Chapter 4

Analytical Study

The purpose of this analytical study is to develop a model that accurately predicts the flexural behavior of reinforced concrete beams strengthened with CFRP. Following previous research (Park & Paulay, 1975; An et. al, 1991), models were developed to track the response of the specimens. The model was also used to predict the nominal capacity and failure mode.

Since ductility is an important property for safe structural design, two procedures were defined to analyze the ductility of the composite beams. One method was based on the index commonly used, and the other was an index proposed by Naaman & Jeong (1995). These will be used to analyze the ductility of the specimens.

4.1 Flexural Behavior Model

The following model is based on derivations presented by Park & Paulay (1975) and An et. al (1991). It employs strain compatibility, force equilibrium, and the following assumptions:

1. Concrete tensile strength is ignored
2. Linear strain distribution through the cross-section
3. Small flexural deformations
4. No shear deformations
5. Perfect bond between different materials
6. Stress-strain curve for concrete is approximated by Figure 4.1 (Park & Paulay, 1975)
7. Stress-strain curve for the reinforcing steel is approximated by Figure 4.2 (Park & Paulay, 1975)

The stress-strain curve for concrete is approximated with the Hognestad curve by using two functions as shown in Figure 4.1 (Park & Paulay, 1975). The first equation is a parabolic function and is used up to the maximum compressive stress ($f_c^{'}$). The other equation is linear and decreases from the maximum compressive stress to the maximum strain. For this model, a maximum concrete strain of 0.003 was used (which is conservative due to the confinement of the concrete caused by the stirrups).

The stress-strain curve for the reinforcing steel was simplified as shown in Figure 4.2. The actual yield stress (f_y) of 70 ksi determined experimentally in Chapter 3 was used for the reinforcing steel. The modulus of elasticity was assumed to be 29000 ksi. The tested modulus (19000 ksi) was not used, since the extensiometer used might have produced inaccurate strain data. Beyond the yield point, the stress was assumed to be constant until failure ignoring the strain hardening effects.

The stress-strain curve for the CFRP was determined experimentally as discussed in Chapter 3. The 0°/90° orientation behaved linearly until failure at an average maximum stress and strain of 49 ksi and 0.012 in/in, respectively. For the ±45° orientation, the stress-strain curve was divided into a tri-linear approximation up to the average maximum stress and strain of 12.9 ksi and 0.039 in/in, respectively. Even though the strain of the CFRP in uniaxial tests exceeded 0.039 in/in, it was assumed that after the material was applied to the concrete, the strain of the CFRP would travel beyond this point.

4.1.1 Model Theory

The model was based on the internal forces, strains, and stresses in the cross-section of the composite beam as shown in Figure 4.3. Using similar triangles, the corresponding strains for the CFRP (ε_{CFRP}), tensile reinforcing steel (ε_S), and compressive reinforcing steel (ε_S') are calculated as shown:

$$\varepsilon_{CFRP} = \varepsilon_{EC}\left(\frac{d_{CFRP} - c}{c}\right) \tag{4-1}$$

$$\varepsilon_S = \varepsilon_{EC}\left(\frac{d_S - c}{c}\right) \tag{4-2}$$

$$\varepsilon_S' = \varepsilon_{EC}\left(\frac{c - d_S'}{c}\right) \tag{4-3}$$

where, ε_{EC} = the concrete compressive strain at the extreme compression fiber

d_{CFRP} = the distance from the centroid of the CFRP to the extreme compression fiber of the concrete

c = the distance from the neutral axis to the extreme compression fiber of the concrete

d_S = the distance from the centroid of the tension reinforcing steel to the extreme compression fiber of the concrete

d_S' = the distance from the centroid of the compressive reinforcing steel to the extreme compression fiber of the concrete

The following are equations for the stresses of the CFRP and reinforcing steel, and are obtained from their stress-strain behavior:

$$f_{CFRP} = E_{CFRP}\,\varepsilon_{CFRP} \tag{4-4}$$

$$If\ \varepsilon_S < \varepsilon_y,\ \ f_S = E_S\,\varepsilon_S\ \ otherwise\ f_S = f_y \tag{4-5}$$

$$If\ \varepsilon_S' < \varepsilon_y,\ \ f_S' = E_S'\,\varepsilon_S'\ \ otherwise\ f_S' = f_y \tag{4-6}$$

where, f_{CFRP} = the stress of the CFRP

E_{CFRP} = the modulus of elasticity of the CFRP

ε_y = the yield strain of the reinforcing steel

f_S = the stress in the tension reinforcing steel

E_S = the modulus of elasticity of the tension reinforcing steel

f_y = the yield stress of the reinforcing steel

f_S' = the stress in the compressive reinforcing steel

E_S' = the modulus of elasticity of the compressive reinforcing steel

Each of the corresponding internal forces can be determined by multiplying the stress by their cross-sectional areas. The forces are as follows:

$$T_{CFRP} = f_{CFRP} A_{CFRP} \quad (4\text{-}7)$$

$$T_S = f_S A_S \quad (4\text{-}8)$$

$$C_S' = f_S' A_S' \quad (4\text{-}9)$$

where, T_{CFRP} = the tension force of the CFRP

A_{CFRP} = the cross-sectional area of the CFRP

T_S = the tension force of the reinforcing steel

A_S = the cross-sectional area of the tension reinforcing steel

C_S' = the compressive force of the reinforcing steel

A_S' = the cross-sectional area of the compressive reinforcing steel

The value of the concrete compressive stress (f_c) is found from the stress-strain curve shown in Figure 4.1. The following equations idealize this curve:

$$If \quad O \le \varepsilon_C < \varepsilon_O, \quad f_C = f_C' \left[2\frac{\varepsilon_C}{\varepsilon_O} - \left(\frac{\varepsilon_C}{\varepsilon_O} \right)^2 \right] \quad (4\text{-}10)$$

$$If \quad \varepsilon_O \le \varepsilon_C < 0.003, \quad f_C = f_C' \left[1 - \frac{0.15}{0.003 - \varepsilon_C} (\varepsilon_C - \varepsilon_O) \right] \quad (4\text{-}11)$$

where, f_C' = the maximum compressive stress of concrete

ε_C = concrete strain at any given point

ε_O = the strain at the maximum compressive stress of concrete, and is defined as:

$$\varepsilon_O = \frac{2 f_C^{/}}{E_C} \qquad (4\text{-}12)$$

where, E_C = the modulus of elasticity of concrete

The concrete compressive force can be defined as follows:

$$C_C = \alpha \, f_C^{/} b \; c \qquad (4\text{-}13)$$

where, C_c = the compressive force of concrete

α = the mean stress factor

b = the width of the concrete beam

The mean stress factor (α) converts the actual stress-strain relationship of concrete into a rectangular stress-strain equivalent (usually is a value between 0-0.85). This parameter is calculated by integrating the area under the stress-strain curve with respect to the compressive strain of concrete as shown in Figure 4-4 (Park & Paulay, 1975). The integration is equated to the mean stress factor as follows:

$$Area \;\; under \;\; stress - strain \;\; curve = \int_O^{\varepsilon_{EC}} f_C d\,\varepsilon_C = \alpha \, f_C^{/} \varepsilon_{EC} \qquad (4\text{-}14)$$

Solving for the mean stress factor yields:

$$\alpha = \frac{\int_O^{\varepsilon_{EC}} f_C d\varepsilon_C}{f_C^{/} \varepsilon_{EC}} \qquad (4\text{-}15)$$

By substituting equations 4-10 and 4-11 into 4-15 and solving for the mean stress factor, the results are the following equations:

$$If \quad O \le \varepsilon_{EC} < \varepsilon_O, \ \ \alpha = \frac{\varepsilon_{EC}}{\varepsilon_O} - \frac{\varepsilon_{EC}^2}{3\varepsilon_O^2} \tag{4-16}$$

$$If \quad \varepsilon_O \le \varepsilon_{EC} < 0.003, \ \ \alpha = 1 - \frac{\varepsilon_O}{3\varepsilon_{EC}} - \frac{0.15}{0.003 - \varepsilon_O}\left(\frac{\varepsilon_{EC}^2 - 2\varepsilon_{EC}\varepsilon_O + \varepsilon_O^2}{2\varepsilon_{EC}}\right) \tag{4-17}$$

The concrete compressive force (C_C) acts at the centroid of the compression zone, which is defined as being a distance γc below the top of the beam (usually a value less than 0.35). The centroid factor (γ) is calculated with the first moment of area under a portion of the concrete stress-strain curve. Taking the first moment of area (M_O) about the origin yields:

$$M_O = A\varepsilon_{CEN} \tag{4-18}$$

where, A = the area under the stress-strain curve

ε_{CEN} = the strain at the centroid of the area under the stress-strain curve

The strain at the centroid of the area (ε_{CEN}) can be defined as:

$$\varepsilon_{CEN} = (1 - \gamma)\varepsilon_{EC} \tag{4-19}$$

By substituting equation 4-19 and the area under the stress-strain curve, as shown in equation 4-14, into equation 4-18 yields:

$$M_O = \left(\int_0^{\varepsilon_{EC}} f_C \, d\varepsilon_C\right)(1 - \gamma)\varepsilon_{EC} \tag{4-20}$$

Also, the first moment of area can be defined as,

$$M_O = \int_0^{\varepsilon_{EC}} f_C \varepsilon_C \, d\varepsilon_C \tag{4-21}$$

After equating 4-20 and 4-21, and solving for the centroid factor,

$$\gamma = 1 - \frac{\int_O^{\varepsilon_{EC}} \varepsilon_C f_C \, d\varepsilon_C}{\varepsilon_{EC} \int_O^{\varepsilon_{EC}} f_C \, d\varepsilon_C} \tag{4-22}$$

Next, by substituting 4-10 and 4-11 into 4-22 and integrating, the centroid factor is as follows:

$$If \quad 0 \le \varepsilon_{EC} < \varepsilon_O, \;\; \gamma = \frac{\frac{1}{3} - \frac{\varepsilon_{EC}}{12\varepsilon_O}}{1 - \frac{\varepsilon_{EC}}{3\varepsilon_O}} \tag{4-23}$$

$$If \quad \varepsilon_O \le \varepsilon_{EC} < 0.003, \;\; \gamma = -0.625 + \frac{3\varepsilon_{EC}^2 - 2A\varepsilon_{EC}^3 + 3A\varepsilon_O\varepsilon_{EC}^2 - 3\varepsilon_O^2 - A\varepsilon_O^3}{\varepsilon_{EC}\left[6\varepsilon_{EC} - 3A\varepsilon_{EC}^2 + 6A\varepsilon_{EC}\varepsilon_O - 6\varepsilon_O - 3A\varepsilon_O^2\right]} \tag{4-24}$$

where, $A = \dfrac{0.15}{0.003 - \varepsilon_O}$

The depth of the neutral axis from the extreme compression fiber (c) is obtained from the equilibrium of the internal forces of the beam. The total compressive forces are equal to the total tensile forces,

$$C_C + C_S^{/} = T_S + T_{CFRP} \tag{4-25}$$

By inserting 4-7, 4-8, 4-9, and 4-13 into 4-25,

$$\alpha f_C^{\prime} b c + f_S^{\prime} A_S^{\prime} = f_S A_S + f_{CFRP} A_{CFRP} \tag{4-26}$$

Note, if no CFRP is used (control beam), zero is inserted for the area of the CFRP (A_{CFRP}).

Next, by substituting equations 4-1 through 4-6 into 4-26, the neutral axis depth can be calculated using the quadratic equation. With this parameter known, the internal resisting moment (M) is obtain by taking the sum of the moments about the middle of the cross-section:

$$M = \alpha f_c^{\prime} b c \left[\frac{h}{2} - \gamma c \right] + f_S^{\prime} A_S^{\prime} \left[\frac{h}{2} - d_S^{\prime} \right] + f_S A_S \left[d_S - \frac{h}{2} \right] + f_{CFRP} A_{CFRP} \left[d_{CFRP} - \frac{h}{2} \right] \tag{4-27}$$

where, h = the height of the beam

The curvature of the beam is determined by considering a small element, dx, subjected to pure bending moments as shown in Figure 4.5 (Park & Paulay, 1975). The radius of curvature (R), the neutral axis from the extreme compression fiber, the concrete strain of the extreme compression fiber, and the tension steel strain, all change under loading. Assuming plane sections remain plane (linear behavior), the rotation between the ends of the element can be described by the following:

$$\frac{dx}{R} = \frac{\varepsilon_{EC} dx}{c} = \frac{\varepsilon_S dx}{d_S - c} \tag{4-28}$$

Therefore,

$$\frac{1}{R} = \frac{\varepsilon_{EC}}{c} \tag{4-29}$$

Since curvature (φ) is the inverse of the radius of curvature, it is defined by the following equation:

$$\varphi = \frac{\varepsilon_{EC}}{c} \tag{4-30}$$

Furthermore, the centerline deflection of the beam can be determined from a curvature (M/EI) diagram of the beam using the moment-area method. Assuming the beam rests on simple supports and four point loading is applied as shown in Figure 3.11, the moment of area under the curvature diagram about the support between midspan and the support yields the following equation for centerline deflection:

$$\Delta = \frac{251\varphi l^2}{2400} \tag{4-31}$$

where, Δ = centerline deflection of the beam
l = distance between supports of the beam

4.1.2 Application of Model

A spreadsheet for each CFRP orientation (0°/90° and ±45°) was created to predict the behavior of the strengthened reinforced concrete specimens. The procedure is based on the equations described in the previous section. When a value for the concrete strain at the extreme compression fiber is entered, the program will calculate the strain for the compressive and tensile steel reinforcement, CFRP strain, curvature, nominal moment, and deflection. The nominal moment and deflection at the points where the steel reinforcement begins to yield and at the maximum capacity are listed in Table 4.1 for each of the specimens.

Table 4.1: Theoretical Values from Flexural Behavior Model

CFRP Layers	Yield Moment, in-kip	Yield Deflection, in.	Maximum Moment, in-kip	Maximum Deflection, in.
None	505	0.72	543	1.83
Two 0°/90°	530	0.73	616	1.65
Three 0°/90°	544	0.73	647	1.58
Four 0°/90°	555	0.73	674	1.52
Two ±45°	518	0.71	581	1.73
Three ±45°	526	0.71	600	1.68
Four ±45°	532	0.71	618	1.64

Plots were generated by increasing the concrete strain in increments of 0.0001 in/in until the ultimate strain of either the concrete or CFRP was reached. Figures 4.6 and 4.7 show the theoretical moment-deflection plots for the beams using 0°/90° and ±45° CFRP orientations, respectively. The figures show very little difference in the behavior of the specimen with CFRP applied until the steel reinforcement begins to yield. After this point, the moment increases and the deflection decreases proportionally to the number of layers that are applied. All plots were terminated due to the concrete reaching the maximum strain of 0.003. Figures 4.8 and 4.9 contain the theoretical moment-CFRP strain plots for the beams using 0°/90° and ±45° orientations, respectively. These plots have the same appearance as the moment-deflection plots. All other theoretical plots (moment versus strains of the compressive steel reinforcement, tensile steel reinforcement, and concrete at the extreme compression fiber) are shown in Appendix C.

4.2 Ductility

Ductility is an important characteristic of any structural element. Since CFRP repair is a fairly new innovation, understanding the effect of this material on the ductility of a reinforced concrete beam is critical. Two methods of measuring this property are discussed in this section. One is based on the ductility index commonly used, and the other is a ductility index based on past research.

4.2.1 Introduction

Ductility is described as the ability of a structural element to sustain inelastic deformation without significant loss in resistance (Naaman, 1986). It is necessary for many reasons including safety, possible redistribution of load or moment, and design of structures subjected to seismic loading (Park & Paulay, 1975).

Safety is the most important consideration for the design of structures. Any type of brittle failure should be avoided, since this could limit warning time and cause lives to be endangered. If the structure possesses ductile behavior, it will be able to experience large deflections while still holding near ultimate loads. (Park & Paulay, 1975)

Redistribution of moments and loads can be important in the design of indeterminate structures. When an ultimate load is approached, a portion of a member in a structure reaches nominal capacity. If a plastic rotation can occur at this point while the load is maintained, additional load can be carried by other sections until their nominal capacity is reached. The ultimate load is finally reached when a suitable amount of plastic hinges are formed and a collapse mechanism is developed. This causes a different distribution of bending moment from that obtained by linear elastic analysis. Most reinforced concrete codes will allow for this type of redistribution if the structure has adequate ductility. (Park & Paulay, 1975)

Where structures are subjected to seismic loading, ductility is a very important consideration. Seismic design is based on energy absorption and dissipation by deformation of the structure after linear behavior. The amount of this inelastic deformation is proportional to the amount of ductility of the member. Therefore, if a structure is ductile enough, design loads can be lower than if the proper ductility is not present. (Park & Paulay, 1975)

4.2.2 Ductility Measurement Methods

4.2.2.1 Conventional Method

Ductility has generally been measured by a ratio called a ductility index or factor (μ). The ductility index is usually expressed as a ratio of rotation (θ), curvature (φ), or deflection (Δ) at failure to the corresponding property at yield, as shown below:

$$\mu = \frac{\theta_U}{\theta_Y} \qquad \mu = \frac{\varphi_U}{\varphi_Y} \qquad \mu = \frac{\Delta_U}{\Delta_Y} \tag{4-32}$$

For this study, deflection will be used as the primary measurement of ductility. Table 4.2 lists the theoretical ductility ratio and percent decrease of ductility with respect to the control beam for each of the specimens.

Table 4.2: Theoretical Ductility Ratios by Conventional Method

CFRP Layers	Ductility Ratio, in/in	% Decrease w/r to Control Beam
None	2.54	N/A
Two 0°/90°	2.26	11
Three 0°/90°	2.16	15
Four 0°/90°	2.08	18
Two ±45°	2.43	4
Three ±45°	2.37	7
Four ±45°	2.31	9

By examining the table, the percent decrease in ductility is 4% to 18% with respect to the control beam depending on the number and type of layers applied. The ±45° strengthened beams seems to behave more ductile than the beams applied with 0°/90° orientation, since the ±45° orientation decreases the ductility by only 4% to 9% while the 0°/90° orientation decreases the ductility by 11% to 18%. The theoretical data obtained with the conventional method are used in Chapter 5 to analyze the experimental beams.

4.2.2.2 Energy Method

The following method is based on experimental testing of prestressed concrete beams with FRP tendons by Naaman and Jeong (1995). They proposed a new ductility index that is expressed as a ratio of the total energy of the beam to the elastic energy released at failure. It is applicable to beams with steel reinforcement, FRP reinforcement, or a combination of both.

Reinforced concrete structures usually behave in a ductile manner if an appropriate amount of steel reinforcement is added. Ductility is achieved by inelastic deformation of the steel before failure. During this period, the concrete beam consumes much of the energy causing the elastic energy released at failure to be reduced. However, this is not the same circumstance for FRP reinforced beams, since this material can usually not attain inelastic deformation. This causes a tremendous amount of elastic strain energy to build up and be released at failure that exceeds that of steel reinforcement, as shown in Figure 4.10 (Naaman & Jeong, 1995). Because of the difference in elastic energy, an alternative ductility index was formulated.

The ductility index proposed is as follows:

$$\mu = \frac{1}{2}\left(\frac{E_{TOT}}{E_{EL}} + 1\right) \tag{4-33}$$

where the total energy (E_{TOT}) is calculated as the area under the moment-deflection curve up to the failure load. The elastic energy (E_{EL}) is the energy released at failure and can be found by investigating unloading tests. Alternatively, the elastic energy released at failure can be estimated by using the area of a triangle formed at the failure load with a weighted average slope of the two initial straight lines of the moment-deflection curve, as shown in Figure 4.11(Naaman & Jeong, 1995). For this study, the estimation method is utilized.

When FRP is applied to a reinforced concrete beam, the energy method should provide a lower value for the ductility index than obtain with the conventional method. However, for a reinforced concrete beam not applied with FRP, the energy method and conventional method should produce similar values for the ductility index. Thus, the energy method provides a common basis for comparison.

Table 4.3 lists the deflection ratio and percent decrease of ductility with respect to the control beam for each of the seven theoretical specimens.

Table 4.3: Theoretical Ductility Index by Energy Method

CFRP Layers	Ductility Index	% Decrease w/r to Control Beam
None	2.37	N/A
Two 0°/90°	1.87	21
Three 0°/90°	1.75	26
Four 0°/90°	1.65	30
Two ±45°	2.11	11
Three ±45°	2.05	14
Four ±45°	1.98	17

The energy method lowers the ductility index of the theoretical strengthened beams 11%-30% with respect to the control beam. As seen with the conventional method, the ±45° CFRP strengthened beams behaves more ductile than the 0°/90° strengthened beams, since the ±45° orientation decreases the ductility by 11% to 17% while the 0°/90° orientation decreases the ductility by 21% to 30%. The theoretical data obtained with the energy method are used in Chapter 5 to analyze the experimental beams.