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Toward diagnosing neutrino non-unitarity through CP phase correlations

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We discuss correlations between the neutrino-mass-embedded Standard Model CP phase δ and the phases that originate from new physics which cause neutrino-sector unitarity violation (UV) at low energies. This study aims to provide one of the building blocks for machinery to diagnose non-unitarity, our ultimate goal. We extend the perturbation theory of neutrino oscillation in matter proposed by Denton et al. (DMP) to include the UV effect expressed by the α parametrization. By analyzing the DMP-UV perturbation theory to first order, we are able to draw a complete picture of the δ–UV phase correlations in the whole kinematical region covered by terrestrial neutrino experiments. Two regions exist with characteristically different patterns of the correlations: (i) the chiral-type $[e^{-i\delta}\alpha_{\mu e},\ e^{-i\delta}\alpha_{\tau e},\ \alpha_{\tau\mu}]$ (Particle Data Group convention) correlation in the entire high-energy region $|\rho E| \gtrsim 6\,(\text{g/cm}^3)$ GeV, and (ii) (blobs of the α parameters)–$e^{\pm i\delta}$ correlation anywhere else. Some relevant aspects for the measurement of the UV parameters, such as the necessity of determining all the $\alpha_{\beta\gamma}$ elements at once, are also pointed out. Subject Index: B52, B54

1. Introduction

At more than forty years after the establishment of the Standard Model (SM) of electroweak interactions [1–3] in the 1970s, people naturally sought physics beyond the SM. Despite that we now know that neutrinos are massive and the lepton flavors mix [4,5], and we have compelling evidence for dark matter in the universe [6,7], we do not yet have a conclusive bigger picture of our fundamental world. Naturally, one asks the question: If we interpret the above open windows as suggestions for the right places to look for new physics beyond the SM, what should we do?

In neutrino physics the possibility of the existence of SM-singlet states, or sterile neutrinos, is widely discussed, as reviewed, e.g., in Refs. [8–10] and the references cited therein. A version of this, sterile leptons with eV-scale masses, might already have been seen at the LSND [11] and MiniBooNE [12] experiments. However, the very recent MicroBooNE data seem to disfavor both the ν_e [13] and photon [14] origins of the low-energy excess of MiniBooNE.[1] On the other hand, IceCube sees a closed contour at 90% confidence level in the mixing angle–Δm^2 space, which is interpreted as a "systematic effect," not a fluke [16]. Thus, the chance of definitively settling the tantalizing question of the existence of eV-scale sterile neutrino(s) is left for the ongoing and upcoming searches, which are to be joined by those in Refs. [17,18]. In more generic contexts, searches for deviation from the SM expectation are performed in the frameworks

[1] For a different interpretation of the same data, see Ref. [15].

of so-called "non-standard interactions" (NSIs) [19] and/or non-unitarity [20,21].[2] See, e.g., Refs. [25–28] for reviews of NSI, Refs. [29–31] for constraints on NSI, and Refs. [20–24,32–44] for a limited list of articles on non-unitarity.

In this paper we address the non-unitarity approach. Let us imagine the environment in which non-unitarity will be studied. If the eV-scale sterile neutrino is the cause of non-unitarity, it is likely that its presence and properties will be found by the advanced searches such as in Refs. [16–18] in the very near future, unless their mixing to the active sector is extremely small. With positive evidence for accessible low-mass sterile(s), one can just go to the experimental data to dig out the correct shape of the sterile lepton model.

Suppose, however, that this does not happen, but nonetheless the results of various precision measurements continue to report small but robust deviations from the νSM, a shorthand notation for the neutrino-mass-embedded SM. Then, the obvious question must be: what is detected? A possible suspect would be non-unitarity in a more generic sense. It may be described by relatively model-independent frameworks, such as the one for physics at high scales [20], $E \gg m_W$, or at low scales [23,32], $E \ll m_W$. If we lack the obvious candidates for such an anomaly, we need to identify the nature of the physics behind non-unitarity either by phenomenological methods or, preferably, via experiment.

In this paper we investigate correlations between the νSM CP phase and the phases that originate from new physics existing at some scale, which is recognized as unitarity violation (UV)[3] at low energies. Using the UV-extended version of Denton et al.'s framework [45], we aim at establishing a unified view of such correlations valid in the whole kinematical region covered by terrestrial neutrino experiments, the "terrestrial region" for short. Typically this is the region of Super-Kamiokande's observation of atmospheric neutrinos, $0.1\,\text{GeV} \lesssim E \lesssim 10\,\text{GeV}$, as reported in Ref. [46, Fig. 3]. A rough sketch of the equiprobability contour of $P(\nu_\mu \to \nu_e)$ in this region is drawn in Ref. [47, Fig. 1]. Our treatment of the phase correlations in this paper surpasses those of the previous works [24,33], which apply only to the two local regions of atmospheric- and solar-scale enhanced oscillations, roughly speaking the resonances [19,48–50].

The ultimate goal in our approach is to diagnose non-unitarity that may originate in new physics beyond the νSM. It is conceivable that the effect of non-unitarity starts to be seen in the interference term $S^*_{\nu\text{SM}} S_{\text{UV}}$, where S_{UV} denotes the UV amplitude. See Appendix A for the more concrete form of S_{UV}. In this approach, what should be reached at the end is to extract the physical properties of S_{UV} by analyzing experimental data that include its effect in the form of $S^*_{\nu\text{SM}} S_{\text{UV}}$, *diagnosing non-unitarity*.

Toward this goal, we attempt here a theoretical investigation of the νSM $e^{\pm i\delta}$–UV phase correlation involved in the interference term $S^*_{\nu\text{SM}} S_{\text{UV}}$. Notice that $e^{\pm i\delta}$ is the unique phase factor in the νSM oscillation amplitude which is written by the fundamental physical parameter of the νSM, the lepton Kobayashi–Maskawa phase δ [51]. Therefore, its correlation with the UV phase factor must contain the key information on the relationship between νSM and UV new

[2] Given the feature of generic NSI with $9 \times 3 = 27$ parameters, non-unitarity in matter can be regarded as a "constrained NSI" with only 9 parameters, in which the production and detection NSI elements are determined by the propagation NSI. See the discussions in Refs. [22–24].

[3] We are aware that in the physics literature UV usually means "ultraviolet." In this paper, however, UV is used as an abbreviation for "unitarity violation" or "unitarity violating."

physics. We will return to the issue of the possible relevance of our investigation of νSM–UV phase correlations to the entire program of diagnosing non-unitarity in Sects. 6 and 7.

This strategy of diagnosing non-unitarity through interference between the νSM and UV-driving new physics presumes that such correlation exists at a detectable level even in the case that the new physics scale is much higher than m_W. Though it may sound unlikely, it is not totally obvious if we can conclude it impossible. We all know that physics must exist at a much higher energy scale like $1/\sqrt{G_N} \sim 10^{19}$ GeV, but almost massless particles around us are allowed to exist. As far as the formulation of high-scale UV is correct in deriving the non-unitary flavor-mixing matrix, the νSM and UV-driving new physics should interfere.[4] We also add that if CP violation has a group-theoretical origin related to strings [52], the correlations between the phases should exist, carrying crucially important information, in such a system.

Despite a slight overlap possibly existing between our descriptions in this paper and those in Refs. [24,33], we will try to make this paper self-contained as far as possible.

2. νSM–UV phase correlations: Now and the next step

To discuss the νSM–UV phase correlations in an unambiguous way, we must first decide how the UV effect is parametrized. We use the so-called the α parametrization [21] in which the non-unitary flavor-mixing matrix N is defined by multiplication of the α matrix with the usual unitary νSM mixing matrix $U \equiv U_{\text{MNS}}$ [53] in the Particle Data Group (PDG) convention [54], see Eq. (5), as

$$N_{\text{PDG}} = (\mathbf{1} - \alpha)\, U_{\text{PDG}} = \left\{ \mathbf{1} - \begin{bmatrix} \alpha_{ee} & 0 & 0 \\ \alpha_{\mu e} & \alpha_{\mu\mu} & 0 \\ \alpha_{\tau e} & \alpha_{\tau\mu} & \alpha_{\tau\tau} \end{bmatrix} \right\} U_{\text{PDG}}. \tag{1}$$

The α parametrization originates in Refs. [55,56]. The α matrix has nine degrees of freedom due to the three real diagonal and the three complex off-diagonal entries.

The problem of correlation between the CP phase factor $e^{\pm i\delta}$ in the νSM and the one in the UV amplitude was investigated in previous papers [24,33] using the α parametrization. These references used UV extensions of the frameworks given in Refs. [57], for the region around the atmospheric resonance, and [58], for the region around the solar resonance. Interestingly, very different types of phase correlation are observed in these two regions. A charming "chiral"-type correlation $[e^{-i\delta}\alpha_{\mu e},\ e^{-i\delta}\alpha_{\tau e},\ \alpha_{\tau\mu}]$ (PDG convention) was found in the former [24], whereas in the latter a less transparent (blobs of the α parameters)–$e^{\pm i\delta}$ correlation [33] is seen. It is good to know that part of the chiral-type correlation, $e^{-i\delta}\alpha_{\mu e}$, in the atmospheric region was observed in the foregoing analyses [21,59,60].

2.1 *Toward νSM–UV phase correlations in the whole terrestrial region*

Thus, to our current knowledge the picture of νSM–UV phase correlation jumps from a local region to another, from the chiral-type correlation in the atmospheric resonance region to the δ–$\alpha_{\beta\gamma}$-blobs correlation in the solar resonance region. Obviously, we need a better treatment of the phase correlations to allow us to draw a global picture of the phase correlation in the whole terrestrial region, i.e. the region covered by the terrestrial neutrino experiments.

[4] If this or the possible other reasonings fail, low-scale UV might be a more natural scenario in which to expect the νSM–UV correlations.

To our knowledge the right theoretical framework for this purpose is one based on the Jacobi method, Denton et al. (DMP) perturbation theory [45] and Agarwalla et al. (AKT) perturbation theory [61] with full coverage of the whole terrestrial region. For a pedagogical introduction to the Jacobi method, see Ref. [61].[5] See Ref. [65, Appendix A] for another favorable feature of the globally valid frameworks, albeit a much less familiar one.

We base our formalism on the DMP perturbation theory and extend it to incorporate the UV effect parametrized by the α parameters, the framework dubbed hereafter as "DMP-UV perturbation theory." This is because DMP is easier to handle, and has a transparent relation to a particular version [57] of the atmospheric-resonance perturbation theory as "half-way" to DMP. In fact, it has been shown analytically [66] that the DMP theory approaches this atmospheric-resonance perturbation theory and the solar-resonance perturbation theory [58] in the appropriate limit. In this way, we can discuss the relationship between our results in this paper and those previously obtained using the atmospheric- and solar-resonance perturbation theories valid in these local regions [24,33].

2.2 *Non-unitarity vs. sterile neutrinos*

It may be worth clarifying what we mean by the non-unitarity approach and its relation to the sterile neutrino hypothesis, both mentioned in the introduction. The existence of sterile neutrinos is a perfect model for non-unitarity, and their separate treatment merely reflects our preferred analysis strategy for physics beyond the νSM. Of course, systems with non-unitarity have a much wider variety, as first raised in Ref. [20] and further studied in many references cited in Sect. 1.

Sterile neutrino states with eV-scale masses can be searched for by LSND–MiniBooNE-type experiments, for example; see, e.g., Ref. [8] for other options. Alternatively, it produces a resonance-like enhancement in $P(\nu_\mu \rightarrow \nu_\mu)$ and $P(\nu_\mu \rightarrow \nu_e)$ at energy $E \sim 1$ TeV [67,68]. As examined in detail by many authors (see the list, e.g., in Ref. [69]) it would allow detection in atmospheric neutrino observations as pursued by IceCube [16]. Once one (or a few) sterile neutrino state(s) have been identified, "diagnosing non-unitarity" is no longer necessary: one can just examine the data to create the model of sterile state(s).

Thus, our non-unitarity approach when applied to sterile neutrinos is meant to treat cases with more elusive sterile states. Let us take the three-active-plus-N_s sterile neutrino model with eV–MeV sterile masses to make more concrete statements. We have argued that such a model can provide a generic model for low-scale UV with the νSM-like three-active-neutrino oscillation probability modified by the non-unitary mixing matrix [23,32]. The condition by which we remain in such a "diagnostics needed" regime is worked out to be $|\rho E| \lesssim 100\,(\text{g/cm}^3)$ GeV [23, Sect. 3.5]. At energies higher than this by a factor of $\sim$50 we meet the $\mathcal{O}(1)$ TeV resonance, and the model moves into the "no need for diagnostics" regime with this unmistakable signature. Therefore, while we aim for the validity of our discussion in this paper in the "terrestrial region," in fact it can extend to $|\rho E| \lesssim 100\,(\text{g/cm}^3)$ GeV.

[5] It has been shown that these Jacobi-method-based approximation schemes provide numerically accurate probability expressions [62]. For further studies of the Jacobi method in neutrino oscillation, see Refs. [63,64].

3. Three-active-neutrino system with non-unitary flavor-mixing matrix

To formulate the DMP-UV perturbation theory using the α parametrization, we follow the method developed in Ref. [24]. We define the system below but defer presentation of a step-by-step formulation of the DMP-UV perturbation theory to Appendix A, where it will be done in a pedagogical manner.

Studies for formulating the three-active-neutrino evolution in matter in the presence of non-unitary flavor mixing appear to have converged on a framework that starts from the Schrödinger equation in the vacuum mass eigenstate basis,

$$i\frac{d}{dx}\check{\nu} = \frac{1}{2E}\left\{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{21} & 0 \\ 0 & 0 & \Delta m^2_{31} \end{bmatrix} + N^{\dagger}\begin{bmatrix} a-b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -b \end{bmatrix} N\right\}\check{\nu}. \tag{2}$$

We just quote Refs. [22] and [23] for high-scale and low-scale UV, respectively, to support our statement. In the latter it is a truncated system from the three-active-plus-N_s sterile model. In this paper we denote the vacuum mass eigenstate basis as the "check basis." In Eq. (2), N denotes the 3×3 non-unitary flavor-mixing matrix which relates the flavor neutrino states to the vacuum mass eigenstates as

$$\nu_{\alpha} = N_{\alpha i}\check{\nu}_i. \tag{3}$$

Hereafter, the subscript Greek indices α, β, or γ run over e, μ, and τ, and the Latin indices i and j run over the mass eigenstate indices 1, 2, and 3. E is the neutrino energy and $\Delta m^2_{ji} \equiv m^2_j - m^2_i$. The usual phase redefinition of the neutrino wave function is done to leave only the mass-squared differences.

The functions $a(x)$ and $b(x)$ in Eq. (2) denote the Wolfenstein matter potential [19] due to charged current (CC) and neutral current (NC) reactions, respectively:

$$\begin{aligned} a(x) &= 2\sqrt{2}G_{\mathrm{F}}N_e E \approx 1.52 \times 10^{-4}\left(\frac{Y_e \rho}{\mathrm{g\,cm^{-3}}}\right)\left(\frac{E}{\mathrm{GeV}}\right)\mathrm{eV}^2, \\ b(x) &= \sqrt{2}G_{\mathrm{F}}N_n E = \frac{1}{2}\left(\frac{N_n}{N_e}\right)a, \end{aligned} \tag{4}$$

where G_{F} is the Fermi constant. N_e and N_n are the electron and neutron number densities in matter; ρ and Y_e denote, respectively, the matter density and number of electrons per nucleon in matter. These four quantities are, in principle, position dependent.

3.1 *The three useful conventions of the lepton flavor-mixing matrix*

We start from the most commonly used form, the PDG convention [54], of the Maki–Nakagawa–Sakata (MNS) matrix,

$$U_{\mathrm{PDG}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix}\begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{bmatrix}\begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

with the obvious notations $s_{ij} \equiv \sin\theta_{ij}$ etc., and δ being the CP-violating phase. Recently, we have started to use the other two conventions, different only by phase redefinitions, called the

ATM (atmospheric) and SOL (solar) conventions:

$$U_{\rm ATM} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\delta} \end{bmatrix} U_{\rm PDG} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} e^{i\delta} \\ 0 & -s_{23} e^{-i\delta} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6}$$

$$U_{\rm SOL} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-i\delta} & 0 \\ 0 & 0 & e^{-i\delta} \end{bmatrix} U_{\rm PDG} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\delta} & 0 \\ 0 & 0 & e^{i\delta} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} e^{i\delta} & 0 \\ -s_{12} e^{-i\delta} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

The reason for our terminology of $U_{\rm ATM}$ and $U_{\rm SOL}$ in Eq. (7) is because the CP phase factor $e^{\pm i\delta}$ is attached to the "atmospheric angle" s_{23} in $U_{\rm ATM}$, and to the "solar angle" s_{12} in $U_{\rm SOL}$, whereas in $U_{\rm PDG}$, δ is attached to s_{13}.

Once the phase convention of the U matrix is changed from $U_{\rm PDG}$ to $U_{\rm ATM}$, a consistent definition of $N_{\rm ATM}$ requires the α matrix to transform as in

$$N_{\rm ATM} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\delta} \end{bmatrix} N_{\rm PDG} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{bmatrix} = \left\{ \mathbf{1} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\delta} \end{bmatrix} \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{bmatrix} \right\} U_{\rm ATM}$$
$$\equiv \left(\mathbf{1} - \alpha^{\rm ATM} \right) U_{\rm ATM}, \tag{8}$$

and therefore the α matrix is convention dependent. In the ATM convention it takes the form

$$\alpha^{\rm ATM} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\delta} \end{bmatrix} \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{bmatrix}$$
$$= \begin{bmatrix} \alpha_{ee} & 0 & 0 \\ \alpha_{\mu e} & \alpha_{\mu\mu} & 0 \\ e^{-i\delta}\alpha_{\tau e} & e^{-i\delta}\alpha_{\tau\mu} & \alpha_{\tau\tau} \end{bmatrix} \equiv \begin{bmatrix} \alpha^{\rm ATM}_{ee} & 0 & 0 \\ \alpha^{\rm ATM}_{\mu e} & \alpha^{\rm ATM}_{\mu\mu} & 0 \\ \alpha^{\rm ATM}_{\tau e} & \alpha^{\rm ATM}_{\tau\mu} & \alpha^{\rm ATM}_{\tau\tau} \end{bmatrix}. \tag{9}$$

Similarly, $N_{\rm SOL} \equiv (\mathbf{1} - \alpha^{\rm SOL}) U_{\rm SOL}$, with

$$\alpha^{\rm SOL} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-i\delta} & 0 \\ 0 & 0 & e^{-i\delta} \end{bmatrix} \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\delta} & 0 \\ 0 & 0 & e^{i\delta} \end{bmatrix}$$
$$= \begin{bmatrix} \alpha_{ee} & 0 & 0 \\ e^{-i\delta}\alpha_{\mu e} & \alpha_{\mu\mu} & 0 \\ e^{-i\delta}\alpha_{\tau e} & \alpha_{\tau\mu} & \alpha_{\tau\tau} \end{bmatrix} \equiv \begin{bmatrix} \widetilde{\alpha}_{ee} & 0 & 0 \\ \widetilde{\alpha}_{\mu e} & \widetilde{\alpha}_{\mu\mu} & 0 \\ \widetilde{\alpha}_{\tau e} & \widetilde{\alpha}_{\tau\mu} & \widetilde{\alpha}_{\tau\tau} \end{bmatrix}, \tag{10}$$

where we have introduced the simplified notation $\alpha^{\rm SOL}_{\beta\gamma} \equiv \widetilde{\alpha}_{\beta\gamma}$ for later convenience.

Therefore, if we talk about the chiral-type correlation between the CP phases in the atmospheric resonance region, it takes three different forms depending upon the U matrix conventions: $[e^{-i\delta}\alpha_{\mu e},\ e^{-i\delta}\alpha_{\tau e},\ \alpha_{\tau\mu}]$ in the PDG convention, $[e^{-i\delta}\alpha^{\rm ATM}_{\mu e},\ \alpha^{\rm ATM}_{\tau e},\ e^{i\delta}\alpha^{\rm ATM}_{\tau\mu}]$ in the ATM convention, and $[\widetilde{\alpha}_{\mu e},\ \widetilde{\alpha}_{\tau e},\ \widetilde{\alpha}_{\tau\mu}]$ in the SOL convention. That is, the phase correlation disappears in the SOL convention. This happens by accident as the convention-dependent change in the α parameters just absorbs the physical phase correlation that existed in the PDG and ATM conventions [24]. Therefore, to our understanding, the phase correlation generically exists and represents the unique physical characteristics of the νSM and UV phases. No phase convention of the U matrix exists which eliminates the phase correlations everywhere in the whole kinematical phase space, as discussed in depth in Ref. [33].

3.2 *We use the SOL convention*

The vanishing $e^{\pm i\delta}$–UV phase correlation in the atmospheric resonance region in the SOL convention suggests that the phase correlation takes the simplest form in wider kinematical phase space in this convention. Therefore, we use the SOL convention for our investigation of the CP phase correlation in the DMP-UV perturbation theory with the notation $\alpha^{\rm SOL} \equiv \widetilde{\alpha}$ for simplicity, as defined in Eq. (10). One must know that the oscillation probability computed using the PDG, ATM, and SOL conventions is exactly identical, because neutrino-state phase redefinition cannot alter physical observables. To transform the probability $P(\nu_\beta \to \nu_\alpha)$ using the SOL convention to the one in the PDG or ATM conventions, however, one must transform the $\widetilde{\alpha}_{\beta\gamma}$ parameters to $\alpha_{\beta\gamma}$ defined in Eq. (1), or $\alpha^{\rm ATM}_{\beta\gamma}$, respectively, following the translation rules in Eqs. (10) and (9).

The SOL convention is used in Ref. [33] to demonstrate physical nature of the CP phase correlations. It is also used in a much more crucial way in Ref. [70] for the transparent formulation of "Symmetry Finder," a powerful tool for symmetry hunting in neutrino oscillations in matter.

4. Computation of the probability $P(\nu_\mu \to \nu_e)$

To focus on the physics discussions of the features of νSM–UV phase correlations in the main text, we postpone our compact formulation of the DMP-UV perturbation theory to Appendix A. In this paper, for simplicity and clarity, we compute the probability to first order in the DMP-UV expansion, and will work with the uniform matter density approximation.

4.1 *Preface to presenting the first-order UV corrections and notations*

The DMP-UV perturbation theory has two kinds of expansion parameters, ϵ and the UV $\widetilde{\alpha}$ parameters. ϵ in the νSM part is defined as

$$\epsilon \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{\rm ren}}, \qquad \Delta m^2_{\rm ren} \equiv \Delta m^2_{31} - s^2_{12}\Delta m^2_{21}, \tag{11}$$

where $\Delta m^2_{\rm ren}$ is the "renormalized" atmospheric Δm^2 used in Ref. [57].[6] In DMP the vacuum mixing angles are elevated to the matter-dressed one, $\theta_{12} \to \psi$ and $\theta_{13} \to \phi$, but θ_{23} and δ are as they are in vacuum [45]. In the expressions of the oscillation probability, we use various

[6] The same quantity is known as the effective $\Delta m^2_{\rm ee}$ in the $\nu_e \to \nu_e$ channel in vacuum [71]. While we prefer to use $\Delta m^2_{\rm ren}$ in the context of the present paper, the question of which symbol should be appropriate here is under debate [57].

simplified notations as follows:

$$h_i \equiv \frac{\lambda_i}{2E}, \qquad \Delta_{\text{ren}} \equiv \frac{\Delta m^2_{\text{ren}}}{2E}, \qquad \Delta_a \equiv \frac{a}{2E}, \qquad \Delta_b \equiv \frac{b}{2E}, \tag{12}$$

where h_i ($i = 1, 2, 3$) denote the eigenvalues of the unperturbed Hamiltonian, the diagonal entries of $\bar{H}_{\nu\text{SM}}$ in Eqs. (A15) or (A21), and a and b are the Wolfenstein matter potentials defined in Eq. (4). We also use $J_{mr} \equiv c_{23}s_{23}c^2_{\phi}s_{\phi}c_{\psi}s_{\psi}$ as the Jarlskog factor [72] in matter.

4.2 *Structure of the S matrix and the probability in DMP-UV perturbation theory*

We decompose the flavor-basis S matrix into the νSM part, $S_{\nu\text{SM}} = S^{(0)}_{\nu\text{SM}} + S^{(1)}_{\nu\text{SM}}$, and the part dependent on the UV α parameter. The latter terms are all first order by definition and consist of the two terms $S^{(1)}_{\text{EV}}$ and $S^{(1)}_{\text{UV}}$. The subscripts "EV" and "UV" respectively imply the α parameter dependent but unitary evolution part and the genuine non-unitary contribution. The latter comes from non-unitary projections at the production and detection points, and the non-unitarity arises solely from it [24]. One can show that

$$S^{(1)}_{\text{UV}} = -\widetilde{\alpha} S^{(0)}_{\nu\text{SM}} - S^{(0)}_{\nu\text{SM}} \widetilde{\alpha}^{\dagger}, \tag{13}$$

see Eq. (A36), Therefore, the flavor basis S matrix can be written to first order as

$$S_{\text{flavor}} = S^{(0)}_{\nu\text{SM}} + S^{(1)}_{\nu\text{SM}} + S^{(1)}_{\text{EV}} - \widetilde{\alpha} S^{(0)}_{\nu\text{SM}} - S^{(0)}_{\nu\text{SM}} \widetilde{\alpha}^{\dagger}. \tag{14}$$

We are now ready to calculate the expressions of the oscillation probability to first order in the expansion parameters. Following Ref. [24], we categorize $P(\nu_\beta \rightarrow \nu_\alpha)$ into the three types of terms:

$$P(\nu_\beta \rightarrow \nu_\alpha) = P(\nu_\beta \rightarrow \nu_\alpha)^{(0+1)}_{\nu\text{SM}} + P(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{EV}} + P(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{UV}}, \tag{15}$$

where

$$\begin{aligned} P(\nu_\beta \rightarrow \nu_\alpha)^{(0+1)}_{\nu\text{SM}} &= \left| \left(S^{(0)}_{\nu\text{SM}} \right)_{\alpha\beta} \right|^2 + 2\text{Re}\left[\left(S^{(0)}_{\nu\text{SM}} \right)^*_{\alpha\beta} \left(S^{(1)}_{\nu\text{SM}} \right)_{\alpha\beta} \right], \\ P(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{EV}} &= 2\text{Re}\left[\left(S^{(0)}_{\nu\text{SM}} \right)^*_{\alpha\beta} \left(S^{(1)}_{\text{EV}} \right)_{\alpha\beta} \right], \\ P(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{UV}} &= -2\text{Re}\left[\left(S^{(0)}_{\nu\text{SM}} \right)^*_{\alpha\beta} \left(\widetilde{\alpha} S^{(0)}_{\nu\text{SM}} + S^{(0)}_{\nu\text{SM}} \widetilde{\alpha}^{\dagger} \right)_{\alpha\beta} \right]. \end{aligned} \tag{16}$$

At the end of Appendix A we obtain the zeroth- and first-order expressions of the flavor-basis S matrix. The νSM part of the oscillation probability $P(\nu_\beta \rightarrow \nu_\alpha)^{(0+1)}_{\nu\text{SM}}$ to first order in ϵ is fully calculated in Ref. [45], and therefore we do not repeat it here. The accuracy of the first-order formulas $P(\nu_\beta \rightarrow \nu_\alpha)^{(0+1)}_{\nu\text{SM}}$ is verified in Ref. [62]. For the explicit expressions of the νSM part of the probabilities, we refer to the arXiv v3 version of Ref. [66] for the formulas in all the relevant channels. Since our interest in this paper is in the νSM–UV phase correlations, we focus on $P(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{EV}}$ and $P(\nu_\beta \rightarrow \nu_\alpha)^{(1)}_{\text{UV}}$ hereafter.[7]

In this paper we only compute the probability in the $\nu_\mu \rightarrow \nu_e$ channel for relative simplicity. In fact, all the $\widetilde{\alpha}$ parameters show up in $P(\nu_\mu \rightarrow \nu_e)^{(1)}$. From our experience in Refs. [24,33], we

[7]One may ask why we do not include the second-order UV effect. We assume, as the objective of our investigation, "small but robust" evidence for departure from the νSM as mentioned in the introduction. In diagnosing non-unitarity via $S^*_{\nu\text{SM}}S_{\text{UV}}$, the leading-order effect is from the first-order terms of S_{UV}. If we assume a $\sim$1% level non-unitarity, the second-order corrections must be negligible. If the UV effect turns out to be at $\sim$10% level it would be a suggestion for revising the zeroth-order paradigm.

are confident that this channel is sufficient to extract the qualitative features of the νSM–UV phase correlations; more comments on this follow in Sect. 6. To back this statement up, we provide the $\widetilde{\alpha}$ parameter dependent part of the flavor basis S matrix in the $\nu_\mu \to \nu_\tau$ channel in Appendix D, which indeed supports the above statement. Of course, if any demand exists, it is straightforward to compute the probabilities $P(\nu_\mu \to \nu_\mu)$ or $P(\nu_\mu \to \nu_\tau)$ using the formulas we provide in this paper.

The probability expressions contain the K matrix elements K_{ij} (for the definition, see Eq. (A11)), which can be written using the $\widetilde{\alpha}$ parameters as

$$
\begin{aligned}
K_{11} &= 2c_\phi^2 \widetilde{\alpha}_{ee} \left(1 - \frac{\Delta_a}{\Delta_b}\right) + 2s_\phi^2 \left[s_{23}^2 \widetilde{\alpha}_{\mu\mu} + c_{23}^2 \widetilde{\alpha}_{\tau\tau} + c_{23}s_{23}\text{Re}\left(\widetilde{\alpha}_{\tau\mu}\right)\right] \\
&\quad - 2c_\phi s_\phi \text{Re}\left(s_{23}\widetilde{\alpha}_{\mu e} + c_{23}\widetilde{\alpha}_{\tau e}\right), \\
K_{12} &= c_\phi \left(c_{23}\widetilde{\alpha}_{\mu e}^* - s_{23}\widetilde{\alpha}_{\tau e}^*\right) - s_\phi \left[2c_{23}s_{23}(\widetilde{\alpha}_{\mu\mu} - \widetilde{\alpha}_{\tau\tau}) + c_{23}^2\widetilde{\alpha}_{\tau\mu} - s_{23}^2 \widetilde{\alpha}_{\tau\mu}^*\right] \\
&= (K_{21})^*, \\
K_{13} &= 2c_\phi s_\phi \left[\widetilde{\alpha}_{ee} \left(1 - \frac{\Delta_a}{\Delta_b}\right) - \left(s_{23}^2 \widetilde{\alpha}_{\mu\mu} + c_{23}^2 \widetilde{\alpha}_{\tau\tau}\right)\right] \\
&\quad + c_\phi^2 \left(s_{23}\widetilde{\alpha}_{\mu e}^* + c_{23}\widetilde{\alpha}_{\tau e}^*\right) - s_\phi^2 \left(s_{23}\widetilde{\alpha}_{\mu e} + c_{23}\widetilde{\alpha}_{\tau e}\right) - 2c_{23}s_{23}c_\phi s_\phi \text{Re}\left(\widetilde{\alpha}_{\tau\mu}\right) \\
&= (K_{31})^*, \\
K_{22} &= 2\left[c_{23}^2\widetilde{\alpha}_{\mu\mu} + s_{23}^2\widetilde{\alpha}_{\tau\tau} - c_{23}s_{23}\text{Re}\left(\widetilde{\alpha}_{\tau\mu}\right)\right], \\
K_{23} &= s_\phi \left(c_{23}\widetilde{\alpha}_{\mu e} - s_{23}\widetilde{\alpha}_{\tau e}\right) + c_\phi \left[2c_{23}s_{23}(\widetilde{\alpha}_{\mu\mu} - \widetilde{\alpha}_{\tau\tau}) + c_{23}^2\widetilde{\alpha}_{\tau\mu}^* - s_{23}^2 \widetilde{\alpha}_{\tau\mu}\right] \\
&= (K_{32})^*, \\
K_{33} &= 2s_\phi^2 \widetilde{\alpha}_{ee} \left(1 - \frac{\Delta_a}{\Delta_b}\right) + 2c_\phi^2 \left[s_{23}^2 \widetilde{\alpha}_{\mu\mu} + c_{23}^2 \widetilde{\alpha}_{\tau\tau} + c_{23}s_{23}\text{Re}\left(\widetilde{\alpha}_{\tau\mu}\right)\right] \\
&\quad + 2c_\phi s_\phi \text{Re}\left(s_{23}\widetilde{\alpha}_{\mu e} + c_{23}\widetilde{\alpha}_{\tau e}\right).
\end{aligned} \tag{17}
$$

4.3 *The probability $P(\nu_\mu \to \nu_e)$: Unitary evolution part*

For convenience, we decompose the unitary evolution part of the probability into the three pieces,

$$
P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}} = P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}\big|_{(\Delta_b x)} + P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}\big|_{K_{11,12,22}} + P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}\big|_{K_{13,23}}, \tag{18}
$$

depending upon the form of the kinematical factors $(\Delta_b x)$ or $\frac{\Delta_b}{h_2 - h_1}$ etc., and on the K matrix elements involved. Here, we note that x denotes the detection point of neutrinos measured from the production point $x = 0$, so that x implies the baseline.

The first term in Eq. (18) is given by

$$
\begin{aligned}
& P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}\big|_{(\Delta_b x)} \\
&= 2(\Delta_b x)\Big[\left\{ \cos 2\psi \,(K_{22} - K_{11}) + \sin 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right\} \\
&\times \left\{ \left[c^2_\phi c^2_\psi s^2_\psi \left(c^2_{23} - s^2_{23} s^2_\phi \right) + J_{mr} \cos\delta \cos 2\psi \right] \sin(h_2 - h_1)x + 2J_{mr} \sin\delta \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \\
&- \left\{ \left(s^2_\psi K_{11} + c^2_\psi K_{22} - K_{33} \right) + c_\psi s_\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right\} \\
&\times \left\{ \left[s^2_{23} c^2_\phi s^2_\phi s^2_\psi - J_{mr} \cos\delta \right] \sin(h_3 - h_2)x + 2J_{mr} \sin\delta \sin^2 \frac{(h_3 - h_2)x}{2} \right\} \\
&- \left\{ \left(c^2_\psi K_{11} + s^2_\psi K_{22} - K_{33} \right) - c_\psi s_\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right\} \\
&\times \left\{ \left[s^2_{23} c^2_\phi s^2_\phi c^2_\psi + J_{mr} \cos\delta \right] \sin(h_3 - h_1)x - 2J_{mr} \sin\delta \sin^2 \frac{(h_3 - h_1)x}{2} \right\} \Big].
\end{aligned} \tag{19}
$$

We should note here that $e^{i\delta} K_{21} + e^{-i\delta} K_{12}$ is a real number. The second term in Eq. (18) reads

$$
\begin{aligned}
& P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}\big|_{K_{11,12,22}} \\
&= 2\Big[\sin 2\theta_{23} c^2_\phi s_\phi \left\{ \cos\delta \,\text{Re}\left(e^{-i\delta} K_{12} \right) - \sin\delta \,\text{Im}\left(e^{-i\delta} K_{12} \right) \right\} \\
&- s^2_{23} c^2_\phi s^2_\phi \sin 2\psi \left\{ \sin 2\psi \,(K_{11} - K_{22}) + \cos 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right\} \\
&+ 2J_{mr} \cos\delta \left\{ \cos 2\psi \,(K_{11} - K_{22}) - \sin 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right\} \Big] \\
&\times \frac{\Delta_b}{h_2 - h_1} \left\{ -\sin^2 \frac{(h_3 - h_2)x}{2} + \sin^2 \frac{(h_3 - h_1)x}{2} \right\} \\
&+ 2\Big[2c_{23} \Big\{ c^2_\phi \left(c_{23} \sin 2\psi + s_{23} s_\phi \cos 2\psi \cos\delta \right) \text{Re}\left(e^{-i\delta} K_{12} \right) \\
&- s_{23} c^2_\phi s_\phi \cos 2\psi \sin\delta \,\text{Im}\left(e^{-i\delta} K_{12} \right) \Big\} \\
&+ c^2_{23} c^2_\phi \sin^2 2\psi \left\{ \cos 2\psi \,(K_{11} - K_{22}) - \sin 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right\} \\
&- s^2_{23} c^2_\phi s^2_\phi \cos 2\psi \sin 2\psi \left\{ \sin 2\psi \,(K_{11} - K_{22}) + \cos 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right\} \\
&+ 2J_{mr} \cos\delta \left\{ \cos 4\psi \,(K_{11} - K_{22}) - \sin 4\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right\} \Big] \frac{\Delta_b}{h_2 - h_1} \sin^2 \frac{(h_2 - h_1)x}{2} \\
&- 4\Big[\sin 2\theta_{23} c^2_\phi s_\phi \left\{ \sin\delta \,\text{Re}\left(e^{-i\delta} K_{12} \right) + \cos\delta \,\text{Im}\left(e^{-i\delta} K_{12} \right) \right\} \\
&+ 2J_{mr} \sin\delta \left\{ \cos 2\psi \,(K_{11} - K_{22}) - \sin 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right\} \Big] \\
&\times \frac{\Delta_b}{h_2 - h_1} \sin \frac{(h_3 - h_1)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_3 - h_2)x}{2}.
\end{aligned} \tag{20}
$$

In the last line we used the trigonometric identity

$$
\begin{aligned}
&[\sin(h_3 - h_2)x - \sin(h_3 - h_1)x + \sin(h_2 - h_1)x] \\
&= 4 \sin \frac{(h_3 - h_1)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_3 - h_2)x}{2}.
\end{aligned}
$$

Here, we need to make a remark about how the δ dependencies are organized in the above formula and its relevance for our later discussion. We are talking about the first line in Eq. (20), the term inside $\{ \cdots \}$, which originates from $K_{12} = e^{i\delta}(e^{-i\delta} K_{12})$, through which an artificial δ dependence may have been created. Later we will encounter the same situation for $K_{32} = e^{i\delta}(e^{-i\delta} K_{32})$. The K_{12} term exists in a term in $(S^{(0)}_{e\mu})^* S^{(1)}_{e\mu}$,

$$
\begin{aligned}
&\left\{ c_{23} c_\phi c_\psi s_\psi e^{-i\delta} \left(e^{i h_2 x} - e^{i h_1 x} \right) + s_{23} c_\phi s_\phi \left[e^{i h_3 x} - \left(c^2_\psi e^{i h_1 x} + s^2_\psi e^{i h_2 x} \right) \right] \right\} \\
&\times c_{23} c_\phi K_{12} \frac{\Delta_b}{h_2 - h_1} \left\{ e^{-i h_2 x} - e^{-i h_1 x} \right\} \\
&= 4 c^2_{23} c^2_\phi c_\psi s_\psi \left(e^{-i\delta} K_{12} \right) \frac{\Delta_b}{h_2 - h_1} \sin^2 \frac{(h_2 - h_1)x}{2} \\
&+ c_{23} s_{23} c^2_\phi s_\phi e^{i\delta} \left(e^{-i\delta} K_{12} \right) \frac{\Delta_b}{h_2 - h_1} \left[2 \left\{ - \sin^2 \frac{(h_3 - h_2)x}{2} \right. \right. \\
&+ \left. \sin^2 \frac{(h_3 - h_1)x}{2} + \cos 2\psi \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \\
&+ \left. i [\sin(h_3 - h_2)x - \sin(h_3 - h_1)x + \sin(h_2 - h_1)x] \right]. \qquad (21)
\end{aligned}
$$

The last two lines of Eq. (21) lead to several terms in Eq. (20) with explicit $\cos\delta$ and $\sin\delta$ dependencies: in the first and fifth lines, and in the third line from the bottom. Though it may look artificial, we believe that this organization is natural. If one goes through the results of $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$ in Eq. (20), the $(e^{i\delta} K_{21} + e^{-i\delta} K_{12}) = 2\text{Re}(e^{-i\delta} K_{12})$ structure is everywhere, and therefore the δ–UV correlation of the type $(e^{-i\delta} K_{12})$ is unmistakable. Notice that since δ lives in the νSM unperturbed part of the DMP perturbation theory, as can be seen in Eq. (A39), δ can sneak into every part of the first-order corrections. However, we must note that in the limit for the atmospheric resonance region the term under discussion survives as one of the δ-independent terms, as will be seen in Sect. 5.

For bookkeeping purpose we divide the last (third) term in Eq. (18) into two pieces, one proportional to $\frac{\Delta_b}{h_3-h_1}$, and the other to $\frac{\Delta_b}{h_3-h_2}$:

$$
\begin{aligned}
& P(\nu_\mu \rightarrow \nu_e)^{(1)}_{\text{EV}}\Big|_{K_{13,23}}\Big|^{\frac{\Delta_b}{h_3-h_1}} \\
&= 4c_{23}c_\phi c_\psi s_\psi \Big[-c_{23}s_\phi s_\psi \,\text{Re}\left(c_\psi K_{31} - s_\psi e^{-i\delta} K_{32}\right) \\
&\quad + s_{23}\cos\delta \left\{ c^2_\psi \text{Re}\left(c^2_\phi K_{13} - s^2_\phi K_{31}\right) - c_\psi s_\psi \text{Re}\left(c^2_\phi e^{i\delta} K_{23} - s^2_\phi e^{-i\delta} K_{32}\right)\right\} \\
&\quad + s_{23}\sin\delta \left\{ c^2_\psi \text{Im}\left(c^2_\phi K_{13} - s^2_\phi K_{31}\right) - c_\psi s_\psi \text{Im}\left(c^2_\phi e^{i\delta} K_{23} - s^2_\phi e^{-i\delta} K_{32}\right)\right\}\Big] \\
&\quad \times \frac{\Delta_b}{h_3-h_1}\left\{ -\sin^2\frac{(h_3-h_2)x}{2} + \sin^2\frac{(h_3-h_1)x}{2} + \sin^2\frac{(h_2-h_1)x}{2}\right\} \\
&\quad + 4s_{23}c_\phi s_\phi \Big[-c_{23}s_\phi s_\psi \left\{\cos\delta \text{Re}\left(c_\psi K_{31} - s_\psi e^{-i\delta}K_{32}\right) - \sin\delta \text{Im}\left(c_\psi K_{31} - s_\psi e^{-i\delta}K_{32}\right)\right\} \\
&\quad + s_{23}\left\{ c^2_\psi \text{Re}\left(c^2_\phi K_{13} - s^2_\phi K_{31}\right) - c_\psi s_\psi \text{Re}\left(c^2_\phi e^{i\delta}K_{23} - s^2_\phi e^{-i\delta}K_{32}\right)\right\}\Big] \\
&\quad \times \frac{\Delta_b}{h_3-h_1}\left\{ (1+c^2_\psi)\sin^2\frac{(h_3-h_1)x}{2} + s^2_\psi \sin^2\frac{(h_3-h_2)x}{2} - s^2_\psi \sin^2\frac{(h_2-h_1)x}{2}\right\} \\
&\quad + 8\Big[c_{23}c_\phi s_\phi s^2_\psi \Big\{ s_{23}s_\phi s_\psi \sin\delta \text{Re}\left(c_\psi K_{31} - s_\psi e^{-i\delta}K_{32}\right) \\
&\quad + \left(-c_{23}c_\psi + s_{23}s_\phi s_\psi \cos\delta\right)\text{Im}\left(c_\psi K_{31} - s_\psi e^{-i\delta}K_{32}\right)\Big\} \\
&\quad - c_{23}s_{23}c_\phi c_\psi s_\psi \sin\delta \left\{ c^2_\psi \text{Re}\left(c^2_\phi K_{13} - s^2_\phi K_{31}\right) - c_\psi s_\psi \text{Re}\left(c^2_\phi e^{i\delta}K_{23} - s^2_\phi e^{-i\delta}K_{32}\right)\right\} \\
&\quad + s_{23}c_\phi \left(-s_{23}s_\phi s^2_\psi + c_{23}c_\psi s_\psi \cos\delta\right)\Big\{ c^2_\psi \text{Im}\left(c^2_\phi K_{13} - s^2_\phi K_{31}\right) \\
&\quad - c_\psi s_\psi \text{Im}\left(c^2_\phi e^{i\delta}K_{23} - s^2_\phi e^{-i\delta}K_{32}\right)\Big\}\Big] \\
&\quad \times \frac{\Delta_b}{h_3-h_1}\sin\frac{(h_3-h_1)x}{2}\sin\frac{(h_2-h_1)x}{2}\sin\frac{(h_3-h_2)x}{2},
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
& P(\nu_\mu \rightarrow \nu_e)^{(1)}_{\text{EV}}\Big|_{K_{13,23}}\Big|^{\frac{\Delta_b}{h_3-h_2}} \\
&= 4c_{23}c_\phi c_\psi s_\psi \Big[c_{23}s_\phi c_\psi \text{Re}\left(s_\psi K_{31} + c_\psi e^{-i\delta}K_{32}\right) \\
&\quad + s_{23}\cos\delta \left\{ s^2_\psi \text{Re}\left(c^2_\phi K_{13} - s^2_\phi K_{31}\right) + c_\psi s_\psi \text{Re}\left(c^2_\phi e^{i\delta}K_{23} - s^2_\phi e^{-i\delta}K_{32}\right)\right\} \\
&\quad + s_{23}\sin\delta \left\{ s^2_\psi \text{Im}\left(c^2_\phi K_{13} - s^2_\phi K_{31}\right) + c_\psi s_\psi \text{Im}\left(c^2_\phi e^{i\delta}K_{23} - s^2_\phi e^{-i\delta}K_{32}\right)\right\}\Big] \\
&\quad \times \frac{\Delta_b}{h_3-h_2}\left\{ -\sin^2\frac{(h_3-h_2)x}{2} + \sin^2\frac{(h_3-h_1)x}{2} - \sin^2\frac{(h_2-h_1)x}{2}\right\} \\
&\quad + 4s_{23}c_\phi s_\phi \Big[c_{23}s_\phi c_\psi \left\{\cos\delta \text{Re}\left(s_\psi K_{31} + c_\psi e^{-i\delta}K_{32}\right) - \sin\delta \text{Im}\left(s_\psi K_{31} + c_\psi e^{-i\delta}K_{32}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ s_{23} \left\{ s^2_\psi \text{Re} \left(c^2_\phi K_{13} - s^2_\phi K_{31} \right) + c_\psi s_\psi \text{Re} \left(c^2_\phi e^{i\delta} K_{23} - s^2_\phi e^{-i\delta} K_{32} \right) \right\} \Bigg] \\
&\times \frac{\Delta_b}{h_3 - h_2} \left\{ (1 + s^2_\psi) \sin^2 \frac{(h_3 - h_2)x}{2} + c^2_\psi \sin^2 \frac{(h_3 - h_1)x}{2} - c^2_\psi \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \\
&+ 8 \Bigg[c_{23} c_\phi s_\phi c^2_\psi \Bigg\{ s_{23} s_\phi c_\psi \sin\delta \text{Re} \left(s_\psi K_{31} + c_\psi e^{-i\delta} K_{32} \right) \\
&+ \left(c_{23} s_\psi + s_{23} s_\phi c_\psi \cos\delta \right) \text{Im} \left(s_\psi K_{31} + c_\psi e^{-i\delta} K_{32} \right) \Bigg\} \\
&- c_{23} s_{23} c_\phi c_\psi s_\psi \sin\delta \left\{ s^2_\psi \text{Re} \left(c^2_\phi K_{13} - s^2_\phi K_{31} \right) + c_\psi s_\psi \text{Re} \left(c^2_\phi e^{i\delta} K_{23} - s^2_\phi e^{-i\delta} K_{32} \right) \right\} \\
&+ s_{23} c_\phi \left(s_{23} s_\phi c^2_\psi + c_{23} c_\psi s_\psi \cos\delta \right) \Bigg\{ s^2_\psi \text{Im} \left(c^2_\phi K_{13} - s^2_\phi K_{31} \right) \\
&+ c_\psi s_\psi \text{Im} \left(c^2_\phi e^{i\delta} K_{23} - s^2_\phi e^{-i\delta} K_{32} \right) \Bigg\} \Bigg] \\
&\times \frac{\Delta_b}{h_3 - h_2} \sin \frac{(h_3 - h_1)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_3 - h_2)x}{2}. \qquad (23)
\end{aligned}
$$

In deriving Eqs. (22) and (23) we encountered the similar issue of using K_{32} or $e^{i\delta}(e^{-i\delta}K_{32})$, which is exactly parallel to the K_{12} vs. $e^{i\delta}(e^{-i\delta}K_{12})$ issue mentioned after Eq. (20). We prefer to make explicit the $(e^{-i\delta}K_{32})$ correlation for the same reason as for the $(e^{-i\delta}K_{12})$ case.

4.4 *The probability $P(\nu_\mu \to \nu_e)$: Non-unitary part*

The non-unitary part of the probability $P(\nu_\mu \to \nu_e)$ is defined in Eq. (16), and it takes the form

$$
\begin{aligned}
P(\nu_\mu \to \nu_e)^{(1)}_{\text{UV}} &= -2\text{Re} \left[\left(S^{(0)}_{e\mu} \right)^* \left\{ \left(\widetilde{\alpha}_{ee} + \widetilde{\alpha}_{\mu\mu} \right) S^{(0)}_{e\mu} + \widetilde{\alpha}^*_{\mu e} S^{(0)}_{ee} \right\} \right] \\
&= -2(\widetilde{\alpha}_{ee} + \widetilde{\alpha}_{\mu\mu}) |S^{(0)}_{e\mu}|^2 - 2\text{Re} \left[\widetilde{\alpha}_{\mu e} (S^{(0)}_{ee})^* S^{(0)}_{e\mu} \right]. \qquad (24)
\end{aligned}
$$

It can be calculated as

$$
\begin{aligned}
P(\nu_\mu \to \nu_e)^{(1)}_{\text{UV}} &= -2(\widetilde{\alpha}_{ee} + \widetilde{\alpha}_{\mu\mu}) \\
&\times \Bigg[s^2_{23} \sin^2 2\phi \left\{ c^2_\psi \sin^2 \frac{(h_3 - h_1)x}{2} + s^2_\psi \sin^2 \frac{(h_3 - h_2)x}{2} \right\} \\
&+ c^2_\phi \sin^2 2\psi \left(c^2_{23} - s^2_{23} s^2_\phi \right) \sin^2 \frac{(h_2 - h_1)x}{2} \\
&+ 4J_{mr} \cos\delta \left\{ \sin^2 \frac{(h_3 - h_1)x}{2} - \sin^2 \frac{(h_3 - h_2)x}{2} + \cos 2\psi \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \\
&- 8J_{mr} \sin\delta \sin \frac{(h_3 - h_1)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_3 - h_2)x}{2} \Bigg] \\
&+ 2c_{23} c_\phi \sin 2\psi \text{Re} \left(\widetilde{\alpha}_{\mu e} e^{i\delta} \right) \left[s^2_\phi \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} - \sin^2 \frac{(h_3 - h_1)x}{2} \right\} \right.
\end{aligned}
$$

$$
\begin{aligned}
&+ c_\phi^2 \cos 2\psi \sin^2 \frac{(h_2 - h_1)x}{2} \Bigg] \\
&+ c_{23} c_\phi \sin 2\psi \,\text{Im}\left(\widetilde{\alpha}_{\mu e} e^{i\delta}\right) \left\{ s_\phi^2 \left[\sin(h_3 - h_2)x - \sin(h_3 - h_1)x \right] - c_\phi^2 \sin(h_2 - h_1)x \right\} \\
&+ s_{23} \sin 2\phi \left\{ \cos\delta \,\text{Re}\left(\widetilde{\alpha}_{\mu e} e^{i\delta}\right) + \sin\delta \,\text{Im}\left(\widetilde{\alpha}_{\mu e} e^{i\delta}\right) \right\} \\
&\times \left[2\cos 2\phi \left\{ c_\psi^2 \sin^2 \frac{(h_3 - h_1)x}{2} + s_\psi^2 \sin^2 \frac{(h_3 - h_2)x}{2} \right\} - c_\phi^2 \sin^2 2\psi \sin^2 \frac{(h_2 - h_1)x}{2} \right] \\
&+ s_{23} \sin 2\phi \left\{ \sin\delta \,\text{Re}\left(\widetilde{\alpha}_{\mu e} e^{i\delta}\right) - \cos\delta \,\text{Im}\left(\widetilde{\alpha}_{\mu e} e^{i\delta}\right) \right\} \left[c_\psi^2 \sin(h_3 - h_1)x + s_\psi^2 \sin(h_3 - h_2)x \right].
\end{aligned}
\tag{25}
$$

In finishing up the computation of the probability in the $\nu_\mu \to \nu_e$ channel, we remark that the qualitative feature of the νSM–UV phase correlations in $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$ (see Eqs. (19), (20), (22), and (23)) is between the K_{ij} blobs of α parameters and $e^{\pm i\delta}$, which is akin to that observed at around the solar-scale enhanced oscillation [33], whereas the one in $P(\nu_\mu \to \nu_e)^{(1)}_{\text{UV}}$ in Eq. (25) is a "chiral type" but with the SOL-convention $\widetilde{\alpha}$ parameters. Though it may look like the one found in Ref. [24], that was under the PDG or ATM conventions of the U matrix.

We also remark, repeating what was said in Sect. 3.2, that if one wants to obtain the expressions for $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$ and $P(\nu_\mu \to \nu_e)^{(1)}_{\text{UV}}$ in the PDG (ATM) convention, the replacement of the $\widetilde{\alpha}_{\beta\gamma}$ parameters by $\alpha_{\beta\gamma}$ ($\alpha^{\text{ATM}}_{\beta\gamma}$) through the translation rule in Eq. (10) (Eq. (9)) suffices.

5. Taking the limit for the atmospheric resonance region

The careful reader must be anxious about the apparent contradiction between our expressions for $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$ given in Eqs. (19), (20), (22), and (23), and the results obtained in our previous paper, Ref. [24], briefly summarized in Sect. 2. In Ref. [24], we claimed that no correlation exists between $e^{\pm i\delta}$ and the complex UV $\widetilde{\alpha}$ parameters in the SOL convention, which is in apparent contradiction to the feature we have seen in Sect. 4. We should be able to resolve this puzzle, but a miracle seems to be needed.

To clear this point up, we examine the limit of the DMP-UV perturbation theory for the region of the atmospheric-scale enhanced oscillations. In fact, such a limit has already been studied [66] for the νSM DMP perturbation theory, and the analysis should apply to our case as the UV effect is treated as perturbation. The suitable limit to approach for the atmospheric-resonance perturbation theory is to take the operational limit $\epsilon \ll 1$, keeping θ_{13} and ϕ finite, and

$$
r_a \equiv \frac{a}{\Delta m^2_{\text{ren}}} \simeq \frac{\Delta_a}{\Delta_{31}} \left(1 + s_{12}^2 \frac{\Delta_{21}}{\Delta_{31}} \right) \sim \mathcal{O}(1). \tag{26}
$$

For convenience, we call this limit the "ATM limit."[8] The key parameter which describes the theory under the ATM limit is the mixing angle ψ, the matter-dressed θ_{12}. It behaves under the

[8] The nature of the ATM limit is "operational" in the sense that $\epsilon \equiv \Delta m^2_{21}/\Delta m^2_{\text{ren}}$ as defined in Eq. (11) is the parameter fixed by nature, and not to vary. But, apparently such a limit is necessary to turn the whole DMP theory [45] into the atmospheric-resonance perturbation theory [57], whose region of validity is restricted to the one around the enhanced atmospheric-scale oscillations. For example, the high-energy limit $\rho E/\Delta m^2_{\text{ren}} \gg 1$, where ρ is the matter density, inevitably sends the angle ϕ to the asymptotic region, $\sin 2\phi \ll 1$, and hence it cannot be the correct limit for the atmospheric-resonance perturbation theory.

limit as

$$\sin 2\psi \simeq \frac{\pm 2\epsilon \sin 2\theta_{12} c_{(\phi-\theta_{13})}}{\left[1 + r_a - \sqrt{1 + r_a^2 - 2r_a \cos 2\theta_{13}}\right]} + \mathcal{O}(\epsilon^2), \qquad \cos 2\psi \simeq \mp 1 + \mathcal{O}(\epsilon^2), \tag{27}$$

where the upper sign is for the normal mass ordering (NMO), and the lower for the inverted mass ordering (IMO); $c_{(\phi-\theta_{13})} \equiv \cos(\phi - \theta_{13})$.

The UV amplitude $S_{\rm UV}$ is already first order in the UV parameter, and therefore the $\sin 2\psi$ terms in it are of order ϵ^2, which means that they can be ignored. Therefore, in taking the ATM limit we can set $\sin 2\psi = 0$ (which also implies $J_{mr} = 0$) and $\cos 2\psi = \mp 1$. This means that $c_\psi = 0$ and $s_\psi = 1$ for the NMO, and $c_\psi = 1$ and $s_\psi = 0$ for the IMO. We discuss below the ATM limit for the IMO case.[9]

Let us start from the non-unitary part of the probability $P(\nu_\mu \to \nu_e)^{(1)}_{\rm UV}$, which has a much simpler expression than $P(\nu_\mu \to \nu_e)^{(1)}_{\rm EV}$. The $e^{\pm i\delta}$–UV $\widetilde{\alpha}$ parameter correlation originates from the second term of Eq. (24), $-2{\rm Re}\left[\widetilde{\alpha}_{\mu e}(S^{(0)}_{ee})^* S^{(0)}_{e\mu}\right]$. In the ATM limit the relevant part tends to

$$\begin{aligned}
&\widetilde{\alpha}_{\mu e}(S^{(0)}_{ee})^* S^{(0)}_{e\mu} \\
&= \widetilde{\alpha}_{\mu e}\Bigg[c_{23}c_\phi \sin 2\psi e^{i\delta}\left[s^2_\phi\left\{-\sin^2\frac{(h_3-h_2)x}{2} + \sin^2\frac{(h_3-h_1)x}{2}\right\} - c^2_\phi \cos 2\psi \sin^2\frac{(h_2-h_1)x}{2}\right] \\
&+ ic_{23}c_\phi c_\psi s_\psi e^{i\delta}\left\{s^2_\phi\left[\sin(h_3-h_2)x - \sin(h_3-h_1)x\right] - c^2_\phi \sin(h_2-h_1)x\right\} \\
&- s_{23}\sin 2\phi \cos 2\phi\left\{c^2_\psi \sin^2\frac{(h_3-h_1)x}{2} + s^2_\psi \sin^2\frac{(h_3-h_2)x}{2}\right\} \\
&+ s_{23}c^3_\phi s_\phi \sin^2 2\psi \sin^2\frac{(h_2-h_1)x}{2} - is_{23}c_\phi s_\phi\left[c^2_\psi \sin(h_3-h_1)x + s^2_\psi \sin(h_3-h_2)x\right]\Bigg] \\
&\to_{\rm ATM} \\
&= \widetilde{\alpha}_{\mu e}\left\{-s_{23}\sin 2\phi \cos 2\phi \sin^2\frac{(h_3-h_1)x}{2} - is_{23}c_\phi s_\phi \sin(h_3-h_1)x\right\}.
\end{aligned} \tag{28}$$

Therefore, the terms with the $e^{\pm i\delta}$–UV $\widetilde{\alpha}$ parameter correlation $e^{i\delta}\widetilde{\alpha}_{\mu e}$ vanish, and only the term with $\widetilde{\alpha}_{\mu e}$ remains without δ. In fact, under the ATM limit, $P(\nu_\mu \to \nu_e)^{(1)}_{\rm UV}$ becomes

$$\begin{aligned}
P(\nu_\mu \to \nu_e)^{(1)}_{\rm UV} &= 2s_{23}\sin 2\phi\left\{\cos 2\phi {\rm Re}\left(\widetilde{\alpha}_{\mu e}\right) - s_{23}\sin 2\phi(\widetilde{\alpha}_{ee} + \widetilde{\alpha}_{\mu\mu})\right\}\sin^2\frac{(h_3-h_1)x}{2} \\
&- s_{23}\sin 2\phi {\rm Im}\left(\widetilde{\alpha}_{\mu e}\right)\sin(h_3-h_1)x,
\end{aligned} \tag{29}$$

which reproduces Eq. (50) in Ref. [24] under the identifications (see Eqs. (10) and (9))

$$\widetilde{\alpha}_{\mu e} = e^{-i\delta}\alpha^{\rm ATM}_{\mu e}, \qquad \widetilde{\alpha}_{\tau e} = \alpha^{\rm ATM}_{\tau e}, \qquad \widetilde{\alpha}_{\tau\mu} = e^{i\delta}\alpha^{\rm ATM}_{\tau\mu}. \tag{30}$$

[9]The simplest interpretation of the probability formulas in Ref. [24] is that the atmospheric resonance level crossing is between the 1–3 states, which implies the IMO according to the state labeling in DMP. The formulas in Ref. [24] are valid for the NMO if we interpret the "λ_1–λ_3 crossing" as "λ_-–λ_+ crossing," as in the original reference [57]. In the DMP language it corresponds to the "λ_2–λ_3 crossing" in NMO. For more details see Ref. [66].

We now turn to the unitary evolution part $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$. In the ATM limit all the terms with $e^{-i\delta}K_{12}$ or $e^{i\delta}K_{23}$ type correlations disappear, and $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$ approaches the form

$$
\begin{aligned}
& P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}} \\
& = -2s^2_{23}c^2_\phi s^2_\phi (\Delta_b x)\,(K_{11} - K_{33}) \sin(h_3 - h_1)x \\
& \quad + 2\sin 2\theta_{23} c^2_\phi s_\phi \text{Re}\,(K_{12}) \frac{\Delta_b}{h_2 - h_1} \left\{ -\sin^2 \frac{(h_3 - h_2)x}{2} + \sin^2 \frac{(h_3 - h_1)x}{2} + \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \\
& \quad + 2\sin 2\theta_{23} c_\phi s^2_\phi \text{Re}\,(K_{32}) \frac{\Delta_b}{h_3 - h_2} \left\{ \sin^2 \frac{(h_3 - h_2)x}{2} + \sin^2 \frac{(h_3 - h_1)x}{2} - \sin^2 \frac{(h_2 - h_1)x}{2} \right\} \\
& \quad + 4s^2_{23} \sin 2\phi \cos 2\phi \text{Re}\,(K_{13}) \frac{\Delta_b}{h_3 - h_1} \sin^2 \frac{(h_3 - h_1)x}{2} \\
& \quad - 2\sin 2\theta_{23} \sin 2\phi c_\phi \text{Im}\,(K_{12}) \frac{\Delta_b}{h_2 - h_1} \sin \frac{(h_3 - h_1)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_3 - h_2)x}{2} \\
& \quad + 2\sin 2\theta_{23} \sin 2\phi s_\phi \text{Im}\,(K_{32}) \frac{\Delta_b}{h_3 - h_2} \sin \frac{(h_3 - h_1)x}{2} \sin \frac{(h_2 - h_1)x}{2} \sin \frac{(h_3 - h_2)x}{2}.
\end{aligned}
\tag{31}
$$

This is dramatically simplified, from a total of 41 lines to only 6 lines as in Eq. (31) under the ATM limit. Notice that the K_{ij} contain only the $\widetilde{\alpha}$ parameters but no δ, see Eq. (17). Therefore, the δ–$\widetilde{\alpha}$ correlation disappears in $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$ under the ATM limit. The expressions for $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$ with explicit usage of the α parameters are given in Eq. (C1) in Appendix C, which reproduces precisely $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$ in Ref. [24, Eq. (49)].

Thus, we were able to resolve the apparent puzzle. Our results for $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$ and $P(\nu_\mu \to \nu_e)^{(1)}_{\text{UV}}$ given in Sects. 4.3 and 4.4 are miraculously fully consistent with Ref. [24].

5.1 *Another miracle?*

The next question we must answer is how wide is the region in which the chiral-type correlation $[e^{-i\delta}\alpha_{\mu e},\ e^{-i\delta}\alpha_{\tau e},\ \alpha_{\tau\mu}]$ (PDG convention) exists. As we have learnt in the foregoing treatment in this section, taking the limit to the region $\sin\psi \simeq \epsilon$ (IMO) and $\sin\psi \simeq 1$ (NMO) in DMP guarantees the chiral-type correlation. In view of Ref. [45, Fig. 1], $\sin 2\psi$ is small, $\simeq 2\epsilon$ (ψ is close to $\pm\pi/2$ or 0), in the entire high-energy region $|Y_e\rho E| \gtrsim 3\,(\text{g/cm}^3)$ GeV. Since ψ is a monotonically increasing function of $Y_e\rho E$ there is no other region where $\sin 2\psi$ is small. Or, in other words, the (blobs of the α parameters)–$e^{\pm i\delta}$ correlation can exist only in the region $|Y_e\rho E| \lesssim 3\,(\text{g/cm}^3)$ GeV, where ψ undergoes a sharp change from the negative to positive regions of $Y_e\rho E$, i.e. the near-vacuum regime.

Now we must ask the final question: Is there any possibility that some other limiting procedures bring us to features of the νSM–UV CP phase correlations quite different from what we already know? We believe that such another miracle is very unlikely to occur. In pursuing such a possibility, we look for a special region of ϕ where it jumps or becomes small. But ϕ is also a monotonically increasing (decreasing) function of $Y_e\rho E$ for the NMO (IMO). Then, our target is practically the regions of small $\sin 2\phi \simeq 2\epsilon$. In Ref. [45, Fig. 1], such a region does exist in $|Y_e\rho E| \gtrsim 40\,(\text{g/cm}^3)$ GeV, but it is inside the small $\sin 2\psi$ region where the chiral-type correlation lives. That is, no new feature of phase correlation is expected. We note that this is the region where all the νSM oscillation modes die away due to the strong matter effect. Nothing interesting happens there for the νSM oscillations and hence the region does not appear to fit our purpose of diagnosing non-unitarity through interference. We note that our treatment is valid in the energy region $|\rho E| \lesssim 100\,(\text{g/cm}^3)$ GeV, as discussed in Sect. 2.2, which is not so far from the region where $\phi \simeq \epsilon$ starts.

Therefore, we believe that no qualitatively new feature of the νSM–UV CP phase correlations is expected beyond the two characteristic patterns that we have already seen. They are the chiral-type correlation in the atmospheric resonance region, and (blobs of the UV α parameters)–δ correlations anywhere else.

6. A completed picture of νSM–UV CP phase correlations

We are now able to draw a complete picture of the $e^{\pm i\delta}$–UV phase correlations in the whole region of the terrestrial neutrino experiments. There are two regions in which the characteristically different patterns of the correlations reside:

- $e^{\pm i\delta}\alpha_{\beta\gamma}$ chiral-type correlations in the region of atmospheric-scale enhanced oscillations, which extends to higher energies, $|\rho E| \gtrsim 6\,\text{g/cm}^3$ GeV.
- K_{ij} (blobs of the UV α parameters)–$e^{\pm i\delta}$ correlations everywhere else.

Let us summarize the features of the correlation in the latter region, as the former region is discussed in detail in Sect. 5. The characteristic features of νSM–UV phase correlations in $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$ can be extracted as the three combinations of the νSM phase $e^{\pm i\delta}$ factor and the K_{ij} blobs of the $\widetilde{\alpha}$ parameters, $e^{-i\delta}K_{12}$, K_{13} (no correlation with δ), and $e^{i\delta}K_{23}$. We recall that the K_{ij} blobs are free from δ and the explicit expressions of these correlated variables are, using Eq. (17):

$$
\begin{aligned}
e^{-i\delta}K_{12} &= e^{-i\delta}\left\{ c_\phi \left(c_{23}\widetilde{\alpha}^*_{\mu e} - s_{23}\widetilde{\alpha}^*_{\tau e} \right) - s_\phi \left[2c_{23}s_{23}(\widetilde{\alpha}_{\mu\mu} - \widetilde{\alpha}_{\tau\tau}) + c^2_{23}\widetilde{\alpha}_{\tau\mu} - s^2_{23}\widetilde{\alpha}^*_{\tau\mu} \right] \right\}, \\
K_{13} &= 2c_\phi s_\phi \left[\widetilde{\alpha}_{ee} \left(1 - \frac{\Delta_a}{\Delta_b} \right) - \left(s^2_{23}\widetilde{\alpha}_{\mu\mu} + c^2_{23}\widetilde{\alpha}_{\tau\tau} \right) \right] \\
&\quad + c^2_\phi \left(s_{23}\widetilde{\alpha}^*_{\mu e} + c_{23}\widetilde{\alpha}^*_{\tau e} \right) - s^2_\phi \left(s_{23}\widetilde{\alpha}_{\mu e} + c_{23}\widetilde{\alpha}_{\tau e} \right) - 2c_{23}s_{23}c_\phi s_\phi \text{Re}\left(\widetilde{\alpha}_{\tau\mu} \right), \\
e^{i\delta}K_{23} &= e^{i\delta}\left\{ s_\phi \left(c_{23}\widetilde{\alpha}_{\mu e} - s_{23}\widetilde{\alpha}_{\tau e} \right) + c_\phi \left[2c_{23}s_{23}(\widetilde{\alpha}_{\mu\mu} - \widetilde{\alpha}_{\tau\tau}) + c^2_{23}\widetilde{\alpha}^*_{\tau\mu} - s^2_{23}\widetilde{\alpha}_{\tau\mu} \right] \right\}.
\end{aligned}
\tag{32}
$$

Notice that the (blobs of the α parameters)–$e^{\pm i\delta}$ correlation in $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$ has been seen in the region of the solar-scale enhanced oscillation [33]; we have shown that its region of validity extends to a much wider region satisfying $|\rho E| \lesssim 6\,\text{g/cm}^3$ GeV.

The νSM–UV phase correlations in $P(\nu_\mu \to \nu_e)^{(1)}_{\text{UV}}$ involve only $\widetilde{\alpha}_{\mu e}$ by definition in Eq. (24). Then, the question is whether this feature is consistent with the above (blobs of the α parameters)–$e^{\pm i\delta}$ correlation. The answer is yes, in the sense that the same $\widetilde{\alpha}_{\mu e}e^{i\delta}$ correlation as in Eq. (25) is buried in the blob-type correlation in $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$, as seen in Eq. (32). Therefore, everything is consistent as far as the (blobs of the UV α parameters)–$e^{\pm i\delta}$ correlations are concerned.

In the alternative region $|\rho E| \gtrsim 6\,\text{g/cm}^3$ GeV there is no correlation between $e^{\pm i\delta}$ and the $\widetilde{\alpha}$ parameters in the SOL convention, or the chiral-type $[e^{-i\delta}\alpha_{\mu e}, e^{-i\delta}\alpha_{\tau e}, \alpha_{\tau\mu}]$ correlation lives in the PDG convention. This picture applies to both $P(\nu_\mu \to \nu_e)^{(1)}_{\text{EV}}$ and $P(\nu_\mu \to \nu_e)^{(1)}_{\text{UV}}$.

The fact that the above picture of the νSM–UV phase correlations comes only from the $\nu_\mu \to \nu_e$ channel may trigger an obvious question if the correlation features are the same in the other channels. In fact, there is ample supporting evidence for this in our previous exercises [24,33]. In addition to these comments it may be worthwhile to add a remark about the $\nu_\mu \to$

ν_τ channel. As the expression of $P(\nu_\mu \to \nu_\tau)$ is lengthy, we just write down the flavor basis S matrix of the $\nu_\mu \to \nu_\tau$ channel in Appendix D. From Eq. (D1), it is clear that the features of the νSM–UV phase correlations discussed above prevail in the $\nu_\mu \to \nu_\tau$ channel.

6.1 *Obtaining the $\widetilde{\alpha}$ parameters in region of the blob correlation*

Outside the atmospheric resonance region, we measure the K_{ij} parameters.[10] In this case, one can invert Eq. (17) for the α parameters as $A = U_{23} F U_{23}^\dagger = U_{23} U_{13}(\phi) K U_{13}^\dagger(\phi) U_{23}^\dagger$, where the A matrix is defined immediately below Eq. (A2). The explicit forms of the inverted expressions are given by

$$
\begin{aligned}
\widetilde{\alpha}_{ee} &= \frac{1}{2}\left(1 - \frac{\Delta_a}{\Delta_b}\right)^{-1} \left\{ c_\phi^2 K_{11} + s_\phi^2 K_{33} + c_\phi s_\phi \left(K_{31} + K_{13}\right) \right\}, \\
\widetilde{\alpha}_{\mu\mu} &= \frac{1}{2}\Big\{ c_{23}^2 K_{22} + s_{23}^2 \left[s_\phi^2 K_{11} + c_\phi^2 K_{33} - c_\phi s_\phi \left(K_{31} + K_{13}\right) \right] \\
&\quad - c_{23} s_{23} \left[s_\phi \left(K_{12} + K_{21}\right) - c_\phi \left(K_{23} + K_{32}\right) \right] \Big\}, \\
\widetilde{\alpha}_{\tau\tau} &= \frac{1}{2}\Big\{ s_{23}^2 K_{22} + c_{23}^2 \left[s_\phi^2 K_{11} + c_\phi^2 K_{33} - c_\phi s_\phi \left(K_{31} + K_{13}\right) \right] \\
&\quad + c_{23} s_{23} \left[s_\phi \left(K_{12} + K_{21}\right) - c_\phi \left(K_{23} + K_{32}\right) \right] \Big\}, \\
\widetilde{\alpha}_{\mu e} &= c_{23} \left(c_\phi K_{21} + s_\phi K_{23}\right) + s_{23} \left[c_\phi^2 K_{31} - s_\phi^2 K_{13} + c_\phi s_\phi \left(K_{33} - K_{11}\right) \right], \\
\widetilde{\alpha}_{\tau e} &= -s_{23} \left(c_\phi K_{21} + s_\phi K_{23}\right) + c_{23} \left[c_\phi^2 K_{31} - s_\phi^2 K_{13} + c_\phi s_\phi \left(K_{33} - K_{11}\right) \right], \\
\widetilde{\alpha}_{\tau\mu} &= s_\phi \left(s_{23}^2 K_{21} - c_{23}^2 K_{12} \right) + c_\phi \left(c_{23}^2 K_{32} - s_{23}^2 K_{23} \right) \\
&\quad + c_{23} s_{23} \left[s_\phi^2 K_{11} + c_\phi^2 K_{33} - K_{22} - c_\phi s_\phi \left(K_{31} + K_{13}\right) \right].
\end{aligned}
\tag{33}
$$

Therefore, by assuming measurement with sufficient precision we can determine all the $\widetilde{\alpha}_{\beta\gamma}$, in principle, in the whole region of the terrestrial experiments.

6.2 *Some remarks about measurement of the α parameters*

Probably the most salient feature of the probability $P(\nu_\mu \to \nu_e)^{(1)}_{\rm EV}$ computed in Sect. 4, as well as the flavor basis S matrix of the $\nu_\mu \to \nu_\tau$ channel in Appendix D, is that all the K_{ij}, and hence the $\widetilde{\alpha}_{\beta\gamma}$ $(\beta, \gamma = e, \mu, \tau)$ parameters with all possible flavor indices, come into the expressions. According to our experience [24,33], this feature holds in the other channels as well, and is even true in $P(\nu_e \to \nu_e)^{(1)}$. This means that a one-by-one strategy, measuring one parameter in one channel and another in another channel, does *not* work, which implies that we need extremely high-precision measurement of the probability to determine, or constrain, all the $\widetilde{\alpha}_{\beta\gamma}$ parameters simultaneously.

To determine practically nine degrees of freedom of the $\widetilde{\alpha}$ parameters at the same time, one must combine all the available channels by utilizing the LBL accelerator, atmospheric, reactor, and solar neutrino observations, hopefully with both neutrino and antineutrino channels if available. We are aware that these requirements are extremely demanding experimentally. In

[10] It may be interesting to note that the form of $[e^{-i\delta} K_{12}, K_{13}, e^{i\delta} K_{23}]$ correlation is somewhat reminiscent of the chiral correlation $[e^{-i\delta}\alpha_{\mu e}, \alpha_{\tau e}, e^{i\delta}\alpha_{\tau\mu}]$, despite the latter being in the ATM convention of the U matrix. This suggests an interesting picture that either the α parameters or the K parameters are the basic elements of the νSM–UV phase correlations, the α- and K-parameter duality, which exists in a way bridging between the different U-matrix conventions.

foreseeing such measurement, it may be worthwhile to examine the $\widetilde{\alpha}_{\beta\gamma}$ dependencies of all the probabilities $P(\nu_\beta \to \nu_\alpha)$ in all the channels. Such an attempt was partially made in Ref. [33], but it must be extended to all the channels.

7. Concluding remarks

After the summary in Sect. 6 of what we have learnt about the νSM–UV phase correlation in this work, only a few remarks are needed to conclude.

The most salient feature of the δ–UV phase correlation we have observed is that it simplifies at high energies, $|\rho E| \gtrsim 6\,\text{g/cm}^3$ GeV. The UV terms are relatively large in this region because it is proportional to the Wolfenstein matter potentials. But, it is highly nontrivial to see that the enhanced UV effect manifests itself in the behavior of the δ–UV phase correlation such a striking manner, altering the (blobs of the α parameters)–$e^{\pm i\delta}$ correlation to the much simpler "chiral-type" correlations $[e^{-i\delta}\alpha_{\mu e},\ e^{-i\delta}\alpha_{\tau e},\ \alpha_{\tau\mu}]$ (PDG convention) [24]. In the remaining region of $|\rho E|$, however, we have found that the (blobs of the α parameters)–$e^{\pm i\delta}$ correlation dominates. The globally valid feature may suggest that the K_{ij} parameters are more natural variables to describe the features of the UV at low energies. This feature, as well as the possible α- and K-parameter duality, leaves us with the question of whether the α or the K_{ij} parameters could appear naturally in some UV models.

An obviously promising strategy for exploring the characteristic features of the δ–UV phase correlation would be to sweep over the energy region $E = 0.1$–10 GeV, where the key variable ψ which controls the phase correlations has a dramatic change, as shown in Ref. [45, Fig. 1]. This would require super-precise measurement throughout the region covered by ESSnuSB [73], T2K-T2HK [74,75], NOvA-DUNE [76,77], and T2KK[11] [78]. If wide-energy-coverage atmospheric neutrino measurement has a promising feature for improving precision, Super- [46] and Hyper-Kamiokande [75], and DUNE [77], as well as IceCube [80] and KM3NeT [81], would become strong candidates. In the realm of natural K_{ij} variables, experimental search for UV may also be pursued in a low-energy LBL set up with the solar-scale enhancement. The possible physics potential in this region was explored in earlier studies [82–84] and revisited in more recent ones [47,58,73,85,86].

We started describing our interest in νSM–UV phase correlations by saying that ultimately we want to construct machinery to diagnose non-unitarity. A natural question would then be in what way knowledge of the νSM–UV phase correlations can help the diagnostic capability. From an experimentalist's view our work may be regarded as a piece for creating "theorists' analysis code" in preparation for the real measurement. By knowing the δ–UV phase correlations we would have a better view of the "migration matrix" which describes variable (and their error) correlations in the δ row. Since the "determination of all $\alpha_{\beta\gamma}$ at once" strategy is required for the α parameters, as discussed in Sect. 6.2, the migration matrix is large and therefore any knowledge of its structure should help.

Thus, even assuming that our discussions in this paper are in the right direction, our phenomenological study may be regarded as just a starting step from one particular side. It is likely that one should approach the nature of UV from various sides, for example by examining illuminating models of neutrino mass, e.g. in Refs. [87,88], and fully consistent models of sterile

[11] A possible acronym used in Ref. [79], but for now the updated name is "Tokai-to-Kamioka observatory-Korea neutrino observatory."

neutrinos, such as in Refs. [8,89], which may reveal what the α or the K parameters imply in more physical contexts.

Acknowledgements

This work has been started not only as a natural continuation of the works initiated and deepened in Refs. [24,33], but it also catapulted out of the revealing results of the phase correlation analysis done in the latter reference. The author thanks Ivan Martinez-Soler for these useful collaborations.

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Appendix A. Formulating the DMP perturbation theory with unitarity violation

We present the formulation of the DMP-UV perturbation theory in the SOL convention to make this paper self-contained. It has some overlaps with Refs. [24,45,57], but the full formulation is worth presenting because inclusion of the UV effect drastically changes the structure of the perturbation theory.

A1. Tilde-basis Hamiltonian

In line with our statement made in Sect. 3 we start from the Schrödinger equation in the vacuum mass eigenstate basis, Eq. (2). In the "tilde basis," which is related to the vacuum mass eigenstate basis $\check{\nu}$ as $\widetilde{\nu} = (U_{13}U_{12})\check{\nu}$, the Hamiltonian in Eq. (2) defined in the check basis is given using the SOL convention $\widetilde{\alpha}$ matrix as

$$
\begin{aligned}
\widetilde{H} &= (U_{13}U_{12})\check{H}(U_{13}U_{12})^{\dagger} \\
&= \frac{\Delta m^2_{\text{ren}}}{2E}\left\{\begin{bmatrix} s^2_{13} & 0 & c_{13}s_{13} \\ 0 & 0 & 0 \\ c_{13}s_{13} & 0 & c^2_{13} \end{bmatrix} + \epsilon \begin{bmatrix} s^2_{12} & 0 & 0 \\ 0 & c^2_{12} & 0 \\ 0 & 0 & s^2_{12} \end{bmatrix}\right. \\
&\quad \left. + \epsilon c_{12}s_{12}\begin{bmatrix} 0 & c_{13}e^{i\delta} & 0 \\ c_{13}e^{-i\delta} & 0 & -s_{13}e^{-i\delta} \\ 0 & -s_{13}e^{i\delta} & 0 \end{bmatrix}\right\} \\
&\quad + U^{\dagger}_{23}\left\{\mathbf{1} - \begin{bmatrix} \widetilde{\alpha}_{ee} & \widetilde{\alpha}^{*}_{\mu e} & \widetilde{\alpha}^{*}_{\tau e} \\ 0 & \widetilde{\alpha}_{\mu\mu} & \widetilde{\alpha}^{*}_{\tau\mu} \\ 0 & 0 & \widetilde{\alpha}_{\tau\tau} \end{bmatrix}\right\}\begin{bmatrix} \Delta_a - \Delta_b & 0 & 0 \\ 0 & -\Delta_b & 0 \\ 0 & 0 & -\Delta_b \end{bmatrix} \\
&\quad \times \left\{\mathbf{1} - \begin{bmatrix} \widetilde{\alpha}_{ee} & 0 & 0 \\ \widetilde{\alpha}_{\mu e} & \widetilde{\alpha}_{\mu\mu} & 0 \\ \widetilde{\alpha}_{\tau e} & \widetilde{\alpha}_{\tau\mu} & \widetilde{\alpha}_{\tau\tau} \end{bmatrix}\right\} U_{23} \\
&\equiv \widetilde{H}_{\text{vac}} + \widetilde{H}_{\text{matt}}.
\end{aligned}
\tag{A1}
$$

The DMP expansion parameter ϵ in Eq. (A1) is defined in Eq. (11), together with Δm^2_{ren}. Δ_a and Δ_b are defined in Eq. (12). In the DMP-UV perturbation theory we use ϵ and the six $\widetilde{\alpha}_{\beta\gamma}$ as the expansion parameters.

$\widetilde{H}_{\text{vac}}$ in Eq. (A1) can de decomposed into zeroth- and first-order terms. We regard the first two terms in $\widetilde{H}_{\text{vac}}$ as $\widetilde{H}^{(0)}_{\text{vac}}$ [57], and the third term as $\widetilde{H}^{(1)}_{\text{vac}}$. $\widetilde{H}_{\text{matt}}$ in Eq. (A1) can be written as

$\widetilde{H}_{\text{matt}} = \widetilde{H}^{(0)}_{\text{matt}} + \widetilde{H}^{(1)}_{\text{UV}} + \widetilde{H}^{(2)}_{\text{UV}}$, where

$$\widetilde{H}^{(0)}_{\text{matt}} = \begin{bmatrix} \Delta_a - \Delta_b & 0 & 0 \\ 0 & -\Delta_b & 0 \\ 0 & 0 & -\Delta_b \end{bmatrix},$$

$$\widetilde{H}^{(1)}_{\text{UV}} = \Delta_b U_{23}^{\dagger} \begin{bmatrix} 2\widetilde{\alpha}_{ee}\left(1 - \frac{\Delta_a}{\Delta_b}\right) & \widetilde{\alpha}^{*}_{\mu e} & \widetilde{\alpha}^{*}_{\tau e} \\ \widetilde{\alpha}_{\mu e} & 2\widetilde{\alpha}_{\mu\mu} & \widetilde{\alpha}^{*}_{\tau\mu} \\ \widetilde{\alpha}_{\tau e} & \widetilde{\alpha}_{\tau\mu} & 2\widetilde{\alpha}_{\tau\tau} \end{bmatrix} U_{23} \equiv \Delta_b F,$$

$$\widetilde{H}^{(2)}_{\text{UV}} = -\Delta_b U_{23}^{\dagger} \begin{bmatrix} \widetilde{\alpha}^{2}_{ee}\left(1 - \frac{\Delta_a}{\Delta_b}\right) + |\widetilde{\alpha}_{\mu e}|^2 + |\widetilde{\alpha}_{\tau e}|^2 & \widetilde{\alpha}^{*}_{\mu e}\widetilde{\alpha}_{\mu\mu} + \widetilde{\alpha}^{*}_{\tau e}\widetilde{\alpha}_{\tau\mu} & \widetilde{\alpha}^{*}_{\tau e}\widetilde{\alpha}_{\tau\tau} \\ \widetilde{\alpha}_{\mu e}\widetilde{\alpha}_{\mu\mu} + \widetilde{\alpha}_{\tau e}\widetilde{\alpha}^{*}_{\tau\mu} & \widetilde{\alpha}^{2}_{\mu\mu} + |\widetilde{\alpha}_{\tau\mu}|^2 & \widetilde{\alpha}^{*}_{\tau\mu}\widetilde{\alpha}_{\tau\tau} \\ \widetilde{\alpha}_{\tau e}\widetilde{\alpha}_{\tau\tau} & \widetilde{\alpha}_{\tau\mu}\widetilde{\alpha}_{\tau\tau} & \widetilde{\alpha}^{2}_{\tau\tau} \end{bmatrix} U_{23}. \quad \text{(A2)}$$

We have defined the F matrix in Eq. (A2). In the same line, we define the matrix removing the 2–3 rotation from F, just full of the α parameters as $A \equiv U_{23} F U_{23}^{\dagger}$. This is for convenience, for use in Sect. 6.2.

A2. Unperturbed and perturbed Hamiltonian in the tilde basis

To formulate the DMP-UV perturbation theory, we decompose the tilde-basis Hamiltonian into zeroth- and first-order terms as $\widetilde{H} = \widetilde{H}^{(0)} + \widetilde{H}^{(1)}$. The unperturbed (zeroth-order) Hamiltonian is given by $\widetilde{H}^{(0)} = \widetilde{H}^{(0)}_{\text{vac}} + \widetilde{H}^{(0)}_{\text{matt}}$. We make the phase redefinition

$$\widetilde{\nu} = \exp\left[i \int^{x} dx' \Delta_b(x') \right] \widetilde{\nu}', \quad \text{(A3)}$$

which is valid even for non-uniform matter density, to get rid of the NC potential term from $\widetilde{H}^{(0)}_{\text{matt}}$. Then, the unperturbed part of the Hamiltonian $(\widetilde{H}^{(0)})'$ is given by

$$(\widetilde{H}^{(0)})' = \Delta_{\text{ren}} \left\{ \begin{bmatrix} \frac{a(x)}{\Delta m^2_{\text{ren}}} + s^2_{13} & 0 & c_{13}s_{13} \\ 0 & 0 & 0 \\ c_{13}s_{13} & 0 & c^2_{13} \end{bmatrix} + \epsilon \begin{bmatrix} s^2_{12} & 0 & 0 \\ 0 & c^2_{12} & 0 \\ 0 & 0 & s^2_{12} \end{bmatrix} \right\}. \quad \text{(A4)}$$

Hereafter, we omit the prime symbol and use Eq. (A4) as the unperturbed part of the Hamiltonian. This is nothing but the zeroth-order Hamiltonian used in Ref. [57].

The perturbed Hamiltonian is then given by

$$\widetilde{H}^{(1)} = \widetilde{H}^{(1)}_{\text{vac}} + \widetilde{H}^{(1)}_{\text{UV}} + \widetilde{H}^{(2)}_{\text{UV}}, \quad \text{(A5)}$$

where $\widetilde{H}^{(1)}_{\text{vac}}$ is the third term in the first line in Eq. (A1), and $\widetilde{H}^{(1)}_{\text{UV}}$ and $\widetilde{H}^{(2)}_{\text{UV}}$ are defined in Eq. (A2). In the following computation we drop the second-order term (the last term) in Eq. (A5) because we confine ourselves to the zeroth- and first-order terms in the νSM and the UV parameters in this paper.

A3. $U_{13}(\phi)$ rotation to the hat basis

We diagonalize the 1–3 sector of $\widetilde{H}^{(0)}$ by doing the $U_{13}(\phi)$ rotation, where

$$U_{13}(\phi) = \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix}, \qquad U_{13}(\phi)^{\dagger} = \begin{bmatrix} c_\phi & 0 & -s_\phi \\ 0 & 1 & 0 \\ s_\phi & 0 & c_\phi \end{bmatrix}. \quad \text{(A6)}$$

After this rotation the neutrino basis becomes the hat basis,

$$|\hat{\nu}\rangle = U_{13}^{\dagger}(\phi)|\widetilde{\nu}\rangle = U_{13}^{\dagger}(\phi)U_{23}^{\dagger}(\theta_{23})|\nu\rangle, \quad \text{(A7)}$$

and the νSM part of the Hamiltonian is given in the SOL convention by

$$\hat{H}_{\nu\text{SM}} = U_{13}^{\dagger}(\phi)\, \widetilde{H}_{\nu\text{SM}}\, U_{13}(\phi)$$
$$= \frac{1}{2E}\begin{bmatrix} \lambda_- & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_+ \end{bmatrix}$$
$$+ \epsilon c_{12} s_{12} \Delta_{\text{ren}} \begin{bmatrix} 0 & \cos(\phi-\theta_{13})e^{i\delta} & 0 \\ \cos(\phi-\theta_{13})e^{-i\delta} & 0 & \sin(\phi-\theta_{13})e^{-i\delta} \\ 0 & \sin(\phi-\theta_{13})e^{i\delta} & 0 \end{bmatrix}. \quad \text{(A8)}$$

In Eq. (A8), the first term $\hat{H}^{(0)}$ is the unperturbed term with the eigenvalues

$$\lambda_- = \frac{1}{2}\left[\left(\Delta m^2_{\text{ren}} + a\right) - \text{sign}(\Delta m^2_{\text{ren}})\sqrt{\left(\Delta m^2_{\text{ren}} - a\right)^2 + 4 s^2_{13} a \Delta m^2_{\text{ren}}}\right] + \epsilon \Delta m^2_{\text{ren}} s^2_{12},$$
$$\lambda_0 = c^2_{12}\, \epsilon\, \Delta m^2_{\text{ren}},$$
$$\lambda_+ = \frac{1}{2}\left[\left(\Delta m^2_{\text{ren}} + a\right) + \text{sign}(\Delta m^2_{\text{ren}})\sqrt{\left(\Delta m^2_{\text{ren}} - a\right)^2 + 4 s^2_{13} a \Delta m^2_{\text{ren}}}\right] + \epsilon \Delta m^2_{\text{ren}} s^2_{12}, \quad \text{(A9)}$$

and the second term is the first-order perturbation. The diagonalization determines ϕ as

$$\cos 2\phi = \frac{\Delta m^2_{\text{ren}} \cos 2\theta_{13} - a}{\lambda_+ - \lambda_-} = \frac{\cos 2\theta_{13} - r_a}{\sqrt{1 + r_a^2 - 2 r_a \cos 2\theta_{13}}},$$
$$\sin 2\phi = \frac{\Delta m^2_{\text{ren}} \sin 2\theta_{13}}{\lambda_+ - \lambda_-} = \frac{\sin 2\theta_{13}}{\sqrt{1 + r_a^2 - 2 r_a \cos 2\theta_{13}}}. \quad \text{(A10)}$$

So far, this is the νSM treatment given in Ref. [57].

The UV part of the Hamiltonian is

$$\hat{H}^{(1)}_{\text{UV}} = U_{13}^{\dagger}(\phi) \widetilde{H}^{(1)}_{\text{UV}}\, U_{13}(\phi) = U_{13}^{\dagger}(\phi) \Delta_b F U_{13}(\phi) \equiv \Delta_b K, \quad \text{(A11)}$$

where we have defined the K matrix as

$$K \equiv U_{13}^{\dagger}(\phi) F U_{13}(\phi) = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$
$$= \begin{bmatrix} c^2_\phi F_{11} + s^2_\phi F_{33} - c_\phi s_\phi (F_{13} + F_{31}) & c_\phi F_{12} - s_\phi F_{32} & c^2_\phi F_{13} - s^2_\phi F_{31} + c_\phi s_\phi (F_{11} - F_{33}) \\ c_\phi F_{21} - s_\phi F_{23} & F_{22} & s_\phi F_{21} + c_\phi F_{23} \\ c^2_\phi F_{31} - s^2_\phi F_{13} + c_\phi s_\phi (F_{11} - F_{33}) & s_\phi F_{12} + c_\phi F_{32} & s^2_\phi F_{11} + c^2_\phi F_{33} + c_\phi s_\phi (F_{13} + F_{31}) \end{bmatrix}. \quad \text{(A12)}$$

The F matrix is defined in Eq. (A2). The explicit expressions of the elements K_{ij} are given in Eq. (17).

A4. $U_{12}(\psi)$ rotation to the bar (energy eigenstate) basis

Since λ_- and λ_0 cross at the solar resonance, $a \approx \epsilon \Delta m^2_{\text{ren}} \cos 2\theta_{12} / \cos^2\theta_{13}$, to describe the physics near this degeneracy we need to diagonalize the (1–2) submatrix of $\hat{H}_{\nu\text{SM}}$ using $U_{12}(\psi)$:

$$U_{12}(\psi) = \begin{bmatrix} c_\psi & s_\psi e^{i\delta} & 0 \\ -s_\psi e^{-i\delta} & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad U_{12}(\psi)^\dagger = \begin{bmatrix} c_\psi & -s_\psi e^{i\delta} & 0 \\ s_\psi e^{-i\delta} & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \text{(A13)}$$

The new neutrino basis is

$$|\bar{\nu}\rangle = U_{12}^{\dagger}(\psi)|\hat{\nu}\rangle = U_{12}^{\dagger}(\psi,\delta)U_{13}^{\dagger}(\phi)U_{23}^{\dagger}(\theta_{23})|\nu\rangle, \tag{A14}$$

and the νSM part of the Hamiltonian is

$$\bar{H}_{\nu\mathrm{SM}} = U_{12}(\psi)^{\dagger}\hat{H}_{\nu\mathrm{SM}}U_{12}(\psi) = \frac{1}{2E}\begin{bmatrix} \lambda_1 & 0 & -s_\psi A_S \\ 0 & \lambda_2 & c_\psi A_S e^{-i\delta} \\ -s_\psi A_S & c_\psi A_S e^{i\delta} & \lambda_3 \end{bmatrix}, \tag{A15}$$

where A_C and A_S characterize the first-order correction effects, and are given by

$$A_C \equiv \epsilon c_{12}s_{12}c_{\phi-\theta_{13}}\Delta m^2_{\mathrm{ren}}, \qquad A_S \equiv \epsilon c_{12}s_{12}s_{\phi-\theta_{13}}\Delta m^2_{\mathrm{ren}}. \tag{A16}$$

The eigenvalues are given by $\lambda_3 = \lambda_+$ and

$$\lambda_1 = \frac{1}{2E}\left(c^2_\psi\lambda_- + s^2_\psi\lambda_0 - 2c_\psi s_\psi \epsilon c_{12}s_{12}c_{\phi-\theta_{13}}\Delta m^2_{\mathrm{ren}}\right),$$
$$\lambda_2 = \frac{1}{2E}\left(s^2_\psi\lambda_- + c^2_\psi\lambda_0 + 2c_\psi s_\psi \epsilon c_{12}s_{12}c_{\phi-\theta_{13}}\Delta m^2_{\mathrm{ren}}\right). \tag{A17}$$

Finally, the UV part of the first-order bar-basis Hamiltonian is given by

$$\bar{H}_{UV} = U_{12}(\psi)^{\dagger}\hat{H}^{(1)}_{UV}U_{12}(\psi) = \Delta_b U_{12}(\psi)^{\dagger}KU_{12}(\psi) \equiv \Delta_b G, \tag{A18}$$

where $\Delta_b \equiv \frac{b}{2E}$ and we have defined the G matrix as

$$G \equiv U_{12}(\psi)^{\dagger}KU_{12}(\psi) = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix}. \tag{A19}$$

The expressions for the G matrix elements in terms of the K_{ij} are:

$$G_{11} = c^2_\psi K_{11} + s^2_\psi K_{22} - c_\psi s_\psi\left(e^{i\delta}K_{21} + e^{-i\delta}K_{12}\right),$$
$$G_{12} = e^{i\delta}\left[c_\psi s_\psi\left(K_{11} - K_{22}\right) + \left(c^2_\psi e^{-i\delta}K_{12} - s^2_\psi e^{i\delta}K_{21}\right)\right] = (G_{21})^*,$$
$$G_{22} = s^2_\psi K_{11} + c^2_\psi K_{22} + c_\psi s_\psi\left(e^{i\delta}K_{21} + e^{-i\delta}K_{12}\right),$$
$$G_{13} = c_\psi K_{13} - s_\psi e^{i\delta}K_{23} = (G_{31})^*,$$
$$G_{23} = e^{-i\delta}\left(s_\psi K_{13} + c_\psi e^{i\delta}K_{23}\right) = (G_{32})^*,$$
$$G_{33} = K_{33}. \tag{A20}$$

Notice that $K_{ji} = K^*_{ij}$, and the K_{ii} are real.

A4.1 The bar-basis Hamiltonian in the SOL convention: Summary To summarize, the unperturbed and perturbed parts of the bar-basis Hamiltonian can be written as $\bar{H} = \bar{H}^{(0)} + \bar{H}^{(1)}$, where

$$\bar{H}^{(0)} = \frac{1}{2E}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$
$$\bar{H}^{(1)} = \epsilon c_{12}s_{12}s_{(\phi-\theta_{13})}\Delta_{\mathrm{ren}}\begin{bmatrix} 0 & 0 & -s_\psi \\ 0 & 0 & c_\psi e^{-i\delta} \\ -s_\psi & c_\psi e^{i\delta} & 0 \end{bmatrix} + \Delta_b G, \tag{A21}$$

with $\Delta_{\mathrm{ren}} \equiv \frac{\Delta m^2_{\mathrm{ren}}}{2E}$ defined in Eq. (12).

A5. Calculation of the bar-basis $\bar{S}$ matrix

To calculate the $\bar{S}$ matrix we define $\Omega(x)$ as

$$\Omega(x) = e^{i\bar{H}^{(0)}x}\bar{S}(x). \tag{A22}$$

Using $i\frac{d}{dx}\bar{S} = \bar{H}(x)\bar{S}$, $\Omega(x)$ obeys the evolution equation

$$i\frac{d}{dx}\Omega(x) = H_1\Omega(x), \tag{A23}$$

where

$$H_1 \equiv e^{i\bar{H}^{(0)}x}\bar{H}^{(1)}e^{-i\bar{H}^{(0)}x}. \tag{A24}$$

Then, $\Omega(x)$ can be computed perturbatively as

$$\Omega(x) = 1 + (-i)\int_0^x dx' H_1(x') + (-i)^2\int_0^x dx' H_1(x')\int_0^{x'} dx'' H_1(x'') + \cdots, \tag{A25}$$

and the $\bar{S}$ matrix is given by

$$\bar{S}(x) = e^{-i\bar{H}^{(0)}x}\Omega(x). \tag{A26}$$

To simplify the expressions we use the notation

$$h_i = \frac{\lambda_i}{2E}, \tag{A27}$$

which leads to

$$e^{\pm i\bar{H}^{(0)}x} = \begin{bmatrix} e^{\pm ih_1x} & 0 & 0 \\ 0 & e^{\pm ih_2x} & 0 \\ 0 & 0 & e^{\pm ih_3x} \end{bmatrix}. \tag{A28}$$

H_1 can be given as the sum of νSM and UV terms, $H_1 = H_1^{\nu\text{SM}} + H_1^{\text{UV}}$, where

$$H_1^{\nu\text{SM}} = \widetilde{\epsilon}\Delta_{\text{ren}}\begin{bmatrix} 0 & 0 & -s_\psi e^{-i(h_3-h_1)x} \\ 0 & 0 & c_\psi e^{-i\delta}e^{-i(h_3-h_2)x} \\ -s_\psi e^{i(h_3-h_1)x} & c_\psi e^{i\delta}e^{i(h_3-h_2)x} & 0 \end{bmatrix},$$

$$H_1^{UV} = \Delta_b\begin{bmatrix} G_{11} & e^{-i(h_2-h_1)x}G_{12} & e^{-i(h_3-h_1)x}G_{13} \\ e^{i(h_2-h_1)x}G_{21} & G_{22} & e^{-i(h_3-h_2)x}G_{23} \\ e^{i(h_3-h_1)x}G_{31} & e^{i(h_3-h_2)x}G_{32} & G_{33} \end{bmatrix},$$

where we have defined $\widetilde{\epsilon}$ as

$$\widetilde{\epsilon} \equiv \epsilon c_{12}s_{12}s_{(\phi-\theta_{13})}. \tag{A29}$$

Then, the first-order terms in $\Omega(x)$ and subsequently $\bar{S}^{(1)}$ can be obtained as

$$\bar{S}^{(1)} = \widetilde{\epsilon}\begin{bmatrix} 0 & 0 & -s_\psi\frac{\Delta_{\text{ren}}}{h_3-h_1}\left\{e^{-ih_3x} - e^{-ih_1x}\right\} \\ 0 & 0 & c_\psi e^{-i\delta}\frac{\Delta_{\text{ren}}}{h_3-h_2}\left\{e^{-ih_3x} - e^{-ih_2x}\right\} \\ -s_\psi\frac{\Delta_{\text{ren}}}{h_3-h_1}\left\{e^{-ih_3x} - e^{-ih_1x}\right\} & c_\psi e^{i\delta}\frac{\Delta_{\text{ren}}}{h_3-h_2}\left\{e^{-ih_3x} - e^{-ih_2x}\right\} & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} G_{11}(-i\Delta_b x)e^{-ih_1x} & G_{12}\frac{\Delta_b}{h_2-h_1}\left\{e^{-ih_2x} - e^{-ih_1x}\right\} & G_{13}\frac{\Delta_b}{h_3-h_1}\left\{e^{-ih_3x} - e^{-ih_1x}\right\} \\ G_{21}\frac{\Delta_b}{h_2-h_1}\left\{e^{-ih_2x} - e^{-ih_1x}\right\} & G_{22}(-i\Delta_b x)e^{-ih_2x} & G_{23}\frac{\Delta_b}{h_3-h_2}\left\{e^{-ih_3x} - e^{-ih_2x}\right\} \\ G_{31}\frac{\Delta_b}{h_3-h_1}\left\{e^{-ih_3x} - e^{-ih_1x}\right\} & G_{32}\frac{\Delta_b}{h_3-h_2}\left\{e^{-ih_3x} - e^{-ih_2x}\right\} & G_{33}(-i\Delta_b x)e^{-ih_3x} \end{bmatrix}. \tag{A30}$$

To summarize, the bar-basis S matrix to first order in ϵ and $\widetilde{\alpha}_{\beta\gamma}$ is given by

$$\bar{S} = \begin{bmatrix} e^{-ih_1 x} & 0 & 0 \\ 0 & e^{-ih_2 x} & 0 \\ 0 & 0 & e^{-ih_3 x} \end{bmatrix} + \begin{bmatrix} \bar{S}^{(1)}_{11} & \bar{S}^{(1)}_{12} & \bar{S}^{(1)}_{13} \\ \bar{S}^{(1)}_{21} & \bar{S}^{(1)}_{22} & \bar{S}^{(1)}_{23} \\ \bar{S}^{(1)}_{31} & \bar{S}^{(1)}_{32} & \bar{S}^{(1)}_{33} \end{bmatrix}, \tag{A31}$$

where the $\bar{S}^{(1)}$ matrix elements are given in Eq. (A30). By using the expressions of the G matrix elements in Eq. (A20), they can be written by using the K matrix elements. The resulting expressions for the $\bar{S}^{(1)}$ elements to first order in the UV $\widetilde{\alpha}$ parameters are as follows:

$$\begin{aligned}
\bar{S}^{(1)}_{11} &= \left[c^2_\psi K_{11} + s^2_\psi K_{22} - c_\psi s_\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right] (-i\Delta_b x) e^{-ih_1 x}, \\
\bar{S}^{(1)}_{22} &= \left[s^2_\psi K_{11} + c^2_\psi K_{22} + c_\psi s_\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right] (-i\Delta_b x) e^{-ih_2 x}, \\
\bar{S}^{(1)}_{33} &= K_{33} (-i\Delta_b x) e^{-ih_3 x}, \\
\bar{S}^{(1)}_{12} &= e^{i\delta} \left[c_\psi s_\psi \left(K_{11} - K_{22} \right) + \left(c^2_\psi e^{-i\delta} K_{12} - s^2_\psi e^{i\delta} K_{21} \right) \right] \frac{\Delta_b}{h_2 - h_1} \left\{ e^{-ih_2 x} - e^{-ih_1 x} \right\}, \\
\bar{S}^{(1)}_{21} &= e^{-i\delta} \left[c_\psi s_\psi \left(K_{11} - K_{22} \right) + \left(c^2_\psi e^{i\delta} K_{21} - s^2_\psi e^{-i\delta} K_{12} \right) \right] \frac{\Delta_b}{h_2 - h_1} \left\{ e^{-ih_2 x} - e^{-ih_1 x} \right\}, \\
\bar{S}^{(1)}_{13} &= \left[\left(c_\psi K_{13} - s_\psi e^{i\delta} K_{23} \right) \Delta_b - \widetilde{\epsilon} s_\psi \Delta_{\text{ren}} \right] \frac{1}{h_3 - h_1} \left\{ e^{-ih_3 x} - e^{-ih_1 x} \right\}, \\
\bar{S}^{(1)}_{31} &= \left[\left(c_\psi K_{31} - s_\psi e^{-i\delta} K_{32} \right) \Delta_b - \widetilde{\epsilon} s_\psi \Delta_{\text{ren}} \right] \frac{1}{h_3 - h_1} \left\{ e^{-ih_3 x} - e^{-ih_1 x} \right\}, \\
\bar{S}^{(1)}_{23} &= e^{-i\delta} \left[\left(s_\psi K_{13} + c_\psi e^{i\delta} K_{23} \right) \Delta_b + \widetilde{\epsilon} c_\psi \Delta_{\text{ren}} \right] \frac{1}{h_3 - h_2} \left\{ e^{-ih_3 x} - e^{-ih_2 x} \right\}, \\
\bar{S}^{(1)}_{32} &= e^{i\delta} \left[\left(s_\psi K_{31} + c_\psi e^{-i\delta} K_{32} \right) \Delta_b + \widetilde{\epsilon} c_\psi \Delta_{\text{ren}} \right] \frac{1}{h_3 - h_2} \left\{ e^{-ih_3 x} - e^{-ih_2 x} \right\}.
\end{aligned} \tag{A32}$$

A6. The relations between various bases

A clarifying note on the relations between the various bases may help. Using the definitions of the tilde and hat bases,

$$\widetilde{H} = (U_{13} U_{12}) \check{H} (U_{13} U_{12})^\dagger, \qquad \hat{H} = U^\dagger_{13}(\phi) \widetilde{H} U_{13}(\phi)$$

the relationship between the bar-basis Hamiltonian $\bar{H}$ and the vacuum mass eigenstate check-basis Hamiltonian $\check{H}$ is given by

$$\begin{aligned}
\bar{H} &= U_{12}(\psi)^\dagger \hat{H} U_{12}(\psi) = U^\dagger_{12}(\psi) U^\dagger_{13}(\phi) U_{13} U_{12} \check{H} U^\dagger_{12} U^\dagger_{13} U_{13}(\phi) U_{12}(\psi), \\
\check{H} &= U^\dagger_{12} U^\dagger_{13} U_{13}(\phi) U_{12}(\psi) \bar{H} U^\dagger_{12}(\psi) U^\dagger_{13}(\phi) U_{13} U_{12}.
\end{aligned} \tag{A33}$$

The bar $\bar{H}$ basis is the basis in which H_0 is diagonalized, and therefore the perturbation theory is formulated using the bar basis. In Eq. (A33) and hereafter, U_{12}, U_{13}, etc. with no specified arguments assume the vacuum angles θ_{12}, θ_{13}, etc., respectively, as arguments.

In our discussion with UV, the non-unitary transformation is involved in the relation between the flavor and the vacuum mass eigenstate (check) basis as

$$\nu_\alpha = N_{\alpha i} \check{\nu}_i = \{(1-\alpha)U\}_{\alpha i}\, \check{\nu}_i. \tag{A34}$$

Then, the relationship between the flavor-basis Hamiltonian H_{flavor} and the bar-basis one $\bar{H}$ becomes

$$\begin{aligned} H_{\text{flavor}} &= \{(1-\alpha)U\}\, \check{H}\, \{(1-\alpha)U\}^\dagger \\ &= (1-\alpha)U_{23}U_{13}(\phi)U_{12}(\psi)\bar{H}U_{12}^\dagger(\psi)U_{13}^\dagger(\phi)U_{23}^\dagger(1-\alpha)^\dagger. \end{aligned} \tag{A35}$$

Then, the flavor-basis S matrix is related to the $\bar{S}$ matrix as

$$S_{\text{flavor}} = (1-\alpha)U_{23}U_{13}(\phi)U_{12}(\psi)\bar{S}U_{12}^\dagger(\psi)U_{13}^\dagger(\phi)U_{23}^\dagger(1-\alpha)^\dagger. \tag{A36}$$

Notice that the structure of the S_{flavor} matrix in Eq. (A36) is very similar to that in the solar-resonance perturbation theory, Ref. [33, Eq. (51)]. That is why we will see similar expressions of the various quantities as functions of the K and G matrices. But, we must note that the similarity is superficial because the energy eigenstate bases in matter by which the perturbation theory is formulated, the bar basis here and the hat basis in Ref. [33], are different from each other. Using the notation φ for θ_{12} in matter, it is given by $\hat{H} = U_{12}^\dagger(\varphi)U_{12}\check{H}U_{12}^\dagger U_{12}(\varphi)$ in Ref. [33], which is very different from Eq. (A33) in our case.

A7. Calculation of the flavor-basis S matrix

Given the expression of the bar-basis $\bar{S}$ matrix in the zeroth and first orders in Eqs. (A31) and (A32), respectively, it is straightforward to compute the flavor-basis S matrix elements using Eq. (A36).

First, we calculate the first-order terms using the $\bar{S}^{(1)}$ matrix. In this calculation we can disregard the $(1-\alpha)$ and $(1-\alpha)^\dagger$ factors because we are interested in up to the first order of the S_{flavor} matrix. Then, what we should do is the three rotations in the 1–2, 1–3, and 2–3 spaces as in Eq. (A36). For the reader's convenience we give an intermediate step, the tilde-basis $\widetilde{S}^{(1)}$ matrix,

$$\widetilde{S}^{(1)} = U_{13}(\phi)\hat{S}^{(1)}U_{13}^\dagger(\phi) = U_{13}(\phi)U_{12}(\psi)\bar{S}U_{12}^\dagger(\psi)U_{13}^\dagger(\phi). \tag{A37}$$

The computed results for the $\widetilde{S}^{(1)}$ matrix elements are given in Appendix B.

Using the $\widetilde{S}^{(1)}$ matrix elements, the flavor-basis $S^{(1)}$ matrix is obtained as

$$\begin{aligned} S^{(1)} &= U_{23}\widetilde{S}^{(1)}U_{23}^\dagger \\ &= \begin{bmatrix} \widetilde{S}_{11}^{(1)} & c_{23}\widetilde{S}_{12}^{(1)} + s_{23}\widetilde{S}_{13}^{(1)} & -s_{23}\widetilde{S}_{12}^{(1)} + c_{23}\widetilde{S}_{13}^{(1)} \\ c_{23}\widetilde{S}_{21}^{(1)} + s_{23}\widetilde{S}_{31}^{(1)} & c_{23}^2\widetilde{S}_{22}^{(1)} + s_{23}^2\widetilde{S}_{33}^{(1)} + c_{23}s_{23}\left(\widetilde{S}_{32}^{(1)} + \widetilde{S}_{23}^{(1)}\right) & c_{23}^2\widetilde{S}_{23}^{(1)} - s_{23}^2\widetilde{S}_{32}^{(1)} + c_{23}s_{23}\left(\widetilde{S}_{33}^{(1)} - \widetilde{S}_{22}^{(1)}\right) \\ -s_{23}\widetilde{S}_{21}^{(1)} + c_{23}\widetilde{S}_{31}^{(1)} & c_{23}^2\widetilde{S}_{32}^{(1)} - s_{23}^2\widetilde{S}_{23}^{(1)} + c_{23}s_{23}\left(\widetilde{S}_{33}^{(1)} - \widetilde{S}_{22}^{(1)}\right) & s_{23}^2\widetilde{S}_{22}^{(1)} + c_{23}^2\widetilde{S}_{33}^{(1)} - c_{23}s_{23}\left(\widetilde{S}_{32}^{(1)} + \widetilde{S}_{23}^{(1)}\right) \end{bmatrix}. \end{aligned} \tag{A38}$$

Notice that the first-order $S^{(1)}$ matrix is necessary to obtain the EV part of the first-order probability as defined in Eq. (16). For the UV part, the $S^{(0)}$ matrix elements suffice.

For the zeroth-order flavor-basis S matrix $S^{(0)}$, we repeat the same calculation using the $\bar{S}^{(0)}$ matrix, the first term in Eq. (A31):

$$
\begin{aligned}
S_{ee}^{(0)} &= c_\phi^2 \left(c_\psi^2 e^{-ih_1 x} + s_\psi^2 e^{-ih_2 x} \right) + s_\phi^2 e^{-ih_3 x}, \\
S_{\mu\mu}^{(0)} &= c_{23}^2 \left(s_\psi^2 e^{-ih_1 x} + c_\psi^2 e^{-ih_2 x} \right) + s_{23}^2 \left[s_\phi^2 \left(c_\psi^2 e^{-ih_1 x} + s_\psi^2 e^{-ih_2 x} \right) + c_\phi^2 e^{-ih_3 x} \right] \\
&\quad - \sin 2\theta_{23} s_\phi c_\psi s_\psi \cos\delta \left(e^{-ih_2 x} - e^{-ih_1 x} \right), \\
S_{\tau\tau}^{(0)} &= s_{23}^2 \left(s_\psi^2 e^{-ih_1 x} + c_\psi^2 e^{-ih_2 x} \right) + c_{23}^2 \left[s_\phi^2 \left(c_\psi^2 e^{-ih_1 x} + s_\psi^2 e^{-ih_2 x} \right) + c_\phi^2 e^{-ih_3 x} \right] \\
&\quad + \sin 2\theta_{23} s_\phi c_\psi s_\psi \cos\delta \left(e^{-ih_2 x} - e^{-ih_1 x} \right), \\
S_{e\mu}^{(0)} &= c_{23} c_\phi e^{i\delta} c_\psi s_\psi \left(e^{-ih_2 x} - e^{-ih_1 x} \right) + s_{23} c_\phi s_\phi \left[e^{-ih_3 x} - \left(c_\psi^2 e^{-ih_1 x} + s_\psi^2 e^{-ih_2 x} \right) \right], \\
S_{\mu e}^{(0)} &= c_{23} c_\phi e^{-i\delta} c_\psi s_\psi \left(e^{-ih_2 x} - e^{-ih_1 x} \right) + s_{23} c_\phi s_\phi \left[e^{-ih_3 x} - \left(c_\psi^2 e^{-ih_1 x} + s_\psi^2 e^{-ih_2 x} \right) \right], \\
S_{e\tau}^{(0)} &= -s_{23} c_\phi e^{i\delta} c_\psi s_\psi \left(e^{-ih_2 x} - e^{-ih_1 x} \right) + c_{23} c_\phi s_\phi \left[e^{-ih_3 x} - \left(c_\psi^2 e^{-ih_1 x} + s_\psi^2 e^{-ih_2 x} \right) \right], \\
S_{\tau e}^{(0)} &= -s_{23} c_\phi e^{-i\delta} c_\psi s_\psi \left(e^{-ih_2 x} - e^{-ih_1 x} \right) + c_{23} c_\phi s_\phi \left[e^{-ih_3 x} - \left(c_\psi^2 e^{-ih_1 x} + s_\psi^2 e^{-ih_2 x} \right) \right], \\
S_{\mu\tau}^{(0)} &= -\left(c_{23}^2 e^{-i\delta} - s_{23}^2 e^{i\delta} \right) s_\phi c_\psi s_\psi \left(e^{-ih_2 x} - e^{-ih_1 x} \right) \\
&\quad + c_{23} s_{23} \left[(s_\phi^2 s_\psi^2 - c_\psi^2) e^{-ih_2 x} + (s_\phi^2 c_\psi^2 - s_\psi^2) e^{-ih_1 x} + c_\phi^2 e^{-ih_3 x} \right], \\
S_{\tau\mu}^{(0)} &= -\left(c_{23}^2 e^{i\delta} - s_{23}^2 e^{-i\delta} \right) s_\phi c_\psi s_\psi \left(e^{-ih_2 x} - e^{-ih_1 x} \right) \\
&\quad + c_{23} s_{23} \left[(s_\phi^2 s_\psi^2 - c_\psi^2) e^{-ih_2 x} + (s_\phi^2 c_\psi^2 - s_\psi^2) e^{-ih_1 x} + c_\phi^2 e^{-ih_3 x} \right].
\end{aligned}
\tag{A39}
$$

Notice that the generalized T invariance, $S_{ij}^{(0)}\big|_{i\leftrightarrow j}(c \to c^*) = S_{ji}^{(0)}(c)$, holds, where c denotes the all complex numbers involved.

Appendix B. Tilde-basis $\widetilde{S}^{(1)}$ matrix: Summary

The first-order tilde basis $\widetilde{S}^{(1)}$ matrix is defined in Eq. (A37) for the expression of the bar-basis $\bar{S}^{(1)}$ matrix given in Eq. (A32). The $\widetilde{S}^{(1)}$ matrix elements obtained are:

$$
\begin{aligned}
\widetilde{S}_{11}^{(1)} &= (-i\Delta_b x)\biggl\{ c_\phi^2 K_{11}\left(s_\psi^4 e^{-ih_2 x} + c_\psi^4 e^{-ih_1 x}\right) + c_\phi^2 c_\psi^2 s_\psi^2 K_{22}\left(e^{-ih_2 x} + e^{-ih_1 x}\right) + s_\phi^2 K_{33} e^{-ih_3 x} \\
&+ c_\phi^2 c_\psi s_\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12}\right)\left(s_\psi^2 e^{-ih_2 x} - c_\psi^2 e^{-ih_1 x}\right)\biggr\} \\
&+ c_\phi^2 c_\psi s_\psi \left[\sin 2\psi \left(K_{11} - K_{22}\right) + \cos 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12}\right)\right] \frac{\Delta_b}{h_2 - h_1}\left\{e^{-ih_2 x} - e^{-ih_1 x}\right\} \\
&+ c_\phi s_\phi c_\psi \biggl\{ c_\psi \left(K_{31} + K_{13}\right)\Delta_b - s_\psi \left(e^{i\delta} K_{23} + e^{-i\delta} K_{32}\right)\Delta_b \\
&- 2\widetilde{\epsilon} s_\psi \Delta_{\text{ren}} \biggr\} \frac{1}{h_3 - h_1}\left\{e^{-ih_3 x} - e^{-ih_1 x}\right\} \\
&+ c_\phi s_\phi s_\psi \biggl\{ s_\psi \left(K_{31} + K_{13}\right)\Delta_b + c_\psi \left(e^{i\delta} K_{23} + e^{-i\delta} K_{32}\right)\Delta_b \\
&+ 2\widetilde{\epsilon} c_\psi \Delta_{\text{ren}} \biggr\} \frac{1}{h_3 - h_2}\left\{e^{-ih_3 x} - e^{-ih_2 x}\right\}, \\
\widetilde{S}_{22}^{(1)} &= (-i\Delta_b x)\biggl\{ c_\psi^2 s_\psi^2 K_{11}\left(e^{-ih_2 x} + e^{-ih_1 x}\right) + K_{22}\left(c_\psi^4 e^{-ih_2 x} + s_\psi^4 e^{-ih_1 x}\right) \\
&+ c_\psi s_\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12}\right)\left(c_\psi^2 e^{-ih_2 x} - s_\psi^2 e^{-ih_1 x}\right)\biggr\} \\
&- c_\psi s_\psi \left[\sin 2\psi \left(K_{11} - K_{22}\right) + \cos 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12}\right)\right] \frac{\Delta_b}{h_2 - h_1}\left\{e^{-ih_2 x} - e^{-ih_1 x}\right\}, \\
\widetilde{S}_{33}^{(1)} &= (-i\Delta_b x)\biggl\{ s_\phi^2 K_{11}\left(s_\psi^4 e^{-ih_2 x} + c_\psi^4 e^{-ih_1 x}\right) + s_\phi^2 c_\psi^2 s_\psi^2 K_{22}\left(e^{-ih_2 x} + e^{-ih_1 x}\right) \\
&+ s_\phi^2 c_\psi s_\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12}\right)\left(s_\psi^2 e^{-ih_2 x} - c_\psi^2 e^{-ih_1 x}\right) + c_\phi^2 K_{33} e^{-ih_3 x}\biggr\} \\
&+ s_\phi^2 c_\psi s_\psi \left[\sin 2\psi \left(K_{11} - K_{22}\right) + \cos 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12}\right)\right] \frac{\Delta_b}{h_2 - h_1}\left\{e^{-ih_2 x} - e^{-ih_1 x}\right\} \\
&- c_\phi s_\phi c_\psi \biggl\{ c_\psi \left(K_{13} + K_{31}\right)\Delta_b - s_\psi \left(e^{i\delta} K_{23} + e^{-i\delta} K_{32}\right)\Delta_b \\
&- 2\widetilde{\epsilon} s_\psi \Delta_{\text{ren}} \biggr\} \frac{1}{h_3 - h_1}\left\{e^{-ih_3 x} - e^{-ih_1 x}\right\} \\
&- c_\phi s_\phi s_\psi \biggl\{ s_\psi \left(K_{13} + K_{31}\right)\Delta_b + c_\psi \left(e^{i\delta} K_{23} + e^{-i\delta} K_{32}\right)\Delta_b \\
&+ 2\widetilde{\epsilon} c_\psi \Delta_{\text{ren}} \biggr\} \frac{1}{h_3 - h_2}\left\{e^{-ih_3 x} - e^{-ih_2 x}\right\},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{S}_{12}^{(1)} &= e^{i\delta}\bigg[c_{\phi} c_{\psi} s_{\psi} (-i\Delta_{b} x) \bigg\{ K_{11} \left(s^2_{\psi} e^{-i h_{2} x} - c^2_{\psi} e^{-i h_{1} x} \right) + K_{22} \left(c^2_{\psi} e^{-i h_{2} x} - s^2_{\psi} e^{-i h_{1} x} \right) \\
&\quad + c_{\psi} s_{\psi} \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \left(e^{-i h_{2} x} + e^{-i h_{1} x} \right) \bigg\} \\
&\quad + c_{\phi} \bigg\{ e^{-i\delta} K_{12} + c_{\psi} s_{\psi} \Big[\cos 2\psi \left(K_{11} - K_{22} \right) \\
&\quad - \sin 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \Big] \bigg\} \frac{\Delta_{b}}{h_{2} - h_{1}} \left\{ e^{-i h_{2} x} - e^{-i h_{1} x} \right\} \\
&\quad - s_{\phi} s_{\psi} \left\{ \left(c_{\psi} K_{31} - s_{\psi} e^{-i\delta} K_{32} \right) \Delta_{b} - \widetilde{\epsilon} s_{\psi} \Delta_{\text{ren}} \right\} \frac{1}{h_{3} - h_{1}} \left\{ e^{-i h_{3} x} - e^{-i h_{1} x} \right\} \\
&\quad + s_{\phi} c_{\psi} \left\{ \left(s_{\psi} K_{31} + c_{\psi} e^{-i\delta} K_{32} \right) \Delta_{b} + \widetilde{\epsilon} c_{\psi} \Delta_{\text{ren}} \right\} \frac{1}{h_{3} - h_{2}} \left\{ e^{-i h_{3} x} - e^{-i h_{2} x} \right\} \bigg], \\
\widetilde{S}_{21}^{(1)} &= e^{-i\delta}\bigg[c_{\phi} c_{\psi} s_{\psi} (-i\Delta_{b} x) \bigg\{ K_{11} \left(s^2_{\psi} e^{-i h_{2} x} - c^2_{\psi} e^{-i h_{1} x} \right) + K_{22} \left(c^2_{\psi} e^{-i h_{2} x} - s^2_{\psi} e^{-i h_{1} x} \right) \\
&\quad + c_{\psi} s_{\psi} \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \left(e^{-i h_{2} x} + e^{-i h_{1} x} \right) \bigg\} \\
&\quad + c_{\phi} \bigg\{ e^{i\delta} K_{21} + c_{\psi} s_{\psi} \Big[\cos 2\psi \left(K_{11} - K_{22} \right) \\
&\quad - \sin 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \Big] \bigg\} \frac{\Delta_{b}}{h_{2} - h_{1}} \left\{ e^{-i h_{2} x} - e^{-i h_{1} x} \right\} \\
&\quad - s_{\phi} s_{\psi} \left\{ \left(c_{\psi} K_{13} - s_{\psi} e^{i\delta} K_{23} \right) \Delta_{b} - \widetilde{\epsilon} s_{\psi} \Delta_{\text{ren}} \right\} \frac{1}{h_{3} - h_{1}} \left\{ e^{-i h_{3} x} - e^{-i h_{1} x} \right\} \\
&\quad + s_{\phi} c_{\psi} \left\{ \left(s_{\psi} K_{13} + c_{\psi} e^{i\delta} K_{23} \right) \Delta_{b} + \widetilde{\epsilon} c_{\psi} \Delta_{\text{ren}} \right\} \frac{1}{h_{3} - h_{2}} \left\{ e^{-i h_{3} x} - e^{-i h_{2} x} \right\} \bigg], \\
\widetilde{S}_{13}^{(1)} &= -(-i\Delta_{b} x) c_{\phi} s_{\phi} \bigg\{ K_{11} \left(s^4_{\psi} e^{-i h_{2} x} + c^4_{\psi} e^{-i h_{1} x} \right) + c^2_{\psi} s^2_{\psi} K_{22} \left(e^{-i h_{2} x} + e^{-i h_{1} x} \right) - K_{33} e^{-i h_{3} x} \\
&\quad + c_{\psi} s_{\psi} \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \left(s^2_{\psi} e^{-i h_{2} x} - c^2_{\psi} e^{-i h_{1} x} \right) \bigg\} \\
&\quad - c_{\phi} s_{\phi} c_{\psi} s_{\psi} \left[\sin 2\psi \left(K_{11} - K_{22} \right) + \cos 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right] \frac{\Delta_{b}}{h_{2} - h_{1}} \left\{ e^{-i h_{2} x} - e^{-i h_{1} x} \right\} \\
&\quad + \bigg\{ c^2_{\psi} \left(c^2_{\phi} K_{13} - s^2_{\phi} K_{31} \right) \Delta_{b} - c_{\psi} s_{\psi} \left(c^2_{\phi} e^{i\delta} K_{23} - s^2_{\phi} e^{-i\delta} K_{32} \right) \Delta_{b} \\
&\quad - \cos 2\phi\, c_{\psi} \widetilde{\epsilon} s_{\psi} \Delta_{\text{ren}} \bigg\} \frac{1}{h_{3} - h_{1}} \left\{ e^{-i h_{3} x} - e^{-i h_{1} x} \right\} \\
&\quad + \bigg\{ s^2_{\psi} \left(c^2_{\phi} K_{13} - s^2_{\phi} K_{31} \right) \Delta_{b} + c_{\psi} s_{\psi} \left(c^2_{\phi} e^{i\delta} K_{23} - s^2_{\phi} e^{-i\delta} K_{32} \right) \Delta_{b} \\
&\quad + \cos 2\phi\, s_{\psi} \widetilde{\epsilon} c_{\psi} \Delta_{\text{ren}} \bigg\} \frac{1}{h_{3} - h_{2}} \left\{ e^{-i h_{3} x} - e^{-i h_{2} x} \right\},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{S}^{(1)}_{31} &= -(-i\Delta_b x) c_\phi s_\phi \bigg\{ K_{11} \left(s^4_\psi e^{-ih_2 x} + c^4_\psi e^{-ih_1 x} \right) + c^2_\psi s^2_\psi K_{22} \left(e^{-ih_2 x} + e^{-ih_1 x} \right) - K_{33} e^{-ih_3 x} \\
&+ c_\psi s_\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \left(s^2_\psi e^{-ih_2 x} - c^2_\psi e^{-ih_1 x} \right) \bigg\} \\
&- c_\phi s_\phi c_\psi s_\psi \left[\sin 2\psi \left(K_{11} - K_{22} \right) + \cos 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \right] \frac{\Delta_b}{h_2 - h_1} \left\{ e^{-ih_2 x} - e^{-ih_1 x} \right\} \\
&+ \bigg\{ c^2_\psi \left(c^2_\phi K_{31} - s^2_\phi K_{13} \right) \Delta_b + c_\psi s_\psi \left(s^2_\phi e^{i\delta} K_{23} - c^2_\phi e^{-i\delta} K_{32} \right) \Delta_b \\
&- \cos 2\phi c_\psi \widetilde{\epsilon} s_\psi \Delta_{\text{ren}} \bigg\} \frac{1}{h_3 - h_1} \left\{ e^{-ih_3 x} - e^{-ih_1 x} \right\} \\
&+ \bigg\{ s^2_\psi \left(c^2_\phi K_{31} - s^2_\phi K_{13} \right) \Delta_b - c_\psi s_\psi \left(s^2_\phi e^{i\delta} K_{23} - c^2_\phi e^{-i\delta} K_{32} \right) \Delta_b \\
&+ \cos 2\phi s_\psi \widetilde{\epsilon} c_\psi \Delta_{\text{ren}} \bigg\} \frac{1}{h_3 - h_2} \left\{ e^{-ih_3 x} - e^{-ih_2 x} \right\},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{S}^{(1)}_{23} &= e^{-i\delta} \bigg[-(-i\Delta_b x) s_\phi c_\psi s_\psi \bigg\{ K_{11} \left(s^2_\psi e^{-ih_2 x} - c^2_\psi e^{-ih_1 x} \right) + K_{22} \left(c^2_\psi e^{-ih_2 x} - s^2_\psi e^{-ih_1 x} \right) \\
&+ c_\psi s_\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \left(e^{-ih_2 x} + e^{-ih_1 x} \right) \bigg\} \\
&- s_\phi \bigg\{ e^{i\delta} K_{21} + c_\psi s_\psi \Big[\cos 2\psi \left(K_{11} - K_{22} \right) \\
&- \sin 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \Big] \bigg\} \frac{\Delta_b}{h_2 - h_1} \left\{ e^{-ih_2 x} - e^{-ih_1 x} \right\} \\
&- c_\phi s_\psi \left\{ \left(c_\psi K_{13} - s_\psi e^{i\delta} K_{23} \right) \Delta_b - \widetilde{\epsilon} s_\psi \Delta_{\text{ren}} \right\} \frac{1}{h_3 - h_1} \left\{ e^{-ih_3 x} - e^{-ih_1 x} \right\} \\
&+ c_\phi c_\psi \left\{ \left(s_\psi K_{13} + c_\psi e^{i\delta} K_{23} \right) \Delta_b + \widetilde{\epsilon} c_\psi \Delta_{\text{ren}} \right\} \frac{1}{h_3 - h_2} \left\{ e^{-ih_3 x} - e^{-ih_2 x} \right\} \bigg],
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{S}^{(1)}_{32} &= e^{i\delta} \bigg[-(-i\Delta_b x) s_\phi c_\psi s_\psi \bigg\{ K_{11} \left(s^2_\psi e^{-ih_2 x} - c^2_\psi e^{-ih_1 x} \right) + K_{22} \left(c^2_\psi e^{-ih_2 x} - s^2_\psi e^{-ih_1 x} \right) \\
&+ c_\psi s_\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \left(e^{-ih_2 x} + e^{-ih_1 x} \right) \bigg\} \\
&- s_\phi \bigg\{ e^{-i\delta} K_{12} + c_\psi s_\psi \Big[\cos 2\psi \left(K_{11} - K_{22} \right) \\
&- \sin 2\psi \left(e^{i\delta} K_{21} + e^{-i\delta} K_{12} \right) \Big] \bigg\} \frac{\Delta_b}{h_2 - h_1} \left\{ e^{-ih_2 x} - e^{-ih_1 x} \right\} \\
&- c_\phi s_\psi \left\{ \left(c_\psi K_{31} - s_\psi e^{-i\delta} K_{32} \right) \Delta_b - \widetilde{\epsilon} s_\psi \Delta_{\text{ren}} \right\} \frac{1}{h_3 - h_1} \left\{ e^{-ih_3 x} - e^{-ih_1 x} \right\} \\
&+ c_\phi c_\psi \left\{ \left(s_\psi K_{31} + c_\psi e^{-i\delta} K_{32} \right) \Delta_b + \widetilde{\epsilon} c_\psi \Delta_{\text{ren}} \right\} \frac{1}{h_3 - h_2} \left\{ e^{-ih_3 x} - e^{-ih_2 x} \right\} \bigg].
\end{aligned}
$$

$\widetilde{\epsilon}$ is defined in Eq. (A29). As in the zeroth-order $\widetilde{S}^{(0)}$ matrix, the generalized T invariance holds as well.

Appendix C. First-order unitary evolution part $P(\nu_\mu \to \nu_e)^{(1)}_{\mathrm{EV}}$ in the atmospheric resonance region

The probability $P(\nu_\mu \to \nu_e)^{(1)}_{\mathrm{EV}}$ in Eq. (31) using the $\widetilde{\alpha}$ parameters is given by

$$
\begin{aligned}
P(\nu_\mu \to \nu_e)^{(1)}_{\mathrm{EV}} &= 2\mathrm{Re}\left[\left(S^{(0)}_{e\mu}\right)^* \left(S^{(1)}_{\mathrm{EV}}\right)_{e\mu}\right] \\
&= s^2_{23}\sin^2 2\phi\Bigg[\cos 2\phi\left\{-\widetilde{\alpha}_{ee}\left(1-\frac{\Delta_a}{\Delta_b}\right)+\left[s^2_{23}\widetilde{\alpha}_{\mu\mu}+c^2_{23}\widetilde{\alpha}_{\tau\tau}+c_{23}s_{23}\mathrm{Re}\left(\widetilde{\alpha}_{\tau\mu}\right)\right]\right\} \\
&\quad + \sin 2\phi\left[s_{23}\mathrm{Re}\left(\widetilde{\alpha}_{\mu e}\right)+c_{23}\mathrm{Re}\left(\widetilde{\alpha}_{\tau e}\right)\right]\Bigg](\Delta_b x)\sin(h_3-h_1)x \\
&\quad + \sin 2\theta_{23}\sin 2\phi\Big\{c^2_\phi\left[c_{23}\mathrm{Re}\left(\widetilde{\alpha}_{\mu e}\right)-s_{23}\mathrm{Re}\left(\widetilde{\alpha}_{\tau e}\right)\right] \\
&\quad - c_\phi s_\phi\left[\sin 2\theta_{23}(\widetilde{\alpha}_{\mu\mu}-\widetilde{\alpha}_{\tau\tau})+\cos 2\theta_{23}\mathrm{Re}\left(\widetilde{\alpha}_{\tau\mu}\right)\right]\Big\} \\
&\quad \times \frac{\Delta_b}{h_2-h_1}\left\{-\sin^2\frac{(h_3-h_2)x}{2}+\sin^2\frac{(h_3-h_1)x}{2}+\sin^2\frac{(h_2-h_1)x}{2}\right\} \\
&\quad + \sin 2\theta_{23}\sin 2\phi\Big\{s^2_\phi\left[c_{23}\mathrm{Re}\left(\widetilde{\alpha}_{\mu e}\right)-s_{23}\mathrm{Re}\left(\widetilde{\alpha}_{\tau e}\right)\right] \\
&\quad + c_\phi s_\phi\left[\sin 2\theta_{23}(\widetilde{\alpha}_{\mu\mu}-\widetilde{\alpha}_{\tau\tau})+\cos 2\theta_{23}\mathrm{Re}\left(\widetilde{\alpha}_{\tau\mu}\right)\right]\Big\} \\
&\quad \times \frac{\Delta_b}{h_3-h_2}\left\{\sin^2\frac{(h_3-h_2)x}{2}+\sin^2\frac{(h_3-h_1)x}{2}-\sin^2\frac{(h_2-h_1)x}{2}\right\} \\
&\quad + 4s^2_{23}\cos 2\phi\sin 2\phi\Bigg\{\sin 2\phi\left[\widetilde{\alpha}_{ee}\left(1-\frac{\Delta_a}{\Delta_b}\right)-\left(s^2_{23}\widetilde{\alpha}_{\mu\mu}+c^2_{23}\widetilde{\alpha}_{\tau\tau}\right)-c_{23}s_{23}\mathrm{Re}\left(\widetilde{\alpha}_{\tau\mu}\right)\right] \\
&\quad + \cos 2\phi\left[s_{23}\mathrm{Re}\left(\widetilde{\alpha}_{\mu e}\right)+c_{23}\mathrm{Re}\left(\widetilde{\alpha}_{\tau e}\right)\right]\Bigg\}\frac{\Delta_b}{h_3-h_1}\sin^2\frac{(h_3-h_1)x}{2} \\
&\quad + 2\sin 2\theta_{23}\sin 2\phi\left\{c^2_\phi\left[c_{23}\mathrm{Im}\left(\widetilde{\alpha}_{\mu e}\right)-s_{23}\mathrm{Im}\left(\widetilde{\alpha}_{\tau e}\right)\right]+c_\phi s_\phi\mathrm{Im}\left(\widetilde{\alpha}_{\tau\mu}\right)\right\} \\
&\quad \times \frac{\Delta_b}{h_2-h_1}\sin\frac{(h_3-h_1)x}{2}\sin\frac{(h_2-h_1)x}{2}\sin\frac{(h_3-h_2)x}{2} \\
&\quad - 2\sin 2\theta_{23}\sin 2\phi\left\{s^2_\phi\left[c_{23}\mathrm{Im}\left(\widetilde{\alpha}_{\mu e}\right)-s_{23}\mathrm{Im}\left(\widetilde{\alpha}_{\tau e}\right)\right]-c_\phi s_\phi\mathrm{Im}\left(\widetilde{\alpha}_{\tau\mu}\right)\right\} \\
&\quad \times \frac{\Delta_b}{h_3-h_2}\sin\frac{(h_3-h_1)x}{2}\sin\frac{(h_2-h_1)x}{2}\sin\frac{(h_3-h_2)x}{2},
\end{aligned}
\tag{C1}
$$

which reproduces Ref. [24, Eq. (49)].

Appendix D. Flavor-basis S matrix in the $\nu_\mu \to \nu_\tau$ channel

The flavor-basis S matrix in the $\nu_\mu \to \nu_\tau$ channel is given in the first order in the $\widetilde{\alpha}$ parameters (ignoring the first-order νSM part) as

$$
\begin{aligned}
& S^{(1)}_{\tau\mu} = c^2_{23}\widetilde{S}^{(1)}_{32} - s^2_{23}\widetilde{S}^{(1)}_{23} + c_{23}s_{23}\left(\widetilde{S}^{(1)}_{33} - \widetilde{S}^{(1)}_{22}\right) \\
= & -s_\phi c_\psi s_\psi \left(c^2_{23}e^{i\delta} - s^2_{23}e^{-i\delta}\right)(-i\Delta_b x)\Big\{K_{11}\left(s^2_\psi e^{-ih_2x} - c^2_\psi e^{-ih_1x}\right) + K_{22}\left(c^2_\psi e^{-ih_2x} - s^2_\psi e^{-ih_1x}\right) \\
& + c_\psi s_\psi \left(e^{i\delta}K_{21} + e^{-i\delta}K_{12}\right)\left(e^{-ih_2x} + e^{-ih_1x}\right)\Big\} \\
& + c_{23}s_{23}(-i\Delta_b x)\Big\{\left[s^2_\psi K_{11} + c^2_\psi K_{22} + c_\psi s_\psi\left(e^{i\delta}K_{21} + e^{-i\delta}K_{12}\right)\right](s^2_\phi s^2_\psi - c^2_\psi)e^{-ih_2x} \\
& + \left[c^2_\psi K_{11} + s^2_\psi K_{22} - c_\psi s_\psi\left(e^{i\delta}K_{21} + e^{-i\delta}K_{12}\right)\right](s^2_\phi c^2_\psi - s^2_\psi)e^{-ih_1x} + c^2_\phi K_{33}e^{-ih_3x}\Big\} \\
& - s_\phi\Big[\cos 2\theta_{23}\left\{\cos\delta \mathrm{Re}\left(e^{-i\delta}K_{12}\right) - \sin\delta \mathrm{Im}\left(e^{-i\delta}K_{12}\right)\right\} \\
& + i\left\{\sin\delta \mathrm{Re}\left(e^{-i\delta}K_{12}\right) + \cos\delta \mathrm{Im}\left(e^{-i\delta}K_{12}\right)\right\}\Big]\frac{\Delta_b}{h_2 - h_1}\left\{e^{-ih_2x} - e^{-ih_1x}\right\} \\
& - \left(c^2_{23}e^{i\delta} - s^2_{23}e^{-i\delta}\right)s_\phi c_\psi s_\psi\Big[\cos 2\psi\,(K_{11} - K_{22}) \\
& - \sin 2\psi\left(e^{i\delta}K_{21} + e^{-i\delta}K_{12}\right)\Big]\frac{\Delta_b}{h_2 - h_1}\left\{e^{-ih_2x} - e^{-ih_1x}\right\} \\
& + c_{23}s_{23}\sin 2\psi\left[\sin 2\psi\,(K_{11} - K_{22}) + \cos 2\psi\left(e^{i\delta}K_{21} + e^{-i\delta}K_{12}\right)\right]\frac{\Delta_b}{h_2 - h_1}\left\{e^{-ih_2x} - e^{-ih_1x}\right\} \\
& - c_\phi s_\psi\Big[\cos 2\theta_{23}\left\{\cos\delta \mathrm{Re}\left(c_\psi K_{31} - s_\psi e^{-i\delta}K_{32}\right) - \sin\delta \mathrm{Im}\left(c_\psi K_{31} - s_\psi e^{-i\delta}K_{32}\right)\right\} \\
& + i\left\{\sin\delta \mathrm{Re}\left(c_\psi K_{31} - s_\psi e^{-i\delta}K_{32}\right) + \cos\delta \mathrm{Im}\left(c_\psi K_{31} - s_\psi e^{-i\delta}K_{32}\right)\right\}\Big]\frac{\Delta_b}{h_3 - h_1}\left\{e^{-ih_3x} - e^{-ih_1x}\right\} \\
& + c_\phi c_\psi\Big[\cos 2\theta_{23}\left\{\cos\delta \mathrm{Re}\left(s_\psi K_{31} + c_\psi e^{-i\delta}K_{32}\right) - \sin\delta \mathrm{Im}\left(s_\psi K_{31} + c_\psi e^{-i\delta}K_{32}\right)\right\} \\
& + i\left\{\sin\delta \mathrm{Re}\left(s_\psi K_{31} + c_\psi e^{-i\delta}K_{32}\right) + \cos\delta \mathrm{Im}\left(s_\psi K_{31} + c_\psi e^{-i\delta}K_{32}\right)\right\}\Big]\frac{\Delta_b}{h_3 - h_2}\left\{e^{-ih_3x} - e^{-ih_2x}\right\} \\
& - c_{23}s_{23}c_\phi s_\phi c_\psi\left[c_\psi\,(K_{13} + K_{31}) - s_\psi\left(e^{i\delta}K_{23} + e^{-i\delta}K_{32}\right)\right]\frac{\Delta_b}{h_3 - h_1}\left\{e^{-ih_3x} - e^{-ih_1x}\right\} \\
& - c_{23}s_{23}c_\phi s_\phi s_\psi\left[s_\psi\,(K_{13} + K_{31}) + c_\psi\left(e^{i\delta}K_{23} + e^{-i\delta}K_{32}\right)\right]\frac{\Delta_b}{h_3 - h_2}\left\{e^{-ih_3x} - e^{-ih_2x}\right\}. \qquad \text{(D1)}
\end{aligned}
$$

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