

# Analysis and Approximation of Viscoelastic and Thermoelastic Joint-Beam Systems

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Dissertation submitted to the Faculty of the  
Virginia Polytechnic Institute and State University  
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy  
in  
Mathematics

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July 21, 2006  
Blacksburg, Virginia

Keywords: Viscoelastic, Thermoelastic, Boltzmann Damping, Beam Networks, Semigroup  
Theory

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(ABSTRACT)

Rigidizable/Inflatable space structures have been the focus of renewed interest in recent years due to efficient packaging for transport. In this work, we examine new mathematical systems used to model small-scale joint dynamics for inflatable space truss structures. We investigate the regularity and asymptotic behavior of systems resulting from various damping models, including Kelvin-Voigt, Boltzmann, and thermoelastic damping. Approximation schemes will also be introduced. Finally, we look at optimal control for the Kelvin-Voigt model using a linear feedback regulator.

This work was supported in part by DARPA/SPO, NASA LaRC and the National Institute of Aerospace under grant VT-03-1, 2535.

# Dedication

To the memory of my father, the greatest man I've ever known. His Christ-like spirit is still cherished in the hearts of his family.

# Acknowledgments

First and foremost, I would like to thank my advisors, Gene Cliff and Terry Herdman, for their patience, guidance, and hard work on my behalf. You have been fun to work with, and I have benefited greatly from your experience. Thanks! I would also like to thank my committee for their patience and advice. A special thanks is in order for Zhuangyi Liu and Ruben Spies. Their work and ideas provided a strong foundation for this dissertation. Thank you both for many fruitful conversations!

My mother and sister have been a constant source of support and encouragement throughout my entire life. Thank you both for believing (albeit wrongly) that I am capable of doing anything I put my mind to. My wife, whom I love, has always been patient with me even when things went badly and I wasn't the easiest person to live with. Without you, I would never have completed this project. Thank you. Last, but certainly not least, I would like to thank Rob Harger for first opening my eyes to the beauty and power of mathematics. If it hadn't been for you, right now I would probably own a business, make five million a year, and be much less happy. Thanks.

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# Chapter 1

## Background

### 1.1 Introduction and Overview

In this paper, we analyze and approximate the dynamic behavior of joint-beam systems with various inherent damping mechanisms. The importance of the joint-beam system lies in the need for a better understanding of the dynamics of large inflatable space structures. Inflatable-rigidizable truss structures composed of advanced materials are of particular interest due to the advantages they offer in packaging and storage (see [12]). These advanced materials require more precise mathematical models in order to capture essential response characteristics.

The joint structure which we use in this paper is taken from [2] and consists of two rigid legs rotating about a joint mass. It allows us to include both the translational and rotational dynamics of the joint. The beam networks considered in [13] allow kinetic energy due to translational motion to be stored in the joint, but rotational energy is not included.

We begin, in Chapter 2, by examining the transversal motions of a single beam with a tip mass and memory damping. The dynamics of the tip mass provide a gentle introduction to the more complex joint structure considered in later chapters. Chapters 3 through 5 deal with the joint-beam system. In Chapter 3, we look at the case of Kelvin-Voigt damping which was considered in [2]. In Chapter 4, we apply the memory damping considered in Chapter 2 to both beams of the joint-beam system. Chapter 5 sees the introduction of thermal effects into the system, including solar radiation, conduction, and internal friction. The triangular joint-beam structure is then presented and analyzed in Chapter 6. The arguments and techniques used to establish well-posedness and exponential stability of the systems described in Chapters 2 through 6 are based upon the techniques used in [15] for

similar systems.

Approximation schemes for the various systems are produced in Chapters 7 and 8. Chapter 7 deals with the single beam from Chapter 2, while Chapter 8 gives approximation schemes for the Kelvin-Voigt and thermoelastic joint-beam systems. Approximation of the triangular structure is also briefly mentioned in the closing of Chapter 8. In Chapter 9, we look at the problem of optimal control for the joint-beam system with Kelvin-Voigt damping. We present a simple numerical example involving a single actuator placed in the joint.

## 1.2 Four Theorems

Throughout this work, the following four theorems will be repeatedly used to establish well-posedness and exponential stability for the various systems considered:

**Theorem 1.2.1** *Let  $\mathcal{A}$  be a linear operator with dense domain  $D(\mathcal{A})$  in a Hilbert space  $\mathcal{H}$ . If  $\mathcal{A}$  is dissipative and  $0 \in \rho(\mathcal{A})$ , then  $\mathcal{A}$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions on  $\mathcal{H}$ .*

**Theorem 1.2.2** *Let  $S(t)$  be a  $C_0$ -semigroup of contractions generated by  $\mathcal{A}$  on a Hilbert space.  $S(t)$  is exponentially stable if and only if*

$$\mathbf{i}\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset \tag{1.2.1}$$

and

$$\overline{\lim}_{|\beta| \rightarrow \infty} \|(i\beta - \mathcal{A})^{-1}\| < \infty \quad \beta \in \mathbb{R}. \tag{1.2.2}$$

**Theorem 1.2.3** *For any function  $u \in H^1(a, b)$ , where  $(a, b)$  is a bounded real interval, we have:*

$$|u(s)| \leq C \|u\|_{H^1(a,b)}^{\frac{1}{2}} \|u\|_{L^2(a,b)}^{\frac{1}{2}}$$

where  $C$  is a positive constant independent of  $u$ .

**Theorem 1.2.4** *For any function  $u \in H^2(\mathbb{R})$ , we have*

$$\|u'\|_{L^2} \leq C \|u''\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}}$$

where  $C$  is a positive constant independent of  $u$ .

Theorem 1.2.1 is a corollary to the Lumer-Phillips Theorem and can be found in [15]. Theorem 1.2.2 is the primary tool we will use to prove exponential stability and is found in [22]. Theorem 1.2.3 is a special case of Theorem 1.4.4 from [15]. It will be used to provide boundary estimates in the proofs for exponential stability. Theorem 1.2.4 is a special case of the Gagliardo-Nirenberg interpolation inequality found in [17] and will be used in the proofs for exponential stability.

# Chapter 2

## Single Beam with a Tip Mass

Before we analyze the joint-beam system, we shall begin by looking at transversal motions of a single, cantilevered beam with a tip mass attached to the free end. The tip mass construction contributes to the system dynamics in a sense roughly similar to that of the joint-leg construction considered in later chapters. We assume a stress-strain relationship of the form  $\sigma(t) = a\epsilon(t) + \int_0^\infty \dot{b}(\zeta)\epsilon(t-\zeta)d\zeta$ , where  $\sigma$  represents stress and  $\epsilon$  denotes strain. This type of relationship is known as viscoelastic memory damping, or Boltzmann-type damping. The presentation in this chapter follows closely that given in [15] for a clamped-clamped beam. A description of the axial motions of a shaft with tip mass and memory damping can be found in [16].

### 2.1 Constitutive Equations

The equation of transversal motion for a beam with tip mass and Boltzmann-type memory damping is the following:

$$\rho A \frac{\partial^2 w(t, s)}{\partial t^2} = -\frac{\partial^2}{\partial s^2} \left[ EI \frac{\partial^2 w(t, s)}{\partial s^2} + \int_0^\infty \dot{g}(z) \frac{\partial^2 w(t-z, s)}{\partial s^2} dz \right], \quad (2.1.1)$$

$$\tilde{\mathbf{M}} \frac{d^2}{dt^2} \begin{bmatrix} y(t) \\ \theta(t) \end{bmatrix} = \tilde{\mathbf{C}} \begin{bmatrix} \hat{N}(t) \\ \hat{M}(t) \end{bmatrix} \quad (2.1.2)$$

for time  $t > 0$  and spatial variable  $s \in [0, L]$ , where

$$\tilde{\mathbf{M}} = \begin{bmatrix} m & mh \\ 0 & I_M \end{bmatrix}, \quad \tilde{\mathbf{C}} = \begin{bmatrix} 1 & 0 \\ -h & -1 \end{bmatrix}. \quad (2.1.3)$$

The notations used in the above system are listed below:

- $w$  - transversal displacement of the beam
- $y$  - transversal displacement of the beam tip
- $\theta$  - rotation angle of the tip mass
- $\rho, A, L, E$  - mass density, cross section area, length, Young's modulus of the beam
- $m$  - mass of the tip
- $I_M$  - moment of inertia of the tip mass about its center of mass
- $h$  - horizontal distance between mass-beam connection and the center of mass for the tip
- $\hat{N}(t) = EI \frac{\partial^3 w}{\partial s^3}(t, L) + \int_0^\infty \dot{g}(z) \frac{\partial^3 w}{\partial s^3}(t - z, L) dz$  - shear force of the beam at the end  $s = L$
- $\hat{M}(t) = EI \frac{\partial^2 w}{\partial s^2}(t, L) + \int_0^\infty \dot{g}(z) \frac{\partial^2 w}{\partial s^2}(t - z, L) dz$  - bending moment of the beam at the end  $s = L$

For later calculations, it is convenient to write (2.1.2) as

$$\mathbf{M} \frac{d^2}{dt^2} \begin{bmatrix} y(t) \\ \theta(t) \end{bmatrix} = \mathbf{C} \begin{bmatrix} \hat{N}(t) \\ \hat{M}(t) \end{bmatrix} \quad (2.1.4)$$

where

$$\mathbf{M} = \begin{bmatrix} m & mh \\ mh & I_M + mh^2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (2.1.5)$$

It is assumed that the kernel function  $g$  satisfies the following conditions:

$$(g1) \quad g \in C^2(0, +\infty) \cap C[0, +\infty), \dot{g} \in L^1(0, +\infty);$$

$$(g2) \quad g(s) > 0, \dot{g}(s) < 0, \ddot{g}(s) > 0 \text{ on } (0, +\infty);$$

$$(g3) \quad \alpha = EI + \int_0^\infty \dot{g}(z)dz > 0;$$

$$(g4) \quad \ddot{g}(s) + \delta \dot{g}(s) \geq 0 \text{ on } (0, +\infty) \text{ for some constant } \delta > 0, \text{ and there exist positive constants } s_1, K \text{ such that for } s \geq s_1, \ddot{g}(s) \leq K|\dot{g}(s)|.$$

In the above, condition (g3) gives assurance that even if the beam has been held steady in a deformed position for a long period of time, a constraining force is still needed to maintain the deformation for the future. Condition (g2) tells us that the recent strain history is weighted more than the earlier strain history. Condition (g4) will be used as a technical aid for the analysis proofs later in this chapter.

The beam is clamped at the end  $s = 0$ . Thus the boundary condition at that end is

$$w(t, 0) = \frac{\partial w}{\partial s}(t, 0) = 0. \quad (2.1.6)$$

At the other end of the beam, we have the geometric compatibility conditions

$$\begin{cases} y(t) = w(t, L) \\ \theta(t) = w_s(t, L). \end{cases} \quad (2.1.7)$$

Now that a description of the system has been given, we need to answer questions regarding well-posedness. Does a unique solution exist? Is it continuously dependent upon initial conditions? To answer these questions, we will set up our system in a state space framework and utilize semigroup theory.

## 2.2 Semigroup Setting

Let

$$\mathcal{H} = H_r^2(0, L) \times L^2(0, L) \times \Sigma_2 \times \mathbb{R}^2 \quad (2.2.1)$$

where

$$H_r^j(0, L) = \{f \in H^j(0, L) : f(0) = f'(0) = \dots = f^{(j-1)}(0) = 0\}, \quad (2.2.2)$$

and

$$\Sigma_2 = L_g^2(0, +\infty; H_r^2(0, L)). \quad (2.2.3)$$

In the above,  $L_g^2(0, +\infty; H_r^2(0, L))$  is the Hilbert space of all  $H_r^2(0, L)$ -valued, square

integrable functions defined on the measure space  $((0, +\infty), |\dot{g}|ds)$ . The prime ( $'$ ) character shall be used to denote the spatial derivative.

We define the inner product in these spaces by

$$\begin{aligned} \langle w, j \rangle_{\mathcal{H}_r^2} &= \alpha \langle w'', j'' \rangle \\ \langle \eta, k \rangle_{L^2} &= \rho A \langle \eta, k \rangle \\ \langle r, b \rangle_{\Sigma_2} &= \int_0^\infty |\dot{g}(\zeta)| \langle r(\zeta)'', b(\zeta)'' \rangle d\zeta, \\ \langle (y, \theta), (x, \omega) \rangle_{\mathbb{R}^2} &= (y, \theta) \mathbf{M} (x, \omega)^T, \end{aligned} \quad (2.2.4)$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual  $L^2$  inner product of functions on  $[0, L]$ , and  $r(\zeta)' \equiv D_s(r(\zeta))$ . From (2.1.5), it is obvious that  $\mathbf{M}$  is strictly positive definite. Thus the inner product on  $\mathbb{R}^2$  is well defined. Note that with the above notation, the induced norm on  $\Sigma_2$  is given by

$$\|r\|_{\Sigma_2}^2 = \int_0^\infty |\dot{g}(\zeta)| \|r(\zeta)''\|^2 d\zeta. \quad (2.2.5)$$

Now, we define the variables

$$\eta = \frac{\partial w}{\partial t}, \quad q = \frac{dy}{dt}, \quad \omega = \frac{d\theta}{dt},$$

$$r(t; \zeta, s) = w(t, s) - w(t - \zeta, s),$$

and

$$z = (w, \eta, r, a)^T$$

where  $a(t) = (q(t), \omega(t))^T \in \mathbb{R}^2$ . The Hilbert space  $\mathcal{H}$  is equipped with the norm induced from the inner products in (2.2.4), i.e.,

$$\|z\|_{\mathcal{H}}^2 = \alpha \|w''\|^2 + \rho A \|\eta\|^2 + \|r\|_{\Sigma_2}^2 + a \mathbf{M} a^T \quad (2.2.6)$$

where  $\|\cdot\|$  denotes the usual  $L^2$  norm.

For the new set of state variables, the compatibility conditions are written

$$\begin{bmatrix} \eta(L) \\ \eta'(L) \end{bmatrix} = \begin{bmatrix} q \\ \omega \end{bmatrix}. \quad (2.2.7)$$

The joint-beam system then can be rewritten as a first order evolution equation

$$\frac{dz}{dt} = \mathcal{A}z \quad (2.2.8)$$

on the state space  $\mathcal{H}$  with

$$\mathcal{A}z = \begin{bmatrix} \eta \\ -\frac{1}{\rho A}(\alpha w'' - \int_0^\infty \dot{g}(\zeta)r(\zeta)''d\zeta)'' \\ \eta - D_\zeta r \\ \mathbf{M}^{-1}\mathbf{C}(\hat{N}, \hat{M})^T \end{bmatrix}, \quad (2.2.9)$$

and

$$\mathcal{D}(\mathcal{A}) = \left\{ z \in \mathcal{H} : \begin{array}{l} w, \eta \in H_r^2, \alpha w'' - \int_0^\infty \dot{g}(\zeta)r(\zeta)''d\zeta \in H^2, \\ D_\zeta r \in \Sigma_2, \quad r(0) = 0, \\ \text{compatibility condition (2.2.7)} \end{array} \right\}. \quad (2.2.10)$$

**Theorem 2.2.1** *The operator  $\mathcal{A}$  generates a  $C_0$  semigroup,  $S(t)$ , of contractions on  $\mathcal{H}$ .*

Proof: The proof we will present is given in [15] for clamped boundary conditions and is repeated here with modifications for the tip mass conditions. It is based on Theorem 1.2.1.

By a straightforward calculation,

$$\operatorname{Re}\langle \mathcal{A}z, z \rangle_{\mathcal{H}} = -\langle D_\zeta r, r \rangle_{\Sigma_2} \leq 0. \quad (2.2.11)$$

The last inequality in the above follows from Lemma 3.3.2 of [15], wherein it is seen that

$$\langle D_\zeta r, r \rangle_{\Sigma_2} = \frac{1}{2} \int_0^\infty \ddot{g}(\zeta) \|r(\zeta)''\|^2 d\zeta. \quad (2.2.12)$$

Hence,  $\mathcal{A}$  is dissipative. The major work is to show  $0 \in \rho(\mathcal{A})$ . Let

$$\tilde{z} = (\tilde{w}, \tilde{\eta}, \tilde{r}, \tilde{q}, \tilde{\omega}) \in \mathcal{H}.$$

Consider the equation

$$\mathcal{A}z = \tilde{z}, \quad (2.2.13)$$

i.e.,

$$\eta = \tilde{w} \in H_r^2, \quad (2.2.14)$$

$$-(\alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta)'' = \rho A \tilde{\eta} \in L^2, \quad (2.2.15)$$

$$\eta - D_\zeta r = \tilde{r} \in \Sigma_2, \quad (2.2.16)$$

$$\mathbf{M}^{-1} \mathbf{C}(\hat{N}, \hat{M})^T = (\tilde{q}, \tilde{\omega})^T \in \mathbb{R}^2. \quad (2.2.17)$$

The solution to (2.2.16), which also satisfies the condition  $r(0) = 0$ , is given by

$$r(\zeta) = \zeta \eta - \int_0^\zeta \tilde{r}(\tau) d\tau. \quad (2.2.18)$$

From this, we have  $D_\zeta r \in \Sigma_2$ . We must now show  $r \in \Sigma_2$ . Let  $T > \epsilon > 0$  be given. By the kernel function property (g4),

$$\begin{aligned} & \int_\epsilon^T |\dot{g}(\zeta)| \|r(\zeta)''\|^2 d\zeta \\ & \leq \frac{1}{\delta} \int_\epsilon^T \ddot{g}(\zeta) \|r(\zeta)''\|^2 d\zeta \\ & = \frac{1}{\delta} \dot{g}(T) \|r(T)''\|^2 - \frac{1}{\delta} \dot{g}(\epsilon) \|r(\epsilon)''\|^2 - \frac{2}{\delta} \int_\epsilon^T \dot{g}(\zeta) \langle r(\zeta)'', (D_\zeta r(\zeta))'' \rangle d\zeta \\ & \leq -\frac{1}{\delta} \dot{g}(\epsilon) \|r(\epsilon)''\|^2 + \frac{1}{2} \int_\epsilon^T |\dot{g}(\zeta)| \|r(\zeta)''\|^2 d\zeta \\ & \quad + \frac{2}{\delta^2} \int_\epsilon^T |\dot{g}(\zeta)| \|D_\zeta r(\zeta)''\|^2 d\zeta. \end{aligned} \quad (2.2.19)$$

This gives

$$\int_\epsilon^T |\dot{g}(\zeta)| \|r(\zeta)''\|^2 d\zeta \leq -\frac{2}{\delta} \dot{g}(\epsilon) \|r(\epsilon)''\|^2 + \frac{4}{\delta^2} \int_\epsilon^T |\dot{g}(\zeta)| \|(D_\zeta r(\zeta))''\|^2 d\zeta. \quad (2.2.20)$$

Now, the proof of Lemma 3.3.1 in [15] shows that as  $\epsilon \rightarrow 0$ ,

$$-\frac{1}{\delta} \dot{g}(\epsilon) \|r(\epsilon)''\|^2 \rightarrow 0. \quad (2.2.21)$$

Therefore, if we let  $T \rightarrow \infty$ , and  $\epsilon \rightarrow 0$ , then we see that

$$\begin{aligned} r \in \Sigma_2, \quad \|r\|_{\Sigma_2}^2 &\leq \frac{4}{\delta^2} \int_0^\infty |\dot{g}(\zeta)| \| (D_\zeta r(\zeta))'' \|^2 d\zeta \\ &\leq K (\|\tilde{r}\|_{\Sigma_2}^2 + \|\tilde{w}\|^2) \end{aligned} \quad (2.2.22)$$

for some  $K > 0$ .

A solution to equation (2.2.15), satisfying the conditions  $w(0) = w'(0) = 0$ , has the form

$$\begin{aligned} w(s) &= -\frac{\hat{N}}{6\alpha}(3Ls^2 - s^3) + \frac{\hat{M}}{2\alpha}s^2 + \frac{1}{\alpha} \int_0^\infty \dot{g}(\zeta)r(\zeta, s)d\zeta \\ &\quad - \frac{1}{\alpha} \int_0^s \int_0^{\tau_4} \int_{\tau_3}^L \int_{\tau_2}^L \rho A \tilde{\eta}(\tau_1) d\tau_1 d\tau_2 d\tau_3 d\tau_4, \end{aligned} \quad (2.2.23)$$

with

$$\begin{aligned} w'(s) &= -\frac{\hat{N}}{2\alpha}(2Ls - s^2) + \frac{\hat{M}}{\alpha}s + \frac{1}{\alpha} \int_0^\infty \dot{g}(\zeta) \frac{d}{ds} r(\zeta, s) d\zeta \\ &\quad - \frac{1}{\alpha} \int_0^s \int_{\tau_3}^L \int_{\tau_2}^L \rho A \tilde{\eta}(\tau_1) d\tau_1 d\tau_2 d\tau_3. \end{aligned} \quad (2.2.24)$$

It is clear that

$$w \in H_r^2, \quad \alpha w'' - \int_0^\infty \dot{g}(\zeta)r(\zeta)'' d\zeta \in H^2. \quad (2.2.25)$$

Note that the solution  $w$  is unique if  $\hat{N}, \hat{M}$  are unique. But, from (2.2.17), we have

$$\begin{bmatrix} \hat{N} \\ \hat{M} \end{bmatrix} = \mathbf{C}^{-1} \mathbf{M} \begin{bmatrix} \tilde{q} \\ \tilde{\omega} \end{bmatrix} = \mathbf{C} \mathbf{M} \begin{bmatrix} \tilde{q} \\ \tilde{\omega} \end{bmatrix}. \quad (2.2.26)$$

This implies that  $(\hat{N}, \hat{M})$  can be solved for uniquely in terms of  $(\tilde{q}, \tilde{\omega})$ . Therefore,  $w$  is uniquely determined.

Finally, from our compatibility condition and (2.2.14), we have the equality

$$\begin{bmatrix} q \\ \omega \end{bmatrix} = \begin{bmatrix} \tilde{w}(L) \\ \tilde{w}'(L) \end{bmatrix}. \quad (2.2.27)$$

In summary, we have thus far obtained a unique solution  $z \in \mathcal{D}(\mathcal{A})$  of equation (2.2.13).

By the expression of  $z$  in terms of  $\tilde{z}$ , it is clear that

$$\begin{cases} \|w''\| \leq K_1(\|\tilde{w}''\| + \|\tilde{\eta}\| + \|\tilde{r}\|_{\Sigma_2} + \|(\tilde{q}, \tilde{\omega})\|_{\mathbb{R}^2}) \\ \|(q, \omega)\|_{\mathbb{R}^2} \leq K_2\|\tilde{w}''\| \\ \|\eta\| \leq K_3\|\tilde{w}''\|, \\ \|r\|_{\Sigma_2} \leq K_4(\|\tilde{w}''\| + \|\tilde{r}\|_{\Sigma_2}), \end{cases}$$

for some constants  $K_1, K_2, K_3, K_4 > 0$ , which leads to

$$\|z\|_{\mathcal{H}} \leq K\|\mathcal{A}z\|_{\mathcal{H}}$$

for some constant  $K$  independent of  $z$ . Therefore,  $0 \in \rho(\mathcal{A})$ ,  $\mathcal{A}^{-1}$  is bounded and  $\mathcal{A}$  is closed. It follows that the range  $R(\lambda - \mathcal{A}) = \mathcal{H}$  for some  $\lambda > 0$ . By Theorem 4.6 in [18],  $\mathcal{D}(\mathcal{A})$  is dense in  $\mathcal{H}$ . The conclusion of this theorem now follows from Theorem 1.2.1.

The semigroup  $S(t)$  generated by  $\mathcal{A}$  is crucial to solving the abstract Cauchy problem (2.2.8). Indeed, if  $z(0) = z_0 \in D(\mathcal{A})$  is the initial condition, then  $S(t)z_0$  defines a unique classical solution to (2.2.8). Note that we are requiring certain smoothness properties of the initial condition in order to guarantee  $S(t)z_0$  is a classical solution. In general, if  $z(0) = z_0 \notin D(\mathcal{A})$ , then the Cauchy problem has no classical solution.  $S(t)z_0$  is called a generalized solution in this case.

## 2.3 Exponential Stability

In the previous section, we established well-posedness for the single beam system with tip-mass provided the initial state lies in the domain of the generator  $\mathcal{A}$ . We will now look at the asymptotic behavior of solutions of the abstract Cauchy problem.

**Theorem 2.3.1**  *$S(t)$  is exponentially stable.*

Proof: The proof of this theorem is similar to that given in [15] for the case of clamped boundary conditions. However, several modifications have to be made to account for the tip mass conditions.

We shall verify the frequency domain conditions (1.2.1) and (1.2.2) from Theorem 1.2.2 via a contradiction argument. If the first condition is false, then there exists a sequence

$$z_n = (w_n, \eta_n, r_n, q_n, \omega_n) \in \mathcal{D}(\mathcal{A})$$

with  $\|z_n\|_{\mathcal{H}} = 1$ , and a sequence  $\beta_n \in \mathbb{R}$  with  $\beta_n \rightarrow b$ ,  $|b| \geq \|\mathcal{A}^{-1}\|^{-1}$ , such that

$$\lim_{n \rightarrow \infty} \|(\mathbf{i}\beta_n I - \mathcal{A})z_n\|_{\mathcal{H}} = 0. \quad (2.3.1)$$

For the simplicity of notation, we omit the subscript  $n$  in the rest of our proof. From (2.3.1) and (2.2.11), we obtain

$$\lim_{n \rightarrow \infty} \operatorname{Re} \langle (\mathbf{i}\beta I - \mathcal{A})z, z \rangle_{\mathcal{H}} = \lim_{n \rightarrow \infty} \frac{1}{2} \int_0^\infty \ddot{g}(\zeta) \|r(\zeta)''\|^2 d\zeta = 0. \quad (2.3.2)$$

With condition (g4), this implies

$$\|r\|_{\Sigma_2} \rightarrow 0. \quad (2.3.3)$$

Our goal is to get a contradiction by showing  $\|z\|_{\mathcal{H}} \rightarrow 0$ . The componentwise version of (2.3.1) is

$$\mathbf{i}\beta w'' - \eta'' \rightarrow 0 \text{ in } L^2(0, L), \quad (2.3.4)$$

$$\mathbf{i}\beta \eta + \frac{1}{\rho A} (\alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta)'' \rightarrow 0 \text{ in } L^2(0, L), \quad (2.3.5)$$

$$\mathbf{i}\beta r - (\eta - D_\zeta r) \rightarrow 0 \text{ in } \Sigma_2, \quad (2.3.6)$$

$$\mathbf{i}\beta \begin{bmatrix} q \\ \omega \end{bmatrix} - \mathbf{M}^{-1} \mathbf{C}(\hat{N}, \hat{M})^T \rightarrow 0 \text{ in } \mathbb{R}^2. \quad (2.3.7)$$

Using conditions (g1) – (g4), it can be deduced that for each kernel function  $g(\zeta)$ , both  $\zeta^2 \dot{g}(\zeta)$  and  $\zeta^2 \ddot{g}(\zeta)$  belong to  $L^1(0, +\infty)$ . It is also easily seen that  $\frac{\zeta \eta}{\beta} \in \Sigma_2$ . Divide (2.3.6) by  $\beta$  and take its inner product with  $\frac{\zeta \eta}{\beta}$  in  $\Sigma_2$ . Since  $r \rightarrow 0$  in  $\Sigma_2$ , we get

$$\left\| \frac{\eta''}{\beta} \right\|^2 \int_0^\infty \zeta |\dot{g}(\zeta)| d\zeta - \frac{1}{\beta} \int_0^\infty \zeta \dot{g}(\zeta) \langle (D_\zeta r(\zeta))'', \frac{\eta''}{\beta} \rangle d\zeta \rightarrow 0. \quad (2.3.8)$$

The goal is now to show that the second term of (2.3.8) converges to zero. To do this, we can first use the Cauchy-Schwartz inequality to get

$$\begin{aligned}
& \left| \zeta \dot{g}(\zeta) \left\langle r(\zeta)'', \frac{\eta''}{\beta} \right\rangle \right| \\
& \leq \zeta |\dot{g}(\zeta)| \|r(\zeta)''\| \left\| \frac{\eta''}{\beta} \right\| \\
& \leq \frac{\zeta |\dot{g}(\zeta)|}{2} \left( \|r(\zeta)''\|^2 + \left\| \frac{\eta''}{\beta} \right\|^2 \right) \rightarrow 0
\end{aligned} \tag{2.3.9}$$

as  $\zeta \rightarrow 0$ . The convergence in (2.3.9) is assured by Lemma 3.3.1 of [2]. We can use the Cauchy-Schwartz inequality again to get

$$\begin{aligned}
& \left| \zeta \dot{g}(\zeta) \left\langle r(\zeta)'', \frac{\eta''}{\beta} \right\rangle \right| \\
& \leq \zeta |\dot{g}(\zeta)| \|r(\zeta)''\| \left\| \frac{\eta''}{\beta} \right\| \\
& \leq \frac{1}{2} |\dot{g}(\zeta)| \|r(\zeta)''\|^2 + \frac{\zeta^2 |\dot{g}(\zeta)|}{2} \left\| \frac{\eta''}{\beta} \right\|^2 \rightarrow 0
\end{aligned} \tag{2.3.10}$$

as  $\zeta \rightarrow \infty$ . The convergence in (2.3.10) is given by Lemma 3.3.2 of [2]. Using (2.3.9) and (2.3.10), we can integrate by parts to establish the following convergence result as  $N \rightarrow \infty$ :

$$\begin{aligned}
& \left| - \int_0^\infty \zeta \dot{g}(\zeta) \langle (D_\zeta r(\zeta))'', \frac{\eta''}{\beta} \rangle d\zeta \right| \\
& = \left| \int_0^\infty \zeta \ddot{g}(\zeta) \left\langle r(\zeta)'', \frac{\eta''}{\beta} \right\rangle d\zeta + \int_0^\infty \dot{g}(\zeta) \left\langle r(\zeta)'', \frac{\eta''}{\beta} \right\rangle d\zeta \right| \\
& \leq \int_0^\infty \sqrt{\ddot{g}(\zeta)} \|r(\zeta)''\| \sqrt{\ddot{g}(\zeta)} \zeta \left\| \frac{\eta''}{\beta} \right\| d\zeta + \|r\|_{\Sigma_2} \left( \int_0^\infty |\dot{g}(\zeta)| d\zeta \right)^{\frac{1}{2}} \left\| \frac{\eta''}{\beta} \right\| \\
& \leq \left( \int_0^\infty \ddot{g}(\zeta) \|r(\zeta)''\|^2 d\zeta \right)^{\frac{1}{2}} \left( \int_0^\infty \ddot{g}(\zeta) \zeta^2 d\zeta \right)^{\frac{1}{2}} \left\| \frac{\eta''}{\beta} \right\| \\
& \quad + \|r\|_{\Sigma_2} \left( \int_0^\infty |\dot{g}(\zeta)| d\zeta \right)^{\frac{1}{2}} \left\| \frac{\eta''}{\beta} \right\| \rightarrow 0
\end{aligned} \tag{2.3.11}$$

because of (2.3.2) and the fact that  $r \rightarrow 0$  in  $\Sigma_2$ . Therefore we have that the second term in (2.3.8) converges to zero. This implies that the first term must also converge to zero. Thus,

we have

$$\|\eta''\| \rightarrow 0. \quad (2.3.12)$$

Since  $\eta$  is an element of  $H_r^2$ , this implies

$$\|\eta'\|, \|\eta\| \rightarrow 0. \quad (2.3.13)$$

Plugging (2.3.12) into (2.3.4), we get

$$\|w''\| \rightarrow 0 \quad (2.3.14)$$

which results in

$$\|w'\|, \|w\| \rightarrow 0. \quad (2.3.15)$$

Regarding the boundary terms, we can use Theorem 1.2.3 and (2.3.12)-(2.3.13) to show

$$|\eta(L)|, |\eta'(L)| \rightarrow 0. \quad (2.3.16)$$

Plugging equation (2.3.16) into equation (2.2.7) yields

$$(q, \omega) \rightarrow 0 \quad (2.3.17)$$

in  $\mathbb{R}^2$ . From equations (2.3.13),(2.3.14),(2.3.16), and (2.3.17), we have our contradiction that  $\|z\|_{\mathcal{H}} \rightarrow 0$ . Therefore, condition (1.2.1) holds.

Now, suppose condition (1.2.2) does not hold. This implies the existence of another sequence

$$z_n = (w_n, \eta_n, r_n, q_n, \omega_n) \in \mathcal{D}(\mathcal{A})$$

with  $\|z_n\|_{\mathcal{H}} = 1$ , and another sequence  $\beta_n \in \mathbb{R}$  with  $|\beta_n| \rightarrow \infty$ , such that

$$\lim_{n \rightarrow \infty} \|(\mathbf{i}\beta_n I - \mathcal{A})z_n\|_{\mathcal{H}} = 0. \quad (2.3.18)$$

Our goal will be to once again demonstrate a contradiction by showing that  $\|z\|_{\mathcal{H}} \rightarrow 0$ . If we omit the subscript  $n$  in future calculations, (2.3.18) gives the same component equations (i.e., (2.3.4) - (2.3.7) ) as before with the notable difference that  $\beta$  is no longer bounded. Notice also that we can still use (2.3.2) to get

$$\|r\|_{\Sigma_2} \rightarrow 0. \quad (2.3.19)$$

With (2.3.19), we can repeat the same procedure we used above to get

$$\left\| \frac{\eta''}{\beta} \right\| \rightarrow 0. \quad (2.3.20)$$

From this, (2.3.4) gives

$$\|w''\| \rightarrow 0. \quad (2.3.21)$$

At this point, however, differences in proving conditions (1.2.1) and (1.2.2) begin to appear. Earlier, we were able to multiply (2.3.20) by  $\beta$  to get  $\eta \rightarrow 0$ . Now, since  $\beta$  is unbounded, we cannot do so. Therefore, let us proceed in a different manner.

The key to showing that  $\|z\|_{\mathcal{H}} \rightarrow 0$  lies in the boundary terms  $\hat{N}$ , and  $\hat{M}$ . By repeated use of Theorem 1.2.3 and the Gagliardo-Nirenberg Inequality, we have the following:

$$\begin{aligned} & \left| \left( \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right)' (L) \right| \\ & \leq C_1 \left\| \left( \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right)' \right\|_{H^1}^{\frac{1}{2}} \left\| \left( \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right)' \right\|_{H^1}^{\frac{1}{2}} \\ & \leq C_2 \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right\|_{H^2}^{\frac{1}{4}} \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right\|_{H^2}^{\frac{1}{4}} \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right\|_{H^2}^{\frac{1}{2}} \\ & = C_2 \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right\|_{H^2}^{\frac{1}{4}} \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right\|_{H^2}^{\frac{3}{4}}. \end{aligned} \quad (2.3.22)$$

Regarding the last term in (2.3.22), we can again use the Gagliardo-Nirenberg Inequality to get

$$\begin{aligned} & \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right\|_{H^2} \\ & \leq \left( \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right\|^2 + C_3 \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right\| \left\| \left( \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right)'' \right\| \right. \\ & \quad \left. + \left\| \left( \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right)'' \right\|^2 \right)^{\frac{1}{2}} \\ & \leq C_4 \left( \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right\| + \left\| \left( \alpha w'' - \int_0^\infty \dot{g}(\zeta) r(\zeta)'' d\zeta \right)'' \right\| \right). \end{aligned} \quad (2.3.23)$$

Now, from (2.3.5) we have that

$$\left\| \frac{(\alpha w'' - \int_0^\infty \dot{g}(\zeta)r(\zeta)''d\zeta)''}{\beta} \right\| \leq C_5. \quad (2.3.24)$$

Therefore, since  $w'' \rightarrow 0$  in  $L^2$  and  $r \rightarrow 0$  in  $\Sigma_2$ , we have that

$$\left\| \frac{\alpha w'' - \int_0^\infty \dot{g}(\zeta)r(\zeta)''d\zeta}{\beta} \right\|_{H^2} \leq C_6. \quad (2.3.25)$$

This implies that

$$\frac{|\hat{N}|}{\beta} \leq C_7 \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta)r(\zeta)''d\zeta \right\|^{\frac{1}{4}}. \quad (2.3.26)$$

Again, we use the convergence of  $w''$  and  $r$  to get

$$\frac{|\hat{N}|}{\beta} \rightarrow 0. \quad (2.3.27)$$

With regard to the other boundary term  $\hat{M}$ , we have

$$\begin{aligned} & \left| (\alpha w'' - \int_0^\infty \dot{g}(\zeta)r(\zeta)''d\zeta)(L) \right| \\ & \leq C_8 \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta)r(\zeta)''d\zeta \right\|^{\frac{1}{2}} \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta)r(\zeta)''d\zeta \right\|_{H_1}^{\frac{1}{2}} \\ & \leq C_8 \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta)r(\zeta)''d\zeta \right\|^{\frac{1}{2}} \left\| \alpha w'' - \int_0^\infty \dot{g}(\zeta)r(\zeta)''d\zeta \right\|_{H^2}^{\frac{1}{2}}. \end{aligned} \quad (2.3.28)$$

Using the same techniques as we did for  $\hat{N}$ , we see that

$$\frac{|\hat{M}|}{\beta} \rightarrow 0. \quad (2.3.29)$$

Plugging (2.3.27) and (2.3.29) into (2.3.7), we get

$$q, \omega \rightarrow 0. \quad (2.3.30)$$

Now, let us take the  $L^2$  inner product of both sides of (2.3.5) with  $\eta$  to get

$$i\beta\|\eta\|^2 + \frac{1}{\rho A} \left( \hat{N}q - \hat{M}\omega + \alpha \langle w'', \eta'' \rangle - \int_0^\infty \dot{g}(\zeta) \langle r(\zeta)'', \eta'' \rangle d\zeta \right) \rightarrow 0. \quad (2.3.31)$$

If we divide (2.3.31) by  $\beta$ , we see that our previous results give

$$\|\eta\|^2 \rightarrow 0. \quad (2.3.32)$$

This is the final result we need for the contradiction  $\|z\|_{\mathcal{H}} \rightarrow 0$ . Therefore, condition (1.2.2) holds and our semigroup is exponentially stable.

# Chapter 3

## Joint-Beam System (Kelvin-Voigt Damping)

We now introduce the joint-beam system, which consists of two flexible beams and a joint (see Figure 3.0.1). The joint configuration consists of a mass and two rigid legs, each of which is connected to the end of a beam. The opposite end of each beam is clamped to a fixed structure. The beam-leg connections are assumed to be "slope-preserving" in the sense that the cross-section of the beam at the connection is parallel to the cross section of the leg.

In this chapter, Kelvin-Voigt damping is assumed to be present in the beams, while no damping occurs in the joint. This system was examined in [2] wherein it was described by an abstract second order differential equation on a Hilbert space  $\mathcal{H}$ . The state at time  $t$  could then be written formally as  $(X(t), \dot{X}(t))$ . The current presentation will take a slightly different approach, though the idea is due to the authors of [2].

### 3.1 Constitutive Equations

The equations of motion for a coupled joint-beam system with Kelvin-Voigt type of damping have been derived in [2] as the following:

$$\rho_i A_i \frac{\partial^2 u_i(t, s_i)}{\partial t^2} = \frac{\partial}{\partial s_i} \left[ E_i A_i \frac{\partial u_i(t, s_i)}{\partial s_i} + \mu_i \frac{\partial^2 u_i(t, s_i)}{\partial s_i \partial t} \right], \quad (3.1.1)$$

$$\rho_i A_i \frac{\partial^2 w_i(t, s_i)}{\partial t^2} = -\frac{\partial^2}{\partial s_i^2} \left[ E_i I_i \frac{\partial^2 w_i(t, s_i)}{\partial s_i^2} + \gamma_i \frac{\partial^3 w_i(t, s_i)}{\partial s_i^2 \partial t} \right], \quad (3.1.2)$$

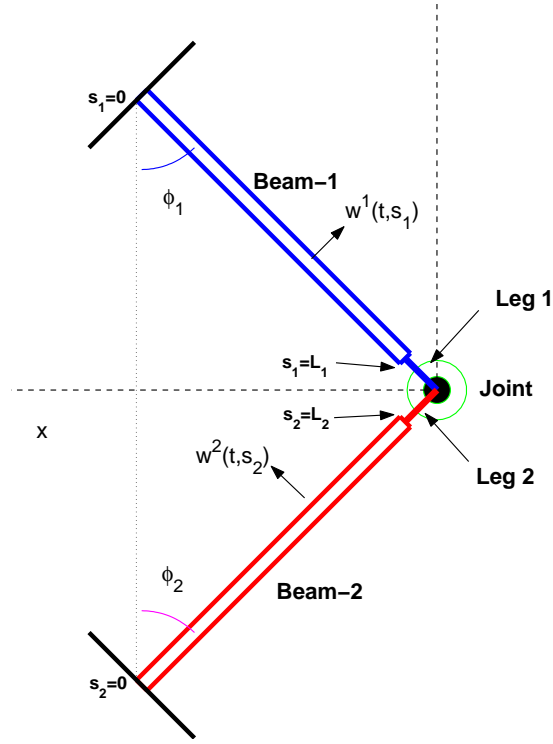


Figure 3.0.1: Joint-Beam System

$$\mathbf{M} \frac{d^2}{dt^2} \begin{bmatrix} x(t) \\ y(t) \\ \theta_1(t) \\ \theta_2(t) \end{bmatrix} = \mathbf{C} \begin{bmatrix} F_1(t) \\ N_1(t) \\ F_2(t) \\ N_2(t) \\ M_1(t) \\ M_2(t) \end{bmatrix} \quad (3.1.3)$$

for time  $t > 0$  and spatial variable  $s_i \in [0, L_i]$ , where

$$\mathbf{M} = \begin{bmatrix} m & 0 & -m_1 d_1 \cos \phi_1 & m_2 d_2 \cos \phi_2 \\ 0 & m & +m_1 d_1 \sin \phi_1 & m_2 d_2 \sin \phi_2 \\ -m_1 d_1 \cos \phi_1 & m_1 d_1 \sin \phi_1 & I_{1\ell} + m_1 d_1^2 & 0 \\ m_2 d_2 \cos \phi_2 & m_2 d_2 \sin \phi_2 & 0 & I_{2\ell} + m_2 d_2^2 \end{bmatrix}, \quad (3.1.4)$$

$$\mathbf{C} = \begin{bmatrix} \sin \phi_1 & -\cos \phi_1 & \sin \phi_2 & \cos \phi_2 & 0 & 0 \\ \cos \phi_1 & \sin \phi_1 & -\cos \phi_2 & \sin \phi_2 & 0 & 0 \\ 0 & \ell_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \ell_2 & 0 & 1 \end{bmatrix}. \quad (3.1.5)$$

The notations used in the above system are listed below:

- $u_i, w_i$  - longitudinal and transversal displacement of the beam  $i$
- $x, y$  - planar displacement of the joint
- $\theta_i$  - rotation angle of the leg  $i$
- $\rho_i, A_i, L_i, E_i, I_i$  - mass density, cross section area, length, Young's modulus, moment of inertia of the beam  $i$
- $\mu_i, \gamma_i$  - damping coefficients
- $m_i, d_i, \ell_i, I_{i\ell}$  - mass, center of mass, length, moment of inertia of leg  $i$
- $m_p$  - mass of the joint,  $m = m_1 + m_2 + m_p$
- $\phi_1$  - initial angle of leg 1 with positive  $y$  axis
- $\phi_2$  - initial angle of leg 2 with negative  $y$  axis
- $F_i(t) = E_i A_i \frac{\partial u_i}{\partial s_i}(t, L_i) + \mu_i \frac{\partial^2 u_i}{\partial s_i \partial t}(t, L_i)$  - extensional force of beam  $i$  at the end  $s_i = L_i$
- $N_i(t) = E_i I_i \frac{\partial^3 w_i}{\partial s_i^3}(t, L_i) + \gamma_i \frac{\partial^4 w_i}{\partial s_i^3 \partial t}(t, L_i)$  - shear force of beam  $i$  at the end  $s_i = L_i$
- $M_i(t) = E_i I_i \frac{\partial^2 w_i}{\partial s_i^2}(t, L_i) + \gamma_i \frac{\partial^3 w_i}{\partial s_i^2 \partial t}(t, L_i)$  - bending moment of beam  $i$  at the end  $s_i = L_i$ .

The beams are clamped at the end  $s_i = 0$ . Thus the boundary conditions are

$$u_i(t, 0) = w_i(t, 0) = \frac{\partial w_i}{\partial s_i}(t, 0) = 0. \quad (3.1.6)$$

At the other end of the beam, we have the geometric compatibility condition

$$\begin{cases} x(t) = -\sin \phi_1 u_1(t, L_1) - \cos \phi_1 w_1(t, L_1) - \ell_1 \cos \phi_1 w_1'(t, L_1) \\ y(t) = -\cos \phi_1 u_1(t, L_1) + \sin \phi_1 w_1(t, L_1) + \ell_1 \sin \phi_1 w_1'(t, L_1) \\ \theta_1(t) = -w_1'(t, L_1), \end{cases} \quad (3.1.7)$$

$$\begin{cases} x(t) = -\sin \phi_2 u_2(t, L_2) + \cos \phi_2 w_2(t, L_2) + \ell_2 \cos \phi_2 w_2'(t, L_2) \\ y(t) = \cos \phi_2 u_2(t, L_2) + \sin \phi_2 w_2(t, L_2) + \ell_2 \sin \phi_2 w_2'(t, L_2) \\ \theta_2(t) = -w_2'(t, L_2), \end{cases} \quad (3.1.8)$$

which is equivalent to

$$\begin{bmatrix} -u_1(t, L_1) \\ w_1(t, L_1) \\ -u_2(t, L_2) \\ w_2(t, L_2) \\ -w_1'(t, L_1) \\ -w_2'(t, L_2) \end{bmatrix} = \begin{bmatrix} x(t) \sin \phi_1 + y(t) \cos \phi_1 \\ -x(t) \cos \phi_1 + y(t) \sin \phi_1 + \ell_1 \theta_1(t) \\ x(t) \sin \phi_2 - y(t) \cos \phi_2 \\ x(t) \cos \phi_2 + y(t) \sin \phi_2 + \ell_2 \theta_2(t) \\ \theta_1(t) \\ \theta_2(t) \end{bmatrix} = \mathbf{C}^T \begin{bmatrix} x(t) \\ y(t) \\ \theta_1(t) \\ \theta_2(t) \end{bmatrix}. \quad (3.1.9)$$

We multiply (3.1.9) by  $\mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}\mathbf{C}$  from the left to arrive at the projection form of (3.1.9),

$$(\mathbf{I} - \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}\mathbf{C}) \begin{bmatrix} -u_1(L_1) \\ w_1(L_1) \\ -u_2(L_2) \\ w_2(L_2) \\ -w_1'(L_1) \\ -w_2'(L_2) \end{bmatrix} = \mathbf{0}. \quad (3.1.10)$$

## 3.2 Semigroup Setting

As in the case with the single beam, our goal will be to set up the joint-beam system as an abstract Cauchy problem on a given state space. We will then utilize semigroup theory to establish well-posedness.

Let

$$\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \mathbb{R}^4 \quad (3.2.1)$$

where

$$\mathcal{H}_i = H_r^1(0, L_i) \times H_r^2(0, L_i) \times [L^2(0, L_i)]^2, \quad i = 1, 2 \quad (3.2.2)$$

and

$$H_r^j(0, L_i) = \{f \in H^j(0, L_i) : f(0) = f'(0) = \dots = f^{(j-1)}(0) = 0\}. \quad (3.2.3)$$

We shall use prime to denote the spatial derivative for both  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . Considering the energy of the system, we define the inner product in these spaces by

$$\langle (u, w, v, \eta), (f, g, h, k) \rangle_{\mathcal{H}_i} = E_i A_i \langle u', f' \rangle + E_i I_i \langle w'', g'' \rangle + \rho_i A_i (\langle v, h \rangle + \langle \eta, k \rangle), \quad (3.2.4)$$

$$\langle (u, w, v, \eta), (f, g, h, k) \rangle_{\mathbb{R}^4} = (f, g, h, k) \mathbf{M} (u, w, v, \eta)^T, \quad (3.2.5)$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual  $L^2$  inner product. From (3.1.4), it can be verified that  $\mathbf{M}$  is strictly positive definite. Thus the inner product on  $\mathbb{R}^4$  is well defined.

We introduce the new variables

$$v_i = \frac{\partial u_i}{\partial t}, \quad \eta_i = \frac{\partial w_i}{\partial t}, \quad p = \frac{dx}{dt}, \quad q = \frac{dy}{dt}, \quad \omega_i = \frac{d\theta_i}{dt},$$

and

$$z = (u_1, w_1, v_1, \eta_1, u_2, w_2, v_2, \eta_2, p, q, \omega_1, \omega_2)^T.$$

The Hilbert space  $\mathcal{H}$  is equipped with the energy norm induced from these inner products in (3.2.4) - (3.2.5), i.e.,

$$\|z\|_{\mathcal{H}}^2 = 2E(t) = \sum_{i=1}^2 [E_i A_i \|u_i'\|^2 + E_i I_i \|w_i''\|^2 + \rho_i A_i (\|v_i\|^2 + \|\eta_i\|^2)] + a \mathbf{M} a^T \quad (3.2.6)$$

where  $a = (p, q, \omega_1, \omega_2)$  and  $\|\cdot\|$  denotes the usual  $L^2$  norm.

For the new set of state variables, the compatibility conditions become

$$\begin{bmatrix} -v_1(L_1) \\ \eta_1(L_1) \\ -v_2(L_2) \\ \eta_2(L_2) \\ -\eta_1'(L_1) \\ -\eta_2'(L_2) \end{bmatrix} = \begin{bmatrix} p \sin \phi_1 + q \cos \phi_1 \\ -p \cos \phi_1 + q \sin \phi_1 + l_1 \omega_1 \\ p \sin \phi_2 - q \cos \phi_2 \\ p \cos \phi_2 + q \sin \phi_2 + l_2 \omega_2 \\ \omega_1 \\ \omega_2 \end{bmatrix} = \mathbf{C}^T \begin{bmatrix} p \\ q \\ \omega_1 \\ \omega_2 \end{bmatrix} \quad (3.2.7)$$

with projection form

$$(\mathbf{I} - \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}\mathbf{C}) \begin{bmatrix} -v_1(L_1) \\ \eta_1(L_1) \\ -v_2(L_2) \\ \eta_2(L_2) \\ -\eta'_1(L_1) \\ -\eta'_2(L_2) \end{bmatrix} = \mathbf{0}. \quad (3.2.8)$$

We now identify our state space as

$$\mathcal{H}_r = \{z \in \mathcal{H} \mid \text{compatibility condition (3.1.10)}\}.$$

For simplicity, we will denote our state space  $\mathcal{H}_r$  by  $\mathcal{H}$  again. The joint-beam system then can be rewritten as a first order evolution equation

$$\frac{dz}{dt} = \mathcal{A}z \quad (3.2.9)$$

on the Hilbert space  $\mathcal{H}$  with

$$\mathcal{A}z = \begin{bmatrix} v_1 \\ \eta_1 \\ \frac{1}{\rho_1 A_1}(E_1 A_1 u'_1 + \mu_1 v'_1)' \\ -\frac{1}{\rho_1 A_1}(E_1 I_1 w''_1 + \gamma_1 \eta''_1)'' \\ v_2 \\ \eta_2 \\ \frac{1}{\rho_2 A_2}(E_2 A_2 u'_2 + \mu_2 v'_2)' \\ -\frac{1}{\rho_2 A_2}(E_2 I_2 w''_2 + \gamma_2 \eta''_2)'' \\ \mathbf{M}^{-1}\mathbf{C}(F_1, N_1, F_2, N_2, M_1, M_2)^T \end{bmatrix}, \quad (3.2.10)$$

and

$$\mathcal{D}(\mathcal{A}) = \left\{ z \in \mathcal{H} : \begin{array}{l} u_i, v_i \in H^1_r, \quad E_i A_i u'_i + \mu_i v'_i \in H^1, \quad w_i, \eta_i \in H^2_r \\ E_i I_i w''_i + \gamma_i \eta''_i \in H^2, \quad \text{compatibility condition (3.2.7)} \end{array} \right\}. \quad (3.2.11)$$

**Theorem 3.2.1** *The operator  $\mathcal{A}$  generates a  $C_0$  semigroup,  $S(t)$ , of contractions on  $\mathcal{H}$ .*

Proof: As in the establishment of well-posedness for the single beam, we will again base our

proof upon Theorem 1.2.1.

By a straightforward calculation,

$$\operatorname{Re}\langle \mathcal{A}z, z \rangle_{\mathcal{H}} = \operatorname{Re}\langle \frac{dz}{dt}, z \rangle_{\mathcal{H}} = \frac{dE(t)}{dt} = - \sum_{i=1}^2 (\mu_i \|v_i'\|^2 + \gamma_i \|\eta_i''\|^2) \leq 0. \quad (3.2.12)$$

Hence,  $\mathcal{A}$  is dissipative. The major work is to show  $0 \in \rho(\mathcal{A})$ . Let

$$\tilde{z} = (\tilde{u}_1, \tilde{w}_1, \tilde{v}_1, \tilde{\eta}_1, \tilde{u}_2, \tilde{w}_2, \tilde{v}_2, \tilde{\eta}_2, \tilde{p}, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2) \in \mathcal{H}.$$

Consider the equation

$$\mathcal{A}z = \tilde{z}, \quad (3.2.13)$$

i.e.,

$$v_i = \tilde{u}_i \in H_r^1, \quad (3.2.14)$$

$$\eta_i = \tilde{w}_i \in H_r^2, \quad (3.2.15)$$

$$(E_i A_i u_i' + \mu_i v_i')' = \rho_i A_i \tilde{v}_i \in L^2, \quad (3.2.16)$$

$$-(E_i I_i w_i'' + \gamma_i \eta_i'')'' = \rho_i A_i \tilde{\eta}_i \in L^2, \quad (3.2.17)$$

$$\mathbf{M}^{-1} \mathbf{C}(F_1, N_1, F_2, N_2, M_1, M_2)^T = (\tilde{p}, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2)^T \in \mathbb{R}^4. \quad (3.2.18)$$

A solution to (3.2.16), which also satisfies the condition  $u_i(0) = 0$ , has the form

$$u_i(s_i) = \frac{F_i}{E_i A_i} s_i - \frac{\mu_i}{E_i A_i} \tilde{u}_i(s_i) - \frac{1}{E_i A_i} \int_0^{s_i} \int_{\tau_2}^{L_i} \rho_i A_i \tilde{v}_i(\tau_1) d\tau_1 d\tau_2. \quad (3.2.19)$$

It is clear that

$$u_i \in H_r^1, \quad E_i A_i u_i' + \mu v_i' \in H^1. \quad (3.2.20)$$

A solution to equation (3.2.17), satisfying the conditions  $w_i(0) = w_i'(0) = 0$ , has the form

$$\begin{aligned} w_i(s_i) = & -\frac{N_i}{6E_i I_i} (3L_i s_i^2 - s_i^3) + \frac{M_i}{2E_i I_i} s_i^2 - \frac{\gamma_i}{E_i I_i} \tilde{w}_i(s_i) \\ & - \frac{1}{E_i I_i} \int_0^{s_i} \int_0^{\tau_4} \int_{\tau_3}^{L_i} \int_{\tau_2}^{L_i} \rho_i A_i \tilde{\eta}_i(\tau_1) d\tau_1 d\tau_2 d\tau_3 d\tau_4, \end{aligned} \quad (3.2.21)$$

with

$$\begin{aligned}
 w'_i(s_i) = & -\frac{N_i}{2E_i I_i}(2L_i s_i - s_i^2) + \frac{M_i}{E_i I_i} s_i - \frac{\gamma_i}{E_i I_i} \tilde{w}'_i(s_i) \\
 & - \frac{1}{E_i I_i} \int_0^{s_i} \int_{\tau_3}^{L_i} \int_{\tau_2}^{L_i} \rho_i A_i \tilde{\eta}_i(\tau_1) d\tau_1 d\tau_2 d\tau_3.
 \end{aligned} \tag{3.2.22}$$

Again, it is clear that

$$w_i \in H_r^2, \quad E_i I_i w_i'' + \gamma_i \eta_i'' \in H^2. \tag{3.2.23}$$

If we can show  $(F_i, N_i, M_i)$  are uniquely determined, then we will have uniqueness of the solutions  $u_i$  and  $w_i$ . With this in mind, let  $s_i = L_i$  in (3.2.19), (3.2.21) and (3.2.22). The matrix form of the resulting equations is

$$\begin{bmatrix} -u_1(L_1) \\ w_1(L_1) \\ -u_2(L_2) \\ w_2(L_2) \\ -w'_1(L_1) \\ -w'_2(L_2) \end{bmatrix} = \mathbf{B} \begin{bmatrix} F_1 \\ N_1 \\ F_2 \\ N_2 \\ M_1 \\ M_2 \end{bmatrix} + \tilde{\mathbf{F}} \tag{3.2.24}$$

where

$$\tilde{\mathbf{F}} = \begin{bmatrix} \frac{\mu_1}{E_1 A_1} \tilde{u}_1(L_1) + \frac{1}{E_i A_i} \int_0^{L_1} \int_{\tau_2}^{L_1} \rho_1 A_1 \tilde{v}_1(\tau_1) d\tau_1 d\tau_2 \\ -\frac{\gamma_1}{E_1 I_1} \tilde{w}_1(L_1) - \frac{1}{E_i I_i} \int_0^{L_1} \int_0^{\tau_4} \int_{\tau_3}^{L_1} \int_{\tau_2}^{L_1} \rho_1 A_1 \tilde{\eta}_1(\tau_1) d\tau_1 d\tau_2 d\tau_3 d\tau_4 \\ \frac{\mu_2}{E_2 A_2} \tilde{u}_2(L_2) + \frac{1}{E_i A_i} \int_0^{L_2} \int_{\tau_2}^{L_2} \rho_2 A_2 \tilde{v}_2(\tau_1) d\tau_1 d\tau_2 \\ -\frac{\gamma_2}{E_2 I_2} \tilde{w}_2(L_2) - \frac{1}{E_i I_i} \int_0^{L_2} \int_0^{\tau_4} \int_{\tau_3}^{L_2} \int_{\tau_2}^{L_2} \rho_2 A_2 \tilde{\eta}_2(\tau_1) d\tau_1 d\tau_2 d\tau_3 d\tau_4 \\ \frac{\gamma_1}{E_1 I_1} \tilde{w}'_1(L_1) + \frac{1}{E_i I_i} \int_0^{L_1} \int_{\tau_3}^{L_1} \int_{\tau_2}^{L_1} \rho_1 A_1 \tilde{\eta}_1(\tau_1) d\tau_1 d\tau_2 d\tau_3 \\ \frac{\gamma_2}{E_2 I_2} \tilde{w}'_2(L_2) + \frac{1}{E_i I_i} \int_0^{L_2} \int_{\tau_3}^{L_2} \int_{\tau_2}^{L_2} \rho_2 A_2 \tilde{\eta}_2(\tau_1) d\tau_1 d\tau_2 d\tau_3, \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} -\frac{L_1}{E_1 A_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{L_1^3}{3E_1 I_1} & 0 & 0 & \frac{L_1^2}{2E_1 I_1} & 0 \\ 0 & 0 & -\frac{L_2}{E_2 A_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{L_2^3}{3E_2 I_2} & 0 & \frac{L_2^2}{2E_2 I_2} \\ 0 & \frac{L_1^2}{2E_1 I_1} & 0 & 0 & -\frac{L_1}{E_1 I_1} & 0 \\ 0 & 0 & 0 & \frac{L_2^2}{2E_2 I_2} & 0 & -\frac{L_2}{E_2 I_2} \end{bmatrix}.$$

After proper row and column exchanging, it is easy to see that

$$\det \mathbf{B} = \frac{(L_1 L_2)^5}{144(E_1 E_2)^3 (I_1 I_2)^2 (A_1 A_2)}.$$

Hence  $\mathbf{B}$  is invertible. This leads to

$$\begin{bmatrix} F_1 \\ N_1 \\ F_2 \\ N_2 \\ M_1 \\ M_2 \end{bmatrix} = \mathbf{B}^{-1} \begin{bmatrix} -u_1(L_1) \\ w_1(L_1) \\ -u_2(L_2) \\ w_2(L_2) \\ -w'_1(L_1) \\ -w'_2(L_2) \end{bmatrix} - \mathbf{B}^{-1} \tilde{\mathbf{F}}. \quad (3.2.25)$$

Substituting (3.2.25) into equation (3.2.18), we get

$$\mathbf{M}^{-1} \mathbf{C} \mathbf{B}^{-1} \begin{bmatrix} -u_1(L_1) \\ w_1(L_1) \\ -u_2(L_2) \\ w_2(L_2) \\ -w'_1(L_1) \\ -w'_2(L_2) \end{bmatrix} = \begin{bmatrix} \tilde{p} \\ \tilde{q} \\ \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{bmatrix} + \mathbf{M}^{-1} \mathbf{C} \mathbf{B}^{-1} \tilde{\mathbf{F}}. \quad (3.2.26)$$

Note that (3.2.26) consists of four linearly independent equations of six variables. However, our compatibility conditions (3.1.10) in the state space provide two more equations linearly independent of the four equations in (3.2.26). Thus,  $(u_i(L_i), w_i(L_i), w'_i(L_i))$  can be solved uniquely as linear combinations of  $(\tilde{p}, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2)$  and  $\tilde{\mathbf{F}}$ . Furthermore,  $(F_i, N_i, M_i)$  can also be solved uniquely as linear combinations of  $(\tilde{p}, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2)$  and  $\tilde{\mathbf{F}}$ . In view of (3.2.19) and (3.2.21), we have obtained unique solution  $(u_i(s_i), w_i(s_i))$ . Finally, from (3.2.14)-(3.2.15), we

can solve the equations

$$\mathbf{C}^T \begin{bmatrix} p \\ q \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} -v_1(L_1) \\ \eta_1(L_1) \\ -v_2(L_2) \\ \eta_2(L_2) \\ -\eta'_1(L_1) \\ -\eta'_2(L_2) \end{bmatrix} = \begin{bmatrix} -\tilde{u}_1(L_1) \\ \tilde{w}_1(L_1) \\ -\tilde{u}_2(L_2) \\ \tilde{w}_2(L_2) \\ -\tilde{w}'_1(L_1) \\ -\tilde{w}'_2(L_2) \end{bmatrix} \quad (3.2.27)$$

to get  $(p, q, \omega_1, \omega_2)$  uniquely as linear combination of  $(\tilde{u}_i(L_i), \tilde{w}_i(L_i), \tilde{w}'_i(L_i))$ .

In summary, we have thus far obtained a unique solution  $z \in \mathcal{D}(\mathcal{A})$  of equation (3.2.13). By the expression of  $z$  in terms of  $\tilde{z}$ , it is clear that

$$\begin{cases} \|u'_i\| + \|w''_i\| \leq K_1 \sum_{i=1}^2 (\|\tilde{u}'_i\| + \|\tilde{w}''_i\| + \|\tilde{v}_i\| + \|\tilde{\eta}_i\| + \|(\tilde{p}, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2)\|_{\mathbb{R}^4}) \\ \|(p, q, \omega_1, \omega_2)\|_{\mathbb{R}^4} \leq K_2 \sum_{i=1}^2 (\|\tilde{u}'_i\| + \|\tilde{w}''_i\|) \\ \|v_i\| + \|\eta_i\| \leq K_3 (\|\tilde{u}'_i\| + \|\tilde{w}''_i\|), \end{cases}$$

for some constants  $K_1, K_2, K_3 > 0$ , which leads to

$$\|z\|_{\mathcal{H}} \leq K \|\mathcal{A}z\|_{\mathcal{H}}$$

for some constant  $K$  independent of  $z$ . Therefore,  $0 \in \rho(\mathcal{A})$ ,  $\mathcal{A}^{-1}$  is bounded and  $\mathcal{A}$  is closed. It follows that the range  $R(\lambda - \mathcal{A}) = \mathcal{H}$  for some  $\lambda > 0$ . By Theorem 4.6 in [18],  $\mathcal{D}(\mathcal{A})$  is dense in  $\mathcal{H}$ . The proof is completed by an application of Theorem 1.2.1.

### 3.3 Analyticity and Exponential Stability

Using the results of the previous section, we see that  $S(t)z_0$  is a classical solution of the abstract Cauchy problem (3.2.9) for any initial condition  $z(0) = z_0 \in D(\mathcal{A})$ . In this section, we shall prove that the semigroup  $S(t)$  is also analytic. Therefore,  $S(t)z_0$  is a classical solution not only for  $z_0 \in D(\mathcal{A})$ , but also for any  $z_0 \in \mathcal{H}$ . Exponential stability, which is important for control design, will be shown as well.

**Theorem 3.3.1**  *$S(t)$  is analytic and exponentially stable.*

Proof: The following frequency domain conditions, which we shall verify using a contradiction argument, are known (Theorem 1.3.3 of [15]) to guarantee analyticity for a  $C_0$  semigroup:

$$\mathbf{i}\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset, \quad (3.3.1)$$

$$\lim_{\beta \rightarrow \infty} \|\beta(\mathbf{i}\beta - \mathcal{A})^{-1}\| < \infty. \quad (3.3.2)$$

If the second condition is false, then there exists a sequence

$$z_n = (u_{1n}, w_{1n}, v_{1n}, \eta_{1n}, u_{2n}, w_{2n}, v_{2n}, \eta_{2n}, p_n, q_n, \omega_{1n}, \omega_{2n}) \in \mathcal{D}(\mathcal{A})$$

with  $\|z_n\|_{\mathcal{H}} = 1$ , and a sequence  $\beta_n \in \mathbb{R}$  with  $\beta_n \rightarrow \infty$  such that

$$\lim_{n \rightarrow \infty} \left\| \left( \mathbf{i}I - \frac{1}{\beta_n} \mathcal{A} \right) z_n \right\|_{\mathcal{H}} = 0. \quad (3.3.3)$$

For the simplicity of notation, we omit the subscript  $n$  in the rest of our proof. It should be noted that, for the remainder of this chapter, the convergence results occur as  $n \rightarrow \infty$ . From (3.3.3) and (3.2.12), we obtain

$$\lim_{n \rightarrow \infty} \operatorname{Re} \langle (\mathbf{i}I - \frac{1}{\beta} \mathcal{A})z, z \rangle_{\mathcal{H}} = \lim_{n \rightarrow \infty} \sum_{i=1}^2 \frac{1}{\beta} (\mu_i \|v'_i\|^2 + \gamma_i \|\eta''_i\|^2) = 0, \quad (3.3.4)$$

which implies that

$$\left\| \frac{v'_i}{\beta^{1/2}} \right\|, \left\| \frac{\eta''_i}{\beta^{1/2}} \right\| \rightarrow 0. \quad (3.3.5)$$

Our goal is to get a contradiction by showing  $\|z\|_{\mathcal{H}} \rightarrow 0$ . The componentwise version of (3.3.3) is

$$\mathbf{i}u'_i - \frac{1}{\beta} v'_i \rightarrow 0, \quad (3.3.6)$$

$$\mathbf{i}w''_i - \frac{1}{\beta} \eta''_i \rightarrow 0, \quad (3.3.7)$$

$$\mathbf{i}\rho_i A_i v_i - \frac{1}{\beta} (E_i A_i u'_i + \mu_i v'_i)' \rightarrow 0, \quad (3.3.8)$$

$$\mathbf{i}\rho_i A_i \eta_i + \frac{1}{\beta} (E_i I_i w''_i + \gamma_i \eta''_i)'' \rightarrow 0, \quad (3.3.9)$$

in  $L^2(0, L_i)$ , and

$$\mathbf{i} \begin{bmatrix} p \\ q \\ \omega_1 \\ \omega_2 \end{bmatrix} - \frac{1}{\beta} \mathbf{M}^{-1} \mathbf{C}(F_1, N_1, F_2, N_2, M_1, M_2)^T \rightarrow 0 \quad (3.3.10)$$

in  $\mathbb{R}^4$ . Combining equations (3.3.5)-(3.3.7), we have

$$\|u_i'\|, \|w_i''\| \rightarrow 0. \quad (3.3.11)$$

The inner product of (3.3.8) and  $v_i$  yields

$$\mathbf{i} \rho_i A_i \|v_i\|^2 - \frac{1}{\beta} (E_i A_i u_i' + \mu_i v_i')(L_i) \bar{v}_i(L_i) + \frac{1}{\beta} \langle E_i A_i u_i' + \mu_i v_i', v_i' \rangle \rightarrow 0. \quad (3.3.12)$$

We use Theorem 1.2.3 and the Gagliardo-Nirenberg inequality to show that the boundary term in (3.3.12) converges to zero. More specifically,

$$\begin{aligned} & \frac{1}{\beta^{3/4}} |(E_i A_i u_i' + \mu_i v_i')(L_i)| \\ & \leq C \frac{1}{\beta^{3/4}} \|E_i A_i u_i' + \mu_i v_i'\|^{1/2} \|E_i A_i u_i' + \mu_i v_i'\|_{H^1}^{1/2} \\ & \leq C \left\| \frac{E_i A_i u_i' + \mu_i v_i'}{\beta^{1/2}} \right\|^{1/2} \left( \left\| \frac{E_i A_i u_i' + \mu_i v_i'}{\beta} \right\|^{1/2} + \left\| \frac{(E_i A_i u_i' + \mu_i v_i)'}{\beta} \right\|^{1/2} \right) \\ & \rightarrow 0, \end{aligned} \quad (3.3.13)$$

and

$$\frac{1}{\beta^{1/4}} |v_i(L_i)| \leq C \|v_i\|^{1/2} \left\| \frac{v_i}{\beta^{1/2}} \right\|_{H^1}^{1/2} \leq C \|v_i\|^{1/2} \left\| \frac{v_i'}{\beta^{1/2}} \right\|^{1/2} \rightarrow 0 \quad (3.3.14)$$

due to (3.3.5), (3.3.8), (3.3.11) and the fact that  $v_i$  is bounded. Clearly, the last term in (3.3.12) also converges to zero due to (3.3.5) and (3.3.11). Therefore,

$$\|v_i\| \rightarrow 0. \quad (3.3.15)$$

Similarly, we take the inner product of (3.3.9) with  $\eta_i$  in  $L^2(0, L_i)$  to get

$$\begin{aligned} \mathbf{i}\rho_i A_i \|\eta_i\|^2 + \frac{1}{\beta} (E_i I_i w_i'' + \gamma_i \eta_i'')(L_i) \bar{\eta}_i(L_i) \\ - \frac{1}{\beta} (E_i I_i w_i'' + \gamma_i \eta_i'')(L_i) \bar{\eta}_i'(L_i) + \frac{1}{\beta} \langle E_i I_i w_i'' + \gamma_i \eta_i'', \eta_i'' \rangle \rightarrow 0. \end{aligned} \quad (3.3.16)$$

We again use Theorem 1.4.4 from [15] and the Gagliardo-Nirenberg inequality to show that the boundary terms in (3.3.16) converge to zero. In fact, we have

$$\begin{aligned} & \frac{1}{\beta^{7/8}} |(E_i I_i w_i'' + \gamma_i \eta_i'')(L_i)| \\ & \leq \frac{C}{\beta^{7/8}} \|(E_i I_i w_i'' + \gamma_i \eta_i'')'\|^{1/2} \|(E_i I_i w_i'' + \gamma_i \eta_i'')'\|_{H^1}^{1/2} \\ & \leq \frac{C}{\beta^{7/8}} \|E_i I_i w_i'' + \gamma_i \eta_i''\|^{1/4} \|E_i I_i w_i'' + \gamma_i \eta_i''\|_{H^2}^{1/4} \|(E_i I_i w_i'' + \gamma_i \eta_i'')'\|_{H^1}^{1/2} \\ & \leq \frac{C}{\beta^{7/8}} \|E_i I_i w_i'' + \gamma_i \eta_i''\|^{1/4} \|E_i I_i w_i'' + \gamma_i \eta_i''\|_{H^2}^{3/4} \\ & \leq C \left\| \frac{E_i I_i w_i'' + \gamma_i \eta_i''}{\beta^{1/2}} \right\|^{1/4} \left( \left\| \frac{E_i I_i w_i'' + \gamma_i \eta_i''}{\beta} \right\|^{3/4} + \left\| \frac{(E_i I_i w_i'' + \gamma_i \eta_i'')''}{\beta} \right\|^{3/4} \right) \\ & \rightarrow 0, \end{aligned} \quad (3.3.17)$$

$$\frac{1}{\beta^{1/8}} |\eta_i(L_i)| \leq \frac{C}{\beta^{1/8}} \|\eta_i\|^{1/2} \|\eta_i\|_{H^1}^{1/2} \leq C \|\eta_i\|^{3/4} \left\| \frac{\eta_i}{\beta^{1/2}} \right\|_{H^2}^{1/4} \leq C \|\eta_i\|^{3/4} \left\| \frac{\eta_i''}{\beta^{1/2}} \right\|^{1/4} \rightarrow 0, \quad (3.3.18)$$

and

$$\begin{aligned} & \frac{1}{\beta^{5/8}} |(E_i I_i w_i'' + \gamma_i \eta_i'')(L_i)| \\ & \leq \frac{C}{\beta^{5/8}} \|E_i I_i w_i'' + \gamma_i \eta_i''\|^{1/2} \|E_i I_i w_i'' + \gamma_i \eta_i''\|_{H^1}^{1/2} \\ & \leq \frac{C}{\beta^{5/8}} \|E_i I_i w_i'' + \gamma_i \eta_i''\|^{3/4} \|E_i I_i w_i'' + \gamma_i \eta_i''\|_{H^2}^{1/4} \\ & \leq C \left\| \frac{E_i I_i w_i'' + \gamma_i \eta_i''}{\beta^{1/2}} \right\|^{3/4} \left( \left\| \frac{E_i I_i w_i'' + \gamma_i \eta_i''}{\beta} \right\|^{1/4} + \left\| \frac{(E_i I_i w_i'' + \gamma_i \eta_i'')''}{\beta} \right\|^{1/4} \right) \\ & \rightarrow 0, \end{aligned} \quad (3.3.19)$$

$$\frac{1}{\beta^{3/8}} |\eta_i'(L_i)| \leq \frac{C}{\beta^{3/8}} \|\eta_i'\|^{1/2} \|\eta_i'\|_{H^1}^{1/2} \leq C \|\eta_i\|^{1/4} \left\| \frac{\eta_i}{\beta^{1/2}} \right\|_{H^2}^{3/4} \leq C \|\eta_i\|^{1/4} \left\| \frac{\eta_i''}{\beta^{1/2}} \right\|^{3/4} \rightarrow 0 \quad (3.3.20)$$

due to (3.3.5), (3.3.9), (3.3.11), and the fact that  $\eta_i$  is bounded. Clearly, the last term in (3.3.16) also converges to zero due to (3.3.5) and (3.3.11). Therefore,

$$\|\eta_i\| \rightarrow 0. \quad (3.3.21)$$

It is useful to point out here that estimates (3.3.13), (3.3.17), and (3.3.19) imply that

$$\frac{1}{\beta^{3/4}} F_i, \frac{1}{\beta^{7/8}} N_i, \frac{1}{\beta^{5/8}} M_i \rightarrow 0. \quad (3.3.22)$$

From this, we can take the inner product of equation (3.3.10) with the vector  $a = (p, q, \omega_1, \omega_2)$  in  $\mathbb{R}^4$  to get

$$\|a\|_{\mathbb{R}^4} = a\mathbf{M}a^T \rightarrow 0. \quad (3.3.23)$$

Combining (3.3.11), (3.3.15), (3.3.21), and (3.3.23), we have arrived at a contradiction. Hence, condition (3.3.2) holds.

To verify the condition (3.3.1) we again use a contradiction argument. Assuming  $i\beta \in \sigma(\mathcal{A})$ , then there exists a sequence  $z_n \in \mathcal{D}(\mathcal{A})$  with  $\|z_n\|_{\mathcal{H}} = 1$  for all  $n$  such that

$$\lim_{n \rightarrow \infty} \| (i\beta I - \mathcal{A}) z_n \|_{\mathcal{H}} = 0.$$

Since  $0 \notin \sigma(\mathcal{A})$  is already known, we can divide  $\beta$  through the above equation. This will result in equation (3.3.4) with  $\beta_n$  equal to  $\beta$  for all  $n$ . An exact repetition of the above argument will then give the contradiction and, hence, analyticity is proven. Finally, the exponential stability is a by-product of conditions (3.3.1) and (3.3.2) since we have the boundedness of  $\|(i\beta I - \mathcal{A})^{-1}\|$ .

# Chapter 4

## Joint-Beam System (Boltzmann Damping)

In this chapter, we return to the Boltzmann damping used in Chapter 2 and apply it to the joint-beam system. In the case of the single beam with tip mass, the abstract Cauchy problem resulting from the system was seen to be well-posed. Also, the memory damping produced exponential stability of solutions to the initial value problem. We will show that memory damping in the beams, with no damping in the joint, produces the same results for the joint-beam system.

### 4.1 Constitutive Equations

The equations of motion for a coupled joint-beam system with viscoelastic damping are the following:

$$\rho_i A_i \frac{\partial^2 u_i(t, s_i)}{\partial t^2} = \frac{\partial}{\partial s_i} \left[ E_i A_i \frac{\partial u_i(t, s_i)}{\partial s_i} + \int_0^\infty \dot{g}_{u_i}(z) \frac{\partial u_i(t - z, s_i)}{\partial s_i} dz \right], \quad (4.1.1)$$

$$\rho_i A_i \frac{\partial^2 w_i(t, s_i)}{\partial t^2} = -\frac{\partial^2}{\partial s_i^2} \left[ E_i I_i \frac{\partial^2 w_i(t, s_i)}{\partial s_i^2} + \int_0^\infty \dot{g}_{w_i}(z) \frac{\partial^2 w_i(t - z, s_i)}{\partial s_i^2} dz \right], \quad (4.1.2)$$

$$\mathbf{M} \frac{d^2}{dt^2} \begin{bmatrix} x(t) \\ y(t) \\ \theta_1(t) \\ \theta_2(t) \end{bmatrix} = \mathbf{C} \begin{bmatrix} F_1(t) \\ N_1(t) \\ F_2(t) \\ N_2(t) \\ M_1(t) \\ M_2(t) \end{bmatrix} \quad (4.1.3)$$

for time  $t > 0$  and spatial variable  $s_i \in [0, L_i]$ , where  $\mathbf{M}$  and  $\mathbf{C}$  are as given in (3.1.4) and (3.1.5).

The kernel functions  $g_{u_i}$  and  $g_{w_i}$  are assumed to satisfy the conditions (g1) - (g4) given in Chapter 2, with the following modification for condition (g3):

$$(g3') \quad \alpha_{u_i} = E_i A_i + \int_0^\infty g_{u_i}(z) dz > 0, \text{ and } \alpha_{w_i} = E_i I_i + \int_0^\infty g_{w_i}(z) dz > 0.$$

The beams are clamped at the end  $s_i = 0$ . Thus the boundary conditions are

$$u_i(t, 0) = w_i(t, 0) = \frac{\partial w_i}{\partial s_i}(t, 0) = 0. \quad (4.1.4)$$

At the other end of the beam, we have the same geometric compatibility conditions as those given for the Kelvin-Voigt system, namely (3.1.7) - (3.1.10).

The notations used in the above system are identical to those given in Chapter 3, with the exception of  $F_i$ ,  $N_i$ , and  $M_i$ . For beam  $i$ , these still represent the extensional force, shear force, and bending moment at the end  $s_i = L_i$ , but they are now given by:

$$\begin{aligned} F_i(t) &= E_i A_i \frac{\partial u_i}{\partial s_i}(t, L_i) + \int_0^\infty g_{u_i}(z) \frac{\partial u_i}{\partial s_i}(t - z, L_i) dz \\ N_i(t) &= E_i I_i \frac{\partial^3 w_i}{\partial s_i^3}(t, L_i) + \int_0^\infty g_{w_i}(z) \frac{\partial^3 w_i}{\partial s_i^3}(t - z, L_i) dz \\ M_i(t) &= E_i I_i \frac{\partial^2 w_i}{\partial s_i^2}(t, L_i) + \int_0^\infty g_{w_i}(z) \frac{\partial^2 w_i}{\partial s_i^2}(t - z, L_i) dz. \end{aligned}$$

## 4.2 Semigroup Setting

Let

$$\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \mathbb{R}^4 \quad (4.2.1)$$

where

$$\mathcal{H}_i = H_r^1(0, L_i) \times H_r^2(0, L_i) \times [L^2(0, L_i)]^2 \times \Sigma_{1_i} \times \Sigma_{2_i}, \quad i = 1, 2 \quad (4.2.2)$$

$$H_r^j(0, L_i) = \{f \in H^j(0, L_i) : f(0) = f'(0) = \dots = f^{(j-1)}(0) = 0\}, \quad (4.2.3)$$

$$\Sigma_{1_i} = L_{\dot{g}_{u_i}}^2(0, +\infty; H_r^1(0, L_i)), \quad (4.2.4)$$

and

$$\Sigma_{2_i} = L_{\dot{g}_{w_i}}^2(0, +\infty; H_r^2(0, L_i)). \quad (4.2.5)$$

In the above,  $L_{\dot{g}}^2(0, +\infty; H_r^i(0, L_i))$  is the Hilbert space of all  $H_r^i(0, L_i)$ -valued, square integrable functions defined on the measure space  $((0, +\infty), |\dot{g}|ds)$ .

We shall use prime to denote the spatial derivative for both  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . The inner products in these spaces are defined as

$$\begin{aligned} \langle (u, w, v, \eta, \sigma, r), (f, j, h, k, q, b) \rangle_{\mathcal{H}_i} &= \alpha_{u_i} \langle u', f' \rangle + \alpha_{w_i} \langle w'', j'' \rangle + \rho_i A_i (\langle v, h \rangle + \langle \eta, k \rangle) \\ &+ \int_0^\infty |\dot{g}_{u_i}(\zeta)| \langle \sigma(\zeta)', q(\zeta)' \rangle d\zeta + \int_0^\infty |\dot{g}_{w_i}(\zeta)| \langle r(\zeta)'', b(\zeta)'' \rangle d\zeta, \end{aligned} \quad (4.2.6)$$

$$\langle (u, w, v, \eta), (f, g, h, k) \rangle_{\mathbb{R}^4} = (f, g, h, k) \mathbf{M}(u, w, v, \eta)^T, \quad (4.2.7)$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual  $L^2$  inner product of functions on  $[0, L_i]$ , and  $\sigma_i(\zeta)' \equiv D_{s_i}(\sigma_i(\zeta))$ .

We now define the variables

$$v_i = \frac{\partial u_i}{\partial t}, \quad \eta_i = \frac{\partial w_i}{\partial t}, \quad p = \frac{dx}{dt}, \quad q = \frac{dy}{dt}, \quad \omega_i = \frac{d\theta_i}{dt},$$

$$\sigma_i(t; \zeta, s_i) = u_i(t, s_i) - u_i(t - \zeta, s_i),$$

$$r_i(t; \zeta, s_i) = w_i(t, s_i) - w_i(t - \zeta, s_i),$$

and

$$z = (u_1, w_1, v_1, \eta_1, \sigma_1, r_1, u_2, w_2, v_2, \eta_2, \sigma_2, r_2, p, q, \omega_1, \omega_2)^T.$$

The Hilbert space  $\mathcal{H}$  is equipped with the norm induced from the inner products (4.2.6)-

(4.2.7), i.e.,

$$\|z\|_{\mathcal{H}}^2 = \sum_{i=1}^2 [\alpha_{u_i} \|u'_i\|^2 + \alpha_{w_i} \|w''_i\|^2 + \rho_i A_i (\|v_i\|^2 + \|\eta_i\|^2) + \|\sigma_i\|_{\Sigma_{1_i}}^2 + \|r\|_{\Sigma_{2_i}}^2] + a\mathbf{M}a^T \quad (4.2.8)$$

where  $a = (p, q, \omega_1, \omega_2) \in \mathbb{R}^4$  and  $\|\cdot\|$  denotes the usual  $L^2$  norm.

For the new set of state variables, the compatibility conditions become

$$\begin{bmatrix} -v_1(L_1) \\ \eta_1(L_1) \\ -v_2(L_2) \\ \eta_2(L_2) \\ -\eta'_1(L_1) \\ -\eta'_2(L_2) \end{bmatrix} = \begin{bmatrix} p \sin \phi_1 + q \cos \phi_1 \\ -p \cos \phi_1 + q \sin \phi_1 + l_1 \omega_1 \\ p \sin \phi_2 - q \cos \phi_2 \\ p \cos \phi_2 + q \sin \phi_2 + l_2 \omega_2 \\ \omega_1 \\ \omega_2 \end{bmatrix} = \mathbf{C}^T \begin{bmatrix} p \\ q \\ \omega_1 \\ \omega_2 \end{bmatrix} \quad (4.2.9)$$

with projection form

$$(\mathbf{I} - \mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1}\mathbf{C}) \begin{bmatrix} -v_1(L_1) \\ \eta_1(L_1) \\ -v_2(L_2) \\ \eta_2(L_2) \\ -\eta'_1(L_1) \\ -\eta'_2(L_2) \end{bmatrix} = \mathbf{0}. \quad (4.2.10)$$

We now take our state space to be

$$\mathcal{H}_r = \{z \in \mathcal{H} \mid \text{compatibility condition (3.1.10)}\}.$$

and denote it by  $\mathcal{H}$  again. The joint-beam system then can be rewritten as a first order evolution equation

$$\frac{dz}{dt} = \mathcal{A}z$$

on the Hilbert space  $\mathcal{H}$  with

$$\mathcal{A}z = \begin{bmatrix} v_1 \\ \eta_1 \\ \frac{1}{\rho_1 A_1}(\alpha_{u_1} u_1' - \int_0^\infty g_{u_1}(\zeta) \sigma_1(\zeta)' d\zeta)' \\ -\frac{1}{\rho_1 A_1}(\alpha_{w_1} w_1'' - \int_0^\infty g_{w_1}(\zeta) r_1(\zeta)'' d\zeta)'' \\ v_1 - D_\zeta \sigma_1 \\ \eta_1 - D_\zeta r_1 \\ v_2 \\ \eta_2 \\ \frac{1}{\rho_2 A_2}(\alpha_{u_2} u_2' - \int_0^\infty g_{u_2}(\zeta) \sigma_2(\zeta)' d\zeta)' \\ -\frac{1}{\rho_2 A_2}(\alpha_{w_2} w_2'' - \int_0^\infty g_{w_2}(\zeta) \rho_2(\zeta)'' d\zeta)'' \\ v_2 - D_\zeta \sigma_2 \\ \eta_2 - D_\zeta r_2 \\ \mathbf{M}^{-1} \mathbf{C}(F_1, N_1, F_2, N_2, M_1, M_2)^T \end{bmatrix}, \quad (4.2.11)$$

and

$$\mathcal{D}(\mathcal{A}) = \left\{ z \in \mathcal{H} : \begin{array}{l} u_i, v_i \in H_r^1, \quad \alpha_{u_i} u_i' - \int_0^\infty g_{u_i}(\zeta) \sigma_i(\zeta)' d\zeta \in H^1 \\ w_i, \eta_i \in H_r^2, \quad \alpha_{w_i} w_i'' - \int_0^\infty g_{w_i}(\zeta) r_i(\zeta)'' d\zeta \in H^2, \\ D_\zeta \sigma_i \in \Sigma_{1_i}, D_\zeta r_i \in \Sigma_{2_i}, \sigma_i(0) = 0, r_i(0) = 0, \\ \text{compatibility condition (4.2.9)} \end{array} \right\}. \quad (4.2.12)$$

**Theorem 4.2.1** *The operator  $\mathcal{A}$  generates a  $C_0$  semigroup,  $S(t)$ , of contractions on  $\mathcal{H}$ .*

Proof: The proof is based on Theorem 1.2.1 and is very similar to the corresponding proof for the single beam system from Chapter 2. Dissipativeness is seen from the inequality

$$\operatorname{Re} \langle \mathcal{A}z, z \rangle_{\mathcal{H}} = - \sum_{i=1}^2 \langle D_\zeta \sigma_i, \sigma_i \rangle_{\Sigma_{1_i}} + \langle D_\zeta r_i, r_i \rangle_{\Sigma_{2_i}} \leq 0. \quad (4.2.13)$$

This inequality follows from Lemma 3.3.2 of [15], wherein it is seen that

$$\begin{aligned} \langle D_\zeta \sigma_i, \sigma_i \rangle_{\Sigma_{1_i}} &= \frac{1}{2} \int_0^\infty \ddot{g}_{u_i}(\zeta) \|\sigma_i(\zeta)'\|^2 d\zeta, \\ \langle D_\zeta r_i, r_i \rangle_{\Sigma_{2_i}} &= \frac{1}{2} \int_0^\infty \ddot{g}_{w_i}(\zeta) \|\sigma_i(\zeta)'\|^2 d\zeta. \end{aligned} \quad (4.2.14)$$

We now need to show  $0 \in \rho(\mathcal{A})$ . Let

$$\tilde{z} = (\tilde{u}_1, \tilde{w}_1, \tilde{v}_1, \tilde{\eta}_1, \tilde{\sigma}_1, \tilde{r}_1, \tilde{u}_2, \tilde{w}_2, \tilde{v}_2, \tilde{\eta}_2, \tilde{p}, \tilde{\sigma}_2, \tilde{r}_2, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2) \in \mathcal{H}.$$

Consider the equation

$$\mathcal{A}z = \tilde{z}, \quad (4.2.15)$$

i.e.,

$$v_i = \tilde{u}_i \in H_r^1, \quad (4.2.16)$$

$$\eta_i = \tilde{w}_i \in H_r^2, \quad (4.2.17)$$

$$(\alpha_{u_i} u_i' - \int_0^\infty g_{u_i}(\zeta) \sigma_i(\zeta)' d\zeta)' = \rho_i A_i \tilde{v}_i \in L^2, \quad (4.2.18)$$

$$-(\alpha_{w_i} w_i'' - \int_0^\infty g_{w_i}(\zeta) r_i(\zeta)'' d\zeta)'' = \rho_i A_i \tilde{\eta}_i \in L^2, \quad (4.2.19)$$

$$v_i - D_\zeta \sigma_i = \tilde{\sigma}_i \in \Sigma_{1_i}, \quad (4.2.20)$$

$$\eta_i - D_\zeta r_i = \tilde{r}_i \in \Sigma_{2_i}, \quad (4.2.21)$$

$$\mathbf{M}^{-1} \mathbf{C}(F_1, N_1, F_2, N_2, M_1, M_2)^T = (\tilde{p}, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2)^T \in \mathbb{R}^4. \quad (4.2.22)$$

The solution to (4.2.20), which also satisfies the condition  $\sigma_i(0) = 0$ , is given by

$$\sigma_i(\zeta) = \zeta v_i - \int_0^\zeta \tilde{\sigma}_i(\tau) d\tau. \quad (4.2.23)$$

From this, we have

$$D_\zeta \sigma_i \in \Sigma_{1_i}. \quad (4.2.24)$$

We must now show  $\sigma_i \in \Sigma_{1_i}$ . To do this, we will duplicate the steps (2.2.19) - (2.2.22) from

Chapter 2. Let  $T > \epsilon > 0$  be given. Using the kernel function property (g4), we have

$$\begin{aligned}
 & \int_{\epsilon}^T |\dot{g}_{u_i}(\zeta)| \|\sigma_i(\zeta)'\|^2 d\zeta \\
 \leq & \frac{1}{\delta} \int_{\epsilon}^T \ddot{g}_{u_i}(\zeta) \|\sigma_i(\zeta)'\|^2 d\zeta \\
 = & \frac{1}{\delta} \dot{g}_{u_i}(T) \|\sigma_i(T)'\|^2 - \frac{1}{\delta} \dot{g}_{u_i}(\epsilon) \|\sigma_i(\epsilon)'\|^2 - \frac{2}{\delta} \int_{\epsilon}^T \dot{g}_{u_i}(\zeta) \langle \sigma_i(\zeta)', (D_{\zeta} \sigma_i(\zeta))' \rangle d\zeta \\
 \leq & -\frac{1}{\delta} \dot{g}_{u_i}(\epsilon) \|\sigma_i(\epsilon)'\|^2 + \frac{1}{2} \int_{\epsilon}^T |\dot{g}_{u_i}(\zeta)| \|\sigma_i(\zeta)'\|^2 d\zeta \\
 & + \frac{2}{\delta^2} \int_{\epsilon}^T |\dot{g}_{u_i}(\zeta)| \|D_{\zeta} \sigma_i(\zeta)'\| d\zeta. \tag{4.2.25}
 \end{aligned}$$

This gives

$$\int_{\epsilon}^T |\dot{g}_{u_i}(\zeta)| \|\sigma_i(\zeta)'\|^2 d\zeta \leq -\frac{2}{\delta} \dot{g}_{u_i}(\epsilon) \|\sigma_i(\epsilon)'\|^2 + \frac{4}{\delta^2} \int_{\epsilon}^T |\dot{g}_{u_i}(\zeta)| \|(D_{\zeta} \sigma_i(\zeta))'\|^2 d\zeta. \tag{4.2.26}$$

Now, the proof of Lemma 3.3.1 in [15] shows that as  $\epsilon \rightarrow 0$ ,

$$-\frac{1}{\delta} \dot{g}_{u_i}(\epsilon) \|\sigma_i(\epsilon)'\|^2 \rightarrow 0. \tag{4.2.27}$$

Therefore, if we let  $T \rightarrow \infty$ , and  $\epsilon \rightarrow 0$ , then we see that

$$\sigma_i \in \Sigma_{1_i}, \quad \|\sigma_i\|_{\Sigma_{1_i}}^2 \leq \frac{4}{\delta^2} \int_0^{\infty} |\dot{g}_{u_i}(\zeta)| \|(D_{\zeta} \sigma_i(\zeta))'\|^2 d\zeta. \tag{4.2.28}$$

We can solve (4.2.21) similarly to get

$$\begin{aligned}
 r_i(\zeta) &= \zeta \eta_i - \int_0^{\zeta} \tilde{r}_i(\tau) d\tau, \quad r_i(0) = 0, \quad r_i \in \Sigma_{2_i}, \\
 \|r_i\|_{\Sigma_{2_i}}^2 &\leq \frac{4}{\delta^2} \int_0^{\infty} |\dot{g}_{w_i}(\zeta)| \|(D_{\zeta} r_i(\zeta))''\|^2 d\zeta. \tag{4.2.29}
 \end{aligned}$$

A solution to (4.2.18), which also satisfies the condition  $u_i(0) = 0$ , has the form

$$u_i(s_i) = \frac{F_i}{\alpha_{u_i}} s_i + \frac{1}{\alpha_{u_i}} \int_0^{\infty} \dot{g}_{u_i}(\zeta) \sigma(\zeta, s_i) d\zeta - \frac{1}{\alpha_{u_i}} \int_0^{s_i} \int_{\tau_2}^{L_i} \rho_i A_i \tilde{v}_i(\tau_1) d\tau_1 d\tau_2. \tag{4.2.30}$$

It is clear that

$$u_i \in H_r^1, \quad \alpha_{w_i} u_i' - \int_0^\infty g_{w_i}(\zeta) \sigma_i(\zeta)' d\zeta \in H^1. \quad (4.2.31)$$

A solution to equation (4.2.19), satisfying the conditions  $w_i(0) = w_i'(0) = 0$ , has the form

$$\begin{aligned} w_i(s_i) = & -\frac{N_i}{6\alpha_{w_i}}(3L_i s_i^2 - s_i^3) + \frac{M_i}{2\alpha_{w_i}} s_i^2 + \frac{1}{\alpha_{w_i}} \int_0^\infty \dot{g}_{w_i}(\zeta) r(\zeta, s_i) d\zeta \\ & - \frac{1}{\alpha_{w_i}} \int_0^{s_i} \int_0^{\tau_4} \int_{\tau_3}^{L_i} \int_{\tau_2}^{L_i} \rho_i A_i \tilde{\eta}_i(\tau_1) d\tau_1 d\tau_2 d\tau_3 d\tau_4, \end{aligned} \quad (4.2.32)$$

with

$$\begin{aligned} w_i'(s_i) = & -\frac{N_i}{2\alpha_{w_i}}(2L_i s_i - s_i^2) + \frac{M_i}{\alpha_{w_i}} s_i + \frac{1}{\alpha_{w_i}} \int_0^\infty \dot{g}_{w_i}(\zeta) \frac{d}{ds_i} r(\zeta, s_i) d\zeta \\ & - \frac{1}{\alpha_{w_i}} \int_0^{s_i} \int_{\tau_3}^{L_i} \int_{\tau_2}^{L_i} \rho_i A_i \tilde{\eta}_i(\tau_1) d\tau_1 d\tau_2 d\tau_3. \end{aligned} \quad (4.2.33)$$

Again, it is clear that

$$w_i \in H_r^2, \quad \alpha_{w_i} w_i'' - \int_0^\infty g_{w_i}(\zeta) r_i(\zeta)'' d\zeta \in H^2. \quad (4.2.34)$$

As in the Kelvin-Voigt case, we would like to show that  $(F_i, N_i, M_i)$  are uniquely determined. We will then have uniqueness of the solutions  $u_i$  and  $w_i$ . Proceeding toward this goal, let  $s_i = L_i$  in (4.2.30), (4.2.32) and (4.2.33). The matrix form of the resulting equations is

$$\begin{bmatrix} -u_1(L_1) \\ w_1(L_1) \\ -u_2(L_2) \\ w_2(L_2) \\ -w_1'(L_1) \\ -w_2'(L_2) \end{bmatrix} = \mathbf{B} \begin{bmatrix} F_1 \\ N_1 \\ F_2 \\ N_2 \\ M_1 \\ M_2 \end{bmatrix} + \tilde{\mathbf{F}} \quad (4.2.35)$$

where

$$\tilde{\mathbf{F}} = \begin{bmatrix} -\frac{1}{\alpha_{u_1}} \int_0^\infty \dot{g}_{u_1}(\zeta) \sigma(\zeta, L_1) d\zeta + \frac{1}{\alpha_{u_1}} \int_0^{L_1} \int_0^{L_1} \rho_1 A_1 \tilde{v}_1(\tau_1) d\tau_1 d\tau_2 \\ \frac{1}{\alpha_{w_1}} \int_0^\infty \dot{g}_{w_1}(\zeta) r(\zeta, L_1) d\zeta - \frac{1}{\alpha_{w_1}} \int_0^{L_1} \int_0^{\tau_2} \int_{\tau_3}^{L_1} \int_{\tau_2}^{L_1} \rho_1 A_1 \tilde{\eta}_1(\tau_1) d\tau_1 d\tau_2 d\tau_3 d\tau_4 \\ -\frac{1}{\alpha_{u_2}} \int_0^\infty \dot{g}_{u_2}(\zeta) \sigma(\zeta, L_2) d\zeta + \frac{1}{\alpha_{u_2}} \int_0^{L_2} \int_0^{L_2} \rho_2 A_2 \tilde{v}_2(\tau_1) d\tau_1 d\tau_2 \\ \frac{1}{\alpha_{w_2}} \int_0^\infty \dot{g}_{w_2}(\zeta) r(\zeta, L_2) d\zeta - \frac{1}{\alpha_{w_2}} \int_0^{L_2} \int_0^{\tau_2} \int_{\tau_3}^{L_2} \int_{\tau_2}^{L_2} \rho_2 A_2 \tilde{\eta}_2(\tau_1) d\tau_1 d\tau_2 d\tau_3 d\tau_4 \\ -\frac{1}{\alpha_{w_1}} \int_0^\infty \dot{g}_{w_2}(\zeta) \frac{d}{ds_i} r(\zeta, L_1) d\zeta + \frac{1}{\alpha_{w_1}} \int_0^{L_1} \int_{\tau_3}^{L_1} \int_{\tau_2}^{L_1} \rho_1 A_1 \tilde{\eta}_1(\tau_1) d\tau_1 d\tau_2 d\tau_3 \\ -\frac{1}{\alpha_{w_2}} \int_0^\infty \dot{g}_{w_2}(\zeta) \frac{d}{ds_i} r(\zeta, L_2) d\zeta + \frac{1}{\alpha_{w_2}} \int_0^{L_2} \int_{\tau_3}^{L_2} \int_{\tau_2}^{L_2} \rho_2 A_2 \tilde{\eta}_2(\tau_1) d\tau_1 d\tau_2 d\tau_3, \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} -\frac{L_1}{\alpha_{u_1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{L_1^3}{3\alpha_{w_1}} & 0 & 0 & \frac{L_1^2}{2\alpha_{w_1}} & 0 \\ 0 & 0 & -\frac{L_2}{\alpha_{u_2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{L_2^3}{3\alpha_{w_2}} & 0 & \frac{L_2^2}{2\alpha_{w_2}} \\ 0 & \frac{L_1^2}{2\alpha_{w_1}} & 0 & 0 & -\frac{L_1}{\alpha_{w_1}} & 0 \\ 0 & 0 & 0 & \frac{L_2^2}{2\alpha_{w_2}} & 0 & -\frac{L_2}{\alpha_{w_2}} \end{bmatrix}.$$

With a little work, we find that

$$\det \mathbf{B} = \frac{(L_1 L_2)^5}{144 \alpha_{u_1} \alpha_{u_2} \alpha_{w_1}^2 \alpha_{w_2}^2}.$$

Hence,  $\mathbf{B}$  is invertible and we can follow the same line of reasoning from Chapter 3 to solve for  $(F_i, N_i, M_i)$  uniquely in terms of  $(\tilde{p}, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2)$ .

Finally, from (4.2.16)-(4.2.17) and the geometric compatibility conditions, we can solve for  $(p, q, \omega_1, \omega_2)$  uniquely in terms of  $(\tilde{u}_i(L_i), \tilde{w}_i(L_i), \tilde{w}'_i(L_i))$ . Drawing everything together, we have obtained a unique solution  $z \in \mathcal{D}(\mathcal{A})$  of equation (4.2.15). By the expression of  $z$

in terms of  $\tilde{z}$ , we see that

$$\left\{ \begin{array}{l} \|u'_i\| + \|w''_i\| \leq K_1 \sum \left( \|\tilde{u}'_i\| + \|\tilde{w}''_i\| + \|\tilde{v}_i\| + \|\tilde{\eta}_i\| + \|\tilde{\sigma}_i\|_{\Sigma_{1_i}} + \|\tilde{r}_i\|_{\Sigma_{2_i}} + \|(\tilde{p}, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2)\|_{\mathbb{R}^4} \right) \\ \|(p, q, \omega_1, \omega_2)\|_{\mathbb{R}^4} \leq K_2 \sum (\|\tilde{u}'_i\| + \|\tilde{w}''_i\|) \\ \|v_i\| + \|\eta_i\| \leq K_3 (\|\tilde{u}'_i\| + \|\tilde{w}''_i\|), \\ \|\sigma_i\|_{\Sigma_{1_i}} + \|r_i\|_{\Sigma_{2_i}} \leq K_4 (\|\tilde{u}'_i\| + \|\tilde{w}''_i\| + \|\tilde{\sigma}_i\|_{\Sigma_{1_i}} + \|\tilde{r}_i\|_{\Sigma_{2_i}}), \end{array} \right.$$

for some constants  $K_1, K_2, K_3, K_4 > 0$ , which leads to

$$\|z\|_{\mathcal{H}} \leq K \|\mathcal{A}z\|_{\mathcal{H}}$$

for some constant  $K$  independent of  $z$ . Therefore,  $0 \in \rho(\mathcal{A})$ ,  $\mathcal{A}^{-1}$  is bounded and  $\mathcal{A}$  is closed. It follows that the range  $R(\lambda - \mathcal{A}) = \mathcal{H}$  for some  $\lambda > 0$ . By Theorem 4.6 in [18],  $\mathcal{D}(\mathcal{A})$  is dense in  $\mathcal{H}$ . We can now apply Theorem 1.2.1 and the proof is complete.

### 4.3 Exponential Stability

As a result of Theorem 4.2.1, we know that the joint-beam system with Boltzmann damping is well-posed, provided that the initial state lies in the domain of  $\mathcal{A}$ . We now look at the asymptotic behavior of the system.

**Theorem 4.3.1**  *$S(t)$  is exponentially stable.*

Proof: As in the case of Theorem 4.2.1, this proof is similar to the corresponding proof of the single beam system in Chapter 2. We shall use a contradiction argument to verify the two conditions from Theorem 1.2.2:

$$\mathbf{i}\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset, \quad (4.3.1)$$

$$\overline{\lim}_{|\beta| \rightarrow \infty} \|(i\beta - \mathcal{A})^{-1}\| < \infty \quad \beta \in \mathbb{R}. \quad (4.3.2)$$

If the first condition is false, then there exists a sequence

$$z_n = (u_{1n}, w_{1n}, v_{1n}, \eta_{1n}, \sigma_{1n}, r_{1n}, u_{2n}, w_{2n}, v_{2n}, \eta_{2n}, \sigma_{2n}, r_{2n}, p_n, q_n, \omega_{1n}, \omega_{2n}) \in \mathcal{D}(\mathcal{A})$$

with  $\|z_n\|_{\mathcal{H}} = 1$ , and a sequence  $\beta_n \in \mathbb{R}$  with  $\beta_n \rightarrow b$ ,  $|b| \geq \|\mathcal{A}^{-1}\|^{-1}$ , such that

$$\lim_{n \rightarrow \infty} \|(\mathbf{i}\beta_n I - \mathcal{A})z_n\|_{\mathcal{H}} = 0. \quad (4.3.3)$$

For the simplicity of notation, we omit the subscript  $n$  in the rest of our proof. Unless otherwise stated, convergence results for the remainder of the proof are understood to occur as  $n \rightarrow \infty$ . From (4.3.3) and (4.2.13), we obtain

$$\lim_{n \rightarrow \infty} \operatorname{Re} \langle (\mathbf{i}\beta I - \mathcal{A})z, z \rangle_{\mathcal{H}} = \lim_{n \rightarrow \infty} \sum_{i=1}^2 \frac{1}{2} \int_0^\infty \ddot{g}_{u_i}(\zeta) \|\sigma_i(\zeta)'\|^2 + \ddot{g}_{w_i}(\zeta) \|r_i(\zeta)''\|^2 d\zeta = 0. \quad (4.3.4)$$

With condition (g4), this implies

$$\|\sigma_i\|_{\Sigma_{1_i}}, \|r_i\|_{\Sigma_{2_i}} \rightarrow 0. \quad (4.3.5)$$

Our goal is to get a contradiction by showing  $\|z\|_{\mathcal{H}} \rightarrow 0$ . The componentwise version of (4.3.3) is

$$\mathbf{i}\beta u_i' - v_i' \rightarrow 0, \quad (4.3.6)$$

$$\mathbf{i}\beta w_i'' - \eta_i'' \rightarrow 0, \quad (4.3.7)$$

$$\mathbf{i}\beta v_i - \frac{1}{\rho_i A_i} (\alpha_{u_i} u_i' - \int_0^\infty \dot{g}_{u_i}(\zeta) \sigma_i(\zeta)' d\zeta)' \rightarrow 0, \quad (4.3.8)$$

$$\mathbf{i}\beta \eta_i + \frac{1}{\rho_i A_i} (\alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta)'' \rightarrow 0, \quad (4.3.9)$$

in  $L^2(0, L_i)$ ,

$$\mathbf{i}\beta \sigma_i - (v_i - D_\zeta \sigma_i) \rightarrow 0 \quad (4.3.10)$$

$$\mathbf{i}\beta r_i - (\eta_i - D_\zeta r_i) \rightarrow 0 \quad (4.3.11)$$

in  $\Sigma_{1_i}$  and  $\Sigma_{2_i}$ , respectively. Also,

$$\mathbf{i}\beta \begin{bmatrix} p \\ q \\ \omega_1 \\ \omega_2 \end{bmatrix} - \mathbf{M}^{-1} \mathbf{C} (F_1, N_1, F_2, N_2, M_1, M_2)^T \rightarrow 0 \quad (4.3.12)$$

in  $\mathbb{R}^4$ . Using conditions (g1) – (g4), it can be deduced that for each kernel function  $g(\zeta)$ , both  $\zeta^2 \dot{g}(\zeta)$  and  $\zeta^2 \ddot{g}(\zeta)$  belong to  $L^1(0, +\infty)$ . It is also easily seen that  $\frac{\zeta v_i}{\beta} \in \Sigma_{1_i}$ . Divide (4.3.10) by  $\beta$  and take its inner product with  $\frac{\zeta v_i}{\beta}$  in  $\Sigma_{1_i}$ . Since  $\sigma_i \rightarrow 0$  in  $\Sigma_{1_i}$ , we get

$$\left\| \frac{v'_i}{\beta} \right\|^2 \int_0^\infty \zeta |\dot{g}_{u_i}(\zeta)| d\zeta - \frac{1}{\beta} \int_0^\infty \zeta \dot{g}(\zeta) \left\langle (D_\zeta \sigma_i(\zeta))', \frac{v'_i}{\beta} \right\rangle d\zeta \rightarrow 0. \quad (4.3.13)$$

The goal is now to show that the second term of (4.3.13) converges to zero. To do this, we employ the Cauchy-Schwartz inequality to obtain

$$\begin{aligned} & |\zeta \dot{g}_{u_i}(\zeta) \langle \sigma_i(\zeta)', \frac{v'_i}{\beta} \rangle| \\ & \leq \zeta |\dot{g}_{u_i}(\zeta)| \|\sigma_i(\zeta)'\| \left\| \frac{v'_i}{\beta} \right\| \\ & \leq \frac{\zeta |\dot{g}_{u_i}(\zeta)|}{2} \left( \|\sigma_i(\zeta)'\|^2 + \left\| \frac{v'_i}{\beta} \right\|^2 \right) \rightarrow 0 \end{aligned} \quad (4.3.14)$$

as  $\zeta \rightarrow 0$ . The convergence in (4.3.14) is assured by Lemma 3.3.1 of [15]. We can use the Cauchy-Schwartz inequality again to get

$$\begin{aligned} & |\zeta \dot{g}_{u_i}(\zeta) \langle \sigma_i(\zeta)', \frac{v'_i}{\beta} \rangle| \\ & \leq \zeta |\dot{g}_{u_i}(\zeta)| \|\sigma_i(\zeta)'\| \left\| \frac{v'_i}{\beta} \right\| \\ & \leq \frac{1}{2} |\dot{g}_{u_i}(\zeta)| \|\sigma_i(\zeta)'\|^2 + \frac{\zeta^2 |\dot{g}_{u_i}(\zeta)|}{2} \left\| \frac{v'_i}{\beta} \right\|^2 \rightarrow 0 \end{aligned} \quad (4.3.15)$$

as  $\zeta \rightarrow \infty$ . The convergence in (4.3.15) is given by Lemma 3.3.2 of [15]. Using (4.3.14) and (4.3.15), and integration by parts, we have

$$\begin{aligned}
 & \left| - \int_0^\infty \zeta \dot{g}(\zeta) \left\langle (D_\zeta \sigma_i(\zeta))', \frac{v'_i}{\beta} \right\rangle d\zeta \right| \\
 = & \left| \int_0^\infty \zeta \ddot{g}_{u_i}(\zeta) \left\langle \sigma_i(\zeta)', \frac{v'_i}{\beta} \right\rangle d\zeta + \int_0^\infty \dot{g}_{u_i}(\zeta) \left\langle \sigma_i(\zeta)', \frac{v'_i}{\beta} \right\rangle d\zeta \right| \\
 \leq & \int_0^\infty \sqrt{\ddot{g}_{u_i}(\zeta)} \|\sigma_i(\zeta)'\| \sqrt{\ddot{g}_{u_i}(\zeta)} \zeta \left\| \frac{v'_i}{\beta} \right\| d\zeta + \|\sigma_i\|_{\Sigma_{1_i}} \left( \int_0^\infty |\dot{g}_{u_i}(\zeta)| d\zeta \right)^{\frac{1}{2}} \left\| \frac{v'_i}{\beta} \right\| \\
 \leq & \left( \int_0^\infty \ddot{g}_{u_i}(\zeta) \|\sigma_i(\zeta)'\|^2 d\zeta \right)^{\frac{1}{2}} \left( \int_0^\infty \ddot{g}_{u_i}(\zeta) \zeta^2 d\zeta \right)^{\frac{1}{2}} \left\| \frac{v'_i}{\beta} \right\| \\
 & + \|\sigma_i\|_{\Sigma_{1_i}} \left( \int_0^\infty |\dot{g}_{u_i}(\zeta)| d\zeta \right)^{\frac{1}{2}} \left\| \frac{v'_i}{\beta} \right\| \rightarrow 0
 \end{aligned} \tag{4.3.16}$$

where the convergence follows from (4.3.4) and the fact that  $\sigma_i \rightarrow 0$  in  $\Sigma_{1_i}$ . Therefore we have that the second term in (4.3.13) converges to zero. This implies that the first term must also converge to zero. Thus, we have

$$\|v'_i\| \rightarrow 0. \tag{4.3.17}$$

Since  $v_i$  is an element of  $H_r^1$ , this implies

$$\|v_i\| \rightarrow 0. \tag{4.3.18}$$

Plugging (4.3.17) into (4.3.6), we get

$$\|u'_i\| \rightarrow 0 \tag{4.3.19}$$

which results in

$$\|u_i\| \rightarrow 0. \tag{4.3.20}$$

Now, if we repeat the same argument using the fact that  $\|r_i\| \rightarrow 0$  in  $\Sigma_{2_i}$ , the similar results will be

$$\|\eta_i\|, \|\eta'_i\|, \|\eta''_i\|, \|w_i\|, \|w'_i\|, \|w''_i\| \rightarrow 0. \tag{4.3.21}$$

Regarding the boundary terms, we can use Theorem 1.2.3 and (4.3.17) – (4.3.18) to show

$$|v_i(L_i)| \leq C \|v_i\|^{\frac{1}{2}} \|v_i\|_{H^1}^{\frac{1}{2}} \rightarrow 0. \tag{4.3.22}$$

Using equation (4.3.21), we can do the same to the other boundary terms to find

$$\eta_i(L_i), \eta'_i(L_i) \rightarrow 0. \quad (4.3.23)$$

Plugging equations (4.3.22) – (4.3.23) into equation (4.2.9) yields

$$(p, q, \omega_1, \omega_2) \rightarrow 0 \quad (4.3.24)$$

in  $\mathbb{R}^4$ . From equations (4.3.18), (4.3.19), (4.3.21), and (4.3.24), we have our contradiction that  $\|z\|_{\mathcal{H}} \rightarrow 0$ . Therefore, condition (4.3.1) holds.

Now, suppose condition (4.3.2) does not hold. This implies the existence of another sequence

$$z_n = (u_{1n}, w_{1n}, v_{1n}, \eta_{1n}, \sigma_{1n}, r_{1n}, u_{2n}, w_{2n}, v_{2n}, \eta_{2n}, \sigma_{2n}, r_{2n}, p_n, q_n, \omega_{1n}, \omega_{2n}) \in \mathcal{D}(\mathcal{A})$$

with  $\|z_n\| = 1$ , and a sequence  $\beta \in \mathbb{R}$  with  $\|\beta\| \rightarrow \infty$ , such that

$$\lim_{n \rightarrow \infty} \|(\mathbf{i}\beta_n I - \mathcal{A})z_n\|_{\mathcal{H}} = 0. \quad (4.3.25)$$

Our goal will be to once again demonstrate a contradiction by showing that  $\|z_n\|_{\mathcal{H}} \rightarrow 0$ . If we omit the subscript  $n$  in future calculations, (4.3.15) gives the same component equations (i.e., (4.3.6) – (4.3.12)) as before with the notable difference that  $\beta$  is no longer bounded. Notice also that we can still use (4.3.4) to get

$$\|\sigma_i\|_{\Sigma_{1_i}}, \|r_i\|_{\Sigma_{2_i}} \rightarrow 0. \quad (4.3.26)$$

With (4.3.26), we can repeat the same procedure we used before to get

$$\left\| \frac{v'_i}{\beta} \right\|, \left\| \frac{\eta''_i}{\beta} \right\| \rightarrow 0. \quad (4.3.27)$$

From this, (4.3.6) and (4.3.7) respectively give

$$\|u'_i\|, \|w''_i\| \rightarrow 0. \quad (4.3.28)$$

Again, as noted in Theorem 2.3, differences in proving conditions (4.3.1) and (4.3.2) begin to appear at this point. The unboundedness of  $\beta$  forces us to proceed in an alternate manner.

The key to showing that  $\|z\| \rightarrow 0$  lies in the boundary terms  $N_i, M_i$ , and  $F_i$ . By repeated use of Theorem 1.2.3 and the Gagliardo-Nirenberg Inequality, we have the following for  $N_i$ :

$$\begin{aligned}
& \left| \left( \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right)' (L_i) \right| \\
& \leq C_1 \left\| \left( \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right)' \right\|_{H^1}^{\frac{1}{2}} \left\| \left( \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right)' \right\|_{H^1}^{\frac{1}{2}} \\
& \leq C_2 \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|_{H^2}^{\frac{1}{4}} \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|_{H^2}^{\frac{1}{4}} \left\| \alpha_{w_i} w_i'' \right. \\
& \quad \left. - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|_{H^2}^{\frac{1}{2}} \\
& = C_2 \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|_{H^2}^{\frac{1}{4}} \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|_{H^2}^{\frac{3}{4}}. \tag{4.3.29}
\end{aligned}$$

Regarding the last term in (4.3.29), we can again use the Gagliardo-Nirenberg Inequality to get

$$\begin{aligned}
& \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|_{H^2} \\
& \leq \left( \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|^2 \right. \\
& \quad + C_3 \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\| \left\| \left( \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right)'' \right\| \\
& \quad \left. + \left\| \left( \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right)'' \right\|^2 \right)^{\frac{1}{2}} \\
& \leq C_4 \left( \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\| + \left\| \left( \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right)'' \right\| \right). \tag{4.3.30}
\end{aligned}$$

Now, from (4.3.9), we have that

$$\frac{\left\| \left( \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right)'' \right\|}{\beta} \leq C_5. \tag{4.3.31}$$

Therefore, since  $w_i'' \rightarrow 0$  in  $L^2$  and  $r_i \rightarrow 0$  in  $\Sigma_{2_i}$ , we have that

$$\frac{\left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|_{H^2}}{\beta} \leq C_6. \quad (4.3.32)$$

This implies that

$$\frac{|N_i|}{\beta} \leq C_7 \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|_{H^2}^{\frac{1}{4}}. \quad (4.3.33)$$

Again, we use the convergence of  $w_i''$  and  $r_i$  to get

$$\frac{|N_i|}{\beta} \rightarrow 0. \quad (4.3.34)$$

With regard to the term  $M_i$ , we have

$$\begin{aligned} & \left| \left( \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right) (L_i) \right| \\ & \leq C_8 \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|_{H^2}^{\frac{1}{2}} \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|_{H^1}^{\frac{1}{2}} \\ & \leq C_8 \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|_{H^2}^{\frac{1}{2}} \left\| \alpha_{w_i} w_i'' - \int_0^\infty \dot{g}_{w_i}(\zeta) r_i(\zeta)'' d\zeta \right\|_{H^2}^{\frac{1}{2}}. \end{aligned} \quad (4.3.35)$$

Using the same techniques as we did for  $N_i$ , we see that

$$\frac{|M_i|}{\beta} \rightarrow 0. \quad (4.3.36)$$

For the last boundary term  $F_i$ , we have

$$\begin{aligned} & \left| \left( \alpha_{u_i} u_i' - \int_0^\infty \dot{g}_{u_i}(\zeta) \sigma_i(\zeta)' d\zeta \right) (L_i) \right| \\ & \leq C_9 \left\| \alpha_{u_i} u_i' - \int_0^\infty \dot{g}_{u_i}(\zeta) \sigma_i(\zeta)' d\zeta \right\|_{H^1}^{\frac{1}{2}} \left\| \alpha_{u_i} u_i' - \int_0^\infty \dot{g}_{u_i}(\zeta) \sigma_i(\zeta)' d\zeta \right\|_{H^1}^{\frac{1}{2}} \\ & = C_9 \left\| \alpha_{u_i} u_i' - \int_0^\infty \dot{g}_{u_i}(\zeta) \sigma_i(\zeta)' d\zeta \right\|_{H^1}^{\frac{1}{2}} \left( \left\| \alpha_{u_i} u_i' - \int_0^\infty \dot{g}_{u_i}(\zeta) \sigma_i(\zeta)' d\zeta \right\|_{H^1}^2 \right. \\ & \quad \left. + \left\| \left( \alpha_{u_i} u_i' - \int_0^\infty \dot{g}_{u_i}(\zeta) \sigma_i(\zeta)' d\zeta \right)' \right\|_{H^1}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (4.3.37)$$

Now, we know from (4.3.8) that

$$\left\| \frac{\left( \alpha_{u_i} u_i' - \int_0^\infty \dot{g}_{u_i}(\zeta) \sigma_i(\zeta)' d\zeta \right)'}{\beta} \right\| \leq C_{10}. \quad (4.3.38)$$

Therefore, since  $u_i \rightarrow 0$  in  $L^2$  and  $\sigma_i \rightarrow 0$  in  $\Sigma_{1_i}$ , we can divide (4.3.37) by  $\beta$  to get

$$\frac{F_i}{\beta} \rightarrow 0. \quad (4.3.39)$$

Plugging (4.3.34), (4.3.36), and (4.3.39) into (4.3.12), we get that

$$p, q, \omega_1, \omega_2 \rightarrow 0. \quad (4.3.40)$$

Thus far, we have each component of  $z$  converging to zero except for  $v_i$  and  $\eta_i$ . If we take the inner product in  $L^2$  of (4.3.9) with  $\eta_i$ , then we have

$$i\beta \|\eta_i\|^2 + \frac{1}{\rho A} \left( N_i \eta_i(L_i) - M_i \eta_i'(L_i) + \alpha_{w_i} \langle w_i'', \eta_i'' \rangle - \int_0^\infty \dot{g}_{w_i}(\zeta) \langle r_i(\zeta)'', \eta_i'' \rangle d\zeta \right) \rightarrow 0. \quad (4.3.41)$$

Divide (4.3.41) by  $\beta$  and use (4.2.9) along with our previously proved convergences to get

$$\eta_i \rightarrow 0. \quad (4.3.42)$$

By taking the inner product in  $L^2$  of (4.3.8) with  $v_i$ , we can repeat the same procedure to get

$$v_i \rightarrow 0. \quad (4.3.43)$$

Putting everything together, we have the contradiction  $z \rightarrow 0$ . Therefore, condition (4.3.2) holds and our semigroup is exponentially stable.

# Chapter 5

## Joint-Beam System (Thermoelastic Damping)

We now remove all structural damping from the joint-beam system, and instead we will introduce thermal effects. This is especially relevant since solar radiation has critical impact on space structures. The thermoelastic joint-beam system results from considering the total strain to be the sum of the mechanical strain and the thermal strain. Stress therefore takes the form  $\sigma = k(\epsilon - \alpha T)$ , where  $T$  denotes the deviation from the reference temperature  $T_0$ . Conduction, radiation, and internal friction are included in the modelling of the thermal equations. These equations are derived in [5] and are based upon a modification of an approximation technique found in [20].

### 5.1 Constitutive Equations

In order to describe a coupled Joint-Beam system with thermoelastic damping, let  $T^i = T^i(t, s_i, \phi_i)$  be the variation of the temperature of beam  $i$  with respect to a reference temperature  $T_0^i$ , where  $s_i$  and  $\phi_i$  are coordinates representing axial location and circumferential angle. Let us now approximate  $T^i$  by

$$T^i(t, s_i, \phi_i) = T^i(t, s_i) + T^{m,i}(t, s_i)g(\phi_i) \quad (5.1.1)$$

where

$$g(\phi_i) = \begin{cases} \cos(\phi_i) - \frac{1}{\pi}, & \text{for } \phi_i \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ -\frac{1}{\pi}, & \text{for } \phi_i \in [-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]. \end{cases}$$

The function  $g$  was chosen for the purpose of approximating solar radiation on a cylindrical beam surface. The addition of the constant  $\frac{-1}{\pi}$  gives  $g$  a "zero average," i.e.  $\int_{-\pi}^{\pi} g(\phi)d\phi = 0$ .

Now define  $\tilde{T}^i(t, s_i) = T^i(t, s_i) - (T^* - T_0^i)$ , where  $T^*$  and  $T_0^i$  are constants to be defined later. Using the variables  $\tilde{T}^i$  and  $T^{m,i}$ , the thermal and mechanical equations for a coupled Joint-Beam system with solar radiation and thermoelastic damping have been derived in [5] as the following:

$$\rho_i A_i \frac{\partial^2 u_i(t, s_i)}{\partial t^2} = E_i A_i \frac{\partial}{\partial s_i} \left( \frac{\partial u_i(t, s_i)}{\partial s_i} - \alpha_i \tilde{T}^i(t, s_i) \right), \quad (5.1.2)$$

$$\rho_i A_i \frac{\partial^2 w_i(t, s_i)}{\partial t^2} = -E_i I_i \frac{\partial^2}{\partial s_i^2} \left( \frac{\partial^2 w_i(t, s_i)}{\partial s_i^2} + \frac{\alpha_i}{2R_i} T^{m,i}(t, s_i) \right), \quad (5.1.3)$$

$$\begin{aligned} \rho_i c_i \frac{\partial \tilde{T}^i(t, s_i)}{\partial t} &= k_a^i \frac{\partial^2 \tilde{T}^i(t, s_i)}{\partial s_i^2} - \frac{4\sigma\epsilon_i(T_0^i + T_s^i)^3}{h_i} \left( \tilde{T}^i(t, s_i) + T^* - T_0^i - T_s^i \right) \\ &\quad - \alpha_i E_i T_0^i \frac{\partial^2 u_i(t, s_i)}{\partial s_i \partial t}, \end{aligned} \quad (5.1.4)$$

$$\begin{aligned} \rho_i c_i \frac{\partial T^{m,i}(t, s_i)}{\partial t} &= k_a^i \frac{\partial^2 T^{m,i}(t, s_i)}{\partial s_i^2} - \left[ \frac{k_c^i \pi^2}{R_i^2 (\pi^2 - 4)} + \frac{4\sigma\epsilon_i(T_0^i + T_s^i)^3}{h_i} \right] T^{m,i}(t, s_i) \\ &\quad + \frac{\alpha_i E_i I_i T_0^i}{2R_i A_i} \frac{\partial^3 w_i(t, s_i)}{\partial s_i^2 \partial t} + \frac{\alpha_s^i S_i}{h_i}, \end{aligned} \quad (5.1.5)$$

$$\mathbf{M} \frac{d^2}{dt^2} \begin{bmatrix} x(t) \\ y(t) \\ \theta_1(t) \\ \theta_2(t) \end{bmatrix} = \mathbf{C} \begin{bmatrix} F_1(t) \\ N_1(t) \\ F_2(t) \\ N_2(t) \\ M_1(t) \\ M_2(t) \end{bmatrix} \quad (5.1.6)$$

where  $\mathbf{M}$  and  $\mathbf{C}$  are as given in (3.1.4)-(3.1.5). The mechanical boundary conditions of the beams are given by (3.1.6)-(3.1.10). The thermal variables are assumed to satisfy Robin

boundary conditions, i.e.

$$\frac{\partial}{\partial s_i} \tilde{T}^i(t, L_i) = -\lambda_R^i \tilde{T}^i(t, L_i) \quad (5.1.7)$$

$$\frac{\partial}{\partial s_i} \tilde{T}^i(t, 0) = \lambda_L^i \tilde{T}^i(t, 0) \quad (5.1.8)$$

$$\frac{\partial}{\partial s_i} T^{m,i}(t, L_i) = -\lambda_R^i T^{m,i}(t, L_i) \quad (5.1.9)$$

$$\frac{\partial}{\partial s_i} T^{m,i}(t, 0) = \lambda_L^i T^{m,i}(t, 0). \quad (5.1.10)$$

Many of the notations in the above system are already defined in Chapter 3. New variables and constants are given below:

- $\epsilon_i, \alpha_s^i, k_a^i, k_c^i$  - surface emissivity, surface absorbtivity, axial thermal conductivity, circumferential thermal conductivity of beam  $i$
- $T_0^i$  - undeformed reference temperature for beam  $i$
- $T_s^i$  - steady-state constant temperature increment produced by solar flux in relation to beam  $i$
- $c_i, \alpha_i, R_i, h_i$  - specific heat, thermal expansion coefficient, radius, wall thickness of beam  $i$
- $S_i$  - solar flux in relation to beam  $i$
- $\sigma$  - Stefan-Boltzmann's constant
- $T^*$  - temperature of the surrounding medium.

The extensional forces, shear forces, and bending moments of the beams at the ends  $s_i = L_i$  are given by

$$F_i(t) = E_i A_i \left( \frac{\partial w_i}{\partial s_i}(t, s_i) - \alpha_i \tilde{T}^i(t, s_i) \right) \Big|_{s_i=L_i} \quad (5.1.11)$$

$$N_i(t) = E_i I_i \frac{\partial}{\partial s_i} \left( \frac{\partial^2 w_i}{\partial s_i^2}(t, s_i) + \frac{\alpha_i}{2R_i} T^{m,i}(t, s_i) \right) \Big|_{s_i=L_i} \quad (5.1.12)$$

$$M_i(t) = E_i I_i \left( \frac{\partial^2 w_i}{\partial s_i^2}(t, s_i) + \frac{\alpha_i}{2R_i} T^{m,i}(t, s_i) \right) \Big|_{s_i=L_i}. \quad (5.1.13)$$

## 5.2 Semigroup Setting

Let

$$\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \mathbb{R}^4 \quad (5.2.1)$$

where

$$\mathcal{H}_i = H_r^1(0, L_i) \times H_r^2(0, L_i) \times [L^2(0, L_i)]^4, \quad i = 1, 2 \quad (5.2.2)$$

We define the inner products in these spaces to be

$$\begin{aligned} \langle (u, w, v, \eta, \tilde{T}, T^m), (f, g, h, k, r, s) \rangle_{\mathcal{H}_i} &= E_i A_i \langle u', f' \rangle + E_i I_i \langle w'', g'' \rangle + \rho_i A_i \left( \langle v, h \rangle + \langle \eta, k \rangle \right) \\ &\quad + \frac{\rho_i c_i A_i}{T_0^i} \left( \langle \tilde{T}, r \rangle + \langle T^m, s \rangle \right) \end{aligned} \quad (5.2.3)$$

$$\langle (u, w, v, \eta), (f, g, h, k) \rangle_{\mathbb{R}^4} = (f, g, h, k) \mathbf{M}(u, w, v, \eta)^T, \quad (5.2.4)$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual  $L^2$  inner product. As in Chapter 3, define the variables

$$v_i = \frac{\partial u_i}{\partial t}, \quad \eta_i = \frac{\partial w_i}{\partial t}, \quad p = \frac{dx}{dt}, \quad q = \frac{dy}{dt}, \quad \omega_i = \frac{d\theta_i}{dt},$$

and

$$z = (u_1, w_1, v_1, \eta_1, \tilde{T}^1, T^{m,1}, u_2, w_2, v_2, \eta_2, \tilde{T}^2, T^{m,2}, p, q, \omega_1, \omega_2)^T.$$

The Hilbert space  $\mathcal{H}$  is equipped with the norm induced from the inner products of its component spaces, i.e.,

$$\|z\|_{\mathcal{H}}^2 = \sum_{i=1}^2 \left[ E_i A_i \|u_i'\|^2 + E_i I_i \|w_i''\|^2 + \rho_i A_i \left( \|v_i\|^2 + \|\eta_i\|^2 \right) + \frac{\rho_i c_i A_i}{T_0^i} \left( \|\tilde{T}^i\| + \|T^{m,i}\| \right) \right] + a \mathbf{M} a^T \quad (5.2.5)$$

where  $a = (p, q, \omega_1, \omega_2)$  and  $\|\cdot\|$  denotes the usual  $L^2$  norm.

Our state space is now defined to be

$$\mathcal{H}_r^T = \{z \in \mathcal{H} \mid \text{compatibility condition (3.1.10)}\}.$$

Again, we will denote this space by  $\mathcal{H}$ . Let us also define the space

$$H_{rb}^2(0, L_i) = \{f \in H^2 \mid f'(0) = \lambda_L^i f(0), \quad f'(L_i) = -\lambda_R^i f(L_i)\}. \quad (5.2.6)$$

Making the substitutions  $\mu_i = \frac{4\sigma\epsilon_i(T_0^i+T_s^i)^3}{\rho_i c_i h_i}$  and  $\gamma_i = \left[ \frac{k_c^i \pi^2}{\rho_i c_i R_i^2 (\pi^2 - 4)} + \frac{4\sigma\epsilon_i(T_0^i+T_s^i)^3}{\rho_i c_i h_i} \right]$ , we can now rewrite the thermoelastic joint-beam system as a first order evolution equation

$$\frac{dz}{dt} = \mathcal{A}z + B$$

on the Hilbert space  $\mathcal{H}$  with

$$\mathcal{A}z = \begin{bmatrix} v_1 \\ \eta_1 \\ \frac{E_1 A_1}{\rho_1 A_1} \left( u_1' - \alpha_1 \tilde{T}^1 \right)' \\ -\frac{E_1 I_1}{\rho_1 A_1} \left( w_1'' + \frac{\alpha_1}{2R_1} T^{m,1} \right)'' \\ \frac{k_a^1}{\rho_1 c_1} \tilde{T}^1'' - \mu_1 \tilde{T}^1 - \frac{\alpha_1 E_1 T_0^1}{\rho_1 c_1} v_1' \\ \frac{k_a^1}{\rho_1 c_1} T^{m,1}'' - \gamma_1 T^{m,1} + \frac{\alpha_1 E_1 I_1 T_0^1}{2\rho_1 c_1 R_1 A_1} \eta_1'' \\ v_2 \\ \eta_2 \\ \frac{E_2 A_2}{\rho_2 A_2} \left( u_2' - \alpha_2 \tilde{T}^2 \right)' \\ -\frac{E_2 I_2}{\rho_2 A_2} \left( w_2'' + \frac{\alpha_2}{2R_2} T^{m,2} \right)'' \\ \frac{k_a^2}{\rho_2 c_2} \tilde{T}^2'' - \mu_2 \tilde{T}^2 - \frac{\alpha_2 E_2 T_0^2}{\rho_2 c_2} v_2' \\ \frac{k_a^2}{\rho_2 c_2} T^{m,2}'' - \gamma_2 T^{m,2} + \frac{\alpha_2 E_2 I_2 T_0^2}{2\rho_2 c_2 R_2 A_2} \eta_2'' \\ \mathbf{M}^{-1} \mathbf{C} (F_1, N_1, F_2, N_2, M_1, M_2)^T \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{-\mu_1 (T^* - T_0^1 - T_s^1)}{\frac{\rho_1 c_1}{\alpha_s^1 S_1}} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{-\mu_2 (T^* - T_0^2 - T_s^2)}{\frac{\rho_2 c_2}{\alpha_s^2 S_2}} \\ 0 \end{bmatrix}, \quad (5.2.7)$$

and

$$\mathcal{D}(\mathcal{A}) = \left\{ z \in \mathcal{H} : \begin{array}{l} v_i \in H_r^1, \quad u_i \in H^2, \quad \eta_i \in H_r^2, \quad w_i \in H^4, \\ \tilde{T}^i, T^{m,i} \in H_{rb}^2, \quad \text{compatibility condition (3.2.7)} \end{array} \right\}. \quad (5.2.8)$$

**Theorem 5.2.1** *The operator  $\mathcal{A}$  generates a  $C_0$  semigroup,  $S(t)$ , of contractions on  $\mathcal{H}$ .*

*Proof:*

Using the geometric constraints (3.2.7), which elements  $z \in \mathcal{D}(\mathcal{A})$  must satisfy, we see

that

$$\begin{aligned} \operatorname{Re}\langle \mathcal{A}z, z \rangle_{\mathcal{H}} &= - \sum_{i=1}^2 \left[ \frac{k_a^i A_i}{T_0^i} \left( \lambda_R^i \tilde{T}^i(L_i)^2 + \lambda_L^i \tilde{T}^i(0)^2 + \lambda_R^i T^{m,i}(L_i)^2 + \lambda_L^i T^{m,i}(0)^2 \right. \right. \\ &\quad \left. \left. + \|\tilde{T}^{i'}\|^2 + \|T^{m,i'}\|^2 \right) + \frac{\rho_i c_i A_i}{T_0^i} \left( \mu_i \|\tilde{T}^i\|^2 + \gamma_i \|T^{m,i}\|^2 \right) \right] \\ &\leq 0. \end{aligned} \tag{5.2.9}$$

Hence,  $\mathcal{A}$  is dissipative. To show  $0 \in \rho(\mathcal{A})$ , let

$$\tilde{z} = (\tilde{u}_1, \tilde{w}_1, \tilde{v}_1, \tilde{\eta}_1, \tilde{r}_1, \tilde{\sigma}_1 \tilde{u}_2, \tilde{w}_2, \tilde{v}_2, \tilde{\eta}_2, \tilde{r}_2, \tilde{\sigma}_2, \tilde{p}, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2) \in \mathcal{H}.$$

Consider the equation

$$\mathcal{A}z = \tilde{z}, \tag{5.2.10}$$

which is given in component form by

$$v_i = \tilde{u}_i \in H_r^1, \tag{5.2.11}$$

$$\eta_i = \tilde{w}_i \in H_r^2, \tag{5.2.12}$$

$$E_i A_i (u_i' - \alpha_i \tilde{T}^i)' = \rho_i A_i \tilde{v}_i \in L^2, \tag{5.2.13}$$

$$-E_i I_i (w_i'' + \frac{\alpha_i}{2R_i} T^{m,i})'' = \rho_i A_i \tilde{\eta}_i \in L^2, \tag{5.2.14}$$

$$k_a^i \tilde{T}^{i''} - \rho_i c_i \mu_i \tilde{T}^i - \alpha_i E_i T_0^i v_i' = \rho_i c_i \tilde{r}_i \in L^2, \tag{5.2.15}$$

$$k_a^i T^{m,i''} - \rho_i c_i \gamma_i T^{m,i} + \frac{\alpha_i E_i I_i T_0^i}{2R_i A_i} \eta_i'' = \rho_i c_i \tilde{\sigma}_i \in L^2 \tag{5.2.16}$$

$$\mathbf{M}^{-1} \mathbf{C}(F_1, N_1, F_2, N_2, M_1, M_2)^T = (\tilde{p}, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2)^T \in \mathbb{R}^4. \tag{5.2.17}$$

Sturm-Liouville theory [19] tells us that we can find unique solutions  $\tilde{T}^i$  and  $T^{m,i}$  to equations (5.2.15) and (5.2.16), respectively. Also, we have the existence of constants  $k_1, k_2$  such that

$$\begin{aligned} \|\tilde{T}^i\| &\leq k_1 (\|\tilde{u}_i'\| + \|\tilde{r}_i\|) \\ \|T^{m,i}\| &\leq k_2 (\|\tilde{w}_i''\| + \|\tilde{\sigma}_i\|). \end{aligned} \tag{5.2.18}$$

The solutions to (5.2.13) and (5.2.14), which also satisfy  $u_i(0) = 0$ ,  $w_i(0) = 0$ , and

$w'_i(0) = 0$ , have the form

$$u_i(s_i) = \frac{F_i}{E_i A_i} s_i + \alpha_i \int_0^{s_i} \tilde{T}^i(\tau) d\tau - \frac{1}{E_i A_i} \int_0^{s_i} \int_{\tau_2}^{L_i} \rho_i A_i \tilde{v}_i(\tau_1) d\tau_1 d\tau_2, \quad (5.2.19)$$

and

$$\begin{aligned} w_i(s_i) = & -\frac{N_i}{6E_i I_i} (3L_i s_i^2 - s_i^3) + \frac{M_i}{2E_i I_i} s_i^2 - \frac{\alpha_i}{2R_i} \int_0^{s_i} \int_0^{\tau_2} T^{m,i}(\tau_1) d\tau_1 d\tau_2 \\ & - \frac{1}{E_i I_i} \int_0^{s_i} \int_0^{\tau_4} \int_{\tau_3}^{L_i} \int_{\tau_2}^{L_i} \rho_i A_i \tilde{\eta}_i(\tau_1) d\tau_1 d\tau_2 d\tau_3 d\tau_4, \end{aligned} \quad (5.2.20)$$

respectively. Note that  $u_i$  and  $w_i$  satisfy

$$u_i \in H_r^1 \cap H^2, \quad w_i \in H_r^2 \cap H^4. \quad (5.2.21)$$

It is now necessary to prove that  $(F_i, N_i, M_i)$  are uniquely defined. We do this by following the same reasoning as given in Chapters 3 and 4. Letting  $s_i = L_i$  in (5.2.19) and (5.2.20), we see that

$$\begin{bmatrix} -u_1(L_1) \\ w_1(L_1) \\ -u_2(L_2) \\ w_2(L_2) \\ -w'_1(L_1) \\ -w'_2(L_2) \end{bmatrix} = \mathbf{B} \begin{bmatrix} F_1 \\ N_1 \\ F_2 \\ N_2 \\ M_1 \\ M_2 \end{bmatrix} + \tilde{\mathbf{F}} \quad (5.2.22)$$

where

$$\tilde{\mathbf{F}} = \begin{bmatrix} -\alpha_1 \int_0^{L_1} \tilde{T}^1(\tau) d\tau + \frac{1}{E_1 A_1} \int_0^{L_1} \int_{\tau_2}^{L_1} \rho_1 A_1 \tilde{v}_1(\tau_1) d\tau_1 d\tau_2 \\ -\frac{\alpha_1}{2R_1} \int_0^{L_1} \int_0^{\tau_2} T^{m,1}(\tau_1) d\tau_1 d\tau_2 - \frac{1}{E_1 I_1} \int_0^{L_1} \int_0^{\tau_4} \int_{\tau_3}^{L_1} \int_{\tau_2}^{L_1} \rho_1 A_1 \tilde{\eta}_1(\tau_1) d\tau_1 d\tau_2 d\tau_3 d\tau_4 \\ -\alpha_2 \int_0^{L_2} \tilde{T}^2(\tau) d\tau + \frac{1}{E_2 A_2} \int_0^{L_2} \int_{\tau_2}^{L_2} \rho_2 A_2 \tilde{v}_2(\tau_1) d\tau_1 d\tau_2 \\ -\frac{\alpha_2}{2R_2} \int_0^{L_2} \int_0^{\tau_2} T^{m,2}(\tau_1) d\tau_1 d\tau_2 - \frac{1}{E_2 I_2} \int_0^{L_2} \int_0^{\tau_4} \int_{\tau_3}^{L_2} \int_{\tau_2}^{L_2} \rho_2 A_2 \tilde{\eta}_2(\tau_1) d\tau_1 d\tau_2 d\tau_3 d\tau_4 \\ \frac{\alpha_1}{2R_1} \int_0^{L_1} T^{m,1}(\tau_1) d\tau_1 + \frac{1}{E_1 I_1} \int_0^{L_1} \int_{\tau_3}^{L_1} \int_{\tau_2}^{L_1} \rho_1 A_1 \tilde{\eta}_1(\tau_1) d\tau_1 d\tau_2 d\tau_3 \\ \frac{\alpha_2}{2R_2} \int_0^{L_2} T^{m,2}(\tau_1) d\tau_1 + \frac{1}{E_2 I_2} \int_0^{L_2} \int_{\tau_3}^{L_2} \int_{\tau_2}^{L_2} \rho_2 A_2 \tilde{\eta}_2(\tau_1) d\tau_1 d\tau_2 d\tau_3 \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} -\frac{L_1}{E_1 A_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{L_1^3}{3E_1 I_1} & 0 & 0 & \frac{L_1^2}{2E_1 I_1} & 0 \\ 0 & 0 & -\frac{L_2}{E_2 A_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{L_2^3}{3E_2 I_2} & 0 & \frac{L_2^2}{2E_2 I_2} \\ 0 & \frac{L_1^2}{2E_1 I_1} & 0 & 0 & -\frac{L_1}{E_1 I_1} & 0 \\ 0 & 0 & 0 & \frac{L_2^2}{2E_2 I_2} & 0 & -\frac{L_2}{E_2 I_2} \end{bmatrix}.$$

After proper row and column exchanging, we find that

$$\det \mathbf{B} = \frac{(L_1 L_2)^5}{144(E_1 E_2)^3 (I_1 I_2)^2 (A_1 A_2)}.$$

Hence  $\mathbf{B}$  is invertible and we can repeat the exact argument given for the Kelvin-Voigt system to solve for  $(F_i, N_i, M_i)$  uniquely in terms of  $(\tilde{p}, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2)$ . Similarly, from (5.2.11)-(5.2.12) and the geometric compatibility conditions, we can get  $(p, q, \omega_1, \omega_2)$  uniquely as a linear combination of  $(\tilde{u}_i(L_i), \tilde{w}_i(L_i), \tilde{w}'_i(L_i))$ .

We therefore have a unique solution  $z \in \mathcal{D}(\mathcal{A})$  of equation (5.2.12), which satisfies

$$\begin{cases} \|u'_i\| + \|w''_i\| \leq K_1 (\|\tilde{u}'_i\| + \|\tilde{w}'_i\| + \|\tilde{v}_i\| + \|\tilde{\eta}_i\| + \|(\tilde{p}, \tilde{q}, \tilde{\omega}_1, \tilde{\omega}_2)\|_{\mathbb{R}^4}) \\ \|(p, q, \omega_1, \omega_2)\|_{\mathbb{R}^4} \leq K_2 (\|\tilde{u}'_i\| + \|\tilde{w}'_i\|) \\ \|v_i\| + \|\eta_i\| \leq K_3 (\|\tilde{u}'_i\| + \|\tilde{w}'_i\|), \\ \|\tilde{T}^i\| + \|T^{m,i}\| \leq K_4 (\|\tilde{u}'_i\| + \|\tilde{w}'_i\| + \|\tilde{r}_i\| + \|\tilde{\sigma}_i\|) \end{cases}$$

for some constants  $K_1, K_2, K_3, K_4 > 0$ . This leads to

$$\|z\|_{\mathcal{H}} \leq K \|\mathcal{A}z\|_{\mathcal{H}}$$

for some constant  $K$  independent of  $z$ . Therefore,  $0 \in \rho(\mathcal{A})$ ,  $\mathcal{A}^{-1}$  is bounded and  $\mathcal{A}$  is closed. It follows that the range  $R(\lambda - \mathcal{A}) = \mathcal{H}$  for some  $\lambda > 0$ . By Theorem 4.6 in [18],  $\mathcal{D}(\mathcal{A})$  is dense in  $\mathcal{H}$ . An application of Theorem 1.2.1 completes the proof.

### 5.3 Exponential Stability

The following theorem is found, in a slightly modified form, in [5].

**Theorem 5.3.1** *The semigroup  $S(t)$ , generated by  $\mathcal{A}$ , is exponentially stable.*

Proof: Just as in the Kelvin-Voigt case, we verify the following conditions:

$$\mathbf{i}\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset, \quad (5.3.1)$$

$$\lim_{\beta \rightarrow \infty} \|(i\beta - \mathcal{A})^{-1}\| < \infty. \quad (5.3.2)$$

First, assume (5.3.1) is true and suppose (5.3.2) is false. This implies the existence of a sequence

$$z_n = (u_{1n}, w_{1n}, v_{1n}, \eta_{1n}, \tilde{T}_n^1, T_n^{m,1}, u_{2n}, w_{2n}, v_{2n}, \eta_{2n}, \tilde{T}_n^2, T_n^{m,2}, p_n, q_n, \omega_{1n}, \omega_{2n}) \in \mathcal{D}(\mathcal{A})$$

with  $\|z_n\|_{\mathcal{H}} = 1$ , and a sequence  $\beta_n \in \mathbb{R}$  with  $\beta_n \rightarrow \infty$  such that

$$\lim_{n \rightarrow \infty} \|(i\beta_n I - \mathcal{A})z_n\|_{\mathcal{H}} = 0. \quad (5.3.3)$$

Throughout the remainder of this proof, the  $n$  subscript will be suppressed for simplicity. Unless otherwise stated, the convergence results in this proof are understood to occur as  $n \rightarrow \infty$ . We will show that (5.3.3) implies  $\|z\|_{\mathcal{H}} \rightarrow 0$ , a contradiction.

By taking the inner product of both sides of (5.3.3) with  $z$ , we see that

$$\operatorname{Re}\langle (iI - \mathcal{A})z, z \rangle_{\mathcal{H}} = \langle \mathcal{A}z, z \rangle_{\mathcal{H}} \rightarrow 0. \quad (5.3.4)$$

Now from (5.2.9), this implies that

$$\|\tilde{T}^i\|_{H^1}, \|T^{m,i}\|_{H^1}, \|\tilde{T}^i(0)\|, \|T^{m,i}(0)\|, \|\tilde{T}^i(L_i)\|, \|T^{m,i}(L_i)\| \rightarrow 0. \quad (5.3.5)$$

Looking at the component version of (5.3.3), we have

$$\mathbf{i}\beta u_i - v_i \rightarrow 0 \text{ in } H_r^1(0, L_i), \quad (5.3.6)$$

$$\mathbf{i}\beta w_i - \eta_i \rightarrow 0 \text{ in } H_r^2(0, L_i), \quad (5.3.7)$$

$$\mathbf{i}\beta v_i - \left( \gamma_i^1 u_i' - \gamma_i^2 \tilde{T}^i \right)' \rightarrow 0 \text{ in } L^2(0, L_i), \quad (5.3.8)$$

$$\mathbf{i}\beta \eta_i - \left( \gamma_i^3 w_i'' + \gamma_i^4 T^{m,i} \right)'' \rightarrow 0 \text{ in } L^2(0, L_i), \quad (5.3.9)$$

$$\mathbf{i}\beta \tilde{T}^i - \gamma_i^5 (\tilde{T}^i)'' + \gamma_i^6 \tilde{T}^i + \gamma_i^7 v_i' \rightarrow 0 \text{ in } L^2(0, L_i), \quad (5.3.10)$$

$$\mathbf{i}\beta T^{m,i} - \gamma_i^8 (T^{m,i})'' + \gamma_i^9 T^{m,i} - \gamma_i^{10} \eta'' \rightarrow 0 \text{ in } L^2(0, L_i), \quad (5.3.11)$$

$$\mathbf{i}\beta(p, q, \omega_1, \omega_2)^T - \mathbf{M}^{-1} \mathbf{C}(F_1, N_1, F_2, N_2, M_1, M_2)^T \rightarrow 0 \quad (5.3.12)$$

where, for simplicity, the  $\gamma_i^j$  denote the appropriate coefficients. Notice that (5.3.5) allows us to rewrite (5.3.10) and (5.3.11) as

$$-\frac{\gamma_i^5}{\beta} (\tilde{T}^i)'' + \frac{\gamma_i^7}{\beta} v_i' \rightarrow 0 \text{ in } L^2(0, L_i) \quad (5.3.13)$$

and

$$-\frac{\gamma_i^8}{\beta} (T^{m,i})'' - \frac{\gamma_i^{10}}{\beta} \eta'' \rightarrow 0 \text{ in } L^2(0, L_i). \quad (5.3.14)$$

We now show that the mechanical components of the system converge to 0 in norm.

- **Part 1:**  $\|u_i'\| \rightarrow 0$

Replace  $v_i'$  in (5.3.13) by  $\mathbf{i}\beta u_i'$  and take the  $L^2$  inner product of both sides with  $u_i'$  to get

$$-\frac{\gamma_i^5}{\beta} \left( \tilde{T}^{i'}(L_i) \bar{u}_i'(L_i) - \tilde{T}^{i'}(0) \bar{u}_i'(0) - \langle \tilde{T}^{i'}, u_i'' \rangle \right) + \mathbf{i}\gamma_i^7 \|u_i'\|^2 \rightarrow 0. \quad (5.3.15)$$

From (5.3.8), we know that  $\| \frac{u_i''}{\beta} \|$  is bounded. Using this fact, along with the inequality

$$\frac{|u_i'(0)|}{\beta}, \frac{|u_i'(L_i)|}{\beta} \leq C \|u_i'\|^{\frac{1}{2}} \frac{\|u_i'\|_{H^1}^{\frac{1}{2}}}{\beta}, \quad (5.3.16)$$

we know  $|u_i'(s)|/\beta$  is bounded on the boundary. Therefore, since  $\tilde{T}^i$  satisfies the Robin boundary conditions, (5.3.5) tells us that (5.3.15) reduces to

$$\mathbf{i}\gamma_i^7 \|u_i'\|^2 \rightarrow 0. \quad (5.3.17)$$

Hence, we have

$$u_i \rightarrow 0 \text{ in } H_r^1. \quad (5.3.18)$$

- **Part 2:**  $\|v_i\| \rightarrow 0$

Take the  $L^2$  inner product of (5.3.8) with  $u_i$ , and then replace  $i\beta u_i$  by  $v_i$ . Since  $\tilde{T}^{i'} \rightarrow 0$  and  $u_i' \rightarrow 0$ , this gives us

$$\|v_i\|^2 - \gamma_i^1 \left( u_i'(L_i) \bar{u}_i(L_i) - u_i'(0) \bar{u}_i(0) \right) \rightarrow 0. \quad (5.3.19)$$

Now, we can use the inequality

$$|u_i'(s) \bar{u}_i(s)| \leq C \|u_i'\|^{\frac{1}{2}} \left( \frac{\|u_i'\|_{H^1}}{\beta} \right)^{\frac{1}{2}} \|u_i'\|^{\frac{1}{2}} \|\beta u_i\|^{\frac{1}{2}} \rightarrow 0 \text{ for } s = 0, L_i \quad (5.3.20)$$

to see that  $\|v_i\| \rightarrow 0$ .

- **Part 3:**  $\|w_i''\| \rightarrow 0$

Replace  $\eta_i''$  in (5.3.14) by  $i\beta w_i''$  and take the  $L^2$  inner product of both sides with  $w_i''$  to get

$$-i\gamma_i^{10} \|w_i''\|^2 - \frac{\gamma_i^8}{\beta} \left( (T^{m,i})'(L_i) \bar{w}_i''(L_i) - (T^{m,i})'(0) \bar{w}_i''(0) - \langle (T^{m,i})', w_i''' \rangle \right) \rightarrow 0. \quad (5.3.21)$$

From (5.3.7) and (5.3.14), we know that  $\frac{(T^{m,i})''}{\beta}$  is bounded. Therefore, from (5.3.9) we have that  $\|w_i^{(4)}/\beta\|$  is bounded. By using the Gagliardo-Nirenberg interpolation inequalities, we can deduce that  $\|w_i''\|_{H^1}/\beta^{1/2}$  is also bounded. Thus, using (5.3.5), we have

$$\frac{\gamma_i^8}{\beta} \langle (T^{m,i})', w_i''' \rangle \rightarrow 0. \quad (5.3.22)$$

Now, use the inequality

$$\frac{|w_i''(s)|}{\beta} \leq C \|w_i''\|^{\frac{1}{2}} \frac{\|w_i''\|_{H^1}^{\frac{1}{2}}}{\beta} \text{ for } s = 0, L_i \quad (5.3.23)$$

to get the boundedness of  $|w_i''(s)|/\beta$  on the boundary. Using the Robin boundary conditions, this implies that

$$\frac{\gamma_i^8}{\beta} \left( (T^{m,i})'(L_i)w_i''(L_i) - (T^{m,i})'(0)w_i''(0) \right) \rightarrow 0. \quad (5.3.24)$$

Plugging (5.3.22) and (5.3.24) into (5.3.21), we have

$$\|w_i''\| \rightarrow 0. \quad (5.3.25)$$

• **Part 4:**  $\|\eta_i\| \rightarrow 0$

We know that  $\|\beta w_i\|$  is bounded from (5.3.7). We also know that  $\|(T^{m,i})''\|/\beta$  converges to zero from (5.3.7), (5.3.14), and (5.3.25). Thus, we have

$$\langle (T^{m,i})'', w_i \rangle \rightarrow 0. \quad (5.3.26)$$

Now, take the  $L^2$  inner product of (5.3.9) with  $w_i$ , and replace  $i\beta w_i$  by  $\eta_i$ . Using (5.3.26), this gives

$$\|\eta''\|^2 - \gamma_i^3 \left( w_i'''(L_i)\bar{w}(L_i) - w_i''(L_i)\bar{w}'_i(L_i) + \langle w_i'', w_i'' \rangle \right) \rightarrow 0. \quad (5.3.27)$$

Regarding the boundary terms, we can once again use the Gagliardo-Nirenberg interpolation inequalities to get

$$\|\beta^{\frac{1}{2}}w_i'\| \leq C\|w_i''\|^{\frac{1}{2}}\|\beta w_i\|^{\frac{1}{2}} \rightarrow 0. \quad (5.3.28)$$

Using (5.3.28), we have

$$|w_i'''(L_i)\bar{w}_i(L_i)| \leq C \left( \frac{\|w_i'''\|}{\beta^{\frac{1}{2}}} \right)^{\frac{1}{2}} \left( \frac{\|w_i'''\|_{H^1}}{\beta} \right)^{\frac{1}{2}} \|\beta w_i\|^{\frac{1}{2}} \|\beta^{\frac{1}{2}}w_i'\|^{\frac{1}{2}} \rightarrow 0, \quad (5.3.29)$$

and

$$|w_i''(L_i)\bar{w}'_i(L_i)| \leq C\|w_i''\|^{\frac{1}{2}} \left( \frac{\|w_i''\|}{\beta^{\frac{1}{2}}} \right)^{\frac{1}{2}} \|\beta^{\frac{1}{2}}w_i'\|^{\frac{1}{2}} \|w_i'\|_{H^1}^{\frac{1}{2}} \rightarrow 0. \quad (5.3.30)$$

Plugging (5.3.25), (5.3.29), and (5.3.30) into (5.3.27), we get

$$\|\eta_i\| \rightarrow 0. \quad (5.3.31)$$

- **Part 5:**  $(p, q, \omega_1, \omega_2) \rightarrow 0$

Using interpolation inequalities and the arguments in Part 3, we have

$$\begin{aligned} \frac{|w_i'''(L_i)|}{\beta} &\leq \frac{C}{\beta} \|w_i'''\|^{1/2} \|w_i'''\|_{H^1}^{1/2} \\ &\leq C \|w_i'''\|^{1/4} \left( \frac{\|w_i'''\|_{H^2}^{3/4}}{\beta} \right) \rightarrow 0. \end{aligned} \quad (5.3.32)$$

The Robin boundary conditions, together with (5.3.5) and (5.3.32), imply that

$$\frac{N_i}{\beta} = \frac{\gamma_i^3 w_i'''(L_i)}{\beta} + \frac{\gamma_i^4 (T^{m,i})'(L_i)}{\beta} \rightarrow 0. \quad (5.3.33)$$

Since  $\|w_i'''\| \rightarrow 0$ , we see from (5.3.23) that

$$\frac{|w_i''(L_i)|}{\beta} \rightarrow 0. \quad (5.3.34)$$

Therefore,

$$\frac{M_i}{\beta} = \frac{\gamma_i^3 w_i''(L_i)}{\beta} + \frac{\gamma_i^4 T^{m,i}(L_i)}{\beta} \rightarrow 0. \quad (5.3.35)$$

Now, since  $\|u_i'''\| \rightarrow 0$ , (5.3.16) tells us that

$$\frac{|u_i'(L_i)|}{\beta} \rightarrow 0, \quad (5.3.36)$$

and hence

$$\frac{F_i}{\beta} = \frac{\gamma_i^1 u_i'(L_i)}{\beta} - \frac{\gamma_i^2 \tilde{T}^i(L_i)}{\beta} \rightarrow 0. \quad (5.3.37)$$

Combining (5.3.33), (5.3.33), (5.3.35), and (5.3.12), we see that

$$(p, q, \omega_1, \omega_2) \rightarrow 0. \quad (5.3.38)$$

Putting together Parts 1 through 5, we have  $\|z\|_{\mathcal{H}} \rightarrow 0$ , a contradiction. Thus, if condition (5.3.1) is true, then condition (5.3.2) is also true.

If we assume condition (5.3.1) is false, then there exists a real sequence  $\beta_n \rightarrow \beta \neq 0$  and a sequence  $z_n \in D(\mathcal{A})$  with  $\|z_n\|_{\mathcal{H}} = 1$  such that

$$\lim_{n \rightarrow \infty} \|(i\beta_n I - \mathcal{A})z_n\|_{\mathcal{H}} = 0. \quad (5.3.39)$$

Now, since the only assumption used above in proving condition (5.3.2) was the fact that  $\beta_n$  was bounded away from 0, an exact repetition of the arguments suffices to prove condition (5.3.1). Therefore, the semigroup  $S(t)$  generated by  $\mathcal{A}$  is exponentially stable, and the theorem is proven.

# Chapter 6

## Triple Joint System

### 6.1 Introduction

In this chapter, we will consider a triangular configuration of three beams connected by three joints. Such a system is a common subcomponent of space trusses. Kelvin-Voigt damping is assumed to be present in each of the three beams. Our model also supposes the presence of two viscously damped springs at each joint which restrain planar motions. These springs serve to counteract rigid body motions. In an actual space truss, this function is performed by the flexural rigidity of other beams outside the triangular system. The rotations of the legs about the joint masses are constrained, with respect to each other, by viscously damped springs. Torsional springs are assumed to affect the averaged rotations of the two legs at each joint.

#### 6.1.1 Equations

The equations of motion for a triangular joint-beam system with Kelvin-Voigt type of damping and viscous springs at the joints are given by the following:

$$\rho_i A_i \frac{\partial^2 u_i(t, s_i)}{\partial t^2} = \frac{\partial}{\partial s_i} \left[ E_i A_i \frac{\partial u_i(t, s_i)}{\partial s_i} + \mu_i \frac{\partial^2 u_i(t, s_i)}{\partial s_i \partial t} \right], \quad (6.1.1)$$

$$\rho_i A_i \frac{\partial^2 w_i(t, s_i)}{\partial t^2} = -\frac{\partial^2}{\partial s_i^2} \left[ E_i I_i \frac{\partial^2 w_i(t, s_i)}{\partial s_i^2} + \gamma_i \frac{\partial^3 w_i(t, s_i)}{\partial s_i^2 \partial t} \right], \quad (6.1.2)$$

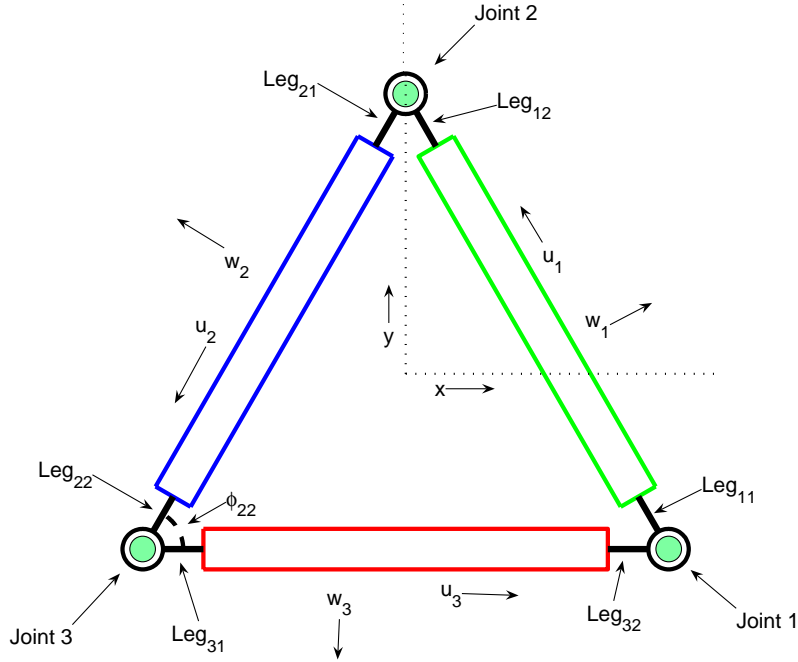


Figure 6.1.1: The Triangular Joint-Beam System

$$\mathbf{M}_1 \frac{d^2}{dt^2} \begin{bmatrix} x_1(t) \\ y_1(t) \\ \theta_{11}(t) \\ \theta_{32}(t) \end{bmatrix} = \mathbf{C}_1 \begin{bmatrix} \hat{F}_{11}(t) \\ \hat{N}_{11}(t) \\ \hat{F}_{32}(t) \\ \hat{N}_{32}(t) \\ \hat{M}_{11}(t) \\ \hat{M}_{32}(t) \end{bmatrix} + \mathbf{K}_1 \begin{bmatrix} x_1(t) \\ y_1(t) \\ \theta_{11}(t) \\ \theta_{32}(t) \end{bmatrix} + \mathbf{D}_1 \frac{d}{dt} \begin{bmatrix} x_1(t) \\ y_1(t) \\ \theta_{11}(t) \\ \theta_{32}(t) \end{bmatrix}, \quad (6.1.3)$$

$$\mathbf{M}_2 \frac{d^2}{dt^2} \begin{bmatrix} x_2(t) \\ y_2(t) \\ \theta_{12}(t) \\ \theta_{21}(t) \end{bmatrix} = \mathbf{C}_2 \begin{bmatrix} \hat{F}_{12}(t) \\ \hat{N}_{12}(t) \\ \hat{F}_{21}(t) \\ \hat{N}_{21}(t) \\ \hat{M}_{12}(t) \\ \hat{M}_{21}(t) \end{bmatrix} + \mathbf{K}_2 \begin{bmatrix} x_2(t) \\ y_2(t) \\ \theta_{12}(t) \\ \theta_{21}(t) \end{bmatrix} + \mathbf{D}_2 \frac{d}{dt} \begin{bmatrix} x_2(t) \\ y_2(t) \\ \theta_{12}(t) \\ \theta_{21}(t) \end{bmatrix}, \quad (6.1.4)$$

$$\mathbf{M}_3 \frac{d^2}{dt^2} \begin{bmatrix} x_3(t) \\ y_3(t) \\ \theta_{22}(t) \\ \theta_{31}(t) \end{bmatrix} = \mathbf{C}_3 \begin{bmatrix} \hat{F}_{22}(t) \\ \hat{N}_{22}(t) \\ \hat{F}_{31}(t) \\ \hat{N}_{31}(t) \\ \hat{M}_{22}(t) \\ \hat{M}_{31}(t) \end{bmatrix} + \mathbf{K}_3 \begin{bmatrix} x_3(t) \\ y_3(t) \\ \theta_{22}(t) \\ \theta_{31}(t) \end{bmatrix} + \mathbf{D}_3 \frac{d}{dt} \begin{bmatrix} x_3(t) \\ y_3(t) \\ \theta_{22}(t) \\ \theta_{31}(t) \end{bmatrix}, \quad (6.1.5)$$

where

$$\mathbf{M}_1 = \begin{bmatrix} m_1 & 0 & -m_{11}d_{11} \sin \phi_{11} & -m_{32}d_{32} \sin \phi_{32} \\ 0 & m_1 & m_{11}d_{11} \cos \phi_{11} & m_{32}d_{32} \cos \phi_{32} \\ -m_{11}d_{11} \sin \phi_{11} & m_{11}d_{11} \cos \phi_{11} & I_Q^{11} & 0 \\ -m_{32}d_{32} \sin \phi_{32} & m_{32}d_{32} \cos \phi_{32} & 0 & I_Q^{32} \end{bmatrix}, \quad (6.1.6)$$

$$\mathbf{C}_1 = \begin{bmatrix} \cos \phi_{11} & -\sin \phi_{11} & \cos \phi_{32} & -\sin \phi_{32} & 0 & 0 \\ \sin \phi_{11} & \cos \phi_{11} & \sin \phi_{32} & \cos \phi_{32} & 0 & 0 \\ 0 & l_{11} & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & l_{32} & 0 & 1 \end{bmatrix}, \quad (6.1.7)$$

$$\mathbf{K}_1 = \begin{bmatrix} -k_{11} & 0 & 0 & 0 \\ 0 & -k_{12} & 0 & 0 \\ 0 & 0 & -k_1 - \frac{d_1}{2} & k_1 - \frac{d_1}{2} \\ 0 & 0 & k_1 - \frac{d_1}{2} & -k_1 - \frac{d_1}{2} \end{bmatrix}, \quad \mathbf{D}_1 = \begin{bmatrix} -\beta_{11} & 0 & 0 & 0 \\ 0 & -\beta_{12} & 0 & 0 \\ 0 & 0 & -\beta_1 & \beta_1 \\ 0 & 0 & \beta_1 & -\beta_1 \end{bmatrix}, \quad (6.1.8)$$

$$\mathbf{M}_2 = \begin{bmatrix} m_2 & 0 & -m_{12}d_{12} \sin \phi_{12} & -m_{21}d_{21} \sin \phi_{21} \\ 0 & m_2 & m_{12}d_{12} \cos \phi_{12} & m_{21}d_{21} \cos \phi_{21} \\ -m_{12}d_{12} \sin \phi_{12} & m_{12}d_{12} \cos \phi_{12} & I_Q^{12} & 0 \\ -m_{21}d_{21} \sin \phi_{21} & m_{21}d_{21} \cos \phi_{21} & 0 & I_Q^{21} \end{bmatrix}, \quad (6.1.9)$$

$$\mathbf{C}_2 = \begin{bmatrix} \cos \phi_{12} & -\sin \phi_{12} & \cos \phi_{21} & -\sin \phi_{21} & 0 & 0 \\ \sin \phi_{12} & \cos \phi_{12} & \sin \phi_{21} & \cos \phi_{21} & 0 & 0 \\ 0 & l_{12} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & l_{21} & 0 & -1 \end{bmatrix}, \quad (6.1.10)$$

$$\mathbf{K}_2 = \begin{bmatrix} -k_{21} & 0 & 0 & 0 \\ 0 & -k_{22} & 0 & 0 \\ 0 & 0 & -k_2 - \frac{d_2}{2} & k_2 - \frac{d_2}{2} \\ 0 & 0 & k_2 - \frac{d_2}{2} & -k_2 - \frac{d_2}{2} \end{bmatrix}, \quad \mathbf{D}_2 = \begin{bmatrix} -\beta_{21} & 0 & 0 & 0 \\ 0 & -\beta_{22} & 0 & 0 \\ 0 & 0 & -\beta_2 & \beta_2 \\ 0 & 0 & \beta_2 & -\beta_2 \end{bmatrix}, \quad (6.1.11)$$

$$\mathbf{M}_3 = \begin{bmatrix} m_3 & 0 & -m_{22}d_{22} \sin \phi_{22} & -m_{31}d_{31} \sin \phi_{31} \\ 0 & m_3 & m_{22}d_{22} \cos \phi_{22} & m_{31}d_{31} \cos \phi_{31} \\ -m_{22}d_{22} \sin \phi_{22} & m_{22}d_{22} \cos \phi_{22} & I_Q^{22} & 0 \\ -m_{31}d_{31} \sin \phi_{31} & m_{31}d_{31} \cos \phi_{31} & 0 & I_Q^{31} \end{bmatrix}, \quad (6.1.12)$$

$$\mathbf{C}_3 = \begin{bmatrix} \cos \phi_{22} & -\sin \phi_{22} & \cos \phi_{31} & -\sin \phi_{31} & 0 & 0 \\ \sin \phi_{22} & \cos \phi_{22} & \sin \phi_{31} & \cos \phi_{31} & 0 & 0 \\ 0 & l_{22} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & l_{31} & 0 & -1 \end{bmatrix}, \quad (6.1.13)$$

and

$$\mathbf{K}_3 = \begin{bmatrix} -k_{31} & 0 & 0 & 0 \\ 0 & -k_{32} & 0 & 0 \\ 0 & 0 & -k_3 - \frac{d_3}{2} & k_3 - \frac{d_3}{2} \\ 0 & 0 & k_3 - \frac{d_3}{2} & -k_3 - \frac{d_3}{2} \end{bmatrix}, \quad \mathbf{D}_3 = \begin{bmatrix} -\beta_{31} & 0 & 0 & 0 \\ 0 & -\beta_{32} & 0 & 0 \\ 0 & 0 & -\beta_3 & \beta_3 \\ 0 & 0 & \beta_3 & -\beta_3 \end{bmatrix}. \quad (6.1.14)$$

## 6.1.2 Compatibility Conditions

At each of the joints, we also have geometric compatibility conditions. These assure continuity of the structure, as well as smoothness at the beam-leg attachments.

### 6.1.3 First Joint

$$\begin{aligned}
 -x_1 &= -u_1(0) \cos \phi_{11} - w_1(0) \sin \phi_{11} + w'_1(0)l_{11} \sin \phi_{11} \\
 -y_1 &= -u_1(0) \sin \phi_{11} + w_1(0) \cos \phi_{11} - w'_1(0)l_{11} \cos \phi_{11} \\
 w'_1(0) &= -\theta_{11}
 \end{aligned} \tag{6.1.15}$$

$$\begin{aligned}
 -x_1 &= u_3(L_3) \cos \phi_{32} + w_3(L_3) \sin \phi_{32} + w'_3(L_3)l_{32} \sin \phi_{32} \\
 -y_1 &= u_3(L_3) \sin \phi_{32} - w_3(L_3) \cos \phi_{32} - w'_3(L_3)l_{32} \cos \phi_{32} \\
 w'_3(L_3) &= -\theta_{32}
 \end{aligned} \tag{6.1.16}$$

which is equivalent to

$$\begin{bmatrix} u_1(t, 0) \\ -w_1(t, 0) \\ -u_3(t, L_3) \\ w_3(t, L_3) \\ w'_1(t, 0) \\ -w'_3(t, L_3) \end{bmatrix} = \mathbf{C}_1^T \begin{bmatrix} x_1(t) \\ y_1(t) \\ \theta_{11}(t) \\ \theta_{32}(t) \end{bmatrix}. \tag{6.1.17}$$

### 6.1.4 Second Joint

$$\begin{aligned}
 -x_2 &= u_1(L_1) \cos \phi_{12} + w_1(L_1) \sin \phi_{12} + w'_1(L_1)l_{12} \sin \phi_{12} \\
 -y_2 &= u_1(L_1) \sin \phi_{12} - w_1(L_1) \cos \phi_{12} - w'_1(L_1)l_{12} \cos \phi_{12} \\
 w'_1(L_1) &= -\theta_{12}
 \end{aligned} \tag{6.1.18}$$

$$\begin{aligned}
-x_2 &= -u_2(0) \cos \phi_{21} - w_2(0) \sin \phi_{21} + w'_2(0)l_{21} \sin \phi_{21} \\
-y_2 &= -u_2(0) \sin \phi_{21} + w_2(0) \cos \phi_{21} - w'_2(0)l_{21} \cos \phi_{21} \\
w'_2(0) &= -\theta_{21}
\end{aligned} \tag{6.1.19}$$

which is equivalent to

$$\begin{bmatrix} -u_1(t, L_1) \\ w_1(t, L_1) \\ u_2(t, 0) \\ -w_2(t, 0) \\ -w'_1(t, L_1) \\ w'_2(t, 0) \end{bmatrix} = \mathbf{C}_2^T \begin{bmatrix} x_2(t) \\ y_2(t) \\ \theta_{12}(t) \\ \theta_{21}(t) \end{bmatrix}. \tag{6.1.20}$$

### 6.1.5 Third Joint

$$\begin{aligned}
-x_3 &= u_2(L_2) \cos \phi_{22} + w_2(L_2) \sin \phi_{22} + w'_2(L_2)l_{22} \sin \phi_{22} \\
-y_3 &= u_2(L_2) \sin \phi_{22} - w_2(L_2) \cos \phi_{22} - w'_2(L_2)l_{22} \cos \phi_{22} \\
w'_2(L_2) &= -\theta_{22}
\end{aligned} \tag{6.1.21}$$

$$\begin{aligned}
-x_3 &= -u_3(0) \cos \phi_{31} - w_3(0) \sin \phi_{31} + w'_3(0)l_{31} \sin \phi_{31} \\
-y_3 &= -u_3(0) \sin \phi_{31} + w_3(0) \cos \phi_{31} - w'_3(0)l_{31} \cos \phi_{31} \\
w'_3(0) &= -\theta_{31}
\end{aligned} \tag{6.1.22}$$

which is equivalent to

$$\begin{bmatrix} -u_2(t, L_2) \\ w_2(t, L_2) \\ u_3(t, 0) \\ -w_3(t, 0) \\ -w'_2(t, L_2) \\ w'_3(t, 0) \end{bmatrix} = \mathbf{C}_3^T \begin{bmatrix} x_3(t) \\ y_3(t) \\ \theta_{22}(t) \\ \theta_{31}(t) \end{bmatrix}. \quad (6.1.23)$$

### 6.1.6 Notations

The notations used in the above system are listed below:

- $u_i, w_i$  - longitudinal and transversal displacement of the beam  $i$
- $x_i, y_i$  - horizontal and vertical displacement of joint  $i$
- $\theta_{ij}$  - perturbed rotation angle of leg  $ij$  with respect to the positive  $x$  axis
- $\rho_i, A_i, L_i, E_i, I_i$  - mass density, cross section area, length, Young's modulus, moment of inertia of the beam  $i$
- $\mu_i, \gamma_i$  - damping coefficients
- $m_{ij}, d_{ij}, \ell_{ij}, I_Q^{ij}$  - mass, center of mass, length, moment of inertia of leg  $ij$
- $m_i$  - mass of joint-leg combination at joint  $i$
- $\phi_{ij}$  - nominal angle of leg  $ij$  with respect to the positive  $x$  axis
- $k_{ij}, \beta_{ij}$  - spring constants and damping coefficients for the linear springs affecting planar motions of joint  $i$
- $k_i, \beta_i$  - spring constant and damping coefficient for the spring describing the internal torque exerted by the two legs upon each other at joint  $i$
- $d_i$  - spring constant of the torsional spring affecting averaged rotations of the two legs at joint  $i$

At the beam-leg interfaces, we denote the bending moment, shear, and extensional forces as

$$\begin{aligned}
\left[ E_i I_i \frac{\partial^2 w_i(t)}{\partial s_i^2} + \gamma_i \frac{\partial^3 w_i(t)}{\partial s_i^2 \partial t} \right] (0) &= M_{i1}(t) \\
\left[ E_i I_i \frac{\partial^2 w_i(t)}{\partial s_i^2} + \gamma_i \frac{\partial^3 w_i(t)}{\partial s_i^2 \partial t} \right] (L_i) &= M_{i2}(t) \\
\left[ E_i I_i \frac{\partial^2 w_i(t)}{\partial s_i^2} + \gamma_i \frac{\partial^3 w_i(t)}{\partial s_i^2 \partial t} \right]' (0) &= N_{i1}(t) \\
\left[ E_i I_i \frac{\partial^2 w_i(t)}{\partial s_i^2} + \gamma_i \frac{\partial^3 w_i(t)}{\partial s_i^2 \partial t} \right]' (L_i) &= N_{i2}(t) \\
\left[ E_i A_i \frac{\partial u_i(t)}{\partial s_i} + \mu_i \frac{\partial^2 u_i(t)}{\partial s_i \partial t} \right] (0) &= F_{i1}(t) \\
\left[ E_i A_i \frac{\partial u_i(t)}{\partial s_i} + \mu_i \frac{\partial^2 u_i(t)}{\partial s_i \partial t} \right] (L_i) &= F_{i2}(t).
\end{aligned}$$

Once again, our goal is to place this system in a state space setting and establish well-posedness. The procedure is similar to that given in the previous chapters, though the presence of three joints adds some variety to the analysis. Once well-posedness is proven, we will examine the asymptotic behavior and regularity of the system.

## 6.2 Semigroup Setting

Let

$$\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \times \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3 \quad (6.2.1)$$

where

$$\mathcal{H}_i = H^1(0, L_i) \times H^2(0, L_i) \times [L^2(0, L_i)]^2, \quad (6.2.2)$$

and

$$\mathcal{G}_i = \mathbb{R}^4 \times \mathbb{R}^4. \quad (6.2.3)$$

We shall use prime to denote the spatial derivative in the following. We define the inner product in these spaces by

$$\langle (u, w, v, \eta), (f, g, h, k) \rangle_{\mathcal{H}_i} = E_i A_i \langle u', f' \rangle + E_i I_i \langle w'', g'' \rangle + \rho_i A_i (\langle v, h \rangle + \langle \eta, k \rangle), \quad (6.2.4)$$

$$\langle (\alpha, \beta), (\gamma, \delta) \rangle_{\mathcal{G}_i} = \alpha \tilde{\mathbf{K}}_i \gamma^T + \beta \mathbf{M}_i \delta^T, \quad (6.2.5)$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual  $L^2$  inner product and

$$\tilde{\mathbf{K}}_i = -\mathbf{K}. \quad (6.2.6)$$

It can be verified that  $\mathbf{M}$  and  $\tilde{\mathbf{K}}_i$  are strictly positive definite. Thus the inner product on  $\mathcal{G}_i$  is well defined.

We introduce the variables

$$v_i = \frac{\partial u_i}{\partial t}, \quad \eta_i = \frac{\partial w_i}{\partial t}, \quad p_i = \frac{dx_i}{dt}, \quad q_i = \frac{dy_i}{dt}, \quad \omega_{ij} = \frac{d\theta_{ij}}{dt},$$

and

$$z = (u_1, w_1, v_1, \eta_1, u_2, w_2, v_2, \eta_2, u_3, w_3, v_3, \eta_3, a_1, b_1, a_2, b_2, a_3, b_3)^T,$$

where

$$a_1 = (x_1, y_1, \theta_{11}, \theta_{32}) \quad (6.2.7)$$

$$b_1 = (p_1, q_1, \omega_{11}, \omega_{32}) \quad (6.2.8)$$

$$a_2 = (x_2, y_2, \theta_{12}, \theta_{21}) \quad (6.2.9)$$

$$b_2 = (p_2, q_2, \omega_{12}, \omega_{21}) \quad (6.2.10)$$

$$a_3 = (x_3, y_3, \theta_{22}, \theta_{31}) \quad (6.2.11)$$

$$b_3 = (p_3, q_3, \omega_{22}, \omega_{31}). \quad (6.2.12)$$

The Hilbert space  $\mathcal{H}$  is equipped with the norm induced from the inner products (6.2.4) - (6.2.5), i.e.,

$$\|z\|_{\mathcal{H}}^2 = 2E(t) = \sum_{i=1}^2 [E_i A_i \|u'_i\|^2 + E_i I_i \|w''_i\|^2 + \rho_i A_i (\|v_i\|^2 + \|\eta_i\|^2)] + a \mathbf{M} a^T \quad (6.2.13)$$

where  $a = (p, q, \omega_1, \omega_2)$  and  $\|\cdot\|$  denotes the usual  $L^2$  norm.

For the new set of state variables, the compatibility conditions become

$$\begin{bmatrix} v_1(t, 0) \\ -\eta_1(t, 0) \\ -v_3(t, L_3) \\ \eta_3(t, L_3) \\ \eta'_1(t, 0) \\ -\eta'_3(t, L_3) \end{bmatrix} = \mathbf{C}_1^T \begin{bmatrix} p_1(t) \\ q_1(t) \\ \omega_{11}(t) \\ \omega_{32}(t) \end{bmatrix}, \quad (6.2.14)$$

$$\begin{bmatrix} -v_1(t, L_1) \\ \eta_1(t, L_1) \\ v_2(t, 0) \\ -\eta_2(t, 0) \\ -\eta'_1(t, L_1) \\ \eta'_2(t, 0) \end{bmatrix} = \mathbf{C}_2^T \begin{bmatrix} p_2(t) \\ q_2(t) \\ \omega_{12}(t) \\ \omega_{21}(t) \end{bmatrix}, \quad (6.2.15)$$

$$\begin{bmatrix} -v_2(t, L_2) \\ \eta_2(t, L_2) \\ v_3(t, 0) \\ -\eta_3(t, 0) \\ -\eta'_2(t, L_2) \\ \eta'_3(t, 0) \end{bmatrix} = \mathbf{C}_1^T \begin{bmatrix} p_3(t) \\ q_3(t) \\ \omega_{22}(t) \\ \omega_{31}(t) \end{bmatrix}. \quad (6.2.16)$$

We now take our state space to be

$$\mathcal{H}_t = \{z \in \mathcal{H} \mid \text{compatibility conditions (6.1.17), (6.1.20), (6.1.23)}\}.$$

Let us again denote this space by  $\mathcal{H}$ . The triple joint system then can be written as the first order evolution equation

$$\frac{dz}{dt} = \mathcal{A}z$$

on the Hilbert space  $\mathcal{H}$  with

$$\mathcal{A}z = \left[ \begin{array}{c} v_1 \\ \eta_1 \\ \frac{1}{\rho_1 A_1} (E_1 A_1 u'_1 + \mu_1 v'_1)' \\ -\frac{1}{\rho_1 A_1} (E_1 I_1 w''_1 + \gamma_1 \eta''_1)'' \\ v_2 \\ \eta_2 \\ \frac{1}{\rho_2 A_2} (E_2 A_2 u'_2 + \mu_2 v'_2)' \\ -\frac{1}{\rho_2 A_2} (E_2 I_2 w''_2 + \gamma_2 \eta''_2)'' \\ v_3 \\ \eta_3 \\ \frac{1}{\rho_3 A_3} (E_3 A_3 u'_3 + \mu_3 v'_3)' \\ -\frac{1}{\rho_3 A_3} (E_3 I_3 w''_3 + \gamma_3 \eta''_3)'' \\ p_1 \\ q_1 \\ \omega_{11} \\ \omega_{32} \\ \mathbf{M}_1^{-1} (\mathbf{C}_1(F_{11}, N_{11}, F_{32}, N_{32}, M_{11}, M_{32})^T + \mathbf{K}_1(x_1, y_1, \theta_{11}, \theta_{32})^T + \mathbf{D}_1(p_1, q_1, \omega_{11}, \omega_{32})^T) \\ p_2 \\ q_2 \\ \omega_{12} \\ \omega_{21} \\ \mathbf{M}_2^{-1} (\mathbf{C}_2(F_{12}, N_{12}, F_{21}, N_{21}, M_{12}, M_{21})^T + \mathbf{K}_2(x_2, y_2, \theta_{12}, \theta_{21})^T + \mathbf{D}_2(p_2, q_2, \omega_{12}, \omega_{21})^T) \\ p_3 \\ q_3 \\ \omega_{22} \\ \omega_{31} \\ \mathbf{M}_3^{-1} (\mathbf{C}_3(F_{22}, N_{22}, F_{31}, N_{31}, M_{22}, M_{31})^T + \mathbf{K}_3(x_3, y_3, \theta_{22}, \theta_{31})^T + \mathbf{D}_3(p_3, q_3, \omega_{22}, \omega_{31})^T) \end{array} \right], \quad (6.2.17)$$

and

$$\mathcal{D}(\mathcal{A}) = \left\{ z \in \mathcal{H} : \begin{array}{l} u_i, v_i \in H_r^1, \quad E_i A_i u'_i + \mu_i v'_i \in H^1, \quad w_i, \eta_i \in H_r^2 \\ E_i I_i w''_i + \gamma_i \eta''_i \in H^2, \quad \text{compatibility conditions (6.2.14)-(6.2.16)} \end{array} \right\}. \quad (6.2.18)$$

### 6.3 Well-Posedness

**Theorem 6.3.1** *The operator  $\mathcal{A}$  generates a  $C_0$  semigroup,  $S(t)$ , of contractions on  $\mathcal{H}$ .*

Proof: By a straightforward calculation,

$$\operatorname{Re}\langle \mathcal{A}z, z \rangle_{\mathcal{H}} = \operatorname{Re}\left\langle \frac{dz}{dt}, z \right\rangle_{\mathcal{H}} = \frac{dE(t)}{dt} = -\sum_{i=1}^3 (\mu_i \|v'_i\|^2 + \gamma_i \|\eta''_i\|^2 + b_i^T \mathbf{D}_i b_i) \leq 0. \quad (6.3.1)$$

Hence,  $\mathcal{A}$  is dissipative. The major work is to show  $0 \in \rho(\mathcal{A})$ . Let

$$\tilde{z} = (\tilde{u}_1, \tilde{w}_1, \tilde{v}_1, \tilde{\eta}_1, \tilde{u}_2, \tilde{w}_2, \tilde{v}_2, \tilde{\eta}_2, \tilde{u}_3, \tilde{w}_3, \tilde{v}_3, \tilde{\eta}_3, \tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2, \tilde{a}_3, \tilde{b}_3)^T.$$

Consider the equation

$$\mathcal{A}z = \tilde{z}, \quad (6.3.2)$$

i.e.,

$$v_i = \tilde{u}_i \in H^1, \quad (6.3.3)$$

$$\eta_i = \tilde{w}_i \in H^2, \quad (6.3.4)$$

$$(E_i A_i u'_i + \mu_i v'_i)' = \rho_i A_i \tilde{v}_i \in L^2, \quad (6.3.5)$$

$$-(E_i I_i w''_i + \gamma_i \eta''_i)'' = \rho_i A_i \tilde{\eta}_i \in L^2, \quad (6.3.6)$$

$$(p_1, q_1, \omega_{11}, \omega_{32}) = \tilde{a}_1 \quad (6.3.7)$$

$$\mathbf{M}_1^{-1} (\mathbf{C}_1(F_{11}, N_{11}, F_{32}, N_{32}, M_{11}, M_{32})^T + \mathbf{K}_1 a_1^T + \mathbf{D}_1 b_1^T) = \tilde{b}_1 \quad (6.3.8)$$

$$(p_2, q_2, \omega_{12}, \omega_{21}) = \tilde{a}_2 \quad (6.3.9)$$

$$\mathbf{M}_2^{-1} (\mathbf{C}_2(F_{12}, N_{12}, F_{21}, N_{21}, M_{12}, M_{21})^T + \mathbf{K}_2 a_2^T + \mathbf{D}_2 b_2^T) = \tilde{b}_2 \quad (6.3.10)$$

$$(p_3, q_3, \omega_{22}, \omega_{31}) = \tilde{a}_3 \quad (6.3.11)$$

$$\mathbf{M}_3^{-1} (\mathbf{C}_3(F_{22}, N_{22}, F_{31}, N_{31}, M_{22}, M_{31})^T + \mathbf{K}_3 a_3^T + \mathbf{D}_3 b_3^T) = \tilde{b}_3. \quad (6.3.12)$$

Using (6.3.3) - (6.3.6), we can solve for  $F_{ij}, M_{ij}, N_{ij}$  in terms of  $u_i(0), u_i(L_i), w_i(0), w_i(L_i), w'_i(0), w'_i(L_i)$  and  $\tilde{z}$  using the same procedure as given in the previous chapters. But then we can use the compatibility conditions to solve for  $u_i(0), u_i(L_i), w_i(0), w_i(L_i), w'_i(0), w'_i(L_i)$  and  $\tilde{z}$  in terms of  $x_i, y_i, \theta_{ij}$ . Therefore, we have  $F_{ij}, M_{ij},$  and  $N_{ij}$  in terms of  $x_i, y_i, \theta_{ij},$  and  $\tilde{z}$ . Using this substitution in (6.3.8), (6.3.10), and (6.3.12), we get twelve equations involving the twelve unknowns  $x_i, y_i, \theta_{ij}$  and the given  $\tilde{z}$ .

The resulting matrix contains more than thirty physical constants. In general, these constants can be chosen to make the matrix singular. However, if restrictions are placed upon the comparative magnitudes of the constants, invertibility can be deduced. For example, if we take as restrictions the following:

- $E_1, E_2, E_3, k_i, k_{ij}, d_i$  are of comparable magnitude
- $\ell_{ij} \approx \frac{L_i}{10}$
- $A_i, I_i \ll L_i$

then invertibility is not difficult to show. Likewise, one can show nonsingularity with other combinations of restrictions.

We shall assume in the following that the physical constants are such that the matrix is invertible. Then we can solve for  $a_1, a_2$ , and  $a_3$  in terms of  $\tilde{z}$ . From this, we can determine  $F_{ij}, N_{ij}, M_{ij}$ . This leads to the solution of

$$u_i, w_i, w'_i$$

in terms of  $\tilde{z}$ . This implies that we have a unique solution  $z \in \mathcal{D}(\mathcal{A})$  of equation (6.3.2). Furthermore, it can be shown that

$$\|z\|_{\mathcal{H}} \leq K \|\mathcal{A}z\|_{\mathcal{H}}$$

for some constant  $K$  independent of  $z$ . Therefore,  $0 \in \rho(\mathcal{A})$ ,  $\mathcal{A}^{-1}$  is bounded and  $\mathcal{A}$  is closed. It follows that the range  $R(\lambda - \mathcal{A}) = \mathcal{H}$  for some  $\lambda > 0$ . By Theorem 4.6 in [18],  $\mathcal{D}(\mathcal{A})$  is dense in  $\mathcal{H}$ . The conditions of Theorem 1.2.1 are satisfied and the proof is complete.

## 6.4 Analyticity and Exponential Stability

In Chapter 3, it was shown that the joint-beam system with Kelvin-Voigt damping resulted in a  $C_0$  semigroup which was analytic and exponentially stable. In this section, we will show the triangular system produces the same results.

**Theorem 6.4.1**  *$S(t)$  is analytic and exponentially stable.*

Proof: We will prove the two conditions given for analyticity in Chapter 3:

$$\mathbf{i}\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset, \quad (6.4.1)$$

$$\overline{\lim}_{|\beta| \rightarrow \infty} \|\beta(i\beta - \mathcal{A})^{-1}\| < \infty. \quad (6.4.2)$$

Suppose that the first condition is true and the second condition is false. Then there exists a sequence

$$z_n = (u_{1n}, w_{1n}, v_{1n}, \eta_{1n}, u_{2n}, w_{2n}, v_{2n}, \eta_{2n}, u_{3n}, w_{3n}, v_{3n}, \eta_{3n}, a_{1n}, b_{1n}, a_{2n}, b_{2n}, a_{3n}, b_{3n}) \in \mathcal{D}(\mathcal{A})$$

with  $\|z_n\|_{\mathcal{H}} = 1$ , and a sequence  $\beta_n \in \mathbb{R}$  with  $\beta_n \rightarrow \infty$  such that

$$\lim_{n \rightarrow \infty} \left\| \left( \mathbf{i}I - \frac{1}{\beta_n} \mathcal{A} \right) z_n \right\|_{\mathcal{H}} = 0. \quad (6.4.3)$$

For the simplicity of notation, we omit the subscript  $n$  in the rest of our proof. The convergence results in this proof are understood to occur as  $n \rightarrow \infty$ , unless otherwise stated. From (6.4.3) and (6.3.1), we obtain

$$\lim_{n \rightarrow \infty} \operatorname{Re} \langle (\mathbf{i}I - \frac{1}{\beta} \mathcal{A}) z, z \rangle_{\mathcal{H}} = \lim_{n \rightarrow \infty} \sum_{i=1}^2 \frac{1}{\beta} (\mu_i \|v'_i\|^2 + \gamma_i \|\eta''_i\|^2 + b_i^T \mathbf{D}_i b_i) = 0, \quad (6.4.4)$$

which implies that

$$\left\| \frac{v'_i}{\beta^{1/2}} \right\|, \left\| \frac{\eta''_i}{\beta^{1/2}} \right\| \rightarrow 0. \quad (6.4.5)$$

Our goal is to get a contradiction by showing  $\|z\|_{\mathcal{H}} \rightarrow 0$ . The componentwise version of (6.3.11) is

$$\mathbf{i}u'_i - \frac{1}{\beta} v'_i \rightarrow 0, \quad (6.4.6)$$

$$\mathbf{i}w''_i - \frac{1}{\beta} \eta''_i \rightarrow 0, \quad (6.4.7)$$

$$\mathbf{i}\rho_i A_i v_i - \frac{1}{\beta} (E_i A_i u'_i + \mu_i v'_i)' \rightarrow 0, \quad (6.4.8)$$

$$\mathbf{i}\rho_i A_i \eta_i + \frac{1}{\beta} (E_i I_i w''_i + \gamma_i \eta''_i)'' \rightarrow 0, \quad (6.4.9)$$

in  $L^2(0, L_i)$ , and

$$\mathbf{i}a_1 - \frac{1}{\beta}(p_1, q_1, \omega_{11}, \omega_{32}) \rightarrow 0 \quad (6.4.10)$$

$$\mathbf{i}b_1 - \frac{1}{\beta}\mathbf{M}_1^{-1}(\mathbf{C}_1(F_{11}, N_{11}, F_{32}, N_{32}, M_{11}, M_{32})^T + \mathbf{K}_1 a_1^T + \mathbf{D}_1 b_1^T) \rightarrow 0 \quad (6.4.11)$$

$$\mathbf{i}a_2 - \frac{1}{\beta}(p_2, q_2, \omega_{12}, \omega_{21}) \rightarrow 0 \quad (6.4.12)$$

$$\mathbf{i}b_2 - \frac{1}{\beta}\mathbf{M}_2^{-1}(\mathbf{C}_2(F_{12}, N_{12}, F_{21}, N_{21}, M_{12}, M_{21})^T + \mathbf{K}_2 a_2^T + \mathbf{D}_2 b_2^T) \rightarrow 0 \quad (6.4.13)$$

$$\mathbf{i}a_3 - \frac{1}{\beta}(p_3, q_3, \omega_{22}, \omega_{31}) \rightarrow 0 \quad (6.4.14)$$

$$\mathbf{i}b_3 - \frac{1}{\beta}\mathbf{M}_3^{-1}(\mathbf{C}_3(F_{22}, N_{22}, F_{31}, N_{31}, M_{22}, M_{31})^T + \mathbf{K}_3 a_3^T + \mathbf{D}_3 b_3^T) \rightarrow 0. \quad (6.4.15)$$

Combining equations (6.4.5)-(6.4.7), we have

$$\|u'_i\|, \|w''_i\| \rightarrow 0. \quad (6.4.16)$$

The inner product of (6.4.8) and  $v_i$  yields

$$\mathbf{i}\rho_i A_i \|v_i\|^2 - \frac{1}{\beta}(E_i A_i u'_i + \mu_i v'_i)(L_i) \bar{v}_i(L_i) + \frac{1}{\beta}(E_i A_i u'_i + \mu_i v'_i)(0) \bar{v}_i(0) + \frac{1}{\beta} \langle E_i A_i u'_i + \mu_i v'_i, v'_i \rangle \rightarrow 0. \quad (6.4.17)$$

We use Theorem 1.2.3 and the Gagliardo-Nirenberg inequality to show that the  $L_i$  endpoint boundary term in (6.4.17) converges to zero as  $N \rightarrow \infty$ . Indeed,

$$\begin{aligned} & \frac{1}{\beta^{3/4}} |(E_i A_i u'_i + \mu_i v'_i)(L_i)| \\ & \leq C \frac{1}{\beta^{3/4}} \|E_i A_i u'_i + \mu_i v'_i\|^{\frac{1}{2}} \|E_i A_i u'_i + \mu_i v'_i\|_{H^1}^{\frac{1}{2}} \\ & \leq C \left\| \frac{E_i A_i u'_i + \mu_i v'_i}{\beta^{1/2}} \right\|^{\frac{1}{2}} \left( \left\| \frac{E_i A_i u'_i + \mu_i v'_i}{\beta} \right\|^{\frac{1}{2}} + \left\| \frac{(E_i A_i u'_i + \mu_i v'_i)'}{\beta} \right\|^{\frac{1}{2}} \right) \\ & \rightarrow 0, \end{aligned} \quad (6.4.18)$$

$$\frac{1}{\beta^{1/4}} |v_i(L_i)| \leq C \|v_i\|^{\frac{1}{2}} \left\| \frac{v_i}{\beta^{1/2}} \right\|_{H^1}^{\frac{1}{2}} \leq C \|v_i\|^{\frac{1}{2}} \left( \left\| \frac{v'_i}{\beta^{1/2}} \right\| + \left\| \frac{v_i}{\beta^{1/2}} \right\| \right)^{\frac{1}{2}} \leq T < \infty \quad (6.4.19)$$

due to (6.4.5), (6.4.8), (6.4.16) and the fact that  $v_i$  is bounded. Likewise, we have that the 0 endpoint boundary term in (6.4.17) converges to zero. Clearly, the last term in (6.4.17)

also converges to zero due to (6.4.5) and (6.4.16). Therefore,

$$\|v_i\| \rightarrow 0. \quad (6.4.20)$$

Similarly, we take the inner product of (6.4.9) with  $\eta_i$  in  $L^2(0, L_i)$  to get

$$\begin{aligned} \mathbf{i}\rho_i A_i \|\eta_i\|^2 + \frac{1}{\beta} (E_i I_i w_i'' + \gamma_i \eta_i'')(L_i) \bar{\eta}_i(L_i) - \frac{1}{\beta} (E_i I_i w_i'' + \gamma_i \eta_i'')(0) \bar{\eta}_i(0) \\ - \frac{1}{\beta} (E_i I_i w_i'' + \gamma_i \eta_i'')(L_i) \bar{\eta}_i'(L_i) + \frac{1}{\beta} (E_i I_i w_i'' + \gamma_i \eta_i'')(0) \bar{\eta}_i'(0) \\ + \frac{1}{\beta} \langle E_i I_i w_i'' + \gamma_i \eta_i'', \eta_i'' \rangle \rightarrow 0. \end{aligned} \quad (6.4.21)$$

We again use Theorem 1.2.3 and the Gagliardo-Nirenberg inequality to show that the  $L_i$  endpoint boundary terms in (6.4.21) converge to zero as  $N \rightarrow \infty$ . In fact, we have

$$\begin{aligned} & \frac{1}{\beta^{7/8}} |(E_i I_i w_i'' + \gamma_i \eta_i'')(L_i)| \\ & \leq \frac{C}{\beta^{7/8}} \|(E_i I_i w_i'' + \gamma_i \eta_i'')'\|^{1/2} \|(E_i I_i w_i'' + \gamma_i \eta_i'')'\|_{H^1}^{1/2} \\ & \leq \frac{C}{\beta^{7/8}} \|(E_i I_i w_i'' + \gamma_i \eta_i'')'\|^{1/2} (\|(E_i I_i w_i'' + \gamma_i \eta_i'')'\| + \|E_i I_i w_i'' + \gamma_i \eta_i''\|)^{1/2} \\ & \leq \frac{C}{\beta^{7/8}} \|(E_i I_i w_i'' + \gamma_i \eta_i'')'\| + \frac{C}{\beta^{7/8}} \|(E_i I_i w_i'' + \gamma_i \eta_i'')'\|^{1/2} \|(E_i I_i w_i'' + \gamma_i \eta_i'')''\|^{1/2} \\ & \leq \frac{C}{\beta^{7/8}} \left( \|E_i I_i w_i'' + \gamma_i \eta_i''\|^{1/2} \|(E_i I_i w_i'' + \gamma_i \eta_i'')''\|^{1/2} + \|E_i I_i w_i'' + \gamma_i \eta_i''\| \right) \\ & \quad + \frac{C}{\beta^{7/8}} \left( \|E_i I_i w_i'' + \gamma_i \eta_i''\|^{1/2} \|(E_i I_i w_i'' + \gamma_i \eta_i'')''\|^{1/2} + \|E_i I_i w_i'' + \gamma_i \eta_i''\| \right)^{1/2} \|(E_i I_i w_i'' + \gamma_i \eta_i'')''\|^{1/2} \\ & \leq C \left( \left\| \frac{E_i I_i w_i'' + \gamma_i \eta_i''}{\beta^{3/4}} \right\|^{1/2} \left\| \frac{(E_i I_i w_i'' + \gamma_i \eta_i'')''}{\beta} \right\|^{1/2} + \left\| \frac{E_i I_i w_i'' + \gamma_i \eta_i''}{\beta^{7/8}} \right\| \right) \\ & \quad + C \left( \left\| \frac{E_i I_i w_i'' + \gamma_i \eta_i''}{\beta^{1/2}} \right\|^{1/2} \left\| \frac{(E_i I_i w_i'' + \gamma_i \eta_i'')''}{\beta} \right\|^{1/2} + \left\| \frac{E_i I_i w_i'' + \gamma_i \eta_i''}{\beta^{3/4}} \right\| \right)^{1/2} \left\| \frac{(E_i I_i w_i'' + \gamma_i \eta_i'')''}{\beta} \right\|^{1/2} \\ & \rightarrow 0, \end{aligned} \quad (6.4.22)$$

$$\begin{aligned}
& \frac{1}{\beta^{1/8}} |\eta_i(L_i)| \\
& \leq \frac{C}{\beta^{1/8}} \|\eta_i\|^{\frac{1}{2}} \|\eta_i\|_{H^1}^{\frac{1}{2}} \\
& \leq \|\eta_i\|^{\frac{1}{2}} \left( \|\eta_i\|^{\frac{1}{2}} + \|\eta_i'\|^{\frac{1}{2}} \right) \\
& \leq \|\eta_i\|^{\frac{1}{2}} \left( \|\eta_i\|^{\frac{1}{2}} + \left( \|\eta_i\|^{\frac{1}{2}} \|\eta_i''\|^{\frac{1}{2}} + \|\eta_i\| \right)^{\frac{1}{2}} \right) \\
& \leq \|\eta_i\|^{\frac{1}{2}} \left( \|\eta_i\|^{\frac{1}{2}} + \|\eta_i\|^{\frac{1}{4}} \|\eta_i\|^{\frac{1}{4}} + \|\eta_i\|^{\frac{1}{2}} \right) \\
& \leq \|\frac{\eta_i}{\beta^{1/8}}\| + \|\eta_i\|^{\frac{3}{4}} \|\frac{\eta_i''}{\beta^{1/2}}\|^{\frac{1}{4}} \\
& \leq T < \infty,
\end{aligned} \tag{6.4.23}$$

$$\begin{aligned}
& \frac{1}{\beta^{5/8}} |(E_i I_i w_i'' + \gamma_i \eta_i'')(L_i)| \\
& \leq \frac{C}{\beta^{5/8}} \|E_i I_i w_i'' + \gamma_i \eta_i''\|^{\frac{1}{2}} \|E_i I_i w_i'' + \gamma_i \eta_i''\|_{H^1}^{\frac{1}{2}} \\
& \leq \frac{C}{\beta^{5/8}} \|E_i I_i w_i'' + \gamma_i \eta_i''\|^{\frac{1}{2}} \left( \|E_i I_i w_i'' + \gamma_i \eta_i''\|^{\frac{1}{2}} + \|(E_i I_i w_i'' + \gamma_i \eta_i'')'\|^{\frac{1}{2}} \right) \\
& \leq \frac{C}{\beta^{5/8}} \|E_i I_i w_i'' + \gamma_i \eta_i''\| + \frac{C}{\beta^{5/8}} \left( \|E_i I_i w_i'' + \gamma_i \eta_i''\|^{\frac{1}{2}} \|(E_i I_i w_i'' + \gamma_i \eta_i'')''\|^{\frac{1}{2}} \right. \\
& \quad \left. + \|E_i I_i w_i'' + \gamma_i \eta_i''\| \right)^{\frac{1}{2}} \|E_i I_i w_i'' + \gamma_i \eta_i''\|^{\frac{1}{2}} \\
& \leq \frac{C}{\beta^{5/8}} \|E_i I_i w_i'' + \gamma_i \eta_i''\| + \frac{C}{\beta^{5/8}} \|E_i I_i w_i'' + \gamma_i \eta_i''\|^{\frac{3}{4}} \|(E_i I_i w_i'' + \gamma_i \eta_i'')''\|^{\frac{1}{4}} \\
& \leq \frac{C}{\beta^{5/8}} \|E_i I_i w_i'' + \gamma_i \eta_i''\| + C \left\| \frac{E_i I_i w_i'' + \gamma_i \eta_i''}{\beta^{1/2}} \right\|^{\frac{3}{4}} \left\| \frac{(E_i I_i w_i'' + \gamma_i \eta_i'')''}{\beta} \right\|^{\frac{1}{4}} \\
& \rightarrow 0,
\end{aligned} \tag{6.4.24}$$

and

$$\begin{aligned}
& \frac{1}{\beta^{3/8}} |\eta'_i(L_i)| \\
& \leq \frac{C}{\beta^{3/8}} \|\eta'_i\|^{\frac{1}{2}} \|\eta'_i\|_{H^1}^{\frac{1}{2}} \\
& \leq \frac{C}{\beta^{3/8}} \|\eta'_i\|^{\frac{1}{2}} \left( \|\eta'_i\|^{\frac{1}{2}} + \|\eta''_i\|^{\frac{1}{2}} \right) \\
& = \frac{C}{\beta^{3/8}} \left( \|\eta'_i\| + \|\eta'_i\|^{\frac{1}{2}} \|\eta''_i\|^{\frac{1}{2}} \right) \\
& \leq \frac{C}{\beta^{3/8}} \left( \|\eta_i\|^{\frac{1}{2}} \|\eta''_i\|^{\frac{1}{2}} + \|\eta_i\| \right. \\
& \quad \left. \left( \|\eta_i\|^{\frac{1}{2}} \|\eta''_i\|^{\frac{1}{2}} + \|\eta_i\| \right)^{\frac{1}{2}} \|\eta_i\|^{\frac{1}{2}} \right) \\
& \leq C \|\frac{\eta_i}{\beta^{1/4}}\|^{\frac{1}{2}} \|\frac{\eta''_i}{\beta^{1/2}}\|^{\frac{1}{2}} + \frac{C}{\beta^{3/8}} \|\eta_i\| + C \|\eta_i\|^{\frac{1}{4}} \|\frac{\eta''_i}{\beta^{1/2}}\|^{\frac{3}{4}} \\
& \leq T < \infty
\end{aligned} \tag{6.4.25}$$

due to (6.4.5), (6.4.9), (6.4.16), and the fact that  $\eta_i$  is bounded. Likewise, we have that the 0 endpoint boundary terms in (6.4.21) converge to zero. Clearly, the last term in (6.4.21) also converges to zero due to (6.4.5) and (6.4.16). Therefore,

$$\|\eta_i\| \rightarrow 0. \tag{6.4.26}$$

It is useful to point out here that the previous estimates are equivalent to the following:

$$\frac{1}{\beta^{3/4}} F_{ij}, \frac{1}{\beta^{7/8}} N_{ij}, \frac{1}{\beta^{5/8}} M_{ij} \rightarrow 0. \tag{6.4.27}$$

Now, since  $z \in \mathcal{D}(\mathcal{A})$ , we know that  $b_1 = (p_1, q_1, \omega_{11}, \omega_{32})$  must satisfy the compatibility condition

$$\begin{bmatrix} v_1(t, 0) \\ -\eta_1(t, 0) \\ -v_3(t, L_3) \\ \eta_3(t, L_3) \\ \eta'_1(t, 0) \\ -\eta'_3(t, L_3) \end{bmatrix} = \mathbf{C}_1^T \begin{bmatrix} p_1(t) \\ q_1(t) \\ \omega_{11}(t) \\ \omega_{32}(t) \end{bmatrix}. \tag{6.4.28}$$

Dividing each side of (6.4.28) by  $\beta$ , our previous analysis gives that the left hand side goes to zero. Therefore, we must also have that

$$\frac{\mathbf{C}_1^T}{\beta} \begin{bmatrix} p_1(t) \\ q_1(t) \\ \omega_{11}(t) \\ \omega_{32}(t) \end{bmatrix} \rightarrow 0. \quad (6.4.29)$$

Since the rank of  $\mathbf{C}_1^T$  is 4, we have

$$\frac{1}{\beta} \begin{bmatrix} p_1(t) \\ q_1(t) \\ \omega_{11}(t) \\ \omega_{32}(t) \end{bmatrix} = \frac{b_1}{\beta} \rightarrow 0. \quad (6.4.30)$$

Plugging (6.4.30) into (6.4.10), we get

$$a_1 \rightarrow 0. \quad (6.4.31)$$

This, in turn, can be plugged into equation (6.4.11) to give

$$b_1 \rightarrow 0. \quad (6.4.32)$$

With similar reasoning, we can show that for each  $i$ , we have

$$a_i, b_i \rightarrow 0. \quad (6.4.33)$$

Combining (6.4.16), (6.4.20), (6.4.26), and (6.4.33), we have arrived at a contradiction. Hence, condition (6.4.2) holds as long as (6.4.1) holds.

To verify the (6.4.1), we again use a contradiction argument. If the condition is false, then there exist sequences  $z_n \in \mathcal{D}(\mathcal{A})$ ,  $\beta_n \in \mathbb{R}$ , and a real number  $\beta$  such that

$$\|z_n\|_{\mathcal{H}} = 1, \quad \beta_n \rightarrow \beta, \quad |\beta| > 0,$$

and

$$\lim_{n \rightarrow \infty} \|(\mathbf{i}\beta I - \mathcal{A}) z_n\|_{\mathcal{H}} = 0. \quad (6.4.34)$$

Since  $\beta_n$  is bounded away from 0 for large  $n$ , we can divide  $\beta_n$  through equation (6.4.34). This will result in equation (6.4.4) with  $\beta_n$  bounded. But nowhere in the verification of condition (6.4.2) did we use the fact that  $|\beta_n| \rightarrow \infty$ . Therefore, an exact repetition of the above argument will give the contradiction which proves condition (6.4.1). Thus, the semigroup  $S(t)$  is analytic. Finally, the exponential stability is a by-product of conditions (6.4.1) and (6.4.2) since we have the boundedness of  $\|(\mathbf{i}\beta_n I - \mathcal{A})^{-1}\|$ .

# Chapter 7

## Single Beam Approximation

The numerical scheme in this chapter is for simulation of the transversal motion of a single, cantilevered beam with a tip mass and Boltzmann damping. It is presented here without proof. For convergence proofs of numerical schemes involving viscoelastic beams with memory, see [7] and [16].

### 7.1 Finite Dimensional Approximation

For the spatial discretization, given a length  $L$  and a positive integer  $N$ , we construct a uniform mesh

$$G(L, N) = \left\{ s_j = \frac{j}{N-1}L \mid j = -1, \dots, N \right\}.$$

Now, let  $S_3^G = \{b_j^G \mid j = -1, \dots, N\}$  be the standard set of cubic splines with  $b_j^G$  centered at  $s_j$ . Since we wish to approximate  $H_r^2(0, L)$  with these functions, we must replace  $b_1^G$  with  $-2b_{-1}^G + b_0^G - 2b_1^G$  to reflect the clamped condition at  $s = 0$ . Note that  $b_N^G$  corresponds to the cubic spline centered at the "phantom" node  $s_N$ . The spline functions will be used directly to approximate the  $w$  and  $\eta$  components of the state space:

$$\begin{aligned} [w^N(t)](s) &= \sum_{j=1}^N c_j(t) b_j^G(s) \\ [\eta^N(t)](s) &= \sum_{j=1}^N k_j(t) b_j^G(s). \end{aligned}$$

To approximate the  $r$  component, we also need a discretization of the time history. To

this end, we construct a non-uniform mesh based on the mass of the kernel function  $\dot{g}$ . Let  $M$  be given and define the sequence  $\{t_j\}_{j=1}^M$  satisfying

$$t_1 = 0, \quad \int_{t_j}^{t_{j+1}} \dot{g}(\zeta) d\zeta = \frac{1}{M} \int_0^\infty \dot{g}(\zeta) d\zeta.$$

If we define the characteristic functions  $E_j = \chi_{(t_j, t_{j+1}]}$ , then we can approximate  $r$  first by

$$r^M = \sum_{i=1}^M r(t_i) E_i.$$

But, remember that  $r(t_i) \in H_r^2$ . So now we approximate  $r^M$  by

$$r^{N,M} = \sum_{i=1}^M r^N(t_i) E_i, \quad r^N(t_i) = \sum_{j=1}^N c_j(t_i) b_j^G(s).$$

We also approximate  $D_\zeta r$  by

$$D^{N,M} r^{N,M} = \sum_{i=1}^M (t_{i+1} - t_i)^{-1} [r^N(t_{i+1}) - r^N(t_i)] E_i.$$

Note that in order to specify  $r^{N,M}$  as just given, it is necessary to give  $N \times M$  unknowns. However, according to the dynamics of our system,  $r(0)$  is always zero. Therefore, it is necessary to determine  $N \times (M - 1)$  unknowns since  $c_j(t_1) = 0$  for  $j = 1, \dots, N$ . If we consider that for each  $t_i$ ,  $r^N(t_i)$  lies in the span of  $S_3^G$ , then we see that  $r^{N,M}$  lies in the span of

$$S_3^{G,1} \otimes S_3^{G,2} \otimes \dots \otimes S_3^{G,M-1}$$

where  $S_3^{G,i} = S_3^G$  is used to represent the space in which  $r^N(t_{i+1})$  lies.

Putting the  $w^N, \eta^N$  and  $r^{N,M}$  components together with the  $q$  and  $\omega$  components, it is seen that they lie in the span of

$$S = S_3^G \otimes S_3^G \otimes S_3^{G,1} \otimes S_3^{G,2} \otimes \dots \otimes S_3^{G,M-1} \otimes \mathbf{e}_1 \otimes \mathbf{e}_2$$

which is an  $NM + N + 2$  dimensional subspace of the space

$$\mathcal{H}_u = H_r^2(0, L) \times H_r^2(0, L) \times \dots \times H_r^2(0, L) \times \mathbb{R}^2.$$

But we must also account for the geometric constraint (2.1.7). The five basis vectors in  $S$  that do not satisfy this constraint are

$$\xi_1 = \begin{bmatrix} 0 \\ b_{N-2}^G \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \xi_2 = \begin{bmatrix} 0 \\ b_{N-1}^G \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \xi_3 = \begin{bmatrix} 0 \\ b_N^G \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \xi_4 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \mathbf{e}_1 \end{bmatrix}, \xi_5 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \mathbf{e}_2 \end{bmatrix}.$$

Define the operator  $C : \mathcal{H} \rightarrow \mathbb{R}^2$  by

$$C \begin{bmatrix} w \\ \eta \\ \vdots \\ q \\ \omega \end{bmatrix} = \begin{bmatrix} \eta(L) \\ \eta'(L) \end{bmatrix} - \begin{bmatrix} q \\ \omega \end{bmatrix}.$$

We seek linear combinations of the  $\xi_k$  that are in the null space of  $C$ . From

$$C\left(\sum_{k=1}^5 \alpha_k \xi_k\right) = \sum_{k=1}^5 \alpha_k C(\xi_k) = 0 \in \mathbb{R}^2$$

we see that the coefficient vector  $\alpha \in \mathbb{R}^5$  is needed. It is apparent that  $\alpha$  must lie in the null space of the matrix  $C_a$ , each of whose columns is an image of  $C(\xi_k)$ ,

$$C_a = \begin{bmatrix} 1 & 4 & 1 & -1 & 0 \\ b & 0 & -b & 0 & -1 \end{bmatrix}$$

where  $b = (b_{N-2}^G)'(L)$ . Using a CAS, we see that three linearly independent vectors in the null space of  $C_a$  are

$$\left(1, -\frac{1}{2}, 1, 0, 0\right), \quad \left(0, \frac{1}{4}, 0, 1, 0\right), \quad \left(\frac{1}{b}, \frac{-1}{4b}, 0, 0, 1\right).$$

Therefore, the linear combinations that satisfy (2.1.7) are

$$\tilde{\xi}_1 = \begin{bmatrix} 0 \\ b_{N-2}^G - \frac{1}{2}b_{N-1}^G + b_N^G \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \tilde{\xi}_2 = \begin{bmatrix} 0 \\ \frac{1}{4}b_{N-1}^G \\ 0 \\ \vdots \\ 0 \\ \mathbf{e}_1 \end{bmatrix}, \quad \text{and} \quad \tilde{\xi}_3 = \begin{bmatrix} 0 \\ \frac{1}{b}b_{N-2}^G - \frac{1}{4b}b_{N-1}^G \\ 0 \\ \vdots \\ 0 \\ \mathbf{e}_2 \end{bmatrix}.$$

We next choose a basis  $\mathbf{S} = \mathbf{b}_1, \dots, \mathbf{b}_{MN+N} \subset \mathcal{H}$  that satisfies our compatibility condition:

$$\mathbf{b}_i = \begin{bmatrix} b_i^G \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad i = 1, \dots, N,$$

$$\mathbf{b}_{N+i} = \begin{bmatrix} 0 \\ b_i^G \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad i = 1, \dots, N-3,$$

$$\mathbf{b}_{2N-2} = \tilde{\xi}_1, \quad \mathbf{b}_{2N-1} = \tilde{\xi}_2, \quad \mathbf{b}_{2N} = \tilde{\xi}_3,$$

$$\mathbf{b}_{2N+i} = \begin{bmatrix} 0 \\ 0 \\ b_i^G \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad i = 1, \dots, N,$$

$$\mathbf{b}_{MN+i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_i^G \\ 0 \\ 0 \end{bmatrix}, \quad i = 1, \dots, N.$$

Leaving aside our basis functions for the moment, let us examine our system after a discretization of the time history. Thus our system becomes

$$M_1 \frac{d}{dt} \begin{bmatrix} w \\ \eta \\ r(t_2) \\ r(t_3) \\ \vdots \\ r(t_M) \\ q \\ \omega \end{bmatrix} = \begin{bmatrix} \eta \\ -[\alpha w'' - \sum_{k=1}^{M-1} g_{k+1}(r(t_{k+1}))]'' \\ \eta - (t_2 - t_1)^{-1} r(t_2) \\ \eta - (t_3 - t_2)^{-1} (r(t_3) - r(t_2)) \\ \vdots \\ \eta - (t_M - t_{M-1})^{-1} (r(t_M) - r(t_{M-1})) \\ \mathbf{C}(\hat{N}^t, \hat{M}^t)^T \end{bmatrix} \quad (7.1.1)$$

where

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho A & 0 & 0 \\ 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{M} \end{bmatrix}, \quad g_k = \int_{t_k}^{t_{k+1}} \dot{g}(\zeta) d\zeta \quad k < M, \quad g_M = \int_{t_M}^{\infty} \dot{g}(\zeta) d\zeta,$$

and

$$\hat{N}^t = [\alpha w'' - \sum_{k=1}^{M-1} g_{k+1}(r(t_{k+1}))]', \quad \hat{M}^t = \alpha w'' - \sum_{k=1}^{M-1} g_{k+1}(r(t_{k+1}))''$$

are approximations to  $\hat{N}^t$  and  $\hat{M}^t$ , respectively.

Taking the inner product of (7.1.1) with  $\phi = (\phi^1, \phi^2, \dots, \phi^{M+1}, \phi^J)^T \in \mathcal{H}_u$  in  $L^2(0, L) \times$

$\dots \times L^2(0, L) \times \mathbb{R}^2$ , we get

$$\begin{aligned}
& \langle w_t, \phi^1 \rangle + \rho A \langle \eta_t, \phi^2 \rangle + \sum_{k=2}^M \langle r_t(t_k), \phi^{k+1} \rangle + \langle \mathbf{M}(\dot{q}, \dot{\omega}), \phi^J \rangle_{\mathbb{R}^2} \\
& = \langle \eta, \phi^1 \rangle - \langle \alpha w'' - \sum_{k=1}^{M-1} g_{k+1}(r(t_{k+1}))'', (\phi^2)'' \rangle + \dots \\
& + \langle \eta - (t_M - t_{M-1})^{-1}(r(t_M) - r(t_{M-1})), \phi^{M+1} \rangle + \langle (-\hat{N}^t, \hat{M}^t)^T, C\phi \rangle
\end{aligned} \tag{7.1.2}$$

If we now write our finite-dimensional system as

$$\begin{bmatrix} w^N \\ \eta^N \\ r^{N,M} \\ q \\ \omega \end{bmatrix} = \sum_{j=1}^{MN+N} z_j(t) \mathbf{b}_j(s),$$

and use the same basis functions to approximate the test functions  $\phi$ , then the last term in (7.1.2) is zero from the construction of the  $\mathbf{b}_j$ . The result is the equation

$$M^N \dot{z}^N(t) = A^N z(t), \tag{7.1.3}$$

where

$$M_{i,j}^N = \langle \mathbf{b}_i^1, \mathbf{b}_j^1 \rangle + \rho A \langle \mathbf{b}_i^2, \mathbf{b}_j^2 \rangle + \langle \mathbf{M} \mathbf{b}_i^J, \mathbf{b}_j^J \rangle_{\mathbb{R}^2} + \sum_{k=3}^{M+1} \langle \mathbf{b}_i^k, \mathbf{b}_j^k \rangle$$

$$\begin{aligned}
A_{i,j}^N = & \langle \mathbf{b}_i^1, \mathbf{b}_j^2 \rangle - \langle \mathbf{b}_i^2, \alpha (\mathbf{b}_j^1)'' - \sum_{k=1}^{M-1} g_{k+1}(\mathbf{b}_j^{2+k})'' \rangle + \langle \mathbf{b}_i^3, \mathbf{b}_j^2 - (t_2 - t_1)^{-1} \mathbf{b}_j^3 \rangle + \dots \\
& + \langle \mathbf{b}_i^{M+1}, \mathbf{b}_j^2 - (t_M - t_{M-1})^{-1} (\mathbf{b}_j^{M+1} - \mathbf{b}_j^M) \rangle.
\end{aligned}$$

## 7.2 Numerical Results

We can now use (7.1.3) to examine the effects of memory damping on our cantilevered beam. Let us momentarily ignore the tip mass and set  $m = I_m = h = 0$ . The beam parameters are

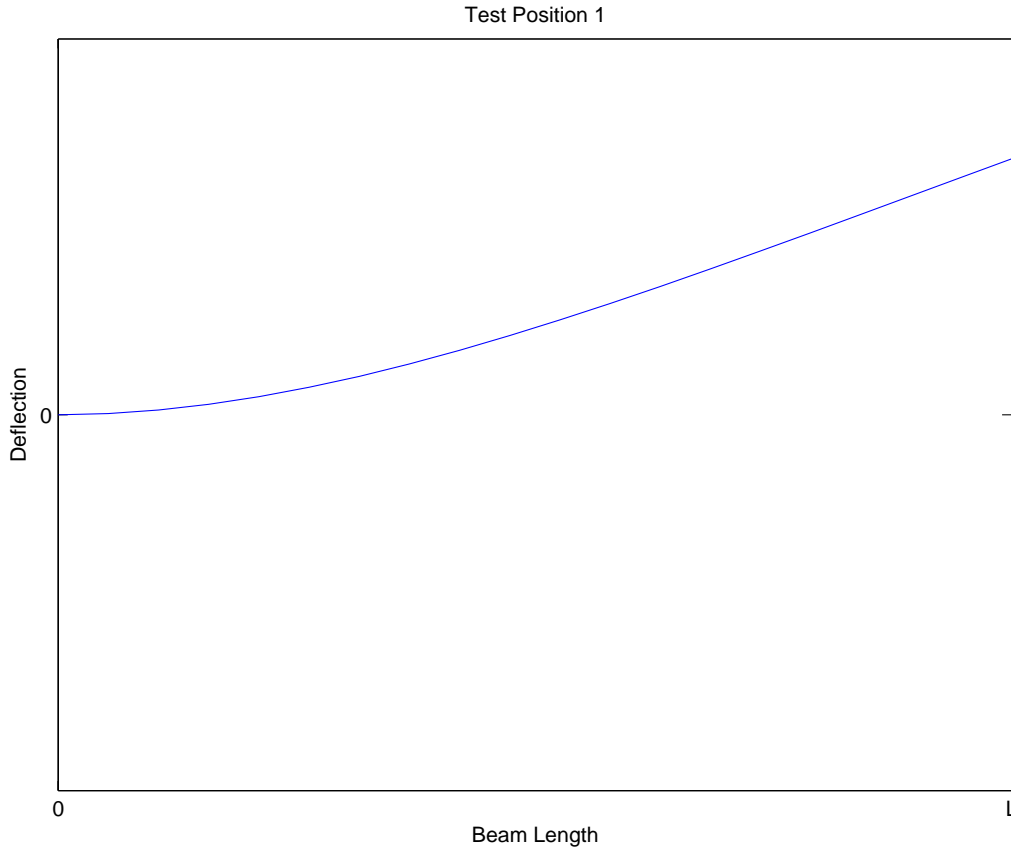


Figure 7.2.1: Initial Position for Experiments 1 and 2

given by  $EI = 100$ ,  $\rho = 1$ ,  $A = 1$ , and  $L = 1$ . For the memory kernel function, we will use

$$g(\zeta) = -50 \frac{e^{-2\zeta}}{\sqrt{(\zeta)}}. \quad (7.2.1)$$

We should point out that these parameter values combined with the chosen kernel function result in a system with very strong memory damping.

Our first experiment is to give the beam the initial displacement shown in Figure 7.2.1. We assume that the beam has been occupying this position for almost no time at all, and was at the neutral position prior to being displaced. The resulting motion is shown in Figure 7.2.2. Note how quickly the deflection magnitudes decay. This is due to the strong memory damping.

For the second experiment, we also displace the beam according to Figure 7.2.1. Only

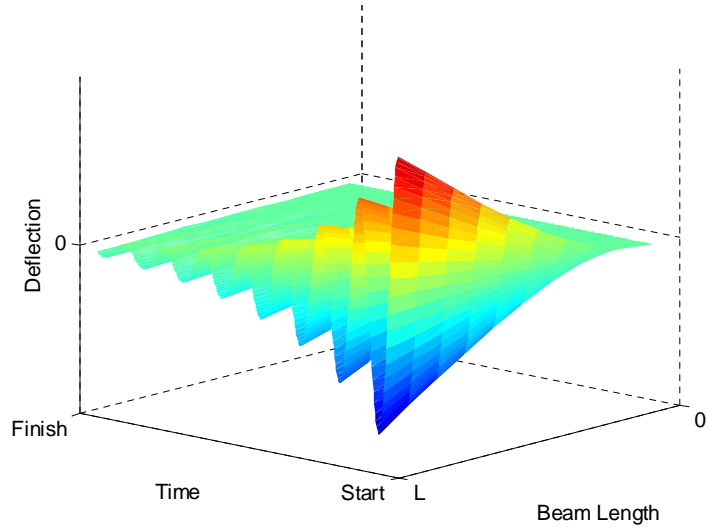


Figure 7.2.2: Deflection Evolution for Experiment 1

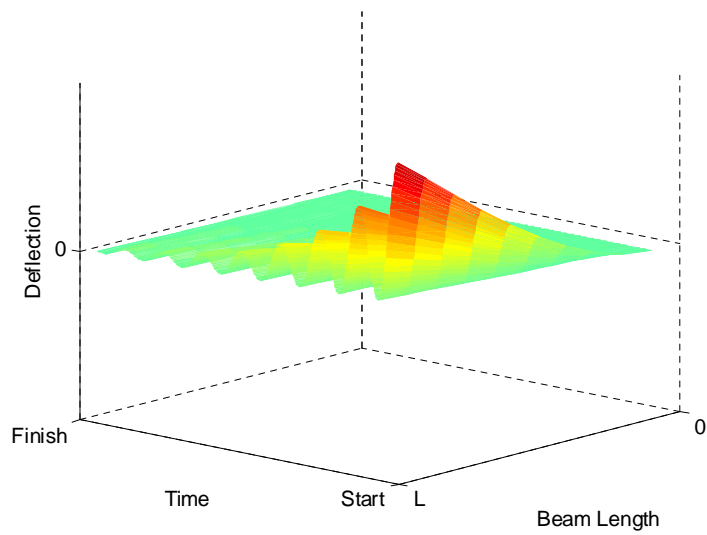


Figure 7.2.3: Deflection Evolution for Experiment 2

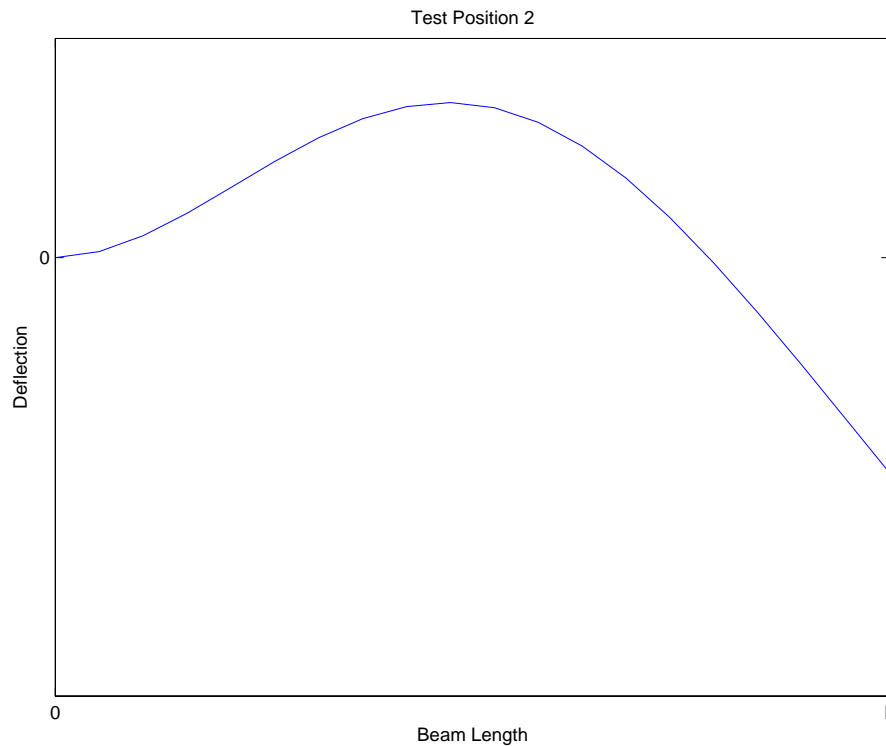


Figure 7.2.4: Initial Position for Experiments 3 and 4

this time, we assume that the beam has been held in this position for an infinitely long period of time. The resulting motion, shown in Figure 7.2.3, is quite different from that of the first experiment. The beam does not completely return to the resting position before it rises up once again. The beam will gradually converge to zero deflection, but it will never have negative deflection at any point in time after it is released.

For the third and fourth experiments, we repeat the processes of the first two experiments using a different initial displacement (see Figure 7.2.4). The results are given in Figures 7.2.5 and 7.2.6.

We now consider the effect of the tip mass on our cantilevered beam. We assume the beam has been at rest in the neutral position for an infinitely long period of time when, suddenly, an initial velocity is imparted to a small area of the beam. Such a situation could arise if the beam is struck with a hammer, for instance. For a beam with no tip mass, the resulting motion is shown in Figure 7.2.7. Note the deflections at the end  $s = L$ . Now, assume a small tip mass ( $m = .1$ ,  $I_m = .1$ ,  $h = .1$ ) is attached to the free end and the

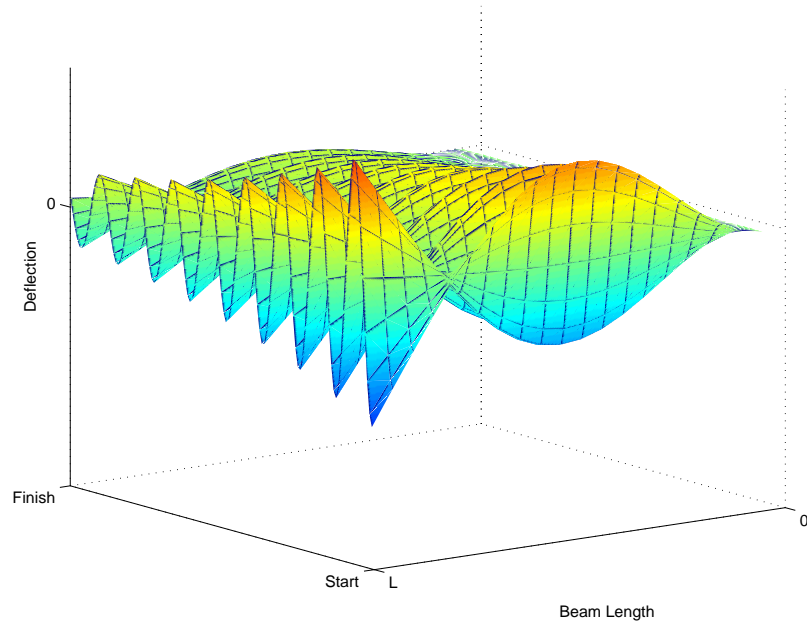


Figure 7.2.5: Deflection Evolution for Experiment 3

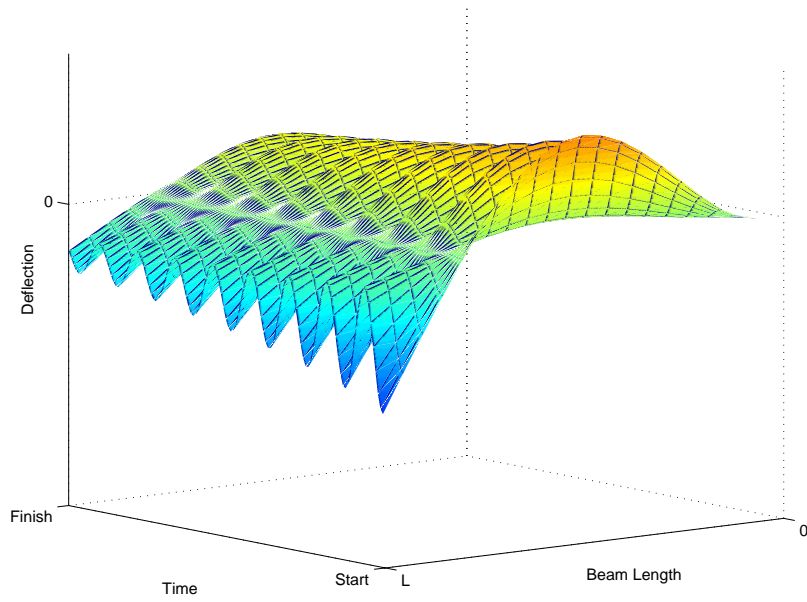


Figure 7.2.6: Deflection Evolution for Experiment 4

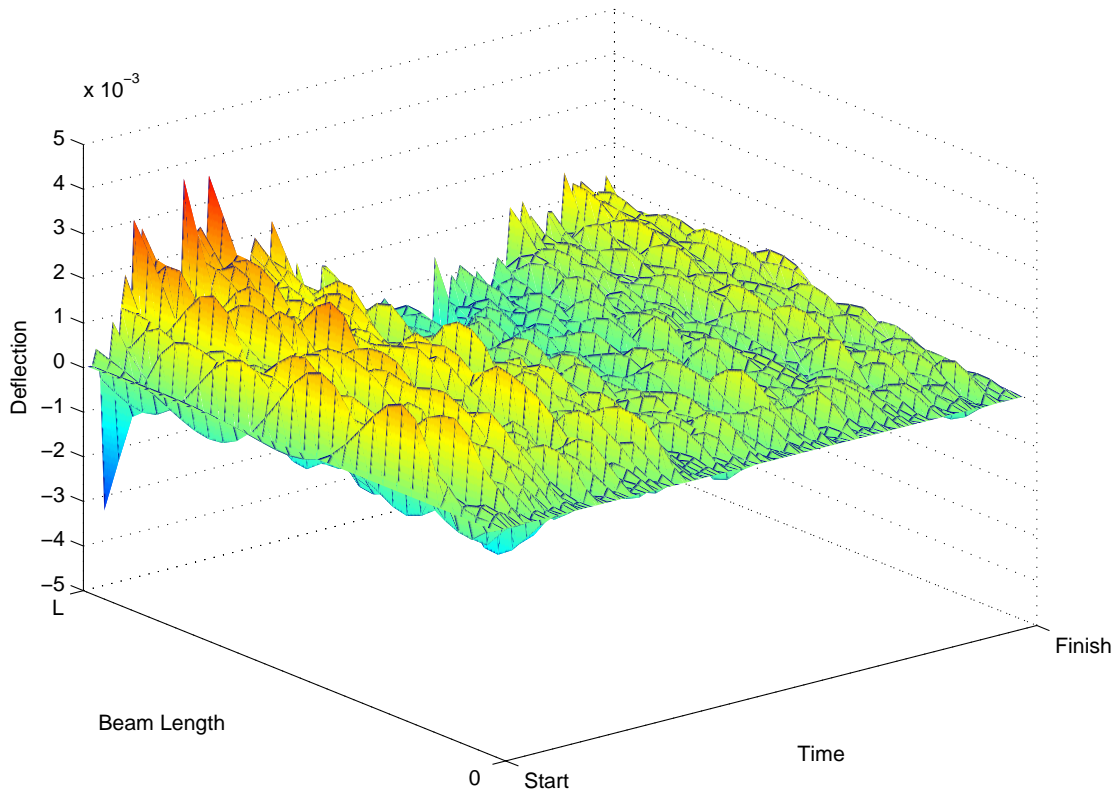


Figure 7.2.7: Deflection Evolution for Initial Velocity and No Tip Mass

experiment is repeated (see Figure 7.2.8). The deflections at the end  $s = L$  are somewhat more subdued than in the free end case. Finally, suppose we have a large tip mass attached to the end ( $m = 10, I_m = 10, h = .1$ ). The resulting motions, shown in Figure 7.2.9, reveal almost no deflection at the end  $s = L$ .

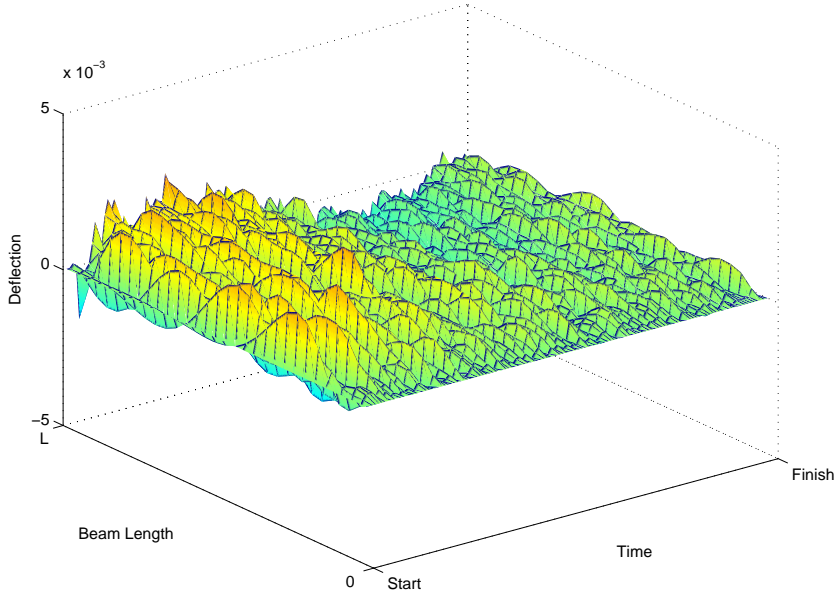


Figure 7.2.8: Deflection Evolution for Initial Velocity and Small Tip Mass

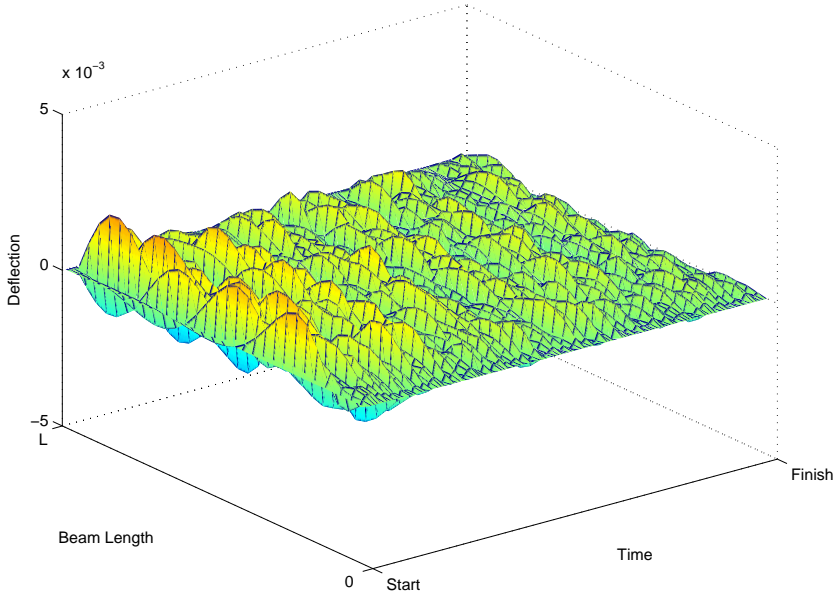


Figure 7.2.9: Deflection Evolution for Initial Velocity and Large Tip Mass

# Chapter 8

## Joint-Beam Approximation

In this chapter, we will focus on approximating the joint-beam systems considered in Chapters 3 and 5. We will start out by proving convergence of the approximations for the system with Kelvin-Voigt damping. This result will be used in Chapter 9 when we consider the optimal control problem. Numerical schemes and results are then produced for the Kelvin-Voigt system and the thermoelastic system. We will conclude the chapter with some statements about approximation of the triangular system considered in Chapter 6.

### 8.1 Finite Dimensional State Space

Let  $U_i^N, W_i^N$  be finite dimensional subspaces of  $H_r^1(0, L_i), H_r^2(0, L_i)$  respectively, such that the following hold:

1.  $P_{1i}^N u_i \rightarrow u_i$  strongly, where  $P_{1i}^N : H_r^1(0, L_i) \rightarrow U_i^N$  is the orthogonal projection of  $u_i$  onto  $U_i^N$  w.r.t. the  $H_r^1(0, L_i)$  inner product.
2.  $P_{2i}^N w_i \rightarrow w_i$  strongly, where  $P_{2i}^N : H_r^2(0, L_i) \rightarrow W_i^N$  is the orthogonal projection of  $w_i$  onto  $W_i^N$  w.r.t. the  $H_r^2(0, L_i)$  inner product.
3. For  $u_i^N \in U_i^N$ ,  $\lim_{s \rightarrow L_i} (u_i^N)'(s)$  exists.
4. For  $w_i^N \in W_i^N$ ,

$$\lim_{s \rightarrow L_i} (w_i^N)''(s)$$

and

$$\lim_{s \rightarrow L_i} (w_i^N)'''(s)$$

exist.

Let  $Z^N = Z_1^N \times Z_2^N \times \mathbb{R}^4$  where  $Z_i^N = U_i^N \times W_i^N \times U_i^N \times W_i^N$ . For each  $N$ , we now define our finite dimensional state space as

$$\mathcal{H}^N = \{z^N \in Z^N \mid z^N \text{ satisfies the geometric constraints}\}. \quad (8.1.1)$$

Let  $P^N : \mathcal{H} \rightarrow \mathcal{H}^N$  be defined as the orthogonal projection onto  $\mathcal{H}^N$  with respect to the inner product inherited from the space  $\tilde{H}_1 \times \tilde{H}_2 \times \mathbb{R}^4$  where  $\tilde{H}_i = H_r^1(0, L_i) \times H_r^2(0, L_i) \times L^2(0, L_i) \times L^2(0, L_i)$ . It can be shown that  $P^N z \rightarrow z$  for  $z \in \mathcal{H}$ .

Before we define our approximation to the operator  $\mathcal{A}$ , let us first define the finite dimensional operators  $D_i^N : U_i^N \rightarrow U_i^N$  and  $K_i^N : W_i^N \rightarrow W_i^N$  by

$$\langle D_i^N u, v \rangle = -\langle u', v' \rangle + \lim_{s \rightarrow L_i} u'(L_i)v(L_i) \quad u, v \in U_i^N$$

$$\langle K_i^N w, \eta \rangle = \langle w'', \eta'' \rangle - \lim_{s \rightarrow L_i} w''(L_i)\eta'(L_i) + \lim_{s \rightarrow L_i} w'''(L_i)\eta(L_i) \quad w, \eta \in W_i^N.$$

Now we can define the operator  $\mathcal{A}^N : \mathcal{H}^N \rightarrow \mathcal{H}^N$  by

$$\mathcal{A}^N \begin{bmatrix} u_1^N \\ w_1^N \\ v_1^N \\ \eta_1^N \\ u_2^N \\ w_2^N \\ v_2^N \\ \eta_2^N \\ a \end{bmatrix} = \begin{bmatrix} v_1^N \\ \eta_1^N \\ \frac{1}{\rho_1 A_1} D_1^N (E_1 A_1 u_1^N + \mu_1 v_1^N) \\ -\frac{1}{\rho_1 A_1} K_1^N (E_1 I_1 w_1^N + \gamma_1 \eta_1^N) \\ v_2^N \\ \eta_2^N \\ \frac{1}{\rho_2 A_2} D_2^N (E_2 A_2 u_2^N + \mu_2 v_2^N) \\ -\frac{1}{\rho_2 A_2} K_2^N (E_2 I_2 w_2^N + \gamma_2 \eta_2^N) \\ \mathbf{M}^{-1} \mathbf{C}(F_1^N, N_1^N, F_2^N, N_2^N, M_1^N, M_2^N)^T \end{bmatrix}, \quad (8.1.2)$$

where

$$\begin{aligned} F_i^N &= \lim_{s \rightarrow L_i} (E_i A_i u_i^N + \mu_i v_i^N)'(s) \\ N_i^N &= \lim_{s \rightarrow L_i} (E_i I_i w_i^N + \gamma_i \eta_i^N)'''(s) \\ M_i^N &= \lim_{s \rightarrow L_i} (E_i I_i w_i^N + \gamma_i \eta_i^N)''(s). \end{aligned}$$

**Lemma 8.1.1** *For each  $N$ ,  $\mathcal{A}^N$  generates a  $C_0$  semigroup,  $T^N(t)$ , on  $\mathcal{H}^N$  such that  $\mathcal{A}^N \in \mathcal{G}(1, 0)$ .*

Proof: It is sufficient to prove that  $\mathcal{A}^N$  is dissipative. But from a straightforward calculation, we have

$$\operatorname{Re}\langle \mathcal{A}^N z^N, z^N \rangle_{\mathcal{H}^N} = - \sum_{i=1}^2 (\mu_i \|(v_i^N)'\|^2 + \gamma_i \|(\eta_i^N)''\|^2) \leq 0. \quad (8.1.3)$$

Therefore  $\mathcal{A}^N$  is dissipative.

**Lemma 8.1.2** *For all  $z \in \mathcal{H}$ ,  $(I - \mathcal{A}^N)^{-1} P^N z \rightarrow (I - \mathcal{A})^{-1} z$  as  $N \rightarrow \infty$ .*

Proof: Given  $z$ , let  $r^N = (I - \mathcal{A}^N)^{-1} P^N z$  and  $r = (I - \mathcal{A})^{-1} z$ , then we have

$$(I - \mathcal{A}^N) r^N = P^N z \rightarrow z = (I - \mathcal{A}) r. \quad (8.1.4)$$

Since  $(I - \mathcal{A}^N)^{-1}$  is uniformly bounded, we have that the sequence  $r^N$  is also bounded. If we denote the components of  $r^N$  and  $r$  by  $(u_1^N, w_1^N, v_1^N, \eta_1^N, u_2^N, w_2^N, v_2^N, \eta_2^N, a^N)^T$  and  $(u_1, w_1, v_1, \eta_1, u_2, w_2, v_2, \eta_2, a)^T$ , respectively, then the component version of (8.1.4) can be written as

$$u_i^N - v_i^N \rightarrow u_i - v_i \quad (8.1.5)$$

$$w_i^N - \eta_i^N \rightarrow w_i - \eta_i \quad (8.1.6)$$

$$v_i^N - \frac{1}{\rho_i A_i} D_i^N (E_i A_i u_i^N + \mu_i v_i^N) \rightarrow v_i - \frac{1}{\rho_i A_i} (E_i A_i u_i' + \mu_i v_i') \quad (8.1.7)$$

$$\eta_i^N + \frac{1}{\rho_i A_i} K_i^N (E_i I_i w_i^N + \gamma_i \eta_i^N) \rightarrow \eta_i + \frac{1}{\rho_i A_i} (E_i I_i w_i'' + \gamma_i \eta_i'') \quad (8.1.8)$$

and

$$\mathbf{M}^{-1} \mathbf{C}(F_1^N, N_1^N, F_2^N, N_2^N, M_1^N, M_2^N)^T \rightarrow \mathbf{M}^{-1} \mathbf{C}(F_1, N_1, F_2, N_2, M_1, M_2)^T. \quad (8.1.9)$$

Now, let  $z^N \in \mathcal{H}^N$  be a sequence such that  $\|z^N\| = 1$ , and denote the components of  $z^N$  by  $(z_u^1, z_v^1, z_w^1, z_\eta^1, z_u^2, z_w^2, z_v^2, z_\eta^2, z_a)$ . We have suppressed the  $N$  dependence of the components of  $z^N$ . Taking the inner product of  $z^N$  with  $(I - \mathcal{A}^N) r^N$  and  $(I - \mathcal{A}) r$  and using (8.1.4), we

see that

$$\begin{aligned}
& \sum_{i=1}^2 \left( \langle (u_i^N)' - (v_i^N)', (z_u^i)' \rangle + \langle (w_i^N)'' - (\eta_i^N)'', (z_w^i)'' \rangle + \langle v_i^N, z_v^i \rangle + \langle E_i A_i (u_i^N)' + \mu_i (v_i^N)', (z_v^i)' \rangle \right. \\
& \quad \left. + \langle \eta_i^N, z_\eta^i \rangle + \langle E_i I_i (w_i^N)'' + \gamma_i (\eta_i^N)'', (z_\eta^i)'' \rangle \right) + (a^N)^T \mathbf{M} z_a \\
\rightarrow & \sum_{i=1}^2 \left( \langle u_i' - v_i', (z_u^i)' \rangle + \langle w_i'' - \eta_i'', (z_w^i)'' \rangle + \langle v_i, z_v^i \rangle + \langle E_i A_i u_i' + \mu_i v_i', (z_v^i)' \rangle \right. \\
& \quad \left. + \langle \eta_i, z_\eta^i \rangle + \langle E_i I_i w_i'' + \gamma_i \eta_i'', (z_\eta^i)'' \rangle \right) + a^T \mathbf{M} z_a. \tag{8.1.10}
\end{aligned}$$

Note that the  $(F_i^N, N_i^N, M_i^N)$  and the  $(F_i, N_i, M_i)$  terms are not present in (8.1.10). This is because  $z^N \in \mathcal{H}^N$  and therefore, due to the geometric constraints on  $z^N$ , the boundary terms have cancelled out. Now, if we suppose that the sequence  $z^N$  with  $\|z^N\| = 1$  is such that  $(z_u^i, z_v^i, z_w^i, z_\eta^i) \rightarrow 0$  in the respective component spaces (this is possible in spite of the geometric constraints on  $z^N$ ), then one can readily see that we have

$$(a^N)^T \mathbf{M} z_a \rightarrow a^T \mathbf{M} z_a.$$

Since this is true for all  $z^N$  satisfying the above conditions, we have

$$a^N \rightarrow a \in \mathbb{R}^4. \tag{8.1.11}$$

Therefore, taking into account the geometric constraints of  $r^N$  and  $r$ , (8.1.11) gives us

$$v_i^N(L_i), w_i^N(L_i), (w_i^N)'(L_i) \rightarrow v_i(L_i), w_i(L_i), w_i'(L_i). \tag{8.1.12}$$

Attacking from another angle, let us consider an element  $z_{v_i}^N \in U_i^N$ . Take the inner product in  $L^2(0, L_i)$  of both sides of (8.1.7) with  $z_{v_i}^N$  to get

$$\begin{aligned}
& \rho_i A_i \langle v_i^N, z_{v_i}^N \rangle + \langle E_i A_i (u_i^N)' + \mu_i (v_i^N)', (z_{v_i}^N)' \rangle - F_1^N z_{v_i}^N(L_i) \rightarrow \\
& \rho_i A_i \langle v_i, z_{v_i}^N \rangle + \langle E_i A_i u_i' + \mu_i v_i', (z_{v_i}^N)' \rangle - F_1 z_{v_i}^N(L_i). \tag{8.1.13}
\end{aligned}$$

Taking the inner product in  $H_r^1(0, L_i)$  of both sides of (8.1.5) with  $z_{v_i}^N$  gives

$$E_i A_i \langle (u_i^N)' - (v_i^N)', (z_{v_i}^N)' \rangle \rightarrow E_i A_i \langle u_i' - v_i', z_{v_i}^N \rangle. \tag{8.1.14}$$

Subtracting (8.1.14) from (8.1.13), we get

$$\begin{aligned} & \rho_i A_i \langle v_i^N, z_{v_i}^N \rangle + (E_i A_i + \mu_i) \langle (v_i^N)', (z_{v_i}^N)' \rangle - F_i^N z_{v_i}^N(L_i) \\ & \rightarrow \rho_i A_i \langle v_i, z_{v_i}^N \rangle + (E_i A_i + \mu_i) \langle v_i', (z_{v_i}^N)' \rangle - F_i z_{v_i}^N(L_i). \end{aligned} \quad (8.1.15)$$

From the fact that  $r^N$  is bounded, we can use (8.1.5) to deduce that  $v_i^N$  is uniformly bounded in  $H_r^1(0, L_i)$ . Therefore, since (8.1.15) holds for all  $z_{v_i}^N \in U_i^N$ , we have that  $F_i^N$  is uniformly bounded. Let us rewrite (8.1.15) as

$$\rho_i A_i \langle v_i^N - v_i, z_{v_i}^N \rangle + (E_i A_i + \mu_i) \langle (v_i^N)' - v_i', (z_{v_i}^N)' \rangle - (F_i^N - F_i) z_{v_i}^N(L_i) \rightarrow 0. \quad (8.1.16)$$

Also, let us define a new inner product on  $H_r^1(0, L_i)$  by

$$\langle f, g \rangle_2 = \rho_i A_i \langle f, g \rangle + (E_i A_i + \mu_i) \langle f', g' \rangle. \quad (8.1.17)$$

Note that the norm on  $H_r^1(0, L_i)$  induced by this inner product is equivalent to the usual norm. Define  $\tilde{P}_i^N$  to be the orthogonal projection operator from  $H_r^1(0, L_i)$  to  $U_i^N$  with respect to the new inner product. Now, if we choose  $z_{v_i}^N$  to be  $\tilde{P}_i^N(v_i^N - v_i) = v_i^N - \tilde{P}_i^N(v_i)$ , then we have  $z_{v_i}^N(L_i) \rightarrow 0$  by (8.1.12). Therefore, since  $F_i^N$  is uniformly bounded, (8.1.16) becomes

$$\rho_i A_i \langle v_i^N - v_i, \tilde{P}_i^N(v_i^N - v_i) \rangle + (E_i A_i + \mu_i) \langle (v_i^N)' - v_i', \tilde{P}_i^N(v_i^N - v_i)' \rangle \rightarrow 0 \quad (8.1.18)$$

which is the same as

$$\langle v_i^N - v_i, \tilde{P}_i^N(v_i^N - v_i) \rangle_2 \rightarrow 0. \quad (8.1.19)$$

But this implies that  $\tilde{P}_i^N(v_i^N - v_i) = v_i^N - \tilde{P}_i^N(v_i) \rightarrow 0$  in  $H_r^1(0, L_i)$ . Due to the assumptions made about the approximating subspaces  $U_i^N$ , we know that  $P_{1i}^N(v_i) \rightarrow v_i$  in  $H_r^1(0, L_i)$ . Because of the equivalence of the norms, this implies that  $\tilde{P}_i^N(v_i) \rightarrow P_{1i}^N(v_i)$ . Therefore,

$$v_i^N \rightarrow P_{1i}^N(v_i) \rightarrow v_i \quad \text{in } H_r^1(0, L_i). \quad (8.1.20)$$

Plugging (8.1.20) into (8.1.5), we have

$$u_i^N \rightarrow u_i \quad \text{in } H_r^1(0, L_i). \quad (8.1.21)$$

A repetition of arguments similar to those given above results in the convergences

$$w_i^N, \eta_i^N \rightarrow w_i, \eta_i \quad \text{in } H_r^2(0, L_i). \quad (8.1.22)$$

Combining (8.1.11), and (8.1.20) - (8.1.22), we have  $r^N \rightarrow r$  and the Lemma is proven.

**Theorem 8.1.1** *For all  $z \in \mathcal{H}^N$ , we have  $T^N(t)P^N(z) \rightarrow T(t)z$  as  $N \rightarrow \infty$ , and the convergence is uniform on bounded intervals of  $t$ .*

Proof: Using Lemmas 8.1.1 and 8.1.2, the theorem follows immediately by an application of the Trotter-Kato Theorem.

## 8.2 Kelvin-Voigt Numerical Scheme

Let  $l_j^{(i)}(s_i)$ ,  $j = 2, \dots, N$  and  $b_j^{(i)}(s_i)$ ,  $j = 1, \dots, N$  be the standard, uniformly-spaced linear and cubic splines on  $[0, L_i]$  satisfying the Dirichlet, clamped boundary conditions at  $s_i = 0$ , and Neumann, free boundary conditions at  $s_i = L_i$  respectively. Make the following definitions

$$\begin{aligned} U_i^N &= \text{Span}\{l_j^{(i)}(s_i)\}, & W_i^N &= \text{Span}\{b_j^{(i)}(s_i)\}, \\ Z_u^N &= U_1^N \times W_1^N \times U_2^N \times W_2^N \times \mathbb{R}^4. \end{aligned}$$

We see that  $H_u^N$  is a finite-dimensional subspace of

$$Z_u = H_r^1(0, L_1) \times H_r^2(0, L_1) \times H_r^1(0, L_2) \times H_r^2(0, L_2) \times \mathbb{R}^4.$$

Also note that  $U_i^N, W_i^N$  satisfy conditions one through four listed at the beginning of Section 1 in this chapter. However, due to the geometric constraint,  $Z_u^N$  must be modified in order to be used in the approximation of the state space  $\mathcal{H}$ . The geometric constraint (3.2.10) affects the following twelve basis elements of  $Z_u^N$ :

$$\xi_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathbf{e}_i \end{bmatrix}, \quad i = 1, \dots, 4, \quad \xi_5 = \begin{bmatrix} l_N^{(1)} \\ 0 \\ 0 \\ 0 \\ \mathbf{0} \end{bmatrix}, \quad \xi_6 = \begin{bmatrix} 0 \\ 0 \\ l_N^{(2)} \\ 0 \\ \mathbf{0} \end{bmatrix},$$

$$\xi_i = \begin{bmatrix} 0 \\ b_{N-(9-i)}^{(1)} \\ 0 \\ 0 \\ \mathbf{0} \end{bmatrix}, \quad i = 7, 8, 9, \quad \xi_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_{N-(12-i)}^{(2)} \\ \mathbf{0} \end{bmatrix}, \quad i = 10, 11, 12.$$

Define  $\mathcal{C} : Z_u^N \rightarrow \mathbb{R}^6$  by

$$\mathcal{C} \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ \mathbf{d} \end{bmatrix} = \begin{bmatrix} -u_1(L_1) \\ w_1(L_1) \\ -u_2(L_2) \\ w_2(L_2) \\ -w_1'(L_1) \\ -w_2'(L_2) \end{bmatrix} - \mathbf{C}^T \mathbf{d} \quad (8.2.1)$$

where  $\mathbf{d}^T = (x, y, \theta_1, \theta_2)$  and  $\mathbf{C}$  is given by (3.1.5). The goal is now to construct linear combinations of the  $\xi_i$  that satisfy the geometric constraint

$$\mathcal{C} \left( \sum_{i=1}^{12} \alpha_i \xi_i \right) = \sum_{i=1}^{12} (\mathcal{C} \xi_i) \alpha_i = \mathbf{0}. \quad (8.2.2)$$

We see that the coefficient vector  $\alpha \in \mathbb{R}^{12}$  must lie in the null space of the matrix

$$\mathbf{C}_d = \begin{bmatrix} -\sin \phi_1 & -\cos \phi_1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos \phi_1 & -\sin \phi_1 & -\ell_1 & 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ -\sin \phi_2 & \cos \phi_2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\cos \phi_2 & -\sin \phi_2 & 0 & -\ell_2 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -b_1 & 0 & b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -b_2 & 0 & b_2 \end{bmatrix} \quad (8.2.3)$$

where  $b_i = (b_{N-2}^{(i)})'(L_i)$ . Using a computer algebra system, we can obtain the following six linear independent vectors in  $\text{Null}(\mathbf{C}_d)$ ,

$$\begin{aligned}
& ( 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ -2 \ 1 \ -2 ), \\
& ( 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ -2 \ 1 \ -2 \ 0 \ 0 \ 0 ), \\
& ( 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \frac{1}{2}(-\frac{1}{b_2} + \ell_2) \ 0 \ \frac{1}{2}(\frac{1}{b_2} + \ell_2) ), \\
& ( 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ \frac{1}{2}(-\frac{1}{b_1} + \ell_1) \ 0 \ \frac{1}{2}(\frac{1}{b_1} + \ell_1) \ 0 \ 0 \ 0 ), \\
& ( 0 \ 1 \ 0 \ 0 \ -\cos \phi_1 \ -\cos \phi_2 \ \frac{1}{2} \sin \phi_1 \ 0 \ \frac{1}{2} \sin \phi_1 \ \frac{1}{2} \sin \phi_2 \ 0 \ \frac{1}{2} \sin \phi_2 ), \\
& \left( 1 \ 0 \ 0 \ 0 \ -\sin \phi_1 \ -\sin \phi_2 \ -\frac{1}{2} \cos \phi_1 \ 0 \ -\frac{1}{2} \cos \phi_1 \ \frac{1}{2} \cos \phi_2 \ 0 \ \frac{1}{2} \cos \phi_2 \right).
\end{aligned}$$

Therefore, we have obtained basis vectors of  $\text{Null}(\mathbf{C}_d)$  :

$$\left\{ \begin{array}{l} -2\xi_{10} + \xi_{11} - 2\xi_{12} \\ -2\xi_7 + \xi_8 - 2\xi_9 \\ \frac{1}{2}[(-\frac{1}{b_2} + \ell_2)\xi_{10} + (\frac{1}{b_2} + \ell_2)\xi_{12}] + \xi_4 \\ \frac{1}{2}[(-\frac{1}{b_1} + \ell_1)\xi_7 + (\frac{1}{b_1} + \ell_1)\xi_9] + \xi_3 \\ -\cos \phi_1 \xi_5 + \cos \phi_2 \xi_6 + \frac{1}{2}[\sin \phi_1 \xi_7 + \sin \phi_1 \xi_9 + \sin \phi_2 \xi_{10} + \sin \phi_2 \xi_{12}] + \xi_2 \\ -\sin \phi_1 \xi_5 - \sin \phi_2 \xi_6 + \frac{1}{2}[-\cos \phi_1 \xi_7 - \cos \phi_1 \xi_9 + \cos \phi_2 \xi_{10} + \cos \phi_2 \xi_{12}] + \xi_1. \end{array} \right. \quad (8.2.4)$$

Let  $Z_c^N$  denote the subspace of  $Z_u^N$  which satisfies the geometric constraint. Using the modified vectors (8.2.4), we can now set up a basis for  $Z_c^N$  as follows:

$$\tilde{\mathbf{b}}_i = \begin{bmatrix} l_{i-1}^{(1)} \\ 0 \\ 0 \\ 0 \\ \mathbf{0} \end{bmatrix}, \quad i = 2, \dots, N-1,$$

$$\tilde{\mathbf{b}}_{N-2+i} = \begin{bmatrix} 0 \\ b_i^{(1)} \\ 0 \\ 0 \\ \mathbf{0} \end{bmatrix}, \quad i = 1, \dots, N-3,$$

$$\tilde{\mathbf{b}}_{2N-4} = \begin{bmatrix} 0 \\ -2b_{N-2}^{(1)} + b_{N-1}^{(1)} - 2b_N^{(1)} \\ 0 \\ 0 \\ \mathbf{0} \end{bmatrix},$$

$$\tilde{\mathbf{b}}_{2N-3} = \begin{bmatrix} 0 \\ \frac{1}{2}[(-\frac{1}{b_1} + \ell_1)b_{N-2}^{(1)} + (\frac{1}{b_1} + \ell_1)b_N^{(1)}] \\ 0 \\ 0 \\ \mathbf{e}_3 \end{bmatrix},$$

$$\tilde{\mathbf{b}}_{2N-4+i} = \begin{bmatrix} 0 \\ 0 \\ l_i^{(2)} \\ 0 \\ \mathbf{0} \end{bmatrix}, \quad i = 2, \dots, N-1,$$

$$\tilde{\mathbf{b}}_{3N-5+i} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_i^{(1)} \\ \mathbf{0} \end{bmatrix}, \quad i = 1, \dots, N-3,$$

$$\tilde{\mathbf{b}}_{4N-7} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2b_{N-2}^{(2)} + b_{N-1}^{(2)} - 2b_N^{(2)} \\ \mathbf{0} \end{bmatrix},$$

$$\tilde{\mathbf{b}}_{4N-6} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2}[(-\frac{1}{b_2} + \ell_2)b_{N-2}^{(2)} + (\frac{1}{b_2} + \ell_2)b_N^{(2)}] \\ \mathbf{e}_4 \end{bmatrix},$$

$$\tilde{\mathbf{b}}_{4N-5} = \begin{bmatrix} -\cos \phi_1 l_N^{(1)} \\ \frac{1}{2} \sin \phi_1 (b_{N-2}^{(1)} + b_{N-2}^{(1)}) \\ \cos \phi_2 l_N^{(2)} \\ \frac{1}{2} \sin \phi_2 (b_{N-2}^{(2)} + b_{N-2}^{(2)}) \\ \mathbf{e}_2 \end{bmatrix},$$

and

$$\tilde{\mathbf{b}}_{4N-4} = \begin{bmatrix} -\sin \phi_1 l_N^{(1)} \\ -\frac{1}{2} \cos \phi_1 (b_{N-2}^{(1)} + b_{N-2}^{(1)}) \\ -\sin \phi_2 l_N^{(2)} \\ \frac{1}{2} \cos \phi_2 (b_{N-2}^{(2)} + b_{N-2}^{(2)}) \\ \mathbf{e}_1 \end{bmatrix}.$$

Now, let  $\tilde{\mathcal{H}}^N$  be the  $8N - 8$  dimensional space  $Z_c^N \times Z_c^N$  and define

$$\mathbf{b}_j = \begin{cases} \begin{bmatrix} \tilde{\mathbf{b}}_j \\ \mathbf{0} \end{bmatrix} & j = 1, \dots, 4N - 4 \\ \begin{bmatrix} \mathbf{0} \\ \tilde{\mathbf{b}}_{j-4N+4} \end{bmatrix} & j = 4N - 3, \dots, 8N - 8. \end{cases} \quad (8.2.5)$$

We can write  $\tilde{z}^N \in \tilde{\mathcal{H}}^N$  as

$$\tilde{z}^N = \begin{bmatrix} u_1^N \\ w_1^N \\ u_2^N \\ w_2^N \\ \mathbf{d} \\ v_1^N \\ \eta_1^N \\ v_2^N \\ \eta_2^N \\ \mathbf{a} \end{bmatrix} = \sum_{j=1}^{8N-8} r_j(t) \mathbf{b}_j(s). \quad (8.2.6)$$

Note that  $\tilde{\mathcal{H}}^N$  is not a subspace of the state space  $\mathcal{H}$  because of the additional  $\mathbf{d}$  component. Define  $\mathcal{H}^N$  to be the space obtained by deleting the  $\mathbf{d}$  component of  $\tilde{\mathcal{H}}^N$ . A basis for  $\mathcal{H}^N$  is similarly obtained from the basis for  $\tilde{\mathcal{H}}^N$  by deleting the  $\mathbf{d}$  component of each basis

element. Note that the dimension of  $\mathcal{H}^N$  is still  $8N - 8$  and we can write  $z^N \in \mathcal{H}^N$  as

$$z^N = \begin{bmatrix} u_1^N \\ w_1^N \\ u_2^N \\ w_2^N \\ v_1^N \\ \eta_1^N \\ v_2^N \\ \eta_2^N \\ \mathbf{a} \end{bmatrix} = \sum_{j=1}^{8N-8} r_j(t) \mathbf{b}_j(s) \quad (8.2.7)$$

where it is understood that the  $\mathbf{b}_j$  have had the  $\mathbf{d}$  component deleted. The reason for working with  $\mathcal{H}^N$  instead of  $\tilde{\mathcal{H}}^N$  is so that the numerical scheme will fit more clearly with the convergence results given earlier in this chapter. It should be mentioned, however, that the actual implementation is unchanged whether we choose to work with  $\mathcal{H}^N$  or  $\tilde{\mathcal{H}}^N$ . Each will result in an equivalent first order system.

Let  $G_i^N$  denote  $U_i^N \times W_i^N$  and set  $G^N = G_1^N \times G_2^N$ . Using the definition of  $\mathcal{A}^N$  as given in (8.1.2), we can now write the finite dimensional system dynamics as

$$\mathcal{M}^N \dot{z}^N = \mathcal{A}^N z^N = \begin{bmatrix} v_1^N \\ \eta_1^N \\ v_2^N \\ \eta_2^N \\ D_1^N (E_1 A_1 u_1^N + \mu_1 v_1^N) \\ -K_1^N (E_1 I_1 w_1^N + \gamma_1 \eta_1^N) \\ D_2^N (E_2 A_2 u_2^N + \mu_2 v_2^N) \\ -K_2^N (E_2 I_2 w_2^N + \gamma_2 \eta_2^N) \\ \mathbf{C}(F_1^N, N_1^N, F_2^N, N_2^N, M_1^N, M_2^N)^T \end{bmatrix} \quad (8.2.8)$$

where

$$\mathcal{M}^N = \begin{bmatrix} \mathbf{I}_{G^N} & & & \\ & \rho_1 A_1 \mathbf{I}_{G_1^N} & & \\ & & \rho_2 A_2 \mathbf{I}_{G_2^N} & \\ & & & \mathbf{M} \end{bmatrix}. \quad (8.2.9)$$

Taking the inner product of both sides of (8.2.8) with  $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \boldsymbol{\phi}_J) \in \mathcal{H}^N$ , we have

$$\begin{aligned}
& \langle (u_1^N)_t, \phi_1 \rangle + \langle (w_1^N)_t, \phi_2 \rangle + \langle (u_2^N)_t, \phi_3 \rangle + \langle (w_2^N)_t, \phi_4 \rangle + \rho_1 A_1 \left( \langle (v_1^N)_t, \phi_5 \rangle + \langle (\eta_1^N)_t, \phi_6 \rangle \right) \\
& + \rho_2 A_2 \left( \langle (v_2^N)_t, \phi_7 \rangle + \langle (\eta_2^N)_t, \phi_8 \rangle \right) + \mathbf{a}_t \mathbf{M} \boldsymbol{\phi}_J \\
= & \langle v_1^N, \phi_1 \rangle + \langle \eta_1^N, \phi_2 \rangle + \langle v_2^N, \phi_3 \rangle + \langle \eta_2^N, \phi_4 \rangle - \langle (E_1 A_1 u_1^N + \mu_1 v_1^N)', \phi_5' \rangle \\
& - \langle (E_1 I_1 w_1^N + \gamma_1 \eta_1^N)'' , \phi_6'' \rangle - \langle (E_2 A_2 u_2^N + \mu_2 v_2^N)', \phi_7' \rangle - \langle (E_2 I_2 w_2^N + \gamma_2 \eta_2^N)'' , \phi_8'' \rangle \\
& - \langle \mathbf{F}, \mathcal{C} \boldsymbol{\phi} \rangle_{\mathbb{R}^6}
\end{aligned} \tag{8.2.10}$$

where  $\mathbf{F} = (F_1^N, N_1^N, F_2^N, N_2^N, M_1^N, M_2^N)$ . Replacing  $\boldsymbol{\phi}$  by  $\mathbf{b}_i$ , we see that the last term in (8.2.10) is zero by the construction of the basis functions. If we use the representation (8.2.7) and let  $\boldsymbol{\phi}$  range over the basis functions  $\mathbf{b}_i$ , then the coefficient vector  $\mathbf{r}(t) = (r_j(t))_{j=1}^{8N-8}$  satisfies the first order system

$$M^N \dot{\mathbf{r}}(t) = A^N \mathbf{r}(t). \tag{8.2.11}$$

If we let  $b_j^k$  denote the  $k$  component of  $\mathbf{b}_j$ , then the matrices  $M$  and  $A$  are given by

$$\begin{aligned}
M_{(i,j)}^N &= \sum_{k=1}^4 \left( \langle \mathbf{b}_j^k, \mathbf{b}_i^k \rangle \right) + \rho_1 A_1 \left( \langle \mathbf{b}_j^5, \mathbf{b}_i^5 \rangle + \langle \mathbf{b}_j^6, \mathbf{b}_i^6 \rangle \right) + \rho_2 A_2 \left( \langle \mathbf{b}_j^7, \mathbf{b}_i^7 \rangle \right. \\
& \left. + \langle \mathbf{b}_j^8, \mathbf{b}_i^8 \rangle \right) + (\mathbf{b}_j^{\mathbf{a}})^T \mathbf{M} \mathbf{b}_i^{\mathbf{a}}
\end{aligned} \tag{8.2.12}$$

and

$$\begin{aligned}
A_{(i,j)}^N &= \sum_{k=1}^4 \left( \langle \mathbf{b}_j^{4+k}, \mathbf{b}_i^k \rangle \right) - \langle (E_1 A_1 \mathbf{b}_j^1 + \mu_1 \mathbf{b}_j^5)', (\mathbf{b}_i^5)' \rangle - \langle (E_1 I_1 \mathbf{b}_j^2 + \gamma_1 \mathbf{b}_j^6)'' , (\mathbf{b}_i^6)'' \rangle \\
& - \langle (E_2 A_2 \mathbf{b}_j^3 + \mu_2 \mathbf{b}_j^7)', (\mathbf{b}_i^7)' \rangle - \langle (E_2 I_2 \mathbf{b}_j^4 + \gamma_2 \mathbf{b}_j^8)'' , (\mathbf{b}_i^8)'' \rangle.
\end{aligned} \tag{8.2.13}$$

We assume the beam material parameters are as given in Table 8.2.1. If we let  $N = 32$ ,  $\mu_i = 10$ , and  $\gamma_i = .1$ , the eigenvalues of the approximating system are shown in Figure 8.2.1. It is interesting to note how these values are affected by the damping parameters. This is shown for axial damping in Figure 8.2.2 and for transversal damping in Figure 8.2.3. A complete examination of the eigenvalues, including convergence, is given in [3].

Table 8.2.1: Material Parameter Values

parameter	value	parameter	value
$L$	2	$\alpha$	$1.0 \times 10^{-6}$
$r$	.064	$L$	1
$h$	.0003	$r$	1
$\rho$	1320	$c$	903
$E$	$0.9 \times 10^{11}$	$\ell$	.2
$\epsilon$	0.4	$d$	.1
$\alpha_s$	0.4	$m$	8% of beam mass
$k_a$	5.75	$m_p$	20% of beam mass
$k_c$	2.34	$\phi_1$	$45^\circ$
$T_0$	280	$\phi_2$	$45^\circ$

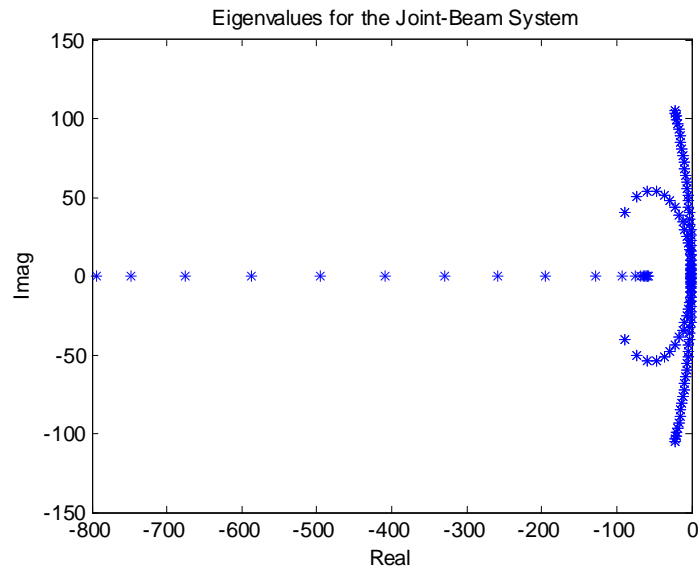


Figure 8.2.1: Eigenvalues for the Joint-Beam System with  $N=32$ ,  $\mu_i = 10$ ,  $\gamma_i = .1$

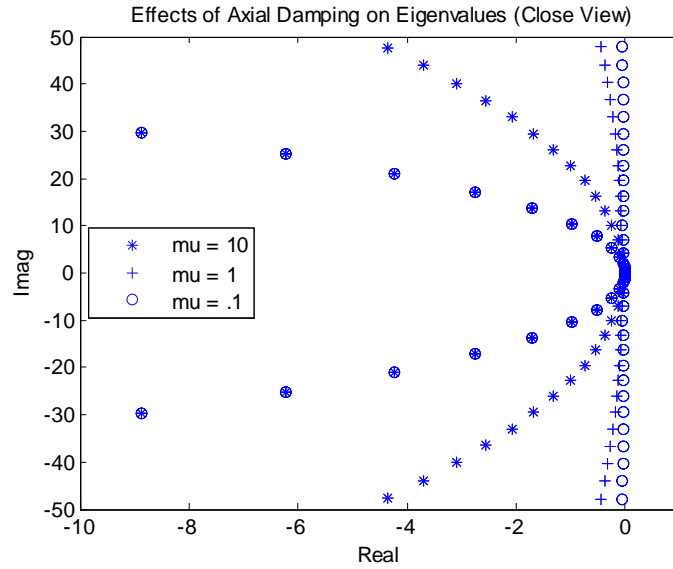


Figure 8.2.2: Effects of Axial Damping for  $N=32$ , and  $\gamma_i$  fixed at .1

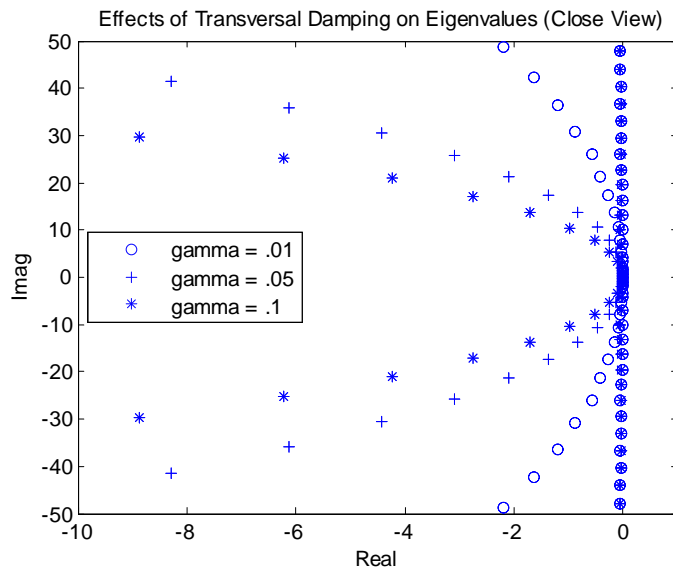


Figure 8.2.3: Effects of Transversal Damping for  $N=32$ , and  $\mu_i$  fixed at .1

### 8.3 Thermoelastic Numerical Scheme

The same spaces and basis functions defined in the previous section can be used to approximate the mechanical motions of the thermoelastic system. In order to approximate the thermal flow, however, new spaces must be defined which account for the Robin boundary conditions. We begin by recalling these conditions as given in Chapter 5:

$$\frac{\partial}{\partial s_i} \tilde{T}^i(t, L_i) = -\lambda_R^i \tilde{T}^i(t, L_i) \quad (8.3.1)$$

$$\frac{\partial}{\partial s_i} \tilde{T}^i(t, 0) = \lambda_L^i \tilde{T}^i(t, 0) \quad (8.3.2)$$

$$\frac{\partial}{\partial s_i} T^{m,i}(t, L_i) = -\lambda_R^i T^{m,i}(t, L_i) \quad (8.3.3)$$

$$\frac{\partial}{\partial s_i} T^{m,i}(t, 0) = \lambda_L^i T^{m,i}(t, 0). \quad (8.3.4)$$

Let  $l_j^{(i)}(s_i)$ ,  $j = 1, \dots, N$  be the standard, uniformly-spaced linear "hat" splines on  $[0, L_i]$  such that

$$l_1^{(i)}(0) = 1, \quad l_N^{(i)}(L_i) = 1. \quad (8.3.5)$$

Note that  $l_1^{(i)}, l_2^{(i)}, l_3^{(i)}$ , and  $l_4^{(i)}$  do not satisfy the Robin boundary conditions. We must therefore modify the splines before we can use them as basis functions for the thermal components. Define

$$\hat{U}_i^N = \text{Span}\{\hat{l}_j^{(i)} \mid j = 1, \dots, N-2\},$$

where

$$\begin{aligned} \hat{l}_1^{(i)} &= l_1^{(i)} + \left(1 + \left(\frac{L_i}{N-1}\right) \lambda_L^i\right) l_2^{(i)}, \\ \hat{l}_{N-2}^{(i)} &= l_N^{(i)} + \left(1 + \left(\frac{L_i}{N-1}\right) \lambda_R^i\right) l_{N-1}^{(i)} \\ \hat{l}_j^{(i)} &= l_{j+1}^{(i)}, \quad j = 2, \dots, N-3. \end{aligned}$$

$\hat{U}_i^N$  satisfies the Robin boundary conditions and is therefore suitable for use in approximating the axial and transversal thermal components of each beam. Recalling the space  $\mathcal{H}^N$  defined

in the previous section, we can now define the space

$$\mathcal{H}_T^N = \mathcal{H}^N \times \left(\hat{U}_1^N\right)^2 \times \left(\hat{U}_2^N\right)^2. \quad (8.3.6)$$

The first  $8N - 8$  basis functions of  $\mathcal{H}_T^N$  can be identified with the basis functions from the Kelvin-Voigt approximation:

$$\mathbf{b}_j = \begin{bmatrix} \mathbf{b}_j \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad j = 1, \dots, 8N - 8. \quad (8.3.7)$$

For the remaining  $4N - 8$  basis functions which must cover the final four components, we have:

$$\mathbf{b}_{8N-8+j} = \begin{bmatrix} \mathbf{0} \\ l_j^1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad j = 1, \dots, N - 2,$$

$$\mathbf{b}_{9N-10+j} = \begin{bmatrix} \mathbf{0} \\ 0 \\ l_j^1 \\ 0 \\ 0 \end{bmatrix} \quad j = 1, \dots, N - 2,$$

$$\mathbf{b}_{10N-12+j} = \begin{bmatrix} \mathbf{0} \\ 0 \\ 0 \\ l_j^1 \\ 0 \end{bmatrix} \quad j = 1, \dots, N - 2,$$

and

$$\mathbf{b}_{11N-14+j} = \begin{bmatrix} \mathbf{0} \\ 0 \\ 0 \\ 0 \\ l_j^1 \end{bmatrix} \quad j = 1, \dots, N-2.$$

Using these basis functions, we can represent any vector  $z^N \in \mathcal{H}_T^N$  as

$$z^N = \begin{bmatrix} u_1^N \\ w_1^N \\ u_2^N \\ w_2^N \\ v_1^N \\ \eta_1^N \\ v_2^N \\ \eta_2^N \\ \mathbf{a} \\ (T^1)^N \\ (T^{m,1})^N \\ (T^2)^N \\ (T^{m,2})^N \end{bmatrix} = \sum_{j=1}^{12N-16} r_j(t) \mathbf{b}_j(s). \quad (8.3.8)$$

Letting  $S_i^N = \hat{U}_i^N \times \hat{U}_i^N$  and using the notations  $D_i^N$  and  $K_i^N$  from Section 8.1, the dynamics for the homogenous (no solar flux) finite-dimensional thermoelastic system can be

written as

$$\mathcal{M}^N \dot{z}^N = \mathcal{A}_T^N z^N = \begin{bmatrix} v_1^N \\ \eta_1^N \\ v_2^N \\ \eta_2^N \\ E_1 A_1 \left( D_1^N u_1^N - \alpha_1 (\tilde{T}_1^N)' \right) \\ -E_1 I_1 \left( K_1^N w_1^N + \frac{\alpha_1}{2R_1} D_1^N (T^{m,1})^N \right) \\ E_2 A_2 \left( D_2^N u_2^N - \alpha_1 (\tilde{T}^{2N})' \right) \\ -E_2 I_2 \left( K_2^N w_2^N + \frac{\alpha_2}{2R_2} D_2^N (T^{m,2})^N \right) \\ \mathbf{C}(F_1^N, N_1^N, F_2^N, N_2^N, M_1^N, M_2^N)^T \\ D_1^N \left( \beta_{11} (\tilde{T}^1)^N \right) - \beta_{12} (\tilde{T}^1)^N - \beta_{13} (v_1^N)' \\ D_1^N \left( \beta_{21} (T^{m,1})^N \right) - \beta_{22} (T^{m,1})^N + \beta_{23} (\eta_1^N)'' \\ D_1^N \left( \beta_{31} (\tilde{T}^2)^N \right) - \beta_{32} (\tilde{T}^2)^N - \beta_{33} (v_2^N)' \\ D_1^N \left( \beta_{41} (T^{m,2})^N \right) - \beta_{42} (T^{m,2})^N + \beta_{43} (\eta_2^N)'' \end{bmatrix} \quad (8.3.9)$$

where

$$\mathcal{M}^N = \begin{bmatrix} \mathbf{I}_{G^N} & & & & & & & \\ & \rho_1 A_1 \mathbf{I}_{G_1^N} & & & & & & \\ & & \rho_2 A_2 \mathbf{I}_{G_2^N} & & & & & \\ & & & \mathbf{M} & & & & \\ & & & & \rho_1 c_1 \mathbf{I}_{S_1^N} & & & \\ & & & & & & \rho_2 c_2 \mathbf{I}_{S_2^N} & \end{bmatrix}, \quad (8.3.10)$$

and the  $\beta_{ik}$  are the appropriate constants as given in Chapter 5. Take the inner product of

both sides of (8.3.9) with  $\phi \in \mathcal{H}_T^N$  to get

$$\begin{aligned}
& \langle (u_1^N)_t, \phi_1 \rangle + \langle (w_1^N)_t, \phi_2 \rangle + \langle (u_2^N)_t, \phi_3 \rangle + \langle (w_2^N)_t, \phi_4 \rangle + \rho_1 A_1 \left( \langle (v_1^N)_t, \phi_5 \rangle + \langle (\eta_1^N)_t, \phi_6 \rangle \right) \\
& + \rho_2 A_2 \left( \langle (v_2^N)_t, \phi_7 \rangle + \langle (\eta_2^N)_t, \phi_8 \rangle \right) + \mathbf{a}_t \mathbf{M} \phi_J + \rho_1 c_1 \left( \langle \tilde{T}_1^N, \phi_9 \rangle + \langle T_{m,1}^N, \phi_{10} \rangle \right) \\
& + \rho_2 c_2 \left( \langle \tilde{T}_2^N, \phi_{11} \rangle + \langle T_{m,2}^N, \phi_{12} \rangle \right) \\
= & \langle v_1^N, \phi_1 \rangle + \langle \eta_1^N, \phi_2 \rangle + \langle v_2^N, \phi_3 \rangle + \langle \eta_2^N, \phi_4 \rangle - \langle E_1 A_1 \left( (u_1^N)' - \alpha_1 \tilde{T}_1^N \right), \phi_5' \rangle \\
& - \langle E_1 I_1 \left( (w_1^N)'' + \frac{\alpha_1}{2R_1} T_{m,1}^N \right), \phi_6'' \rangle - \langle E_2 A_2 \left( (u_2^N)' - \alpha_2 \tilde{T}_2^N \right), \phi_7' \rangle \\
& - \langle E_2 I_2 \left( (w_2^N)'' + \frac{\alpha_2}{2R_2} T_{m,2}^N \right), \phi_8'' \rangle - \beta_{11} \langle (\tilde{T}_1^N)', \phi_9' \rangle - \beta_{12} \langle \tilde{T}_1^N, \phi_9 \rangle \\
& - \beta_{13} \langle (v_1^N)', \phi_9 \rangle - \lambda_R^1 \tilde{T}_1^N(L_1) \phi_9(L_1) - \lambda_L^1 \tilde{T}_1^N(0) \phi_9(0) - \beta_{21} \langle (T_{m,1}^N)', \phi_{10}' \rangle \\
& - \beta_{22} \langle T_{m,1}^N, \phi_{10} \rangle + \beta_{23} \langle (\eta_1^N)'', \phi_{10} \rangle - \lambda_R^1 T_{m,1}^N(L_1) \phi_{10}(L_1) - \lambda_L^1 T_{m,1}^N(0) \phi_{10}(0) \\
& - \beta_{31} \langle (\tilde{T}_2^N)', \phi_{11}' \rangle - \beta_{32} \langle \tilde{T}_2^N, \phi_{11} \rangle - \beta_{33} \langle (v_2^N)', \phi_{11} \rangle - \lambda_R^2 \tilde{T}_2^N(L_2) \phi_{11}(L_2) \\
& - \lambda_L^2 \tilde{T}_2^N(0) \phi_{11}(0) - \beta_{41} \langle (T_{m,2}^N)', \phi_{12}' \rangle - \beta_{42} \langle T_{m,2}^N, \phi_{12} \rangle + \beta_{43} \langle (\eta_2^N)'', \phi_{12} \rangle \\
& - \lambda_R^2 T_{m,2}^N(L_2) \phi_{12}(L_2) - \lambda_L^2 T_{m,2}^N(0) \phi_{12}(0) - \langle \mathbf{F}, \mathcal{C}\phi \rangle_{\mathbb{R}^6}. \tag{8.3.11}
\end{aligned}$$

As in the previous section, if we replace  $\phi$  by  $\mathbf{b}_i$ , we see that the last term in (8.3.11) is zero by the construction of the basis functions. If we use the representation (8.3.8) and let  $\phi$  range over the basis functions  $\mathbf{b}_i$ , then the coefficient vector  $\mathbf{r}(t) = (r_j(t))_{j=1}^{12N-16}$  satisfies the first order system

$$M^N \dot{\mathbf{r}}(t) = A^N \mathbf{r}(t) \tag{8.3.12}$$

where

$$\begin{aligned}
M_{(i,j)}^N &= \sum_{k=1}^4 \left( \langle \mathbf{b}_j^k, \mathbf{b}_i^k \rangle \right) + \rho_1 A_1 \left( \langle \mathbf{b}_j^5, \mathbf{b}_i^5 \rangle + \langle \mathbf{b}_j^6, \mathbf{b}_i^6 \rangle \right) + \rho_2 A_2 \left( \langle \mathbf{b}_j^7, \mathbf{b}_i^7 \rangle \right. \\
& \left. + \langle \mathbf{b}_j^8, \mathbf{b}_i^8 \rangle \right) + \rho_1 c_1 \left( \langle \mathbf{b}_j^9, \mathbf{b}_i^9 \rangle + \langle \mathbf{b}_j^{10}, \mathbf{b}_i^{10} \rangle \right) + \rho_2 c_2 \left( \langle \mathbf{b}_j^{11}, \mathbf{b}_i^{11} \rangle + \langle \mathbf{b}_j^{12}, \mathbf{b}_i^{12} \rangle \right) \\
& + (\mathbf{b}_j^{\mathbf{a}})^T \mathbf{M} \mathbf{b}_i^{\mathbf{a}} \tag{8.3.13}
\end{aligned}$$

Table 8.3.2: Real Parts of Eigenvalues Closest to the Imaginary Axis

N	$\alpha = 10^{-6}$	$\alpha = 10^{-5}$	$\alpha = 10^{-4}$
8	-1.6544e-9	-1.6555e-7	-1.9017e-5
16	-1.5494e-9	-1.5513e-7	-1.7641e-5
32	-1.1704e-9	-1.1712e-7	-1.2573e-5
64	-1.0764e-9	-1.0767e-7	-1.1171e-5
128	-1.0825e-9	-1.0827e-7	-1.1028e-5

and

$$\begin{aligned}
A_{(i,j)}^N = & \sum_{k=1}^4 \left( \langle \mathbf{b}_j^{4+k}, \mathbf{b}_i^k \rangle \right) - \langle E_1 A_1 ((\mathbf{b}_j^1)' - \alpha_1 \mathbf{b}_j^9), (\mathbf{b}_i^5)' \rangle - \langle E_1 I_1 \left( (\mathbf{b}_j^2)'' + \frac{\alpha_1}{2R_1} \mathbf{b}_j^{10} \right), (\mathbf{b}_i^6)'' \rangle \\
& - \langle E_2 A_2 ((\mathbf{b}_j^3)' - \alpha_2 \mathbf{b}_j^{11}), (\mathbf{b}_i^7)' \rangle - \langle E_2 I_2 \left( (\mathbf{b}_j^4)'' + \frac{\alpha_2}{2R_2} \mathbf{b}_j^{12} \right), (\mathbf{b}_i^8)'' \rangle \\
& - \beta_{11} \langle (\mathbf{b}_j^9)', (\mathbf{b}_i^9)' \rangle - \beta_{12} \langle \mathbf{b}_j^9, \mathbf{b}_i^9 \rangle - \beta_{13} \langle (\mathbf{b}_j^5)', \mathbf{b}_i^9 \rangle \\
& - \lambda_L^1 \mathbf{b}_j^9(L_1) \mathbf{b}_i^9(L_1) - \lambda_L^1 \mathbf{b}_j^9(0) \mathbf{b}_i^9(0) - \beta_{21} \langle (\mathbf{b}_j^{10})', (\mathbf{b}_i^{10})' \rangle \\
& - \beta_{22} \langle \mathbf{b}_j^{10}, \mathbf{b}_i^{10} \rangle + \beta_{23} \langle (\mathbf{b}_j^6)''', \mathbf{b}_i^{10} \rangle - \lambda_R^1 \mathbf{b}_j^{10}(L_1) \mathbf{b}_i^{10}(L_1) - \lambda_L^1 \mathbf{b}_j^{10}(0) \mathbf{b}_i^{10}(0) \\
& - \beta_{31} \langle (\mathbf{b}_j^{11})', (\mathbf{b}_i^{11})' \rangle - \beta_{32} \langle \mathbf{b}_j^{11}, \mathbf{b}_i^{11} \rangle - \beta_{33} \langle (\mathbf{b}_j^7)', \mathbf{b}_i^{11} \rangle - \lambda_R^2 \mathbf{b}_j^{11}(L_2) \mathbf{b}_i^{11}(L_2) \\
& - \lambda_L^2 \mathbf{b}_j^{11}(0) \mathbf{b}_i^{11}(0) - \beta_{41} \langle (\mathbf{b}_j^{12})', (\mathbf{b}_i^{12})' \rangle - \beta_{42} \langle \mathbf{b}_j^{12}, \mathbf{b}_i^{12} \rangle + \beta_{43} \langle (\mathbf{b}_j^8)''', \mathbf{b}_i^{12} \rangle \\
& - \lambda_R^2 \mathbf{b}_j^{12}(L_2) \mathbf{b}_i^{12}(L_2) - \lambda_L^2 \mathbf{b}_j^{12}(0) \mathbf{b}_i^{12}(0). \tag{8.3.14}
\end{aligned}$$

Unless otherwise stated, we assume the values of the material parameters are given in Table 8.3.2. The eigenvalues for the isolated thermoelastic system are highly dependent upon  $\alpha$ , the coefficient of thermal expansion. In Table 8.3.2 we have computed the real parts of the eigenvalues closest to the imaginary axis for varying values of  $N$  and  $\alpha$ . The larger  $\alpha$  values allow mechanical energy to be transformed into thermal energy more easily. Since energy is only dissipated in the form of thermal energy, larger  $\alpha$  values give faster energy decay for vibrations.

This decay can be seen quite clearly from Figure 8.3.1, where we have subjected our system to an initial axial vibration in Beam 1. Note that the energy decay is very slow for both values of  $\alpha$ . If, on the other hand, we give Beam 1 an initial thermal distribution, the energy decay is much faster (see Figure 8.3.2).

Thus far, our thermoelastic simulations have dealt with the isolated system. It is inter-

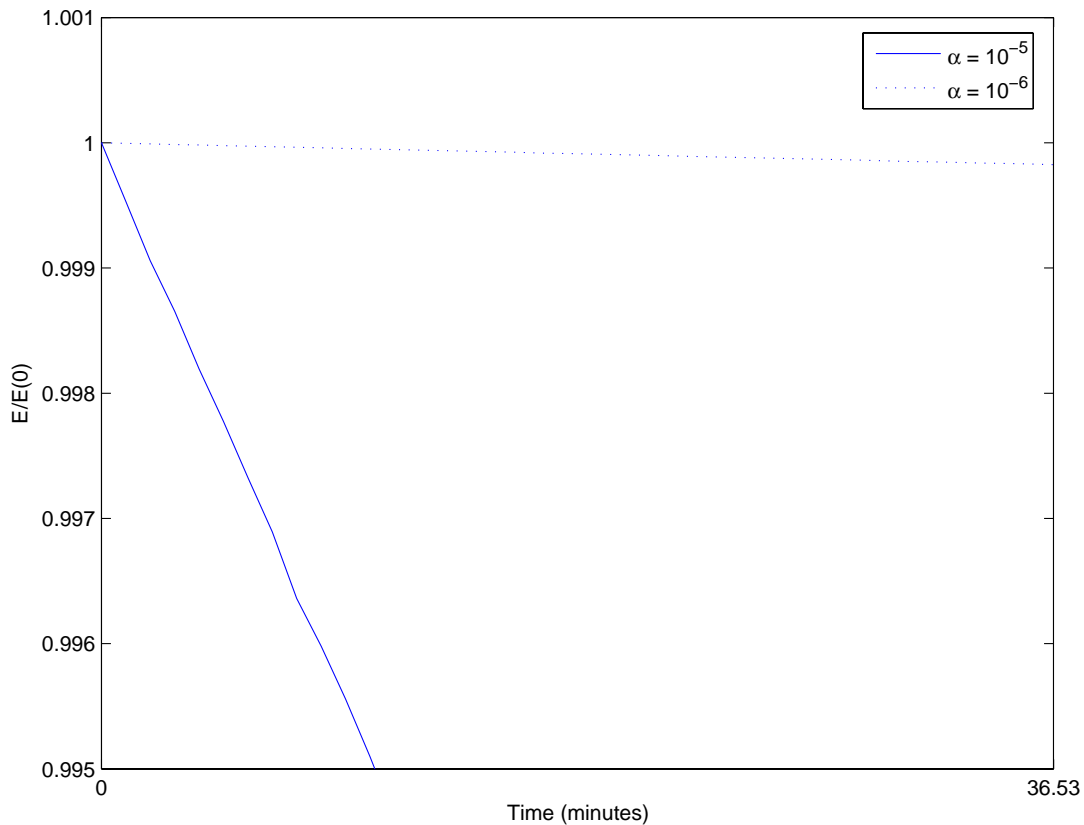


Figure 8.3.1: Energy Dissipation from an Initial Mechanical Vibration

esting to view the response when our system is subjected to solar radiation. Initially, we assume that both beams have the material parameters given in Table 8.2.1. Suppose solar radiation is hitting Beam 1 orthogonally so that the top of the beam (the side in the direction of positive  $w_1$ ) is heated. For this case, the deflections after a short period of time are shown for the positional components  $w_1$  and  $u_2$  in Figure 8.3.3. Note that, although the radiation tries to bend Beam 1 downward, the pressure is not enough to overcome the axial stiffness of Beam 2. There is some compression of Beam 2, but not much. We should mention here that when this simulation is continued for a longer period of time, constant radiation results in a temperature gradient in Beam 1 which is strong enough to force the beam downward in spite of Beam 2's stiffness.

In order to illustrate the case further, let the modulus of Beam 2 be weakened to  $E_2 = .9e9$ . Then, as seen in Figure 8.3.4, the transversal bending of Beam 1 almost immediately overwhelms the axial stiffness of Beam 2.

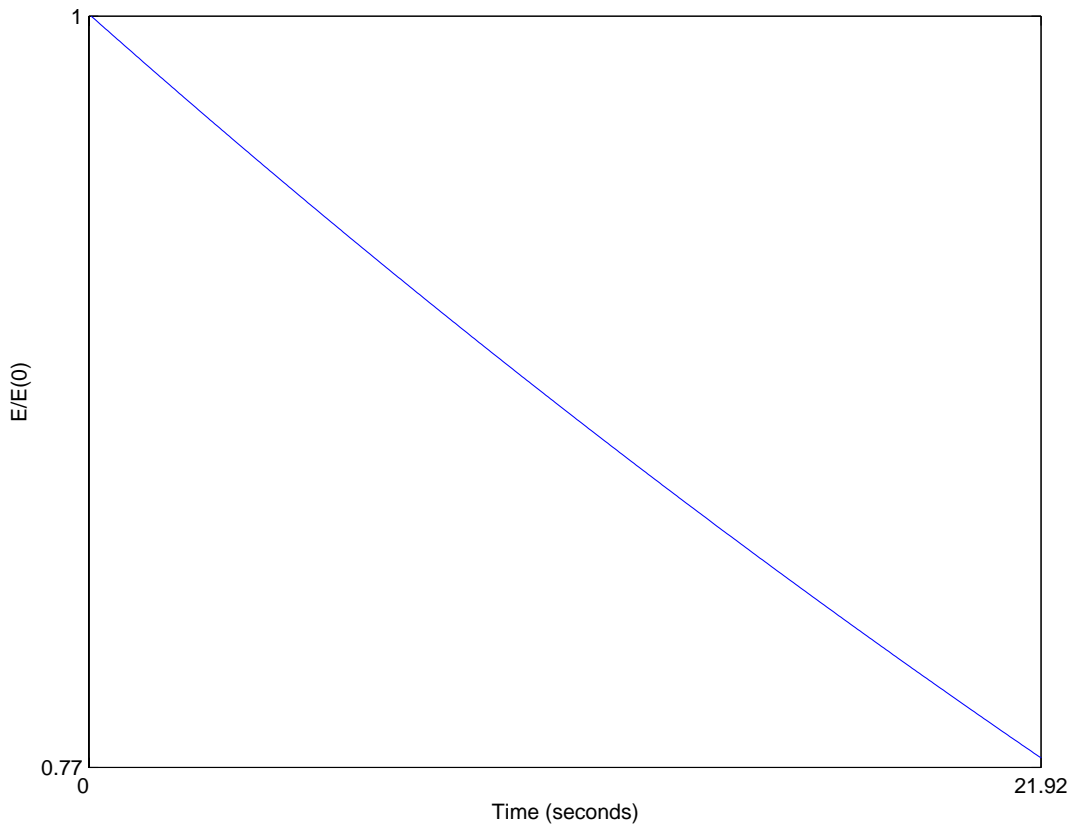


Figure 8.3.2: Energy Dissipation from an Initial Thermal Distribution

Before closing this chapter, we would like to briefly mention the approximation of a triangular joint-beam system. The process of modifying the basis functions to account for joint dynamics is identical to that for the Kelvin-Voigt joint-beam system. However, the complexity is greatly magnified by the fact that eighteen modified basis functions are needed instead of just six, as in the case for the single joint. Using 30 elements for each beam and material parameters similar to those listed for the thermoelastic system, the eigenvalues for the triangular system are shown in Figure 8.3.5. It also interesting to look at the evolution in time of the strain distribution. For example, suppose that Beam 1 is given an initial transversal velocity in a small area near the center of the beam. Assume that the velocity is directed outward from the center of the triangle. Figures 8.3.6 - 8.3.8 then show how the disturbance is propagated throughout the triangle.

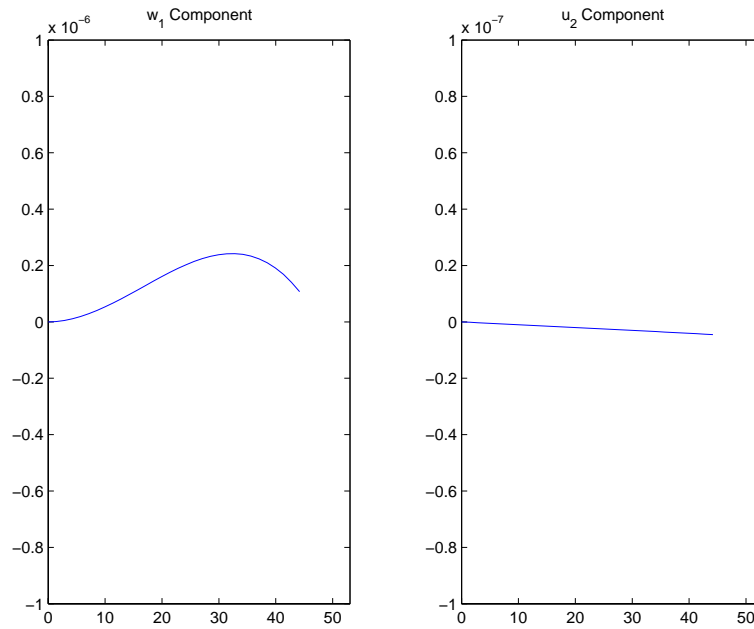


Figure 8.3.3: Deflections of Equally Stiff Beams after .006576 Seconds of Solar Radiation - Axes are in the Length Scale of 1 = .0453 meters

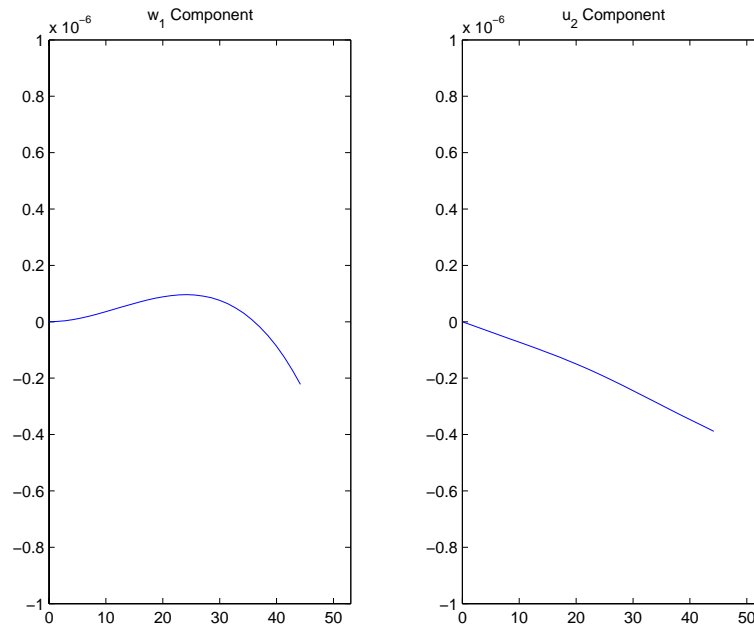


Figure 8.3.4: Deflections of Unequally Stiff Beams after .006576 Seconds of Solar Radiation - Axes are in the Length Scale of 1 = .0453 meters

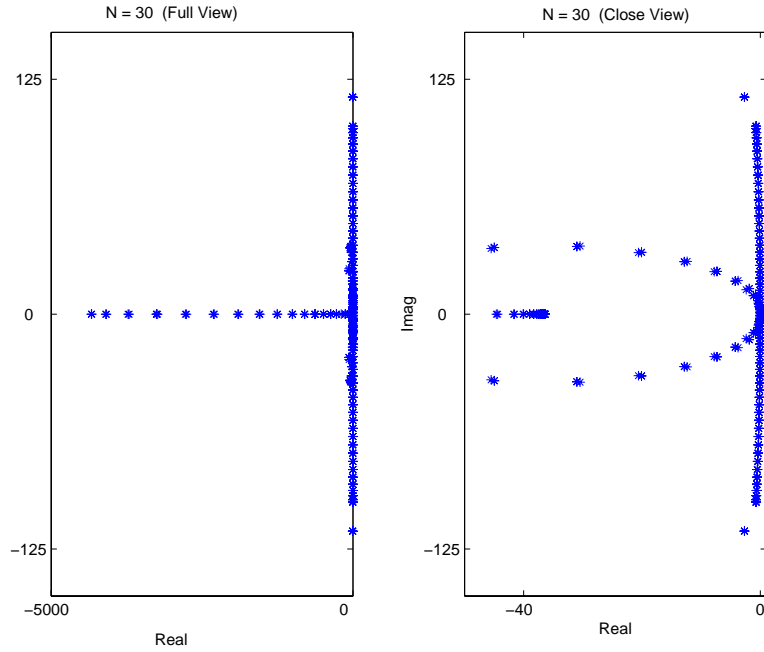


Figure 8.3.5: Eigenvalues for the Triple-Joint System N=30

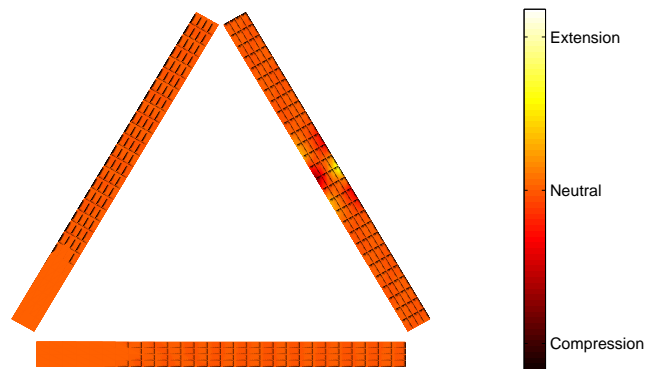


Figure 8.3.6: Strain Distribution after 4 microseconds

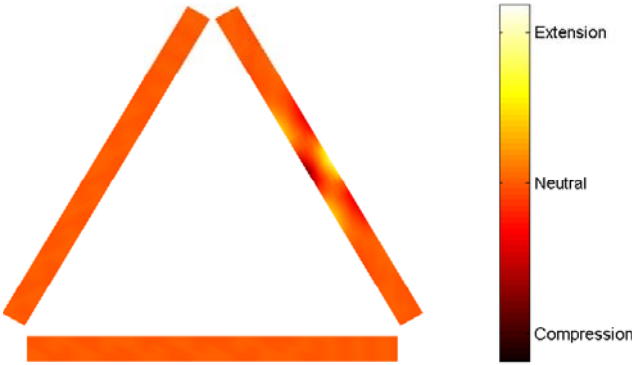


Figure 8.3.7: Strain Distribution after 15 microseconds

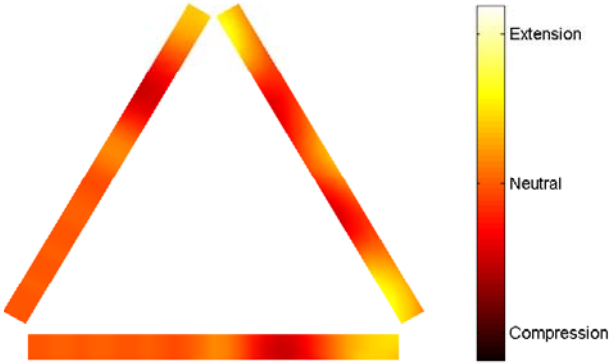


Figure 8.3.8: Strain Distribution after 187 microseconds

# Chapter 9

## Joint-Beam (Kelvin-Voigt Damping) Control

### 9.1 The Linear Quadratic Regulator Problem

In this chapter, we look at the issue of optimal control for the joint-beam system with Kelvin-Voigt damping. Let  $\mathcal{A}$  be as given in (3.2.10) and let  $B : U \rightarrow \mathcal{H}$  be a bounded operator from the finite-dimensional Hilbert space  $U$  to the state space. Let  $Q$  be a bounded, nonnegative, self-adjoint operator on  $\mathcal{H}$ . Also, let  $R$  be a bounded operator on  $U$  such that  $R$  is self-adjoint and strictly positive. For  $u \in L^2(0, \infty; U)$ , define the functional

$$J(z_0, u) = \int_0^\infty \left( \langle Qz(t), z(t) \rangle + \langle Ru(t), u(t) \rangle \right) dt \quad (9.1.1)$$

where  $z$  satisfies

$$\frac{dz(t)}{dt} = \mathcal{A}z(t) + Bu(t), \quad z(0) = z_0. \quad (9.1.2)$$

The problem is to find a control  $u_{min} \in L^2(0, \infty; U)$  such that

$$J(z_0, u_{min}) \leq J(z_0, u), \quad \forall u \in L^2(0, \infty; U).$$

Since  $\mathcal{A}$  has been shown to generate an exponentially stable semigroup (3.3.1), [6] tells us that a unique optimal control does exist and can be written as

$$u_{min}(t) = -R^{-1}B^*\Pi z(t), \quad (9.1.3)$$

where  $\Pi$  is the unique self-adjoint, nonnegative solution of the algebraic Riccati equation

$$\Pi A + A^* \Pi - \Pi B R^{-1} B^* \Pi + Q = 0. \quad (9.1.4)$$

Rather than trying to solve (9.1.4) directly, we will first approximate the infinite-dimensional control problem with a finite-dimensional version, and then prove convergence of the approximate solutions to the true solution. For each  $N$ , let  $\mathcal{H}^N, P^N, \mathcal{A}^N$ , and  $T^N(t)$  be as given in Chapter 8. Also, let  $B^N : U \rightarrow \mathcal{H}^N$  and  $Q^N : \mathcal{H}^N \rightarrow \mathcal{H}^N$  be linear operators such that  $Q^N$  is self-adjoint and nonnegative. The finite-dimensional optimization problem is to find  $u_{min}^N \in L^2(0, \infty; U)$  that minimizes

$$J^N(z_0, u) = \int_0^\infty \left( \langle Q^N z^N(t), z^N(t) \rangle + \langle Ru(t), u(t) \rangle \right) dt \quad (9.1.5)$$

where  $z^N$  satisfies

$$\frac{dz^N(t)}{dt} = \mathcal{A}^N z^N(t) + B^N u(t), \quad z^N(0) = P^N z_0. \quad (9.1.6)$$

Now from (8.1.1), we know that each  $\mathcal{A}^N$  generates a  $C_0$  semigroup of contractions for each  $N$ . In Section 2 of this chapter, it will be shown that each of the semigroups  $T^N(t)$  are exponentially stable. Therefore, [6] again assures us of the existence of a unique optimal control  $u_{min} \in L^2(0, \infty; U)$  given by

$$u_{min}^N(t) = -R^{-1}(B^N)^* \Pi^N z^N(t). \quad (9.1.7)$$

Here,  $\Pi^N$  is the unique self-adjoint, nonnegative solution of the finite-dimensional algebraic Riccati equation

$$\Pi^N \mathcal{A}^N + (\mathcal{A}^N)^* \Pi^N - \Pi^N B^N R^{-1} (B^N)^* \Pi^N + Q^N = 0. \quad (9.1.8)$$

In general, the optimal controls found in (9.1.7) may not produce acceptable results when applied to the infinite-dimensional system (9.1.2). Therefore, certain conditions must be met in order to guarantee that, for large  $N$ , the controls  $u_{min}^N$  will achieve near-optimal performance when applied to the original system. In [10] and [9] we find that the following conditions are more than sufficient to assure acceptable performance of the  $u_{min}^N$ :

1.  $Q^N P^N z \rightarrow Qz$ ,  $B^N u \rightarrow Bu$ ,  $(B^N)^* P^N z \rightarrow B^* z$ , as  $N \rightarrow \infty$  for all  $z \in \mathcal{H}$  and all

$u \in U$ .

2.  $T^N(t)P^N z \rightarrow T(t)z$ ,  $(T^N)^*(t)P^N z \rightarrow T^*(t)z$ , as  $N \rightarrow \infty$  for all  $z \in \mathcal{H}$ .
3. There exist positive constants  $M$ ,  $\alpha$  independent of  $N$  such that  $\|T^N(t)\| \leq Me^{-\alpha t}$  for all  $N$ .

The convergence of the approximating semigroups  $T^N(t)$  was shown in Chapter 8. Sections 2 and 3 of this chapter will show that the adjoint convergence and uniform exponential stability conditions are also satisfied. Section 4 will consider an example of a simple control scheme satisfying Condition (1) and some numerical results will be presented in the final section.

## 9.2 Approximation of the Adjoint Semigroup

In this section, we will characterize the adjoints  $\mathcal{A}^*$ ,  $(\mathcal{A}^N)^*$  of the operators  $\mathcal{A}$  and  $\mathcal{A}^N$ , respectively. We will then be able to see the strong convergence of the semigroups  $(T^N)^*(t)P^N \rightarrow T^*(t)$  generated by these adjoints.

### 9.2.1 Adjoint Operator $\mathcal{A}^*$

Define the operator  $K : D(K) \rightarrow \mathcal{H}$  as

$$K \begin{bmatrix} u_1 \\ w_1 \\ v_1 \\ \eta_1 \\ u_2 \\ w_2 \\ v_2 \\ \eta_2 \\ a \end{bmatrix} = \begin{bmatrix} -v_1 \\ -\eta_1 \\ \frac{1}{\rho_1 A_1}(-E_1 A_1 u_1' + \mu_1 v_1')' \\ -\frac{1}{\rho_1 A_1}(-E_1 I_1 w_1'' + \gamma_1 \eta_1'')'' \\ -v_2 \\ -\eta_2 \\ \frac{1}{\rho_2 A_2}(-E_2 A_2 u_2' + \mu_2 v_2')' \\ -\frac{1}{\rho_2 A_2}(-E_2 I_2 w_2'' + \gamma_2 \eta_2'')'' \\ \mathbf{M}^{-1} \mathbf{C}(\tilde{F}_1, \tilde{N}_1, \tilde{F}_2, \tilde{N}_2, \tilde{M}_1, \tilde{M}_2)^T \end{bmatrix}, \quad (9.2.1)$$

$$D(K) = \{z \in \mathcal{H} | v_i \in H_r^1, \eta_i \in H_r^2, (-E_i A_i u_i' + \mu_i v_i') \in H^1, (E_i I_i w_i'' - \gamma_i \eta_i'') \in H^2, \text{geometric conditions are satisfied}\} \quad (9.2.2)$$

where

$$\begin{aligned}\tilde{F}_i &= \lim_{s \rightarrow L_i} (-E_1 A_1 u_1' + \mu_1 v_1')(s) \\ \tilde{N}_i &= \lim_{s \rightarrow L_i} (-E_1 I_1 w_1'' + \gamma_1 \eta_1'')(s) \\ \tilde{M}_i &= \lim_{s \rightarrow L_i} (-E_1 I_1 w_1'' + \gamma_1 \eta_1'')(s).\end{aligned}$$

**Theorem 9.2.1** *The adjoint  $\mathcal{A}^*$  of the infinitesimal generator  $\mathcal{A}$  is given by the operator  $K$  defined above.*

Proof: Suppose  $z_1 \in D(\mathcal{A}^*)$ . This implies that, for all  $z \in D(\mathcal{A})$ , we have

$$\langle \mathcal{A}z, z_1 \rangle = \langle z, \mathcal{A}^* z_1 \rangle. \quad (9.2.3)$$

Let  $z = (u_1, w_1, v_1, \eta_1, u_2, w_2, v_2, \eta_2, a)$ ,  $z_1 = (\tilde{u}_1, \tilde{w}_1, \tilde{v}_1, \tilde{\eta}_1, \tilde{u}_2, \tilde{w}_2, \tilde{v}_2, \tilde{\eta}_2, \tilde{a})$ , and  $\mathcal{A}^* z_1 = (f_1, g_1, h_1, j_1, f_2, g_2, h_2, j_2, b)$ . Computing the inner products in (9.2.3), we have

$$\begin{aligned}& \sum_{i=1}^2 \left( E_i A_i \langle v_i', \tilde{u}_i' \rangle + E_i I_i \langle \eta_i'', \tilde{w}_i'' \rangle + \langle (E_i A_i u_i' + \mu_i v_i')', \tilde{v}_i \rangle - \langle (E_i I_i w_i'' + \gamma_i \eta_i'')'', \tilde{\eta}_i \rangle \right) \\ & \quad + \left[ \mathbf{C} (F_1, N_1, F_2, N_2, M_1, M_2)^T \right]^T \tilde{a} \\ = & \sum_{i=1}^2 \left( E_i A_i \langle u_i', f_i' \rangle + E_i I_i \langle w_i'', g_i'' \rangle + \rho_i A_i \langle v_i, h_i \rangle + \rho_i A_i \langle \eta_i, j_i \rangle \right) + a^T \mathbf{M} b.\end{aligned} \quad (9.2.4)$$

Now, since  $z$  is arbitrary in  $D(\mathcal{A})$ , we can consider the effects of restricting  $z$  in the following ways:

- **Case 1:**  $\mathbf{u}_i \in \mathbf{H}_0^2(\mathbf{0}, \mathbf{L}_i)$

Using (9.2.4), we see that we have

$$\langle u_i'', \tilde{v}_i \rangle - \langle u_i', f_i' \rangle = 0 \quad (9.2.5)$$

which can be written in integral form as

$$\int_0^{L_i} u_i'' \tilde{v}_i - u_i' f_i' ds = 0. \quad (9.2.6)$$

Since this is true for all  $u_i \in H_0^2(0, L_i)$ , the Fundamental Lemma of the Calculus of Variations (FLCV) gives the existence of constants  $\beta_i$  such that

$$\tilde{v}_i(s) = \beta_i - \int_0^s f_i'(\tau) d\tau. \quad (9.2.7)$$

Therefore,  $\tilde{v}_i \in H^2(0, L_i)$ , and  $\tilde{v}_i' = -f_i'$ .

- **Case 2:**  $\mathbf{w}_i \in \mathbf{H}_0^4(\mathbf{0}, \mathbf{L}_i)$

This assumption results in the integral form

$$\int_0^{L_i} w_i'' g_i'' + w_i'''' \tilde{\eta}_i ds = 0. \quad (9.2.8)$$

Integration by parts followed by an application of the FLCV gives us

$$\tilde{\eta}_i \in H^2(0, L_i), \tilde{\eta}_i'' = -g_i''. \quad (9.2.9)$$

- **Case 3:**  $\mathbf{v}_i \in \mathbf{H}_0^2(\mathbf{0}, \mathbf{L}_i)$

By considering the integral form and then using the FLCV as above, we get

$$h_i = \frac{1}{\rho_i A_i} (-E_i A_i \tilde{u}_i' + \mu_i \tilde{v}_i')'. \quad (9.2.10)$$

- **Case 4:**  $\eta_i \in \mathbf{H}_0^4(\mathbf{0}, \mathbf{L}_i)$

Proceeding as in the previous cases, we have

$$j_i = -\frac{1}{\rho_i A_i} (-E_i I_i \tilde{w}_i'' + \gamma_i \tilde{\eta}_i'')''. \quad (9.2.11)$$

Now, if we plug the results of these cases into (9.2.4) and integrate by parts, we have

$$\begin{aligned} & \sum_{i=1}^2 \left( E_i A_i \langle v'_i, \tilde{u}'_i \rangle + E_i I_i \langle \eta''_i, \tilde{w}''_i \rangle - \langle E_i A_i u'_i + \mu_i v'_i, \tilde{v}'_i \rangle - \langle E_i I_i w''_i + \gamma_i \eta''_i, \tilde{\eta}''_i \rangle \right. \\ & \quad \left. + F_i \tilde{v}_i(L_i) - F_i^* \tilde{v}_i(0) - N_i \tilde{\eta}_i(L_i) + N_i^* \tilde{\eta}_i(0) + M_i \tilde{\eta}'_i(L_i) - M_i^* \tilde{\eta}'_i(0) \right) \\ & \quad + \left[ \mathbf{C} (F_1, N_1, F_2, N_2, M_1, M_2)^T \right]^T \tilde{a} \end{aligned} \quad (9.2.12)$$

$$\begin{aligned} = & \sum_{i=1}^2 \left( E_i A_i \langle u'_i, -\tilde{v}'_i \rangle + E_i I_i \langle w''_i, -\tilde{\eta}''_i \rangle - \langle v'_i, -E_i A_i \tilde{u}'_i + \mu_i \tilde{v}'_i \rangle - \langle \eta''_i, -E_i I_i \tilde{w}''_i + \gamma_i \tilde{\eta}''_i \rangle \right. \\ & \quad \left. + \tilde{F}_i v_i(L_i) - \tilde{N}_i \eta_i(L_i) + \tilde{M}_i \eta'_i(L_i) \right) + a^T \mathbf{M} b, \end{aligned} \quad (9.2.13)$$

where

$$\begin{aligned} F_i^* &= (E_i A_i u'_i + \mu_i v'_i)'(0) \\ N_i^* &= (E_i I_i w''_i + \gamma_i \eta''_i)'(0) \\ M_i^* &= (E_i I_i w''_i + \gamma_i \eta''_i)(0). \end{aligned}$$

If we cancel out the identical terms on each side of (9.2.13), the result is

$$\begin{aligned} & \sum_{i=1}^2 \left( F_i \tilde{v}_i(L_i) - F_i^* \tilde{v}_i(0) - N_i \tilde{\eta}_i(L_i) + N_i^* \tilde{\eta}_i(0) + M_i \tilde{\eta}'_i(L_i) - M_i^* \tilde{\eta}'_i(0) \right) \\ & \quad + \left[ \mathbf{C} (F_1, N_1, F_2, N_2, M_1, M_2)^T \right]^T \tilde{a} \\ = & \sum_{i=1}^2 \left( \tilde{F}_i v_i(L_i) - \tilde{N}_i \eta_i(L_i) + \tilde{M}_i \eta'_i(L_i) \right) + a^T \mathbf{M} b. \end{aligned} \quad (9.2.14)$$

Let us now consider the result of one more case:

- **Case 5:**  $(\mathbf{F}_i, \mathbf{N}_i, \mathbf{M}_i, \mathbf{F}_i^*, \mathbf{N}_i^*, \mathbf{M}_i^*) = \mathbf{0}$

This assumption implies that the left hand side of the equality (9.2.14) is zero. Using the geometric constraints on elements in  $D(\mathcal{A})$ , we see that (9.2.14) becomes

$$0 = \left[ -\mathbf{C} \left( \tilde{F}_1, \tilde{N}_1, \tilde{F}_2, \tilde{N}_2, \tilde{M}_1, \tilde{M}_2 \right)^T \right]^T a + b^T \mathbf{M} a. \quad (9.2.15)$$

Therefore, we have

$$b = \mathbf{M}^{-1} \mathbf{C} \left( \tilde{F}_1, \tilde{N}_1, \tilde{F}_2, \tilde{N}_2, \tilde{M}_1, \tilde{M}_2 \right)^T. \quad (9.2.16)$$

Now, using the result of Case 5, (9.2.14) becomes

$$\begin{aligned} & \sum_{i=1}^2 \left( F_i \tilde{v}_i(L_i) - F_i^* \tilde{v}_i(0) - N_i \tilde{\eta}_i(L_i) + N_i^* \tilde{\eta}_i(0) + M_i \tilde{\eta}'_i(L_i) - M_i^* \tilde{\eta}'_i(0) \right) \\ & \quad + \left[ \mathbf{C} (F_1, N_1, F_2, N_2, M_1, M_2)^T \right]^T \tilde{a} \\ = & \sum_{i=1}^2 \left( \tilde{F}_i v_i(L_i) - \tilde{N}_i \eta_i(L_i) + \tilde{M}_i \eta'_i(L_i) \right) \\ & \quad + \left[ \mathbf{C} \left( \tilde{F}_1, \tilde{N}_1, \tilde{F}_2, \tilde{N}_2, \tilde{M}_1, \tilde{M}_2 \right)^T \right]^T a. \end{aligned} \quad (9.2.17)$$

But, again because of the geometric constraints, the right hand side of (9.2.17) is zero. If we let  $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4)^T$  and perform the matrix multiplication, then (9.2.17) becomes

$$\begin{aligned} & -F_1^* \tilde{v}_1(0) + N_1^* \tilde{\eta}_1(0) - M_1^* \tilde{\eta}'_1(0) - F_2^* \tilde{v}_2(0) + N_2^* \tilde{\eta}_2(0) - M_2^* \tilde{\eta}'_2(0) \\ & + F_1 \left( \tilde{v}_1(L_1) + \tilde{a}_1 \sin \phi_1 + \tilde{a}_2 \cos \phi_1 \right) + N_1 \left( -\tilde{\eta}_1(L_1) - \tilde{a}_1 \cos \phi_1 + \tilde{a}_2 \sin \phi_1 + \tilde{a}_3 \ell_1 \right) \\ & + F_2 \left( \tilde{v}_2(L_2) + \tilde{a}_1 \sin \phi_2 - \tilde{a}_2 \cos \phi_2 \right) + N_2 \left( -\tilde{\eta}_2(L_2) + \tilde{a}_1 \cos \phi_2 + \tilde{a}_2 \sin \phi_2 + \tilde{a}_4 \ell_2 \right) \\ & + M_1 \left( \tilde{\eta}'_1(L_1) + \tilde{a}_3 \right) + M_2 \left( \tilde{\eta}'_2(L_2) + \tilde{a}_4 \right) = 0. \end{aligned} \quad (9.2.18)$$

Since  $F_i, N_i, M_i, F_i^*, N_i^*, M_i^*$  are arbitrary, their coefficients in (9.2.18) must be zero. This

gives us

$$\tilde{v}_1(L_1) = -\tilde{a}_1 \sin \phi_1 - \tilde{a}_2 \cos \phi_1 \quad (9.2.19)$$

$$-\tilde{\eta}_1(L_1) = \tilde{a}_1 \cos \phi_1 - \tilde{a}_2 \sin \phi_1 - \tilde{a}_3 \ell_1 \quad (9.2.20)$$

$$\tilde{v}_2(L_2) = -\tilde{a}_1 \sin \phi_2 + \tilde{a}_2 \cos \phi_2 \quad (9.2.21)$$

$$-\tilde{\eta}_2(L_2) = -\tilde{a}_1 \cos \phi_2 - \tilde{a}_2 \sin \phi_2 - \tilde{a}_4 \ell_2 \quad (9.2.22)$$

$$\tilde{\eta}'_1(L_1) = -\tilde{a}_3 \quad (9.2.23)$$

$$\tilde{\eta}'_2(L_2) = -\tilde{a}_4 \quad (9.2.24)$$

$$\tilde{v}_i(0) = 0 \quad (9.2.25)$$

$$\tilde{\eta}_i(0) = 0 \quad (9.2.26)$$

$$\tilde{\eta}'_i(0) = 0. \quad (9.2.27)$$

Therefore,  $z_1$  satisfies the geometric constraints. Putting this together with the results of Case 1 - Case 5, we see that  $D(\mathcal{A}^*) = D(K)$  and  $\mathcal{A}^*z = Kz$  for  $z \in D(\mathcal{A}^*)$ . Thus, the Theorem is proven.

**Theorem 9.2.2** *The operator  $\mathcal{A}^*$  generates a  $C_0$  semigroup of contractions  $T^*(t)$  on  $\mathcal{H}$ .*

Proof: It is easily shown that  $\mathcal{A}^*$  is dissipative. Therefore, since both  $\mathcal{A}$  and  $\mathcal{A}^*$  are dissipative, the theorem is proven.

## 9.2.2 Approximate Adjoint Operators $A^{N*}$

Given  $N$ , define the operator  $K^N : \mathcal{H}^N \rightarrow \mathcal{H}^N$  by

$$K^N \begin{bmatrix} u_1^N \\ w_1^N \\ v_1^N \\ \eta_1^N \\ u_2^N \\ w_2^N \\ v_2^N \\ \eta_2^N \\ a \end{bmatrix} = \begin{bmatrix} -v_1^N \\ -\eta_1^N \\ \frac{1}{\rho_1 A_1} D_1^N (-E_1 A_1 u_1^N + \mu_1 v_1^N) \\ -\frac{1}{\rho_1 A_1} K_1^N (-E_1 I_1 w_1^N + \gamma_1 \eta_1^N) \\ -v_2^N \\ -\eta_2^N \\ \frac{1}{\rho_2 A_2} D_2^N (-E_2 A_2 u_2^N + \mu_2 v_2^N) \\ -\frac{1}{\rho_2 A_2} K_2^N (-E_2 I_2 w_2^N + \gamma_2 \eta_2^N) \\ \mathbf{M}^{-1} \mathbf{C} (\tilde{F}_1^N, \tilde{N}_1^N, \tilde{F}_2^N, \tilde{N}_2^N, \tilde{M}_1^N, \tilde{M}_2^N)^T \end{bmatrix}, \quad (9.2.28)$$

where

$$\begin{aligned}\tilde{F}_i^N &= \lim_{s \rightarrow L_i} (-E_i A_i u_i^N + \mu_i v_i^N)'(s) \\ \tilde{N}_i^N &= \lim_{s \rightarrow L_i} (-E_i I_i w_i^N + \gamma_i \eta_i^N)'''(s) \\ \tilde{M}_i^N &= \lim_{s \rightarrow L_i} (-E_i I_i w_i^N + \gamma_i \eta_i^N)''(s).\end{aligned}$$

**Theorem 9.2.3** *For any given  $N$ , the adjoint operator  $(\mathcal{A}^N)^*$  is equal to the operator  $K^N$ . Also,  $(\mathcal{A}^N)^*$  generates a  $C_0$  semigroup of contractions  $(T^N)^*(t)$  on  $\mathcal{H}^N$ .*

Proof: Routine calculations show that

$$\langle \mathcal{A}^N z_1^N, z_2^N \rangle = \langle z_1^N, K^N z_2^N \rangle \quad (9.2.29)$$

for all  $z_1^N, z_2^N \in \mathcal{H}^N$ . Also, we have

$$\langle \mathcal{A}^N z^N, z^N \rangle = - \sum_{i=1}^2 (\mu_i \|(v_i^N)'\|^2 + \gamma_i \|(\eta_i^N)''\|^2) \leq 0. \quad (9.2.30)$$

Therefore, both assertions of the theorem follow.

**Theorem 9.2.4**  *$(T^N)^*(t)P^N z \rightarrow T^*(t)z$  on  $\mathcal{H}$  with uniform convergence on bounded intervals of  $t$ .*

Proof: The proof is identical to the proof for the convergence of  $T^N(t)P^N z \rightarrow T(t)z$ .

### 9.3 Uniform Exponential Stability

The problem in this section is to show the existence of positive constants  $M_1$  and  $M_2$  independent of  $N$  such that  $\|T^N(t)\| \leq M_1 e^{-M_2 t}$  for all  $N$ . Gibson, in [10], solved this problem for a similar system using Lyapunov functionals. We will follow the argument in [10] with slight modification.

Define the spaces

$$\begin{aligned}Q &= (L^2(0, L_1))^2 \times (L^2(0, L_2))^2 \times \mathbb{R}^4, \\ T &= H_r^1(0, L_1) \times H_r^2(0, L_1) \times H_r^1(0, L_2) \times H_r^2(0, L_2)\end{aligned} \quad (9.3.1)$$

with the respective inner products

$$\left\langle \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} \right\rangle_Q = \rho_1 A_1 \langle \alpha_1, \beta_1 \rangle + \rho_1 A_1 \langle \alpha_2, \beta_2 \rangle + \rho_2 A_2 \langle \alpha_3, \beta_3 \rangle + \rho_2 A_2 \langle \alpha_4, \beta_4 \rangle + \alpha_5^T \mathbf{M} \beta_5, \quad (9.3.2)$$

$$\left\langle \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} \right\rangle_T = \frac{\mu_1}{2} \langle \alpha'_1, \beta'_1 \rangle + \frac{\gamma_1}{2} \langle \alpha''_2, \beta''_2 \rangle + \frac{\mu_2}{2} \langle \alpha'_3, \beta'_3 \rangle + \frac{\gamma_2}{2} \langle \alpha''_4, \beta''_4 \rangle. \quad (9.3.3)$$

Now, for each  $N$ , let  $z^N$  denote  $(u_1^N, w_1^N, v_1^N, \eta_1^N, u_2^N, w_2^N, v_2^N, \eta_2^N, a^N)^T \in \mathcal{H}^N$ . Let  $b^N$  denote  $(x, y, \theta_1, \theta_2)$ , where  $x, y, \theta_1$ , and  $\theta_2$  are determined by the  $L_i$  boundary values of  $u_i, w_i$ , and  $w'_i$  as in (3.1.9). Define the functional

$$V^N(z^N) = K_1 \langle z^N, z^N \rangle_{\mathcal{H}^N} + \left\langle \begin{bmatrix} v_1^N \\ \eta_1^N \\ v_2^N \\ \eta_2^N \\ a^N \end{bmatrix}, \begin{bmatrix} u_1^N \\ w_1^N \\ u_2^N \\ w_2^N \\ b^N \end{bmatrix} \right\rangle_Q + \left\langle \begin{bmatrix} u_1^N \\ w_1^N \\ u_2^N \\ w_2^N \end{bmatrix}, \begin{bmatrix} u_1^N \\ w_1^N \\ u_2^N \\ w_2^N \end{bmatrix} \right\rangle_T. \quad (9.3.4)$$

If  $T^N(t)$  is the semigroup generated by  $\mathcal{A}^N$ , then define

$$\dot{V}^N(z^N) = \liminf_{t \searrow 0} \frac{1}{t} \left[ V^N(T^N(t)z^N) - V^N(z^N) \right]. \quad (9.3.5)$$

Using (8.1.2), we have

$$\begin{aligned}
 \dot{V}^N(z^N) &= 2K_1 \langle \mathcal{A}^N z^N, z^N \rangle_{\mathcal{H}^N} + \left\langle \begin{bmatrix} \frac{1}{\rho_1 A_1} D_1^N (E_1 A_1 u_1^N + \mu_1 v_1^N) \\ -\frac{1}{\rho_1 A_1} K_1^N (E_1 I_1 w_1^N + \gamma_1 \eta_1^N) \\ \frac{1}{\rho_2 A_2} D_2^N (E_2 A_2 u_2^N + \mu_2 v_2^N) \\ -\frac{1}{\rho_2 A_2} K_2^N (E_2 I_2 w_2^N + \gamma_2 \eta_2^N) \\ \mathcal{F}^N \end{bmatrix}, \begin{bmatrix} u_1^N \\ w_1^N \\ u_2^N \\ w_2^N \\ b^N \end{bmatrix} \right\rangle_Q \\
 &\quad + \left\langle \begin{bmatrix} v_1^N \\ \eta_1^N \\ v_2^N \\ \eta_2^N \\ a^N \end{bmatrix}, \begin{bmatrix} v_1^N \\ \eta_1^N \\ w_2^N \\ \eta_2^N \\ a^N \end{bmatrix} \right\rangle_Q + 2 \left\langle \begin{bmatrix} v_1^N \\ \eta_1^N \\ v_2^N \\ \eta_2^N \end{bmatrix}, \begin{bmatrix} u_1^N \\ w_1^N \\ u_2^N \\ w_2^N \end{bmatrix} \right\rangle_T \tag{9.3.6}
 \end{aligned}$$

where  $\mathcal{F}^N = \mathbf{M}^{-1} \mathbf{C} (F_1^N, N_1^N, F_2^N, N_2^N, M_1^N, M_2^N)^T$ . Routine calculation, along with the geometric compatibility conditions (3.1.9) and (3.2.7), yields

$$\begin{aligned}
 \dot{V}^N(z^N) &= 2K_1 \langle \mathcal{A}^N z^N, z^N \rangle_{\mathcal{H}^N} - \left\langle (E_1 A_1 u_1^N + \mu_1 v_1^N)', (u_1^N)' \right\rangle - \left\langle (E_1 I_1 w_1^N + \gamma_1 \eta_1^N)'', (w_1^N)'' \right\rangle \\
 &\quad - \left\langle (E_2 A_2 u_2^N + \mu_2 v_2^N)', (u_2^N)' \right\rangle - \left\langle (E_2 I_2 w_2^N + \gamma_2 \eta_2^N)'', (w_2^N)'' \right\rangle \\
 &\quad + \left\langle \begin{bmatrix} v_1^N \\ \eta_1^N \\ v_2^N \\ \eta_2^N \\ a^N \end{bmatrix}, \begin{bmatrix} v_1^N \\ \eta_1^N \\ w_2^N \\ \eta_2^N \\ a^N \end{bmatrix} \right\rangle_Q + 2 \left\langle \begin{bmatrix} v_1^N \\ \eta_1^N \\ v_2^N \\ \eta_2^N \end{bmatrix}, \begin{bmatrix} u_1^N \\ w_1^N \\ u_2^N \\ w_2^N \end{bmatrix} \right\rangle_T \\
 &= -2K_1 \sum_{i=1}^2 \left( \mu_i \|(v_i^N)'\|^2 + \gamma_i \|(\eta_i^N)''\|^2 \right) - \sum_{i=1}^2 \left( E_i A_i \|(u_i^N)'\|^2 + E_i I_i \|(w_i^N)''\|^2 \right) \\
 &\quad + \rho_1 A_1 \|v_1^N\|^2 + \rho_1 A_1 \|\eta_1^N\|^2 + \rho_2 A_2 \|v_2^N\|^2 + \rho_2 A_2 \|\eta_2^N\|^2 + (a^N)^T \mathbf{M} a^N. \tag{9.3.7}
 \end{aligned}$$

The fact that  $V^N$  is a Lyapunov function follows immediately. Note that the compatibility condition (3.2.7) guarantees the existence of  $\alpha_1 > 0$  such that

$$\|a^N\|_{\mathbb{R}^4} < \alpha_1 \left( \|v_1^N\| + \|\eta_1^N\| + \|v_2^N\| + \|\eta_2^N\| \right). \tag{9.3.8}$$

It should also be remarked that  $\alpha_1$  is independent of  $N$  and  $z^N \in \mathcal{H}^N$ . Thus, we have the

existence of  $\alpha_2 > 0$  such that, for  $K_1$  large enough and for all  $N$ ,

$$\dot{V}^N(z^N) \leq -\alpha_2 \|z^N\|^2, \quad z^N \in \mathcal{H}^N. \quad (9.3.9)$$

Now, looking back at  $V^N$  in (9.3.4), we also have the existence of  $\alpha_3 > 0$  such that, for  $K_1$  large enough and for all  $N$ ,

$$V^N(z^N) \geq \alpha_3 \|z^N\|^2, \quad z^N \in \mathcal{H}^N. \quad (9.3.10)$$

Let us fix the value of  $K_1$  so that both (9.3.9) and (9.3.10) are satisfied. This implies the existence of  $\alpha_4 > 0$  such that for all  $N$

$$\dot{V}^N(z^N) \leq -\alpha_4 V^N(z^N), \quad z^N \in \mathcal{H}^N. \quad (9.3.11)$$

Therefore, we can use Theorem IV.1.1 from [21] to get

$$V^N\left(T^N(t)z^N\right) \leq \exp^{-\alpha_4 t} V^N(z^N), \quad z^N \in \mathcal{H}^N \quad (9.3.12)$$

for all  $N$ . Using this result with (9.3.10), we have

$$\|T^N(t)z^N\|^2 \leq \alpha_5 \exp^{-\alpha_4 t} V^N(z^N), \quad z^N \in \mathcal{H}^N \quad (9.3.13)$$

for some  $\alpha_5 > 0$  and for all  $N$ . Now, from (9.3.4) we also have the existence of  $\alpha_6 > 0$  such that

$$V^N(z^N) \leq \alpha_6 \|z^N\|^2, \quad z^N \in \mathcal{H}^N \quad (9.3.14)$$

for all  $N$ . Plugging this into (9.3.13), we obtain

$$\|T^N(t)z^N\| \leq \alpha_7 \exp^{-\frac{\alpha_4}{2}t} \|z^N\|, \quad z^N \in \mathcal{H}^N \quad (9.3.15)$$

for some  $\alpha_7 > 0$  and for all  $N$ . Therefore the approximation systems preserve exponential stability uniformly in  $N$ .

## 9.4 Joint Actuator Example

The previous two sections have shown that the approximating semigroups  $T^N(t)$  satisfy Conditions (2) and (3) of Section 9.1. In this section, we will look at a specific control problem such that Condition (1) is also satisfied.

Let us consider the joint-beam system with a single actuator present within the joint. This actuator is assumed to provide an internal moment which affects the rotational velocities of the two rigid legs. The system can be written as

$$\frac{dz(t)}{dt} = \mathcal{A}z(t) + Bu(t), \quad z(0) = z_0 \quad (9.4.1)$$

where  $\mathcal{A}$  is defined in (3.2.10) and  $B : \mathbb{R} \rightarrow \mathcal{H}$  is given by

$$Bu = \left[ \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, [0, 0, u, -u]^T \right]^T. \quad (9.4.2)$$

Letting  $Q = I_{\mathcal{H}}$  and  $R = 1$  in (9.1.1), our goal is to find  $u_{\min} \in L^2(0, \infty; \mathbb{R})$  that minimizes

$$J(z_0, u) = \int_0^\infty \left( \|z(t)\|_{\mathcal{H}}^2 + |u(t)|^2 \right) dt \quad (9.4.3)$$

where  $z$  satisfies (9.4.1). If we define  $Q^N = P^N Q|_{\mathcal{H}^N} = I_{\mathcal{H}^N}$  and  $B^N = P^N B$ , then it follows that Condition (1) of Section 9.1 is satisfied. Therefore, we can effectively approximate  $u_{\min}$  by  $u_{\min}^N$  where

$$u_{\min}^N(t) = -R^{-1}(B^N)^* \Pi^N z^N(t) = -(B^N)^* \Pi^N z^N(t) \quad (9.4.4)$$

and  $z^N$  satisfies

$$\frac{dz^N(t)}{dt} = \mathcal{A}^N z^N(t) + B^N u_{\min}^N(t), \quad z^N(0) = P^N z_0. \quad (9.4.5)$$

On the surface, the control problem appears to be solved. However, it should be noted that (9.4.4) is the operator form of the solution. In order to numerically solve for  $u_{\min}^N$ , we need the matrix forms of the operators involved. Hence, we also need the matrix form of the finite-dimensional Ricatti equation (9.1.8). This will be done by following the approach given in [10]. It should be noted that we are using the functions  $(\mathbf{b}_i)_1^{8N-8}$ , defined in Section 8.2, as the basis for  $\mathcal{H}^N$ . Also, we use the notation  $[K]$  to denote the matrix form of any finite-dimensional operator  $K$ .

By straightforward calculation, we have

$$\begin{aligned} [(B^N)^*] &= [B^N]^T W^N, \\ [(\mathcal{A}^N)^*] &= (W^N)^{-1} [\mathcal{A}^N]^T W^N, \\ [Q^N] &= (W^N)^{-1} \tilde{Q}^N, \end{aligned}$$

where  $W^N$  and  $\tilde{Q}^N$  are the matrices whose  $ij^{\text{th}}$  entries are given by  $\langle \mathbf{b}_i, \mathbf{b}_j \rangle_{\mathcal{H}^N}$  and  $\langle \mathbf{b}_i, Q^N \mathbf{b}_j \rangle_{\mathcal{H}^N}$ , respectively. Thus we see that the Riccati operator equation (9.1.8) is equivalent to the matrix equation

$$(W^N)^{-1} [\mathcal{A}^N]^T W^N [\Pi^N] + [\Pi^N] [A^N] - [\Pi^N] [B^N]^T R^{-1} [B^N]^T W^N [\Pi^N] + (W^N)^{-1} \tilde{Q}^N = 0. \quad (9.4.6)$$

If we make the substitution  $\tilde{\Pi}^N = W^N [\Pi^N]$  and multiply both sides of (9.4.6) on the left by  $W^N$ , we have

$$[\mathcal{A}^N]^T \tilde{\Pi}^N + \tilde{\Pi}^N [A^N] - \tilde{\Pi}^N [B^N]^T R^{-1} [B^N]^T \tilde{\Pi}^N + \tilde{Q}^N = 0. \quad (9.4.7)$$

Using our specific choices of  $R$  and  $Q^N$ , (9.4.7) becomes the matrix Riccati equation

$$[\mathcal{A}^N]^T \tilde{\Pi}^N + \tilde{\Pi}^N [A^N] - \tilde{\Pi}^N [B^N]^T [B^N]^T \tilde{\Pi}^N + W^N = 0, \quad (9.4.8)$$

which can be solved numerically.

Since our control space is defined to be  $U = \mathbb{R}$ , we have only one functional gain which can be written as

$$f = \sum_{i=1}^N \beta_i \mathbf{b}_i. \quad (9.4.9)$$

From [10], we see that the  $\beta_i$  coefficients are given by

$$\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} = (W^N)^{-1} \tilde{\Pi}^N [B^N] R^{-1} = (W^N)^{-1} \tilde{\Pi}^N [B^N]. \quad (9.4.10)$$

In order to show the gain converges in  $\mathcal{H}^N$  as  $N \rightarrow \infty$ , we must have convergence in all eight functional components as well as the  $\mathbb{R}^4$  component. This convergence for  $\mu_i = \gamma_i = 1$  can be seen in Figures 9.4.1, 9.4.2, and Table 9.4.1. The effects of adding a joint actuator

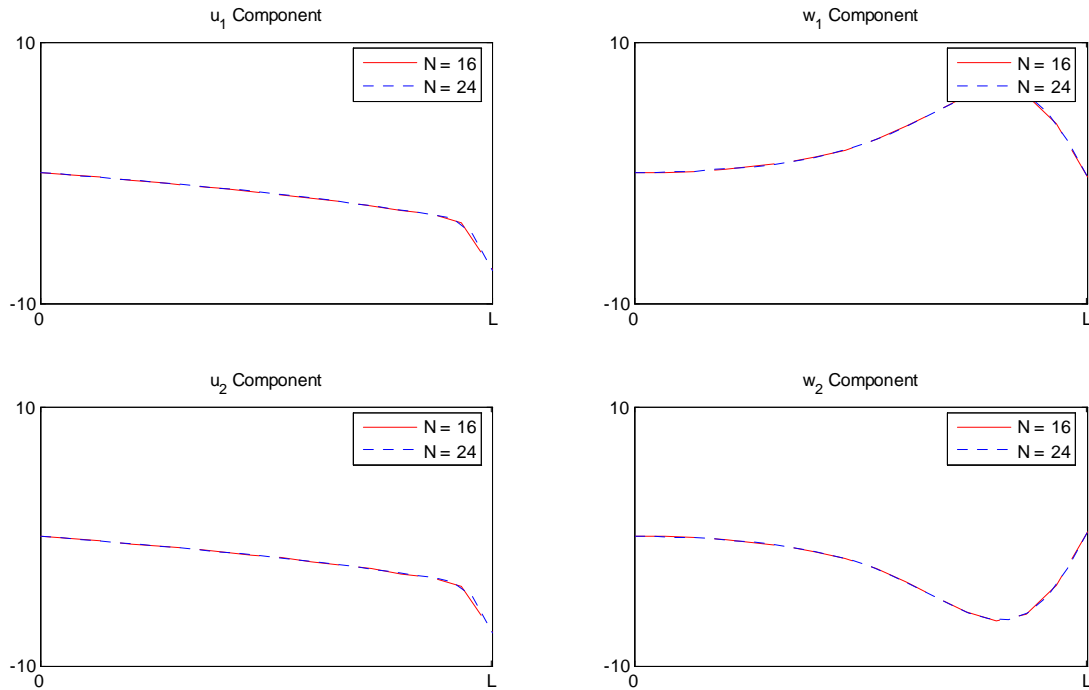


Figure 9.4.1: Positional Components of Functional Gain

to the original Kelvin-Voigt system can be seen in the time evolution of the system energy. In Figure 9.4.3, we have subjected one of the beams to an axial velocity impulse starting near the fixed end of the beam and traveling toward the joint. The uncontrolled system steadily dissipates energy as expected from the theory in Chapter 3. The controlled system briefly mirrors the uncontrolled system, then adds energy as the controller acts to counter the motions of the beam. As the two motions cancel each other, energy is quickly dissipated until the controlled system contains less energy than the uncontrolled system. It should be noted that  $N = 32$ ,  $\mu_i = \gamma_i = 1$  in this example.

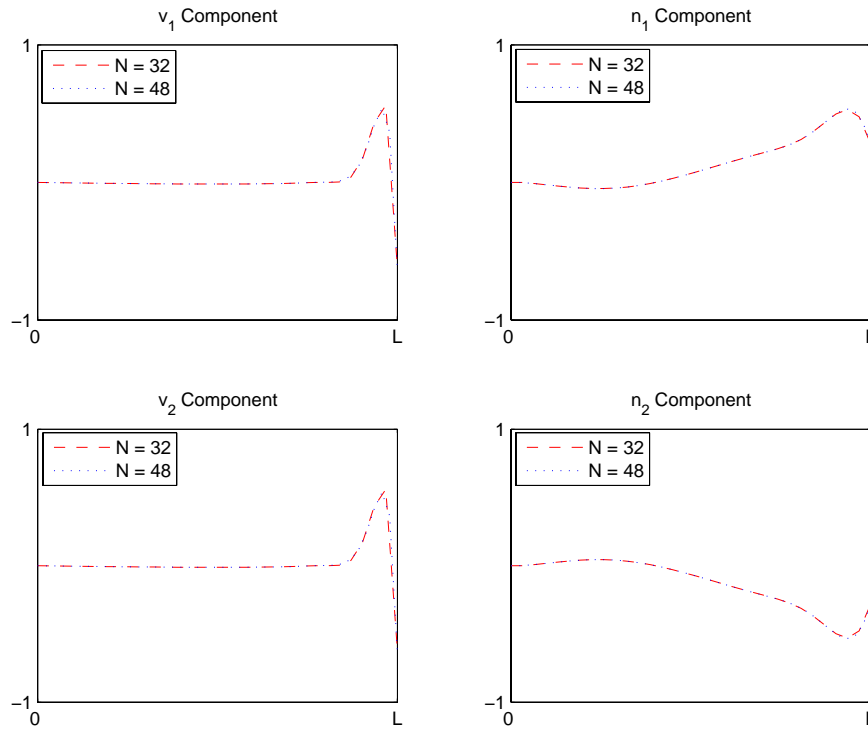


Figure 9.4.2: Velocity Components of Functional Gain

Table 9.4.1: Joint Components of Functional Gain

	N=16	N=32	N=64	N=128
$p$	.7845	.8833	.9126	.9202
$q$	.0000	.0000	.0000	.0000
$\omega_1$	.1880	.2045	.2095	.2108
$\omega_2$	-.1880	-.2045	-.2095	-.2108

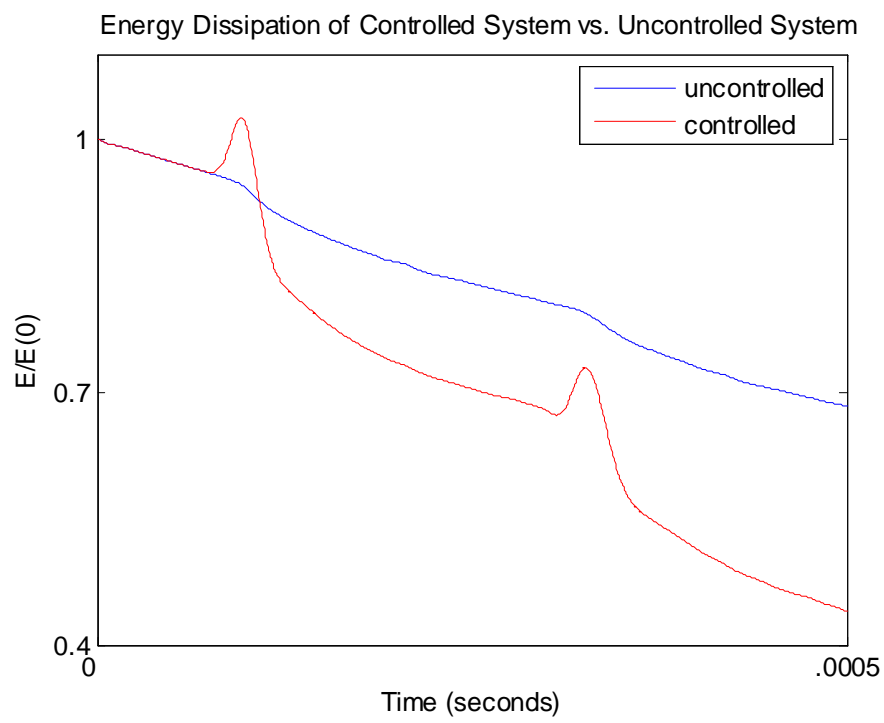


Figure 9.4.3: Effects of Control on System Energy

# Bibliography

- [1] B.A. Boley and J.H. Weiner, *Theory of Thermal Stresses*, Wiley, 1960.
- [2] J.A. Burns, E.M. Cliff, Z. Liu, and R.D. Spies, *On Coupled Transversal and Axial Motions of Two Beams with a Joint*, submitted, ICAM report 05-03-01, 2005.
- [3] J.A. Burns, E.M. Cliff, Z. Liu, and R.D. Spies, *Results on Transversal and Axial Motions of a System of Two coupled to a Joint through Two Legs*, submitted, 2006.
- [4] J.A. Burns, Z.Y. Liu, and R.E. Miller, *Approximations of Thermoelastic and Viscoelastic Control Systems*, ICASE Report no. 90-43, Institute for Computer Application in Science and Engineering, NASA Langley Research Center, Hampton, VA 1990.
- [5] E.M. Cliff, B. Fulton, T. Herdman, Z. Liu, and R.D. Spies, *Well Posedness and Exponential Stability of a Thermoelastic Joint-Leg-Beam System with Robin Boundary Conditions*, 2006, submitted.
- [6] R.F. Curtain and H.J. Zwart, *An Introduction to Infinite-Dimensional Linear systems Theory*, Springer-Verlag, 1995.
- [7] R.H. Fabiano and K. Ito, *Semigroup Theory and Numerical Approximation for Equations in Linear Viscoelasticity*, SIAM J. Math. Anal., 21(2), 1990, pp. 374-393.
- [8] A. Friedman, *Partial Differential Equations*, Holt, Rinehart, and Winston, 1969.
- [9] J.S. Gibson, *Linear-Quadratic Optimal Control of Hereditary Differential Systems: Infinite Dimensional Riccati Equations and Numerical Approximations*, SIAM J. Control and Optimization, 21(1), 1983, pp. 95-139.
- [10] J.S. Gibson and A. Adamian, *Approximation theory for LQG optimal control of flexible structures*, ICASE Report No. 88-48, Institute for Computer Application in Science and Engineering, NASA Langley Research Center, Hampton, VA 1988.

- [11] F.L. Huang, *Characteristic Condition for Exponential Stability of Linear Dynamical Systems in Hilbert Spaces*, Ann. of Diff. Eqs., 1(1), 1985, pp 43-56.
- [12] C.H.M. Jenkins (ed.), *Gossamer Spacecraft: Membrane and Inflatable Technology for Space Applications*, AIAA Progress in Aeronautics and Astronautics, (191), 2001.
- [13] J.E. Langnese, G. Laugering, and E.J.P.G. Schmidt, *Modeling, Analysis and Control of Dynamic Elastic Multi-Link Structures*, Birkhauser, 1994.
- [14] Z. Liu, *Approximation and Control of a Thermoviscoelastic System*. 1989.
- [15] Z. Liu and S. Zheng, *Semigroups Associated with Dissipative Systems*, 398 Research Notes in Mathematics, Chapman & Hall/CRC, 1999.
- [16] R. Miller, *Approximation of the LQR Control Problem for Systems Governed by Partial Functional Differential Equations*, 1988.
- [17] L. Nirenberg, *On elliptic partial differential equations*, Annali della Scuola Norm. Sup. Pisa, 13 (1959), 115-162.
- [18] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Second Edition, Springer Verlag, 1983.
- [19] M. Renardy and R. Rogers, *An Introduction to Partial Differential Equations*, Springer, 2004.
- [20] E.A. Thornton and R.S. Foster, *Dynamic Responce of Rapidly Heated Space Structures*, AIAA Journal 1992-2207, pp.1185-1203.
- [21] J.A. Walker, *Dynamical Systems and Evolution Equations*, Plenum Press, 1980.
- [22] A. Wyler, *Stability of wave equations with dissipative boundary conditions in a bounded domain*, Differential and Integral Equations, Vol. 7, No. 2 (1994), pp.345-366.