COMPENSATOR DESIGN FOR A SYSTEM OF TWO CONNECTED BEAMS

by

WEI HUANG

Dissertation submitted to the faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

APPROVED:

Prof. J. A. Burns, Chairman

Prof. t. L. Herdman

Prof. E. M. Cliff

Prof. K. B. Hannsgen

Prof. M. D. Gunzburger

August, 1994

Blacksburg, Virginia

COMPENSATOR DESIGN FOR A SYSTEM OF TWO CONNECTED BEAMS

by

WEI HUANG

Committee Chairman: Prof. J. A. Burns

Department of Mathematics

(Abstract)

The goal of this paper is to study the LQG problem for a class of infinite dimensional systems. We investigate the convergence of compensator gains for such systems when standard finite element schemes are used to discretize the problem. We are particularly interested in the analysis of the uniformly exponential stability of the corresponding closed - loop systems resulting from the finite dimensional compensators. A specific multiple component flexible structure is used to focus the analysis and to test problem in numerical simulations.

An abstract framework for analysis and approximation of the corresponding dynamics system is developed and used to design finite - dimensional compensators. Linear semigroup theory is used to establish that the systems are well posed and to prove the convergence of generic approximation schemes. Approximate solutions of the optimal regulator and optimal observer are constructed via Galerkin - type approximations. Convergence of the scheme is established and numerical results are presented to illustrate the method.

Acknowledgements

First, I would like to express my sincere appreciation and gratitude to my advisor and thesis committee chairman, Professor John A. Burns, for his support, constant encouragement, guidanceand suggestions which made this work possible. I wish to thank Professors Terry L. Herdman, Max D. Gunzburger and Kenneth B. Hannsgen of the Mathetics Department and Professor Eugene M. Cliff of the Aerospace and Ocean Engineering Department for serving on my committee, and for their comment and suggestions. I would also like to thank Professor David L. Russell and the group at ICAM for their friendship and kindness.

I wish to acknowledge the financial support I received from the Air Force Office of Scientific Research under Grants F49620 - 93 - 1 - 0280 and from the Mathematics Department.

Finally, I want to thank my family for their support and I would especially like to thank my father, Kexin Huang, my mother, Zhenqi Fu, my wife, Xiangsui, and my sweet daughter, Nancy for their endless love, support and encouragement. God bless all of you. I love you.

Wei Huang

August, 1994

Table of Contents

page

LIST OF TABLES AND FIGURES

Chapter I Introduction

1.1 Problem Statement

In this paper, we investigate an approximation scheme for Linear - Quadratic - Gaussian (LQG) optimal control of flexible structures . We concentrate on a multiple component stucture composed of two beams, a rigid hub at the left end of the structure and distributed masses at the right end of both beams as shown in Figure 1 . Although damping is an important issue in such systems, we limit our study to Kelvin - Voigt damping.

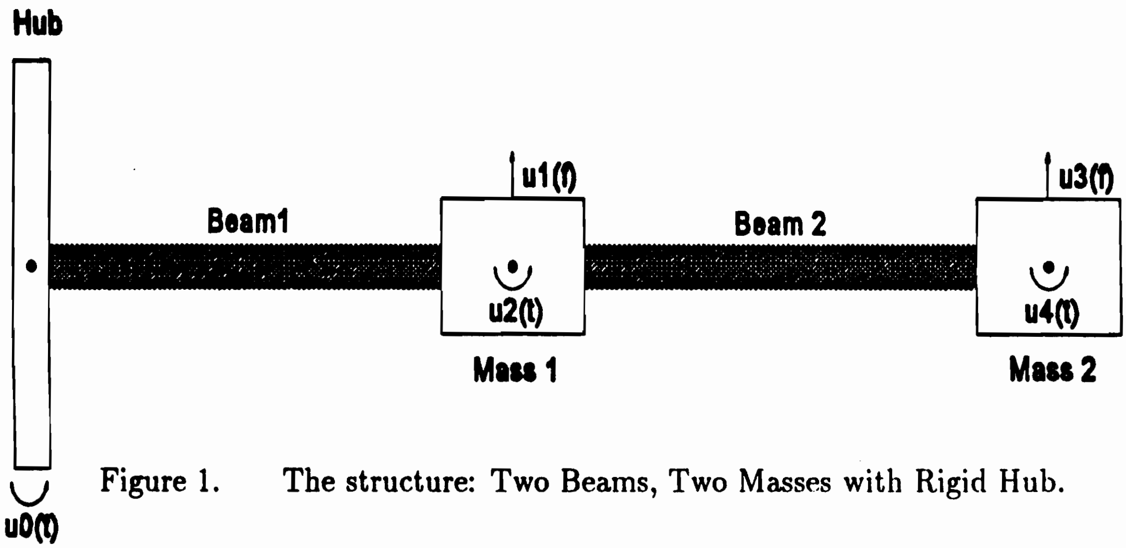


Figure 1. The structure: Two Beams, Two Masses with Rigid Hub.

The following notation is used: l_i is the length of beam $i,(i = 1,2)$, l_{m_i} is the length of mass $i,(i = 1,2)$,I_i is the moment of inertia of beam $i,(i = 1,2)$, J_i is moment of inertia for mass $i, i = 0,1,2$, where 0 refers to the hub, r_i is the coefficient of Kelvin- Voigt damping for beam i, $\theta(t)$ is angle of rotation of the hub, $u_0(t)$ is the external torque applied to the hub, $u_1(t)$ is the moment applied at mass 1 ,$u_2(t)$ is the thrust applied at mass 1 ,$u_3(t)$ is the moment applied at mass 2 and $u_4(t)$ is the thrust applied at mass 2.

To derive the mathematical model corresponding to this structure, define $V_1(t, x_1)$ to be the vertical displacement of beam 1 measured from the undeflected position of the beam as shown above, at time t, position $x_1, 0 \leq x_1 \leq l_1$. Define $V_2(t, x_2)$ to be the vertical displacement of beam 2 measured from the undeflected position of beam 2 as determined from the deflected position of beam 1 at time t, position $x_2, 0 \leq x_2 \leq l_2$ (see Figure 2). Measurement of the displacement of beam 2 with respect to this coordinate system rather than the equilibrium position in Figure 1 results in cantilevered boundary conditions on beam 2.

The system of equations describing this structure consists of the standard beam equations including angular acceleration terms due to the hub rotation, and boundary conditions which include the dynamics of distributed masses and the hub.

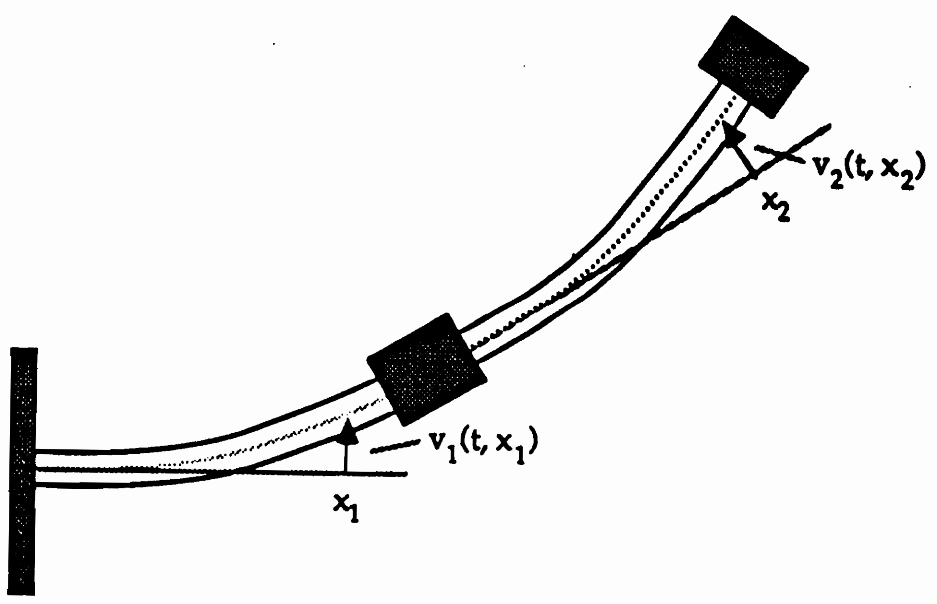


Figure 2. Coordinate System for the Structure.

The equations describing elastic deformations of the two beams for $t \geq 0, 0 \leq$

$x_i \leq l_i$, are given by

$$(\rho A)_1[\frac{\partial^2 V_1}{\partial t^2}(t,x_1) + x_1\ddot{\theta}(t)] + (rI)_1\frac{\partial^5 V_1}{\partial t \partial x^4}(t,x_1) + (EI)_1\frac{\partial^4 V_1}{\partial x^4}(t,x_1) = 0, \tag{1.1.1}$$

$$(\rho A)_2[\frac{\partial^2 V_2}{\partial t^2}(t,x_2) + \frac{\partial^2 V_1}{\partial t^2}(t,l_1) + (x_2 + l_{m_1})\frac{\partial^3 V_1}{\partial x \partial t^2}(t,l_1) + (x_2 + l_1 + l_{m_1})\ddot{\theta}(t)]$$
$$+ (rI)_2\frac{\partial^5 V_2}{\partial t \partial x^4}(t,x_2) + (EI)_2\frac{\partial^4 V_2}{\partial x^4}(t,x_2) = 0, \tag{1.1.2}$$

where ρ is the density of the beam material, A is the cross sectional area of the beam, r is a positive constant, I is the moment of inertia of the cross section with respect to the neutral axis, $I = \int\limits_A y^2\, dA$, and E is Young's modulus.

At the hub there are three boundary conditions. The conditions are the two clamped conditions for beam 1, and one boundary condition for bending moment which states that the external applied torque (or control) is equal to the difference between the moment on the hub and the moment at the left end of beam 1. The conditions are given by

$$V_1(t,0) = 0, \frac{\partial V_1}{\partial x}(t,0) = 0, \tag{1.1.3}$$

$$J_0\ddot{\theta}(t) - (rI)_1\frac{\partial^3 V_1}{\partial t \partial x^2}(t,0) - (EI)_1\frac{\partial^2 V_1}{\partial x^2}(t,0) = u_0(t). \tag{1.1.4}$$

At the connection between the two beams, i.e., at mass 1, there are four boundary conditions. These conditions are the two cantilevered conditions for beam 2 and two conditions on the shear force and bending moment. The boundary condition for shear force states that the thrust applied at mass 1 equals the force generated by the acceleration of mass 1 plus the difference between the shear forces on the left end of beam 1 and the right end of beam 2. The boundary condition for bending momemt states that the moment applied at mass 1 is equal to the moment generated by the rotation of mass 1 plus the difference between moments on the left end of beam 2 and the right end of beam 1. Thus, one has

$$V_2(t,0) = 0, \qquad \frac{\partial V_2}{\partial x}(t,0) = 0, \tag{1.1.5}$$

$$\begin{aligned}
&m_1[\frac{\partial^2 V_1}{\partial t^2}(t,l_1) + \frac{l_{m_1}}{2}\frac{\partial^3 V_1}{\partial x \partial t^2}(t,l_1) + (l_1 + \frac{l_{m_1}}{2})\ddot{\theta}(t)] \\
&+ (rI)_1 \frac{\partial^4 V_1}{\partial t \partial x^3}(t,l_1) + (EI)_1 \frac{\partial^3 V_2}{\partial x^3}(t,l_1) \\
&- (rI)_2 \frac{\partial^4 V_2}{\partial t \partial x^3}(t,0) - (EI)_2 \frac{\partial^3 V_2}{\partial x^3}(t,0) \\
&= u_2(t),
\end{aligned} \tag{1.1.6}$$

and

$$\begin{aligned}
&J_1[\frac{\partial^3 V_1}{\partial x \partial t^2}(t,l_1) + \ddot{\theta}(t)] \\
&+ (rI)_1[\frac{\partial^3 V_1}{\partial t \partial x^2}(t,l_1) + \frac{l_{m_1}}{2}\frac{\partial^4 V_1}{\partial t \partial x^3}(t,l_1)] \\
&+ (EI)_1[\frac{\partial^2 V_1}{\partial x^2}(t,l_1) + \frac{l_{m_1}}{2}\frac{\partial^3 V_1}{\partial x^3}(t,l_1)] \\
&- (rI)_2[\frac{\partial^3 V_2}{\partial t \partial x^2}(t,0) - \frac{l_{m_1}}{2}\frac{\partial^4 V_2}{\partial t \partial x^2}(t,0)] \\
&- (EI)_2[\frac{\partial^2 V_2}{\partial x^2}(t,0) - \frac{l_{m_1}}{2}\frac{\partial^3 V_2}{\partial x^3}(t,0)] \\
&= u_1(t).
\end{aligned} \tag{1.1.7}$$

At mass 2 there are two boundary conditions. These conditions describe the shear force and bending moment and are similar to those described above. The boundary condition for shear force states that the thrust applied at mass 2 equals the difference between the force generated by the acceleration of mass 2 and the shear force on the right end of beam 2. The boundary condition for bending moment states that the moment applied on mass 2 equals the difference between the moment generated by the rotation of mass 2 and the moment on the right end of beam 2. These conditions are given by

$$m_2[\frac{\partial^2 V_2}{\partial t^2}(t,l_2) + \frac{\partial^2 V_1}{\partial t^2}(t,l_1) + \frac{l_{m_2}}{2}\frac{\partial^3 V_2}{\partial x \partial t^2}(t,l)$$

$$+ (l_{m_1} + \frac{l_{m_2}}{2} + l_2)\frac{\partial^3 V_1}{\partial x \partial t^2}(t, l_1) + (l_1 + l_{m_1} + \frac{l_{m_2}}{2} + l_2)\ddot{\theta}(t)]$$

$$- (rI)_2 \frac{\partial^4 V_2}{\partial t \partial x^3}(t, l_2) - (EI)_2 \frac{\partial^3 V_1}{\partial x^3}(t, l_2)$$

$$= u_4(t), \tag{1.1.8}$$

and

$$J_2[\frac{\partial^3 V_2}{\partial x \partial t^2}(t, l_2) + \frac{\partial^3 V_1}{\partial x \partial t^2}(t, l_1) + \ddot{\theta}(t)]$$

$$+ (rI)_2[\frac{\partial^3 V_2}{\partial t \partial x^2}(t, l_2) + \frac{l_{m_2}}{2}\frac{\partial^4 V_2}{\partial t \partial x^3}(t, l_2)]$$

$$+ (EI)_2[\frac{\partial^2 V_2}{\partial x^2}(t, l_2) + \frac{l_{m_2}}{2}\frac{\partial^3 V_2}{\partial x^3}(t, l_2)]$$

$$= u_3(t). \tag{1.1.9}$$

We consider the following outputs

$$y_1(t) = \theta(t),$$

$$y_2(t) = V_1(t, l_1),$$

$$y_3(t) = \frac{\partial V_1}{\partial t}(t, l_1),$$

$$y_4(t) = V_2(t, l_2),$$

$$y_5(t) = \frac{\partial V_2}{\partial t}(t, l_2). \tag{1.1.10}$$

The first goal of this paper is to develop a framework for the analysis and approximation of linear control systems defined by dynamic compensations. We investigate finite dimensional approximations (Galerkin - type) of the infinite - dimensional compensators and show that these finite - dimensional compensators, when applied to the original model, produce near optimal performance. This is to say that the approximating closed loop system converges to the original one. Stability of the resulting closed loop system is established.

The material of this paper is organized as follows. In Chapter 2, we recall fundamental results on existence for a second order system and establish the well - posedness of the system with two connected beams (TCB). In Chapter 3, we present an abstract framework for compensator design. Our approach is to approximate an ideal infinite-dimensional LQG compensator with a sequence of finite-dimensional compensators. We study both the convergence of the approximating compensators and the performance of the closed-loop systems. especially, the uniformly exponential stability of the infinite dimensional closed - loop systems when suboptimal feedback is used. In Chapter 4, we apply the theory developed in Chapter 3 to design a finite-dimensional compensator for the connected beams. Convergence of the approximating feedback controller is established by showing that both control and estimator gains converge. The performance of dynamic compensation (the LQG controller) is compared to that of full state feedback.

1.2 Notation

Standard notation is used throughout the paper. The field of all real numbers is denoted by R. For $n \geq 1, R^n$ denotes n-dimensional Euclidean space with norm $|x|$ and inner product $x.y$. If H is a Hilbert space with inner-product $< .,. >_H$, then the norm of an element $y \in H$ is given by $||y||_H = (< y, y >_H)^{1/2}$. Given a Banach space E, its topological dual is denoted E^* and the duality pairing between an element x^* of E^* and x of E by $< x^*, x >_{E^* \times E}$. Let a and b , $a < b$, be two real numbers and E be a Banach space . The Banach space of equivalence classes of Lebesgue measurable function from [a, b] into E which are P- integrable $(1 \leq p < \infty)$ or essentially bounded $(P = \infty)$ is defined by $L^P([a, b], E)$. The space of m- times boundedly continuously differentiable functions from [a, b] into E is defined by $C^m([a,b],E)$. For $m \geq 1$ and an open interval I, $H^m(I)$ is the sobolev space of all functions in $L^2(I;R)$ whose first m distributional derivatives belong to $L^2(I;R)$. We use the superscript T to denote the transpose of a matrix (A^T).

Chapter II Well-Posedness

2.1 Basic Definition and Preliminaries

Let V and H be complex Hilbert spaces with norms $\|.\|_V$ and $\|.\|_H$, respectively. We denote the inner product on H by $< .,. >$. Assume that V is densely and continuously embeded in H. As is standard, the elements of the dual H^* of H are identified with the elements of H. Hence, we have the standard Gelfand triple denoted by

$$V \longrightarrow H \cong H^* \longrightarrow V^*.$$

Let $a(.,.)$ be a continuous sesquilinear form on V. Then there is $A \in L(V, V^*)$ such that for all $z \in V$,

$$a(z, \Phi) = (Az)(\Phi) =< Az, \Phi >_{V^*,V} . \tag{2.1.1}$$

Let the (restricted) domain of A be defined by

$$Dom(A) = \{z \in V : Az \in H\},$$

we define the operator $\mathbf{A}$ by

$$\mathbf{A} = A|_{Dom(A)}, Dom(\mathbf{A}) = Dom(A)$$

and for $z \in Dom(\mathbf{A}), \Phi \in V$,

$$a(z, \Phi) =< \mathbf{A}z, \Phi >_H . \tag{2.1.2}$$

A sesquilinear form $a(.,.)$ on V is V -elliptic if there exists a constant $C > 0$ such that for all $\Phi \in V$,

$$\text{Re}A(\Phi, \Phi) \geq C\|\Phi\|_V^2, \tag{2.1.3}$$

or V-coercive if there exist constants $C > 0$ and $\lambda \geq 0$ such that for all $\Phi \in V$,

$$\text{Re}a(\Phi, \Phi) \geq C|\Phi|_V^2 - \lambda|\Phi|_H^2. \tag{2.1.4}$$

THEOREM 2.1[6,Theorem 2.8]. *If $a(.,.)$ is a continuous, V-coercive sesquilinear form on V, then $D(\mathbf{A})$ is dense in V and hence dense in H.*

Thus, any continuous V-coercive sesquilinear form $a(.,.)$ on V is associated with a densely defined operator $\mathbf{A}$ on $Dom(\mathbf{A})$ with $a(z, \Phi) = < \mathbf{A}z, \Phi >_H$.

THEOREM 2.2[6,Theorem 2.8]. *If $\mathbf{A}$ is the operator given by (2.1.2) and $a(.,.)$ is continuous and V- coercive, then $-\mathbf{A}$ is the infinitesimal generator of an analytic semigroup $T(t) \in L(H), t \geq 0$.*

Before considering the second order problem, we consider the weak formulation of the first order initial value problem by

$$< \dot{z}(t), \Phi >_H + a(z(t), \Phi) = < f(t), \Phi >_H,$$
$$z(0) = z_0, \qquad \text{for all } \Phi \in V, \tag{2.1.5}$$

where $f : [0, T] \to H$. This problem gives rise to the abstract nonhomogeneous initial value problem on H,

$$\dot{z}(t) = -\mathbf{A}z(t) + f(t)$$
$$z(0) = z_0, \tag{2.1.6}$$

where $-\mathbf{A}$ is the infinitesimal generator of an analytic semigroup $T(t) \in L(H), t \geq 0$.

Define a mild solution of (2.1.6) as

$$z(t) = T(t)z_0 + \int_0^t T(t-\tau)f(\tau)\,d\tau, \tag{2.1.7}$$

whenever f is sufficiently smooth so that this function is well defined. A strong solution of (2.1.6) in H is a function $z \in L_2([0,T],V)$ such that $\dot{z} \in L_2([0,T],H)$, $z(0) = z_0$ and (2.1.6) holds almost everywhere on [0, T]. The following theorem gives the existence and form of the solution of (2.1.6).

THEOREM 2.3[5,Theorem 3.1.3]. *If* $-\mathbf{A}$ *is the generator of an analytic semigroup* $T(t)$ *and* $f : [0,T] \to H$ *is uniformly Hölder continuous, i. e.,*

$$|f(t) - f(s)|_H \leq K(t-s)^\alpha, 0 \leq s \leq t, 0 < \alpha \leq 1, \tag{2.1.8}$$

then for each $z_0 \in H$, *(2.1.6) has a unique strong solution* $z \in C([0,T];H) \cap C^1([0,1];H)$ *given by (2.1.7) . This solution depends continuously on* z_0 *and f.*

To obtain analogous results for second order systems, our approach is to write them in first order form on a product space and then to apply the previous theorems. Let $d(.,.)$ be a continuous sesquilinear form on V with

$$d(z,\Phi) = (Dz)(\Phi) = < Dz, \Phi >_{V^*,V} . \tag{2.1.9}$$

Similarly, we can define the operator $\mathbf{D}$. That is , if the restricted domain of D is defined by

$$Dom(D) = \{\Phi \in V : D\Phi \in H\},$$

we define

$$\mathbf{D} = D|_{Dom(D)}, Dom(\mathbf{D}) = Dom(D).$$

Then $\mathbf{D} : Dom(\mathbf{D}) \subset V \subseteq H \to H$, and for $z \in Dom(\mathbf{D}), \Phi \in V$,

$$d(z, \Phi) = < \mathbf{D}z, \Phi >_H . \tag{2.1.10}$$

A weak formulation of the second order initial value problem can be written as

$$< \ddot{z}(t), \Phi >_H + d(\dot{z}(t), \Phi) + a(z(t), \Phi) = < f(t), \Phi >_H$$

$$z(0) = z_0, \dot{z}(0) = z_1 \qquad \text{for all } \Phi \in V. \tag{2.1.11}$$

Assuming $a(.,.)$ and $d(.,.)$ are continuous and V-coercive, the corresponding abstract equation is given by

$$\ddot{z}(t) + \mathbf{D}\dot{z}(t) + \mathbf{A}z(t) = f(t)$$

$$z(0) = z_0, \dot{z}(0) = z_1, \tag{2.1.12}$$

where $\mathbf{A}$ and $\mathbf{D}$ are given by (2.1.2) and (2.1.10), respectively.

We define the product spaces $\Theta = V \times V, \Omega = V \times H$ with inner products $< .,. >_\Theta = < .,. >_V + < .,. >_V$ and $< .,. >_\Omega = < .,. >_V + < .,. >_H$, respectively, and give a sesquilinear form $\hat{a}$ on $\Theta \times \Theta$ by

$$\hat{a}((\phi, \psi), (\xi, \zeta)) = - < \psi, \xi >_V + a(\phi, \zeta) + d(\psi, \zeta). \tag{2.1.13}$$

Then (2.1.11) can be written as

$$< \dot{y}(t), x >_\Omega + \hat{a}(y, x) = < F(t), x >_\Omega,$$

$$y(0) = y_0 \qquad \text{for all} \qquad x \in \Theta, \tag{2.1.14}$$

where $y(t) = (z(t), \dot{z}(t)), y_0 = (z_0, z_1), x = (\phi, \psi)$ and $F(t) = (0, f(t))$. Using this notation, we can write (2.1.11) in first order form on Ω.

$$\dot{y}(t) = -\hat{A}y(t) + F(t)$$

$$y(0) = y_0, \tag{2.1.15}$$

where

$$\hat{A} = \begin{bmatrix} 0 & -I \\ \mathbf{A} & \mathbf{D} \end{bmatrix} \tag{2.1.16}$$

is defined on the restricted domain

$$D(\hat{A}) = \{(\phi, \psi) \in \Theta : \mathbf{A}\phi + \mathbf{D}\psi \in H\} \subseteq \Theta.$$

If $\hat{a}(.,.)$ is Θ -coercive, $D(\hat{A})$ is dense in Ω, by Theorem 2.1. The operator $\hat{A}$ is associated with $\hat{a}$ as

$$\hat{a}(x, \eta) = < -\hat{A}x, \eta >_{\Omega}, x \in D(\hat{A}), \eta \in \Theta. \tag{2.1.17}$$

By Theorem 2.2, if $\hat{a}$ is continuous and Θ-coercive, $-\hat{A}$ is the generator of an analytic semigroup $s(t) \in L(H), t \geq 0$. The following theorem gives the well-posedness and the form of the solution for (2.1.12).

THEOREM 2.4[6,Theorem 2.17]. *If $a(.,.)$ and $d(.,.)$ are continuous and V-coercive and $f : [0,T] \to H$ is uniformly Hölder continuous, then for each $y_0 \in \Theta$, (2.1.12) has a unique strong solution $y = (z, \dot{z})$ such that $z \in C([0,T], V) \cap C^1([0,T], V)$ and $\dot{z} \in C((0,T), V) \cap C^1((0,T), V)$ given by*

$$y(t) = T(t)y_0 + \int_0^t T(t-s)F(s)\, ds. \tag{2.1.18}$$

In above results, we see that $\mathbf{A}$ operator must be V-coercive. If not , then a self-adjoint linear operator $A_1 \in L(H, H)$ can be chosen so that $\hat{A} = \mathbf{A} + A_1$ is V- coercive. This choice can be made by selecting an operator whose null space is the orthogonal complement in H of the eigenspace of $\mathbf{A}$ corresponding to nonpositive eigenvalues. Any A_1 making $\hat{A}$ coercive must be positive definite on the nonpositive eigenspace of A.

Once A_1 is chosen , V is defined to be the completion of $D(A)$ with respect to the inner product $< z, \hat{z} >_V = < \hat{A}z, \hat{z} >_H$ for $z, \hat{z} \in D(A)$. Consequently, $V = D(\hat{A}^{1/2})$ and $< z, \hat{z} >_V = < \hat{A}^{1/2}z, \hat{A}^{1/2}\hat{z} >_H$ for $z, \hat{z} \in V$. Since the square root of an operator cannot always be written explicitly , neither can V. Once it is established that the abstract system corresponding to $\hat{A}$ is wellposed , the wellposedness of the original system is obtained by observing that the original system given by A is a bounded perturbation of the wellposed system, i. e.,$A = \hat{A} - A_1$. Applying the following theorem , we see that the original problem is wellposeded.

THEOREM 2.5[23,p.76]. *Let X be a Banach space and let A be the infinitesimal generator of a C_0 -semigroup $T(t)$ on X, satisfying $|T(t)| \leq Me^{\omega t}$. If B is a bounded linear operator on X, then $A + B$ is the infinitesimal generator of a C_0- semigroup $S(t)$ on X satisfying*

$$|S(t)| \leq Me^{(\omega + M|B|)t}.$$

2.2 Well-Posedness

The system governing the two beam problem is given by

$$J_0\ddot{\theta}(t) - (rI)_1\dot{V}_{1xx}(t,0) - (EI)_1V_{1xx}(t,0) = u_0(t), \tag{2.2.1}$$

$$\begin{aligned}&(\rho A)_1[\ddot{V}_1(t,x_1) + x_1\ddot{\theta}(t)] + (rI)_1\dot{V}_{1xxxx}(t,x_1) + (EI)_1V_{1xxxx}(t,x_1) = 0,\end{aligned} \tag{2.2.2}$$

$$\begin{aligned}&(\rho A)_2[\ddot{V}_2(t,x_2) + \ddot{V}_1(t,l_1) + (x_2 + l_{m1})\ddot{V}_{1x}(t,l_1) + (x_2 + l_1 + l_{m1})\ddot{\theta}(t)]\\ &+ (rI)_2\dot{V}_{2xxxx}(t,x_2) + (EI)_2V_{2xxxx}(t,x_2) = 0,\end{aligned} \tag{2.2.3}$$

$$J_1[\ddot{V}_{1x}(t,l_1) + \ddot{\theta}(t)] + (rI)_1[\dot{V}_{1xx}(t,l_2) + \frac{l_{m1}}{2}\dot{V}_{1xxx}(t,l_1)]$$
$$- (rI)_2[\dot{V}_{2xx}(t,0) - \frac{l_{m1}}{2}\dot{V}_{2xxx}(t,0)] + (EI)_1[V_{1xx}(t,l_1) + \frac{l_{m1}}{2}V_{1xxx}(t,l_1)]$$
$$- (EI)_2[V_{2xx}(t,0) - \frac{l_{m1}}{2}V_{2xxx}(t,0)] = u_1(t), \tag{2.2.4}$$
$$m_1[\ddot{V}_1(t,l_1) + \frac{l_{m1}}{2}\ddot{V}_{1x}(t,l_1) + (l_1 + \frac{l_{m1}}{2})\ddot{\theta}(t)] + (rI)_1\dot{V}_{1xxx}(t,l_1)$$
$$+ (EI)_1V_{1xxx}(t,l_1) - (rI)_2\dot{V}_{2xxx}(t,0) - (EI)_2V_{2xxx}(t,0) = u_2(t). \tag{2.2.5}$$

And

$$J_2[\ddot{V}_{2x}(t,l_2) + \ddot{V}_{1x}(t,l_1) + \ddot{\theta}(t)] + (rI)_2[\dot{V}_{2xx}(t,l_2)$$
$$+ \frac{l_{m2}}{2}\dot{V}_{2xxx}(t,l_2)] + (EI)_2[V_{2xx}(t,l_2) + \frac{l_{m2}}{2}V_{2xxx}(t,l_2)] = u_3(t), \tag{2.2.6}$$
$$m_2[\ddot{V}_2(t,l_2) + \ddot{V}_1(t,l_1) + \frac{l_{m2}}{2}\ddot{V}_{2x}(t,l_2) + (l_{m1} + \frac{l_{m2}}{2} + l_2)\ddot{V}_{1x}(t,l_1)$$
$$+ (l_1 + l_{m1} + \frac{l_{m2}}{2} + l_2)\ddot{\theta}(t)] - (rI)_2\dot{V}_{2xxx}(t,l_2) - (EI)_2V_{2xxx}(t,l_2) = u_4(t), \tag{2.2.7}$$
$$V_1(t,0) = 0, V_{1x}(t,0) = 0, V_2(t,0) = 0, V_{2x}(t,0) = 0. \tag{2.2.8}$$

The equations have been rewritten in order to facilitate the identification of the "natural state", the operator **A**, etc. The state space H is chosen as $H = \mathbf{R} \times L_2[0,l_1] \times L_2[0,l_2] \times R^4$, and the state is defined by $z(t) = (z_1(t), z_2(t), z_3(t), z_4(t), z_5(t), z_6(t),$

$z_7(t))$ in H, where

$$
\begin{aligned}
& z_1(t) = \theta(t), \\
& z_2(t) = V_1(t,.) + i_1\theta(t), \\
& z_3(t) = V_2(t,.) + V_1(t,l_1) + (i_2 + l_{m1})V_{1x}(t,l_1) + (i_2 + l_1 + l_{m1})\theta(t), \\
& z_4(t) = V_{1x}(t,l_1) + \theta(t), \\
& z_5(t) = V_1(t,l_1) + \frac{l_{m1}}{2}V_{1x}(t,l_1) + (l_1 + \frac{l_{m1}}{2})\theta(t), \\
& z_6(t) = V_{2x}(t,l_2) + V_{1x}(t,l_1) + \theta(t), \\
& z_7(t) = V_2(t,l_2) + V_1(t,l_1) + \frac{l_{m2}}{2}V_{2x}(t,l_2) + (l_{m1} + \frac{l_{m2}}{2} + l_2)V_{1x}(t,l_1) \\
& + (l_1 + l_{m1} + \frac{l_{m2}}{2} + l_2)\theta(t).
\end{aligned}
\tag{2.2.9}
$$

and i_j is the identity mapping on $[0, l_j]$,

$$i_j(x_j) = x_j, \qquad \text{for} \qquad x_j \in l_j, j = 1, 2. \tag{2.2.10}$$

Taking the inner product of equations (2.2.1) - (2.2.7) with $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7)$, we obtain an abstract form of the equation in H,

$$\ddot{z}(t) + D\dot{z}(t) + Az(t) = B_0u(t). \tag{2.2.11}$$

The inner product on H is taken to be

$$
\begin{aligned}
< z, \hat{z} >_H = & J_0 z_1 \hat{z}_1 + < (\rho A)_1 z_2, \hat{z}_2 >_{L_2[0,l_1]} \\
& + < (\rho A)_2 z_3, \hat{z}_3 >_{L_2[0,l_2]} + J_1 z_4 \hat{z}_4 \\
& + m_1 z_5 \hat{z}_5 + J_2 z_6 \hat{z}_6 + m_2 z_7 \hat{z}_7.
\end{aligned}
\tag{2.2.12}
$$

Therefore, it follows that

$$
\mathbf{A}z = \begin{bmatrix}
-\frac{(EI)_1}{J_0} z_{2xx}(0), \\
\frac{(EI)_1}{(\rho A)_1} z_{2xxxx}(.), \\
\frac{(EI)_2}{(\rho A)_2} z_{3xxxx}(.), \\
\frac{1}{J_1}[(EI)_1(z_{2xx}(l_1) + \frac{l_{m1}}{2} z_{2xxx}(l_1)) - (EI)_2(z_{2xx}(0) - \frac{l_{m1}}{2} z_{3xxx}(0))], \\
\frac{1}{m_1}[(EI)_1 z_{3xxx}(0) - (EI)_2 z_{2xxx}(l_1)], \\
\frac{(EI)_2}{J_2}[z_{3xx}(l_2) + \frac{l_{m2}}{2} z_{3xxx}(l_2)], \\
-\frac{(EI)_2}{m_2} z_{3xxx}(l_2)
\end{bmatrix}, \tag{2.2.13}
$$

where

$$
\begin{aligned}
D(\mathbf{A}) = \{z \in H : \; & z_2 \in H^4[0, l_1], z_3 \in H^4[0, l_2], z_2(0) = 0, \\
& z_2(l_1) = z_5 - \frac{l_{m1}}{2} z_4, z_3(0) = z_5 + \frac{l_{m1}}{2} z_4, z_3(l_2) = z_7 - \frac{l_{m2}}{2} z_0, \\
& z_{2x}(0) = z_1, z_{2x}(l_1) = z_4 = z_{3x}(0), z_{3x}(l_2) = z_6\}
\end{aligned} \tag{2.2.14}
$$

and

$$
\mathbf{D}z = \begin{bmatrix}
-\frac{(rI)_1}{J_0} z_{2xx}(0) \\
\frac{(rI)_1}{(\rho A)_1} z_{2xxxx}(.) \\
\frac{(rI)_2}{(\rho A)_2} z_{3xxxx}(.) \\
\frac{1}{J_1}[(rI)_1(z_{2xx}(l_1) + \frac{l_{m1}}{2} z_{2xxx}(l_1)) - (rI)_2(z_{3xx}(0) - \frac{l_{m1}}{2} z_{3xxx}(0))] \\
\frac{1}{m_1}[(rI)_2 z_{3xxx}(0) - (rI)_1 z_{2xxx}(l_1)] \\
\frac{(rI)_2}{J_2}[z_{3xx}(l_2) + \frac{l_{m2}}{2} z_{3xxx}(l_2)] \\
-\frac{(rI)_2}{m_2} z_{3xxx}(l_2)
\end{bmatrix}, \tag{2.2.15}
$$

where

$$
Dom(\mathbf{D}) = Dom(\mathbf{A}).
$$

Also,

$$B = \begin{bmatrix} \frac{1}{J_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{J_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{J_2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{m_2} \end{bmatrix}. \tag{2.2.16}$$

The control is given by

$$u(t) = [u_0(t), u_1(t), u_2(t), u_3(t), u_4(t)]^T$$

and the output becomes

$$\begin{aligned} y_1(t) &= z_1(t); \\ y_2(t) &= -l_1 z_1(t) - \frac{l_{m_1}}{2} z_4(t) + z_5(t); \\ y_3(t) &= -l_1 \dot{z}_1(t) - \frac{l_{m_1}}{2} \dot{z}_4(t) + \dot{z}_5(t); \\ y_4(t) &= -(l_2 + \frac{l_{m_1}}{2}) z_4(t) - z_5(t) - \frac{l_{m_2}}{2} z_6(t) + z_7(t); \\ y_5(t) &= -(l_2 + \frac{l_{m_1}}{2}) \dot{z}_4(t) - \dot{z}_5(t) - \frac{l_{m_2}}{2} \dot{z}_6(t) + \dot{z}_7(t). \end{aligned} \tag{2.2.17}$$

The rigid body rotation of the hub induces a zero eigenvalue for $\mathbf{A}$. Hence, $\mathbf{A}$ is not V- coercive. Define a bounded operator $\mathbf{A}_1$ so that $\hat{\mathbf{A}} = \mathbf{A} + \mathbf{A}_1$ is V-coercive. As discussed above, there are infinite possibilities for $\mathbf{A}_1$. We choose $\mathbf{A}_1$ so that

$$< \mathbf{A}_1 z, \hat{z} >_H = J_0 z_1 \hat{z}_1.$$

Then $V = D(\tilde{A}^{1/2}) = D(A^{1/2})$ and is contained in the set

$$\begin{aligned}
&\{z \in H : z_2 \in H^2[0,l_1], z_3 \in H^2[0,l_2], z_2(0) = 0,\\
&z_2(l_1) = z_5 - \frac{l_{m1}}{2} z_4, z_3(0) = z_5 + \frac{l_{m1}}{2} z_4,\\
&z_3(l_2) = z_7 - \frac{l_{m2}}{2} z_6, z_{2x}(0) = z_1,\\
&z_{2x}(l_1) = z_4 = z_{3x}(0), z_{3x}(l_2) = z_6\}. \qquad (2.2.18)
\end{aligned}$$

The inner product on V is taken to be

$$\begin{aligned}
< z, \hat{z} >_V = &< \tilde{A}^{1/2} z, \tilde{A}^{1/2} \hat{z} >_H\\
&= < (EI)_1 z_{2xx}, \hat{z}_{2xx} >_{L_2[0,l_1]} + < (EI)_2 z_{3xx}, \hat{z}_{3xx} >_{L_2[0,l_2]}\\
&\quad + < A_1 z, \hat{z} >_H . \qquad (2.2.19)
\end{aligned}$$

Define $\hat{\mathbf{D}} = \mathbf{D} + \mathbf{A}_1$ where $\mathbf{A}_1$ is defined above. To prove that the problem is well-posed, we need to determine the sesquilinear forms associated with $\hat{\mathbf{A}}$ and $\hat{\mathbf{D}}$. Consider $< \hat{\mathbf{A}} z, \Phi >_H$ for $z \in D(\mathbf{A}), \Phi \in V$. We have

$$\begin{aligned}
< \hat{\mathbf{A}} z, \Phi >_H = &- (EI)_1 z_{2xx}(0)\phi_1 + < (EI)_1 z_{2xxxx}, \phi_2 >_{L_2[0,l_1]}\\
&+ < (EI)_2 z_{2xxxx}, \phi_3 >_{L_2[0,l_2]}\\
&+ [(EI)_1 (z_{2xx}(l_1) + \frac{l_{m1}}{2} z_{2xxx}(l_1)) - (EI)_2 (z_{2xx}(0) - \frac{l_{m1}}{2} z_{3xxx}(0))]\phi_4\\
&+ [(EI)_2 z_{3xxx}(0) - (EI)_1 z_{2xxx}(l_1)]\phi_5\\
&+ (EI)_2 [z_{3xx}(l_2) + \frac{l_{m2}}{2} z_{3xxx}(l_2)]\phi_6\\
&- (EI)_2 z_{3xxx}(l_2)\phi_7 + J_0 z_1 \phi_1. \qquad (2.2.20)
\end{aligned}$$

Integrating by parts twice and rearranging terms yields

$$\begin{aligned} < \hat{\mathbf{A}}z, \Phi >_H = &< (EI)_1 z_{2xx}, \phi_{2xx} >_{L_2[0,l_1]} + < (EI)_2 z_{3xx}, \phi_{3xx} >_{L_2[0,l_2]} \\ &+ (EI)_1[z_{2xxx}(x_1)\phi_2(x_1)|_0^{l_1} - z_{2xx}(x_1)\phi_{2x}(x_1)|_0^{l_1}] \\ &+ (EI)_2[z_{3xxx}(x_2)\phi_3(x_2)|_0^{l_2} - z_{3xx}(x_2)\phi_{3x}(x_2)|_0^{l_2}] \\ &- (EI)_1 z_{2xx}(0)\phi_1 + [(EI)_1(z_{2xx}(l_1) + \frac{l_{m1}}{2} z_{2xxx}(l_1)) \\ &- (EI)_2(z_{3xx}(0) - \frac{l_{m1}}{2} z_{3xxx}(0))]\phi_4 \\ &+ [(EI)_2 z_{3xxx}(0) - (EI)_1 z_{2xxx}(l_1)]\phi_5 \\ &+ (EI)_2[z_{3xx}(l_2) + \frac{l_{m2}}{2} z_{3xxx}(l_2)]\phi_6 - (EI)_2 z_{3xxx}(l_2)\phi_7 + J_0 z_1 \phi_1. \end{aligned}$$

Using the fact that $\Phi \in V$, we can make substitutions into the boundary terms arising from integration by parts and regroup terms to obtain

$$\begin{aligned} < \hat{\mathbf{A}}z, \phi >_H = &< (EI)_1 z_{2xx}, \phi_{2xx} >_{L_2[0,l_1]} + < (EI)_2 z_{3xx}, \phi_{3xx} >_{L_2[0,l_2]} \\ &+ (EI)_1[-z_{2xx}(0)\phi_1 + z_{2xxx}(l_1)(\phi_5 - \frac{l_{m1}}{2}\phi_4) - z_{2xx}(l_1)\phi_4 \\ &+ z_{2xx}(0)\phi_1 + z_{2xx}(l_1)\phi_4 + \frac{l_{m1}}{2} z_{2xxx}(l_1)\phi_4 - z_{2xxx}(l_1)\phi_5] \\ &+ (EI)_2[z_{3xxx}(l_2)(\phi_7 - \frac{l_{m2}}{2}\phi_6) - z_{3xxx}(0)(\phi_5 + \frac{l_{m1}}{2}\phi_4) \\ &- \phi_6 z_{3xx}(l_2) + \phi_4 z_{3xx}(0) - \phi_4 z_{3xx}(0) + \frac{l_{m1}}{2} z_{3xxx}(0)\phi_4 \\ &+ z_{3xxx}(0)\phi_5 + z_{3xx}(l_2)\phi_6 + \frac{l_{m2}}{2} z_{3xxx}(l_2)\phi_6 - z_{3xxx}(l_2)\phi_7] + J_0 z_1 \phi_1. \end{aligned}$$

Cancelling like terms leaves

$$\begin{aligned} \hat{\mathbf{a}}(z, \Phi) = &< (EI)_1 z_{2xx}, \phi_{2xx} >_{L_2[0,l_1]} \\ &+ < (EI)_2 z_{3xx}, \phi_{3xx} >_{L_2[0,l_2]} + J_0 z_1 \phi_1 \\ =& \mathbf{a}(z, \phi) + < \mathbf{A}_1 z, \phi >_H . \end{aligned} \tag{2.2.21}$$

To write the sesquilinear form correspoding to $\mathbf{D}$ and hence $\hat{\mathbf{D}}$, we use the part of $\hat{a}$ corresponding to $\mathbf{A}$.

Separate $\mathbf{a}$ into $\mathbf{a}_1$ and $\mathbf{a}_2$ where

$$\mathbf{a}_1(z,\phi) = < (EI)_1 z_{2xx}, \phi_{2xx} >_{L_2[0,l_1]}$$

$$\mathbf{a}_2(z,\phi) = < (EI)_2 z_{3xx}, \phi_{3xx} >_{L_2[0,l_2]} .$$

Then,

$$\mathbf{d}(z,\phi) = \frac{r_1}{E_1}\mathbf{a}_1(z,\phi) + \frac{r_2}{E_2}\mathbf{a}_2(z,\phi) \qquad \text{for all } z,\phi \in V,$$

and

$$\hat{\mathbf{d}}(z,\phi) = \mathbf{d}(z,\phi) + < \mathbf{A}_1 z, \phi >_H \tag{2.2.22}$$

$$= < (rI)_1 z_{2xx}, \phi_{2xx} >_{L_2[0,l_1]} + < (rI)_2 z_{3xx}, \phi_{3xx} >_{L_2[0,l_2]} + J_0 z_1 \phi_1 .$$

The weak formulation of the system is given by

$$< \ddot{z}(t), \psi >_H + \mathbf{d}(\dot{z}(t), \psi) + \mathbf{a}(z(t), \psi) = < \mathbf{B}u(t), \psi >_H$$

for all $\psi \in V$. To show it is wellposed, we consider the system

$$< \ddot{z}(t), \psi >_H + \hat{\mathbf{d}}(\dot{z}(t), \psi) + \hat{\mathbf{a}}(z(t), \psi) = < \mathbf{B}u(t), \psi >_H,$$

where $\hat{\mathbf{a}}(z,\psi), \hat{\mathbf{d}}(\dot{z},\psi)$ are given by (2.2.10), (2.2.11) , respectively .

In [5, Theorem 3.5.1 and Theorem 3.5.2], King proved that $\hat{\mathbf{a}}(z,\psi), \hat{\mathbf{d}}(\dot{z},\psi)$ are both continuous and V- elliptic . By Theorem 2.4, the problem

$$\ddot{z}(t) + \hat{\mathbf{D}}\dot{z}(t) + \hat{\mathbf{A}}z(t) = \mathbf{B}u(t)$$

is wellposed. Therefore, by Theorem 2.5, we know the original model for the systems given by (2.2.2) through (2.2.8) is wellposed.

Chapter III The LQG Optional Control Problem

3.1 The Optional Infinite-Dimensional Compensator and Closed-Loop Systems

Consider the evolution eqution on the Hilbert space E

$$\dot{z}(t) = -\mathbf{A}z(t) + f(t),$$
$$z(0) = z_0 \in E, \qquad t > 0, \tag{3.1.1}$$

where $-\mathbf{A}$ is the infinitesmial of strongly continuous semigroup $T(t)$ on E. The mild solution to (3.1.1) is given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)f(s)\,ds. \tag{3.1.2}$$

We consider input of the form

$$f(t) = Bu(t) + \tilde{B}\gamma(t), t > 0, \tag{3.1.3}$$

and an output given by

$$y(t) = Cz(t) + \nu(t), t > 0, \tag{3.1.4}$$

where z is the mild solution to (3.1.1) , $u(t) \in R^m, \gamma(t) \in R^l, y(t) \in R^P, \nu(t) \in R^P, B \in \mathbf{B}(R^{mxn}, E), \tilde{B} \in \mathbf{B}(R^l, E)$ and $C \in \mathbf{B}(H, R^P)$. Also, γ and ν are stationary zeromean Gaussian white noise processes with covariance matrices Γ and $\hat{R}$, respectively, and $\hat{R}$ is positive definite.

Consider the problem of minimizing the cost functional

$$J(u) = \lim_{t_f \to \infty} E\{\frac{1}{t_f}\int_0^{t_f} [< Qz(t), z(t) > +u^T(t)Ru(t)]\,dt\} \tag{3.1.5}$$

subject to (3.1.2) where Q is a nonnegative (definite), bounded and self-adjoint operator on E and $R \in R^{mxm}$ is positive definite and self-adjoint.

The basic theory for the infinite-dimensional LQG optimal control problem with bounded input and output operators can be found in [2], [3], [4] , [5] ,[9] and [10]. We briefly summarize the relevant results and essential feature of theory. The LQG problem can be seperated into a deterministic linear-quadratic regulator problem on the infinite interval and a dual state-estimator, or filtering problem.

First, we consider the regulator problem, which is to choose the control u to minimize the integral in (3.1.5) when both noise processes in (3.1.3), (3.1.4) are zero, the output operator C is the identity, and $t_f = \infty$.

The pair (A, B) is said to be exponentially stabilizable if there exists an operator $K \in L(E, R^m)$ such that $A - BK$ generates an exponentially stable semigroup. And the pair (A, C) is exponentially detectable if there exists an operator $G \in L(R^P, E)$ such that $A - GC$ generates an exponentially stable semigroup on E .

If (A, B) is exponentially stabilizable and (Q, A) is exponentially detectable, there exists a unique nonnegative self-adjoint solution $\Pi \in \mathbf{B}(E, E)$ to the operator algebraic Riccati equation

$$A^*\Pi + \Pi A - \Pi B R^{-1} B^* \Pi + Q = 0, \tag{3.1.6}$$

with $\Pi(Dom(A)) \subset Dom(A^*)$. The optimal control for infinite-time linear-quadratic regulator problem has the feedback form

$$u(t) = -Kz(t), t \geq 0 \tag{3.1.7}$$

where

$$K = R^{-1}B^*\Pi \in B(H, R^m). \tag{3.1.8}$$

For the filtering problem, we define

$$\hat{Q} = \tilde{B}\Gamma\tilde{B}^*.$$

If the pair (C, A) is exponentially detectable and the pair $(A, \hat{Q})$ is exponentially stabilizable, the operator algebraic Riccati equation

$$A\hat{\Pi} + \hat{\Pi}A^* - \hat{\Pi}C^*\hat{R}^{-1}C\hat{\Pi} + \hat{Q} = 0, \tag{3.1.9}$$

admits a unique nonnegative self-adjoint solution $\hat{\Pi} \in B(H, H)$ with $\hat{\Pi}(Dom(A^*)) \subset Dom(A)$. The minimum-variance estimate of $z(t)$ given $y(\tau), (\tau \leq t)$, is a mild solution $\hat{z}(t)$ to the evolution equation

$$\dot{\hat{z}}(t) = A\hat{z}(t) + Bu(t) + \hat{K}\{y(t) - C\hat{z}(t)\} \tag{3.1.10}$$

where

$$\hat{K} = \hat{\Pi}C^*\hat{R}^{-1} \in B(R^P, H). \tag{3.1.11}$$

The optimal LQG compensator consists of the filter, or state estimator, (3.1.10) and the control law

$$u(t) = -K\hat{z}(t), t \geq 0 \tag{3.1.12}$$

with the control and filter gain operators given by (3.1.8) and (3.1.11), respectively .

The optimal closed-loop system then takes the form

$$\begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix} = S_{\infty\infty}(t) \begin{bmatrix} z(0) \\ \hat{z}(0) \end{bmatrix}, t \geq 0, \tag{3.1.13}$$

where $S_{\infty\infty}(t)$ is the semigroup generated on E x E by the operator

$$A_{\infty\infty}(t) = \begin{bmatrix} \mathbf{A} & -\mathbf{B}K \\ \hat{K}C & [\mathbf{A} - \mathbf{B}K - \hat{K}C] \end{bmatrix}, \qquad t \geq 0, \tag{3.1.14}$$

with

$$Dom(\mathbf{A}_{\infty\infty}) = Dom(\mathbf{A})\text{x}Dom(\mathbf{A}).$$

If $S(t)$ and $\hat{S}(t)$ are the semigroups of bounded linear operators generated on E by infinitesimal generator $\mathbf{A}-\mathbf{B}K$ and $\mathbf{A}-\hat{K}\mathbf{C}$, respectively, then it is easy to establish the following

THEOREM 3.1[3,Theorem 7.2]. *Suppose that* $\|S(t)\| \leq M_1 e^{-\alpha_1 t}$, *and* $\|\hat{S}(t)\| \leq M_2 e^{-\alpha_2 t}, t \geq 0$. *Then for any* $\alpha_3 < \min\{\alpha_1, \alpha_2\}$, *there is* $M_3 > 0$, *such that*

$$\|S_{\infty\infty}\| \leq M_3 e^{-\alpha_3 t}, t \geq 0 \tag{3.1.15}$$

and

$$\sigma(\mathbf{A}_{\infty\infty}) = \sigma(\mathbf{A}-\mathbf{B}K) \cup \sigma(\mathbf{A}-\hat{K}C).$$

Now we consider the system (2.1.13) . Since $\Pi \in L(E,E)$ and $E = V$ x H, we can write

$$\Pi = \begin{bmatrix} \Pi_0 & \Pi_1 \\ \Pi_1^* & \Pi_2 \end{bmatrix}, \tag{3.1.16}$$

where $\Pi_0 \in L(V,V), \Pi_1 \in L(H,V), \Pi_2 \in L(H,H)$ and Π_0, Π_2 are nonnegative and self - adjoint. With $z = (x, \dot{x})$, (3.1.2) becomes

$$u(t) = -R^{-1} B_0^* [\Pi_1^* x(t) + \Pi_2 \dot{x}(t)]. \tag{3.1.17}$$

Since $B_0 \in L(R^m, H)$, there exists vector $b_i \in H, 1 \leq i \leq m$, such that

$$B_0 u = \sum_{i=1}^{m} b_i u_i, \qquad \text{for} \qquad u = [u_1, u_2, ..., u_m]^T \in R^m. \tag{3.1.18}$$

Also, for $h \in H$,

$$\mathbf{B}_0^* h = [< b_1, h >_H, < b_2, h >_H, ..., < b_m, h >_H]^T. \tag{3.1.19}$$

Since $\Pi_1^* x(t)$ and $\Pi_2 \dot{x}(t)$ are elements of H, we see from (3.1.17) and (3.1.19) that the components of the optimal control have the feedback form

$$u_i(t) = - < f_i, x(t) >_V - < g_i, \dot{x}(t) >_H, i = 1, ..., m, \tag{3.1.20}$$

where $f_i \in V$ and $g_i \in H$ are given by

$$f_i = \sum_{j=1}^{m} (R^{-1})_{ij} \Pi_1 b_j, g_i = \sum_{j=1}^{m} (R^{-1})_{ij} \Pi_2 b_j, \tag{3.1.21}$$

for $i = 1, 2, ..., m$. The measurement operator C in(3.14) has the form

$$C = [C_1, C_2] \tag{3.1.22}$$

where $C_1 \in L(V, R^P)$ and $C_2 \in L(H, R^P)$. Hence, if we denote by $(C(x, \dot{x}))_i$ the ith component to the vector $C(x, \dot{x})$ for $(x, \dot{x}) \in E$, then there must exist $C_{1i} \in V$ and $C_{2i} \in H$ such that

$$(C(x, \dot{x}))_i =< C_{1i}, x >_V + < C_{2i}, \dot{x} >_H, \tag{3.1.23}$$

for $i = 1, 2, ..., P$.

Also, the estimator gain operator $\hat{K}$ is given by

$$\hat{K} y = \sum_{i=1}^{P} (\hat{f}_i, \hat{g}_i) y_i \tag{3.1.24}$$

for $y = [y_1, y_2, ..., y_P]^T \in R^P$, where the functional estimator gains $\hat{f}_i$ and $\hat{g}_i$ are elements of V and H, respectively.

For the optimal estimator gain, we partition $\hat{\Pi}$ as

$$\hat{\Pi} = \begin{bmatrix} \hat{\Pi}_0 & \hat{\Pi}_1 \\ \hat{\Pi}_1^* & \hat{\Pi}_2 \end{bmatrix} \tag{3.1.25}$$

and use (3.1.11) and (3.1.23) to obtain

$$\hat{f}_i = \sum_{j=1}^{P} (\hat{R}^{-1})_{ij} (\hat{\Pi}_0 C_{1j} + \hat{\Pi}_1 C_2 j),$$

$$\hat{g}_i = \sum_{j=1}^{P} (\hat{R}^{-1})_{ij} (\hat{\Pi}_1^* C_{1j} + \hat{\Pi}_2 C_{2j}), \tag{3.1.26}$$

for $i = 1, 2, ..., P$.

3.2 Approximation and Convergence

The approximation and convergence theory for infinite dimensional LQG problem was initiated in [8], [3] , [9], [10] and [12]. Here, we briefly sumerize the known results. Let H_n be a sequence of finite - dimensional subspaces of H and let P_n denote a sequence of orthogonal projections, $P_n : H \to H_n \subset H$.

Consider the approximating system $(\mathbf{A}_n, \mathbf{B}_n, \mathbf{Q}_n)$, where $\mathbf{A}_n = P_n\mathbf{A} \in L(H_n)$ is the infinitesimal generator of the C_0 - semigroup $S_n(t) = P_n S(t)$ on $H_n, \mathbf{B}_n = P_n\mathbf{B} \in L(R^m, H_n), \hat{\mathbf{B}}_n = P_n\hat{\mathbf{B}} \in L(R^l, H_n), C_n = P_n C$ and $Q_n = (C_n)^* C_n = P_n Q P_n \in L(H, H_n)$. Take $\hat{Q}_n = \hat{\mathbf{B}}_n \Gamma(\hat{\mathbf{B}}_n) = P_n \hat{Q} P_n \in L(H_n)$.

HYPOTHESIS 1.. *The finite - dimensional algebraic Riccati equations*

$$(\mathbf{A}_n)^*\Pi_n + \Pi_n\mathbf{A}_n - \Pi_n\mathbf{B}_n R^{-1}(\mathbf{B}_n)^*\Pi_n + Q_n = 0 \tag{3.2.1}$$

and

$$\mathbf{A}_n\hat{\Pi}_n + \hat{\Pi}_n(\mathbf{A}_n)^* - \hat{\Pi}_n(C_n)^*\hat{R^{-1}}C_n\hat{\Pi}_n + \hat{Q}_n = 0 \tag{3.2.2}$$

admit unique nonnegative self-adjoint solutions $\Pi_n \in L(H_n)$ *and* $\hat{\Pi}_n \in L(H_n)$*,respectively.*

The associated sequence of finite dimensional LQR problem is given by

$$\text{Minimize} \quad J_n(z_n(0), u) \text{ over all} \quad u \in L_2((0, \infty), u) \tag{3.2.3}$$

subject to

$$\dot{z}_n(t) = \mathbf{A}_n z_n(t) + \mathbf{B}_n u(t),$$
$$z_n(0) = z_{n0} = P_n z_0,$$
$$y_n(t) = C_n z_n(t), t \geq 0, \tag{3.2.4}$$

where

$$J_n(z_n(0), u) = \int_0^\infty (< Q_n z_n(t), z_n(t) >_H + < Ru(t), u(t) >_U)\, dt. \tag{3.2.5}$$

Since $B_n : U \rightarrow H_n$, the trajectories of (3.2.4) evolve in H_n. Thus, (3.2.3) is an LQR problem on the finite dimensional state space H_n. The optimal control $u_n(t)$ has the feedback form

$$u_n(t) = -K_n z_n \tag{3.2.6}$$

and

$$K_n = R^{-1} B_n^* \Pi_n \tag{3.2.7}$$

where $\Pi_n \in L(H_n)$ is the unique nonnegative self- adjoint solution of (3.2.1) on H_n. The trajectory $z_n(t)$ is the corresponding solution of (3.2.4) with $u = u_n$.

Similarly, we can apply the same procedure as above to the dual problem in order to obtain a convergence $\hat{K}_n$ to $\hat{K}$, where the Kalman filter gain $\hat{K}$ is given by (3.1.8). The nth compensator is

$$\dot{\hat{z}}_n = \mathbf{A}_n \hat{z}_n + \mathbf{B}_n u + \hat{K}_n(y - C_n \hat{z}_n) \tag{3.2.8}$$

where the estimator gain operator $\hat{K}_n$ is

$$\hat{K}_n = \hat{\Pi}_n C_n^* \hat{R}^{-1} \tag{3.2.9}$$

and $\hat{\Pi}_n$ is the nonnegative and self-adjoint solution to the Riccati equation (3.2.2).

The resulting closed-loop system is by

$$\begin{bmatrix} z(t) \\ \hat{z}_n(t) \end{bmatrix} = S_{\infty n}(t) \begin{bmatrix} z(0) \\ \hat{z}_n(0) \end{bmatrix} \tag{3.2.10}$$

where $S_{\infty n}(t)$ is the C_0 - semigroup of bounded linear operator on $E \times E_n$ with infinitesimal generator

$$\mathbf{A}_{\infty n} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}K_n \\ \hat{K}_n C & [\mathbf{A}_n - \mathbf{B}_n K_n - \hat{K}_n C_n] \end{bmatrix} \tag{3.2.11}$$

and

$$Dom(\mathbf{A}_{\infty n}) = Dom(\mathbf{A}) \times E_n.$$

Note that $\mathbf{A}_{\infty n}$ has compact resolvent if and only if $\mathbf{A}$ does.

HYPOTHESIS 2.. *For each n, there exists a linear mapping P_n from H onto H_n such that*

$$\lim_{n \to \infty} P_n x = x, x \in H. \tag{3.2.12}$$

For each $x \in H$ and each $t \geq 0$, we have

$$\lim_{n \to \infty} T_n(t) P_n x = T(t)x, \tag{3.2.13}$$

$$\lim_{n \to \infty} T_n^*(t) P_n x = T^*(t)x, \tag{3.2.14}$$

where, in each case, the convergence is uniformly in t for t in bounded intervals. Also,

$$\lim_{n \to \infty} B_n u = Bu, u \in R^m, \tag{3.2.15}$$

$$\lim_{n \to \infty} Q_n P_n x = Qx, x \in H, \tag{3.2.16}$$

and

$$\lim_{n \to \infty} C_n P_n x = Cx, x \in H. \tag{3.2.17}$$

We have the following result.

THEOREM 3.2[10,Theorem 3.1]. *Suppose Hypothesis 2 holds and assume the following:*

(A1) (A_n, B_n) *is uniformly exponentially stabilizable, i. e. , there exists a sequence* $K_n \in L(H_n, R^m)$ *such that sup* $\|K_n\| < \infty$ *and for* $M_1 \geq 1$ *and* $\omega_1 > 0$,

$$\|S_n P_n\| \leq M_1 e^{-\omega_1 t}, t \geq 0. \tag{3.2.18}$$

(A_n, C_n) *is uniformly exponentially detectable, i. e., there exists a sequence* $\hat{K}_n \in L(R^m, H_n)$ *such that sup*$\|\hat{K}_n\| < \infty$ *and for* $M_2 \geq 1$, $\omega_2 > 0$,

$$\|\hat{S}_n P_n\| \leq M_2 e^{-\omega_2 t}, t \geq 0. \tag{3.2.19}$$

(A2) (A_n, Q_n) *is uniformly exponentially detactable and* $(A_n, \hat{Q}_n)$ *is uniformly exponentially stabilizable.*

Then, (3.2.1) has the unique nonnegative solution Π_n, *for* $\hat{M}_1 \geq 1$ *and* $\hat{\omega}_1 > 0$,

$$\|e^{(A_n - BB_n^* \Pi_n)t} P_n\| \leq \hat{M}_1 e^{-\hat{\omega}_1 t}, t \geq 0 \tag{3.2.20}$$

Also, (3.2.2) has the unique nonnegative solution $\hat{\Pi}_n$, *for* $\hat{M}_2 \geq 1$ *and* $\hat{\omega}_2 > 0$,

$$\|e^{(A_n - \hat{\Pi}_n C_n^* C_n)t} P_n\| \leq \hat{M}_2 e^{-\hat{\omega}_2 t}, t \geq 0. \tag{3.2.21}$$

Moreover, if

(A3) (A, B) *is exponentially stabilizable and* (A, C) *is exponentially detectable.* (A, Q) *is exponentially detectable and* $(A, \hat{Q})$ *is exponentially stabilizable .*

Then, (3.6) has the unique nonnegative solution Π,

$$\lim_{n \to \infty} \Pi_n P_n x = \Pi x, x \in H, \tag{3.2.22}$$

and (3.9) has the unique nonnegative solution $\hat{\pi}$,

$$\lim_{n\to\infty} \hat{\Pi}_n P_n x = \hat{\Pi} x, x \in H. \tag{3.2.23}$$

Remark. Obviously, if Q and $\hat{Q}$ are uniformly positive definite, then (A2) is satisfied, (A, Q) is detectable, and $(A, \hat{Q})$ is stabilizable .

COROLLARY 3.3[10,Theorem 3.3]. *Under the assumption of Theorem 3.2, let $K = B^*\Pi$ and $\hat{K} = \hat{\Pi}C^*$ and for each n, let $K_n = B_n^*\Pi_n$ and $\hat{K}_n = \hat{\Pi}_n C_n^*$, then*

$$\lim_{n\to\infty} \|K_n P_n - K\|_{L(H,R^n)} = 0 \tag{3.2.24}$$

and

$$\lim_{n\to\infty} \|\hat{K}_n - \hat{K}\|_{L(R^p,H)} = 0. \tag{3.2.25}$$

If we denote the projection of $E \times E$ onto $E \times E_n$ by

$$P_{EE_n} = \begin{bmatrix} I & 0 \\ 0 & P_n \end{bmatrix}$$

then we obtain the following result.

THEOREM 3.4[3,Theorem 9.3]. *For $t \geq 0$, we have*

$$\lim_{n\to\infty} S_{\infty n}(t) P_{EE_n} z = S_{\infty\infty}(t) z, z \in E \times E, \tag{3.2.26}$$

uniformly on bounded t - intervals .

Remark: In Theorem 3.4, we know that $S_{\infty n}(t)P_{EE_n}$ converges strongly to $S_{\infty\infty}(t)$. However, when $S_{\infty\infty}(t)$ is uniformly exponentially stable, we don't know if $S_{\infty n(t)}$ till preserves the stability property for n sufficiently large or not. Gibson and

Adaimian pointed out in [3] that although numerical results for numerous examples with various kind of damping and approximation suggested that this was usually true but they were unable to prove it in general. Only in a special case, they obtained the following results.

THEOREM 3.5[3,Theorem 9.4].

(i) Suppose that the basis vectors of the approximation scheme are the natural modes of undamped free vibration and that the structure damping does not couple the modes.

Then

$$\lim_{n\to\infty} S_{\infty n}(t)P_{EE_n}z = S_{\infty\infty}(t)z, z \in ExE, \tag{3.2.27}$$

uniformly in bounded t - intervals.

(ii) If , additionally, $S_{\infty\infty}(t)$ is uniformly exponentially stable, then $S_{\infty n}(t)$ is uniformly exponentially stable for n sufficiently large .

In order to extend the above results to general case, we will follow the approaches of Triggiani [11], Schumacher [12] and Ito [10] . For an infinitesmal generator **A** , we have the growth constant :

$$\begin{aligned}\omega_0 = \omega_0(A) &= \inf_{t\in[0,\infty)} \frac{1}{t}\log\|S_A(t)\| \\ &= \lim_{t\to\infty}\frac{1}{t}\log\|S_A(t)\| < \infty,\end{aligned} \tag{3.2.28}$$

and for each $\omega > \omega_0$, there is a constant M_ω such that $\|S_A(t)\| \le M_\omega e^{\omega t}, t \ge 0$, see [11, p.386].

The semigroup $S(t)$ is said to be exponentially stable if the growth constant ω_0 is negative. We say that **A** satisfied the spectrum determined growth assumption in case

$$\omega_0(\mathbf{A}) = \sup \mathrm{Re}\sigma(\mathbf{A}). \tag{3.2.29}$$

PROPOSITION 3.6[11,Proposition 2.2]. *Let*

(i) $\mathbf{A}$ *satisfy the spectrum determined growth assumption (3.2.29);*

(ii) $\sup Re\sigma(\mathbf{A}) < 0$ *and hence* $\sup Re\sigma(\mathbf{A}) < -\alpha$, *for some* $\alpha > 0$. *Then*

$$\|S_A(t)\| \leq M_\alpha e^{-\alpha t}, \qquad t \geq 0$$

and hence,

$$\|S_A(t)x_0\| \leq M_\alpha e^{-\alpha t}\|x_0\|, \qquad t \geq 0$$

for all $x_0 \in \mathbf{X}$.

HYPOTHESIS 3.3. *For any matrices* $\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, K_c$ *and* G_c,

$$\mathbf{H}_c = \begin{bmatrix} \mathbf{A} & -\mathbf{B}K_c \\ G_c\mathbf{C} & F_c \end{bmatrix} \tag{3.2.30}$$

with $F_c = \mathbf{A}_c - \mathbf{B}_cK_c - G_c\mathbf{C}_c$ *and dom (H_c) = dom (A) x W. H_c satisfies the spectrum -determined growth assumption*

$$\omega_0(H_c) = \sup\{Re\lambda : \lambda \in \sigma(H_c)\}. \tag{3.2.31}$$

<u>Remark.</u> [11,p.387] . The spectrum determined growth assumption holds in particular if: (i) $\mathbf{A}$ is bounded on X; (ii) more generally, $\mathbf{A}$ generates an analytic semigroup $S_A(t), t > 0$; (iii) even more generally , $\mathbf{A}$ generates a differentiable semigroup for $t > t_0$; (iv) there is some $t > 0$, such that

$$\begin{aligned} r(S_A(\bar{t})) &= \sup\{|\lambda| : \lambda \in \sigma(S_A(\bar{t}))\} \\ &= \sup\{|\lambda| : \lambda \in \sigma_\rho(S_a(\bar{t})) \cup \sigma_r(S_A(\bar{t})) \cup \{0\}\}. \end{aligned}$$

We have the following results:

THEOREM 3.7[10, Theorem 3.3]. *Assume that the assumption (A1) - (A3) of Theorem 3.2 are satisfied and that* $(\mathbf{A}, Q)$ *is exponentially detectable and* $(\mathbf{A}, \hat{Q})$ *is exponentially stabilizable. Define an operator* $\hat{A}_{\infty n}$ *defined on* ExE_n *by*

$$\hat{\mathbf{A}}_{\infty n} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}K^n \\ \hat{K}^n C & \hat{F}^n \end{bmatrix}$$

where $\hat{F}^n = A^n - B^n K^n - \hat{K}^n C^n$.

Then for an $\delta > 0$, *there exists an integer* N_δ *such that for* $N \geq N_\delta$,

$$\{\lambda : Re\lambda \geq -\min(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3) + \delta\}$$

is contained in $\rho(\hat{A}_{\infty n})$. *Here* $-\omega_3$ *is the growth constant of semigroup generated by* $\mathbf{A} - \mathbf{B}K$.

Moreover, if Hypothesis 3 holds, the $\hat{A}_{\infty n}$ *generates a uniformly stable semigroup* $S_{\infty n}(t)$ *on* E x E_n *for N sufficiently large.* ■

Define the approximating conditions:

(c1) For each $z \in V$, there exists a sequence $\hat{z}^N \in H^N$ with

$$\lim_{n\to\infty} \|\hat{z}^N - z\|_V = 0.$$

(c2) The embedding of V onto H is compact.

Remark. Note that it is required that projections onto H^N converge in the V-norm. In [10], it is required an inverse approximation property:

$$\inf_{x\in H^N} \|R(s;A)z - x\|_V \leq \varepsilon_1(n)\|z\|_H$$

$$\inf_{x\in H^N} \|R(s;A^*)z - x\|_V \leq \varepsilon_2(n)\|z\|_H,$$

where $\varepsilon_1(N), \varepsilon_2(N) \to 0$ as $N \to \infty$. These conditions are stronger and more difficult to verify than (c1), (c2). Furthermore, the above conditions can only be satisfied if $R(s; \mathbf{A})$ is compact.

THEOREM 3.8. *If (c1), (c2) hold, then Hypothesis 2 is satisfied.*

PROOF: Conditions (c1), (c2) clearly imply (H2): (i), (iii). We next argue that

$$T_n(t)P_n x \to T(t)x \tag{3.2.32}$$

$$T_n^*(t)P_n x \to T^*(t)x \tag{3.2.33}$$

for $x \in H$ with the convergence uniform in t on bounded subset of $[0, \infty)$. First we note that

$$\mathrm{Re}a(z, z) + K < z, z > \geq c\|z\|_V^2, z \in V, \tag{3.2.34}$$

$$a(z, v) = < -\mathbf{A}_n z, v >, z, v \in H_n, \tag{3.2.35}$$

$$\overline{a(z, v)} = < -\mathbf{A}_n^* z, v >, z, v \in H_n \tag{3.2.36}$$

and the fact that $H_n \subset V$, we shall show that the following convergence statement hold:

$$R(\lambda; \mathbf{A}_n)P_n z \to R(\lambda; \mathbf{A})z, \tag{3.2.37}$$

$$R(\lambda; \mathbf{A}_n^*)P_n z \to R(\lambda; \mathbf{A}^*)z. \tag{3.2.38}$$

We then may use the Trotter - Kato Approximation Theorem to conclude that (3.2.32) and (3.2.33) are obtained.

Since $\mathbf{A}_n$ defined through $a(.,.)$, by Lumer - Phillips Theorem [23] , we obtain

$$\|T_n(t)\| \leq e^{Kt}. \tag{3.2.39}$$

Choose $\lambda > K, z \in H$, define $W_n := R(\lambda; A_n)P_n x$ and $W := R(\lambda; A)x$. We shall show that $W_n - W \to 0$ as $n \to \infty$. Let $z_n := W_n - \hat{W}_n$, where $\hat{W}_n$ is an approximation for W chosen according to (c1), we obtain the

$$\begin{aligned} a(W, z_n) &= < -\mathbf{A}W, z_n > \\ &= < (\lambda - \mathbf{A})W, z_n > -\lambda < W, z_n > \\ &= < z, z_n > -\lambda < W, z_n > . \end{aligned}$$

Similarly,

$$a(W_n, z_n) =< z, z_n > -\lambda < W_n, z_n >$$

and then

$$a(W_n, z_n) = a(W, z_n) + \lambda < W - W_n, z_n > .$$

Using (3.2.34), we have

$$\begin{aligned} c\|z_n\|_V^2 \leq& \mathrm{Re}a(z_n, z_n) + \lambda < z_n, z_n > \\ =& \mathrm{Re}a(W_n, z_n) - \mathrm{Re}a(\hat{W}_n, z_n) + \lambda < z_n, z_n > \\ =& \mathrm{Re}a(W - \hat{W}, z_n) + \lambda[\mathrm{Re} < W - W_n, z_n > + < z_n, z_n >] \\ \leq& |a(W - \hat{W}_n, z_n)| + \lambda| < W - \hat{W}_n, z_n > |. \end{aligned}$$

Thus, there exists $C_1 > 0$ such that

$$\|z_n\|_V \leq C_1\|W - \hat{W}_n\|_V.$$

and

$$W_n - W = z_n + \hat{W}_n - W.$$

It follows from the triangle inequality that $W_n \to W$ as $n \to \infty$.

Turning to (3.2.38), we recall that $\overline{a(z,v)} = < -\mathbf{A}^* v, z >$ and define $b : V\text{x}V \rightarrow C$ by $b(z,v) = \overline{a(z,v)}$. Then $b(.,.)$ satisfies the same equalities as $a(.,.)$. We may therefore verify (3.2.38) by referring to the analysis for (3.2.37). ∎

THEOREM 3.9. *Suppose the following conditions are satisfied:*

(i) $(\mathbf{A}, \mathbf{B})$ *is exponentially stabilizable;* $(\mathbf{A}, \mathbf{C})$ *is exponentially detectable .*

(ii) $(\mathbf{A}, \mathbf{Q})$ *is exponentially detectable;* $(\mathbf{A}, \hat{\mathbf{Q}})$ *is exponentially stabilizable.*

Using Galerkin approximation, define an operator $\hat{A}_{\infty n}$ *on* E x E_n *by*

$$\hat{\mathbf{A}}_{\infty n} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}K_n \\ \hat{K}_n C & \hat{F}_n \end{bmatrix}$$

where $\hat{F}_n = \mathbf{A}_n - \mathbf{B}_n K_n - \hat{K}_n C_n$. *Then ,* $\hat{A}_{\infty n}$ *generates a uniformly stable semigroup* $S_{\infty n}(t)$ *on* E x E_n *for N sufficiently large and*

$$\lim_{n\to\infty} S_{\infty n}(t) P_{EE_n} z = S_{\infty\infty}(t) z, z \in ExE,$$

uniformly in bounded t*- intervals.*

PROOF.: First, we note that if $(\mathbf{A}, \mathbf{B})$ is exponentially stabilizable, then there is a $K \in L(E, R^n)$ such that $\mathbf{A} - \mathbf{B}K$ generate a stable semigroup. We want to show there exists an integer N_1 such that for all $n \geq N_1$, the pairs $(A_n, P_n B)$ are uniformly exponentiallly stabilizable by the operator K, i. e., there exist positive constants M_1 and α such that the semigroups $S_n(t)$ generated by $A_n - B_n K P_n$ uniformly stable,i. e. ,

$$\|S_n(t)\| \leq M_1 e^{-\alpha t}, \qquad n > N_1. \tag{3.2.40}$$

We define a sesquilinear form $a_B : V\text{x}V \rightarrow C$ by

$$a_B(z,v) = a(z,v) + < BKz, v > + k_1 < z, v >$$

where k_1 is chosen so that

$$\text{Re}a_B(z,z) \geq C_1\|z\|_V^2$$

and

$$\|a_B(z,v)\| \leq C_2\|z\|_V\|v\|_V$$

for some positive constants $0 < C_1 \leq C_2$. Then, we have

$$\frac{|\text{Im}a_B(z,z)|}{|\text{Re}a_B(z,z)|} \leq \frac{C_2}{C_1}$$

and define

$$\theta = tan^{-1}\frac{C_2}{C_1}.$$

It follws that the numerical range of a_B is contained in the sector

$$\sum_{\theta} := \{\lambda \in C||arg\lambda| \leq \theta\}$$

[1] and the spectrum of $\mathbf{A} - \mathbf{B}K - k_1 I$ is contained in the left sector

$$\hat{\sum}_{\theta} = \{\lambda \in C, |arg(-\lambda)| \leq \theta\}.$$

We next consider the restriction of a_B to H_nxH_n. Similarly, the spectrum of $\mathbf{A}_n - \mathbf{B}_n KP_n - k_2 I$ is also contained in $\hat{\sum}_{\theta}$, uniformly in n. Let $A_n^B = A_n - B_n KP_n$, then the spectrum of A_n^B is contained in $\hat{\sum}_{\theta} + k_1$. It follows that for any fixed $\delta > 0$, the set of all eigenvalues λ of $\mathbf{A}_n^B$ with $\text{Re}\lambda > -\delta$ is bounded, uniformly in n .

Next, we claim that for some positive integer $N_0, \bigcup_{n=N_0}^{\infty} \sigma(\mathbf{A}_n^B)$ is contained in the left half plane $\text{Re}(\lambda) \leq -\epsilon, \epsilon > 0$. If not, then there exists a sequence λ_j, eigenvalues of $\mathbf{A}_n^B$, satisfying $\text{Re}\lambda_j > -\delta$. Since $\{\lambda_j\}$ are bounded, there exists a subsequence $\{\lambda_{n_j}\}$ satisfying $\lambda_{n_j} \rightarrow \hat{\lambda}$, with $\text{Re}(\hat{\lambda}) \geq 0$. Let ϕ_{n_j} be an eigenvector with $\|\phi\| = 1$ and $\mathbf{A}_{N_j}^B \phi_{n_j} = \lambda_{n_j} \phi_{n_j}$ for all n sufficiently large. Since $\{\phi_{n_j}\}$ are bounded in V,there exists a subsequence $\{\phi_{n_{j_i}}\}$ with $\phi_{n_{j_i}} \to \hat{\phi} \in V$. For convenience, we relable the notation and assume that $\mathbf{A}_n^B \phi_n = \lambda_n \phi_n$ with $\lambda_n \to \hat{\lambda}$ and $\phi_n \to \hat{\phi} \in V$. Choose $\psi \in V$ and $\{\psi_n\}, \psi_n \in H_n : \psi_n \to \psi$. Since $a(., \psi) \in V'$,

$$\lim_{n \to \infty} a(\phi_n, \psi) = a(\hat{\phi}, \psi),$$

.

$$\lim_{n \to \infty} a(\phi_n, \psi) + < BK\phi_n, \psi_n >= a(\hat{\phi}, \psi) + < BK\hat{\phi}, \psi >,$$

and

$$a(\phi_n, \psi_n) + < \mathbf{B}K\phi_n, \psi_n >$$
$$= < -\mathbf{A}_n^B \phi_n, \psi_n >$$
$$= - \lambda_n < \phi_n, \psi_n > .$$

Then

$$a(\hat{\phi}, \psi) + < \mathbf{B}K\hat{\phi}, \psi >= -\hat{\lambda} < \hat{\phi}, \psi >$$

for all $\psi \in V$, or

$$< (\mathbf{A} - \mathbf{B}K - \hat{\lambda})\hat{\phi}, , \psi >= 0$$

for all $\psi \in V$. It follows that $\hat{\phi}$ is an eigenvector corresponding to the eigenvalue $\hat{\lambda}$ for $\mathbf{A} - \mathbf{B}K$ and this contridicts the fact $\mathbf{A} - \mathbf{B}K$ generate a stable semigroup.

Hence, $\sum_{n=N_0}^{\infty} \sigma(\mathbf{A}_n^B)$ is contained in the interior of the left sector $\hat{\sum}$ with vertex $-\dfrac{\varepsilon}{2}$.

For some $C = C(C_1, C_2)$,

$$\|R(\lambda, \mathbf{A}_n^B - k_1 I)\| \leq \frac{C}{|\lambda|}$$

or

$$\|R(\lambda, \mathbf{A}_n^B)\| \leq \frac{C}{|\lambda - k_1|}.$$

The uniform exponential bound (3.2.40) of $S_n(t)$ from

$$S_n(t) = \int_{\Gamma} e^{\lambda t} R(\lambda, \mathbf{A}_n^B)\, d\lambda$$

where Γ is the positively oriented boundary of the sector $\hat{\sum}$. An estomate of the contour integral for $t \in [1, \infty)$ along with the observation that $|S_n(t)| \leq e^{k_1}$ for $t \in [0, 1]$ yields the bound (3.2.40).

Similarly, if $(\mathbf{A}, \mathbf{C})$ is exponentially detectable, i. e., there exists a $G \in L(R^r, E)$ such that $\mathbf{A} - GC$ generate a stable semigroup, then there exists an integer N_2 such that the semigroup $\hat{S}_n(t)$ generated by $\mathbf{A}_n - G_n P_n C_n$ unformly stable, i. e. ,

$$\|\hat{S}_n(t)\| \leq M_2 e^{-\beta t}, \qquad n > N_2,$$

for some positive constants $M_2 \geq 1$ and $\beta > 0$. Hence, condition (A2) of Theorem 3.2 is satisfied. If $(\mathbf{A}, Q)$ is exponentially detectable and $(\mathbf{A}, \hat{Q})$ is exponentially stabilizable, we also can prove condition (A3) of Theorem 3.2 is satisfied.

Since

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & 0 \end{bmatrix}$$

generates an analytic semigroup on Ex E_n and

$$\mathbf{A}_{\infty n} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}K_n \\ \hat{K}_n C & \hat{F}_n \end{bmatrix}$$

where $\hat{F}_n = \hat{A}_n - \hat{B}_n K_n - \hat{K} C_n$ is bounded perturbation of A_0. It follows from [11] that the Hypothesis 3 is satisfied.

Therefore, the theorem is a consequence of Theorem 3.7. ∎

Remark. For a large class of dynamics systems with bounded input operators, Theorem 3.9 guarantees the stability properties of the resulting closed loop system are preserved uniformly. In case of dynamics systems with unbounded control operators, Lasiecka recently obtains the corresponding stability results in [18] and [19] .

Chapter IV Numerical Examples and Simulations

4.1 Matrix Representations of Approximating Operators

Since H^N is of dimension N, it is spanned by N linearly independent elements in $H^N, e_i^N, i = 1, 2, ..., N$.

For $N \geq 1$, assume a general Galerkin finite element approximation of the form

$$z^N(t) = \sum_{j=1}^{N} \zeta_j(t) e_i^N \tag{4.1.1}$$

and use $z^N(t)$ to approximate $z(t)$.

It is clear that $\zeta(t) = [\zeta_1(t), \zeta_2(t), ..., \zeta_N(t)]^T$ satisfies a system of the form

$$M^N \ddot{\zeta}(t) + D^N \dot{\zeta}(t) + K^N \zeta(t) = B_0^N u(t) \tag{4.1.2}$$

where the mass matrix M^N, damping matrix D^N, stiffness matrix K^N , and actuator influence matrix B_0^N are given by

$$\begin{aligned} M^N &= [M_{ij}^N] = [< e_i^N, e_j^N >_H], \\ D^N &= [D_{ij}^N] = [d(e_i^N, e_j^N)], \\ K^N &= [K_{ij}^N] = [a(e_i^N, e_j^N)], \\ B_0^N &= [B_{0ij}^N] = [< e_i^N, B_{0j} >_H]. \end{aligned}$$

Let $\eta = (\zeta, \dot{\zeta})$,

$$\dot{\eta} = A^n \eta + B^n u \tag{4.1.3}$$

where A^n is the matrix representation of the operator A_n,

$$A^n = \begin{bmatrix} 0 & I \\ -(M^n)^{-1} K^n & -(M^n)^{-1} D^n \end{bmatrix} \tag{4.1.4}$$

and B^n is the matrix representation of the operator B_n ,

$$B^n = \begin{bmatrix} 0 \\ (M^n)^{-1} B_0^n \end{bmatrix}. \tag{4.1.5}$$

The matrix representation of the operator C is

$$C^n = [[c_1 e_i^n], [c_2 e_i^n]]. \tag{4.1.6}$$

To discuss the matrix representations of the operators $Q_n, \hat{Q}_n, \Pi_n$ and $\hat{\Pi}_n$, it is convenient to define the block-diagonal matrix

$$W^n = \begin{bmatrix} K^n & 0 \\ 0 & M^n \end{bmatrix}. \tag{4.1.7}$$

Since $Q = Q^* \in L(E)$ and $E = V$ x H, we can write

$$Q = \begin{bmatrix} Q_0 & Q_1 \\ Q_1^* & Q_2 \end{bmatrix},$$

where $Q_0 = Q_0^* \in L(V), Q_1 \in L(H, V)$, and $Q_2 = Q_2^* \in L(H)$.

The matrix representation of Q_n is

$$Q^n = (W^n)^{-1} \begin{bmatrix} Q_0^n & Q_1^n \\ (Q_1^n)^T & Q_2^n \end{bmatrix} \tag{4.1.8}$$

where

$$Q_0^n = [< e_i^n, Q_0 e_j^n >_V],$$

$$Q_1^n = [< e_i^n, Q_1 e_j^n >_V],$$

$$Q_2^n = [< e_i^n, Q_2 e_j^n >_H].$$

Similarly, we partition $\hat{Q}$ as

$$\hat{Q} = \begin{bmatrix} \hat{Q}_0 & \hat{Q}_1 \\ \hat{Q}_1^* & \hat{Q}_2 \end{bmatrix}.$$

The matrix representation of $\hat{Q}_n$ is

$$\hat{Q}^n = (W^n)^{-1} \begin{bmatrix} \hat{Q}_0^n & \hat{Q}_1^n \\ (\hat{Q}_1^n)^T & \hat{Q}_2^n \end{bmatrix} \tag{4.1.9}$$

where

$$\hat{Q}_0^n = [< e_i, \hat{Q}_0 e_j >],$$

$$\hat{Q}_1^n = [< e_i, \hat{Q}_1 e_j >],$$

$$\hat{Q}_2^n = [< e_i, \hat{Q}_2 e_j >].$$

The matrix representation Π^n and $\hat{\Pi}^n$ of Π_n and $\hat{\Pi}_n$ are determined by solving Riccati matrix equations equivalent to the operator equations (3.2.1) and (3.2.2). In general, the form of Π^n is not symmetric, but the matrix $W^n\Pi^n$ is symmetric and nonnegative. Hence, rather than solving the matrix representation of (3.2.1) directly, it is preferable to premultiply the matrix representation of (3.2.1) by W^n to obtain a Riccati matrix equation that can be solved for the symmetric matrix $W^n\Pi^n$. Also , instead of solving the matrix representation of (3.2.2), it is preferable to postmultiply the matrix representation of (3.2.2) by $(W^n)^{-1}$ to obtain a Riccati matrix equation that can be solved for the symmetric matrix $\hat{\Pi}^n(W^n)^{-1}$.

Let

$$\tilde{\Pi}^n = W^n \Pi^n.$$

Then

$$u_n(t) = [u_{1n}(t), u_{2n}(t), ..., u_{mn}(t)]^T$$

and

$$u_{in}(t) = - < f_{in}, x_n(t) >_V - < g_{in}, \dot{x}_n(t) >_H$$

where

$$f_{in} = \sum_{j=1}^{n} \beta_j^{f_i} e_j,$$
$$g_{in} = \sum_{j=1}^{n} \beta_j^{g_i} e_j \tag{4.1.10}$$

are the approximating functional control gains and

$$\begin{bmatrix} \beta^{f_1} & \beta^{f_2} & \dots & \beta^{f_m} \\ \beta^{g_1} & \beta^{g_2} & \dots & \beta^{g_m} \end{bmatrix} = W^{-n}\tilde{\Pi}^n B^n R^{-1}, \tag{4.1.11}$$

for

$$\beta^{f_i}, \beta^{g_i} \in R^n, (i = 1, ..., n).$$

Similarly, let

$$\tilde{\hat{\Pi}}^n = \hat{\Pi}^n (W^n)^{-1}$$

and consider

$$\hat{K}_n y = \sum_{i=1}^{P} (\hat{f}_{in}, \hat{g}_{in}) y_i$$

for $y = [y_1, y_2, ..., y_P]^T \in R^P$. The matrix representation of $\hat{K}_n$ is

$$\hat{K}^n = \begin{bmatrix} \hat{\beta}^{f_1} & \hat{\beta}^{f_2} & \dots & \hat{\beta}^{f_P} \\ \hat{\beta}^{g_1} & \hat{\beta}^{g_2} & \dots & \hat{\beta}^{g_P} \end{bmatrix} = \tilde{\hat{\Pi}}^n (W^n)^{-1} (C^n)^T \hat{R}^{-1} \tag{4.1.12}$$

where the columns $\hat{\beta}^{f_i}, \hat{\beta}^{g_i} \in R^n$. Then

$$\hat{f}_{in} = \sum_{j=1}^{n} \hat{\beta}_j^{f_i} e_j, \qquad i = 1, ..., P,$$
$$\hat{g}_{in} = \sum_{j=1}^{n} \hat{\beta}_j^{g_i} e_j, \qquad i = 1, ..., P. \tag{4.1.13}$$

We have the following result:

THEOREM 3.8[3,Theorems 5.10, 8.6]. *Under the assumption of Theorem 3.2, we have*

$$\lim_{n\to\infty} f_{in} = f_i, \qquad i = 1, 2, ..., m,$$
$$\lim_{n\to\infty} g_{in} = g_i, \qquad i = 1, 2, ..., m \tag{4.1.14}$$

and

$$\lim_{n\to\infty} \hat{f}_{in} = \hat{f}_i, \qquad i = 1, 2, ..., p$$
$$\lim_{n\to\infty} \hat{g}_{in} = \hat{g}_i, \qquad i = 1, 2, ..., p, \tag{4.1.15}$$

where f_i and g_i are the functional control gains in (3.1.22) and $\hat{f}_i$ and $\hat{g}_i$ are the functional estimator gains in (3.1.26).

Consider the Galerkin approximation to the state

$$z^n(t) = \sum_{j=1}^{n} \zeta_j(t) e_i^n$$

where $\{e_1^n, e_2^n, ..., e_n^n\}$ is the basis for the approximating space $H^N \in V$. Since $e_i^n \in V$, it is a vector of seven components denoted by

$$e_i^N = (e_{i1}^N, e_{i2}^N, e_{i3}^N, e_{i4}^N, e_{i5}^N, e_{i6}^N, e_{i7}^N) \in R\text{x}H^2[0, l_1]\ \text{x}H^2[0, l_2]\text{x}R^4$$

with

$$e_{i2}^N(0) = 0, e_{i2}^N(l_1) = e_{i5}^N - \frac{l_{m1}}{2} e_{i4}^N, e_{i3}^N(0) = e_{i5}^N + \frac{l_{m1}}{2} e_{i4}^N,$$
$$e_{i3}^N(l_2) = e_{i7} - \frac{l_{m2}}{2} e_{i6}^N, e_{i2x}^N(0) = e_{i1}^N, e_{i2x}^N(l_1) = e_{i4}^N = e_{i3x}^N(0),$$
$$e_{i3x}^N(l_2) = e_{i6}^N. \tag{4.1.16}$$

We choose Cubic B - splines to represent the displacements of beams 1 and 2. Given an interval $I = [0, L]$ and $h = \frac{L}{N}$. Defined the uniform partition of I, $\triangle^N = \{0, h, 2h, ..., Nh = L\}$ and the set

$$S_3^N(I) = \{\phi \in C^2[0, L] : \phi \quad \text{is a cubic polynomial on } [jk, (j+1)h], j = 0, ..., N-1\}.$$

To approximate $H^2[0,l_i]$ by $S_3^N(I)$ where $I_i = [0,l_i]$, let $B_i(x)$ be the i^{th} cubic B spline defined as :

$$B_i(x) = \begin{cases} \frac{1}{h^3}(x - x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}] \\ \frac{1}{h^3}[h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3], & x \in [x_{i-1}, x_i] \\ \frac{1}{h^3}[h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3], & x \in [x_i, x_{i+1}] \\ \frac{1}{h^3}(x_{i+2} - x)^3, & x \in [x_{i+1}, x_{i+2}] \\ 0, & \text{otherwise.} \end{cases}$$

Then $B_{-1}^N, B_0^N, ..., B_{N+1}^N$ provides a basis for $S_3^N(I)$ where is $N + 3$ dimensions.

Let a_i and b_i denote the $n+3$ cubic B - splines on the intervals $[0, l_1]$ and $[0, l_2]$, respectively. By the essential boundary conditions for beams, both sets of splines and their derivatives must be zero at $x_i = 0$. Taking a linear combination of the first three splines gives a set of $n + 1$ splines which meets these two conditions. The elements of the basis functions corresponding to the displacements of beams 1 and 2 are

$$\alpha_i = \begin{cases} a_0 - 2a_{-1} - 2a_1, & i = 1, \\ a_i, & i = 2, ..., n+1, \end{cases} \tag{4.1.17}$$

and

$$\beta_i = \begin{cases} b_0 - 2b_{-1} - 2b_1, & i = 1, \\ b_i, & i = 2, ..., n+1. \end{cases} \tag{4.1.18}$$

Denoting the space spanned by α_i as H_1^N and the space spanned by β_i as H_2^N, $H_1^N = S_3^N(I_1) \cap H^2[0, l_1]$ with the essential boundary comditions for beam 1, and $H_2^N = S_3^N(I_2) \cap H^2[0, l_2]$ with the essential boundary conditions for beam 2.

To approximate $V, 2n + 3$ total basis functions $\{e_1, ..., e_{2n+3}\}$ are chosen as follows:

$$e_1 = \begin{bmatrix} 1 \\ i_1 \\ i_2 + l_1 + l_{m1} \\ 1 \\ l_1 + \frac{l_{m1}}{2} \\ 1 \\ l_1 + l_{m1} + l_2 + \frac{l_{m2}}{2} \end{bmatrix},$$

$$e_i = \begin{bmatrix} 0 \\ \alpha_i \\ \alpha_i(l_1) + (i_2 + l_{m1})\alpha_{ix}(l_1) \\ \alpha_{ix}(l_1) \\ \alpha_i(l_1) + \frac{l_{m1}}{2}\alpha_{ix}(l_1) \\ \alpha_{ix}(l_1) \\ \alpha_i(l_1) + (l_{m1} + l_2 + \frac{l_{m2}}{2})\alpha_{ix}(l_1) \end{bmatrix} \quad \text{for} \quad i = 2, ..., n+2,$$

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \beta_i \\ 0 \\ 0 \\ \beta_{ix}(l_2) \\ \beta_i(l_2) + \frac{l_{m2}}{2}\beta_{ix}(l_2) \end{bmatrix} \quad \text{for} \quad i = n+3, ..., 2n+3,$$

where i_i is the identity mapping on I_i as

$$i_i(x_i) = x_i, \qquad \text{for} \qquad x_i \in I_i, i = 1, 2.$$

Thus, the approximating space is

$$H^n = R\, \text{x} H_1^N \text{x} H_2^N \text{x} R^4.$$

The matrix representations of $\mathbf{A}_n$ and $\mathbf{B}_n$ can be constructed as following

$$A^N = \begin{bmatrix} 0 & I \\ -(M^N)^{-1}K^N & -(M^N)^{-1}D^N \end{bmatrix}, \qquad B^N = \begin{bmatrix} 0 \\ -(M^N)^{-1}B_0^N \end{bmatrix} \qquad (4.1.19)$$

where

$$M_{ij}^N = < e_i^N, e_j^N >_H,$$

$$D_{ij}^N = d(e_i^N, e_j^N),$$

$$K_{ij}^N = a(e_i^N, e_j^N),$$

$$B_{0ij}^N = < e_i^N, B_0 >_H .$$

The mass matrix is formed by computing the H inner product of the basis functions . It has the form

$$M^N = \begin{bmatrix} M_{11}^N & M_{12}^N & M_{13}^N \\ (M_{12}^N)^T & M_{22}^N & M_{23}^N \\ (M_{13}^N)^T & (M_{23}^N)^T & M_{33}^N \end{bmatrix} \tag{4.1.20}$$

where M_{11}^N is a 1×1 matrix, M_{12}^N and M_{13}^N are $1 \times (n+1)$ row matrices, and M_{22}^N, M_{23}^N, and M_{33}^N are $(n+1) \times (n+1)$ matrices . We have

$$M_{11}^N = < e_1^N, e_1^N >_H = J_0 + (\rho A)_1 \frac{l_1^3}{3} + (\rho A)_2 [\frac{l_2^3}{3} + l_2(l_1 + l_{m1})(l_2 + l_1 + l_{m1})] + J_1$$
$$+ m_1(l_1 + \frac{l_{m1}}{2})^2 + J_2 + m_2(l_1 + l_{m1} + l_2 + \frac{l_{m2}}{2})^2.$$

For j : $\quad 2 \leq j \leq n+2$,

$$M_{12}^N = < e_1^N, e_j^N >_H = (\rho A)_1 < x_1, \alpha_j >_{L_2[0,l_1]} + \alpha_j(l_1)[(\rho A)_2 l_2 (\frac{l_2}{2} + (l_1 + l_{m1}))$$
$$+ m_1(l_1 + \frac{l_{m1}}{2}) + m_2(l_1 + l_{m1} + l_2 + \frac{l_{m2}}{2})]$$
$$+ \alpha_{jx}(l_1)[(\rho A)_2 l_2^2 (\frac{l_{m1}}{2} + \frac{l_2}{3} + (\frac{l_{m1}}{l_2} + \frac{1}{2})(l_1 + l_{m1})) + J_1$$
$$+ m_1 \frac{l_{m1}}{2}(l_1 + \frac{l_{m1}}{2}) + J_2$$
$$+ m_2(l_1 + l_{m1} + l_2 + \frac{l_{m2}}{2})(l_{m1} + l_2 + \frac{l_{m2}}{2})].$$

For j : $\quad n+3 \leq j \leq 2n+3$,

$$M_{13}^N = < e_1^N, e_j^N >_H = (\rho A)_2 (< x_2, \beta_j >_{L_2[0,l_2]} + (l_1 + l_{m1}) < 1, \beta_j >_{L_2[0,l_2]})$$

$$+ \beta_j(i_2)[m_2(l_1 + l_{m1} + l_2 + \frac{l_{m2}}{2})]$$

$$+ \beta_{jx}(l_2)[J_2 + m_2(l_1 + l_{m1} + l_2 + \frac{l_{m2}}{2})\frac{l_{m2}}{2}].$$

For i,j : $\quad 2 \le i,j \le n+2$,

$$M_{22}^N = < e_i^N, e_j^N >_H = (\rho A)_1 < \alpha_i, \alpha_j >_{L_2[0,l_1]} + \alpha_i(l_1)\alpha_j(l_1)((\rho A)_2 l_2 + m_1 + m_2)$$

$$+ \alpha_{ix}(l_1)\alpha_j(l_1)[(\rho A)_2 l_2(l_{m1} + \frac{l_2}{2}) + m_1\frac{l_{m1}}{2}$$

$$+ m_2(l_{m1} + l_2 + \frac{l_{m2}}{2})]$$

$$+ \alpha_i(l_1)\alpha_{jx}(l_1)[(\rho A)_2 l_2(l_{m1} + \frac{l_2}{2}) + m_1\frac{l_{m1}}{2}$$

$$+ m_2(l_{m1} + l_2 + \frac{l_{m2}}{2})]$$

$$+ \alpha_{ix}(l_1)\alpha_{jx}(l_1)[(\rho A)_2 l_2(\frac{l_2^2}{2} + l_{m_1}l_2 + l_{m_1}^2) + J_1 + m_1(\frac{l_{m1}}{2})^2 + J_2$$

$$+ m_2(l_{m1} + l_2 + \frac{l_{m2}}{2})^2].$$

For i and j , $2 \le i \le n+2$ and $n+3 \le j \le 2n+3$,

$$M_{23}^N = < e_i^N, e_j^N >_H = (\rho A)_2(\alpha_{ix}(l_1) < x_2, \beta_j >_{L_2[0,l_2]} + (\alpha_i(l_1) + l_{m1}\alpha_{ix}(l_1)) < 1, \beta_j >_{L_2[0,l_2]})$$

$$+ \alpha_i(l_1)\beta_j(l_2)m_2 + \alpha_i(l_i)\beta_{jx}(l_2)m_2\frac{l_{m2}}{2}$$

$$+ \alpha_{ix}(l_1)\beta_j(l_2)m_2(l_{m1} + l_2 + \frac{l_{m2}}{2})$$

$$+ \alpha_{ix}(l_1)\beta_{jx}(l_2)[J_2 + m_2(l_{m1} + l_2 + \frac{l_{m2}}{2})\frac{l_{m2}}{2}]$$

and for $i,j : n+3 \le i,j \le 2n+3$,

$$M_{33}^N = < e_i^N, e_j^N >_H = (\rho A)_2 < \beta_i, \beta_j >_{L_2[0,l_2]} + \beta_i(l_2)\beta_j(l_2)m_2 + \beta_{ix}(l_2)\beta_j(l_2)m_2\frac{l_{m2}}{2}$$

$$+ \beta_i(l_2)\beta_{jx}(l_2)m_2\frac{l_{m2}}{2} + \beta_{ix}(l_2)\beta_{jx}(l_2)[J_2 + m_2(\frac{l_{m2}}{2})^2].$$

The stiffness matrix, K^N, is given by

$$K_{ij}^N = a(e_i^N, e_j^N) = < (EI)_1 e_{i2xx}^N, e_{j2xx}^N >_{L_2[0,l_1]} + < (EI)_2 e_{i3xx}^N, e_{j3xx}^N >_{L_2[0,l_2]} .$$

Since e^N_{12xx}, e^N_{13xx} are zeros, K^N has zeros in row one and column one. Thus,

$$K^N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K^N_{11} & 0 \\ 0 & 0 & K^N_{22} \end{bmatrix} \tag{4.1.21}$$

where

$$K^N_{11} = a(e^N_i, e^N_j) =< (EI)_1\alpha_{ixx}, \alpha_{jxx} >_{L_2[0,l_1]}, 2 \le i,j \le n+2,$$

and

$$K^N_{22} = a(e^N_i, e^N_j) =< (EI)_2\beta_{ixx}, \beta_{jxx} >_{L_2[0,l_2]}, \qquad n+3 \le i,j \le 2n+3.$$

The damping matrix D^N has the same form as K^N ,

$$D^N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D^N_{11} & 0 \\ 0 & 0 & D^N_{22} \end{bmatrix} \tag{4.1.22}$$

where

$$D^N_{11} = d(e^N_i, e^N_j) =< (\gamma I)_1\alpha_{ixx}, \alpha_{jxx} >_{L_2[0,l_1]}, \qquad 2 \le i,j \le n+2,$$

and

$$D^N_{22} = d(e^N_i, e^N_j) =< (\gamma I)_2\beta_{ixx}, \beta_{jxx} >_{L_2[0,l_2]}, \qquad n+3 \le i,j \le 2n+3.$$

Since

$$\begin{aligned} B_0 u &= \begin{bmatrix} \frac{1}{J_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{J_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{J_2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \\ &= \begin{bmatrix} b_{01} & b_{02} & b_{03} & b_{04} & b_{05} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \\ &= \sum_{i=1}^{5} b_i u_{i-1}, \end{aligned}$$

where

$$B_{01} = \begin{bmatrix} \frac{1}{J_0} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad B_{02} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_1} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$B_{03} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix}, \qquad B_{04} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{J_2} \\ 0 \end{bmatrix}$$

and

$$B_{05} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix},$$

the actuator influence matrix B_0^N is given by

$$B_0^N = [B_{01}^N, B_{02}^N, B_{03}^N, B_{04}^N, B_{05}^N] \tag{4.1.23}$$

where

$$B_{01}^N = [< e_i^N, B_{02} >_H] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}_{(2n+3)\times 1},$$

$$B_{02}^N = [< e_i^N, B_{02} >_H] = \begin{bmatrix} 1 \\ \alpha_{2x} \\ \alpha_{3x} \\ . \\ . \\ . \\ \alpha_{n+2x} \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}_{(2n+3)\times 1} ,$$

$$B_{03}^N = [< e_i^N, B_{03} >_H] = \begin{bmatrix} l_1 + \frac{l_{m1}}{2} \\ \alpha_2(l_1) + \frac{l_{m1}}{2}\alpha_{2x}(l_1) \\ . . \\ . \\ \alpha_{n+2}(l_1) + \frac{l_{m1}}{2}\alpha_{n+2x}(l_1) \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}_{(2n+3)\times 1} ,$$

$$B_{04}^N = [< e_i^N, B_{04} >_H] = \begin{bmatrix} 1 \\ \alpha_{2x}(l_1) \\ . \\ . \\ . \\ \alpha_{n+2x}(l_1) \\ \beta_{n+3x}(l_2) \\ . \\ . \\ . \\ \beta_{2n+3x}(l_2) \end{bmatrix}_{(2n+3)\times 1} .$$

and

$$B_{05}^N = [< \epsilon_i^N, B_{05} >] = \begin{bmatrix} l_1 + l_{m1} + l_2 + \frac{l_{m2}}{2} \\ \alpha_2(l_1) + (l_{m1} + l_2 + \frac{l_{m2}}{2})\alpha_{2x}(l_1) \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{n+2}(l_1) + (l_{m1} + l_2 + \frac{l_{m2}}{2})\alpha_{n+2x}(l_1) \\ \beta_{n+3}(l_2) + \frac{l_{m2}}{2}\beta_{n+3x}(l_2) \\ \cdot \\ \cdot \\ \cdot \\ \beta_{2n+3}(l_2) + \frac{l_{m2}}{2}\beta_{2n+3x}(l_2) \end{bmatrix}_{(2n+3)\times 1}.$$

We also have

$$\begin{aligned} y_1(t) &= z_1(t), \\ y_2(t) &= -l_1 z_1(t) - \frac{l_{m_1}}{2} z_4(t) + z_5(t), \\ y_3(t) &= -l_1 \dot{z}_1(t) - \frac{l_{m_1}}{2} \dot{z}_4(t) + \dot{z}_5(t), \\ y_4(t) &= -(l_2 + \frac{l_{m_1}}{2}) z_4(t) - z_5(t) - \frac{l_{m_2}}{2} z_6(t) + z_7(t), \\ y_5(t) &= -(l_2 + \frac{l_{m_1}}{2}) \dot{z}_4(t) - \dot{z}_5(t) - \frac{l_{m_2}}{2} \dot{z}_6(t) + \dot{z}_7(t). \end{aligned} \tag{4.1.24}$$

and

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -l_1 & 0 & 0 & -\frac{l_{m_1}}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -l_1 & 0 & 0 & -\frac{l_{m_1}}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & -(l_2 + \frac{l_{m_1}}{2}) & -1 & -\frac{l_{m_2}}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(l_2 + \frac{l_{m_1}}{2}) & -1 & -\frac{l_{m_2}}{2} & 1 \end{bmatrix}$$

$$* \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_7 \\ \dot{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \dot{z}_7 \end{bmatrix}$$

$$= [c_1, c_2] \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_7 \\ \dot{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \dot{z}_7 \end{bmatrix} . \tag{4.1.25}$$

The matrix representation of the operator C is

$$C^N = [[c_1 e_i^N], [c_2 e_i^N]] = [[c_{ji}^*]_{4\times N}, [c_{ji}^{**}]_{4\times N}] \tag{4.1.26}$$

where

$$e_i^N = (e_{i1}^N, e_{i2}^N, e_{i3}^N, e_{i4}^N, e_{i5}^N, e_{i6}^N, e_{i7}^N)^T$$

and for $i = 1, 2, ..., 2n + 3 \equiv N$,

$$c_{1i}^* = e_{i1}^N,$$

$$c_{2i}^* = -l_1 e_{i1}^N - \frac{l_{m_1}}{2} e_{i4}^N + e_{i5}^N,$$

$$c_{3i}^* = 0,$$

$$c_{4i}^* = -(l_2 + \frac{l_{m_1}}{2}) e_{i4}^N - e_{i5}^N - \frac{l_{m_2}}{2} e_{i6}^N + e_{i7}^N,$$

$$c_{5i}^* = 0,$$

and

$$c_{1i}^{**} = 0,$$

$$c_{2i}^{**} = 0,$$

$$c_{3i}^{**} = -l_1 e_{i1}^N - \frac{l_{m_1}}{2} e_{i4}^N + e_{i5}^N,$$

$$c_{4i}^{**} = 0,$$

$$c_{5i}^{**} = -(l_2 + \frac{l_{m_2}}{2})e_{i4}^N - e_{i5}^N - \frac{l_{m_2}}{2}e_{i6}^N + e_{i7}^N.$$

Thus, we have

$$W = C^N\eta = [[C_{ji}^*]_{4\times N}, [C_{ji}^{**}]_{4\times N}] \begin{bmatrix} \zeta \\ \dot{\zeta} \end{bmatrix}. \tag{4.1 .27}$$

We choose $R = 1$ and

$$Q = \begin{bmatrix} Q_0 & Q_1 \\ Q_1^* & Q_2 \end{bmatrix} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} = I. \tag{4.1.28}$$

The matrix representation of Q_n is

$$Q^n = (W^n)^{-1} \begin{bmatrix} Q_0^n & Q_1^n \\ (Q_1^n)^T & Q_2^n \end{bmatrix}. \tag{4.1.29}$$

Similarly, we let

$$\hat{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.1.30}$$

and

$$\hat{Q} = \begin{bmatrix} \hat{Q}_0 & \hat{Q}_1 \\ \hat{Q}_1^* & \hat{Q}_2 \end{bmatrix} = I. \tag{4.1.31}$$

The matrix representation of $\hat{Q}_n$ is

$$\hat{Q}^n = (W^n)^{-1} \begin{bmatrix} \hat{Q}_0^n & \hat{Q}_1^n \\ (\hat{Q}_1^n)^T & \hat{Q}_2^n \end{bmatrix}. \tag{4.1.32}$$

4.2 Numerical Results

In this section we discuss numerical results obtained by implementing the Galerkin finite elements approximation scheme developed in section 3.3 for the two-connect-beam system (TCB). First, we fix certain parameters and use these selected values for all computations. For the initial structure, an aluminum beam and a steel beam are chosen for beam 1 and beam 2, respectively . The specific beam dimensions are chosen to satisfy the basic assumptions of the Euler-Bernouli model of a long and slender beam. The physical constants for density and Young's modulus can be found in any standard materials reference.

Using different parameters for the structure, two problems are investigated to test and illustrate the method. graphs of the feedback control gains and estimator gains are given for each example. Open and closed eigenvalues are plotted for Example 1 only. Two simulations are also performed for different initial deformations of the structure by using the LSIM routine from the Matlab Control Toolbox. Given an array for time t, LSIM computes the time response w and the state time history z to the input time history u, for a given system of the form

$$\dot{z} = Az + Bu,$$

$$w = Cz + Du. \tag{4.2.1}$$

To verify the effectiveness of the compensator scheme for TCB,we performed the simulations using full state feedback and dynamic compensation. Closed - loop responses are computed by simulating (4.2.1) with A being the closed - loop system matrix defined by $A^n - B^n K^n$ and

$$\hat{A}^n = \begin{bmatrix} A^n & -B^n K^m \\ \hat{K}^m C^n & [A^m - B^m K^m - \hat{K}^m C^m] \end{bmatrix}, m \leq n, \tag{4.2.2}$$

respectively.

The matrix representation for the first order system approximating TCB is

$$A^n = \begin{bmatrix} 0 & I \\ -(M^n)^{-1}K^n & -(M^n)^{-1}D^n \end{bmatrix}, \qquad B^n = \begin{bmatrix} 0 \\ -(M^n)^{-1}b_0^n \end{bmatrix} \tag{4.2.3}$$

and $\eta = (\zeta, \dot{\zeta})$ where $\zeta = (\zeta_1, ..., \zeta_{2n+3})$. The approximation for the state $z(t)$ is

$$z^{2n+3}(t) = \sum_{j=1}^{2n+3} \zeta_j(t) e_j, \tag{4.2.4}$$

where the basis vectors, $e_j, j = 1, ..., 2n+3$, are given in section 3.3 .

The initial conditions, $y(0)$, to be supplied to LSIM are the same as $\zeta(0)$ in the notation above. Thus, for a particular initial state, $y(0)$, the initial conditions for the simulation are the coefficients of the basis functions which yield the approximation for $z(0)$ in (4.2.4). Once the simulation data were obtained, the state history of the deformation of the structure was plotted. In each figure,the beams are shown by lines with beam 1 between 0 and 2 meters and beam 2 between 2.25 and 5.25 meters.

4.2.1. Example 1 — The Initial Structure

In Example 1, the specific parameters are listed in Table 1. The moment of inertia for a beam can be computed as $I = \dfrac{Ah^2}{12}$, where $A = wh$. The initial parameters for the rigid bodies are given in Table 2.

Using the chosen parameters, the optimal controls for the finite dimensional LQG problems are computed using the Matlab Control Toolbox. For $n = 8$, open and closed loop eigenvalues are plotted in Figure 1.1 and Figure 1.2 with different damping parameters. The optimal control functions , $u(t) = [u0(t), u1(t), u2(t), u3(t), u4(t)]^T$, , are plotted in Figure 1.3 for estimator and full order respectively with $t = 0 : 2.5 : 100$

(second). We choose $\theta(0) = \frac{\pi}{12}$ and all other initial states as well as all initial velocities to be zero for full order and all initial states and all initial velocities to be zero for estimator. The resulting feedback control gain matrix and estimator gain matrix are expressed in terms of constant and functional gains. The constant gains are listed in Tables 3 and 4. The functional control gains of bending and velocity for $n = 3, 4$ and 8 are plotted with the x-axis representing meters. The corresponding plots can be found in Figure 2.1 - Figure 2.20 . The functional estimator gains are plotted in a similar way. The corresponding plots can be found in Figure 2.21 - Figure 2.40.

The convergence results for approximations to the infinite-dimensional LQG problem guarantee that all functional gains do converge. The numerical results show that this convergence is fast. Note that all three gains resulting from the values of $n = 3, 4$ and 8 are plotted, though in some case only one graph is evident, since the gains converge rapidly. In general, the difference in the gain values at the nodes for $n = 4$ and 8 is on the order of 10^{-4} or 10^{-5}.

For the control bending gains, the largest (positive) or smallest (negative) values occur where the greatest strain is expected to exist. The values close to zero indicate the points of least strain. In Figure 2.1 – Figure 2.5 , the bending gains for beam 1 are linear or almost-linear. We can expect that the greatest strain occures at the end points of beam 1. In Figure 2.6 — figure 2.10, the greatest strain occured at the points between the end points of beam 2. Similar analysis can also be applied to the estimator gains L2 and L3(see Figure 2.21 — Figure 2.30).

For the control velocity gains, the largest value (positive) or smallest value (negative) occures at the point of the greatest expected velocity. The value close to zero indicates the point of the least velocity (see Figure 2.11 - Figure 2.20). The plots for control velocity gains show that both beams and masses have associated velocity gains. As expected, the control velocity gain at the hub is zero which reflects the fact that beam 1 is fixed at the hub (see Figure 2.11 — Figure 2.15). Similar analysis can also be applied to the estimator gains L5 and L6 (see Figure 2.31 — Figure 2.40).

4.2.2. Example 2 — The Effect of Parameter Variation

In Example 2, the mass length, mass variation and mass moment of inertia for mass 1 are increased to 0.5 m (from 0.25m), 5kg (from 1 kg) and $5kgm^2$(from $1kgm^2$) respectively, while all others are fixed at the values in Tables 1 and 2.

Constant gains are shown in Tables 5 and 6. Since the variation of parameters for mass 1, some changes are expected. The bending gain(5:4) for beam 1 has an obvious change in shape. Except for the bending gain(5:4) for beam 1, the rest have little change in shape (see Figure 3.1 — Figure 3.5). The greatest strain still occured at the end points of beam 1 near mass 1. The largest (positive) value of the bending gain (5:1) for beam 1 is decreased to 0.0275 (from 0.205). The smallest (negative) value of the bending gain (5:2) for beam 1 is decreased to -2.49 (from -2.1). The smallest (negative) value of the bending gain(5:3) for beam 1 is increased to -0.97 (from -1.18). The value of the bending gain(5:4) at the end point of beam 1 (close to mass 1) is decreased to the smallest (negative) value -0.88 (from the largest value 0.4). The largest (positive) value of bending gain (5:5) for beam 1 is increased to 1.28 (from 0.21). The bending gain (5:4) for beam 2 has an obvious change in shape. The rest of control bending gains for beam 2 also have some small changes in shape (see Figure 3.6 — Figure 3.10). For the bending gain (5:1) of beam 2, the value at the point where the greatest strain exists for Example 1 still remains the the same for Example 2. However, the value at the end point near mass 1 is increased to -0.0126 (from -0.32). The largest value (positive) of the bending gain (5:2) is at the end point near mass 1, which increases to 9.2 (from 3.75). The largest value (positive) of the bending gain (5:3) for beam 2 is also at the end point near mass 1, increasing to 3.6 (from 2.05). For the bending gain (5:4) of beam 2, the value at the end point near mass 1 is increased to 3.35 (from -0.66), becoming the largest value (positive). The smallest (negative) value of bending gain (5:5) for beam 2 is decreased to -4.4 (from -0.28) at the end point near mass 1.

For beam 1 , the control velocity gain (5:4) has a different shape (concave up) compared to that (concave down) in Example 1. Except for the velocity gain (5:4) of beam 1 , the rest have little change in shape (see Figure 3.11 — Figure 3.15). The smallest value (negative) of the velocity gain (5:1) for beam 1 is decreased to -1.02 (from -0.92). The smallest value (negative) of the velocity gain (5:2) for beam 1 is decreased to -7.1 (from -1.52). The largest value (positive) of the velocity gain(5:3) for beam 1 is at the end point of beam 1 near mass 1, decreasing from 3.35 to 2.15. The change of the velocity gain (5:4) for beam 1 is that the value at the end point of beam 1 near mass 1 is increased from -1.06 (smallest value) to 1.58 (largest value). For the velocity gain (5:5) of beam 1, the smallest value is also at the end point near mass 1, decreasing to -15.4 (from -0.48).

The control velocity gains (5:1 — 5:5) for beam 2 have different shapes (concave up or concave down) compared with those (concave up for one half of beam 2 and concave down for other half) in Example 1. The largest (positive) or smallest (negative) value is reached at the point between the end points of beam 2 instead of the end points(see Figure 3.16 — Figure 3.20). The largest value (positive) of velocity gains (5:1) for beam 2 is decreased to 0.115 (from 0.51). The smallest value (negative) of velocity gain (5:2) for beam 2 is sharply decreased to -58 (from -6.6) . The smallest value of velocity gain (5:3) for beam 2 is also sharply decreased to -22 (from -3.2). The smallest velocity gain (5:4) for beam 2 is decreased to -21 (from -2.1). The largest value of velocity gain (5:5) for beam 2 is increased to 28 (from 1.9).

One can conclude that varying all three parameters for mass 1 can affect the shapes and the quentities of all functional control gains. Especially, the variation of parameters will affect the bending gain (5:4) and velocity gains significantly. A similar conclusion for functional estimator gains, (L2, L3, L5 and L6), can also be obtained by the same analysis (see Figure 3.21 — Figure 3.40).

4.2.3 Simulation 1: Rotation of the Hub

We consider the nth control system

$$\dot{z}_n = A_n z_n + B_n u_n,$$
$$y_n = C_n z_n. \tag{4.2.3.1}$$

and the nth estimator

$$\dot{\hat{z}}_n = A_n \hat{z}_n + B_n u_n + \hat{K}_n (y_n - C_n \hat{z}_n),$$
$$u_n = -K_n \hat{z}_n. \tag{4.2.3.2}$$

The closed loop system is equivalent to

$$\begin{bmatrix} \dot{z}_n \\ \dot{\hat{z}}_n \end{bmatrix} = A_{n,n} \begin{bmatrix} z_n \\ \hat{z}_n \end{bmatrix} \tag{4.2.3.3}$$

where the operator

$$A_{n,n} = \begin{bmatrix} A_n & -B_n K_n \\ \hat{K}_n C_n & [A_n - B_n K_n - \hat{K}_n C_n] \end{bmatrix} \tag{4.2.3.4}$$

generates the closed - loop semigroup $S_{n,n}(t)$ on $E_n \times E_n$ and

$$\begin{bmatrix} z_n(t) \\ \hat{z}_n(t) \end{bmatrix} = S_{n,n}(t) \begin{bmatrix} z_n(0) \\ \hat{z}_n(0) \end{bmatrix}. \tag{4.2.3.5}$$

If we let $\hat{S}_n(t)$ be the semigroup generated by $A_n - \hat{K}_n C_n$ and state

$$e(t) = z_n(t) - \hat{z}_n(t). \tag{4.2.3.6}$$

It is easy to obtain

$$e(t) = \hat{S}_n(t) e(0), \qquad t \geq 0. \tag{4.2.3.7}$$

To see the effect of the estimation scheme under the initial condition,we choose with $\theta(0) = \frac{\pi}{12}$ and all other states to be zero. Initial velocities of all states are chosen to be zero. This corresponds to the initial condition for the matrix representation:

$$\zeta_1(0) = \frac{\pi}{12},$$
$$\zeta_2(0) = \zeta_3(0) = ... = \zeta_{35}(0) = 0,$$
$$\dot{\zeta}_1(0) = \dot{\zeta}_2(0) = ... = \dot{\zeta}_{35}(0) = 0.$$

We choose two sets of different initial conditions, Initial Conditon 1 and Initial Condition 2, for the corresponding nth compensator. Initial Condition 1 consists of

$$\hat{\zeta}_1(0) = 0.3,$$
$$\hat{\zeta}_2(0) = \hat{\zeta}_3(0) = ... = \hat{\zeta}_{35}(0) = 0,$$
$$\dot{\hat{\zeta}}_1(0) = \dot{\hat{\zeta}}_2(0) = ... = \dot{\hat{\zeta}}_{35}(0) = 0, \tag{4.2.3.8}$$

and Initial Condition 2 is given by

$$\hat{\zeta}_1(0) = \hat{\zeta}_2(0) = ... = \hat{\zeta}_{35}(0) = 0,$$
$$\dot{\hat{\zeta}}_1(0) = \dot{\hat{\zeta}}_2(0) = ... = \dot{\hat{\zeta}}_{35}(0) = 0. \tag{4.2.3.9}$$

The behavior of closed loop systems for the control scheme and the compensator scheme with different initial conditions are shown in Figures 4.1 and 4.2. Referring to Figure 4.1, the initial condition for the compensator is Initial Condition 1 and the initial deflection of the structure is shown by the top line. The plots represent the motion of the structure between 0 and 120 seconds with each plot given at 3 second interval. We compare the behavior of both the closed loop systems. As time proceeds, the external applied torque rotates the structure towards equilibrium and the difference of solutions in the simulation for both closed loop systems approaches zero. In Figure 2, Initial Condition 2 is used and a similar analysis can be applied. We determine that the difference of solutions in both simulation for the closed loop systems approaches zero and that the more difference between the initial conditions we choose, the more difference in behavior of two closed systems we will obtain.

4.2.4 Simulation 2: Displacement of Beams

To see the effect of the estimation scheme on an initial displacement of beams, we choose the initial conditions, $\theta(0) = \frac{\pi}{12}$,$V_1(0, x_1) = \frac{1}{24}x_1^3$,$V_2(0, x_2) = \frac{1}{27}x_2^3$. All other states (including velocities) are chosen to be zero. These particular states are chosen to support the theoretical assumption that displacements are not too large. This gives a relatively small displacement at the end of beams and at the end of the structure. The corresponding initial conditions for the matrix representation are given by

$$\zeta_1(0) = \frac{\pi}{12}, \zeta_2(0) = 0,$$

$$\zeta_3(0) = 0.000651, \zeta_4(0) = 0.002604,$$

$$\zeta_5(0) = 0.006333, \zeta_6(0) = 0.013021,$$

$$\zeta_7(0) = 0.022787, \zeta_8(0) = 0.036458,$$

$$\zeta_9(0) = 0.054687, \zeta_{10}(0) = 0.078125,$$

$$\zeta_{11}(0) = 0, \zeta_{12}(0) = 0.001953,$$

$$\zeta_{13}(0) = 0.007811, \zeta_{14}(0) = 0.019529,$$

$$\zeta_{15}(0) = 0.039063, \zeta_{16}(0) = 0.068359,$$

$$\zeta_{17}(0) = 0.109374, \zeta_{18}(0) = 0.164063,$$

$$\zeta_{19}(0) = 0.234374$$

and

$$\dot{\zeta}_1(0) = \dot{\zeta}_2(0) = ... = \dot{\zeta}_{19}(0) = 0.$$

There are two sets of initial conditions , Initial Condition 1 and Initial Condition

2, fot the nth compensator. Initial Condition 1 is given by

$$\hat{\zeta}_1(0) = 0.3, \hat{\zeta}_2(0) = 0,$$

$$\hat{\zeta}_3(0) = 0.0006, \hat{\zeta}_4(0) = 0.002,$$

$$\hat{\zeta}_5(0) = 0.006, \hat{\zeta}_6(0) = 0.01,$$

$$\hat{\zeta}_7(0) = 0.02, \hat{\zeta}_8(0) = 0.03,$$

$$\hat{\zeta}_9(0) = 0.05, \hat{\zeta}_{10}(0) = 0.07,$$

$$\hat{\zeta}_{11}(0) = 0, \hat{\zeta}_{12}(0) = 0.002,$$

$$\hat{\zeta}_{13}(0) = 0.008, \hat{\zeta}_{14}(0) = 0.02,$$

$$\hat{\zeta}_{15}(0) = 0.04, \hat{\zeta}_{16}(0) = 0.07,$$

$$\hat{\zeta}_{17}(0) = 0.1, \hat{\zeta}_{18}(0) = 0.2,$$

$$\hat{\zeta}_{19}(0) = 0.2,$$

and

$$\dot{\hat{\zeta}}_1(0) = \dot{\hat{\zeta}}_2(0) = ... = \dot{\hat{\zeta}}_{19}(0) = 0.$$

Initial Condition 2 consists of

$$\hat{\zeta}_1(0) = \hat{\zeta}_2(0) = ... = \hat{\zeta}_{19}(0) = 0,$$

$$\dot{\hat{\zeta}}_1(0) = \dot{\hat{\zeta}}_2(0) = ... = \dot{\hat{\zeta}}_{19}(0) = 0.$$

The behavior of both the closed loop systems (compensator and control) with different initial estimator conditions are shown in Figure 5.1 and Figure 5.2. Referring to Figure 5.1, the initial estimator condition is Initial Condition 1 and the initial deflection of the structure shows a convex flexure of beams by the top line . The plots represent the motion of the structure between 0 and 120 seconds with each plot given at 3 second interval. With the initial deflection , the structure returns the position of a straight line quickly and then keeps the beam straight. As time proceeds, the motions of the structure by both schemes approach the equilibrium quickly. In Figure 5.2, the compensator closed loop system is plotted with Initial Condition 2 and a similar anlysis can also be applied.

In Figures 5.1 and 5.2, we found that the more difference between the initial conditions for control closed loop system and estimator closed loop system , the more difference in the simulations for two closed loop system even though the difference approaches zero.

Chapter V Conclusions

5.1 Summary of Results

One of the important problems in control theory is the design of feedback control systems under the assumption that only partial observation of the state is available. This is typically done by constructing an appropriate observer (estimator) of the original state,whereby the desired feedback control law — called the compensator — is based on the information available from the observed (estimated) state variable. In recent years, the problem of consistent and convergent approximations of infinite - dimensional compensators has attracted a lot of attention (see [3] , [10], [12], [13] and [19]).

The goal of this paper is to study the LQG problem for a class of infinite dimensional systems. We present an abstract framework for compensator design. We investigate both the convergence of the approximating compensators and the performance of the closed - loop systems, especially, the uniformly exponential stability of the closed - loop systems. We concentrate on a specific Two - Connected - Beam (TCB) structure in order to focus the analysis by using a test problem in numerical simulations.

We develop a Galerkin finite element approximation scheme for TCB and prove the convergence of the resulting sequence of approximating LQG problems. Convergence is demonstrated via control and estimator gains of two numerical examples. Two simulations also illustrate the performance of both the control scheme and the estimator scheme.

When applying compensator design procedure to flexible structures, stability properties of the observer and of the resulting closed loop system are of obvious importance. Even with the stable closed loop system by the infinite dimensional compensator, it doesn't guarantee, in general, that the finite - dimensional compensator produces a uniformly stable closed loop system. While numerical experiments indicate that the uniform stability of the compensator should be preserved for a large class of approximations (including finite elements), however, the proof of this property has been recognized as an open problem (see [3]). In [3], such conclusions hold only in a very special case when the selected approximation is modal and certain decoupling condition holds for the original system (see Theorem 9.3 in [3]).

In order to extend the above results to a general case, we follow the approaches of Triggiani [11], Schumacher [12] and Ito [10]. For a large class of dynamics satisfying the so - called " spectrum determined growth " conditions, we not only introduce finite - dimensional approximations (based on finite elements) of the original infinite - dimensional compensator and show the finite - dimensional compensators, when applied to the the original model, produce near optimal performance, but also prove that the stability properties of the resulting closed loop system are preserved uniformly.

The main results obtain in this paper may be summarized as follows:

* We present a comprehensive study of the LQG problem for a class of infinite dimensional systems. In particular, we apply the schemes we developed to the Two - Connected - Beam (TCB) structure to illustrate and test the methods.

* Theorem 3.9 guarantees that the stability properties of the resulting closed - loop system are preserved for a large class of dynamics systems with bounded input operators.

* We use the approximating conditions (c1) and (c2) to weaken and simplify the corresponding conditions used in [10].

5.2 Future Reseach Opportunities

There are several areas that still requir further work. In particular, we have following problems to solve.

* The wellposedness theory in Chapter II needs to be extended to a second order system with weaker types of damping. In particular, a theory is needed for a flexible structure with weak structure damping.
* To establish the theory for the problem of obtaining the lowest - order compensator that closely approximates the infinite dimensional compensator.
* For a flexible structure with bounded input operators, the approximation theory develop in this paper is reasonably complete. The most important extensions should be to the corresponding problem with a unbounded input operator, for which there exists little approximation theory. Due to the different kinds of unbounded input operators, stiffness operators and structural damping, all of which must be considered in detail.

Table 1. Beam Parameters for Example 1

Parameter	*Beam 1: Aluminum*	*Beam 2: Steel*
L	2 m	3 m
w	2.54 cm	3.81 cm
h	0.3175 cm	0.15875 cm
A	$8.0645 \times 10^{-5}\, m^2$	$6.0483 \times 10^{-5}\, m^2$
ρ	$2700\, kg / m^3$	$7800\, kg / m^3$
ρA	$2.1774 \times 10^{-1}\, kg / m$	$4.7177 \times 10^{-1}\, kg / m$
E	$7.0 \times 10^{10}\, N / m^2$	$2.0 \times 10^{11}\, N / m^2$
I	$6.7746 \times 10^{-11}\, m^4$	$1.27 \times 10^{-11}\, m^4$
EI	$4.74222\, Nm^2$	$2.54\, Nm^2$
γI	.0002 kg / m sec	.0002 kg / m sec

Table 2. Rigid Body Parameters for Example 1

Parameter	*Hub*	*Mass 1*	*Mass 2*
J	$1\, kg\, m^2$	$1\, kg\, m^2$	$1\, kg\, m^2$
m		1 kg	1 kg
l_m		.25 m	.25 m

Table 3. Control Constant Gains for Example 1

Control Constant Gains for N=3:

K1:	2.89613e-01	-3.47794e-02	4.52314e-02	1.72517e-01	9.39737e-01
K4:	1.77492e+00	-1.21181e+00	-2.41052e+00	2.56539e-01	-5.97187e-01
K7:	-1.21186e+00	1.43633e+01	7.27307e+00	-2.46491e+00	-1.11176e+00
K8:	-4.13734e-01	5.90978e+00	4.28723e+00	-1.36735e+00	-5.47770e-01
K9:	2.56539e-01	-2.46491e+00	-1.65596e+00	1.72949e+00	2.98278e-01
K10:	-6.77246e-01	2.92203e+01	1.25677e+01	-4.13851e+00	-3.02776e+00

Control Constant Gains for N=4:

K1:	2.89635e-01	-3.51844e-02	4.49690e-02	1.72646e-01	9.39704e-01
K4:	1.77793e+00	-1.20941e+00	-2.41518e+00	2.56574e-01	-5.98429e-01
K7:	-1.20941e+00	1.43600e+01	7.26655e+00	-2.46630e+00	-1.10819e+00
K8:	-4.15009e-01	5.90596e+00	4.29957e+00	-1.36814e+00	-5.48583e-01
K9:	2.56574e-01	-2.46630e+00	-1.65678e+00	1.73176e+00	2.95762e-01
K10:	-4.29850e-01	2.82250e+01	1.20129e+01	-4.22690e+00	-2.84149e+00

Control Constant Gains for N=8:

K1:	2.89644e-01	-3.53579e-02	4.48562e-02	1.72702e-01	9.39690e-01
K4:	1.77912e+00	-1.20845e+00	-2.41695e+00	2.56603e-01	-5.98962e-01
K7:	-1.20845e+00	1.43582e+01	7.26401e+00	-2.46700e+00	-1.10650e+00
K8:	-4.15443e-01	5.90449e+00	4.30464e+00	-1.36852e+00	-5.48949e-01
K9:	2.56603e-01	-2.46700e+00	-1.65720e+00	1.73264e+00	2.94825e-01
K10:	-1.52920e-01	2.72865e+01	1.14224e+01	-4.36380e+00	-2.60928e+00

Table 4. Estimator Constant Gains for Example 1

Estimator Constant Gains for N=3:

L1:	1.17303e+00	-4.44799e-02	-9.55244e-02	-1.57210e-02	8.80150e-02
L4:	1.97550e-01	1.72204e-02	-2.73612e+00	8.69776e-02	5.18414e-01
L7:	8.78528e-02	5.87820e-02	4.35167e-01	-5.29955e-01	-5.71483e+00
L8:	3.10558e-01	4.33866e-02	6.10181e-01	-1.43302e-01	-1.68160e+00
L9:	8.56021e-02	2.41041e-02	-1.69766e-02	-7.78698e-02	1.62278e-01
L10:	6.81994e-01	1.53969e-01	-1.60782e+00	-1.05074e+00	-1.24468e+01

Estimator Constant Gains for N=4:

L1:	1.17303e+00	-4.44824e-02	-9.55345e-02	-1.57157e-02	8.79912e-02
L4:	1.97550e-01	1.72167e-02	-2.74387e+00	8.69730e-02	5.16800e-01
L7:	8.78563e-02	5.88272e-02	4.28411e-01	-5.29978e-01	-5.72436e+00
L8:	3.10548e-01	4.34239e-02	6.23666e-01	-1.43465e-01	-1.67979e+00
L9:	8.56161e-02	2.40769e-02	-1.62192e-02	-7.76736e-02	1.58847e-01
L10:	6.85126e-01	1.49245e-01	-2.02090e+00	-1.01828e+00	-1.18631e+01

Estimator Constant Gains for N=8:

L1:	1.17303e+00	-4.44834e-02	-9.55388e-02	-1.57134e-02	8.79804e-02
L4:	1.97549e-01	1.72152e-02	-2.74674e+00	8.69705e-02	5.16268e-01
L7:	8.78581e-02	5.88468e-02	4.25936e-01	-5.29987e-01	-5.72730e+00
L8:	3.10543e-01	4.34402e-02	6.29023e-01	-1.43537e-01	-1.67952e+00
L9:	8.56221e-02	2.40653e-02	-1.59411e-02	-7.75899e-02	1.58024e-01
L10:	6.87936e-01	1.45044e-01	-2.54510e+00	-9.89817e-01	-1.10858e+01

Table 5. Control Constant Gains for Example 2

Control Constant Gains for N=3:

K1:	2.79911e-01	4.10070e-01	2.43203e-01	1.95688e-01	8.09969e-01
K4:	1.50202e+00	-1.11768e-01	-1.76265e+00	8.05963e-02	-5.73320e-01
K7:	-1.11768e-01	3.81790e+01	1.46823e+01	1.38346e+01	-1.85891e+01
K8:	1.14876e-01	1.45426e+01	5.78737e+00	5.25942e+00	-7.19532e+00
K9:	8.05963e-02	1.38346e+01	5.15867e+00	6.42563e+00	-6.65323e+00
K10:	1.26529e+00	7.33044e+01	2.64957e+01	2.72567e+01	-3.60785e+01

Control Constant Gains for N=4:

K1:	2.79857e-01	4.08900e-01	2.42836e-01	1.94976e-01	8.10860e-01
K4:	1.50507e+00	-1.20070e-01	-1.76961e+00	7.77095e-02	-5.70459e-01
K7:	-1.20070e-01	3.80945e+01	1.46629e+01	1.37583e+01	-1.84978e+01
K8:	1.11720e-01	1.45128e+01	5.78147e+00	5.23135e+00	-7.16230e+00
K9:	7.77095e-02	1.37583e+01	5.13421e+00	6.38202e+00	-6.59870e+00
K10:	1.43927e+00	7.49190e+01	2.69203e+01	2.76025e+01	-3.67492e+01

Control Constant Gains for N=8:

K1:	2.79834e-01	4.08403e-01	2.42681e-01	1.94677e-01	8.11237e-01
K4:	1.50631e+00	-1.23630e-01	-1.77252e+00	7.64885e-02	-5.69234e-01
K7:	-1.23630e-01	3.80575e+01	1.46543e+01	1.37258e+01	-1.84586e+01
K8:	1.10373e-01	1.44998e+01	5.77887e+00	5.21939e+00	-7.14813e+00
K9:	7.64885e-02	1.37258e+01	5.12378e+00	6.36359e+00	-6.57552e+00
K10:	1.65607e+00	7.76469e+01	2.77023e+01	2.83835e+01	-3.79865e+01

Table 6. Estimator Constant Gain for Example 2

Estimator Constant Gains for N=3:

L1:	1.14831e+00	-4.34687e-02	-7.78445e-02	-2.38390e-02	5.73603e-02
L4:	1.65221e-01	2.00838e-02	-2.37615e+00	3.16685e-02	1.23808e-03
L7:	6.75855e-02	1.88342e-02	4.39172e-01	-4.71009e-01	-1.39966e+01
L8:	2.69494e-01	1.94408e-02	1.21450e-01	-1.43516e-01	-5.21911e+00
L9:	4.24205e-02	3.57282e-02	2.87472e-01	-2.67377e-01	-6.10170e+00
L10:	5.69246e-01	6.24101e-02	-1.16959e+00	-8.63888e-01	-2.71743e+01

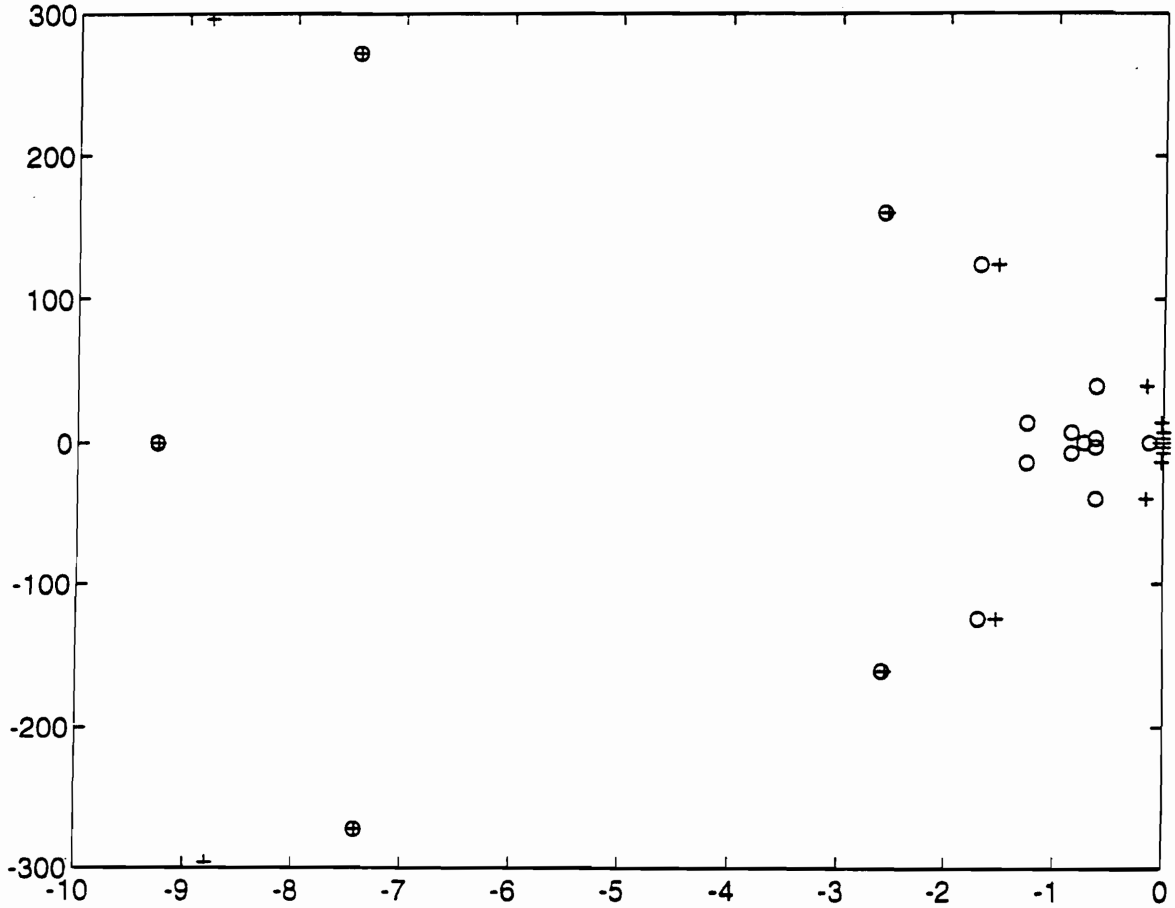
Estimator Constant Gains for N=4:

L1:	1.14831e+00	-4.34709e-02	-7.78632e-02	-2.38199e-02	5.74478e-02
L4:	1.65228e-01	2.00752e-02	-2.38217e+00	3.17971e-02	4.28645e-03
L7:	6.75534e-02	1.89094e-02	4.40019e-01	-4.71200e-01	-1.39733e+01
L8:	2.69483e-01	1.94681e-02	1.22261e-01	-1.43572e-01	-5.21180e+00
L9:	4.24337e-02	3.57280e-02	2.87803e-01	-2.67011e-01	-6.07689e+00
L10:	5.68215e-01	6.46264e-02	-1.54133e+00	-8.99348e-01	-2.79541e+01

Estimator Constant Gains for N=8:

L1:	1.14832e+00	-4.34718e-02	-7.78713e-02	-2.38118e-02	5.74853e-02
L4:	1.65232e-01	2.00715e-02	-2.38458e+00	3.18523e-02	5.61872e-03
L7:	6.75398e-02	1.89420e-02	4.40396e-01	-4.71280e-01	-1.39614e+01
L8:	2.69478e-01	1.94800e-02	1.22596e-01	-1.43597e-01	-5.20809e+00
L9:	4.24390e-02	3.57285e-02	2.87895e-01	-2.66861e-01	-6.06538e+00
L10:	5.66256e-01	6.78568e-02	-1.99790e+00	-9.50474e-01	-2.88638e+01

Figure 1.1 Eigenvalues Near Imaginary Axis,Example 1(rl=.0002)



N=8, Open Loop : + , Closed Loop : o

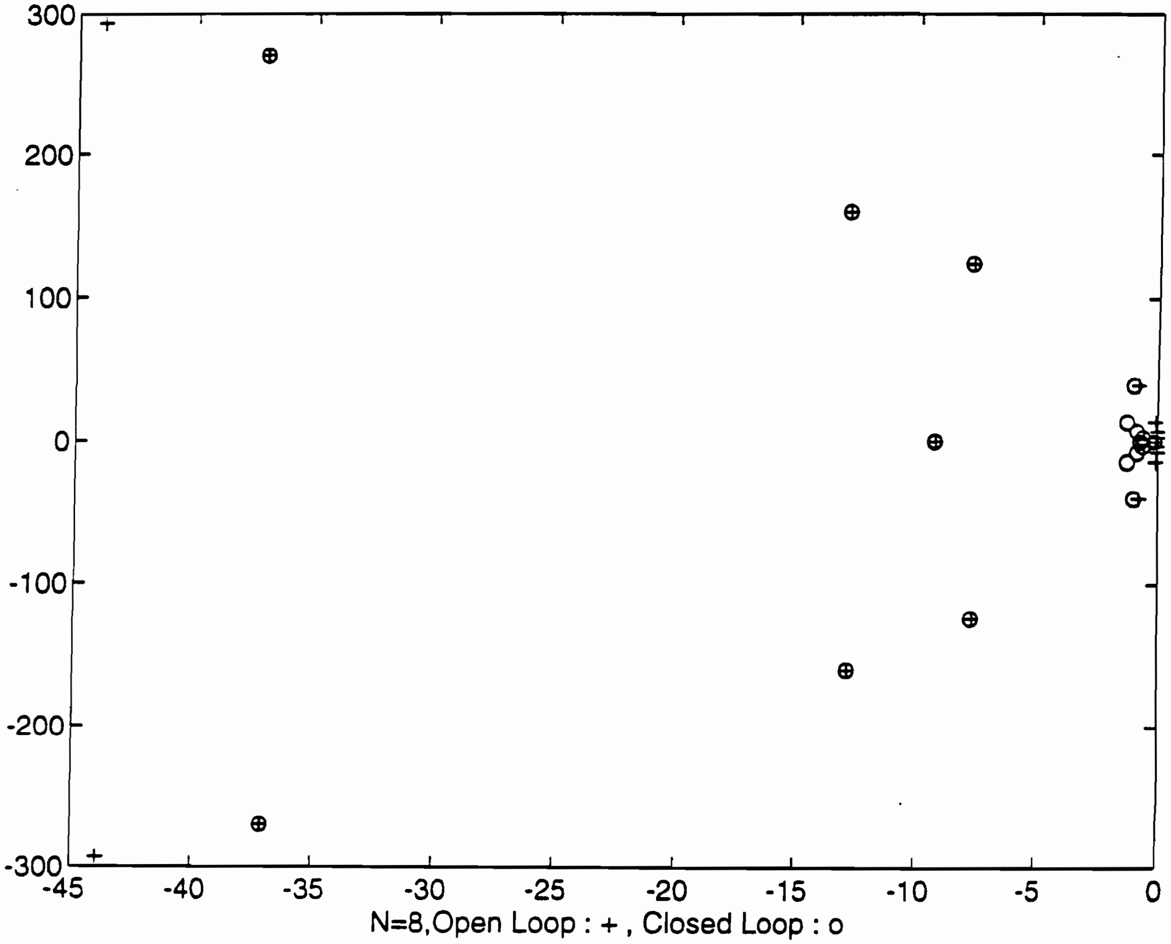


Figure 1.2 Eigenvalues Near Imaginary Axis,Example 1(rl=.001)

Figure 1.3 Optimal Control u(t)=[u0(t),u1(t),u2(t),u3(t),u4(t)]

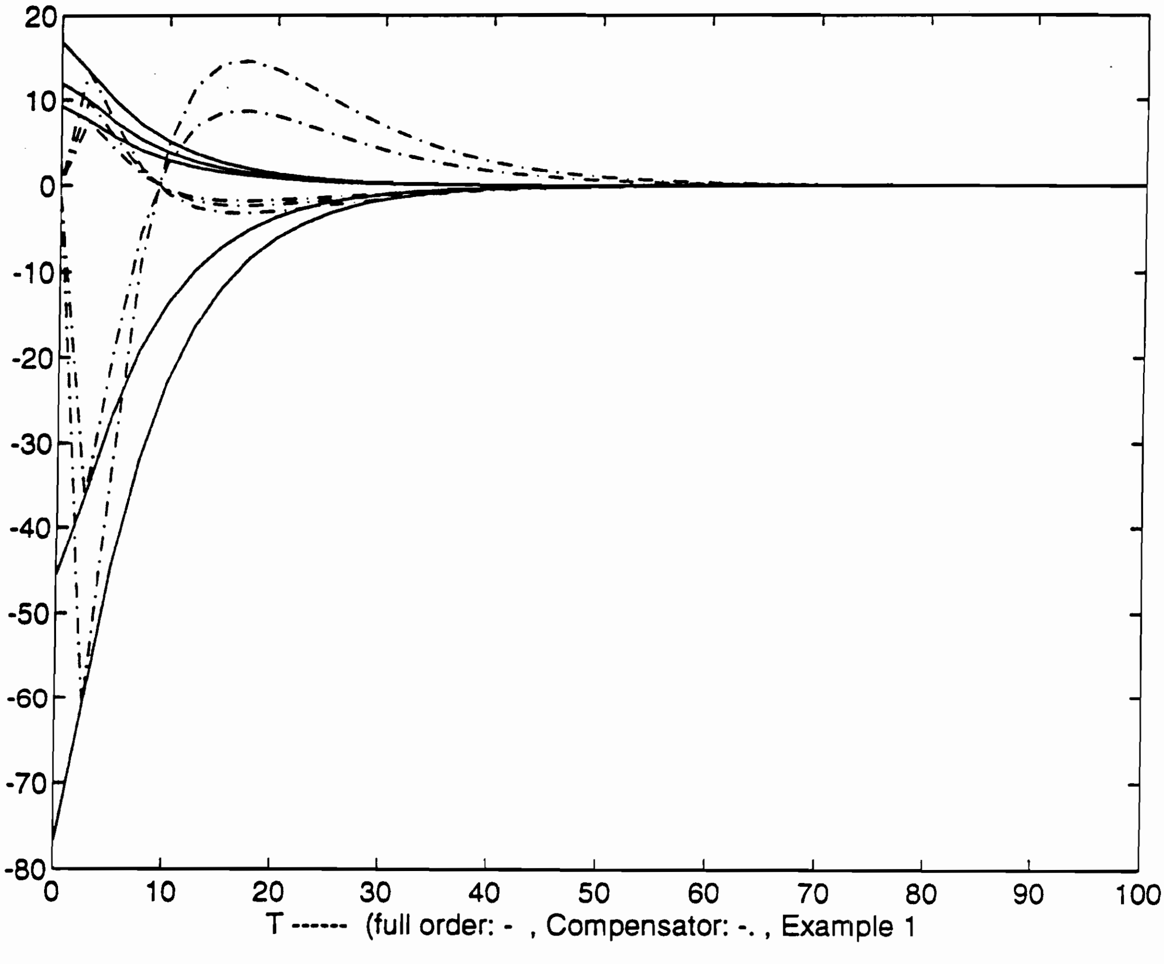
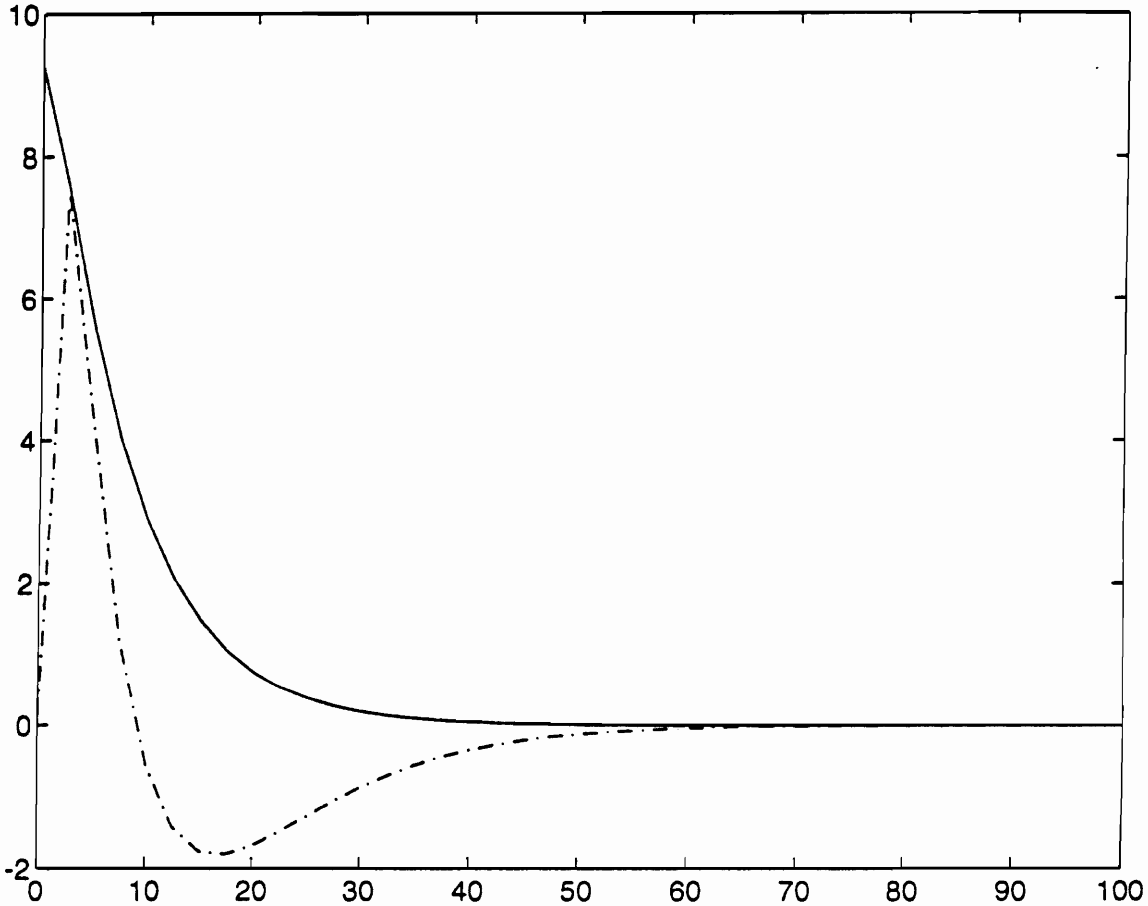


Figure 1.4 Control Function u0(t), Example 1



T ------ (full oreder: -   , compensator: -.)

Figure 1.5 Control Function u1(t), Example 1

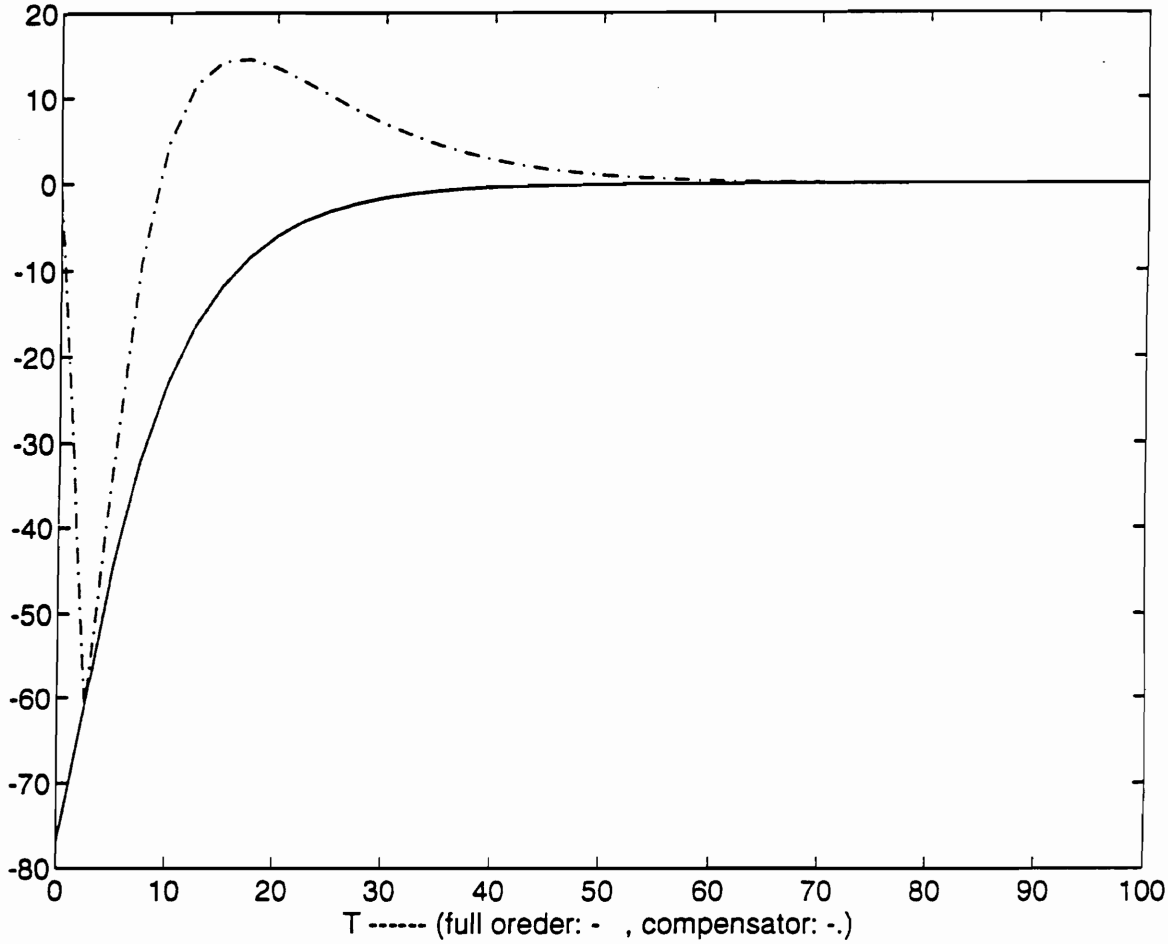


Figure 1.6 Control Function u2(t), Example 1

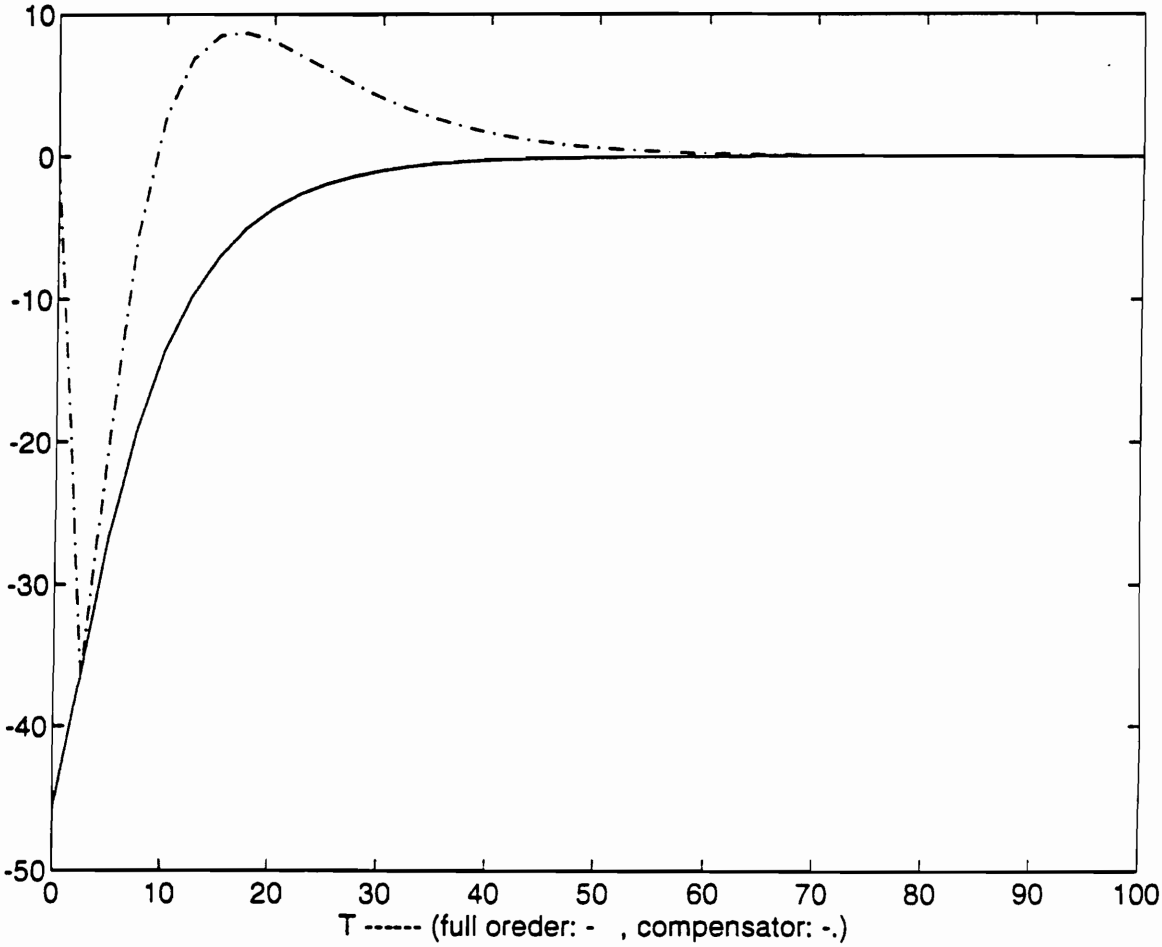
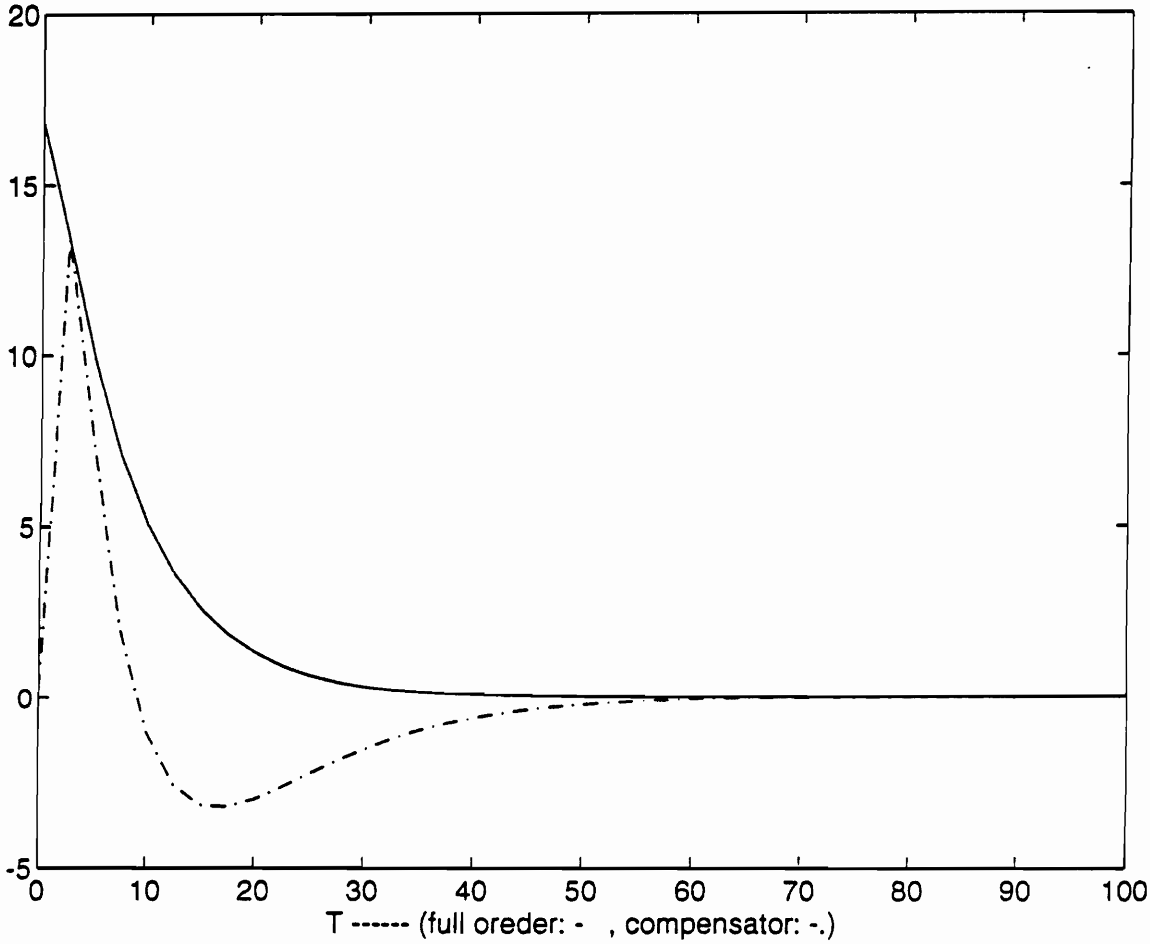


Figure 1.7 Control Function u3(t), Example 1



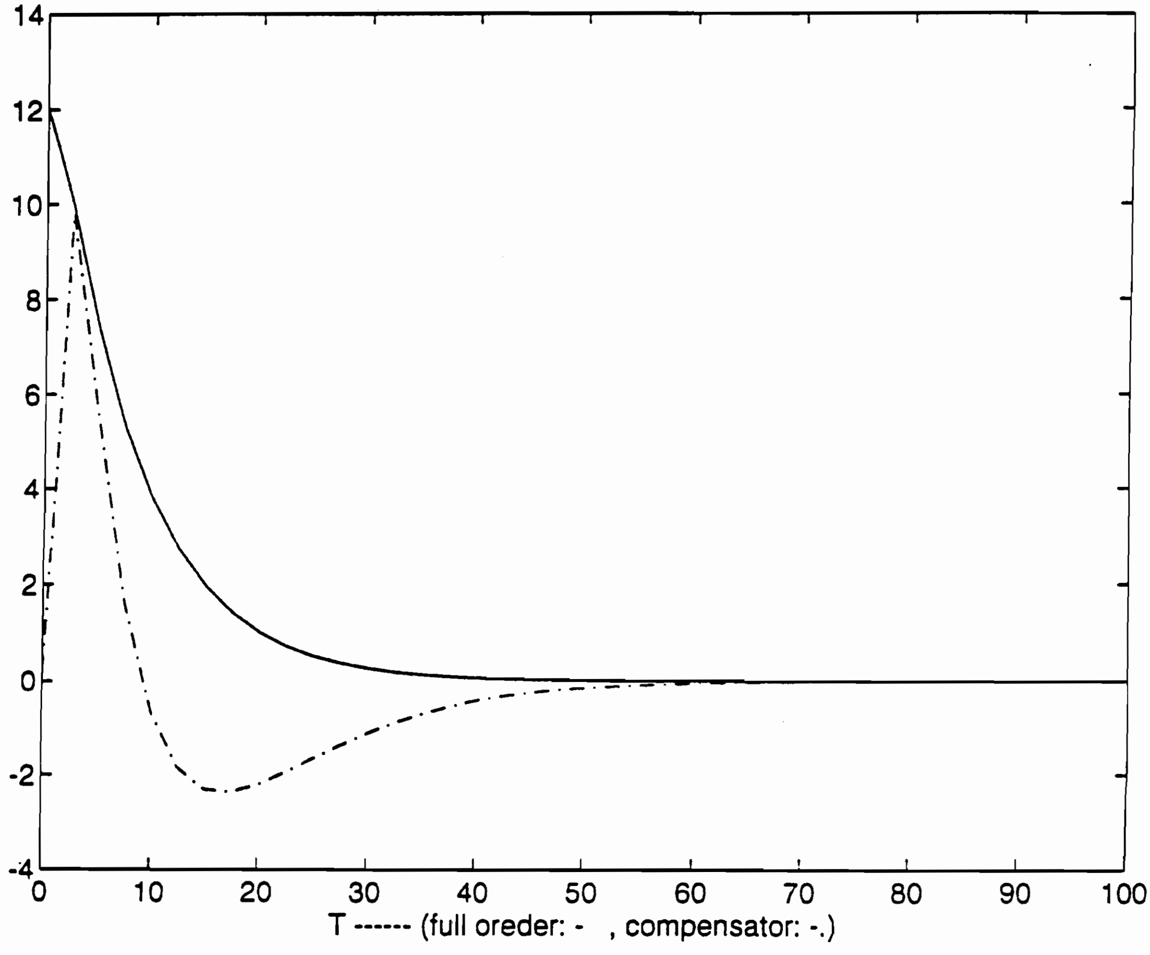
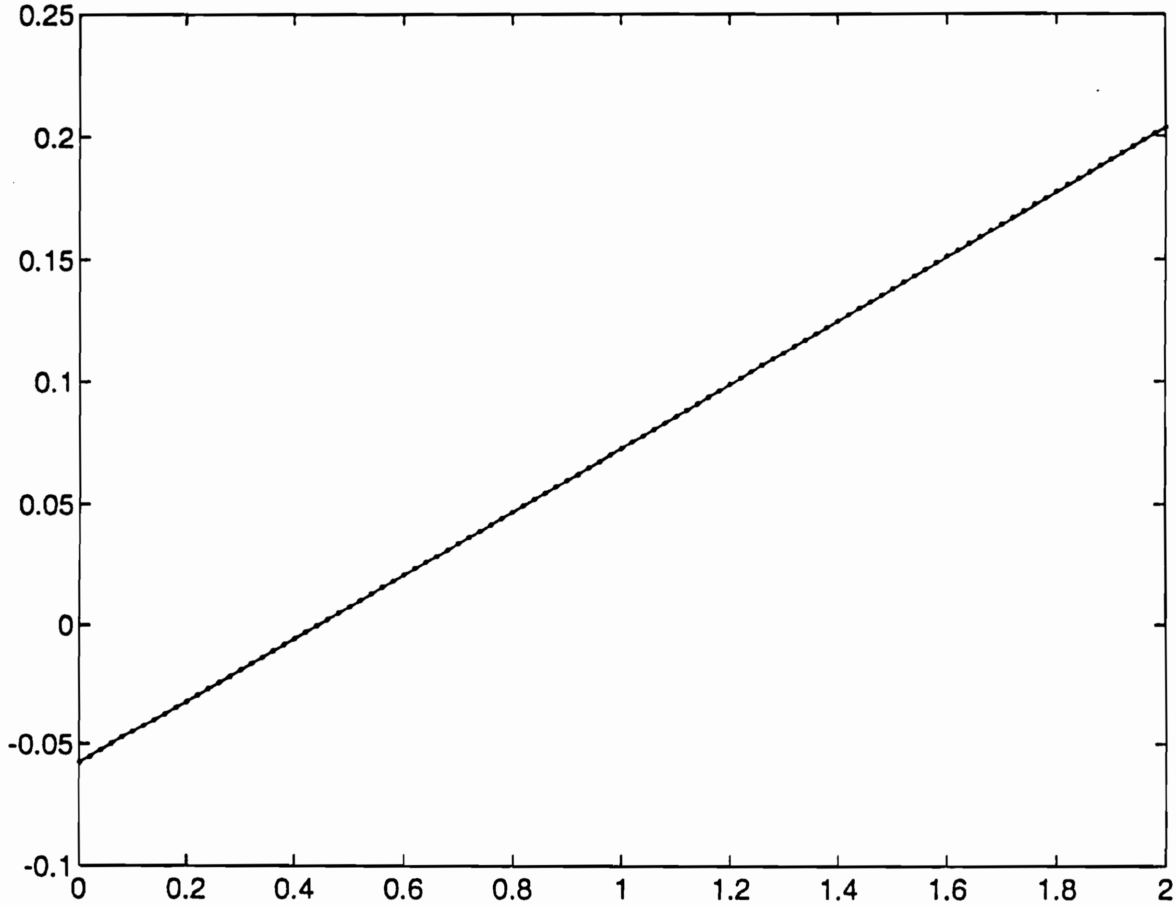


Figure 1.8 Control Function u4(t), Example 1

Figure 2.1 The Bending Gains(5:1) for Beam 1:n=3(.),n=4(-.),n=8(-)



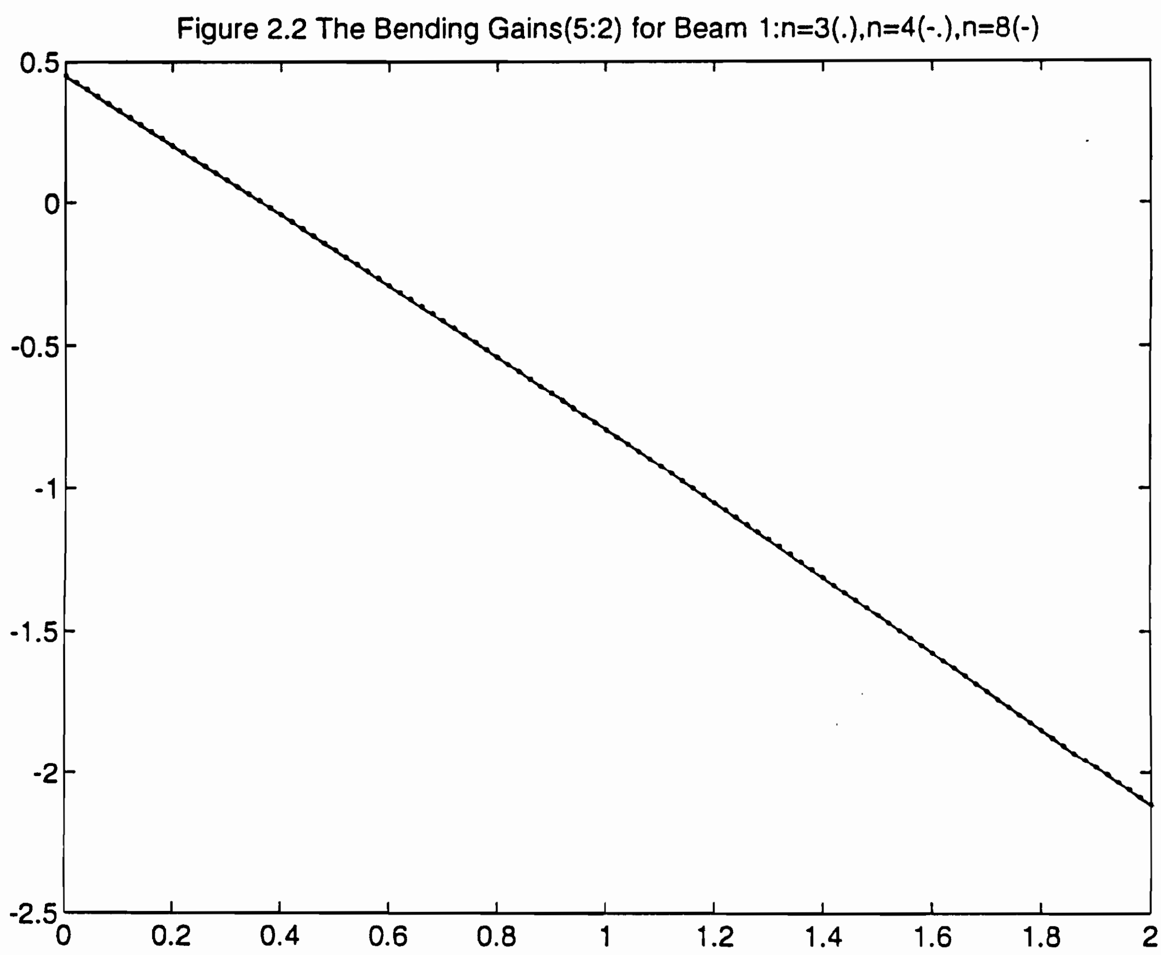


Figure 2.2 The Bending Gains(5:2) for Beam 1:n=3(.),n=4(-.),n=8(-)
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Figure 2.3 The Bending Gains(5:3) for Beam 1:n=3(.),n=4(-.),n=8(-)

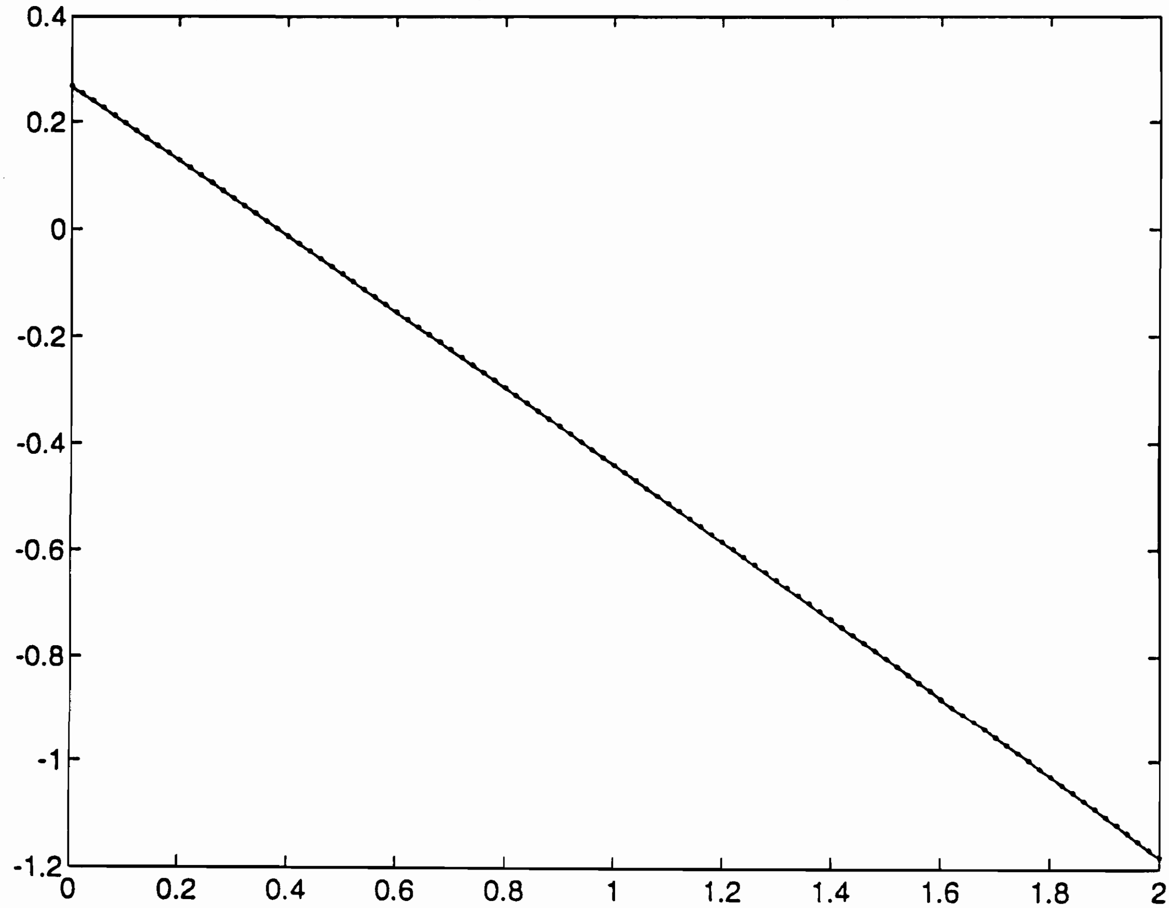


Figure 2.4 The Bending Gains(5:4) for Beam 1:n=3(.),n=4(-.),n=8(-)

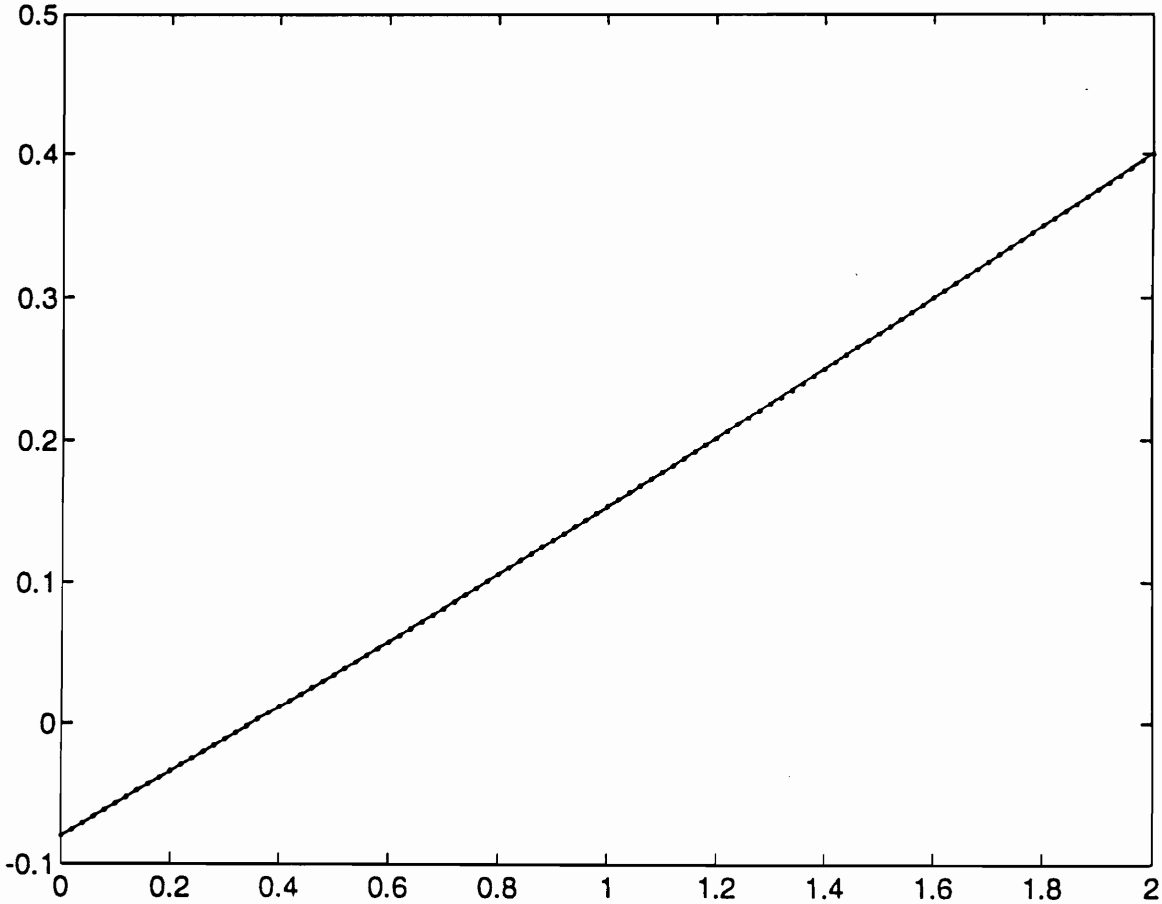


Figure 2.5 The Bending Gains(5:5) for Beam 1:n=3(.),n=4(-.),n=8(-)

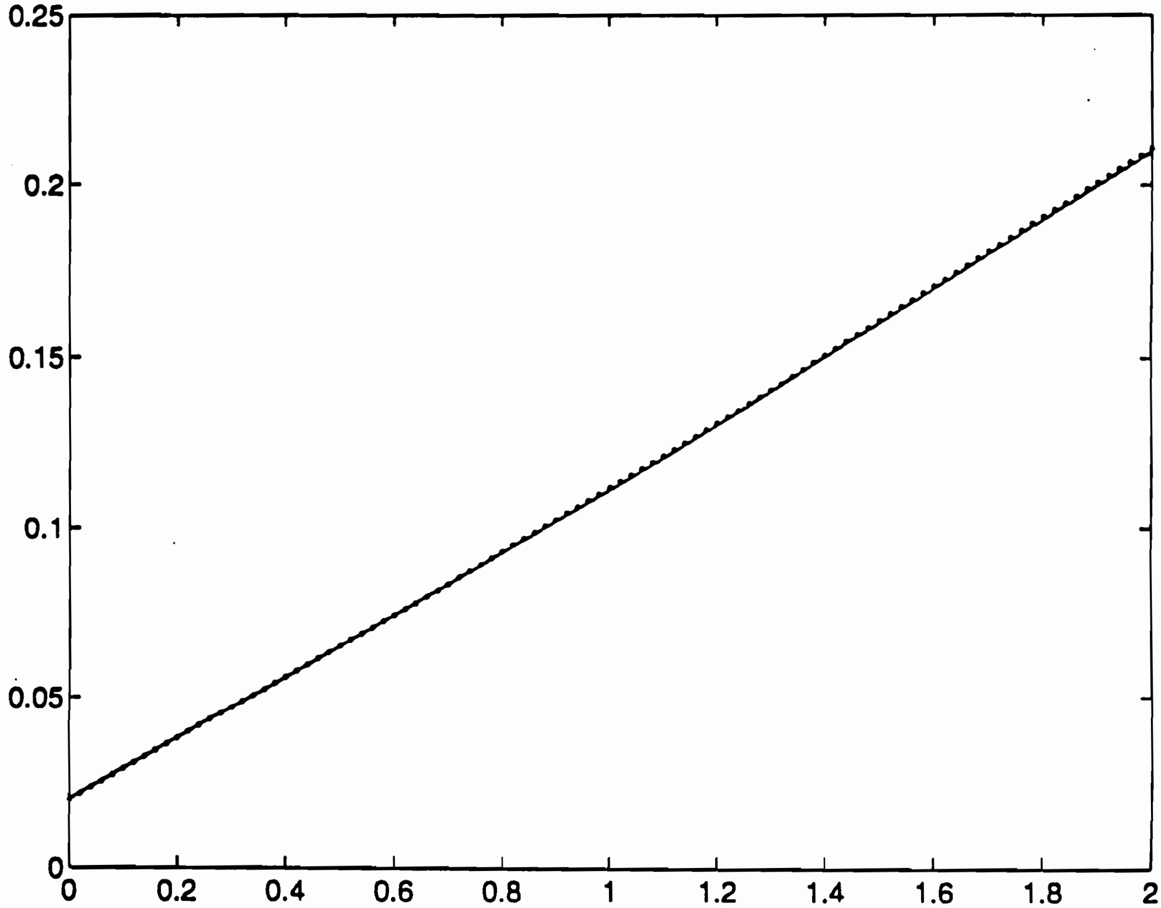


Figure 2.6 The Bending Gains(5:1) for Beam 2:n=3(.),n=4(-.),n=8(-)

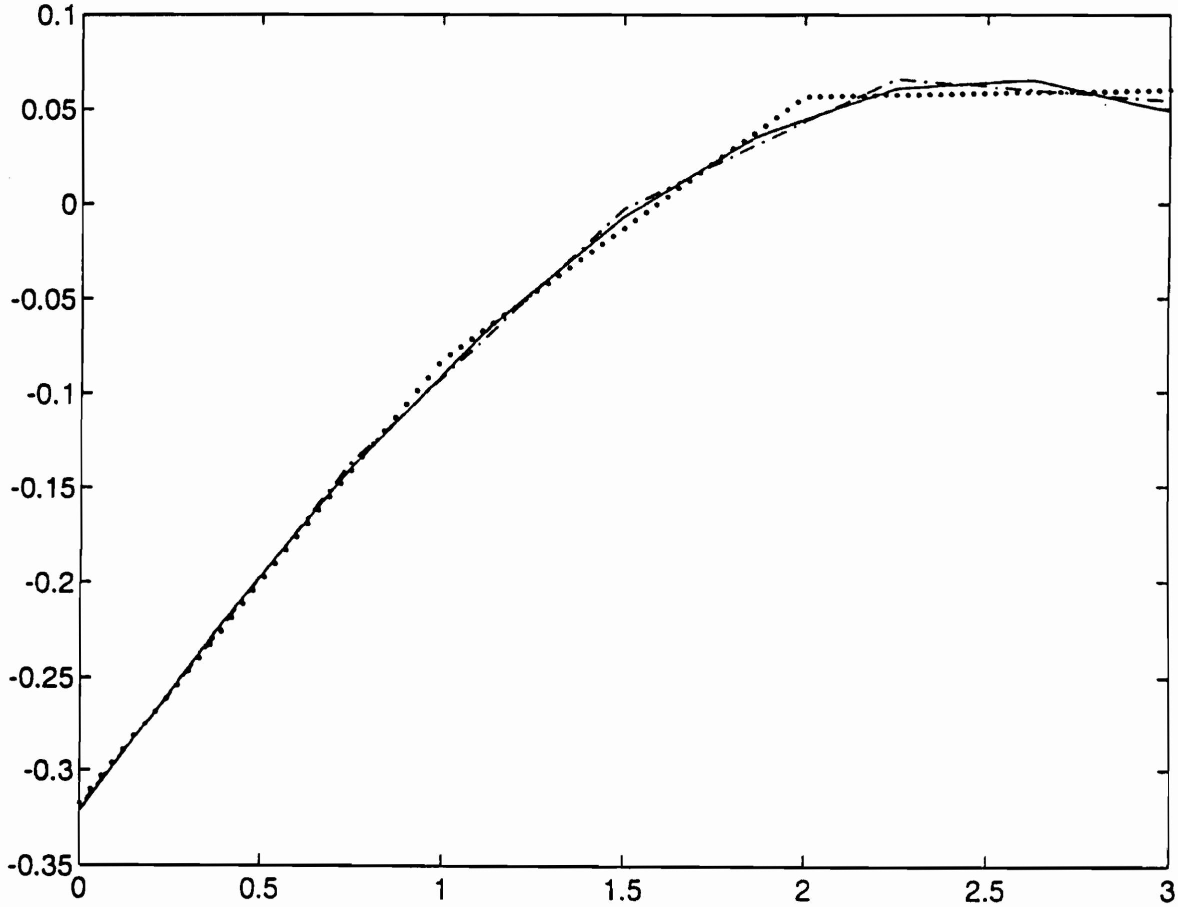


Figure 2.7 The Bending Gains(5:2) for Beam 2:n=3(.),n=4(-.),n=8(-)

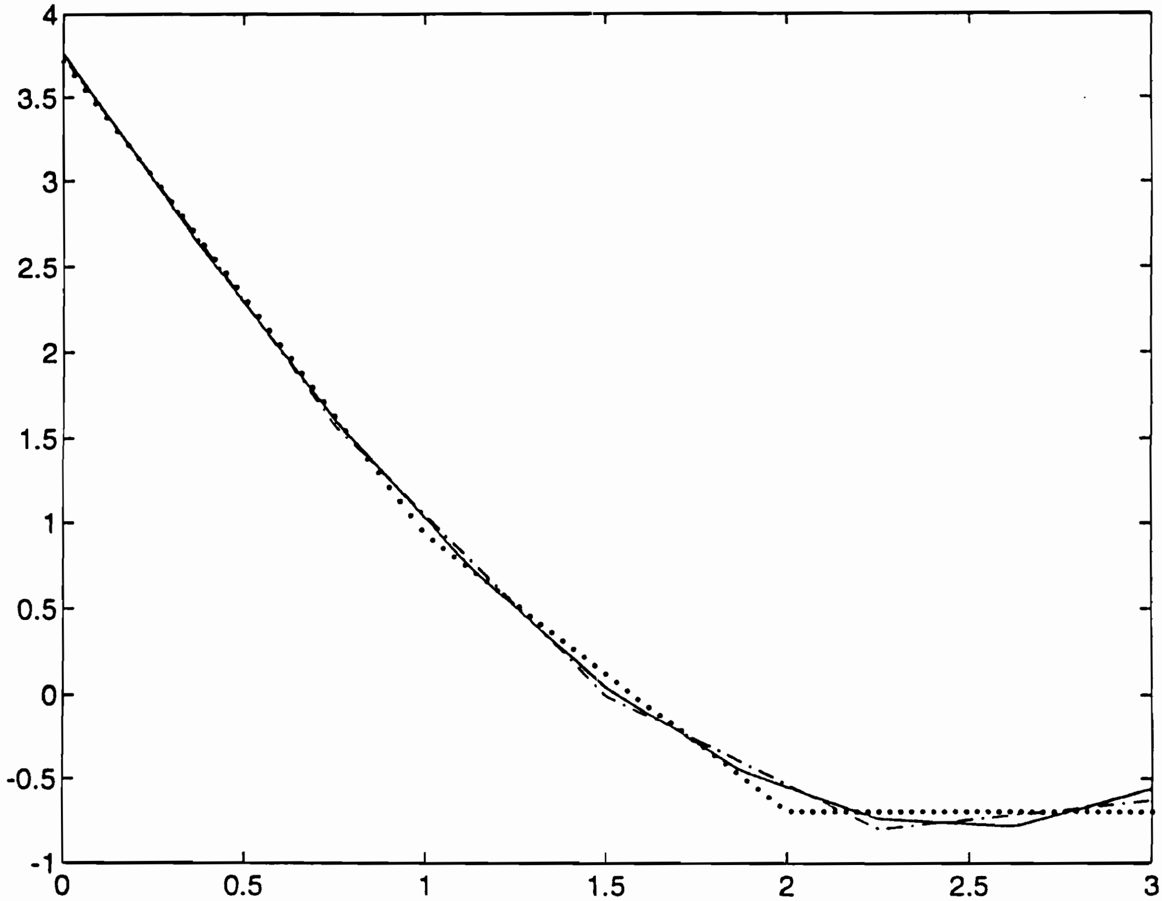


Figure 2.8 The Bending Gains(5:3) for Beam 2:n=3(.).n=4(-.),n=8(-)

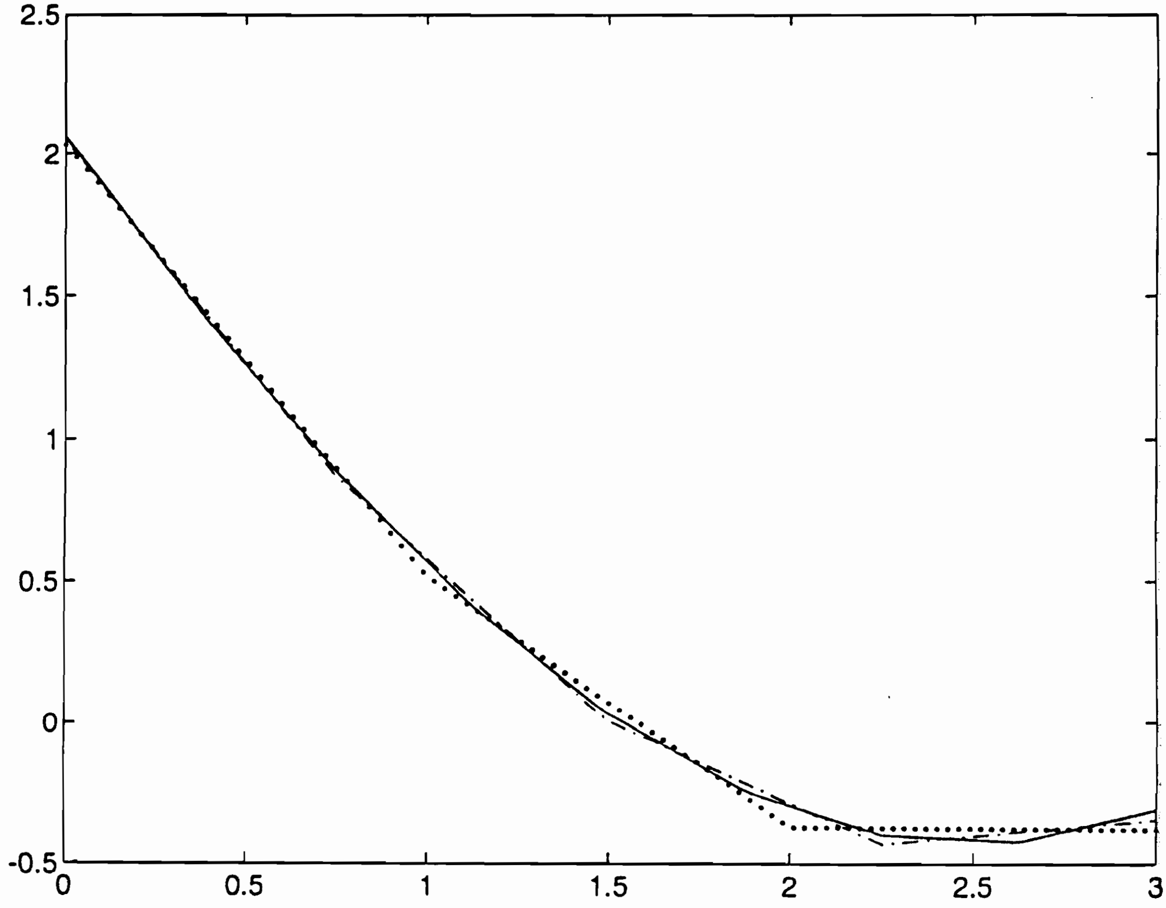


Figure 2.9 The Bending Gains(5:4) for Beam 2:n=3(.),n=4(-.),n=8(-)

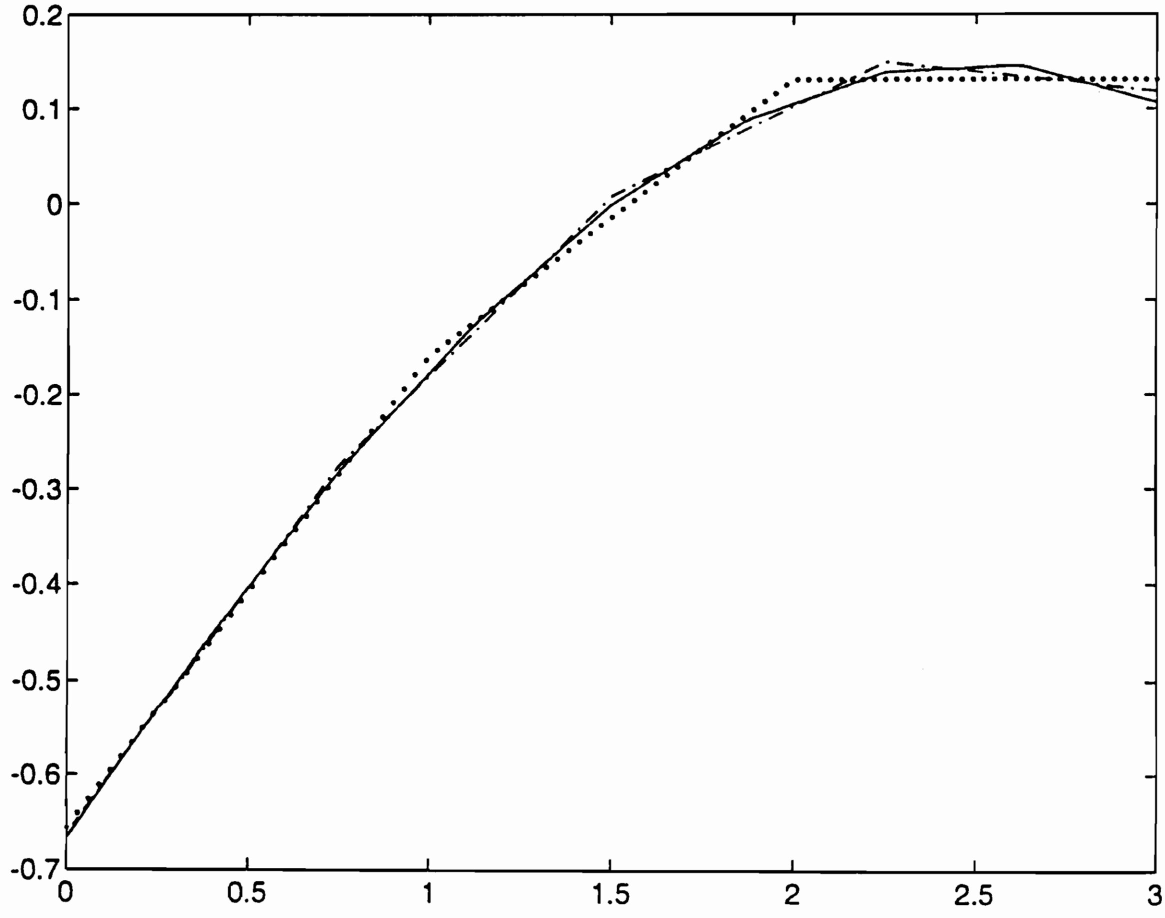


Figure 2.10 The Bending Gains(5:5) for Beam 2:n=3(.),n=4(-.),n=8(-)

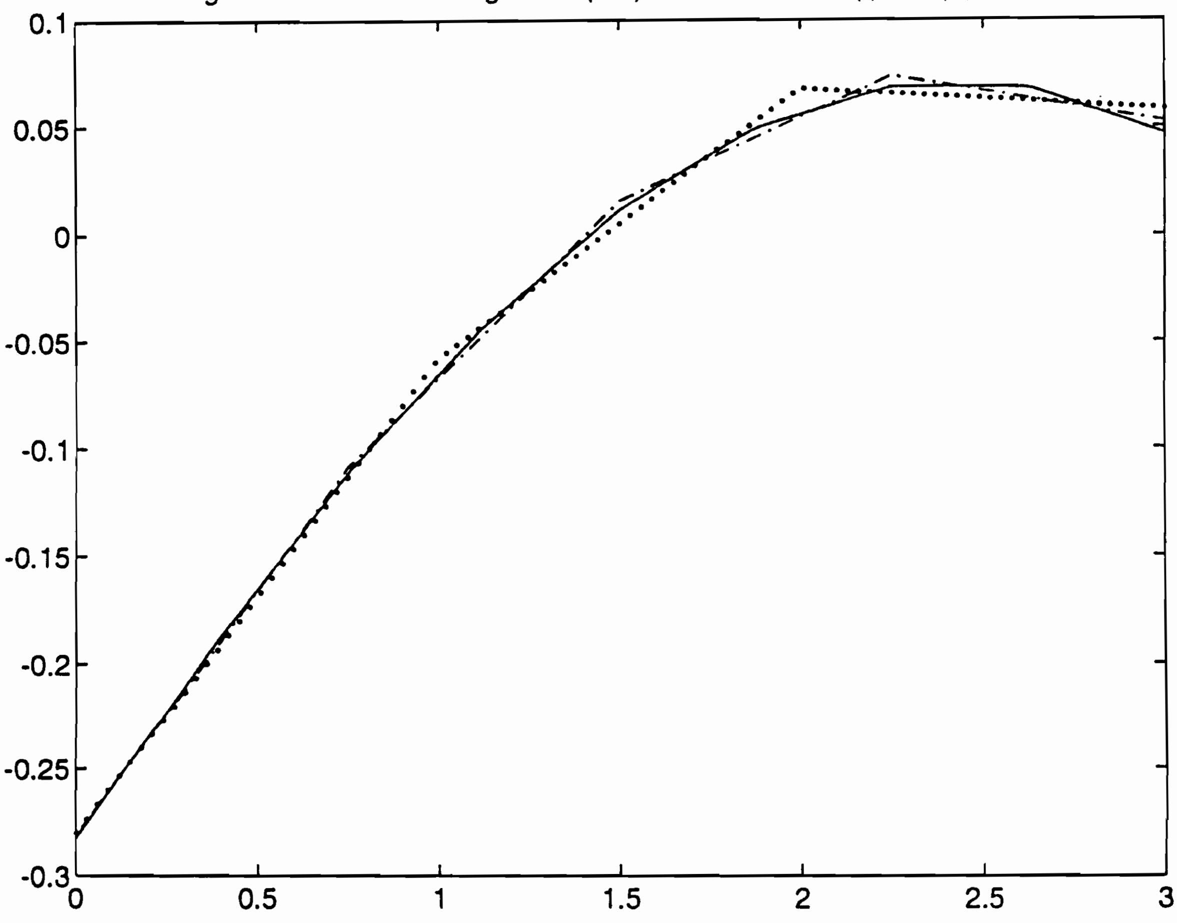


Figure 2.11 Velocity Gains(5:1) for Beam 1:n=3(.),n=4(-.),n=8(-)

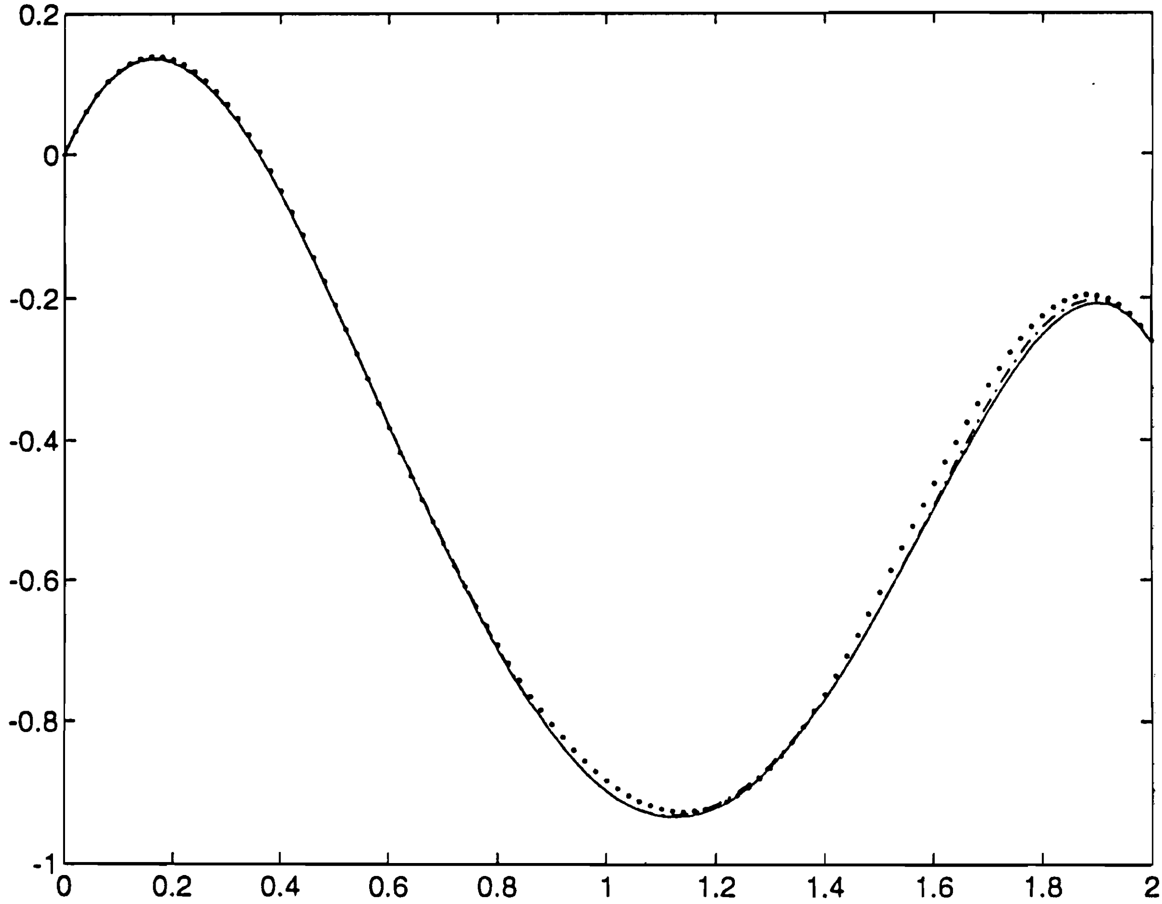


Figure 2.12 Velocity Gains(5:2) for Beam 1:n=3(.),n=4(-.),n=8(-)

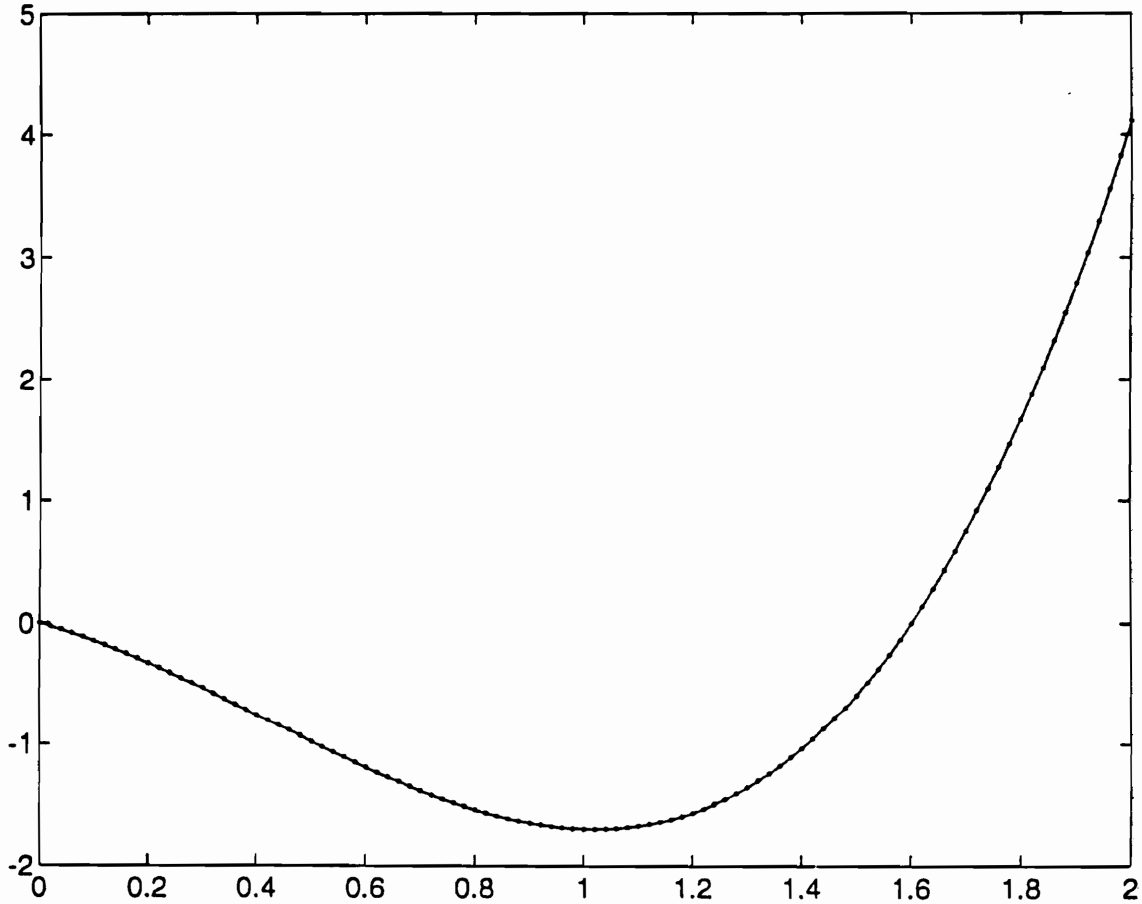
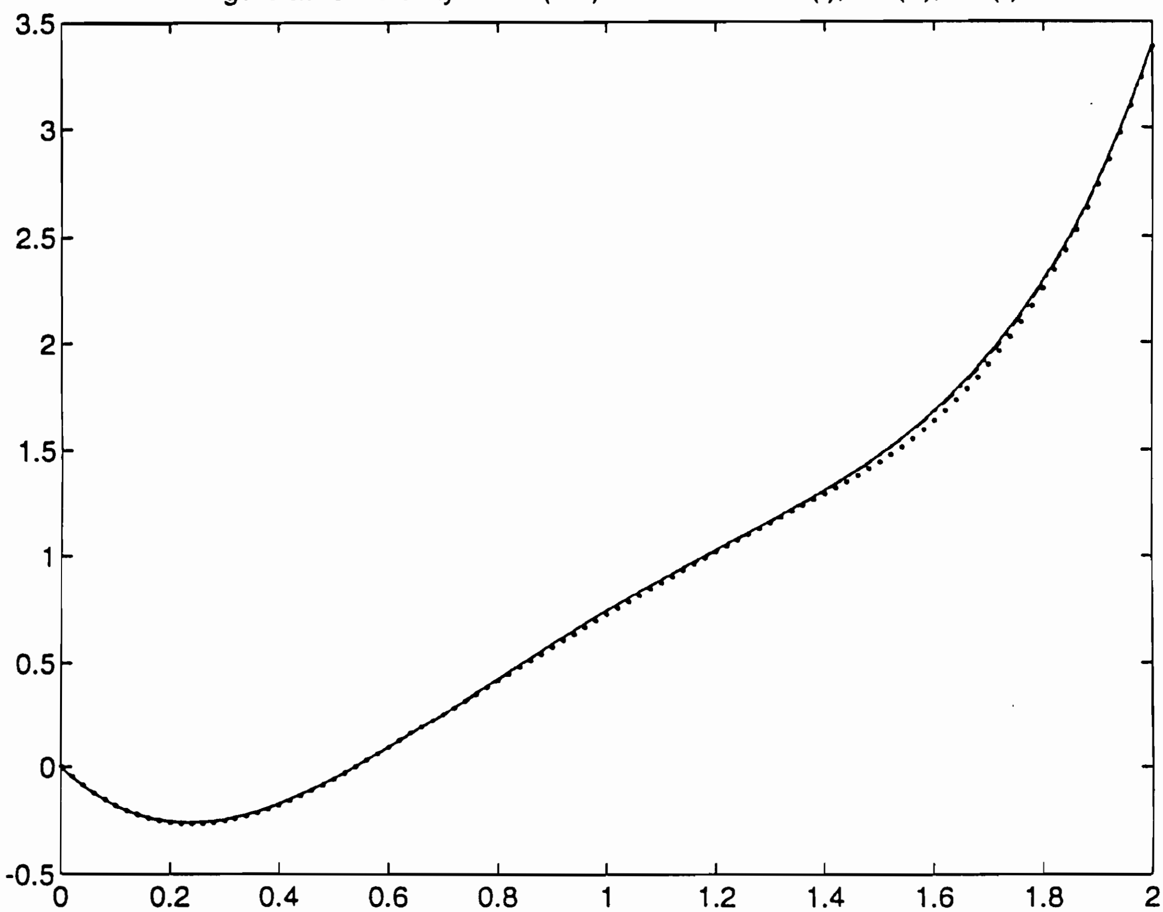


Figure 2.13 Velocity Gains(5:3) for Beam 1:n=3(.),n=4(-.),n=8(-)



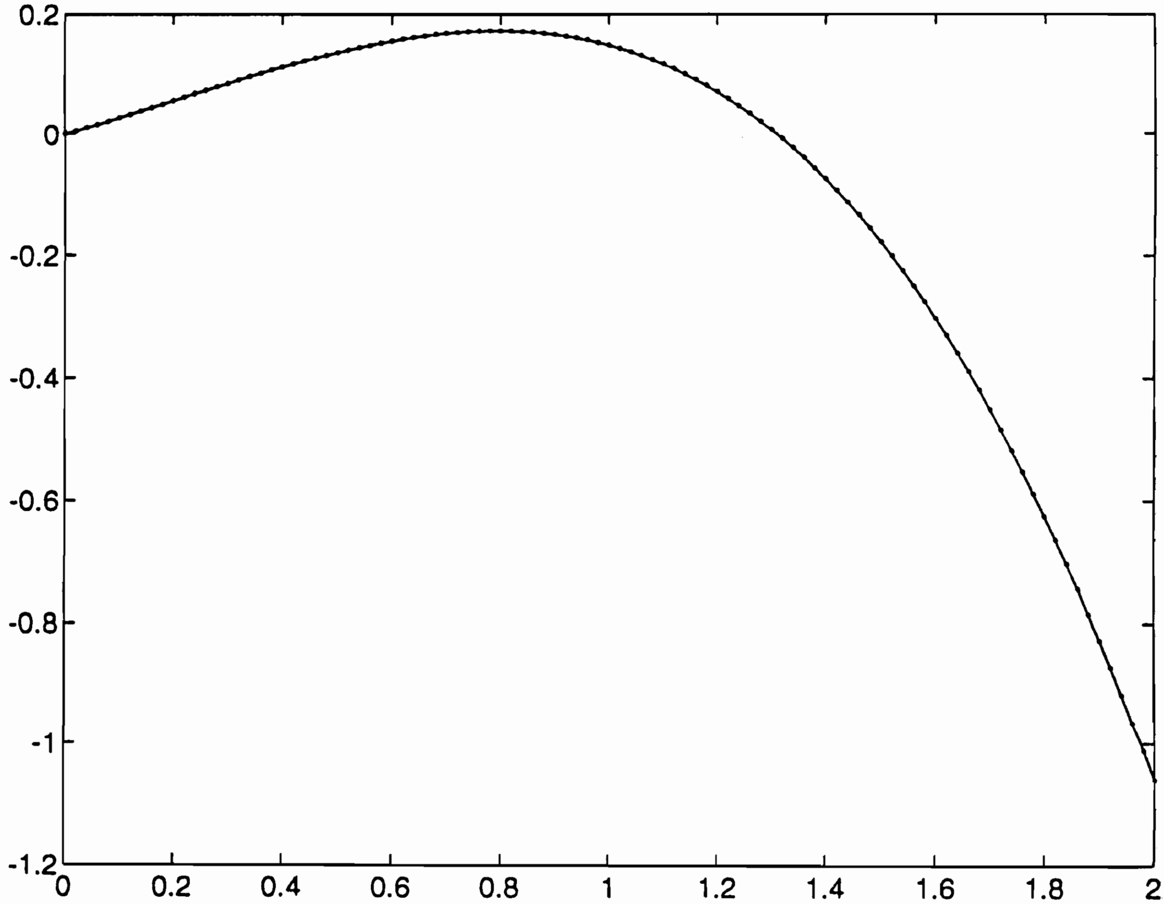


Figure 2.14 Velocity Gains(5:4) for Beam 1:n=3(.),n=4(-.),n=8(-)

Figure 2.11 Velocity Gains(5:1) for Beam 1:n=3(.),n=4(-.),n=8(-)

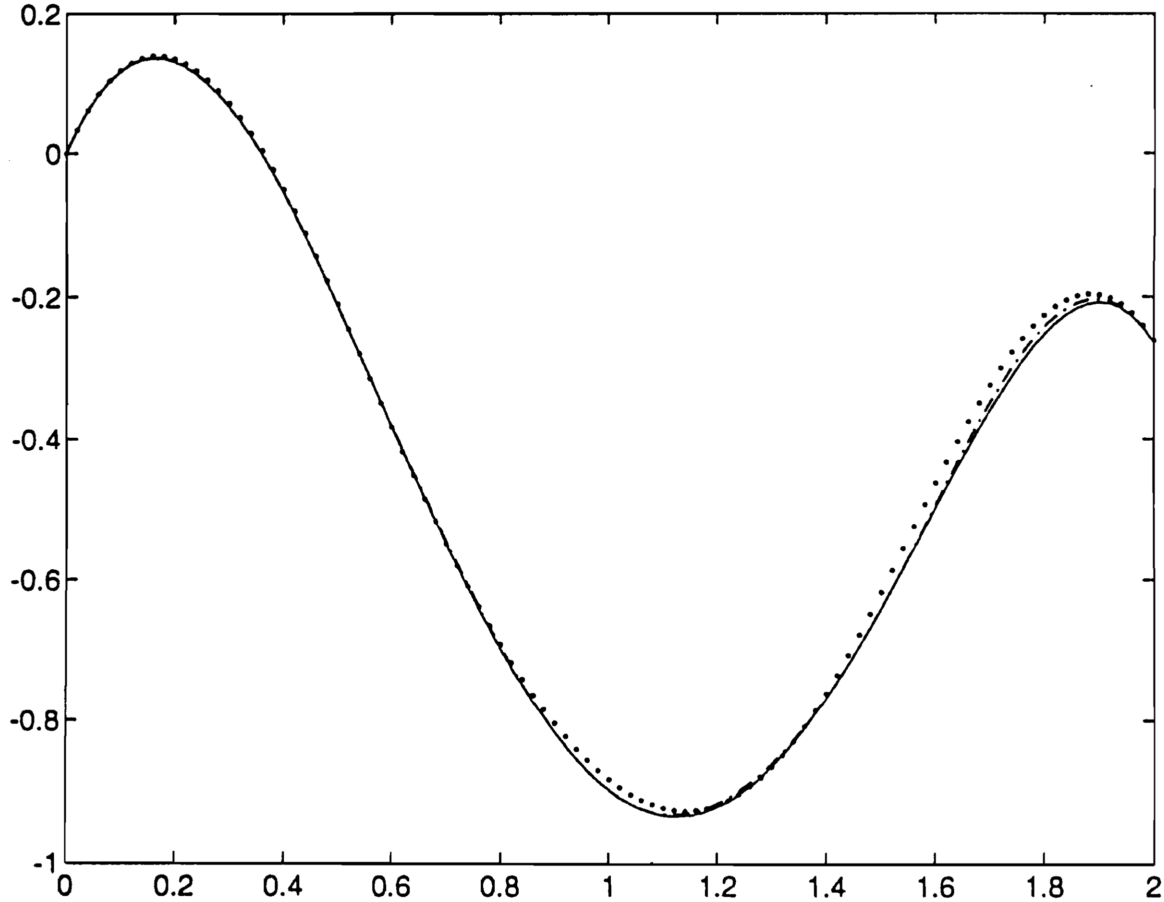


Figure 2.15 Velocity Gains(5:5) for Beam 1:n=3(.),n=4(-.),n=8(-)

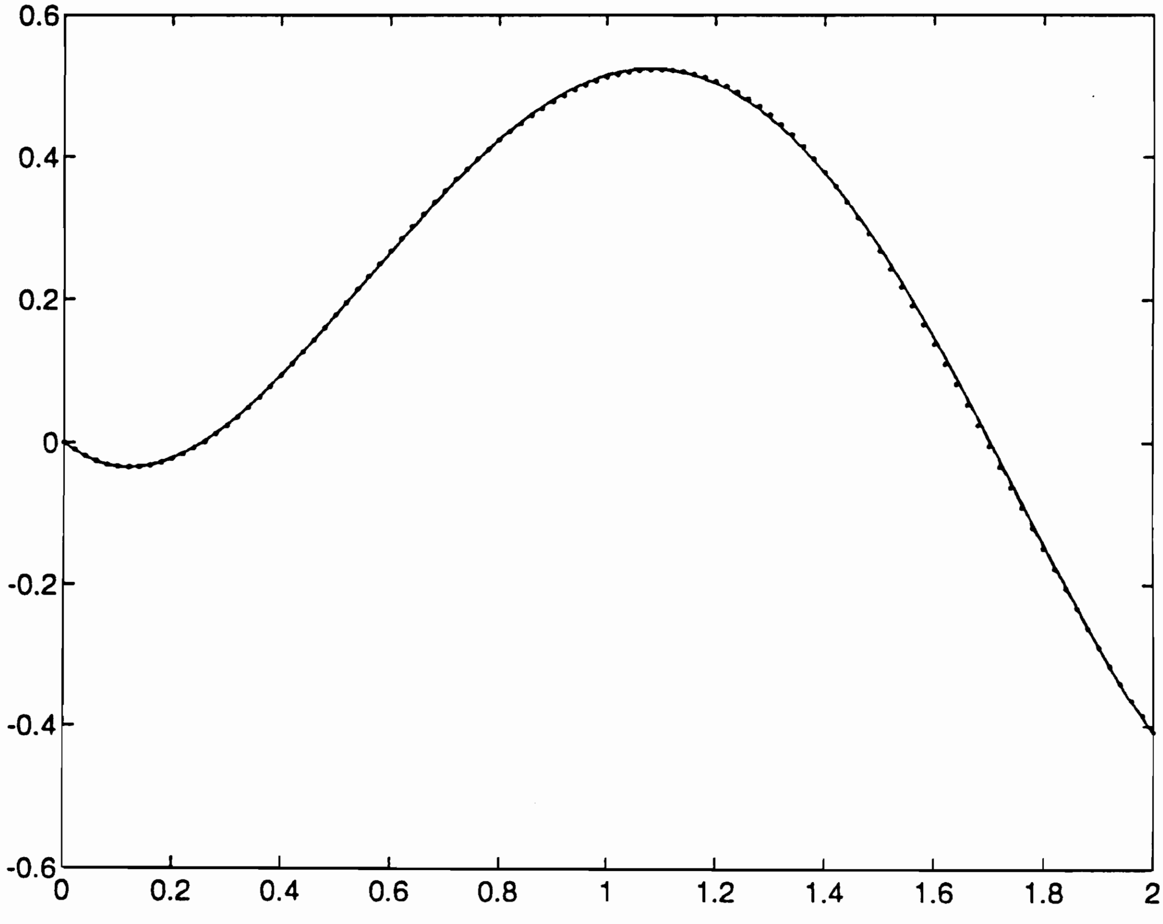


Figure 2.16 Velocity Gains(5:1) for Beam 2:n=3(.),n=4(-.),n=8(-)

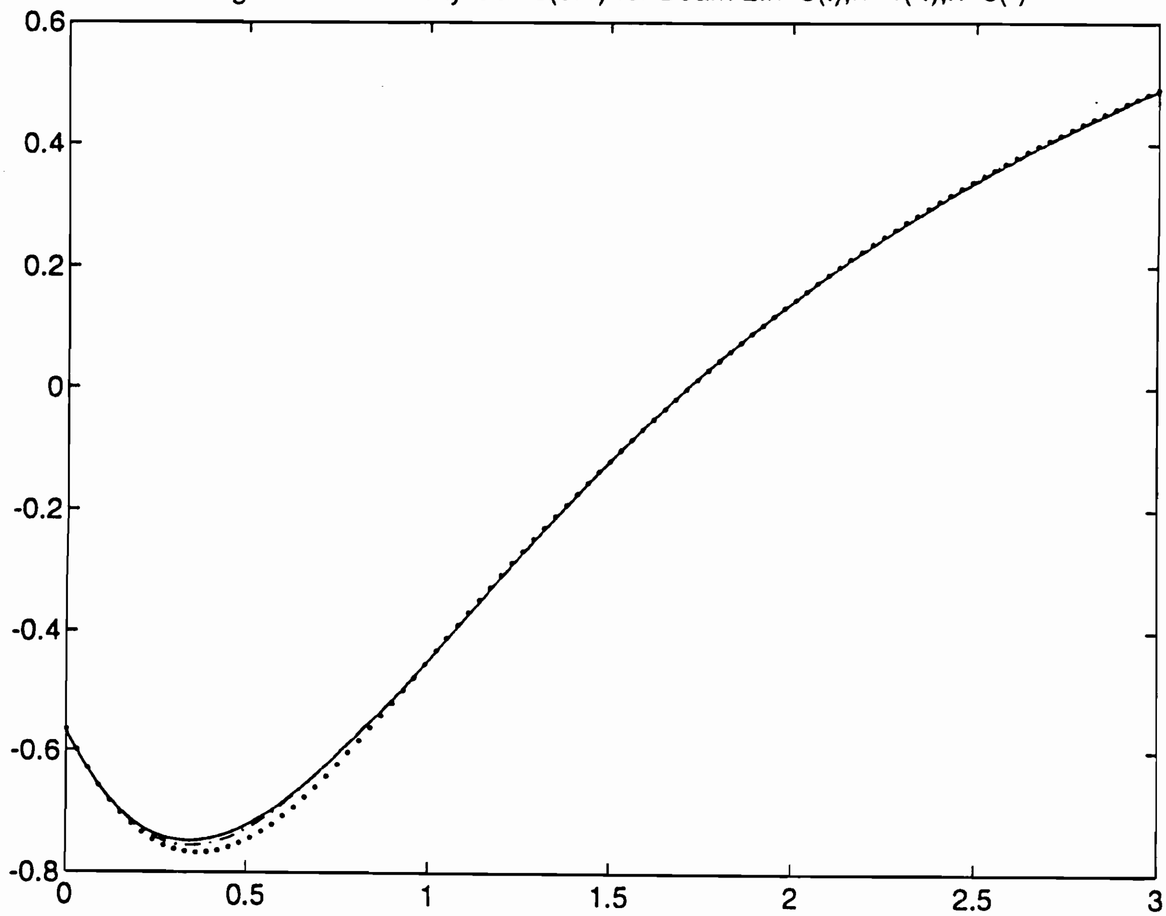


Figure 2.17 Velocity Gains(5:2) for Beam 2:n=3(.),n=4(-.),n=8(-)

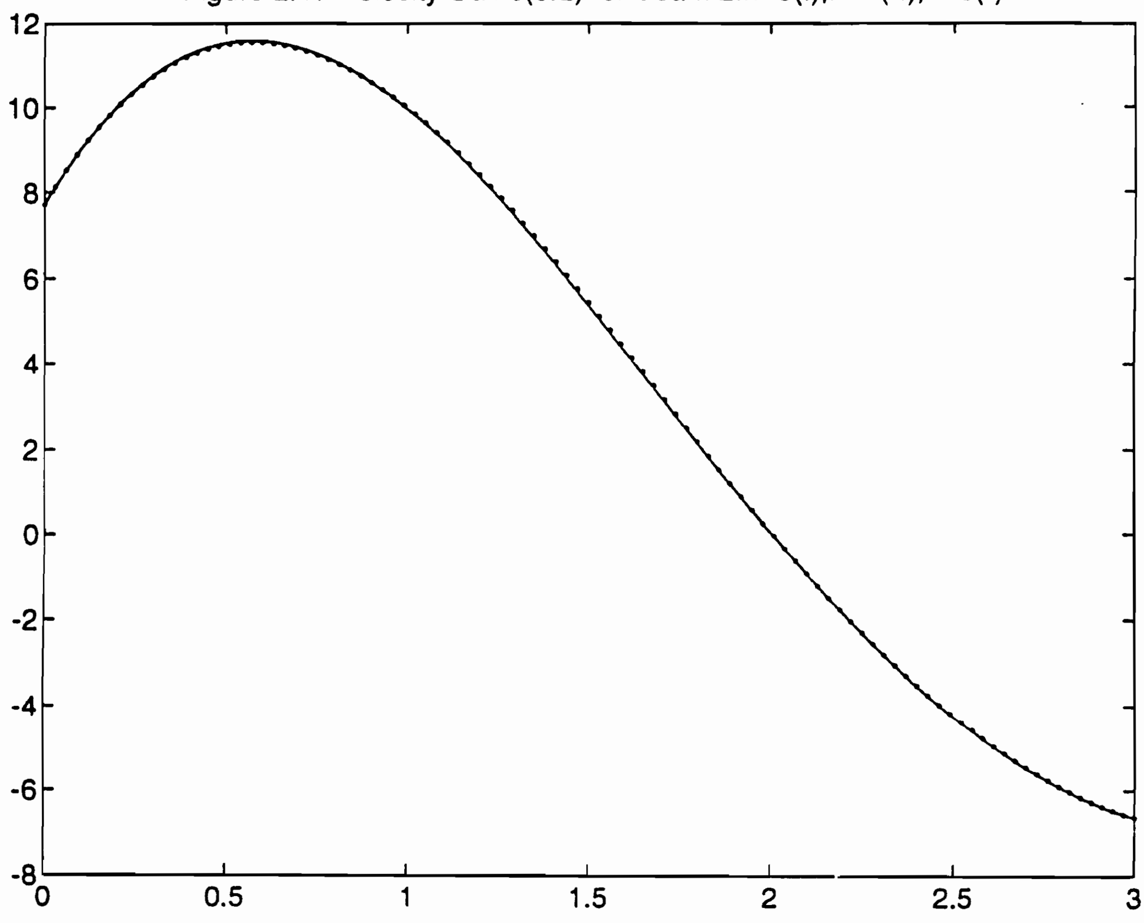


Figure 2.18 Velocity Gains(5:3) for Beam 2:n=3(.),n=4(-.),n=8(-)

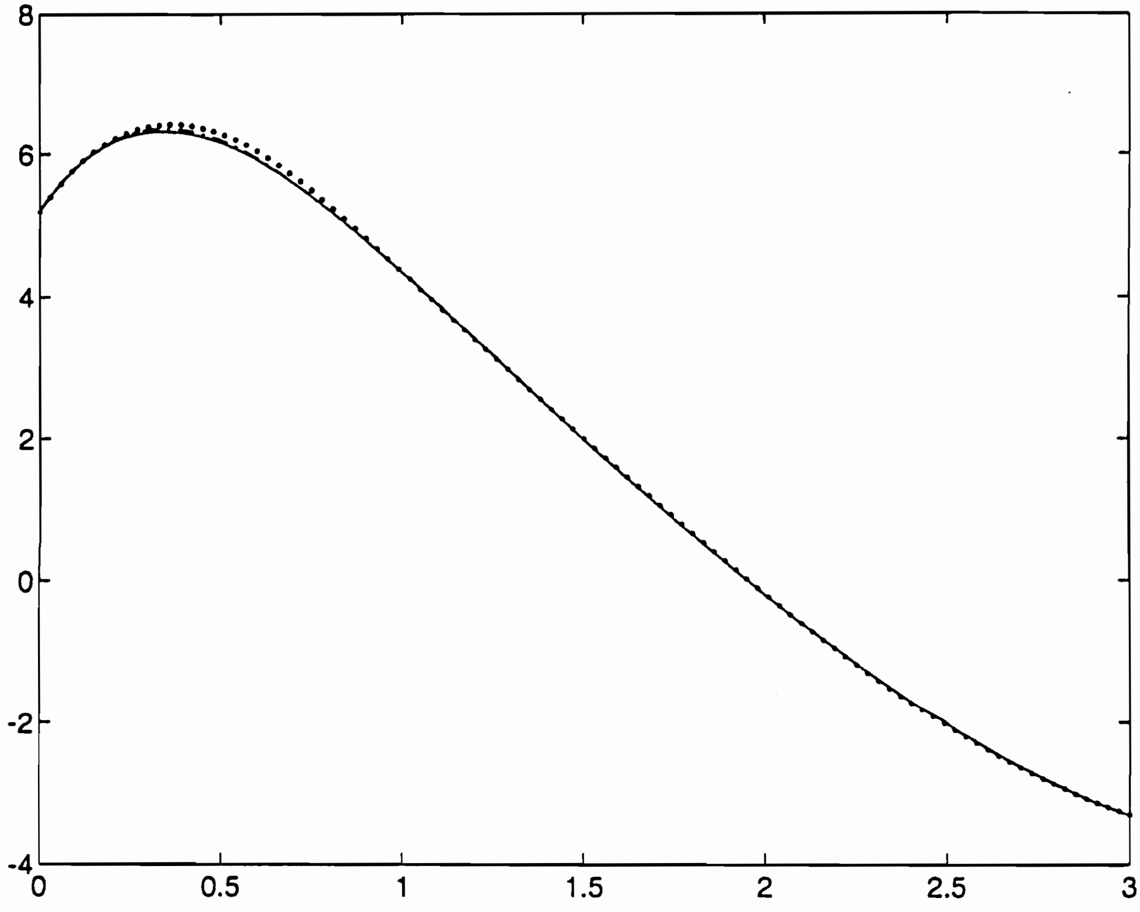


Figure 2.19 Velocity Gains(5:4) for Beam 2:n=3(.),n=4(-.),n=8(-)

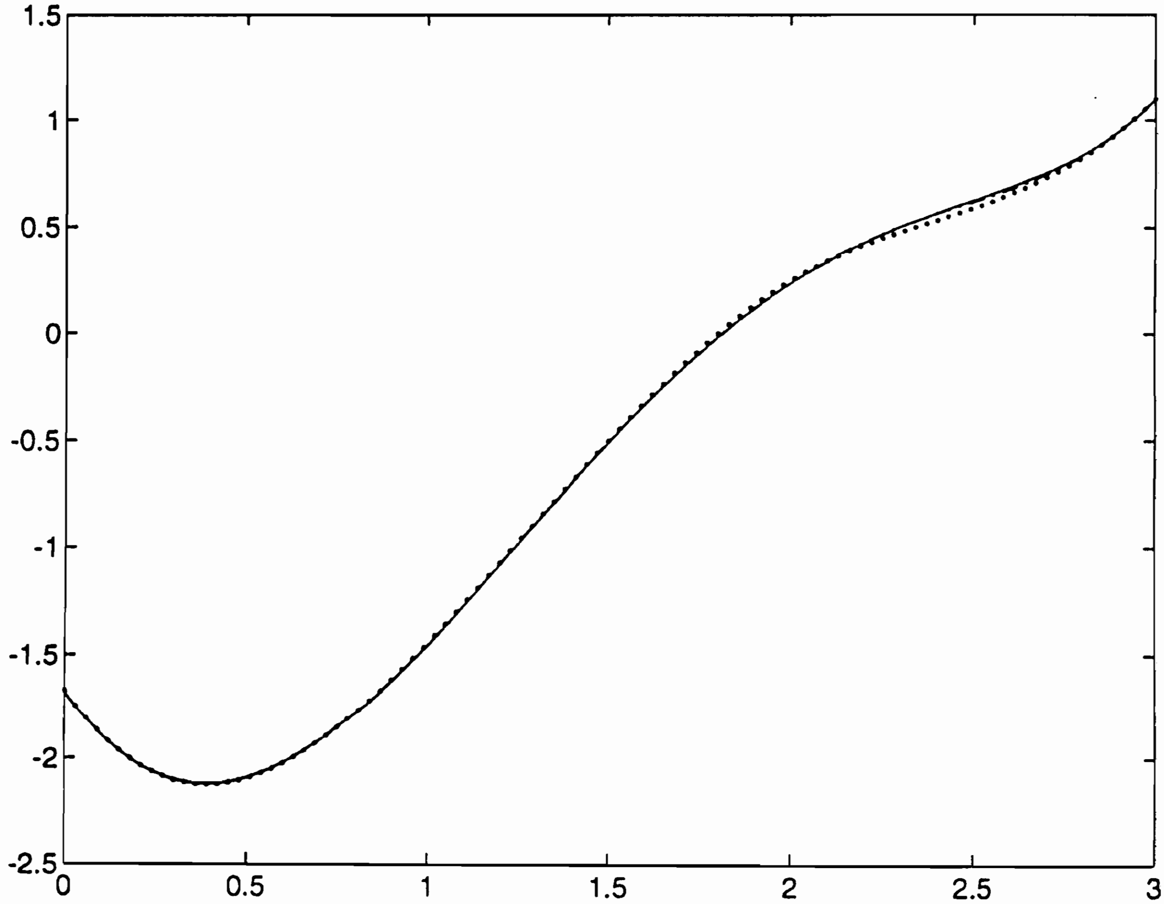


Figure 2.20 Velocity Gains(5:5) for Beam 2:n=3(.),n=4(-.),n=8(-)

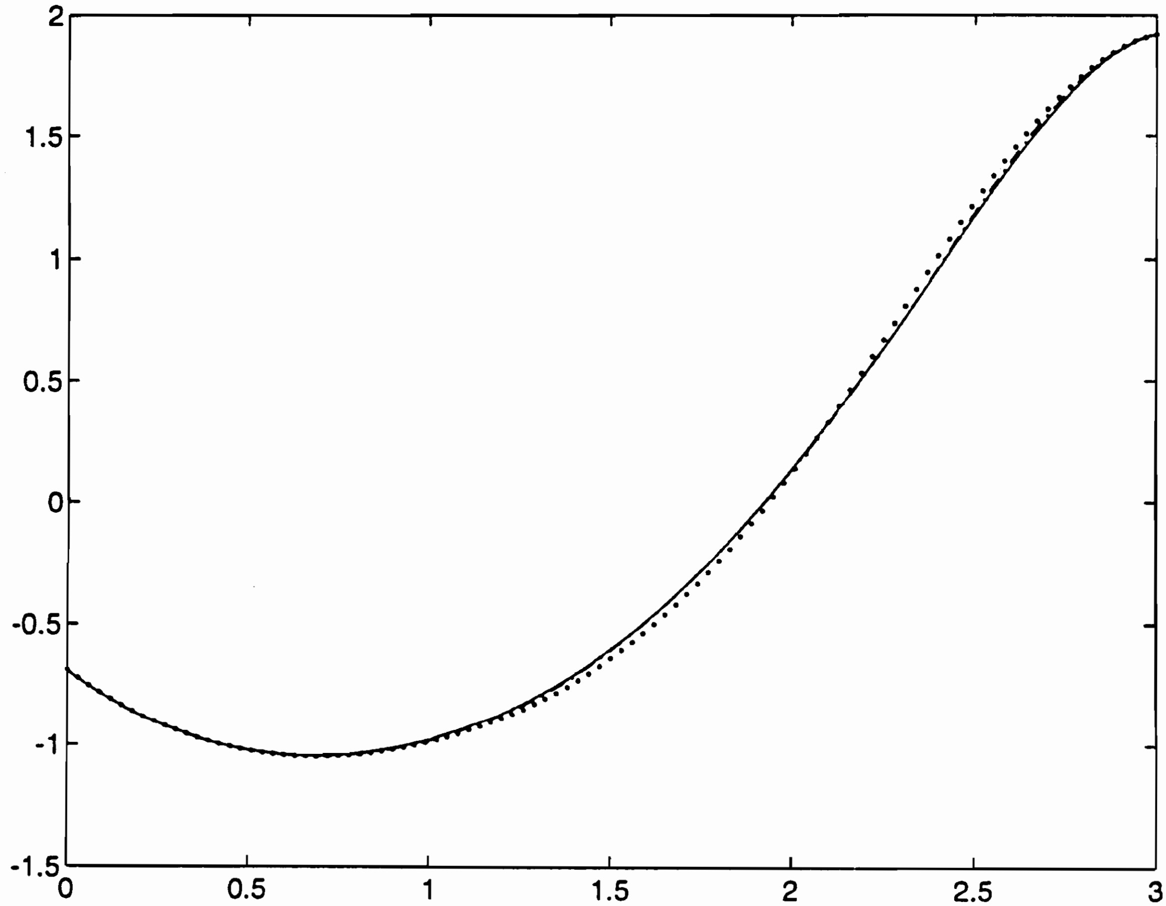


Figure 2.21 Estimator Gains L2 (5:1): n=3(.),n=4(-.),n=8(-)

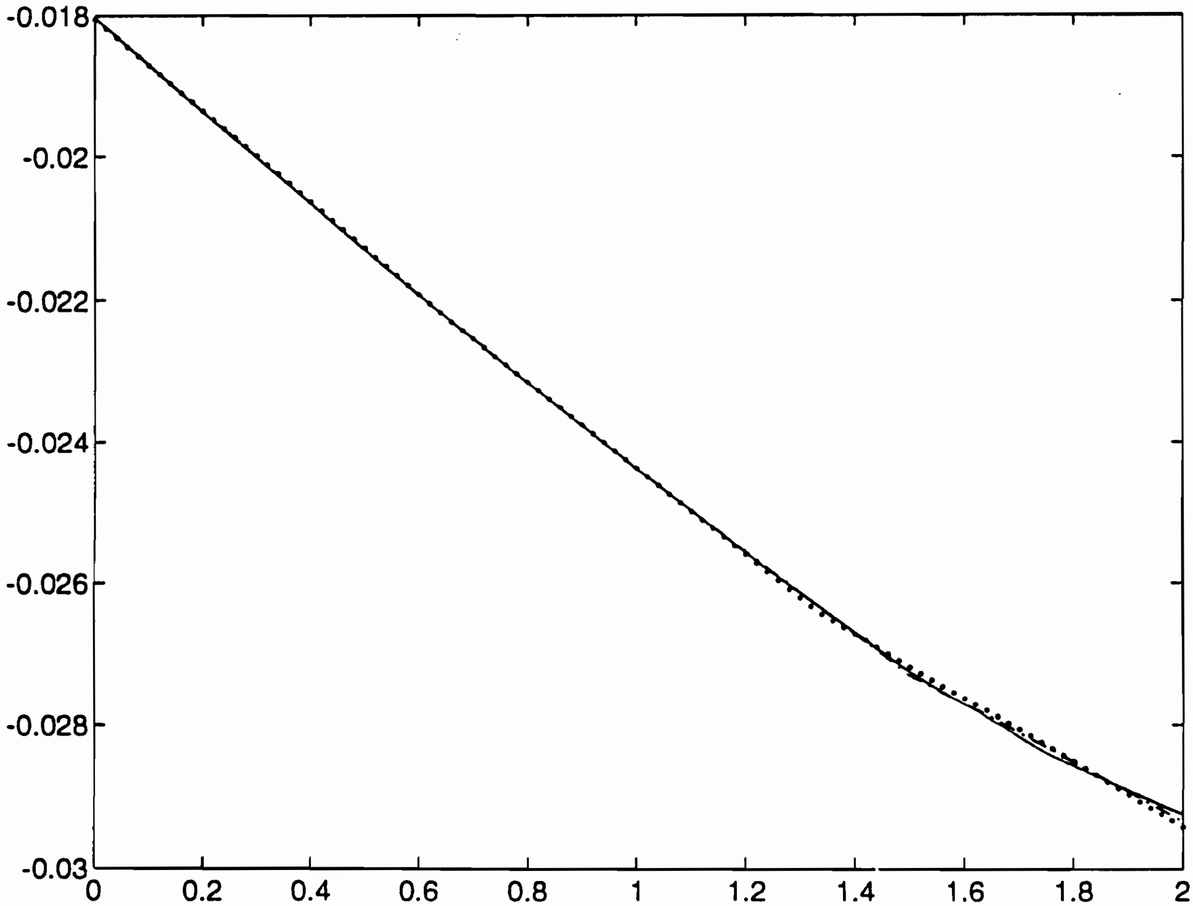


Figure 2.22 Estimator Gains L2 (5:2): n=3(.),n=4(-.),n=8(-)

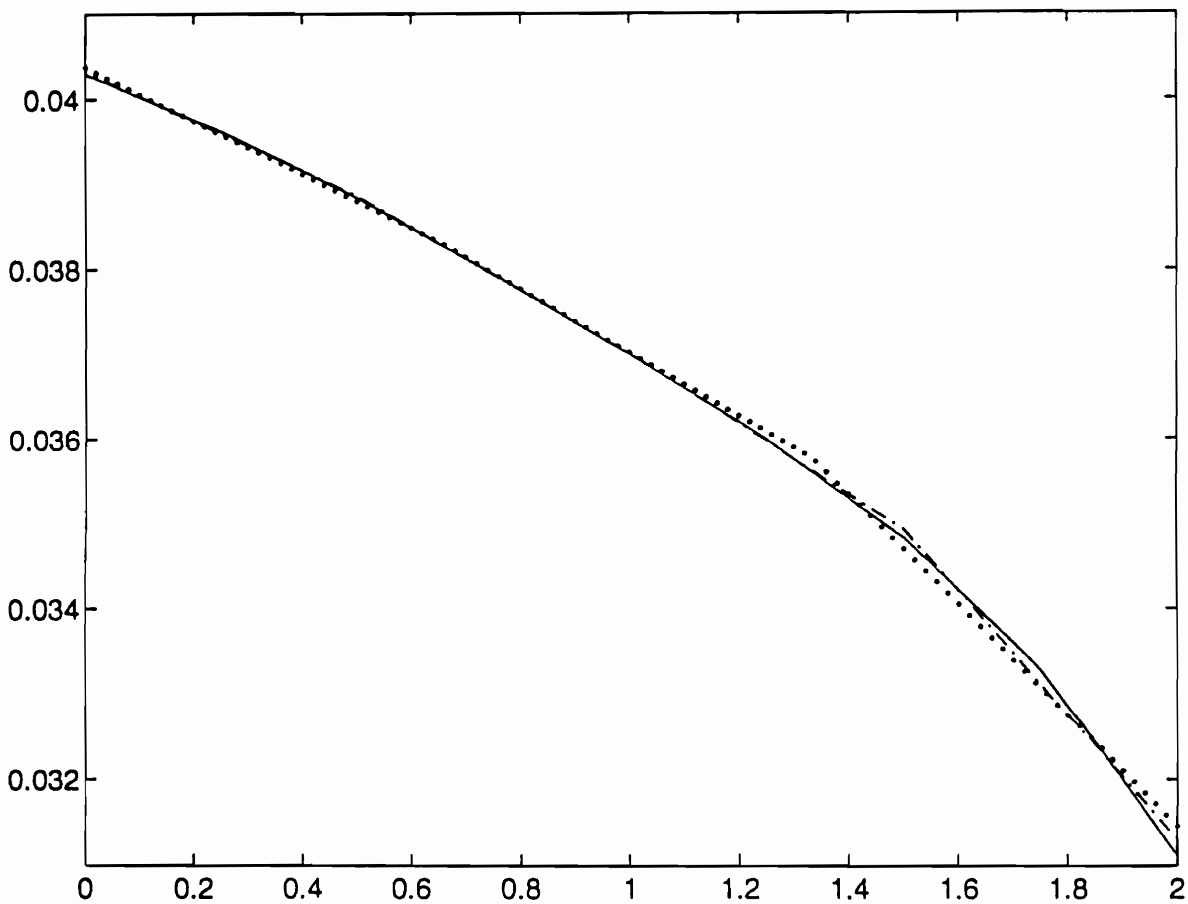
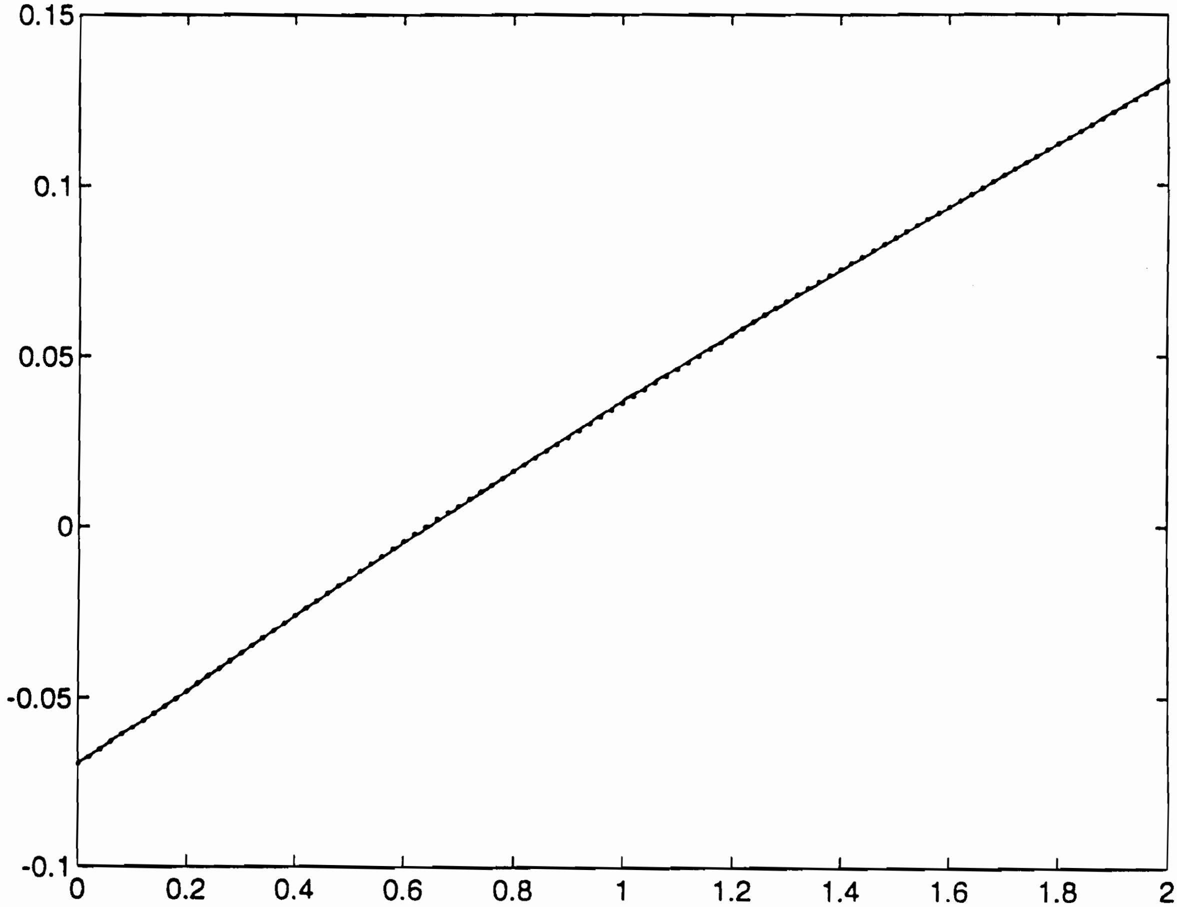


Figure 2.23 Estimator Gains L2 (5:3): n=3(.),n=4(-.),n=8(-)



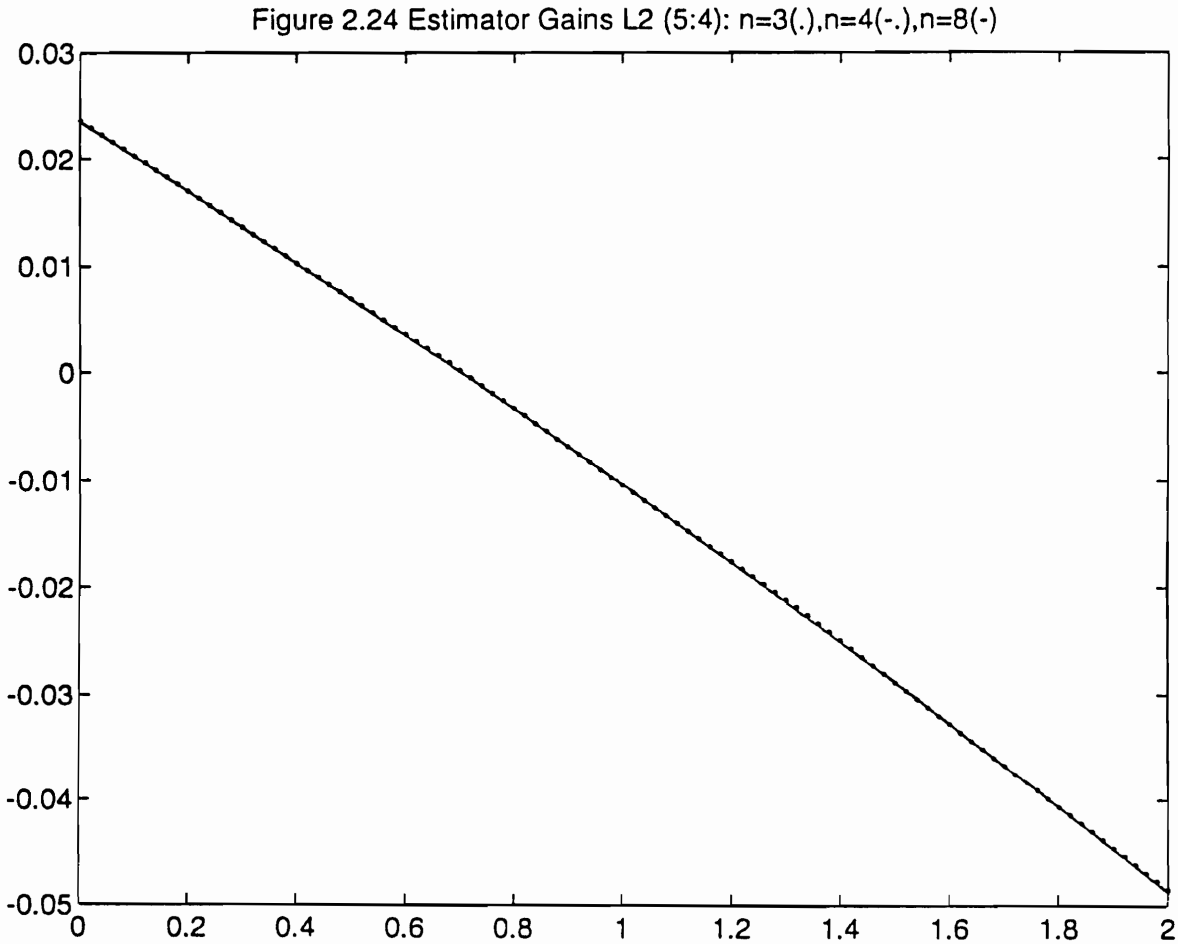

0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Figure 2.25 Estimator Gains L2 (5:5): n=3(.),n=4(-.),n=8(-)

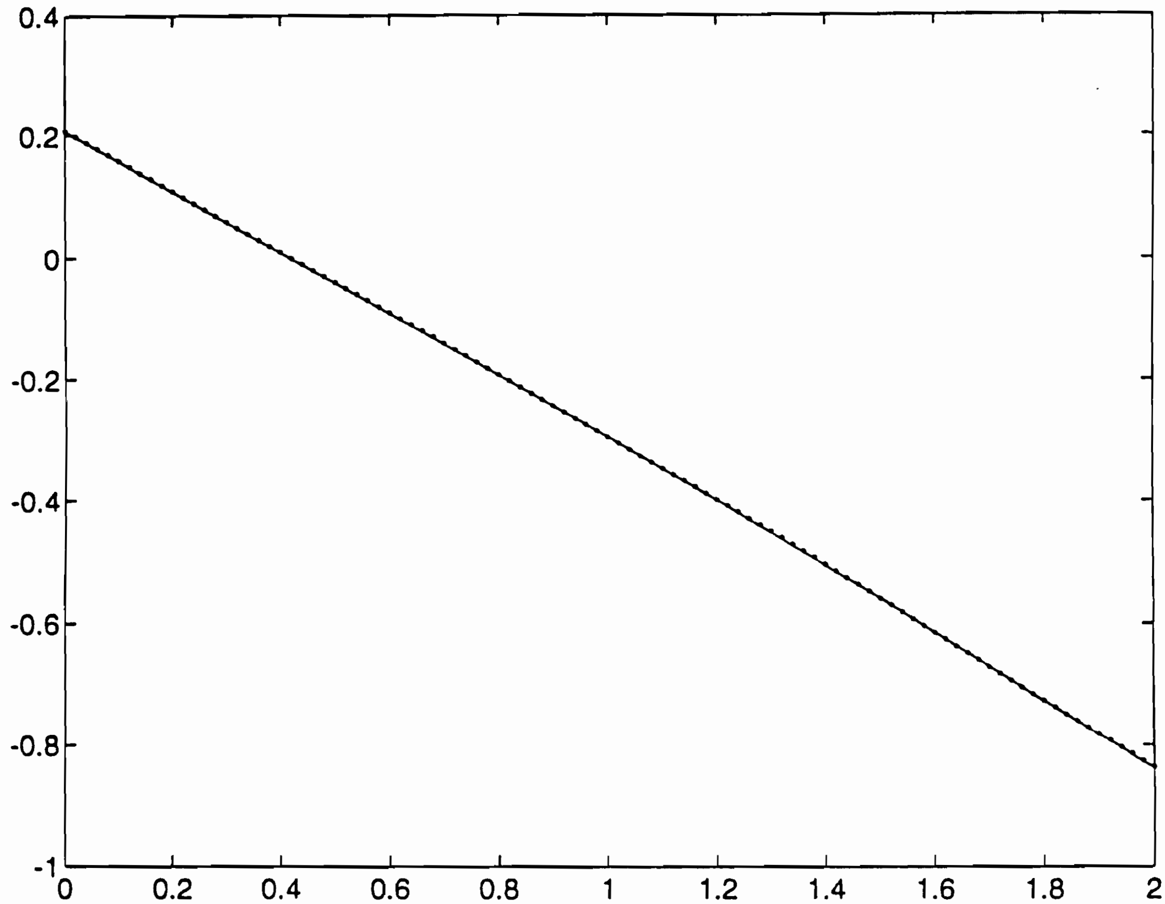
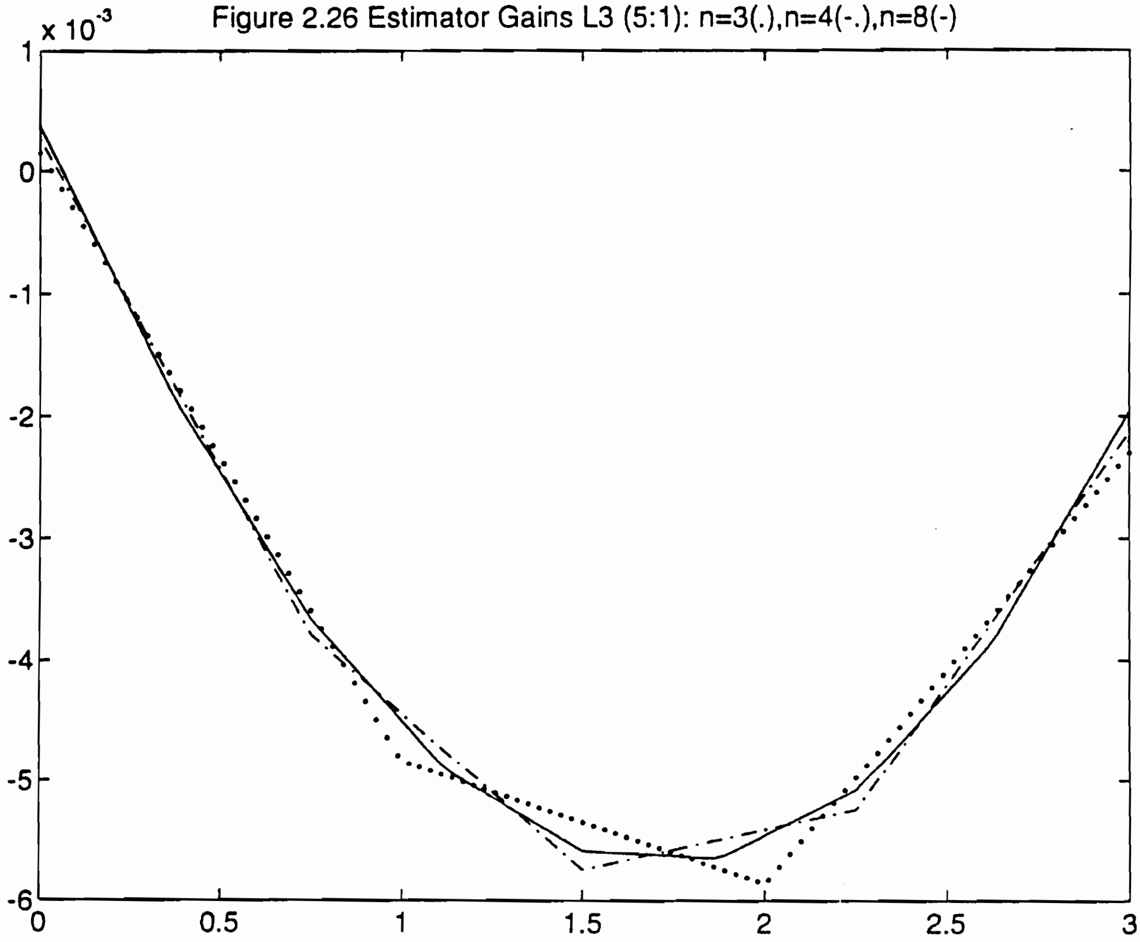


Figure 2.26 Estimator Gains L3 (5:1): n=3(.),n=4(-.),n=8(-)



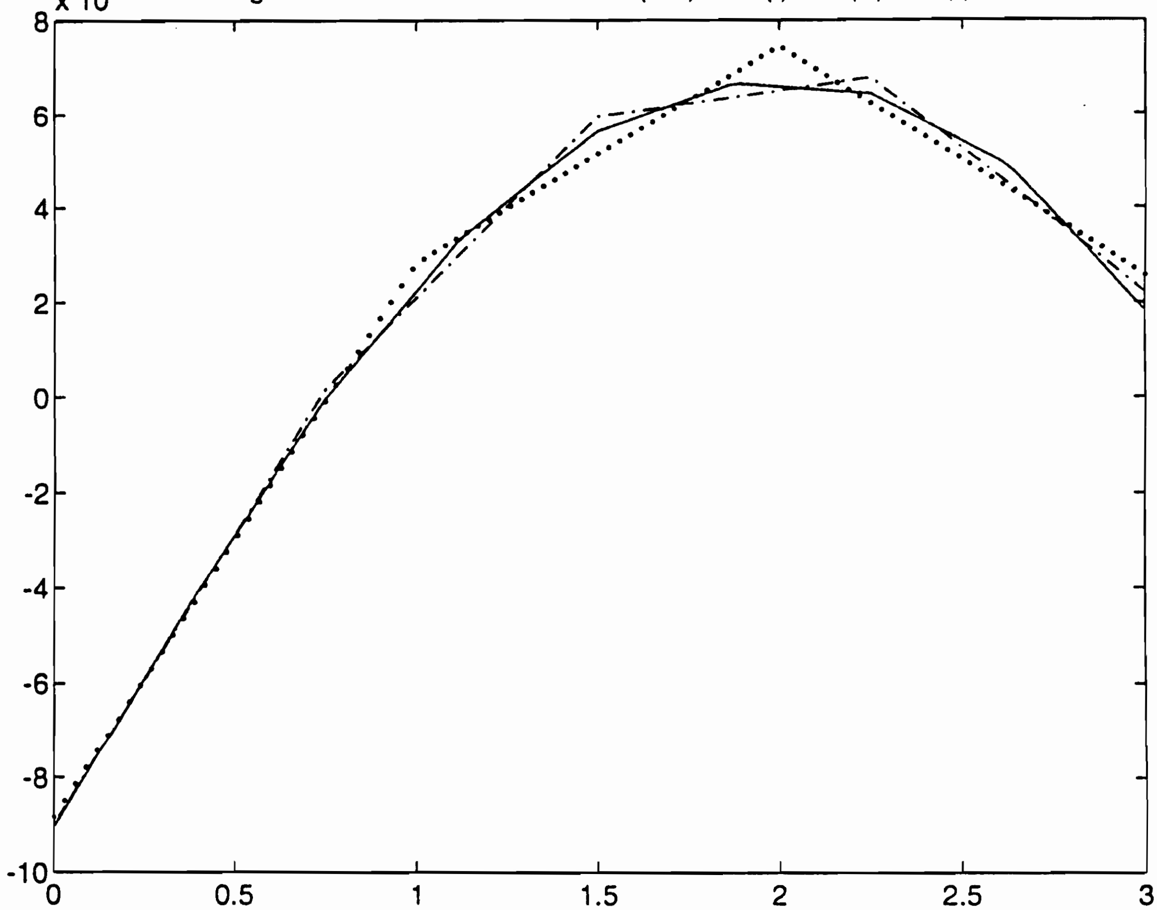


Figure 2.27 Estimator Gains L3 (5:2): n=3(.),n=4(-.),n=8(-)

Figure 2.28 Estimator Gains L3 (5:3): n=3(.),n=4(-.),n=8(-)

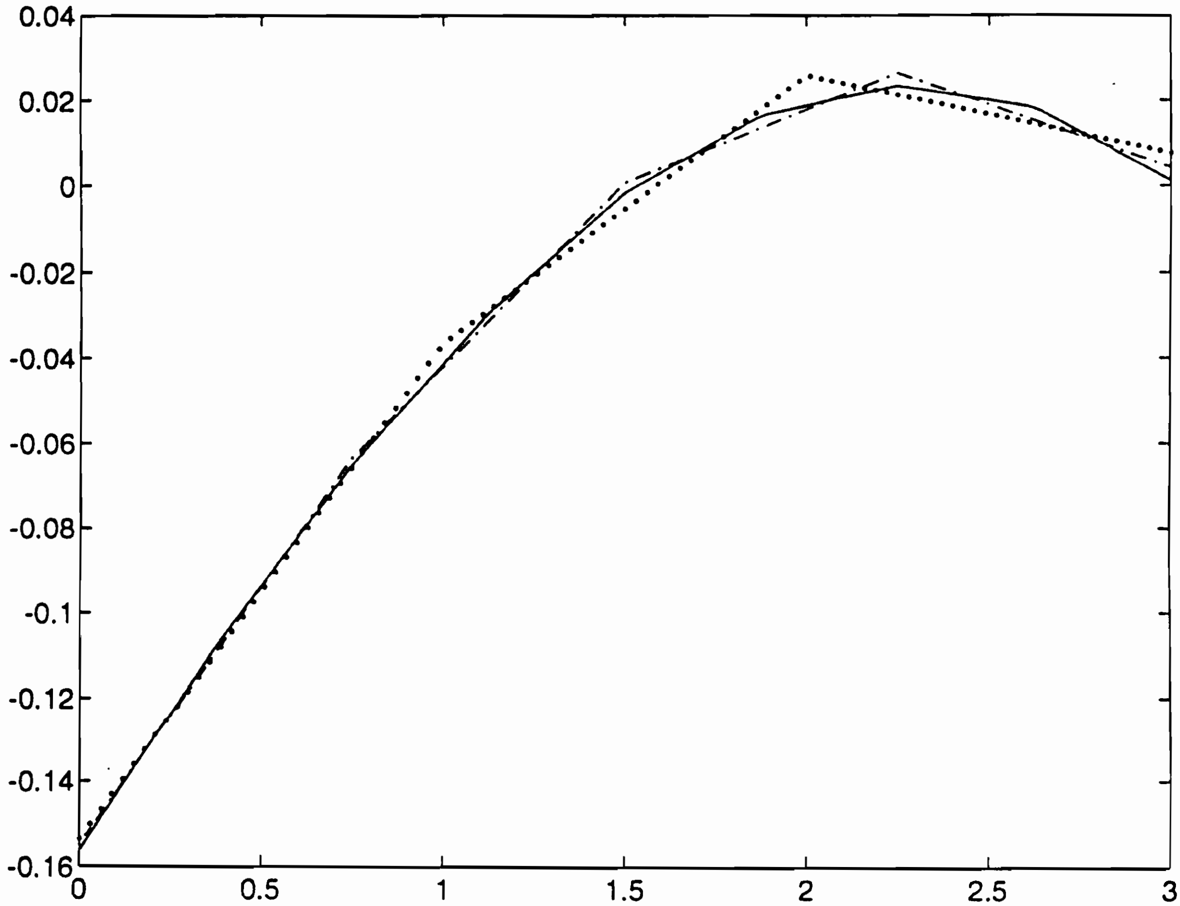


Figure 2.29 Estimator Gains L3 (5:4): n=3(.),n=4(-.),n=8(-)

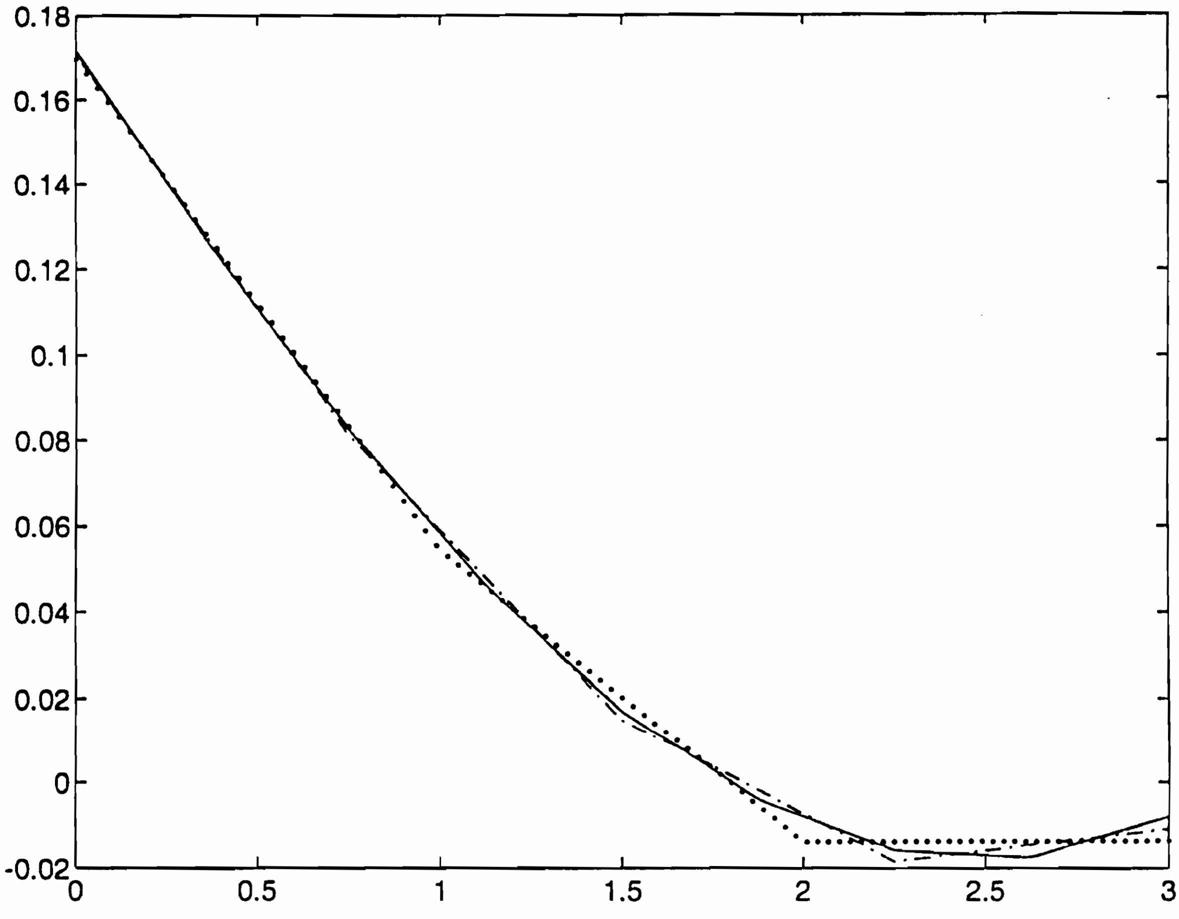


Figure 2.30 Estimator Gains L3 (5:5): n=3(.),n=4(-.),n=8(-)

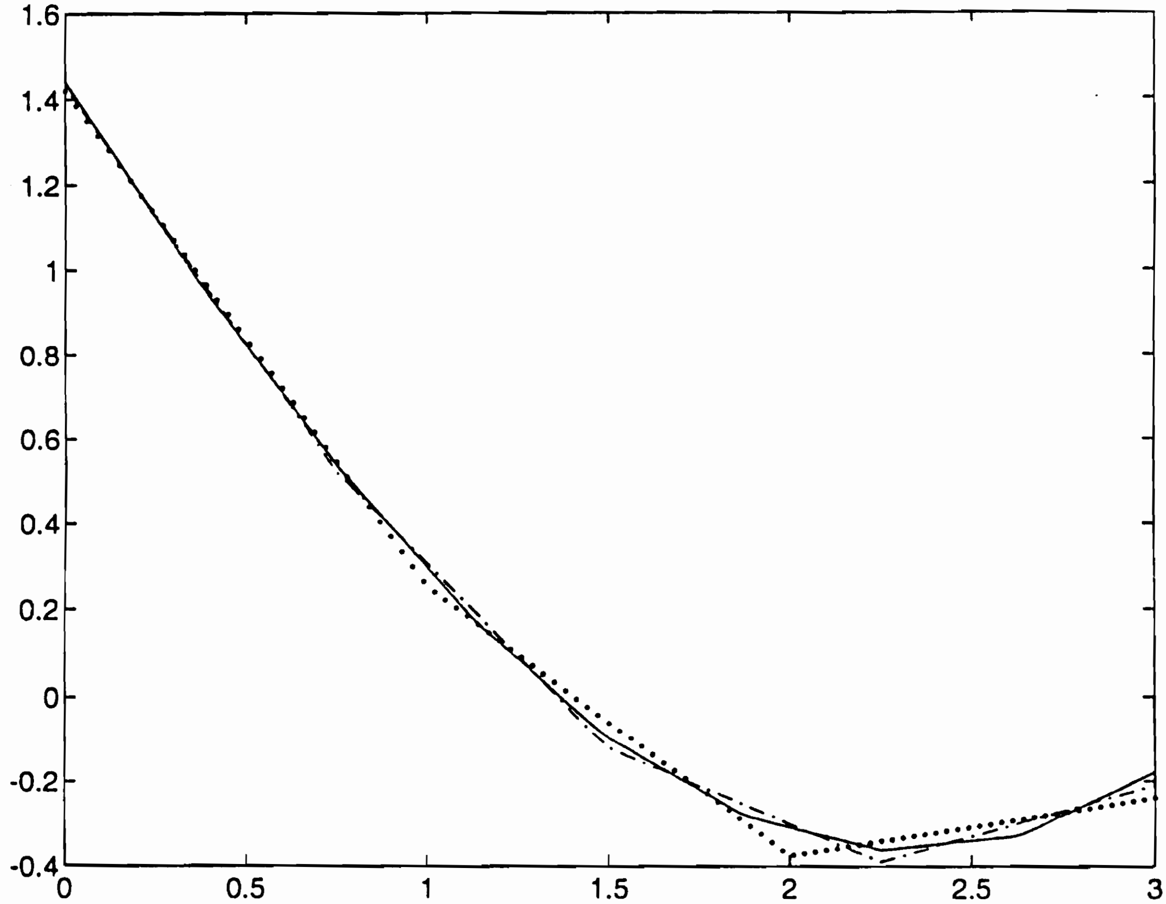


Figure 2.31 Estimator Gains L5 (5:1): n=3(.),n=4(-.),n=8(-)

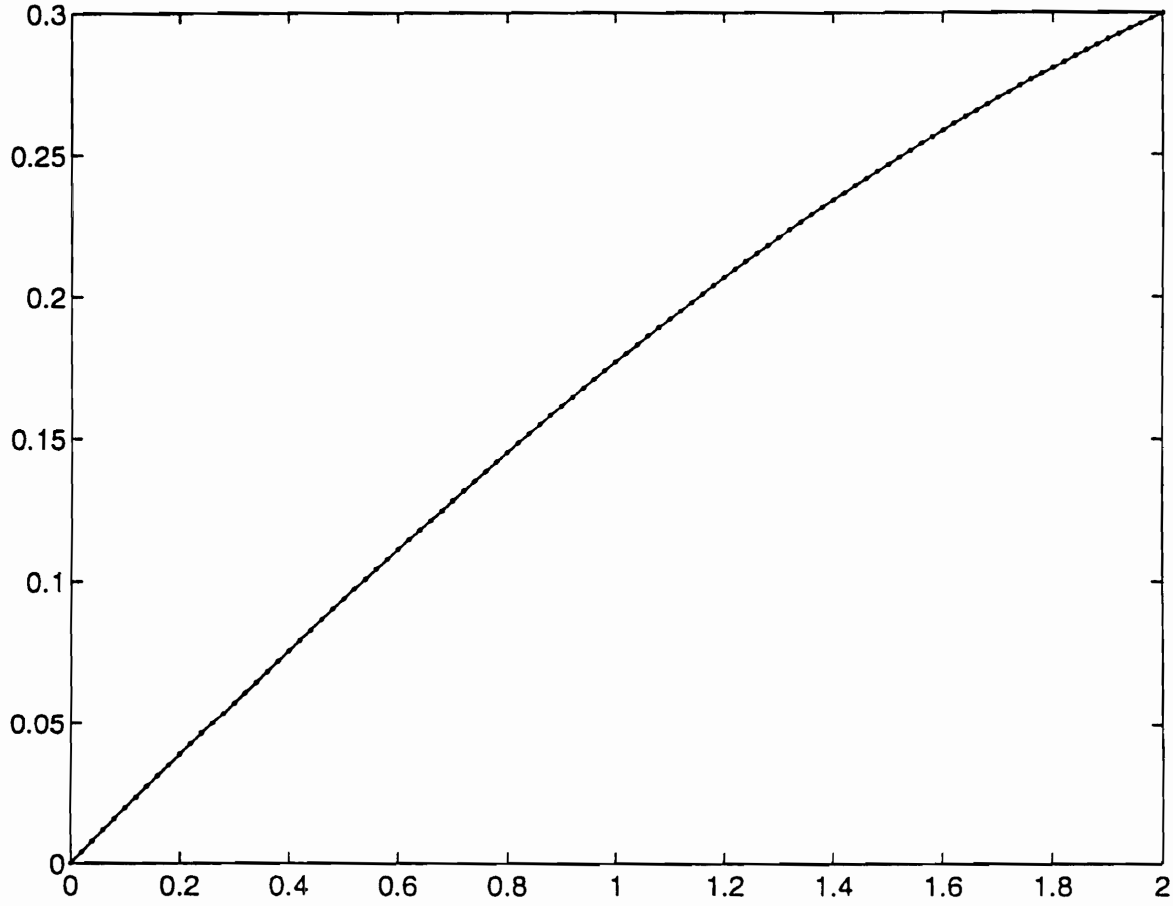


Figure 2.32 Estimator Gains L5 (5:2): n=3(.),n=4(-.),n=8(-)

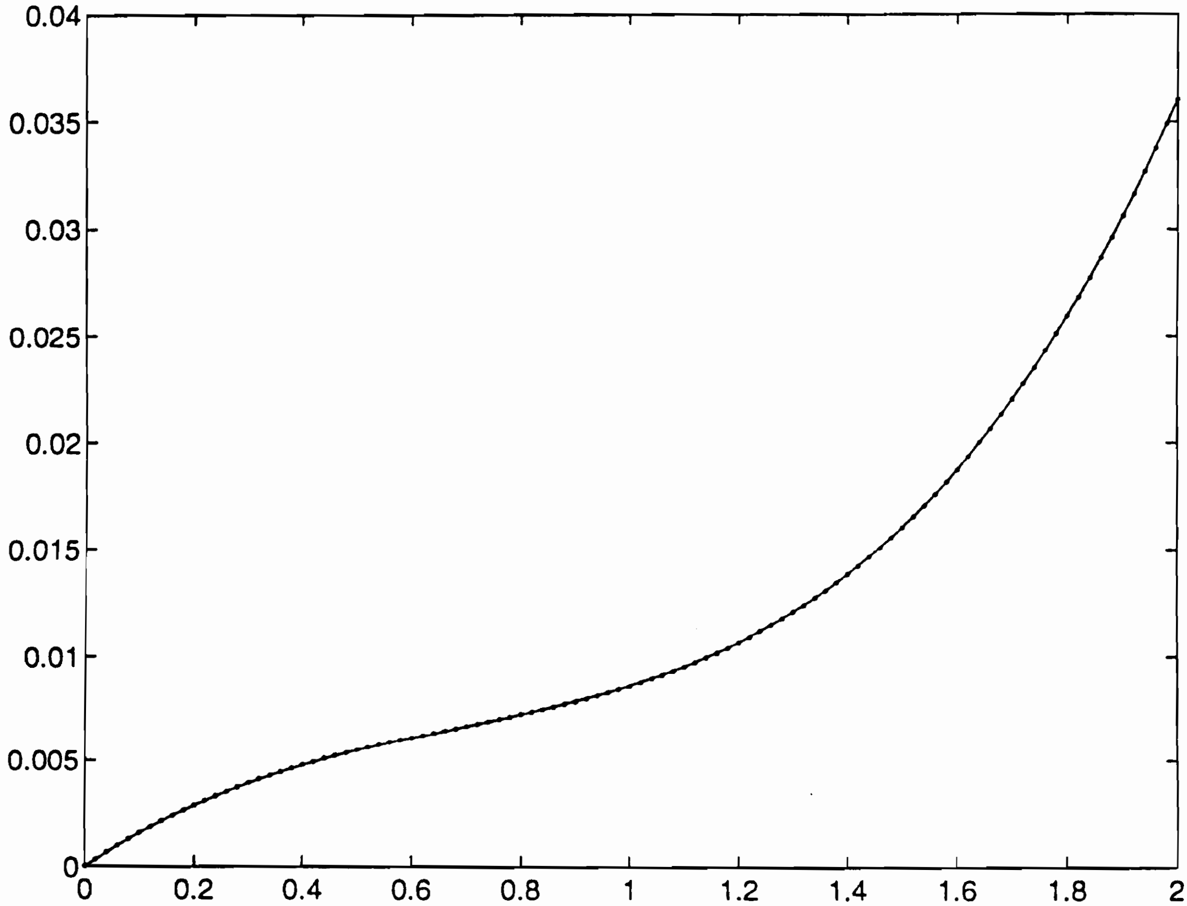


Figure 2.33 Estimator Gains L5 (5:3): n=3(.),n=4(-.),n=8(-)

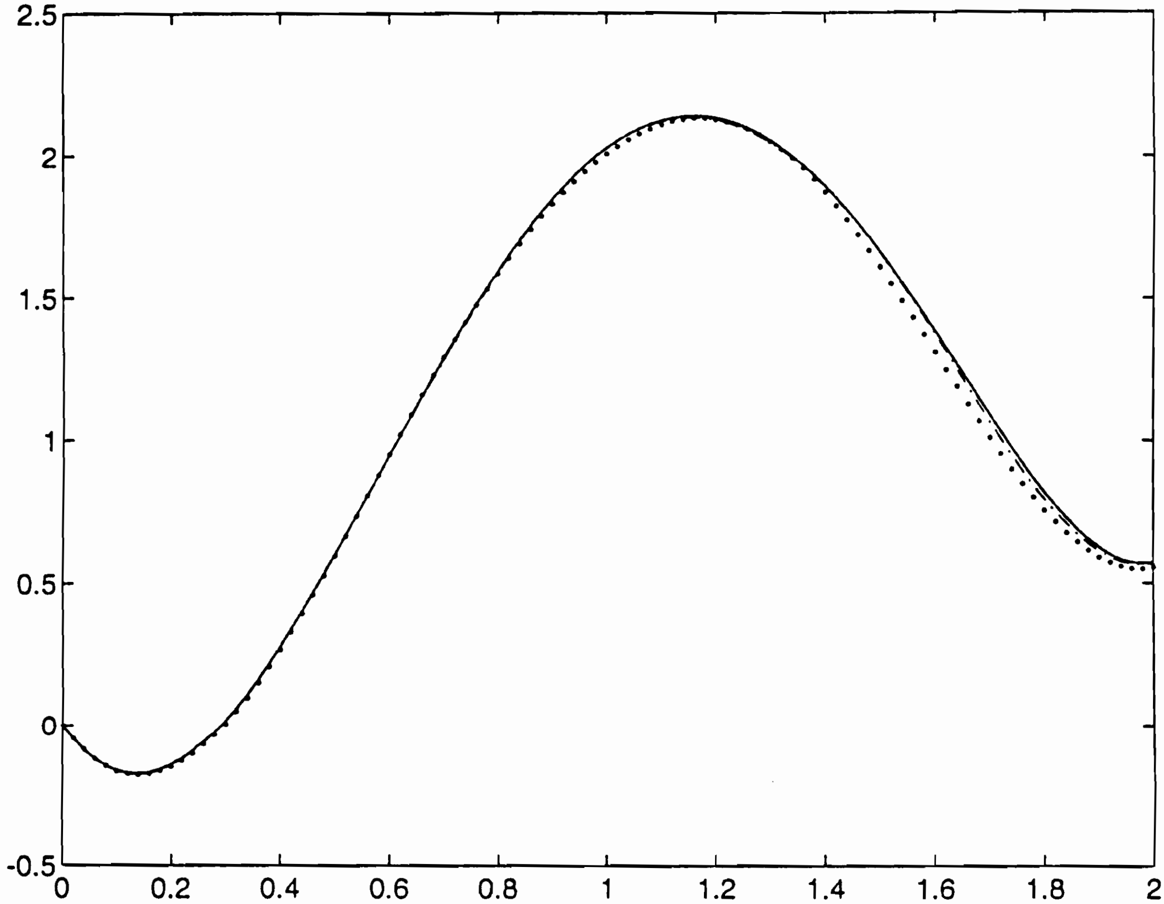


Figure 2.34 Estimator Gains L5 (5:4): n=3(.),n=4(-.),n=8(-)

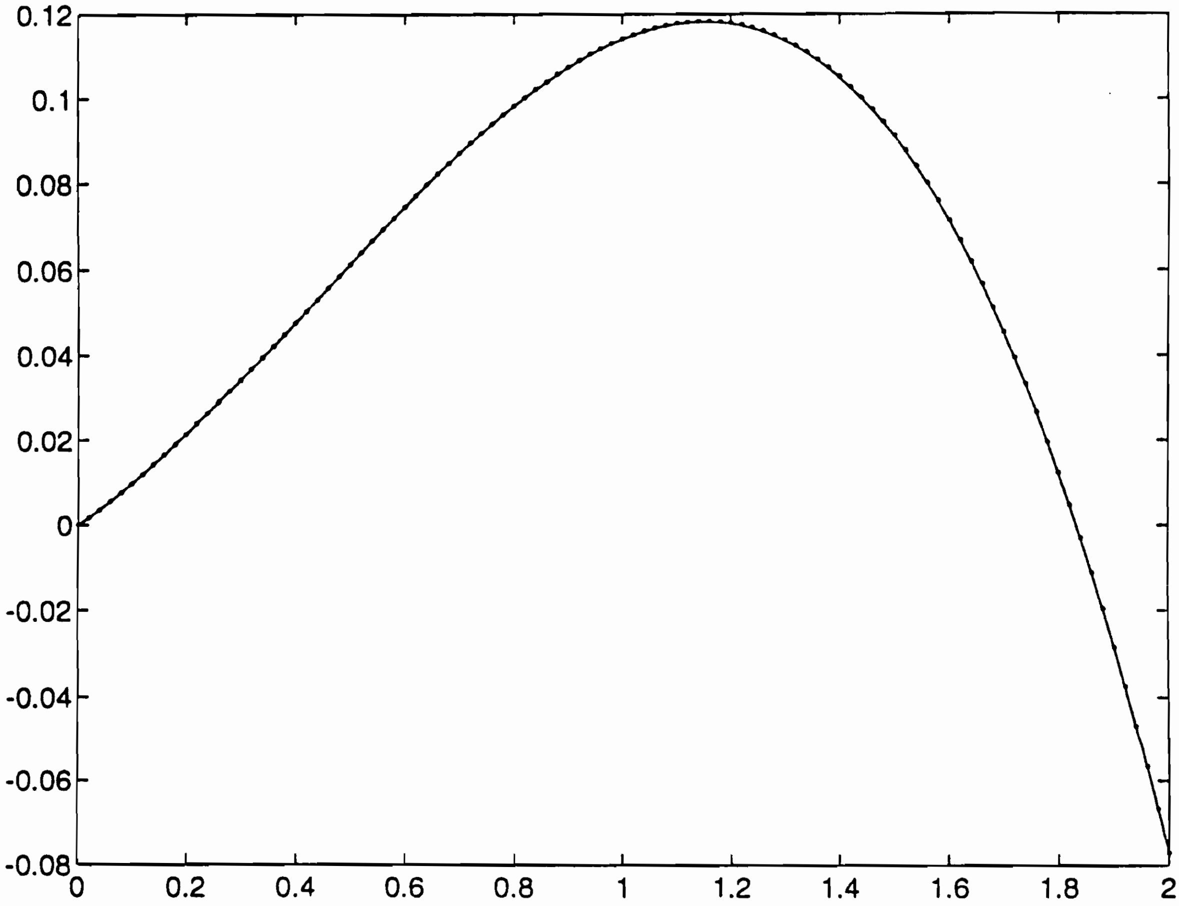


Figure 2.35 Estimator Gains L5 (5:5): n=3(.),n=4(-.),n=8(-)

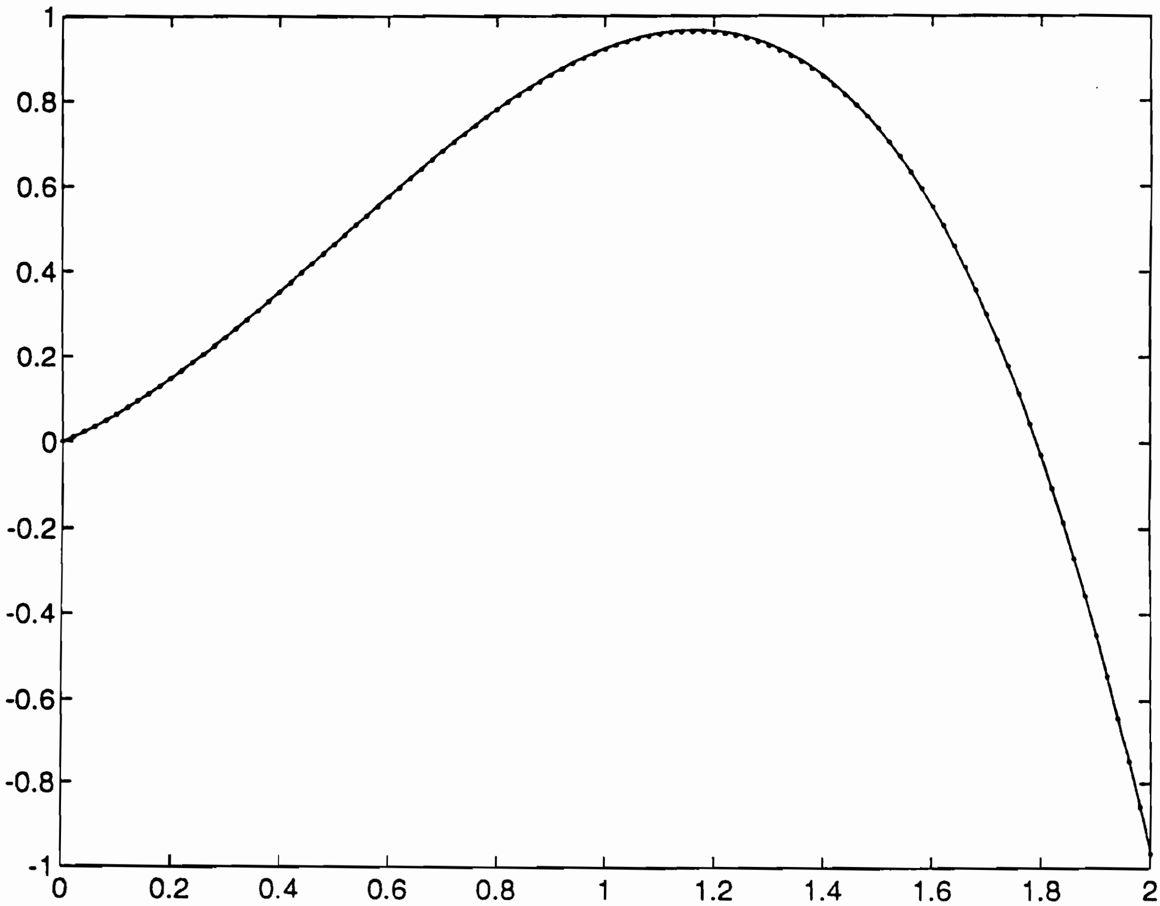


Figure 2.36 Estimator Gains L6 (5:1): n=3(.),n=4(-.),n=8(-)

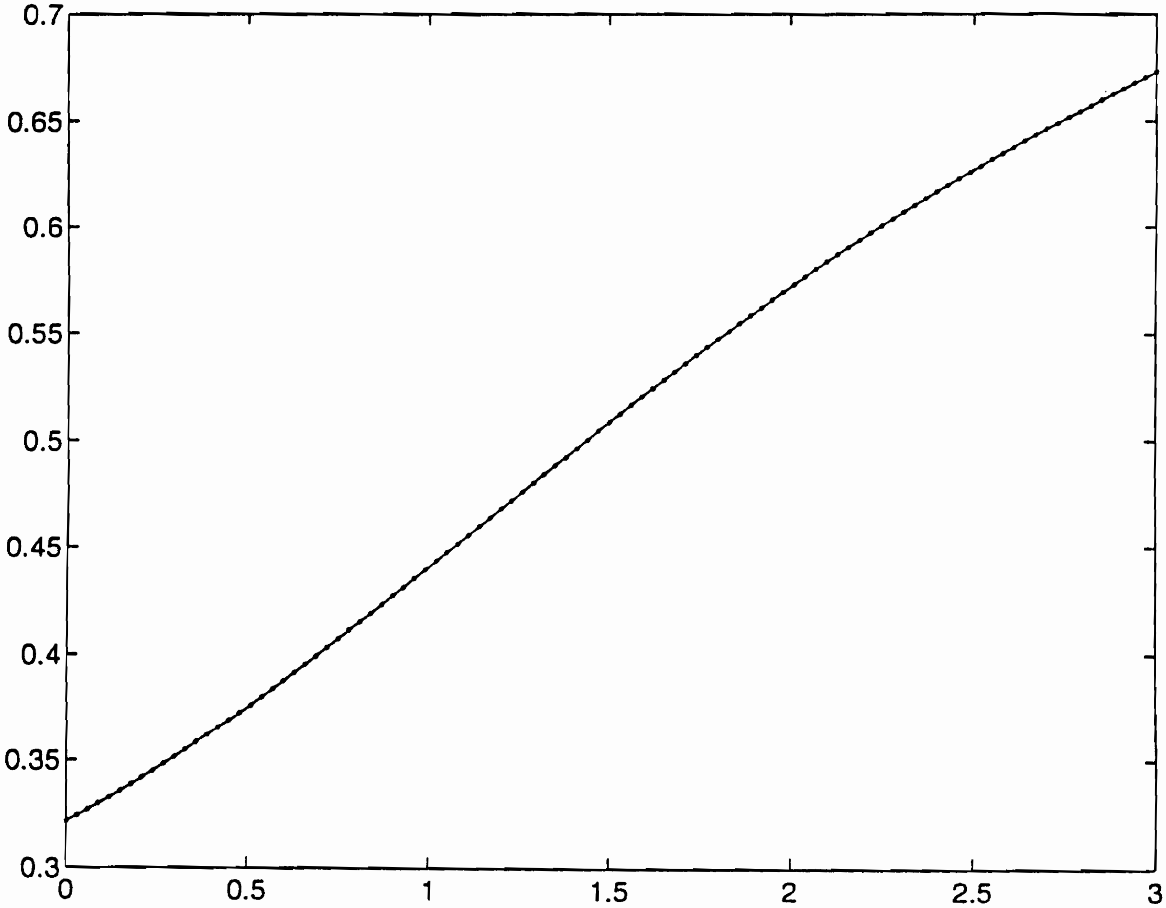


Figure 2.37 Estimator Gains L6 (5:2): n=3(.),n=4(-.),n=8(-)

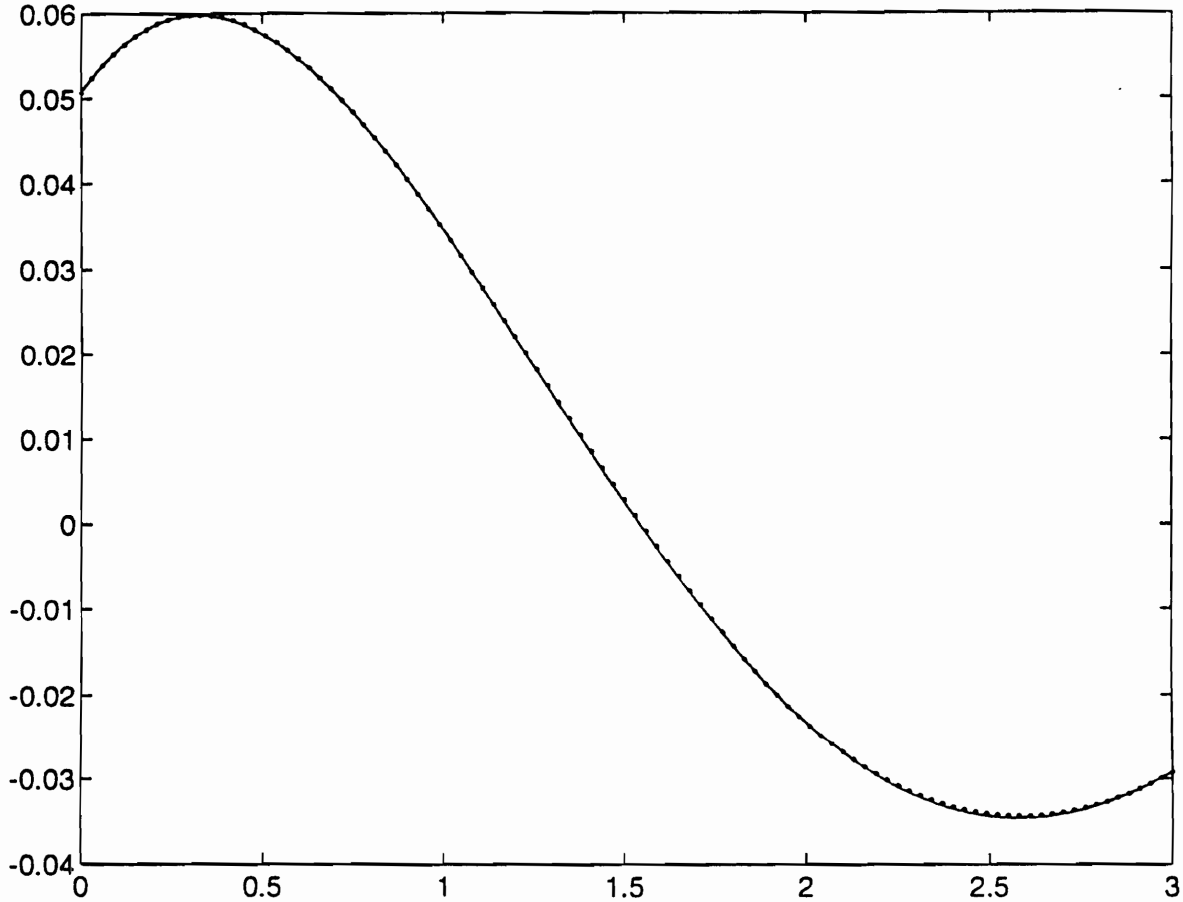


Figure 2.38 Estimator Gains L6 (5:3): n=3(.),n=4(-.),n=8(-)

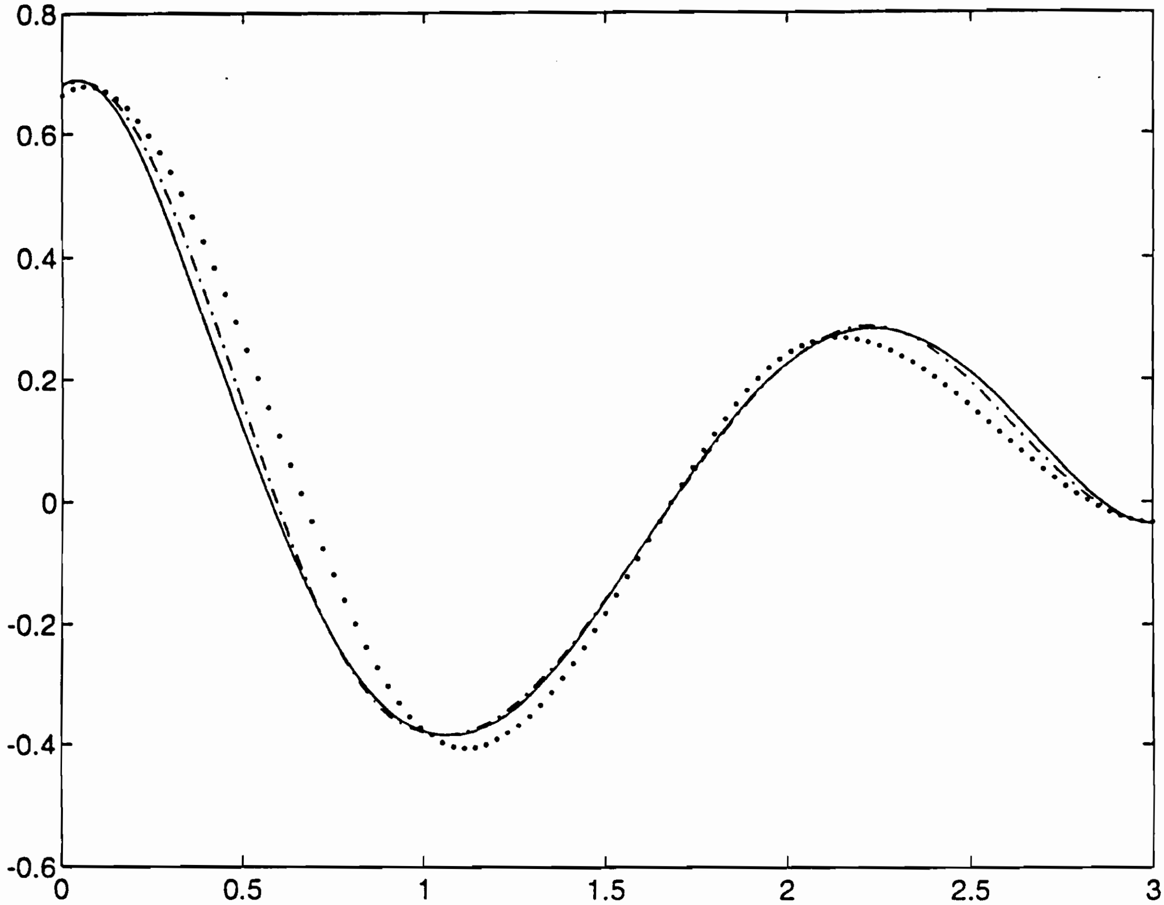


Figure 2.39 Estimator Gains L6 (5:4): n=3(.),n=4(-.),n=8(-)

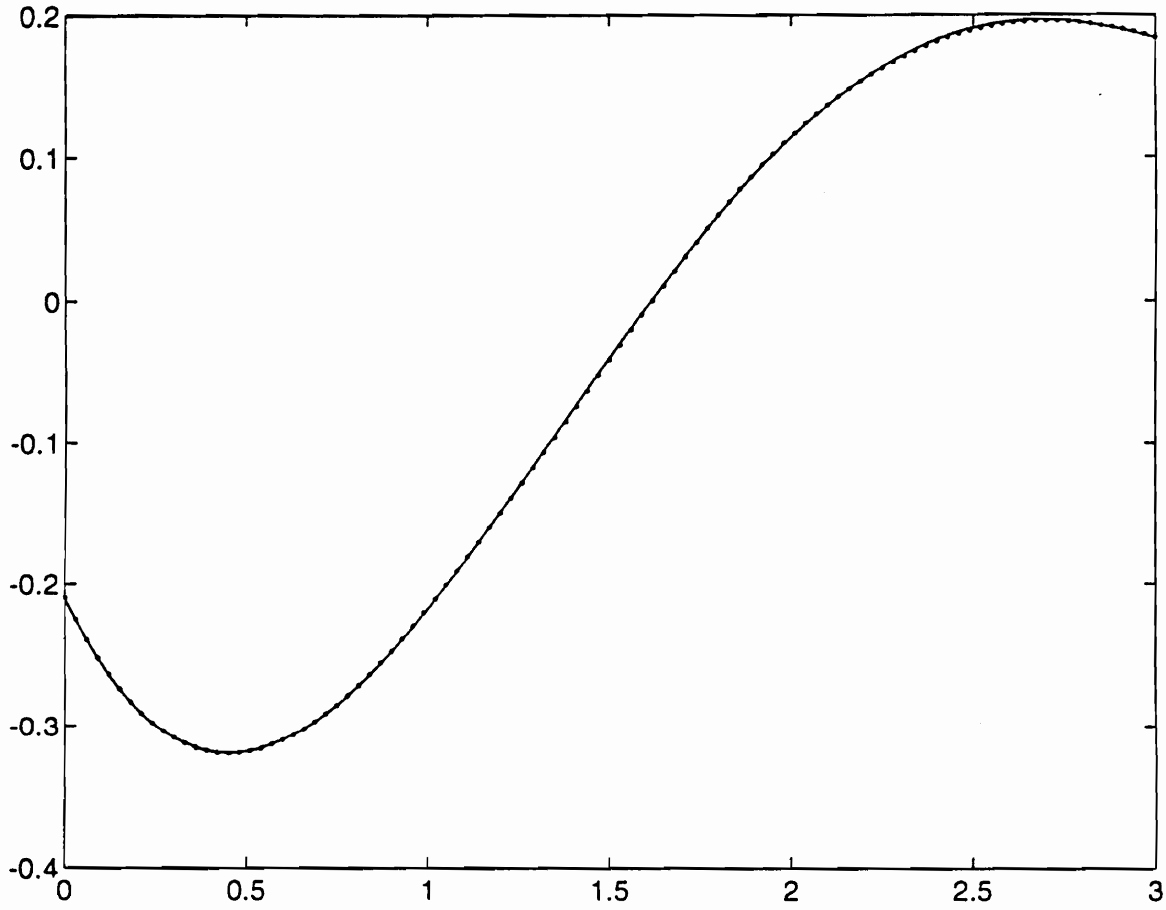


Figure 2.40 Estimator Gains L6 (5:5): n=3(.),n=4(-.),n=8(-)

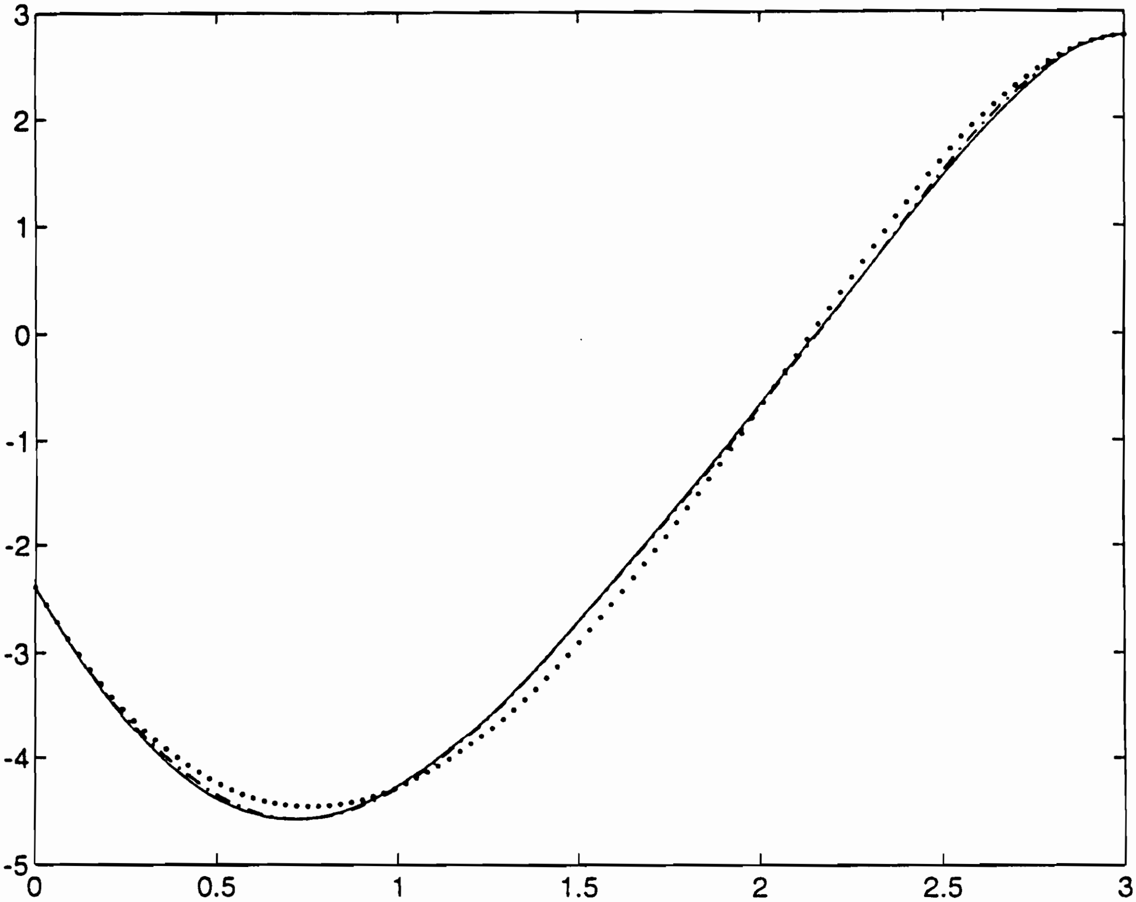


Figure 3.1 Bending Gains(5:1) for Beam 1:n=3(.),n=4(-.),n=8(-)

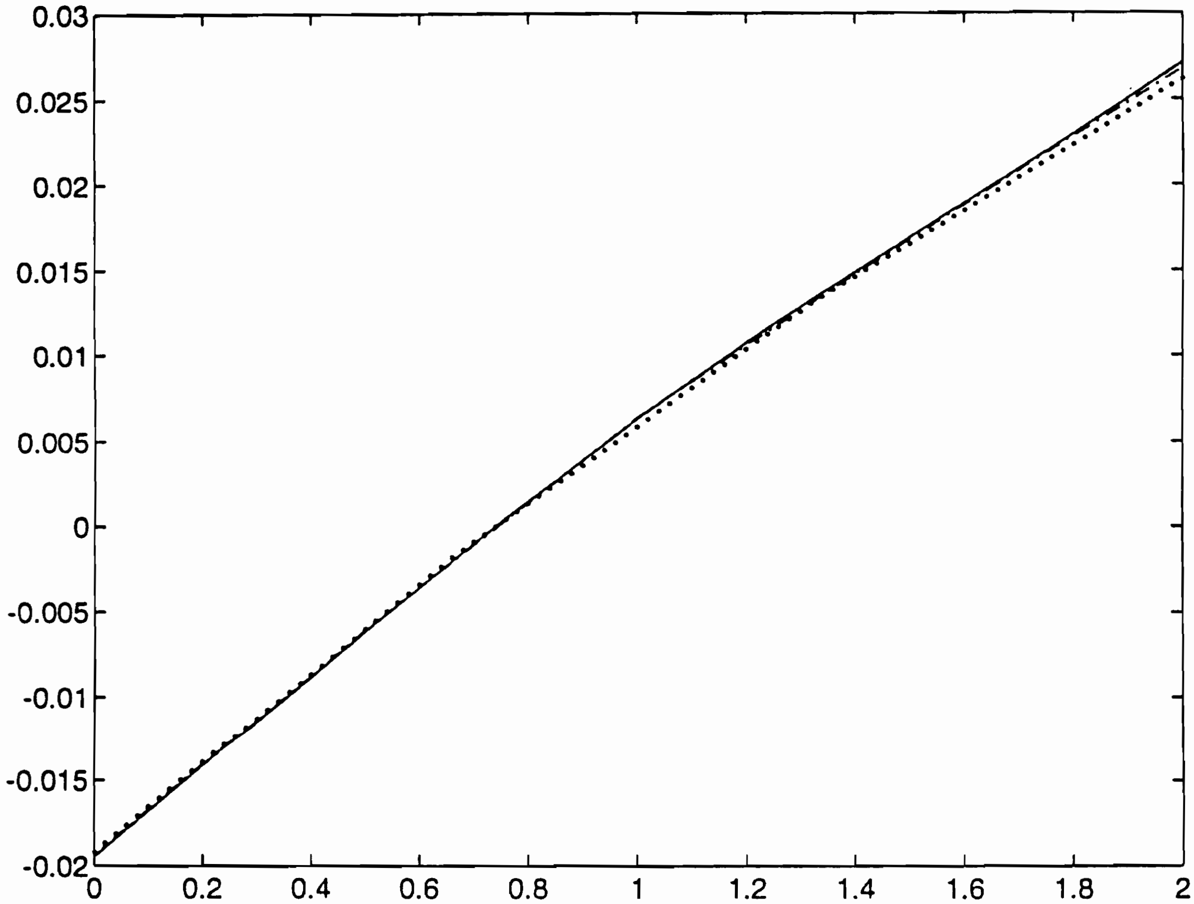
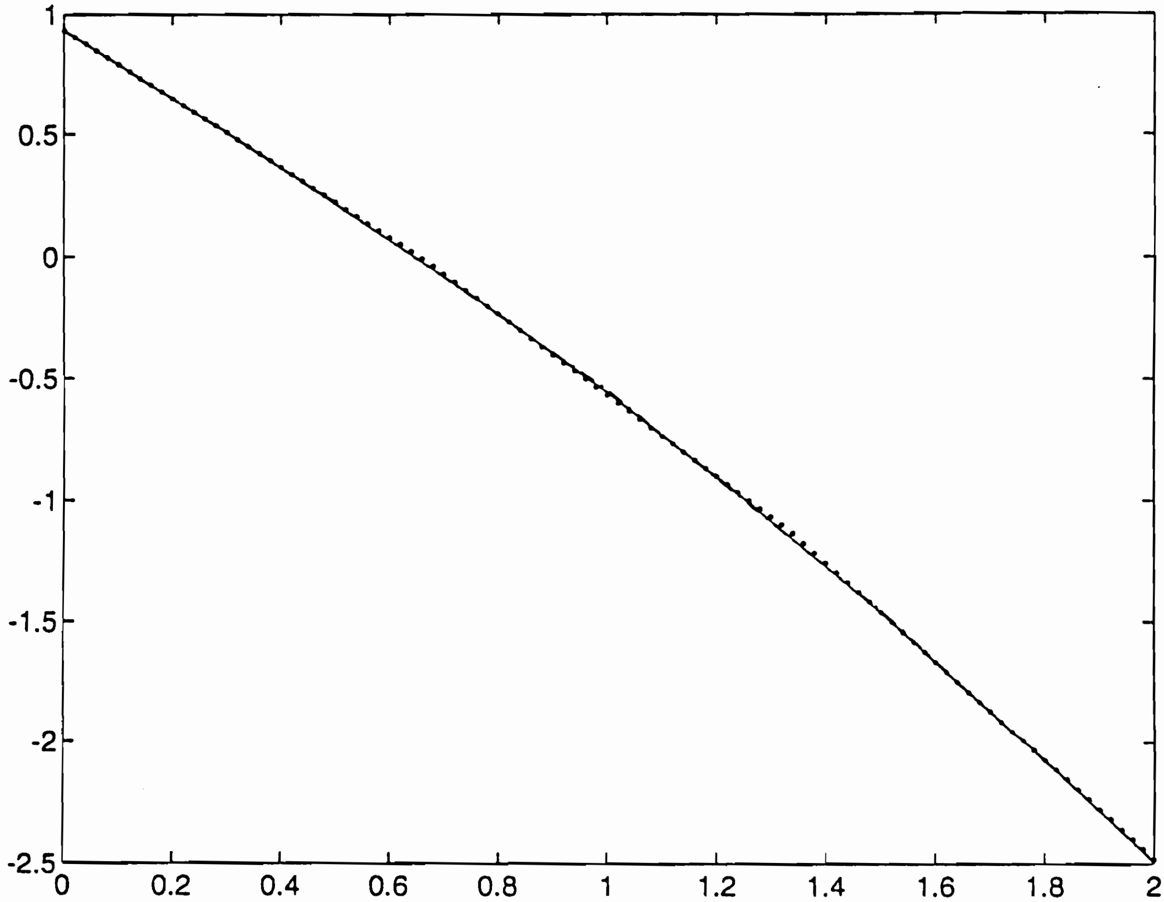


Figure 3.2 Bending Gains(5:2) for Beam 1:n=3(.),n=4(-.),n=8(-)



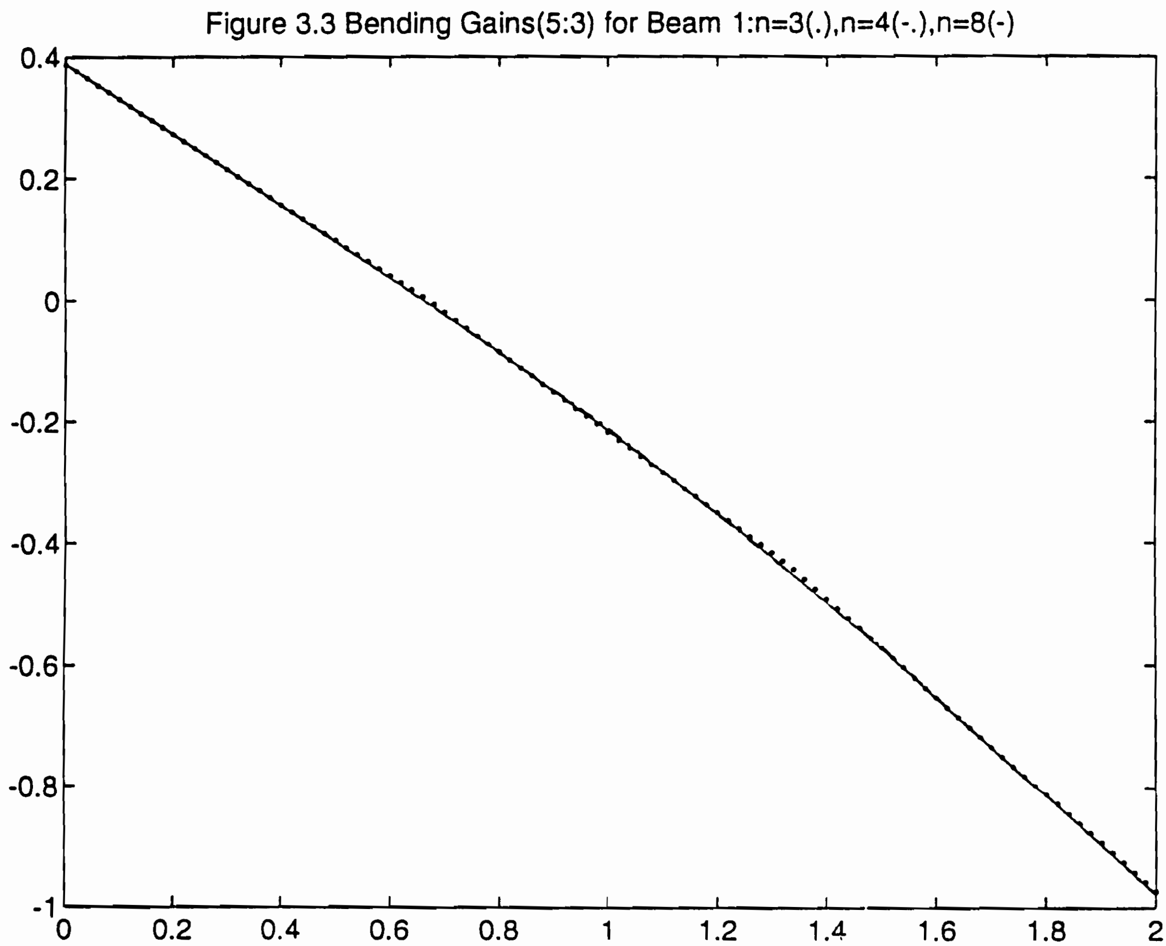


Figure 3.3 Bending Gains(5:3) for Beam 1:n=3(.),n=4(-.),n=8(-)
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Figure 3.4 Bending Gains(5:4) for Beam 1:n=3(.),n=4(-.),n=8(-)

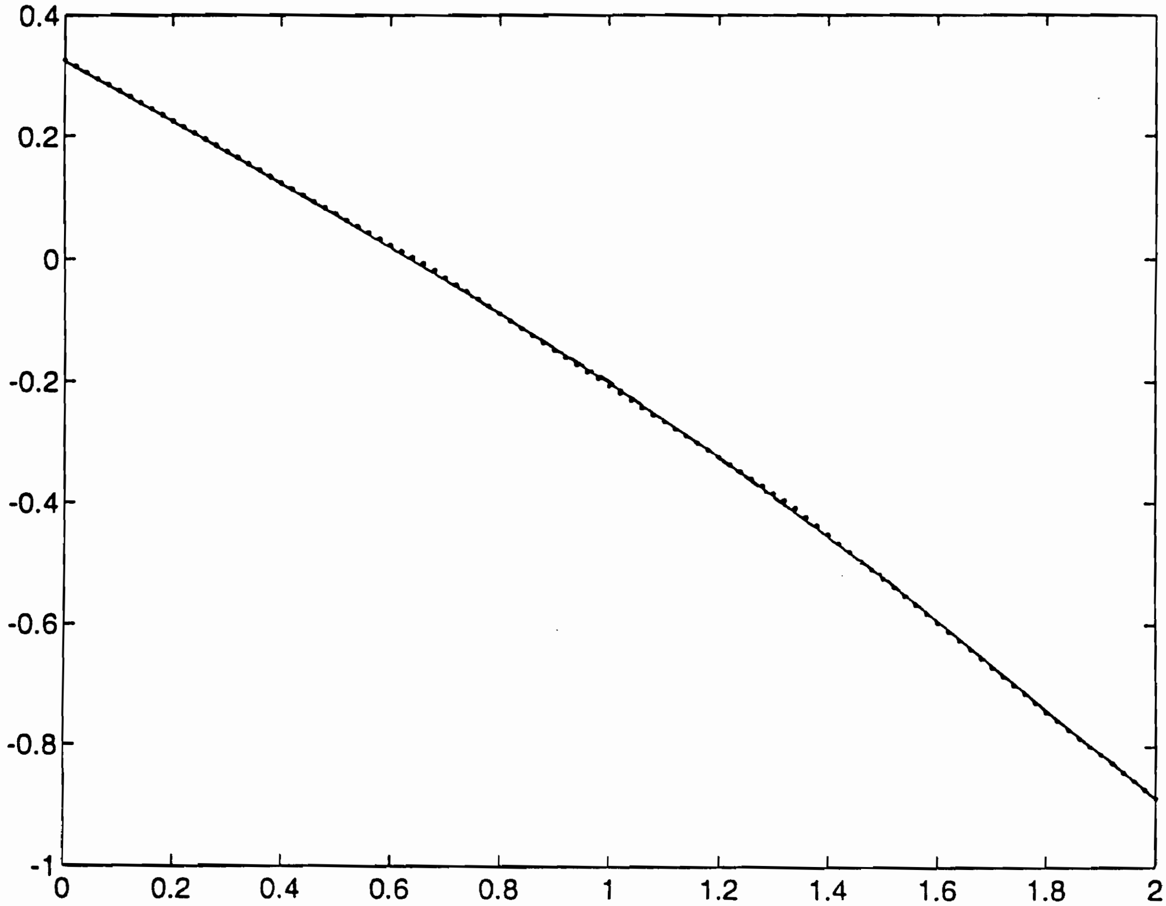


Figure 3.5 Bending Gains(5:5) for Beam 1:n=3(.),n=4(-.),n=8(-)

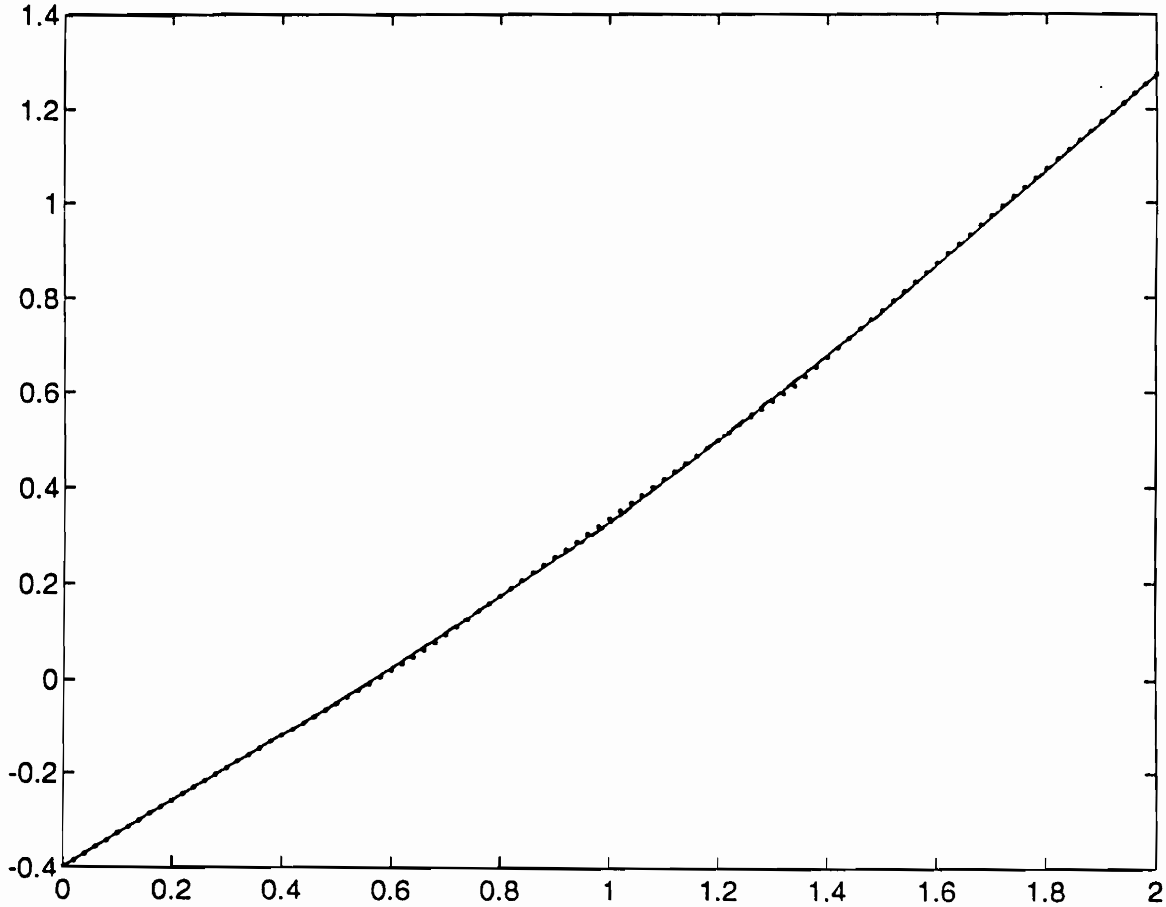


Figure 3.6 Bending Gains(5:1) for Beam 2:n=3(.),n=4(-.),n=8(-)

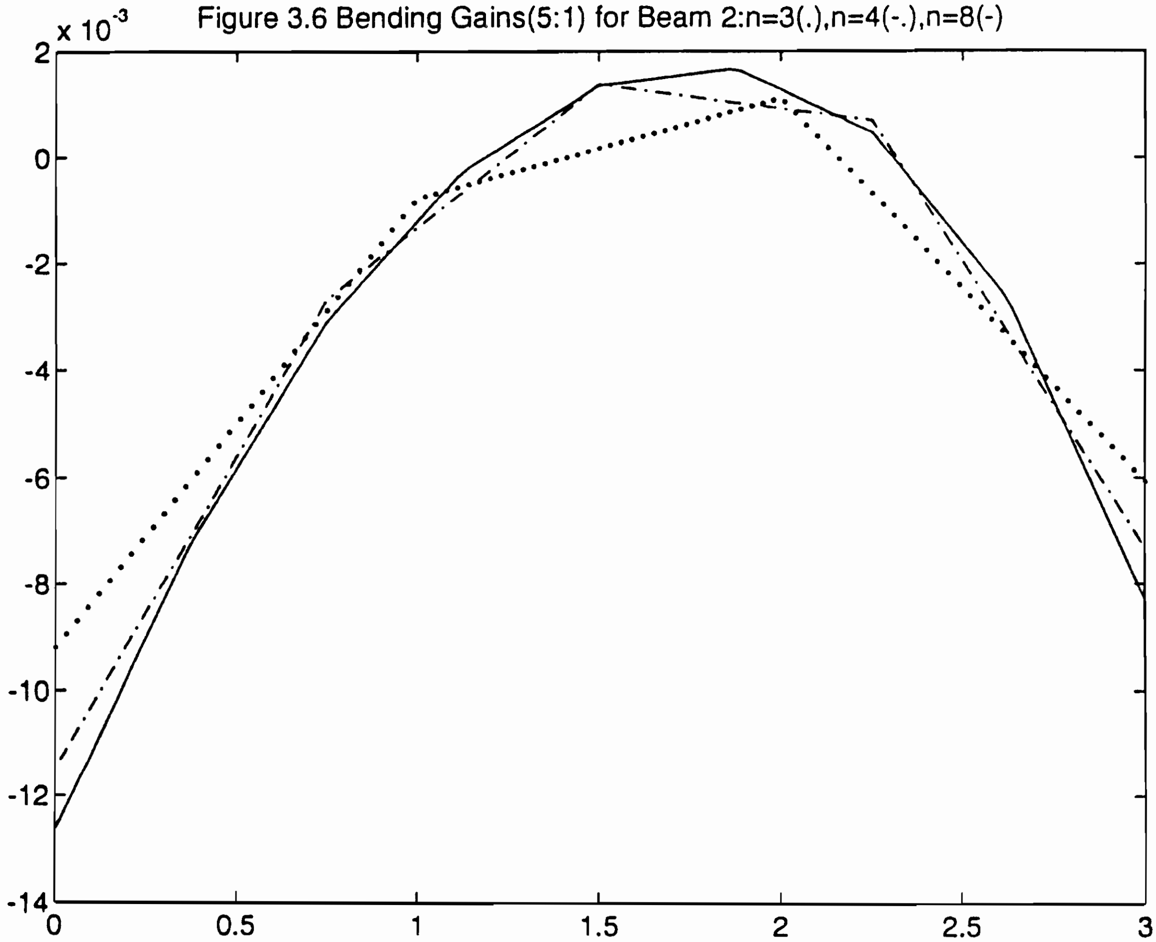


Figure 3.7 Bending Gains(5:2) for Beam 2:n=3(.),n=4(-.),n=8(-)

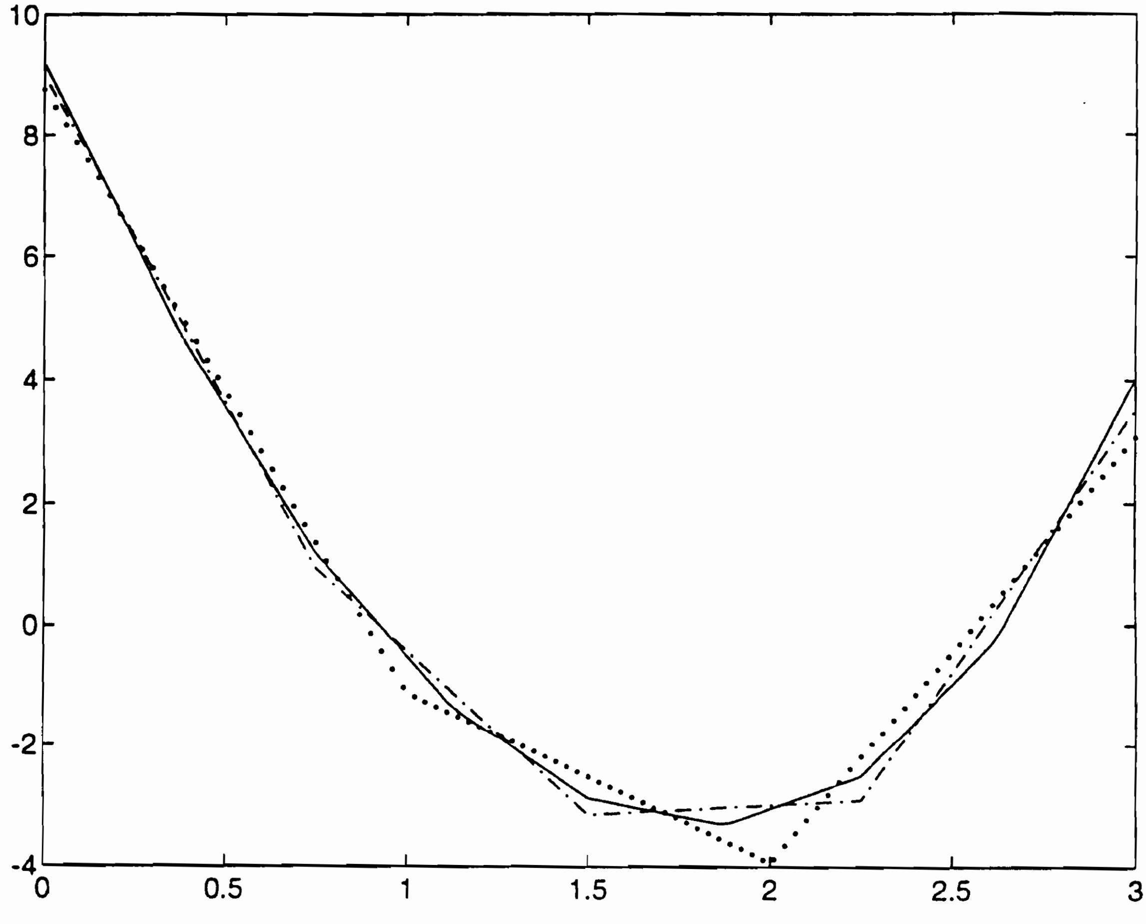


Figure 3.8 Bending Gains(5:3) for Beam 2:n=3(.),n=4(-.),n=8(-)

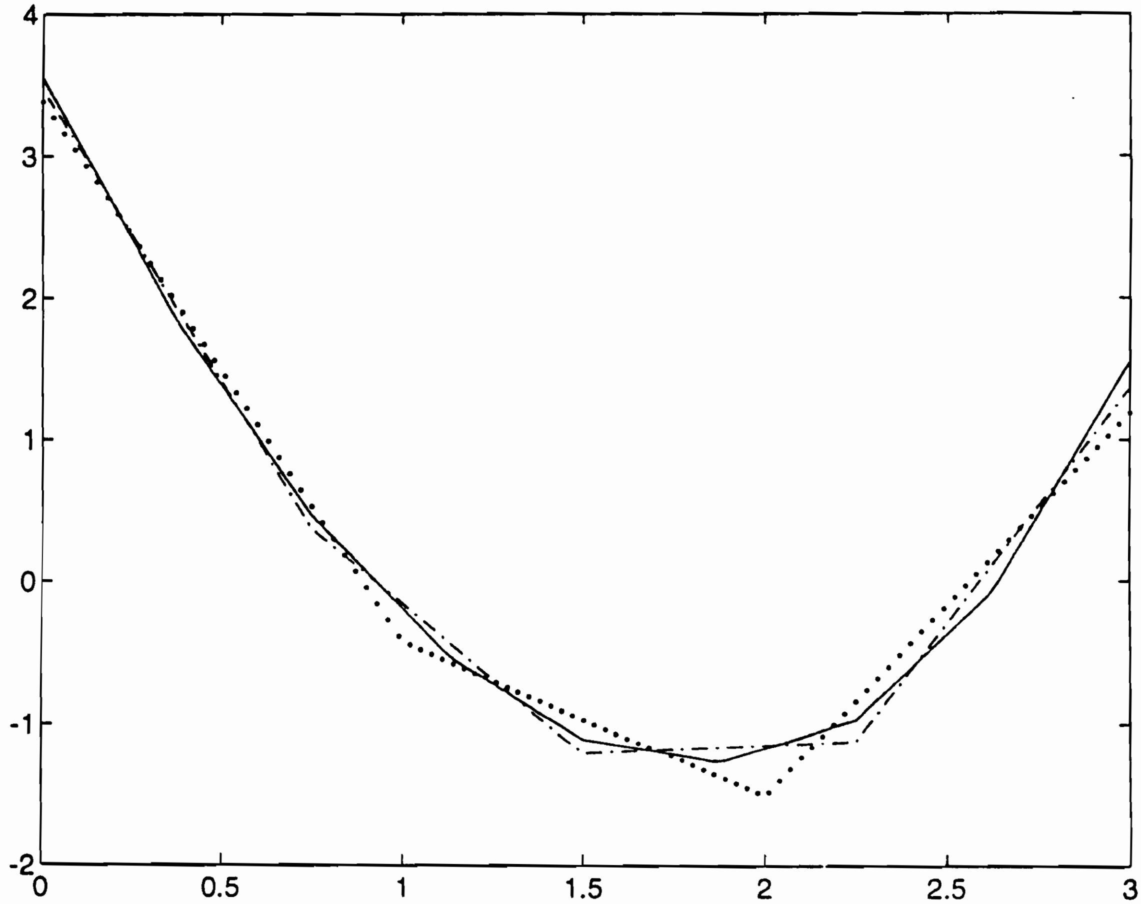


Figure 3.9 Bending Gains(5:4) for Beam 2:n=3(.),n=4(-.),n=8(-)

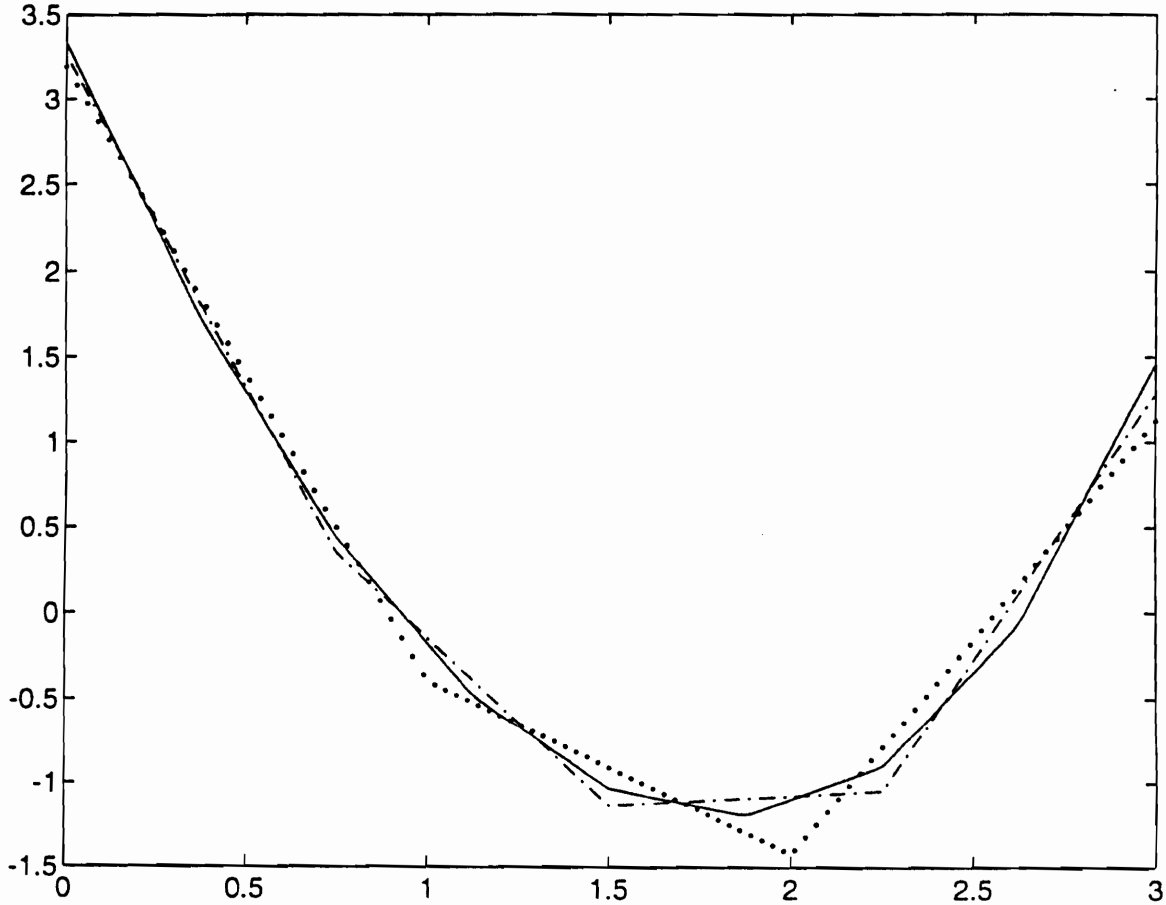


Figure 3.10 Bending Gains(5:5) for Beam 2:n=3(.),n=4(-.),n=8(-)

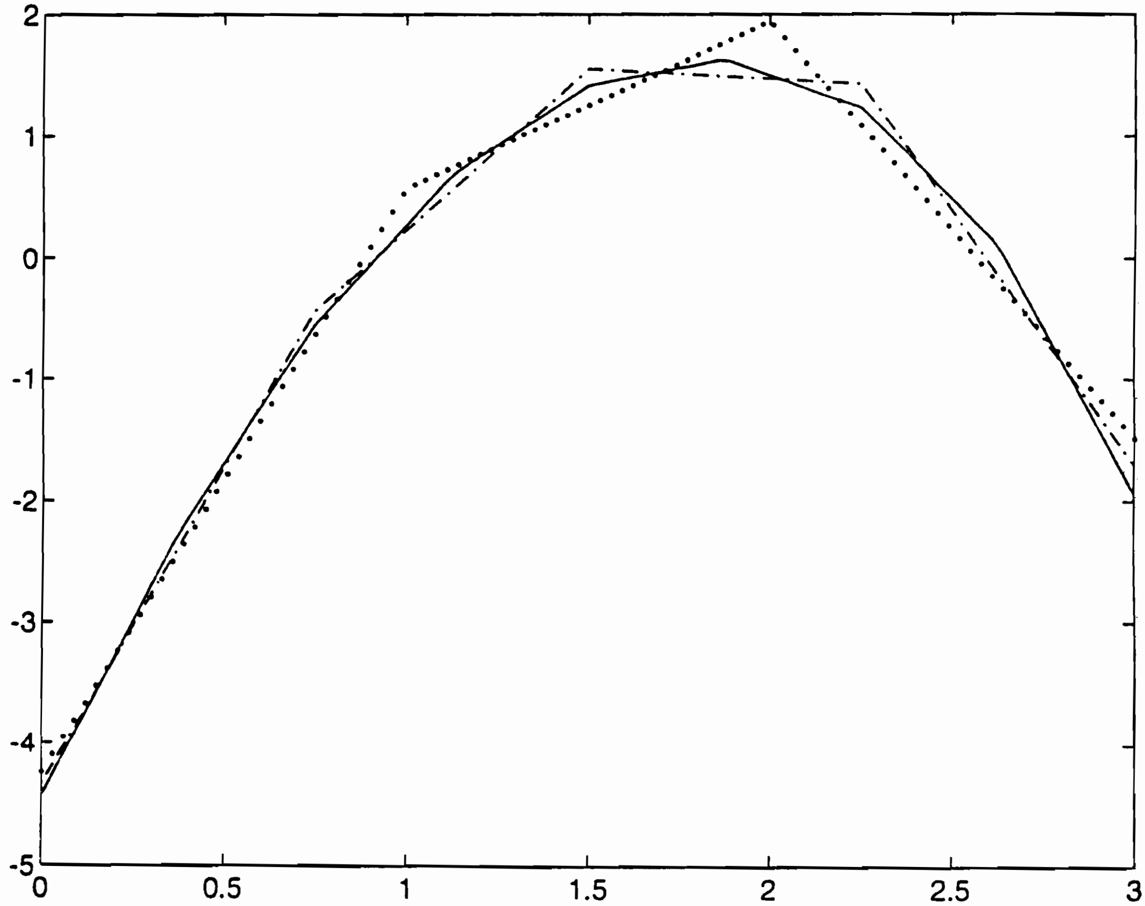


Figure 3.11 Velocity Gains(5:1) for Beam 1:n=3(.),n=4(-.),n=8(-)

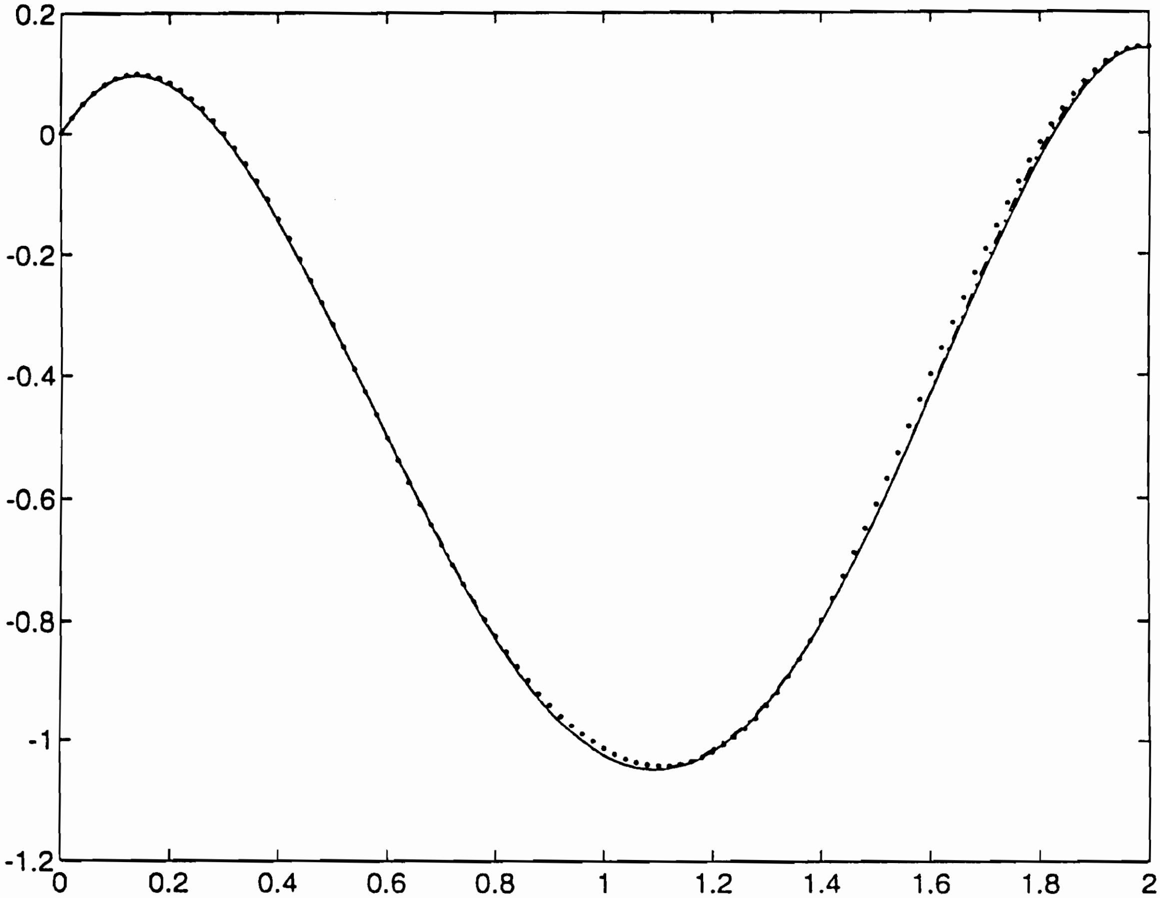


Figure 3.12 Velocity Gains(5:2) for Beam 1:n=3(.),n=4(-.),n=8(-)

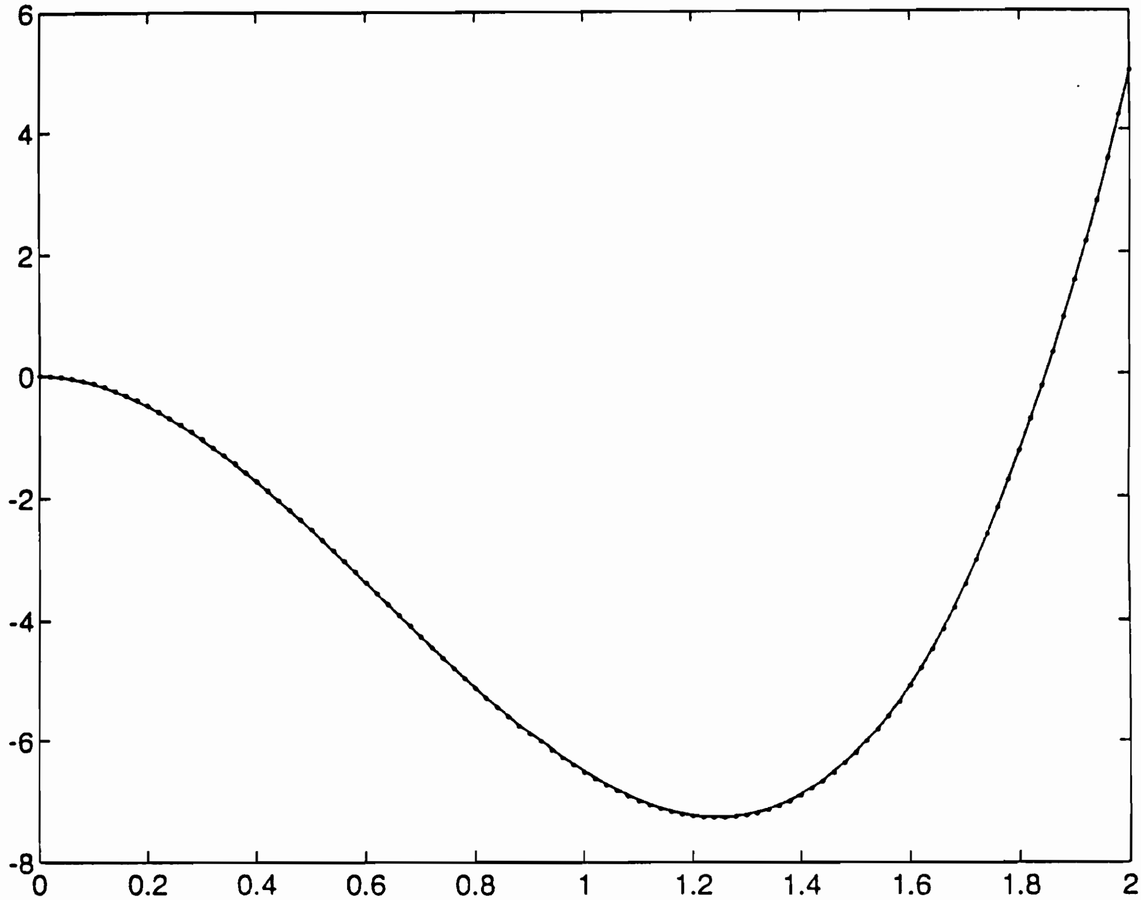
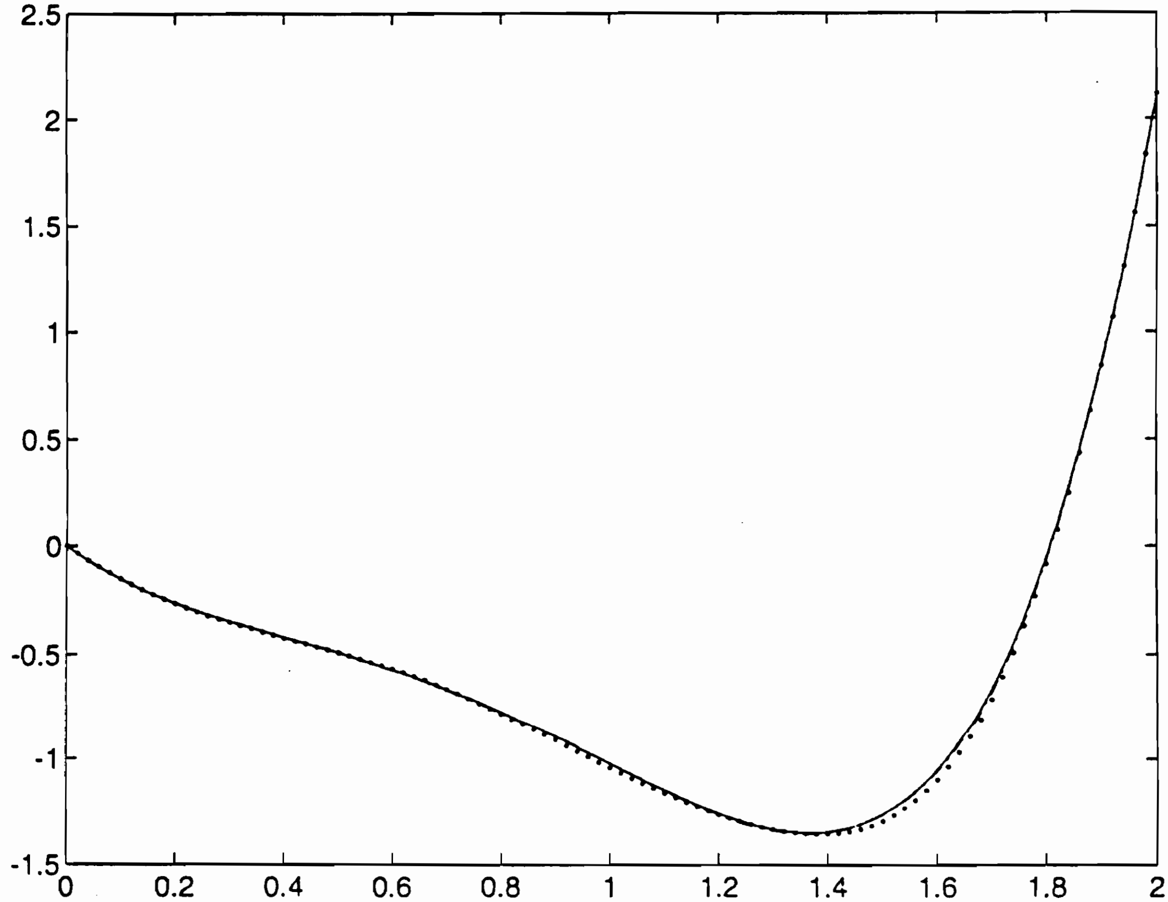


Figure 3.13 Velocity Gains(5:3) for Beam 1:n=3(.),n=4(-.),n=8(-)



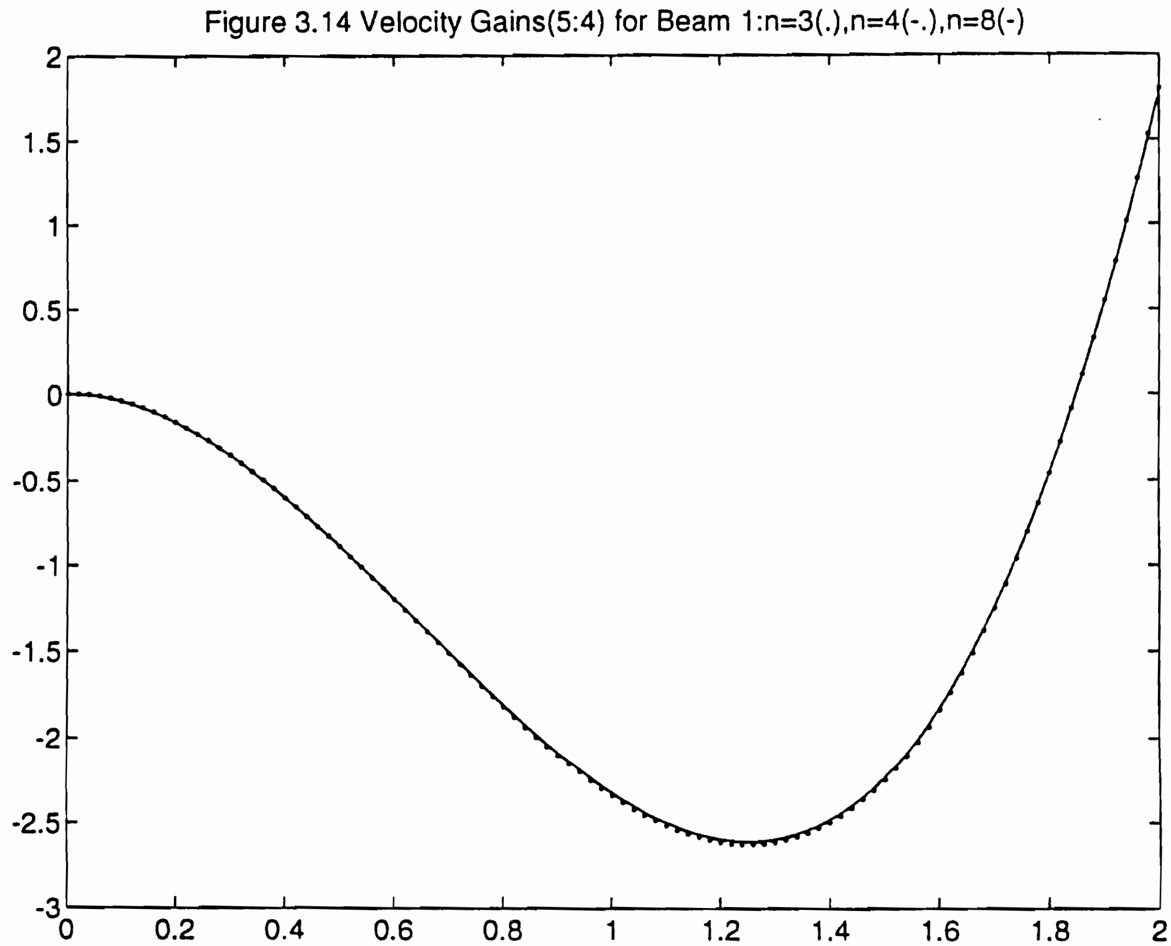


Figure 3.14 Velocity Gains(5:4) for Beam 1:n=3(.),n=4(-.),n=8(-)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Figure 3.15 Velocity Gains(5:5) for Beam 1:n=3(.),n=4(-.),n=8(-)

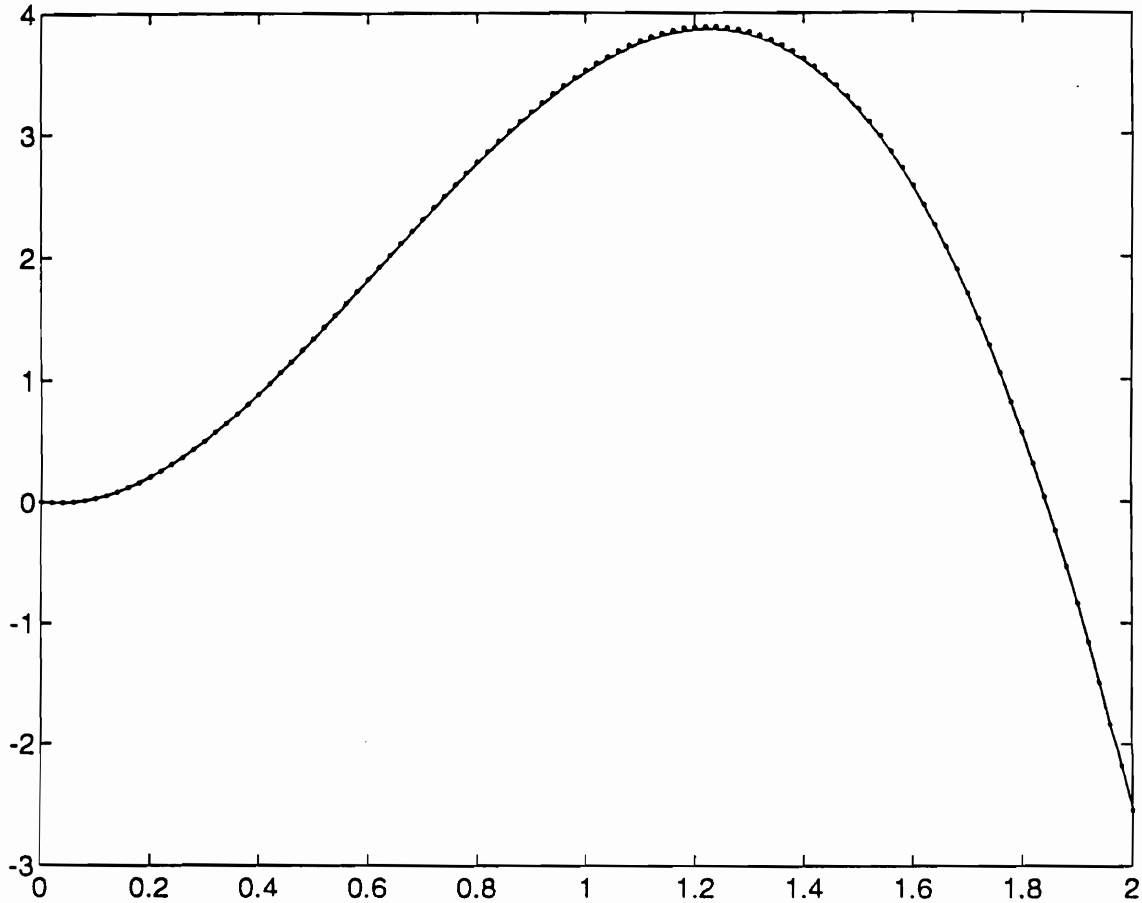


Figure 3.16 Velocity Gains(5:1) for Beam 2:n=3(.),n=4(-.),n=8(-)

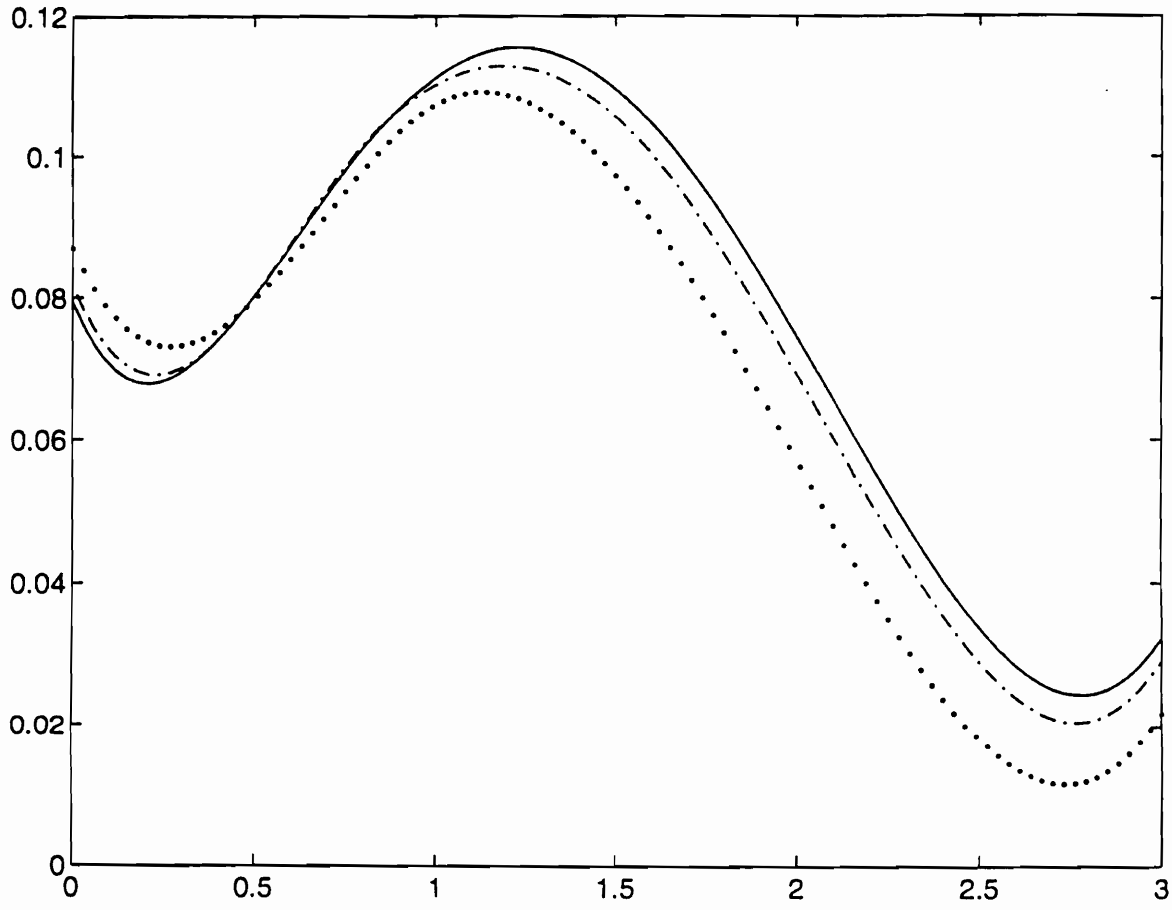


Figure 3.17 Velocity Gains(5:2) for Beam 2:n=3(.),n=4(-.),n=8(-)

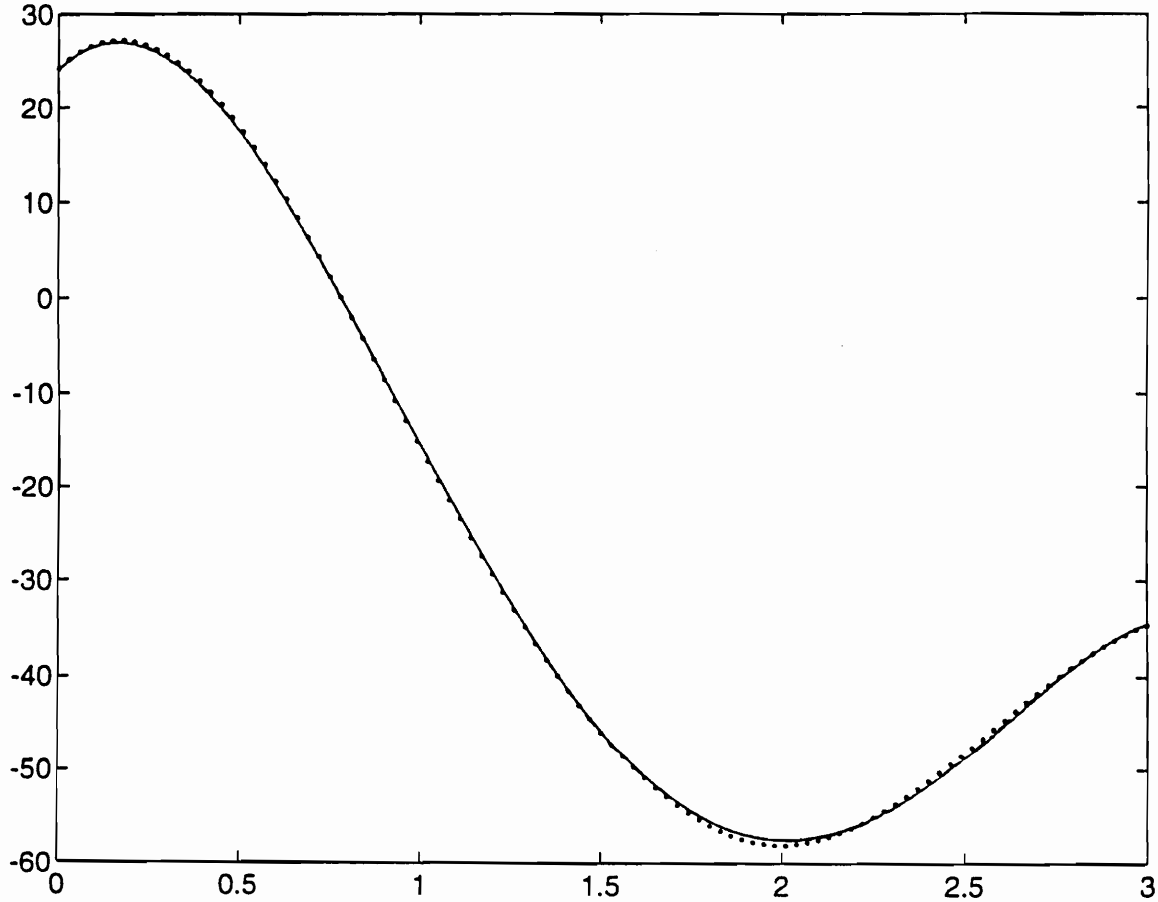


Figure 3.18 Velocity Gains(5:3) for Beam 2:n=3(.),n=4(-.),n=8(-)

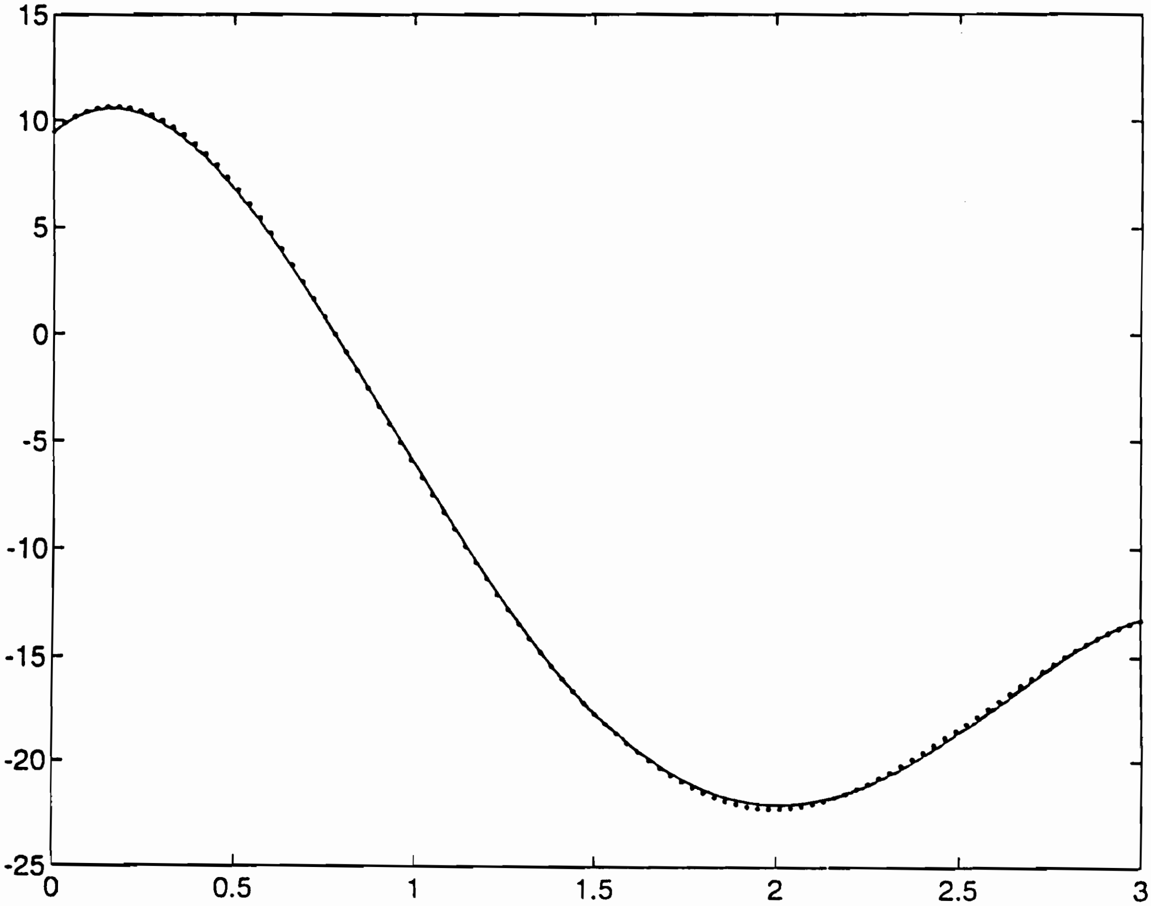


Figure 3.19 Velocity Gains(5:4) for Beam 2:n=3(.),n=4(-.),n=8(-)

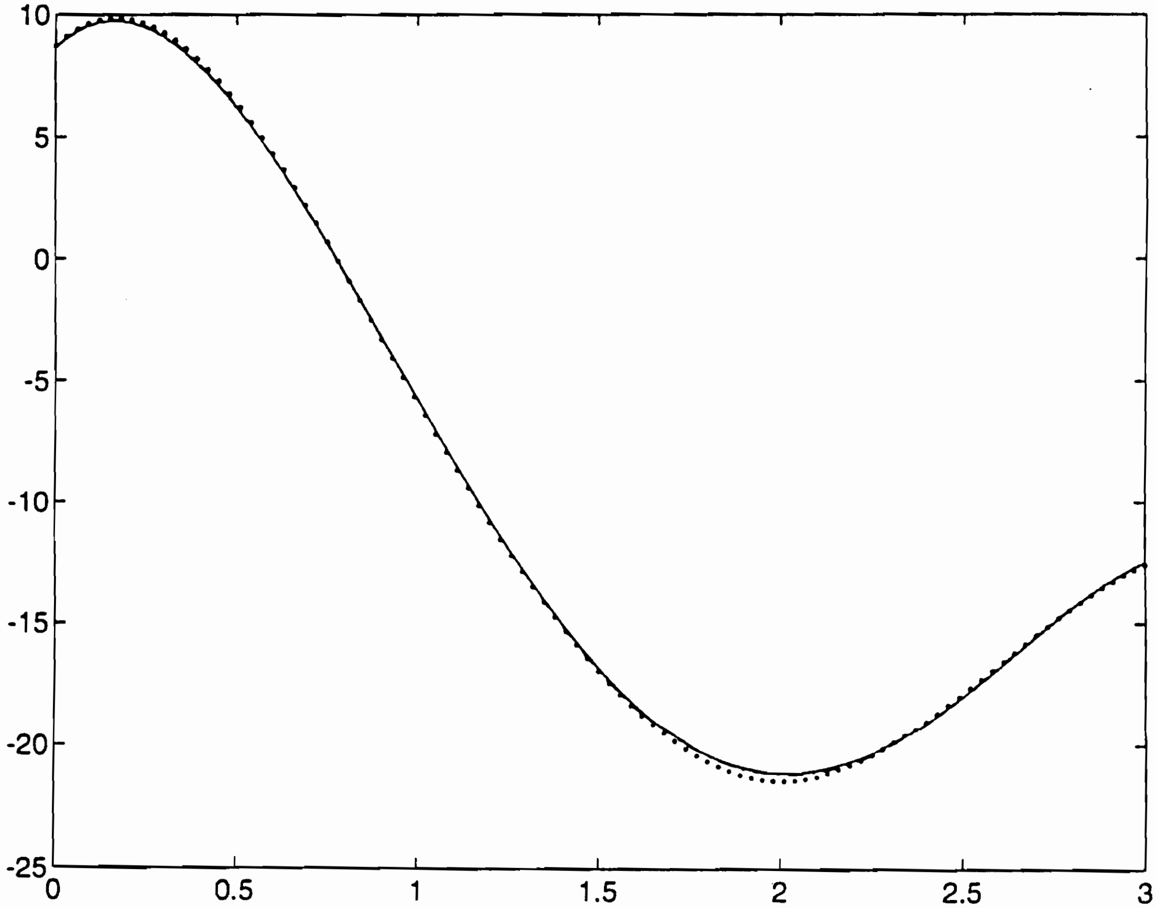


Figure 3.20 Velocity Gains(5:5) for Beam 2:n=3(.),n=4(-.),n=8(-)

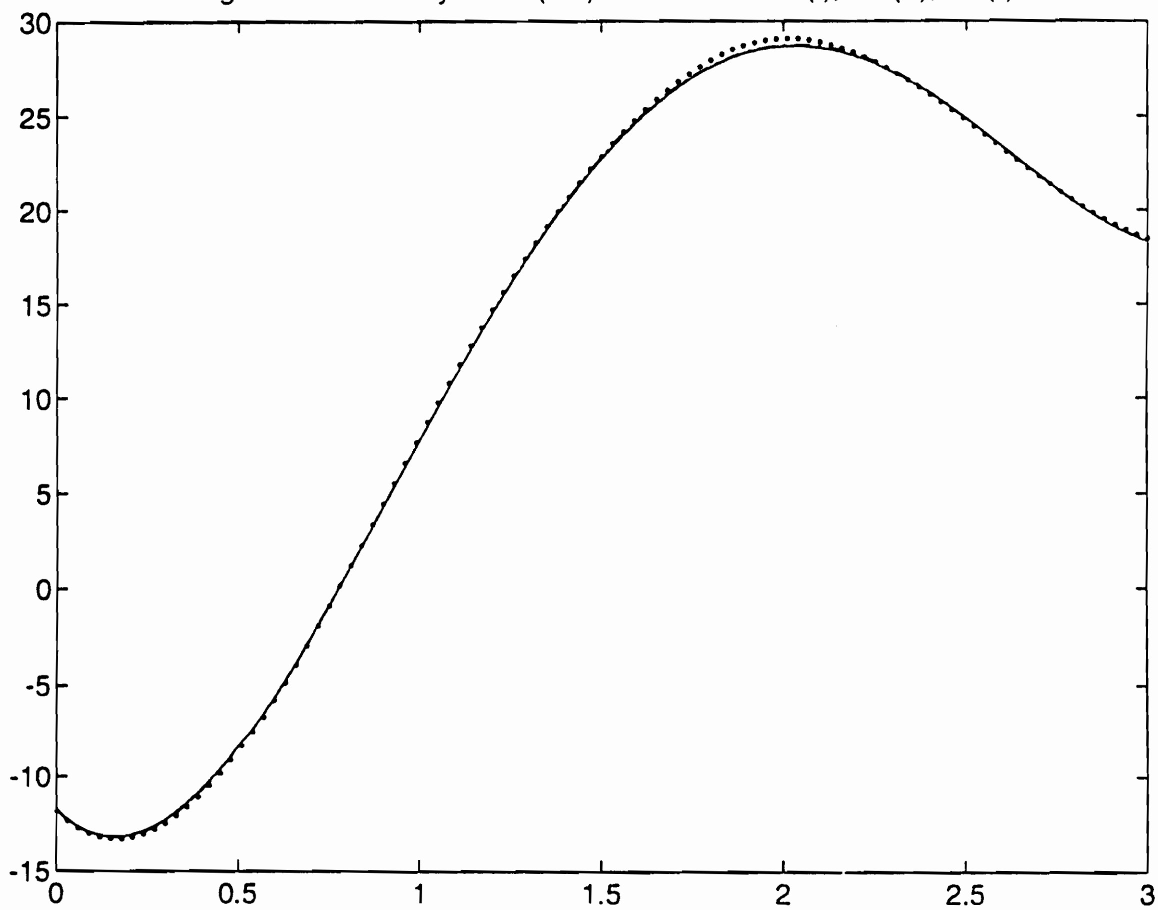


Figure 3.21 Estimator Gains L2 (5:1): n=3(.),n=4(-.),n=8(-)

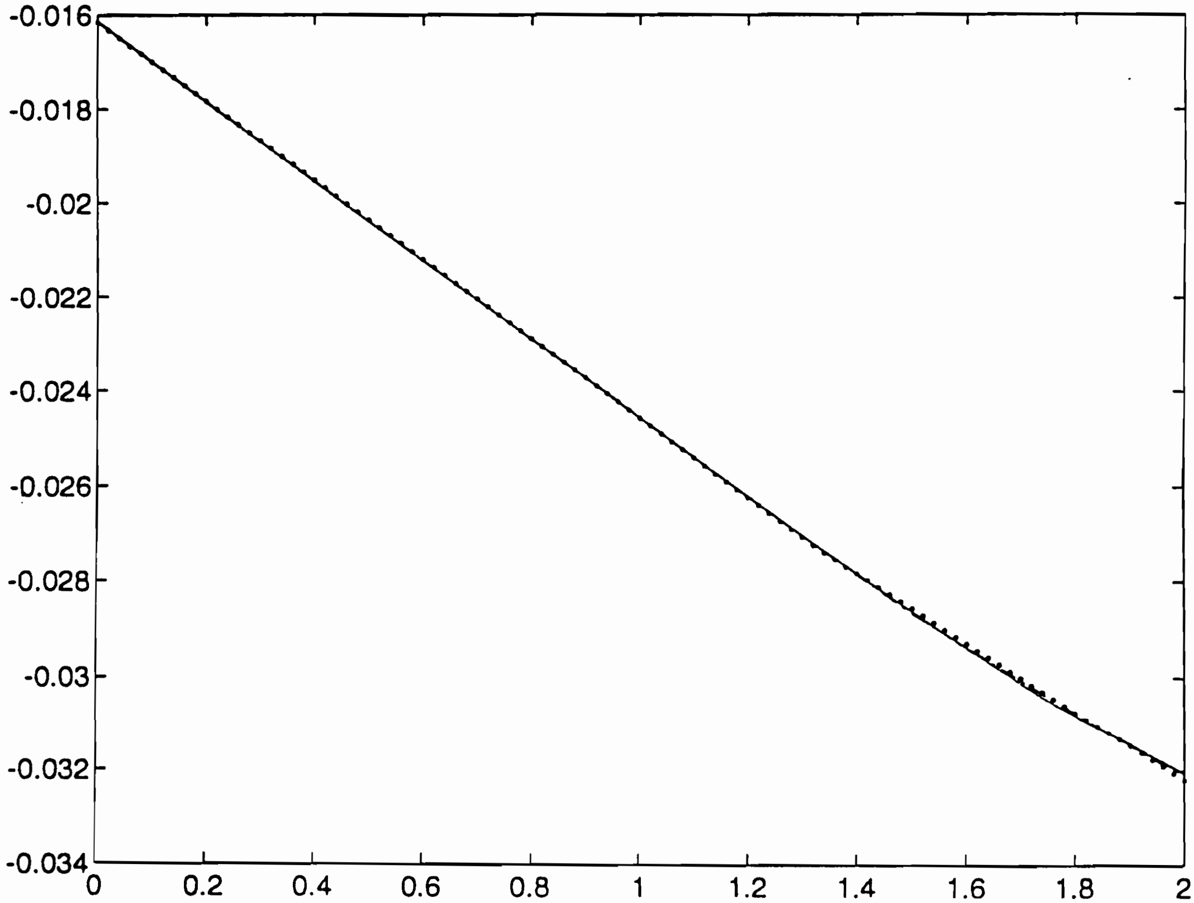


Figure 3.22 Estimator Gains L2 (5:2): n=3(.),n=4(-.),n=8(-)

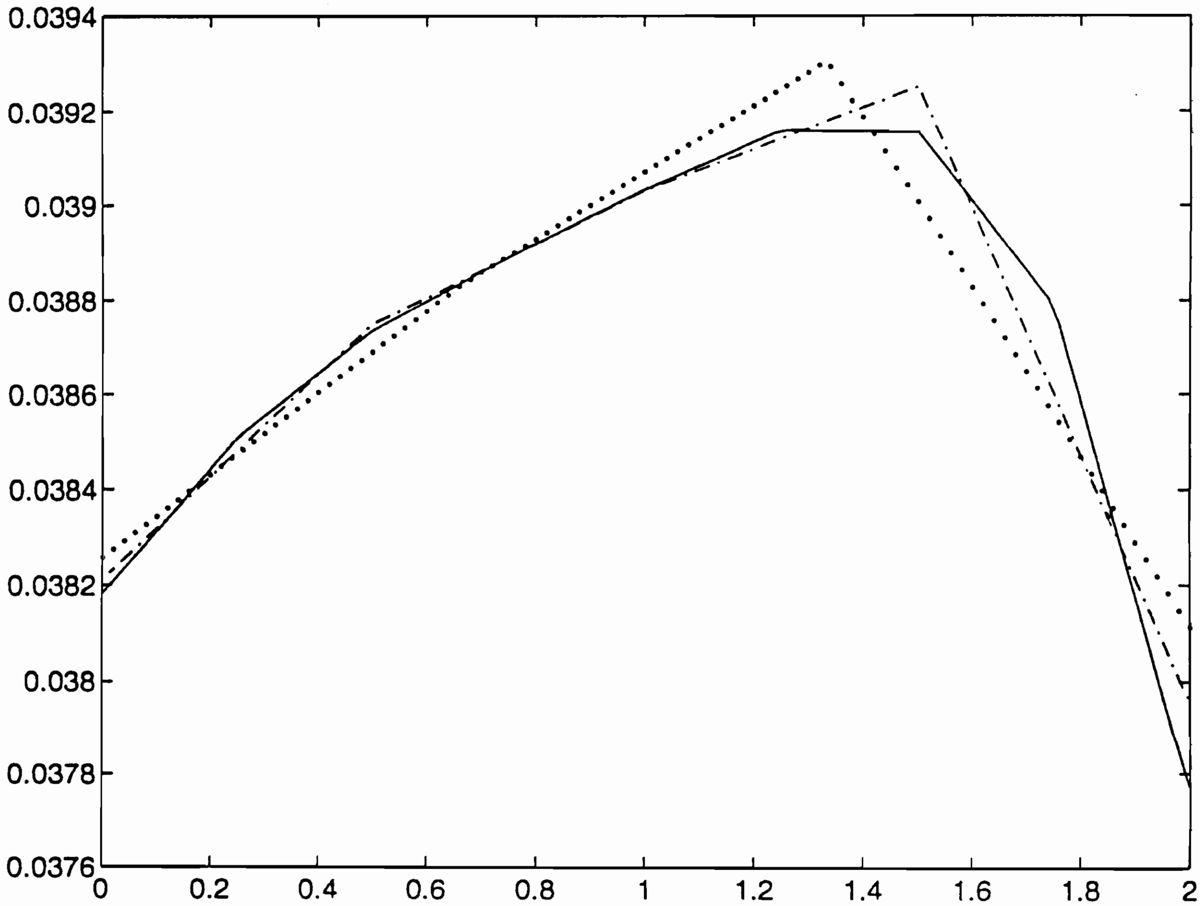


Figure 3.23 Estimator Gains L2 (5:3): n=3(.),n=4(-.),n=8(-)

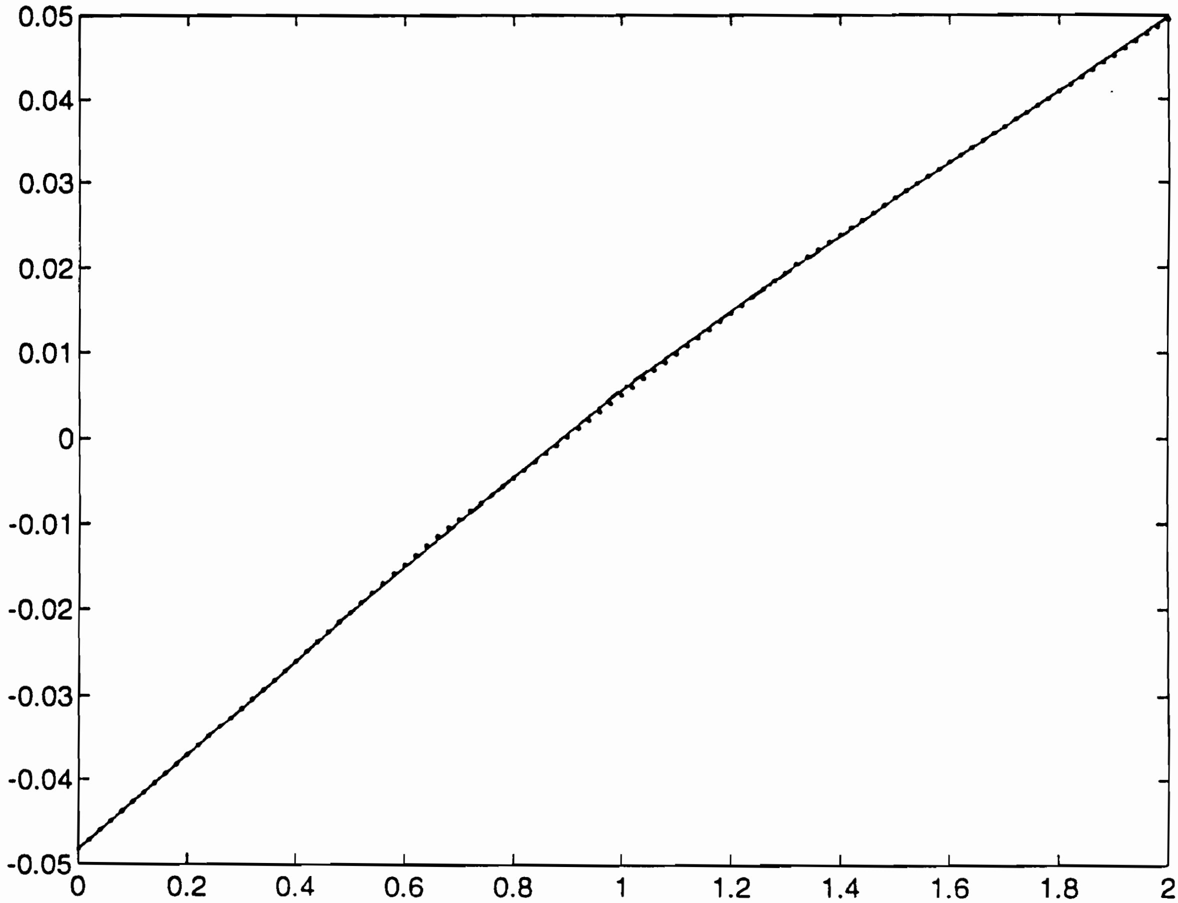


Figure 3.24 Estimator Gains L2 (5:4): n=3(.),n=4(-.),n=8(-)

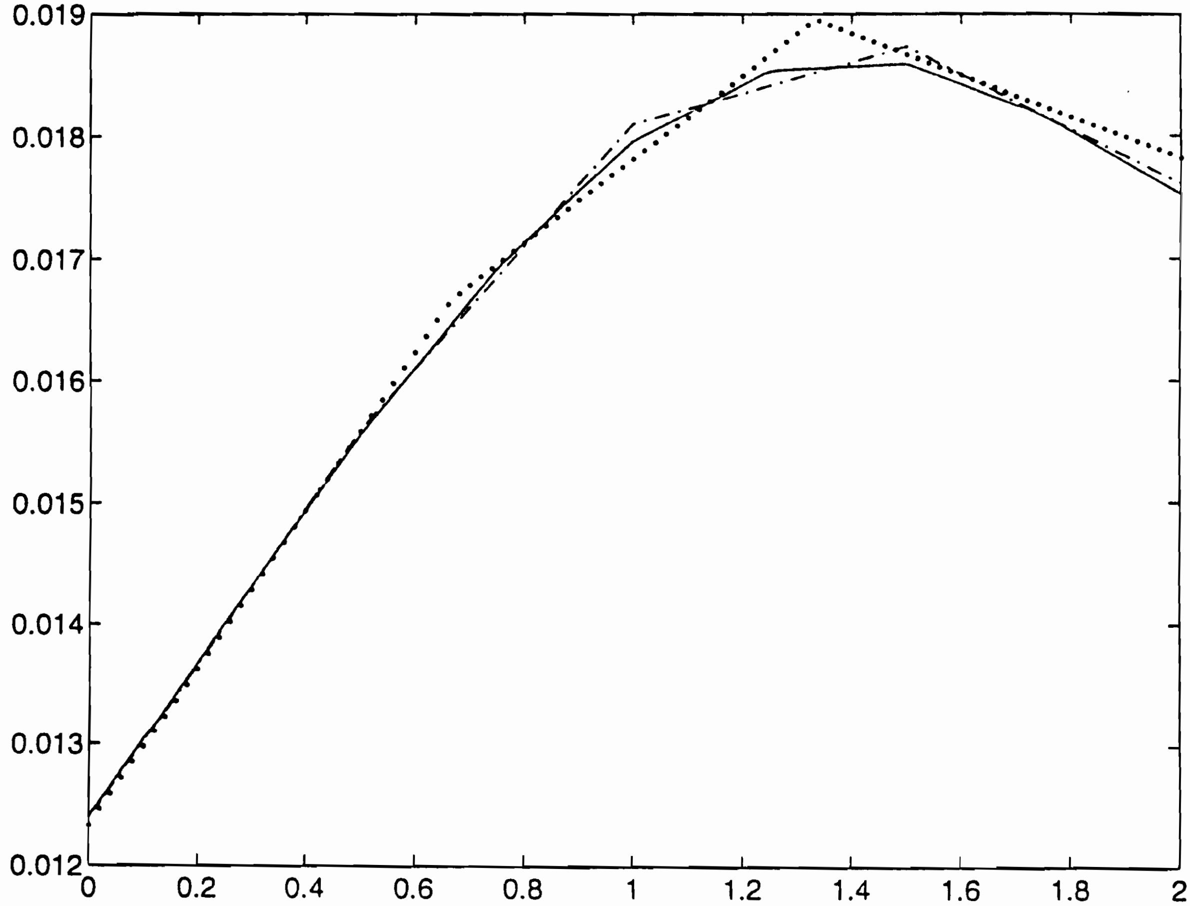


Figure 3.25 Estimator Gains L2 (5:5): n=3(.),n=4(-.),n=8(-)

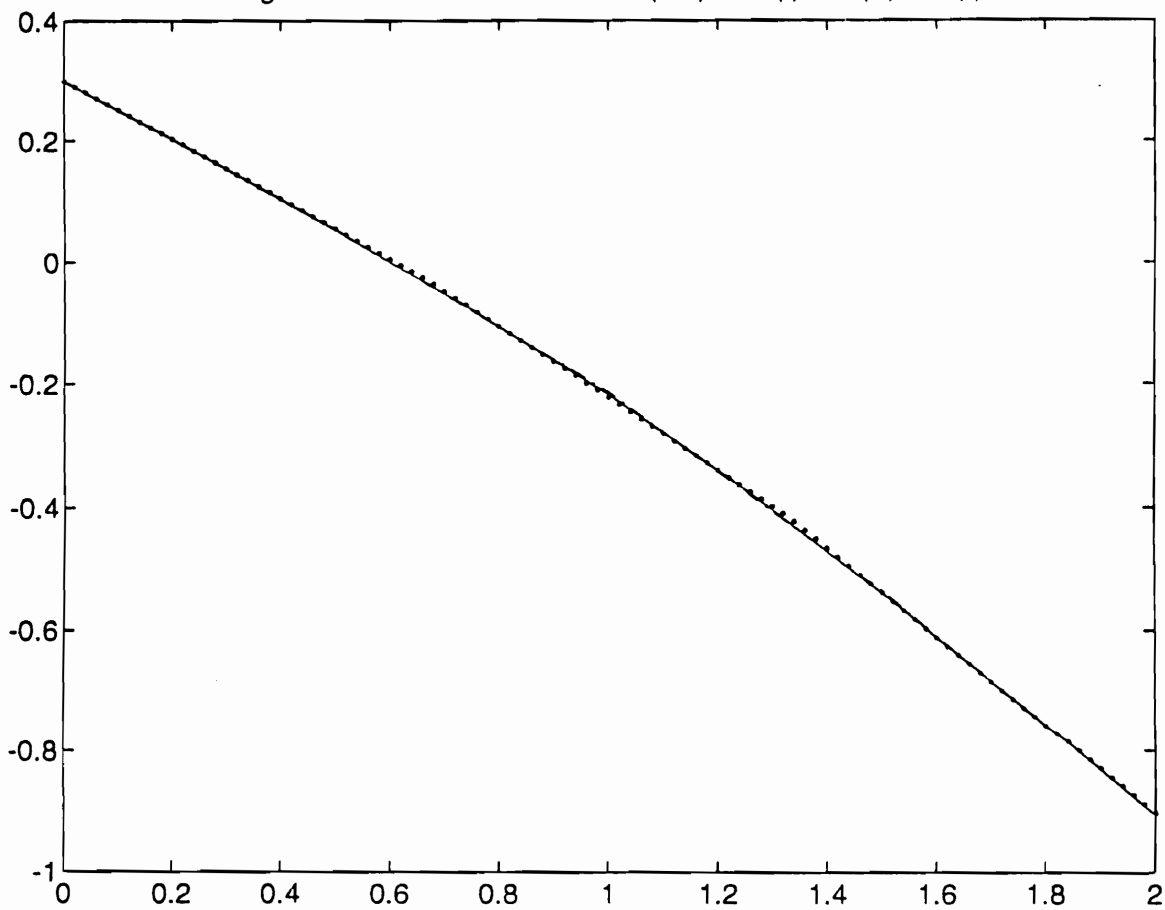
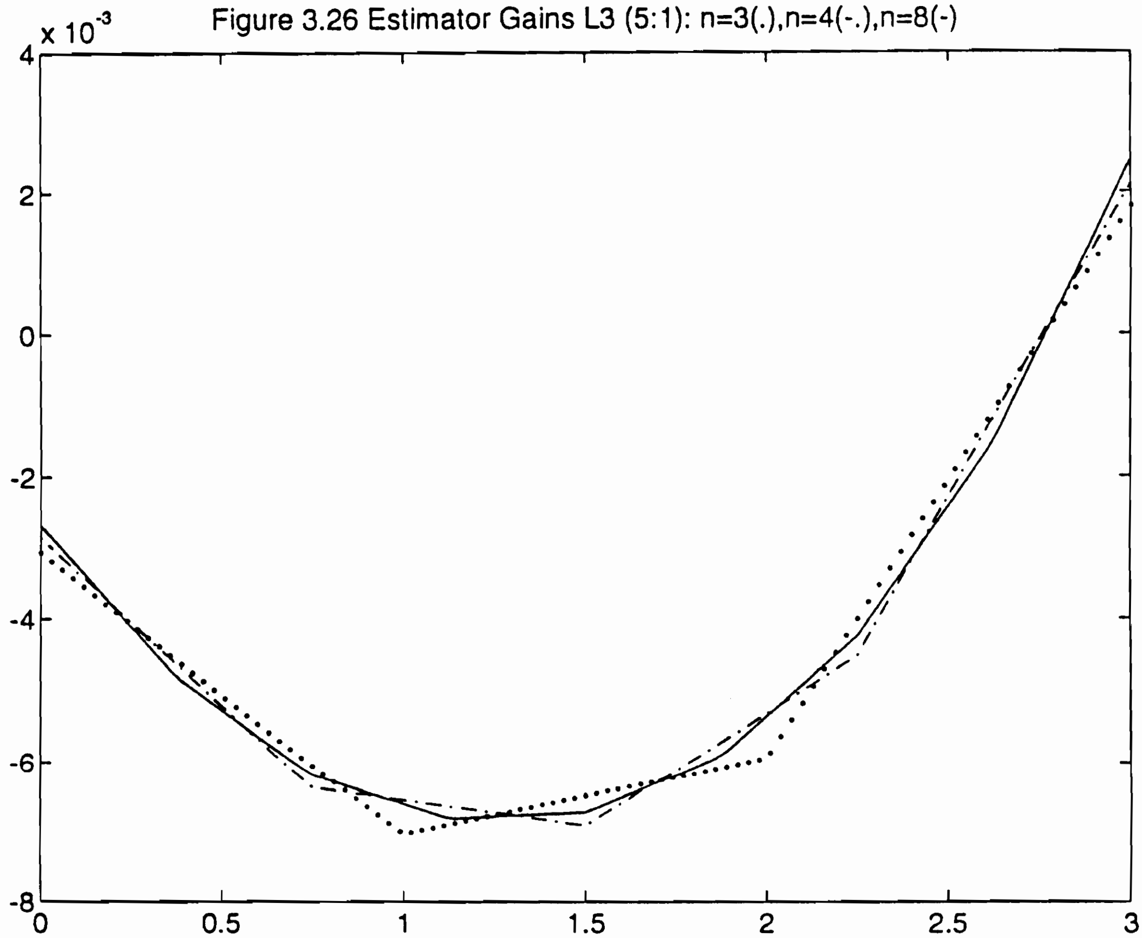


Figure 3.26 Estimator Gains L3 (5:1): n=3(.),n=4(-.),n=8(-)



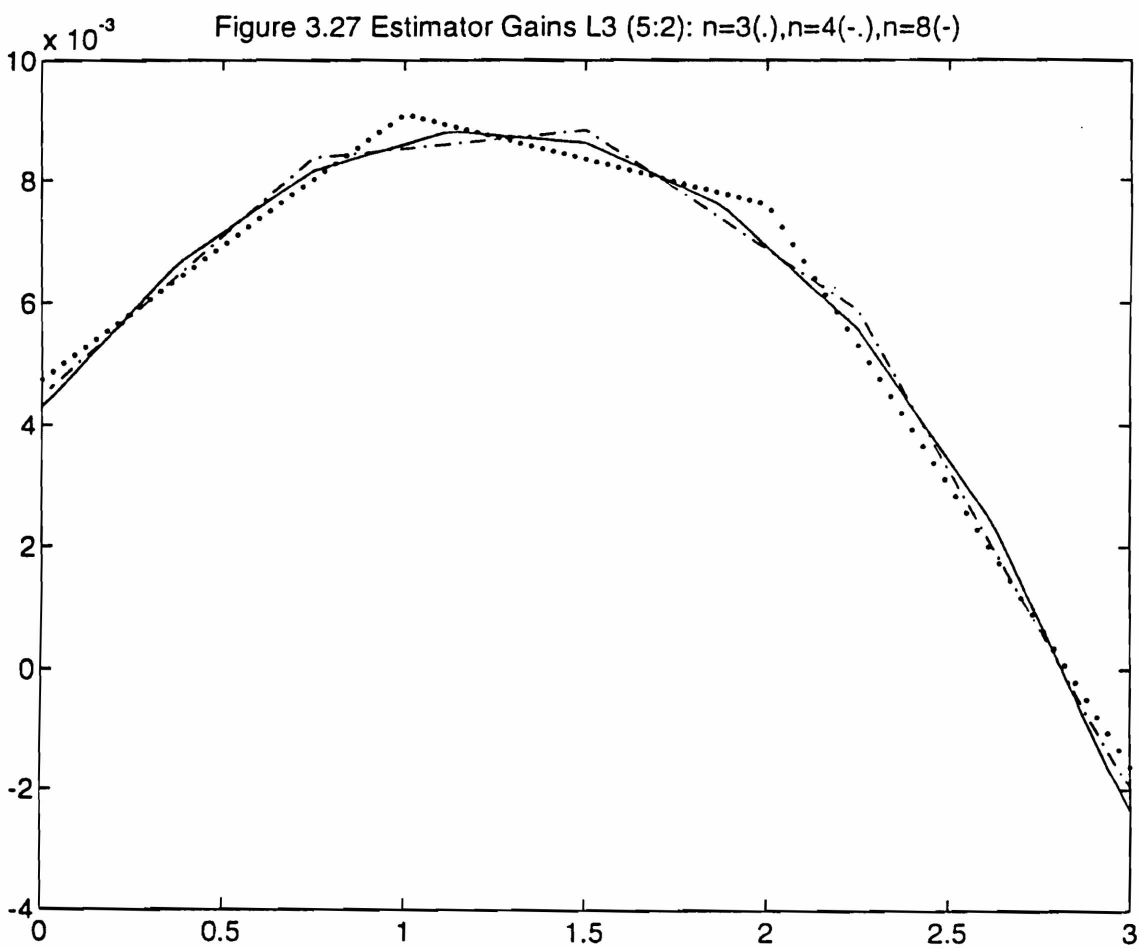


Figure 3.27 Estimator Gains L3 (5:2): n=3(.),n=4(-.),n=8(-)
x 10^-3
10
8
6
4
2
0
-2
-4
0
0.5
1
1.5
2
2.5
3

Figure 3.28 Estimator Gains L3 (5:3): n=3(.),n=4(-.),n=8(-)

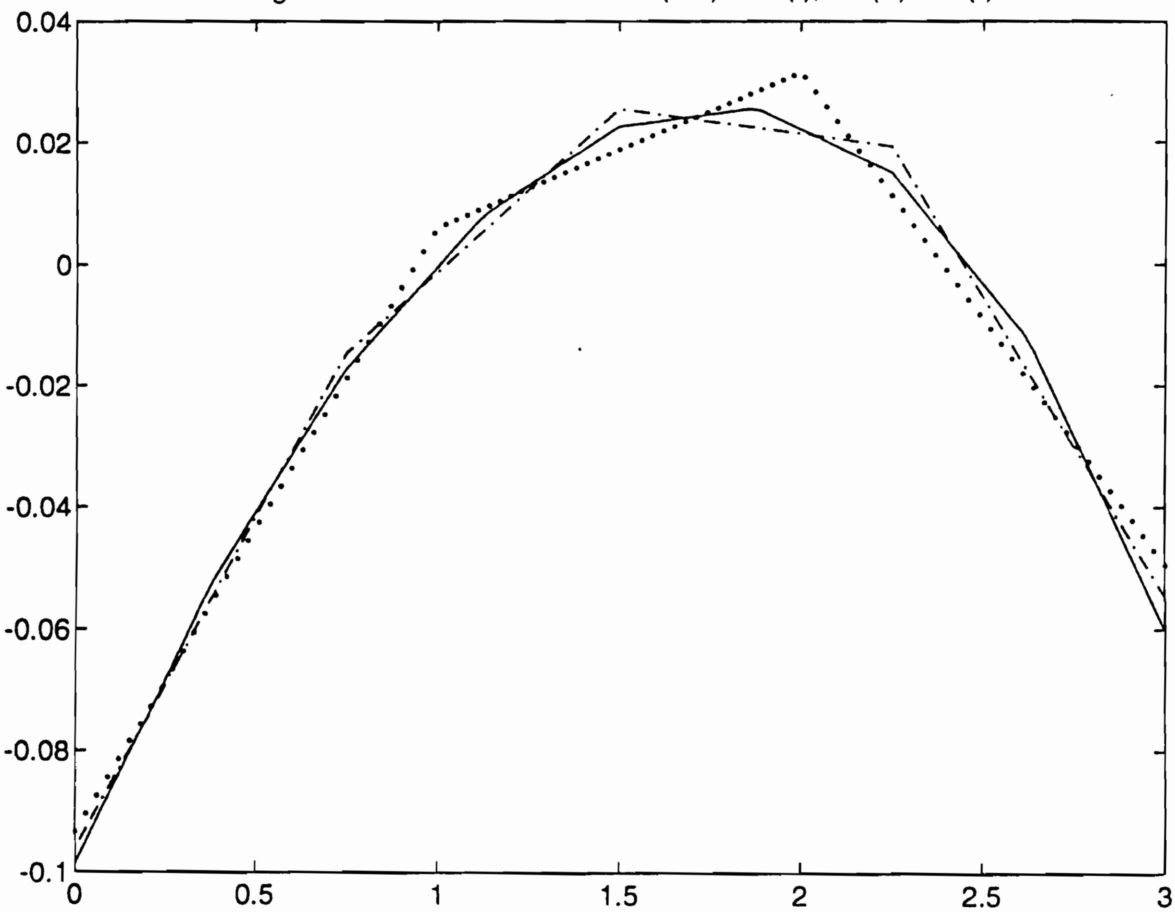


Figure 3.29 Estimator Gains L3 (5:4): n=3(.),n=4(-.),n=8(-)

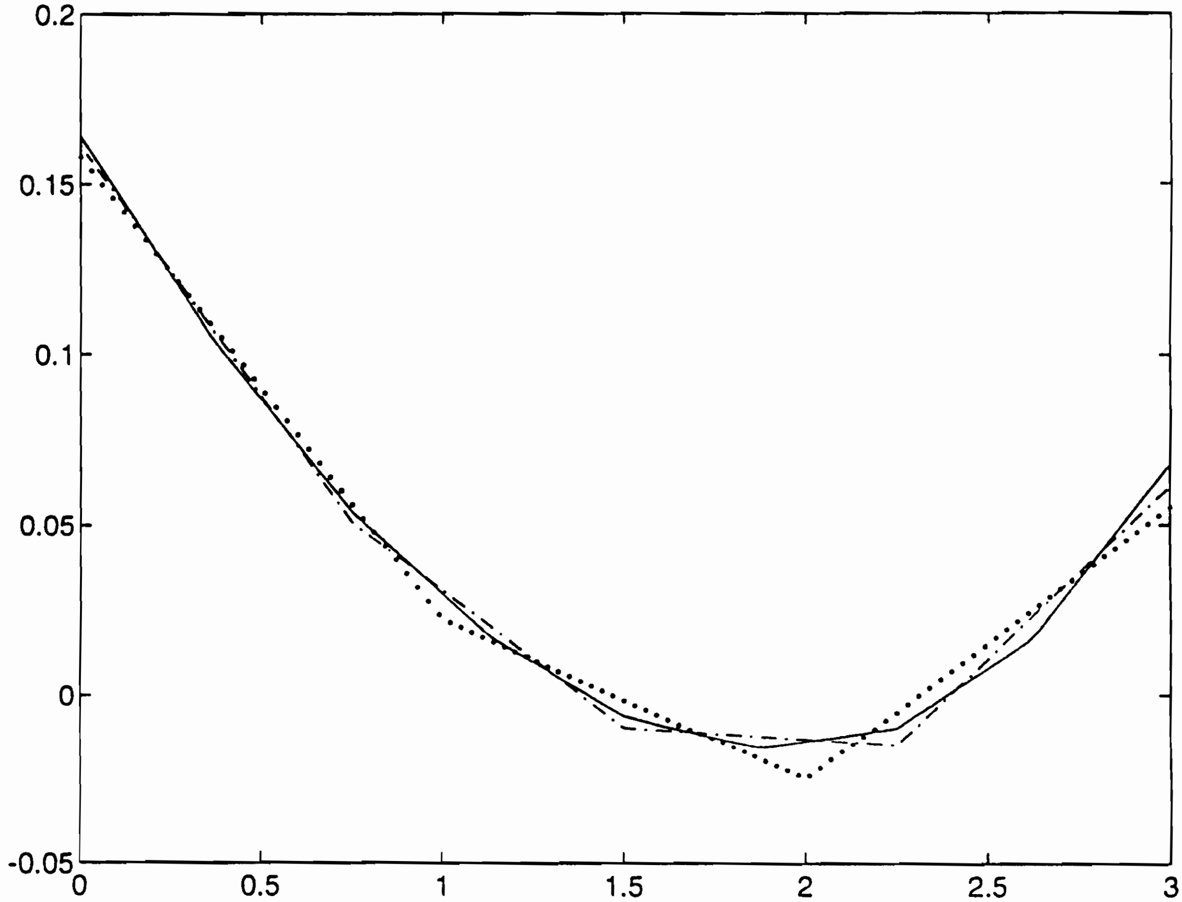
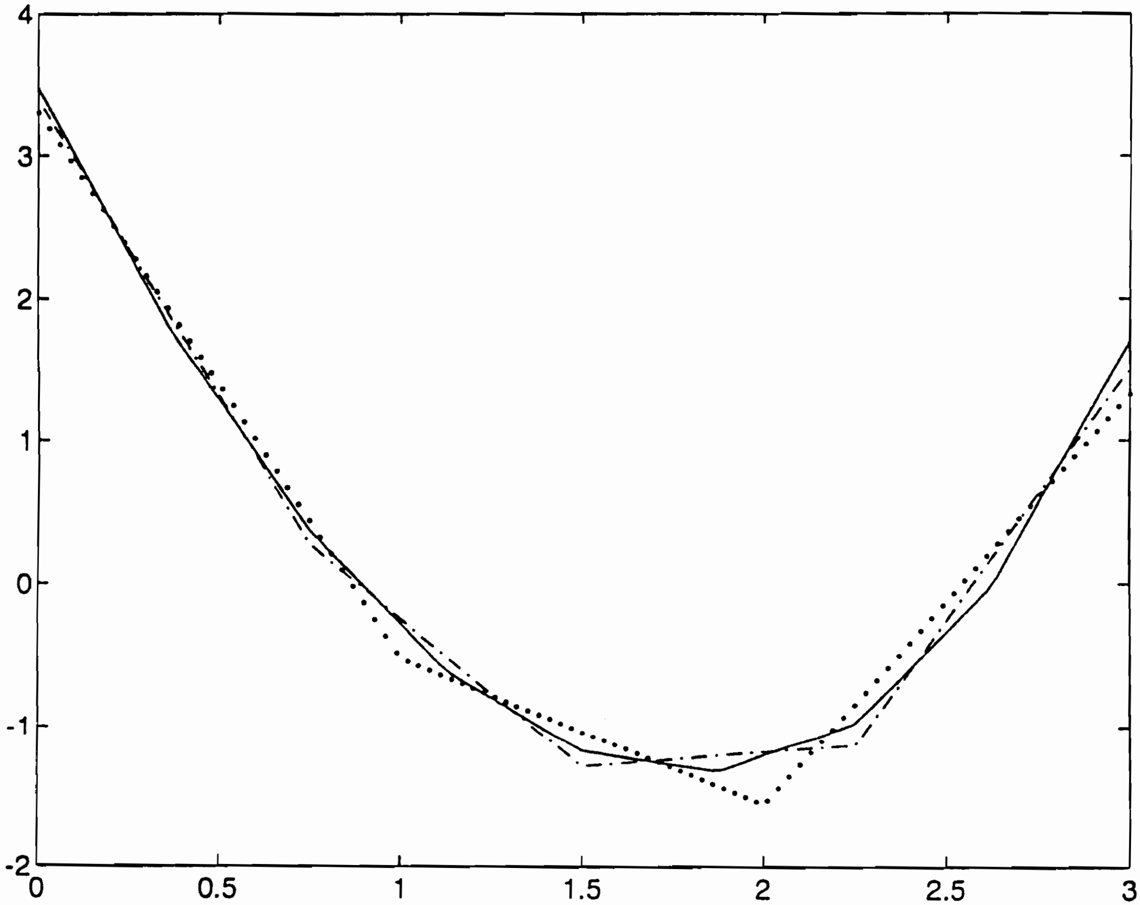


Figure 3.30 Estimator Gains L3 (5:5): n=3(.),n=4(-.),n=8(-)



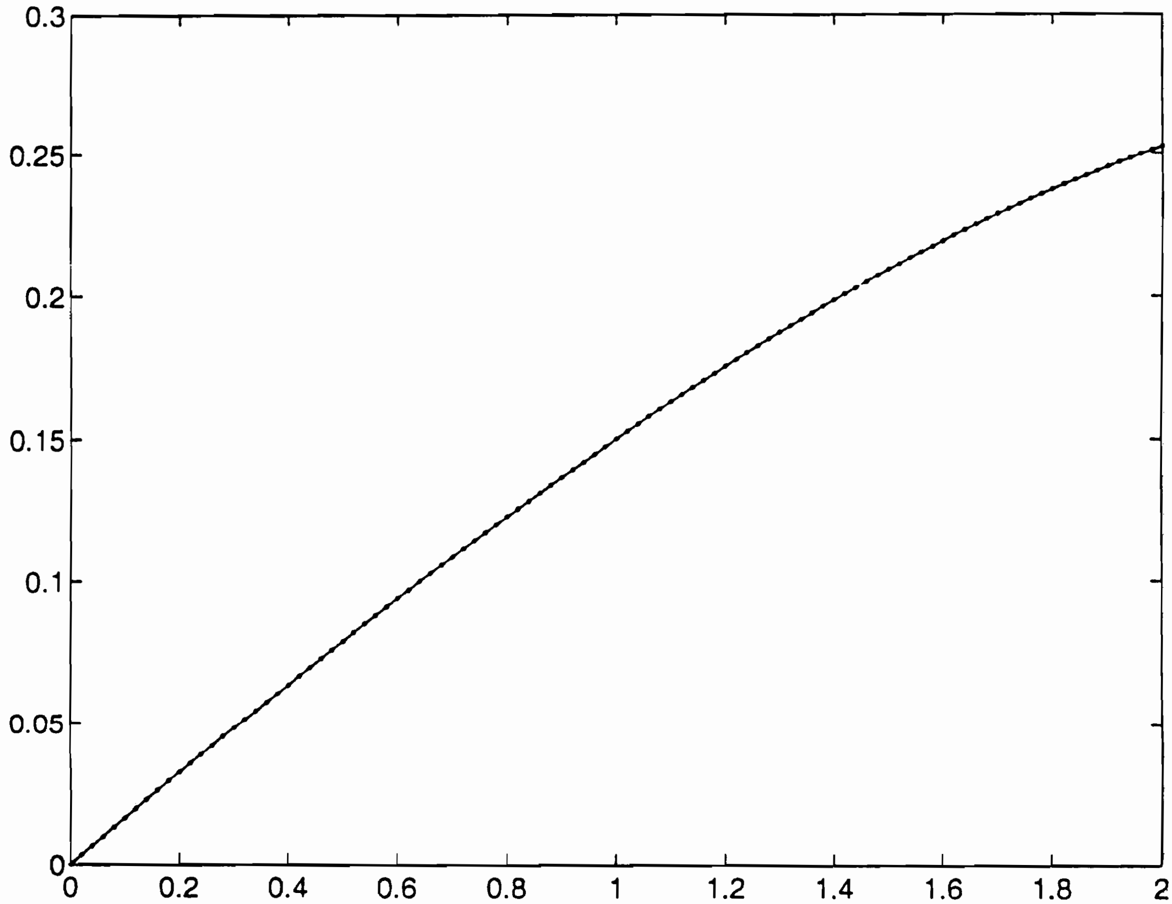


Figure 3.31 Estimator Gains L5 (5:1): n=3(.),n=4(-.),n=8(-)

Figure 3.32 Estimator Gains L5 (5:2): n=3(.),n=4(-.),n=8(-)

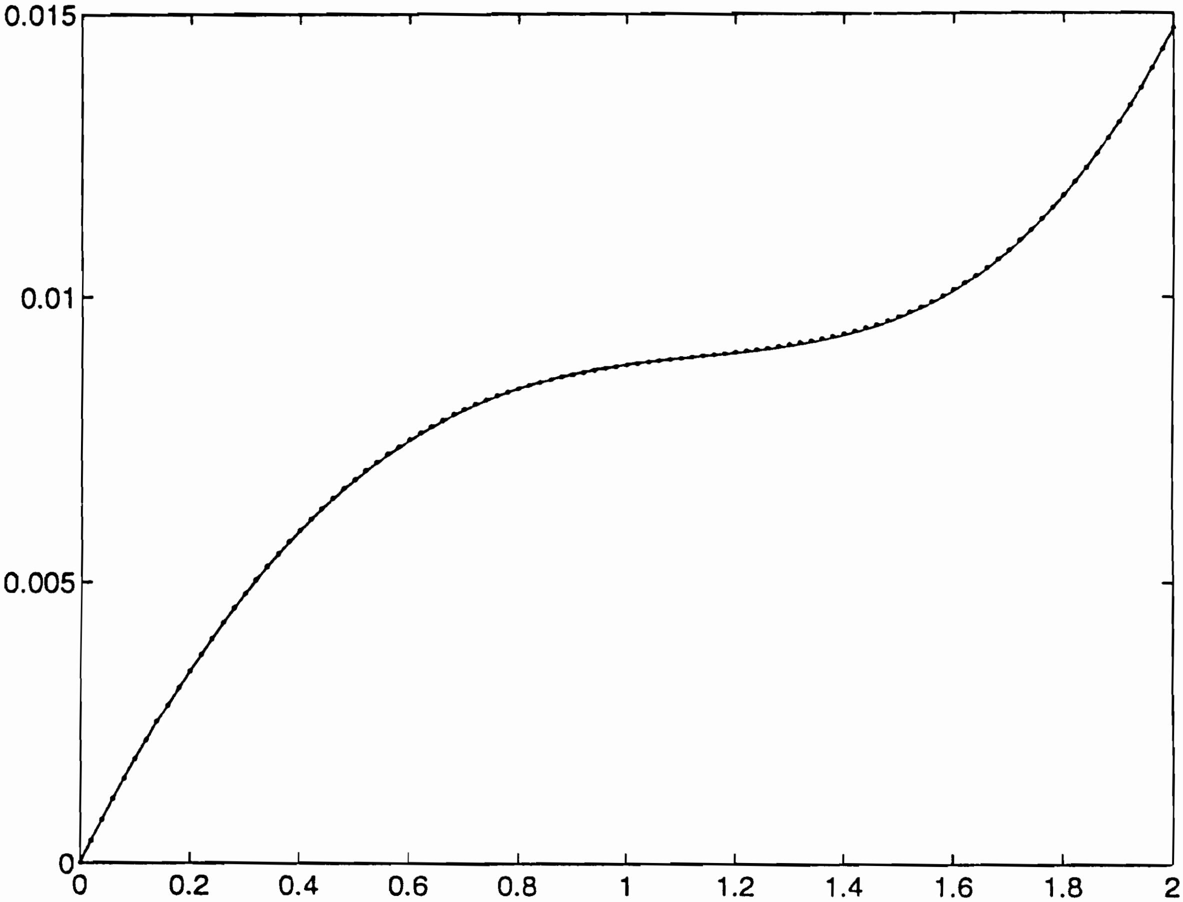


Figure 3.33 Estimator Gains L5 (5:3): n=3(.),n=4(-.),n=8(-)

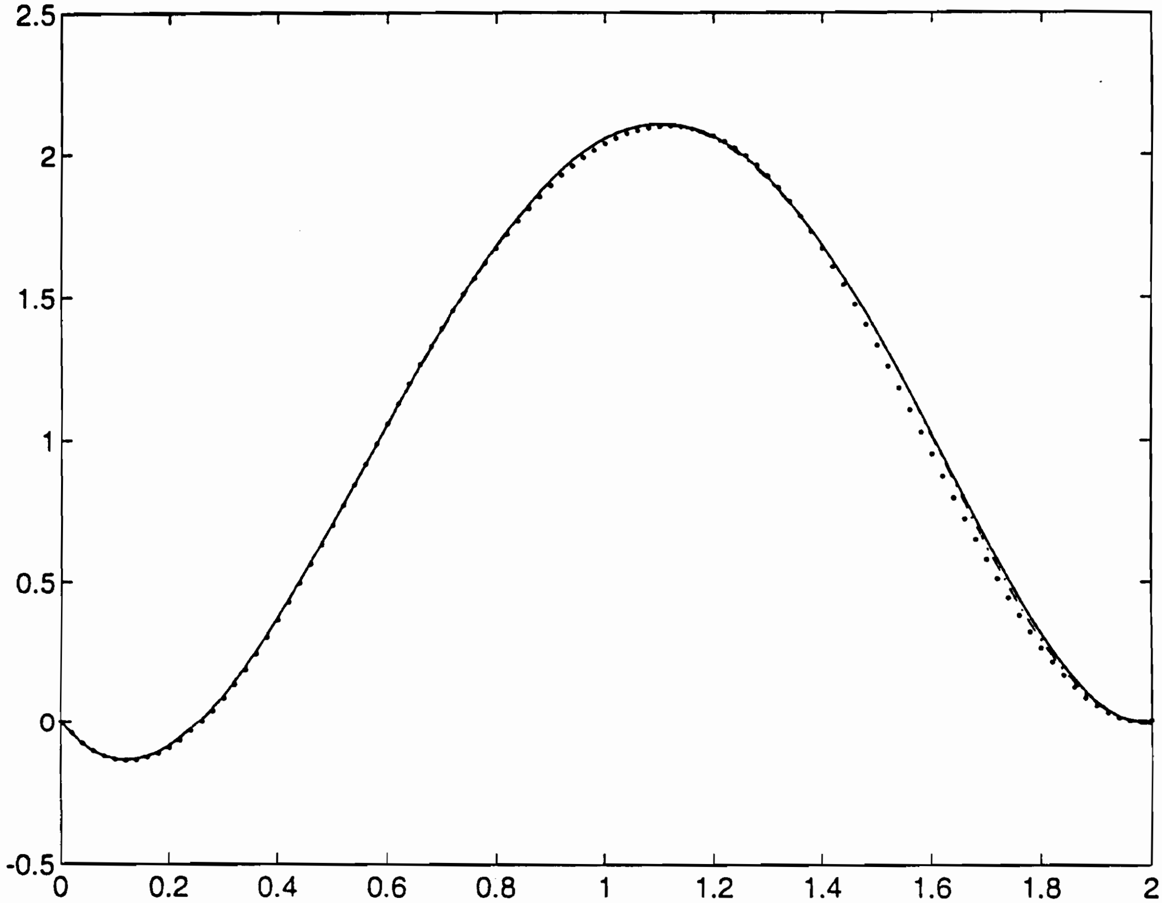


Figure 3.34 Estimator Gains L5 (5:4): n=3(.),n=4(-.),n=8(-)

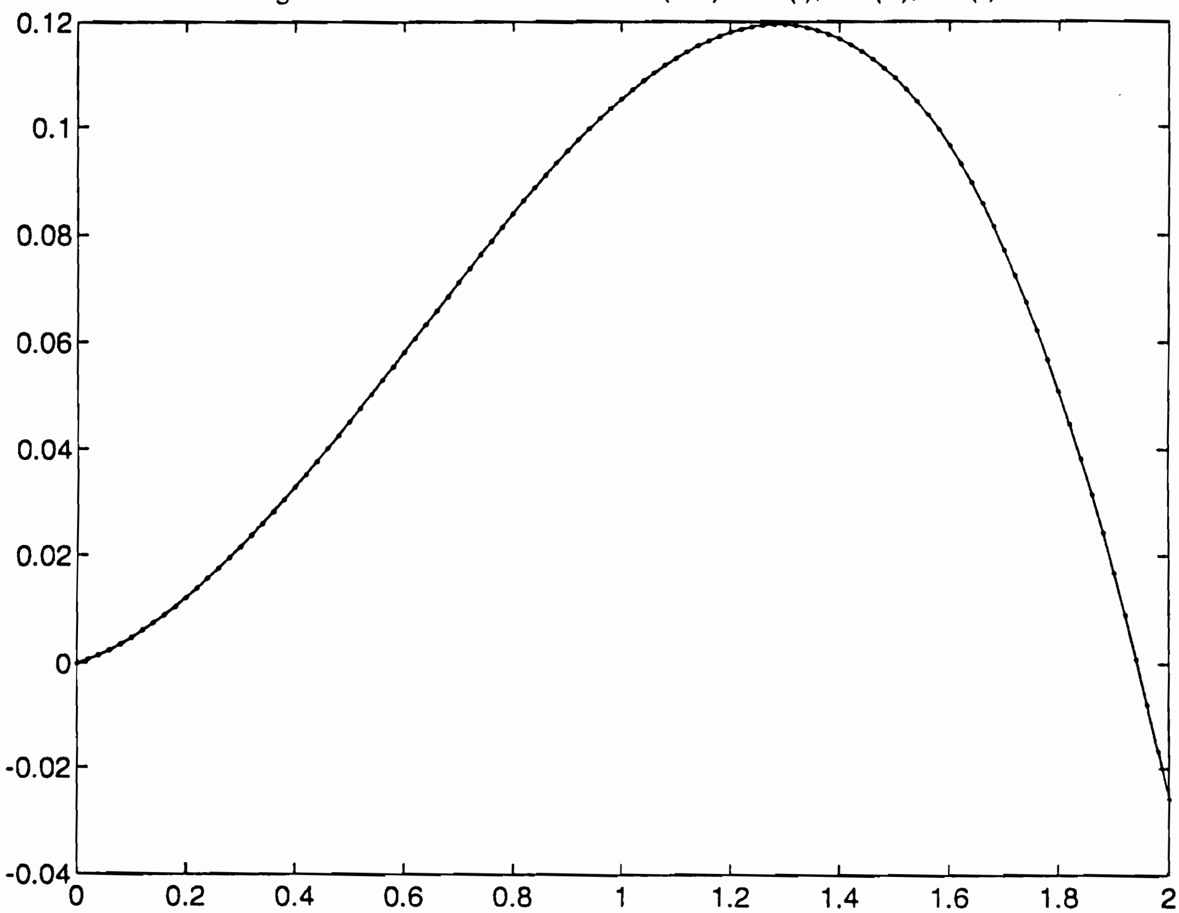


Figure 3.35 Estimator Gains L5 (5:5): n=3(.),n=4(-.),n=8(-)

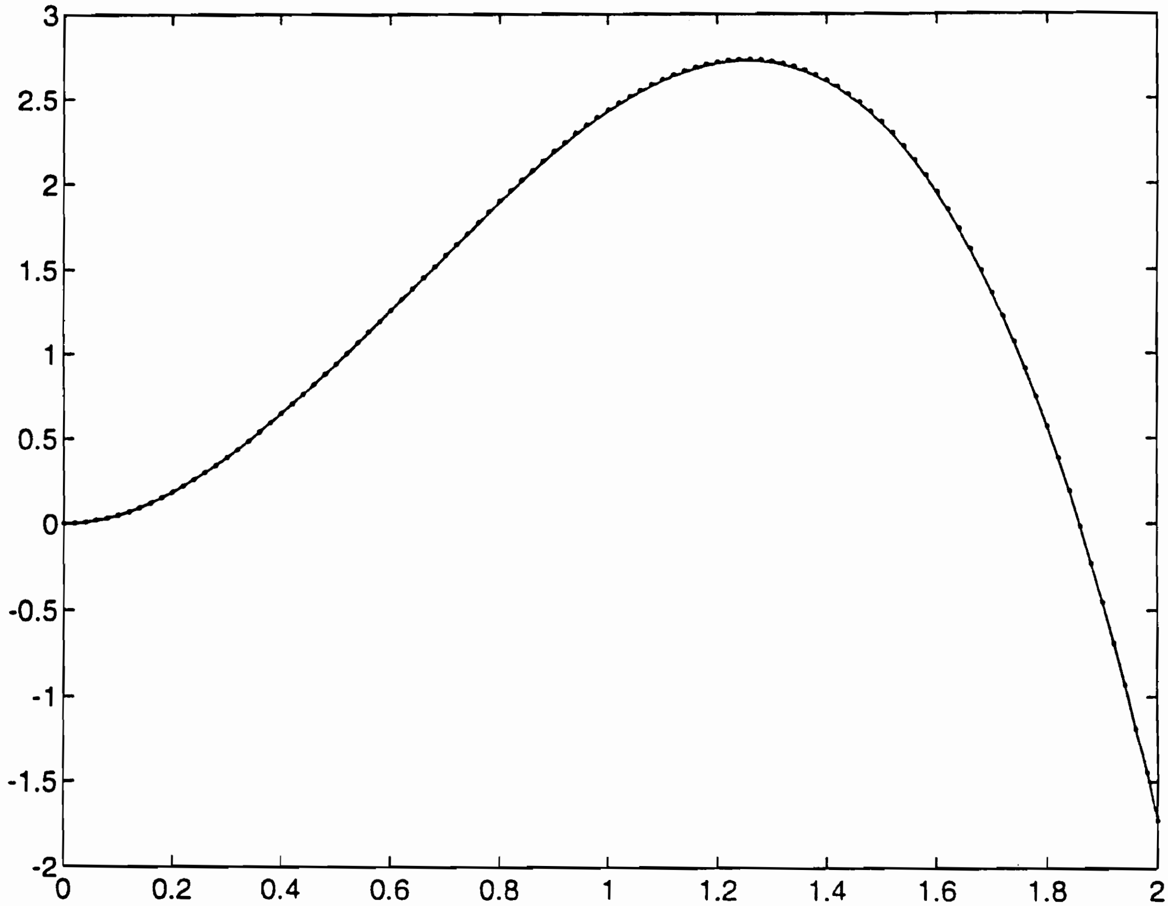


Figure 3.36 Estimator Gains L6 (5:1): n=3(.),n=4(-.),n=8(-)

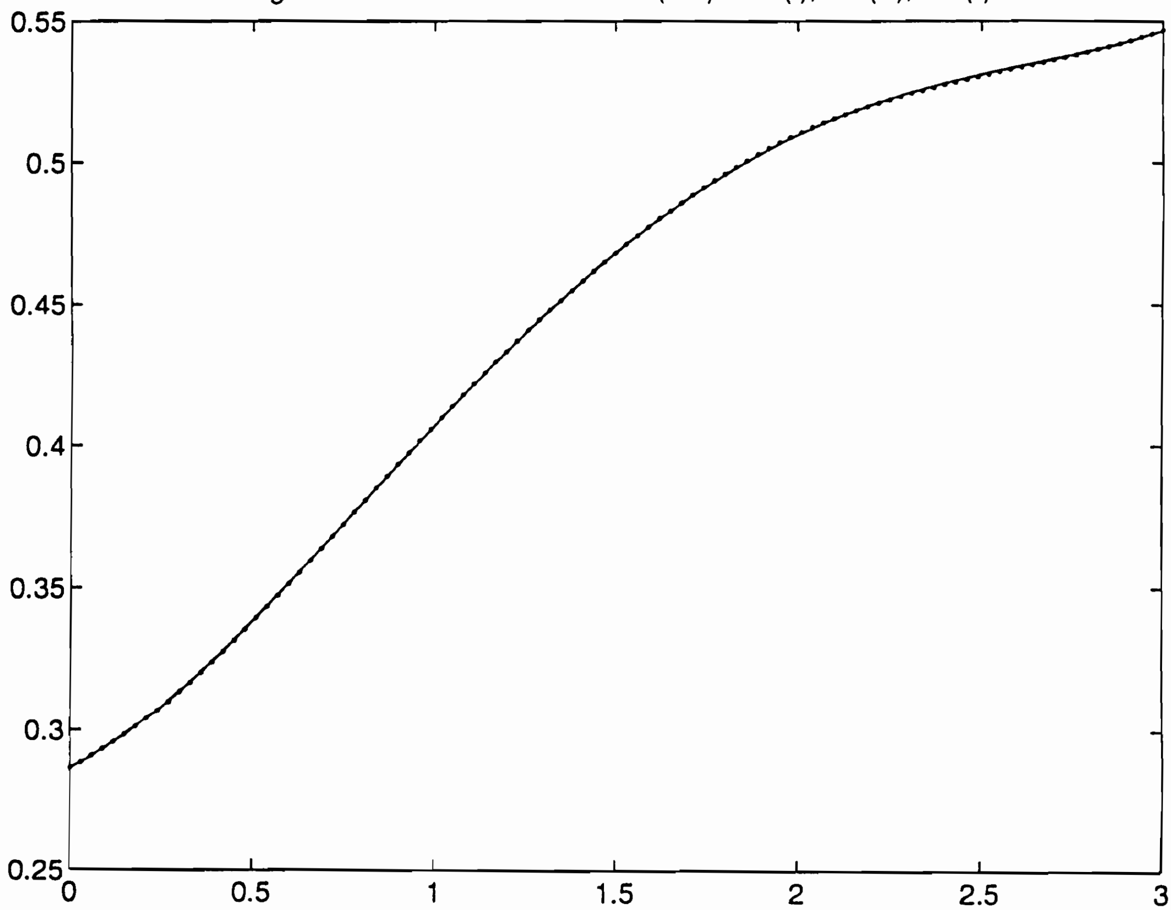


Figure 3.37 Estimator Gains L6 (5:2): n=3(.),n=4(-.),n=8(-)

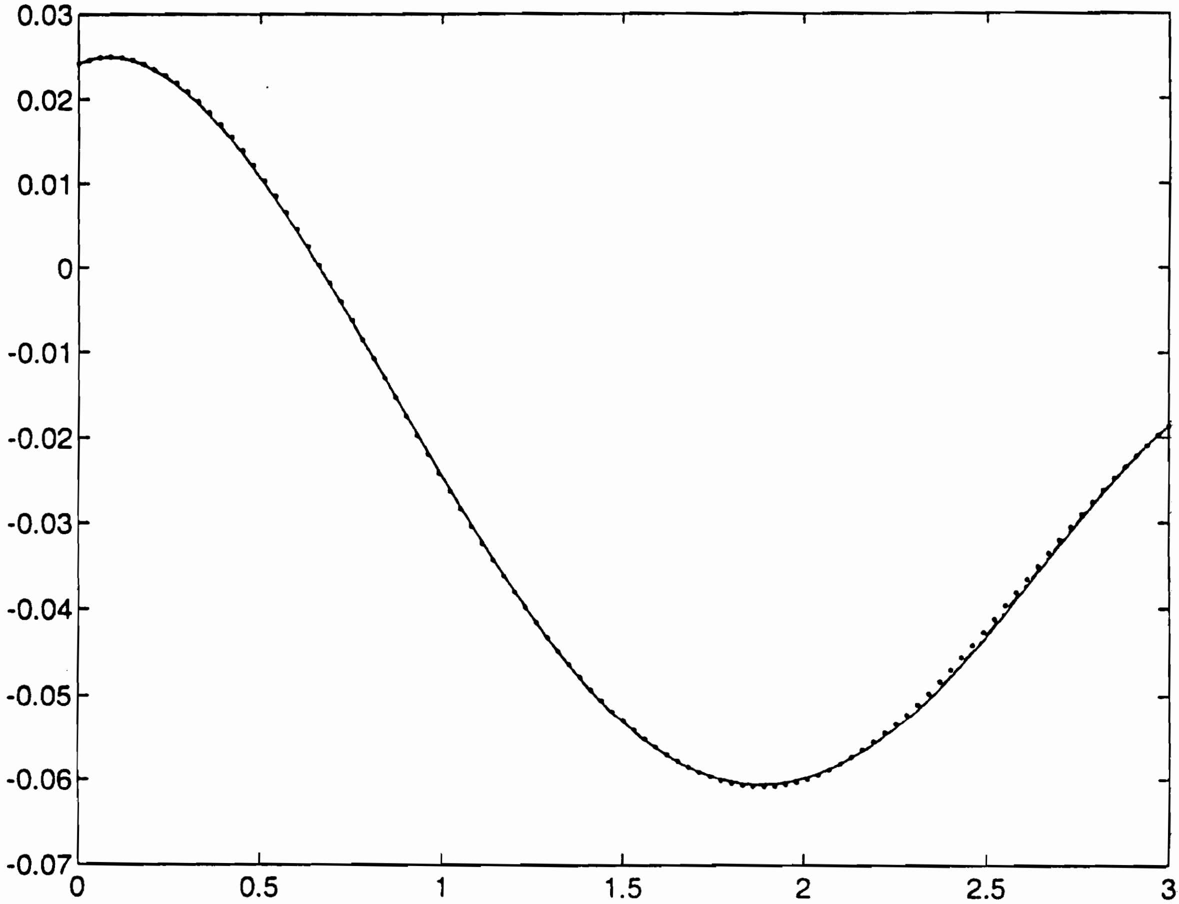


Figure 3.38 Estimator Gains L6 (5:3): n=3(.),n=4(-.),n=8(-)

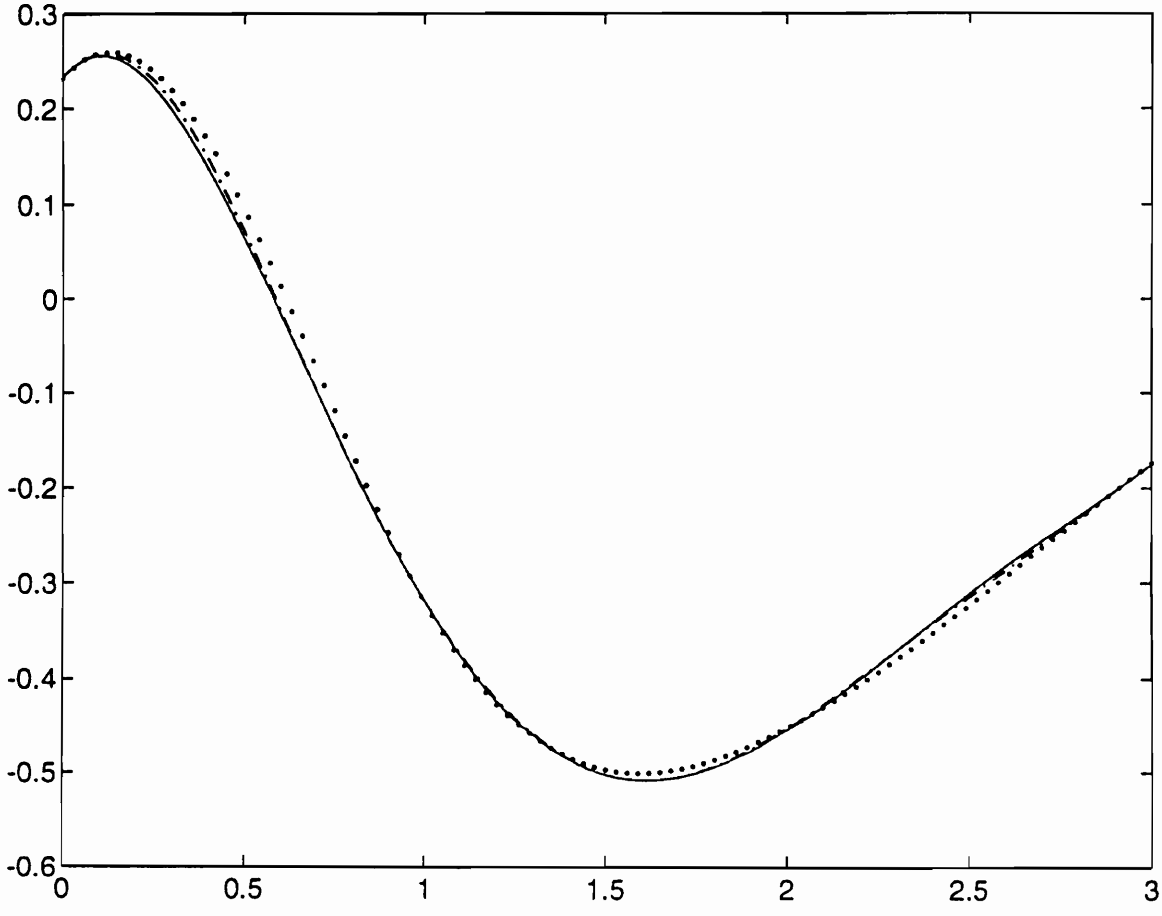


Figure 3.39 Estimator Gains L6 (5:4): n=3(.),n=4(-.),n=8(-)

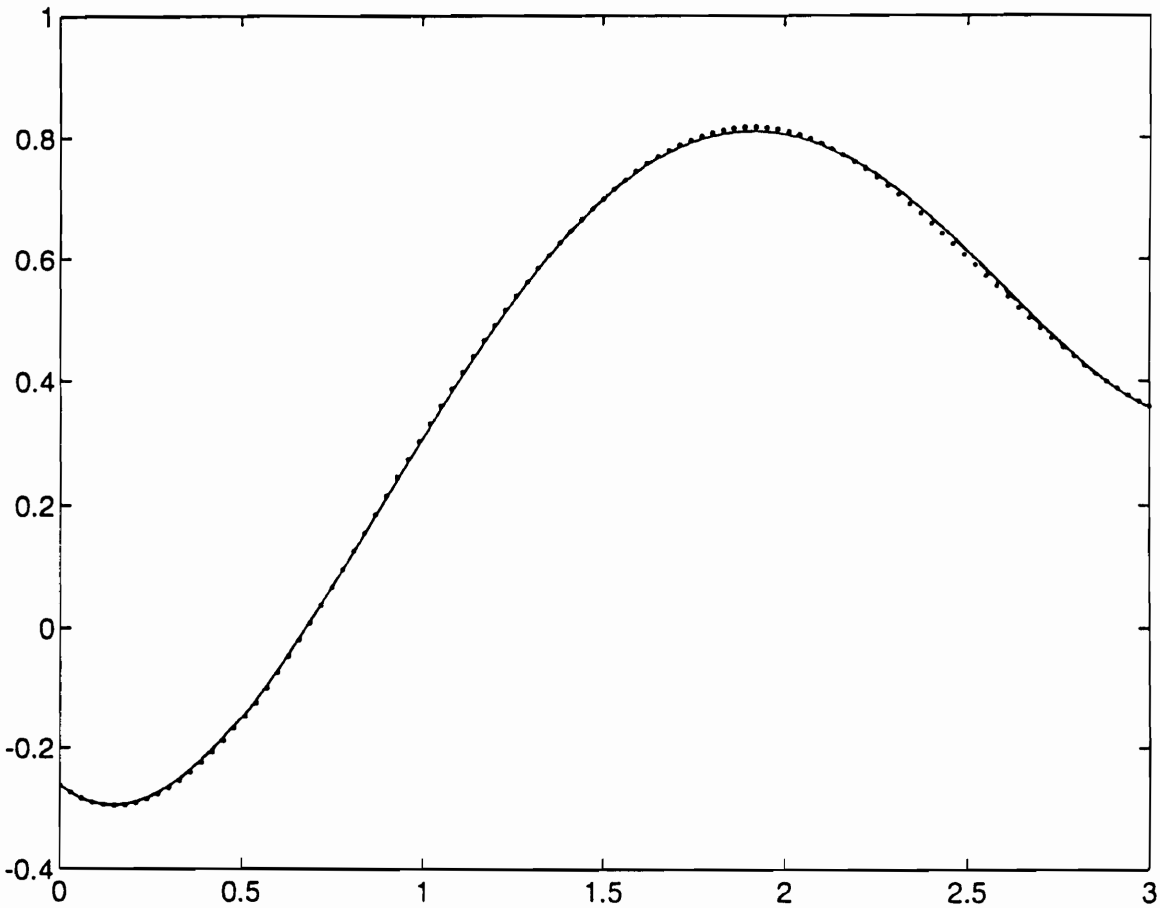


Figure 3.40 Estimator Gains L6 (5:5): n=3(.),n=4(-.),n=8(-)

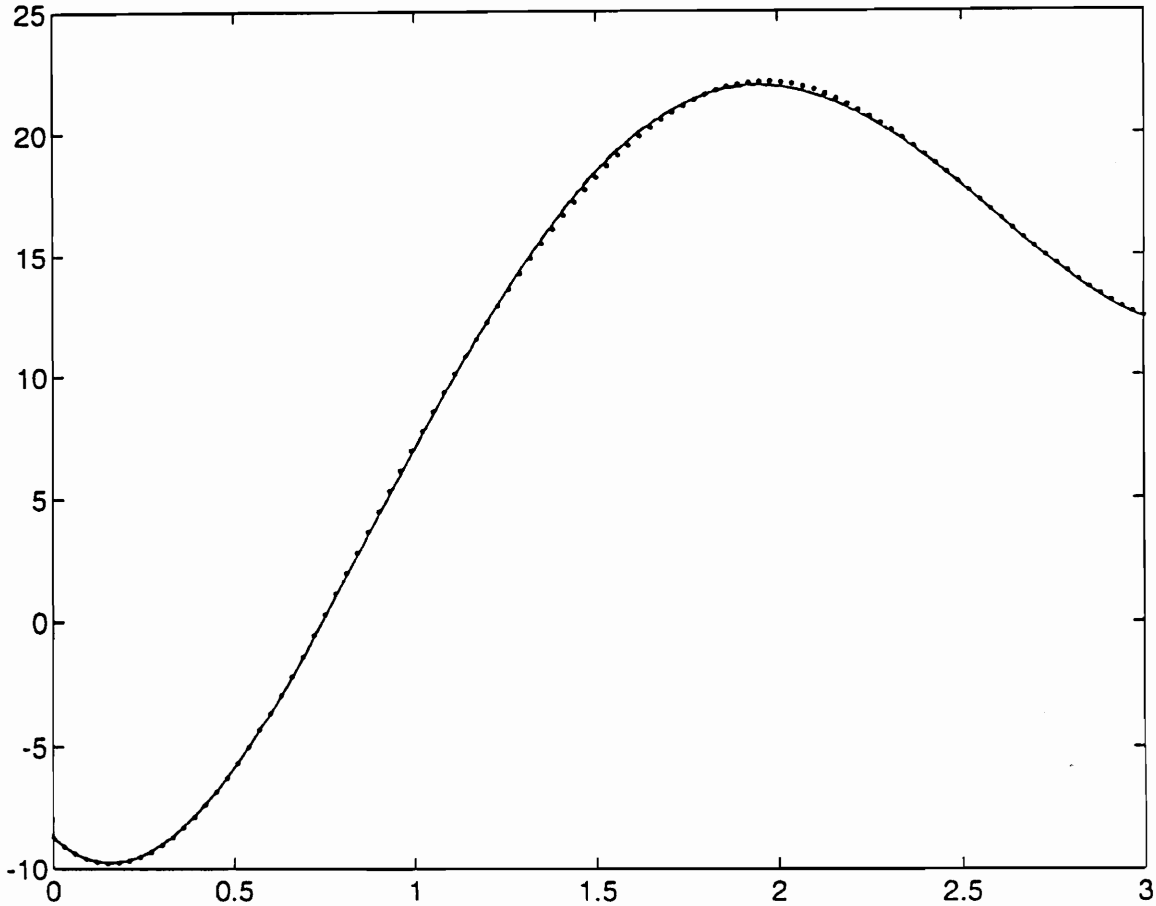
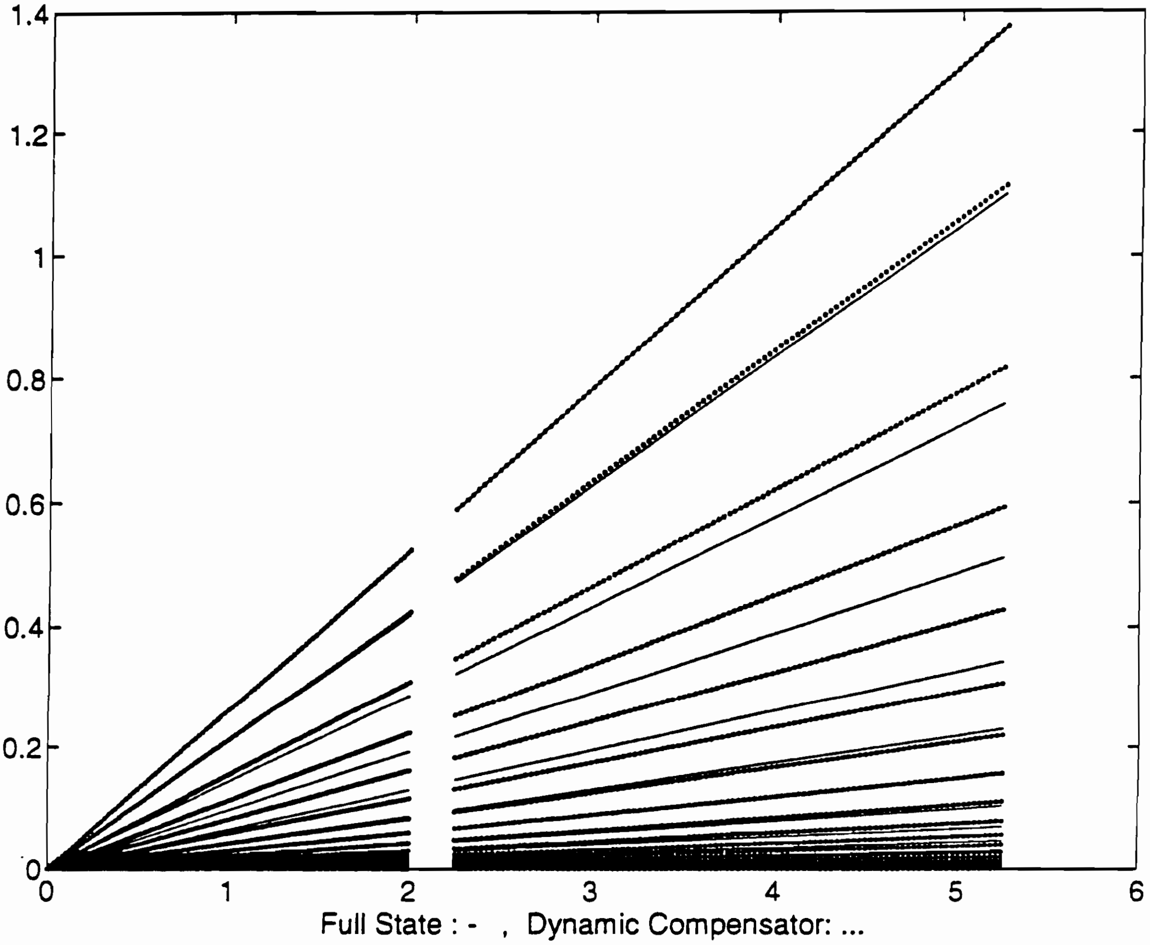


Figure 4.1 Simulation 1 (Initial Condition 1): t=0:3:120



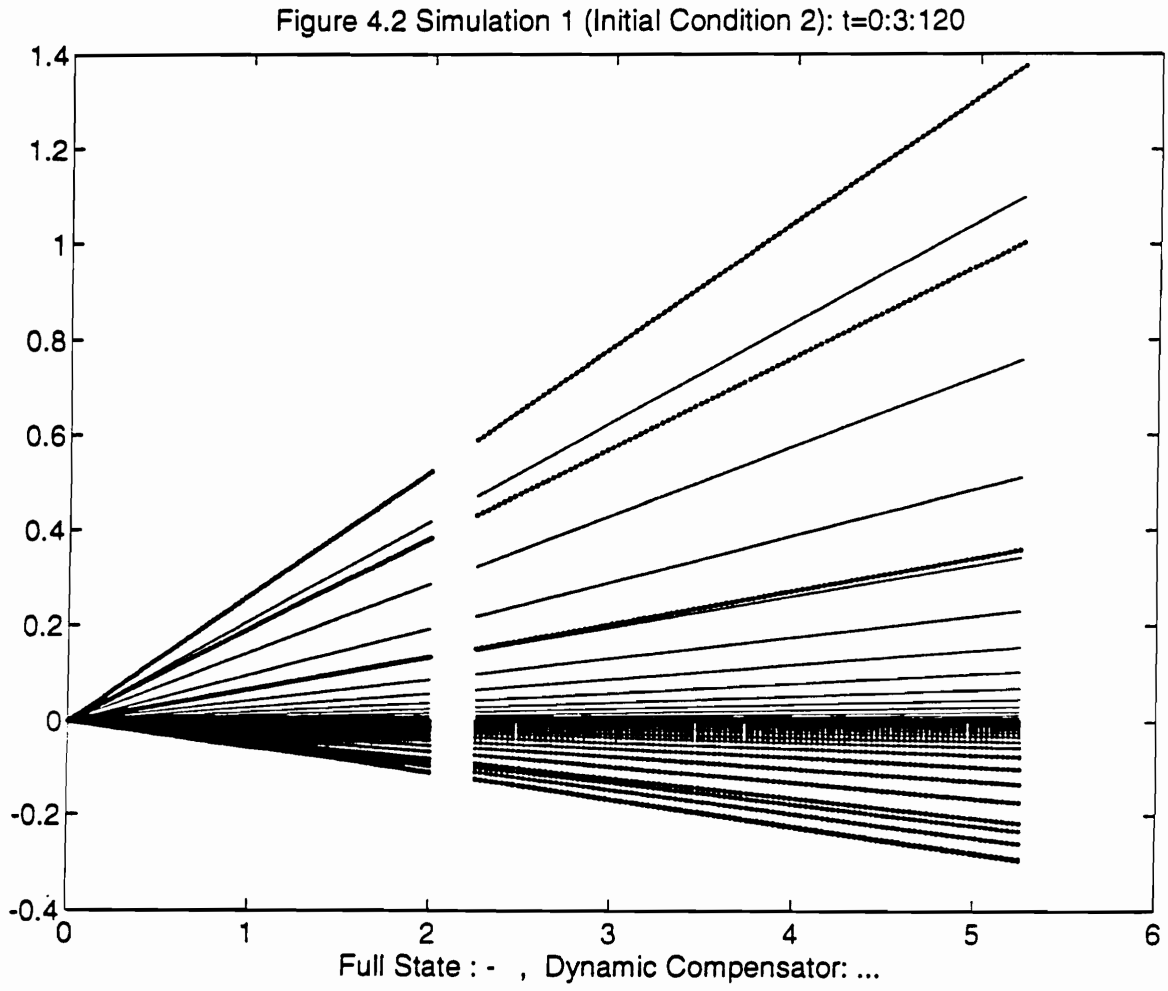
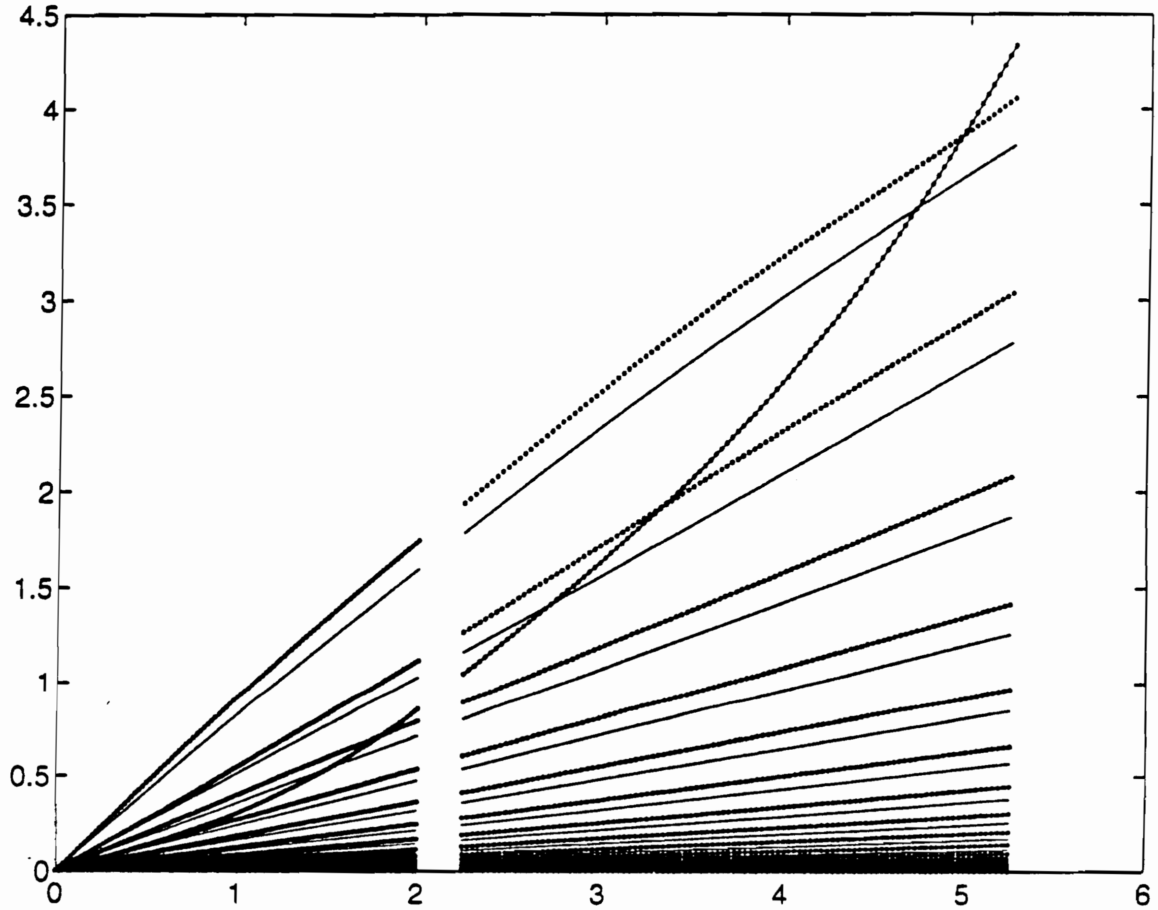


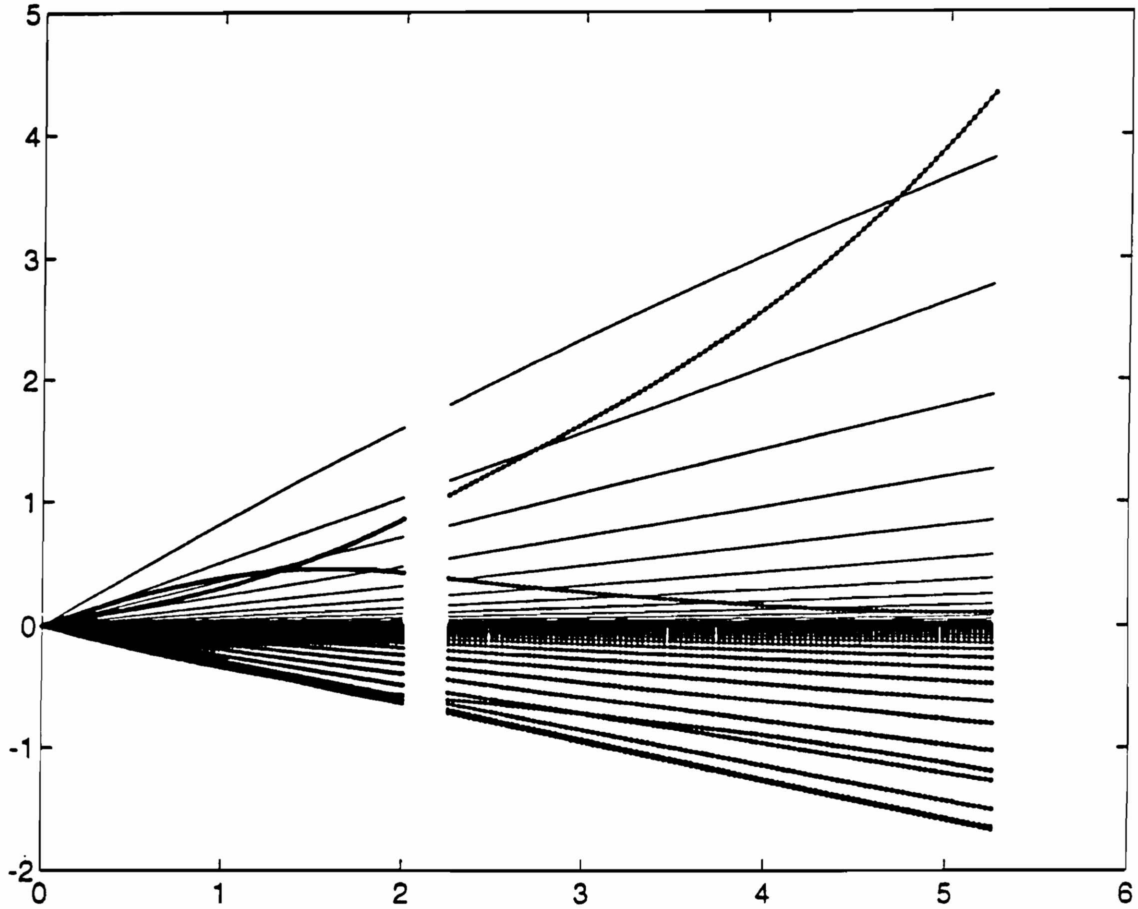
Figure 4.2 Simulation 1 (Initial Condition 2): t=0:3:120
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
1
2
3
4
5
6
Full State : - , Dynamic Compensator: ...

Figure 5.1 Simulation 2 (Initial Condition 1): t=0:3:120



Full State: - , Dynamic Compensator: ..

Figure 5.2 Simulation 2 (Initial Condition 2): t=0:3:120



Full State: - , Dynamic Compensator: ..

Reference

[1] T. Kato, Perturbation Theory for Linear Operators, 2nd ed., Springer - Verlag, New York 1980.

[2] R. Curtain and A. Pritchard, Infinite Dimensional Linear Systems Theory, Springer - Verlag, New York, 1978.

[3] J. S. Gibson and A. Adamian, Approximation Theory for Linear - Quadratic - Gaussian Optimal Control of Flexible Structures. SIAM J. Control and Optimation, Vol. 29, No. 1 pp 1 - 37, 1991.

[4] J. Gibson, The Riccati Integral equations for optimal control problems on Hilbert Spaces, SIAM J. Control and Optim.,17 (1979), pp 537 - 565.

[5] B. B. King, Modeling and Control of Multiple Component Structures, Dissertation of Ph. D., Clemson University, 1992.

[6] H. T. Bank and J. A. Burns, Introduction to Control of Distibuted parameter systems, to appear, Birkhauser, Boston, 1992.

[7] R. E. Showalter, Hilbert Space Methods for Partial Differential Equations, Pitman Publishing, London, 1977.

[8] H. T. Banks, K. Ito, and Y. Wang, Computational Methods for Identification and Feedback Control in Structures with Piezoceramic Actuators and Sensors. To appear.

[9] H. T. Banks and K. Kunisch, The Linear Regulator Problem for Parabolic Systems, SIAM. J. Control and Opti., 22, pp 684 - 698, 1984.

[10] K. Ito. Finite Dimensional Compensators for Infinite Dimensional Systems via Galerkin - Type Applications, SIAM J. control and Opti., 28, pp1251 - 1269, 1990.

[11] R. Triggiani, On the Stabilizability Problem in Banach Space, J. Math. Anal.

Appl., 52(1975), pp383 - 403.

[12] J. M. Schumacher, A Direct Approach to Compensator Design for Distributed Parameter Systems, SIAM J. Control and Opti., 21(1983), pp 823 - 836.

[13] H. T. Banks and K. Kunisch, Estimation Techniques for Distributed Parameter Systems, Birkhauser, Boston, 1989.

[14] D. Russell, Mathematical Models for the Elastic Beam and Their Control - Theoretic implications in Semigroups, Theory and Applications, 2, Brezis, et. al eds. p. 177, Longman Scientific Technical, 1986.

[15] G. Strang and G. J. Fix, An analysis of the Finite Element Method, Prentice - Hall, Inc, Englewood Cliffs, NJ, 1973.

[16] F. Conrad, Stabilization of Beams by Pointwise Feedback Control, SIAM J. Control and Optim., 28, 1990, pp. 423 - 437.

[17] G. Da. Prato., I. Lasiecka , R. Triggiani, A Direct Study of Riccati Equations Arising in Hyperbolic Boundary Control Problems, J. Diff. Eq. , 64 (1) (1986), 26 -47.

[18] I. Lasiecka, Galerkin Approximations of Infinite - Dimensional Compensators for Flexible Structures with Unbounded Control Action, Acta Applicandae Mathematicae 28: 101 - 133, 1992.

[19] I Lasiecka, Finite Element Approximations of Compensator Design for Analytic Generator with Fully Unbounded Controls / Observations, to appear in SIAM J. Control and Optim..

[20] R. Curtain, Finite Dimensional Compensators For Parabolic Distributed Systems with Unbounded Control and Observation, SIAM J. Control 22(1984),255 -277.

[21] J. S. Gibson, Approximation Theory for Linear Quadratic Gaussian Control of

Flexible Structures, SIAM J. Control and Optim., 29 (1990), pp 1 - 38.

[22] V. Ihomee, Galerkin Finite Element Methods for Parabolic Problems, Lecture Notes in Math., Vol. 1054, springer, Berlin, 1984.

[23] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer - Verlag, 1983.

[24] S. Chen and R. Triggiani, Proof of Extensions of Two Conjectures on Structural Damping for Elastic Systems, Pacific J. Math., 136 (1989), pp. 15 - 55.

Vita

Wei Huang was born on March 14, 1955 in Changsha, China. He graduated from the Changsha 10th High School in 1972. His education was interupted for six years during the Cultural Revolution. He received the B. S. degree in Mathematics from Hunan University in Changsha, China in 1982. He taught calculus at Medical College of Hongyang, China, from 1982 to 1984. He received the M. S. in Mathematics from Hunan University in Changsha, China in 1986. He taught Mathematics at Hunan University from 1986 to 1989. He received the Ph. D. in Mathematics in 1994 from Virginia Polytechnic Institute and State University. He is a member of AMS.