with the existing numerical schemes in novel ways, support the construction of fraction schemes (Steffe, 2010b). Therefore, as students develop an increasingly complex ability to understand whole number situations, they can also develop schemes and operations for fractional situations. The trajectory of students’ construction of fraction schemes begins with a part-whole scheme, and advances to the set of partitive fraction schemes, the unit fraction composition scheme (Hackenberg & Tillema, 2009), and the iterative fraction scheme (Norton & Wilkins, 2009, 2012). Included in the following discussion of these schemes, and students’ behaviors when operating with these schemes, I will also discuss partitioning, iterating, and splitting operations, as they play an important role in student conceptions of fractions.

Fragmenting, segmenting, and partitioning. In their colloquial use, fragmenting, segmenting, and partitioning may be synonymous; however, Steffe (2010a, 2010c, 2010d) distinguishes between the three. Between fragmenting and segmenting, he differentiates that fragmenting is used

> to refer to simultaneity in breaking without the restriction of there being equal parts. … [and] ‘segmenting’ [is used] to refer to sequentiality in breaking without restriction on the size of the parts. … I advance the hypothesis that neither fragmenting nor segmenting is the more primitive and that both are involved in the construction of number sequences as well as in the construction of fraction schemes. (Steffe, 2010a, p. 6)

Accordingly, fragmenting and segmenting are operations that can coexist, and are related to students’ counting and fraction schemes. A synthesis of the relationships between each of the levels of fragmenting, the equisegmenting operation and the partitioning operations are displayed in Table 2.1. This synthesis is based on the research of Hackenberg, Norton, and Wright (2016), Piaget, Inhelder, and Szeminska (1960), and Steffe (2010a).

Table 2.1. Alignment of Students’ Fragmenting, Segmenting, and Partitioning Operations

Fragmenting	Segmenting	Partitioning
Level 1: Division in 2	--	--
Level 2: Division in 3	Equisegmenting	--
Level 3: Coordinate 2 Goals (small numbers)	--	Simultaneous Partitioning
Level 4: Coordinate 2 Goals (all numbers)	--	Equipartitioning
Level 5: *n* items among *m*	--	Distributive Partitioning

Furthermore, there are five levels of fragmenting (Steffe, 2010c) that are based on Piaget, Inhelder, and Szeminska’s (1960) stages of subdivision of area. The first level of fragmenting is marked by children’s ability to divide an area into two equal parts, and is available to students who have yet to construct an INS. The second level of fragmenting requires the operations of an INS (Hackenberg, Norton, & Wright, 2016; Steffe, 2010c) because at this level students apply “numerical composites to project units into continuous units” (Steffe, 2010c, p. 69). The second level is marked by students’ ability to subdivide an area into three parts, however, at this level students will have difficulty coordinating the goals of making a specified number of parts greater than three and exhausting the whole (Steffe, 2010c).

As Steffe (2010a) notes, fragmenting and segmenting are not necessarily more or less sophisticated than one another. In particular, the equisegmenting operation bears similarity to the second level of fragmenting (Table 2.1). Equisegmenting is marked by the student's lack of anticipation of the result, which generally leads to an inaccurate division adjusted by trial and error (Steffe, 2010d). A student who is equisegmenting and attempts to divide a line segment into five equal parts would first attempt to mark one-fifth of the line segment, and would then attempt to construct a second one-fifth, a third, *etc*. Upon completion, the pieces constructed are unlikely to be acceptable estimates of one-fifth, and they may or may not exhaust the entire line segment. Upon completion, the equisegmenting student understands that her divisions are inaccurate, but because they are constructed sequentially, rather than simultaneously (Ulrich, 2016b), this task is challenging.

When students have constructed the third level of fragmenting, they are considered to be partitioning for the first time (Table 2.1; Hackenberg et al., 2016). At this level, students apply "composite units to project units into continuous units" (Steffe, 2010c, p. 69), which implies that they coordinate the goals of making equal parts and exhausting the whole, for a small number of parts (Hackenberg et al., 2016). Students at the third level of fragmenting can also construct the simultaneous partitioning operation, with which, the task of equally dividing a line segment into five parts, for example, becomes anticipatory, and thus, a simultaneous act (Ulrich, 2016b). In this example, a student operating in this way is assimilating the task using their composite unit for five (Steffe, 2010d), which indicates that they set the goal of dividing a line segment into five equal parts prior to beginning the activity. Although simultaneous partitioning implies the application of the composite unit, these students are limited by their understanding of the task as dividing the line segment into five equal, but not identical, parts (Ulrich, 2016b).

This limitation is resolved in the equipartitioning operation, which allows the student to not only assimilate the task with a composite unit, but additionally implies the understanding that any of the one-fifths created could be mentally disembedded from the whole and iterated five times to recreate the whole (Steffe, 2010d). This indicates a multiplicative relationship among any one of the five parts and the whole (Ulrich, 2016b), and each one-fifth piece is identical to the others. Steffe (2010d) found that many students who are hypothesized to be equipartitioning will use this understanding to check their work. Although the behavior of dividing a line segment into five equal parts may appear similar for all three of these partitioning stages, the underlying mental constructs (failing to anticipate the results, anticipating the results through the application of a composite unit, and applying an iterable unit of one) are distinct and facilitate the student's construction of differing fraction schemes. Furthermore, Steffe (2010c) states that students who are equipartitioning have likely constructed the fourth level of fragmenting.

Finally, the fifth level of fragmenting requires the distribution operation and indicates that children can partition n items into m parts (Steffe, 2010c). This operation will be further illuminated in the discussion about the distributive partitioning operation because the schemes and operations that have been discussed up to this point are insufficient to support students' construction of the fifth level of fragmenting and the distributive partitioning operation. It is mentioned at this point only for the purposes of completeness within Steffe's (2010b) levels of fragmenting.

Part-whole fraction scheme. The part-whole fraction scheme is fairly primitive in terms of a student's fractional understanding, and it precedes what is technically the first fraction scheme, the partitive unit fraction scheme (Norton & Hackenberg, 2010). The operations of the part-whole fraction scheme are partitioning and disembedding (Steffe, 2010d). Hence, a student must have constructed the simultaneous partitioning operation, and apply this operation in conjunction with the disembedding operation in order to be considered as having constructed the part-whole fraction scheme. With a part-whole fraction scheme, a child's conception of a proper fraction, say, three-fifths, is three parts out of a total five parts. This requires the student to partition a whole into five equal parts and disembed three of them from the whole, and limits the student's work with fractions significantly because the student does not perceive of unit fractions as identical pieces that can be iterated to recreate the whole (Norton & Hackenberg, 2010).

Partitive fraction schemes. The existing research on students' understanding of fractions finds two major transitions students must make in their reasoning. The first of these is the transition from a part-whole conception of fractions to a partitive conception of fractions (Norton & Hackenberg, 2010). The partitive fraction schemes can be divided into a hierarchy of three schemes: the partitive unit fraction scheme, the partitive fraction scheme, and the reversible partitive fraction scheme (Norton & Wilkins, 2009, 2012).

Partitive unit fraction scheme. The partitive unit fraction scheme (PUFS; Steffe, 2010d) is the first within the partitive fraction schemes, and is therefore the first advancement students make beyond a part-whole fraction scheme. The distinction between these two schemes is that the PUFS involves the operations of partitioning and disembedding, as did the part-whole fraction scheme, but also requires students to use the iterating operation (Steffe, 2010d). The actions of the partitive unit fraction scheme align closely with the key indicators of the ENS: disembedding and an iterable unit of one (Ulrich, 2016b). Because the PUFS incorporates the iterating operation, a student assimilating fraction tasks with her PUFS understands one-fourth, for example, to be one piece out of the total four, which can be disembedded from the whole, and iterated four times to recreate the whole; iterating the unit fraction to recreate the whole is the critical distinction between the part-whole fraction scheme and the PUFS. This inclusion of iterating the unit fraction parallels the student's construction of the iterable unit of one, an indicator of ENS, within their number sequences.

Despite the ENS being one of the two more sophisticated number sequences, having only constructed an ENS limits a student's fractional understanding. Steffe (2010d) outlines the difficulties two ENS students have (one of whom has constructed a part-whole fraction scheme and the other a partitive unit fraction scheme). They have visually drawn a line segment of length 24 units, called a 24 stick, by iterating a six stick four times. When asked what fraction of the whole (24 stick) a single six stick represents, the students both reply, "six-fourth." In interpreting this problem, the students are limited by their units coordinating activity. Although they are both aware that the six stick was iterated four times to make the 24 stick, they could not take the three levels of units necessary as material for further operating, and correctly identify the six stick as one-fourth of the 24 stick. In this case, the three levels of units involved are the 24 stick, which is a composite unit containing four composite units, each of which contains six units. Solving this task requires the student to assimilate the task with those three levels of units, and operate on them by disembedding one of the six sticks and then compare the six stick to the whole; taking three levels of units as material for further operating is only available to students assimilating with three levels of units, and correspondingly, students who have constructed a GNS.

Partitive fraction scheme. Norton and Wilkins (2009) note that for students operating within the partitive fraction schemes, assimilating tasks with a unit fraction and tasks with non-unit fractions requires different levels of understanding. So, as students with a PUFS understand four-fourths to be the iteration of one-fourth, four times, they have likely not generalized a non-unit fraction, such as three-fourths, to be three iterations of one-fourth. This generalization is the distinction between students operating with the PUFS and the partitive fraction scheme (PFS; Steffe, 2010d). Although a PFS is more powerful than a PUFS, there are two important limitations to note for students operating with a PFS. First, a PFS student cannot yet reverse the operations of the scheme; this reversal of the iterating operation is characteristic of the next partitive fraction scheme, the reversible partitive fraction scheme (RPFS; Steffe, 2010e). The second limitation is that the understanding of a unit fraction as iterable to create a non-unit fraction extends only to proper fractions. That is to say that the student who can create three-fourths by iterating a unit three times cannot necessarily make sense of five-fourths as five iterations of one fourth. To work with improper fractions requires an iterative fraction scheme (IFS; Steffe, 2010d). I will discuss the next two fraction schemes (RPFS and IFS), but it is important to first take an aside and explain the significance of the splitting operation as it relates to the construction of these two more powerful fraction schemes.

***Splitting operation*.** Thus far, the fraction schemes discussed have relied largely on the operations of partitioning and iterating. Simultaneous partitioning is a necessary operation for even the part-whole fraction scheme, and the iterating operation supports the construction of the partitive fraction schemes. But, just as Norton and Hackenberg (2010) identified constructing a partitive conception of fractions to be the first significant transition in students' fractional understanding, they explain the second to be the construction of the splitting operation, which is constructed prior to constructing the RPFS (Norton & Wilkins, 2009, 2012).

Splitting is defined to be the simultaneous "*composition* of partitioning and iterating" (Steffe, 2010d, p. 122, emphasis in original). Steffe (2010d) further explains that composing these two operations is qualitatively different than applying them sequentially, as students with a PUFS are likely to do (cf. Wilkins & Norton, 2011), and implies the understanding of a multiplicative relationship between the unit fraction and the whole. Norton and Wilkins (2012) refer to this critical operation as "two sides of the same coin" (p. 261) that allows students to solve more advanced problems. One example of a splitting task is as follows: "The stick shown below is 5 times as long as another stick. Draw the other stick" (Norton & Wilkins, 2012, p. 562). An arbitrary line segment is shown below the task to represent the stick. Solving this task requires students to "posit a piece that, when iterated five times, reproduces the whole; but to produce that piece, students need to partition" (Norton & Wilkins, 2012, p. 562). Consider the counterintuitive nature of the necessary thinking to complete this task. The language of the task includes "5 times," which for some students will trigger the operation of iterating, or making the stick five times bigger, however, this is not what the task requires at all. The task, although devoid of fractional language, requires the student to draw a stick that is one-fifth the length of the original. This solution requires the creation of a hypothetical stick, generated by partitioning the original stick into five pieces. Within the process of partitioning, however, the student must also understand from the task's wording that the iteration of this piece five times will regenerate the original stick. This line of reasoning exemplifies Steffe's (2002, 2010d) splitting operation.

Steffe (2002) hypothesized that the splitting operation is constructed through the interiorization of equipartitioning. The interiorization of equipartitioning indicates that the student has reprocessed the equipartitioning operation to the extent that it is now material for further operating. Without this interiorization, the composition of partitioning with iterating would not be possible. This hypothesis has been confirmed and expanded upon in more recent literature. Norton and Wilkins (2011) quantitatively confirmed the hypothesis by demonstrating that the majority of sixth and seventh grade students in their research had constructed the PUFS, the operation of which is equipartitioning, prior to constructing the splitting operation. They (Wilkins & Norton, 2011) furthermore quantitatively confirmed that "the relationship between partitioning and splitting is indirect and mediated by the construction of PUFS" (p. 411). Together, these findings act as a quantitative verification of Steffe's hypothesis that the interiorization of equipartitioning supports the construction of splitting.

Reversible partitive fraction scheme. The reversible partitive fraction scheme (RPFS; Steffe, 2010e) is the next scheme that students construct following the PFS and the splitting operation (Norton & Wilkins, 2009, 2012), and Steffe (2010f) regards the splitting operation as instigating the reversibility of the scheme. Reversibility within a scheme is indicated by the student's ability to take the results of a scheme's operations and use them as "input for producing a situation of the scheme" (Steffe, 2010e, p. 125). Speaking in terms of the RPFS, the results of the PFS taken as input are a proper fraction, such as three-fourths, which is created by iterating one-fourth three times. In reversing the iterating operation, the RPFS allows students to understand that three-fourths can be partitioned into three identical unit fractions, each of which is one-fourth, and could be iterated four times to generate the whole; as students operating with an RPFS have constructed the splitting operation, the partitioning and iterating can be accomplished simultaneously.

Summary of the partitive fraction schemes. The onset of the hierarchy of partitive fraction schemes is marked by the inclusion of iterating as a fraction operation, along with partitioning and disembedding, in the PUFS. An abstraction of the results of the PUFS leads to the generalization that unit fractions can be iterated to recreate not only the whole, but other proper fractions, as well; this marks the PFS. Splitting is an important construction in the trajectory of partitive fraction schemes because it allows students to partition and iterate simultaneously, rather than sequentially, which in turn allows students enough operational power to reverse the operations of the PFS to construct the RPFS. As Norton and Hackenberg (2010) indicate, the construction of a partitive conception of fractions and of the splitting operation are two arduous and critical constructions for students' fractional knowledge, which make the partitive fraction schemes of great importance in students' overall understanding of rational numbers. Although the operations of the partitive fraction schemes are available to students who have constructed an ENS, there are still some limitations to these schemes. These limitations are resolved with the ability to assimilate with three levels of units and the subsequent construction of the GNS.

Recursive partitioning and the unit fraction composition scheme. Recursive partitioning is marked by a student's ability to partition "a partition in service of a non-partitioning goal" (Hackenberg & Tillema, 2009, p. 9). For example, asking a student to find one fifth of one seventh requires that the student will partition a unit into seven pieces, and then divide one of those sevenths into five smaller pieces. Although the task does not specifically direct the student to partition all sevenths into five equal parts (resulting in 35ths), a student who is recursively partitioning anticipates the repetition of this action across each seventh. In partitioning a partition, the student must assimilate the task with at least two levels of units and coordinate a third level of units in activity (Hackenberg & Tillema, 2009). Steffe (2003) similarly describes recursive partitioning as first constructing a composite unit (fifths), copying the composite unit (seven times), and joining the seven composite units of five "into a unit of units of units" (p. 240). Furthermore, recursive partitioning is a necessary operation of the unit fraction composition scheme (UFCS; Steffe, 2003, 2010f), the action of which is applying recursive partitioning to the results of the RPFS (Steffe, 2003, 2010f).

In their research with sixth grade students, Hackenberg and Tillema (2009) found that with an MC1 students can solve recursive partitioning tasks, but only with significant support. They describe Sara, an MC1 student, who solves the task of dividing a sub sandwich into 17 parts and then sharing *each* part among three people. For Sara, this triggers multiplication and she determines that the sub will be split into 51 pieces. She then applies a fraction scheme to determine that one piece is one fifty-first. However, when Sara is only prompted to divide a unit into 15 parts and then share one part between two people, she does not partition each fifteenth into two parts. She is only able to express her resulting piece as one half of one fifteenth or as one sixteenth (because the unit is broken into sixteen unequal parts); she cannot express the result as one thirtieth. Sara is unable to construct recursive partitioning because she has only constructed an MC1, and therefore can only coordinate two levels of units in activity. Thus, Sara solves a multiplication problem involving two levels of units and separately solves a fraction problem involving two levels of units. It is not until she is able to coordinate three levels of units in activity that she will be able to apply her multiplicative and fractional schemes simultaneously to construct the recursive partitioning operation, and subsequently, the UFCS.

Iterative Fraction Scheme. Once students have constructed splitting and the reversible partitive fraction scheme (Norton & Wilkins, 2012), and have interiorized three levels of units (Hackenberg, 2007), the iterative fraction scheme (IFS; Steffe, 2010d) is within the ZPC for students. The IFS, simply stated, allows students the capability to work with fractions larger than one. Hackenberg (2007) specifies that students with an IFS are able to operate on improper fractions as "numbers 'in their own right'" (p. 27), which she explains to mean that students interpret $\frac{7}{4}$, for example, as more than just a whole and three parts of another whole. Without constructing an IFS, students may conceive of improper fractions as mixed numbers (a whole and some more parts), however, this is not as powerful mathematically as is a conception of the improper fraction as a number in and of itself (Hackenberg, 2007). A more powerful understanding, and the understanding defined by the IFS, is that one-fourth iterated seven times produces $\frac{7}{4}$.

Originally, Steffe (2010d) hypothesized that upon constructing the splitting operation, students were able to construct the IFS. However, further research has determined (Hackenberg, 2007) and quantitatively confirmed (Norton & Wilkins, 2012) an additional construction that is necessary to support construction of the IFS – the interiorization of three levels of units. In terms of whole numbers, three levels of units implies a unit of units of units (e.g., 18 is a unit, comprised of six units, each of which is comprised of three units of one). Three levels of units in fractions, on the other hand, implies a unit within units within units (e.g., $\frac{7}{4}$ is a unit, comprised of seven units, each of which is one-fourth of the whole). So for students to assimilate a fractional task with three levels of units, as is required to construct an IFS, students must be able to keep track of the relationship between the improper fraction, the whole, and the unit fraction prior to operating (Hackenberg, 2007); assimilating with three levels of units aligns with a student's construction of a GNS, implying that an IFS is not within the ZPC of students until they have constructed a GNS.

Relationships Among the Schemes and Operations

As outlined in the previous sections, students' construction of quantitative concepts has been studied as it relates to units coordination, number sequences, multiplicative concepts, and fraction schemes. There has also been extensive research into the relationships among the development of these schemes. Figure 2.2 depicts the relationships that have been identified in the research. As I detail specific relationships I will introduce truncated versions of the figure. First, notice in Figure 2.2 that units coordination (Ulrich, 2015, 2016a) lies at the top of the diagram, as the level of units with which students can assimilate tasks directly influences the definitions of the other constructs within the figure. Each horizontal row outlines the hierarchical trajectory of students' constructions within a particular area of development–number sequences (Steffe & Cobb, 1988; Ulrich, 2015, 2016a), fraction schemes (Norton & Wilkins, 2009, 2012; Steffe & Olive, 2010) and operations (Steffe, 2010), and multiplicative concepts (Hackenberg & Tillema, 2009). The solid arrows moving from left to right denote the increasingly powerful nature of each of these constructs.

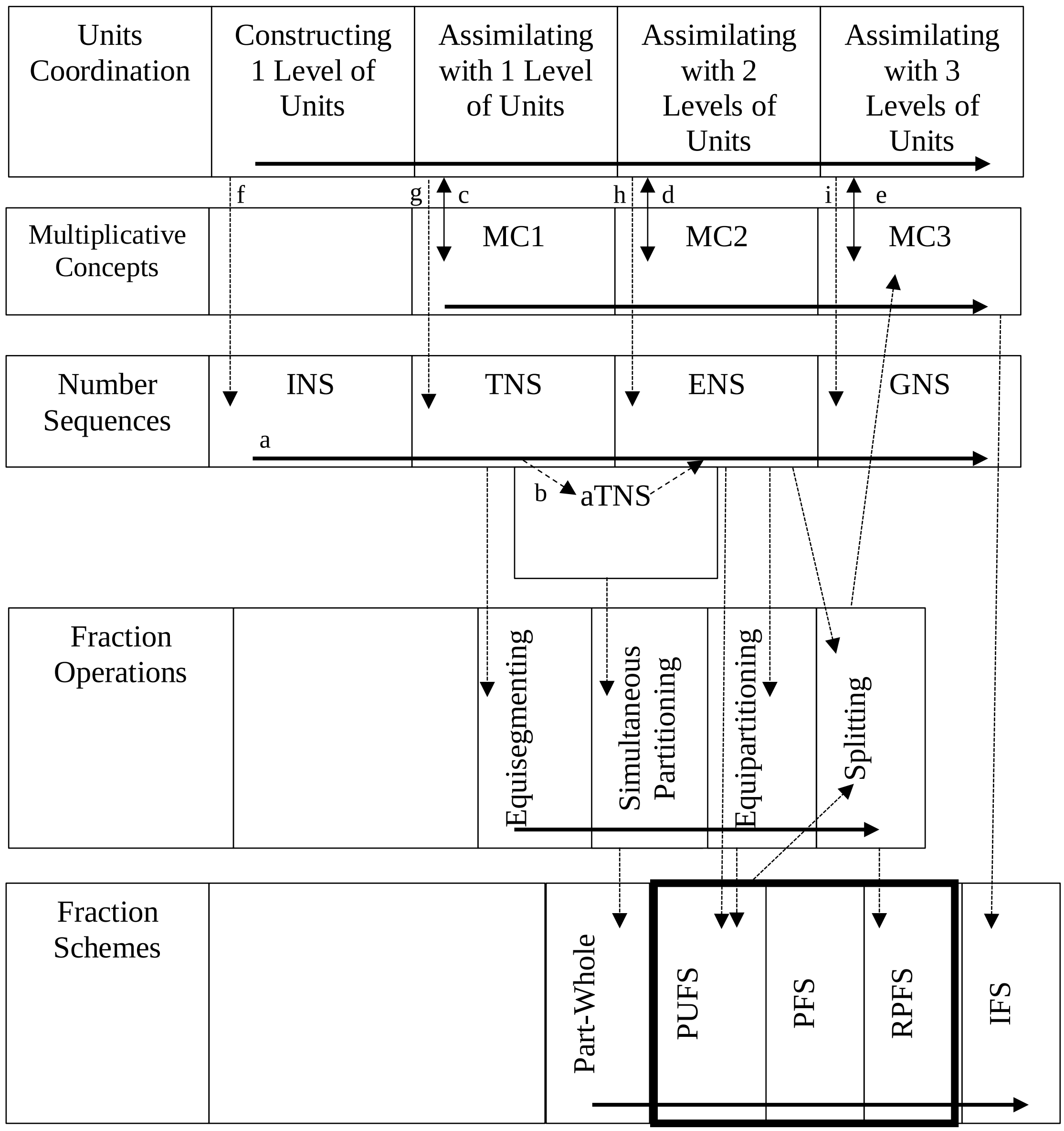

Figure 2.2. Relationships among units coordination, multiplicative concepts, number sequences, fraction operations, and fraction schemes are aligned in rows, and the relationships identified by research are denoted with arrows.

When considering number sequences, recall that Steffe and Cobb's (1988) original research identified a progression of students' number sequences through INS, TNS, ENS, and GNS for students through grade five (Figure 2.2, Arrow a). Ulrich (2016b), however, defines an additional construct, aTNS, which characterizes the assimilatory structure of some middle grades students. To depict this relationship, there are dashed arrows showing the path that some students may take from TNS through aTNS and then to ENS (Figure 2.2, Arrow b). These arrows are dashed because it is not understood whether or not aTNS is a number sequence or a phase within the TNS. It is additionally unknown whether all students will construct an aTNS or whether this "outgrowth" (Ulrich, 2016b) will only occur for some students who struggle to move beyond a TNS.

In the row depicting the fraction schemes, notice that there is a bold box enclosing all of the partitive fraction schemes, but that each of the partitive fraction schemes are then listed individually within that box. This is to visually represent the relationship of the partitive schemes to one another while maintaining the progression that students have been shown to go through (Norton & Wilkins, 2009, 2012) in constructing these schemes.

Reading the diagram vertically, there are bi-directional, vertical arrows between the units coordination and multiplicative concepts (Figure 2.2, Arrows c, d, and e) because each multiplicative concept is defined in terms of the levels of units with which students can assimilate tasks (Hackenberg & Tillema, 2009). The arrows are bi-directional because assimilating with one level of units and constructing MC1, for example, are synonymous—each implies the other. The remaining vertical arrows are dotted and uni-directional to imply the schemes and operations that are potentially available to students. For example, it is not until students can construct one level of units in activity that an INS is within their ZPC (Ulrich, 2015; Figure 2.2, Arrow f). Constructing one level of units necessarily precedes the construction of an INS, which is why the arrow begins at constructing one level of units and ends at INS, and furthermore, because constructing one level of units is not constructed simultaneously with an INS, the arrow is not solid. Moving to the next column, students who can assimilate with one level of units and construct another in activity can also construct a TNS (Ulrich, 2015; Figure 2.2, Arrow g); this relationship is shown with the dashed arrow beginning at assimilating with one level of units and ending at TNS. Similarly, students assimilating with two levels of units can construct an ENS (Ulrich, 2016a; Figure 2.2, Arrow h), and students assimilating with three levels of units can construct a GNS (Ulrich, 2016; Figure 2.2, Arrow i).

Number sequences and fraction operations. Vertical or diagonal dotted arrows to denote the additional relationships identified in the literature, and will discuss each of those arrows in the following paragraphs, beginning by outlining the identified relationships between students' number sequences and fraction operations (Figure 2.3). The operations of students' number sequences are identified in the research to be direct predictors of the types of fraction operations that students are able to construct. Students operating with a TNS are able to construct a composite unit (Ulrich, 2015). The ability to construct composite units also allows TNS students to engage in equisegmenting (Ulrich, 2016b; Figure 2.3, Arrow a). Equisegmenting is marked by a student's lack of anticipation of their actions (Steffe, 2010d)—the inability to anticipate the results of their partitioning activity, and instead, create the partitions sequentially is an indication that they are applying their composite unit to the whole in activity, rather than as an assimilatory construct. Thus, the equisegmenting operation requires a student to have constructed a TNS (Ulrich, 2016). As with the entire diagram, the uni-directional arrow indicates that the TNS is constructed prior to the equisegmenting operation, and the arrow is dotted because equisegmenting cannot be assumed for all students with a TNS; rather, it is within these students' ZPCs (Figure 2.3, Arrow a).

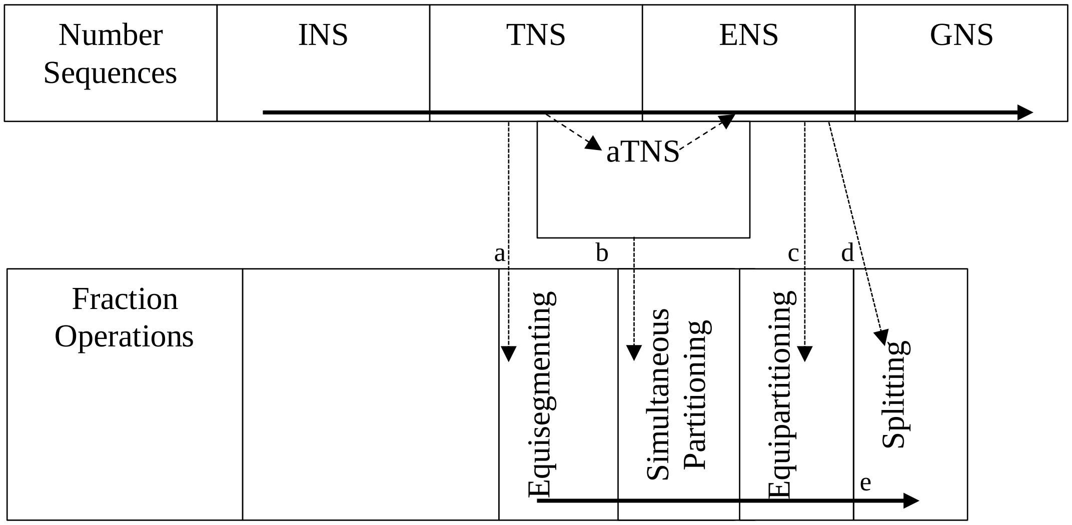

Figure 2.3. A truncated version of Figure 2.2, which includes only the relationships between the number sequences and fraction operations.

Students who are simultaneously partitioning, rather than equisegmenting, tend to have an "uncanny ability" (Steffe, 2010d) to create equal partitions. This is the result of their ability to assimilate the task of partitioning with a composite unit, which affords the student the ability to project a composite unit onto the whole prior to action, and anticipate the act of partitioning the whole into a certain number of pieces (Ulrich, 2016b). Being able to assimilate with composite units, rather than construct them in activity, is the marker of at least an aTNS student (Ulrich, 2016b), if not an ENS student (Steffe, 2010b), and therefore, students will not construct the simultaneous partitioning operation without first constructing at least an aTNS (Ulrich, 2016b). In Figure 2.3, simultaneous partitioning is indicated to be in the ZPC of students who have constructed at least an aTNS by the uni-directional dotted arrow beginning at aTNS (Figure 2.3, Arrow b).

Although students who have constructed the simultaneous partitioning operation can make excellent approximations of their partitions, the mental operations behind simultaneous partitioning are qualitatively different than those behind equipartitioning. Equipartitioning implies that the student assimilates the task with a composite unit, but can additionally disembed any of their partitioned pieces and iterate them to check the accuracy of their partitions (Steffe, 2010d). The understanding of the student that all of the pieces of the partitioned whole are not only equal, but identical, is what prompts them to iterate one piece to recreate the whole, and the operations of disembedding and iterating are not available until students have constructed an ENS (Steffe, 2010b); accordingly, equipartitioning is not available to students until they have constructed an ENS (Ulrich, 2016). Because equipartitioning is within the ZPC of ENS students, Figure 2.3 shows a uni-directional, dotted arrow from ENS to equipartitioning (Figure 2.3, Arrow c).

In addition to equipartitioning, the splitting operation is not available to students who have not constructed at least an ENS (Ulrich, 2016b), because splitting is the "*composition* of partitioning and iterating" (Steffe, 2010d, p. 122). Thus, as the iterating operation is not available prior to constructing an ENS, nor is the splitting operation (Figure 2.3, Arrow d). Furthermore, splitting has been demonstrated by several researchers (Steffe, 2002; Wilkins & Norton, 2011; Ulrich, 2016b) to be a reorganization of equipartitioning. It follows logically that if equipartitioning cannot be constructed without the operations of an ENS (Figure 2.3, Arrow e), than neither can splitting.

Counter to this conclusion, however, is the case of Adam (Ulrich, 2016b). In initial interviews, Adam solved a splitting task, which led researchers to attribute to him an ENS. Upon further consideration, however, it was determined that Adam failed to make comparisons between the unit and the whole, and he was neither disembedding nor iterating units, all of which should have been available operations to an ENS student. Ultimately, Ulrich (2016b) found that Adam's solutions to the splitting task was "only possible because Adam was able to endure high levels of perturbation in problem solving and had constructed assimilatory composite units that he could utilize strategically in operating" (p. 17). To summarize, Adam was able to solve a splitting task because he was assimilating with a composite unit. He had not, however, constructed the disembedding or iterating operations. Therefore, Ulrich (2016b) concluded that he was not splitting in the sense that Steffe (2002) defines it – by composing the partitioning and iterating operations. Ulrich's (2016b) analysis concluded that because Adam could assimilate with composite units, and was a perseverant problem solver, he solved a splitting task; he was not, however, a splitter. As a result of this case study, it can be understood that some middle grades students who have constructed an assimilatory composite unit, but not an ENS, may be able to solve splitting tasks. However, because these students are not mentally compositing their partitioning and iterating operations, they are not splitting prior to constructing an ENS.

Fraction schemes and fraction operations. The fraction operations used for the activity of each fraction scheme were previously discussed, but are included in Figure 2.4, as well. To reiterate, the part-whole fraction scheme requires students to have constructed at least the simultaneous partitioning (Figure 2.4, Arrow a) and disembedding operations (Steffe, 2010d), the PUFS and PFS rely on equipartitioning (Figure 2.4, Arrow b), disembedding, and iterating (Steffe, 2010d), and RPFS relies on the splitting operation (Steffe, 2010f; Figure 2.4, Arrow d), and IFS relies on the splitting operation and an MC3 (Hackenberg, 2007; Figure 2.4, Arrow e). This is not to say that students who have constructed splitting can be assumed to have constructed the RPFS or IFS, for example. Rather, once the student has constructed the operations necessary for the scheme, the construction of the scheme is within their ZPC; that is the reason that dotted arrows are used to mark these relationships. Additionally, the arrows are one directional, with the construction of the fraction operations preceding that of the schemes.

Notice also that the PUFS and PFS both utilize the equipartitioning operation, and the RPFS and IFS both utilize the splitting operation. Regardless of this similarity, the order in which these schemes are constructed has been well documented (Norton & Wilkins, 2009, 2012;

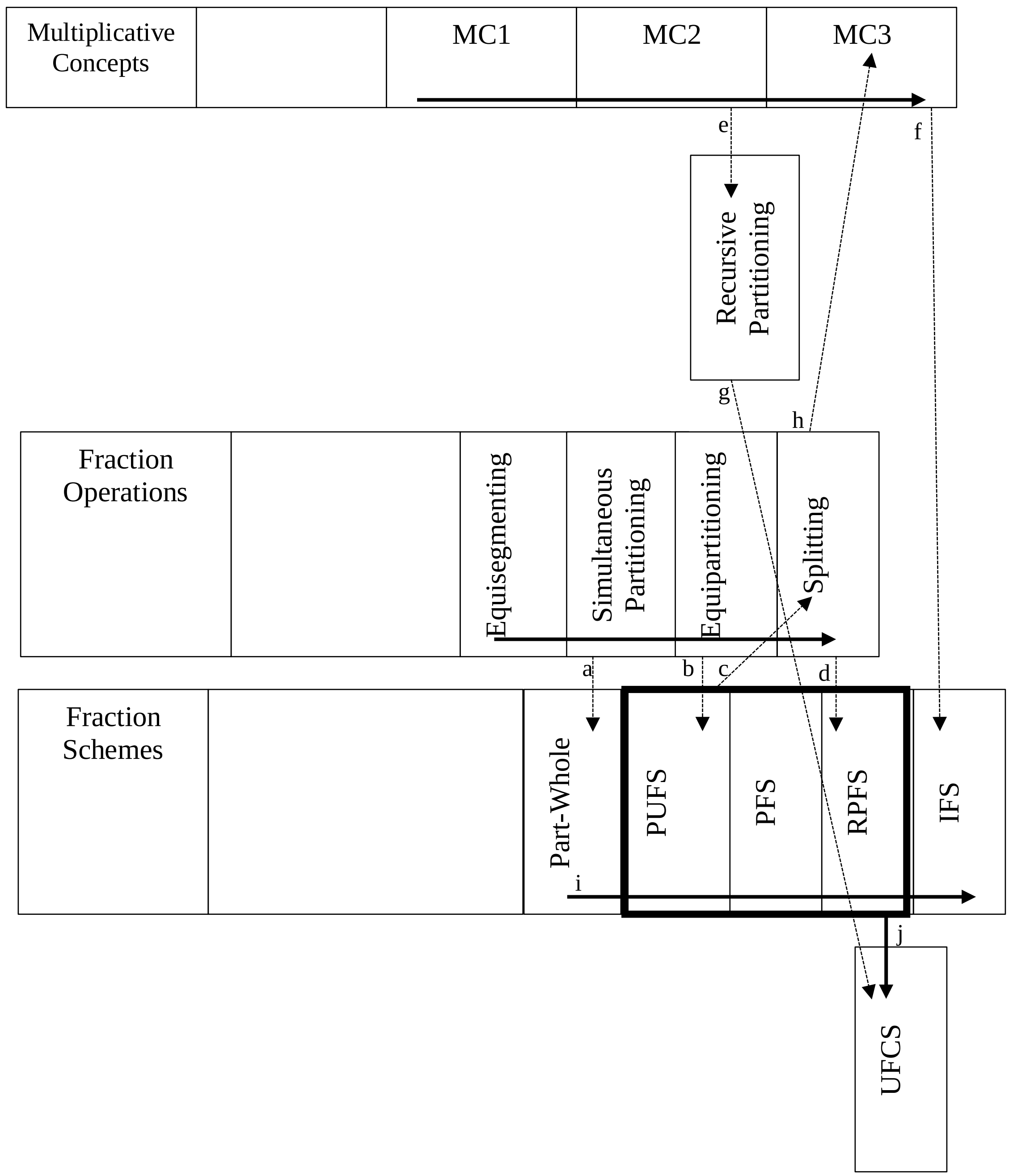

Figure 2.4. A modified version of Figure 2.2, which includes only the relationships between the fraction operations and fraction schemes and adds recursive partitioning and the unit

fraction composition scheme (UFCS).

Figure 2.4, Arrow i), with the PUFS preceding the PFS and the RPFS preceding the IFS. Consider first the relationship between the PUFS and the PFS. Once students have constructed the equipartitioning operation, the PUFS is within their ZPC (Ulrich, 2016b; Figure 2.4, Arrow b). The construction of the PFS, then, does not require an additional operation, but rather a generalization on the student's part that not only can a unit fraction be iterated to recreate the whole (e.g., one-fourth iterated four times generates four-fourths), but also that a unit fraction can be iterated a certain number of times to create another proper fraction (e.g., one-fourth iterated three times generates three-fourths). For this reason, no new operations are engaged in the PFS compared to the PUFS, but the result of assimilating a task with the PFS can be either a proper fraction or the whole, whereas the result of assimilating a task with the PUFS can be only the whole unit—a proper fraction is not a possible result within the PUFS.

As the PUFS and the PFS utilize the equipartitioning operation, so do the RPFS and the IFS both utilize the splitting operation. However, a student's construction of the IFS does involve additional constructions, namely the ability to assimilate with three levels of units (Norton & Wilkins, 2012), which implies an MC3 (Hackenberg, 2007). A one-directional arrow beginning at MC3 and leading to IFS (Figure 2.4, Arrow f) indicates the need for students to construct MC3 prior to constructing IFS. Steffe (2010f) hypothesizes that constructing the splitting operation makes possible the reorganization of the PFS into the RPFS. However, in order to construct the IFS, students must construct the splitting operation and be able to assimilate with three levels of units (Norton & Wilkins, 2012). In addition to needing to construct both splitting and three levels of units prior to constructing the IFS, splitting must be constructed before three levels of units (Figure 2.4, arrow h). Norton and Wilkins (2012) quantitatively tested for a relationship between the construction of three levels of units and splitting, and found a strong association. In particular, all but one student constructed splitting prior to constructing three levels of units. They conclude that splitting is likely to be constructed first, which confirms Hackenberg's (2007) conclusions.

In additional research, Wilkins and Norton (2011) tested for the relationship between partitioning, splitting, and PUFS. In this case, it was determined that the relationships between partitioning and splitting, and between iterating and splitting, were mediated by PUFS. This relationship is demonstrated by the only arrow in Figure 2.4 originating with a fraction scheme and terminating with a fraction operation (Arrow c). The implication of this finding is that for students who have constructed the equipartitioning operation, PUFS is within their ZPC; this corresponds with Steffe's (2010d) finding that equipartitioning precedes the construction of PUFS. However, the determination that PUFS mediates the relationship between partitioning and splitting, and between iterating and splitting provides further evidence of the order in which students construct these schemes and operations.

Finally, added to Figure 2.4 are the recursive partitioning operation and the unit fraction composition scheme (UFCS). Hackenberg and Tillema (2009) found that recursive partitioning is within the ZPC of students who have constructed the second multiplicative concept (Figure 2.4, Arrow e). They determined that with an MC1, students may solve recursive partitioning tasks with significant supports because they can apply their multiplication and fraction schemes sequentially. MC2 students, on the other hand, can apply them simultaneously as a result of being able to assimilate the task with two levels of units (Hackenberg & Tillema, 2009). Having constructed the recursive partitioning operation, students can apply recursive partitioning (Figure 2.4, Arrow g) to the results of their RPFS (Figure 2.4, Arrow j) to construct the UFCS (Steffe, 2003). In Figure 2.4, arrow g connecting the recursive partitioning operation to the UFCS is dotted because the UFCS is within the ZPC of students who have constructed the recursive partitioning operation. However, arrow j connecting the RPFS to the UFCS is a solid arrow because its meaning is similar to that of arrow i. Arrow i indicates the hierarchy of fraction schemes from least to most sophisticated. Once students have constructed an RPFS, they can construct each of two more sophisticated schemes: the IFS and the UFCS. To construct the IFS, students apply three levels of units coordination to the results of their RPFS (Hackenberg, 2007). To construct the UFCS, students apply recursive partitioning to the results of their RPFS. Therefore, unlike the other fraction schemes in Figure 2.4, the results of the RPFS can be applied in two different ways to construct distinct fraction schemes.

Application of Schemes to Algebraic Reasoning

In the previous section, I defined each of the schemes and operations that have been examined in the existing research and additionally outlined the relationships among each of these constructs. Examining only the relationships among the schemes described here, however, leave glaring holes in the applicability of schemes to mathematics in its entirety. In the next section, I will address the gap in part by demonstrating the explanatory power of these schemes for students' algebraic reasoning. To begin, I will define algebraic reasoning as it has been discussed in the research and as it will be used in this study, as well as components of algebraic reasoning such as equality, variable, and writing equations.

Algebraic reasoning. The study of algebra comprises a large portion of middle and high school mathematics curricula, and has been referred to as a "gatekeeper" in mathematics (Cai & Knuth, 2011, p. vii). Algebra is persistently characterized as being difficult for students to master (e.g., Kieran, 2007). But, it is a staple in middle and high school curricula due to its importance within mathematics as a whole (NCTM, 2000). Furthermore, the National Council of Teachers of Mathematics (NCTM, 2000) and the Common Core State Standards Initiative (CCSSI, 2010) agree that algebra should be a strand of mathematics beginning in the primary grades. Thus, while "it is agreed that the means for developing algebraic ideas in earlier grades is not to simply push the traditional secondary school curriculum down into elementary school... [it is necessary to develop] a better understanding of the various factors that make the transition from arithmetic to algebra difficult for students" (Cai & Knuth, 2011, p. viii). The purpose of this research study aligns with this goal by examining the algebraic reasoning of students in the middle grades, and identifying cognitive structures that support students' algebraic reasoning.

Some research into algebraic thinking and learning has focused on the relationship between arithmetic and algebraic thinking, and how algebra can be introduced in earlier grades by presenting algebra as an extension of arithmetic thinking. Russell, Schifter, and Bastable (2011), for example, have identified four ideas that underlie both arithmetic and algebraic thinking, and therefore, constitute a potential link between the two. These ideas include: (1) an understanding of operations, such as the relationship between addition and subtraction, or how squaring a sum differs from squaring a product; (2) generalizing and justifying, which include verbalizing patterns and constructing informal proofs or explanations to support the generalizations; (3) extending the number system to include, for example, negatives and decimals; and (4) using symbolic notation in a meaningful way. Russell and her colleagues (2011) have identified situations in which these four concepts can be developed through instruction of arithmetic, and further, how they contribute to early algebraic reasoning.

The conceptual bridge between arithmetic and algebraic reasoning (Russell et al., 2011) sheds light on how students too cognitively immature to engage in formal algebraic reasoning (e.g., Filloy & Rojano, 1989; Herscovics & Linchevski, 1994) can still begin to develop algebraic ideas. Refocusing research of algebraic reasoning to include the work of younger students may address not only the difficulties students have in learning algebra, but may also help to establish the instantiation of those difficulties. Hackenberg (2013) operates under a definition of algebraic reasoning that is broad, and encompasses both "generalizing and abstracting arithmetical and quantitative relationships, and systematically representing those generalizations in some way… [and] learning to reason with algebraic notation in lieu of quantities" (pp. 541–542). Although Hackenberg's definition mentions the use of algebraic notation as an eventual goal of algebraic reasoning, she does not characterize the use of algebraic notation as the onset of algebraic reasoning as some other researchers do (e.g., Filloy & Rojano, 1989; Herscovics & Linchevski, 1994).

Hackenberg's definition is supported by Radford's (2011) definition of algebraic reasoning, in which he maintains that "the use of notations … is neither a necessary nor a sufficient condition for thinking algebraically" (p. 310). Radford continues that "what characterizes algebraic thinking is that it deals with *indeterminate* quantities conceived of in *analytic* ways" (p. 310, emphasis in original). Thus, algebraic reasoning need not be restricted to reasoning that involves formal algebraic notation. Moreover, the use of these more restrictive definitions ignores the types of algebraic reasoning in which younger students may engage (Brizuela & Schiliemann, 2004). Therefore, because the purpose of this research study is to examine the algebraic reasoning of middle-grades students, some of whom have not taken a formal algebra course, a broader characterization of algebraic reasoning will be used.

Radford (2011) defines a hierarchy of non-symbolic algebraic thinking in elementary children (Table 2.2). These begin with commonality, which is simply noticing patterns. Next, children progress to generalizing, or finding terms in a pattern. The first stage of generalizing that Radford characterizes as algebraic reasoning is generalizing through particular examples, which implies that students generalize a pattern by calculating a term in the pattern that could not be found recursively. For example, if students have identified the first, second, and third terms in a pattern, they might generalize through particular examples by calculating the 100^{th} term in the pattern. This is identified as a form of algebraic reasoning because students reason abstractly about unknown quantities, although they do so numerically rather than using algebraic notation. Generalizing patterns through particular examples "deals with *indeterminate* quantities conceived of in *analytic* ways" (Radford, 2011, p. 310, emphasis in original). In this form of algebraic reasoning, the variable is still considered to be tacit, and "indeterminacy as such does not reach the level of symbolization, not even the level of discourse" (p. 311).

Table 2.2. Radford's (2011) Hierarchy of Non-Symbolic Algebraic Thinking.

Level of Algebraic Thinking	Description
Commonality	Noticing similarities between figures or terms in a pattern.
Generalizing	Extending the similarities noticed to find additional figures or terms in the pattern.
Generalizing through particular examples*	Applying a rule to a larger term or figure that is not calculated recursively.
Verbalizing	Explicitly stating a rule in terms of an unknown quantity.

*Generalizing through particular examples is noted as the onset of algebraic reasoning.

More sophisticated forms of non-symbolic algebraic reasoning include verbalizing patterns to abstract the pattern to any term, which involves an explicit indeterminacy (Radford, 2011). These stages of generalizing patterns have been shown by Radford to extend to the thinking of elementary children who have not taken an algebra course. This demonstrates that this hierarchy of non-symbolic algebraic thinking is appropriate for children who have not taken algebra and provides a clear identification of the level at which children are perceived to be reasoning algebraically.

Using the level of generalizing patterns through particular examples (Radford, 2011) provides a clear demarcation between arithmetic and algebraic reasoning. Radford's characterization of algebraic reasoning, however, encompasses some informal strategies for solving equations that other research has disregarded. Namely, unwinding. Knuth, Stephens, McNeil, and Alibali (2006) refer to both guess and check and unwinding as pre-algebraic because they do "not emphasize the symmetry of an equation" (p. 310). Guess and check is the process of guessing a solution to an equation or problem, and then substituting the guessed number into the equation or problem to check if it is a correct solution. This does not involve an analysis of indeterminate quantities, making it pre-algebraic by Radford's characterization. However, an unwinding strategy (Knuth et al., 2006) involves the reasoning about an indeterminate quantity analytically because the student conceives of increasing or decreasing an unknown quantity based on information given in the problem. Thus, unwinding can be considered as a form of algebraic reasoning, albeit an informal strategy for solving equations. Therefore, based on these characterizations, the present research study will define algebraic reasoning as dealing with indeterminate quantities in analytic ways (Radford, 2011), and will characterize unwinding as an algebraic strategy and guess and check as a pre-algebraic strategy.

Equality. In addition to determining what constitutes algebraic reasoning, it is important to also consider what concepts act in support of algebraic reasoning. Research has identified students' concept of the equal sign as one such concept (e.g., Carpenter et al., 2003; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Matthews et al., 2012). As early as preschool, children can have two distinct notions of equality (Kieran, 1981). A relational concept indicates an understanding of the equal sign as indicating a balance, or the equality of expressions on either side of the equation. This is the more sophisticated of the two notions. However, many children perceive of the equal sign operationally (e.g., Baroody & Ginsberg, 1983), indicating a conception of the equal sign as indicating an operation. This is detrimental to students' ability to solve equations because "virtually all manipulations on equations require understanding that the equal sign represents a relation" (Carpenter et al., 2003). And, although most students begin to conceive of the equal sign as indicating a relation more so than an operation around the age of 13, many mistakes in high school mathematics can be attributed to incorrect uses of the equal sign (Kieran, 1981).

A relational concept of the equal sign has been specifically tied to students' success in solving equations. Middle school students' concept of the equal sign has been found to be positively related to their ability to solve equations (Knuth et al., 2006) and having a relational concept of the equal sign, in particular, has been shown to predict students' success in solving algebraic equations (Alibali et al., 2007). Simply put, "without a relational understanding [of the equal sign], the algebraic principle of maintaining equality is nonsensical" (Byrd et al., 2015, p. 61). Therefore, characterizing students' concept of the equal sign as operational or relational is important in developing a holistic understanding of their ability to reason algebraically.

Matthews and his colleagues (2012) developed a framework to understand students' concept of the equal sign (Table 2.3). The framework is hierarchical, and includes four leveled concepts of the equal sign through which students progress. First, is the rigid operational level. Within this level, students are likely to perceive the equation $5 + 2 = 7$ to be true, but $7 = 5 + 2$ to be false, because seven does not equal five (Matthews et al., 2012). In other words, students conceive of the equal sign as indicating that the number immediately to its right must be equal to the expression on the left. The second level of students' concept of the equal sign is flexible operational, and although students still conceive of the equal sign as an operator, they understand $7 = 5 + 2$ to be true. The third level is basic relational, and with such a concept students can determine that $6 + 1 = 5 + 2$ by summing the expressions on both sides of the equal sign. This is the more basic of the two relational concepts of the equal sign, but within Matthews and his colleagues' (2012) framework, is the first indication that students conceive of the equal sign as demonstrating a relationship between two quantities, rather than an indicator to act on an expression or quantity. Finally, at level four, comparative relational, students can reason strategically (cf. Steffe, 2010b) about quantities to the extent that they can determine the truth of equations without calculating the value on each side of the equation. A typical question evaluating whether or not students have constructed the comparative relational concept of the equal sign is "without adding … can you tell if the number sentence '67+86=68+85' is true or false?" (Matthews et al., 2012, p. 320).

Table 2.3. Matthews and Colleagues' (2012) Hierarchy of Concepts of the Equal Sign

Level 1	Rigid Operational	An expression on the left equals the number immediately to the right of the equal sign. e.g., $5 + 2 = 7$ but $7 \neq 5 + 2$
Level 2	Flexible Operational	An expression can be on the left or right of the equal sign, but must equal one number. e.g., $5 + 2 = 7$ or $7 = 5 + 2$, but $6 + 1 \neq 5 + 2$
Level 3	Basic Relational	Both sides of the equal sign can contain expressions. The student must calculate their value to determine equality. e.g., $6 + 1 = 5 + 2$ because $7 = 7$
Level 4	Comparative Relational	Students can determine equality without calculating the

	value of the expressions. They instead reason strategically. e.g., $6 + 1 = 5 + 2$ because 6 is one more than 5 and 1 is one less than 2.

Students concept of equality is thought to be more dependent on instruction and context than it is upon cognitive limitations (e.g., Baroody & Ginsberg, 1983; Cobb, 1987; Knuth et al., 2011). For example, in a cross-sectional study of students in grades six through nine, Knuth and his colleagues (2011) determined that students' understanding of the equal sign as relational increased from less than 30% of students in sixth grade to 46% of students in ninth grade. Despite this apparent growth, a relational concept of the equal sign does not replace an operational one; rather, a relational concept of the equal sign compliments an operational one (Matthews et al., 2012). The implication of this result is that students who have constructed a relational concept of the equal sign may still apply an operational concept at times. The effect of applying an operational concept to algebraic situations, despite having constructed a relational concept, are unknown.

Variable. In addition to students' concept of the equal sign, students' concept of variable also plays a role in their algebraic reasoning. Matthews and his colleagues' (2012) found that students' concept of variable was related to their concept of the equal sign. Specifically, students with a more sophisticated concept of the equal sign performed better on algebraic tasks involving variables. In related research conducted by Knuth and his colleagues (2006), students with a relational concept of the equal sign were significantly more likely to solve algebraic equations correctly, even when controlling for math ability and grade level. Furthermore, students without formal algebra instruction had a better chance of solving equations if they demonstrated a relational concept of the equal sign, than if they had an operational concept (Knuth et al., 2006).

Thus, students' concept of the equal sign is related to their ability to reason algebraically, and to operate with variables.

Variable misconceptions are common among students. Lucariello, Tine, and Ganley (2014) gave a survey to students in grades 6 through 12. The purpose of the survey was to identify students' misconceptions related to variables. They found that 28% of students chose three or more answers that demonstrated a misconception about variables, and moreover, high school students had more misconceptions about variables than did middle school students. Although Lucariello and colleagues do not posit a reason for the increase in students' misconceptions, MacGregor and Stacey (1997) have noted that two of the sources for students' misconceptions about variable are interference from the learning of new mathematical ideas and poor instruction. These two rationales for misconceptions related to variable could account for increased instances of misconceptions among older students.

In addition to noting the existence of misconceptions, MacGregor and Stacey (1997) also identify what students' conceptions of variable are likely to be. Their conceptions of variable (Table 2.4) build on Küchemann's (1981) hierarchy of variables. The concepts of variable identified in MacGregor and Stacey's research include abbreviated word, alphabetical value, numerical value, use of different letter, and letter ignored; these conceptions were evident among students across Australian school years seven through ten (ages 11-15) in their research. However, three conceptions related to variable were also evident among only students in years eight through ten. These included using a variable as a label, assuming a variable equals one, and using a variable as a general referent.

Table 2.4. MacGregor and Stacey's (1997) Concepts of Variable

Misconceptions in Years 7-10	
Abbreviated word*	Using U*h* represents an unknown height

Alphabetical value	Letting h=8 because h is the eighth letter of the alphabet
Numerical value*	Assuming a reasonable value for a variable
Use of different letter	Choose different letters to represent related unknown quantities, such as g and h to represent related heights
Letter ignored*	Do not incorporate a letter, or do not acknowledge a letter in an expression
Misconceptions in Years 8-10	
Label	Associating a label with the name of an object. D represents David's height, for example.
Variable equals 1	Assuming a variable is equal to one, so $10 + h = 11$, for example.
General referent	Using the same variable to represent different quantities, e.g., h is David's height and Con's height.

Note: Examples are taken from MacGregor and Stacey (1997).

*These categories were originally included in Küchemann's (1981) classifications.

Hackenberg, Jones, Eker, and Creager (2017) posit that some difficulty related to operating on unknowns is related to students' multiplicative concepts. Specifically, they found that "conceiving of an unknown … means that students need to be able to take a unit of units as given … [which] requires operating with at least the second multiplicative concept" (p. 42). This is because a quantitative unknown consists of "a unit of some number of units" (Hackenberg et al., 2017, p. 43). In other words, an unknown quantity constitutes a two-level unit structure because it is a composite unit containing an unknown number of units of one. The implication of this result is that to operate on or reflect on an unknown quantity, students must assimilate tasks with at least a two-level unit structure and construct a third level of units in activity. Without an assimilatory composite unit, an unknown is meaningless.

Equations. The importance of having constructed an assimilatory composite unit to support students' reasoning about unknown quantities extends to students' ability to write equations. Hackenberg and her colleagues (2017) found that even students who had constructed an assimilatory composite unit struggled to represent the multiplicative relationship between two unknown quantities algebraically. One problem used in this research is to represent the relationship between the unknown height of a corn stalk that is five times the unknown height of a tomato plant[6]. On this problem, Paige, an MC2 student, built the equation $y = 5x$ through the activity of working with numerical examples prior to representing it algebraically. This is attributed to her assimilatory composite unit; Paige assimilates the task as a two-level unit structure (the height of the tomato plant within the height of the corn stalk), and constructs the unknown quantity in activity. Thus, she simplifies “the units coordinations involved in the problem” (Hackenberg et al., 2017, p. 50) by assimilating the task using numerical examples, and then builds the unknown aspect in activity.

On the same corn stalk and tomato problem in Hackenberg and her colleagues’ (2017) research, Connor, another MC2 student, represented the multiplicative relationship correctly using known quantities ($20/5 = 4$) but was unable to represent the multiplicative relationship using unknowns ($C/B = D$). In a follow up interview, Connor still could not represent the multiplicative relationship between two unknowns algebraically. Instead, he represented it additively as $A + B + B + B = D$, where D represents the answer. Finally, a third MC2 student, Tim, insisted that the equation representing a five to one multiplicative relationship was “approximate” (Hackenberg et al., 2017, p. 45) when working with unknown quantities. In other words, the unknown height of a cornstalk was “approximately” five times the unknown height of a tomato plant, despite the problem explicitly stating the corn stalk was five times the height of the tomato plant.

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[6] This problem is congruent to the phone cords problem (Appendix B, A1).

All of the MC2 students' limitations are attributed to their inability to maintain the three-level unit structure that results from operating on an unknown quantity following activity (Hackenberg et al., 2017). "With *x* maintained as an unknown, there's no way to evaluate *5x* to reduce the levels of units to coordinate" (Hackenberg et al., 2017, p. 54). This difficulty makes it impossible for MC2 students to maintain, for example, five iterations of a composite unit, *x*, consisting of an unknown number of units following activity. Based on these results, it seems reasonable to conclude that while an assimilatory composite unit is a necessary mental structure for representing multiplicative relationships algebraically, it is not sufficient.

Clement (1982) has also researched students' difficulty in writing equations representing multiplicative relationships with the students and professors problem. In this task, which is perhaps the most widely known task related to students' equation writing behaviors (Bush & Karp, 2013), students are asked to write an equation to represent there being six times more students than professors at a university. Clement (1982) found that even among 150 freshmen engineering majors, approximately 25% made a reversal error by representing the multiplicative relationship between the unknown number of professors and students as some form of $6S = P$, where S represents the number of students and P represents the number of professors. He goes on to explain that this reversal error stemmed from two non-standard ways of thinking about the problem. Some students engaged in word order matching, which implies that they simply matched the components of the equation to the order of the words in the problem. Other students made a static comparison, which indicates the students believed that because there are more students at the university than professors, the students should be multiplied by six so that the resulting number of students is greater than the resulting number of professors.

These students' erroneous attempts at representing the students and professors problem provide insight into students' difficulty representing the multiplicative relationship between two unknowns algebraically. The focus of Clement's (1982) research, however, was not on providing a rationale for why such a large percentage of undergraduate students engaged in word order matching or a static comparison method on the students and professors problem. It is entirely possible that these students made a mistake; they are undergraduates who, based on their major of study, have presumably taken calculus. However, it is also possible that these reversal errors are necessary (Steffe, 2010d) errors that are regularities in the students' reasoning resulting from not having constructed sufficient cognitive structures to support writing equations that represent multiplicative relationships between two unknowns. Hackenberg and her colleagues' (2017) determination that many students with an assimilatory composite unit struggle to represent such relationships provides a possible framing of word order matching and the static comparison approach as necessary errors. In other words, it is possible that students who have constructed only an assimilatory composite unit are more likely to make reversal errors when representing the multiplicative relationships between two unknowns.

In additional research relating students' units coordination to their algebraic reasoning, Hackenberg (2013) and Hackenberg and Lee (2015) model middle-grades students' algebraic reasoning using the multiplicative concepts they have constructed. One of the tasks Hackenberg (2013) uses to analyze students' algebraic reasoning is the phone cords problem (Appendix B, A1), which incorporates a splitting task. In this problem, students are asked to draw a picture to represent the length of Rebecca's iPhone cord as it compares to Stephen's, and to represent this situation algebraically. A correct solution includes a picture in which Rebecca's cord length is one-fifth that of Stephen's, and an equation similar to $S = 5R$, where R and S are the lengths of

Rebecca's and Stephen's cords, respectively. Interestingly, one MC1 student, Henry, was able to solve this task. Hackenberg (2013) explains

> In Henry's initial picture for A1, the segment representing Rebecca's cord length was longer than the segment representing Stephen's … *Without any intervention from the interviewer*, Henry reinitiated his activity and drew a small segment. … Henry repeated one cord length five times, and the new segment represented the other cord length … Henry called the long segment Rebecca's and the short segment Stephen's. However, when the interviewer restated the problem, Henry switched these meanings. (p. 552, emphasis in original)

Henry's solution to this task is interesting because splitting is not within the ZPC of MC1 students because they have not yet constructed the disembedding operation (Steffe and Olive, 2010), and in fact, Hackenberg (2013) notes that Henry had not constructed the splitting operation. Her description of Henry's solution to a splitting task joined with her indication that he was not splitting can potentially be explained by Ulrich's (2016b) explanation of an aTNS student's ability to solve a splitting task by sequentially partitioning and iterating.

Of further interest from Hackenberg's (2013) analysis is that Henry was the only MC1 student to write a correct algebraic expression to represent the splitting task. Based on Ulrich's (2016b) description of aTNS students, it could be argued that his ability to do so was a result of his ability to assimilate tasks with a composite unit. Ulrich (2016b) points out that assimilating tasks with a composite unit is what allows aTNS students to solve splitting tasks without having constructed the splitting operation. It is possible that Henry's assimilatory composite unit (Rebecca's cord within Stephen's cord) also facilitated his algebraic representation of the situation. Perhaps for students constructing the second level of units in activity (TNS and MC1 students) the algebraic representation would prove too difficult because the relationship between Rebecca's and Stephen's cords decays following activity. This points to a potential advantage of aTNS students' algebraic reasoning over that of TNS students.

Hackenberg (2013) also analyzes the algebraic reasoning of MC1 students on the border problem (Appendix B, A4) and she, again, analyzes Henry's work. In this problem, students are asked to generalize a pattern that would allow them to calculate the number of squares on the border of a square grid with any side length. Henry generalizes that to find the perimeter of a square with a side length of 10, he would add 10 + 10 + 8 + 8, and for a side length of six he would add 6 + 6 + 4 + 4. He does not verbalize the relationship between 10 and eight or between six and four in the concrete examples he examines; this can be interpreted as his perception of the quantities as being unrelated. Hackenberg (2013) attributes this to his not having constructed the disembedding operation. Henry was not able to understand the side length of eight, for example, as both a part of and separate from the side length of 10.

Courtney had also constructed an MC1 but unlike Henry could not solve the splitting task. Like Henry, she makes progress on the border problem, although in a different way. Courtney explains that to find the perimeter of a square with a side length of 10, she would add 10 + 10 + 10 + 10 – 4. (Note that Courtney's solution is also correct.) To accompany this verbalization she writes, "10 + 10 = 20 + 10 = 30 + 10 = 40" (Hackenberg, 2013, p. 556). Based on her written work, Hackenberg determines that the tens are absorbed within the sum as she adds. In other words, she cannot reflect upon the composite unit she constructed in activity. This is typical of a TNS student (Ulrich, 2015). This constraint limits her algebraic reasoning because the interviewer's suggestion that she represent the side length as *x* does not make sense to

Courtney (Hackenberg, 2013) – the side length has decayed from her unit structure and she is left only to operate on the sum.

In contrast to Henry, there is no evidence in Hackenberg's (2013) research that Courtney had constructed an aTNS. It is more likely that she had constructed only a TNS, which is the number sequence assumed to align with an MC1 (see Figure 2.2). Therefore, she may have less sophisticated schemes and fewer operations on which to draw when reasoning algebraically, compared to Henry. In particular, Courtney is assumed to be assimilating tasks with one level of units and constructing a second level in activity. This explains her constrained reasoning on the border task.

On the other hand, Hackenberg (2013) suggests that although Henry did not represent the border problem algebraically, if he did, his representation would likely be something like $x + x + y + y$. (Hackenberg notes that the students had limited time to solve the border problem.) I further suggest that the ability to represent this situation algebraically (as $x + x + y + y$) should be within the ZPC of students who have constructed an aTNS because they can assimilate the task as a unit of units (the side length within the border), and can reflect upon the relationship between the side lengths and the border. This hypothesis requires further examination with students who have been clearly identified as having constructed an aTNS.

In related research, Hackenberg and Lee (2015) analyze how the second and third multiplicative concepts can explain students' algebraic reasoning and how the reasoning of these students is more sophisticated than that of MC1 students. In particular, they examine how middle-grades students use fractions as multipliers on an unknown. In this research, they used two tasks in which students were asked to represent the relationship between two unknowns with a fractional coefficient ($y = \frac{1}{5}x$ and $y = \frac{2}{5}x$). On the first problem, no MC2 students used $\frac{1}{5}$ as a multiplier on *x*, however, two MC2 students did represent the relationship as $y = \frac{x}{5}$. On the second problem, one MC2 student represents the relationship with a fractional unknown while retaining a quantitative meaning for the fraction. Hackenberg and Lee (2015) argue that students must construct an IFS, which must be preceded by an MC3 (see Figure 2.2), prior to using fractions as multipliers on an unknown. This explains why MC2 students were generally unable to do so. Instead, MC2 students represented a fractional multiplier as division, if at all. The implication of this finding is that it is possible for ENS students[7] to represent the relationship between two unknowns using a whole number divisor, but probably not a fraction multiplier. It follows that it is unlikely that aTNS students will be able to represent this relationship algebraically using a fraction multiplier or a whole number divisor. Although this hypothesis requires testing, it may provide a useful indicator to distinguish between aTNS and ENS students.

In addition to Hackenberg and her colleagues' (Hackenberg, 2013; Hackenberg & Lee, 2015; Hackenberg et al., 2017) research, Olive and Çaglayan (2008) examine the algebraic reasoning of four, eighth-grade students in a teaching experiment, and analyze their reasoning in terms of units coordination. They specifically examine the algebraic reasoning of the students in the context of the coin problem (Appendix B, A5), which requires students to write an equation representing the value of an unknown number of nickels, dimes, and quarters that sum to $5.40. One student, Ben, is able to assimilate tasks with two levels of units and to construct a third level of units in activity. When he attempts to solve the coin problem, he quickly writes an equation that represents the relationship between the number and value of each type of coin to the total

[7] Recall that constructing an ENS is within the ZPC of students who have constructed an MC2, and that it is unclear from existing research whether aTNS students' reasoning will be more consistent with that of MC1 or MC2 students (see Figure 2.2).

amount: $(.05N) + (.1D) + (.25Q) = \$5.40$, and explains that the N, D, and Q represent the numbers of nickels, dimes, and quarters; to do this requires a three level unit structure (the value of a single coin within the number of that type of coin within the value of all the coins; Olive & Çaglayan, 2008). However, when pressed to substitute representations of D and Q in terms of N ($D = N + 3$ and $Q = N - 2$), he begins to conflate the number of dimes and nickels as well as their values (Olive & Çaglayan, 2008). This is because the third level of units has decayed.

For Ben, his inability to assimilate the task with three levels of units was severely limiting to his algebraic reasoning. Although Olive and Çaglayan (2008) do not attribute to Ben a number sequence, his units coordination implies that an ENS would be within his ZPC (Ulrich, 2016a), thus, it is likely that Ben has constructed the disembedding operation. I hypothesize that it is the disembedding operation that allowed Ben to write the literal equations representing the number of dimes and quarters in terms of the number of nickels ($D = N + 3$ and $Q = N - 2$). Disembedding is hypothesized to be a necessary operation for writing these literal equations because they require the student to understand the number of quarters, for example, to be a quantity embedded within the number of nickels that can also be removed from and compared to the number of nickels without destroying the number of nickels as a unit in and of itself. Despite the algebraic reasoning afforded to Ben by his application of the disembedding operation, he was still restricted because he was unable to substitute expressions for D and Q that would have allowed him to solve the problem independently. His inability to make those substitutions was, by my assessment, a limitation of only having assimilated the task with two levels of units.

In addition to Ben, Olive and Çaglayan (2008) also analyze the algebraic reasoning of Maria with regard to the same coin problem. In contrast to Ben, Maria is identified as being able to assimilate tasks with three levels of units. As a result, Olive and Çaglayan (2008) describe the "ease with which [she] established the second equation [$.05N + .1(N+3) + .25(N-2) = 5.40$]" (p. 280). In other words, because Maria assimilated the task with three levels of units, she was able to relate the value of a single coin, the number of each type of coin, and the value of all the coins in a single equation. Also, because the equation was material for further operating she could substitute expressions for D and Q in terms of N, which Ben was unable to do.

Again, Olive and Çaglayan (2008) did not analyze Maria's reasoning in terms of the number sequences, but because she assimilates tasks with three levels of units, it is reasonable to assume that she had constructed a GNS (Ulrich, 2016a). So a GNS student solved the coin problem "with ease," and an ENS student constructed the three level unit structure in activity which allowed him to write the first equation but not the second. Olive and Çaglayan (2008) did not include any TNS students in their analysis. Because of TNS students' abilities to only construct two levels of units in activity, it seems clear that Olive and Çaglayan chose not to include these students because they would be unlikely to make much, if any, meaningful progress on the coin problem. However, aTNS students are now known to assimilate tasks with a composite unit (Ulrich, 2016b), similar to Ben. Thus, it stands to reason that aTNS students would potentially be able to represent the first equation algebraically. However, due to aTNS students' inability to disembed (Ulrich, 2016b), I suggest that they would not be able to write the literal equations that relate the number of dimes and quarters to the number of nickels. Therefore, the algebraic reasoning of aTNS students on problems similar to the coin problem requires further consideration.

Difficulties reasoning algebraically. The previous sections have discussed literature that defines algebraic reasoning, that identifies the importance of particular components of algebraic reasoning, including equality, variable, and writing equations, and how students' schemes and operations have been used to model students' algebraic reasoning in extant literature. Although several of these discussions throughout the literature review incorporated students' difficulties with one particular aspect of algebraic reasoning, there is also literature that discusses students' difficulty with algebraic reasoning as a whole. Stacey and MacGregor (1997) found that students' difficulties can generally be included in five categories: (1) seeing the operation and not just the answer to a problem, (2) conceptualizing the equal sign as operational rather than relational, (3) understanding the properties of numbers, (4) adapting to the use of all numbers, and (5) working on tasks that do not necessarily involve a practical context.

Seeing the answer and not the operation, for example, was particularly problematic for students (Stacey & MacGregor, 1997). In their research, 14-year-old students were asked to identify a pattern and to represent it algebraically. Most could identify numerical values within the pattern, three-fourths of the students could describe the relationship in words, but only half were able to represent the pattern algebraically. Stacey and MacGregor conclude that representing such patterns algebraically is difficult because students tend to see the answer, not the operation. Furthermore, in arithmetic, most problems can be solved using more than one operation (e.g., subtraction can be solved by adding up). In algebra, however, this is generally not the case (Stacey & MacGregor, 1997). This difficulty was also identified to be a part of students' difficulty working without a practical context. In arithmetic, students may be able to reason about whether to subtract five from 25 or 25 from five based on the context of the problem, and an understanding of whether the answer should be positive or negative; in algebra, the same type of context is unhelpful (Stacey & MacGregor, 1997). In general, the difficulties with algebra noted by Stacey and MacGregor incorporate some of the components of algebraic reasoning that have been previously discussed, but additionally bring to light additional ways in which many students struggle to reason algebraically.

Yerushalmy (2006) has also studied students' difficulties reasoning algebraically. His research used a graphing software intervention with students in grades seven through nine to study the effects of the intervention on students' use of algebraic notation. He concluded that struggling algebra students in grade seven were still struggling in grade nine because they did not engage in the use of algebraic notation fluently until two-thirds of the way through their ninth-grade year. In contrast, the highest achieving students used algebraic notation across all three years of the study (Yerushalmy, 2006). This is one piece of evidence that suggests the difficulty some students have in applying algebraic notation. Furthermore, this research suggests that some groups of students tend to struggle algebraically, despite interventions in their learning.

Purpose of the Research Study

Researchers have used units coordination and multiplicative concepts as explanatory tools for students' algebraic reasoning, but these constructs do not align with students' construction of an aTNS. This leaves an opening to examine the differences between the underlying mental structures of TNS, aTNS, and ENS students, and how these differing structures impact their algebraic reasoning. Furthermore, students' constructions of more sophisticated multiplicative concepts (Hackenberg, 2013; Hackenberg & Lee, 2015; Hackenberg et al., 2017) have been shown to influence their algebraic reasoning, which constitutes a large portion of the typical middle and high school mathematics curriculum. Therefore, an examination of whether students' schemes are dependent upon their grade level will provide insight into whether their constructions of schemes is being supported by existing curricula. Consequently, the purpose of this research study is to examine the algebraic reasoning of aTNS students, specifically as it compares to that of TNS and ENS students, and additionally, to analyze whether students' construction of the aTNS is dependent upon their grade level. To address these purposes, the research questions are:

1. How do aTNS students reason algebraically?
2. How is the algebraic reasoning of aTNS students constrained compared to that of ENS students, or advantaged over that of TNS students?
3. What is the relationship between the numbers of students who have constructed each number sequence and their grade levels, from grades six through nine?

Chapter 3: Methodology

To address the dual purpose of this research study, a mixed methodology was used. According to Creswell and Plano Clark (2007), mixed methods are best applied when "the use of quantitative and qualitative approaches, in combination, provides better understanding of research problems than either approach alone" (p. 5). That is to say that the qualitative and quantitative strands of the research study should seek to achieve complementarity, which refers to the use of qualitative and quantitative analysis to create a deeper understanding of each strand that would not be possible without the combination (Greene, Caracelli, & Graham, 1989). The present research study utilized a sequential (Creswell & Plano Clark, 2011) design, and engaged in mixing to achieve complementarity (Greene et al., 1989) with qualitative priority (Morse, 2003).

Rationale for Mixed Methods

Creamer (2017) states that the use of mixed methods can include *mixing* in up to four phases of the research study—research design, data collection, data analysis, and discussion, where mixing is "the linking, merging, or embedding of qualitative and quantitative strands" (p. 5), and higher levels of mixing are understood to maximize the potential of the mixed methods approach to research (Woolley, 2009). In the present research study, the mixing of qualitative and quantitative methods allowed for a fully integrated research design (Creamer, 2017), which implies mixing was accomplished in all four possible phases of the research study. Mixing was accomplished in the research design by the inclusion of a purpose statement and research questions featuring aspects of both qualitative and quantitative reasoning. In data collection, mixing was achieved through *nested sampling*, which indicates the sample in one strand of the research study was a subset of the other (Creamer, 2017). Mixing was accomplished in the analysis and the discussion by the simultaneous analysis of qualitative and quantitative data, which led to *meta-inference*, which is an inference that will "compare and contrast, infuse, link, [and] modify … sets of inferences generated by the two strands of the study" (Teddlie & Tashakkori, 2009, p. 300).

In phase one of data collection, survey data was coded and analyzed with the purpose of attributing to students a number sequence. The results of this analysis and subsequent number sequence attribution were used for two purposes. First, to conduct nested sampling in the second phase of data collection. That is to say, a subset of students surveyed in phase one were selected to participate in phase two based on three criteria, one of which was the number sequence attributed to them through analysis of the survey data. The second purpose was to conduct a statistical analysis of the survey data. By mixing in the data collection, additional mixing was possible in the data analysis and discussion.

Mixing in the data analysis was accomplished by the simultaneous examination of qualitative interview results and quantitative statistical analysis. Such analysis of the qualitative and quantitative results in tandem allowed for the construction of a meta-inference in the discussion. Considering both the changes in the number of students with each number sequence across the middle grades and the strengths and limitations of algebraic reasoning of students with each number sequence allowed for conclusions that built upon both quantitative and qualitative results; tying together the results of studying the changes in the numbers of students to the qualitative results of their algebraic reasoning allowed for a better understanding of the mathematical growth of these students. This exemplifies the inclusion of meta-inference in the research study, and completes the justification of a fully integrated mixed methods research study.

Mixed methods is appropriate as a research methodology when the linking of qualitative and quantitative strands strengthen the study's design, data collection, analysis, or discussion. The present research study mixed in all four of these phases, constituting a fully integrated mixed methods study, by including a mixed purpose, using nested sampling, simultaneously analyzing both strands of data, and constructing meta-inference. As Woolley (2009) writes, the mixing of qualitative and quantitative strands within one research study should seek to "be mutually illuminating, thereby producing findings that are greater than the sum of the parts" (p. 7). In this research study, by linking the results of quantitative and qualitative data, the understanding of middle grades students' algebraic reasoning is strengthened.

Phase One

In the first phase of this research study, survey data was collected. The purpose of the survey was to attribute to each subject a number sequence (Steffe & Cobb, 1988; Ulrich, 2016b). The reason for identifying students' number sequences was two-fold. The first reason was to inform the selection of participants in phase two of the research study. The second reason was to understand how the percentage of students with each number sequence differs across grades six through nine.

Participants. The participants were selected from the middle and high schools in a small town in the rural southeastern United States. The population of the town is approximately 8,100 people (United States Census Bureau, 2016). In the semester that data was collected for this research study, there were 134 sixth graders, 116 seventh graders, 117 eighth graders, and 106 ninth graders. Students in grades six through eight attend the middle school and students in grades nine through twelve attend the high school. At the high school, 93% of students earned a score of proficient on the 2016-2017 standardized state tests, and at the middle school, 87% were proficient; the schools are both fully accredited (Virginia Department of Education [VDOE], 2016). The student population at the high school is 80.6% white, 12.8% black, and 3.3% Hispanic, and at the middle school is 84.1% white, 9.8% black, and 2.1% Hispanic. Less than one percent of the students at each school are English Language Learners. 41.3% of students at the high school are considered to be economically disadvantaged and 43.6% of students at the middle school are economically disadvantaged. This includes students who receive free/reduced meals, receive temporary assistance for needy families, qualifies for Medicaid, or are experiencing homelessness (VDOE, 2016).

For the first phase of this research study, all students enrolled in grades six through nine who were enrolled in a math class during the semester in which data was collected were considered eligible to participate. All eligible students completed the survey during their normally scheduled math class, and each student was given the choice as to whether or not their survey would be used as part of the research study. A large number of students were sampled in phase one because it was expected that in middle and high school there would be small numbers of students who had constructed only an INS or a TNS; by surveying all students in grades six through nine, it was more likely that students with these number sequences would be identified.

Of the students surveyed, 100 sixth graders, 92 seventh graders, 74 eighth graders, and 60 ninth graders gave assent to have their surveys scored for use in the research study, for a total of 326 participants (Table 3.1). This represents approximately 75% of sixth graders, 79% of seventh graders, and 63% of eighth graders. There were additionally three students in sixth grade, 11 students in seventh grade, and eight students in eighth grade who did not give assent for their surveys to be scored as a part of the research study. The remaining 31 students in sixth grade, 13 students in seventh grade, and 35 students in eighth grade were not surveyed due to absence.

The ninth-grade class in this school district had a total of 106 students; 60 of those students were surveyed and gave assent for their surveys to be used as a part of the research study. This represents approximately 57% of the ninth-grade population. Five students in ninth grade were

Table 3.1. Total numbers of student participants in phase one of data collection across grades six through nine.

	# (% of grade) of student subjects	# (% of grade) of students who did not give assent to participate	# (% of grade) of students who were not surveyed*	# (%) totals
Grade 6	100 (74.6%)	3 (2.2%)	31 (23.1%)	134 (100%)
Grade 7	92 (79.3%)	11 (9.5%)	13 (11.2%)	116 (100%)
Grade 8	74 (63.2%)	8 (6.8%)	35 (29.9%)	117 (100%)
Grade 9	60 (56.6%)	5 (4.7%)	41 (38.7%)	106 (100%)
Totals	326 (68.9%)	27 (5.7%)	120 (25.4%)	473 (100%)

*Students not surveyed in grades six, seven, and eight were absent from school. Students not surveyed in grade nine were either absent from school or not enrolled in a math class during the spring semester in which data collection took place.

surveyed but did not give assent for their surveys to be scored as a part of the research study. This represents approximately 5% of the ninth-grade population. The proportion of ninth-grade students who completed the survey was lower than that in the middle school due to the high school's semester block schedule. Classes at the high school are scheduled either during the fall or the spring semester, but not both. Therefore, there are four groups of ninth graders, with regard to their math classes (Table 3.2).

Group A did not take Algebra 1 in middle school, or are repeating Algebra 1 as a ninth grader. These students took Algebra 1 Part 1 in the fall of ninth grade and Algebra 1 Part 2 in the spring of ninth grade when data was collected for this research study. Twenty students from this group participated in the research study, and four completed the survey but did not give assent to have their surveys scored for the research study. Academically speaking, students in group A are likely the typically lowest achieving math students in ninth grade. Group B completed Algebra 1 in middle school, completed Geometry in the fall of ninth grade, and were enrolled in Algebra 2 during the spring of ninth grade when data was collected for this research study. Twenty-five

Table 3.2. Four groups of students based on their math class enrollment in ninth grade.

Groups by Math Class Enrollment	Grade Level in which Algebra 1 was Taken	Math Class Enrollment in Fall 2017	Math Class Enrollment in Spring 2018	# Participants in Group (% of 9th Grade Participants)	# Students in Group (% of 9th Grade)
Group A	9th grade	Algebra 1 Part 1	Algebra 1 Part 2	20 (33.3%)	30 (28.3%)
Group B	8th grade	Geometry	Algebra 2	25 (41.7%)	19 (17.9%)
Group C	8th grade	None	Geometry	15 (25.0%)	27 (25.5%)
Group D	8th grade	Geometry	None	0 (0.0%)	30 (28.3%)
Totals	--	--	--	60 (100%)	106 (100%)

students from this group participated in the research study, and none completed the survey but did not give assent to have their surveys scored for the research study. Academically speaking, students in group B are likely the typically highest achieving math students in ninth grade. Group C completed Algebra 1 in middle school, did not enroll in a math class in the fall of ninth grade, and were enrolled in Geometry during the spring of ninth grade when data was collected for this research study. Fifteen students from this group participated in the research study, and one completed the survey but did not give assent to have their surveys scored for the research study. Finally, group D completed Algebra 1 in middle school, completed Geometry in the fall of ninth grade, and were not enrolled in a math class in the spring of ninth grade when the data for this research study was collected. The school reported that there were 30 ninth graders in this group. Students in groups C and D are likely the typically average achieving math students in ninth grade. Thus, a limitation of this research study is that students from group D, and likely typically average achieving math students, were under represented in the sample.

Survey description. The survey (Ulrich & Wilkins, 2017) used for data collection in phase one assesses students' units coordination and construction, and ultimately, was used to attribute to each student a number sequence. The survey consists of 25 questions, is given in two parts, and is expected to require at most 50 minutes to complete. Survey questions and an initial coding guide were developed based on existing theory related to students' units coordination and construction (Ulrich & Wilkins, 2015). Following the initial survey design, each task on the survey was evaluated by an expert in mathematics education. Together, this suggests both face and content validity. Furthermore, predictive validity was established by verifying the number sequences assigned through survey analysis with the number sequences assigned through clinical interview analysis. The comparison of number sequences assigned by these two methods was conducted with nine students, and a moderately strong relationship was found between the two classification systems (Ulrich & Wilkins, 2017).

Through a quantitative analysis of the survey data, students were grouped into four, ordinal categories: Potentially pre-numerical or INS students, TNS students, aTNS students, and ENS or potentially GNS students. Ulrich and Wilkins (2017) refer to these categories as students' *stage classifications*, and they refer to the formula used to determine the stage classifications as the *scoring rubric*. Note that although there are five number sequences in total, the survey does not distinguish between students' construction of an ENS and a GNS.

In scoring the survey, the written work on each task is determined to give evidence of either an indication or a contraindication of a particular number sequence (Ulrich & Wilkins, 2015). The indications and contraindications were based on prior research. Indications are assigned positive values and contraindications are assigned negative values, and both indications and contraindications are measured as being strong (+/- 1), moderate (+/- 0.6), or weak (+/- 0.3). Ulrich and Wilkins (2017) explain how these scores can be applied through one example in which a student had three weak contraindications to having constructed an aTNS (each scored as -0.3). However, the same student's survey was determined to have one weak indication (+0.3), one moderate indication (+0.6), and one strong indication (+1) of having constructed an aTNS. Thus, the indications of having constructed an aTNS outweighed the weak contraindications, resulting in the example student's stage classification being an aTNS.

The weighted values providing indications of or contraindications of each number sequence are inputted into the scoring rubric (Ulrich & Wilkins, 2015). The stage classification is understood to be a "lower bound" (Ulrich & Wilkins, 2017, p. 7) of students' number sequences because the researchers' interpretations of indications and contraindications are limited to students' written work. For instance, a student who has constructed an ENS can solve problems in a manner that is consistent with that of an aTNS student. Although this manner of solving problems is less sophisticated than the ENS student's full capability, it is entirely possible, and this may lead to the attribution of an aTNS to this student. This is not measurement error – the ENS student *has* constructed the mental structures of an aTNS (namely, the ability to assimilate tasks with a composite unit (Ulrich, 2016b). However, the ENS student can also iterate units of one and disembed composite units (Steffe, 2010b). On the other hand, because the scoring rubric gives a lower bound for students' stage classifications, it is highly unlikely that an aTNS student will be attributed an ENS. In other words, because the aTNS student has not constructed the iterating and disembedding operations, they are not able to solve tasks in a manner consistent with ENS students even though an ENS student may solve tasks in a manner consistent with an aTNS student.

Quantitative data collection. The survey (Ulrich & Wilkins, 2017) was administered in one middle school and one high school, and to all students who were enrolled in a math class in grades six through nine at the time of data collection. The survey was administered during the students' normal math class. In the middle school, quantitative data was collected over three days. In the high school, quantitative data was collected over two days. Students who were absent on their class's scheduled day for data collection did not complete the survey at a later date, so as to minimize class disruptions and missed instructional time. All classes at the middle and high school were 90 minutes long. As such, it was left up to the teacher's discretion whether or not students would be allowed to work on the survey for more than the anticipated 50 minute time period. At the middle school, the teachers allowed students to work on the survey for up to one hour; only two students did not complete their survey within the hour. At the high school, all students finished within the anticipated 50 minutes.

Interrater reliability. Following data collection, surveys were grouped by grade and all students who did not give assent for their surveys to be included in the research study were removed. A random selection of roughly 10% of surveys from each grade level were selected and scored by two researchers using the scoring codes and formula (Ulrich & Wilkins, 2015); the purpose of selecting and scoring these surveys first was to calibrate the use of the scoring rubric and ensure accuracy and reliability in the scoring of the surveys. This resulted in the scoring of 10 surveys from sixth grade, nine surveys from seventh grade, seven surveys from eighth grade, and six surveys from ninth grade. After each researcher independently scored the randomly selected surveys from one grade level, they met to discuss the scoring code assigned to each item on each survey. All disagreements were reconciled.

Next, the researchers independently scored the remaining surveys. After independently scoring all surveys from one grade level, the researchers met to discuss the scoring codes assigned to each item on any surveys for which they disagreed on the stage classification. The researchers disagreed on the stage classifications for five students in sixth grade, eight students in seventh grade, seven students in eighth grade, and three students in ninth grade, for a total of 23 disagreements (Table 3.3).

In addition to calculating percent agreements and disagreements, weighted Kappa (κ_{ω}; Cohen, 1968) scores were also calculated (Table 3.3). Cohen's Kappa (κ; Cohen, 1960) is

Table 3.3. Interrater Reliability

	# Stage Classification Discrepancies (% out of Grade)	# Stage Classification Agreements (% out of Grade)	Weighted Kappa Score
Grade 6	5 (5.0%)	95 (95.0%)	0.922
Grade 7	8 (8.8%)	83 (91.2%)	0.870
Grade 8	7 (9.6%)	66 (90.4%)	0.880
Grade 9	3 (5.0%)	57 (95.0%)	0.938
Totals	23 (7.1%)	301 (92.9%)	0.906

appropriate to evaluate the interrater reliability of categorical data because it takes into account not only the percent of times that raters agree or disagree on items, but also the amount of agreement that results from chance. κ_ω, on the other hand, is a measure of interrater reliability that is appropriate with ordinal data because in addition to taking into account agreement, disagreement, and agreement by chance, it additionally considers the level of disagreement between raters (Cohen, 1968). Cohen (1968) refers to κ_ω as a "chance-corrected proportion of weighted agreement" (p. 215). Interpreted in terms of the present research study, if one researcher attributes a TNS to a student and the other attributes an aTNS, this is only one degree of disagreement because the two stage classifications are contiguous. However, if one researcher were to attribute an INS to a student and the other attributes an ENS, this is three degrees of disagreement. Accordingly, in calculating κ_ω, one degree of disagreement is weighted less heavily than are two or three degrees of disagreement.

$\kappa_\omega$scores for each individual grade level were 0.922, 0.870, 0.880, and 0.938 for sixth, seventh, eighth, and ninth grades, respectively. According to Landis and Koch (1977), $\kappa_\omega$scores between 0.81 and 1.00 are considered "almost perfect" agreement (p. 165). As such, the $\kappa_\omega$scores for all individual grade levels are in almost perfect agreement. For the overall sample including grades sixth through nine, the κ_{ω}statistic was 0.906, which is also indicative of almost perfect agreement (Landis & Koch, 1977).

Quantitative analysis. To address the research question, how does the number of students who have constructed an aTNS change across the middle grades, the quantitative data were analyzed using a chi-square test of independence (Howell, 2013). The purpose of this test was to identify the existence of a possible number sequence by grade interaction (Siegel & Castellan, 1988). Thus, the null hypothesis under examination was that the number of students who have constructed each number sequence does not vary as a function of grade, and the alternative hypothesis is that the number of students who have constructed each number sequence does vary as a function of grade.

The chi-square test is appropriate for the analysis of ordinal data (Howell, 2013), however, its appropriateness relies upon two underlying assumptions. The first of these is the independence of observations (Howell, 2013), which implies that no two observations are related. In the present research study, this criteria is satisfied because each survey and subsequent number sequence attribution is not related to or affected by the others.

The second assumption of the chi-square test is that of a sufficiently large sample size (Agresti, 2002). Although there is no hard and fast rule for defining the necessary sample size, the validity of the test is understood to be compromised when more than 20% of the cells have an expected frequency of less than five or if any cells are zero (Siegel & Castellan, 1988). The sample of students in the present research study, however, did not violate the assumption of a sufficiently large sample (Table 3.4). There were fewer than five TNS students in eighth grade, and there were fewer than five INS students in ninth grade. However, this is only two cells out of the total 16 cells, or 12.5%; thus, a chi-square test of independence is an appropriate test.

Table 3.4. Stage Classification by Grade

	Stage Classification				
	INS	TNS	aTNS	ENS+	Total
Grade 6	10	7	54	29	100
Grade 7	5	6	51	30	92
Grade 8	5	2	26	41	74
Grade 9	4	5	21	30	60
Total	24	20	152	130	326

Following the chi-square test of independence, the Goodman and Kruskal's gamma statistic (G) was also calculated. Gamma is a non-parametric test that indicates the strength of an association between two ordinal variables (Siegel & Castellan, 1988), making it appropriate for examining the potential association between students' grade and stage classification. The assumptions underlying the use of the gamma statistic are that the variables are ordinal, and that there is at least a weak monotonic relationship between the variables (Siegel & Castellan, 1988); these criteria are met by the variables used to measure students' number sequences and grade level.

With a statistically significant result to the omnibus chi-square test, thus leading to the rejection of the null hypothesis, *post hoc* testing was also necessary to further examine the dependency of students' number sequences on their grade level. Thompson (1988) indicates that subjectively evaluating the cell frequencies when an omnibus chi-square statistic is statistically significant to determine which frequencies may have contributed to the significant result is inappropriate. To avoid this error in analysis, *post hoc* testing was conducted in the form of odds ratios. The odds ratios were calculated between each grade level and stage classification, and 95% confidence intervals were calculated to determine between which grade levels and number sequences there was a statistically significant difference in the number of students.

Reporting odds ratios is comparable to reporting effect sizes, in that it facilitates the interpretation of the magnitude of the results (Keith, 2015). By calculating the odds ratios for having constructed each number sequence from one grade to another, it was possible to compare the probabilities of having constructed each number sequence at each grade. The odds ratios were calculated in a binary fashion by calculating the odds of having constructed an ENS in each grade, compared to the odds of having not yet constructed an ENS (i.e., having constructed only an INS, TNS, or aTNS). This comparison was made across all grade levels. Similarly, the odds ratios were calculated for having constructed an aTNS in each grade compared to the odds of having not yet constructed an aTNS (i.e., having constructed only an INS or TNS).

Limitations

One limitation of this research study is its cross-sectional nature. Although this is not a violation of the underlying assumptions of the chi-square test, it presents limitations in terms of the conclusions that can be drawn when compared to the conclusions afforded by a longitudinal study. The chi-square test examines the dependence of students' number sequences on their grade level; this is in comparison to measuring the effect of students' grade level on their number sequence construction. The latter is available only by following individual students over an extended period of time (Singer & Willett, 2003). While collecting data at only one point in time will potentially limit the conclusions of the research study, it is recognized to be beyond the scope of the project, and is considered as an area for future research.

Phase Two

In the second phase of the study, data was collected through semi-structured clinical interviews (Clement, 2000). Participants for the interviews were selected by nested sampling, in which a subset of a larger group is used in the second phase of a research study (Creamer, 2017).

Students were selected to participate in qualitative interviews based on three criteria. First, all students who elected to participate in phase one survey were asked to participate. Interview participants were selected from the students who returned both a parent permission form and a student assent form indicating their willingness to be interviewed. Then, from the returned permission forms, students were selected across all four grades. Finally, as much as possible, students from different number sequence attributions were selected; these included students who had constructed a TNS, an aTNS, and at least an ENS. The reason for taking a sample of students from each number sequence in each grade was to represent the algebraic reasoning of students across all of the middle grades. This distribution of students begins to paint a picture of the algebraic reasoning of students with varying levels of exposure to an algebra curriculum. Also, students who were determined to have constructed at most an INS were not selected for interviews because their algebraic reasoning does not pertain to the research study's purpose of examining how the algebraic reasoning of aTNS students compares to that of TNS and ENS students.

The ideal distribution of students for interviews would have been one TNS student from each grade[8], two aTNS students from each grade, and one ENS student from each grade. However, these distributions of students varied slightly based upon the results of the survey, and the parents' and students' willingness to participate in the interview. The interview participants for phase two are described briefly in Table 3.5. In sixth grade, two students were selected with a stage classification of aTNS and two with a stage classification of ENS. Two additional students were selected from sixth grade who had been attributed an aTNS based on the results of the survey, but there was reason to believe that these students had potentially only constructed a

[8] Note that grade is being used as a proxy for age in this research study.

TNS. Thus, because no sixth grade students who were attributed a TNS returned permission forms, these two additional students were interviewed (Tabitha and Theresa). In seventh and eighth grades, no students who were attributed a TNS returned permission forms. Thus, three students who had constructed an aTNS and one student who had constructed at least an ENS were interviewed in seventh grade, and one with an aTNS and three with at least an ENS in eighth grade. In ninth grade, one student who had constructed a TNS, one student who had constructed an aTNS, and two students who had constructed at least an ENS were interviewed; this distribution was again, based on limitations due to the number of permission forms returned.

Table 3.5. Summary of students selected to participate in phase two clinical interviews.

	TNS	aTNS	ENS	GNS	Total
Grade 6	*Math 6*: Tabitha*	*Math 6*: Abby Aaron Ann	*Math 6*: Elle Evan	--	6
Grade 7	--	*Math 7*: Ava Alyssa Andy	--	*Pre-Algebra*: Greg**	4
Grade 8	--	*Algebra 1*: Amanda	*Pre-Algebra*: Emily *Algebra 1*: Erin	*Algebra 1*: Gavin**	4
Grade 9	*Algebra 1, Part 2*: Travis	*Algebra 1, Part 2:* Alex	*Algebra 1, Part 2*: Elizabeth *Algebra 2*: Emma	--	4
Total	2	8	6	2	18

* Tabitha was attributed an aTNS based on the results of the survey, but based on her interview, it was determined that she had more likely constructed only a TNS. Thus, she is classified as a TNS student for the qualitative portion of the research study. Evidence of Abbie having constructed only a TNS is given in the results.

**Greg and Gavin were attributed at least an ENS based on the results of the survey, but based on their interviews, it was determined that they had more likely constructed a GNS. Thus, they are classified as GNS students for the qualitative portion of the research study. Evidence of their construction of a GNS is given in the results.

Qualitative data collection. Semi-structured clinical interviews were used to collect qualitative data in the second phase of this research study. Clinical interviews were initially used in research by Piaget to examine the structure of children's knowledge and the means by which they reason. From this initial use, they have developed to include two distinct approaches that are applicable in a variety of fields of research (Clement, 2000). According to Clement (2000), the clinical interview can collect data through various methods, including open-ended questioning or think-aloud problem solving interviews. Because of the diversity of methods, the clinical interview can be widely applied to a breadth of research questions that seek to understand student thinking at a single point in time, or over a more extended period of time.

Students were interviewed on two days, for approximately 30 to 40 minutes on each day, with the exception of one student. Emma was only interviewed once due to several weather related school closings. The eighth-grade students had the shortest amount of time available for interviews because they lose part of a period to ride the bus to the high school for elective classes. The interview began with number sequence screening tasks, the purpose of which was to confirm students' number sequence attribution from the phase one survey and also to determine the level with which students conceptualize the equal sign on tasks that were not algebraic in nature. The number sequence screening tasks were taken directly from the phase one survey (Appendix A). The questions assessing students' concept of the equal sign were adapted based on Mathews and colleagues' (2012) framework for students' concepts of the equal sign (Appendix A). Eleven algebra tasks were selected or adapted from existing research, or were designed for the interviews (Appendix B), although no students answered all eleven tasks. The order of tasks in the interviews and rationale for inclusion in the interview is discussed below.

Number sequence screening tasks. Ulrich and Wilkins (2017) found that the majority of ENS students could solve bar tasks four and five (Appendix A), but that TNS and aTNS students generally could not. Furthermore, the majority of aTNS and ENS students could solve bar tasks two and three, but TNS students generally could not. The bar tasks were used to begin each students' interview; they were presented in the order of the flow chart shown in Figure 3.1.

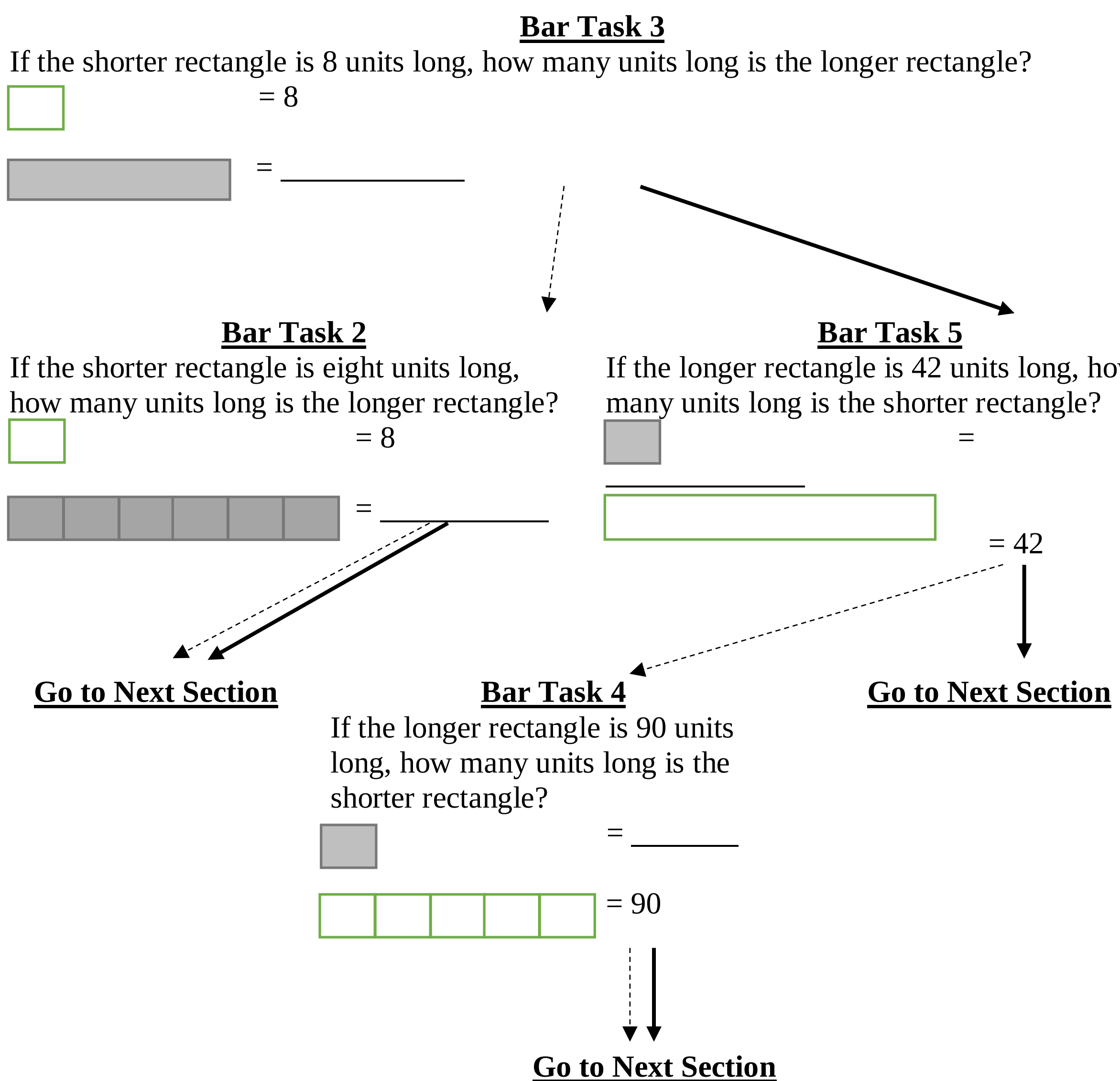


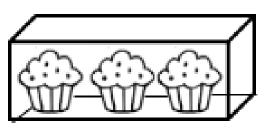
Figure 3.1 Number sequence screening tasks began with bar task 3. Bold arrows indicate the

next question asked if students responded correctly. Dashed arrows indicate the next question asked if students responded incorrectly.

Following the bar tasks, students were asked either cupcake task A, B, or both (Figure 3.2). Students who were successful on at least bar task 3 were given cupcake task B, first. If students made a reasonable attempt at cupcake task B, then they moved on to the next section. If students were not successful on at least bar task 3, or if they struggled to make meaningful progress on cupcake task B, then they were asked cupcake task A. Based on the results of Ulrich and Wilkins' (2017) research, cupcake task A was accessible to students who had constructed only a TNS, and to students who had constructed more sophisticated number sequences. Moreover, Ulrich and Wilkins give suggestions for how students who have constructed each number sequence are likely to complete the task. Thus, it was used as a number sequence screening task due to its accessibility to all students and the ability to analyze students' use of a composite unit using their drawings and solution methods.

Cupcake Task A

You have baked 39 cupcakes and you will put the cupcakes in boxes of three. How many boxes will you fill?



Cupcake Task B

There are 3 rows of 6 cupcakes that are unboxed. If there are 9 rows of cupcakes in all, how many cupcakes are hidden in the box?

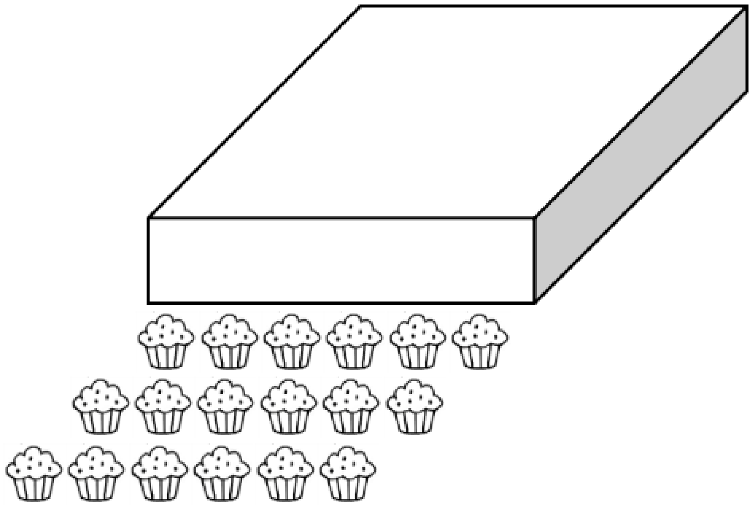


Figure 3.2. Cupcakes tasks A and B were used as screening tasks in the interviews (from Ulrich and Wilkins, 2017).

After the cupcake tasks, students were asked to complete either splitting task A, B, or both (Figure 3.3). Students who were either not given cupcake task B or who did not make meaningful progress on cupcake task B, were given splitting task A. Students who made meaningful progress on cupcake task B were given splitting task B. If students struggled on splitting task B, or if the interviewer was not able to discern the method by which students were attempting to solve splitting task B, then students were also given splitting task A to allow the interviewer to ask more questions about their thinking. The purpose of incorporating the splitting tasks was to screen for students who may be solving splitting tasks by sequentially applying the partitioning and iterating operations, as opposed to students who were splitting by simultaneously applying the partitioning and iterating operation. Also, the phone cords problem and the modified splitting problem (Appendix B) incorporate a contextualized splitting task; thus, including two de-contextualized splitting tasks in the number sequence screening portion of the interview provided a point of comparison.

Splitting Task A

The stick shown below is 4 times as long as another stick. Draw the other stick.

Splitting Task B

The stick shown below is 6 times as long as another stick. Draw the other stick.

Figure 3.3. Splitting tasks A and B were used as screening tasks in the interviews (from Ulrich and Wilkins, 2017).

Equality screening tasks. Following the number sequence screening tasks, students were asked a series of questions designed to determine the level of sophistication with which students conceptualize the equal sign (Figure 3.4). According to Mathews and his colleagues (2012), there is a hierarchy of concepts of the equal sign, which include rigid operational (level 1), flexible operational (level 2), basic relational (level 3), and comparative relational (level 4). All students were first asked to respond to task L3, which was modeled from Mathews et al.'s assessment. If students determined that the equation was false, and furthermore explained that the value on each side of the equal sign must be the same for the equation to be true, it was decided that they could at least conceptualize the equal sign at level 3; these students then progressed to task L4. If students did not calculate the values of the expressions on either side of the equal sign on task L3 to determine it was false, or could not explain that both sides of the equation need to be the same in order for the equation to be true, it was decided that they did not conceptualize the equal sign at level 3; these students then progressed to task L2. Tasks L2 and L4 were also based on the work of Mathews and his colleagues (2012), and were used to assess whether students could conceptualize the equal sign at the flexible operational or comparative relational levels, respectively.

Equality Task L2

Can you tell if the equation is true or false?

17 = 53 - 36

Equality Task L3

Can you tell if the equation is true or false?

37 + 29 - 5 = 48 + 14

Equality Task L4

Without adding, can you tell if the equation is true or false?

67 + 86 = 68 + 85

Figure 3.4. Equality tasks were used as screening tasks in the interviews (modified from Matthews et al., 2012).

Algebra tasks. The algebra tasks were presented to students, one at a time, on a full sheet of paper, and were read aloud by the interviewer. Each task consists of an initial prompt, which was given to all students verbally and in writing, as well as follow up questions that were designed to probe students' thinking, or to help students find an entry point to solving the task if they were struggling. The follow up questions were only given verbally to students, at the interviewer's discretion, based upon the students' progress completing the task. The first four tasks were selected or adapted from the research of Hackenberg (2013) and Hackenberg and Lee (2015), the fifth and sixth tasks were selected or adapted from the research of Olive and Çaglayan (2008), and tasks seven through ten were designed for the present research. Each of the prompts is explained below, and justification for selecting or adapting each task is given.

In previous research, Hackenberg (2013) and Hackenberg and Lee (2015) have studied how students' multiplicative concepts can explain their algebraic reasoning. Four tasks from their research will be included in the interview protocol for the present research study (Appendix B, tasks A1-A4). These tasks were selected based on Hackenberg's (2013) analysis of the underlying mental structures necessary to solve each task. The phone cords problem (A1) and the modified splitting problem (A2) are splitting tasks. These are important tasks for the present research study because splitting is within the ZPC of students who have constructed an ENS, and not attainable for TNS students (Ulrich, 2016a). However, aTNS students have been shown to solve splitting tasks in a way that does not require the disembedding operation, and is therefore qualitatively distinct from splitting (Ulrich, 2016b). With this understanding, it was interesting to consider how aTNS students might respond differently to representing a splitting task algebraically as compared to TNS and ENS students. The phone cords problem was given to all students in the interview. The modified splitting problem was given only to students who did not make meaningful progress on the phone cords problem. This decision was made to preserve time, but also because both are contextualized splitting tasks, however, the modified splitting problem uses smaller numbers and has a more intuitive relationship between the two proposed values. Thus, if students were successful on the phone cords problem, time was not spent on the modified splitting problem. Conversely, if students were not able to make meaningful progress on the phone cords problem, they were asked to attempt the modified splitting problem.

The recursive partitioning problem (Appendix B, A3) was modified from Hackenberg's research; this problem tests a student's recursive partitioning operation and asks them to apply algebraic reasoning to the result. While recursive partitioning is within the ZPC of MC2 students (Hackenberg & Tillema, 2009) who have likely constructed an ENS (Ulrich, 2016a), it has not yet been documented whether or not these will be within the ZPC of aTNS students. Furthermore, the manner in which aTNS students will reason algebraically about these tasks remains unclear. The recursive partitioning problem was given to the first four students who were interviewed, all of whom were ninth-grade students. Because of the large amount of time that this task required, and the minimal amount of progress that even ENS students made on representing the recursive partitioning problem algebraically, this task was only used with students in grades six through eight if time allowed at the conclusion of the second day of interviews.

The border problem is adapted from Hackenberg's (2013) research (Appendix B, A4). In this task, students are asked to generalize a pattern regarding the length of the perimeter of the square, and then to represent the perimeter algebraically. Hackenberg notes that while MC1 students were able to verbalize a pattern about the perimeter of a square given the side length, they were unable to represent it algebraically in terms of a square with side *x*. She attributes this difficulty to MC1 students' lack of the disembedding operation. As aTNS students have not constructed the disembedding operation, their algebraic reasoning on the border problem may be consistent with that of a TNS student, however, it must be considered that their ability to assimilate tasks with a composite unit advantages their reasoning in some way. This task was attempted by all students. The adaptation to this problem is based on Radford's (2011) framework for students' generalizing behavior, which includes generality by example. Radford indicates that if students engage in generality by example, then this is the onset of algebraic reasoning. As such, a prompt was added to Hackenberg's protocol for the border problem in which students were asked to find the number of squares on the border of a 100-by-100 grid.

In addition to drawing from Hackenberg's research with middle grades students, the coin problem from Olive and Çaglayan's (2008) research with eighth-grade students was also selected (Appendix B, A5). By Olive and Çaglayan's analysis, solving the coin problem algebraically requires three levels of units coordination—the value of a single coin, the number of each coin, and the total value of the coins. In their research, students who assimilated the task with two levels of units and constructed the third in activity had limitations; this should be consistent with the reasoning of ENS students and possibly aTNS students. Because the present research study is focused on the reasoning of aTNS students, and because aTNS students may not be able to solve this problem, the modified coin problem was also created (Appendix B, A6). By my own analysis, I anticipate that solving this task requires two levels of units coordination—the number of each coin and the total number of coins—which indicates that it may be accessible to TNS students. While there is some documentation of the algebraic reasoning of students on the coin problem, it remains to be seen whether aTNS students will be able to solve and represent algebraically the original coin problem, and how their algebraic reasoning on the modified coin problem compares to that of TNS students. During interviews, it was decided that students who had constructed only a TNS would not attempt the coin problem (A5) to avoid exceeding their threshold for frustration. Instead, the TNS students only attempted the modified coin problem (A6). There were also two aTNS students, Samantha and Edward, who did not have time to complete the coin problem on day one, and after observing their attempts at the modified coin problem, the coin problem was skipped all together to avoid exceeding their threshold for frustration.

The remaining tasks, A7 through A11 (Appendix B), were designed for this research study. The block pattern problems (A7 and A8) were designed to elicit students' abilities to generalize and abstract recursive and explicit definitions. The patterns in each of these tasks are linear in nature, and purposefully included a rate of increase of one block per figure so as to avoid any complications that may have prohibited the algebraic abstraction of the pattern by students who did not reason multiplicatively or could not nest algebraic expressions. These tasks were given to all students, except Emma, who was never able to complete her second interview. The football problem (A9), the soccer problem (A10), and the substitute problem (A11) were designed to understand students' abilities and limitations in writing equations that include nested quantities. For example, the football problem requires students to write an equation that nests the number of touchdowns scored within the number of points scored. The soccer problem was designed as a foil to the football problem. It is a nearly identical situation, the distinction being that the soccer problem does not involve a nested quantity. The football problem was eliminated from the TNS students' interviews to avoid exceeding their perceived threshold of frustration; three other students also did not complete the football problem due to time constraints: Emma, Alex, and Andy. The soccer problem was given to all students, except Emma. The substitute problem was only given to students who had time during their second interview.

Qualitative Analysis. The analysis of data collected in clinical interviews can be either generative or convergent, and while these two methods of analysis serve distinct purposes, they are complementary (Clement, 2000). In the convergent approach, data analysis is conducted by coders who use a very specific coding scheme to analyze units as small as sentence clauses (Clement, 2000). This approach to data analysis is appropriate when there is an existing theory that can be applied to understanding the phenomenon under examination. The second method of data analysis in clinical interviews is the generative approach, in which transcripts of interviews are analyzed by a single researcher who focuses on interpreting large chunks of data (Clement, 2000). Clement (2000) points out that the generative approach is more appropriate when no theory exists to frame the research; thus, researchers interpret a large chunk of transcript data without necessarily having a preexisting conception of what codes they may use, or how the participants' thinking and reasoning may fit into the big picture.

For the purposes of the present research study, data analysis was both convergent and generative. The results of tasks A1 through A6 were analyzed convergently for TNS and ENS students. Hackenberg (2013) has outlined the manner in which MC1 students reason algebraically, and Hackenberg and Lee (2015) have similarly outlined the manner in which MC2 students are capable of reasoning algebraically; because a TNS is within the ZPC of students who have constructed an MC1 and an ENS is within the ZPC of students who have constructed an MC2, convergent analysis was appropriate. Similarly on the coin problem, Olive and Çaglayan's (2008) research provided a foundation for convergent analysis of TNS and ENS students' reasoning. Alternatively, generative analysis was used to analyze the clinical interviews of the aTNS students because the algebraic reasoning of these students has not previously been studied, and it is therefore unknown in what ways their reasoning will be similar or dissimilar to that of TNS and ENS students. Furthermore, a generative approach was taken in the analysis of all students' interviews on tasks A7 through A11, as these tasks have not been utilized in previous research.

Regardless of the approach taken in the analysis of phase two interviews, all interviews were video and audio recorded. All videos were transcribed, and written work was digitized. In the first stage of qualitative analysis, each students' responses to the number sequence screening tasks were analyzed and compared to their phase one stage classification. Students' responses on the bar tasks and the cupcake tasks were compared to Ulrich and Wilkins' (2017) analyses of those tasks to determine whether it was reasonable to retain students' stage classification from the survey as their number sequence designation. Furthermore, students' responses to the splitting tasks in the interview were analyzed to determine whether they were or were not able to solve splitting tasks; this was also compared to their stage classification. Finally, for all students given a stage classification of ENS, which includes all ENS and GNS students, responses were analyzed for the existence of reasoning that might be more characteristic of a GNS than an ENS. Following the analysis of students' number sequence attribution, students' responses to the equality tasks were also analyzed to determine whether their concept of the equal sign, in decontextualized situations, was consistent with level 1, 2, 3, or 4.

Analysis of students' algebraic reasoning task responses was completed by task. In other words, task A1 was analyzed across all students, then task A2, *etcetera*, and patterns were studied within tasks by number sequence, grade level, and math class enrollment. For tasks in which convergent analysis was used, codes were determined *a priori*. For tasks in which generative analysis was used, *a priori* codes were used, as appropriate, and other codes were generated *a posteriori* (Appendix C).

Chapter 4: Results

The results section will first present the results of the quantitative data analysis, including exploratory data analysis, the chi-square test, the Gamma test, and *post hoc* odds ratios. These tests were used in conjunction with one another to understand the distribution of students in each grade level and who have constructed each number sequence. Then, the qualitative results are presented. The qualitative results are organized by the problems in the qualitative interviews, and are summarized to highlight similarities in students' algebraic reasoning by number sequence.

Exploratory Data Analysis

The quantitative phase of the research study included a survey of 326 students in grades six through nine. The purpose of the survey was to attribute stage classifications to each student, which include four categories of number sequences, including: Pre-numerical or INS, TNS, aTNS, and ENS or higher (Table 4.1). The results of the survey indicate that of 100 sixth grade students, 10% (10) have constructed at most an INS or are pre-numerical, 7% (7) have constructed at most a TNS, 54% (54) have constructed at most an aTNS, and 29% (29) have constructed at most an ENS or GNS. Of 92 seventh grade students, 5.4% (5) have constructed at most an INS or are pre-numerical, 6.5% (6) have constructed at most a TNS, 55.4% (51) have constructed at most an aTNS, and 32.6% (30) have constructed at most an ENS or a GNS. Of 74 eighth grade students, 6.8% (5) students have constructed at most an INS or are pre-numerical, 2.6% (2) have constructed at most a TNS, 35.1% (26) have constructed at most an aTNS, and 55.4% (41) have constructed at most an ENS or a GNS. Of 60 ninth grade students, 6.7% (4) have constructed at most an INS or are pre-numerical, 8.3% (5) have constructed at most a TNS, 35% (21) have constructed at most an aTNS, and 50% (30) have constructed at most an ENS or a GNS. The largest number and percentage of INS students were sixth graders; the largest number of TNS students were sixth graders, but the largest percentage of TNS students were eighth grade; the largest number of aTNS students were sixth graders, but the largest percentage of aTNS students were seventh graders; the largest number and percentage of ENS students were eighth graders.

Table 4.1. Results of the Survey

	Grade 6	Grade 7	Grade 8	Grade 9	Total
INS	10.0% (10)	5.4% (5)	6.8% (5)	6.7% (4)	7.4% (24)
TNS	7.0% (7)	6.5% (6)	2.6% (2)	8.3% (5)	6.1% (20)
aTNS	54.0% (54)	55.4% (51)	35.1% (26)	35.0% (21)	46.6% (152)
ENS	29.0% (29)	32.6% (30)	55.4% (41)	50.0% (30)	39.9% (130)
Total	100% (100)	100% (92)	100% (74)	100% (60)	100% (326)

In total, 46.6% of students surveyed across grades six through nine had constructed at most an aTNS. The percentage of students having constructed an aTNS is comparable from sixth to seventh grade, and from eighth to ninth grade, but the percentages decrease approximately 20% in the number of aTNS students from seventh to eighth grade (Figure 4.1). A complementary pattern emerges in the percentages of students who have constructed an ENS. The number of ENS students are similar in grades six and seven, and in grades eight and nine; however, from grade seven to eight there is an increase of approximately 23% in the number of ENS students. There are no such changes in the numbers of INS and TNS students surveyed. In other words, the numbers of students who had constructed only an INS or a TNS remained relatively equal across the middle grades.

Recall that 46 ninth-grade students did not participate in the study, 30 of whom did not participate because they were not enrolled in a math class in the semester of data collection (see Table 3.2). These students are presumed to be academically similar to the 15 ninth-grade students who were enrolled in Geometry during the semester of data collection. With this in mind, an extrapolation of the predicted percentages of students to have constructed each number sequence across the middle grades was calculated (Figure 4.1, dotted lines). Of the 15 ninth-

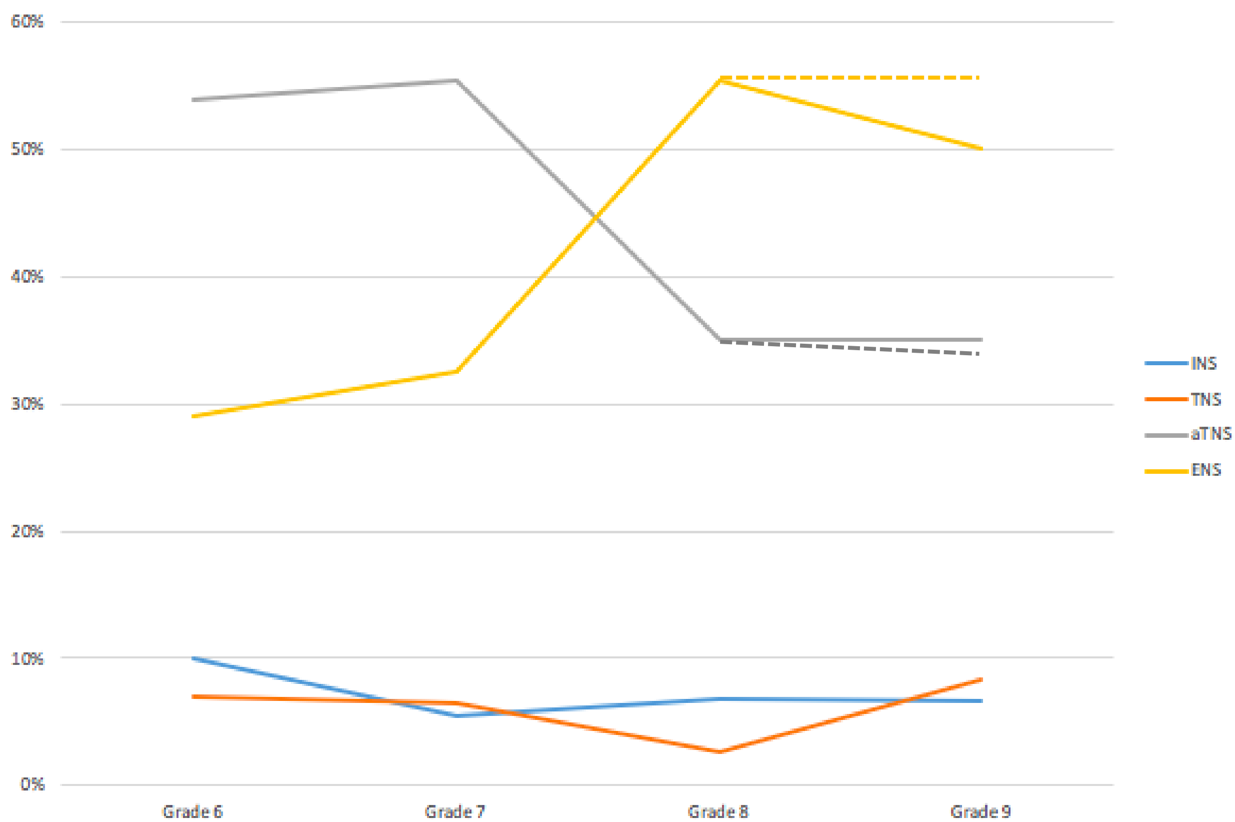


Figure 4.1. The percentage of aTNS students decreases from grade seven to eight, while the percentage of ENS students increases. There are not similar changes in the percentages of INS and TNS students. Dotted lines indicate predicted percentages for ninth grade.

grade Geometry students surveyed, 33.3% had constructed an aTNS, and 66.6% an ENS; these percentages were applied to the additional 30 ninth-graders not enrolled in a math class. As a result of this extrapolation, the percentage of ninth-grade aTNS students who have constructed an aTNS decreases from 35% (21 out of 60) to 34.4% (31 out of 90). The percentage of ninth-grade ENS students is predicted to increase from 50% (30 out of 60) to 55.6% (50 out of 90). This extrapolation provides a potential explanation for the drop from eighth to ninth grade students who had constructed an ENS.

Relationship between Students' Number Sequences and Grade Levels

The decrease in the number of aTNS students and the increase in the number of ENS students suggests the possibility of a relationship between students' number sequence construction and their grade level. A chi-square test was used to examine the possibility of this relationship. The results of the chi-square test indicate a statistically significant difference in the distribution of number sequences across the grades ($\chi^2(9, N = 326) = 20.26, p = .016$). As a result of the indication that there is a difference in number sequences by grade, Goodman and Kruskal's Gamma statistic was calculated to test the strength and direction of the association between students' number sequences and grades. There was a moderate, positive correlation between students' number sequences and their grade level, which was statistically significant ($G = 0.23, p = .001$). Thus, increases in students' grade level are moderately related to increases in their number sequence.

To further examine the nature of this relationship, the odds ratios and 95% confidence intervals were calculated to compare the odds of students having an aTNS to an ENS, compared to students who had not yet constructed that number sequence across all four grades. The odds ratios demonstrate that the odds of students in eighth and ninth grades having constructed an ENS or GNS, compared to any number sequence less than an ENS, are significantly greater than the odds of having constructed an ENS or GNS in sixth and seventh grades (Table 4.2). Specifically, the odds of having constructed an ENS or GNS in eighth grade are 3.04 times the odds of having constructed an ENS or GNS in sixth grade, and are 2.57 times the odds of having constructed an ENS or GNS in seventh grade. The odds of ninth-grade students having constructed an ENS or GNS are 2.45 times and 2.07 times the odds of having constructed an ENS or GNS in sixth and seventh grades, respectively. Seventh graders are no more or less likely to have constructed an ENS than are sixth graders (1.18; 0.64, 2.19), nor are ninth graders any more or less likely than are eighth graders (0.80; 0.41, 1.59).

Table 4.2. Odds Ratios comparing Students who have Constructed an ENS or GNS to those who have Not, By Grade

	Grade 6	Grade 7	Grade 8	Grade 9
Grade 6	--	--	--	--
Grade 7	1.18 (0.64,2.19)	--	--	--
Grade 8	3.04* (1.62,5.71)	2.57* (1.36,4.83)	--	--
Grade 9	2.45* (1.26,4.76)	2.07* (1.06, 4.03)	0.80 (0.41, 1.59)	--

Note: 95% confidence intervals are listed in parenthesis

Table 4.3. Odds Ratios comparing Students who have Constructed an aTNS to those who have Not, By Grade

	Grade 6	Grade 7	Grade 8	Grade 9
Grade 6	--	--	--	--
Grade 7	1.06 (0.60, 1.87)	--	--	--
Grade 8	0.46* (0.25, 0.86)	0.44* (0.23, 0.82)	--	--
Grade 9	0.46* (0.24, 0.89)	0.43* (0.22, 0.85)	0.99 (0.49, 2.03)	--

Note: 95% confidence intervals are listed in parenthesis

Complementary trends were found by examining the odds ratios and 95% confidence intervals for students who had constructed an aTNS, compared to students who have constructed less than an aTNS, across all four grades (Table 4.3). The odds of eighth-grade students having constructed an aTNS is roughly half the odds of sixth-grade students (0.46; 0.25, 0.86) and seventh-grade students (0.44; 0.23, 0.82). The odds of ninth-grade students having constructed an aTNS are also approximately half the odds of sixth-grade students (0.46; 0.24, 0.89) and seventh-grade students (0.43; 0.22, 0.85) having constructed an aTNS. As with the likelihood of having constructed an ENS, sixth graders were no more or less likely to have constructed an aTNS than were seventh graders (1.06; 0.60, 1.87), nor were eighth graders any more or less likely to have constructed an aTNS than ninth graders (0.99; 0.49, 2.03).

Number Sequence Attributions

The qualitative phase of the research study involved clinical interviews with 18 students in grades six through nine, two of whom had constructed a TNS, eight of whom had constructed an aTNS, and eight of whom had constructed at least an ENS (Table 4.4). At the beginning of the clinical interview, students were asked a series of screening tasks. Although an aTNS was attributed to Tabitha based on the results of her survey, the results of the screening tasks indicate that it is more likely that she had only constructed a TNS. Tabitha was the only student whose responses to the screening tasks at the beginning of the interview led to a change in her stage classification, making the categorization based on the survey 94.4% accurate for the 18 participants in phase two.

For the qualitative phase of the results only, Tabitha is attributed a TNS, rather than an aTNS as indicated by the survey. One piece of evidence to support this classification is in Tabitha's solution to the third bar task (Appendix A). Tabitha solved bar task 3 during the interview, arriving at an answer of 32 units. When asked how, she said that she counted by eights, and then demonstrated that "I count on my fingers. 8; 9, 10, 11, 12, 13, 14, 15, 16; 17, 18, 19, 20, 21, 22, 23, 24; 25, 26, 27, 28, 29, 30, 31, 32." Tabitha's solution to bar task 3 indicates that she is not counting with a composite unit of eight. Instead, she uses figurative material to keep track of counting by eight, four times. This is more characteristic of a TNS student than an aTNS student, but it is not a contraindication of an aTNS because aTNS students are capable of acting in a manner consistent with a TNS student. However, Tabitha specifically stated that she had counted by eights, indicating that she was aware counting by eights would shortcut the counting process. Despite this awareness, she still counted by ones, which is taken as evidence that Tabitha was not able to count using a composite unit; this supports the attribution of a TNS.

Table 4.4. Phase Two Participant Summaries

Pseudonym[9]	Math Class	Number Sequence
	Grade 6	
Tabitha	Math 6	TNS*
Ann	Math 6	aTNS
Abby	Math 6	aTNS
Aaron	Math 6	aTNS
Elle	Math 6	ENS
Evan	Math 6	ENS
	Grade 7	
Ava	Math 7	aTNS
Alyssa	Math 7	aTNS
Andy	Math 7	aTNS
Greg	Pre-Algebra	GNS**
	Grade 8	
Emily	Pre-Algebra	ENS
Amanda	Algebra 1	aTNS
Erin	Algebra 1	ENS
Gavin	Algebra 1	GNS**
	Grade 9	
Travis	Algebra 1, Part 2	TNS
Alex	Algebra 1, Part 2	aTNS

[9] Pseudonyms were assigned based on students' number sequences. For example, students attributed a TNS were assigned pseudonyms beginning with T. This was done to ease the reader's recall of students' number sequences throughout the results and discussion.

Elizabeth	Algebra 1, Part 2	ENS
Emma	Algebra 2	ENS

* Tabitha was attributed an aTNS based on the results of the survey, but based on her interview, it was determined that she had more likely constructed only a TNS. Thus, she is classified as a TNS student.
**Greg and Gavin were attributed at least an ENS based on the results of the survey, but based on their interviews, it was determined that they had more likely constructed a GNS. Thus, they are classified as GNS students.

Tabitha also solved cupcake task A (Appendix A) in a manner that is more consistent with a TNS student than an aTNS student (Figure 4.2) On this task, she responded that she would fill 12 boxes, but it is her method, not her incorrect answer, that is of interest. She explained, "I started with this box (the box of cupcakes printed with the task[10]), so I put three squares as cupcakes (in each row) and I counted them all the way down to 39. And I counted each row." In this excerpt, Tabitha is explaining that she first drew 36 individual cupcakes (she included the three cupcakes printed with the task as part of her total), and counted them individually to 39. Then, she went back and switched to counting each row of 3 cupcakes. This is characteristic of TNS students' solutions to this task (Ulrich & Wilkins, 2017; Wilkins & Ulrich, 2017). The implication of this solution method that makes it more typical of a TNS student than an aTNS student is that Tabitha does not assimilate the task as a composite unit of three cupcakes. Rather, she assimilates the task as a unit of one cupcake. Then, after ensuring she has represented all 39 cupcakes, she counts the number of boxes as if it were a separate problem. This response is evidence that Tabitha's reasoning is more consistent with a TNS students, rather than an aTNS student.

Tabitha's response to cupcake task A was 12, not 13. She included the three cupcakes printed on the task as a part of her 39, and when she counted the individual cupcakes, she

[10] Throughout the results, text within the parentheses of a quotation is my own addition. I added this text to clarify students' meanings, based on pointing or gesturing seen in video recordings, or based on my own inferences from the broader context of the quotation.

included these three in her count. However, when she went back and counted the rows, she lost track of her intention to include the row of cupcakes printed on the task and counted only the 12 rows she had drawn. This is further indication that to Tabitha, the task of creating 39 individual cupcakes (units of one) and counting boxes of three cupcakes (composite units of three) were two separate tasks because in switching from units of one to a composite unit of three, she lost

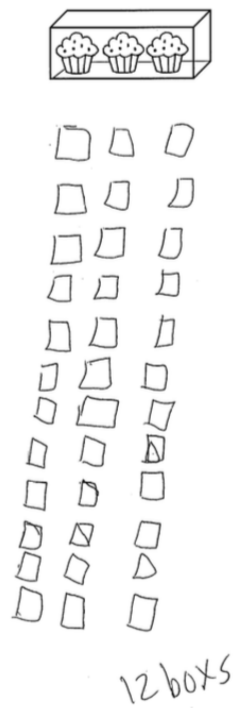


Figure 4.2. Tabitha’s solution to Cupcake Task A.

track of which units of one she had counted. This is additional support of the conclusion that Tabitha had constructed only a TNS.

Two other students, Greg and Gavin, had been attributed at least an ENS as a result of the survey, though it was deemed more likely, as a result of their interviews, that they had constructed a GNS. This does not contradict the results of the survey, as the survey pools ENS and GNS students into one category. This is worth noting, however, because their algebraic reasoning was significantly more advanced than that of the ENS students, and a distinction in their number sequence attribution is helpful in explaining the apparent ease of their algebraic reasoning.

The primary piece of evidence indicating that a GNS was at least within the zone of potential construction (ZPC; Norton & D'Ambrosio, 2008) of both Greg and Gavin was in their response to the phone cords problem (A1)[11]. On this task, both students correctly represented the situation algebraically with a variation of the equation $y = 5x$. Immediately after writing this equation, Greg then also wrote $x = \frac{1}{5}y$ without being prompted to do so. Gavin first rewrote $S = 5R$ as $R = \frac{S}{5}$, and when asked if the second equation could be represented in any other way, he wrote $\frac{1}{5}S = R$, as well. According to Hackenberg and Lee (2015), this is evidence of reciprocal reasoning, which is only available to students who have constructed the third multiplicative concept (MC3), and a GNS is within the ZPC of students who have constructed an MC3 (Ulrich, 2016a). It is also worth noting that while Gavin was enrolled in Algebra 1 during the semester in which data was collected, making it possible that he had learned this manner of representing the equation in school, no other students enrolled in Algebra 1, Algebra 1 Part 2, or Algebra 2 demonstrated reciprocal reasoning.

Equality

A series of three questions in the first interview were included to identify students' conceptions of the equal sign (Table 4.5; Matthews et al., 2012). In total, two students were determined to be reasoning at level two, which indicates a flexible operation concept of the equal sign; five students were reasoning at level three, which indicates a basic relational concept of the

[11] There were no tasks in the screening portion of the interview designed to distinguish between ENS and GNS students, so the students' solution to an algebra task is taken as evidence that they had likely constructed a GNS. Because Greg and Gavin's number sequence was likely not an ENS, their algebraic reasoning is not discussed with the other ENS students.

equal sign; and 11 students were reasoning at level four, which indicates a comparative relational concept of the concept sign. These levels were determined during the screening tasks of the interviews, and the levels assigned are judged to be the most sophisticated level with which the student can reason about the equal sign.

Table 4.5. Participants' Attributed Concept of the Equal Sign

	Grade 6	Grade 7	Grade 8	Grade 9
Level 1	--	--	--	--
Level 2	Tabitha Ann	--	--	--
Level 3	Aaron	Ava	Amanda	Travis Emma
Level 4	Abby Elle Evan	Alyssa Andy Greg	Emily Erin Gavin	Alex Elizabeth

Tabitha and Ann were determined to be reasoning at level 2, which is consistent with a flexible operational concept of the equal sign; their reasoning and explanations on the equality tasks were similar. First, the students were asked to determine whether the equation 37 + 29 - 5 = 48 + 14 was true or false. Both Tabitha and Ann found it to be false. Although this is correct, their reasoning was consistent with an operational concept of the equal sign. Tabitha said, "I added 37 plus 29 and I got 66. So I subtracted 66 by 5 and I got 61. And on here, it said that 37 plus 29 minus 5 equals 48 but actually it equals 61." The implication of Tabitha's explanation is that she conceptualizes the equal sign to indicate an operation because she calculated the value of the expression to the left of the equal sign, but then thought it should be equal to the number immediately following the equal sign (48); the expression on the left should result in 48. This is in contrast to a relational view (levels 3 and 4), in which students conceptualize the equal sign to indicate the relationship between the values of the expressions to its left and right.

Students who were determined to have a concept consistent with level 3, a basic relational concept of the equal sign, determined that the equation 37 + 29 - 5 = 48 + 14 was false by calculating the value of each expression, indicating that they did not have the same value ($61 \neq 62$). This is evidence of a relational concept of the equal sign. However, when the students were asked whether or not they could determine the truth of the equation 67 + 86 = 68 + 85 without adding, students with a level 3 concept of the equal sign indicated that they could not. Emma, a ninth-grade ENS student, explained,

> you would have to add 67 plus 86 and 68 plus 85 to see if they would be equal numbers. Like if both the numbers were equal. …unless it was like *a* plus *b* and then it was like *b* plus *a* and it was both the same numbers. Like 85 plus 75 equals 75 plus 85.

Emma's response indicates that she holds a relational view of the equal sign, because she understands that the equal sign indicates a relationship between two expressions. However, she is unable to reason about the relationship between the two given expressions without calculating their sums. Ava, a seventh-grade aTNS Math 7 student, was also found to have a basic relational concept of the equal sign. On the same task, she responded that "I still kind of could tell, a little bit" that the equation was false without adding. She elaborated that "67 is lower than 68, and 85 is lower than 86," but she maintained that the expression to the right of the equal sign would be bigger than the expression to the left because she was not able to reason flexibly enough to compare the expressions without calculating their values.

Students who were determined to have a comparative relational concept of the equal sign (level 4), on the other hand, were able to determine that the equation 67 + 86 = 68 + 85 was true, without adding. Alex responded:

> that one is lower than this one (67 compared to 68) and that one is lower than this one (85 compared to 86). … I think it'd be true because if it's 67 plus 86 you want to get that

> number (the sum) but if you get 68 plus 85 you're going to get the same number (sum) because that one (a one from the 68) pretty much just came over here (to the 85)… on both sides one (addend) is higher only by one so you'll get the same number (sum) if you add them together.

Alex is explaining that equality can be determined without calculating the sums because on each side of the equation, one of the addends is one more than the other. Thus, he can see that the sums would be equal by taking one away from the 68 and moving it to the 85. In other words, $a + (b + 1) = (a + 1) + b$. This is evidence of a comparative relational concept of the equal sign. Alex's explanation was characteristic of all of the students who were determined to be reasoning at level 4, regardless of grade level, math class enrollment, and number sequence attribution.

There was not a clear relationship between students' number sequence construction and their concept of the equal sign. The students who reasoned at level 2 had constructed either a TNS or an aTNS. The students who reasoned at level 3 had constructed a TNS, an aTNS or an ENS. The students who reasoned at level 4 had constructed an aTNS, an ENS, or a GNS. Thus, it is not clear whether a student's number sequence construction prohibits them from constructing a more sophisticated concept of the equal sign; even with a TNS, Travis was able to construct a basic relational concept of the equal sign (level 3). Conversely, despite having constructed an ENS, Emma had only constructed a basic relational concept of the equal sign. Both GNS students had constructed a comparative relational concept of the equal sign (level 4), which may indicate that students with a more sophisticated number sequence are more likely to construct a more sophisticated concept of the equal sign, but a GNS is certainly not necessary to conceptualize the equal sign as comparative relational, because four aTNS students and five ENS students also reasoned in a manner consistent with level 4. On the other hand, no TNS students had a comparative relational concept of the equal sign.

It is interesting to note that no seventh-, eighth-, or ninth-grade students demonstrated an operational concept of the equal sign; only two sixth-grade students had a flexible operational concept of the equal sign (level 2). The students who reasoned at levels 3 and 4 were spread across all four grades, and had constructed either a TNS, an aTNS, an ENS, or a GNS.

Algebraic Reasoning

The interview tasks were designed to elicit students' algebraic reasoning as it pertains to their abilities to write and solve linear equations and systems of equations, and to generalize and represent patterns. The results of the participants' algebraic reasoning will be arranged by task number to allow for comparison across students. After presenting and analyzing selected students' responses, overall similarities and differences in students' responses will be summarized based on their number sequence attribution, and at times, their math class enrollment.

It is also worth noting that in general, the purpose of this research was to examine the algebraic reasoning of aTNS students and how their algebraic reasoning compared to that of their TNS and ENS peers. Because both Greg and Gavin are suggested to have constructed a GNS, the discussion of their algebraic reasoning is not as detailed as that of the other students. Greg's and Gavin's solutions to the interview tasks are included in summary tables throughout the results section and in the descriptive statistics provided, but are otherwise largely disregarded.

The phone cords problem (A1). In response to task A1, the phone cords problem (Appendix B), 12 students solved the splitting task[12] within the phone cords problem (Table 4.6). The 12 students who solved the splitting task were spread across the four grades; four had constructed an aTNS, six had constructed an ENS, and two had constructed a GNS. No TNS students solved the splitting task and four aTNS students did not solve the splitting task. Aaron, a sixth-grade student, did not draw a picture, so it was unclear whether or not he solved the splitting task. He correctly created examples of the two cord lengths (e.g., 3 feet and 15 feet, 5 feet and 25 feet), and was aware that Rebecca's was the shorter and Steven's was the longer. He also solved the splitting task during the screening tasks in the interview. Without a picture, however, it is difficult to state definitively whether or not he solved the splitting task, and so he is categorized as not solving the splitting task.

Table 4.6. Results of Contextualized Splitting Task

	Grade 6	Grade 7	Grade 8	Grade 9	Total
TNS	0/1 (0%)	--	--	0/1 (0%)	0/2 (0%)
aTNS	2/3* (66.7%)	0/3 (0%)	1/1 (100%)	1/1 (100%)	4/8 (50%)
ENS	2/2 (100%)	--	2/2 (100%)	2/2 (100%)	6/6 (100%)
GNS	--	1/1 (100%)	1/1 (100%)	--	2/2 (100%)
Total	4/6 (66.7%)	1/4 (25%)	4/4 (100%)	3/4 (75%)	12/18 (66.7%)

*It was unclear whether Aaron solved the splitting task; within this table, his solution is counted as incorrect.

Neither of the TNS students solved the splitting task that was a part of the phone cords problem. Although they were both aware that Steven's cord was longer than Rebecca's cord, and represented it as such in their pictures, they did not verbally or visually represent the relationship between the two cord lengths. Both were prompted to use an example to try to verbalize the

[12] With respect to task A1, I make no distinction between solving the splitting task (sequentially partitioning and iterating) and splitting (simultaneously partitioning and iterating).

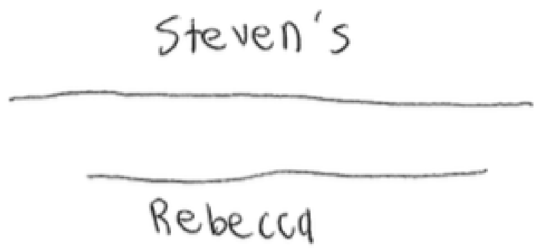
relationship between the cord lengths. Travis determined that if Rebecca's cord was five feet, Steven's would be 25. Tabitha could not generate the related cord lengths.

> Interviewer (I): Let's just try Rebecca's cord being two feet long. How long do you think Steven's cord would be?
>
> Tabitha (T): … Maybe this long?
>
> I: OK, great. Are you able to figure out how long that is? If we're pretending that this is two feet, or that this is representing two feet, are you able to figure out how long Steven's is?
>
> T: (thinks for 44 s.) Seven feet?
>
> I: Excellent. How did you get seven feet?
>
> T: I went up from two, and then, I went up to five, and I counted. Like, 2; 3, 4, 5, 6, 7.

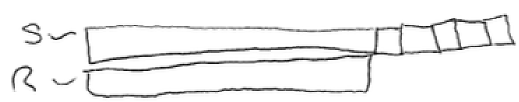
Tabitha's reasoning is additive in nature, rather than the multiplicative reasoning necessary on this task. On other attempted numerical examples, Tabitha did not even clearly articulate how she generated her guess for Steven's cord length, and there was not always a difference of five.

Three aTNS students also did not solve the splitting task. Ava, for example, drew Rebecca's cord shorter than Steven's (Figure 4.3), and said, "So it's [Steven's] five times the length, so hers would probably be just a little bit shorter." Ava was not sure how much shorter Rebecca's should be, and there was no visible relationship between the lengths. Alyssa also did not solve the splitting task. To represent the relationships between Steven's and Rebecca's cords, she drew two equal cords and then added five small squares onto Steven's to demonstrate that it was five times the length of Rebecca's. In another variation, Andy could not represent the multiplicative relationship between the two cord lengths, and made Steven's cord roughly half the length of Rebecca's cord.

Of the 12 students who solved the splitting task, nine also represented the situation algebraically. An additional two students, Aaron and Alyssa, did not solve the splitting task but did represent the situation algebraically (Table 4.7). An algebraic representation of the relationship between the two cord lengths was considered correct if the students wrote an equation similar to $y = 5x$, and explained that Rebecca's cord was represented by x, and Steven's was represented by y. A solution was not considered correct if they could not explain



(a) Ava's Solution



(b) Alyssa's Solution

(c) Andy's Solution

Figure 4.3. aTNS students' incorrect solutions to the splitting task within the phone cords problem.

the meaning of the two variables, or if they reversed the meaning of the two variables. Of the nine students who solved the splitting task and produced a correct equation, two were in sixth grade and had constructed an ENS; one was in seventh grade and had constructed a GNS; four were in eighth grade and had constructed either an aTNS, an ENS, or a GNS; and two were in ninth grade and had constructed an ENS. Neither Aaron nor Alyssa are considered to have solved the splitting task, but both represented the situation algebraically.

Table 4.7. Results of The Phone Cords Problem

	Grade 6	Grade 7	Grade 8	Grade 9	Total
TNS	0/1 (0%)	--	--	0/1 (0%)	0/2 (0%)
aTNS	1/3 (33.3%)	1/3 (33.3%)	1/1 (100%)	0/1 (0%)	3/8 (32.5%)
ENS	2/2 (100%)	--	2/2 (100%)	2/2 (100%)	6/6 (100%)
GNS	--	1/1 (100%)	1/1 (100%)	--	2/2 (100%)
Total	3/6 (50%)	2/4 (50%)	4/4 (100%)	2/4 (50%)	11/18 (61.1%)

Furthermore, the ability of students to represent the phone cords problem (A1) algebraically was more closely associated with their number sequence than their grade level. By grade, 50% of sixth, seventh, and ninth graders represented the task algebraically, and 100% of eighth graders did. Comparatively, by number sequence, there is an increasing trend that is associated with the increasing sophistication of students' number sequences; 0% of TNS students represented the task algebraically, 32.5% of aTNS students did, and 100% of ENS and GNS students did. These patterns suggest that students' number sequences were more closely related to their ability to represent a multiplicative relationship between two unknowns algebraically than was their grade level.

Table 4.8 represents the relationship between the students' solutions to the contextualized splitting task and the algebraic representation of the phone cords problem (A1). Four students did not solve the splitting task or represent the phone cords problem algebraically. Two of these students had constructed a TNS and two had constructed an aTNS. In comparison, nine students correctly solved both portions of the phone cords problem by solving the splitting task and writing a correct algebraic representation of the problem. Of these students, one had constructed an aTNS and eight had constructed at least an ENS. Thus, the main diagonal in Table 4.8 shows that in general, for students who had constructed a TNS or at least an ENS there was a relationship between their solution to the splitting task and the phone cords problem; they solved both parts or neither. aTNS students, were less consistent. All students in the off-diagonal of Table 4.8 are aTNS students. Three of the aTNS students solved the splitting task but did not represent the phone cords problem algebraically, and two did not solve the splitting task but did represent the phone cords problem algebraically. The disconnect between the solutions to the two parts of the task was only present among aTNS students in this research study.

Table 4.8. Number of Students by Number Sequence who Solved the Splitting Task and Represented A1 Algebraically

	Solution to the Contextualized Splitting Task		
Algebraic Representation	Correct	Incorrect	Total
Correct	**9 Total** 0 TNS 1 aTNS 8 ENS+	**2 Total** 0 TNS 2 aTNS 0 ENS+	**11**
Incorrect	**3 Total** 0 TNS 3 aTNS 0 ENS+	**4 Total** 2 TNS 2 aTNS 0 ENS+	**7**
Total	**12**	**6**	**18**

All ENS students represented the phone cords problem (A1) algebraically, regardless of whether or not they were enrolled in an Algebra course[13]. Five of the six ENS students who represented the situation algebraically first wrote an incorrect equation or confused the cord lengths before arriving at a correct equation. Elizabeth, for example, explained the following when asked if she could represent the relationship between the cord lengths algebraically.

> Elizabeth (Eli): I believe so. [I: OK.] It's going to be, like, five *n* [writes $5 \cdot n$]. It's longer than Rebecca's, so, to find out we would have to know, like, how long it is… No, I don't think you can.
>
> I: Well, what does *n* stand for?
>
> El: How long it is. …
>
> I: How long whose is?

[13] An Algebra course is considered a course at least equal to Algebra 1, and for the participants in this study, included Algebra 1, Algebra 1 Part 2, and Algebra 2. Pre-Algebra was not considered an Algebra course.

El: Umm, his is. It's five times. Oh, Rebecca's. That's Rebecca's and his is five times longer [I: OK.] than Rebecca's, but I guess we would have to know the number.

I: OK, I see. So n is the length of…

El: n is the length of his. Of hers. I'm sorry, my bad. Of Rebecca's.

In this data excerpt, Elizabeth is struggling to determine whether the n in the expression $5n$ represents Steven's or Rebecca's cord length. At first, she indicates that n is the length of Steven's cord when she says "Umm, his is. It's five times." Later in the interview, through the use of a number example, Elizabeth determined that the equation $5 \cdot n = S$ represents the relationship between the cord lengths, if n represents Rebecca's cord length and S represents Steven's cord length.

> If her cord was three feet long, and his consists of, like, five little sections of that, then his would be fifteen feet long. So then five times three does equal 15, so 15 equals 15. So that's true. … yeah, it (the equation) works.

This quotation occurred after Elizabeth had written $5 \cdot n = S$, but she was confused as to whether her equation should be multiplicative ($5n$) or additive ($n+4$). A number example helped her decide which was more appropriate. Ultimately, Elizabeth's algebraic representation of the relationship between the two cord lengths was considered correct because she generated the correct equation and explained the meaning of each component of the equation.

The necessity of a number example was common among ENS students. Both Elizabeth and Erin used a number example to determine which equation, of more than one equation they had written or described, was correct; Emily used a number example to correct the reversal of

Steven's and Rebecca's cord lengths in her equation; and Evan used a number example to build up his equation.

Although Erin eventually wrote a correct equation to represent the relationship between Steven's and Rebecca's cord lengths, she initially attempted to write an inequality. She wrote an inequality because "we don't know exactly how long they (the cords) are." Erin was then prompted to think about writing an equation instead of an inequality, and the following discussion occurred.

> I: Can we then use an equation with those two variables and represent how much longer Steven's is in comparison to Rebecca's?
>
> Erin (Er): Yeah. [I: OK.] I don't know if I can do that. …
>
> I: Well, what variables do you want to use? Let's start here.
>
> Er: R and, I don't want to use S, so I'll do X. …
>
> I: OK. So what are you trying to say, what are you trying to say about their cord lengths?
>
> Er: His is five times bigger. … Oh! That makes sense [writing].
>
> I: OK. Tell me what that says.
>
> Er: x + 5 is greater than R.

Even after the interviewer suggested an equation, Erin wrote an inequality. Following this exchange, Erin reasoned about a numerical example in which Steven's cord was five feet and Rebecca's cord was one foot, and then wrote $5R = x$, but she needed to study a numerical example to express the exact relationship between the cord lengths algebraically.

No TNS students represented the situation algebraically, regardless of whether or not they were enrolled in an Algebra course. This is not surprising, considering they did not solve the splitting task. Both TNS students were also given the opportunity to work on the modified splitting problem (task A2) because they made limited progress on the phone cords problem (task A1). Despite the introduction of the modified task with smaller numbers and a more intuitive length comparison, neither TNS student solved the splitting task or algebraically represented the modified splitting task, either.

Three of the eight aTNS students who participated in interviews represented the phone cords problem (task A1) algebraically. Interestingly, only one of those three students definitively solved the splitting task involved in the problem. Two commonalities among the solutions of these three aTNS students were that they created an equation that was based upon the use of number examples, and that they reverted to an operational concept of the equal sign in their equation writing. Both of these behaviors can be observed in Alyssa's solution to the task, and each will be discussed as it relates to Alyssa's solution following the data excerpt. First, recall that Alyssa did not solve the splitting task. Instead, she drew two equal cord lengths and then added five small units to the end of Steven's cord (Figure 4.3). However, she did represent the situation algebraically. At first, Alyssa wrote $x5 - y$, and stated

> Alyssa (A): So Steven's would be *x* and Rebecca's would be *y*. And Steven's is five times as long as hers so you do this (multiply *x* by 5) and subtract hers (*y*) and you get the answer, I guess.
>
> I: OK. And what would the answer be?
>
> A: I don't know because there's no numbers. It just has 5.
>
> I: That's OK, but what would it, what would the answer represent? …
>
> A: The amount of cord that was timesed onto it. …
>
> I: So what if we say, for example, that Rebecca's cord is three feet long. Can you figure out how long Steven's cord would be then?

A: His would be 15, cause hers, his is five times as long as hers, so three times five is 15.

I: Oh, very good. So if Rebecca's is three feet, Steven's is 15 feet. Does that work in our equation? …

A: I don't think it would work with that equation ($x5 - y$) because you can't get these two (three and five) times each other. So it would probably be x equals five times three. And then you would get 15 and that'd be his. …

I: Can we put a variable for Rebecca's cord length also?

A: Yeah. x equals five times y.

It was not until Alyssa considered a numerical example that she rewrote her equation. Prior to using a numerical example, she tried to multiply five by the length of Steven's cord, which she represented as $x5$, because the problem states that his cord length is five times the length of Rebecca's; accordingly, Alyssa multiplied his cord length times five. After determining numerically that Rebecca's cord length should be multiplied by five, the product of which yield's Steven's cord length, she was able to generate the equations "x equals five times three" and then "x equals five times y."

In addition to using numerical examples to generate algebraic representations of the relationship between the two cord lengths, all three aTNS students also reverted to an operational concept of the equal sign at least once during their equation writing on task A1. In the above data excerpt from Alyssa, she initially indicates that the result of $x5 - y$ would be "the answer, I guess," and explained that you don't know what the answer is because you don't know the values of x and y. Alyssa was focused on finding a numerical result to her expression, $x5 - y$, and not being able to calculate this value was a stumbling point for the aTNS students.

Aaron explained his equation, $Y \times 5 = Z$, and indicated that "you wouldn't know the answer to it," meaning, you don't know what Z represents. It was not until Aaron was asked if Steven's cord length was represented somewhere in his equation that he changed his explanation and said that Z represented Steven's cord length "Cause it's five times longer than Rebecca's." Amanda wrote the equation $5x = y$, and said that "y represents the total length." All three of these excerpts demonstrate that these students conceptualized the equal sign as being operational, rather than relational; this is despite the fact that they each demonstrated a relational concept of the equal sign (levels 3 and 4) during the screening tasks of the interview.

Five aTNS students did not represent the relationship between the two cord lengths algebraically; these five students demonstrated some difficulties with the task that were similar to their more successful aTNS peers, as well as some additional difficulties. Alex, for example, introduced the idea of using a numerical example to scaffold his equation writing, which is similar to the three aTNS students described previously. However, he also represented the relationship between the cord lengths additively and ignored the variables he included in his equation when he wrote $B5 - x1 = 4$ft. He explained his equation by saying "this is one foot (*x1*, or Rebecca's cord). This is five feet (*B5*, or Steven's cord). You do five feet minus one foot, you get four feet. That's how much longer it (Steven's) is than this one (Rebecca's)." In this explanation, Alex applies additive reasoning by indicating that Steven's cord is four more than Rebecca's, and he ignores the variables, *B* and *x*, that he had included in the equation, which is evidenced by their disappearance in the result of four feet.

The border problem (A4). On the border problem (task A4; Appendix B), students were asked to determine the number of squares on the border of a ten-by-ten grid, and then were asked to generalize by determining the number of squares on the border of a 100-by-100 grid, to verbalize the method of finding the number of squares on the border of any grid, and to write an expression to represent the number of squares on the border of an *n*-by-*n* grid. In response to this series of questions, 17 students extended their method for finding the number of squares on the border of a grid to include a 100-by-100 grid. Seventeen students verbalized their method for any grid, and 11 wrote an algebraic expression to communicate a method for finding the number of squares on the border of an *n*-by-*n* grid (Table 4.9).

Table 4.9. Results of Task A4

	Grade 6	Grade 7	Grade 8	Grade 9	Total
			TNS		
Generalize	1/1 (100%)	--	--	0/1* (0%)	1/2 (50%)
Verbalize	1/1 (100%)	--	--	n/a	1/1 (100%)
Algebraic	0/1 (0%)	--	--	n/a	0/1 (0%)
			aTNS		
Generalize	3/3 (100%)	3/3 (100%)	1/1 (100%)	1/1 (100%)	10/10 (100%)
Verbalize	3/3 (100%)	3/3 (100%)	1/1 (100%)	1/1 (100%)	10/10 (100%)
Algebraic	1/3 (33.3%)	0/3 (0%)	1/1 (100%)	1/1 (100%)	3/10 (30%)
			ENS		
Generalize	2/2 (100%)	--	2/2 (100%)	2/2 (100%)	6/6 (100%)
Verbalize	2/2 (100%)	--	2/2 (100%)	2/2 (100%)	6/6 (100%)
Algebraic	2/2 (100%)	--	2/2 (100%)	2/2 (100%)	6/6 (100%)
			GNS		
Generalize	--	1/1 (100%)	1/1 (100%)	--	2/2 (100%)
Verbalize	--	1/1 (100%)	1/1 (100%)	--	2/2 (100%)
Algebraic	--	1/1 (100%)	1/1 (100%)	--	2/2 (100%)
			Total		
Generalize	6/6 (100%)	4/4 (100%)	4/4 (100%)	3/4 (75%)	17/18 (94.4%)
Verbalize	6/6 (100%)	4/4 (100%)	4/4 (100%)	3/4 (75%)	17/18 (94.4%)
Algebraic	3/6 (50%)	1/4 (25%)	4/4 (100%)	3/4 (75%)	11/18 (61.1%)

*Travis was not asked to verbalize or algebraically represent a method for finding the number of squares on the border of the grid because he did not successfully find the number of squares on a 100-by-100 grid.

The only student who did not generalize through the use of particular examples was Travis. During his interview, Travis struggled to understand why there were 36 squares, and not 40, on the border of a 10-by-10 grid. Although the problem's directions instruct the students not to count the squares on the border one by one, Travis eventually counted all 36 squares one by one to convince himself that there were not 40. After doing so, he concluded that "you count nine and nine and nine and nine," but this was the only progress he made on this task. He did not determine the number of squares on the border of a six-by-six grid, nor a 100-by-100, and because of his limited progress, he was not asked to verbalize or write an algebraic expression.

Although Tabitha had also only constructed a TNS, she was able to determine the number of squares on the border of a 100-by-100 grid using the same method that she used to determine the number of squares on a 10-by-10 grid and a six-by-six grid. This method was to add the given side length twice, and then to add the given side length minus two, twice; the algebraic equivalent of this is $2n + 2(n - 2)$. She also explained how she could tell her math teacher to find the number of squares on the border of any grid, although this was difficult for her.

> T: You could still subtract the top part by 2 and you could get the side length (the number of squares on the two vertical sides after removing the corner squares)
>
> I: Very good. And then what would you do with those numbers?
>
> T: And then you would add the top and bottom part together ($n + n$), and once you get that you would add it by the side part ($n – 2$) and then after you do that you would do the second side ($n – 2$). And you'll get your answer for the border.

Although Tabitha's description of how to find the border is a bit crude, she was significantly more successful on this task than was Travis, despite their both having constructed a TNS and despite Tabitha being only in sixth grade.

Students used four different methods to find the number of squares on the borders of the grids on the border problem (Table 4.10). Ten students' method involved adding the given side twice, and the given side minus the two corners twice, the algebraic equivalent of which is $2n + 2(n-2)$. Of those ten students, four represented the method algebraically. All four of those students had constructed either an ENS or a GNS. One TNS, four aTNS, and one ENS student

Table 4.10. Students' Methods for The Border Problem

	Method One	Method Two	Method Three	Method Four	Total
	2n + 2(n-2)	n + n – 1 + n – 1 + n – 2	4n – 4	4(n-2)+4	
Verbalize*					
	10 Total	**3 Total**	**6 Total**	**1 Total**	**20 Total****
	1 TNS	-- TNS	-- TNS	-- TNS	1 TNS
	4 aTNS	1 aTNS	3 aTNS	1 aTNS	9 aTNS
	3 ENS	2 ENS	3 ENS	-- ENS	8 ENS
	2 GNS	-- GNS	-- GNS	-- GNS	2 GNS
Represent Algebraically					
	4 Total	**2 Total**	**5 Total**	**1 Total**	**12 Total*****
	0 TNS	-- TNS	-- TNS	-- TNS	0 TNS
	0 aTNS	0 aTNS	2 aTNS	1 aTNS	3 aTNS
	2 ENS	2 ENS	3 ENS	-- ENS	7 ENS
	2 GNS	-- GNS	-- GNS	-- GNS	2 GNS

*Travis is not included in the table because he did not verbalize a method.

**This total includes Alex twice (grade 9, aTNS) because he verbalized methods one and three. This total also includes Erin (grade 8, ENS) and Emma (grade 9, ENS) twice because they verbalized methods one and three.

***This total includes Emma twice because she represented methods one and three algebraically.

who verbalized this method could not represent it algebraically. Three students described a method of adding the given side, the given side minus one corner twice, and the given side minus two corners once, the algebraic equivalent of which is $n + (n - 1) + (n - 1) + (n - 2)$. Of those three students, two represented it algebraically, and both of those students had constructed an ENS. The one aTNS student who verbalized this method could not represent it algebraically. Six students used a method of adding the four given sides and then subtracting the four corners, the algebraic equivalent of which is $4n - 4$. Of those six students, five represented the method algebraically; two of those five had constructed an aTNS and three had constructed an ENS. One aTNS student verbalized method three but did not represent it algebraically. Finally, Amanda used a method that was different than the other students. She described subtracting two from the given side length, multiplying that by four, and then adding the four corners back in, which equates to $4(n - 2) + 4$. Amanda verbalized this method and represented it algebraically.

All students who had constructed at least an ENS were able to represent at least one method algebraically on the border problem, regardless of their grade level or math class enrollment. Erin initially attempted to represent method one algebraically, but was unsuccessful. She then generated method three and correctly represented it algebraically. On the other hand, only two aTNS students were successful in representing their method algebraically, and both of these students used method three.

One particular difficulty students encountered in representing their method algebraically revolved around relating the given side length to the given side length minus one or two corners. For example, Alex attempted to represent method one algebraically but when he was unsuccessful, instead generated method three.

Alex (Al): OK. You have *n, n, n, n* (the four sides). Can it be, like, $n + n + n + n$? [I: OK, awesome.] No, because of this (the corners).

I: OK. You said $n + n + n + n$, and then you said, no. Because of what?

Al: Well, you would be adding 10, 10, 10, 10. You would be adding one (corner) each time. One extra, I think. Yeah. … I could do $n + n$ (add the top and bottom), but then these (vertical sides) would have to be different. So I could express that better. [Writes $n + n + b + b$]

I: Yeah, those would have to be different. I agree with what you're thinking there. So, how could we represent this? …

Al: I want to say use a different variable, but it has to be a certain amount, number, in here, too. [I: Ahh, I see.] If you do $n + n$, you might be able to subtract, um, four? I want to say $n + n – 4$, and then you could just do the sides together.

I: OK. Write down what that would be.

Al: $n + n – 4 + n + n$.

In this data excerpt, Alex is attempting to algebraically represent method one. He expresses an understanding that the two lengths, n and n-2, within method one are related but not the same when he decides against adding $n + n + n + n$ and states that "it has to be a certain amount." Alex consistently subtracted two from the given side length in each of the numerical examples he completed earlier, and he verbalized this relationship earlier, but he cannot represent it algebraically. To resolve this perturbation, Alex first attempts to use a separate, and unrelated variable, b, to represent the vertical side lengths in lieu of $n – 2$, because b is the second letter of the alphabet. Then, because he recognizes that using b does not reflect the relationship between the two amounts, he generates a new method of adding the four sides and subtracting the four corners all at once.

Alyssa and Andy made a similar adjustment in their algebraic representations on the border problem. Like Alex, Andy wrote $n + n + x + x$ to represent the number of squares on the border of an n-by-n grid. Unlike Alex, he did not demonstrate any awareness of or perturbation by a relationship between the n and the x, despite verbalizing the relationship as "subtract two" earlier on the task. Alyssa, on the other hand, wrote the expression $n + n + L + L$ to algebraically represent method one. In doing so, she specifically selected L to represent the shorter side length because L is two letters before N in the alphabet, and she recognized the relationship between the two amounts n and $n - 2$, and she explained that "You could do n, since we're doing, like, the alphabet you could take two letters… before it." Of the four aTNS students who attempted to algebraically represent method one, all four were unsuccessful, and three of those algebraic representations included the use of variables that were numerically unrelated to the side length, n. No students who had constructed more than an aTNS used unrelated variables to represent related quantities.

While Alex, Andy, and Alyssa attempted to use unrelated variables to represent related quantities, two other aTNS students, Ann and Ava, used the same variable to represent unrelated quantities in their incorrect algebraic solutions. Ann was attempting to algebraically represent method three. Her final representation was $n \cdot 4 = n - 4 = n$. With this, she explained that her equation meant: "n could be any number, times four, equals n minus four, equals n." She further explained that you multiply by four because squares have four sides, which results in "an unknown number (the second n)." Four is then subtracted to eliminate the corner squares, and the resulting n represents the number of squares on the border. While her method is correct, its algebraic representation utilizes one variable for the side length, the product of the side length and four, and the number of squares on the border.

Similarly, Ava represented method two using the variable n to represent multiple things. Her final algebraic representation was written as three separate expressions: $n + 9$; $n + 9$; $n + 8$. She explained that to find the border, you add the side length, n, plus nine; take that sum, n, and add nine again; finally, take the new sum, n, and add eight. In this solution, Ava uses n to represent the side length, and the first and second intermediate sums. Ava also relies on the initial numerical example presented in the task—a grid with a side length of 10—in her solution, which is why she adds nine, nine, and eight.

Abby was the final student who used the same variable to represent multiple quantities. She used method one throughout the interview, but in her attempt to represent the number of squares on the border of an n-by-n grid algebraically, she used n to represent both the side length and the corner squares. Her final equation was $N + N - 2N + N + N - 2N$. She explained this as adding one side (n) and then adding a second side but subtracting the two corners on the second side ($n - 2n$), and then repeating that process a second time to account for all four sides. Thus, to Abby, the N in her expression represents both the given side length and a label denoting the corners.

Tabitha had a similar difficulty to that experienced by Alyssa, Andy, and Alex. Instead of using two unrelated variables to represent related quantities, however, Tabitha used one variable at a time and used numbers to represent the related quantity. To begin, Tabitha wrote $n + n$, to represent summing the top and bottom of an n-by-n grid, however, she then said she would add 2 + 2 to incorporate the vertical sides. In explaining, she said:

T: You subtract four by two, and you get two. …

I: So where does the two come into your square?

T: In the side lengths.

I: Hmm. OK. So is there a way we could represent the side lengths with a variable?

T: I think so?

I: What do you think? …

T: I don't know.

In this data excerpt, Tabitha is attempting to incorporate the vertical side lengths into her expression, but she cannot do so with a variable. To resolve this difficulty, she reverts to using a number example that we had previously discussed—a grid with a side length of four squares—the border of which she had calculated by adding 4 + 4 + 2 + 2. Tabitha's incorporation of one variable and one number into her expression is reminiscent of Ava's string of expressions, $n + 9$; $n + 9$; $n + 8$.

Of the eight aTNS students interviewed, six of them struggled as a result of either using unrelated variables to represent related quantities or as a result of using the same variable to represent multiple quantities. Two aTNS students represented their method algebraically, and one aTNS student abandoned method one when he was unsuccessful and then correctly represented method three algebraically. In comparison, the one TNS student who made progress on this task engaged in an unfruitful method of representing method one algebraically, in which she used a variable to represent the horizontal side lengths and a number example to represent the vertical side lengths.

The coin problem (A5). According to their teachers, students who were enrolled in Algebra 1, Algebra 1 Part 2, or Algebra 2 had received instruction on solving systems of equations algebraically using the substitution method, which is what is required on the coin problem (A5; Appendix B) and the modified coin problem (A6), if they are to be solved algebraically. Despite this instruction, students' solution methods were heavily dependent upon their number sequence construction (Tables 4.11). Students who had constructed only an aTNS guessed and checked on the coin problem, regardless of whether or not they were enrolled in at least an algebra class. ENS and GNS students attempted to solve the coin problem algebraically, regardless of whether or not they were enrolled in an algebra class. The coin problem was the only algebra problem on which aTNS students were more successful than ENS students. Presumably, because aTNS students were willing to persevere through tedious guess and check solutions whereas ENS students tended to become overwhelmed by an algebraic solution method. The details of students' solutions to the coin problem are presented in the remainder of this section.

Of the six aTNS students who attempted the coin problem, three were successful. These students were Abby, a sixth-grade student enrolled in Math 6, Alyssa, a seventh-grade student

Table 4.11. Results of The Coin Problem by Course Enrollment and Solution Method, within each Number Sequence

	Less Than Algebra I		At Least Algebra I		
	Guess & Check	Algebra	Guess & Check	Algebra	Total
aTNS	2/4 (50%)	--	1/2 (50%)	--	3/6 (50%)
ENS	--	0/3 (0%)	--	1/3 (33.3%)	1/6 (16.7%)
GNS	--	1/1 (100%)	--	1/1 (100%)	2/2 (100%)
Total	2/4 (50%)	1/4 (25%)	1/2 (50%)	2/4 (50%)	6/14 (42.9%)

Note: Neither TNS student the coin problem, nor did two aTNS students; this was due to time and the students' perceived threshold for frustration.

enrolled in Math 7, and Amanda, an eighth-grade student enrolled in Algebra 1. Although Amanda had studied systems of equations in math class, she and the other two aTNS students who were successful on the coin problem used guess and check. Guess and check seemed like the most intuitive method for these students, but all three were asked to try to represent the situation algebraically. Amanda wrote three equations: $d = n + 3$, $q = n - 2$, and $T = n + q + d$; in each of these equations, *d* represents the number of dimes, *n* represents the number of nickels, q represents the number of quarters, and *T* represents the total[14]. When asked if *T* stood for the total number of coins or the total value of the coins, Amanda was ambiguous.

> I: OK, so you wrote $T = n + q + d$. Is that right?
>
> Amanda (Am): Uh hmm.
>
> I: OK. So T is the total value. I see that here (written at the top of the page).
>
> Am: Oh, haha. Tricked myself there. … Because it's the total. T is the total. And to get the total, you have to add up all three—nickels, dimes, and quarters—to get the total.
>
> I: Ahh, OK. I've got you now. OK. So is this (n) the number of nickels you have, though, or is that the value of the nickels?
>
> Am: Umm… well, I believe it's the value.

In the first two equations that she wrote, $d = n + 3$ and $q = n - 2$, the variables represent the number of each type of coin. In the third equation, however, $T = n + q + d$, she is indicating that they represent the value of the type of coin. Amanda was unable to resolve this perturbation through further questioning, and shortly after this exchange, she abandoned the equations in favor of guess and check.

Alyssa was also successful in solving the coin problem, although she had difficulty similar to Amanda's in distinguishing between the value and the number of the coins. She began

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[14] These were the variables and meanings explained by all students on tasks A5 and A6, unless otherwise noted.

the problem by guessing and checking several values for the number of nickels, dimes, and quarters. When prompted to represent the situation algebraically, she wrote three equations: $d + 3 = n$, $2 - n = q$, and $d + 3 + n - 2q = 5.40$. The first two equations are Alyssa's attempt to represent the relationship between the number of nickels and dimes, and the number of nickels and quarters, respectively. However, both include a reversal, since the problem states there are three more dimes than nickels and two fewer quarters than nickels. Then, in her third equation, she attempts to represent that the sum of the coins' values is five dollars and forty cents, but she explains the difficulty she is having in conceptualizing the problem algebraically:

> A: I put *d* times[15] three because *d*, the dimes, have three more than the nickels. Plus *n* minus two quarters, *2q*. So, there's the nickels minus two fewer quarters. And then it equals five-forty, but I don't know how you could actually figure how many of each you would have.
>
> I: OK. So *d* times three plus *n* minus *2q* is five dollars and forty cents. OK. So is *d* the number of dimes you have, or is it the value of the dimes?
>
> A: It's just representing dimes. *d* is the dimes, *n* is the nickels, *q* is the quarters.

In this excerpt, it is first worth noting that Alyssa's utterance that *2q* represents two quarters is evidence of her conceptualizing the variable, *q*, as a label, rather than an unknown quantity. Furthermore, she has attempted to relate the total number of coins with the total value of the coins, despite an awareness that the variables represent the number of each type of coin, and not their value. Finally, Alyssa explains that part of her difficulty in conceptualizing this task algebraically is that she doesn't know how to proceed without knowing the value of *n*. This is

[15] At this point in the task, Alyssa was attempting to multiply the number of dimes times three. We resolved this issue later, and she corrected the equation to include addition, rather than multiplication.

evidenced when she says, "I don't know how you could actually figure out how many of each you would have." Although she made progress representing the task algebraically, her inability to conceptualize how the value of each coin could be nested within the number of each type of coin, and how the number of each type of coin could be nested within the total value of $5.40, limited her ability to represent those three quantities simultaneously in one equation.

The final aTNS student who solved the coin problem was Abby. Abby briefly attempted to write equations to represent the problem algebraically, but she was unable to make meaningful progress using the equations, so she quickly reverted to guess and check. Abby's initial equations were: $d - 3n =$, and $n - 2n = q$.

> Abby (Ab): OK. So since there's three more dimes than nickels, you put the dimes minus three, um, nickels equals. That's how you find out how many nickels you have.
>
> I: So, you take the dimes and subtract three nickels from that? That's how you get the number of nickels?
>
> Ab: Um, actually you subtract three dimes and then you would get the nickels [changes equation to $d - 3 = n$].
>
> I: Oh, OK. I'm with you. And then explain the next one to me.
>
> Ab: The next one, let's see. Alright, so, the, since the, there's two fewer quarters than nickels. You have to find out how many nickels there are, and then if you subtract two nickels out of the nickels, the groups of nickels, you get, you get your qu… how many quarters you have.
>
> I: OK, so if this is three dimes (the 3 in $d - 3 = n$), what is this (the *2n* in $n - 2n = q$)?

Abby never answered. She did not attempt any more work on her equations, either. After that, she spent approximately seven minutes finishing the problem using guess and check. Abby's difficulty in representing the coin problem algebraically is similar to Alyssa's in that she expresses a need to know the number of nickels before proceeding with the task.

Only one of the six ENS students who participated in interviews correctly solved the coin problem. That student was Emma, a ninth-grade student enrolled in Algebra 2, and she needed fairly significant support in her solution, despite having studied systems of equations in Algebra 1 and Algebra 2. Working independently, Emma wrote the equations $d = 3 + n$, $q = n - 2$, $n = d - 3$, and $.10d + .25q + .05n = 5.40$. This was the extent of her ability to work through the task independently. She was then asked "is there something you could plug in down there (pointing to the fourth equation) for *d*?", to which she responded by writing a fifth equation: $.10(3 + n) + .25(n - 2) + .05(q - 3) = 5.40$. Then, with support, she changed the *q – 3* back to an *n*, and solved to find the number of each type of coin.

Elizabeth, a ninth grade student in Algebra 1 Part 2, was unsuccessful on the coin problem, but was able to write three correct, and important equations in working toward a solution: $n + 3 = d$, $n - 2 = q$, and $d.10 + q.25 + n.05 = 5.40$. Elizabeth also wrote the equation $n + 3 + n - 2 + n = 5.40$. Despite discussion about the meanings of the variables, Elizabeth was not able to incorporate substituted expressions for *d* and *q* into an equation, while also representing the values of each type of coin, .10, .25, and .05. Due to this limitation, she did not complete the task; however, she demonstrated awareness that her algebraic work was productive, but that she was somehow missing the link between her equations that would allow her to complete the task. Elle, Emily, and Erin, three other ENS students, also struggled to understand how to incorporate the substituted expressions for *d* and *q* into an equation while also representing the value of each type of coin.

The main difference between ENS students' solutions and aTNS students' solutions is that all six of the ENS students attempted to solve the task algebraically, even if they were not enrolled in an algebra class. Moreover, each of them demonstrated understanding that an algebraic method would be more productive and efficient than a guess and check method. In other words, while aTNS students made their best efforts at an algebraic solution to the coin problem, they were unable to combine the equations representing the relationship between the numbers of dimes and quarters to the number of nickels ($d = n + 3$ and $q = n - 2$) and the equation relating the value and number of each type of coin with the total value ($.10(n + 3) + .05n + .25(n - 2) = 5.40$). Due to this inability, their efforts on the problem resulted in equations that did not incorporate the number of each type of coin, the value of each type of coin, and the total value.

Despite this limitation for aTNS students, the coin problem (A5) is the only task from the algebra interview on which the aTNS students had a higher success rate than the ENS students. Three aTNS students solved A5 in comparison to one ENS student. Thus, while the aTNS students solved the task in a less sophisticated manner, they were more persistent in their problem solving efforts; when algebra was not productive, aTNS students sought out a different method.

The modified coin problem (A6). In total, 17 students solved the modified coin problem correctly (A6; Appendix B). Similar to the results of the coin problem (A5), students' use of algebra or guess and check to solve the problem was more closely related to their number sequence than to their enrollment in an algebra course, with the exception of ENS students (Table 4.12). Of the seven students enrolled in an algebra course, all solved the modified coin problem correctly, and four of them solved the task algebraically. Each of the four students enrolled in an algebra course who solved the task correctly using algebra had constructed at least an ENS; the other three students enrolled in an algebra course who solved the task using guess and check had constructed either a TNS or an aTNS. Both TNS students solved this task using guess and check, although one was enrolled in an algebra course and the other was not. Of the eight aTNS students interviewed, seven were successful on the modified coin problem and used a guess and check method, while the one student who was unsuccessful attempted to solve the task algebraically; the one unsuccessful student was not enrolled in an algebra course. Both GNS students solved this task algebraically, although one was enrolled in an algebra course and the other was not. The only discernable pattern of solution method based on math class enrollment was for ENS students; the three ENS students enrolled in an algebra course solved the modified coin problem algebraically, and the three ENS students not enrolled in an algebra course solved the modified coin problem using guess and check.

Table 4.12. Results of The Modified Coin Problem by Course Enrollment and Solution Method, within each Number Sequence

	Less Than Algebra I		At Least Algebra I		
	Guess & Check	Algebra	Guess & Check	Algebra	Total
TNS	1/1 (100%)	--	1/1 (100%)	--	2/2 (100%)
aTNS	5/5 (100%)	0/1 (0%)	2/2 (100%)	--	7/8 (87.5%)
ENS	3/3 (100%)	--	--	3/3 (100%)	6/6 (100%)
GNS	--	1/1 (100%)	--	1/1 (100%)	2/2 (100%)
Total	9/9 (100%)	1/2 (50%)	3/3 (100%)	4/4 (100%)	17/18 (94.4%)

Algebraic solutions to the modified coin problem (A6). Travis, a ninth-grade student enrolled in Algebra 1 Part 2, began the modified coin problem using guess and check, but was prompted to try to represent the situation algebraically. He wrote the equation: $6d + n - 1q = 17$, and explained that

> Travis (Tr): The *d* is dimes, the *n* is nickels, so it would be 6 dimes, uh, 6 more dimes than, 6 more dimes than nickels and one fewer, so it'd be minus *q* for quarters equals 17.
>
> I: What does the *q* stand for again?
>
> Tr: Quarters.
>
> I: OK. What about the quarters?
>
> Tr: There's one fewer than the nickels.

Travis's use of *6d* and *1q* in his equation, along with his explanation of their meanings, is evidence of his use of the variables *d* and *q* as abbreviated words standing in for dimes and quarters, rather than his conceptualization of them as unknown quantities. At this point in the interview, Travis was asked if he could write any other equations that might be helpful in his solution; he stated that he could go back to guessing numbers. This indicates that he was unable to conceptualize the relationships between the numbers of each type of coins (e.g., six more dimes than nickels) while simultaneously conceptualizing the total number of coins to be 17.

Of the eight aTNS students interviewed, seven of them correctly solved the modified coin problem using guess and check. Of these students, four wrote at least one correct equation when they were prompted to do so. Alex, a ninth-grade Algebra 1 Part 2 student, wrote the equation $d + n + q = 17$, but was unsure how to represent the relationships between the number of dimes and nickels, and quarters and nickels, in equations. When asked about this he responded that “I wanted to say that there’s already six dimes because I was going to see if we could add that into there.” This excerpt demonstrates that Alex was unable to conceptualize the relationship between an unknown number of dimes and nickels.

Abby and Amanda were able to represent the relationship between the numbers of each type of coin, but could not correctly write an equation to represent the total number of coins. Abby, a sixth-grade Math 6 student, wrote the equations $n + 6 = d$ and $n - 1 = q$, to start. She had also attempted to represent the sum of the dimes, nickels, and quarters algebraically with the expression $6 + n - 1$, but then struggled to finish the problem algebraically.

I: What do we know about all three of them (the variables)?

Ab: That they all have to equal, they all have to combine to make 17.

I: Great. How could you write that out in an equation?

Ab: 17. 17. OK. Nickels plus six. Ohhh! You have, OK, so 13 nickels. That would be six more.

Abby worked quietly for about two more minutes, before stating that “there are four nickels, three quarters, and ten dimes.” In writing the expression $6 + n - 1$, Abby was attempting to sum the number of dimes, nickels, and quarters by incorporating the information that there are six more dimes than nickels and one fewer quarter than nickels. Despite having represented this information accurately in her initial two equations ($n + 6 = d$ and $n - 1 = q$), she could not incorporate it into a sum. She was also unable to conceptualize the role of the 17 in an equation, despite stating that "they all have to combine to make 17."

Amanda's attempt at solving the modified coin problem had some similarities to Abby's. Amanda initially wrote the same two equations as Abby, representing the relationships between the numbers of nickels, dimes, and quarters. Throughout her equation writing, however, when asked what the *d* represented, she said "what I've not figured out yet … and this all hangs in the balance because I don't know nickels yet… . Because if I find out nickels, then the rest of it will solve itself." In these statements, Amanda is demonstrating an inability to conceptualize $n + 6$ as a quantity in and of itself, and rather, thinks of it as two separate quantities—an amount of nickels and six—which she must literally sum before relating their value to the number of dimes.

Amanda wrote an equation to represent the sum of the coins, as well: $T = 6 + n \cdot 6 - 1$. In this equation, *T* represents the total, 17. This equation is similar in nature to Abby's last equation, because in an attempt to incorporate the idea that there are six more dimes than nickels and one fewer quarter than nickels, they both simply incorporated a plus six and a minus one. This is evidence of both Abby and Amanda's inability to conceptualize the relationship between the numbers of coins while simultaneously conceptualizing the sum of the number of each type of coin. Aaron and Andy, both of whom have constructed only an aTNS, are the only other students who wrote equations similar to those of Amanda and Abby. Although not all ENS students were successful in solving the modified coin problem algebraically, no ENS student made such an error in their equation writing; this was unique to these aTNS students.

As previously noted, three ENS students solved the modified coin problem algebraically and three used guess and check. The students who used guess and check were asked to write equations to represent the problem, although for the ENS students who were not in an algebra course, solving these equations was not as accessible as guess and check. All ENS students, who were asked to write equations,[16] represented the relationship between the numbers of coins using some variation on the equations $n + 6 = d$ and $n - 1 = q$, as well as some variation of $d + n + q = 17$. Four of the five ENS students who wrote equations also correctly substituted expressions from the first two equations into the third to form an equation similar to: $(n + 6) + n + (n - 1) = 17$. Evan, a sixth-grade student enrolled in Math 6, was the only ENS student asked to work on this problem algebraically who was unable to make such substitutions.

Although ENS students who were not enrolled in at least Algebra 1 were not successful in solving the modified coin problem algebraically, all ENS students who were asked to represent the task algebraically were able to write at least three correct equations, and four of them could write all four of the correct equations necessary to solve the task algebraically. This is different than the aTNS students, who, regardless of whether or not they were enrolled in an algebra course, could not successfully generate the fourth equation necessary to solve the modified coin problem algebraically.

Guess and check solutions to the modified coin problem (A6). Twelve students used guess and check to find a correct solution to the modified coin problem: two TNS students, seven aTNS students, and three ENS students. All students who used guess and check were successful.

Travis immediately attempted to guess and check a solution to the task. He spent approximately two minutes thinking before he expressed confusion about guessing a number of

[16] Emily is the only ENS student who was not asked to write equations because of time limitations.

dimes while maintaining the relationship between the numbers of nickels and quarters. "I'm trying to get a number that's six more, um, than um, dimes that have six more than nickels so that I can see how many quarters, or um… ." This demonstrates that to Travis, calculating the relationships between the numbers of dimes and nickels, and quarters and nickels, respectively, could not be conceptualized simultaneously. Instead, they were treated as two separate but related tasks. Travis then talked through one numerical example, in which there were 13 dimes, seven nickels, and six quarters. These represent the correct relationships between the numbers of coins, but Travis lost track of there being only 17 coins in all. Evidence of this is that when asked if he had found the correct combination of coins, he first found the sum of the coins in his example, and then thought for 10 more seconds before saying his solution was too large.

At the conclusion of the task, Travis explained his use of guess and check.

> Tr: I would uh, pick a number (for dimes) and see, um, then see what would be six less than that (nickels), and then one less than that (quarters). Then I would add them all up and see if they would equal 17.

Travis's step-by-step explanation for solving the modified coin problem characterizes his inability to conceptualize the number of each type of coin embedded within the total number; due to this limitation, he worked through the task as if it were three separate but related problems.

Tabitha, on the other hand, first tried to exhaust the 17 on the modified coin problem before attending to the relationships between the numbers of coins. She initially said "I'm thinking about what would add up to 17" and guessed six dimes, six nickels, and five quarters. At the interviewer's suggestion, Tabitha then tried adjusting the relationships between the numbers of coins first and checking the sum of the numbers of coins second. Although she was more successful following this suggestion, the processes of determining the numbers of coins and checking them against the sum of 17 were still two steps for Tabitha, as it was for Travis.

The seven aTNS students who used guess and check on the modified coin problem did so more fluidly than the TNS students. Abby, for example, started the task by drawing 17 circles to represent the 17 coins. Then, she began filling in the coins with N's, D's, and Q's to represent each type of coin. While filling in the coins, she attended to the relationship between the numbers of coins. Ann's solution was different, although she also used a picture to support her guessing and checking behavior (Figure 4.4). She started by drawing three circles with N's written inside to represent nickels, then explained that "we always have to have six more dimes than nickels. … And then you'd have to have one fewer quarter than nickels, so you'd have to have two." She drew nine circles with D's inside to represent dimes, and two circles with Q's inside to represent quarters. When she determined this was only 14 coins in all, she made an incremental adjustment to all of the types of coins. Ann drew one more of each type of coin on the page, and after doing so did not need to check that the relationships between the numbers of coins held; it was assumed. Of the seven aTNS students who solved the modified coin problem using guess and check, six of them made similar incremental adjustments to their guesses. Ann and Abby both used pictures to support their guessing and checking. The other four aTNS students who used guess and check wrote down each guess in the form of an ordered triple or an informal chart. All seven of them made incremental adjustments to their guesses.

In addition to the seven aTNS students who used guess and check on the modified coin problem (A6), three ENS students also guessed and checked. In contrast to the aTNS students, however, Elle and Emily both used an unwinding strategy (Knuth et al., 2006) to begin task A6. They used the fact that there were six more dimes than nickels, and 17 coins in all to reduce the number of total coins to 11; this enabled them to think about there being the same number of dimes and nickels. Also, all three ENS students who solved task A6 using guess and check guessed the solution on the first try. Evan stated that he “accidentally” solved the task because

Figure 4.4 Ann’s use of a drawing on the modified coin problem. “Coins” enclosed in a black rectangle are the coins Ann added using an incremental adjustment. (Rectangles were added digitally following the interview.)

the interviewer was trying to encourage him to work on an algebraic solution, but he guessed the correct answer before he could make progress on the task algebraically. The ENS students’ guess and check strategies were more sophisticated than those of the aTNS students. No aTNS students used an unwinding strategy to simplify the number of relationships in the problem, nor did any aTNS students guess the solution to the problem on the first try.

Some students with a TNS, an aTNS, and an ENS all used guess and check on the modified coin problem, but their guessing and checking behaviors were distinct. TNS students worked through the task sequentially by guessing a number of nickels, calculating the number of dimes and quarters, and then retrospectively comparing the number of coins to 17. Or, by guessing an arbitrary combination of three numbers that summed to 17 and then retrospectively checking the relationships between the number of dimes, nickels, and quarters. aTNS students made incremental adjustments to their guesses by adding or subtracting one from the number of each type of coin, which demonstrates that they were able to simultaneously maintain the goals of exhausting all 17 coins while maintaining the relationships between the numbers of coins. ENS students' guess and check methods were unique in comparison to TNS and aTNS students. Two ENS students used unwinding strategies to reason about the task and all three ENS students who guessed and checked guessed the answer on the first try. Furthermore, whereas TNS and aTNS students tended to guess and check regardless of whether or not they were enrolled in an algebra course, only ENS students who were not enrolled in an algebra class relied on a guess and check method. This suggests that algebraically solving systems of equations by operating on a two-level unit structure is a concept on which ENS students are prepared to accept instruction.

The visual block pattern (A7) and the block pattern (A8). All students who were presented with the block pattern problems (tasks A7 and A8; Appendix B) successfully generalized using examples (Table 4.13). The majority of students also verbalized the patterns on each task, with the exception of Tabitha, Travis, and Abby, who did not verbalize explicit patterns on either of the block pattern problems. Neither TNS student was successful in representing the task algebraically; three of seven, and four of eight aTNS students successfully represented tasks A7 and A8 algebraically, respectively, and all ENS and GNS students did.

On each task, students were asked to find the first several terms, and then to skip ahead to find the one-hundredth term. Students were asked to look for an explicit pattern in order to find the one-hundredth term. Both TNS students needed support to engage in this form of algebraic reasoning. Travis, for example, applied a recursive pattern of adding one block to each figure on the visual block pattern problem (A7). He was then asked to determine the number of blocks in the one-hundredth figure.

> Tr: You would keep, um, adding up one block at a time, so… (thinking for 2 minutes, 7 seconds)

Table 4.13. Results of The Block Problems

	Grade 6		Grade 7		Grade 8		Grade 9		Total	
	A7	A8	A7	A8	A7	A8	A7	A8	A7	A8
TNS										
Generality	1/1	1/1	--	--	--	--	1/1	1/1	2/2 (100%)	2/2 (100%)
Verbalize	0/1	0/1	--	--	--	--	0/1	0/1	0/2 (0%)	0/2 (0%)
Algebraic	0/1	0/1	--	--	--	--	0/1	0/1	0/2 (0%)	0/2 (0%)
aTNS										
Generality	3/3	3/3	3/3	3/3	1/1	1/1	--	1/1	7/7 (100%)	8/8 (100%)
Verbalize	2/3	2/3	3/3	3/3	1/1	1/1	--	1/1	6/7 (85.7%)	7/8 (82.5%)
Algebraic	0/3	1/3	2/3	2/3	1/1	1/1	--	0/1	3/7 (42.9%)	4/8 (50%)
ENS										
Generality	2/2	2/2	--	--	2/2	2/2	--	1/1	4/4 (100%)	5/5 (100%)
Verbalize	2/2	2/2	--	--	2/2	2/2	--	1/1	4/4 (100%)	5/5 (100%)
Algebraic	2/2	2/2	--	--	2/2	2/2	--	1/1	4/4 (100%)	5/5 (100%)
GNS										
Generality	--	--	1/1	1/1	1/1	1/1	--	--	2/2 (100%)	2/2 (100%)
Verbalize	--	--	1/1	1/1	1/1	1/1	--	--	2/2 (100%)	2/2 (100%)
Algebraic	--	--	1/1	1/1	1/1	1/1	--	--	2/2 (100%)	2/2 (100%)
Total										
Generality	6/6	6/6	4/4	4/4	4/4	4/4	1/1	3/3	15/15 (100%)	17/17 (100%)
Verbalize	4/6	4/6	4/4	4/4	4/4	4/4	0/1	2/3	12/15 (80%)	14/17 (82.3%)
Algebraic	2/6	3/6	¾	¾	4/4	4/4	0/1	1/3	9/15 (60%)	11/17 (64.7%)
Percentages										
Generality	100	100	100	100	100	100	100	100	100	100
Verbalize	66.7	66.7	100	100	100	100	0	66.7	80	82.3
Algebraic	33.3	50	75	75	100	100	0	33.3	60	64.7

Note: Each cell represents the number of students who completed that portion of the task correctly out of the number of students who attempted that portion of the task, in that category.

I: Tell me what you're thinking about, or what you're trying to figure out right now.

Tr: Something to, um, help me add up to, uh, figure 100.

Travis was attempting to determine how many figures he was skipping to get from the last known figure to figure 100 so that he would know how many times to add one more block; with support, he found the number of missing figures. He eventually applied the explicit pattern of adding two to the given figure number to determine the number of blocks, but when asked to verbalize the explicit pattern, he could not; he could only apply it to specific examples. Additionally, when asked to represent the explicit pattern algebraically for figure *f*, he assigned a value to *f* in order to make sense of the situation.

> I: How many blocks are there in figure *f*. Can you explain how you would do that?
>
> Tr: (thinking for 71 seconds)
>
> I: Seems like you're thinking about something. What are you thinking about?
>
> Tr: Um, how many, uh, blocks would be in figure *f*.
>
> I: OK. Do you have any thoughts about it so far? Doesn't have to be an answer, could just be something you're thinking about.
>
> Tr: I'm trying to figure out if *f* could equal any number, like, uh, six.
>
> I: OK. And so if *f* equals six, um, how many blocks would be in that figure, then?
>
> Tr: Eight.
>
> I: Awesome. Good job. What made you think to use *f* equals six?
>
> Tr: *f* is the sixth letter in the alphabet.

This excerpt demonstrates the difficulty that Travis experienced moving beyond the stage of generality through particular examples. Rather than write an expression for the number of blocks in figure *f*, he selected a value for *f* and calculated the number of blocks in the sixth term, which is evidence of his conceptualizing the variable as an alphabetical value. Although Travis could maintain the "plus two" relationship between the figure number and number of blocks when calculating numerical examples, he could not extend this relationship to think abstractly about any figure.

On the block pattern problem (A8), Alex was able to generalize and verbalize his pattern, but failed to write an explicit formula.

> I: Can you write an expression to tell me how many blocks would be in figure *f*?
>
> Al: Hmm. *f* plus six equals something.
>
> I: OK… that's good … Would that ($f + 6$) represent the number of blocks in figure *f*?
>
> Al: Uhh, figure *f*. I'm not sure how to add six to *f* because if you have an equal sign isn't that an equation?
>
> I: Yeah, yeah, yeah. We only need an expression though, we don't need an equation, so we don't have to have an equal sign.
>
> Al: So it's just $f + 6$.
>
> I: OK. Does that work?
>
> Al: No, I don't think it does because it's the number of blocks you need. … Well, if you're taking any figure, if *f* means figure and you have the number beside it plus six, that's the number of blocks you have. Yeah. So, like, if you have *f* and then your number plus six, you'd have your answer.
>
> I: Hmm. OK. So that's good. So can that represent the number of blocks? … Is there any figure number you can think of that this wouldn't work for?
>
> Al: A letter? Yes. [It will work] if it's not a letter.

Alex wrote the expression $f + 6$ while solving this task, but did not demonstrate understanding of what the expression meant. He continually referred to putting the figure number after *f*; in this excerpt, he specifically said, "if *f* means figure and you have the number beside it…" which shows that he did not conceptualize *f* as an unknown quantity. Also, despite going through several numerical examples to try to help Alex understand the role of *f* in the expression, he continued to be hesitant about allowing a letter to stand in for a number. Ava demonstrated a similar difficulty in writing the expression $x + 2$ on the visual block pattern problem (A7). When asked if she could represent the number of blocks in figure *x*, she responded, "No. Cause there's not a number, so you'd have, you wouldn't know how many. You wouldn't know what you need to add. You would have nothing to add two to." Ava's declaration that *x* is not a number is reminiscent of Alex's insistence that the expression doesn't work for a letter; both situations are evidence of students conceptualizing a variable as a specific value.

While only about half of aTNS students represented their generalized pattern algebraically on the block pattern problems (A7 and A8), all ENS students were able to do so. ENS students also had greater ease in identifying explicit patterns. Elle, for example, initially identified a recursive pattern on the visual block pattern problem (A7). The interviewer gave her two prompts, however, that were sufficient to point her toward an explicit pattern. The interviewer suggested she make a chart to arrange the information, and suggested she look for a pattern "across the table," meaning between the figure numbers and the numbers of blocks. At these suggestions, Elle immediately stated "four minus two equals two, and three minus one equals two." These values correspond to the second figure containing four blocks and the first figure containing three blocks. Then, when asked how she could explain the pattern to her math teacher, she said, "if she had a figure number, I'd tell her, add two to that to get her answer." These two prompts were sufficient in helping Elle transition from a recursive pattern to an explicit pattern; the same two prompts helped the other sixth grade ENS student, Evan, as well. The support provided to Elle and Evan on the visual block pattern problem is the highest level of interviewer intervention that was provided to any ENS students on tasks A7 and A8; ENS students identified explicit patterns and formulas more swiftly than did aTNS students, and some ENS students did not use recursive patterns at all, whereas aTNS students tended to identify recursive patterns first and explicit patterns second, if at all.

Another trend among aTNS students was that finding an explicit pattern on the block pattern problem (A8) was more easily accomplished. All students who completed both tasks completed A7 first and A8 second. One TNS student found an explicit pattern on both tasks, and the other did not find an explicit pattern on either task. Of the ENS students, all four who completed both tasks found explicit patterns and formulas on both tasks. aTNS students, however, tended to struggle to identify an explicit formula on task A7, but not on A8. Aaron and Ann identified an explicit pattern on task A8 without making any mention of a recursive pattern, despite both of them struggling to identify an explicit pattern on task A7. Ann, Amanda, and Andy also struggled to find an explicit pattern on task A7. On task A8, each of these students initially identified a recursive pattern, as they had on A7; however, when asked by the interviewer to recall how they had determined the one-hundredth figure, they quickly and accurately identified an explicit pattern on A8. It is unclear, of course, whether the aTNS students were learning throughout the interview because there is no evidence as to whether or not their newfound ease in finding explicit patterns is lasting. Regardless, these five aTNS students, more so than any others in the interviews, identified the explicit patterns on tasks A7 and A8 with differing levels of ease.

The football (A9) problem. Of the thirteen students who attempted the football problem (A9; Appendix B), six were able to represent the situation algebraically (Table 4.14). No aTNS students who attempted the football problem were able to correctly represent the situation algebraically. Five of the aTNS students were not enrolled in an algebra course, and one aTNS student was enrolled in an algebra course. Four out of five ENS students who attempted the football problem represented the relationship algebraically. Two of those four students were not enrolled in an algebra course, and two were enrolled in an algebra course. The ENS student who did not represent the situation algebraically was not enrolled in an algebra course. Both GNS students represented the football problem algebraically; one of those students was enrolled in an algebra course and the other was not. Furthermore, there is no apparent relationship between the number of students who represented the football task algebraically and the students' grade. However, the percentage of students who represented the task algebraically increases by number sequence; No aTNS students solved the task, 80% of ENS students did, and 100% of GNS students did.

Table 4.14. Results of The Football Problem

	Grade 6	Grade 7	Grade 8	Grade 9	Total
aTNS	0/3 (0%)	0/2 (0%)	0/1 (0%)	--	0/6 (0%)
ENS	2/2 (100%)	--	1/2 (50%)	1/1 (100%)	4/5 (80%)
GNS	--	1/1 (100%)	1/1 (100%)	--	2/2 (100%)
Total	2/5 (40%)	1/3 (33.3%)	2/4 (50%)	1/1 (100%)	6/13 (46.2%)

Note: Each cell represents the number of students who completed that portion of the task correctly out of the number of students who attempted that portion of the task, in that category.

Although no aTNS students solved the football problem, several of them made progress toward a solution. Abby, for example, attempted to represent the situation algebraically using two separate equations. She wrote the equations $x - 3 = s$ and $7/s = td$, where *x* represents the team's score last week, and *s* is team's score this week, and the two equations are meant to be executed in succession. Abby explained that after using the first equation to calculate the team's score this week, the second equation is used to determine the number of touchdowns (*td*) scored this week. Despite other errors in the equations, representing the task using two equations is evidence that Abby conceptualizes the situation as involving two separate problems, rather than one.

Similarly, Aaron and Alyssa, both of whom were unsuccessful on the football problem, represented the task using two equations, rather than one. Aaron began the task by correctly working through two number examples. He then wrote the equations: $\frac{x}{7} = y - 3 = z$. In this string of equations, he stated that *x* stood for last week's score and *y* for this week's score. Aaron explained his equation using a number example, in which the team scored 35 points last week.

> Aaron (Aa): Last week their score was 35, and that's how many points they have, so you have divided by seven to get how many touchdowns. And then since we did 35, I got, you'd get five. … So once you got five here, you subtracted it by three and got two.
>
> I: OK. So this *z* at the end, what does that stand for?
>
> Aa: That stands for the two.

This excerpt provides evidence that Aaron conceptualizes the task as two separate, but related, problems; dividing the team's score by seven is one problem, and subtracting three touchdowns is the second. Also, the manner by which Aaron explains his equation using a specific number example, and then defines the variable, *z*, as "two," instead of as an unknown quantity, is evidence that Aaron conceptualizes the variables in his equations as specific numbers. Finally, Aaron wrote a string of equations, in which the first expression, $\frac{x}{7}$, is equivalent to *y*, and the second expression, $y - 3$, is equivalent to *z*. This is evidence of an operational concept of the equal sign, which is inconsistent with Aaron's concept of the equal sign during the screening tasks.

The final difficulty demonstrated by aTNS students was the use of inverse arithmetic operations to work through number examples. For instance, when working through a number example, in which the team scored 28 points last week, Alyssa was asked to remind the interviewer how she arrived at four touchdowns. Alyssa responded, "Because seven times four is 28. You had seven points for each one, and there's four. 28." In this instance, Alyssa multiplied four times seven to get 28, despite four being the answer. Using this number example to facilitate the writing of an equation was relatively useless, because she did not conceptualize the problem as involving division by seven. Instead, she conceptualized it as figuring out what number times seven would result in 28; this was a struggle for both Alyssa and Ava.

ENS students demonstrated some difficulties in representing the football problem algebraically that were similar to those of the aTNS students. Three of the four ENS students who represented the task algebraically initially wrote two separate equations. Those students were Elizabeth, Elle, and Evan. The distinction for these three students from the aTNS students, is that although they began the problem by writing two equations, these three ENS students were all able to combine their two equations into one. Elle, for example, initially wrote two equations: $s + 21 = a$ and $a/7 = d$. Similar to aTNS students Abby and Aaron, she indicated that s, this week's score, plus 21 results in a, or last week's score. Then, dividing a by seven results in d, or the number of touchdowns scored last week. When asked if the two equations could be combined into one, she confessed that "I know how to do it, but I just don't know how to write it." Elle was able to work through numerical examples, but struggled to think about the situation singularly. The interviewer prompted Elle "Explain it to me in words." Elle then explained and wrote simultaneously: “So basically we take *c* plus two, wait hold on. *c* plus three, which would equal the total number of touchdowns. We times that by seven because that’s how much each touchdown’s worth, which would give you 35.” The corresponding equation was $(c + 3)7 = b$, in which *c* was the number of touchdowns scored last week, and *b* was the score this week. Although Elle’s equation represents this week’s score in terms of last week’s number of touchdowns, instead of last week’s number of touchdowns in terms of this week’s score, she used a number example to support her equation writing in a way that aTNS students were not able to do.

Elizabeth tried to use an inequality to represent the relationships in the football problem before moving on to an equation. After clarifying the specifics of the problem, she asked

> El: Do we have to have an equal sign?
>
> I: No. … See what you can come up with.
>
> El: Well, I was thinking that I don’t know what to do, so I was like, well, maybe it’s there. (wrote $L > D$ indicating the number of touchdowns scored last week is greater than the number of touchdowns scored this week.)
>
> I: Ahh, you’re going to represent it as an inequality. Very good. I’d like to know how *much* bigger it is, though.
>
> El: It’s three times bigger.

When Elizabeth stated that “It’s three times bigger,” she was referring to the three more touchdowns that the football team scored last week compared to this week. She mistakenly applied a multiplicative comparison, rather than an additive one, and moreover, wrote that the number of touchdowns from last week was greater than the number of touchdowns from this week, when in fact, it was equal to three times that amount. Following this exchange, the interviewer prompted Elizabeth to go back to the two equations she had previously written, and Elizabeth successfully combined them into one, correct equation.

The soccer problem (A10). The soccer problem (A10; Appendix B) was more accessible to students within each number sequence than the football problem. Sixteen students attempted task A10, and 13 represented the situation algebraically (Table 4.15). The three students who did not represent the soccer problem algebraically were Tabitha, Travis, and Ann.

Table 4.15. Results of The Soccer Problem

	Grade 6	Grade 7	Grade 8	Grade 9	Total
TNS	0/1 (0%)	--	--	0/1 (0%)	0/2 (0%)
aTNS	2/3 (66.7%)	3/3 (100%)	1/1 (100%)	1/1 (100%)	7/8 (87.5%)
ENS	2/2 (100%)	--	1/1 (100%)	1/1 (100%)	4/4 (100%)
GNS	--	1/1 (100%)	1/1 (100%)	--	2/2 (100%)
Total	4/6 (66.7%)	4/4 (100%)	3/3 (100%)	2/3 (66.7%)	13/16 (81.3%)

Note: Each cell represents the number of students who completed that portion of the task correctly out of the number of students who attempted that portion of the task, in that category.

Tabitha did not represent the situation algebraically. She considered two numerical examples to make sense of the task. Then she wrote the expression $z - 3$, and turned that into the equation $z - 3 = 6$, with z representing last week's score.

I: What does that ($z - 3$) equal when we take last week's score minus 3?

T: Umm…

I: What was the three and the two? (The team's scores this week in the two number examples)

T: Your this week's score.

I: Yeah, that's exactly right. What should we set this (z – 3) equal to, then?

T: Six?

I: Where are you getting six?

T: Because I added two and three right here[17].

I: Ohh, OK. Go ahead and write that down. … Tell me what your equation says and then explain it to me.

T: z minus three equals six. And… you could do nine minus three equals six.

I: Oh, good. Perfect. And what does that tell us, then?

T: It tells us that, um, last week's score minus the three fewer points that they made this week equals the this week's score.

In this data excerpt, Tabitha relies heavily on numerical examples both to build her equation and to interpret the meaning of the equation. First, despite having defined a variable, b, earlier during the problem to represent the team's score this week, she does not use it in her equation. Instead, she selects the two numerical values that represented the team's score this week, three and two, and erroneously adds them to decide that the team's score this week is six. Although she seems aware that six is standing in for the team's score this week in the last phrase of the data excerpt, she does not equate the value with b at any point. This is evidence that to Tabitha, the variables are standing in for specific numerical values.

Travis was able to write the correct equation to represent the additive relationship on the soccer problem, but like Tabitha, was reliant upon a numerical example. He initially wrote $L = T - 3$, which represents last week's score being equal to this week's score minus three; this is a reversal error. Upon checking his equation with a numerical example, he became aware of his

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[17] It's unclear whether Tabitha meant she multiplied three and two, or whether she added incorrectly; this mistake is inconsequential to the point at hand.

mistake, and re-wrote the equation as $T = L - 3$. Without a numerical example, Travis would not have been perturbed by the incorrect equation he initially wrote. Furthermore, even after rationalizing his correct equation, he explained its meaning using numerical examples.

Ann is the only aTNS student who did not solve the soccer problem correctly. She, like Travis and Tabitha, relied on numerical examples in writing her equation.

> Ann (An): Let's say five [I: OK.] was last week. And this week, since they scored three fewer, it'd be five minus three equals … two.
>
> I: Very good.
>
> An: So that would, so two would be this week.
>
> I: So we want to represent their score this week in terms of their score from last week. So what variables might you want to use?
>
> An: Since we have what the numbers could be, then you could take two, hmm, plus three equals five.
>
> I: Very good. And could we use variables in our equation since we don't know if five and two were actually their scores?
>
> An: Yes. You could go, like, *x* plus three equals *y*.
>
> I: Very good. So in that equation, what do the *x* and the *y* represent?
>
> An: So, since we did that and we already, we found out what it could be, *x* could represent two plus three would equal five, so *y*.

Ann is indicating that in the equation $x + 3 = y$, *x* is standing in for two and *y* is standing in for five; she is considering the variables in her equation as standing in for the specific numerical examples she used to represent the situation. Accordingly, despite the correctness of the actual equation, Ann's solution is not considered to represent the situation algebraically.

Of the aTNS students who represented the soccer problem algebraically, four were almost immediate in their response and wrote some variation of the equation $L - 3 = T$, with L representing the team's score last week and T representing the team's score this week. These four students did not use any number examples prior to writing their equations and were able to communicate the meanings of their variables and the equation as a whole. These students were Abby, Aaron, Alyssa, and Amanda. Amanda is the only one of these four students to be enrolled in an algebra course.

Summary of students' algebraic reasoning. Students' algebraic reasoning will be summarized in this section by examining students' abilities and weaknesses. These will be disaggregated by number sequence for purposes of comparison, and will also identify patterns in students' algebraic reasoning between algebraic tasks.

Students' algebraic abilities. Across the algebraic tasks, indications were noted of students' abilities to engage in the following types of algebraic reasoning: generalizing through particular examples, verbalizing generalized patterns, representing additive relationships algebraically, representing multiplicative relationships algebraically, and nesting algebraic expressions (Table 4.16). The types of algebraic reasoning are arranged from left to right in an approximate order of increasing difficulty, based on the results of extant literature and the results of the present research study. The numbers listed below each type of algebraic reasoning are the task numbers on which that type of reasoning was observed by at least one student. The check marks listed next to each students' name indicate the task numbers on which the student demonstrated each particular type of algebraic reasoning. Cells that are greyed indicate the student attempted the type of algebraic reasoning noted, but was unsuccessful. Cells marked with a dash (-) indicate the student did not attempt the indicated problem during their interview.

Finally, cells that are blank indicate that the student’s solution method did not require the noted form of algebraic reasoning. This was particularly prevalent on the border problem (A4) and the coin problems (A5, and A6) because students used a variety of methods.

All students, regardless of number sequence, were able to generalize through particular examples. This form of algebraic reasoning was observed by students on the border problem (A4), the visual block pattern problem (A7) and the block pattern problem (A8). Travis only engaged in these two forms of algebraic reasoning on the block pattern problems (A7 and A8), as he made very little progress on the border problem (A4). Tabitha verbalized a generalized pattern

Table 4.16. Students' Algebraic Abilities across All Tasks

	Generalize			Verbalize			Represent Additive Relationships Algebraically							Represent Multiplicative Relationships Algebraically			Nest Algebraic Relationships			
	4	7	8	4	7	8	4	5	6	7	8	9	10	1	4	5	4	5	6	9
TNS																				
Tabitha	✓	✓	✓	✓				--				--			--	--		--		--
Travis		✓	✓	--			--	--				--			--	--	--	--		--
aTNS																				
Ann	✓	✓	✓	✓	✓	✓			✓		✓				✓					
Abby	✓	✓	✓	✓					✓			✓	✓							
Aaron	✓	✓	✓	✓	✓	✓			✓				✓	✓	✓					
Ava	✓	✓	✓	✓	✓	✓		--					✓			--		--		
Alyssa	✓	✓	✓	✓	✓	✓				✓	✓	✓	✓	✓						
Andy	✓	✓	✓	✓	✓	✓		--		✓	✓	--	✓			--		--		--
Amanda	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓			✓			
Alex	✓	--	✓	✓	--	✓				--		--	✓							--
ENS																				
Elle	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓		✓		✓*	✓*	✓
Evan	✓	✓	✓	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓		✓			✓*
Emily	✓	✓	✓	✓	✓	✓				✓	✓		--	✓	✓	✓				
Erin	✓	✓	✓	✓	✓	✓			✓	✓	✓	✓	✓	✓	✓				✓*	✓
Elizabeth	✓	--	✓	✓	--	✓		✓	✓	--	✓	✓	✓	✓	✓	✓	✓	✓*	✓*	✓*
Emma	✓	--	--	✓	--	--	✓	✓	✓	--	--	--	--	✓	✓	✓	✓	✓	✓*	--

Checked cells (✓) indicate that the student correctly engaged in the specified type of algebraic reasoning. Greyed cells indicate the student's incorrect attempt to engage in the specified type of algebraic reasoning. A dash (--) indicates that the student did not attempt the specified task. An empty cell indicates that the student solved the task in a manner that did not require that form of algebraic reasoning.

*On these tasks, the students nested an algebraic expression in a sense, but their nested expression was not multiplicative in nature, as were the other demonstrations of nesting algebraic expressions.

on the border problem, but neither TNS student represented an additive relationship algebraically. In fact, Table 4.16 demonstrates that for the two TNS students interviewed, generalizing through particular examples was the extent of their abilities to engage in algebraic reasoning.

aTNS students were able to generalize through particular examples and to verbalize generalized patterns. There were opportunities to represent the additive relationship between two unknown quantities algebraically on all tasks from A4 through A10, and all aTNS students represented at least one of those relationships during their interview. Alex and Ava represented one additive relationship algebraically, Ann and Aaron represented two, Abby and Andy represented three, Alyssa represented four, and Amanda represented additive relationships algebraically on all seven of the tasks. In contrast, aTNS students were unable to correctly represent an additive relationship algebraically on between zero and four tasks.

Additionally, there were opportunities to represent a multiplicative relationship on the phone cords problem (A1), the border problem (A4), and the coin problem (A5), although on the border problem and the coin problem, this was dependent on the student's solution method. For example, students who guessed and checked on the coin problem did not represent a multiplicative relationship algebraically, but that form of algebraic reasoning did not follow from their solution method. Accordingly, those cells in Table 4.16 were left blank. On the other hand, students who attempted to solve the coin problem algebraically, but were unable to generate an equation demonstrating a multiplicative relationship between quantities were marked with a grey cell in Table 4.16. All aTNS students attempted to represent a multiplicative relationship algebraically on at least the phone cords problem. Four of the eight aTNS students represented a multiplicative relationship on at least one of the noted tasks, and four were unable to represent a multiplicative relationship on the noted tasks. Only one aTNS student, Aaron, represented a multiplicative relationship on more than one task. Thus, Table 4.16 demonstrates that for aTNS students, generalizing through particular examples and verbalizing patterns are readily accessible forms of algebraic reasoning. aTNS students represented additive relationships algebraically, although with inconsistency, and some students represented multiplicative relationships algebraically, but again, inconsistently. Finally, nesting algebraic relationships was generally beyond the ability of aTNS students.

Similar to aTNS students, all ENS students generalized through particular examples, verbalized generalized patterns, and represented the additive relationship between two unknowns algebraically. Above the algebraic reasoning of some aTNS students, all ENS students also represented multiplicative relationships algebraically on at least one task, and five out of six ENS students nested an algebraic expression within another equation or expression, in such a way that the nested expression is operated upon either multiplicatively or additively. Nesting an algebraic expression was observed on the border problem (A4), the coin problems (A5 and A6), and the football problem (A9), and only one aTNS student demonstrated this form of algebraic reasoning. On these tasks, all five ENS students who nested an expression wrote equations that operated on another algebraic expression additively and multiplicatively. Thus, while aTNS students could not generally nest algebraic expressions, ENS students generally could. Also, ENS students represented additive relationships with almost 100% accuracy, and represented multiplicative relationships on at least two out of the three tasks noted.

Students' conceptions and behaviors limiting algebraic reasoning. In addition to the types of algebraic reasoning students exhibited throughout the interview, students also engaged in reasoning indicative of non-normative conceptions and behaviors that were either non-standard or counterproductive to their algebraic reasoning (Table 4.17). One category of these behaviors was "building" equations. Evidence of students building equations was their reliance upon numerical examples in order to write an equation. Students who built equations using numerical examples did so to construct additive (a or A) or multiplicative (m or M) equations, where lower case letters denote students who built equations incorrectly or that could not be explained normatively, and capital letters denote students who built equations correctly and explained them normatively. Students' conceptions of variables are based on Stacy and MacGregor's (1997) framework. This framework outlines five conceptions of a variable that students may hold that are non-normative. These include a variable as a numerical value (#), as an abbreviated word or label (L), or as an alphabetical value (a). Within this framework, students may also use unrelated variables to represent related quantities (u), may ignore variables all together (i), or may use the same variable as a general referent for unrelated quantities (s). The equality category indicates tasks on which students reverted to an operational concept of the equal sign despite reasoning in a manner consistent with a relational concept of the equal sign during the screening tasks in the interview. Finally, miscellaneous non-normative and non-standard forms of reasoning are noted. These include:

- Students using additive reasoning to write an equation that represents a multiplicative situation (+),
- Reversal errors made in equation writing (R),
- Students expressing a "need" to know the value of an expression before completing an equation (N),
- Students writing two or more separate equations or expressions, or a string of equations, to represent a situation instead of writing one equation (E),

- Students struggling to incorporate appropriate arithmetic operations in an equation because they completed numerical examples using the inverse operation (/),
- Students writing an inequality rather than an equation to represent the relationship between two unknown quantities (<).

The four overarching categories and the subcategories within each, are not suggested to be hierarchical.

There are a very small number of limiting conceptions and behaviors for the TNS students. This is because they engaged in such a limited amount of algebraic reasoning throughout the interview. Recall that Tabitha generalized by particular examples and verbalized on pattern, and Travis only generalized. The TNS students did not write any correct equations, which made fewer opportunities for them to build an equation using numerical examples, or to revert to an operational concept of the equal sign.

There was evidence of many more limiting conceptions and behaviors among aTNS students, compared to TNS students. All aTNS students built at least one equation throughout their interview. Six aTNS students used numerical examples to build an equation involving a multiplicative relationship. The two aTNS students who did not, Ava and Andy, were also not successful in writing an equation including a multiplicative relationship. The three students who successfully represented the multiplicative relationship on the phone cords problem (A1) algebraically, however, built the equation using numerical examples. Although this may be a non-standard method of writing an equation, it was potentially beneficial to these three aTNS students. There was also evidence that all aTNS students, except Amanda, reasoned about variables in a non-normative manner at some point throughout the interview. While four aTNS students conceptualized variables in only one non-normative way, five aTNS students conceptualized variables in multiple non-normative ways throughout the interview, and three of those five aTNS students reasoning using more than one non-standard conception of a variable

Table 4.17. Conceptions and Behaviors Limiting Students' Algebraic Reasoning

	"Build" Equation	Conceptions of Variable	Equality	Miscellaneous
		TNS		
Tabitha		4#, 10#		1+
Travis		6L, 7a, 8#, 10#		10R
		aTNS		
Ann	1m, 10a	4s, 7u, 10#		1+, 4E
Abby	1m	4s, 7s		5N, 9E
Aaron	1m	7u, 9#	1, 9	6R, 9E
Ava	1a	4#s, 7#	6	4E, 9/
Alyssa	1m	4au, 5L	1, 8	5RN, 6R, 9E/
Andy	7A	4u	7, 10	
Amanda	1m		1	1+, 6N
Alex	1m, 10A	1i, 4au, 8#, 10L		1+
		ENS		
Elle				9E
Evan	1M			9E
Emily	1M			
Erin	1M	4#s		1+<
Elizabeth	1M		1	9+E<
Emma	1M			

"Build" Equation Codes: a = additive equation; m = multiplicative equation. Capital letters indicate a correct algebraic representation. Lower case letters indicate an incorrect algebraic representation.

Conceptions of Variable: # = numerical value; L = abbreviated word; a = alphabetical value; u = unrelated variables; i = letter ignored; s = same variable

Equality: Student reverted to an operational concept of the equal sign

Miscellaneous: +: Student engaged in additive reasoning rather than multiplicative; R: Reversal error; N: Student expressed a "need" to know the value of an algebraic expression; E: Student wrote two or more separate equations or a string of equations instead of one equation; /: Student struggled to incorporate appropriate arithmetic operations in an equation; <: Student represented an exact relationship using an inequality within the same algebraic task. Additionally, five students reverted to an operational concept of the equal sign when reasoning algebraically at least once during the interview, and seven aTNS students engaged in at least one miscellaneous non-normative or non-standard manner of reasoning throughout the interview.

The picture of ENS students' limiting conceptions and behaviors is much different than that of aTNS students. Five ENS students also built equations in algebraic situations involving a multiplicative relationship, but unlike aTNS students, all five of those students were successful in representing the phone cords problem (A1) algebraically after building the equation using numerical examples. All ENS students who built equations with numerical examples were successful in writing those equations. On the other hand, no ENS students built an equation involving an additive relationship, suggesting that this was not necessary for ENS students to be successful in representing additive relationships algebraically, as it was for some aTNS students.

Only one ENS student demonstrated a non-normative concept of variable throughout the interview. Also, only one ENS student reverted to an operational concept of the equal sign during the interview. Despite evidence of these limiting conceptions and behaviors, each of these students was able to move beyond these limitations to solve the noted tasks correctly. Erin initially used the same variable to represent different values on the border problem (A4), but she was aware that this did not produce a correct equation; when she was unable to correct her equation, she generated a new method and represented it algebraically instead. Elizabeth reverted to an operational concept of the equal sign in her equation writing on the phone cords problem (A1), but used numerical examples to facilitate her equation writing and successfully completed the task.

Finally, three ENS students began the football problem (A9) by writing multiple equations, rather than one equation, to represent the situation algebraically, but all three students were able to combine their equations into a single equation. Elle and Elizabeth did so by building their nested equation using numerical examples. In its entirety, it seems as though for ENS students, the same conceptions and behaviors that were rampant and at times limiting for aTNS students, were overcome by ENS students to successfully complete algebraic tasks.

Chapter 5: Discussion

The Number Sequences of Middle Grades Students

The results of the survey indicate that of the 100 sixth-grade students surveyed, 54 had constructed only an aTNS; this constitutes 54% of the sixth-grade students surveyed. Considering students across all grades six through nine, the percentage with an aTNS decreases to 46.6%. These results are dissimilar from those of Ulrich and Wilkins (2017), who found that approximately 36% of sixth-grade students had constructed an aTNS. Their 36%, however, indicates the number of students who took the assessment, gave assent for their survey to be used in the study, and were found to have constructed only an aTNS. Thus, to facilitate comparison, Figure 5.1 shows the percentage of students who had constructed (from left to right) an aTNS, INS, TNS, ENS, or who did not participate in the research study. Note that the portion farthest to the left represents aTNS students, which is out of order of the hierarchy of number sequences; the graph was arranged in this way to ease the visual comparison of the percentages of students who had constructed an aTNS. With this inclusion, the present research found that 40.3% of sixth-graders, and 32.1% of all middle grades students had constructed an aTNS. These results more closely align with those of Ulrich and Wilkins. In both research studies, the sample was one of convenience, indicating that the results may not represent a larger population.

Even after including students who did not participate in the research study, the percentage of ENS students identified in the present research is still noticeably less than the percentage identified by Ulrich and Wilkins (2017; Figure 5.1, yellow bands). Ulrich and Wilkins found that approximately 40.4% of sixth-graders had constructed at least an ENS, while the present research identified only 21.6% of sixth-graders had constructed at least an ENS and only 27.5%

of all middle grades students had constructed at least an ENS. The difference in the numbers of ENS students identified in sixth grade could hypothetically be explained by classroom-level or

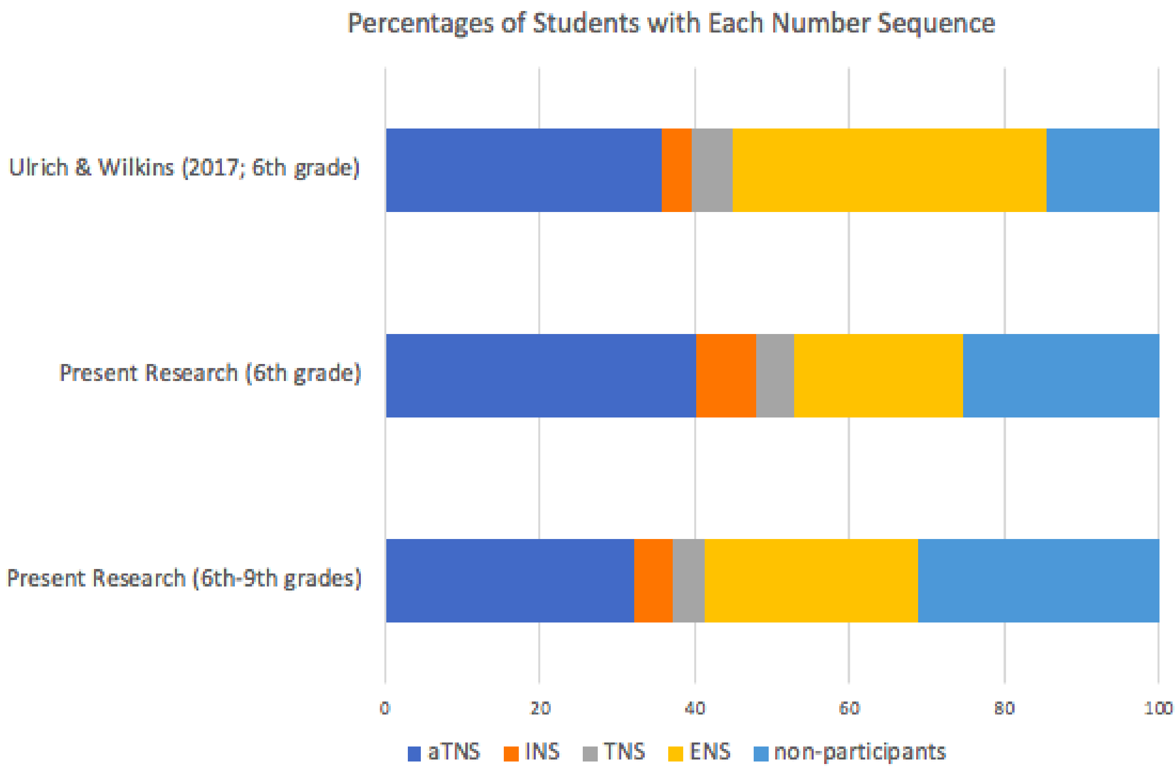

Figure 5.1. Percentages of students with each number sequence in past and present research. Notice that the aTNS is positioned farthest to the left, followed by the INS, TNS, ENS, and non-participants.

school-level differences in the populations. The difference could also be explained by the larger percentage of students who did not participate in the present research study (Figure 5.1, light blue bands). Only approximately 14.7% of sixth-graders did not participate in Ulrich and Wilkins' study, compared to 25.4% of sixth graders and 31.1% of all middle grades students in the present research study.

Recall that 30 ninth-grade students did not participate in the present research study because they were not enrolled in a math class during the semester in which data was collected. Accordingly, Figure 5.2 shows the predicted percentages of aTNS and ENS students in ninth grade. The inclusion of these predicted percentages facilitates the interpretation of the Chi-square and corresponding *post hoc* tests. Exploratory data analysis indicated that the numbers of students who had constructed an INS or TNS were similar across the four grade levels, which indicates that students who have constructed only an INS or a TNS in sixth grade are not significantly likely to construct a more sophisticated number sequence by ninth grade. This conclusion is limited by the cross-sectional nature of the data collected in this research study, and longitudinal data is necessary to draw additional conclusions about the changes in individual students' number sequences across the middle grades. However, this preliminary data highlights that the numbers of students who have constructed an INS or a TNS are not decreasing significantly across the middle grades.

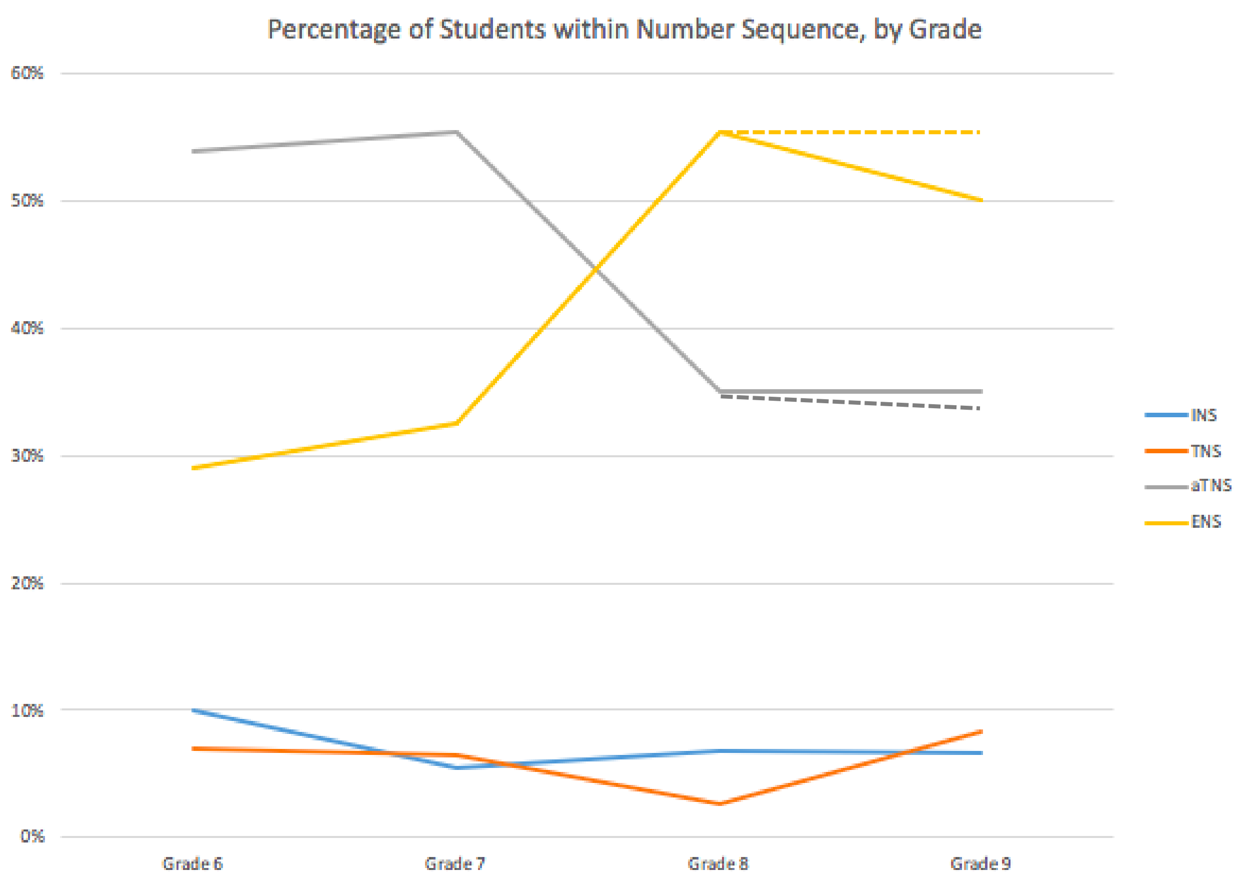

Figure 5.2. Solid lines indicate the actual percentages of students with each number sequence.

The dotted extensions include the predicted number sequences of 30 ninth graders who were not enrolled in a math class during the semester of data collection.

Although the numbers of students with an INS or TNS remain stagnant across the middle grades, there was a significant decrease in the numbers of students who had constructed an aTNS in sixth and seventh grades compared to eighth and ninth grades. Conversely, there was a significant increase in the numbers of students who had constructed an ENS in eighth and ninth grades compared to sixth and seventh grades. However, the percentage of students in ninth grade who had constructed an ENS is less than the percentage of students in eighth grade who had constructed an ENS. This is counterintuitive because if the number of INS, TNS, and aTNS students do not change when comparing eighth graders to ninth graders, then the number of ENS students should not change either; where have those ninth graders gone?

The cross-sectional data, again, limits the conclusions that can be drawn regarding this peculiarity. Despite the continuity of the students in this study attending the same elementary, middle, and high school, it is possible that the ninth grade class is less sophisticated in their constructions of number sequences than is the eighth grade class. It is also possible that the non-participation of ninth grade students who were not enrolled in a math class during the semester of data collection has affected the percentage of ENS students identified. Thus, Figure 5.2 includes predicted percentages of students who had constructed each number sequence. These results, while hypothetical, are more intuitive than those produced by the actual data. Furthermore, they provide a plausible rationale for the disappearing ninth-grade ENS students.

Finally, the intersection of the lines indicating the numbers of aTNS and ENS students between grades seven and eight indicates an interaction between the students' number sequences and grade levels. This is supported by the odds ratios, which indicated that there were significantly more ENS than aTNS students in eighth grade compared to seventh grade. A hypothesized rationale for this difference is that in the middle school that participated in the current study, an algebra one course is not offered until eighth grade. Therefore, in eighth grade, approximately 44 students enrolled in algebra one, making it possible that course enrollment and subsequent instruction supported the construction of more sophisticated number sequences for some students. Evidence of students' preparedness to accept instruction on concepts of algebra, as they relate to the algebraic tasks included in this study, will be discussed in the next sections. Even with this evidence, however, this hypothesis is speculative and requires future empirical consideration.

Algebraic Reasoning

In this section, the algebraic reasoning of the participants is discussed, and is organized by students' number sequences.

TNS students. TNS students were the most limited in their algebraic reasoning, compared to the aTNS and ENS students. In general, the most prevalent limitations are attributed to them not assimilating with a composite unit. Therefore, because unknowns constitute a composite unit (Hackenberg et al., 2017), TNS students could only operate on unknowns in activity and the results of those operations were not available for reflection following activity. Additional limitations can be attributed to their additive reasoning (Ulrich, 2015, 2016a), and their inability to disembed (Steffe, 2010b).

Generality through particular examples. TNS students were successful at generalizing through particular examples, as demonstrated on the border problem (A4; Appendix B), and the pattern block problems (A7 and A8; Appendix B). In this form of algebraic reasoning, students determine a pattern by examining several consecutive terms, and then apply the pattern to a non-consecutive term that could not be easily calculated recursively (Radford, 2011). Radford (2011) identifies this as the first true stage of algebraic reasoning in young children because of the "indeterminacy and analyticity" (p. 311) required to reason in this manner, and further explains that when students engage in generality through particular examples, the unknown quantity is implicit. To accomplish this form of generalizing on the border problem, for example, Tabitha's reasoning suggests using a method of comparison (Hackenberg, Norton, Wright, 2016) to determine a pattern for calculating the borders of the grids. She explained the relationship between the numbers she was summing when she said, "You could still subtract the top part by 2 and you could get the side length." Then, she constructed an additive comparison in activity to determine the border of a 100-by-100 grid. The implication of constructing an additive comparison in activity for TNS students is that it describes a "transformation" (Ulrich, 2016a, p. 38) from one addend to the sum, and because this was done in activity, the addends are not material for reflection (Ulrich, 2016a). Thus, the result of Tabitha's generality through particular examples was a border length that consisted of a counting sequence beginning at one. This is consistent with the results of Hackenberg (2013), who found that because MC1 students construct composite units in activity, the side lengths of the grid were absorbed by the sum. Although this "absorption" (Hackenberg, 2013, p. 556) is a limitation, it did not prohibit TNS students from generalizing through particular examples (Radford, 2011).

Travis, on the visual block pattern problem (A7), similarly made generalizations through particular examples when he determined the number of blocks in several non-consecutive figure numbers. However, he first attempted to do so recursively. At a point when he had calculated the number of blocks in figures zero through four, he was asked to calculate the number of blocks in figure 100. He sat silently for 127 seconds before stating that he was trying to figure out "something to help me add up to figure 100." Later in the task he identified the explicit pattern of adding two to the figure number to determine the number of blocks, and explained, “you would say that there are, um, two more blocks than that (the figure) number.” Similar to Tabitha, engaging in generality through particular examples was possible for Travis because he constructed additive comparisons in activity. In other words, Travis generated a pattern of adding two to the figure number to determine the number of blocks, and then transformed other given figure numbers by adding two. Tabitha and Travis constructed additive comparisons in activity to engage in generality through particular examples, which is evidence that TNS students can engage in some forms of algebraic reasoning.

Verbalizing. According to Radford’s (2011) stages of generalizing, generalizing through particular examples is followed by verbalizing patterns. When students verbalize patterns, the indeterminacy becomes explicit (Radford, 2011). Although TNS students could generalize through particular examples, they were unable to verbalize patterns consistently. On the border problem (A4), Tabitha referred to specific values when verbalizing the relationship between the side lengths of the grid, and on the block pattern problems (A7 and A8), Travis verbalized his pattern in terms of a specific figure. On A7, he referred to a figure having “two more blocks than that (the figure) number,” but based on his hand gestures and inflection, “that” is interpreted to refer to the specific numerical example he was working on at the time. On task A8, he verbalized the pattern he had found in terms of the first figure in the pattern: “if 7 equals figure 1, then 7 has to be 6 more than 1.” In each of these instances, the students relied on numerical examples to explain their pattern. Their reliance on numerical values is evidence that they cannot maintain the relationship between two unknown quantities, even when algebraic notation is not required. This can be attributed to their inability to assimilate with composite units, which implies that the composite, in this case an unknown quantity, decays following activity. Thus, the TNS students could not verbalize these patterns as they relate to *any* figure number or *any* number of blocks; they were tied to a relationship between two known values.

On additional tasks that required the students to verbalize a generalized pattern, A7 and A8, Tabitha did not even verbalize the pattern using known values. This is interpreted as evidence that there is an additional limiting factor influencing TNS students' ability to verbalize patterns. On the visual block pattern problem (A7), for example, Tabitha generalized when she determined that there would be 102 blocks in figure 100, but reverted to a recursive pattern when she tried to verbalize the relationship. She explained that to find any figure number you would need to "add it by the first figure's blocks," meaning that she would add beginning with the number of blocks in the first figure. Although she used an explicit pattern to generalize, the explicit pattern was not available for reflection because it was the result of constructing an additive comparison in activity; without a disembedding operation, Tabitha was unable to apply the same pattern verbally to an unknown value.

These results indicate that additive reasoning, the construction of an unknown composite unit, and the disembedding operation were critical in these beginning stages of algebraic reasoning. TNS students' construction of additive comparisons in activity allowed them to generalize through particular examples, but did not support further algebraic reasoning in the form of verbalizing patterns. Verbalizing patterns was limited because the TNS students could not operate on an unknown quantity, nor could they disembed one quantity from another without destroying the relationship between the two. For these reasons, TNS students' algebraic reasoning was limited to generalizing through particular examples.

The results of verbalizing patterns in the present study are presented in comparison to Hackenberg's (2013) results, in which she found that five out of six MC1[18] students were able to verbalize the method for counting the number of squares on the border of a grid with a known side length. In comparison, only one of the two TNS students in the present study found the border of a grid other than the original 10-by-10 grid verbalized the pattern in general terms. Furthermore, neither TNS student verbalized patterns on additional tasks requiring them to do so. Hackenberg attributes this limitation to MC1 students not disembedding; although TNS students are also not disembedding, making this a plausible explanation for their difficulty verbalizing patterns, aTNS students verbalized patterns with the exception of one student on two out of three tasks. Because aTNS students also do not disembed, this makes it necessary to conclude that mental constructs in addition to disembedding act in support of verbalizing patterns. Those additional constructs are the assimilatory composite unit, which is not available for TNS students, and additive reasoning, which is qualitatively different for TNS students than for aTNS students. The roles of these constructs in supporting students' verbalizing patterns is discussed in more depth as they relate to aTNS students' reasoning on the border problem, and the block pattern problems.

Units coordination and construction. On tasks in which students were required to represent an additive relationship between two unknowns, TNS students continued to struggle. Constructing a composite unit constituting an unknown quantity in activity continued to limit TNS students' equation writing, similar to how it limited their verbalizing patterns. On the visual block pattern problem (A7), for example, Travis wrote the expression $f + 2$ to represent the number of blocks in figure *f*, but explained that there would be eight blocks in figure *f* because "f

[18] Recall that MC1 students and TNS students both assimilate tasks with one level of units, and that a TNS is within the ZPC of students who have constructed an MC1 (see Figure 2.1).

is the sixth letter in the alphabet." This is interpreted as Travis's attempt to construct an unknown figure number in activity, but once the unknown decayed, he was left searching for meaning. He was unable to maintain a conceptualization of f as an unknown quantity, so he applied an alpha-numeric code instead. By conceptualizing f as six, he reduced the level of complexity of the unit structure; this was necessary for Travis, because he could not maintain the composite unit following activity.

Tabitha and Travis both suffered a similar limitation on the soccer problem (A10; Appendix B), although it manifested differently because this task did not ask them to generalize a pattern, as the visual block pattern problem (A7) did; rather, they were asked to write a one-step additive equation after reading the task. Tabitha wrote the equation $z - 3 = 6$, but was reliant on numerical examples to explain the equation. Despite having defined a second variable, she did not use it. Also, despite having indicated what z was meant to represent earlier in the task, after writing the equation, she explained that z could be nine. These behaviors are interpreted as her constructing an unknown quantity in activity to represent the score of last week's game (z). The limitation of her units coordination and construction, however, is that she cannot operate on a composite unit. Her use of a numerical example is interpreted as a compensation for the inability to maintain a conceptualization of the unknown quantity. In other words, as the unknown nature of z decayed from her unit structure, she relied instead on a numerical example.

Travis's behavior on the soccer problem (A10) initially seems distinct; he wrote the equation $T = L - 3$, which is a reversal. Similar to Tabitha, however, because Travis was unable to operate on an unknown quantity representing last week's score (L), his behaviors during the interview seem to indicate he engaged in word order matching (Clement, 1982). Although

Clement (1982) specifically discussed this method as it relates to writing multiplicative equations, Travis's equation writing behavior on the soccer problem is consistent with Clement's description of the student's belief that the elements of the equation should match the wording of the task. Thus, it seems that for Travis, word order matching was used to compensate for his inability to conceptualize operating on an unknown quantity because it was constructed in activity.

Disembedding. Additional limitations of TNS students' algebraic reasoning can be attributed to their not having constructed a disembedding operation. This was evident on the border problem (A4) and the modified coin problem (A6; Appendix B), and are attributed to their inability to disembed. On the border problem, Tabitha attempted to represent her method for finding the number of squares on the border of an n-by-n grid, but she relied on numerical examples. As was previously discussed, this was attributed to a limitation of her ability to operate on an unknown quantity. This reliance extended to her equation writing. Tabitha wrote $n + n$ in a beginning attempt to represent the border algebraically, but when asked how she could represent the shorter sides, she assigned them a numerical length. Later during the same task, she indicated that she could assign a numerical length to the longer side length and a variable to the shorter side lengths, which is the opposite of her original statement. She could not, however, conceptualize both the shorter and the longer side lengths as having an unknown length simultaneously. This limitation is slightly different than that which was previously discussed as a limitation of her inability to operate on an unknown quantity. This limitation is evidence of Tabitha's inability to conceptualize two related quantities as unknowns simultaneously, which is attributed to her not having constructed the disembedding operation. This result is consistent with that of Hackenberg (2013), who hypothesized that based on her results, an MC1[19] student would represent related quantities using unrelated variables because he had not constructed the disembedding operation. However, Tabitha's difficulty seems to go one step beyond that hypothesized by Hackenberg because she did not even conceive of the two side lengths as two unrelated variables, but instead conceptualized only one unknown at a time.

On the modified coin problem (A6), Travis's equation writing was limited by not disembedding, and by his units coordination and construction. Travis was attempting to represent that the sum of the number of dimes, nickels, and quarters was 17, and furthermore, that there were six more dimes than nickels and one fewer quarter than nickels. Travis had completed numerical examples demonstrating the additive relationship between the numbers of dimes and nickels, and quarters and nickels. However, when he attempted to represent this information in an equation, he wrote $6d + n - 1q = 17$, and his explanation indicates that he conceptualized the variables as labels for objects rather than unknown quantities. First, this is considered to be a limitation of his units coordination and construction because the equation requires the coordination of the number of each type of coin within the total number of coins; this assumption is based on the analysis of Olive and Çaglayan (2008). Extending their analysis to the present research, as TNS students only construct two-level unit structures in activity, to operate on the equation by substituting expressions for d and q in terms of n is beyond Travis's capabilities. However, in addition to this limitation of not operating on a two-level unit structure, to write this equation also requires him to simultaneously conceptualize the relationship between the numbers of dimes and nickels, and quarters and nickels. Regardless of his units coordination and construction, such a simultaneous concept would require the disembedding operation.

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[19] Recall that MC1 students and TNS students both assimilate tasks with one level of units, and that a TNS is within the ZPC of students who have constructed an MC1 (see Figure 2.1).

TNS students were also limited in their ability to write an equation representing the multiplicative relationship between two unknowns, as evidenced by their inability to make progress on the phone cords problem (A1; Appendix B) or the modified splitting problem (A2; Appendix B). Neither student drew a correct picture. Travis worked through numerical examples but could not conceptualize one of the cords as having an unknown length. Tabitha did not correctly determine numerical examples, and did not represent the situation algebraically. It is not surprising that the TNS students could not write an equation on tasks A1 and A2, considering their inability to represent additive tasks algebraically. However, Hackenberg (2013) reported that three out of six MC1[20] students drew a correct picture on tasks A1 and A2, and two were able to write an equation for one of the tasks with coaching; she attributes this difficulty to their not disembedding. The small number of students in each study makes it impossible to determine whether the results of the present study are distinct from Hackenberg's. Regardless, this research contributes to an understanding of the severity of students' struggles to reason algebraically without having constructed the disembedding operation.

Units coordination and disembedding as limiting factors. Previous sections have outlined the ways in which TNS students' inability to operate on composite units and not having constructed a disembedding operation limited their algebraic reasoning. However, on the modified coin problem (A6), these two limitations were both prevalent in TNS students' guessing and checking behaviors. Although guessing and checking is not generally considered a form of algebraic reasoning (Knuth, Stephens, McNeil & Alibali, 2006), it is relevant to the present research because it was a method used by many students to solve systems of equations tasks. The students' guessing and checking behavior can be attributed to their ability to operate

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[20] Recall that MC1 students and TNS students both assimilate tasks with one level of units, and that a TNS is within the ZPC of students who have constructed an MC1 (see Figure 2.1).

on composite units and to disembed or not disembed. Specifically for TNS student, their guessing and checking was limited by their inability to operate on a composite unit and their inability to disembed.

Tabitha, for example, began by guessing that there were six nickels, six dimes, and five quarters because six, six, and five sum to 17; she disregarded the information about the relationships between the numbers of nickels, dimes, and quarters. This can be attributed to TNS students' inability to operate on composite units. Tabitha's guesses are evidence that although she was able to construct a composite unit of 17 coins in activity, she was unable to maintain awareness of the 17 coins as a single, composite unit while also maintaining the relationships between the numbers of coins; she focused only on one or the other because to focus on both would have required her to reflect on the 17 coins as a composite unit.

In a slight variation, Travis maintained the relationships between the numbers of dimes, nickels, and quarters using figurative material but did not simultaneously maintain that there needed to be 17 coins in total; he treated these as two separate problems. Travis's treatment of the modified coin problem as two separate tasks is evidence of his inability to operate on a composite unit and his inability to disembed. Travis assimilated the task as a one-level unit structure of some number of nickels, and then in activity, determined the appropriate number of dimes and subsequently the appropriate number of quarters. TNS students have not constructed a disembedding operation, however, and as a result, Travis relied on figurative material to keep track of the numbers of coins as he guessed. Furthermore, TNS students could construct a composite unit of 17 in activity, which might facilitate their conception of 17 as a set of coins, but because the composite unit decays following activity, the second level of units also decays leaving them to reflect only upon 17 individual coins. For Travis, this decay manifested itself in his treatment of the modified coin problem as two tasks – first he attended to the relationships between the numbers of coins, then he attended to the sum of the coins.

Summary. The results of this analysis indicate that both Tabitha and Travis were limited in their algebraic reasoning. Both generalized using particular numerical examples, which indicates that they were reasoning algebraically in some capacity (Radord, 2011). This was dependent upon their construction of additive comparisons. Algebraic reasoning that included verbalizing patterns and representing unknown quantities using algebraic notation both required students to reflect and operate on unknowns. Hackenberg and her colleagues (2017) indicate that unknown quantities represent a two-level unit structure, which provides a rationale for the difficulty TNS students had on these tasks. Both Travis and Tabitha relied on numerical examples to explain algebraic expressions and equations that they wrote when the unknown quantity decayed and they were left searching for the meanings of their equations. The lack of a disembedding operation was also limiting for TNS students, as it precluded them from reasoning simultaneously about two related unknown quantities. Both TNS students applied non-standard conceptions of variable to compensate for this limitation. Although guessing and checking is not considered a form of algebraic reasoning (Knuth et al., 2006), it was a strategy that TNS students applied and found some success in using. Their success on the modified coin problem was encouraging, but even their guessing and checking behavior was limited by their inability to operate on composite units, and by not having constructed a disembedding operation. Without more sophisticated cognitive structures to support algebraic reasoning, TNS students were able to do little more than generalize.

aTNS students. aTNS students more successfully verbalized generalized patterns and represented those patterns algebraically than did TNS students. aTNS students also wrote linear equations representing additive relationships and multiplicative relationships more successfully than did TNS students. These advantages in their algebraic reasoning are attributed to their assimilatory composite unit. Because unknowns constitute a composite unit (Hackenberg et al., 2017), aTNS students were advantaged over TNS students because unknowns were an assimilatory structure that could be operated and reflected on by aTNS students. This allowed aTNS students to make more progress than TNS students on several of the tasks. aTNS students' progress on the tasks, however, made the limitations of not having constructed a disembedding operation and not having constructed a splitting operation more observable throughout the interview; TNS students did not make enough progress on the tasks involving splitting for this constraint to be a point of discussion. Also similar to TNS students, they were constrained by their additive reasoning, although in a qualitatively different way. Each of these constructs that define the number sequences will be discussed in this section as they afforded and constrained aTNS students' algebraic reasoning.

***Units coordination and construction*.** aTNS students' units coordination and construction was less limiting to their algebraic reasoning than it was to that of TNS students; however, there were still situations in which it was apparent. On the border problem (A4), Ann wrote a string of expressions to represent the border of an *n* by *n* grid: $n \times 4 = n - 4 = n$. In this equation, n is used as a general referent for the side length, the sum of the four side lengths, and the total border. This result is consistent, although distinct, from that of Hackenberg (2013), who described an MC1[21] student whose units coordination was limiting because when she summed the sides of the grid, the sum "absorbed" (p. 556) each addend. Ann's algebraic

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[21] It is unclear from existing research whether aTNS students' reasoning is more similar to that of MC1 or MC2 students (see Figure 2.1).

representation of the border of the grid is more sophisticated than that described by Hackenberg, specifically because her inclusion of an unknown quantity implies operation on an assimilatory composite unit, however, the result of that operation was still absorbed.

aTNS students' units coordination was also limiting on the coin problem (A5; Appendix B). Olive and Çaglayan (2008) report that this task requires students to operate on a three-level unit structure, which they concluded made it difficult for students who assimilate with only two-levels of units. The results of the present research similarly found that aTNS students struggled on the coin problem as a result of their units coordination. Amanda, for example, tried to represent the three-level unit structure in an equation, but when asked to explain her equation, changed the meaning of her variables from the number of coins to the value of the coins. This is considered a limitation of her units coordination because although she was tacitly aware of the need to incorporate both the number and value of the coins, she could not represent both in relation to the total value. As she progressed through the task, the meaning of the variables changed. Similarly, Alyssa wrote an equation representing two of the three levels in the unit structure, but could not incorporate the third. Thus, although aTNS students were advantaged over TNS students because of their assimilatory composite unit, there were still limitations to their algebraic reasoning as a result of their units coordination and construction.

Although guessing and checking is not necessarily an algebraic strategy (Knuth et al., 2006), it was a prevalent method for aTNS students on the coin problems (A5 and A6), and guessing and checking was more productive for aTNS students than it was for TNS students. Three aTNS students solved the coin problem by guessing and checking, and seven solved the modified coin problem by guessing and checking. The success with which aTNS students applied a guess and check method on these tasks is attributed to their ability to operate on composite units. This was observed in two ways. First, aTNS students maintained awareness of the total number of coins as consisting of three sets of coins. Second, they made incremental adjustments to their guesses.

Abby, for example, drew 17 circles to represent the coins. Then, filled in the circles with N's, D's, and Q's to represent the types of coins. While filling in the coins, she attended to the relationship between the numbers of coins. This is in contrast to Tabitha, a TNS student who arbitrarily guessed a combination of three numbers that added to 17. This comparison demonstrates the manner by which Abby anticipated the need to use all 17 coins, and maintained the relationships between the numbers of coins prior to operating.

Also on the modified coin problem, Ann drew three circles with N's written inside to represent her first guess of there being three nickels. She then drew nine dimes because "we always have six more dimes than nickels." And she drew two quarters because "you'd have to have one fewer quarter than nickels…" After completing her diagram, Ann counted her coins and determined there were only 14 in all, so she made an incremental adjustment by adding one to each type of coin. This is different than the adjustments TNS students made because Ann drew one more dime, one more nickel, and one more quarter, whereas TNS students started from scratch when their guesses were incorrect. Six of the aTNS students who solved the modified coin problem using guess and check made incremental adjustments to their guesses.

aTNS students were able to add one to each type of coin because they understood this would maintain the relationship of six more dimes than nickels, and one fewer quarter than nickels. After making this incremental adjustment, Ann, for example, did not go back and check that these relationships had been upheld. She knew without checking that there would still be six more dimes than nickels and one fewer quarter than nickels. This incremental adjustment is evidence that aTNS students operated on embedded composite units with the help of figurative material. In other words, when they realized their initial guess had not exhausted the 17 coins, they increased the number of each type of coin by one. Ann and Abby both used pictures to support their operations on embedded composite units. The other four aTNS students who made incremental adjustments to their guesses wrote down each guess in the form of an ordered triple, or an informal chart. Either way, making note of their guesses is hypothesized to have eased the mental operations for aTNS students, and to have been a productive activity in their solution to a system of equations.

This evidence that aTNS students anticipated the need to exhaust all 17 coins with their guesses, and their ability to incrementally adjust their guesses by adding one to each type of coin is consistent with operating on embedded units within a composite unit. In comparison, TNS students did not make incremental adjustments to their guesses. After realizing that a guess was either too large or too small, TNS students made a new guess and re-calculated the six more and one fewer relationships between the numbers of coins. TNS students also did not anticipate the goal of exhausting all 17 coins prior to operation. On the other hand, ENS students were advantaged in their guess and check by the disembedding operation; this will be examined in the section discussing ENS students.

Disembedding. Similar to the algebraic constraints of TNS students, aTNS students were also constrained because they had not constructed a disembedding operation. On the border problem, two of the three aTNS students who attempted to represent the border of an n by n grid as $4n - 4$ were successful in doing so; this method does not require the disembedding operation because there is no need to maintain the relationship between two side lengths. However, no aTNS student successfully represented the border of an n by n grid as $2n + 2(n - 2)$ or as $n +$

$(n-1)+(n-1)+(n-2)$. This is consistent with Hackenberg's (2013) result, in which she indicates that to relate n to $n-2$ requires the disembedding operation. Although Hackenberg's results do not include the second method, consistent reasoning allows for the conclusion that to relate the shorter side lengths to the longer requires the disembedding operation. Without it, the relationship between the two side lengths would decay. Alex, Alyssa, and Andy each had erroneous solutions to task A4 as a result of this limitation. Andy wrote the expression $x+x+y+y$ to represent the border; this is what Hackenberg (2013) predicts an MC1 student would do as a result of the limitation of not disembedding.

Alyssa and Alex, however, maintained an awareness of the relationship between the side lengths while writing their expressions. Alex wrote $n+n+b+b$, using b because it is the second letter of the alphabet and there is a difference of two between the side lengths. He knew this was not correct, though, and abandoned this method is search of one that he could better represent algebraically. He eventually represented the border as $n+n-4+n+n$, which did not require him to disembed. Alyssa wrote $n+n+L+L$, and stated that she chose L because it was two letters before n in the alphabet. Both Alex and Alyssa are demonstrating a tacit awareness of the need to represent the shorter side as two units shorter than the longer side, but are unsure how to do so. To compensate for the inability to relate the two side lengths abstractly, they applied an alpha-numeric code to their variables, which is concluded to be a manifestation of not disembedding in their algebraic reasoning.

In another interesting solution, Amanda represented the border of an n by n grid as $4(n-2)+4$. At first glance, and considering the results of Hackenberg (2013), this was interpreted as an application of the disembedding operation. Upon further consideration of the interview, however, there was no evidence that Amanda maintained awareness of the relationship between the shorter and longer side lengths after expressing the shorter as $n - 2$. Accordingly, Amanda's response is considered as a compliment to that of Hackenberg (2013), rather than a contradiction; Amanda applied her assimilatory composite unit to conceptualize the unknown side length, n, and operate on it to represent the shorter side length, $n - 2$. Following activity, Amanda seems to have taken the shorter side length as material for further operating. For this to be possible for an aTNS student, two assumptions must be made; evidence of the validity of these assumptions is given next.

The first assumption is that she has lost track of the longer side length; this assumption is necessary because to keep track of both side lengths would require the disembedding operation. Evidence of the validity of this assumption is that before writing $n - 2$, Amanda said, "Now I have to make up another variable," which is taken as evidence that she did not relate the two side lengths as unknown quantities. Furthermore, after writing $n - 2$, she did not again reference its relationship to the longer side length. Second, Amanda took the shorter side length as material for further operating when she multiplied it by four. This operation is assumed to be made possible because she conceptualized the shorter side length as an unknown quantity – not an unknown quantity that has been acted on by decreasing by two. Evidence of the validity of this second assumption is that when she wrote her expression, she left out the parentheses, writing $4 \cdot n - 2 + 4$. It is possible this was an oversight; the evidence is in her explanation. When asked if she meant to multiply n by four, or $n - 2$ by four, she said, "I meant this whole part to be together," meaning $n - 2$ was one "whole part" to Amanda. The implication of this is that Amanda considers the shorter side length as an unknown quantity with no explicit relationship to n. Thus, she produced an expression to represent the border of the grid that at face value appeared to have required the disembedding operation, but upon further consideration can be understood without disembedding.

The implication of these results for all of the aTNS students on the border problem (A4) is that some students were limited because they have not constructed a disembedding operation whereas others were not; some students were successful in writing an expression to represent the border of the grid because their method did not require the disembedding operation. However, aTNS students whose method required them to maintain the relationship between two or more side lengths algebraically were constrained. In comparison, all ENS students correctly represented the border of the grid algebraically regardless of their method; methods relating two side lengths were not problematic for ENS students because they have constructed the disembedding operation. Thus, on task A4 in particular, where multiple methods were possible, several aTNS students applied creative problem solving to circumvent their cognitive shortcomings.

aTNS students also applied creative methods to solving the coin problem and the modified coin problem (A5 and A6) when they were unable to reason algebraically through the tasks. Three of six aTNS students who attempted the coin problem were successful using guess and check methods. No aTNS students solved the coin problem algebraically. Difficulty on this task is consistent with Olive and Çaglayan's (2008) indication that an algebraic solution to this task requires students to operate on an equation representing a three-level unit structure. However, the results of the present research indicate that not having constructed a disembedding operation was an additionally limiting factor for aTNS students. Furthermore, the creative manner by which aTNS students solved the border problem (A4) and the coin problems (A5 and A6) is characteristic of Ulrich's (2016b) definition of aTNS students as persistent problem solvers. These results contribute to the understanding of the manner by which aTNS students find ways to be successful on tasks that they have not constructed sufficient mental structures to support in a more traditional algebraic way.

Abby and Alyssa both attempted to algebraically represent the relationship between the number of dimes and nickels, and quarters and nickels, but did so incorrectly. This is explained by their not having constructed a disembedding operation. They operated on an unknown quantity erroneously and without perturbation because they could not conceive of the relationship between the numbers of the two types of coins simultaneously. Alyssa, for example, wrote $d + 3 = n$ to represent there being three more dimes than nickels. This type of mistake could potentially be just that – a mistake. It is also possible, however, that because Alyssa was unable to simultaneously conceptualize the relationship between the number of dimes and nickels, she resorted to word order matching (Clement, 1982) to represent the situation algebraically.

Continuing to work on the coin problem, Alyssa algebraically represented the relationships between the number of dimes and nickels, and quarters and nickels. Then, she attempted to represent the number of dimes, nickels, and quarters in one equation. In doing so, she lost track of the additive relationships between the number of dimes and nickels, and quarters and nickels, and wrote $d3 + n - 2q = 5.40$. Her explanation that "I put d times three because d, the dimes, have three more than the nickels. Plus n minus two quarters, $2q$. So there's nickels minus two fewer quarters," is an indication that the variables d and q have become labels, rather than representations of unknown quantities. This is taken as a limitation of not having constructed a disembedding operation because she is unable to conceptualize the additive relationship between the number of dimes and nickels, and quarters and nickels, simultaneously;

the relationship is destroyed when they are combined into a single equation. This is an example of how the inability to disembed was a limiting factor above and beyond students' units coordination and construction on the coin problem.

The modified coin problem (A6) was designed to complement the coin problem (A5); it eliminates the value of the coins, thus requiring students to operate on a two-level, rather than a three-level, unit structure. Although seven of eight aTNS students solved the modified coin problem, all of them did so using guess and check. The one aTNS student who persisted with an algebraic strategy was the one aTNS student who was unsuccessful on the modified coin problem. This is further evidence that not having constructed the disembedding operation is a limiting factor in students' solutions to tasks such as the coin problems, in addition to only having constructed an assimilatory composite unit. In their algebraic attempts to solve the modified coin problem (A6), Abby, Amanda, Aaron, and Andy all made similar equation errors to Alyssa's error described on the coin problem (A5). For example, despite having written the equations $n + 6 = d$ and $n - 1 = q$ to represent the relationships between the numbers of dimes and nickels, and quarters and nickels, respectively, Abby then wrote $6 + n + 1$. In this equation, she did not incorporate variables for the number of dimes or quarters. In total between tasks A5 and A6, five aTNS students wrote similar equations in which they either applied a label concept of variable or did not incorporate variables, despite having appropriate concepts of variable earlier in the same task. This is taken as a limitation of their not having constructed the disembedding operation; aTNS students were unable to incorporate the relationships between the numbers of dimes and nickels, and quarters and nickels, in one equation.

aTNS students' solutions on the coin problems (A5 and A6) demonstrate two limitations of their algebraic reasoning. First, not having constructed the disembedding operation was limiting to aTNS students on the coin problems. This is a limitation that was not discussed by Olive and Çaglayan (2008), presumably because they were conducting research with students who had constructed the disembedding operation. However, it is important to consider the limitation of aTNS students not having constructed a disembedding operation in solving systems of equations. aTNS students were unable to maintain the relationships between two variables simultaneously within one equation, which is why they could not substitute expressions. Second, aTNS students problem solved creatively when their algebraic reasoning was limited. As noted, seven of eight aTNS students recognized that their algebraic methods were not productive and abandoned them for guess and check methods. These seven students were able to successfully guess and check to find a solution on the modified coin problem, and three were successful at guessing and checking on the coin problem.

Units coordination and disembedding as limiting factors. The limitations of aTNS students having constructed only an assimilatory composite unit and not having constructed a disembedding operation have been discussed thus far as separate limitations to their algebraic reasoning. There were also situations throughout the interviews, however, when the aTNS students' inability to operate on a three-level unit structure and inability to disembed acted as limiting factors in their algebraic reasoning.

On the border problem (A4), Ava wrote three expressions to represent the border of an *n*-by-*n* grid: $n + 9$; $n + 9$; $n + 8$. This is evidence that Ava was using n as a general referent, and can be consistently explained by Hackenberg's (2013) analysis that the addends are being "absorbed" (p. 556) by the sum. However, Ava also uses nine and eight, which are the shorter side lengths in the first numerical example considered on task A4, in which students find the border of a 10-by-10 grid. The indication of this inclusion of a specific numerical example in her expressions is that she is unable to maintain the relationships between the side lengths of the method she described. As a result, she reverted to the incorporation of a numerical example. This solution method suggests a situation in which Ava was bound by her level of units coordination and by not having constructed a disembedding operation. An assimilatory composite unit allowed her to conceptualize and operate on an unknown by adding nine, but the result of that operation was absorbed by the sum, which she referred to as a second *n*. In addition, she referred to adding nines and eights to the side lengths because she could not maintain the "subtract one" and "subtract two" relationships between the lengths that she had previously described.

Operating on an unknown quantity was supported by the aTNS students' assimilatory composite unit. This is consistent with Hackenberg and her colleagues' (2017) research that indicates that an unknown quantity constitutes a two-level unit structure. However, aTNS students were only successful in representing an additive relationship between two unknowns algebraically approximately half of the time they attempted to do so on tasks A4, A5, A6, A7, A8, A9, and A10[22]. This result suggests there is a limiting factor, which may be attributable to the decay of the third-level of units and the lack of a disembedding operation. This is consistent with Ava's reasoning on the border problem, which was just described, and will inform the discussion of students' other responses throughout this section.

For Abby, Amanda, and Alyssa, being limited to only an assimilatory composite unit and without having constructed a disembedding operation were limiting factors on the coin problems. In this situation, it manifested as their indication that they could not continue algebraically on the

[22] This fraction represents only students' solutions when they attempted to algebraically represent an additive relationship. It does not include the responses of students who used a different method, or who did not attempt the task. Nor does this fraction represent the success of the students on the task in its entirety – only in representing an additive relationship between two unknowns algebraically.

task without knowing the number of one type of coin. Each of these aTNS students represented the relationships between the numbers of dimes and nickels, and quarters and nickels. Because these students are not disembedding, writing these equations is explained by their ability to operate on an assimilatory composite unit. However, following activity, the relationship between the numbers of coins decays, and they are left to reflect upon the number of dimes as unknown quantities, but not the relationship between the number of dimes to nickels, for example. The evidence of this is that Abby said, “You have to find out how many nickels there are” before finding the number of quarters. Amanda said, when speaking about writing an equation representing both the number of nickels and quarters that “we don’t really know q yet, so that wouldn’t really make sense.” Finally, Alyssa stated that “we don’t know how much, how much dimes or nickels she has. If we knew those two we could figure out the quarters.” These are all indications that the aTNS students could not proceed with the task without knowing the number of at least one type of coin. This is a manifestation of them applying their assimilatory composite unit, but not disembedding, because after operating on an unknown quantity, the relationship between the two unknown quantities decayed and the students could only conceptualize the numbers of coins simultaneously if they had numerical values to compare.

On the block pattern problems (A7 and A8), in which students abstracted an explicit pattern, having constructed only an assimilatory composite unit and not having constructed the disembedding operation were again limiting factors for aTNS students. There was evidence that Alex (task A8) and Ava (task A7) conceived of the variables they were using as specific numbers; this is a limitation of their not being able to reflect upon the relationship between the unknown quantities following activity, which is the manifestation of not disembedding and not assimilating with three levels of units. Alex insisted that his expression, $f + 6$, needed to be set equal to something which indicates his need to simplify the unit structure in order to reflect upon its result. When he was told that it was not necessary to write an equation, and is asked if he thinks the expression is correct, he responds, “No, I don’t think it is because it’s the number of blocks you need.” This statement indicates that he “needs” the number of blocks as an output, but to Alex, this expression does not relate to the number of blocks. In other words, he was unable to keep track of how $f + 6$ relates to the number of blocks because the number of blocks was the result of the activity of adding six to an unknown quantity. Then, he uses several numerical examples to try to prove the correctness of his expression but finally says, that it is correct “If it’s (*f*) not a letter.” This final declaration is evidence that Alex cannot conceptualize the relationship between the unknown number of blocks and the unknown figure number – only the relationship between the two facilitated by numerical comparisons. That Alex could not conceptualize the relationship between the unknowns demonstrates the limitations of his algebraic reasoning because he was only assimilating with a composite unit and because he had not constructed a disembedding operation.

Abby was the only aTNS student who did not verbalize an explicit pattern between the figure number and the number of blocks on the block pattern problems (A7 and A8), which can also be attributed to the limitation of having only an assimilatory composite unit without a disembedding operation. This is similar to the behavior of Tabitha, a TNS student. Tabitha’s behavior was attributed to her inability to disembed; although she constructed an additive comparison in activity to engage in generality by example, she was unable to abstract the pattern verbally because she could not disembed the figure number from the number of blocks. On the other hand, when TNS students successfully verbalized patterns, they seem to have applied specific numerical values rather than an unknown quantity. aTNS students were more successful at verbalizing patterns with this one exception provided by Abby. Similar to Tabitha, Abby could not speak generally about an explicit pattern and insisted that “first you would have to figure out, like, which one’s in front of it. … what figure’s in front of it.” Abby clung to the idea that she needed to know how many blocks were in the preceding figure, which is indicative of recursive, not explicit, thinking.

If failing to verbalize an explicit pattern is to be attributed to Abby not having constructed a disembedding operation, then that calls into question the success of the other aTNS students who successfully verbalized explicit patterns because they, too, have not constructed the disembedding operation. Also, whereas TNS students verbalized patterns as numerical examples, thereby relying on numerical comparisons rather than disembedding, only one aTNS student, Ann, verbalized a pattern using a numerical example. The success of aTNS students in verbalizing patterns, then, is attributed to their operating on an unknown quantity.

It is suggested that aTNS students conceptualized the figure number as an unknown quantity and operated on it by applying the pattern they observed through numerical comparisons. The result of this is an unknown number of blocks. It is important to note that this explanation holds only if, to the aTNS students, the unknown relationship between the number of blocks and the figure number then decays. Alex’s response can be interpreted as evidence of such decay. When asked to verbalize the pattern on task A8, he said first, “Add six.” When asked to what, he clarified, “The number you have”; this is taken as evidence that he was conceiving of adding six to a known quantity rather than an unknown one. Other aTNS students used similarly specific language such as “each” figure or “that” figure, which seems to indicate their inability to conceptualize both the figure number and the number of blocks as related unknown quantities; this is evidence that they were attempting to reflect upon the result of activity on a composite unit, or unknown quantity, but that they could not conceptualize this as a relationship between unknowns because they were not disembedding.

The limiting effect of aTNS students not having constructed a disembedding operation and having constructed only an assimilatory composite unit was also observed on the football problem (A9; Appendix B). No aTNS students represented this task algebraically as one equation. Three students made meaningful progress on the task, however, and all three represented the situation as either two related, but separate equations, or as one string of equations. Abby, for example, wrote two related equations to represent the situation: $x - 3 = s$ and $7/s = td$, but could not join them together[23]; Alyssa's two equations were comparable. Aaron wrote a string of equations: $x/7 = y - 3 = z$. Each of these responses are reminiscent of Hackenberg and Tillema's (2009) indication of how MC2 students might conceptualize tasks requiring three-levels of units following activity. They describe MC2 students as solving a "*separate* problem" and the result losing its "status" (p. 4, emphasis in original) as a unit of units of units. Comparably, these aTNS students represented the football problem as two separate three-level unit structures, constructed in activity, that cannot be joined together without first simplifying the level of complexity.

Aaron accomplished this simplification by applying a numerical example. He explained his string of equations in the following way:

> Last week their score was 35 and that's how many points they have (*x*), so you have divided by seven to get how many touchdowns (*y*). And then since we did 35, I got, you'd get five … So once you got five here (*y*), you subtracted it by three and got two.

[23] That Abby reversed the *s* and the 7 in this equation is not pertinent to the current discussion.

Aaron conceptualizes the changes on the football team's score last week as a string of operations. He was able to simplify the complexity of the task by inserting a specific numerical example rather than attempting to conceptualize the task as including an unknown number of points and touchdowns. No other aTNS students were able to make the same level of progress on task A9 as Aaron did, which indicates the level of difficulty that this involved, even after reducing the level of complexity of the unit structure by eliminating the unknown.

As indicated, this task was challenging for aTNS students. Their need to write two separate equations can be attributed to their level of units coordination; to represent this task as one equation without first conceiving of it as two separate equations would have required them to assimilate with three-levels of units. The lack of a disembedding operation was also perceived as a limiting factor for aTNS students on this task. This will be discussed further in conjunction with ENS students' solutions because the evidence is the most clear in a comparison of these two groups of students.

Splitting. The final characterization of aTNS students' algebraic reasoning that is important to make is related to their ability to split. This was observed on task A1, in which students were asked to solve a splitting task and then represent the relationship between the two unknown quantities algebraically. Of the eight aTNS students who attempted to represent the splitting task algebraically, three were successful. In comparison, Hackenberg (2013) found that two of six MC1 students were successful on this task and Hackenberg and Lee (2015) found that four of six MC2[24] students were successful.

Interestingly, only one of those three students also successfully solved the splitting task. aTNS students are indicated to solve splitting tasks by sequentially, rather than simultaneously,

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[24] It is unclear from existing research whether aTNS students' reasoning is more similar to that of MC1 or MC2 students (see Figure 2.1).

partitioning and iterating (Ulrich, 2016b). The implication of these sequential actions is a disconnect between the partitioning and iterating, and the conceptualization of the relationship between the two lengths. Thus, while on the surface, the result of a splitting task may appear to be the same for aTNS students compared to students who are splitting, that aTNS students' solutions to task A1 were unrelated to solving the splitting task suggests that the results of a splitting task for an aTNS student are not congruent to the results for an ENS or GNS student.

Of the three aTNS students who successfully represented task A1 algebraically by writing some form of the equation $y = 5x$ and explaining the meaning of each component of the equation, all three built the equation using numerical examples. This is consistent with the results of Hackenberg et al. (2015, 2017) who found that MC2 students were likely to use numerical examples to build equations involving a multiplicative comparison. Hackenberg and her colleagues (2017) conclude that this is a result of the students' units coordination because although they assimilate the task with a composite unit, they must construct the third level of units in activity. Therefore, building algebraic equations using numerical examples is a behavioral manifestation of their inability to assimilate the task with three-levels of units. That is to say, the third-level is constructed in activity.

During her equation writing, Alyssa began with the equation $x5 - y$. Her explanation of the expression indicated that she multiplied Steven's cord length (x) by five because it was the longer cord length. This is consistent with Clement's (1982) static comparison approach of equation writing, in which students erroneously perceive that the larger unknown quantity should be multiplied by the constant. In this instance, Alyssa's behavior is a limitation that results from not splitting; she was unable to reflect upon the result of the initial splitting task, so she searched for contextual clues to facilitate her equation writing.

In comparison is the solution of Alex, who presumably solved the splitting task by sequentially partitioning and iterating, but did not represent the situation algebraically. Alex wrote an additive equation, $b5 - x1 = 4$. This is taken as evidence that despite solving the splitting task by supposedly sequentially partitioning and iterating, Alex was unable to maintain the resulting relationship between the two lengths in his equation writing. Instead, he relied on his assimilatory composite unit to conceptualize an unknown length, but following operation, the unknown aspect of his unit structure seems to have decayed, leaving him with only known values: five and one. Thus, when he subtracted, the variables were no longer relevant, and were ignored.

Additional evidence that not having constructed a splitting operation limited aTNS students' algebraic reasoning is observed in their explanations of their equations. The three aTNS students who represented the phone cords problem (A1) algebraically, Aaron, Amanda, and Alyssa, all reverted to an operational concept of the equal sign when they explained each component of the equation. Aaron indicated that "you wouldn't know the answer to it," Alyssa said that her expression represented "the answer, I guess," and Amanda stated that the isolated variable was "the total length." Each of these utterances are taken as evidence that the students' unit structures decayed following activity leaving them with an unknown result that could not be conceptualized in terms of the related quantity. Without the ability to reflect on the results of the splitting operation, these aTNS students were at a loss to explain the meaning of their equations. The work summarized in this section is provided as evidence that sequentially partitioning and iterating does not produce a mentally congruent result to the splitting operation, the implied simultaneity of which would have allowed aTNS students to reflect upon its result. This will be discussed further as it relates to the results of ENS students.

Summary. aTNS students assimilate with a composite unit, which allowed them to operate on unknown quantities in some situations. However, aTNS students were limited because they have not constructed a disembedding operation or a splitting operation, and additive comparisons are not assimilatory structures for aTNS students. These limitations led to inconsistencies in aTNS students' algebraic reasoning. aTNS students attempted to operate on embedded composite units, rather than disembedding, and their algebraic representation of a splitting task was unrelated to their solution to the splitting task due to the sequential nature by which they apply the partitioning and iterating operations (Ulrich, 2016b). Due to these limitations, aTNS students could not reflect on the results of disembedding or splitting. This resulted in aTNS students explaining expressions and equations by applying non-standard concepts of variables and an operational concept of the equal sign. Guessing and checking was more efficient for aTNS students than for TNS students, because of aTNS students' assimilatory composite unit, which was an advantage on the systems of equations tasks. Overall, aTNS students made more progress on algebraic tasks than did their TNS peers, regardless of algebra course enrollment, but their algebraic reasoning was inconsistent.

ENS students. ENS[25] students' algebraic reasoning was more consistent and productive than that of their TNS and aTNS peers. These students represented linear patterns with explicit formulas algebraically, wrote linear equations representing both additive and multiplicative relationships, and represented and solved systems of equations more consistently than did aTNS students. These advantages in their algebraic reasoning are largely attributed to their disembedding and splitting operations. However, ENS students' algebraic reasoning was limited

[25] Recall that two students, Greg and Gavin, were assigned a stage classification of ENS or GNS, but that during their interviews it was determined to be more likely that they had constructed a GNS, as opposed to only an ENS. Thus, their algebraic reasoning is not included in the descriptions of ENS students' algebraic reasoning.

in similar ways to aTNS students due to having constructed only an assimilatory composite unit. Each of these constructs that characterize the ENS will be discussed in this section as it advantaged ENS students' algebraic reasoning.

Disembedding. Having constructed a disembedding operation was an advantage to ENS students, particularly on the border problem (A4). On this task, all ENS students who attempted to represent the border of the grid as $4n - 4$ did so correctly; this method does not rely on the disembedding operation because students do not have to maintain the relationship between side lengths to conceptualize the border in this way. Two of the three ENS students also represented the border of an n-by-n grid as some variation of $2n + 2(n - 2)$, and three out of three ENS students represented the border as some variation on $n + (n - 1) + (n - 1) + (n - 2)$. Each of these latter two methods require the disembedding operation because the students must algebraically represent the relationship between related side lengths.

Also on the border problem (A4), all six ENS students successfully represented at least one method for finding the border of an n-by-n grid algebraically, regardless of their enrollment in an algebra class. This result is in contrast to TNS and aTNS students. No TNS and three aTNS students represented the border of the grid algebraically, regardless of course enrollment. This suggests that the students' number sequence was more influential in their ability to produce a correct algebraic representation than was their math course enrollment.

ENS students' disembedding operation also advantaged their algebraic reasoning on tasks involving systems of equations. The evidence of this is that four of the five ENS students who attempted to solve the modified coin problem (A6) algebraically substituted expressions representing the relationship between the numbers of dimes and nickels, and quarters and nickels into an equation representing all of the coins. Algebraically, this was represented as something similar to $(n+6)+n+(n-1)=17$. This demonstrates the ability of ENS students to simultaneously conceptualize the relationships between the numbers of dimes and quarters as they relate to the number of nickels. To conceptualize this relationship also requires them to apply their units coordination, but the disembedding operation advantaged their reasoning in comparison to their aTNS peers, none of whom made correct substitutions. As a result of their substitutions, the ENS students who were enrolled in an algebra course were successful in completing the task algebraically. The ENS students who were not enrolled in an algebra course did not complete the task algebraically; despite making substitutions, they resorted to guess and check. Thus, it seems that perhaps simplifying and solving this type of equation is an algebraic topic that ENS students are prepared to accept instruction on.

Additionally, ENS students' disembedding operation was an advantage to their algebraic reasoning on the block pattern problems (A7 and A8), in which they were asked to generalize and algebraically represent an explicit formula. All ENS students were able to generalize through particular examples, verbalize, and algebraically represent these tasks. This is attributed to their application of the disembedding operation, which advantaged their reasoning by allowing them to conceive of the unknown figure number as both a part of and separate from the unknown number of blocks. Evan's visual representation of the visual block pattern problem (A7; Figure 5.3) is taken as evidence of the disembedding operation. In this work, Evan draws the number of blocks in each figure, and visually demonstrates the relationship between the figure number and the number of blocks by drawing a rectangle around the number of blocks that equals the figure number. Alone, this could be taken as evidence that Evan is merely comparing the figure number to the number of blocks in a visual manner, because he does not visually demonstrate that he maintains the relationship between the figure number and the number of blocks. Regardless, supported by his algebraic representation, it can be concluded that Evan is in fact disembedding the figure number from the number of blocks in his reasoning. This is evidence that like aTNS students, ENS students were able to operate on an unknown quantity, but following activity ENS students could maintain the relationship between the quantities, which aTNS students often could not.

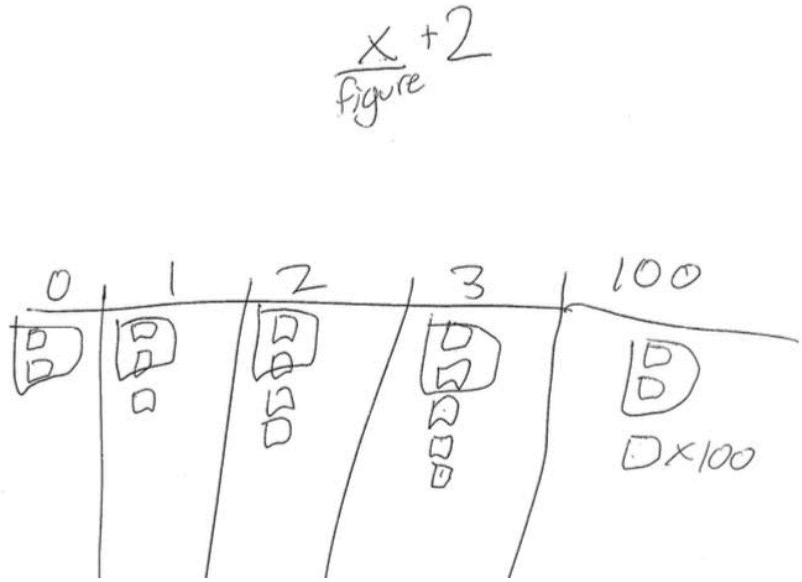

Figure 5.3 Evan's drawing on the visual block pattern problem

Units coordination and disembedding as supporting factors. ENS students' units coordination is similar to that of aTNS students in that students who have constructed both number sequences assimilate tasks with a composite unit (Ulrich, 2016a, 2016b). The implication is that ENS students can assimilate tasks with an unknown quantity, constituted by a composite unit (Hackenberg et al., 2017), and operate on the unknown. Following activity, however, the third level of units constructed in activity decays (Ulrich, 2016a). In these ways, the mental constructs of ENS students are similar to aTNS students, but ENS students are also described to have constructed a disembedding operation (Steffe, 2010b). ENS students' disembedding and coordination of units was found to be a significant advantage on tasks in which the two operations supported the students' algebraic reasoning.

On its own, ENS students having constructed only an assimilatory composite unit limited their success on the coin problem (A5), as only one ENS student solved the task. This is consistent with Olive and Çaglayan's (2008) result that students must operate on a three-level unit structure to solve this task algebraically. Emma is the only ENS student who solved this task. She independently wrote the equation $.10d + .25q + .05n = 5.40$, which represents a three-level unit structure (Olive & Çaglayan, 2008), and with interviewer intervention substituted expressions representing the relationship between the numbers of dimes and nickels, and quarters and nickels, respectively. In this way, she operated on the three-level unit structure by applying her disembedding operation, but she did not initially see how this action would be productive. Of her substitutions, she said, "I'm pretty much just simplifying it further, but I don't know how much that would help." Further evidence that she did not understand why making substitutions was a productive method, is that she initially substituted an expression replacing the variable n with an expression in terms of q. This created the equation $.10(n+3) + .25(n-2) + .05(q+2) = 5.40$. Again, following interviewer intervention, she returned the expression $q+2$ to n, and was then able to solve for the numbers of nickels, dimes, and quarters. Thus, although Emma could make substitutions into her equation for the numbers of dimes and quarters, which eventually allowed her to solve, she required interviewer intervention, and Emma's solution method was not without detours. This demonstrates that she did not conceptualize the task in its entirety. As such, she was advantaged by her disembedding operation, but despite being enrolled in Algebra 2 and having studied systems of equations, her assimilatory composite unit still limited her success because it was very difficult for her to operate on an equation representing a three-level unit structure, despite interviewer support.

Four other ENS students also made progress on task A5 that was qualitatively distinct from aTNS students, and can be attributed to their disembedding operation. Elizabeth, for example, wrote two equations: $d.10 + q.25 + n.05 = 5.40$ and $n + 3 + n - 2 + n = 5.40$. The first equation represents a three-level unit structure because it incorporates the value of each type of coin, within the number of each type of coin, within the total value. Elizabeth constructed this three-level unit structure in activity, but was unable to operate on the equation by substituting expressions for *d* and *q*, respectively, in terms of *n*. In comparison, the second equation represents a two-level unit structure because it incorporates the number of each type of coin within the total value of the coins. Elizabeth was able to apply the disembedding operation to operate on this equation by substituting expressions for *d* and *q*, respectively, in terms of *n*. Elizabeth, and the other three ENS students who made similar substitutions, were then unable to solve the task because they could not operate on the equation representing a three-level unit structure. However, their solutions were distinct from aTNS students as a result of their disembedding operation.

Interestingly, no ENS students attempted to find a solution by guessing and checking. Admittedly, such a solution is cumbersome, but recall that three aTNS students solved task A5 by guess and check. Regardless of their algebra class enrollment, all ENS students attempted to solve the coin problem (A5) algebraically. This is evidence that ENS students understand the productivity of their algebraic attempts, even if their units coordination and construction limited their success because they could not operate on an equation representing a three-level unit structure. Therefore, these results contribute an area of algebraic reasoning on which ENS students may be prepared to accept instruction if tasks are appropriately situated within the capabilities of their assimilatory composite unit.

Although no ENS students attempted to solve the coin problem (A5) using guess and check, three ENS students out of six did attempt to solve the modified coin problem (A6) using guess and check, in comparison to seven aTNS students out of eight. Elle and Emily both used an unwinding strategy (Knuth et al., 2006) to begin the modified coin problem, which is considered to be a pre-algebraic strategy (Knuth et al., 2006). These students used the fact that there were six more dimes than nickels and 17 coins in all to reduce the total number of coins to 11, eliminating the need to think about the numbers of dimes and nickels as separate quantities. This is evidence that they understood the units to be identical, rather than just equal, which is characteristic of an ENS student's iterable unit of one. As a whole, their unwinding strategy is taken as evidence that they disembedded and operated on the number of dimes, and then re-embedded the number of dimes into the total number of coins.

Also, the three ENS students who solved the modified coin problem (A6) using guess and check were able to guess the solution on the first try. The ease with which the three ENS students applied guess and check to find a solution on task A6 is attributed to their ability to operate on composite units, and to disembed. This is in comparison to aTNS students who can operate on composite units, but cannot disembed, so instead operated on embedded composite units. Operating on embedded composite units is mentally taxing for students (Ulrich, 2016b), and aTNS students eased the mental load by recording their guesses with figurative material. This allowed them to capitalize on their guess and check strategies by making incremental adjustments with better efficiency than TNS students, however, no aTNS students "accidentally" solved the coin problems, as Evan stated that he did. The ease with which ENS students guessed a solution to the modified coin problem is evidence of their application of a disembedding operation to the composite unit.

On the football problem (A9), ENS students struggled, but four of the five ENS students who attempted the task represented it algebraically, and their method was to write two separate equations and combine them. This is interpreted as a limitation of the ENS students' only having constructed an assimilatory composite unit, which is again, reminiscent of Hackenberg and Tillema's (2009) description of MC2 students solving a multiplicative task requiring the coordination of three-levels of units by treating them as two distinct problems. This is also similar to the attempted solutions of three aTNS students on the football problem. However, four ENS students were successful on the football problem whereas no aTNS students were successful. This difference is attributed to ENS students' disembedding operation. Although aTNS students wrote two separate equations, they could not join them together by substituting an expression for one into the other because they had not constructed a disembedding operation. ENS students had constructed a disembedding operation and as such, were not limited in this way. Although it was difficult for them, they operated on one equation by substituting into another.

Elle's solution to this task was particularly similar to that of Aaron, an aTNS student. She, like Aaron, began by writing two equations, which places the task within her capabilities because it involves the construction of a three-level unit structure. Then, like Aaron, she applied a numerical example to facilitate the joining of the equations. After completing number examples and writing two separate equations, she said, "I know how to do it, but I just don't know how to write it." This indicates that although she could mentally run through the process of completing the calculations, she was unable to conceptualize the entire relationship as a single unit structure involving unknowns. She was then prompted to explain her process, and she simultaneously explained and wrote that "So basically we take *c* plus 2, wait hold on. *C* plus 3, which would equal the total number of touchdowns. We times that by 7 because that's how much each touchdown's worth, which would give you 35." Despite saying 35, Elle wrote *b*, which is evidence that in order to combine the equations she resorted to conceptualizing the variables as a specific numerical example in which the team had scored 35 points the previous week. Conceptualizing the variables as specific examples rather than unknown quantities reduced the complexity of the unit structure, allowing her to operate on a three-level unit structure, albeit in a non-standard way.

Although this is similar to Aaron's solution, Elle's is distinct in two ways. First, she successfully substituted an expression representing the number of touchdowns the team scored this week into the equation; Aaron never made that substitution. This is explained by Elle's application of her disembedding operation. Also, Aaron never moved beyond explaining the result of his equation as a constant. Aaron could only explain his resulting variable, *z*, as representing 2. Elle, on the other hand, initially indicated that b was 35, but when asked, interpreted it as an unknown. This is also evidence of the importance of the disembedding operation on this task because despite having exhausted her ability to construct additional levels of units, she could still reflect on the relationship between the two variables as a result of her disembedding operation.

Units coordination and splitting as supporting factors. On the phone cords problem (A1), ENS students' assimilatory composite unit, in conjunction with their splitting operation, advantaged their reasoning. All six ENS students represented the phone cords problem algebraically by writing an equation similar to $y = 5x$, and explained each component of the equation. Five of the six ENS students built up their equation using numerical examples, which is consistent with the results of Hackenberg and her colleagues (Hackenberg & Lee, 2015; Hackenberg et al., 2017), who found this behavior typical of MC2[26] students, and who explain it to be a limitation of the students' assimilatory composite unit. However, Hackenberg and Lee (2015) did find that two of the six MC2 students they interviewed using this task were unable to write a correct equation to represent the situation, whereras all ENS students interviewed in the present research study wrote a correct equation.

Although both aTNS and ENS students built their equations on the phone cords problem (A1) using numerical examples, the three successful aTNS students then reverted to an operational concept of the equal sign when asked to explain their equations; this was taken as evidence that following the decay of the third level of units, the sequential partitioning and iterating that aTNS students applied to solve the splitting task did not allow them to reflect on the relationship between the lengths of the cords and interpret it appropriately. On the other hand, ENS students, who simultaneously apply the partitioning and iterating operations, in other words, splitting, were able to use the results of the splitting task to interpret the components of the equations they wrote and did not regress to an operational concept of the equal sign, or to non-standard concepts of variable.

Summary. The algebraic reasoning of ENS students was the most advanced of the students in this study. ENS students applied their assimilatory composite unit to operate on unknowns; in contrast to aTNS students, however, algebraic reasoning was also supported by having constructed a disembedding operation, a splitting operation, and having constructed additive comparisons that were assimilatory. Thus, when the third-level of units constructed in

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[26] Recall that MC2 students and ENS students both assimilate tasks with two level of units, and that an ENS is within the ZPC of students who have constructed an MC2 (see Figure 2.1).

activity decayed, ENS students could reflect upon the results of disembedding, splitting, and making additive comparisons to give normative explanations of their equations and expressions. Also, the guessing and checking of ENS students was the most advanced, and was supported by an assimilatory composite unit and a disembedding operation. However, the only ENS students who used guess and check to solve the modified coin problem were those who were not enrolled in an algebra class; ENS students who were enrolled in an algebra class did not use guess and check at all, and instead solved systems of equations algebraically. The ENS students' algebraic solutions to the coin problems (A5 and A6), both of which represented a system of equations, involved them applying the substitution method for solving. This method was facilitated by having constructed a disembedding operation. However, on A5, students also needed to operate on a three-level unit structure which is beyond the operations of ENS students because they have only constructed an assimilatory composite unit. Thus, while ENS students were advantaged on systems of equations tasks by having constructed a disembedding operation, they were limited because they could only apply disembedding within the constraints of their units coordination and construction. Operating on a three-level unit structure was also necessary on the football problem (A9). Accordingly, ENS students struggled but were ultimately able to apply their disembedding operation on this task to combine equations using numerical examples. In these ways, ENS students were more prepared to reason algebraically than their aTNS peers.

Chapter 6: Conclusions

The purpose of the present research study is to examine how middle-grades students' number sequences can be used to explain and predict their algebraic reasoning. This purpose was addressed using both qualitative and quantitative methods, and focused on the following three research questions.

1. What are the number sequences of students in the middle grades?
2. What are the algebraic capabilities of middle-grades students who have constructed only an aTNS?
3. What are the similarities and differences in the algebraic reasoning of middle-grades students who have constructed a TNS or an ENS, in comparison to students who have constructed an aTNS?

Each of these questions is addressed in the following sections.

Effect of Additive Reasoning on Algebraic Reasoning

One of the primary points for discussion in the present research regarding aTNS students' algebraic reasoning has been related to their intermittent success on tasks requiring them to make an additive comparison between two related unknown quantities. This was a main focus because it was a key distinction between the algebraic reasoning of TNS and aTNS students. It was also found that aTNS students were only successful on approximately half of the tasks when they attempted to represent an additive relationship algebraically.

These inconsistencies can be understood by examining students' ability to construct additive comparisons (Table 6.1). TNS students construct composite units in activity, which implies they are not material for further operating (Ulrich, 2015). Considering this in combination with Hackenberg and her colleagues' (2017) result that an unknown quantity constitutes a composite unit, implies that TNS students cannot operate on or reflect upon unknown quantities. Therefore, to "transform" (Ulrich, 2016a, p. 38) one unknown quantity into another is likely nonsensical. A TNS student would have to conceptualize beginning their counting acts at an unknown starting point, increasing the unknown starting point by some predetermined amount, and terminating counting at an unknown ending point. This is an overarching synthesis of the manner by which additive reasoning limits TNS students in representing additive relationships between two unknowns algebraically.

Table 6.1. Summary of Mental Constructs Supporting Students' Algebraic Representation of Additive Comparisons, by number sequence (Adapted from Ulrich, 2016a)

	Description	Effect on algebraic reasoning
	TNS	
Units coordination and construction	Constructs composite units, and thus unknowns, in activity.*	Unknowns decay following activity and are not material for further operations
Additive reasoning	Constructs additive comparisons between two quantities as a description of a transformation.**	Transforming one unknown into another is nonsensical because the unknown cannot be acted upon.
	aTNS	
Units coordination and construction	Assimilates with a composite unit, and thus unknowns.*	Unknowns are available for reflection following activity and are material for further operations.
Additive reasoning	Constructs additive comparisons between two quantities as a description of a transformation.**	Transforms one unknown into another by operating on embedded unknown.
	ENS	
Units coordination and construction	Assimilates with a composite unit*, and thus unknowns.	Unknowns are available for reflection following activity and are material for further operations.
Additive reasoning	Constructs additive comparisons as an assimilatory quantity.**	Disembeds, operates on, and re-embeds unknowns.

*Paraphrased from Ulrich, 2015, 2016.
**Quoted from Ulrich, 2016a, p. 38.

For ENS students, on the other hand, an unknown quantity is one that can be operated on because it is supported by an assimilatory composite unit (Hackenberg et al., 2017). Furthermore, ENS students can assimilate tasks with an additive comparison (Ulrich, 2016a).

This implies that they are able to reflect upon the results of their additive comparisons, because one unknown can be disembedded from the other without destroying the relationship between the two. Hackenerg and colleagues' (2017) and Ulrich's (2016a) results support the conclusions of the present research study, which indicate that although the three-level unit structure formed by operating on an unknown quantity in activity decays for ENS students, they are still able to make sense of the results of their algebraic reasoning on additive tasks because of their additive reasoning. This synthesizes how additive reasoning supported ENS students' consistency in representing additive relationships algebraically.

Although aTNS students in this study were not as successful as ENS students on representing additive relationships algebraically, they were more successful than TNS students. An assimilatory composite unit was presented as the first mental structure characteristic of an aTNS that the aTNS students' abilities to represent additive relationships algebraically. This allowed aTNS students to conceptualize unknown quantities. Not having constructed the disembedding operation was found to be a delimiting factor for aTNS students, however. The inability to disembed limited aTNS students' ability to conceptualize the relationship between unknown quantities following activity. aTNS students' success can be interpreted, then, by examining the case of Adam (Ulrich, 2016b).

Ulrich (2016b) explains that Adam, the first identified student who had constructed an aTNS, engaged in what she initially believed to be strategic reasoning. However, strategic reasoning is characteristic of ENS students (Steffe, 2010b), and Adam's behaviors were inconsistent with having constructed an ENS. Ulrich concluded that Adam was operating on an embedded composite unit, and not reasoning strategically. That is to say, Adam was not disembedding and operating on composite units then re-embedding them, as ENS students do when they reason strategically. Instead, Ulrich concluded that Adam was operating on embedded composite units, which was mentally taxing for Adam, which made his reasoning inconsistent (Ulrich, 2016b). I liken the equation writing behaviors of aTNS students in the present research study on tasks involving an additive comparison, such as the border problem (A4), the coin problems (A5 and A6), the block pattern problems (A7 and A8), the football problem (A9), and the soccer problem (A10), to Adam's operation on embedded composite units.

Relating this back to the observed behaviors of aTNS students in this study, aTNS students applied their composite unit to unknown quantities allowing them to operate on unknowns. However, without disembedding, the relationship between the related unknowns were often lost. This manifested itself in many ways, including aTNS students' use of numerical examples and statements of the need to evaluate expressions with unknowns, using alpha-numeric codes, using unrelated variables to represent related quantities, applying variables as labels on objects, and using variables as general referents. Just as often as aTNS students applied one of these non-normative means of operating or explaining their algebraic representations of additive relationships, they applied their expressions and equations normatively. This is concluded to be the result of their operation on an unknown as an embedded quantity.

In summary, TNS students algebraic reasoning was virtually non-existent; they generalized using particular numerical examples, but did not verbalize patterns or represent additive or multiplicative relationships algebraically. In general, these limitations to their algebraic reasoning can be attributed to not having constructed an assimilatory composite unit, and not having constructed the disembedding operation. However, on several tasks, the limitations of not having constructed an assimilatory composite unit and the disembedding operation were mediated by TNS students' additive reasoning. In Figure 6.1, the inability of TNS students to represent additive relationships algebraically is represented with a greyed box. Furthermore, the mental structures that were found in this study to be necessary to support such algebraic reasoning are also greyed because they have not been constructed by TNS students;

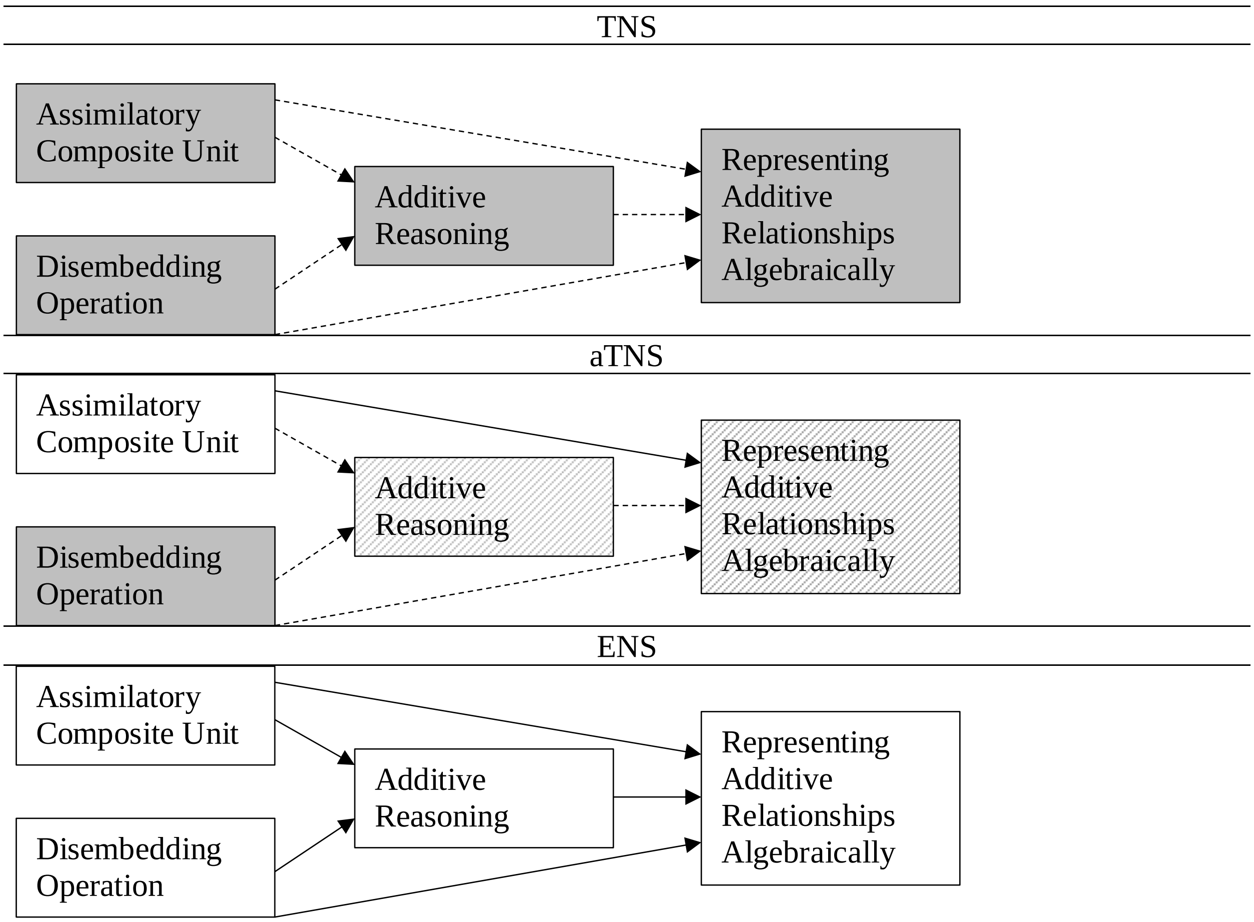

Figure 6.1. Mental structures and coordinations required of students to represent additive relationships algebraically, arranged by number sequences. Grey boxes represent mental structures that have not been constructed, and algebraic reasoning that is subsequently unavailable. Striped boxes represent mental structures that are not assimilatory and the subsequent algebraic reasoning that is available inconsistently. Dotted lines represent coordinations that are not possible, due to limitations of mental structures.

these include an assimilatory composite unit, the disembedding operation, and additive reasoning as an assimilatory structure. Finally, because these mental structures have not been constructed, it is not possible for TNS students to coordinate them in support of the algebraic goal of representing additive relationships algebraically. This is represented by dashed lines.

In comparison, ENS students' algebraic reasoning was advantaged by their assimilatory composite unit and disembedding operation; furthermore, because these mental constructs support additive comparisons as an assimilatory structure, ENS students demonstrated ease on algebraic tasks involving the representation of an additive relationship between unknowns. This is represented in the bottom row of Figure 6.1. The mental structures that this research study found to act in support of representing additive relationships algebraically are not greyed. This is because ENS students have constructed each of these mental structures. Accordingly, ENS students can coordinate these mental structures in support of representing additive relationships algebraically. These coordinations are represented by solid arrows, and the resulting algebraic reasoning is also not greyed.

Finally, aTNS students were advantaged by an assimilatory composite unit but limited by their lack of a disembedding operation. Representing additive relationships between unknown quantities was cognitively demanding for aTNS students, which is attributed to a need to operate on embedded composite units to represent additive relationships algebraically. The result is that algebraic representations of additive relationships were inconsistent for aTNS students. This is represented in Figure 6.1 by a striped box. Additionally, the only solid arrow in Figure 6.1 for aTNS students connects an assimilatory composite to representing additive relationships algebraically. The solid arrow represents the manner by which aTNS students in the present research study operated on an assimilatory composite unit, constituting an unknown (Hackenberg et al., 2017), to represent additive relationships algebraically. The dashed arrows indicate that aTNS students did not coordinate an assimilatory composite unit with a disembedding operation to leverage additive reasoning in their algebraic representations of additive relationships. This is because aTNS students have not yet constructed a disembedding operation (Steffe, 2010b), and therefore, additive reasoning is not yet an assimilatory structure for these students (Ulrich, 2016a). In the present research study, aTNS students' inconsistency in representing additive relationships algebraically is attributed to not having constructed a disembedding operation and not having constructed additive reasoning as an assimilatory structure. In other words, aTNS students could only represent additive relationships algebraically by operating on an assimilatory composite unit as an embedded quantity. For this reason, aTNS students inconsistently represented additive relationships algebraically.

Effect of Splitting on Algebraic Reasoning

Three out of eight aTNS students (32.5%) represented the multiplicative relationship between two unknowns algebraically on the phone cords problem (A1), which is in comparison to six out of six ENS students (100%). Moreover, both aTNS and ENS students who represented a multiplicative relationship algebraically did so by building the equation using numerical examples. This is consistent with Hackenberg and her colleagues' (2017) results for MC2[27] students, which she attributes to their need to build the equation in activity. The students in this research demonstrated similar reasoning, but the inconsistency of aTNS students' responses seemed to be mediated by splitting.

Figure 6.2 presents a visual representation of the cognitive structures and operations acted in supports of the algebraic representations of multiplicative relationships in the present research study. aTNS students relied on an assimilatory composite unit to represent these

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[27] Recall that MC2 students and ENS students both assimilate tasks with two level of units, and that an ENS is within the ZPC of students who have constructed an MC2 (see Figure 2.1).

relationships, which is represented by the solid line connecting assimilatory composite unit to representing multiplicative relationships algebraically. The ability to operate on an assimilatory composite unit allowed aTNS students to conceive of the multiplicative relationship between two unknowns as a three-level unit structure, however, following activity, the third-level of units decayed. This was observed in the inability of aTNS students to explain their equations as a relationship between two unknowns following activity; they all resorted to an operational concept of the equal sign. For this reason, the box indicating the representation of multiplicative relationships is striped for aTNS students in Figure 6.2. Furthermore, although three aTNS students out of eight solved the splitting task on the phone cords problem, aTNS students have not constructed sufficient mental structures to support the splitting operation (Ulrich, 2016b). Therefore, aTNS students in the present research study are presumed to have solved the splitting task by sequentially partitioning and iterating, which implies that the splitting operation is not assimilatory for those students. This is represented in Figure 6.2 by the striped splitting box.

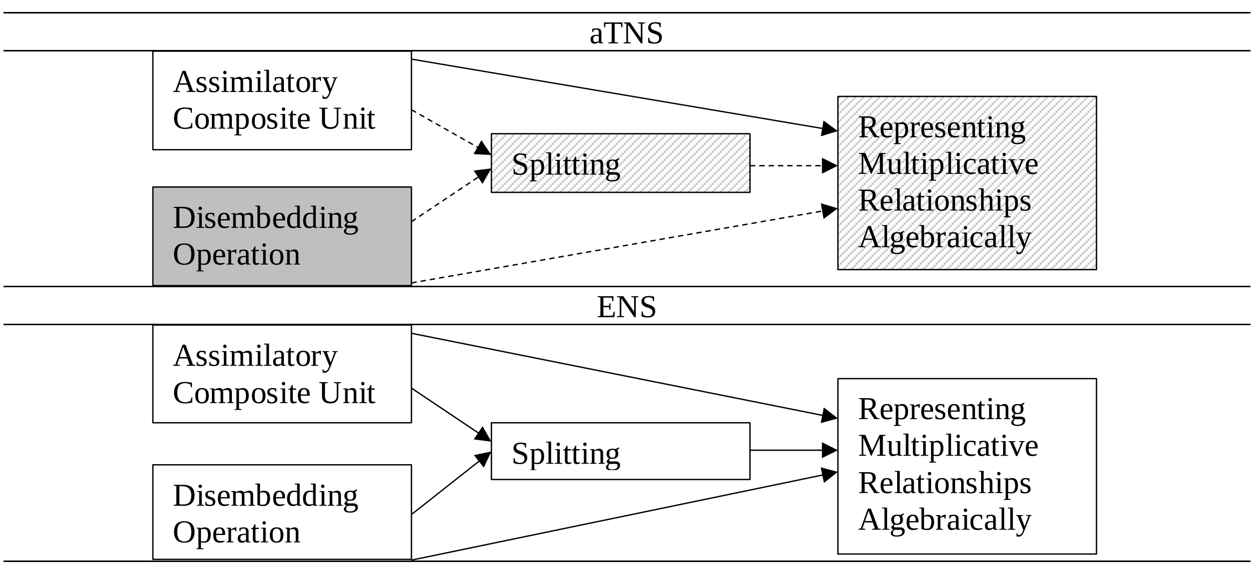


Figure 6.2. Mental structures and coordinations required of students to represent multiplicative relationships algebraically, arranged by number sequences. Grey boxes represent mental structures that have not been constructed. Striped boxes represent mental structures that are not assimilatory and the subsequent algebraic reasoning that is available

inconsistently. Dotted lines represent coordinations that are not possible, due to limitations of mental structures.

Finally, because aTNS students have not constructed a splitting operation as an assimilatory structure, it did not support their algebraic representations of multiplicative relationships. This was evidenced by the inconsistency between aTNS students' solution to the splitting task and their solution to the algebraic portion of the phone cords problem. One aTNS student solved the splitting task and represented the problem algebraically. Two aTNS students did not solve the splitting task but did represent the problem algebraically. Two aTNS students solved the splitting task but did not represent the problem algebraically. These inconsistencies are attributed to aTNS students' inability to reflect on the results of the splitting task, and manifested in aTNS students' regression to an operational concept of the equal sign when they explained their equations.

Students' Concepts of Variable and Equality

Students from each number sequence demonstrated non-standard concepts of variable. These included conceptualizations of variables as a specific numerical example, labels for objects, alpha-numeric codes, unrelated quantities, ignored variables, and general referents (Küchemann, 1981; MacGregor and Stacey, 1997). MacGregor and Stacey (1997) indicate that students in algebra classes demonstrated non-standard concepts of variable that were the result of interference from instruction. The present research demonstrates the application of each of these non-standard concepts of variable across grade levels and course enrollment. aTNS students most frequently applied non-standard concepts of variable. Ostensibly, TNS students did not demonstrate as many because they did not engage in comparable levels of algebraic reasoning as did aTNS students, and ENS students did not demonstrate as many because algebraic reasoning was better supported by an assimilatory composite unit, a disembedding operation, a splitting operation, and additive reasoning as an assimilatory structure in comparison to aTNS students, who relied solely on an assimilatory composite unit.

These results support the possibility of non-standard concepts of variable being applied in compensation for insufficient mental structures to support the writing of equations and reflection on equations following activity. Hackenberg and her colleagues (2017) similarly found that MC2 students may apply non-standard concepts of variable when cognitive structures are insufficient to support reasoning with regard to unknown quantities. Findings from the present research also document non-standard conceptualizations of variables as necessary errors in which students engage when their mental structures decay, for example, leaving them unable to reflect on variables as unknown quantities. However, this research also demonstrates that students may apply different non-standard conceptions of variable depending on the task and the delimiting mental constructs.

One non-standard conception of variable that was applied by two ENS students was the writing of inequalities, rather than equations, to represent exact multiplicative relationships. This is compared to Hackenberg and her colleagues' (2017) description of an MC2 student who stated that a multiplicative relationship between two unknowns was approximate, unless he was substituting numerical values. Although this is a variation on an operational concept of the equal sign, it is an additional demonstration of the manner in which students' limited cognitive structures may manifest in their behaviors.

Additionally, students in the present research study demonstrated a regression in their concept of the equal sign to compensate for insufficient mental structures to support algebraic reasoning. This was particularly prevalent among aTNS students. On task A1, aTNS students applied an operational concept of the equal sign because the results of the splitting operation were not available for their reflection, and on tasks involving additive reasoning because they were unable to reflect upon their operations on embedded composite units. Thus, it is concluded that in addition to applying non-standard conceptions of variable to situations in which students' cognitive structures do not support algebraic reasoning, they may also revert to an operational concept of the equal sign.

Students' regression to an operational concept of the equal sign is supported by previous literature (Jones et al., 2013; Matthews et al., 2012; Stephens et al., 2013); in particular, Matthews et al. (2012) found that a relational concept of the equal sign does not replace, but rather compliments, an operational concept. The present research builds upon this result by framing the application of an operational concept of the equal sign as a necessary error in the student's reasoning that is applied to compensate for the decay of mental activity related to splitting, disembedding, or the coordination of units.

Additional research has identified that students' conception of the equal sign is not so much the result of mental structures but of instruction (Baroody & Ginsburg, 1983; Byrd et al., 2015; Cobb, 1987; Knuth et al., 2011). This result is supported by the present research; only sixth graders were found to have constructed only an operational concept of the equal sign. However, it is also notable that no TNS students constructed a flexible relational concept of the equal sign (level 4), which is the most sophisticated (Matthews et al., 2012), regardless of their grade or course enrollment. Therefore, it is possible for TNS students to construct a relational concept of the equal sign, as Travis did (level 3), but it is also possible that students must construct at least an aTNS in order to support a flexible relational concept of the equal sign (level 4). This hypothesis is based on the reasoning of students who were determined to have constructed a flexible operational concept; each of them applied some form of a transfer compensatory strategy (Steffe & Cobb, 1988) to reason about the equality of two expressions without calculating the sum.

Sameness substitution and systems of equations. The research of Jones (2008) indicates that a relational concept of the equal sign includes sameness and a substitution component. The present research utilized the equality framework of Matthews and his colleagues (2012), which does not distinguish between sameness and substitution. Accordingly, this was not considered during analysis. However, aTNS students were unable to substitute additive expressions in place of a variable in solving systems of equations. This was attributed to not disembedding, implying their inability to maintain the relationships between two related quantities during substitution. Further research should consider mental constructs that support the substitution component of equality, and what the relationship is between the substitution component and the flexible relational concept utilized in the present study.

Explicit and Recursive Formulas

As noted, aTNS students were inconsistent in their representations of additive relationships between two unknowns. Tasks A7 and A8, however, were distinct from other tasks requiring the representation of an additive relationship because on these tasks, students were required to abstract an explicit pattern. All students, regardless of their number sequence, were able to make a numerical comparison to engage in generality by example and determine the one-hundredth term in the pattern; this is considered to be the determination of an explicit pattern. aTNS students were also able to verbalize the pattern in general terms, however, relying on numerical comparisons, rather than disembedding, did not allow them to consistently represent these patterns algebraically. ENS students, on the other hand, consistently represented additive explicit formulas as a result of their disembedding operation. Further research should examine to what extent the disembedding operation facilitates ENS students' reasoning with regard to other explicit formulas, in particular, linear patterns with a rate of change other than one, and non-linear patterns.

Number Sequences of Middle Grades Students

The number of students in eighth and ninth grade who had constructed an ENS was significantly larger than the number of students in sixth and seventh grade who had constructed an ENS. The number of students in eighth and ninth grade who had constructed an aTNS was significantly smaller than the number of students in sixth and seventh grade who had constructed an aTNS. This demonstrates that students who have constructed at least an aTNS in sixth grade are likely to advance their concept of number by the time they enter high school. However, the percentage of students in eighth and ninth grade who had constructed only an INS or a TNS was comparable to the percentages of students who had constructed only an INS or a TNS in sixth and seventh grade. This suggests that students with only an INS or a TNS entering middle school are not constructing more sophisticated number sequences by the time they enter high school.

This result is presented in comparison to those of Norton and Wilkins (2013) who concluded that students who have not constructed the splitting operation by the end of sixth grade are unlikely to have constructed the PUFS by the end of seventh grade. Students with an INS and a TNS have not constructed the splitting operation; that the INS and TNS students in the present research study are not likely to construct more sophisticated number sequences supports Norton and Wilkins' result. In other words, because the number of students who have constructed only an INS and a TNS does not change from sixth grade to seventh grade confirms Norton and Wilkins' result that students who have not constructed the splitting operation in sixth grade, measured in the present study as having constructed an INS or a TNS, are unlikely to do so by seventh grade.

aTNS students have also not constructed the splitting operation. Thus, Norton and Wilkins' (2013) result would predict that aTNS students are also no more likely to have constructed the splitting operation by the end of seventh grade. In the present research study, aTNS students were no more likely to construct a more sophisticated number sequence in seventh grade compared to sixth grade. This is consistent with Norton and Wilkins' result. However, there were approximately three times as many eighth-grade ENS students in the present research study compared to sixth-grade ENS students, and approximately two and a half times as many eighth-grade ENS students compared to seventh-grade ENS students. The splitting operation is within the ZPC of ENS students (Steffe, 2010d). The likeliness of students' construction of the splitting operation in sixth grade compared to eighth grade needs to be more clearly delineated in future research.

Using Number Sequences as a Predictor of Algebra Readiness

The qualitative results of this study suggest that students number sequences can be used to explain and predict their algebraic reasoning as it relates to generalizing patterns, and writing and solving linear equations and systems of equations. In particular, students' units coordination and construction, disembedding operation, splitting operation, and additive reasoning were shown to be mental constructs that contribute to algebraic reasoning. Students with an ENS were shown to be successful on many of the algebraic tasks, as a result of these mental structures, whereas aTNS students were inconsistent in their algebraic representations, and TNS students were largely unsuccessful, in comparison. This indicates that while ENS students may be prepared to engage meaningfully with the mathematical content in a high school algebra course, aTNS students and TNS students likely are not.

Considering this result in conjunction with the changes in students' number sequences across the middle grades provides clarity into students' algebra readiness. Students are approximately two and a half times more likely to have constructed an ENS than an aTNS in eighth grade compared to sixth, indicating that aTNS students are likely constructing more sophisticated concepts of number by the time they enter algebra. Being more likely to have constructed an ENS has implications for their algebraic reasoning, thus, making it probable that sixth grade aTNS students' concept of number will advance sufficiently by high school to prepare them to engage in some forms of algebraic reasoning necessary in an algebra class (cf. Norton & Wilkins, 2013). In contrast, INS and TNS students are no more likely to have constructed a more sophisticated number sequence by eighth and ninth grade than they were in sixth and seventh grade. Having constructed only a TNS was shown in this study to have negative implications for their algebraic reasoning. Thus, INS and TNS students are no more prepared for an algebra course in ninth grade than they were in sixth grade.

It has been said that

> If there is a heaven for school subjects, algebra will never go there. It is the one subject in the curriculum that has kept children from finishing high school… It has caused more family rows, more tears, more heartaches, and more sleepless nights than any other school subject. (Cai & Knuth, 2011, p. vii)

The results of this study identify the students for whom these family rows, tears, heartaches, and sleepless nights are likely a reality by determining that for TNS students in the middle grades, algebraic tasks as simple as verbalizing and writing one step additive equations can prove too difficult. Of equal importance, this study demonstrates that middle grades students' grade level and algebra course enrollment are less closely related to their algebraic reasoning on many tasks than are their number sequences. For example, no TNS students algebraically represented additive or multiplicative relationships. aTNS students algebraically represented additive relationships approximately half of the time and multiplicative relationships approximately 38% of the time (see Table 4.15). ENS students algebraically represented additive and multiplicative relationships approximately 88% of the time (see Table 4.15). With each increasingly sophisticated number sequence, the students successfully represented algebraic relationships more consistently. In contrast, the numbers of students from each grade who algebraically represented additive (6^{th} grade 63%; 7^{th} grade 47%; 8^{th} grade 83%; 9^{th} grade 56%) and multiplicative (6^{th} grade 58%; 7^{th} grade 25%; 8^{th} grade 75%; 9^{th} grade 75%) tasks did not have a clear trend (see Table 4.15). This finding brings relevance to the importance of understanding how to support students' construction of more sophisticated concepts of number, which in turn supports their algebraic reasoning. Such a focus in the middle grades curriculum may help prevent the heartache and sleepless nights that dismally characterize algebra for so many students.

Future Research

Further research needs to examine the means by which middle grades students' concepts of number can be addressed so as to increase their algebra readiness in a meaningful way. Additive reasoning is proposed to mediate the relationship between units coordination and disembedding in the algebraic representation of additive relationships (see Figure 6.1). Similarly, the splitting operation is proposed to mediate the relationships between units coordination and disembedding in the algebraic representation of multiplicative relationships (See Figure 6.2).

Quantitative validation of these models is necessary, and understanding these relationships can drive instructional design and decision making.

Longitudinal research is also necessary to strengthen the results of this study. The cross-sectional nature of the study allows only for the examination of between student changes in number sequences, but neglects within student changes. In other words, what are the individual growth trajectories of students' number sequence constructions? Knowing only that students in ninth grade are three times more likely to have constructed an ENS than an aTNS in comparison to students in sixth grade says nothing to how many students' number sequences progressed from an aTNS to an ENS. A longitudinal design accounts for such student-level changes.

Furthermore, a constructivist teaching experiment (Steffe & Thompson, 2000; Steffe & Ulrich, 2013) that incorporates longitudinal design will allow for more precise model building. This type of analysis will allow for a more in-depth understanding of consistencies in behavioral patterns, and the underlying mental constructs that support these behaviors, for each individual student, and subsequently, for groups of students by number sequence. This type of research can be used to further inform the modeling of the relationships between students units coordination and construction, disembedding, splitting, additive reasoning, and algebraic representations of additive and multiplicative relationships.

The results of this study indicate that ENS students may be prepared to accept instruction in solving linear systems of equations algebraically. All students who had constructed an ENS attempted to solve the systems of equations tasks algebraically, regardless of their course enrollment, which demonstrates that they understood the productivity of this algebraic behavior. This was particularly true on the modified coin problem (A6), which required students to operate on a two-level unit structure. Future research should consider to what extent ENS students are able to solve systems of equations involving two-level unit structures, and whether their solutions extend to graphical methods and methods of elimination. The results of task A5, on the other hand, demonstrate that when operating on a three-level unit structure is required, systems of equations are still beyond the algebraic reasoning of ENS students. Future research should also examine to what extent graphical and elimination methods of solving systems of equations involving three-level unit structures are available to ENS students.

References

Agresti, A. (2002). *Categorical data analysis* (2nd ed.). Hoboken, NJ: Wiley-Interscience.

Alibali, M. W., Knuth, E. J., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning, 9*, 221–247.

Baroody, A. J., & Ginsberg, H. P. (1983). The effects of instruction on children's understanding of the "equals" sign. *Elementary School Journal, 84*, 199–212.

Brizuela, B., & Schliemann, A. D. (2004). Ten-year old students solving linear equations. *For the Learning of Mathematics, 24*(2), 33–40.

Byrd, C. E., McNeil, N. M., Chesney, D. L., & Matthews, P. G. (2015). A specific misconception of the equal sign acts as a barrier to children's learning of early algebra. *Learning and Individual Differences*, *38*, 61–67.

Cai, J., & Knuth, E. (2011). [Introduction]. In J. Cai & E. Knuth (Eds.) *Early algebraization: A global dialogue from multiple perspectives* (pp. vii–xi). New York, NY: Springer.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Portsmouth, NH: Heinemann.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13*(1), 16–30.

Clement, J. 2000. Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.) *Handbook of research design in mathematics and science education* (pp. 547–589). Mahwah, NJ: Erlbaum.

Cobb, P. (1987). An investigation of young children's academic arithmetic contexts. *Educational Studies in Mathematics, 18*(2), 109–124.

Cohen, J. (1960). A coefficient of agreement for nominal scales. *Educational and Psychological Measurement*, *20*, 37–46.

Cohen, J. (1968). Weighted kappa: Nominal scale agreement with provision for scaled disagreement or partial credit. *Psychological Bulletin, 70*, 213–220.

College Preparatory Mathematics (Algebra 1), 2nd Edition (2002). T. Salle, J. Kysh, E. Kasimatis and B. Hoey (Program Directors). Sacramento, CA: CPM Educational Program.

Common Core State Standards for Mathematics. (2010). *Common Core State Standards Initiative.* Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Creamer, E. G. (2017). *An introduction to fully integrated mixed methods research*. Thousand Oaks, CA: Sage.

Creswell, J. W., & Plano Clark, V. L. (2007). *Designing and conducting mixed methods research* (1st ed.). Thousand Oaks, CA: Sage.

Creswell, J. W., & Plano Clark, V. L. (2011). *Designing and conducting mixed methods research* (2nd ed.). Thousand Oaks, CA: Sage.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, *9*(2), 19–25.

Glasersfeld, E. v. (1981). An attentional model for the conceptual construction of units and number. *Journal for Research in Mathematics Education*, *12*(2), 83–94.

Glasersfeld, E. v. (1995). Growing up constructivist: Languages and thoughtful people. In *Radical constructivism: A way of knowing and learning* (pp. 1–23). New York, NY: RoutledgeFalmer.

Greene, J. C., Caracelli, V. J., & Graham, W. F. (1989). Toward a conceptual framework for mixed-method evaluation designs. *Educational Evaluation and Policy Analysis*, *11*(3), 255–274.

Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *Journal of Mathematical Behavior, 26*, 27–47.

Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior, 32,* 538–563.

Hackenberg, A. J., Jones, R., Eker, A., & Creager, M. (2017). "Approximate" multiplicative relationships between quantitative unknowns. *Journal of Mathematical Behavior, 48,* 38–61.

Hackenberg, A. J., & Lee, M. Y. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education, 46*(2), 196–243.

Hackenberg, A. J. & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior, 28,* 1–18.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*(1), 59–78.

Howell, D. C. (2013). *Statistical methods for psychology* (8th ed.). Belmont, CA: Wadsworth Cengage Learning.

MacDonald, P. L., & Gardner, R. C. (2000). Type I error rate comparisons of post hoc procedures for I x J chi-square tables. *Educational and Psychological Measurement, 60*(5), 735–754.

Jones, I. (208). A diagrammatic view of the equals sign: Arithmetical equivalence as a means, not an end. *Research in Mathematics Education, 10*(2), 151–165.

Keith, T. Z. (2015). *Multiple regression and beyond: An introduction to multiple regression and structural equation modeling* (2nd ed.). New York, NY: Routledge.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317–326.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester Jr. (Eds.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). Charlotte, NC: New Age Publishing; Reston, VA: National Council of Teachers of Mathematics.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*, 297–312.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2011). Middle school students' understanding of core algebraic concepts: Equivalence & variable. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 259–276). New York, NY: Springer-Varlag.

Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 102–119). London: Murray.

Landis, J. R., & Koch, G. G. (1977). The measurement of observer agreement for categorical data. *Biometrics, 33*, 159–174.

Lucariello, J., Tine, M. T., & Ganley, C. M. (2014). A formative assessment of students' algebraic variable misconceptions. *The Journal of Mathematica Behavior, 33*, 30–41.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. *Educational Studies in Mathematics, 33*(1), 1–19.

Matthews, P., Rittle-Johnson, B., McEldoon, K., & Taylor, R. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education*, *43,* 316–350.

Morse, J. M. (2003). Principles of mixed methods and multimethod research design. In A. Tashakkori & C. Teddlie (Eds.), *Handbook of mixed methods in social and behavioral research* (pp. 189–208). Thousand Oaks, CA: Sage.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Norton, A., Boyce, S., Phillips, N., Anwyll, T., Ulrich, C., & Wilkins, J. L. M. (2015). A written instrument for assessing students' units coordination structures. *Mathematics Education, 10*(2), 111–136.

Norton, A., & D'Ambrosio, B. S. (2008). ZPC and ZPD: Zones of teaching and learning. *Journal for Research in Mathematics Education, 39*(3), 220–246.

Norton, A., & Hackenberg, A. J. (2010). Continuing research on students' fraction schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 341–352). New York, NY: Springer.

Norton, A., & Wilkins, J. L. M. (2009). A quantitative analysis of children's splitting operations and fraction schemes. *Journal of Mathematical Behavior, 28,* 150–161.

Norton, A., & Wilkins, J. L. M. (2012). The splitting group. *Journal for Research in Mathematics Education, 43*(5), 557–583.

Norton, A., & Wilkins, J. L. M. (2013). Supporting students' constructions of the splitting operation. *Cognition and Instruction, 31*(1), 2–28.

Olive, J. (2001). Children's number sequences: An explanation of Steffe's constructs and an extrapolotation to rational numbers of arithmetic. *The Mathematics Educator, 11*(1), 4–9.

Olive, J., & Çaglayan, G. (2008). Learners' difficulties with quantitative units in algebraic word problems and the teacher's interpretation of those difficulties. *International Journal of Science and Mathematics Education, 6*, 269–292.

Piaget, J. (1970). *Genetic Epistemology* (E. Duckworth, Trans.). New York, NY: Norton.

Piaget, J. (2001). Abstraction, differentiation, and integration in the use of elementary arithmetic operations. In R. L. Campbell (Ed., Trans.), *Studies in reflecting abstraction* (pp. 33–53). New York, NY: Taylor Francis. (Original work published 1977)

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry*. Oxford, England: Basic Books.

Radford, L. (2011). Grade 2 students' non-symbolic algebraic thinking. In J. Cai and E. Knuth (Eds.), *Early algebraization* (pp. 303–322). New York, NY: Springer.

Russell, S. J., Schifter, D., & Bastable, V. (2011). Developing algebraic thinking in the context of arithmetic. In J. Cai & E. Knuth (Eds.) *Early algebraization: A global dialogue from multiple perspectives* (pp. 43–70). New York, NY: Springer.

Siegel, S., & Castellan, N. J., Jr. (1988). *Nonparametric statistics for the behavioral sciences* (2nd ed.). New York, NY: McGraw-Hill.

Singer, J. D., & Willett, J. B. (2003). *Applied longitudinal data analysis: Modeling change and event occurrence*. New York, NY: Oxford.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259–309.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 20,* 267–307.

Steffe, L. P. (2003). Fraction commensurate, composition, and adding schemes: Learning trajectories of Jason and Laura: Grade 5. *Journal of Mathematical Behavior, 22,* 237–295.

Steffe, L. P. (2007). *Problems in mathematics education*. Paper presented for the Senior Scholar Award of the Special Interest Group for Research in Mathematics Education (SIG-RME) at the annual conference of the American Educational Research Association in Chicago, Illinois.

Steffe, L. P. (2010a). A new hypothesis concerning children's fractional knowledge. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 1–12). New York, NY: Springer.

Steffe, L. P. (2010b). Operations that produce numerical counting schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 27–48). New York, NY: Springer.

Steffe, L. P. (2010c). Articulation of the reorganization hypothesis. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 49–74). New York, NY: Springer.

Steffe, L. P. (2010d). The partitive and the part-whole schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 75–122). New York, NY: Springer.

Steffe, L. P. (2010e). The unit composition and the commensurate schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 123–170). New York, NY: Springer.

Steffe, L. P. (2010f). The partitioning and fraction schemes. In L. P. Steffe & J. Olive (Eds.), *Children's fractional knowledge* (pp. 315–340). New York, NY: Springer.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies.* New York, NY: Springer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillsdale, NJ: Erlbaum.

Steffe, L. P., & Tzur, R. (1994). Interaction and children's mathematics. *Journal of Research in Childhood Education, 8*(2), 99–116.

Steffe, L. P. & Ulrich, C. (2013). Constructivist teaching experiment. In S. Lerman (Ed.), *Encyclopedia of mathematics education: SpringerReference* (pp. 102–107). Berlin, Germany: Springer-Verlag.

Teddlie, C., & Tashakkori, A. (2009). *Foundations of mixed methods research: Integrating qualitative and quantitative approaches in the social and behavioral sciences.* Thousand Oaks, CA: Sage.

Thompson, B. (1988). Misuse of chi-square contingency-table test statistics. *Educational and Psychological Research*, *8*(1), 39–49.

Tzur, R., & Simon, M. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education, 2,* 287–304.

Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively (part 1). *For the Learning of Mathematics, 35*(3), 2–7.

Ulrich, C. (2016a). Stages in constructing and coordinating units additively and multiplicatively (part 2). *For the Learning of Mathematics, 36*(1), 34–39.

Ulrich, C. (2016b). The tacitly nested number sequence in sixth grade: The case of Adam. *The Journal of Mathematical Behavior, 43*, 1–19.

Ulrich, C., & Wilkins, J. L. M. (2015). *A written instrument for assessing children's number sequences*. Blacksburg, VA: Virginia Tech.

Ulrich, C., & Wilkins, J. L. M. (2017). Using written work to investigate stages in sixth-grade students' construction and coordination of units. *International Journal of STEM Education, 23*(4). Retrieved from https://stemeducationjournal.springeropen.com/articles/10.1186/s40594-017-0085-0

United States Census Bureau. (2016). *Population and housing estimates.* Washinton, DC: Author.

Virginia Department of Education. (2016). *School quality profiles.* Richmond, VA: Author.

Vygotsky, L. S. (1986). *Thought and language: Revised edition.* Cambridge, MA: MIT Press.

Wilkins, J. L. M., & Norton, A. (2011). The splitting loope. *Journal for Research in Mathematics Education, 42*(4), 386–416.

Wilkins, J. L. M., Norton, A., & Ulrich, C. (2017). Activating a fourth level of units coordination. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 271–274). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Wilkins, J. L. M., & Ulrich, C. (2017). The role of skip counting in children's reasoning. *Virginia Mathematics Teacher, 43*(2). Retrieved from http://www.vctm.org/The-Role-of-Skip-Counting-and-Figurative-Reasoning

Woolley, C. M. (2009). Meeting the mixed methods challenge of integration in a sociological study of structure and agency. *Journal of Mixed Methods Research, 3*(1), 7–25.

Appendix A: Number Sequence Screening Tasks

Bar Task 2[28]

If the shorter rectangle is eight units long, how many units long is the longer rectangle?

= 8

= __________

Bar Task 3

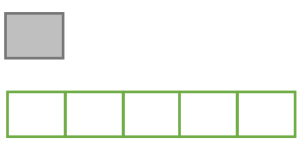
If the shorter rectangle is 8 units long, how many units long is the longer rectangle?

= 8

= __________

Bar Task 4

If the longer rectangle is 90 units long, how many units long is the shorter rectangle?

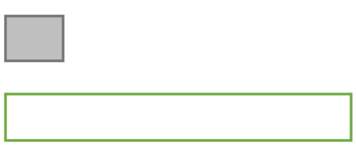


= __________

= 90

Bar Task 5

If the longer rectangle is 42 units long, how many units long is the shorter rectangle?



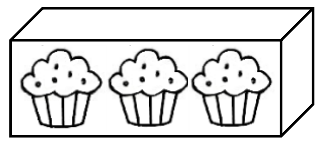
= __________

= 42

[28] Bar tasks, cupcake tasks, and splitting tasks taken from Ulrich and Wilkins (2017).

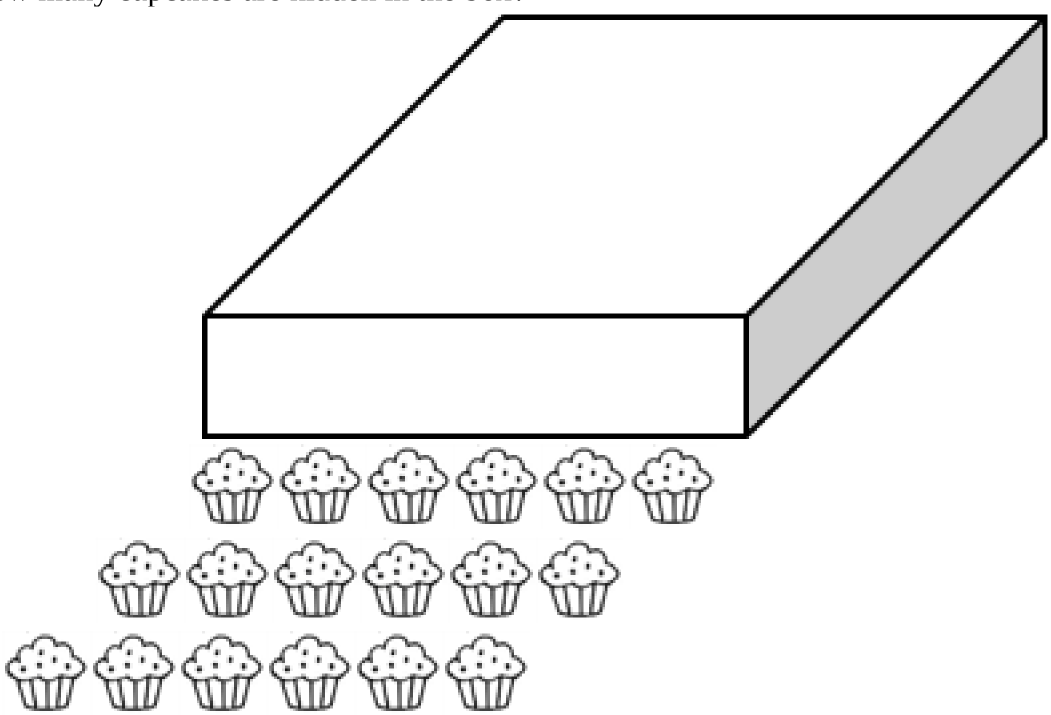
Cupcake Task A

You have baked 39 cupcakes and you will put the cupcakes in boxes of three. How many boxes will you fill?



Cupcake Task B

There are 3 rows of 6 cupcakes below that are unboxed. If there are 9 rows of cupcakes in all, how many cupcakes are hidden in the box?



Splitting Task A

The stick shown below is 4 times as long as another stick. Draw the other stick.

Splitting Task B

The stick shown below is 6 times as long as another stick. Draw the other stick.

Equality Task L2[29]

Can you tell if the equation is true or false?

17 = 53 - 36

Equality Task L3

Can you tell if the equation is true or false?

37 + 29 - 5 = 48 + 14

Equality Task L4

Without adding, can you tell if the equation is true or false?

67 + 86 = 68 + 85

[29] Equality tasks adapted from Matthews et al. (2012).

Appendix B: Algebra Tasks

The Phone Cords Problem: Algebra Task A1[30]

Stephen has a cord for his iPhone that is some number of feet long. His cord is five times the length of Rebecca's cord.

(a) Could you draw a picture of this situation? Describe what your picture represents.
(b) Can you write an equation for this situation? Can you tell me in words what your equation means?
    a. *As necessary*. Can you check your equation with your picture?
    b. *As necessary*. Check your equation using this question: who has a longer cord, Stephen or Rebecca?
    c. *As necessary*. If Rebecca's cord is three feet long, how long is Steven's cord?
(c) Can you write more than one equation?
    a. *As necessary* (if they have only written something like t = 5*q, where t represents Stephen's cord length and q represents Rebecca's cord length): can you write an equation to express Rebecca's cord length in terms of Stephen's?
    b. *As necessary* (if they have written something like t=q/5): can you write this equation using multiplication?
(d) Let's say that Stephen's cord is 15 feet long. Explain how to find the length of Rebecca's cord.

The Modified Splitting Problem: Algebra Task A2

How tall do you think your math teacher (or principal) is? It's not a value that we know exactly, right? But we could ask him or have him stand by a measuring tape and find out the value. Let's say we know that he is three times the height of a little boy who's 1-year-old. We don't know the 1-year-old's height either.

(a) Could you draw a picture of this situation? Describe to me what your picture represents.
(b) Can you write an equation for this situation? What does your equation mean in words?
    a. *As necessary*. Can you check your equation with your picture?
(c) Can you write more than one equation?
    a. *As necessary* Can you write an equation to determine the height of the baby in terms of the height of the adult?
    b. *As necessary* Can you write this equation using multiplication?
(d) Let's say that the teacher/principal is 6 feet tall. How tall is the baby?
(e) Let's say that the teacher/principal is 7 feet tall. How tall is the baby?
(f) Let's say that the baby is 2 and a half feet tall. How tall is the principal/teacher?

[30] Algebra tasks A1 through A4 taken from Hackenberg (2013) and Hackenberg and Lee (2015).

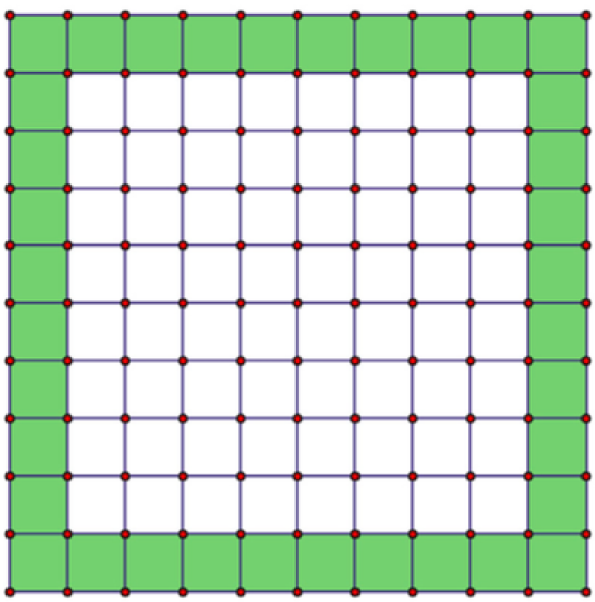
The Candy Bar Problem: Algebra Task A3

You share this candy bar with seven of your friends. Now you share your piece with your two brothers. (Show picture and ask the student to mark off only their share.) Let's say that h is the weight of the whole bar. How much does your bar weigh?

(a) If this question is hard, Can you show me how you would share the candy bar among you and your seven friends?
(b) Can you show me how you would share your piece among you and your two brothers?
(c) What fraction is your piece of the entire candy bar?
(d) Can you write down an expression for the weight of your piece of candy?
(e) What if the whole candy bar weighed 48 ounces. How much would your piece weigh?

The Border Problem: Algebra Task A4

Here is a picture of a 10x10 grid with the square son the border highlighted.



(a) Without counting one-by-one, and without writing anything down, can you find a way to determine how many squares are on the border? Elicit and probe reasoning.
(b) Now, let's say your square was 6x6. Could you use your method to determine the number of squares on the border?
(c) What if we said your square was 100x100. Could you use your method to determine the number of squares on the border?
(d) Let's pretend that your math teacher came here, and he/she has not seen this problem before. Can you explain to him/her how you would find the number of squares on the border of *any* grid?
(e) Can you write an expression to communicate your method for finding the number of squares on the border of an *n*x*n* grid?

The Coin Problem: Algebra Task A5[31]

Ms. Speedy keeps coins for paying the toll crossing on her commute to and from work. She presently has three more dimes than nickels and two fewer quarters than nickels. The total value if $5.40. Find the number of each type of coin she has.

(a) How would you use algebra to write an equation to represent this problem?
(b) *As necessary*. What variables would you need to use to write the equation?
(c) *As necessary* (if students conflate the number of each type of coin with the value of each type of coin). What does each variable represent? What about each dime does it represent? The value of a dime or the number of dimes?
(d) *As necessary* (if students write equations N, D, and Q). Can you solve an equation that has three variables in it? Let's try writing an equation for D in terms of N and an equation for Q in terms of N. Can those be used to solve the problem?

The Modified Coin Problem: Algebra Task A6

Ms. Speedy has 17 coins. She has six more dimes than nickels and one fewer quarter than nickels. Find the number of each type of coin she has.

(a) How would you use algebra to write an equation to represent this problem?
(b) *As necessary*. What variables would you need to use to write the equation? What would each variable represent?
(c) *As necessary* (if students write equations N, D, and Q). Can you solve an equation that has three variables in it? Let's try writing an equation for D in terms of N and an equation for Q in terms of N. Can those be used to solve the problem?
(d) *As necessary* (if students are not able to make progress solving the problem algebraically). Give students coins and ask the students if they can figure it out.

[31] Algebra task A5 taken from CPM Algebra 1, UNIT 4, CP-16, 2002

The Visual Block Pattern Problem: Algebra Task A7

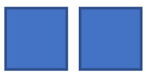
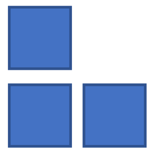
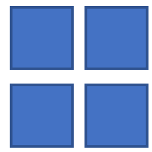
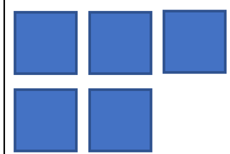
			

Figure 0	Figure 1	Figure 2	Figure 3

These blocks are arranged in a pattern.

(a) Can you draw figure 4?
(b) How would you find figure 5? How many blocks are in figure 5?
(c) How many blocks would be in Figure 2.100?
  a. *As Necessary*. Is there a relationship between the figure number and the number of blocks?
  b. *As Necessary*. Try making a chart that shows the figure number and the number of blocks.
  c. As necessary. In figure 3, for example, where is the 3 in the blocks? What happens to the 3 to get the total number of blocks. Does that work for figure 4? Can we use that to figure out figure F?
(d) Let's pretend your math teacher walks in the door, and hasn't seen this pattern before. Explain to her how to find the number of blocks in any figure of this pattern.
  a. If the student explains a recursive pattern, ask: What if the previous figure isn't drawn? How many blocks would you tell your teacher to draw on the top row? The bottom row? How many blocks would you tell your teacher there are all together?
(e) Can you write an algebraic expression to communicate the number of blocks in figure x?
  a. As necessary. How does the figure number help determine the number of blocks in each figure?
  b. As necessary. Remind me how you determined the number of blocks in Figure 2.100. Can you use that to help write an equation?

The Block Pattern Problem: Algebra Task A8

Figure Number	**Number of Blocks**
Figure 1	7
Figure 2	8
Figure 3	9
Figure 4	
Figure 5	
Figure 6	
Figure 7	
Figure 8	
Figure 9	
Figure 10	16
…	…
Figure 100	
…	
Figure *f*	

The numbers in this table represent a new pattern. On the left is the figure number and on the right is the number of blocks. Can you complete the table?

(a) How many blocks would be in Figure 2.100?
   a. *As necessary.* Is there a relationship between the figure number and the number of blocks? Does that relationship work for every figure number?

(b) Can you write an expression to communicate the number of blocks in figure *f*?
   a. *As necessary*. What pattern did you use to find the number of blocks in Figure 2.100?
   b. *As necessary*. Can you explain how to find the number of blocks in any figure?

The Football Problem: Algebra Task A9

Each time the school's football team scores a touchdown, they earn 7 points. Last week, they had their best game of the season, and this week they scored three fewer touchdowns than last week. What was their total score *last* week in terms of the number of touchdowns scored *this* week.

(a) Can you use Algebra to represent last week's score in terms of this week's number of touchdowns?
    a. *As Necessary.* Can you try out a few number examples to help you make sense of the problem?
    b. *As Necessary.* For example, if they scored 28 points last week, how many touchdowns did they score last week? How many touchdowns would they have scored this week?
    c. *As Necessary.* What variables do you need to define?

(b) Can you represent this week's number of touchdowns in terms of last week's score?

The Soccer Problem: Algebra Task A10

The school's soccer team had their best game of the season last week. This week, they scored 3 fewer points than last week. How many points did they score this week?

(a) Can you use algebra to represent this week's score in terms of last week's score?
    - *As Necessary*. What if they scored five points last week. How many points did they score this week?
    - *As Necessary.* What is the relationship between last week's score and this week's score?
    - *As* Necessary. What variables could you use to represent last week's score and this week's score?

(b) Can you write an equation to represent this week's score in terms of last week's score?

The Substitute Problem: Algebra Task A11

Your teacher is absent for the day, and the substitute does not know what to have you do during class, so she assigns what she thinks is busy work. She says, "Please find the sum of all the numbers from 1 to 100." One student finishes the problem in less than a minute, so the substitute tells her to find the sum of all the numbers from 1 to 1000. She finishes this problem just as quickly.

What might this student be doing? How might she find the sum of all the numbers from 1 to n?

(a) *As* Necessary. Ask the student to find the sum of the numbers from 1 to 6.
(b) *As* Necessary. Ask the student how they might find the sum of numbers from 1 to 100 using a similar idea.
(c) *As* Necessary. Ask the student to explain in words and pictures what their method is.
(d) Ask the student to write an expression to find the sum from 1 to n using that method.

Appendix C: Coding Dictionary

A Priori Codes	Definition	Reference[32]
Building Equation (multiplicative)	Students reason about a multiplicative relationship using numerical examples, and then substitute variables in place of numbers. In essence, creating a third-level of units (the unknown) in activity.	Hackenberg et al. (2017)
Variable as numerical example	Students write an equation, but explain the meaning of the equation using specific numerical examples.	MacGregor and Stacey (1997)
Variable as label	Students treat an unknown quantity as a label on an object.	MacGregor and Stacey (1997)
Variable as alphanumeric code	Students apply a number to an unknown based on the letters of the alphabet.	MacGregor and Stacey (1997)
Variable as general referent	Students use the same variable to represent unrelated or non-equivalent quantities.	MacGregor and Stacey (1997)
Variable as unrelated quantity	Students use unrelated unknowns to represent related quantities.	MacGregor and Stacey (1997)
Operational concept of the equal sign	Students refer to the equal sign as producing “the answer” or as an operator.	Matthews et al. (2012)
Disembedding	Evidence that the student is operating on one quantity in relation to another	Hackenberg (2013)
~Disembedding	Evidence that the student is unable to operate on one	Hackenberg (2013)

[32] The references listed are the citations that informed my development of these codes, not necessarily the research that originally defined the term. E.g., Stacy and Macgregor (1997) relied on the work of Kuchemann (1981), who originally created a the hierarchy of unknowns. Similarly, Hackenberg (2013) did not define the disembedding operation, but demonstrated how the disembedding operation is related to students’ algebraic reasoning.

	quantity in relation to another	
Units Coordination (2, 3, or 4 levels)	Students demonstrate the coordination of two, three, or four levels of units.	Olive & Çaglayan (2008)
Reversal – word order matching	Students reverse the pairing of an unknown with a known quantity due to the working of the problem.	Clement (1982)
Reversal – static comparison	Students reverse the pairing of an unknown with a known quantity due to a conception that the larger unknown should be paired with the known quantity.	Clement (1982)

A Posteriori Codes	Definition	Inspiring Reference
Operating on embedded composite units	Students who have not constructed the disembedding operation operate on composite units.	Ulrich (2016b): Describes Adam's strategic reasoning as operating on embedded composite units.
Build Equation (additive)	Students reason about an additive relationship using numerical examples, and then substitute variables in place of numbers. In essence, creating a second-level of units (the unknown) in activity.	Hackenberg et al. (2017): Describes MC2 students as building multiplicative equations.
Write two equations	Students represent an algebraic relationship using two separate equations, rather than one.	Hackenberg and Tillema (2009): Describe students solving multiplicative tasks as two separate problems as a limitation of units coordination.
Need to know	Students express the "need to know" the value of an expression containing an unknown before they can continue the problem.	Stacey and MacGregor (1997): Describe difficulty students have in seeing the answer and not just the operation.

	Demonstrates the inability to operate on the expression.	
Inverse operation	Students conceptualize a numerical relationship using the inverse operation (e.g., how many touchdowns in 35 point? And the student solves using multiplication), making it difficult for them to represent the relationship algebraically.	Stacey and MacGregor (1997): Describe difficulty students have transitioning to algebra because thinking of subtraction as adding up, for example, is no longer as helpful as it is in arithmetic.
Inequality	Students write an inequality rather than an equation to represent the relationship between two unknowns.	Hackenberg et al. (2017): Describe MC2 students statement that multiplicative relationships between two unknowns are "approximate."
~Splitting	Students struggle to represent the results of a splitting task algebraically, potentially because they do not simultaneously partition and iterate.	Ulrich (2016b): Describes the manner by which aTNS students solve splitting tasks as sequential partitioning and iterating.
Equi-Segmenting	Students attempt to partition a whole without anticipation of exhausting the whole.	Steffe (2010d)
Simultaneous Partitioning	Students attempt to partition a whole, anticipate the goal of exhausting the whole, but do not conceptualize the parts as identical.	Steffe (2010d)
Equi-Partitioning	Students partition a whole, anticipate the goal of exhausting the whole, and conceptualize the parts as identical.	Steffe (2010d)