STOCHASTIC FLOW SHOP SCHEDULING

by

S. Suresh

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APPROVED:

R. D. Foley, Chairman

H. D. Sherali

Y. D. Mittal

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ABSTRACT

In this thesis we present new results for the makespan and the flowtime in a flow shop without intermediate storage between machines. We consider m machines and n jobs with random processing times. Since there is no intermediate storage between machines, a job which has finished processing at one machine may have to stay at that machine until the next machine is free. This phenomenon is known as blocking. Our goal is to select the optimal schedule; in our case, the schedule which in some sense minimizes the makespan or the flowtime. Makespan is the total time required to process a set of jobs and flowtime is sum of all the times at which jobs are completed.

Our results require various stochastic orderings on the processing time distributions. Some of these orderings minimize the expected flowtime or expected makespan, and some stochastically minimize the makespan. The stochastic minimization results are much stronger. The optimum sequence in these cases not only minimize the expected makespan, but

also maximize the probability of completing a set of jobs by time t for any t.

Our last result resolves the conjecture of Pinedo (1982a) that in a stochastic flow shop with m machines, n-2 deterministic jobs with unit processing time, and two stochastic jobs each with mean one, the sequence which minimizes the expected makespan has one of the stochastic jobs first and the other last. We prove that Pinedo's conjecture is almost true. We prove that either the sequence suggested by Pinedo or a sequence in which the stochastic jobs are adjacent at one end of the sequence minimizes the expected makespan. Our result does not require the stochastic jobs to have an expected value of one. Furthermore, we show that our result cannot be improved in the sense that in some cases one sequence is strictly optimal and in other cases the other is strictly optimal.

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To My Sister

Malathy

Chapter I

INTRODUCTION

For many years Industrial Engineers have been confronted with the problem of scheduling jobs in production shops in an efficient manner. In some cases, particular schedules have been proven to be optimal. In other cases, heuristics have been used to locate good schedules. In this thesis we have proven particular schedules to be optimal under certain assumptions.

Although we use the industrial engineering terminology of scheduling 'jobs' on 'machines', the results can be applied directly to many other areas such as scheduling jobs in computer systems and the scheduling of operating rooms for surgery.

If the exact processing time of each job on each machine is known, the model is deterministic. A large amount of literature exists in the area of deterministic sequencing and scheduling. Unfortunately in reality, the processing times may be random and must be estimated. Hence, we are confronted with a stochastic scheduling problem. Technological manufacturing problems, in processes involving transport and storage, and the behavior of computer systems are good examples where a probabilistic model is necessary to analyze these systems. Few results exist in the area of stochastic scheduling and sequencing due to the difficulty of analyzing these systems.

1.1 SYSTEM

Different models have been used to analyze different production shop situations. We consider flow shops with infinite intermediate storage and flow shops with zero intermediate storage. These can be considered as the extreme cases of a flow shop with finite intermediate storage.

A flow shop with infinite intermediate storage consists of m machines in tandem and n jobs to be processed on each machine. The n jobs are processed in the same order on each machine. There is an infinite amount of storage space between the machines.

A flow shop with zero intermediate storage, as the name indicates, has no storage space between machines. Thus, if a job finishes processing on machine k but machine k+1 is still busy, the job must wait on machine k until the job on machine k+1 is finished. This phenomenon is known as blocking; hence, this model is also known as a flow shop with blocking.

1.2 CRITERIA

There are many scheduling criteria: makespan, flowtime, maximum lateness, etc.. Generally, we consider the makespan or the flowtime as the scheduling criterion of interest. The makespan is the total length of time needed to process all the jobs. Since we assume that the first job starts processing at time zero, the makespan is just the departure time of the last job from the last machine. The flowtime is the sum of all the times at which jobs are completed. The makespan is clearly the criterion of interest if we wish to finish the jobs as quickly as possible. The flowtime is of interest when one wishes to minimize the total in-process inventory cost. The total in-process inventory cost is directly related to the flowtime. For example, if the flowtime is t hours and the in-process inventory cost is $x/hour/unit then the total in-process inventory cost is $tx/unit. Thus, for a fixed number of jobs the total in-process inventory cost can be minimized by minimizing the flowtime.

1.3 THE NEED FOR ORDERINGS OF RANDOM VARIABLES

In deterministic models, it is simple to compare the processing times of two jobs. However, in stochastic models, a problem arises. The question is how can we compare the processing times of two jobs since they are random variables.

Similarly, how can we compare the makespan or flowtime of two different schedules since they are random variables. In fact, there are many different ways that this can be done. These different ways are referred to as stochastic orderings.

Definition

(1) A random variable X_i is said to be nonoverlapping and bigger than X_j if $Pr[X_j \geq X_i] = 1$. It is represented by $X_i \leq X_j$.

(2) Another possibility is that X_i is stochastically smaller than X_j if $Pr[X_i > t] \leq Pr[X_j > t]$ for every value of t. That is, the complimentary distribution function of X_j lies above that of X_i. This is represented by $X_i \leq_d X_j$ and will be referred to as d-ordering or as stochastic dominance. Here X_i is stochastically dominated by X_j.

(3) Yet another possibility is $E[X_i] \leq E[X_j]$ where E[.] represents the expected value. This ordering will be referred to as the expected value ordering. There are many other orderings like $\geq_c$, $\geq_{cc}$, $\geq_v$, all of which are described in Appendix A.

The makespan and the flowtime are also random variables depending on the schedule. A sequence stochastically minimizes the makespan (flowtime) if the makespan (flowtime) of any other sequence is larger with respect to d-ordering.

Note that stochastic dominance implies the dominance of moments of any order, i.e., $X_i \leq_d X_j$ implies $E[X_i^k] \leq E[X_j^k]$ for all k when the moments exist. This is a strong stochastic ordering, the importance of which is explained in Appendix A.

1.4 NOTATION

We use $S_{i,j}$ to represent the processing time of job i on machine j. The processing times are said to be d-ordered if $S_{1,j} \geq_d S_{2,j} \geq_d \ldots\ldots \geq_d S_{n,j}$ for all j and the jobs are sequenced 1,2,....,n. The processing times are said to be nonoverlapping and longest expected processing time first if $S_{1,j} \geq S_{2,j} \geq \ldots\ldots \geq S_{n,j}$ for all j and the jobs are sequenced 1,2,....,n. Similiar notation is used for other orderings.

A makespan (flowtime) is said to be stochastically minimized if the makespan (flowtime) of this sequence is stochastically smaller than the makespan (flowtime) of any other sequence.

In general a sequence is represented by the Arabic numerals, i.e., 1,2,....n. We let $D^a_{i,j}$ represent the departure of job i from machine j for a sequence (a).

Definition (1) A sequence 1,2,....,n is a SEPT sequence (shortest expected processing time first) if

$E[S_{1,j}] \leq E[S_{2,j}] \leq \ldots\ldots \leq E[S_{n,j}]$ for all j.

(2) A sequence 1,2,.....,n is a LEPT sequence (longest expected processing time first) if

$E[S_{1,j}] \geq E[S_{2,j}] \geq \ldots\ldots \geq E[S_{n,j}]$ for all j.

(3) A sequence 1,2,...,n is a SEPT-LEPT sequence if there is a k such that

$$E[S_{1,j}] \leq E[_{2,j}] \leq \ldots\ldots\ldots \leq E[S_{k,j}]$$

$$E[S_{k,j}] \geq E[S_{k+1,j}] \geq \ldots\ldots \geq E[S_{n,j}]$$

for all j.

Note that 1,2,...,k is a SEPT sequence and k,k+1,....,n is a LEPT sequence. A SEPT or a LEPT sequence is also a SEPT-LEPT sequence.

1.5 RESULTS

We have worked with two different assumptions.

Assumption IM. Here we assume that all the machines are identical (IM). The processing time of job i on machine j depends on job i and not on machine j. Thus if $S_{i,j}$ represents the processing time of job i on machine j then $Pr[S_{i,j} \leq t] = F_i(t)$ for all machines. That is $S_{i,j}$ for j=1,2,...,m are independent and indentically distributed.

Assumption IJ. Here we assume that all the jobs are identical (IJ). Specifically, here $Pr[S_{i,j} \leq t] = F_j(t)$ for all jobs where $S_{i,j}$ represents the processing time of job i on machine j. Here $S_{i,j}$ for i=1,2,...,n are independent and indentically distributed.

Flowtime results. We have three flowtime results. One of them is with Assumption IJ and the other two are with Assumption IM.

The first result with Assumption IJ states that in a two machine flow shop the expected flowtime is also independent of machine ordering. This shows that for all stochastic orderings the expected flowtime is independent of the machine and the job ordering.

The second result with Assumption IM shows that in an m machine flow shop with nonoverlapping processing times, the flowtime is stochastically minimized by a SEPT sequence. Here the expected flowtime is dependent on the job ordering.

The third result with Assumption IM shows that in a two machine flow shop with the processing times d-ordered the expected flowtime is minimized by a SEPT sequence. Notice that by weakening the hypothesis in the third result compared to the second result the result on the flowtime weakens from stochastic minimization on m machines to expected value minimization on two machines.

Makespan results. We have three makespan results; all of them with Assumption IM. The first result states that on m machines if the processing times are nonoverlapping then the makespan is stochastically minimized if and only if the sequence is SEPT-LEPT. For the system described here the SEPT sequence stochastically minimizes both the flowtime and makespan. This is not true in general.

The second result shows that on m machines if we have two jobs with nonoverlapping and faster processing times than the other n-2 jobs which have identically distributed processing times, the makespan is stochastically minimized if one of the fast jobs is placed first and the other last. The fast jobs need not have identical processing time distribution. This sequence is SEPT-LEPT. Again in this case notice that instead of n nonoverlapping jobs as in the previous result, the hypothesis is slightly relaxed to n-2 jobs being nonoverlapping with the other two. Then we have a particular SEPT-LEPT sequence stochastically minimizing the makespan and not any SEPT-LEPT sequence. This leads us to believe that for stochastically minimizing the makespan the slow jobs should be together.

Our third result shows that with m machines, n-2 deterministic jobs and two stochastic jobs, the optimal sequence to minimize the expected makespan is either to schedule one of the stochastic jobs first and the other last or to schedule the stochastic jobs together at one end of the sequence. We also show that this result cannot be improved in the sense that in some cases one sequence is strictly optimal and in other cases the other sequence is strictly optimal. Pinedo (1982a) had conjectured that for the system described in this result and with the stochastic jobs having a mean of one, the optimal sequence to minimize the expected makespan is to schedule one of the stochastic jobs first and the other last. Our examples, which prove that the results cannot be improved any further, all use stochastic jobs with mean one; thus disproving Pinedo's conjecture.

1.6 ORGANIZATION

This thesis is divided into five chapters and two appendices. In Chapter 2 we review the literature that has appeared so far not only about stochastic models but also about deterministic models. We also point out why a separate study with our assumptions is needed in this area.

We recognize that Chapter 3 titled Negative Results is quite unusual. Here we list intuitively reasonable conjectures with their counterexamples. These conjectures are based on previous results which have appeared in the literature. We feel that it will save time and give insight to future researchers.

In Chapter 4 we list our main results with the proofs. The results give optimal schedules for minimizing the expected makespan and expected flowtime. In Chapter 5 titled Summary we draw conclusions from our results. We also give some discussions for further research.

In Appendix A we give a detailed discussion of results on stochastic orderings which are used in this thesis. We also discuss the importance of the orderings and their relation to each other. The reader is encouraged to read Appendix A for a thorough understanding of the results and implications in this thesis. Appendix B contains some lemmas used in proving the main results.

Chapter II

LITERATURE REVIEW

We use sequence and schedule interchangably, since in our cases the sequence specifies the schedule completely; that is the order of processing remains the same at every machine.

2.1 DETERMINISTIC FLOW SHOP WITH BLOCKING

Reddi and Rammorthy (1972) and Wismer (1972) considered deterministic flow shops with zero intermediate storage. They prove that this problem is 'N. P. Complete'. Thus when the problem size is quite big it is nearly impossible to obtain the optimal solution. For further details on 'N.P. Complete' see Garey and Johnson (1979).

Since, even in a deterministic case, the problem is so complex, it will be impossible to deal with the stochastic case without suitable assumptions. Hence we have introduced Assumptions IM and IJ. Furthermore, we need some way of comparing the processing times since they are random variables. The different ways of comparing the processing times are referred to as stochastic orderings and have been explained in detail in the Appendix A. The orderings used here have been used by Barlow and Proshan (1965), Pinedo (1982 a & b) and

Stoyan (1977) in queueing and other areas. See Bawa (1982) for other applications on stochastic orderings.

2.2 STOCHASTIC FLOW SHOPS

Literature on stochastic flow shops generally use either Assumption IM or IJ as mentioned in section 1.4.

Pinedo and Schrage (1981) provide a nice survey on stochastic shop scheduling. With Assumption IM we are concerned with the sequencing of jobs. With Assumption IJ we are concerned with the sequencing of the machines.

2.2.1 Duality

Pinedo (1982a), in the case of flow shop without blocking, established a flow shop which he called dual flow shop. As in section 1.5 let $S_{i,j}$ denote the processing time of job i on machine j where i = 1,....m and j = 1,....n for a flow shop called 'primal flow shop'. Now consider a second flow shop where there are m machines and n jobs. Denote the processing time on machine i of job j be $S_{i,j}$ to be called 'dual flow shop'. Pinedo shows that with Assumption IJ in the primal flow shop and with Assumption IM in the dual flow shop, $D^p_{i,j} =_d D^d_{j,i}$ where $D^p_{i,j}$ is the departure time of job i from machine j in the primal flow shop and $D^d_{j,i}$ is the departure time of job j from machine i in the dual flow shop.

2.2.2 Stochastic Flow Shops without Blocking

Tembe and Wolff (1974) show that with Assumption IJ and two machines, one deterministic and the other stochastic, the departure epoch of every job is stochastically minimized by placing the deterministic machine first. Pinedo (1982a) shows that with Assumption IM and two jobs, one being deterministic and the other stochastic, the makespan is stochastically minimized by sequencing the deterministic job first. Tembe's and Wolff's result shows that if we had n jobs the departure epoch of the n^{th} job is stochastically minimized thus stochastically minimizing the makespan. Tembe and Wolff had two machines, n jobs, and used Assumption IJ. Pinedo had two jobs, n machines, and used Assumption IM. So, the system used by Pinedo was the dual of the system used by Tembe and Wolff. The final results consequently follow from Pinedo's (1982a) proof on duality.

Tembe and Wolff (1974) also show that with Assumption IJ and nonoverlapping processing time on m machines the LEPT sequence stochastically minimizes the departure epoch of every job. Pinedo (1982b) extended this result showing that any SEPT-LEPT sequence stochastically minimizes the departure epoch of every job. Pinedo (1982a) shows that with Assumption IM and nonoverlapping processing time of m jobs any SEPT-LEPT sequence minimizes the expected makespan. Again note that the system of Pinedo (1982b) is in the same sense dual of the system of Pinedo (1982a). Though Pinedo proves that any SEPT-LEPT sequence minimizes the expected makespan, the result can be extended and can be shown that any SEPT-LEPT stochastically minimizes the makespan based on his own results on duality.

Pinedo (1982b) also shows that with Assumption IJ, m-2 deterministic machines, and two stochastic machines, the departure epoch of every job is stochastically minimized if one stochastic machine is placed first and the other last. Pinedo (1982a) shows that with Assumption IM, n-2 deterministic jobs, and the other two stochastic jobs, the makespan is stochastically minimized if one stochastic job is placed first and the other last. Not surprisingly, the system of Pinedo (1982a) and Pinedo (1982b) are again duals of one another. The results can be thought of as duals of one another.

Muth (1979) shows that with Assumption IJ the makespan for any sequence of machines is stochastically the same as the makespan if the sequence of machines is reversed. This is an extremely important result, since it holds irrespective of the storage capacity. As a consequence of duality Pinedo (1982a) gave an important result based on Muth's result. In a flow shop without blocking with Assumption IM the makespan for any sequence of jobs is stochastically the same as the makespan of the reversed sequence of jobs.

2.2.3 Stochastic Flow Shop with Blocking

Dattatreya (1978) shows that with Assumption IJ and two machines, one deterministic and the other stochastic, the departure epoch of job is stochastically minimized by placing the deterministic job first. He also shows that with the same assumption and nonoverlapping processing time on m machines the LEPT sequence stochastically minimizes the departure epoch of every job. The reader may recall that these were the same results Tembe and Wolff (1974) obtained in a flow shop without blocking.

As we mentioned in section 2.2.2 Muth's result on makespan is true even in the case of stochastic flow shop with blocking.

Pinedo (1982b) working with Assumption IJ was interested in the output rate in steady state, i.e., the reciprocal of the mean interdeparture time. This is called the capacity, i.e., the long run output rate with infinite number of jobs awaiting processing at the first machine. Though capacity and departure epoch are not the same quantities it is clear that if a sequence stochastically minimizes the departure epoch of every job then the capacity is maximized. However, if the capacity is maximized by a sequence of machines the departure epoch of every job is not necessarily minimized. Pinedo shows that if m-1 stations are identical and have deterministic service times and one station has an arbitrary stochastic service time distribution, the capacity does not depend on the sequence. He also shows that when the processing time are nonoverlapping on the machines the capacity does not depend on the sequence. The capacity in this system is the reciprocal of the expected service time of the slowest machine.

The capacity of a system with m-2 deterministic machines with processing time of one unit and two stochastic machines not necessary identical, both with mean one and symmetrical density functions, is maximized when either one of the stochastic machines is set up at the beginning and the other at the end of the sequence. Finally, he shows that the capacity of the system with m-2 identical machines with identical processing time distribution and faster than the other two nonidentical machines is maximized when one of the slow stations is set up first and the other one last.

Notice in the first two results the capacity did not depend on the sequence of machines. In the last two results the capacity did depend on the sequence of machines.

Pinedo (1982a) worked with the Assumption IM and was interested in minimizing the expected makespan. He shows that if the processing time of the jobs are nonoverlapping then the expected makespan is minimized if and only if the sequence is SEPT-LEPT. He also shows that when the processing time of the jobs are d-ordered for a sequence 1,2,....,n with the slowest job first, the expected makespan on two machines is minimized by the sequence n,n-2,....4,2,1,3,.......n-1 (or by Muth's result its reverse). Note that this sequence is a SEPT-LEPT sequence but not any SEPT-LEPT sequence. Comparing the two results the hypothesis was weakened from nonoverlapping to d-ordered and the final result was weakened from any SEPT-LEPT sequence on m machines to a particular SEPT-LEPT sequence on two machines.

Pinedo (1982a) wanted to investigate the influence of variance on the processing times. If the jobs are v-ordered for a sequence 1,2,...,n with the least variable job first, the expected makespan on two machines is minimized by the same sequence, namely, n,n-2,....4,2,1,3,...n-1 (or its reverse by Muth's result).

2.3 LACK OF DUALITY

Pinedo (1982a) pointed out that unfortunately the same kind of duality relationship described in section 2.2.1 for a stochastic flow shop without blocking does not exist in a flow shop with blocking. This can be seen by comparing the three flowtime results as well as the results in our Summary. It is also easy to establish counterexamples to prove that this same dual relationship does not exist.

In systems without blocking, due to the duality proved by Pinedo (1982a), results with Assumption IJ are dual to results with Assumption IM in dual systems. Unfortunately this cannot be done in stochastic flow shops with blocking. Thus, a separate study is necessary with Assumption IM.

Until now, only Pinedo's (1982a) work with Assumption IJ has appeared in the literature. Except for Baker's (1974, p.107) work on flowtime on single machine no other papers have appeared in the literature on flowtime in the whole area of stochastic flow shop with blocking. In this thesis we have studied this relatively new area of stochastic flow shop with blocking with Assumption IM and obtained some new results.

Chapter III

NEGATIVE RESULTS

As mentioned earlier it is unusual to have a chapter entitled 'Negative Results'. Here we present reasonable conjectures and their counterexamples. They show that the sufficient assumptions on the past results are very close to necessary. We advise the reader who is not familiar with convex and concave stochastic orderings to see Appendix A befor reading this chapter.

3.1 CONJECTURE 1

If $S_{1,j} \geq_c S_{2,j} \geq_c \ldots \geq_c S_{n,j}$ for all j on two machines then the expected makespan on two machines is minimized by the sequence n,n-2,...,4,2,1,3,5,...,n-1 (or its reverse).

Motivation for this Conjecture. Pinedo (1982a) proved that if $S_{1,j} \geq_d S_{2,j} \geq_d \ldots \geq_d S_{n,j}$ for all j then the expected makespan on two machines is minimized by a sequence n,n-2,...4,2,1,3,5,...n-1 (or its reverse). He also proved that the same sequence minimizes the expected makespan if $S_{1,j} \geq_v S_{2,j} \geq_v \geq \ldots \geq_v S_{n,j}$ for all j. If $S_{1,j} \geq_c S_{2,j} \geq_c \ldots \geq_c S_{n,j}$ for all j, have equal means, and symmetric density functions then it implies $S_{1,j} \geq_v S_{2,j} \geq_v \geq \ldots \geq_v S_{n,j}$. Based on these two results we slightly weakened the hypothesis to conjecture that the result is also true for a c-ordering.

The result is trivially true for two jobs because by Muth's result the sequence 1,2 and 2,1 have the same expected makespan. The counterexample presents a case with three jobs.

Counterexample: We have three jobs and two machines. $S_{1,j} \geq_c S_{2,j} \geq_c S_{3,j}$ for all j. So according to our conjecture the sequence 2,1,3 should be optimal.

Let $\Pr[S_{3,j}=1/2]$ be equal to one.

i	1/8	2
$\Pr[S_{1,j}=i]$	1/4	3/4

i	1/4	1
$\Pr[S_{2,j}=i]$	1/4	3/4

Expected makespan for 2,1,3 = 297/64 .

Expected makespan for 1,2,3 = 295/64 .

Clearly, in this case 2,1,3 is not the optimal sequence. However, we know that if $S_{1,j} \geq_d S_{2,j} \geq_d S_{3,j}$ then $S_{1,j} \geq_c S_{2,j} \geq_c S_{3,j}$. It is already proved by Pinedo (1982a) that if $S_{1,j} \geq_d S_{2,j} \geq_d S_{3,j}$ then 2,1,3 is optimal.

Conclusion. If the jobs are c-ordered it is not necessarily true that a specific sequence is optimal without additional information.

3.2 CONJECTURE 2

If $S_{1,j} \geq_{cc} S_{2,j} \geq_{cc} \ldots \geq_{cc} S_{n,j}$ for all j then the expected makespan on two machines is minimized by a sequence n,n-2,...4,2,1,3,5,...,n-1 (or its reverse).

Motivation for this Conjecture. The basis for this conjecture is similar to conjecture 1. Here if $S_{1,j} \geq_{d} S_{2,j} \geq_{d} \ldots \geq_{d} S_{n,j}$ for all j, then $S_{1,j} \geq_{cc} S_{2,j} \geq_{cc} \ldots \geq_{cc} S_{n,j}$. Again note that in this case the hypothesis is slightly weakened.

As explained before the result is trivially true for two jobs by Muth's result. Interestingly enough, we could prove that the conjecture is true for three jobs. The counterexample presents a case with four jobs.

Counterexample. We have four jobs and two machines. $S_{1,j} \geq_{cc} S_{2,j} \geq_{cc} S_{3,j} \geq_{cc} S_{4,j}$ for all j. So according to our conjecture the sequence 4,2,1,3 should be optimal.

i	7/4
$Pr[S_{1,j}=i]$	1

i	3/4	7/4
$Pr[S_{3,j}=i]$	1/2	1/2

i	5/4
$Pr[S_{2,j}=i]$	1

i	1/4	1
$Pr[S_{4,j}=i]$	1/4	3/4

Expected makespan for 4,2,1,3 = 113/16 .

Expected makespan for 4,1,2,3 = 109/16 .

Clearly, in this case 4,2,1,3 is not the optimal sequence.

We know that if $S_{1,j} \geq_d S_{2,j} \geq_d S_{3,j} \geq_d S_{4,j}$ then $S_{1,j} \geq_{cc} S_{2,j} \geq_{cc} S_{3,j} \geq_{cc} S_{4,j}$ and in that case Pinedo (1982a) has proved that 4,2,1,3 is optimal.

Conclusion. If the jobs are cc-ordered it is not possible to prove that a specific sequence is optimal without additional information.

3.3 DEFINITION

A j-k 'pairwise interchange' of a sequence is a sequence obtained by reversing the order of jobs from j through k. For example, a 3-6 pairwise interchange of the sequence 1,2,3,4,5,6,7,8,9 yields the sequence 1,2,6,5,4,3,7,8,9.

3.3.1 Conjecture 3

For any stochastic ordering of jobs on two machines, if there exists a sequence from which each possible pairwise interchange yields a sequence higher in expected makespan, then this sequence minimizes the expected makespan.

Motivation for this Conjecture: In many of Pinedo's proofs, he takes a general sequence and keeps improving it by this pairwise interchange. Once the optimal sequence is reached the sequence cannot be improved any further. We are trying to obtain a simpler criterion for the optimality of a particular schedule. Specifically, we are trying to establish a criterion that if a sequence is locally optimal, in the sense that if it cannot be improved by any pairwise interchange, then it is globally optimal, i.e., it is the optimal sequence.

Counterexample. We have two machines and four jobs. $E[S_{1,j}] \geq E[S_{2,j} \geq E[S_{3,j}] \geq E[S_{4,j}]$ for all j.

The cumulative distribution fuctions are:

t	t<1	t<4	t=4
$Pr[S_{1,j} \leq t]$	1/2t	1/2	1

t	t<1	t=1
$Pr[S_{3,j} \leq t]$	0	1

t	t<1	t<4	t=4
$Pr[S_{2,j} \leq t]$	3/4t	1/2	1

t	t<1	t=1
$Pr[S_{4,j} \leq t]$	1/4t	1

Let us consider a sequence 2,1,3,4. The following are the possible pairwise interchanges: 1,2,3,4; 3,1,2,4; 4,3,1,2; 2,3,1,4; 2,4,3,1; 2,1,4,3. By Muth's result the expected makespan of 2,1,3,4 and 4,3,1,2 are equal.

With a little algebra, we have

expected makespan of 2,1,3,4 = 296/32

expected makespan of 1,2,3,4 = 300/32

expected makespan of 3,1,2,4 = 297/32

expected makespan of 4,3,1,2 = 296/32

expected makespan of 2,3,1,4 = 308/32

expected makespan of 2,4,3,1 = 297/32

expected makespan of 2,1,4,3 = 296/32.

So according to our conjecture the sequence 2,1,3,4 being locally optimal should be an optimal sequence.

Expected makespan of 1,2,4,3 = 293/32 .

Clearly the sequence 2,1,3,4 is not optimal. Yet every sequence which can be possibly obtained by a j-k pairwise interchange of 2,1,3,4 has a higher expected makespan. Note that it is possible to obtain 1,2,4,3 from 2,1,3,4 in a series of pairwise interchanges with the expected makespan nonincreasing. It is not known whether it is always possible to obtain the optimal sequence from any sequence with the expected makespan nonincreasing.

Conclusion. The local optimal implying global optimal in the same sense described in this section does not exist in general.

Chapter IV

POSITIVE RESULTS

In this chapter we present our main results and their proofs. We first present three results on flowtime and then three results on makespan. The notation is consistent with that used in Chapter 1.

4.1 FLOWTIME RESULTS

First, we present an expression for the flowtime of n jobs on two machines. The flowtime of n jobs can be written as the sum of makespans of the first; first and second; first, second and third; and so on. For example, in the case of five jobs ordered as 1,2,3,4,5:

Flowtime = Makespan of 1 + Makespan of 1,2
+ Makespan of 1,2,3
+ Makespan of 1,2,3,4 + Makespan of 1,2,3,4,5.

This is true for any number of machines. The expected makespan of n jobs sequenced 1,2,...,n and machines sequenced 1,2 is as follows,

$$\text{Expected makespan} = E[S_{1,1}] + E[\max(S_{1,2}, S_{2,1})] + \ldots + E[\max(S_{n-1,2}, S_{n,1})] + E[S_{n,2}].$$

So the expression for expected flowtime is as follows,

$$E[FT] = E[S_{1,1}] + E[S_{1,2}]$$

$$+ E[S_{1,1}] + E[\max(S_{1,2},S_{2,1})] + E[S_{2,2}]$$

.

.

.

$$+ E[S_{1,1}] +\ldots+ E[\max(S_{n-1,2},S_{n,1}] + E[S_{n,2}].$$

Combining these terms yields

$$\begin{aligned}E[FT] = {} & nE[S_{1,1}] \\ & + E[S_{1,2}] + E[S_{2,2}] + \ldots + E[S_{n,2}] \\ & + (n-1)E[\max(S_{1,2},S_{2,1})] + (n-2)E[\max(S_{2,2},S_{3,1})] \\ & +\ldots \\ & + 2E[\max(S_{n-2,2},S_{n-1,1})] + E[\max(S_{n-1,2},S_{n,1}). \quad (1)\end{aligned}$$

<u>Theorem 1</u>. With Assumption IJ on two machines the expected flowtime is stochastically independent of job ordering and of the machine ordering.

<u>Proof</u>. With Assumption IJ, it is clear that $S_{1,i},S_{2,i},\ldots,S_{n,i}$ have the same distribution for any i. Thus it is clear that the expected flowtime is independent of the job ordering. Furthermore, the expected flowtime for a job ordering 1,2,....n and machine ordering 1,2 we have from equation 1,

$$\begin{aligned}E[FT] &= nE[S_{1,1}] + nE[S_{1,2}] + \sum_{i=1}^{n-1} iE[\max(S_{1,2},S_{1,1})] \\ &= nE[S_{1,1}] + nE[S_{1,2}] + n(n-1)/2\ E[\max(S_{1,2},S_{1,1})].(2)\end{aligned}$$

Reversing the machines, we have the expected flowtime as

$$E[FT] = nE[S_{1,2}] + nE[S_{1,1}] + n(n-1)/2\ E[\max(S_{1,1},S_{1,2})].(3)$$

It is clear that equations (2) and (3) are equal, thus proving that the expected flowtime is independent of machine ordering. □

Theorem 2. With Assumption IM, if the n jobs have nonoverlapping processing times with $S_{1,j} \geq S_{2,j} \geq \ldots\ldots \geq S_{n,j}$ for all j then the SEPT sequence independent of the order of the machines stochastically minimizes the flowtime.

Proof. Since $S_{i,1}, S_{i,2}, \ldots . S_{i,n}$ are identically distributed for all i it is clear that the flowtime is stochastically independent of the order of the machines. Hence we assume that the machines are ordered 1,2,....,m.

Foley and Suresh (1984) have proved that any SEPT-LEPT sequence stochastically minimizes the makespan for the system described in this theorem. As mentioned in section 4.1 the flowtime is the sum of the makespans. By applying Lemma 5 from the Appendix B repeatedly, in order to stochastically minimize the flowtime it is sufficient that we stochastically minimize each term in the sum of the makespans. To minimize the first term the n^{th} job should be placed first. Similiarly, the second term can be minimized by placing the n^{th} and $n\text{-}1^{st}$ jobs as first and second jobs respectively. The third term can be minimized by the n^{th}, $n\text{-}1^{st}$, $n\text{-}2^{nd}$ jobs. Proceeding in this manner the SEPT sequence is obtained. □

Theorem 3. With Assumption IM, in a two machine flow shop, if the n jobs are d-ordered as $S_{1,j} \geq_d S_{2,j} \geq_d \cdots \geq_d S_{n,j}$ for j = 1,2 then on two machines the SEPT sequence minimizes the expected flowtime.

Proof. Since $S_{i,1}$ and $S_{i,2}$ are identically distributed for all i, it is clear that the expected flowtime is the same for both orderings of the machines. Hence we assume the machines are ordered 1,2.

$$E[FT] = nE[S_{1,1}] \tag{4}$$

$$+ E[S_{1,2}] + E[S_{2,2}] + \ldots + E[S_{n,2}] \tag{5}$$

$$+ (n-1)E[\max(S_{1,2},S_{2,1})] + (n-2)E[\max(S_{2,2},S_{3,1})] + \ldots + 2E[\max(S_{n-2,2},S_{n-1,1})] + E[\max(S_{n-1,2},S_{n,2})].$$

It is also clear that for any ordering of the jobs the expression marked (5) will be the same. Now, consider a sequence 1,2,.....,n. We will show that the sequence n,1,2,....,n-1 has a smaller expected flowtime. The difference in the expected flowtime between 1,2,...,n and n,1,2,...,n-1 is

$$E[S_{1,1}] - E[S_{n,1}] + \sum_{i=1}^{n-1} [E[S_{1,1}] - E[S_{n,1}] - E[\max(S_{n,2},S_{1,1})] + E[\max(S_{i,2},S_{i+1,1})]].$$

Using the lemmas in Appendix B, it is shown that each term in the series is positive, and, therefore the whole quantity is positive. Thus, n,1,2,...,n-1 has lower expected flowtime than 1,2,....,n.

Next, we show that n,n-1,1,2,...,n-2 is better than n,1,2,...,n-1. Now the difference between n,1,2,...,n-1 and the n,n-1,1,2,...,n-2 is

$$E[\max(S_{n,2},S_{1,1})] - E[\max(S_{n,2},S_{n-1,1})]$$
$$+ \sum_{i=1}^{n-2} [E[\max(S_{n,2},S_{1,1})] + E[\max(S_{i,2},S_{i+1,1})]$$
$$- E[\max(S_{n,2},S_{n-1,1})] - E[\max(S_{n-1,2},S_{1,1})]].$$

Again, using the lemmas in Appendix B, we can show that each term in the series is positive, and, therefore, the whole quantity is positive. Thus n,n-1,1,2,...,n-2 is better than n,1,2,...,n-1.

Consider an arbitrary sequence $n,n-1,...,n-k,j_1,...,j_\ell,n-k-1,j_{\ell+1},...,j_{n-k-2}$. We will show that $n,n-1,...,n-k,n-k-1,j_1,...,j_\ell,j_{\ell+1},...,j_{n-k-2}$ is better than the earlier sequence. The difference in the expected flowtime between the two sequences is

$$(n-k-1)\ E[\max(S_{n-k,2},S_{j_1,1})] - (n-k-1)\ E[\max(S_{n-k,2},S_{n-k-1,1})]$$
$$+ E[\max(S_{j_1,2},S_{j_2,1})] + E[\max(S_{j_2,2},S_{j_3,1})] +...$$
$$+ (n-k-\ell-1)\ E[\max(S_{j_\ell,2},S_{n-k-1,1})]$$
$$+ (n-k-\ell-2)\ E[\max(S_{j_{\ell+1},2},S_{n-k-1,1})]$$
$$- (n-k-2)\ E[\max(S_{n-k-2,2},S_{j_1,1})]$$
$$- (n-k-\ell-2)\ E[\max(S_{j_\ell,2},S_{j_{\ell+1},1})].$$

Using the lemmas in Appendix B we can prove that the above quantity is positive. Thus proving that $n, n-1, \ldots, n-k, n-k-1, j_1, \ldots, j_\ell, j_{\ell+1}, \ldots, j_{n-k-2}$ is better than the original sequence. Proceeding in a similiar manner, we can prove that any arbitrary sequence can be improved through a series of changes until the SEPT sequence is obtained. □

4.2 MAKESPAN RESULTS

Theorem 4. With Assumption IM, if there are n jobs with nonoverlapping processing time distributions, the makespan is stochastically minimized if and only if the sequence is SEPT-LEPT.

Proof. Let the k^{th} job be the slowest job. For any SEPT-LEPT sequence the makespan on m machines is

$$S_{1,1} + S_{2,1} + \ldots + S_{k-1,1} \tag{7}$$

$$+ S_{k,1} + S_{k,2} + S_{k,3} + \ldots + S_{k,n} \tag{8}$$

$$+ S_{k+1,m} + S_{k+2,m} + \ldots + S_{n,m}. \tag{9}$$

For any sequence, not necessarily SEPT-LEPT, it can be verified that the portion of the equation marked (8) will be the same. For sequences other than SEPT-LEPT at least one term from either portion marked (7) or (9) will be replaced by a term which has a higher processing time than was originally present. For example, by interchanging jobs 3 and 2 the

SEPT-LEPT configuration will be disturbed. The makespan for this new sequence is

$$S_{1,1} + S_{3,1} + S_{3,2} + \dots + S_{k-1,1}$$
$$+ S_{k,1} + S_{k,2} + S_{k,3} + \dots + S_{k,n}$$
$$+ S_{k+1,m} + S_{k+2,m} + \dots + S_{n,m}.$$

Since $S_{3,2} \geq S_{2,1}$ it is clear that the SEPT-LEPT sequence is stochastically better than the second sequence.

Using the same argument it can be proved that the SEPT-LEPT sequence stochastically minimizes the makespan. □

In Theorem 5 we use $j_1, j_2, \dots, j_{n-2}$ to identify identical stochastic jobs whose processing times are identically distributed on all the machines. That is, these jobs have Assumption IJ and Assumption IM at the same time. Then we use X and Y to denote the other two jobs which are not necessarily identical. X_i (Y_i) then denotes the processing time of job X (Y) on machine i. $\{X_1, X_2, \dots, X_m\}$ is independent and identically distributed. The same is true for $\{Y_1, Y_2, \dots, Y_m\}$. $\{X_1, X_2, \dots, X_m\}$ and $\{Y_1, Y_2, \dots, Y_m\}$ are independent but X_i and Y_j need not be indentically distributed for any i and j.

<u>Theorem 5</u>. With Assumption IM, for n-2 jobs having identical processing time distributions which are nonoverlapping and slower than the other two nonidentical jobs on m machines, the makespan is stochastically minimized by placing one of the fast jobs first and the other one last.

Proof. Let us assume the machines are ordered 1,2,...m. The nonidentical jobs are represented by X and Y. We will show that any sequence which does not have X and Y in the first and last positions can be improved through a series of changes which results in having X and Y jobs first and last. Assume X to be in the k^{th} position, $k > 1$, and Y in the ℓ^{th} position with $k < \ell$. Let the identically distributed jobs be represented by $j_1, j_2, \ldots j_{n-2}$. Let us represent the job sequence $j_1, j_2, \ldots, j_{k-1}, X, j_k, \ldots, j_{\ell-1}, Y, j_\ell, \ldots, j_{n-2}$ by 'a'.

Moving job X to the front yields the sequence $X, j_1, j_2, \ldots, j_{k-1}, j_k, \ldots, j_{\ell-1}, Y, j_\ell, \ldots, j_{n-2}$ which will be represented by 'b'.

Now, we wish to show that

$$X_1 + D^a_{k-1,j} = D^b_{k,j} \quad \text{for} \quad 1 \leq j \geq m. \qquad (10)$$

To see this, note that after the first job in 'b', the sequence is exactly the same as 'a' until the k^{th} job is reached. Since the jobs are nonoverlapping, after time X_1, 'b' behaves exactly as 'a' until the k^{th} job is reached. Thus, the second job in 'b' departs at the time the first job in 'a' departs plus X_1. Referring to the k^{th} job in the first case

$$D^a_{k,1} = \max(D^a_{k-1,1} + X_1, D^a_{k-1,2}) = D^a_{k-1,2}$$

$$D^a_{k,1} = \max(D^a_{k,1} + X_2, D^a_{k-1,3}) = D^a_{k-1,3}$$

$$D^a_{k,j} = D^a_{k-1,j+1} \tag{11}$$

From (11) we have

$$X_1 + D^a_{k-1,j+1} = D^b_{k,j+1}$$

$$X_1 + D^a_{k,j} = D^b_{k,j+1} \tag{12}$$

We also know

$$D^b_{k,j+1} - D^b_{k,j} \geq S_{k-1,j+1}.$$

Substituting for D^b_{kj+1}, we have

$$X_1 + D^a_{k,j} - D^b_{k,j} \geq S_{k-1,j+1}$$

$$D^a_{k,j} - D^b_{k,j} \geq S_{k-1,j+1} - X_1 \geq 0$$

$$D^a_{k,j} - D^b_{k,j} \geq 0 \quad \text{for } 1 \leq j \leq m-1.$$

For j = m

$$D^a_{k,m} = D^a_{k-1,m} + X_m. \tag{13}$$

From (10)

$$D^b_{k,m} = D^a_{k-1,m} + X_1. \tag{14}$$

From (13) and (14), we see that $D^a_{k,m}$ and $D^b_{k,m}$ are stochastically equal. Since $D^a_{k,j}$ is stochastically greater than $D^b_{k,j}$ and since the last n-k jobs are in the same order in both sequences, $D^a_{i,j}$ is greater than $D^b_{i,j}$ for $k \leq i \leq n$ and for all j. Thus schedule 'a' is worse than 'b', and we have proved that X should be placed first.

Now we show that Y should be at the other end. By Muth's (1979) result the reverse of this sequence has stochastically the same makespan as the original sequence. In the reverse sequence if Y appears in the middle, by the same arguments Y should be placed first. In this sequence X is at the other end. □

In Theorem 6 we have n-2 deterministic jobs with unit processing times on all machines and two stochastic jobs X and Y. Here a sequence is represented by $I_k X I_\ell Y I_h$ denoting k adjacent deterministic jobs, then job X, then ℓ adjacent deterministic jobs, then job Y, then finally h adjacent deterministic jobs. k, ℓ, h are all nonnegative. If $\ell = 0$ then X and Y are adjacent.

Theorem 6. With Assumption IM, n-2 identical deterministic jobs with unit processing times and two stochastic jobs, X and Y, one of the following sequences is optimal

(a) $I_{n-2}XY$

(b) $I_{n-2}XY$

(c) $XI_{n-2}Y$

The following two lemmas are needed to prove the main result.

Lemma 1. The sequence $I_{k+\ell}XYI_h$ is stochastically better than the sequence $I_k X I_\ell Y I_h$ provided $k > 0$.

Proof. If $\ell = 0$, then both sequences are the same. Thus trivially the result is true. Now assume $\ell > 0$.

Let us denote the sequence $I_k X I_\ell Y I_h$ by 'a' and the sequence $I_{k+1} X I_{\ell-1} Y I_h$ by 'b'. Until the k^{th} job, both sequences are exactly the same. After the k^{th} job, the sequence 'b' is delayed by one unit of time compared to sequence 'a'. Thus for the $k+2^{nd}$ job in sequence 'b' we have

$$D^b_{k+2,j} = 1 + D^a_{k+1,j} \qquad \text{for } 1 \le j \le m$$

On the other hand for sequence 'a', we have

$$D^a_{k+2,1} = \max(D^a_{k+1,1} + 1,\ D^a_{k+1,2})$$
$$D^a_{k+2,1} = \max(D^a_{k+1,1} + 2,\ D^a_{k+1,2} + 1,\ D^a_{k+2,3})$$
$$\vdots$$
$$D^a_{k+2,j} = \max(D^a_{k+1,1} + j,\ D^a_{k+1,2} + j-1,\ \ldots,\ D^a_{k+1,j} + 1,\ D^a_{k+1,j+1}).$$

Since $D^a_{k+2,j} \ge D^b_{k+2,j}$ for all j, and the last $\ell+h$ jobs are the same in both sequences, sequence 'b' is stochastically better than sequence 'a'. Applying this argument repeatedly we can prove that $I_{k+\ell} XYI_h$ is stochastically better than $I_k X I_\ell Y I_h$. □

Lemma 2. A sequence $I_{k+h}XY$ is stochastically better than I_kXYI_h provided $k > 0$.

Proof. The proof is exactly the same as Lemma 1. □

Proof of the Theorem 6. Let us take a general sequence $I_kXI_\ell YI_h$.

(a) If $k > 0$, then using Lemmas 1 and 2 we can show that $I_{k+\ell+h}XY$ is stochastically better than $I_kXI_\ell YI_h$.

(b) If $k = 0$ then reverse the sequence to obtain $I_hYI_\ell X$. By Muth's (1979) result we know that $XI_\ell YI_h$ and $I_hYI_\ell X$ have stochastically the same makespan. Now if $h > 0$ then using Lemmas 1 and 2 we can prove that $I_{h+\ell}YX$ is stochastically better than $I_hYI_\ell X$.

(c) If $k = 0$ and $h = 0$ then we have $XI_\ell Y$ which is one of the possibly optimal sequences. □

Now we show that this theorem cannot be improved in the sense that sometimes XI_kY is strictly optimal, sometimes I_kXY is strictly optimal and sometimes I_kYX is strictly optimal.

Example 1. Let the number of machines be 3 and the number of jobs be 4. Here we will give an example where XI_2Y is strictly optimal.

The random variables are distributed as follows:

i	0	2
Pr[X=i]	1/2	1/2

i	1/2	1 1/2
Pr[Y=i]	1/2	1/2

The expected makespan for XI_2Y = 392/64.

The expected makespan for I_2YX = 393/64.

The expected makespan for I_2XY = 412/64.

Example 2. Let the number of machines be 3 and the number of jobs be 4. Here we will give an example where I_2YX is strictly optimal.

The random variables are distributed as follows:

i	0	3
Pr[X=i]	2/3	1/3

i	0	2
Pr[Y=i]	1/2	1/2

The expected makespan for XI_2Y = 741/108.

The expected makespan for I_2YX = 738/108.

The expected makespan for I_2XY = 766/108.

For an example in which I_2XY is strictly optimal, interchange the distribution of X and Y in Example 2.

Although in Example 2 the sequence is strictly optimal, it is not stochastically better than the sequence XI_2Y. It is easy to verify that

$$\Pr[\text{makespan of } XI_2Y > 4] < \Pr[\text{makespan of } I_2YX > 4]$$

which shows that I_2YX is not stochastically better than XI_2Y.

With some additional restrictions, we can narrow the field of choices given in the theorem as follows.

Proposition 1. If $X_i \geq Y_j$ for all i and j, then I_kXY is stochastically better than I_kYX.

Proof. The makespan for I_kYX is

$$\begin{aligned}(k-1) + \max(&1 + Y_1 + X_1 + X_2 + \ldots + X_m,\ 2 + X_1 + X_2 + \ldots + X_m \\ &3 + X_2 + X_3 + \ldots\ldots + X_m,\ 4 + X_3 + X_4 + \ldots + X_m, \\ &\vdots \\ &m + X_{m-1} + X_m)\end{aligned} \tag{15}$$

The makespan for I_kXY is

$$\begin{aligned}(k-1) + \max(&1 + X_1 + X_2 + \ldots + X_m + Y_m,\ 2 + X_2 + \ldots + X_m + Y_m \\ &3 + X_3 + \ldots\ldots + X_m + Y_m,\ 4 + X_4 + \ldots + X_m + Y_m, \\ &\vdots \\ &m + X_m + Y_m)\end{aligned} \tag{16}$$

Letting Y_m in (16) equal Y_1 in (15), it implies that the makespan of I_kXY is stochastically better than the makespan of I_kYX. □

Proposition 2. If $X_i \geq Y_j$ and $X_i \geq 1$ for all i and j, then I_kXY is stochastically optimal.

Proof. The makespan for I_kYX is

$$k + X_1 + X_2 + \ldots\ldots + X_m + Y_m.$$

The makespan for XI_kY depends on whether the number of machines, m, is greater than k.

Case (1) Let $k \geq m$.

The makespan is

$$\begin{aligned} X_1 + X_2 + \ldots + X_m + k + Y_m \\ + \max(&Y_1 + Y_2 + Y_3 + \ldots + Y_{m-1} - m + 1, \\ &Y_2 + Y_3 + \ldots + Y_{m-1} - m + 2, \\ &Y_3 + \ldots + Y_{m-1} - m + 3, \\ &\vdots \\ &Y_{m-1} - 1, 0) \end{aligned}$$

Case (2) Let $k < m$ and $m - k = b$.

The makespan is

$$X_1 + X_2 + \dots + X_m + k + Y_m$$

$$\max(Y_b + Y_{b+1} + Y_{b+2} + \dots + Y_{m-1} - k,$$
$$1 + Y_{b+1} + Y_{b+2} + \dots + Y_{m-1} - k,$$
$$2 + Y_{b+2} + Y_{b+3} + \dots + Y_{m-1} - k,$$
$$\vdots$$
$$k - 1 + Y_{m-1} - k, 0)$$

Note that if $\Pr[Y_i > 1] > 0$, I_kXY is strictly stochastically optimal. However, if $\Pr[Y_i > 1] = 0$, both I_kXY and XI_kY are stochastically equal. Hence both are optimal. □

In Proposition 2 the sequence not only minimizes the expected makespan of I_kXY but any moment of the makespan of I_kXY than ahy moment of the makespan of any other sequence.

If $X_i \leq 1$ and $Y_j \leq 1$ for all i and j, from Theorem 5 we know XI_kY is stochastically optimal. With this we have shown cases where each one of the possible optimal sequences of Theorem 6 is strictly stochastically optimal. In Theorem 6 we have resolved the Pinedo's (1982a) conjecture. Even with the stochastic jobs having mean one the Theorem cannot be improved as shown in examples 1, 2, 3.

We also note that Pinedo and Schrage (1981, p. 13) made a related conjecture that the sequence XI_kY is optimal when the stochastic jobs have symmetric distributions with expectation one. The results of Theorem 6 apply; hence, either XI_kY, I_kXY, or I_kYX should be optimal. However, a counterexample has been presented to this conjecture in Suresh, Foley, and Dickey (1983). Again note that an optimal sequence may not be the only optimal sequence in many cases. For example, if we had X and Y uniformly distributed between 1 and 2 and two deterministic jobs, then as a special case of Theorem 5 an optimum sequence is IXYI, Initially, this appears to contradict our results of Theorem 6; however, both the sequences IXYI and I_2XY are optimal and have stochastically the same makespan.

Chapter V

SUMMARY

Here we will compare our results with the results that have already appeared in the literature. Then we discuss some areas of further research.

5.1 LACK OF DUALITY

In Theorem 1 with Assumption IJ the expected flowtime on two machines is independent of the machine ordering. In Theorems 2 and 3 with Assumption IM the expected flowtime is minimized by a SEPT sequence of the jobs; and thus not independent of the job ordering. This shows that there does not exist a dual relationship as described in Chapter 2.

Pinedo (1982a) has proved that with Assumption IJ, m-2 identical deterministic machines, and two stochastic machines having symmetric processing time distributions, the capacity is maximized by placing one stochastic machine first and the other last. Suresh, Foley and Dickey (1983) have given a counterexample to the conjecture that with Assumption IM, m-2 deterministic jobs and two stochastic jobs having symmetric processing time distributions, the expected makespan is always minimized by placing one stochastic job first and the other last. Though capacity and makespan are different quantities, this is another example which shows that the dual relationship explained in Chapter 2 does not exist in stochastic flow shop with blocking.

5.2 COMPARISONS WITH OTHER RESULTS

In Theorem 1 with Assumption IJ, for any stochastic ordering of the processing time it is quite surprising to note that the expected flowtime is independent of the machine ordering. In Theorem 2 & 3 we prove that with Assumption IM the expected flowtime depends on the job ordering. Theorem 2 is a unique result because for this system we have shown in Theorem 4 that the SEPT sequence stochastically minimizes both the flowtime and makespan. This is not true in general.

Theorem 3 here and Theorem 3 of Pinedo (1982a) have the same system. Pinedo proves that the sequence n,n-2,......4,2,1,3,.....,n-1 (or its reverse) minimizes the expected makespan. We have proved that n,n-1,n-2,.....2,1, i.e., SEPT, minimizes the expected flowtime. Thus, in general, the same sequence does not minimize the expected makespan and the expected flowtime.

Referring to Theorem 5, it is clear that a sequence with job X in the first place and Y in the last place has stochastically the same makespan as the sequence with Y in the first place and X in the last place. Hence, there are two optimal sequences. With some additional information available about jobs X and Y note that there may be other optimal sequences. For example, if it is given that jobs X and Y are nonoverlapping with each other and X is faster than Y then

$X, Y, j_1, j_2, \ldots, j_{n-2}$

$j_{n-2}, \ldots, j_2, j_1, Y, X$

$X, j_1, j_2, \ldots, j_{n-2}, Y$

$Y, j_{n-2}, \ldots, j_2, j_1$

are all optimal sequences.

Pinedo (1982b) has shown that to maximize capacity with Assumption IJ, the slower machines have to be placed at the ends. We have proved in Theorem 5 that to stochastically minimize the makespan with Assumption IM, the faster jobs have to go to the ends. Though makespan and capacity are not the same quantities, they are closely related as mentioned in section 2.2.3.

5.3 AREAS OF FURTHER RESEARCH

Baker (1974; p. 107) gives a sequence for minimizing the weighted expected flowtime on a single machine. If we assume equal weights as in this paper, the optimum sequence of Baker is SEPT. Since a SEPT sequence minimizes the expected flowtime under these conditions on both one and two machines, a SEPT sequence might be optimal for m machines.

Maximizing capacity seems an important criterion when there are infinite jobs and minimizing the makespan is important when there are finite jobs. Let us take four jobs, two of them having identically distributed processing times forming one set and the other two having identically distributed processing times forming the second set. Let one set be nonoverlapping over the other set. For minimizing the expected makespan we have proved by Theorem 5 that the faster jobs should be in the ends and the slower jobs in the middle.

Assume the same situation, but instead of jobs let us take four machines. Pinedo (1982b) has proved that in order to maximize the capacity the slower machines should be placed at the ends and faster machines in the middle. This leads one to believe that if the jobs are nonidentical and have to be ordered it is better to place the faster jobs at the ends and the slower jobs in the middle; whereas in the case of machines being nonidentical it is better to place the slower machines in the ends and faster machines in the middle.

In Theorem 6 we presented a result with n-2 identical deterministic jobs. We believe that the results can be extended to n-2 deterministic jobs not necessarily identical. The optimal sequence may have the deterministic jobs in a

SEPT sequence. We also believe that with j stochastic jobs and n-j identical deterministic jobs an optimal sequence minimizing the expected makespan will always have the deterministic jobs together. This has been proved in Theorem 6 with j equals to two.

An area of further investigation would be to extend Theorem 5 to a case where jobs X and Y are d-ordered with the other n-2 identical jobs.

Appendix A

APPENDIX

When one tries to establish a relationship between random variables, there are many possible ways of doing it. These relationships are referred as stochastic orderings.

A.1 NONOVERLAPPING SMALLER

$X_i \le X_j$ almost surely if any value of X_i is less than X_j. If X_i and X_j are independent, then the maximum value of X_i should be less than the minimum value of X_j. If X_i and X_j are dependent, then they are nonoverlapping if $Pr[X_i \ge X_j] = 0$. This is an extremely strong ordering because one random variable is always less than the other.

A.2 D-ORDERING

$X_i \le_d X_j$ if $E[f(X_i)] \le E[f(X_j)]$ for any f which is nondecreasing monotone function. We call this a d-ordering; however it has been referred as type 1 ordering by Stoyan (1977) and stochastic dominance by Bawa (1982). It is easy to show in the literature that $E[f(X_i)] \le E[f(X_j)]$ for any nondecreasing monotone fuction f if and only if $Pr[X_i > t] \le Pr[X_j > t]$. Therefore, as a consequence we can define $X_i \le_d X_j$ if $Pr[X_i > t] \le Pr[X_j > t]$ for any t. Notice that the complimentary distribution function of X_j will always lie above that of X_i. By choosing $f(t)= t^n$ in $E[f(X_i)] \le E[f(X_j)]$ we see that all moments of X_i are dominated by corresponding moments of X_j. The d-ordering has a nice economic interpretation. If $X_i \le_d X_j$ then for any utility function f which is always monotone nondecreasing we know that the expected utility of X_i is always less than the expected utility of X_j.

A.3 CONVEX ORDERING

$X_i \le_c X_j$ if $E[f(X_i)] \le E[f(X_j)]$ for any nondecreasing nonegative convex function f. This is known as c-ordering. It is also equivalent to $E[\max(0,X_i-a)] \le E[\max(0,X_j-a)]$ and to $\int_a^\infty g_{X_i}(t)dt \le \int_a^\infty g_{X_j}(t)dt$ for any a where $g_{X_i}(t)$ and $g_{X_j}(t)$ are probability density functions of X_i and X_j respectively. Economically, if $X_i \le_c X_j$ then for any utility function which is convex the utility of X_i is always less than that of X_j.

A.4 CONCAVE ORDERING

$X_i \le_{cc} X_j$ if $E[f(X_i)] \le E[f(X_j)]$ for any nondecreasing nonnegative concave function f. This is known as cc-ordering. It is also equivalent to $E[\min(0,X_i-a)] \le E[\min(0,X_j-a)$ and to $\int_0^a g_{X_i}(t)dt \le \int_0^a g_{X_j}(t)dt$ for any a where

$g_{X_i}(t)$ and $g_{X_j}(t)$ are probability density functions of X_i and X_j respectively. Economically, if $X_i \le_{cc} X_j$ then for any utility function which is concave the expected utility of X_i is always less than that of X_j.

A.5 V-ORDERING

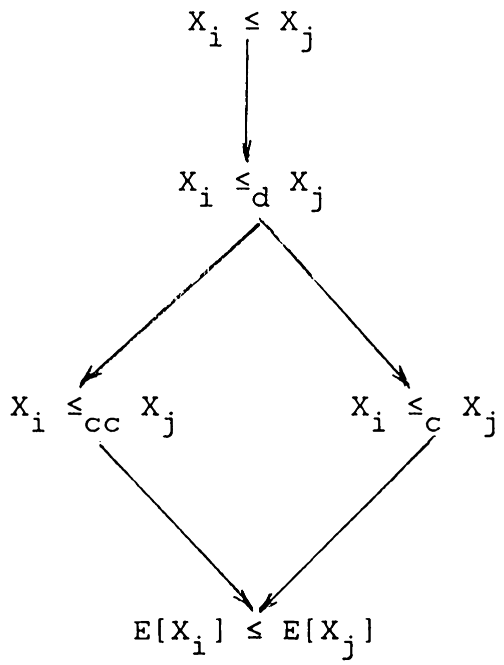
$X_i \le_v X_j$ if $E[X_i] = E[X_j] = \mu$, X_i and X_j have symmetric probability density functions, $Pr[X_i \le t] \le Pr[X_j \le t]$ for $0 < t < \mu$ and $Pr[X_i \le t] \ge Pr[X_j \le t]$ for $\mu < t < 2\mu$. This is known as v-ordering. Pinedo (1982a) wanted to investigate the influence of the variance in the processing times.

A.6 EXPECTED VALUE ORDERING

$E[X_i] \le E[X_j]$ if the expected value of X_i is less than the expected value of X_j. The other orderings described earlier are in a sense partial orderings, i.e., it is possible to have $X_i \ge_d X_j$ and $X_j \ge_d X_i$. On the contrary, the expected value ordering is a total ordering because it is not possible to have $E[X_i] \ge E[X_j]$ and $E[X_j] \ge E[X_i]$ unless $E[X_i] = E[X_j]$. Economically, if $E[X_i] \le E[X_j]$ then for any linear utility function the expected utility of X_i is less than that of X_j.

A.7 RELATIONS BETWEEN STOCHASTIC ORDERINGS

The stochastic orderings discussed here are related to one another. For example, nonoverlapping implies any other ordering, i.e., if $X_i \leq X_j$ then $X_i \leq_d X_j$, $X_i \leq_{cc} X_j$, $X_i \leq_c X_j$, $E[X_i] \leq E[X_j]$. The following tree has been constructed to show the relation between the stochastic orderings. Here an ordering at the top implies all the orderings below it.



An interesting relation exists between v-ordering, the concave and convex ordering. If $X_i \leq_c X_j$, X_i and X_j have symmetric density functions around the mean and $E[X_i] = E[X_j]$, then $X_i \leq_v X_j$. On the other hand if $X_i \leq_{cc} X_j$, X_i and X_j have symmetric density functions and $E[X_i] = E[X_j]$, then

$X_i \geq_v X_j$. Notice that for concave ordering the implication is reversed for the v-ordering. Many other stochastic orderings are possible. For more on stochastic orderings and their relations with one another, we refer the reader to Stoyan (1977).

It is clear that both flowtime and makespan are random variables which depend on the schedule. The processing times of different jobs on every machine are also a random variables. We would like our hypothesis to be as weak as possible and our result to be as strong as possible. An excellent result would be where the processing times are expected value ordered and the makespan of the optimal sequence is nonoverlapping compared to any other sequence. Clearly, this is impossible. For example, it is almost impossible to get a nonoverlapping result for makespan even if we make our assumption about the processing time as strong as that for nonoverlapping.

The work done so far in this area has optimal schedules which minimize only the expected makespan. As rightly pointed out in Dempster et al (1981), the expected value criterion is not always a good measure of optimality where a 'once for all' decision must be taken. Sometimes, we want optimality criterion for any expected utility criterion involving a monotone utility function. (As mentioned an section A.2

$E[f(X_i)] \le E[f(X_j)]$ for any monotone utility function f if and only if $Pr[X_i>t] \le Pr[X_j>t]$.) The expected value criterion is appropriate for these models in which relatively small costs or gains are involved per unit time. In the contrary situation, where decisions have to be taken in the face of uncertainities involving relatively large gains or losses the expected utility criterion is appropriate. Thus we would like to get a schedule in which the makespan is stochastically minimized. By this we mean that the makespan of the optimal sequence is stochastically less than any other sequence. Therefore $Pr[A^*>t] \le Pr[A>t]$ where A^* is the makespan of the optimal sequence and A is the makespan of any other sequence. Note that by this optimality criterion the optimal sequence maximizes the probability a set of jobs completes processing within a time 't'. Thus not only is this sequence optimal for minimizing the expected makespan but also for other scheduling criteria like due date. Due date is the date by which a given set of jobs has to be processed.

Appendix B

The background results are presented in the form of some lemmas. In particular, Lemmas 3, 4 and 5 are quite useful. The following lemmas are used to prove the theorems. Assume for all the lemmas that $F_i(t)$ is the complimentary distribution of the random variable X_i, i.e., $\bar{F}_i(t) = Pr[X_i>t]$ and $F_i(t) = 1 - \bar{F}_i(t)$.

B.1 LEMMAS FOR FLOWTIME RESULTS

Lemma 1. Let X_1 and X_2 be independent random variables then $E[\max(X_1,X_2)] = \int_0^\infty (\bar{F}_1(t) + \bar{F}_2(t) - \bar{F}_1(t)\bar{F}_2(t))\, dt$

Proof.

$$E[\max(X_1,X_2)] = \int_0^\infty (1 - (1 - \bar{F}_1(t))(1 - \bar{F}_2(t))dt$$
$$= \int_0^\infty \bar{F}_1(t) + \bar{F}_2(t) - \bar{F}_1(t)\bar{F}_2(t)dt.$$

Lemma 2. Let X_1, X_2, X_3 be independent random variables and $X_2 \geq_d X_3$ then
$E[\max(X_1,X_2)] \geq E[\max(X_1,X_3)]$.

Proof. First, by Lemma 1,

$$E[\max(X_1,X_2)] = \int_0^\infty [\bar{F}_1(t) + \bar{F}_2(t) - \bar{F}_1(t)\bar{F}_2(t)]dt$$
$$= \int_0^\infty [\bar{F}_1(t) + \bar{F}_2(t)(1 - \bar{F}_1(t))]dt$$
$$= \int_0^\infty [\bar{F}_1(t) + \bar{F}_2(t)F_1(t)]dt.$$

Similiarly,

$$E[\max(X_1,X_3)] = \int_0^\infty [\bar{F}_1(t) + \bar{F}_3(t)F_1(t)]dt.$$

Since $\bar{F}_2(t) \geq \bar{F}_3(t)$ for all t, we have

$E[\max(X_1,X_2)] \geq E[\max(X_1,X_3)]$.

Lemma 3. Let X_1, X_2, X_3 be independent random variables and $X_1 \geq_d X_3$, $X_2 \geq_d X_3$ then

$$E[X_1] + E[\max(X_2,X_3)] \geq E[X_3] + E[\max(X_2,X_1)].$$

Proof. First, by Lemma 1

$$E[X_1] + E[\max(X_2,X_3)] = \int_0^\infty [\bar{F}_1(t) + \bar{F}_2(t) + \bar{F}_3(t) - \bar{F}_2(t)\bar{F}_3(t)]dt.$$

Similiarly,

$$E[X_3] + E[\max(X_2,X_1)] = \int_0^\infty [\bar{F}_3(t) + \bar{F}_2(t) + \bar{F}_1(t) - \bar{F}_2(t)\bar{F}_1(t)]dt.$$

Since, $\bar{F}_1(t) \geq \bar{F}_3(t)$ for all t, we have

$$E[X_1] + E[\max(X_2,X_3)] \geq E[X_3] + E[\max(X_2,X_1)].$$

Lemma 4. Let X_1, X_2, X_3, X_4 be indepedent random variables $X_1 \geq_d X_2 \geq_d X_3 \geq_d X_4$ then both of the following hold:

(i) $E[\max(X_1,X_2)] + E[\max(X_3,X_4)] \leq E[\max(X_1,X_3)] + E[\max(X_2,X_4)]$

(ii) $E[\max(X_1,X_3)] + E[\max(X_2,X_4)] \leq E[\max(X_1,X_4)] + E[\max(X_2,X_3)]$

Proof. The result has been used by Pinedo (1982a) but we include a proof to make this thesis self-contained.

Expanding the left hand side of (i) using Lemma 1 yields,

$$E[\max(X_1,X_2)] + E[\max(X_3,X_4)] = \int_0^\infty [\bar{F}_1(t) + \bar{F}_2(t) + \bar{F}_3(t) + \bar{F}_4(t) - \bar{F}_1(t)\bar{F}_2(t) - \bar{F}_3(t)\bar{F}_4(t)dt.$$

Similiarly, the right hand side yields,

$$E[\max(X_1,X_3)] + E[\max(X_2,X_4)]$$
$$= \int_0^\infty [\bar{F}_1(t) + \bar{F}_3(t) + \bar{F}_2(t) + \bar{F}_4(t) - \bar{F}_1(t)\bar{F}_3(t) - \bar{F}_2(t)\bar{F}_4(t)dt.$$

Since,

$$(\bar{F}_1(t) - \bar{F}_3(t))(\bar{F}_2(t) - \bar{F}_4(t)) \ge 0,$$

we have

$$\bar{F}_1(t)\bar{F}_2(t) + \bar{F}_3(t)\bar{F}_4(t) \ge \bar{F}_1(t)\bar{F}_3(t) + \bar{F}_2(t)\bar{F}_4(t).$$

Using this in equation (3) and (4), we have

$$E[\max(X_1,X_2)] + E[\max(X_3,X_4)] \le E[\max(X_1,X_3)] + E[\max(X_2,X_4)].$$

The second result can be proved in a similiar manner by noting that

$$(\bar{F}_1(t) - \bar{F}_2(t))(\bar{F}_3(t) - \bar{F}_4(t)) \ge 0.$$

Lemma 5 Let $Z_{ij} = X_i + Y_j$. Then Z_{ij} is stochastically minimized if X_i and Y_j are stochastically minimized.

Proof. Here we use the following result (Stoyan, pp. 4-5,1977): $Z_1 \le_d Z_2$ if and only if $E[f(Z_1)] \le E[f(Z_2)]$ for all monotone non-decreasing nonnegative function f.

Suppose $X_{i*} \le_d X_i$ for all i and similiarly $Y_{j*} \le_d Y_j$ for all j. It suffices to show that

$$E[f(X_{i*} + Y_{j*}) \le E[f(X_i + Y_j)] \qquad (17)$$

for all non-decreasing functions f. Since $X_{i*} \le_d X_i$, for all non decreasing f and real numbers x

$$E[f(X_{i*} + x) \le E[f(X_i + x)]$$

Hence,

$$E[f(X_{i*} + Y_j)/Y_j] \le E[f(X_i + Y_j/Y_j)].$$

Now, the left hand side is an increasing function of Y_j, hence since $Y_j \le_d Y_j$

$$E[f(X_{i*} + Y_{j*})/Y_{j*}] \le E[f(X_{i*} + Y_j/Y_j)]$$

$$\le E[f(X_i + Y_j)/Y_j)]$$

Taking expectations, yields (17).

B.2 REVERSIBILITY

The following lemma has been stated by Pinedo (1982a) but no proof has been given by him. He has attributed the proof to Datttreya (1978) and Muth (1979). Muth (1979) has proved the lemma with Assumption IJ. As pointed out in Chapter 2.3 in 'Lack of Duality' a separate proof is neccesary with Assumption IM.

Lemma 6. With Assumption IM, the total time required to process a given sequence of n jobs has the same distribution as the total time required to process the same n jobs in the reverse order.

Proof. We let M_1 be the matrix whose elements are the service times $S_{i,j}$; for n jobs ordered 1,2,....,m. Thus,

$$M_1 = \begin{bmatrix} S_{1,1} & \cdots & S_{1,m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ S_{n,1} & \cdots & S_{n,m} \end{bmatrix} .$$

The matrix corresponding to job reversal alone, that is jobs ordered as n,n-1,......,2,1 is

$$M_2 = \begin{bmatrix} S_{n,1} \cdot & \cdot & \cdot\ S_{n,m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ S_{1,1} \cdot & \cdot & \cdot\ S_{1,m} \end{bmatrix}$$

The matrix corresponding to job reversal and line reversal, that is jobs ordered as n,n-1,.....2,1 and machines ordered m,m-1,....2,1 is

$$M_3 = \begin{bmatrix} S_{n,m} \cdot & \cdot & \cdot\ S_{n,1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ S_{1,m} \cdot & \cdot & \cdot\ S_{1,1} \end{bmatrix}$$

The makespan as developed by Muth (1979) is a function f{.} of the matrix of service times, namely

$$P_i = f\{M_i\},\ i = 1,2,3.$$

It was shown in Muth (1979) that $P_1 = P_3$. In the stochastic case, the distribution fuction of P_i

$$F_i(x) = Pr[P_i \le x],\ i=1,2,3,$$

is some function of the joint distribution of the elements of the random matrix M_i. All the elements in any given row are independent and identically distributed random variables; thus the order of the elements in any given row may be rearranged without changing the probabilistic properties of M_i. Since M_2 is obtained by reversing all rows of M_3 it follows that M_2 and M_3 are equivalent, in the sense that their probability measures are equal. Thus we obtain

$$F_3(x) = F_2(x).$$

We already know

$$F_3(x) = F_1(x).$$

Thus,

$$F_1(x) = F_2(x).$$

Therefore, stochastically the makespan is invariant under job reversal.

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