Model reduction of linear time-periodic dynamical systems

Caleb C. Magruder

Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Serkan Gugercin, Chair
Christopher Beattie
John Rossi

May 1, 2013
Blacksburg, Virginia

Keywords: Model Reduction, Time-varying Systems

Model order reduction of linear time-periodic dynamical systems

Caleb C. Magruder

(ABSTRACT)

Few model reduction techniques exist for dynamical systems whose parameters vary with time. We have particular interest here in linear time-periodic dynamical systems; we seek a structure-preserving algorithm for model reduction of linear time-periodic (LTP) dynamical systems of large scale that generalizes from the linear time-invariant (LTI) model reduction problem.

We extend the familiar LTI system theory to analogous concepts in the LTP setting. First, we represent the LTP system as a convolution operator of a bivariate periodic kernel function. The kernel suggests a representation of the system as a frequency operator, called the Harmonic Transfer Function. Second, we exploit the Hilbert space structure of the family of LTP systems to develop necessary conditions for optimal approximations.

Additionally, we show an *a posteriori* error bound written in terms of the $\mathcal{H}_2$ norm of related LTI multiple input/multiple output system. This bound inspires an algorithm to construct approximations of reduced order.

To verify the efficacy of this algorithm we apply it to three models: (1) fluid flow around a cylinder by a finite element discretization of the Navier-Stokes equations, (2) thermal diffusion through a plate modeled by the heat equation, and (3) structural model of component 1r of the Russian service module of the International Space Station.

Dedication

To my parents, Clarke and Lucille Magruder, for their steadfast guidance in times of celebration and trial alike. Their unconditional support has had been an immeasurable blessing on my life.

Acknowledgments

First and foremost, I would like to thank my advisors, Serkan Gugercin and Christopher Beattie, who have served as my official and unofficial advisors during my last several years at Virginia Tech. These two have had an unequivocal influence on my mathematical training, the direction of my studies and introduction to the academic community. Both have spent innumerable hours in research meetings guiding this project and others to completion, generously providing feedback and criticism. Indeed, my undergraduate research with them sparked my interest in mathematics as a career. Their enthusiasm for research is contagious and has made working for them fun and exciting.

Many thanks to John Rossi for serving on my committee. His insight was invaluable when I was stuck on several proofs in this paper. Additionally, Dr. Rossi's courses in real analysis and complex analysis have been my most challenging and most rewarding coursework to date. His instruction style has served as a model for me as I have begun teaching my own classes.

I would also like to thank Camron Withrow for tolerating my many brainstorming sessions, inspired and ridiculous alike. Although an algebraist, Camron is more adept at applied mathematics than he will ever admit.

Thanks to Jeff Borggaard for teaching me numerical analysis and providing me with the fluid dynamics model in this paper. Thanks also to Chris Jarvis for providing the data set I used in the simulation.

Finally, thanks to my officemates Craig Powers, George Kuster, Will Frey, Kiyho Moon, Tiger Wang, Greg Marx, Dave Wells, and Ryan Shifler for their camaraderie and for keeping the office space alive. Our many conversations ranged from inane to completely absurd, offering a healthy distraction from my work.

Contents

List of Figures

List of Acronyms

LTI	Linear time-invariant
LTP	Linear time-periodic
LTV	Linear time-varying
SISO	Single input/single output
MIMO	Multiple input/multiple output
EMP	Exponentially modulated signal
ODE	Ordinary differential equation
LO	Local oscillator
IRKA	Iterated Rational Krylov Algorithm
POD	Proper Orthogonal Decomposition (a.k.a. Principle Component Analysis)

List of Symbols

$\delta(t)$	Dirac delta function
G	linear dynamical system
$\tilde{G}$	reduced order linear dynamical system
ω	frequency is radians
ω_0	fundamental frequency of a periodic system
$g(t,\tau)$	impulse response of dynamical system G, $u(t) = \delta(t-\tau)$
$\tilde{g}(t,\tau)$	reduced order impulse response of dynamical system $\tilde{G}$, $u(t) = \delta(t-\tau)$
$g_k(t-\tau)$	Fourier coefficient of $g(t,\tau)$, referred to subsystems of G
$\tilde{g}_k(t-\tau)$	Fourier coefficient of $\tilde{g}(t,\tau)$, referred to subsystems of reduced order $\tilde{G}$
$\hat{g}_k(s)$	Laplace transform of subsystem $g_k(t)$
$\tilde{\hat{g}}_k(s)$	Laplace transform of subsystem $\tilde{g}_k(t)$
$*$	convolution operator, discrete or continuous
$\mathbf{A}$	finite dimensional matrix
$(\mathbf{A})^*$	conjugate transpose of matrix $\mathbf{A}$
$[\mathbf{A}]_{m,n}$	(m,n)th element of matrix $\mathbf{A}$
$\mathbf{e}_j$	unit column vector with scalar 1 in the jth element, 0 elsewhere
$\Vert\mathbf{A}\Vert_F$	Frobenium norm of matrix $\mathbf{A}$
$\mathbf{A}(t)$	periodic finite dimensional matrix
$\mathbf{A}_n$	Fourier coefficient of symbol $\mathbf{A}(t)$
$\mathcal{T}(\mathbf{A}(t))$	Toeplitz form of matrix $\mathbf{A}(t)$
$\mathrm{diag}(a_1,\ldots,a_k)$	diagonal matrix with elements $a_1,\ldots,a_k$
$\mathrm{blkdiag}(\{\mathbf{A}_i\})$	block diagonal matrix with elements $\{\mathbf{A}_i\}$
$\mathrm{Res}(g,\lambda)$	the residue of a meromorphic function, g, at complex value λ
$\mathcal{A}$	infinite dimensional vector or doubly infinite dimensional matrix
$\hat{\mathbf{x}}(s)$	Laplace transform of $\mathbf{x}(t)$
$\mathcal{G}(s)$	harmonic transfer function of dynamical system G
$\tilde{\mathcal{G}}(s)$	harmonic transfer function of reduced-order dynamical system, $\tilde{G}$
$\mathfrak{g}_{m,n}$	(m,n)th element of harmonic transfer function, $[\mathcal{G}(s)]_{m,n}$
$\mathcal{H}_p$	Hardy space defined on the open right half plane

Chapter 1

Introduction

1.1 Problem Formulation

We consider the single input / single output (SISO) linear dynamical systems described by a state space representation,

$$G : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)u(t) \\ y(t) = \mathbf{c}^T(t)\mathbf{x}(t) \end{cases} \tag{1.1}$$

where $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$ and $\mathbf{b}(t), \mathbf{c}(t) \in \mathbb{R}^{n\times 1}$ with no feed through ($d(t) \equiv 0$). We choose the state space parameters to be periodic, $\mathbf{A}(t) = \mathbf{A}(t+T)$, $\mathbf{b}(t) = \mathbf{b}(t+T)$ and $\mathbf{c}(t) + \mathbf{c}(t+T)$ for some period $T > 0$. Further, we consider input functions $u(t)$ defined on the entire real line with bounded energy: $\int_{-\infty}^{\infty} |u(t)|^2 dt < \infty$. We assume that sufficient conditions for stability over all $u \in \mathcal{L}_2$ are satisfied.

We call $\mathbf{A}(t)$, $\mathbf{b}(t)$ and $\mathbf{c}(t)$ the *state space parameters* of the system G. For convenience, we can order the parameters in the following way to write the system as

$$G = \left[\begin{array}{c|c} \mathbf{A}(t) & \mathbf{b}(t) \\ \hline \mathbf{c}^T(t) & \mathbf{0} \end{array}\right]. \tag{1.2}$$

Our goal is to find, for any order $r \ll n$, a reduced-order model represented as

$$\tilde{G} : \begin{cases} \dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}(t)\tilde{\mathbf{x}}(t) + \tilde{\mathbf{b}}^T(t)u(t) \\ \tilde{y}(t) = \tilde{\mathbf{c}}^T(t)\tilde{\mathbf{x}}(t) \end{cases} \tag{1.3}$$

where $\tilde{\mathbf{A}}(t) \in \mathbb{R}^{r\times r}$ and $\tilde{\mathbf{b}}(t), \tilde{\mathbf{c}}(t) \in \mathbb{R}^{r\times 1}$ are also T-periodic and such that $\tilde{y}(t) \approx y(t)$ for a variety of inputs $u(t)$.

1.2 Previous Work

A considerable amount of literature has been written to discuss this problem in the linear time-invariant (LTI) setting. For an introduction to the topic of model reduction, see the seminal work, *Approximation of Large-Scale Dynamical Systems*, Antoulas (2009), [2]. Discussion of continuous-time LTP systems, however, is lacking in the model reduction community. Existing literature on LTP systems focuses on control, spectral analysis, and harmonic response.

Major contributions include:

- *Wereley and Hall* - Expand state-space systems with periodic matrices and exponentially modulated periodic inputs into Fourier series and harmonics are equated. This is labeled the *harmonic balance method* and yields the *Harmonic Transfer Function*, a generalization of the frequency response operator. The operator is infinite-dimensional but consists of a countable number of time-invariant LTI components. [19, 21, 22, 23]
- *Sandberg* - Expand the impulse response using Fourier analysis. The frequency response of the system is understood as a countable collection of infinite-dimensional LTI transfer functions. Sandberg coins the term *Floquet-Fourier representation*, an equivalent realization of the state-space parameters that make analytic expressions of frequency responses considerably easier to represent.

  Develop an algorithm to approximate these infinite-dimensional LTI transfer functions by a truncation of power series centered at $z = \infty$. To the writer's knowledge, this is the only existing method for LTP model approximation that exploits the harmonic nature of LTP system dynamics. [16, 17, 18]
- *Zhou and Hagiwara* - Address dynamical system norm computation using truncations of the Harmonic Transfer Function. Truncation error is addressed and convergence of the truncations is proven. Additionally, Zhou and Hagiwara discuss the spectral characteristics of LTP systems as input/output operator. [24, 25, 26, 27, 28]
- *Mollerstedt* - Applied LTP system theory to the following problems: (1) power distribution networks with nonlinear and switching dynamics, (2) diode converter locomotive dynamics, specifically linearization around a nominal solution from numerical simulation, and (3) power converter modeling, applied to a micro-turbine line side converter. [12, 17, 18]

This list is not exhaustive, but summarizes some relevant work that served as a primer for the creation of this thesis. This list constitutes a collection of references frequently cited throughout the following chapters.

1.3 Thesis Organization and Contribution

This project was initiated with the goal of developing a model reduction approach in the LTP setting. Chapters 6 and 7 focus primarily on the author's solution to the original question. In the process of tackling this problem, numerous connections between existing LTP system analysis frameworks have been made. These results have been distributed throughout chapters 3, 4, and 5 to provide the reader with a narrative of the essential concepts within LTP system theory. Proofs when provided are the original work of the author unless otherwise noted.

- *Chapter 2* reviews background LTI system theory. In particular, we discuss the $\mathcal{H}_2$ dynamical system norm and best approximation. We discuss necessary conditions for best approximations in the $\mathcal{H}_2$ sense and a fixed-point algorithm that converges to best approximations, called the Iterated Rational Krylov Algorithm (IRKA).

- *Chapter 3* reviews background LTP system theory. We develop a preliminary understanding of the steady-state response as a frequency operator. Additionally, we review the Floquet and Toeplitz transformations, necessary for discussions in future chapters.

- *Chapter 4* introduces the Harmonic Transfer Function from Wereley and Hall's seminal paper, [22]. The harmonic balancing method is implemented and dynamical system norms are written in term of the Harmonic Transfer Function.

- *Chapter 5* analyzes the input/output system as a convolution operator whose kernel is the time-periodic impulse response. A Fourier expansion of the impulse response enables us to work with each harmonic response individually. Finally, the family of LTP system is shown to exist within a Hilbert space and an inner product and corresponding norm is developed.

- *Chapter 6* proposes a projection-based algorithm for structure-preserving approximation. We show an *a posteriori* error bound exists that can be written in terms of the $\mathcal{H}_2$ norm of related LTI multiple input/multiple output system. Additionally, we develop necessary conditions for optimal approximations that generalize from the LTI setting by exploiting structured orthogonality conditions of the underlying Hilbert space.

- *Chapter 7* tests the efficacy of the proposed algorithm by reducing three LTP dynamical systems: (1) fluid flow around a cylinder by a finite element discretization of Navier-Stokes, (2) thermal diffusion through a plate modeled by the heat equation, and (3) structural model of component 1r (Russian service module) of the International Space Station.

Chapter 2

Linear Time-Invariant Model Reduction

Define continuous time linear time-invariant (LTI) dynamical systems as any system of the form of (1.1) such that $\mathbf{A}(t)$, $\mathbf{b}(t)$ and $\mathbf{c}(t)$ are time-invariant. That is, (1.1) reduces to

$$G : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ y(t) = \mathbf{c}^T\mathbf{x}(t) \end{cases} \tag{2.1}$$

and the reduced order approximation we seek is of the form

$$\tilde{G} : \begin{cases} \dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{b}}^T u(t) \\ \tilde{y}(t) = \tilde{\mathbf{c}}^T\tilde{\mathbf{x}}(t) \end{cases} \tag{2.2}$$

with $\mathbf{x}(t) \in \mathbb{R}^n$, $\tilde{\mathbf{x}} \in \mathbb{R}^r$, $r \ll n$ and $\tilde{y}(t) \approx y(t)$ for a variety of bounded energy inputs $u(t)$. We define the dynamical system G to be **asymptotically stable** if the state space parameter $\mathbf{A}$ is Hurwitz, that is if its eigenvalues are in the open left-half plane.

2.1 Transfer Function

Define the **impulse response** of a system to be the output of the system driven with the impulse function, $\delta(t)$. Denote $\hat{y}(s) = \mathcal{L}\{y(t)\}$, $\hat{u}(s) = \mathcal{L}\{u(t)\}$ as the Laplace transforms of $y(t)$ and $u(t)$ respectively. Then we can write the input/output mapping $G : \hat{u} \mapsto \hat{y}$ as $\hat{y}(s) = G(s)\hat{u}(s) = \mathbf{c}^T[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b}\hat{u}(s)$, where $G(s)$ is the Laplace transform of the impulse response, $g(t)$.

We define $G(s) = \mathbf{c}^T[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b}$ to be the **transfer function** of the dynamical system G. As the system is finite dimensional, it is easy to see that $G : \mathbb{C} \to \mathbb{C}$ forms a proper rational meromorphic function with poles at the eigenvalues of $\mathbf{A}$.

2.2 Projection-Based Model Reduction

Let $\mathcal{V}$ and $\mathcal{W}$ be r-dimensional subspaces of $\mathbb{R}^n$ such that $\mathcal{V} \bigcap \mathcal{W}^{\perp} = \{0\}$. (No vector in $\mathcal{V}$ is orthogonal to any vector in $\mathcal{W}$ except the trivial 0 vector.) Choose matrices $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}$ so that $\mathcal{V} = \mathrm{Ran}(\mathbf{V})$ and $\mathcal{W} = \mathrm{Ran}(\mathbf{W})$. Then we know $\mathbf{W}^T\mathbf{V}$ is nonsingular. Without loss of generality we assume $\mathbf{W}^T\mathbf{V} = \mathbf{I}$. We can approximate the state space $\mathbf{x}(t) \in \mathbb{R}^n$ with an r-dimensional state, $\tilde{\mathbf{x}}(t) \in \mathbb{R}^r$ such that $\mathbf{x}(t) \approx \mathbf{V}\tilde{\mathbf{x}}(t)$. If we set the error $\mathbf{V}\dot{\tilde{\mathbf{x}}}(t) - \mathbf{A}\mathbf{V}\tilde{\mathbf{x}}(t) - \mathbf{b}u(t)$ to be perpendicular to the $\mathcal{W}$ space we have constructed a Petrov-Galerkin approximation to the linear dynamical system. Then

$$\mathbf{W}^T(\mathbf{V}\dot{\tilde{\mathbf{x}}}(t) - \mathbf{A}\mathbf{V}\tilde{\mathbf{x}}(t) - \mathbf{b}u(t)) = 0 \tag{2.3}$$

$$\tilde{y}(t) = \mathbf{c}^T\mathbf{V}\dot{\tilde{\mathbf{x}}}(t). \tag{2.4}$$

Then our reduced model state-state parameters become $\tilde{\mathbf{A}} = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\tilde{\mathbf{b}} = \mathbf{W}^T\mathbf{b}$, and $\tilde{\mathbf{c}} = \mathbf{c}^T\mathbf{V}$. Clearly our choice of $\mathbf{V}$ and $\mathbf{W}$ determine the accuracy of our reduced order approximation, $\tilde{G} = \left[\begin{array}{c|c} \tilde{\mathbf{A}} & \tilde{\mathbf{b}} \\ \hline \tilde{\mathbf{c}} & \end{array}\right]$.

2.3 Multiple input/multiple output systems

We now consider a system with an *input vector* $\mathbf{u}(t) \in \mathbb{R}^m$ and *output vector* $\mathbf{y}(t) \in \mathbb{R}^p$, so that $\mathbf{B} \in \mathbb{R}^{n \times m}$ and $\mathbf{C} \in \mathbb{R}^{p \times n}$ for some $m, p \geq 1$. We write the dynamical system,

$$G : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases} \qquad \Longrightarrow \qquad \mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}. \tag{2.5}$$

The transfer function here becomes matrix valued, $\mathbf{G}(s) \in \mathbb{C}^{p \times m}$. Approximations to $\mathbf{G}$ should share input/output dimensions. Indeed, if $\mathbf{W}, \mathbf{V} \in \mathbb{R}^{n \times r}$ with $\mathbf{W}^T\mathbf{V} = \mathbf{I}$ we can define a (matrix-valued) reduced order transfer function

$$\tilde{\mathbf{G}}(s) = \mathbf{C}\mathbf{V}(s\mathbf{I} - \mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{B}. \tag{2.6}$$

Then the reduced order matrix-valued transfer function $\tilde{\mathbf{G}}(s) \in \mathbb{R}^{p \times m}$ is of the same dimension as the full order system, $\mathbf{G}(s)$.

2.4 The $\mathcal{H}_2$ Hilbert Space

Let $\mathcal{H}_2$ denote the Hardy space defined on the open right-half plane. That is, $\mathcal{H}_2$ is the set of functions, $f(s)$, that are analytic for s in the open right-half plane, $\mathrm{Re}(s) > 0$, and such that for each fixed $Re(s) = x > 0$, $f(x + \imath y)$ is square integrable as a function of $y \in (-\infty, \infty)$ in such a way that

$$\sup_{x>0} \int_{-\infty}^{\infty} |f(x + \imath y)|^2 dy < \infty. \tag{2.7}$$

For a discussion of the following results, see Gugercin et al. (2008), [9]. The $\mathcal{H}_2$ space holds our interest because transfer functions of stable SISO finite-dimensional dynamical systems, $G(s) = \mathbf{c}^T[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b}$, are elements of $\mathcal{H}_2$. If $G(s)$ and $H(s)$ are in $\mathcal{H}_2$ then we define the inner product to be

$$\langle G, H\rangle_{\mathcal{H}_2} := \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G(\imath\omega)}H(\imath\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(-\imath\omega)H(\imath\omega)d\omega \tag{2.8}$$

and hence with a norm

$$\|G\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\imath\omega)|^2 d\omega\right)^{1/2}. \tag{2.9}$$

The $\mathcal{H}_2$ norm is of particular interest to because it bounds the the dynamical system output for bounded energy inputs, $u(t)$:

$$\|y\|_{\mathcal{L}_\infty} \leq \|G\|_{\mathcal{H}_2}\|u\|_{\mathcal{L}_2}. \tag{2.10}$$

Additionally, we can view the $\|G\|_{\mathcal{H}_2}$ as a measure of the energy of the impulse response, $g(t)$:

$$\|G\|_{\mathcal{H}_2} = \int_0^{\infty} \overline{g}(t)g(t)dt. \tag{2.11}$$

Both of these results utilize Parseval's theorem which states that the Fourier transformation is an isometry between square-integrable time-domain and frequency-domain functions. A discussion of these results can be found in Gugercin et al. (2008), [9].

MIMO Dynamical System Norms

In the case that G and H are MIMO and of the same input/output dimensions, $p \times m$, we generalize the inner product and norm equations above to

$$\langle \mathbf{G}, \mathbf{H}\rangle_{\mathcal{H}_2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left(\overline{\mathbf{G}(\imath\omega)}\mathbf{H}^T(\imath\omega)\right) d\omega \tag{2.12}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left(\mathbf{G}(-\imath\omega)\mathbf{H}^T(\imath\omega)\right) d\omega \tag{2.13}$$

and

$$\|\mathbf{G}\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|\mathbf{G}(\imath\omega)\|_F^2 d\omega\right)^{1/2} \tag{2.14}$$

$$= \int_0^{\infty} \text{trace}(\boldsymbol{g}^*(t)\boldsymbol{g}(t))dt \tag{2.15}$$

where $\boldsymbol{g}(t) \in \mathbb{R}^{p\times m}$ is the matrix-valued impulse response of the MIMO dynamical system, G.

2.5 $\mathcal{H}_2$ Pole-Residue Formulation

Suppose $G(s)$ and $H(s)$ are transfer functions of stable, SISO systems. The next result is used frequently throughout this paper as an alternative representation of the inner product which follows from residue calculus.

Theorem 2.5.1 *(Gugercin et al. (2008), [9]) Suppose that $G(s)$ has poles at $\lambda_1, \lambda_2 \ldots, \lambda_n$ and $H(s)$ has poles $\mu_1, \mu_2, \ldots, \mu_m$ both sets contained in the open left-half plane. Then*

$$\langle G, H\rangle_{\mathcal{H}_2} = \sum_{k=1}^{m} Res[G(-s)H(s), \mu_k] = \sum_{k=1}^{n} Res[H(-s)G(s), \lambda_k] \tag{2.16}$$

In particular,

- *if μ_k is a simple pole of $H(s)$, then*

  $$Res[G(-s)H(s), \mu_k] = G(-\mu_k)Res[H(s), \mu_k]$$

- *if μ_k is a double pole of $H(s)$, then*

  $$Res[G(-s)H(s), \mu_k] = G(-\mu_k)Res[H(s), \mu_k] - G'(-\mu_k)\cdot h_0(\mu_k)$$

  where $h_0(\mu_k) = \lim_{s\to\mu_k}((s-\mu_k)^2 H(s))$.

2.6 Optimal $\mathcal{H}_2$ Model Reduction

In this paper, we concern ourselves with the $\mathcal{H}_2$ dynamical system norm for two reasons:

- $\langle\cdot,\cdot\rangle_{\mathcal{H}_2}$ forms a Hilbert space, providing convenient necessary conditions for optimal approximations.

- The system output $y(t)$ is bounded by the $\mathcal{H}_2$ system norm,

$$\max|y(t)-\tilde{y}(t)| \leq \|G-\tilde{G}\|_{\mathcal{H}_2}\|u(t)\|_{\mathcal{L}_2}$$

The optimal $\mathcal{H}_2$ model reduction problem is stated as follows: we seek $\tilde{G}$ that minimizes

$$\|G-\tilde{G}\|_{\mathcal{H}_2} = \min_{\substack{\dim(\tilde{H})=r \\ \tilde{H}:\ \text{stable}}} \|G-\tilde{H}\|_{\mathcal{H}_2}. \tag{2.17}$$

Define a rational function to be **proper** if its denominator is of a greater order than its numerator. We will have occasion to use the following theorems exploiting the Hilbert space structure of $\mathcal{H}_2$:

Theorem 2.6.1 (Gugercin et al. (2008), [9]) ***Structured Orthogonality Conditions*** *Let $\mu_1, \mu_2, \ldots, \mu_r \subset \mathbb{C}$ be distint points int he open left-half plane and define $\mathcal{M}(\mu)$ to be the set of all proper rational functions that have simple poles exactly at $\mu_1, \mu_2, \ldots, \mu_r$. Then*

- *$H \in \mathcal{M}(\mu)$ implies that H is the transfer function of a stable dynamical system with dim$(H) = r$;*
- *$\mathcal{M}(\mu)$ is an $(r-1)$-dimensional subspace of $\mathcal{H}_2$;*
- *$\tilde{G} \in \mathcal{M}(\mu)$ solves*

$$\|G-\tilde{G}\|_{\mathcal{H}_2} = \min_{\tilde{H}\in\mathcal{M}(\mu)} \|G-\tilde{H}\|_{\mathcal{H}_2} \tag{2.18}$$

if and only if

$$\langle G-\tilde{G}, H\rangle_{\mathcal{H}_2} = 0 \qquad \textit{for all } H \in \mathcal{M}(\mu). \tag{2.19}$$

Furthermore the solution, $\tilde{G}$, to (2.18) *exists and is unique.*

Theorem 2.6.2 (Meier and Luenberger (1967), [11]) ***Interpolation-based optimality conditions****. Given a stable SISO system $G(s) = \mathbf{c}^T[s\mathbf{I}-\mathbf{A}]^{-1}\mathbf{b}$, let $\tilde{G}(s) = \tilde{\mathbf{c}}^T[sI-\tilde{\mathbf{A}}]^{-1}\tilde{\mathbf{b}}$ be a local minimizer of dimension r for the optimal $\mathcal{H}_2$ model reduction problem,* (2.17)*, and suppose that $\tilde{G}(s)$ has simple poles at $\tilde{\lambda}_i$, $i = 1, \ldots, r$. Then $\tilde{G}(s)$ interpolates both $G(s)$ and its first derivative at $-\lambda_i$, $i = 1, \ldots, r$:*

$$\tilde{G}(-\tilde{\lambda}_i) = G(-\tilde{\lambda}_i) \quad \textit{and} \quad \tilde{G}'(-\tilde{\lambda}_i) = G'(-\tilde{\lambda}_i) \quad \textit{for } i = 1, \ldots, r. \tag{2.20}$$

Optimal model reduction then can be achieved via algorithms that search within the candidate space of low rank dynamical systems, $\tilde{G}$, that satisfy the interpolation conditions above. One such algorithm, Iterated Rational Krylov Algorithm (IRKA), was published in Gugercin et al. (2008), [9].

Theorem 2.6.3 (Gugercin et al. (2008), [9]) *Assume that* $\tilde{\mathbf{G}}(s)$ *is an optimal MIMO reduced order model minimizing* $\|\mathbf{G} - \tilde{\mathbf{G}}\|_{\mathcal{H}_2}$ *and suppose further that* $\tilde{\mathbf{G}}$ *has simple poles* $\tilde{\lambda}_i$ *with* $\tilde{\mathbf{G}}(s) = \sum_{k=1}^{n} \frac{1}{s - \tilde{\lambda}_k} \tilde{\mathbf{c}}_k \tilde{\mathbf{b}}_k^T$ *and the residue of* $\tilde{\mathbf{G}}(s)$ *at* $\tilde{\lambda}_k$ *is matrix valued and rank one:* $Res[\tilde{\mathbf{G}}(s), \tilde{\lambda}_k] = \tilde{\mathbf{c}}_k \tilde{\mathbf{b}}_k^T$*. Then necessary conditions for the MIMO optimal* $\mathcal{H}_2$ *reduction problem are:*

$$\mathbf{G}(-\tilde{\lambda}_k)\tilde{\mathbf{b}}_k = \tilde{\mathbf{G}}(-\tilde{\lambda}_k)\tilde{\mathbf{b}}_k \tag{2.21}$$

$$\tilde{\mathbf{c}}_k^T \mathbf{G}(-\tilde{\lambda}_k) = \tilde{\mathbf{c}}_k^T \tilde{\mathbf{G}}(-\tilde{\lambda}_k) \tag{2.22}$$

$$\tilde{\mathbf{c}}_k^T \mathbf{G}'(-\tilde{\lambda}_k)\tilde{\mathbf{b}}_k = \tilde{\mathbf{c}}_k^T \tilde{\mathbf{G}}'(-\tilde{\lambda}_k)\tilde{\mathbf{b}}_k \tag{2.23}$$

2.7 The Iterative Rational Krylov Algorithm

Algorithm: MIMO $\mathcal{H}_2$ Optimal Tangential Interpolation Method
(Antoulas et al. (2010), [3])

```
1. Make an initial $r$-fold shift selection: $\{\sigma_1, \ldots, \sigma_r\}$ that is closed under conjugation and initial tangent directions $\hat{\mathbf{b}}_1, \ldots, \hat{\mathbf{b}}_r$ and $\hat{\mathbf{c}}_1, \ldots, \hat{\mathbf{c}}_r$, also closed under conjugation.
2. Choose $\mathbf{V}$ and $\mathbf{W}$ so that $\text{Ran}(\mathbf{V}) = \text{span}\left\{(\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{b}}_1 \ \cdots \ (\sigma_r\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{b}}_r\right\}$, $\text{Ran}(\mathbf{W}) = \text{span}\left\{(\sigma_1\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{C}^T\hat{\mathbf{c}}_1 \ \cdots \ (\sigma_r\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{C}^T\hat{\mathbf{c}}_r\right\}$, and $\mathbf{W}^T\mathbf{V} = \mathbf{I}$.
3. while (not converged)
    (a) $\tilde{\mathbf{A}} = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\tilde{\mathbf{B}} = \mathbf{W}^T\mathbf{B}$ and $\tilde{\mathbf{C}} = \mathbf{C}\mathbf{V}$
    (b) Compute $\mathbf{Y}^*\tilde{\mathbf{A}}\mathbf{X} = \text{diag}(\tilde{\lambda_i})$ and $\mathbf{Y}^*\mathbf{X} = \mathbf{I}$ where $\mathbf{Y}^*$ and $\mathbf{X}$ are the left and right eigenvectors of $\lambda\mathbf{I} - \tilde{\mathbf{A}}$.
    (c) $\sigma_i \leftarrow -\lambda_i(\tilde{\mathbf{A}})$ for $i = 1, \ldots, r$, $\hat{\mathbf{b}}_i^* \leftarrow \mathbf{e}_i^T\mathbf{Y}^*\tilde{\mathbf{B}}$ and $\hat{\mathbf{c}}_i \leftarrow \tilde{\mathbf{C}}\mathbf{X}\mathbf{e}$.
    (d) Choose $\mathbf{V}$ and $\mathbf{W}$ so that $\text{Ran}(\mathbf{V}) = \text{span}\left\{(\sigma_1\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{b}}_1 \ \cdots \ (\sigma_r\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{b}}_r\right\}$, $\text{Ran}(\mathbf{W}) = \text{span}\left\{(\sigma_1\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{C}^T\hat{\mathbf{c}}_1 \ \cdots \ (\sigma_r\mathbf{I} - \mathbf{A}^T)^{-1}\mathbf{C}^T\hat{\mathbf{c}}_r\right\}$, and $\mathbf{W}^T\mathbf{V} = \mathbf{I}$.
4. $\tilde{\mathbf{A}} = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\tilde{\mathbf{B}} = \mathbf{W}^T\mathbf{B}$ and $\tilde{\mathbf{C}} = \mathbf{C}\mathbf{V}$
```

Upon convergence, the reduced order system, $\tilde{G} = \left[\begin{array}{c|c} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \hline \tilde{\mathbf{C}} & \end{array}\right]$ will satisfy the necessary conditions according to Theorem 2.6.3.

This algorithm is also known as the Iterative Rational Krylov Algorithm (IRKA). Introduced in Gugercin et al. (2008), [9], the process converges to locally optimal approximations in the $\mathcal{H}_2$ sense very rapidly. Since it only requires large-scale linear solves and eigendecompositions of $(r \times r)$-dimensional matrices, this algorithm is appealing in large-scale settings. Its primary competitor in the LTI model reduction community, Balanced Truncation, requires solutions to large-scale Lypanov equations, an operation that scales very poorly.

This paper seeks to develop analogous theory for a broader class of dynamical systems, those with linear time-periodic coefficients in particular.

Chapter 3

Linear Time-Periodic Dynamical Systems

We return to considering the family of linear time-periodic (LTP) dynamical systems with finite-dimensional state-space realization:

$$G : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)u(t) \\ y(t) = \mathbf{c}^T(t)\mathbf{x}(t) \end{cases} \tag{3.1}$$

We make a set of assumptions about the linear operator $G : \mathcal{L}_2 \to \mathcal{L}_2$ that serves as a map from input signals $u \in \mathcal{L}_2$ to output signals $y \in \mathcal{L}_2$.

(a) G is bounded

(b) G has a finite-dimensional state-space realization (3.1), whose parameters $\mathbf{A}(t)$, $\mathbf{b}(t)$, and $\mathbf{c}(t)$ are all T-periodic.

(c) G has a time-domain representation

$$y(t) = Gu = \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau \tag{3.2}$$

where $g(t,\tau)$ is the impulse response of the system excited by the input $\delta(t-\tau)$.

(d) G is causal. Explicitly,

$$g(t,\tau) = 0, \qquad \text{for all} \quad t < \tau \tag{3.3}$$

(e) g has uniform exponential decay. That is, there exist positive constants κ_1, κ_2 such that,

$$|g(t,\tau)| \le \kappa_1 \cdot e^{-\kappa_2(t-\tau)}, \qquad \text{for all} \quad t > \tau \tag{3.4}$$

For a discussion of the class of systems that satisfy these constraints, see Sandberg et al. (2004), [17] and Sandberg (2006), [16].

Remark In the case that G is linear-time invariant, the impulse response satisfies

$$g(t,\tau) = g(t-\tau, 0), \qquad \text{for all} \quad t > \tau \tag{3.5}$$

Like the time-invariant case, the system G represents an input-output map via a kernel convolution of the system's impulse response.

Remark Since the state-space parameters are T-periodic, so is the impulse response:

$$g(t+T, \tau+T) = g(t,\tau), \quad forall \quad t \geq \tau. \tag{3.6}$$

Remark The impulse response of the system G is given by

$$g(t,\tau) = \mathbf{c}^T(t)\Phi_{\mathbf{A}}(t,\tau)\mathbf{b}(\tau), \ t > \tau \tag{3.7}$$

where $\Phi_{\mathbf{A}}(t,\tau)$ is the transition matrix for $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$.

3.1 Exponentially Modulated Periodic Signals

Definition $u(t)$ is an *exponentially modulated periodic* (EMP) signal if it can be written in the form

$$u(t) = e^{st} \sum_{n=-\infty}^{\infty} u_n e^{\imath n\omega_0 t} = \sum_{n=-\infty}^{\infty} u_n e^{s_n t}, \qquad t \geq 0 \tag{3.8}$$

with $s_n = s + \imath n\omega_0$ and $s \in \mathbb{C}$.

Here, the real part of s, $\text{Re}(s)$ determines the exponential decay/growth of u. Clearly the series representation of $u(t)$ is not unique.

Theorem 3.1.1 (Wereley (1991), [21]) *(LTP System Response to EMP Signals). Given* $G = \left[\begin{array}{c|c} \mathbf{A}(t) & \mathbf{B}(t) \\ \hline \mathbf{C}(t) & \end{array}\right]$*, there exists a similarity transformation (Floquet) such that* $G = \left[\begin{array}{c|c} \mathbf{Q} & \overline{\mathbf{B}}(t) \\ \hline \overline{\mathbf{C}}(t) & \end{array}\right]$*. Denote* y_{ss} *and* y_{tr} *to be the steady state and transient responses* *respectively of an EMP input* $u(t)$ *and initial condition* ξ_0*. Then,*

$$y_{ss}(t) = \sum_{n=-\infty}^{\infty} \left\{ \sum_{\ell,m=-\infty}^{\infty} \overline{\mathbf{C}}_{n-\ell}(s_\ell \mathbf{I} - \mathbf{Q})^{-1} \overline{\mathbf{B}}_{\ell-m} u_m \right\} e^{s_n t} \tag{3.9}$$

$$y_{tr}(t) = \mathbf{C}(t)\Phi(t,0) \left\{ \xi_0 - \sum_{\ell,m=-\infty}^{\infty} (s_\ell \mathbf{I} - \mathbf{Q})^{-1} \overline{\mathbf{B}}_{\ell-m} u_m \right\} \tag{3.10}$$

Additionally, $y_{tr}(t) \to 0$ *if the above system* Σ *is stable.*

Remark It's important to realize here that the steady state response, $y_{ss}(t)$ is also a EMP signal. Hence, we can write

$$y_{ss}(t) = \sum_{n=-\infty}^{\infty} y_n e^{s_n t} \qquad \text{where} \qquad y_n = \sum_{\ell,m=-\infty}^{\infty} \overline{\mathbf{C}}_{n-\ell}(s_\ell \mathbf{I} - \mathbf{Q})^{-1} \overline{\mathbf{B}}_{\ell-m} u_m \tag{3.11}$$

This is explained in Wereley's thesis, [21],

> The fundamental result shown above is that when an EMP test signal is injected into an LTP system with a time periodic dynamics matrix, the steady state output response is also an EMP signal. Thus, the steady state output response leads to the concept of a transfer function for LTP systems because the input and output signal spaces are equal. Hence, the LTP frequence response can be stated: *in steady state, an LTP system maps an EMP input signal, to an EMP output signal of the same frequency, but with possibly different amplitude and phase (as long as s is not in Q). The LTP frequency response notion is completely analogous to the LTI frequency response notion.*

3.2 Frequency Coupling

An important characteristic of stable LTI systems is that their steady state response to a single-frequency sinusoid is another sinusoid of the same frequency. That is, if $G(s) = \mathbf{c}^T[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b}$ is the transfer function of the LTI system, G, then

$$u(t) = \sin(\omega t) \qquad \overset{y=Gu}{\Longrightarrow} \qquad y(t) \to |G(\imath\omega)| \sin[\omega t + \arg(G(\imath\omega))] \tag{3.12}$$

Note that we write $y(t) \to$ to describe the steady-state behavior or $y(t)$ after the transient response has decayed.

LTP systems however, produce a countable number of harmonics of the input frequency, $\omega + \ell\omega_0$ for integer ℓ where $\omega_0 = \dfrac{2\pi}{T}$ is the fundamental frequency of the system G.

$$u(t) = \sin(\omega t) \qquad \overset{y=Gu}{\Longrightarrow} \qquad y(t) \to \sum_{\ell} |g_\ell(\imath\omega)| \sin[(\omega + \ell\omega_0)t + \arg(g_\ell(\imath\omega))] \tag{3.13}$$

Therefore for a single frequency input, an LTP system will have output frequencies $\{\omega_\ell = \omega + \ell\omega_0\}$. [Sandberg, 10]

We call these $\{g_\ell\}$ the subsystems of G, and define them by a Fourier expansion on the impulse response $g(t, \tau)$. Let $r = t - \tau$. Since $g(t, t - r)$ is T-periodic in t, the impulse response can be expanded into a Fourier series for r:

$$g(t, t - r) = \sum_{\ell=-\infty}^{\infty} g_\ell(r) e^{\imath\ell\omega_0 t} \tag{3.14}$$

$$g_\ell(r) = \frac{1}{T} \int_0^T e^{-\imath\ell\omega_0 t} g(t, t - r) dt \tag{3.15}$$

These subsystems are discussed at length in chapter 5.

3.3 Floquet Transformation

The Floquet transformation is a change of variable that transforms the time-periodic differential equation

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}, \qquad \mathbf{A}(t) = \mathbf{A}(t + T) \tag{3.16}$$

to an equation of constant coefficients, $\dot{\mathbf{z}} = \mathbf{Q}\mathbf{z}(t)$ via a periodic change of variables, $\mathbf{z}(t) = \mathbf{P}^{-1}(t)\mathbf{x}(t)$.

To accomplish this, first note that if $\mathbf{X}(t)$ is a fundamental matrix solution for (3.16), then so is $\mathbf{X}(t + T)$ and there exists a constant $\mathbf{R}$ such that $\mathbf{X}(t + T) = \mathbf{X}(t)\mathbf{R}$ for all t. (This is stated as a theorem in most ODE texts. For a proof see Grimshaw (1993), [8]). We compute $\mathbb{R}$ by solving $\mathbf{R} = \mathbf{X}^{-1}(0)\mathbf{X}(T)$.

We define the eigenvalues of $\mathbf{R}$, $\rho_1, \ldots, \rho_n$, to be the **Floquet multipliers** of (3.16) and the **Floquet exponents** $\{\mu_1, \ldots, \mu_n\}$ defined by $\rho_i = e^{\mu_i T}$. The Floquet multipliers and exponents are also called the characteristic multipliers and exponents respectively.

Note that μ_i are not unique (if μ_i is a characteristic exponent, then so is $\mu_i + 2\pi\imath k/T$). Note also that the multipliers do not depend on our choice of $\mathbf{X}(t)$. Additionally, from Gantmacher (1959), [7], we know that a $\mathbf{Q}$ exists such that,

$$\mathbf{R} = e^{\mathbf{Q}T}. \tag{3.17}$$

$\mathbf{Q}$ becomes important in the Floquet's existence theorem below which can be found in Grimshaw (1993), [8].

Theorem 3.3.1 (Floquet's Theorem) *In the system*

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} \tag{3.18}$$

let $\mathbf{A}(t)$ *be a continuous* $n \times n$ *matrix for* $-\infty < t < \infty$ *which is periodic of period* T

$$\mathbf{A}(t+T) = \mathbf{A}(t). \tag{3.19}$$

Then any fundamental matrix $\mathbf{X}(t)$ *of* (3.18), *has a representation of the form*

$$\mathbf{X}(t) = \mathbf{P}(t)e^{\mathbf{Q}t}, \qquad \text{where } \mathbf{P}(t+T) = \mathbf{P}(t) \tag{3.20}$$

and $\mathbf{Q}$ *is a constant matrix (and* $\mathbf{P}(t)$, $\mathbf{Q}$ *are* $n \times n$ *matrices).*

Unfortunately, the Floquet transformation is intractable for large order systems. Computational methods have been developed recently to make these transformations more affordable. The reader is encouraged to see Moore (2005), [14] and Cai et al. (2001), [6] for a discussion of these methods as they are outside the scope of this paper.

Utilizing the above theorem, we can determine an alternate realization of LTP systems:

Corollary 3.3.2 *Given a LTP system, G, a Floquet transformation can be performed so that system parameter* $\mathbf{A}(t) = \mathbf{Q}$ *becomes time-invariant via the periodic change of variable:* $\mathbf{z}(t) = \mathbf{P}^{-1}(t)\mathbf{x}(t)$.

$$G : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)u(t) \\ y(t) = \mathbf{c}^T(t)\mathbf{x}(t) \end{cases} \quad \Longrightarrow \quad \begin{cases} \dot{\mathbf{z}}(t) = \mathbf{Q}\mathbf{z}(t) + \mathbf{P}^{-1}(t)\mathbf{b}(t)u(t) \\ y(t) = \mathbf{c}^T(t)\mathbf{P}(t)\mathbf{z}(t) \end{cases} \tag{3.21}$$

where $\mathbf{Q} = \mathbf{P}^{-1}(t)[\mathbf{A}(t)\mathbf{P}(t) - \dot{\mathbf{P}}(t)]$.

Proof From Floquet's theorem, we know $\mathbf{X}(t) = \mathbf{P}(t)e^{\mathbf{Q}t}$ where $\mathbf{P}(t)$ is a T-periodic matrix and $\mathbf{Q}$ satisfies $e^{\mathbf{Q}t} = \mathbf{X}(T)$.

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) = \mathbf{A}(t)\mathbf{P}(t)e^{\mathbf{Q}t} \tag{3.22}$$

$$= \frac{d}{dt}\{\mathbf{P}(t)e^{\mathbf{Q}t}\} = \dot{\mathbf{P}}(t)e^{\mathbf{Q}t} + \mathbf{P}(t)\mathbf{Q}e^{\mathbf{Q}t}. \tag{3.23}$$

This gives us the equality

$$\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{Q} = \mathbf{A}(t)\mathbf{P}(t). \tag{3.24}$$

By the change of variable $\mathbf{x}(t) = \mathbf{P}(t)\mathbf{z}(t)$,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)u(t) \tag{3.25}$$

$$\dot{\mathbf{P}}(t)\mathbf{z}(t) + \mathbf{P}(t)\dot{\mathbf{z}}(t) = \mathbf{A}(t)\mathbf{P}(t)\mathbf{z}(t) + \mathbf{b}(t)u(t) \tag{3.26}$$

$$\dot{\mathbf{z}}(t) = \mathbf{P}^{-1}(t)[\mathbf{A}(t)\mathbf{P}(t) - \dot{\mathbf{P}}(t)]\mathbf{z}(t) + \mathbf{P}^{-1}(t)\mathbf{b}(t)u(t) \tag{3.27}$$

$$\dot{\mathbf{z}}(t) = \mathbf{Q}\mathbf{z}(t) + \mathbf{P}^{-1}(t)\mathbf{b}(t)u(t). \tag{3.28}$$

□

3.4 Toeplitz Transform

Here we define Toeplitz transformations on matrix-valued symbols, $\mathbf{A}(t)$. These block Toeplitz matrices prove important in harmonic analysis later in the paper.

We denote the Fourier coefficients of $\mathbf{A}(t)$ by $\{\mathbf{A}_n\}$

$$\mathbf{A}(t) = \sum_{n=-\infty}^{\infty} \mathbf{A}_n e^{\imath n\omega_0 t} \qquad\qquad \mathbf{A}_n = \frac{1}{T}\int_0^T \mathbf{A}(t)e^{-\imath n\omega_0 t}dt \tag{3.29}$$

where $\omega_0 = \dfrac{2\pi}{T}$ is the fundamental frequency of the underlying system, i.e., $\mathbf{A}(t+T) = \mathbf{A}(t)$.

Theorem 3.4.1 (Wereley (1991), [21]) *Let $\mathbf{A}(t) \in L_2^{n\times n}[0,T]$ be piecewise smooth, that is, $\mathbf{A}(t)$ is continuous and its first derivative is piecewise continuous. Also, assume that $\mathbf{A}(0) = \mathbf{A}(T)$. Then the convergence of the complex Fourier series*

$$\mathbf{A}(t) = \sum_{n=-\infty}^{\infty} \mathbf{A}_n e^{\imath n\omega_0 t} \tag{3.30}$$

to $\mathbf{A}(t)$ on the fundamental interval is absolute and uniform with respect to t on the fundamental interval.

The Toeplitz transform of $\mathbf{A}(t)$, denoted by $\mathcal{T}\{\mathbf{A}(t)\}$, maps the set of complex Fourier coefficients, $\{\mathbf{A}_n\}$, into a doubly infinite block Toeplitz matrix, $\mathcal{A}$, of the form

$$\mathcal{T}\{\mathbf{A}(t)\} = \mathcal{A} = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \mathbf{A}_0 & \mathbf{A}_{-1} & \mathbf{A}_{-2} & \\ \ddots & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{A}_{-1} & \ddots \\ & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{3.31}$$

Theorem 3.4.2 (Wereley (1991), [21]) *The Toeplitz transform exhibits the following properties:*

1. $\mathcal{T}\{\mathbf{A}(t)\mathbf{B}(t)\} = \mathcal{T}\{\mathbf{A}(t)\}\mathcal{T}\{\mathbf{B}(t)\} = \mathcal{A}\mathcal{B}$
2. $\mathcal{T}\{\mathbf{A}(t) + \mathbf{B}(t)\} = \mathcal{T}\{\mathbf{A}(t)\} + \mathcal{T}\{\mathbf{B}(t)\} = \mathcal{A} + \mathcal{B}$
3. *Suppose* $\dot{\mathbf{P}}(t) \in L_2^{n\times n}[0, T]$. *Then*

$$\mathcal{T}\{\dot{\mathbf{P}}(t)\} = \mathcal{N}\mathcal{P} - \mathcal{P}\mathcal{N} = \mathcal{N}\mathcal{P} + \mathcal{P}\mathcal{N}^* \tag{3.32}$$

where $\mathcal{N} = \text{blkdiag}\{\imath n\omega_0 \mathbf{I}\}$

Chapter 4

Harmonic Transfer Function

The transfer function is an essential tool in dynamical systems theory in the linear time-invariant case. In particular, we construct linear spaces of dynamical systems whose norms are defined by the systems' respective transfer functions. Further, we interpret the transfer function as a frequency operator mapping sinusoidal inputs to sinusoidal steady-state responses according to transfer function evaluations along the imaginary axis:

$$u(t) = \sin(\omega t) \qquad \overset{y=Gu}{\Longrightarrow} \qquad y(t) \to |G(\imath\omega)| \sin[\omega t + \arg(G(\imath\omega))] \tag{4.1}$$

Transfer functions are so essential to the analysis of LTI systems that we seek an analogous tool for LTP systems. Harmonic transfer functions (HTF) were introduced by Wereley (1991), [21] and Wereley and Hall (1990), [22] who applied harmonic analysis to state-space systems with EMP inputs.

Definition $\mathcal{G}(s)$ is called the *harmonic transfer function* of dynamical system $G = \left[\begin{array}{c|c} \mathbf{A}(t) & \mathbf{b}(t) \\ \hline \mathbf{c}^T(t) & \end{array}\right]$ if $\mathcal{G}(s)$ is of the form

$$\mathcal{G}(s) = \mathcal{C}[s\mathcal{I} - (\mathcal{A} - \mathcal{N})]^{-1}\mathcal{B}, \tag{4.2}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are the Toeplitz transformations of $\mathbf{A}(t)$, $\mathbf{b}(t)$, and $\mathbf{c}^T(t)$, respectively, i.e.

$$\mathcal{A} = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \mathbf{A}_0 & \mathbf{A}_{-1} & \mathbf{A}_{-2} & \\ \ddots & \mathbf{A}_1 & \mathbf{A}_0 & \mathbf{A}_{-1} & \ddots \\ & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

and $\mathcal{N}$ is the block diagonal matrix $\mathcal{N} = [\ldots, -\imath 1\omega_0\mathbf{I}, \imath 0\omega_0\mathbf{I}, \imath 1\omega_0\mathbf{I}, \ldots]$.

4.1 Harmonic Balancing Method

To understand the construction of the HTF, $\mathcal{G}(s) = \mathcal{C}[s\mathcal{I} - (\mathcal{A} - \mathcal{N})]^{-1}\mathcal{B}$, consider the harmonic balancing method below:

For simplicity we assume the system is SISO. Define $\mathcal{U} = [\ldots, u_{-1}, u_0, u_1, \ldots]^T$, $\mathcal{Y} = [\ldots, y_{-1}, y_0, y_1, \ldots]^T$, and $\mathcal{X} = [\ldots, \mathbf{x}_{-1}, \mathbf{x}_0, \mathbf{x}_1, \ldots]^T$ to be the ordered Fourier coefficients of $u(t)$, $y(t)$ and $\mathbf{x}(t)$ respectively. Given a finite-dimensional linear continuous-time periodic system

$$G : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)u(t) \\ y(t) = \mathbf{c}^T(t)\mathbf{x}(t), \end{cases} \tag{4.3}$$

perform a Fourier expansion of each term and simplify. We introduce $s_n = s + \imath n\omega_0$ to simplify notation.

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} s_n\mathbf{x}_n e^{s_n t} &= \sum_{m\in\mathbf{Z}} \mathbf{A}_m e^{\imath m\omega_0 t} \sum_{n\in\mathbb{Z}} \mathbf{x}_n e^{s_n t} + \sum_{m\in\mathbb{Z}} \mathbf{b}_m e^{\imath m\omega_0 t} \sum_{n\in\mathbb{Z}} u_n e^{s_n t} && (4.4)\\
&= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbf{Z}} \mathbf{A}_{n-m} e^{\imath(n-m)\omega_0 t} \mathbf{x}_m e^{s_m t} + \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \mathbf{b}_{n-m} e^{\imath(n-m)\omega_0 t} u_m e^{s_m t} && (4.5)\\
&= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbf{Z}} \mathbf{A}_{n-m}\mathbf{x}_m e^{s_n t} + \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \mathbf{b}_{n-m} u_m e^{s_n t} && (4.6)\\
\sum_{n\in\mathbb{Z}} y_n e^{s_n t} &= \sum_{m\in\mathbf{Z}} \mathbf{c}_m^T e^{\imath m\omega_0 t} \sum_{n\in\mathbb{Z}} \mathbf{x}_n e^{s_n t} && (4.7)\\
&= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbf{Z}} \mathbf{c}_{n-m}^T e^{\imath(n-m)\omega_0 t} \mathbf{x}_m e^{s_m t} && (4.8)\\
&= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbf{Z}} \mathbf{c}_{n-m}^T \mathbf{x}_m e^{s_n t}. && (4.9)
\end{aligned}$$

The second equality is a Cauchy product. Then:

$$0 = \sum_{n\in\mathbb{Z}} \left\{ s_n \mathbf{x}_n - \sum_{m\in\mathbb{Z}} \mathbf{A}_{n-m}\mathbf{x}_m - \sum_{m\in\mathbb{Z}} \mathbf{b}_{n-m} u_m \right\} e^{s_n t} \tag{4.10}$$

$$0 = \sum_{n\in\mathbb{Z}} \left\{ y_n - \sum_{m\in\mathbb{Z}} \mathbf{c}^T_{n-m}\mathbf{x}_m \right\} e^{s_n t} \tag{4.11}$$

Since $\{e^{\imath n\omega_0 t}\}$ forms a complete orthogonal basis then each of the inside terms must be zero:

$$s_n \mathbf{x}_n = \sum_{m\in\mathbb{Z}} \mathbf{A}_{n-m}\mathbf{x}_m + \sum_{m\in\mathbb{Z}} \mathbf{b}_{n-m} u_m \tag{4.12}$$

$$y_n = \sum_{m\in\mathbb{Z}} \mathbf{c}^T_{n-m}\mathbf{x}_m. \tag{4.13}$$

A more concise representation of the above harmonic transfer function is

$$s\mathcal{X} = (\mathcal{A} - \mathcal{N})\mathcal{X} + \mathcal{B}\mathcal{U} \tag{4.14}$$

$$\mathcal{Y} = \mathcal{C}\mathcal{X}. \tag{4.15}$$

Let $\mathfrak{g}_{m,n}(s)$ denote the (m,n)th element of the harmonic transfer function. That is, $\mathfrak{g}_{m,n}(s) = [\mathcal{G}(s)]_{m,n}$. To calculate the elements of the harmonic transfer function, we are required to either solve the (m,n)th matrix block of the inverse infinite-dimensional matrix $[s\mathcal{I} - (\mathcal{A} - \mathcal{N})]^{-1}$ or perform a Floquet similarity transformation. The first option is intractable; therefore, we explore the latter despite it's computational cost.

Remark If the dynamical system G has a Floquet-transformed representation

$$G = \left[\begin{array}{c|c} \mathbf{Q} & \mathbf{b}(t) \\ \hline \mathbf{c}^T(t) & \mathbf{0} \end{array} \right] \tag{4.16}$$

then we can represent elements of the transfer function, $\mathcal{G}(s)$:

$$\mathfrak{g}_{m,n}(s) = [\mathcal{G}(s)]_{m,n} = \sum_{\ell\in\mathbb{Z}} \mathbf{c}^T_{m-\ell}[s_\ell \mathbf{I} - \mathbf{Q}]^{-1}\mathbf{b}_{\ell-n}. \tag{4.17}$$

For a discussion of this result, see Wereley and Hall (1990), [22].

Note here that elements along the diagonal of the HTF are equal up to a purely imaginary shift $\imath k\omega_0$.

$$\mathfrak{g}_{m+k,n+k}(s) = \mathfrak{g}_{m,n}(s + \imath k\omega_0). \tag{4.18}$$

This can be shown using the a change of variable $\hat{\ell} = \ell - k$.

Remark What's remarkable here is that each element of the HTF is a linear time-invariant system. Thus we can treat the LTP system as a collection of LTI system for which analysis is considerably more familiar. We view the infinite-dimemsional size of this new operator as the cost of removing the time dependency of the system response.

4.2 The HTF as an ℓ_2 Operator

We can think of an LTP system as an operator on the space of EMP signals. Then the HTF is an explicit representation of the mapping $y = Gu$ as an infinite-dimensional matrix-vector multiplication:

Given harmonic transfer function, $\mathcal{G}(s)$ and EMP input $u(t) = \sum_{n=-\infty}^{\infty} u_n e^{s_n t}$ with $s_n = s + \imath n\omega_0$, we can express the mapping $y = Gu$ as

$$\mathcal{Y} = \mathcal{G}(s)\mathcal{U} \tag{4.19}$$

where $\mathcal{U} = [\ldots, u_{-1}, u_0, u_1, \ldots]^T$ and $\mathcal{Y} = [\ldots, y_{-1}, y_0, y_1, \ldots]^T$ are the ordered Fourier coefficients of $u(t)$ and $y(t)$ respectively.

Definition We say a system G is **unstable** if there exists $s \in \mathbb{C}$ in the open right half-plane for which the ℓ_2 operator $\mathcal{G}(s)$ is unbounded. Otherwise the system is **stable**.

In Zhou et al. (2004), [28], it is shown that the harmonic transfer function is a closed, densely defined Fredholm operator on the Hilbert space ℓ_2, and its spectrum contains only a finite number of elements in any bounded neighborhood of the origin.

4.3 $\mathcal{H}_2$-Norm

Definition We define the the $\mathcal{H}_2$ norm of a LTP system, G,

$$\|G\|_{\mathcal{H}_2}^2 = \frac{1}{T}\int_0^T \int_{-\infty}^{+\infty} \overline{g(t,\tau)} g(t,\tau) dt d\tau \tag{4.20}$$

where $g(t,\tau)$ is the impulse response of the system when the impulse occurs at time $t = \tau$.

Remark Note that this definition is a generalization of the LTI case. If G is LTI, then $g(t,\tau) = g(t-\tau, 0)$ and the above equation generalizes (2.11),

$$\|G\|_{\mathcal{H}_2}^2 = \int_0^\infty g^*(t,0)g(t,0)dt. \tag{4.21}$$

We can express $\|G\|_{\mathcal{H}_2}$ in terms of the HTF under certain conditions:

Theorem 4.3.1 (Zhou and Hagiwara (2002), [24]) *If $\mathbf{A}(t)$ is piecewise continuous and differentiable at a.e. $t \in [0,T]$ and that $\mathbf{b}(t)$ and $\mathbf{c}(t)$ are continuous and the Fourier series expansion are absolutely convergent, then*

$$\|G\|_{\mathcal{H}_2}^2 = \frac{1}{2\pi}\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} trace[\mathcal{G}(\imath\varphi)^*\mathcal{G}(\imath\varphi)]d\varphi. \tag{4.22}$$

Remark Note that in the case that G is SISO and LTI, $G = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \\ \hline \mathbf{c}^T & 0 \end{array}\right]$:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \text{trace}[\mathcal{G}(\imath\varphi)^*\mathcal{G}(\imath\varphi)]d\varphi &= \frac{1}{2\pi}\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \sum_k |\mathfrak{g}_{k,k}(\imath\varphi)|^2 d\varphi && (4.23)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{c}^T[\imath\varphi\mathbf{I} - \mathbf{A}]^{-1}\mathbf{b}|^2 d\varphi && (4.24)\\
&= \|G\|_{\mathcal{H}_2}^2. && (4.25)
\end{aligned}$$

Thus (4.22) generalizes our definition for the $\mathcal{H}_2$ norm for LTI dynamical systems, (2.11).

4.4 Computation of $\mathcal{H}_2$ Norm

In a paper by Zhou and Hagiwara (2002), [24], a method for computing the $\mathcal{H}_2$ norm of a Floquet-transformed LTP system is developed using truncations of the Fourier summations of the symbols $\mathbf{b}(t)$ and $\mathbf{c}(t)$. Their paper also discusses a method for computing the $\mathcal{H}_\infty$-norm but is outside the scope of this paper. Below is an abbreviated version of their discussion in their paper.

4.4.1 Skew Truncation

Consider the Floquet-transformed HTF for (4.3),

$$\mathcal{G}(s) = \mathcal{C}[s\mathcal{I} - (\mathcal{Q} - \mathcal{N})]^{-1}\mathcal{B}. \tag{4.26}$$

Definition Define G_N to be the **skew truncation** of G

$$G_N : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{Q}\mathbf{x}(t) + \hat{\mathbf{b}}(t)u(t) \\ y(t) = \hat{\mathbf{c}}^T(t)\mathbf{x}(t) \end{cases} \tag{4.27}$$

with $\hat{\mathbf{b}}(t) = \sum_{m=-N}^{N} \mathbf{b}_m e^{\imath m\omega_0 t}$, $\hat{\mathbf{c}}(t) = \sum_{m=-N}^{N} \mathbf{c}_m e^{\imath m\omega_0 t}$ where $\{\mathbf{b}_m\}$, $\{\mathbf{c}_m\}$ are the Fourier coefficients of $\mathbf{b}(t)$ and $\mathbf{c}(t)$. Write $\mathcal{G}_N$ as the HTF of G_N. Then

$$\mathcal{G}_N(s) = \mathcal{C}_N[s\mathcal{I} - (\mathcal{Q} - \mathcal{N})]^{-1}\mathcal{B}_N \tag{4.28}$$

where $\mathcal{B}_N := \mathcal{T}\left\{\sum_{m=-N}^{N} \mathbf{b}_m e^{\imath m\omega_0 t}\right\}$ and $\mathcal{C}_N := \mathcal{T}\left\{\sum_{m=-N}^{N} \mathbf{c}_m e^{\imath m\omega_0 t}\right\}$, each of which are banded block-Toeplitz infinite-dimensional matrices.

Definition Define the LTI system $\mathfrak{G}_N = \left[\begin{array}{c|c} \mathfrak{A}_N & \mathfrak{B}_N \\ \hline \mathfrak{C}_N & 0 \end{array}\right]$ where

$$\mathfrak{C}_N := \begin{bmatrix} \mathbf{c}_{-N}^T & & & & 0 \\ \vdots & \ddots & & & \\ \mathbf{c}_0^T & & \mathbf{c}_{-N}^T & & \\ \vdots & \ddots & \vdots & \ddots & \\ \mathbf{c}_N^T & \cdots & \mathbf{c}_0^T & \cdots & \mathbf{c}_{-N}^T \\ & \ddots & \vdots & \ddots & \vdots \\ & & \mathbf{c}_N^T & & \mathbf{c}_0^T \\ & & & \ddots & \vdots \\ 0 & & & & \mathbf{c}_N^T \end{bmatrix} \qquad \mathfrak{B}_N = \begin{bmatrix} \mathbf{b}_{-N} \\ \vdots \\ \mathbf{b}_{-1} \\ \mathbf{b}_0 \\ \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_N \end{bmatrix} \tag{4.29}$$

$$\mathfrak{A}_N = \begin{bmatrix} \mathbf{Q} - \imath N\omega_0 & & \\ & \ddots & \\ & & \mathbf{Q} + \imath N\omega_0 \end{bmatrix}. \tag{4.30}$$

4.4.2 Computing the $\mathcal{H}_2$-Norm of a Skew Truncated System

Theorem 4.4.1 (Zhou and Hagiwara (2002), [24])

- $\|\mathcal{G} - \mathcal{G}_N\|_{\mathcal{H}_2} \to 0$ *as* $N \to \infty$

- $\|\mathcal{G}_N\|_{\mathcal{H}_2} = \|\mathfrak{G}_N\|_{\mathcal{H}_2}$
- $\lim\limits_{N\to\infty} trace(\mathfrak{B}_N^* \mathcal{V}_N \mathfrak{B}_N) = \lim\limits_{N\to\infty} trace(\mathfrak{C}_N \mathcal{W}_N \mathfrak{C}_N^*) = \|\mathfrak{G}\|_2^2$
  where $\mathcal{V}_N$ and $\mathcal{W}_N$ are, respectively, the solutions of the finite-dimensional Lyapunov equations

$$\mathfrak{A}_N^* \mathcal{V}_N + \mathcal{V}_N \mathfrak{A}_N + \mathfrak{C}_N \mathfrak{C}_N^* = 0 \tag{4.31}$$

$$\mathfrak{A}_N \mathcal{W}_N + \mathcal{W}_N \mathfrak{A}_N^* + \mathfrak{B}_N \mathfrak{B}_N^* = 0. \tag{4.32}$$

The method proposed by Zhou and Hagiwara for computation of $\mathcal{H}_2$ norm for LTP systems is then:

1. Construct the Floquet transformed dynamical HTF $\mathcal{G}(s) = \mathcal{C}[s\mathcal{I} - (\mathcal{Q} - \mathcal{N})]^{-1}\mathcal{B}$
2. Truncate Fourier expansions of symbols $\mathbf{b}(t)$ and $\mathbf{c}(t)$.
3. Construct associated LTI dynamical system $\mathfrak{G}_N$.
4. Compute $\|\mathfrak{G}_N\|_{\mathcal{H}_2}$ via large-scale Lyapunov solvers as an approximation to $\|G\|_{\mathcal{H}_2}$.

Chapter 5

Kernel Representation

An essential tool to LTI dynamical system theory is the ability to treat the family of LTI stable systems as a Hilbert space whose associated transfer functions form a subspace in $\mathcal{H}_2$. See theorem 2.6.2. Two major consequences of this result are:

1. we can represent the $\mathcal{H}_2$ inner product in terms of the poles and residues of the respective systems,
2. we can apply structured orthogonality conditions to determine candidates for optimal approximations.

We seek a generalization of the Hilbert space structure to the family of dynamical systems with time-periodic coefficients.

We return to the convolution representation of the input-output operator G via a Fourier decomposition on the kernel, $g(t, \tau)$, i.e.

$$y(t) = Gu = \int_{-\infty}^{t} g(t, \tau) u(\tau) d\tau. \tag{5.1}$$

In chapter 3 we understand LTP systems to be a frequency operator mapping sinusoidal inputs to a steady-state output consisting of a countable number of harmonics of the input frequency:

$$u(t) = \sin(\omega t) \qquad \overset{y=Gu}{\Longrightarrow} \qquad y(t) \to \sum_{\ell} |g_\ell(\imath\omega)| \sin[(\omega + \ell\omega_0)t + \angle g_\ell(\imath\omega)]. \tag{5.2}$$

The harmonics then occur at frequencies $\omega + \ell\omega_0$ for integer ℓ where $\omega_0 = \dfrac{2\pi}{T}$ is the fundamental frequency of the system G.

Definition We define these $\{g_\ell(r)\}$ to be the **subsystems** of G, and define them via a Fourier expansion on the impulse response $g(t, t-r)$ where $r = t - \tau$. Since $g(t, t-r)$ is T-periodic in t, the impulse response can be expanded into a Fourier series for r:

$$g(t, t-r) = \sum_{\ell=-\infty}^{\infty} g_\ell(r) e^{\imath\ell\omega_0 t} \qquad \text{where} \qquad g_\ell(r) = \frac{1}{T}\int_0^T e^{-\imath\ell\omega_0 t} g(t, t-r)dt. \tag{5.3}$$

Frequently we will work with the Laplace transform of the subsystems. We write these as $\{\hat{g}_\ell(s)\}$ where $\hat{g}_\ell(s) = \mathcal{L}\{g_\ell(t)\}$ where $\mathcal{L}$ denotes the Laplace transform.

5.1 Kernel Subsystems and the Harmonic Transfer Function

The subsystems defined above indeed are elements of the HTF. To prove this, we first need the following theorem:

Theorem 5.1.1 (Convergence of Skew Truncations, Sandberg et al. (2005), [18]) *Let* $g_N(t, t-r) = \sum_{k=-N}^{N} g_k(t-r)e^{\imath k\omega_0 t}$ *denote the truncation of* $g(t, t-r) = \sum_{k=-\infty}^{\infty} g_k(t-r)e^{\imath k\omega_0 t}$ *and let* G_N *be the associated system. Define* $y = Gu$ *and* $y_N = G_N u$*. Assume that a periodic system* G *has impulse response that is uniformly exponentially stable,*

$$|g(t, \tau)| \leq \kappa_1 e^{-\kappa_2(t-\tau)}.$$

Then for all inputs $u \in \mathcal{L}_\infty(\mathbb{R})$*, the output* $y_N(t)$ *converges, uniformly in* t*, to* $y(t)$ *as* $N \to \infty$*.*

Theorem 5.1.2 *Assume that a system,* G*, is both causal and* $\|G\|^2_{\mathcal{H}_2} < \infty$*. Additionally, assume that the impulse response of* G*,* $g(t, \tau)$*, is uniformly exponentially stable. If* $\{g_k(t)\}$ *and* $\mathcal{G}(s)$ *are, respectively, the associated subsystems and harmonic transfer function and* $u(t) \in \mathcal{L}_2$ *is an EMP signal, then*

$$g_k(t) = \mathcal{L}^{-1}\{\mathfrak{g}_{k,0}(s)\} \qquad \textit{where} \qquad \mathfrak{g}_{m,n}(s) = [\mathcal{G}(s)]_{m,n} \tag{5.4}$$

Proof Define $g_{[N]}(t, t-r) = \sum_{k=-N}^{N} g_k(t-r)e^{\imath k\omega_0 t}$ to be the truncated Fourier expansion of $g(t, t-r) = \sum_{k=-\infty}^{\infty} g_k(t-r)e^{\imath k\omega_0 t}$ and let G_N be the associated dynamical system. Indeed, it happens that G_N is the skew truncated system from Chapter 4, but we need the next several theorems in this chapter to prove it.

Suppose $u(t)$ is an EMP signal. Then we write $u(t) = e^{st}\sum_m u_m e^{\imath m\omega_0 t} = \sum_m u_m e^{s_m t}$.

$$y(t) = \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau \tag{5.5}$$

$$= \int_{-\infty}^{t} \sum_k g_k(t-\tau)e^{\imath k\omega_0 t}u(\tau)d\tau \tag{5.6}$$

We can interchange the summation and integral because $G_N u \to Gu$ uniformly by Theorem 5.1.1.

$$y(t) = \sum_k e^{\imath k\omega_0 t}\int_{-\infty}^{t} g_k(t-\tau)u(\tau)d\tau \tag{5.7}$$

$$= \sum_k e^{\imath k\omega_0 t}[g_k(t) * u(t)] \tag{5.8}$$

$$= \sum_k e^{\imath k\omega_0 t}[g_k(t) * \sum_m u_m e^{s_m t}] \tag{5.9}$$

$$= \sum_k e^{\imath k\omega_0 t}\sum_m u_m[g_k(t) * e^{s_m t}] \tag{5.10}$$

Define $\hat{g}_k(s) = \mathcal{L}\{g_k(t)\}$. Then

$$y(t) = \sum_k e^{\imath k\omega_0 t}\sum_m u_m \hat{g}_k(s_m)e^{s_m t} \tag{5.11}$$

$$= \sum_k e^{\imath k\omega_0 t}\sum_m u_m g_{k,0}(s_m)e^{s_m t} \tag{5.12}$$

$$= \sum_k \sum_m u_m g_{k+m,m}(s)e^{s_{m+k}t} \tag{5.13}$$

$$= \sum_k \left[\sum_m g_{k,m}(s)u_m\right] e^{s_k t} \tag{5.14}$$

$$= \sum_k [\mathcal{G}\mathcal{U}]_k e^{s_k t} \tag{5.15}$$

where $y_k = \sum_m \mathfrak{g}_{k,m}(s)u_m = [\mathcal{Y}]_k = [\mathcal{G}\mathcal{U}]_k$. □

Remark We can write

$$\mathcal{G}(s) = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \mathfrak{g}_{-1,-1}(s) & \mathfrak{g}_{-1,0}(s) & \mathfrak{g}_{-1,1}(s) & \\ \ddots & \mathfrak{g}_{0,-1}(s) & \mathfrak{g}_{0,0}(s) & \mathfrak{g}_{0,1}(s) & \ddots \\ & \mathfrak{g}_{1,-1}(s) & \mathfrak{g}_{1,0}(s) & \mathfrak{g}_{1,1}(s) & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.16}$$

$$= \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \hat{g}_0(s - \imath\omega_0) & \hat{g}_{-1}(s) & \hat{g}_{-2}(s + \imath\omega_0) & \\ \ddots & \hat{g}_1(s - \imath\omega_0) & \hat{g}_0(s) & \hat{g}_{-1}(s + \imath\omega_0) & \ddots \\ & \hat{g}_2(s - \imath\omega_0) & \hat{g}_1(s) & \hat{g}_0(s + \imath\omega_0) & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.17}$$

5.2 The Family of LTP Systems as a Hilbert Space

The major result in this section is that the family of LTP systems form a subspace of the Hilbert space, $\mathfrak{H} = \bigoplus_{\ell=-\infty}^{\infty} \mathcal{H}_2$. Define

$$\langle G, H \rangle_{\mathfrak{H}} = \sum_{\ell=-\infty}^{\infty} \langle \hat{g}_\ell(\cdot), \hat{h}_\ell(\cdot) \rangle_{\mathcal{H}_2} \tag{5.18}$$

That is, the family of LTP systems that have a Floquet-transformed representation of the form,

$$G : \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{Q}\mathbf{x}(t) + \mathbf{b}(t)u(t) \\ y(t) = \mathbf{c}^T(t)\mathbf{x}(t) \end{cases} \tag{5.19}$$

with $\mathbf{b}(t)$ and $\mathbf{c}(t)$ piecewise continuous form a subspace of $\mathfrak{H}$. Additionally, we can express the $\mathcal{H}_2$ norm of the infinite dimensional subsystems in terms of their poles and residues. That is,

$$\langle \hat{g}_\ell(\cdot), \hat{h}_\ell(\cdot) \rangle^2_{\mathcal{H}_2} = \sum_{k=-\infty}^{\infty} \sum_{j=1}^{n} \hat{g}_\ell(-\lambda_j^{(k)}) \mathrm{Res}[\hat{h}_\ell(s), \lambda_j^{(k)}] \tag{5.20}$$

where $\lambda_j^{(k)} = \lambda_j - \imath k\omega_0$ with $\{\lambda_j\}_{j=1}^r = \boldsymbol{\lambda}(\mathbf{Q})$.

We begin by introducing several lemmas and conclude by proving equations (5.18) and (5.20). In a standard abuse of notation, we write $\langle\cdot,\cdot\rangle_{\mathfrak{H}} = \langle\cdot,\cdot\rangle_{\mathcal{H}_2}$ to be understood as the LTP $\mathcal{H}_2$ norm.

Lemma 5.2.1 (Borwein and Kratz (2005), [5]) *Let* $a \in \ell_1$ *and* $y_n = (a*x)_n = \sum a_{n-k}x_k = \sum a_k x_{n-k}$ *be a discrete convolution. Then*

- $\|y\|_2 \leq \|a\|_1\|x\|_2$ *for every* $x \in \ell_2$
- $\|y\|_1 \leq \|a\|_1\|x\|_1$ *for every* $x \in \ell_1$

Lemma 5.2.2 *Let* $\mathbf{x}(t)$ *be piecewise smooth and* T*-periodic. Then if* $\mathbf{x}(t) = \sum_{k\in\mathbb{Z}} \mathbf{x}_k e^{jk\omega_0 t}$ *is its Fourier series, then* $\{\mathbf{x}_k\} \in \ell_1$*, i.e.* $\displaystyle\sum_{k\in\mathbb{Z}} \|\mathbf{x}_k\|_2 < \infty$.

Proof This proof follows Bob Rogers' manuscript, Rogers (2012), [15], with differing notation.

Without loss of generality assume $\omega_0 = 1$. Then $\mathbf{x}(t) = \sum_{k\in\mathbb{Z}} \mathbf{x}_k e^{jkt}$, $\mathbf{x}_k = \frac{2}{T}\int_0^T \mathbf{x}(t)e^{-jkt}dt$. Define $\mathbf{y}_k = \frac{2}{T}\int_0^T \dot{\mathbf{x}}(t)e^{-jkt}dt$. Then via Bessel's inequality,

$$\sum_{k\in\mathbb{Z}} \|\mathbf{y}_k\|_2^2 \leq \frac{2}{T}\int_0^T |\dot{\mathbf{x}}(t)|dx < \infty.$$

Since $\|\mathbf{y}_k\|_2 = |k|\|\mathbf{x}_k\|_2$. Then

$$\sum_{\substack{k=-N\\k\neq 0}}^{N} \|\mathbf{x}_k\|_2 = \sum_{\substack{k=-N\\k\neq 0}}^{N} \frac{1}{|k|}\|\mathbf{y}_k\|_2 \tag{5.21}$$

$$\leq \sqrt{\sum_{\substack{k=-N\\k\neq 0}}^{N} \frac{1}{|k|^2}}\sqrt{\sum_{\substack{k=-N\\k\neq 0}}^{N} \|\mathbf{y}_k\|_2^2} \tag{5.22}$$

Both factors on the right converge as $N \to \infty$. Therefore $\displaystyle\sum_{k\in\mathbb{Z}} \|\mathbf{x}_k\|_2 < \infty$. □

5.2.1 The $\mathcal{H}_2$ Inner Product

Theorem 5.2.3

$$\langle G, H\rangle_{\mathcal{H}_2}^2 = \sum_{\ell\in\mathbb{Z}} \langle \hat{g}_\ell(s), \hat{h}_\ell(s)\rangle_{\mathcal{H}_2} \tag{5.23}$$

where $\hat{g}_n(s) = \mathcal{L}\{g_n(t)\}$ *and* $\hat{h}_n(s) = \mathcal{L}\{h_n(t)\}$ *where* $\hat{g}_n$ *and* $\hat{h}_n$ *are the subsystems of* G *and* H *respectively.*

Proof Using equation (4.18) and theorem 5.1.2 we can express elements of the harmonic transfer functions in terms of the subsystems,

$$\mathfrak{g}_{m,n}(s) = [\mathcal{G}(s)]_{m,n} = \sum_{\ell\in\mathbb{Z}} \mathbf{c}^T_{m-\ell}[s_\ell \mathbf{I} - \mathbf{Q}]^{-1}\mathbf{b}_{\ell-n} = \hat{g}_{m-n}(s - \imath n\omega_0) \tag{5.24}$$

Next if $s_n = s + \imath n\omega_0$, note that

$$\begin{aligned}
[\mathcal{G}(s)^*\mathcal{H}(s)]_{n,n} &= \sum_{k\in\mathbb{Z}} [\mathcal{G}^*]_{n,k}[\mathcal{H}]_{k,n} && (5.25)\\
&= \sum_{k\in\mathbb{Z}} \overline{\mathfrak{g}_{k,n}(s)}\mathfrak{h}_{k,n}(s) && (5.26)\\
&= \sum_{k\in\mathbb{Z}} \overline{\hat{g}_{k-n}(s_{-n})}\hat{h}_{k-n}(s_{-n}) && (5.27)\\
&= \sum_{\hat{k}\in\mathbb{Z}} \overline{\hat{g}_{\hat{k}}(s_{-n})}\hat{h}_{\hat{k}}(s_{-n}) && (5.28)
\end{aligned}$$

$$\begin{aligned}
\langle G, H\rangle_{\mathcal{H}_2} &= \frac{1}{2\pi}\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \operatorname{trace}[\mathcal{G}(\imath\varphi)^*\mathcal{H}(\imath\varphi)]d\varphi && (5.29)\\
&= \frac{1}{2\pi}\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \overline{\hat{g}_k(\imath\varphi - \imath n\omega_0)}\hat{h}_k(\imath\varphi - \imath n\omega_0)d\varphi && (5.30)\\
&= \sum_{k\in\mathbb{Z}} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{\hat{g}_k(\imath\varphi)}\hat{h}_k(\imath\varphi)d\varphi && (5.31)\\
&= \sum_{k\in\mathbb{Z}} \langle \hat{g}_k, \hat{h}_k\rangle_{\mathcal{H}_2} && (5.32)
\end{aligned}$$

□

5.2.2 The $\mathcal{H}_2$ Space

From theorem 5.2.3, it follows that the dynamical system norm is written,

$$\|G\|^2_{\mathcal{H}_2} = \sum_{\ell\in\mathbb{Z}} \|\hat{g}_\ell\|^2_{\mathcal{H}_2} \tag{5.33}$$

In case where $\mathbf{b}(t)$ and $\mathbf{c}^T(t)$ have finite dimensional Fourier series expansions, there exist a finite number of nontrivial subsystems, each of which have only finite spectra.

Theorem 5.2.4 *Given system* $G = \left[\begin{array}{c|c} \mathbf{A}(t) & \mathbf{B}(t) \\ \hline \mathbf{C}(t) & \end{array}\right]$ *with* $\mathbf{A}(t)$*,* $\mathbf{B}(t)$ *and* $\mathbf{C}(t)$ *T-periodic and piecewise smooth. If the Floquet-transformed state space matrix* $\mathbf{Q}$ *is Hurwitz, then*

$$\|G\|_{\mathcal{H}_2} < \infty \tag{5.34}$$

Proof Without loss of generality, assume G is in the Floquet form. Write $\|G\|^2_{\mathcal{H}_2} = \sum_{\ell \in \mathbb{Z}} \|\hat{g}_\ell\|^2_{\mathcal{H}_2}$ with $\hat{g}_\ell = \sum_{k\in\mathbb{Z}} \mathbf{c}^T_{\ell-k}[s_k\mathbf{I} - \mathbf{Q}]^{-1}\mathbf{b}_k$.
First, show that $\|\hat{g}_\ell\|_{\mathcal{H}_2} < \infty$:

$$\|\hat{g}_\ell\|_{\mathcal{H}_2} \leq \sum_{k\in\mathbb{Z}} \|\mathbf{c}^T_{\ell-k}[s_k\mathbf{I} - \mathbf{Q}]^{-1}\mathbf{b}_k\|_{\mathcal{H}_2} \tag{5.35}$$

$$\leq \sum_{k\in\mathbb{Z}} \|\mathbf{c}_{\ell-k}\|_2 \|\mathbf{b}_k\|_2 \|[s_k\mathbf{I} - \mathbf{Q}]^{-1}\|_{\mathcal{H}_2} \tag{5.36}$$

$$= \|[s\mathbf{I} - \mathbf{Q}]^{-1}\|_{\mathcal{H}_2} \sum_{k\in\mathbb{Z}} \|\mathbf{c}_{\ell-k}\|_2 \|\mathbf{b}_k\|_2 \tag{5.37}$$

Note that $\{\|\mathbf{c}_k\|_2\}, \{\|\mathbf{b}_k\|_2\} \in \ell_1$ by lemma 5.2.2. Using lemma 5.2.1 we see that the sequence

$$\left\{ \sum_{k\in\mathbb{Z}} \|\mathbf{c}_{\ell-k}\|_2 \|\mathbf{b}_k\|_2 \right\}_{\ell=-\infty}^{\infty} = \{\|\mathbf{c}_\ell\|_2\}_{\ell=-\infty}^{\infty} * \{\|\mathbf{b}_\ell\|_2\}_{\ell=-\infty}^{\infty} \in \ell_1. \tag{5.38}$$

Since $\mathbf{Q}$ is Hurwitz, then $\|[s\mathbf{I} - \mathbf{Q}]^{-1}\|_{\mathcal{H}_2} < \infty$. Therefore $\{\|\hat{g}_\ell\|^2_{\mathcal{H}_2}\}$ forms a ℓ_1 sequence indexed in ℓ. Since $\ell_1 \subset \ell_2$, $\|G\|^2_{\mathcal{H}_2} = \sum_{\ell\in\mathbb{Z}} \|\hat{g}_\ell\|^2_{\mathcal{H}_2} < \infty$. □

5.2.3 Pole-Residue Inner Product

Theorem 5.2.5

$$\langle \hat{g}_\ell, \hat{h}_\ell \rangle_{\mathcal{H}_2} = \sum_{k\in\mathbb{Z}} \sum_{j=1}^{n} \hat{g}_\ell(-\lambda_j^{(k)}) Res[\hat{h}_\ell(s), \lambda_j^{(k)}] \tag{5.39}$$

Proof The following proof shows

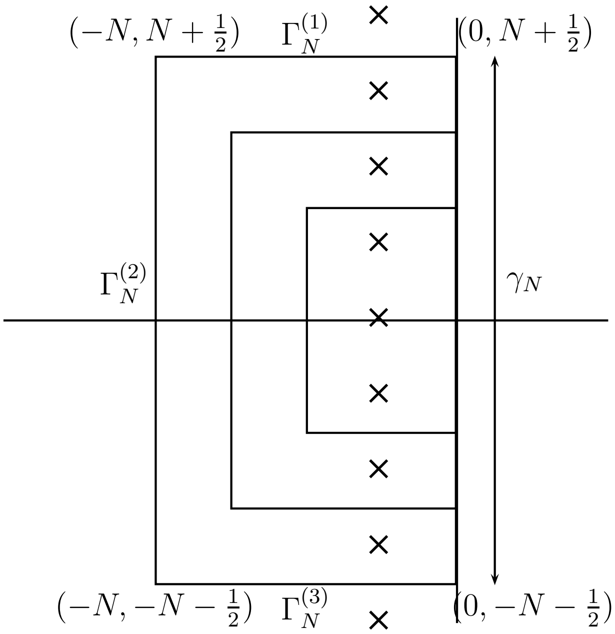
$$\|\hat{g}_\ell\|^2_{\mathcal{H}_2} = \sum_{k\in\mathbb{Z}} \sum_{j=1}^{n} \hat{g}_\ell(-\lambda_j^{(k)}) \mathrm{Res}[\hat{g}_\ell(s), \lambda_j^{(k)}]. \tag{5.40}$$

The proof naturally extends however to show (5.39).

Fix ℓ and set $f = \hat{g}_\ell$. Then without loss of generality assume $n = 1$, $Q = -1$, $\omega_0 = 1$. Then $\lambda_k = -1 + k\imath$ and $r_k = c_{\ell-k} b_k$. So $f(s) = \sum_{k \in \mathbb{Z}} \frac{r_k}{s - (-1 + k\imath)}$ with $\{r_k\} \in \ell_1$.

Consider the rectangular contour, Γ_N, defined by vertices $(0, N+\frac{1}{2})$, $(0, -N-\frac{1}{2})$, $(-N, -N-\frac{1}{2})$, $(-N, N+\frac{1}{2})$ for integer $N \geq 2$. Define $\Gamma_N^{(2)}$, $\Gamma_N^{(2)}$, $\Gamma_N^{(3)}$, γ_N to represent the the top, left, bottom, and right contours of the rectangle respectively. (See Figure 5.1). Finally, write $\Gamma_N = \Gamma_N^{(2)} + \Gamma_N^{(2)} + \Gamma_N^{(3)} + \gamma_N$ to represent the entire contour.

Figure 5.1: Contour integration path, Γ_N



Then from Residue Calculus we know

$$\int_{\Gamma_N} f(s) f(-s) ds = \sum_{k=-N}^{N} f(-\lambda^{(k)}) \text{Res}[f(s), \lambda^{(k)}]$$

- We first show that $\int_{\Gamma_N^{(2)}} f(s) f(-s) ds \to 0$ as $N \to \infty$:

$\Gamma_N^{(2)} = \{z = -N + \imath y : -N - \frac{1}{2} \leq y \leq N + \frac{1}{2}\}$. Then

$$|f(-N+\imath y)f(N-\imath y)| = \left|\sum_{k\in\mathbb{Z}} \frac{r_k}{-N+1+\imath(y-k)}\right| \left|\sum_{k\in\mathbb{Z}} \frac{r_k}{N+1+i(-y-k)}\right| \tag{5.41}$$

$$\leq \frac{1}{N^2-1}\left|\sum_{k\in\mathbb{Z}} r_k\right|^2 . \tag{5.42}$$

Further, $\text{length}(\Gamma_N^{(2)}) = 2N+1$. Therefore using the usual estimate, $\left|\int_{\Gamma_N^{(2)}} f(s)f(-s)ds\right| \leq \frac{2N+1}{N^2-1}\left|\sum_{k\in\mathbb{Z}} r_k\right|^2 \to 0$ as $N \to \infty$.

- Next, we show that $\int_{\Gamma_N^{(1)}} f(-s)f(s)ds \to 0$ as $N \to \infty$:

  $\Gamma_N^{(1)} = \{z = x + \imath(N+\frac{1}{2}) : -N \leq x \leq 0\}$. Note that $\text{length}(\Gamma_N^{(1)} = N$. Then for $s \in \Gamma_N^{(1)}$,

$$|f(s)f(-s)| = |f(x+\imath(N+1/2))f(-x-\imath(N+1/2))| \tag{5.43}$$

$$= \left|\sum_{k\in\mathbb{Z}} \frac{r_k}{x+1+\imath(N+\frac{1}{2}-k)}\right| \left|\sum_{m\in\mathbb{Z}} \frac{r_m}{-x+1+\imath(-N-\frac{1}{2}-k)}\right| \tag{5.44}$$

$$\leq \sum_{k\in\mathbb{Z}} \frac{|r_k|}{|N+\frac{1}{2}-k|} \sum_{m\in\mathbb{Z}} \frac{|r_m|}{|N+\frac{1}{2}+m|} \tag{5.45}$$

$$= \sum_{k\in\mathbb{Z}} \frac{|r_k|}{|N+\frac{1}{2}-k|} \sum_{m\in\mathbb{Z}} \frac{|r_{-m}|}{|N+\frac{1}{2}-m|} \tag{5.46}$$

  Define

$$a_k = \frac{1}{|k+\frac{1}{2}|} \in \ell_2 \tag{5.47}$$

$$c_N = \sum_{k\in\mathbb{Z}} \frac{|r_k|}{|N+\frac{1}{2}-k|} = \sum_{k\in\mathbb{Z}} |r_k| a_{N-k} = |r_k| * a_k \tag{5.48}$$

$$d_N = \sum_{m\in\mathbb{Z}} \frac{|r_{-m}|}{|N+\frac{1}{2}-m|} = \sum_{m\in\mathbb{Z}} |r_{-m}| a_{N-m} = |r_{-k}| * a_k \tag{5.49}$$

  Since $\{|r_k|\}, \{|r_{-k}|\} \in \ell_1$ and $\{a_k\} \in \ell_2$, then $c_k = |r_k| * a_k, d_k = |r_{-k}| * a_k \in \ell_2$. Therefore on $s \in \Gamma_N^{(1)}$, $|f(s)f(-s)| \leq c_N d_N$. Note that $\{c_N d_N\} \in \ell_1$. Finally, using ML-estimate,

$$\left|\int_{\Gamma_N^{(1)}} f(-s)f(s)ds\right| \leq Nc_N d_N \to 0 \text{ as } N \to \infty$$

To conclude the proof we can write

$$\begin{aligned}
\|f\|_{\mathcal{H}_2}^2 &= \int_{-\infty}^{\infty} f(\imath\omega)f(-\imath\omega)d\omega && (5.50)\\
&= \lim_{N\to\infty}\int_{\gamma_N} f(s)f(-s)ds + \int_{\Gamma_N^{(1)}+\Gamma_N^{(2)}+\Gamma_N^{(3)}} f(s)f(-s)ds && (5.51)\\
&= \lim_{N\to\infty}\int_{\Gamma_N} f(s)f(-s)ds && (5.52)\\
&= \lim_{N\to\infty}\sum_{k=-N}^{N} f(-\lambda^{(k)})\mathrm{Res}[f(s),\lambda^{(k)}] && (5.53)\\
&= \sum_{k\in\mathbb{Z}} f(-\lambda^{(k)})\mathrm{Res}[f(s),\lambda^{(k)}] && (5.54)
\end{aligned}$$

□

With the necessary inner product space framework setup, we are ready to begin discussing the problem of approximating LTP system.

Chapter 6

Model Reduction

We seek a low-dimensional approximation to the large-scale LTP system G. Let G and $\tilde{G}$ be n-dimensional and r-dimensional systems respectively with $r \ll n$.

$$G:\begin{cases}\dot{\mathbf{x}}(t)=\mathbf{A}(t)\mathbf{x}(t)+\mathbf{b}(t)u(t)\\ y(t)=\mathbf{c}^T(t)\mathbf{x}(t)\end{cases}\qquad\Longrightarrow\qquad \tilde{G}:\begin{cases}\dot{\tilde{\mathbf{x}}}(t)=\tilde{\mathbf{A}}(t)\tilde{\mathbf{x}}(t)+\tilde{\mathbf{b}}^T(t)u(t)\\ \tilde{y}(t)=\tilde{\mathbf{c}}^T(t)\tilde{\mathbf{x}}(t)\end{cases}\tag{6.1}$$

with $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$, $\mathbf{b}(t), \mathbf{c}(t), \mathbf{x}(t) \in \mathbb{R}^n$ and $\tilde{\mathbf{A}}(t) \in \mathbb{R}^{r\times r}$, $\tilde{\mathbf{b}}(t), \tilde{\mathbf{c}}(t), \tilde{\mathbf{x}}(t) \in \mathbb{R}^r$.

We can express both systems in the concise forms:

$$G=\left[\begin{array}{c|c}\mathbf{A}(t) & \mathbf{b}(t)\\ \hline \mathbf{c}^T(t) & \end{array}\right]\qquad\Longrightarrow\qquad \tilde{G}=\left[\begin{array}{c|c}\tilde{\mathbf{A}}(t) & \tilde{\mathbf{b}}(t)\\ \hline \tilde{\mathbf{c}}^T(t) & \end{array}\right]\tag{6.2}$$

The following methods discussed assume the availability of a Floquet-transformed equivalent system. Since we know such a transformation exists, without loss of generality, we assume $\mathbf{A}(t) = \mathbf{Q}$ in this chapter.

6.1 An Example of the Floquet-Fourier Representation and its Structure

As an illustration, consider a n-dimensional dynamical system G that has a Floquet-transformed representation $G = \left[\begin{array}{c|c}\mathbf{Q} & \mathbf{b}(t)\\ \hline \mathbf{c}^T(t) & \mathbf{0}\end{array}\right]$ with $\mathbf{Q} \in \mathbb{R}^{n\times n}$ and symbols $\mathbf{b}(t) = \sum_{-1}^{1} \mathbf{b}_k e^{\imath k\omega_0 t} \in \mathbb{R}^n$ and $\mathbf{c}(t) = \sum_{-1}^{1} \mathbf{c}_k e^{\imath k\omega_0 t} \in \mathbb{R}^n$. with three non-zero terms each. Then G has 5 non-trivial subsystems, $\{\hat{g}_i\}_{i=-2}^{2}$, can be written in terms of the Fourier coefficients of $\mathbf{b}(t)$ and $\mathbf{c}(t)$:

$$\begin{aligned}
\hat{g}_2(s) &= \mathbf{c}_1^T[s_1\mathbf{I}-\mathbf{Q}]^{-1}\mathbf{b}_1 \\
\hat{g}_1(s) &= \mathbf{c}_1^T[s\mathbf{I}-\mathbf{Q}]^{-1}\mathbf{b}_0 + \mathbf{c}_0^T[s_1\mathbf{I}-\mathbf{Q}]^{-1}\mathbf{b}_1 \\
\hat{g}_0(s) &= \mathbf{c}_1^T[s_{-1}\mathbf{I}-\mathbf{Q}]^{-1}\mathbf{b}_{-1} + \mathbf{c}_0^T[s\mathbf{I}-\mathbf{Q}]^{-1}\mathbf{b}_0 + \mathbf{c}_{-1}^T[s_1\mathbf{I}-\mathbf{Q}]^{-1}\mathbf{b}_1 \\
\hat{g}_{-1}(s) &= \mathbf{c}_0^T[s_{-1}\mathbf{I}-\mathbf{Q}]^{-1}\mathbf{b}_{-1} + \mathbf{c}_{-1}^T[s\mathbf{I}-\mathbf{Q}]^{-1}\mathbf{b}_0 \\
\hat{g}_{-2}(s) &= \mathbf{c}_{-1}^T[s_{-1}\mathbf{I}-\mathbf{Q}]^{-1}\mathbf{b}_{-1}
\end{aligned} \tag{6.3}$$

Each subsystem then, is a finite summation of n-dimensional LTI terms. We can write the dynamical system norm

$$\|G\|_{\mathcal{H}_2}^2 = \sum_{\ell=-2}^{2} \|\hat{g}_\ell\|_{\mathcal{H}_2}^2 \tag{6.4}$$

and the harmonic transfer function, $\mathcal{G}(s)$:

$$\mathcal{G}(s) = \begin{bmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \hat{g}_0(s-2\imath\omega_0) & \hat{g}_{-1}(s-\imath\omega_0) & \hat{g}_{-2}(s) & 0 & 0 & \ddots \\
\ddots & \hat{g}_1(s-2\imath\omega_0) & \hat{g}_0(s-\imath\omega_0) & \hat{g}_{-1}(s) & \hat{g}_{-2}(s+\imath\omega_0) & 0 & \ddots \\
\ddots & \hat{g}_2(s-2\imath\omega_0) & \hat{g}_1(s-\imath\omega_0) & \hat{g}_0(s) & \hat{g}_{-1}(s+\imath\omega_0) & \hat{g}_{-2}(s+2\imath\omega_0) & \ddots \\
\ddots & 0 & \hat{g}_2(s-\imath\omega_0) & \hat{g}_1(s) & \hat{g}_0(s+\imath\omega_0) & g_{-1}(s+\imath\omega_0) & \ddots \\
\ddots & 0 & 0 & \hat{g}_2(s) & \hat{g}_1(s+\imath\omega_0) & \hat{g}_0(s+2\imath\omega_0) & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix} \tag{6.5}$$

Note here that the HTF is banded doubly infinite-dimensional with translated subsystems repeating themselves along the diagonals.

6.2 Structured Orthogonality Conditions

Theorem 6.2.1 *Suppose* $\tilde{G}$ *solves the minimization problem,* $\min_{dim(\tilde{G})=r} \|G-\tilde{G}\|_{\mathcal{H}_2}$. *Write* $G = \left[\begin{array}{c|c} \Lambda & \mathbf{b}(t) \\ \hline \mathbf{c}^T(t) & \mathbf{0} \end{array}\right]$ *and* $\tilde{G} = \left[\begin{array}{c|c} \tilde{\Lambda} & \tilde{\mathbf{b}}(t) \\ \hline \tilde{\mathbf{c}}^T(t) & \mathbf{0} \end{array}\right]$ *(the diagonalized equivalent form of the Floquet-transformed systems* G *and* $\tilde{G}$ *respectively) where* $\mathbf{b}(t)$, $\tilde{\mathbf{b}}(t)$, $\mathbf{c}(t)$ *and* $\tilde{\mathbf{c}}(t)$ *all have* *finite Fourier expansions (that is,* $\mathbf{b}(t) = \sum_{-N}^{N} \mathbf{b}_k e^{\imath k\omega_0 t}$*, etc.). Define* $\hat{g}_\ell(s)$ *and* $\tilde{\hat{g}}_\ell(s)$ *to be the subsystems of* G *and* $\tilde{G}$ *respectively. Then the following equations hold for* $-N \leq \hat{k} \leq N$ *and* $-N \leq \hat{m} \leq N$*:*

- $$\sum_{\ell\in\mathbb{Z}} \left[\hat{g}_\ell(-\lambda_j + \imath\hat{k}\omega_0) - \tilde{\hat{g}}_\ell(-\lambda_j + \imath\hat{k}\omega_0)\right] [\tilde{\mathbf{c}}_{\ell-\hat{k}}]_j = 0$$
- $$\sum_{\ell\in\mathbb{Z}} \left[\hat{g}_\ell(-\lambda_j + \imath(\ell-\hat{m})\omega_0) - \tilde{\hat{g}}_\ell(-\lambda_j + \imath(\ell-\hat{m})\omega_0)\right] [\tilde{\mathbf{b}}_{\ell-\hat{m}}]_j = 0$$

We label these as the necessary conditions for optimal LTP approximations in the $\mathcal{H}_2$ *sense.*

Proof Denote $\mathcal{M}_{\mathbf{b}} = \left\{ H = \left[\begin{array}{c|c} \tilde{\Lambda} & \overline{\mathbf{b}}(t) \\ \hline \tilde{\mathbf{c}}^T(t) & \mathbf{0} \end{array}\right], \quad \overline{\mathbf{b}}(t) = \sum_{-N}^{N} \overline{\mathbf{b}}_k e^{\imath k\omega_0 t}, \quad \tilde{\Lambda}, \tilde{\mathbf{c}}(t) \text{ fixed} \right\}$ as a finite dimensional subspace of $\bigoplus_{\ell\in\mathbb{Z}} \mathcal{H}_2$ where $\tilde{G} \in \mathcal{M}_{\mathbf{b}}$. By the Hilbert space structure, for all $H \in \mathcal{M}_{\mathbf{b}}$ we know

$$0 = \langle G - \tilde{G}, H \rangle = \sum_{\ell\in\mathbb{Z}} \langle \hat{g}_\ell - \tilde{\hat{g}}_\ell, \hat{h}_\ell \rangle \tag{6.6}$$

Choose $H = \left[\begin{array}{c|c} \tilde{\Lambda} & \overline{\mathbf{b}}(t) \\ \hline \tilde{c}^T(t) & \mathbf{0} \end{array}\right]$ such that $\overline{\mathbf{b}}_k = \begin{cases} \mathbf{e}_j, & k = \hat{k} \\ \mathbf{0} & \text{otherwise} \end{cases}$. Then

$$\hat{h}_\ell(s) = \sum_{-N}^{N} \sum_{j=1}^{r} \frac{[\tilde{c}_{\ell-k}]_j [\overline{\mathbf{b}}_k]_j}{s - \lambda_j + \imath k\omega_0} = \frac{[\tilde{c}_{\ell-\hat{k}}]_j}{s - \lambda_j + \imath\hat{k}\omega_0} \tag{6.7}$$

Therefore,

$$\begin{aligned}
0 &= \sum_{\ell\in\mathbb{Z}} \langle \hat{g}_\ell - \tilde{\hat{g}}_\ell, \hat{h}_\ell \rangle & (6.8)\\
&= \sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sum_{j=1}^{r} \left[\hat{g}_\ell(-\lambda_j + \imath k\omega_0) - \tilde{\hat{g}}_\ell(-\lambda_j + \imath k\omega_0)\right] \operatorname{Res}[\hat{h}_\ell(s), \lambda_j - \imath k\omega_0] & (6.9)\\
&= \sum_{\ell\in\mathbb{Z}} \left[\hat{g}_\ell(-\lambda_j + \imath\hat{k}\omega_0) - \tilde{\hat{g}}_\ell(-\lambda_j + \imath\hat{k}\omega_0)\right] [\tilde{\mathbf{c}}_{\ell-\hat{k}}]_j & (6.10)
\end{aligned}$$

for each integer $\hat{k} \in [-N, N]$.

We have a similar proof for the second condition. Define

$$\mathcal{M}_{\mathbf{c}} = \left\{ H = \left[\begin{array}{c|c} \tilde{\Lambda} & \tilde{\mathbf{b}}(t) \\ \hline \overline{\mathbf{c}}^T(t) & \mathbf{0} \end{array}\right], \quad \overline{\mathbf{c}}(t) = \sum_{-N}^{N} \overline{\mathbf{c}}_k e^{\imath k\omega_0 t}, \quad \tilde{\Lambda}, \tilde{\mathbf{b}}(t) \text{ fixed} \right\}$$

and $\overline{\mathbf{c}}_m = \begin{cases} \mathbf{e}_j, & m = \hat{m} \\ \mathbf{0}, & \text{otherwise} \end{cases}$ where $-N \leq \hat{m} \leq N$. Then

$$\hat{h}_\ell(s) = \sum_{k=\ell-N}^{\ell+N} \sum_{j=1}^{r} \frac{[\overline{\mathbf{b}}_{\ell-k}]_j [\tilde{b}_k]_j}{s - \lambda_j + \imath k\omega_0} = \frac{[\tilde{\mathbf{b}}_{\hat{k}}]_j}{s - \lambda_j + \imath \hat{k}\omega_0} = \frac{[\tilde{\mathbf{b}}_{\ell-\hat{m}}]_j}{s - \lambda_j + \imath(\ell - \hat{m})\omega_0} \tag{6.11}$$

where $\ell - \hat{k} = \hat{m}$. Then

$$0 = \sum_{\ell\in\mathbb{Z}} \langle \hat{g}_\ell - \tilde{\hat{g}}_\ell, \hat{h}_\ell \rangle_{\mathcal{H}_2} \tag{6.12}$$

$$= \sum_{\ell\in\mathbb{Z}} \Big[\hat{g}_\ell(-\lambda_j + \imath(\ell - \hat{m})\omega_0) - \tilde{\hat{g}}_\ell(-\lambda_j + \imath(\ell - \hat{m})\omega_0) \Big] [\tilde{\mathbf{b}}_{\ell-\hat{m}}]_j \tag{6.13}$$

for integers $\hat{m} \in [-N, N]$. □

Remark We can represent these necessary conditions as a linear system of equations:

$$\begin{aligned}
&\begin{bmatrix} [\tilde{\mathbf{c}}_N]_j & \cdots & [\tilde{\mathbf{c}}_{-N}]_j \end{bmatrix}
\begin{bmatrix}
\cdots & \hat{g}_0(-\lambda_j + \imath N\omega_0) & \cdots & \hat{g}_{-N}(-\lambda_j) & \cdots & \hat{g}_{-2N}(-\lambda_j - \imath N\omega_0) & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
\cdots & \hat{g}_N(-\lambda_j + \imath N\omega_0) & \cdots & \hat{g}_0(-\lambda_j) & \cdots & \hat{g}_{-N}(-\lambda_j - \imath N\omega_0) & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
\cdots & \hat{g}_{2N}(-\lambda_j + \imath N\omega_0) & \cdots & \hat{g}_N(-\lambda_j) & \cdots & \hat{g}_0(-\lambda_j - \imath N\omega_0) & \cdots
\end{bmatrix} \\
&= \begin{bmatrix} [\tilde{\mathbf{c}}_N]_j & \cdots & [\tilde{\mathbf{c}}_{-N}]_j \end{bmatrix}
\begin{bmatrix}
\cdots & \tilde{\hat{g}}_0(-\lambda_j + \imath N\omega_0) & \cdots & \tilde{\hat{g}}_{-N}(-\lambda_j) & \cdots & \tilde{\hat{g}}_{-2N}(-\lambda_j - \imath N\omega_0) & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
\cdots & \tilde{\hat{g}}_N(-\lambda_j + \imath N\omega_0) & \cdots & \tilde{\hat{g}}_0(-\lambda_j) & \cdots & \tilde{\hat{g}}_{-N}(-\lambda_j - \imath N\omega_0) & \cdots \\
 & \vdots & & \vdots & & \vdots & \\
\cdots & \tilde{\hat{g}}_{2N}(-\lambda_j + \imath N\omega_0) & \cdots & \tilde{\hat{g}}_N(-\lambda_j) & \cdots & \tilde{\hat{g}}_0(-\lambda_j - \imath N\omega_0) & \cdots
\end{bmatrix}
\end{aligned} \tag{6.14}$$

$$
\begin{bmatrix}
\vdots & & \vdots & & \vdots \\
\hat{g}_0(-\lambda_j + \imath N\omega_0) & \cdots & \hat{g}_{-N}(-\lambda_j) & \cdots & \hat{g}_{-2N}(-\lambda_j - \imath N\omega_0) \\
\vdots & & \vdots & & \vdots \\
\hat{g}_N(-\lambda_j + \imath N\omega_0) & \cdots & \hat{g}_0(-\lambda_j) & \cdots & \hat{g}_{-N}(-\lambda_j - \imath N\omega_0) \\
\vdots & & \vdots & & \vdots \\
\hat{g}_{2N}(-\lambda_j + \imath N\omega_0) & \cdots & \hat{g}_N(-\lambda_j) & \cdots & \hat{g}_0(-\lambda_j - \imath N\omega_0) \\
\vdots & & \vdots & & \vdots
\end{bmatrix}
\begin{bmatrix}
\left[\tilde{\mathbf{b}}_N\right]_j \\
\vdots \\
\left[\tilde{\mathbf{b}}_{-N}\right]_j
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\vdots & & \vdots & & \vdots \\
\tilde{\hat{g}}_0(-\lambda_j + \imath N\omega_0) & \cdots & \tilde{\hat{g}}_{-N}(-\lambda_j) & \cdots & \tilde{\hat{g}}_{-2N}(-\lambda_j - \imath N\omega_0) \\
\vdots & & \vdots & & \vdots \\
\tilde{\hat{g}}_N(-\lambda_j + \imath N\omega_0) & \cdots & \tilde{\hat{g}}_0(-\lambda_j) & \cdots & \tilde{\hat{g}}_{-N}(-\lambda_j - \imath N\omega_0) \\
\vdots & & \vdots & & \vdots \\
\tilde{\hat{g}}_{2N}(-\lambda_j + \imath N\omega_0) & \cdots & \tilde{\hat{g}}_N(-\lambda_j) & \cdots & \tilde{\hat{g}}_0(-\lambda_j - \imath N\omega_0) \\
\vdots & & \vdots & & \vdots
\end{bmatrix}
\begin{bmatrix}
\left[\tilde{\mathbf{b}}_N\right]_j \\
\vdots \\
\left[\tilde{\mathbf{b}}_{-N}\right]_j
\end{bmatrix}
\tag{6.15}
$$

6.3 LTP Connection to MIMO Systems

Theorem 6.3.1 *Let* $G = \left[\begin{array}{c|c} \mathbf{Q} & \mathbf{b}(t) \\ \hline \mathbf{c}^T(t) & \mathbf{0} \end{array}\right]$ *be a LTP system such that* $\mathbf{b}(t)$ *and* $\mathbf{c}(t)$ *have finite Fourier expansions written*

$$
\mathbf{c}(t) = \sum_{\ell=-N}^{N} \mathbf{c} e^{-\imath\ell\omega_0 t} \tag{6.16}
$$

$$
\mathbf{b}(t) = \sum_{\ell=-N}^{N} \mathbf{b} e^{-\imath\ell\omega_0 t} \tag{6.17}
$$

We define an associated LTI MIMO system

$$
H = \left[\begin{array}{c|c} \mathbf{Q} & [\mathbf{b}_{-N}, \ldots, \mathbf{b}_N] \\ \hline \begin{bmatrix} \mathbf{c}_{-N}^T \\ \vdots \\ \mathbf{c}_N^T \end{bmatrix} & \mathbf{0} \end{array}\right] \tag{6.18}
$$

This system has $2N+1$ *inputs formed by the Fourier coefficients of* $\mathbf{b}(t)$ *and* $2N+1$ *outputs formed by the Fourier coefficients of* $\mathbf{c}(t)$*. Then the* $\mathcal{H}_2$ *norm of the LTP dynamical system* G *can be bounded in terms of the* $\mathcal{H}_2$ *norm of the LTI dynamical system* H*, namely*

$$
\|G\|_{\mathcal{H}_2} \leq \sqrt{2N+1}\ \|H\|_{\mathcal{H}_2} \tag{6.19}
$$

Proof The above system, G, has subsystems that can be written

$$\hat{g}_n(s) = \sum_{\ell=-N+\ell}^{N+\ell} \mathbf{c}_{n-\ell}^T [s_\ell \mathbf{I} - \mathbf{Q}]^{-1} \mathbf{b}_\ell \tag{6.20}$$

Each of these terms are finite-dimensional LTI systems.

$$\begin{aligned} \|G\|_{\mathcal{H}_2}^2 &= \sum_{\ell=-2N}^{2N} \|\hat{g}_\ell(s)\|_{\mathcal{H}_2}^2 && (6.21) \\ &= \sum_{\ell=-2N}^{2N} \left\| \sum_{\ell=-N+\ell}^{N+\ell} \mathbf{c}_{n-\ell}^T [s_\ell \mathbf{I} - \mathbf{Q}]^{-1} \mathbf{b}_\ell \right\|_{\mathcal{H}_2}^2 && (6.22) \end{aligned}$$

Now we use that $\|a_1 + \ldots + a_n\|^2 \leq \frac{n}{2}(\|a_1\|^2 + \ldots \|a_n\|^2)$ (application of the triangle inequality and the property, $2ab \leq a^2 + b^2$ for $a, b \in \mathbb{R}$) to show

$$\left\| \sum_{n=-N+\ell}^{N+\ell} \mathbf{c}_{n-\ell}^T [s_\ell \mathbf{I} - \mathbf{Q}]^{-1} \mathbf{b}_\ell \right\|_{\mathcal{H}_2}^2 \leq (2N+1) \sum_{n=-N+\ell}^{N+\ell} \|\mathbf{c}_{n-\ell}^T [s_\ell \mathbf{I} - \mathbf{Q}]^{-1} \mathbf{b}_\ell\|_{\mathcal{H}_2}^2 \tag{6.23}$$

Then we have

$$\begin{aligned} \|G\|_{\mathcal{H}_2}^2 &= (2N+1) \sum_{\ell=-2N}^{2N} \sum_{n=-N+\ell}^{N+\ell} \|\mathbf{c}_{n-\ell}^T [s_\ell \mathbf{I} - \mathbf{Q}]^{-1} \mathbf{b}_\ell\|_{\mathcal{H}_2}^2 && (6.24) \\ &= (2N+1) \sum_{i,j=-N}^{N} \|\mathbf{c}_i^T [s_j \mathbf{I} - \mathbf{Q}]^{-1} \mathbf{b}_j\|_{\mathcal{H}_2}^2 && (6.25) \end{aligned}$$

After rearranging indices and dropping trivial terms. If

$$H = \left[\begin{array}{c|c} \mathbf{Q} & [\mathbf{b}_{-N}, \ldots, \mathbf{b}_N] \\ \hline \begin{bmatrix} \mathbf{c}_{-N}^T \\ \vdots \\ \mathbf{c}_N^T \end{bmatrix} & \mathbf{0} \end{array} \right] \tag{6.26}$$

Then

$$\|G\|_{\mathcal{H}_2}^2 = (2N+1) \int_{-\infty}^{\infty} \operatorname{trace}(H(j\omega) H(j\omega)^*) d\omega = (2N+1) \|H\|_{\mathcal{H}_2}^2 \tag{6.27}$$

□

This result inspires a model order reduction algorithm for LTP systems outlined in the next section.

6.4 A Model Reduction Method for LTP Systems

Theorem 6.4.1 *Given* $G = \left[\begin{array}{c|c} \mathbf{Q} & \mathbf{b}(t) \\ \hline \mathbf{c}^T(t) & \mathbf{0} \end{array}\right]$ *where* $\mathbf{b}(t)$ *and* $\mathbf{c}^T(t)$ *have finite Fourier expansions written* $\mathbf{b}(t) = \sum_{-N}^{N} \mathbf{b}_k e^{\imath k \omega_0 t}$ *and* $\mathbf{c}(t) = \sum_{-N}^{N} \mathbf{c}_k e^{\imath k \omega_0 t}$*. Let* $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}$ *be projection matrices such that* $\mathbf{W}^T\mathbf{V} = \mathbf{I}$ *and define the reduced system,*

$$\tilde{G} = \left[\begin{array}{c|c} \mathbf{W}^T\mathbf{Q}\mathbf{V} & \mathbf{W}^T\mathbf{b}(t) \\ \hline \mathbf{c}^T(t)\mathbf{V} & \mathbf{0} \end{array}\right] = \left[\begin{array}{c|c} \tilde{\mathbf{Q}} & \tilde{\mathbf{b}}(t) \\ \hline \tilde{\mathbf{c}}^T(t) & \mathbf{0} \end{array}\right] \tag{6.28}$$

Then

$$\|G - \tilde{G}\|_{\mathcal{H}_2} \leq \sqrt{2N+1}\,\|H - \tilde{H}\|_{\mathcal{H}_2} \tag{6.29}$$

where

$$H = \left[\begin{array}{c|c} \mathbf{Q} & [\mathbf{b}_{-N}, \ldots, \mathbf{b}_N] \\ \hline \begin{bmatrix} \mathbf{c}_{-N}^T \\ \vdots \\ \mathbf{c}_N^T \end{bmatrix} & \mathbf{0} \end{array}\right] \qquad \tilde{H} = \left[\begin{array}{c|c} \tilde{\mathbf{Q}} & [\tilde{\mathbf{b}}_{-N}, \ldots, \tilde{\mathbf{b}}_N] \\ \hline \begin{bmatrix} \tilde{\mathbf{c}}_{-N}^T \\ \vdots \\ \tilde{\mathbf{c}}_N^T \end{bmatrix} & \mathbf{0} \end{array}\right] \tag{6.30}$$

Proof Suppose that systems G and $\tilde{G}$ have dimensions n and r respectively. Then we can write the error system $G_{err} = G - \tilde{G}$ can be written as an $n + r$-dimensional LTP system,

$$G_{err} = G - \tilde{G} = \left[\begin{array}{c|c} \begin{bmatrix} \mathbf{Q} & \\ & \tilde{\mathbf{Q}} \end{bmatrix} & \begin{bmatrix} \mathbf{b}(t) \\ -\tilde{\mathbf{b}}(t) \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{c}^T(t) & \tilde{\mathbf{c}}^T(t) \end{bmatrix} & \mathbf{0} \end{array}\right] \tag{6.31}$$

Then the associated MIMO system, H_{err} is written

$$H_{err} = \left[\begin{array}{c|c} \begin{bmatrix} \mathbf{Q} & \\ & \tilde{\mathbf{Q}} \end{bmatrix} & \begin{bmatrix} \mathbf{b}_{-N} & \cdots & \mathbf{b}_N \\ -\tilde{\mathbf{b}}_{-N} & \cdots & -\tilde{\mathbf{b}}_N \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{c}_{-N}^T & \tilde{\mathbf{c}}_{-N}^T \\ \vdots & \vdots \\ \mathbf{c}_N^T & \tilde{\mathbf{c}}_N^T \end{bmatrix} & \mathbf{0} \end{array}\right] \tag{6.32}$$

Then

$$\|G - \tilde{G}\|_{\mathcal{H}_2} = \|G_{err}\|_{\mathcal{H}_2} \leq \sqrt{2N+1}\,\|H_{err}\|_{\mathcal{H}_2} = \sqrt{2N+1}\,\|H - \tilde{H}\|_{\mathcal{H}_2} \tag{6.33}$$

□

Algorithm: Model Reduction of an LTP System

Given dynamical system $G = \left[\begin{array}{c|c} \mathbf{A}(t) & \hat{\mathbf{b}}(t) \\ \hline \hat{\mathbf{c}}^T(t) & \mathbf{0} \end{array}\right]$

1. Perform a Floquet-transformation so what we have a realization $G = \left[\begin{array}{c|c} \mathbf{Q} & \mathbf{b}(t) \\ \hline \mathbf{c}^T(t) & \mathbf{0} \end{array}\right]$

2. Truncate Fourier seriers $\mathbf{b}(t) \approx \sum_{k=-N}^{N} \mathbf{b}_k e^{\imath k\omega_0 t}$ and $\mathbf{c}(t) \approx \sum_{k=-N}^{N} \mathbf{c}_k e^{\imath k\omega_0 t}$. Define

$$G_{[N]} = \left[\begin{array}{c|c} \mathbf{Q} & \displaystyle\sum_{k=-N}^{N} \mathbf{b}_k e^{\imath k\omega_0 t} \\ \hline \displaystyle\sum_{k=-N}^{N} \mathbf{c}_k^T e^{\imath k\omega_0 t} & \mathbf{0} \end{array}\right] \tag{6.34}$$

3. Construct the associated MIMO system,

$$H_{[N]} = \left[\begin{array}{c|c} \mathbf{Q} & [\mathbf{b}_{-N}, \ldots, \mathbf{b}_N] \\ \hline \begin{bmatrix} \mathbf{c}_{-N}^T \\ \vdots \\ \mathbf{c}_N^T \end{bmatrix} & \mathbf{0} \end{array}\right] \tag{6.35}$$

   from the Fourier coefficients of $G_{[N]}$.

4. Find projection matrices, $\mathbf{V}$ with $\mathbf{W}$, $\mathbf{W}^T\mathbf{V} = \mathbf{I}$, that minimize $\|H_{[N]} - \tilde{H}_{[N]}\|_{\mathcal{H}_2}$ for r-dimensional

$$\tilde{H}_{[N]} = \left[\begin{array}{c|c} \mathbf{W}^T\mathbf{Q}\mathbf{V} & \mathbf{W}^T[\mathbf{b}_{-N}, \ldots, \mathbf{b}_N] \\ \hline \begin{bmatrix} \mathbf{c}_{-N}^T \\ \vdots \\ \mathbf{c}_N^T \end{bmatrix} \mathbf{V} & \mathbf{0} \end{array}\right] = \left[\begin{array}{c|c} \tilde{\mathbf{Q}} & [\tilde{\mathbf{b}}_{-N}, \ldots, \tilde{\mathbf{b}}_N] \\ \hline \begin{bmatrix} \tilde{\mathbf{c}}_{-N}^T \\ \vdots \\ \tilde{\mathbf{c}}_N^T \end{bmatrix} & \mathbf{0} \end{array}\right]. \tag{6.36}$$

5. Construct approximating LTP system $\tilde{G}$ from the coefficients of $\tilde{H}_{[N]}$

$$\tilde{G} = \left[\begin{array}{c|c} \tilde{\mathbf{Q}} & \displaystyle\sum_{k=-N}^{N} \tilde{\mathbf{b}}_k e^{\imath k\omega_0 t} \\ \hline \displaystyle\sum_{k=-N}^{N} \tilde{\mathbf{c}}_k^T e^{\imath k\omega_0 t} & \mathbf{0} \end{array}\right] \tag{6.37}$$

Then the dynamical system error, $\|G - \tilde{G}\|_{\mathcal{H}_2}$ is bounded by

$$\begin{aligned} \|G - \tilde{G}\|_{\mathcal{H}_2} &\leq \|G - G_{[N]}\|_{\mathcal{H}_2} + \|G_{[N]} - \tilde{G}\|_{\mathcal{H}_2} && (6.38) \\ &\leq \|G - G_{[N]}\|_{\mathcal{H}_2} + \sqrt{2N+1}\|H_{[N]} - \tilde{H}_{[N]}\|_{\mathcal{H}_2} && (6.39) \end{aligned}$$

Notice here that there is a trade-off in our choice of N. The error from truncation

$$\|G - G_{[N]}\|_{\mathcal{H}_2} \to 0 \qquad \text{as} \qquad N \to \infty.$$

Meanwhile, the error from our low-rank approximation

$$\sqrt{2N+1}\|H_{[N]} - \tilde{H}_{[N]}\|_{\mathcal{H}_2} \to \infty \qquad \text{as} \qquad N \to \infty.$$

Various model reduction methods can be applied here such as Proper Orthogonal Decomposition, Balanced Truncation, Iterative Rational Krylov Algorithm (IRKA), etc. Since IRKA produces locally optimal approximations in the $\mathcal{H}_2$ sense, it is a natural candidate for producing $\tilde{H}_{[N]}$ to minimize $\|H_{[N]} - \tilde{H}_{[N]}\|_{\mathcal{H}_2}$.

Chapter 7

Results

We apply the algorithm above to three models: (1) a fluid mechanical model describing fluid flow around a cylinder formulated as a finite element solution to Navier-Stokes, (2) a structural model of the component of the International Space Station, and (3) a heat diffusion model of a plate described by the heat equation. To learn more about either of these models, see Antoulas et al. (2001), [1].

7.1 Fluid Mechanical Model

We model two-dimensional fluid flows around a square cylinder with a finite element solution to the Navier-Stokes equation. For information about how this model was constructed, see Hay et al. (2009), [10].

This model forms a quadratic ODE

$$\dot{\mathbf{x}} = F(\mathbf{x}) \tag{7.1}$$

where $F(\mathbf{x})$, written element-wise is

$$\mathbf{e}_i^T[F(\mathbf{x})] = \mathbf{e}_i^T\mathbf{b} + \mathbf{e}_i^T\mathbf{R}\mathbf{x} + [\mathbf{x}^T\mathbf{B}_i\mathbf{x}] \tag{7.2}$$

where $\mathbf{b}$ is a diffusive term of mean flow, $\{\mathbf{B}_i\}$ is the nonlinear convection term of the fluctuation around the mean, and $\mathbf{A}$ represents all linear terms acting convection and diffusion of the fluctuation.

The solution to the quadratic model, (7.2), is periodic. Suppose $\mathbf{x}(t)$ is a T-periodic solution to the ODE and consider a perturbation of the solution, $\mathbf{b} \mapsto \mathbf{b} + \mathbf{b}_\epsilon u(t)$ where $\|\mathbf{b}_\epsilon u(t)\|$ is small. Define the perturbed solution $\mathbf{x}(t) + \mathbf{x}_\epsilon(t)$ and linearize around $\mathbf{x}(t)$:

$$\mathbf{e}_i^T[\dot{\mathbf{x}}(t)+\dot{\mathbf{x}}_\epsilon(t)] = \mathbf{e}_i^T[\mathbf{b}+\mathbf{b}_\epsilon u(t)+\mathbf{R}(\mathbf{x}(t)+\mathbf{x}_\epsilon(t))]+[\mathbf{x}(t)+\mathbf{x}_\epsilon(t)]^T\mathbf{B}_i[\mathbf{x}(t)+\mathbf{x}_\epsilon(t)] \tag{7.3}$$

Removing higher-order terms and the nominal solution we get a LTP ODE:

$$\dot{\mathbf{x}}_\epsilon(t) = \mathbf{R}\mathbf{x}_\epsilon(t) + \begin{bmatrix} \mathbf{x}(t)^T(\mathbf{B}_1+\mathbf{B}_1^T) \\ \vdots \\ \mathbf{x}(t)^T(\mathbf{B}_n+\mathbf{B}_n^T) \end{bmatrix} \mathbf{x}_\epsilon(t)+\mathbf{b}_\epsilon u(t) \tag{7.4}$$

We construct an output by averaging the average rate of flow across all states of the system, $\mathbf{c}^T = [\frac{1}{n},\ldots,\frac{1}{n}]$ where n is the dimension of system. Define $\mathbf{A}(t) = \mathbf{R} + \begin{bmatrix} \mathbf{x}(t)^T(\mathbf{B}_1+\mathbf{B}_1^T) \\ \vdots \\ \mathbf{x}(t)^T(\mathbf{B}_n+\mathbf{B}_n^T) \end{bmatrix}$ as the T-periodic state-space matrix. Then we can write the LTP dynamical system,

$$G : \begin{cases} \dot{\mathbf{x}}_\epsilon(t) = \mathbf{A}(t)\mathbf{x}_\epsilon(t)+\mathbf{b}_\epsilon u(t) \\ y(t) = \mathbf{c}^T(t)\mathbf{x}_\epsilon(t) \end{cases} \tag{7.5}$$

For the following simulations, we ran our algorithm on a system of order $n = 12$ constructed by a POD based approximation to a $n = 22,600$ order system. For more information about this model, see Hay et al. (2009), [10]. First we perform a Floquet transformation on the ODE and truncate the number of Fourier coefficients at $N = 3$. Then the Floquet-Fourier transformed system has $4N + 1 = 13$ subsystems.

To evaluate our algorithm's efficacy, we begin with a system, G, of the form (7.5), with dimension $n = 12$. After a Floquet transformation, we construct reduced order models of dimensions $r = 7, 8, 9, 10$ and compare their respective error system $\mathcal{H}_2$ norms: $\|G - \tilde{G}\|_{\mathcal{H}_2}$. See figure 7.2.

For the order $r = 10$ system we have run simulations of the full and reduced order output as well as generated bode plots for the full, reduced and error systems. See figures 7.1 and 7.3 respectively.

Figure 7.1: Time simulation of the system in Figure 7.4 of full ($n = 270$) and reduced ($r = 30$) systems fed with sinusoidal input, $u(t) = \sin(19.2875t)$

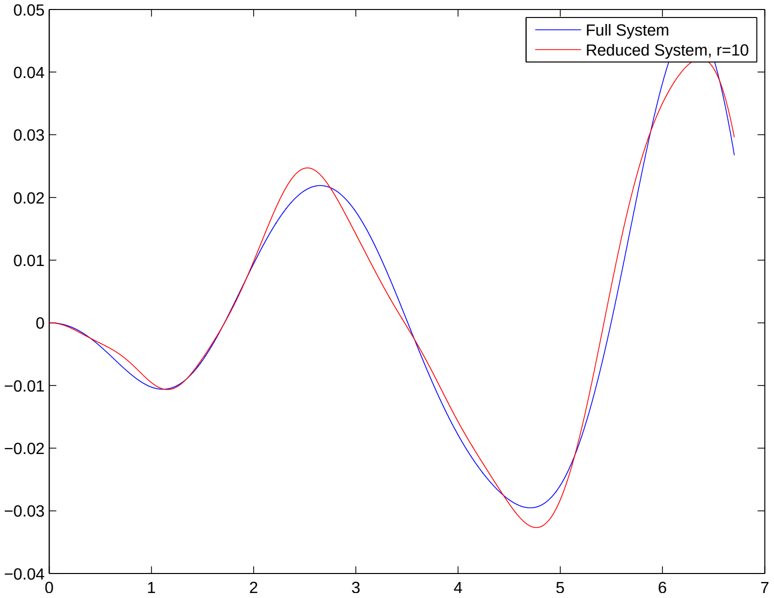


Figure 7.2: Dynamical system error for various sizes of reduced systems

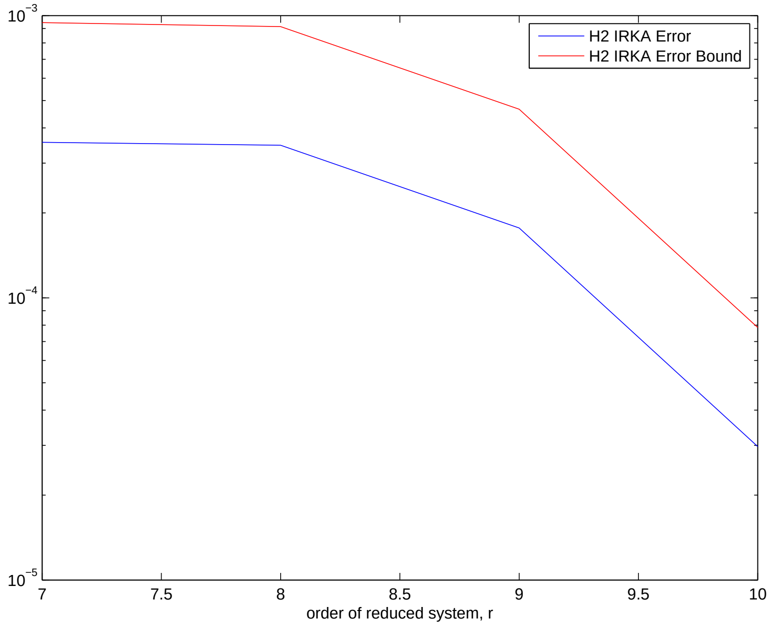
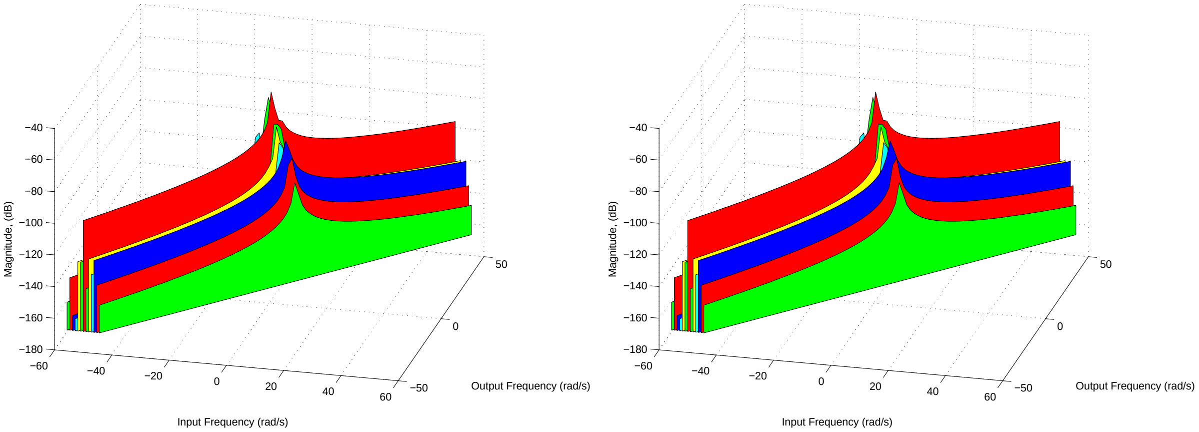
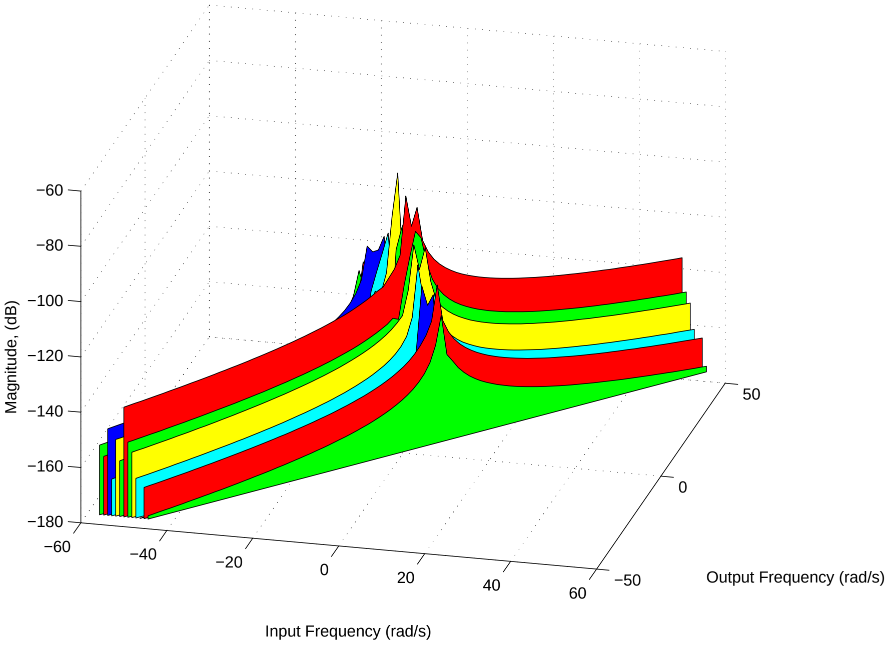


Figure 7.3: LTP Bode plots: full, reduced, and error systems, $r = 10$



(a) Full order

(b) Reduced order



(c) Error

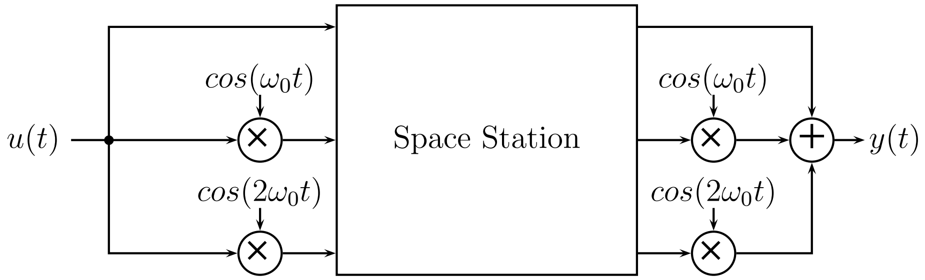
7.2 Structural Model

The structural model has 270 states, 3 inputs and 3 outputs. We make this model LTP by placing modulators on inputs 2 and 3. We do the same for outputs 2 and 3. Define $\mathbf{B} = \begin{bmatrix} \mathbf{b}_0 & \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$ to be the original input vector and $\mathbf{C} = \begin{bmatrix} \mathbf{c}_0^T \\ \mathbf{c}_1^T \\ \mathbf{c}_2^T \end{bmatrix}$ to be the original output vector. Then we construct a SISO system by feeding the input, $u(t)$, into two modulators with local oscillator (LO) frequencies of ω_0 and $2\omega_0$ respectively.

$$G = \left[\begin{array}{c|c} \mathbf{Q} & \mathbf{b}_0 + \mathbf{b}_1\cos(\omega_0 t) + \mathbf{b}_2\cos(2\omega_0 t) \\ \hline \mathbf{c}_0 + \mathbf{c}_1\cos(\omega_0 t) + \mathbf{c}_2\cos(2\omega_0 t) & \mathbf{0} \end{array}\right] \tag{7.6}$$

To see this abstracted into a diagram see Figure 7.4

Figure 7.4: Structural Model of Component 1r(Russian service module) of the International Space Station with inputs and outputs modulated by local oscillator frequencies ω_0 and $2\omega_0$



We construct the associated MIMO system,

$$H = \left[\begin{array}{c|c} \mathbf{Q} & [\frac{1}{2}\mathbf{b}_2\ \frac{1}{2}\mathbf{b}_1\ \mathbf{b}_0\ \frac{1}{2}\mathbf{b}_1\ \frac{1}{2}\mathbf{b}_2] \\ \hline \begin{bmatrix} \frac{1}{2}\mathbf{c}_2^T \\ \frac{1}{2}\mathbf{c}_1^T \\ \mathbf{c}_0 \\ \frac{1}{2}\mathbf{c}_1^T \\ \frac{1}{2}\mathbf{c}_2^T \end{bmatrix} & \mathbf{0} \end{array}\right] \tag{7.7}$$

We compute an optimal approximation $\tilde{H}$, to the MIMO model, H, by IRKA. We construct our low dimensional approximation, $\tilde{G}$, from the same projection matrices $\mathbf{V}$ and $\mathbf{W}$ that we used to construct $\tilde{H}$:

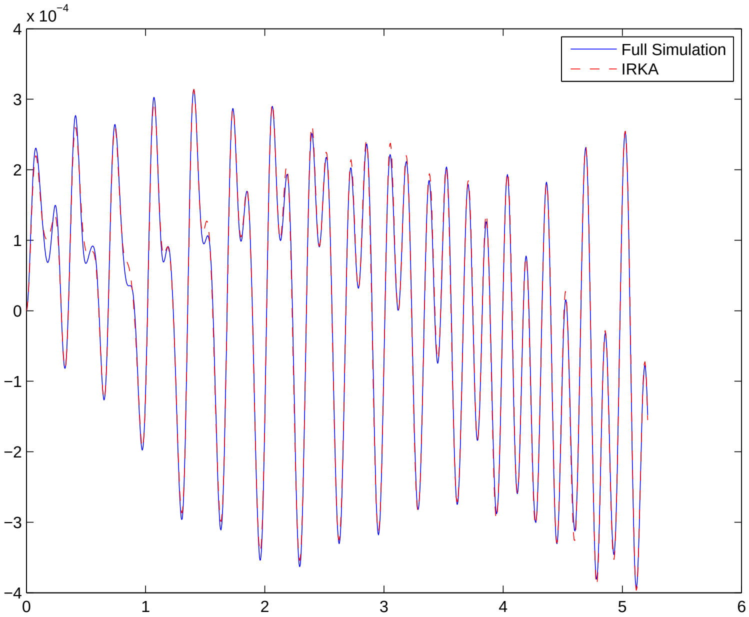
$$\tilde{G} = \left[\begin{array}{c|c} \mathbf{W}^T\mathbf{Q}\mathbf{V} & \mathbf{W}^T\mathbf{b}_0 + \mathbf{b}_1\cos(\omega_0 t) + \mathbf{b}_2\cos(2\omega_0 t) \\ \hline \mathbf{c}_0 + \mathbf{c}_1\cos(\omega_0 t) + \mathbf{c}_2\cos(2\omega_0 t)\mathbf{V} & \mathbf{0} \end{array}\right] \tag{7.8}$$

$$= \left[\begin{array}{c|c} \tilde{\mathbf{Q}} & \tilde{\mathbf{b}}_0 + \tilde{\mathbf{b}}_1\cos(\omega_0 t) + \tilde{\mathbf{b}}_2\cos(2\omega_0 t) \\ \hline \tilde{\mathbf{c}}_0 + \tilde{\mathbf{c}}_1\cos(\omega_0 t) + \tilde{\mathbf{c}}_2\cos(2\omega_0 t) & \mathbf{0} \end{array}\right]. \tag{7.9}$$

The system G is already in the Floquet-Fourier form of dimension $n = 270$ and $N = 2$ since the Fourier expansions of $\mathbf{b}(t)$ and $\mathbf{c}(t)$ have 5 nontrivial terms. We construct reduced order models of dimensions $r = 4$ to $r = 80$ and compare their respective error system $\mathcal{H}_2$ norms: $\|G - \tilde{G}\|_{\mathcal{H}_2}$. See figure 7.6.

For the order $r = 30$ system we have run simulations of the full and reduced order output as well as generated bode plots for the full, reduced and error systems. See figures 7.5 and 7.7 respectively.

Figure 7.5: Time simulation of the system in Figure 7.4 of full ($n = 270$) and reduced ($r = 30$) systems fed with sinusoidal input, $u(t) = \sin(19.2875t)$



For comparison, we tried to reduce the model by Proper Othogonal Decomposion (POD), with a test function of $u(t) = \sin(19.2875t)$ but each reduced model was unstable. As a result we omitted POD from these comparisons.

Figure 7.6: Dynamical system error for various sizes of reduced systems

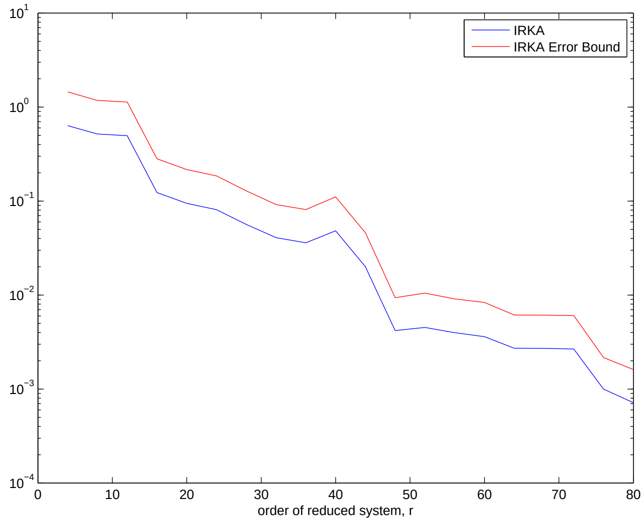
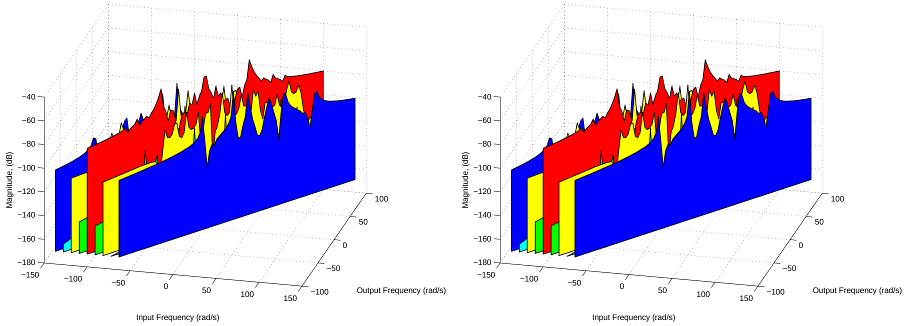
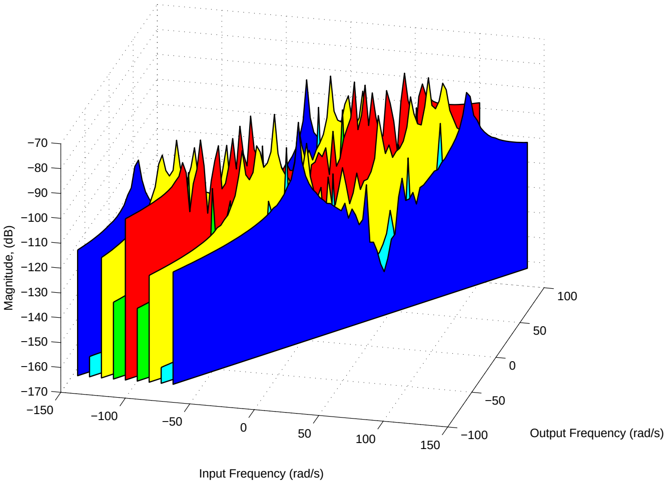


Figure 7.7: LTP Bode plots for Figure 7.4: full, reduced, and error systems, $r = 30$



(a) Full order (b) Reduced order



(c) Error

7.3 Heat Model

We begin with a two input, two output model describing the heat diffusion on a plate with two heat sources and two points of measurements. It is described by the heat equation. A model of order 197 is obtained by spatial discretization. Define $\mathbf{B} = \begin{bmatrix} \mathbf{b}_0 & \mathbf{b}_1 \end{bmatrix}$ to be the original input vector and $\mathbf{C} = \begin{bmatrix} \mathbf{c}_0^T \\ \mathbf{c}_1^T \end{bmatrix}$ to be the original output vector.

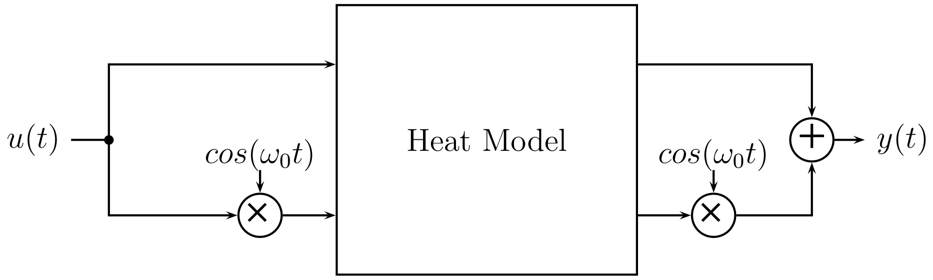
We construct a SISO system by feeding the input, $u(t)$ through a modulator with local oscillator frequency of ω_0 into the second input and output.

$$G = \left[\begin{array}{c|c} \mathbf{Q} & \mathbf{b}_0 + \mathbf{b}_1 \cos(\omega_0 t) \\ \hline \mathbf{c}_0 + \mathbf{c}_1 \cos(\omega_0) & \mathbf{0} \end{array}\right] \tag{7.10}$$

We construct the associate MIMO system,

$$H = \left[\begin{array}{c|c} \mathbf{Q} & [\frac{1}{2}\mathbf{b}_1 \ \mathbf{b}_0 \ \frac{1}{2}\mathbf{b}_1] \\ \hline \begin{bmatrix} \frac{1}{2}\mathbf{c}_1^T \\ \mathbf{c}_0 \\ \frac{1}{2}\mathbf{c}_1^T \end{bmatrix} & \mathbf{0} \end{array}\right] \tag{7.11}$$

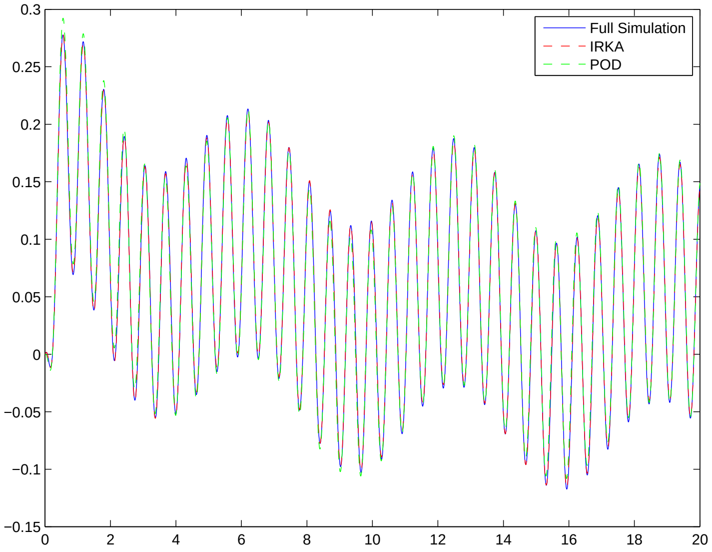
Figure 7.8: Heat model of a plate with inputs and outputs modulated by local oscillator frequency ω_0



The system G is already in the Floquet-Fourier form of dimension $n = 197$ and $N = 1$ since the Fourier expansions of $\mathbf{b}(t)$ and $\mathbf{c}(t)$ have 3 nontrivial terms. We construct reduced order models of dimensions $r = 2$ to $r = 20$ and compare their respective error system $\mathcal{H}_2$ norms: $\|G - \tilde{G}\|_{\mathcal{H}_2}$. See figure 7.10.

For the order $r = 6$ system we have run simulations of the full and reduced order output as well as generated bode plots for the full, reduced and error systems. See figures 7.9 and 7.11 respectively.

Figure 7.9: Time simulation of the system in Figure 7.8 of full ($n = 197$) and reduced ($r = 6$) systems fed with sinusoidal input, $u(t) = \sin(10t)$



For comparison, we construct an additional reduced order model to compare by proper orthogonalization decomposition (POD) on the full order system, G. As a "test function" we chose $u(t) = \sin(10t)$ from Figure 7.9. The reduced model works very well in Figure 7.9 when fed with the test function but poorly for other inputs. Hence, POD underperforms our algorithm in the $\mathcal{H}_2$ sense. See Figure 7.10.

Figure 7.10: Dynamical system error for various sizes of reduced systems by MIMO IRKA

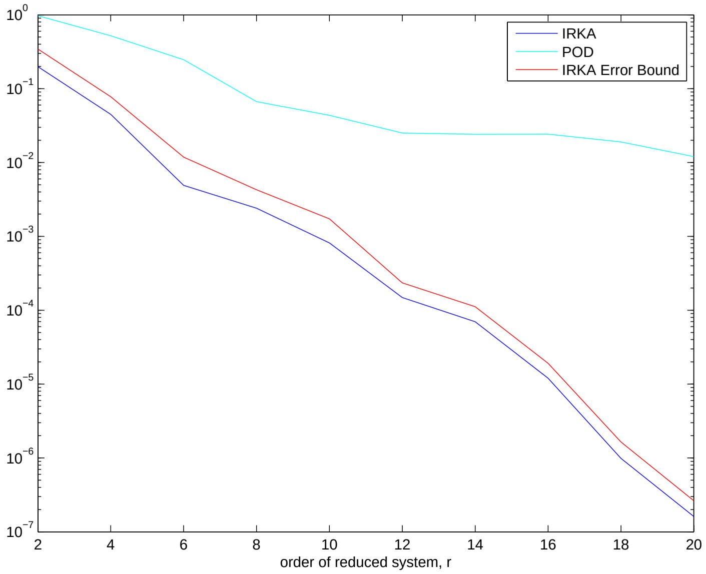
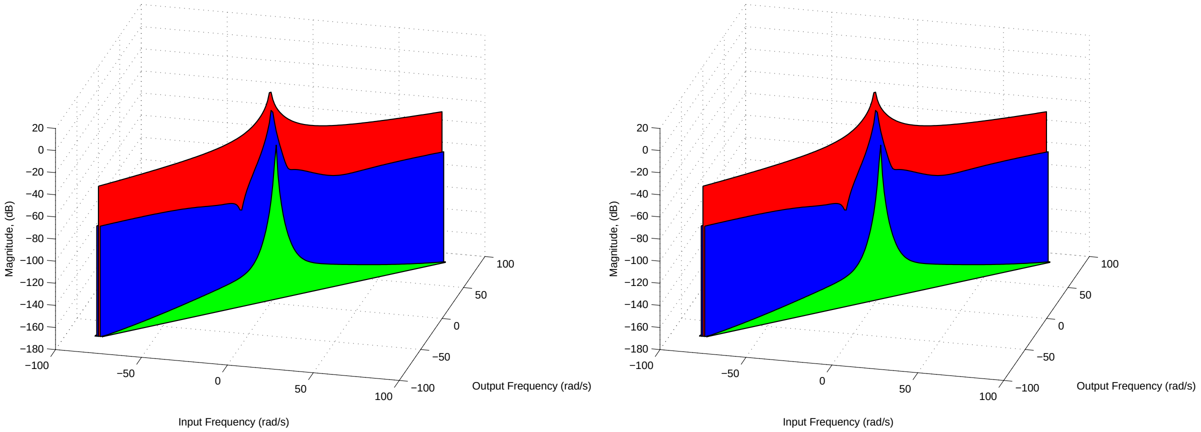
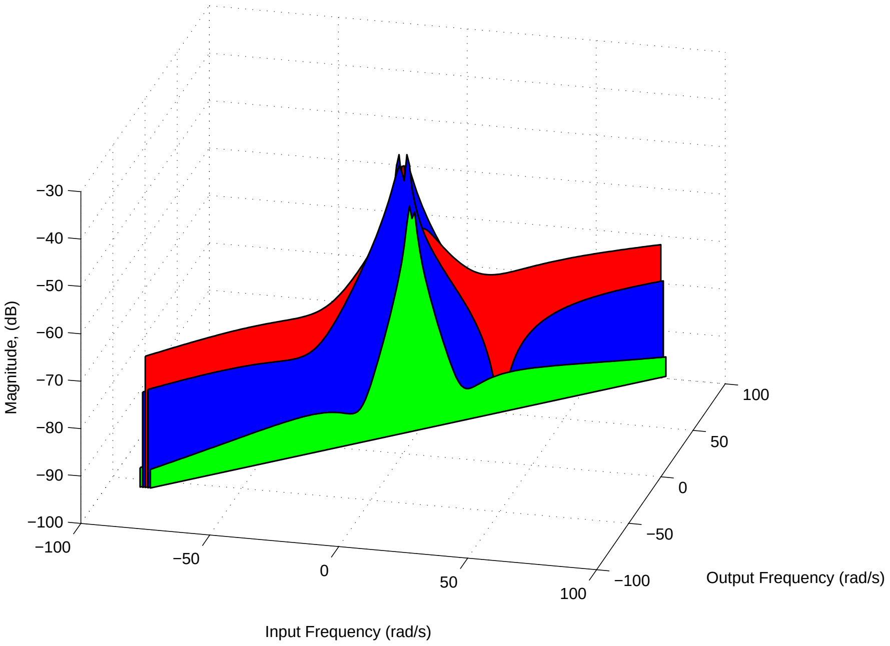


Figure 7.11: LTP Bode plots for Figure 7.8: full, reduced, and error systems, $r = 6$



(a) Full order (b) Reduced order



(c) Error

In conclusion, while our algorithm is not optimal, it performs very well for only the cost of reduction of a MIMO system.

Chapter 8

Conclusion

We conclude this paper with a summary of it's contributions to the LTP system theory and model reduction of LTP systems.

- *Theorem 5.1.2*: Connection between subsystems (Fourier coefficients of the impulse response) and the elements of the HTF written in the Floquet-Fourier form is demonstrated.
- *Theorem 5.2.3*: Dynamical system $\mathcal{H}_2$-norm and inner product are formulated in terms of the subsystems instead of in term of the Harmonic Transfer Function.
- *Theorem 5.2.5*: The $\mathcal{H}_2$ inner product for infinite-dimensional transfer functions is formulated in terms of poles and residues. This generalizes the familiar inner product calculation in the finite-dimensional setting.
- *Theorem 6.2.1*: Necessary conditions are developed for best approximations by exploiting the structured orthogonality conditions of the underlying Hilbert space. These are written in terms of tangential interpolation conditions of the full and reduced systems respectively.
- *Theorem 6.3.1*: We bound our $\mathcal{H}_2$ dynamical system norm by the familiar $\mathcal{H}_2$ norm of a MIMO system.

In summary, we have extended LTI system theory to the LTP setting, including necessary conditions for best approximations, dynamical system $\mathcal{H}_2$-norm and inner product, and the conceptualization of the system as a frequency response operator. Finally, we proposed an algorithm that we apply to three models to verify the efficacy of our approach.

Bibliography

[1] A C Antoulas, D C Sorensen, and S Gugercin. A survey of model reduction methods for large-scale systems. *Contemporary mathematics*, 2001.

[2] Athanasios C Antoulas. *Approximation of Large-Scale Dynamical Systems.* Society for Industrial and Applied Mathematics, June 2009.

[3] Athanasios C Antoulas, Christopher A Beattie, and Serkan Gugercin. Interpolatory Model Reduction of Large-Scale Dynamical Systems. In *... Control of Large-Scale Systems*, pages 3–58. Springer US, Boston, MA, May 2010.

[4] U Baur, C Beattie, P Benner, and S Gugercin. Interpolatory projection methods for parameterized model reduction. *SIAM Journal on Scientific ...*, 2011.

[5] D Borwein and W Kratz. Weighted Convolution Operators on lp. *Canadian Mathematical Bulletin*, 2005.

[6] Z Cai, Y Gu, and W Zhong. ScienceDirect.com - Computers & Structures - A new approach of computing Floquet transition matrix. *Computers & Structures*, 2001.

[7] F R Gantmacher. *Theory of Matrices (Vol 1).* Chelsea Publishing Company, New York, N.Y., January 1959.

[8] R Grimshaw. Nonlinear Ordinary Differential Equaitons - R. Grinshaw - Google Books. 1993.

[9] S Gugercin, A C Antoulas, and C Beattie. $\mathcal{H}_2$ Model Reduction for Large-Scale Linear Dynamical Systems. *SIAM Journal on Matrix Analysis and Applications*, 30(2):609, 2008.

[10] Alexander Hay, Jeffrey T Borggaard, and Dominique Pelletier. Local improvements to reduced-order models using sensitivity analysis of the proper orthogonal decomposition. *Journal of Fluid Mechanics*, 629:41–72, 2009.

[11] Lewis Meier and David G Luenberger. Approximation of linear constant systems. *Automatic Control, IEEE Transactions on*, 12(5):585–588, 1967.

[12] E Mollerstedt. Dynamic analysis of harmonics in electrical systems. 2000.

[13] Pierre Montagnier, Christopher C. Paige, and Raymond J. Spiteri. Real Floquet factors of linear time-periodic systems. *Systems & Control Letters*, 50(4):251–262, 2003.

[14] G Moore. JSTOR: SIAM Journal on Numerical Analysis, Vol. 42, No. 6 (2005), pp. 2522-2568. *SIAM journal on numerical analysis*, 2005.

[15] Robert Rogers. *Fourier Series and Partial Differential Equations.* August 2012.

[16] H Sandberg. On Floquet-Fourier Realizations of Linear Time-Periodic Impulse Responses. In *Decision and Control, 2006 45th IEEE Conference on*, pages 1411–1416, 2006.

[17] H Sandberg, E Mollerstedt, and B Bernhardsson. Frequency-domain analysis of linear time-periodic systems. In *American Control Conference, 2004. Proceedings of the 2004*, pages 3357–3362, 2004.

[18] H Sandberg, E Mollerstedt, and Bernhardsson. Frequency-domain analysis of linear time-periodic systems. *IEEE Transactions on Automatic Control*, 50(12):1971–1983, December 2005.

[19] A. Siddiqi and SR Hall. Identification of the harmonic transfer functions of a helicopter rotor. 2001.

[20] A. Varga. An overview of recent developments in computational methods for periodic systems. *Proceedings of IFAC Workshop on Periodic Control Systems*, 2007.

[21] N.M. Wereley. Analysis and control of linear periodically time varying systems. 1991.

[22] N.M. Wereley and S.R. Hall. Frequency response of linear time periodic systems. *Decision and Control, 1990., Proceedings of the 29th IEEE Conference on*, pages 3650–3655 vol. 6, 1990.

[23] Norman M Wereley and Steven R Hall. Linear Time Periodic Systems: Transfer Function, Poles, Transmission Zeroes and Directional Properties. In *American Control Conference, 1991*, pages 1179–1184, 1991.

[24] J. Zhou and T. Hagiwara. H 2 and H norm computations of linear continuous-time periodic systems via the skew analysis of frequency response operators. *Automatica*, 38 (8):1381–1387, 2002.

[25] J. Zhou and T. Hagiwara. Existence conditions and properties of the frequency response operators of continuous-time periodic systems. *SIAM Journal on Control and Optimization*, 40:1867, 2002.

[26] J. Zhou and T. Hagiwara. Finite-dimensional models in avaluating the H2 norm of continuous-time periodic systems. *Proc. of IFAC 2005 World Congress, Prague, Czech Republic*, 2005.

[27] J. Zhou, T. Hagiwara, and M. Araki. Stability analysis of continuous-time periodic systems via the harmonic analysis. *IEEE Transactions on Automatic Control*, 47(2): 292–298, 2002.

[28] J. Zhou, T. Hagiwara, and M. Araki. Spectral characteristics and eigenvalues computation of the harmonic state operators in continuous-time periodic systems. *Systems & Control Letters*, 53(2):141–155, 2004.