A Flexible Construction and Improvement Heuristic For The Quadratic Assignment Problem

by

P Rajgopal

Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of
Master of Science
in
Industrial Engineering and Operations Research

APPROVED:

Dr. H. D. Sherali (Chairman)

Dr. S. Sarin

Dr. T. J. Greene

May 1985
Blacksburg, Virginia

A Flexible Construction and Improvement Heuristic For The Quadratic Assignment Problem

by

P Rajgopal

Dr. H. D. Sherali (Chairman)

Industrial Engineering and Operations Research

(ABSTRACT)

This thesis is concerned with the development of heuristic algorithms for the popular Quadratic Assignment Problem (QAP) which finds a wide variety of applications in various fields. This discrete optimization problem, which seeks the placement of m facilities on m locations in order to minimize a quadratic interactive cost, is well known to be NP-hard and turns out to be computationally intractable for even moderately sized problems. Hence, problems involving more than 12-15 facilities usually need to be analysed by approximate solution procedures.

The more successful heuristic procedures which exist for problem QAP are computationally intensive, some of these resulting from a premature termination of exact solution procedures. The motivation here is to develop a polynomial time heuristic which is effective with respect to the quality of solutions obtained, while at the same time not being computationally very expensive.

The method proposed herein is flexible in that one can operate it to suitably trade solution quality against effort as desired, and is portable in that the modules used as building blocks can be employed in conjunction with other heuristics as well. Computational experience on test problems found in the literature is provided to evaluate the worth of this method.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to Dr. Hanif D. Sherali under whose supervision this thesis was completed. He is a remarkable educator and without his inspiring guidance, encouragement and patience, I doubt I would have ever completed.

I would like to thank my committee, Dr. Timothy J. Greene and Dr. Subhash Sarin. Each of these people have helped me with their valuable suggestions and have helped me develop as a graduate student.

I would like to express my gratitude to my parents for their continued encouragement and interest.

Finally I would like to thank my wife Meena, for the love, patience and understanding she has shown and especially the long hours she has put in along with me towards the typing of this thesis.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

CHAPTER I

INTRODUCTION

The Quadratic Assignment Problem (QAP) seeks the assignment of some m indivisible entities to some m mutually exclusive locations in order to minimize a total quadratic interaction cost. It first came into existence when Koopmans and Beckmann (1957) formulated problems concerning the allocation of plants to sites on a one to one basis, where there was an interchange between every pair of plants. Their objective was to minimize the total interaction cost proportional to the flow of material between plants and the distance between the plants. Considerable work has been done since then by mathematicians and engineers alike on problems with this structure. However, the problem still remains of considerable interest due to the variety of possible solution approaches and the number of applications it has in various fields.

Mathematically, defining the binary variable,

$$x_{ij} = \begin{cases} 1 \text{ if facility } i \text{ is situated at location } j \\ & \text{for } i,j=1, \ldots, m, \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

this problem can be stated as follows.

$$\underline{\text{QAP}}\text{: minimize } \{1/2 \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{\ell=1}^{m} c_{ijk\ell}\, x_{ij}\, x_{k\ell} : x \in X\} \qquad (2)$$

where

$$X = \{x = (x_{11}, \ldots, x_{mm}) : \sum_{i=1}^{m} x_{ij} = 1 \text{ for } j=1, \ldots, m$$

$$\sum_{j=1}^{m} x_{ij} = 1 \text{ for } i=1, \ldots, m$$

$$x_{ij} \text{ binary, for } i,j=1, \ldots, m\} \qquad (3)$$

is the set of feasible assignments, and where c_{ijkl} is the interaction cost resulting from the simultaneous location of facility i at location j and of facility k at location l. Often, this interaction cost results from a flow of material that must occur between facilities i and k, wherever they are ultimately placed, with the cost being proportional to the distance travelled. Hence, one usually assumes (as we do in this thesis) that,

$$c_{ijk\ell} = f_{ik}\, d_{j\ell}, \quad i,j,k,\ell=1, \ldots, m \qquad (4)$$

where $f_{ik} = f_{ki}$ is the cost of separating facilities i and k by a unit distance, based on the flow that must occur between them, and $d_{jl} = d_{lj}$ is the distance between location sites j and l, measured using any appropriate distance metric. Furthefmore, since an assignment solution corresponds to a permutation of locations among the facilities (or vice versa), a solution to Problem QAP is often represented as

$$\{\ell(i),\ i=1,\ \ldots,\ m\} \tag{5}$$

where l(i) is the location at which facility i is placed, for i=1, ..., m.

In general the facilities could represent machines, plants, departments, circuit components etc., and similarly the locations could represent appropriate sites for these facilities.

CHARACTERISTICS OF THE QAP

The total cost of assigning facilities to sites is a product of two quantities - the interaction between pairs of facilities and the distance between these pairs of facilities summed over all such pairs, which gives the problem its name. It is this nonlinearity and discrete nature of the problem which creates difficulties in developing efficient solution techniques. One exact way of solving problems of this nature would be to enumerate all possible permutations. For a problem with m facilities and m locations, the number of permutations is m!.

In fact Horowitz and Sahni (1978) have shown that for all values of $\varepsilon > 0$, the optimization problem of determining a solution to QAP within ε of the optimum is NP-hard. This fact is computationally manifested rather assertively in that problems of size $m > 12$ remain usually unsolved by even the best of exact (branch and bound) algorithms after considerable computational effort. Although the objective function can be made convex, the combinatorial nature of the problem which presents $m!$ solutions of the type (5) to be compared (with respect to a nonlinear objective), renders the task of solving QAP exactly prohibitive. For example, Nugent, et al (1968) have estimated that for $m = 12$, for which $m!$ is approximately $4.79 * 10^8$, a total enumeration on an IBM 7090 computer would consume 3 years of cpu time. Consequently, heuristic algorithms are indispensable in analyzing quadratic assignment problems.

APPLICATIONS OF THE QUADRATIC ASSIGNMENT PROBLEM.

Ever since this problem was first formulated, it has been a source of considerable interest principally because of the wide range of problems which can be formulated as quadratic assignment problems. These applications include facility location problems, building layout problems, control panel layout problems, wiring problems, and suburban land-use planning problems, among others. Francis and White (1974) and Sherali (1979) discuss these in detail and these are summarized below.

1. The layout problem of allocating m machines to m sites on a shop floor on a one to one basis is an example of the quadratic assignment problem. In this example each machine can be alloted to one and only one location and likewise, each location can accommodate only one machine. The inter flow of products between a pair of machines can be denoted as f_{ij} and the distance between this pair can be denoted as d_{kp} . The total cost due to this interaction between machine i and j is then given by $f_{ij}d_{kp}$. Based on this a total flow times distance expression would need to be minimized.
2. The allocation of departments in the various rooms of a plant where there is interaction between the departments in the form of mail or people walking between the pairs of departments is an example of the Quadratic Assignment Problem. Elshafei (1977) discusses one such problem in the context of a hospital layout design, and his data comprises one of the test problems considered in this thesis. Of course as pointed out by Elshafei (1977), some modifications to the quadratic assignment solution may be required based on ventilation, plumbing and wiring considerations.

3. Design of control panels to minimize the expected time required to execute a sequence of operations is another example. If the sequence of operations involves the adjustments of control knob i, located at position k, followed by the adjustment of control knob j, located at position p, then t_{ikjp} would represent the time required to go from position k to position p. The objective would then be to minimize a measure of total time. Dorris (1971) discusses such applications with the objective of reducing total eye travel or hand motion between the instruments during the course of performing different maneuvers.
4. The location of items in a storage bin in a storeroom is another example. A stock keeper wants the items placed in such a way that the time required to fill a customer's order is minimized. Thus if the order required the selection of item i located in bin k followed by the selection of item j located in bin p, the time required to walk from bin k to bin p is given by t_{ikjp}. Based on this, a total time expression would need to be minimized.
5. A problem faced with the rapid advance of electronics is popularly known as the backboard wiring problem. This was initially researched by Loberman and Weinberger (1957) and Steinberg (1961). The problem here is to place electronic chips on a board such that the length of wire interconnecting these chips is minimized. The purpose

here, besides reducing construction and maintainance costs, is to provide a good circuit design with a minimal amount of stray capacitance, delay line effects, cross talk and shielding requirements. The flow here would be the number of wires interconnecting chips i and j, and distances would be straight line distances between location positions. One of Steinberg's problem will be considered in this thesis.

6. Another application is in suburban land use planning. This problem is concerned with allocating residential units, neighborhood centers, high schools, community and recreational areas, commercial centers, offices, industries, etc. to potential sites. Here the net utility of allocating a particular facility to a particular site depends on the locations chosen for the other facilities. Sharpe and Brotchie (1972) discuss the role playing by problem QAP in analytically addressing this type of problem.
7. Several other miscellaneous applications of problem QAP exist. Examples include the production scheduling problem of Geoffrion and Graves (1976), the scheduling of the U.S Coast Guard's high endurance cutter ships in the Pacific area as considered by Sibre (1977), the problem of scheduling classes and examinations in an educational system as formulated by Carlson and Nemhauser (1966), and

the problem of minimizing latency in magnetic drums on disc storage computers as discussed by Lawler (1963).

SCOPE OF THE THESIS

This thesis involves the study of a heuristic algorithm for the QAP. The method is mainly motivated by QAPs over grid layouts. The facilities in such problems are restricted to lie on grid points over rectangular regions as shown in Figure 1. The measure of distance will be rectilinear. Problems with this structure are very popular, since a majority of facility location problems have a rectangular layout. Most of the test problems available in literature possess this structure. (e.g. see Nugent et al (1968) and Steinberg (1961)).

Figure 1 represents a layout with nine locations, represented by the dots. It is emphasized here, that although this study considers many problems with this structure, the methodology can be applied to general situations as well.

THESIS ORGANIZATION

Chapter II gives a review of the existing methods for solving the quadratic assignment problem, Chapter III describes the proposed heuristic, along with a step by step solution of an illustrative example, and finally Chapter IV

1•   2•   3•

4•   5•   6•

7•   8•   9•

Figure 1. Example of a problem with a grid structure

gives the computational results on test problems in the literature.

CHAPTER II

A REVIEW OF EXISTING TECHNIQUES TO SOLVE THE Q.A.P

All approaches to solve Problem QAP can be classified under two headings namely Exact and Heuristic methods.

1. Exact Techniques
    a. Branch and Bound
        1) Single Assignment Algorithms
        2) Pair Assignment Algorithms
        3) Pair Exclusion Algorithms
    b. Techniques Not Employing Branch and Bound
2. Heuristic Techniques
    a. Construction Techniques
    b. Improvement Techniques
        1) Methods employing pairwise interchange
        2) Methods not employing pairwise interchange

These are discussed briefly below.

BRANCH AND BOUND TECHNIQUE (1.A)

This is an exact procedure of locating m facilities on m locations by enumerating in a controlled manner all the permutations of locating the m facilities on the locations. Many of the permutations are handled implicitly, thus reducing computational effort. This idea was first introduced by Little et al (1963) in the context of solving the travelling salesman problem.

The general method employed is as follows. Starting from node zero at which stage no facility is fixed, branches are created. Each branch represents the allocation of a facility to a location or vice versa and essentially represents a further partitioning of the problem. Each branch terminates in a node. A node in the enumeration tree represents a subproblem at which the solution is restricted to satisfy the conditions imposed on the branches connecting it to node zero. Thus, following a path from node zero up to an end of the tree determines the allocation of facilities to locations. Facilities or locations restricted or assigned at any particular node are called fixed. The unassigned or unalloted facilities or locations are called free. Creating branches until all the ends of the tree, i.e., until all the facilities are fixed, is equivalent to enumerating all the permutations of locating m facilities on m locations. This is however avoided by computing lower bounds. This is defined as follows. The lower bound at any partial placement defined at a node, is a value which indicates that every possible combination of the free facilities will yield an objective value higher than this value.

The lower bound at any partial placement can be broken down into three components:

1. A value b_1 which gives the interaction cost between the assigned facilities themselves.

2. A value b_2 which is a lower bound on the interaction between the assigned facilities and the unassigned facilities.

3. A value b_3 which is a lower bound on the interaction between the unassigned facilities themselves.

Thus if the objective value of some known feasible complete placement is lower than the lower bound of a partial placement then that node corresponding to the partial placement and its descendents can be eliminated from further consideration. This process of eliminating further branching from a node is called fathoming the node. The best known objective value is referred to as the upper bound or the incumbent value. Thus the importance of lower bounds in a branch and bound procedure is tremendous, since better (larger) lower bounds can fathom a greater percentage of the enumeration tree, thus reducing the overall computational effort. The different techniques presented in literature differ mainly in the computation of the lower bound. The different approaches using the branch and bound approach are briefly discussed below.

Single Assignment Algorithms (1.A.1). All algorithms in this class, associate a particular location or facility to correspond to each level in the branching process. For each location, the distance from all other locations is computed and this is arranged in a nonincreasing order. The facilities/locations are then assigned to each level in this order. A depth first approach is usually followed, i.e a last in first out node selection strategy is followed. Backtracking, which is the process of moving up the tree after reaching a level from which no more branching is possible, therefore occurs such that the earliest possible level immediately preceding the present level is selected for branching.

The interested reader is referred to the following papers for details on the different procedures existing in the literature.

1. Gilmore (1962)
2. Lawler (1963)
3. Graves and Whinston (1970)
4. Bazaraa (1975a)
5. Roucairol (1978)

Pair Assignment Algorithms (1.A.2). Unlike the previous class of algorithms, pair assignment algorithms assign pairs of facilities, i and j, to locations k and p. For a symmetric case where the flow from facility i to j and j to i are identical, there are $\frac{m(m-1)}{2}$ pairs of facilities and $\frac{m(m-1)}{2}$ pairs of locations. For the asymmetric case, there are m(m-1) pairs of facilities and locations. Thus if a four facility problem is considered with facilities A,B,C,D and locations 1,2,3 and 4 then the following combination of pairs exist:

Facility pairs (A,B), (A,C), (A,D), (B,C), (B,D), (C,D)

Location pairs (1,2), (1,3), (1,4), (2,3), (2,4), (3,4)

Moreover the cost of assigning pairs of facilities to pairs of locations can be computed easily as follows. If f_{AB} represents the total flow between facilities A and B, and d_{12} represents the distance between location 1 and 2, then the cost of locating the pair of facilities A and B at locations 1 and 2 is given by $f_{AB}d_{12}$. However, a feasible assignment of these pairs does not guarantee a feasible solution to the original problem. To illustrate this, assigning facility pair (A,B) to location pair (1,2) and facility pair (A,C) to (3,4) is a possible assignment of pairs. However, this does not give a one to one assignment of facilities to locations. Thus some additional constraints have to be added to the linear pair assignment problem such that a solution to this linear assignment problem, gives a feasible assignment for the original problem. The constraints that have to be added on ensure that if a pair of facilities i,j are allotted to locations k,p, then future allotments involving facilities i or j should involve locations k or p consistently and similarly locations k or p should be involved with facilities i or j.

Hence the enumeration occurs over the set of discrete variables. Thus,

$$Y_{ijkp} = \begin{cases} 1 & \text{if fac. i and k are assigned to loc j and p resp.} \\ 0 & \text{otherwise} \end{cases}$$

The solution technique proceeds level by level, fixing a new pair of facilities to a pair of locations. Whenever a node is fathomed backtracking occurs to the highest point in the tree having an active node. Rules are given to assign pairs of facilities to locations depending on the approach. The following papers give details on approaches using pair assignment algorithms.

1. Land (1963)
2. Gavett and Plyter (1966)
3. Pierce and Crowston (1971)
4. Smith (1975)

Pair Exclusion Algorithm (1.A.3). In the pair exclusion algorithms, solving a problem entails the exclusion of some solution at each stage of the step wise process. The solution procedure uses a technique similar to the pair assignment algorithm to create the pair assignments and also imposes the additional constraints as above. Now if the relaxed problem of assigning pairs produces an infeasible assignment as discussed, then there exist certain conflicting assignments which do not exist in the optimal solution. Hence the search tree branches into as many nodes as there are conflicting assignments, removing one conflicting assignment at each node created henceforth. Having branched, lower bounds are computed at each descendent node. For this purpose, an assignment problem with the additional pair exclusion constraints is solved at each node. If this does not result in a feasible solution, and if the lower bound does not exceed the upper bound at any node, then further branching is necessary. However, if a feasible solution is obtained, then that node is fathomed. At any stage, the procedure selects another node with the lowest bound and proceeds as before. When all nodes are fathomed the method stops.

The following papers deal with this class of methods.

1. Pierce and Crowston (1971)
2. Smith (1975)

COMPUTATIONAL CAPABILITIES OF THE BRANCH AND BOUND APPROACH.

Branch and Bound techniques are computationally feasible only for problems of size twelve or less. Even the most sophisticated of techniques such as those by Bazaraa and Elshafei (1979) cannot handle problems of greater size. Some computational results for the problems in the literature handled by the branch and bound technique are presented in Appendix I. However, it can be said that these techniques still present the most efficient, exact ways for solving small sized problems such as those discussed. In case the location site layout is a rectangular grid, the method proposed by Bazaraa and Kirca (1983) is the most attractive. They use a single assignment procedure, with the main feature being the elimination of mirror images in the search tree, thus reducing the computational effort.

EXACT TECHNIQUES NOT EMPLOYING BRANCH AND BOUND (1.B)

These approaches use techniques like integer programming, quadratic integer programming, cutting plane algorithms and decomposition principles to solve the problem QAP.

The following references cover literature on these techniques.

1. Cabot and Francis (1970)
2. Bazaraa and Goode (1975)
3. Love and Wong (1976)
4. Van de Panne (1975)
5. Kaufman and Broeckx (1978)
6. Bazaraa and Sherali (1980)

A detailed description of such methods is available in Sherali (1979).

HEURISTICS (2)

A heuristic is considered as a common sense approach to providing approximate solutions to problems. Heuristic techniques are very popular for solving quadratic assignment problems. This is not only because heuristics provide intuitive, easy to comprehend techniques, but also, they remain the only way of solving large sized problems which exist in reality.

A heuristic however is not an exact approach and there is no way to guarantee or verify optimality. At best, one may compare the heuristic value with some lower bound on the problem objective value. Typically, better solutions can be obtained by spending greater time on a problem.

The Heuristics existing in literature can be classified as follows.

2.a Construction Procedures

2.b Improvement Procedures

   1 Those using mainly pairwise interchanges

   2 Those using other improvement schemes

We remark here that most procedures combine both these strategies, although they tend to rely more heavily on one of these strategies.

Construction Methods (2.A). These methods were the earliest to be adopted due to their simplicity and their intuitive appeal. As the name suggests this method selects facilities to be placed on locations one at a time sequentially and proceeds until all the facilities have been placed using certain rules. Since the aim is usually to minimize the total distance travelled by the units, the rule involves the placement of heavily connected facilities together. Although these are typically quick in producing a solution, they do not produce very competitive solutions, unless some improvement scheme is used subsequently. The following articles elaborate on the techniques used by some methods.

1. Wimmert (1958)
2. Conway and Maxwell (1961)
3. Gilmore (1962)
4. Hillier and Connors (1966)

5. Heider (1972)
6. Edwards, et al (1970)
7. Parker (1976)
8. Burkard and Stratmann (1978)

Improvement Procedures (2.B). As shown earlier, Improvement Procedures can be classified into the following.

2.b.1 Those using pairwise interchanges
2.b.2 Those using other improvement schemes

Pairwise interchanges (2.B.1). These techniques start with a feasible arrangement of facilities to locations, and then systematically improve solutions by checking if an interchange between two pairs of facilities produces any improvement. The initial placement may either be random or by some easy rule involving very less computational effort. This solution technique, however, only yields solutions which are known as 2-opt solutions. Some techniques may be however more involved, attempting to imitate some higher-order exchange scheme. As pointed out by Los (1977) and by Burkard and Stratmann (1978), attempting all three-way or higher-order exchanges usually involves a prohibitive amount of effort. Hence, as in Burkard and Stratmann (1978), Mirchandani and Obata (1979) and Bazaraa and Kirca (1983), higher-order exchanges which are limited in number and scope or extent are prescribed. However, it should be noted that the success of an improvement routine is heavily dependent on the starting solution which is constructed.

The following is a list of articles dealing with this subject.

1. Hillier (1963)
2. Armour and Buffa (1963)
3. Pegels (1966)
4. Hillier and Connors (1966)
5. Nugent et al (1968)
6. Vollman et al (1968)
7. Heider (1972)
8. Bazaraa and Elshafei (1979)
9. Elshafei (1977)
10. Patel et al (1976)
11. Los (1978)
12. Obata (1979)
13. Liggett (1981)

Since the approach offered uses the pairwise interchange technique for improving solutions, some of the more important ones are discussed in greater detail.

Hillier (1963) suggested the following procedure. Working on problems with the grid structure, he determined the maximum movement of a facility either right, left, up or down. He called this maximum number of steps p and determined the p step movements for all the facilities. The cost of all these p step movements to the corresponding sites were determined without the counter movement of the facility occupying the site. Interchange of facilities was considered with the facility that offered the maximum cost reduction. This was implemented only if the interchange resulted in a decreased cost. The move numbers were then updated and the procedure continued until no more p step improvements were found. Now p was decremented by one and the process was repeated.

In an improvement by Hillier and Connors (1966) only those improvements were considered in the preliminary moves which decreased the cost by at least a specified amount.

Armour Buffa and Vollman (1963) provided the well known method of CRAFT. From an arbitrary starting solution, they computed the decrease in cost by interchanging each pair of facilities. Interchange between only that pair of facilities that resulted in the minimum cost was implemented. This method terminated when no more improvement was possible.

Vollman Nugent and Zartler (1966) suggested the following two phase technique which was a considerable improvement over CRAFT. In the first phase two facilities which contributed the greatest to the cost were chosen. All pairwise interchanges with one of the chosen facilities was computed and the largest reduction was implemented. The facility interchanged was then disregarded from further consideration. When no further improvements were possible with the first facility chosen, the procedure was repeated with the second facility. Again two facilities with the greatest contributions to cost were chosen and the process was repeated. This continued until no further improvements resulted.

The procedure then switched over to the second phase where all possible pairwise interchanges were attempted. This whole procedure was repeated once more and then the method terminated.

Pegels (1966) used a discrete optimization technique adopted by Reiter and Sherman (1965), and developed a heuristic based on the following concept. All possible pairwise interchanges were done on a random layout giving a local (2-opt) minimum. This was used to generate a second starting layout giving a second local (2-opt) minimum. The user was asked to provide a value for the probability of not finding the optimal solution at termination and this was used to determine the number of iterations to be performed.

Los (1978) proposed the following technique to improve upon CRAFT. Here the evaluation of all possible exchanges was done only once. At each exchange the computed exchange values were updated recursively, thus reducing computational effort for all subsequent iterations.

Patel, Dewald and Cote (1976) used the following concept to reduce computational effort. They created algorithms to partition the facilities into groups of facilities in such a way that the interactions between the partitions was minimized. The partitions attempted to have almost equal number of facilities in each group. Highly interconnected facilities were put together and referred to as groups or supernodes. This was done until the required number of supernodes were formed. The locations were then divided into superlocations and supernodes were alloted to superlocations and .improved by pairwise interchange. After this placement, pairwise interchange were conducted within the elements of each supernode only, thus reducing the computational effort considerably.

Some other methods are those of Obata (1979) and Liggett (1981) Obata (1979) adopted a scheme in which he examined the effects of all pair wise exchanges along with some three way and four way exchanges. Liggett (1981) determined that an initial constructive procedure followed by an improvement procedure such as those discussed produced good solutions in reasonable time.

Other Improvement Schemes (2.B.2). These improvement techniques are considerably more complex than the pairwise interchange technique. The following are some of the articles on this method.

1. Steinberg (1961)

2. Gaschutz and Ahrens (1968)
3. Ireland (1974)
4. Burkard and Stratman (1978)
5. Bazaraa and Sherali (1980)
6. Murtagh et al (1982)

Steinberg (1961) formulated what is now popularly known as the backboard wiring problem. He divided the problems into sets such that the flow between elements of a set was zero. To introduce the idea behind his technique consider the following. Let U_1 be a set such that the interconnections between elements in it is zero. Let s_1 be any solution set i.e. a feasible assignment of facilities to locations. Let S_1 be a set of all placements which leave the placement of facilities in the set U_1 unchanged with respect to the solution s_1. Among all such placements in S_1 determine that which minimizes the cost of an appropriate linearized program and call this solution s_2. Now use s_2 on another unconnected set, say U_2 to find a solution s_3. Continue until all sets have been selected. Thereafter repeat again with U_1 and the placement last generated. This process continues until no further improved solution results.

The heuristic of Gaschutz and Ahrens (1968) inbeds the problem into a graph theoretic framework in which the nodes represent the facilities and in which facilities with a positive interaction are connected by links having weights proportional to the level of interaction. A continuous relaxation technique is then employed to determine coordinates of the nodes such that the sum of weighted square deviations of the nodes from each other is minimized, subject to a normalization constraint that the sum of the x and the sum of the y coordinates should each equal zero, and the sum of their squares should each equal unity. An iterative scheme is used to solve this problem. The resulting coordinates are then used to match the facilities onto the location grid.

Burkard and Stratmann (1978) alternated between a branch and bound approach and an improvement exchange routine, they named VERBES, until no further improvement was possible. This yielded extremely good results.

Bazaraa and Sherali (1980,1982) got at least as good as, and some better quality solutions for test problems as did Burkard and Stratmann (1978). They implemented Benders partitioning method, and a disjunctive cutting plane method on a mixed integer formulation of the problem QAP. They empirically demonstrated the prohibitiveness of using these methods as exact algorithms, developing theoretical lower bounds on computational complexity in some cases. They hence converted these algorithms to heuristic procedures which were able to produce better quality solutions on test problems than any of the previous methods.

Mirchandani and Obata (1979) also employed Gaschutz and Ahrens (1968) method along with Hall's (1970) algorithm for solving the squared deviation problem as a preprocessor in a rotating grid algorithm. Here, several trial solutions were obtained by attempting to match the locations of the placement-grid as it rotated into several positions superimposed upon Gaschutz and Ahrens solution. This matching was performed by using an assignment problem to obtain a least deviation from the Gaschutz and Ahrens solution. A pair wise exchange was then performed over each resulting solution. In addition, the mixed-exchange scheme discussed earlier was used to improve the solution.

Murtagh, Jefferson and Sornprasit used a simple procedure to obtain a good starting solution and then solved the problem as a nonlinear program using MINOS. The almost near integer solutions were converted to integer solutions by the use of a rounding heuristic.

COMPUTATIONAL CAPABILITIES OF HEURISTICS.

Heuristic techniques, unlike branch and bound methods, are used to solve larger problems of sizes 15 and greater. With respect to the quality of solutions produced, the most competitive existing heuristics are those proposed by Burkard and Stratmann (1978), Bazaraa and Sherali (1980,1982), Mirchandani and Obata (1979) and Bazaraa and Kirca (1983).

Of these, the methods of Mirchandani and Obata (1979) and Bazaraa and Kirca (1983) are designed to solve quadratic assignment problems on symmetric, rectangular grid location layouts, whereas the other methods mentioned above are more generally applicable. The above algorithms when operated for a sufficient number of "iterations" in order to produce good quality solutions, are all computationally intensive, but produce solutions of roughly comparable quality. The best known results are listed in Appendix I.

CHAPTER III

APPROACH TO THE PROBLEM

In this chapter the heuristic scheme is proposed. The motivation for this effort stems from the undeniable fact that most algorithms presented so far in the literature do extremely well for small problems. However they get bogged down and become highly computationally intensive while solving problems of sizes twenty and greater. The principal contributing factor used in designing the heuristic is the fact that quadratic assignment problems involving the location of up to 8-10 facilities can be solved exactly rather quickly. The exponential growth in effort, as seen by the growth in m!, typically occurs beyond m=10.

Hence, the construction and improvement of solutions in the proposed method is based on the exact solution of several suitable subproblems not exceeding m=10 in size, coupled with pairwise exchange. (The exact algorithm employed is a branch and bound algorithm described in Sherali (1979).) By controlling the number of such subproblems solved, a polynomial time procedure results. Moreover, implicit enumeration used to obtain the best configuration of 8-10 suitably chosen facilities at a time can be expected to be preferable to an explicit, partially restricted 3 or 4-way interchange scheme. The improvement routine used is portable and can be successfully implemented for several other methods too.

The main heuristic $H(M_1,M_2)$ described below, where the arguments denote certain parameter values, is generally applicable to Problems QAP with cost data given in the flow and distance format embodied by (4). However, some of the steps are indeed motivated from the viewpoint of a rectangular grid location layout. We also assume here that $m >= 12$; for smaller sized problems, an exact branch and bound algorithm may be employed.

The details of the heuristic employed are now discussed. The concepts employed will be illustrated with a step by step solution for a problem of size twelve, which will henceforth be referenced as QAP12. The parameters $M_1=8$ and $M_2=8$ will be used for this problem. (The problem used for this purpose is one of the test problems of Nugent et al (1968)). The motivation for the various steps will also be presented along with the statement of the heuristic. The flow and distance matrix for the example problem is as shown in Figure 2.

In order to facilitate the statement of the heuristic, the following definition of a certain specific quadratic assignment subproblem is given. Consider a situation in which some facilities are located on certain designated sites. (It may be that all the m facilities are located over the m location sites.) Let S_f and S_d denote subsets of the facilities and locations, respectively, with

$$m >= |S_d| \equiv n_d >= |S_f| \equiv n_f,$$

1	2	3	4
5	6	7	8
9	10	11	12

distance matrix

0	1	2	3	1	2	3	4	2	3	4	5
1	0	1	2	2	1	2	3	3	2	3	4
2	1	0	1	3	2	1	2	4	3	2	3
3	2	1	0	4	3	2	1	5	4	3	2
1	2	3	4	0	1	2	3	1	2	3	4
2	1	2	3	1	0	1	2	2	1	2	3
3	2	1	2	2	1	0	1	3	2	1	2
4	3	2	1	3	2	1	0	4	3	2	1
2	3	4	5	1	2	3	4	0	1	2	3
3	2	3	4	2	1	2	3	1	0	1	2
4	3	2	3	3	2	1	2	2	1	0	1
5	4	3	2	4	3	2	1	3	2	1	0

Flow matrix

0	5	2	4	1	0	0	6	2	1	1	1
5	0	3	0	2	2	2	0	4	5	0	0
2	3	0	0	0	0	0	5	5	2	2	2
4	0	0	0	5	2	2	10	0	0	5	5
1	2	0	5	0	10	0	0	0	5	1	1
0	2	0	2	10	0	5	1	1	5	4	0
0	2	0	2	0	5	0	10	5	2	3	3
6	0	5	10	0	1	10	0	0	0	5	0
2	4	5	0	0	1	5	0	0	0	10	10
1	5	2	0	5	5	2	0	10	0	5	0
1	0	2	5	1	4	3	5	10	5	0	2
1	0	2	5	1	0	3	0	10	0	2	0

Figure 2. Nugent et al problem of size 12

and such that if any fixed facility (or location) is included in S_f (or S_d), then the corresponding assigned location (or facility) is also included in S_d (or S_f). Then, QAP (S_f,S_d) means the quadratic assignment subproblem in which the n_f facilities in S_f are optimally allocated to some n_f of the $n_d >= n_f$ locations in S_d, using n_d-n_f dummy facilities to solve this problem, while holding all the fixed facilities (locations) not in $S_f(S_d)$ as fixed, and considering all interaction costs associated with the facilities in S_f U {fixed facilities}, but ignoring the interaction costs associated with any other facilities. Note that although QAP(S_f,S_d) depends on the current configuration, no additional notation is used to specify this, since the associated subproblem will be clear from the context.

HEURISTIC H(M1,M2) (Assume m >= 12)

Parameter Selection

Select appropriate values of the parameters M_1 and M_2. Guidelines for picking these values, as well as their connotations, are given below. Other parameters whose values are prescribed below were fixed empirically based on some trial runs.

PHASE I: Construction of an Initial Solution.

STEP 1

This step attempts to rank the facilities in decreasing order of their interaction with other facilities. The intention is to place facilities with greater interaction centrally in the layout. An approach sometimes employed is to add the interaction of a facility with respect to all other facilities and place those facilities with larger values in central positions. However, one problem which may be encountered could be the following. Consider two facilities i and j which have a low interaction with each other. Hence, myopically, they have a low affinity towards each other. However, the interaction of facilities i and j with common facilities may be high. Thus facilities i and j can be expected to be close together because of these common high interactions. However, an approach using only the interaction f_{ij} between i and j would not detect this affinity between i and j. To accomplish this, the following artificial interaction F_{ij} is computed.

For each $i,j=1, \ldots, m$,

$$F_{ij} = f_{ij} + \frac{1}{2} \sum_{\substack{k=1 \\ k\neq i,j}}^{m} \text{minimum} \{ f_{ik}, f_{jk} \} , \qquad (6)$$

Thus the value F_{ij} determines the affinity between facilities i and j, considering all other facilities as well. The sum F_i of the new flows F_{ij} between a facility i and all other facilities is then computed as follows.

$$\text{and let } F_i = \sum_{j=1}^{m} F_{ij}, \text{ for } i=1, \ldots, m. \tag{7}$$

The F_i values are then arranged in descending order. This list gives a ranking of facilities indicating the priority each facility has, in occupying a central position in the grid. Let α_i be the rank of facility i, using average ranks in case of ties in the F_i values.

The adjusted sum of the flows for each facility arranged in descending order for QAP12 is shown in Figure 3.

However, this ranking may not be sufficient. Consider the case of a facility having a very large flow with some facility as compared to all other flows, and having no interaction at all with the other facilities. The above technique would tend to rank it topmost, though it may not be necessary for it to occupy a central position. To avoid this problem, the following is done. Determine the mean f and standard deviation σ of all the interactions f_{ij}. Compute,

$$\bar{f} = \max\{1, f - \frac{1}{2}\sigma\} \tag{8}$$

If the flow of a facility with any other facility is greater than this value $\bar{f}$, replace the flow value with unity. Otherwise, give it an interaction flow of zero. This scheme will tend to place facilities with significant flow

FACILITY	$\sum_j F_{ij}$	Rank
11	119.5	1
7	100.5	2
4	96.0	3
9	95.0	4
8	93.5	5
6	88.5	6
10	82.5	7
12	79.0	8
1	78.5	9
2	75.5	10
3	74.5	11
5	74.0	12

Figure 3. Facilities arranged in nonincreasing order after Ist modification

connections centrally in the layout. This can be represented mathematically as follows.

Define

$$\bar{f}_{ij} = \begin{cases} 1 & \text{if } f_{ij} \geq \bar{f} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } i,j=1, \ldots, m . \qquad (9)$$

The idea here is to generate the ability to simply count significant flow connections. Define $\bar{f}_i$ from $\bar{f}_{ij}$ as follows.

$$\bar{f}_i = \sum_{j=1}^{m} \bar{f}_{ij} \quad \text{for } i=1, \ldots, m. \qquad (10)$$

Rank the facilities in descending order of these $\bar{f}$ values. Let β be the rank of facility i, again using average ranks in case of ties in the $\bar{f}_i$ values. Figure 4 indicates this for the problem QAP12.

Two rankings of the facilities, constructed in the above two ways are now available. A way to incorporate these into one list was devised. This is done on the basis of average ranks over the two lists. Namely, having found the ranks of facilities in each list (where average ranks are assigned for ties), the mean rank of the two list ranks is computed for

FACILITY	$\bar{f}_i$	Rank
11	6	1
4	5	3.5
7	5	3.5
8	5	3.5
9	5	3.5
2	4	7
6	4	7
10	4	7
1	3	10.5
3	3	10.5
5	3	10.5
12	3	10.5

Figure 4. Facilities arranged in nonincreasing order after IInd modification.

FACILITY (LIST L_f)	MEAN RANK
11	1
7	2.75
4	3.25
9	3.75
8	4.25
6	6.50
10	7.00
2	8.50
12	9.25
1	9.75
3	10.75
5	11.25

Figure 5. Average Ranking of facilities in nondecreasing order

each facility. Let $\gamma_i = (\alpha + \beta_i)/2$ be the average of the two ranks.

Thus facility eight has a rank of five in Figure 3 and 3.5 in Figure 4. Its average rank is,

$$\frac{5+3.5}{2} = 4.25$$

as shown in Figure 5. Construct a final list of facilities in nondecreasing order of the γ_i ranks, i=1, ... ,m. This will be denoted as list L_f. This list now incorporates both the ideas discussed and is a better indicator of the tendency of each facility to occupy a central position. Figure 5 illustrates this for problem QAP12. Assume that the facilities are henceforth renumbered in order of their appearance in L_f.

STEP 2

Rank each location in the layout based on the distance of a location in the layout from all other locations in the layout. Thus if d_{ij} represents the rectilinear distance, say, between location i and j as given in Figure 1, then d_i is computed as shown below and is used as an index for centrality.

$$d_i = \sum_{j=1}^{m} d_{ij} \quad \text{for each } i=1, \ldots, m \tag{11}$$

Construct a list L_d of location arranged in nondecreasing order of their d_i values, i=1, .. ,m. Assume that the locations are henceforth renumbered in order of their appearance in L_d. Figure 6 indicates the d_i values arranged in ascending order.

STEP 3

Steps 3 to 5 now create and optimally solve certain specific subproblems in order to construct an initial assignment. The first subproblem is created by choosing locations from the list L_d created in step 2. Up to a maximum of six locations from the top of the list L_d are candidates to form the first subproblem location set with the following restriction. If a location is part of the first subproblem set then all locations i which do not have a larger D_i value than this location also form part of the first subproblem set. If this restriction causes more than six locations to be selected, choose the top six locations from the L_d list. Let the number of locations selected in this manner be denoted by n_d Note that n_d <= 6. This can be represented mathematically as follows.

$$n_d = \begin{cases} \text{largest integer: } 3 \leq n_d \leq 6,\ d_{(n_d+1)} > d_{n_d}, \text{ if it exists} \\ 6 \quad \text{otherwise} \end{cases} \tag{12}$$

Let S_d be the (ordered) set of the top n_d locations in L_d. Let $n_f = n_d$ and let S_f be the set of top n_f facilities in L_f. Solve QAP(S_f,S_d). Let $\bar{L}_f \equiv S_f$ denote the (ordered) set

LOCATION (LIST L_d)	TOTAL DISTANCE
6	20
7	20
2	24
3	24
10	24
11	24
5	26
8	26
1	30
4	30
9	30
12	30

Figure 6. Locations in nondecreasing order based on their distance from others

of fixed facilities, and let the other facilities be called free. Similarly, let $\bar{L}_d$ denote the (ordered) set of free locations, where $\bar{L}_d = L_d - S_d$. Further, let l(p) denote the (fixed) locations corresponding to each fixed facility $p \varepsilon \bar{L}_f$, and let $m_f = |\bar{L}_d|$ denote the number of free facilities/locations.

The motivation for taking facilities and locations from top of lists L_f and L_d is as follows. Facilities which have greater interaction with other facilities and should consequently be placed centrally are located at the top of list L_f. Likewise the central locations at which the above facilities should be placed are those at the top of list L_d.

For QAP12 locations 6,7,2,3,10 and 11 form the location set as is evident from Figure 6 and facilities 11,7,4,9,8 and 6 form the facility set based on Figure 5. The optimum solution for this six facility-six location subproblem is as shown in Figure 7.

For QAP12 the free locations and facilities are shown in Figure 8.

STEP 4

Determine the size and configuration of the quadratic assignment subproblem to be solved next as follows. Define

-	11	9	-
-	8	7	-
-	4	6	-

Figure 7. Location of six facilities after Branch & Bound (Obj=83)

Free locations	Free facilities
5	10
8	2
1	12
4	1
9	3
12	5

Figure 8. Free locations and facilities

$$n_f = \begin{cases} 3 \text{ if } m_f > 10 \\ m_f \text{ otherwise} \end{cases} \quad . \tag{13}$$

Let S_f be the set of the top n_f free facilities in L_f. In order to define the location set S_d to be considered, let

$$L_d' = \{\text{first } \min\{m_f, 20\} \text{ locations in } \bar{L}_d\}. \tag{14}$$

If $|L_d'| <= 10$, let $n_d = |L_d'|$ and let $S_d = L_d'$. Otherwise, put $n_d = 10$ and select a subset of 10 locations from L_d' as follows. Compute

$$f_p = \sum_{i \varepsilon S_f} f_{pi} \quad \text{for each } p \varepsilon \bar{L}_f \ . \tag{15}$$

For each candidate location $j \varepsilon L_d'$, compute

$$\nu(j) = \sum_{p \varepsilon \bar{L}_f} f_p \, d_{j\ell(p)} \ . \tag{16}$$

Rank order the locations in L_d' in nondecreasing order of their $\nu(j)$ values and choose as S_d the first $n_d = 10$ locations from this ordered list.

Solve QAP(S_f,S_d). Perform a pairwise interchange[1] improvement on all the presently fixed facilities, including the $|S_d|-|S_f|$ dummy facilities just located. Update the ordered subsets $\bar{L}_f$, $\bar{L}_d$, let the number of free facilities/locations be $m_f \equiv m - |\bar{L}_f|$, and update the locations $l(p)$ of the fixed facilities $p \varepsilon \bar{L}_f$.

For QAP12 the number of free facilities remaining is six, which is less than the cut off value of ten. Thus the facility set contains the facilities (10,2,12,1,3,5). Also since the number of locations available is less than ten all the available locations, namely (8,5,4,9,1,12) are selected to form the location set. The solution after the application of the exact method is shown in Figure 9 and the solution after the pairwise interchange routine on the 12 facilities is shown in Figure 10 .

STEP 5

Given $8 <= M_1 <= 10$, let $n_f = \min \{|\bar{L}_f|, M_1\}$. Choose the last n_f facilities from the ordered list $\bar{L}_f$ as a facility set S_f.

---

[1] Whenever an improvement results, recommence the consideration of pairwise interchange from the start. In order to maintain the polynomial nature of the heuristic, a maximum of 10 restarts is permitted.

3	11	9	12
2	8	7	10
1	4	6	5

Figure 9. Location of 12 facilities after Branch & Bound (Obj=313)

3	7	9	12
1	8	11	4
2	10	6	5

Figure 10. Loc. of 12 facs. after pairwise exchange,(Obj=291)

Let $S_d=\{ j: j = l(p) \text{ for } p \in S_f\}$ be the set of $n_d=n_f$ locations corresponding to the facilities in S_f. Solve $QAP(S_f,S_d)$, and if an improvement results, perform a pairwise interchange on all the fixed facilities. Update $l(p)$, $p \in L_f$ if necessary. If $m_f = 0$, go to step 6. Otherwise, return to step 4.

For QAP12 this did not result in any improvement. The last 8 facilities were (8,6,10,2,12,1,3,5) and these were already placed optimally relative to the four fixed facilities (11,7,4,9).

Phase II (a): Improvement Routine Using L_d

In steps 6, 7 and 8 subsets of locations of sizes up to M_2 are selected along with the facilities assigned to these locations, and these facilities are optimally relocated among the associated sites. The location subsets chosen overlap partially so that the subproblems solved are not disjoint, and the subproblems are selected in order of importance, viz, from central locations onwards, towards off central locations.

STEP 6

Initialize $S=L_d$, $n_s=m$ and denote by $f(j)$ the facility which is currently fixed on location site j, $j=1, \ldots ,m$.

STEP 7

Given $8 <= M_2 <= 10$, let $n_d = \min\{n_s, M_2\}$, and denote by S_d the top n_d locations in S. Accordingly, define

$$S_f = \{p: p=f(j) \text{for } j \in S_d\}$$

as the set of $n_f = n_d$ facilities corresponding to the locations in S_d. Solve $QAP(S_f, S_d)$. If an improvement results, perform a pairwise interchange improvement over all the facilities. Update $f(j), j=1,...,m$, if necessary.

For QAP12 the optimization of facilities (7,9,1,8,11,4,10,6) placed on locations (2,3,5,6,7,8,10,11) respectively yields an improved solution as shown in Figure 11.

STEP 8

If $n_s <= M_2$, go to step 9. Otherwise, remove the top 2 locations from s, reduce n_s by 2, and return to step 7.

Phase II(b): Improvement Routine Using L_f

The procedure now enters another similar improvement phase defined by steps 9, 10 and 11, wherein the choice of the quadratic assignment subproblem to be solved is based on the ordered facility list L_f, as opposed to the ordered location list L_d as in phase II(a) above. Because of possibly

3	9	7	12
1	11	8	4
2	10	6	5

Figure 11. Loc. of 12 facs. after step 8 (Obj=289)

diminishing returns, fewer subproblem are solved in phase II(b) than in phase II(a).

STEP 9

Initialize $S=L_f$, $n_s = m$ and denote by $l(i)$ the location site currently occupied by facility i, $i=1,..,m$.

STEP 10

Given $8 <= M_2 <= 10$, let $n_f = \min\{n_s, M_2\}$, and denote by S_f the top n_f facilities in S. Accordingly, define

$$S_d = \{j: j=l(i) \text{ for } i \in S_f\}$$

in the set of $n_d = n_f$ locations corresponding to the facilities in S_f. Solve $QAP(S_f, S_d)$. If an improvement results, perform a pairwise interchange improvement over all the facilities. Update $l(i), i=1,...,m$, if necessary.

STEP 11

If $n_s <= M_2$, go to step 12. Otherwise, remove the top 4 facilities from S, reduce n_s by 4, and return to step 10.

Note that the number of loops through each of the phase I, II(a) and II(b) is linear in m, with each such loop being of order $O(m^2)$ and hence the heuristic embodied by phase I, II(a) and II(b) is polynomial of order $O(m^3)$.

Phase III: Location-Site Dependent Improvement Routine.

STEP 12

This is an improvement procedure in which blocks of eight to ten facilities are picked, and subjected to the branch and bound improvement. This step can be tailored to the specific location-site distribution. Cluster analysis as in Patel, et al (1976) or Gaschutz and Ahrens (1968), or simply common sense based rules, may be used to predesignate location sites for solving quadratic assignment subproblems. Other partial higher-order exchange method such as those employed by Burkard and Stratmann (1978), Mirchandani and Obata (1979) or Bazaraa and Kirca (1983) may also be used. In addition, by controlling the number and size of such subproblems solved, the polynomial nature of the heuristic may be maintained. Some ways of picking these blocks are suggested below. Let A_i denote the set of locations the facilities of which will be placed optimally with respect to the other facilities. Some A_i sets over which optimization could be performed for a 12 facility problem as shown in Figure 12 are given below.

1. $A_1 = \{ (1, 5, 9), (2, 6, 10), (3, 7, 11) \}$

   The above is a block consisting of columns 1, 2 and 3 of the grid, the elements of which are shown in paren-

1• 2• 3• 4•
5• 6• 7• 8•
9• 10• 11• 12•

Figure 12. A grid with 12 facilities and locations

1• 2• 3• 4• 5•
6• 7• 8• 9• 10•
11• 12• 13• 14• 15•

Figure 13. A grid with 15 facilities and locations

thesis. Other permutations of the columns should also be included.

2. $A_2 = \{ (1, 2, 3, 4), (5, 6, 7, 8) \}$

These consist of elements of row 1 and 2 of the grid. Other permutations of rows should also be included.

3. $A_3 = \{ 1, 2, 3, 4, 5, 8, 9, 10, 11, 12 \}$

These are all but the two central locations of the grid.

4. $A_4 = \{ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \}$

These consist of all but the 2 diagonally opposite corners. The other possibility includes all but the 2 diagonally opposite corners 4 and 9.

For a 15 facility problem as shown in Figure 13, some A_i sets are shown below.

1. $A_1 = \{ (1, 6, 11), (2, 7, 12), (3, 8, 13) \}$

The above is a block consisting of columns 1, 2 and 3 of the grid, the elements of which are shown in parenthesis. Similarly, other permutations of the columns should be included.

2. $A_2 = \{ (1, 2, 3, 4, 5), (6, 7, 8, 9, 10) \}$

These consist of elements of row 1 and 2 of the grid. Other permutations of rows should also be included.

3. $A_3 = \{ 2, 3, 4, 6, 7, 9, 10, 12, 13, 14 \}$

4. $A_4 = \{ 1, 2, 4, 5, 6, 10, 11, 12, 14, 15 \}$

This pattern of choosing facilities for placing optimally with respect to the other facilities can be easily extended to other problems.

Whenever any such subproblem results in an improvement, the pairwise interchange routine is applied to the resulting improved overall solution.

CONCLUSION

We conclude this section by suggesting certain variations that may be attempted in the context of the above heuristic. Firstly, note that one could control the values of the parameters M_1, M_2 in order to trade of between the overall efforts and the quality of the solutions obtained. Secondly, in the same spirit, if one were willing to expend more effort, then one may wish to construct more then one initial solution in phase I by looping through steps 4 and 5 with different values of M_1. The improvement schemes may then be applied to all these initial solutions, or only to the best of these solutions. Thirdly, one may wish to operate only phase II(a) or phase II(b) as desired. Finally, the number of subproblems solved in phase II(a) or II(b) may be controlled by changing the reduction in n_s at each visit to step 8 or 11, respectively.

CHAPTER IV

COMPUTATIONAL RESULTS

Since optimal solutions can be verified only for problems of sizes no more than 12, the task of testing the performance of heuristics can be quite difficult. For the quadratic assignment problem, the literature contains certain test problems in the papers of Nugent,et al (1968), Steinberg (1961) and Elshafei (1977). These problems have evolved as benchmarks for proposed heuristic procedures. If one examines the quality of solutions obtained by the different heuristics for these problems, as compiled by Obata (1979) for instance, one may observe that these problems are quite formidable and only the recent procedures of Burkard and Stratmann (1978), Mirchandani and Obata (1979), Bazaraa and Sherali (1980,1982) and Bazaraa and Kirca (1983) produce close to the best quality solutions. However as indicated earlier, these procedures must be operated in a computationally intensive fashion to produce the best results (for example, see Bazaraa and Kirca (1983)). These test problems will be used to evaluate the proposed heuristics. The performance criteria used includes both the quality of the solutions obtained and the CPU time taken to determine the solutions.

Figure 14, provides results for different variations of heuristic $H(M_1,M_2)$, namely, H(8,8), H(8,10), and H(10,10).

Here a final variation H(8-10,10) denotes a run in which the loops for step 4 and 5 are made twice, once with a value of $M_1 = 8$ and then again with $M_1 = 10$. The better of the two resulting initial solutions was then processed through phase II. Results for these can be deduced from H(8,8) and H(10,10). In order to avoid duplicating tasks while making loops through steps 4 and 5 with $M_1=10$, step 3 was skipped and the pairwise exchange at the end of step 4 was omitted. Column H(8-10,10) summarizes results for this run. For each of these variations, and for each of the six test problems, the table gives three two-tuples of the form (γ,t), corresponding to phase I, phase II(a) and phase II(b) respectively, where γ denotes the best objective value found at the end of each phase, and t denotes the cpu time in seconds on an IBM/3084 computer from the start until the end of that phase. These results do not include the phase III improvement scheme in step 12 of the heuristic procedure. Some runs of this type are demonstrated below. The final column in Figure 14 gives the objective values of the best known solutions in the literature for these problems.

By studying the entries in Figure 14, the reader may observe how one may strike a trade-off between solution quality and solution effort. Solution H(8,8) is computationally less intensive than H(8,10) and H(10,10). H(8-10,10) a computationally more intensive variation than H(8,10) or H(10,10) yields however much better solutions.

Problem	m	H(8,8)	H(8,10)	H(10,10)	H(8-10,10)	Best Phase III Solution	Best Known Solution
Nugent, et al's [1968] Problems	12	(291 , 0.19) (289 , 0.62) (289 , 1.02)	(291 , 0.19) (289 , 6.14) (289 , 13.87)	(289 , 3.61) (289 , 10.50) (289 , 18.22)	(291 , 7.19) (289 , 13.28) (289 , 20.99)	289	289
	15	(578 , 14.56) (578 , 14.86) (578 , 15.13)	(578 , 14.65) (578 , 18.92) (575 , 21.00)	(578 , 16.32) (578 , 20.56) (575 , 22.64)	(578 , 23.13) (578 , 27.44) (575 , 29.53)	575	575
	20	(1333 , 1.91) (1285 , 3.01) (1285 , 3.46)	(1333 , 1.92) (1285 , 12.96) (1285 , 20.73)	(1312 , 11.46) (1312 , 29.32) (1298 , 40.13)	(1333 , 13.54) (1285 , 32.28) (1285 , 40.05)	1285	1285
	30	(3132 , 9.03) (3132 , 10.71) (3132 , 11.67)	(3132 , 9.07) (3122 , 22.09) (3122 , 32.33)	(3176 , 112.97) (3164 , 128.46) (3149 , 153.11)	(3098 , 81.92) (3067 , 92.01) (3067 , 98.84)	3067	3064
Steinberg's [1961] Problem	36	(5022 , 50.68) (5022 , 51.83) (5022 , 52.37)	(5022 , 50.70) (5022 , 55.73) (5022 , 58.43)	(5134 , 53.65) (5115 , 61.86) (5115 , 63.86)	(5022 , 105.12) (5022 , 112.33) (5022 , 115.06)	4979	4800
Elshafei's [1977] Problem	19	(11290194 , 1.04) (11290194 , 1.52) (11290194 , 3.07)	(11290194 , 1.04) (11290194 , 4.82) (11290194 , 44.32)	(10562428 , 1.21) (8606274 , 8.51) (8606274 , 10.21)	(10562428 , 1.96) (8606274 , 8.89) (8606274 , 10.60)	8606274	8606274

Figure 14. Computational Results for proposed Heuristic

The remainder of this section illustrates some applications of Phase III (step 12) of the proposed heuristic. The aim here is to demonstrate how this improvement routine can readily improve on heuristics, without adding excessive computational burden. To show portability this routine was applied on the best result obtained by Murtagh, Jefferson and Sornprasit (1982) on the problem of size 12. The best solution obtained by them was 295. The Improvement routine gave the optimum answer of 289 after the application of this routine as shown in Figure 15.

The improvement routine was also applied to the best solution for QAP30 obtained by Obata (1979). The solution of 3074 was improved to 3068 by allowing facilities in column 1 and 2 to be relocated optimally with respect to all other facilities. The solutions before and after the improvement are shown in Figure 16.

The Improvement routine was applied on the best solutions obtained for problems of size 30 and 36. The solution was improved from 3122 to 3115 as shown in Figure 17 for QAP30, and from 5022 to 4979 as shown in Figure 18 QAP36.

Note that since most of the facilities are almost optimally placed at this stage, the branch and bound algorithm consumes very little effort. Hence, at a minimal additional effort, improvements in solutions may be obtained by solving suitable subproblems based on the location-site configuration. Appendix II lists the solutions obtained by the proposed heuristics.

5	10	2	3
6	7	11	9
4	8	1	12

Best solution for 12 facilities (Obj=295)

2	1	8	3
10	7	11	9
5	6	4	12

Improvement after fixing locations 6 and 7 (Obj=293)

2	10	6	5
1	11	8	4
3	9	7	12

Improvement after fixing locations 1 & 12 or 4 & 9 (Obj=289)

Figure 15. Improvement on best solution obtained by Murtagh et al 1982

24	26	25	28	2	5
12	6	10	13	9	21
1	22	7	19	29	3
17	18	8	16	27	20
15	23	11	30	4	14

Best Solution obtained by Obata (1978) (Obj=3074)

26	6	25	28	2	5
24	12	10	13	9	21
1	22	7	19	29	3
17	18	8	16	27	20
15	23	11	30	4	14

Solution after Improvement (Obj=3068)

Figure 16. Improvement on best solution obtained by Obata 1978

5	2	6	21	13	28
29	16	3	7	9	25
4	30	11	19	10	26
20	27	22	8	1	24
14	15	18	23	12	17

Solution before Improvement (Obj=3122)

5	2	21	6	13	28
29	16	3	7	9	25
4	30	19	11	10	26
20	27	22	8	1	24
14	15	18	23	12	17

Improved solution for above (Obj=3115)

Figure 17. An Improvement on the Heuristic H(8,10) solution for QAP30

29	19	14	11	12	1	27	24	25
30	28	15	7	13	10	9	2	26
31	32	23	20	4	8	18	21	-
34	33	16	6	5	3	17	22	-

Solution before Improvement (Obj=5022)

31	29	14	11	12	1	27	24	25
30	28	15	7	13	10	9	2	26
32	19	23	20	4	8	18	21	-
34	33	16	6	5	3	17	22	-

Solution After First Improvement (Obj=5115)

31	29	14	11	12	1	27	24	25
30	28	15	7	13	10	9	2	26
34	19	23	20	4	8	18	21	-
33	32	16	6	5	3	17	22	-

Solution After Second Improvement (Obj=4979)

Figure 18. An Improvement on the Heuristic H(8,10) solution for QAP36

SUMMARY AND CONCLUSIONS

This thesis has been concerned with the development of a flexible construction and improvement heuristic for the popular quadratic assignment problem. The key idea behind designing the proposed heuristic has been to replace the explicit enumeration of a restricted set of combinations of 2 or 3 or 4 facilities, with an efficient, optimal relocation of judicially chosen sets of several more (8 or 10) facilities via implicit enumeration. Hopefully, this strategy will prompt the development of an enhanced class of heuristic procedures for the quadratic assignment problem. The proposed heuristic has been shown to be flexible in which parameters which determine the size and number of quadratic assignment subproblems to be solved, as well as the number of loops through the improvement routines, may be varied as desired in order to trade-off between solution quality and computational effort. The Improvement scheme, as embodied by Steps 6-12 of $H(M_1,M_2)$ may be employed in conjunction with any other heuristic which constructs reasonably good initial placements. Hence, one may use Gaschutz and Ahren's (1968) method as in Mirchandani and Obata (1979), or use other continuous relaxations of QAP such as in Murtagh, et al (1982), in order to arrive at initial solutions which may be improved by the applications of Steps 6-12 to devise attractive heuristics. The quality of the final solution obtained is strongly related to the initial solution to which the improvement routine is applied. Portability has been demonstrated by improving solutions obtained through other heuristics. Further enhancements may result by combining the different phases of the proposed heuristic in different ways in order to produce different variations in the procedure. Additionally, one may wish to refine step 12, by employing, for example, some suitable cluster analysis technique to choose the subproblems being solved optimally.

APPENDIX I

Exact Solutions

Problem	Size	Optimal Obj.
QAP5	5	25
QAP6	6	43
QAP7	7	74
QAP8	8	107
QAP12	12	289
	Heuristic Solutions	
QAP15	15	575
QAP20	20	1285
QAP30	30	3064
QAP19	19	8606274
QAP36	34	4800

Figure 19. Best known solutions for test problems

APPENDIX II

Prob	Obj	Best Facilities for Locations 1,2,3..n
QAP12	289	(3,9,7,12,1,11,8,4,2,10,6,5)
QAP15	575	(9,8,13,2,1,11,7,14,3,4,12,5,6,15,10)
QAP20	1285	(17,5,7,1,6,19,15,20,8,13,4,2,12,11,16,18 14,10,3,9)
QAP30	3067	(15,23,11,30,4,14,17,18,8,16,27,20,1,22,7, ,19,3,29,24,26,25,10,9,2,12,6,28,13,21,5
QAP19	3606274	(9,10,7,18,14,19,13,17,6,11,4,5,12,8,15,1 ,1,2,3)
QAP36	5022	(29,19,14,11,12,1,27,24,25,30,28,15 7,13,10,9,2,26,31,32,23,20,4,8,18 21,-,34,33,16,6,5,3,17,22,-)

Figure 20. Best solutions by given heuristics for test problems

Problem	Obj	Best Locations for facilities 1,2,3...n
QAP36	4979	(31,29,14,11,12,1,27,24,25,30,28,15,7, 13,10,9,2,26,34,19,23,20,4,8,18,21,35, 33,32,16,6,5,3,17,22,36)

Figure 21. Best solutions by Improvement Routine

BIBLIOGRAPHY

1. Armour, G. C., and E. S. Buffa," A Heuristic and Simulative Approach to Relative Location of Facilities,; Management Science, Volume 9 pp. 294-309, 1963.
2. Bazaraa, M. S, " The Quadratic Set Covering (Assignment) Problem: Applications and Computations ", Report Presented to the National Science Foundation, under grant GK-38337, 1975.
3. Bazaraa, M. S., and A. N. Elshafei," Exact and Heuristic Procedures for the Quadratic Assignment Problem," Naval Research Logistics Quarterly, 1979.
4. Bazaraa, M. S., and J. J. Goode, " The Travelling Salesman Problem: A Duality Approach," Mathematical Programming, Volume 13, pp.221-237,1977
5. Bazaraa, M. S., and O. Kirca, " A Branch and Bound Based Heuristic for Solving the Quadratic Assignment Problem," Naval Research Logistics Quarterly, Volume 30, Number 2, pp.287-304, 1983.
6. Bazaraa, M. S., and H. D. Sherali, " Benders Partitioning Scheme Applied to a New Formulation of the Quadratic Assignment Problem," Naval Research Logistics Quarterly, Volume 27, pp. 29-41, 1980.
7. Bazaraa, M. S., and H. D. Sherali, " On the Use of Exact and Heuristics Cutting Methods for the Quadratic Assign-

ment Problem," Journal of Operations Research Society, Volume 33, pp. 991-1003, 1982.

8. Burkard, R. E., and K. H. Stratmann," Numerical Investigation on the Quadratic Assignment Problem," Naval Research Logistics Quarterly, Volume 25, Number 1, pp. 129-248, 1978.
9. Cabot, V. A., and R. L. Francis, " Solving Certain Nonconvex Quadratic Minimization Problems by Ranking the Extreme Points," Operations Research, Volume 18, pp. 82-86, 1970.
10. Carlson, R. G., and G. L. Nemhauser, " Scheduling to Minimize Interactive Costs," Operations Research, Volume 14, pp. 52-58, 1966.
11. Conway, R. W., and W. L. Maxwell, " A Note on the Assignment of Facility Location," Journal of Industrial Engineering, Volume 12, Number 1, pp. 34-36, 1961.
12. Dorris, A. L., " The Utility of Optimization Techniques in the Design of Man-Machine Systems," Unpublished M.S Thesis, Georgia Institute of Technology, 1971,
13. Edwards, H. K., B. E. Gillett, and M. E. Hale, " Modular Allocation Technique (MAT)," Management Science, Volume 17, pp. 161-169, 1970.
14. Elshafei, A. N., " Hospital Layout as a Quadratic Assignment Problem," Operations Research Quarterly, Volume 28, pp. 167-179, 1977.

15. Francis, R. L.,and J. A. White," Facility Layout and Location: An Analytical Approach, Prentice-Hall, 1974.

16. Gaschutz, G. K., and J. H. Ahrens, " Suboptimal Algorithm for the Quadratic Assignment Problem," Naval Research Logistics Quarterly, Volume 15, pp. 49-62, 1968.

17. Gavett, J. W., and N. V. Plyter," The Optimal Assignment of Facilities to Locations by Branch and Bound," Operations Research, Volume 14, pp. 210-232, 1966.

18. Geoffrion, A. M., and G. W. Graves," Multicommodity Distribution System Design by Benders' Decomposition," Management Science, Volume 20 5, pp. 822-844, 1974.

19. Gilmore, P. C.," Optimal and Suboptimal Algorithms for the Quadratic Assignment Problem," SIAM Journal on Applied Mathematics, Volume 10, pp. 305-313, 1962.

20. Graves, G. W., and A. B. Whinston," An Algorithm for the Quadratic Assignment Problem," Management Science, Volume 17, pp. 453-471, 1970.

21. Hall, K. M.," An r-Dimensional Quadratic Placement Algorithm," Management Science, Volume 17, pp.219-229, 1970.

22. Heider, C. H.," Decomposition Procedure for the Quadratic Assignment Problem," Professional Paper Number 100, 1972.

23. Heider, C. H., " A Computationally Simplified Pair Exchange Algorithm for the Quadratic Assignment Problem," Professional Paper Number 101, 1972.

24. Hillier, F. S., " Quantitative Tools for Plant Layout Analysis," The Journal of Industrial Engineering, Volume 14, pp. 33-40, 1963.

25. Hillier, F. S., and M. M. Connors, "Quadratic Assignment Problem Algorithms and The Location of Indivisible Facilities, Management Science, Volume 13, pp. 42-57, 1966.

26. Horowitz, E, and S. Sahni, "Fundamentals of Computer Algorithms," Computer Science Press, Rockville, Maryland, 1984.

27. Ireland, C. T.," An Eigen-Vector Structure for a Facilities Location Problem and a Related Algorithm," Report for Office of Naval Research Project Number NR-042-267, 1974.

28. Kaufman, L., and F. Broeckx, " An Algorithm for the Quadratic Assignment Problem Using Benders' Decomposition," European Journal of Operational Research, Volume 2, pp. 207-211, 1978.

29. Koopmanns, P. C., and M. Beckmann, "Assignment Problems and the Location of Economic Activities," Econometrica, Volume 25, pp. 53-76, 1957.

30. Land, A. N., "A Problem of Assignment with Interrelated Costs" Operational Research Quarterly, Volume 14, pp. 185-198, 1963.

31. Lawler, E. L.,"The Quadratic Assignment Problem," Management Science, Volume 19, pp. 586-599, 1963.

32. Liggett, R. S. " Strategies," Management Science, Volume , pp. 442-457, 1981.

33. Loberman, H., and A. Weinberges," Formal Procedures for Connected Terminals with a Minimum Total Wire Length," Journal of the Association of Computing Machinery, Volume 4, pp. 428-437, 1957.

34. Los, M., "Simultaneous Optimization of Land Use and Transportation: A Synthesis of the Quadratic Assignment Problem and of the Optimal Network Problem," Regional Science and Urban Economics, 1977.

35. Love, R. L., and J. Y. Wong, " Solving Quadratic Assignment Problems with Rectangular Distances and Integer Programming," Naval Research Logistics Quarterly, Volume 23, Number 3, pp. 623-628, 1976.

36. Mirchandani, P. B., and P. Obata, " Locational Decisions With Interactions Between Facilities: The Quadratic Assignment Problem - A Review," Working Paper PS-79-1, Rensselaer Polytechnic Institute, 1979.

37. Murtagh, B. A., T. R. Jefferson, and V. Sornprasit, " A Heuristic Procedure for Solving the Quadratic Assignment Problem," European Journal of Operations Research, Volume 9, pp. 71-76, 1982.

38. Nugent, C. E., T. E. Vollmann, and J. Ruml, "An Experimental Comparison of Techniques for the Assignment of Facilities to Locations," Operations Research, Volume 16, pp. 150-173, 1968.

39. Obata, T., " The Quadratic Assignment Problem: Evaluation of Exact and Heuristic Algorithms," unpublished Doctoral Dissertation, Rensselaer Polytechnic Institute, July 1979.

40. Panne, Van de, "Methods for Linear and Quadratic Programming, North Holland Publishing Company, Amsterdam, Oxford-American Elsevier Publishing Company, Inc., New York, 1975.

41. Parker, C. S., " An Experimental Investigation of some Heuristic Strategies for Component Placement," Operational Research Quarterly, Volume 27, pp. 71-81, 1976.

42. Patel, A. M., C. G. DeWald, and L. C. Cote, " Prepartioning as a Means of Enhancing Pair Wise Interchange Solutions to Quadratic Assignment Problems," Computer and Operations Research, Volume 3, pp. 49-56, 1976.

43. Pegels, C. C, " Plant Layout and Discrete Optimization," International Journal of Production Research, Volume 5, pp. 81-92,1966.

44. Pierce, J. F., and W. B. Crowston, " Tree Search Algorithms for the Quadratic Assignment Problems," Naval Research Logistics Quarterly, Volume 18, pp. 1-36, 1971.

45. Roucairol, Catherine, " A Reduction Method for the Quadratic Assignment Problems," III Symposium Uber Operations Research, Universitat Mannheim, October 1978.

46. Sharpe, R., and J. Brotchie, " An Urban System Study, " Royal Australian Planning Institute Journal, Volume 10, pp. 3-12, 1972.

47. Sherali, H. D., " The Quadratic Assignment Problem: Exact and Heuristic Methods," Doctoral Dissertation, Georgia Institute of Technology, June 1979.

48. Sibre, C. E., " A Quadratic Assignment Linear Programming Approach to Ship Scheduling for the U.S Coast Guard," Unpublished M.S thesis, Naval Postgraduate School, 1977.

49. Smith, H. C. T., " A Computational Comparison of an Improved Pair Assignment Algorithm and a Pair Exclusion Algorithm for the Quadratic Assignment Problem," AD-A019799, 1975.

50. Steinberg, L.,"The Backboard Wiring Problem: A Placement Algorithm," SIAM Review, Volume 3, pp. 37-50, 1961.

51. Vollmann, T. E., C. E. Nugent, and R. L. Zartler, " A Computerized Model for Office Layout," The Journal of Industrial Engineering, Volume 19, pp. 321-329, 1968.

52. Wimmert, R. J. "A Mathematical Method of Equipment Location," The Journal of Industrial Engineering, Volume 9, Number 6, pp. 498-505, 1958.