A Growth and Yield Prediction Model for Thinned Stands of Yellow-Poplar

by

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Thesis submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

Forestry

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November, 1982
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ACKNOWLEDGEMENTS

I wish to thank Drs. Thomas Burk and James Burger for their suggestions and comments regarding this study. I am especially indebted to Dr. Harold Burkhart for his patience and guidance during this work.

I also wish to thank the United States Forest Service, Southeastern Forest Experiment Station for supplying the data used in this study. Special thanks are extended to who provided the necessary background information and details regarding the Forest Service study, and who made many valuable suggestions during this data analysis.

Finally, I thank my family, and especially my parents, for their patience, support, and encouragement during my years in school.

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Chapter I

INTRODUCTION

Yellow-poplar (Liriodendron tulipifera L.) is an important commercial species that is often cut for lumber and veneer. Because tree size and quality have an impact on the yields of these products, thinning is an important silvicultural tool in yellow-poplar management. Most stands of yellow-poplar can produce a number of lumber- and veneer-size trees without thinning; however, thinning concentrates growth on the best and largest trees (Beck and Della-Bianca 1975). Therefore, reliable estimates of stand growth and yield are needed to determine optimal thinning regimes.

In 1972, Beck and Della-Bianca (1972) published equations for predicting basal-area growth and cubic-foot volume growth and yield in stands thinned to various levels of basal area. Subsequently, they published equations to predict board-foot growth and yield and residual quadratic mean stand diameter growth (Beck and Della-Bianca 1975). The equations were based on measurements taken five years after the initial thinnings on a series of 141 permanent plots.

Since the initial remeasurements, two additional assessments have been taken at 10 and 15 years after the initial thinning. The plots were thinned again at the time of the first 5-year remeasurement, thus stand characteristics and tree vigor were somewhat different for the second and third 5-year growth periods as compared to the first period. Consequently, the coefficients derived by Beck and Della-Bianca (1972, 1975) may not be appropriate for predicting growth and yield in yellow-poplar stands that have been thinned more than once.

The purpose of this project was to evaluate the Beck and Della-Bianca equations, and, if deemed necessary, to re-estimate coefficients in their equations. From the equations, a computer simulation model would be developed to describe the development of yellow-poplar stands given a set of initial conditions, a thinning regime, and a rotation age. This simulation model can be employed both at the whole stand and the diameter distribution level.

Chapter II

LITERATURE REVIEW

A number of models have been developed to predict growth and yield of various tree species. However, a large portion of the studies, and consequently the literature, have been directed towards pure stands of even-aged southern pines, particularly loblolly pine (*Pinus taeda* L.) (Farrar 1979). This review will attempt to relate the methods and procedures presented in the southern pine growth and yield literature to those used by Beck and Della-Bianca (1972, 1975) in their studies with yellow-poplar. For the most part, the underlying methods and assumptions are the same.

2.1 WHOLE STAND MODELS

The first yield predictions in the U. S. were made by constructing normal yield tables for unmanaged even-aged stands of a given species. Temporary plots and the concept of normal stocking were used. Thus only stands dense enough to produce wood at the fullest capacity for that species, age, climate, and soil were sampled. Normal yield tables constructed using graphical techniques were developed by Bruce (1926), Reineke (1927), and Osborne and Schumacher (1935). The earliest comprehensive predictions of yields for the South were presented in Miscellaneous Publication 50 (U.S. Forest Service 1929). Volume and yield tables for yellow-poplar in the southern Appalachians were presented by McCarthy (1933). These tables provide,for a given species, the per acre yield of wood in some specified volume unit as a function of age and site index. Age and site index were allowed to vary with these types of yield tables, but density was not. In addition, the definition of full or normal stocking is often vague. For these reasons the approach was unsatisfactory for stands with non-normal densities, and this resulted in an interest in variable density yield tables.

MacKinney, et al. (1937) suggested the use of multiple regression to estimate variable-density yield, and later, it was used to construct a yield prediction equation for loblolly pine stands of varying ages, site indicies, and densities (MacKinney and Chaiken 1939). Since the 1950's, computers have made data reduction and model fitting easier, allowing the study of larger, more detailed data sets (Farrar 1979).

Following an approach similar to that of MacKinney and Chaiken, many investigators have used multiple regression to construct stand aggregate growth and/or yield expressions. These models provide estimates for the whole stand as a function of stand level attributes such as age,density and site index.

Schumacher and Coile (1960) constructed yield models for natural stands of the four major southern pines, and Coile and Schumacher (1964) presented yield models for thinned and unthinned plantations of slash and loblolly pine. This approach, with certain modifications, was used by Goebel and Warner(1969) for loblolly pine plantations and by Burkhart et.al. (1972a, 1972b) to predict yield for natural stands and plantations of loblolly pine.

Until the early 1960's, seperate independent equations were developed to predict growth and yield. Predictions based on independently constructed growth and yield equations have often produced inconsistent and illogical results. In 1962, Buckman (1962) introduced a model for red pine where yield was obtained through mathematical integration of the growth equation over time. This concept of compatibility between growth and yield prediction was discussed in detail by Clutter(1963). In this case, a volume function for natural loblolly pine stands was expressed as the integral of the growth function, indicating the logical relationship which should exist between growth and yield equations.

Sullivan and Clutter(1972) generalized this concept and refined Clutter's equations to develop a simultaneous growth and yield model for loblolly pine that provided not only analytically, but also numerically consistent growth and yield predictions. They also recognized the difficulties which arise when data from permanent plots are used to estimate the parameters of equations from models such as Buckman (1962) and Clutter (1963). There are two main problems. First, the parameters in any one equation are not independent of those in other equations of the system. This leads to numerically inconsistent equations when the parameter estimates are inserted in the model. Second, the successive measurements of variables on the same plot do not constitute statistically independent observations (Sullivan and Clutter 1972). A more detailed explanation of this problem and possible solutions are discussed by Sullivan and Reynolds (1976).

To overcome these problems, Sullivan and Clutter (1972) developed a single linear model which related projected stand volume to initial stand age, projected age, site index, and initial basal area. When projected age was set equal to initial age, the model simplified to a conventional yield equation. Through further algebraic manipulation, a basal area projection model was also developed.

Their equations were as follows:

1) Cubic-foot yield was given by,

$$\hat{V} = \exp[b_0 + b_1(S) + b_2(1/A) + b_3(\ln(B)] \qquad (2.1.1)$$

2) Projected cubic-foot volume was given by,

$$\hat{V} = \exp[b_0 + b_1(S) + b_2(1/A_2) + b_3(A_1/A_2)(\ln B_1) + b_4(1-A_1/A_2) + b_5(S)(1-A_1/A_2)] \qquad (2.1.2)$$

3) Projected basal area was given by,

$$\hat{B}_2 = \exp[(A_1/A_2)(\ln B_1) + b_1(1-A_1/A_2) + b_2(S)(1-A_1/A_2)] \qquad (2.1.3)$$

where,

S = site index in feet,

A_i = stand age in years at the ith measurement,

$\ln B_i$ = logarithm to the base e of basal area per acre in square feet at the ith measurement.

This growth and yield model has been sucessfully used for loblolly pine (Brender and Clutter,1970, Sullivan and Williston, 1977, and Murphy and Sternitzke,1979), shortleaf pine (Murphy and Beltz,1981), slash pine (Bennett,1970), and yellow-poplar (Beck and Della-Bianca,1972).

Brender and Clutter (1970) predicted yields of even-aged, natural stands of loblolly pine by fitting both initial and remeasurement data from all plots with the model developed by Sullivan and Clutter (1972). Again, when current age equalled projected age (A1 = A2), a conventional yield equation resulted. Their model was given as follows.

$$\mathrm{Log}(CV_2) = b_0 + b_1(S) + b_2(1/A_2) + b_3(1-A_1/A_2) + b_4(\log B_1)(A_1/A_2) \qquad (2.1.4)$$

where,

S = site index,

A_1 = current stand age,

B_1 = current basal area,

A_2 = projected stand age,

CV_2 = projected per acre cubic-foot volume at age, A_2.

Bennett (1970) estimated yield in natural slash pine stands using the same equation form as Brender and Clutter (1970). He believed the equations could be applied with confidence to thinned stands throughout the range of slash pine in Georgia and Florida.

Beck and Della-Bianca (1972) based their analysis of yellow-poplar on the system of compatible growth and yield models developed by Clutter (1963) and later improved on by

Sullivan and Clutter (1972). The following growth and yield prediction models were used by Beck and Della-Bianca (1972):

$$\ln Y_2 = b_0 + b_1(1/S) + b_2(1/A_2) + b_3(A_1/A_2)(\ln B_1) + b_4(1-A_1/A_2) + b_5(S)(1-A_1/A_2) \qquad (2.1.5)$$

$$\ln B_2 = (A_1/A_2)(\ln B_1) + (b_4/b_3)(1-A_1/A_2) + (b_5/b_3)(S)(1-A_1/A_2) \qquad (2.1.6)$$

$$\ln Y = b_0 + b_1(1/S) + b_2(1/A) + b_3(\ln B) \qquad (2.1.7)$$

(when projection period is zero years;
i.e. $A_2 = A_1 = A$, $B_2 = B_1 = B$)

where,

Y_2 = stand volume at projected age, A_2,

B_2 = basal area at projected age, A_2,

Y = present stand volume,

S = site index,

B_1 = present basal area,

A_1 = present age,

ln represents the natural (Naperian) logarithm.

The only difference between these models and the ones proposed by Sullivan and Clutter (1972) was in the site index term. Sullivan and Clutter (1972) used site index without transformation, whereas Beck and Della-Bianca (1972) used the inverse of site index. The reason for the inverse transformation may have been that an upper bound on the range of values that the variable could take on may have been desired. Perhaps this was necessary when dealing with high site index study areas.

By taking the first derivatives of the basal-area and cubic-foot yield models, the following compatible growth models were obtained.

The basal area growth model was:

$$dB/dA = (B/A)\left[(b_4/b_3) + (b_5/b_3)(S) - \ln(B)\right] \qquad (2.1.8)$$

The cubic-foot growth model was:

$$dY/dA = y^*\left[-b_2(1/A_2) + b_3(1/B)(dB/dA)\right] \qquad (2.1.9)$$

where y^* is total cubic-foot yield calculated with equation (2.1.7).

The fits of the yield equations and the growth equations were found to be comparable to those obtained by Sullivan and Clutter (1972) for loblolly pine. In fitting the equations, Beck and Della-Bianca (1972) used two sets of measurements from the same plots. One set was taken following thinning at plot establishment and the other was taken five years after the thinning. When the least squares regressions were fitted, the two sets were combined and treated as independent observations. However, as was the case with Sullivan and Clutter (1972), the consequences of this independence assumption did not appear serious. Sullivan and Clutter (1972) found that estimation of parameters for their models under non-independence of observations assumptions using alternative estimation techniques did not produce parameter estimates that were significantly different, from a practical standpoint, from ordinary least squares estimates.

Sullivan and Williston (1977) also fitted equations using the Sullivan and Clutter (1972) models to predict growth and yield of thinned loblolly pine plantations in loessial soil areas. Again, the models provided a consistent set of prediction equations for cubic-foot volume and basal area projection when dependent observations from remeasured plot data were used.

Growth and yield equations have also been developed for board-foot volumes, however, they are not as numerous and are generally not as precise as the cubic-foot models. The following are examples of these types of models.

Leak, et.al. (1970) related board-foot volume of Eastern white pine (Pinus strobus) to age, site, and stand density with the same variables that were used to characterize cubic-foot volume. The use of this form resulted in a lower correlation and higher standard error of estimation for board-foot volume.

Brender and Clutter (1970) also developed board-foot yield tables based on the Sullivan and Clutter (1972) model. Like Leak, et al. (1970), the same variables that were used in the cubic-foot model were also used in the board-foot model. A similar reduction in precision of fit resulted. Their model is given by:

$$\mathrm{LogBV} = b_0 + b_1(S) + b_2(1/A_2) + b_3(1-A_1/A_2) + b_4(\log B_1)(A_1/A_2) \qquad (2.1.10)$$

where,

BV = projected per acre board-foot volume at age, A_2,

S = site index,

A_1 = initial age,

logB = the natural logarithm of the initial basal area, B_1.

Bennett (1970) related board-foot volume to basal area and cubic-foot volume which were determined from the Sullivan and Clutter (1972) model forms. The board-foot equation is given by:

$$\mathrm{BFV} = b_0 + b_1(B) + b_2(\mathrm{CFV}) \qquad (2.1.11)$$

where,

BFV = board-foot yield,

B = basal area,

CFV = cubic-foot stocking.

Board-foot volume prediction based on this method appears to give satisfactory results.

Through a preliminary analysis, Beck and Della-Bianca (1975) determined that some measure of stand structure was needed to adequately express board-foot stand volume in thinned stands of yellow-poplar. Their model related board-foot volume to dominant stand height, residual quadratic mean stand diameter, and residual stand basal area. The coefficients for the equation were determined by using the ratio of International 1/4-inch board-foot stand volume to residual stand basal area as the dependent variable. The equation is given by:

$$BFV/B_1 = b_0 + b_1(D^{\frac{1}{2}}) + b_2(D) + b_3(H*D^{\frac{1}{2}}) \qquad (2.1.12)$$

where,

BFV = International 1/4-inch board-foot stand volume per acre of all trees 11.0 inches d.b.h. and over.

B1 = Residual stand basal area in square feet per acre of all trees 4.6 inches d.b.h. and over.

H = Height of the dominant stand in feet; measured on a sample of 15-20 dominant and codominant trees per acre. This is equivalent to the height used in determining site index.

D = Residual quadratic mean stand diameter in inches computed as,

$$\sqrt{\frac{B_1}{\text{Residual number of trees/acre}} / 0.005454}$$

Board-foot growth and future volume were obtained by projecting stand height, basal area, and residual quadratic mean stand diameter with suitable equations for all combinations of site indices, ages, residual stand basal areas, and a range of residual quadratic mean stand diameters.

2.2 DIAMETER DISTRIBUTION MODELS

The models discussed so far have been whole stand projection models. Another approach to growth and yield prediction is through diameter distribution models.

An early diameter distribution approach was carried out by Buell (1945) where he predicted growth in uneven-aged timber stands of mixed hardwood and pine species on the basis of diameter distributions. However, it was several years later before diameter distribution methods and techniques were studied in any great detail.

Often it is assumed that the underlying diameter distribution of the stand can be adequately characterized by a probability density function (pdf). Many different probability distributions have been used to describe the diameter distributions of stands.

Clutter and Bennett (1965) fitted the beta distribution to observed diameter distribution data from old-field slash pine plantations, and from this, developed variable density stand tables. The beta distribution is very flexible in shape and therefore can approximate a wide range of diameter distributions. Also, the pdf has finite limits which constrain all diameters to be within upper and lower bounds. One disadvantage of this distribution, however, is that the pdf must be numerically integrated to obtain probabilities over various ranges of the the random variable ,i.e. to obtain the proportion of trees in each diameter class, as the cummulative distribution function (cdf) does not exist in closed form.

Bennett and Clutter (1968) used the beta distribution as a basis for the construction of yield tables in slash pine and obtained reliable and consistent estimates of board-foot, cordwood, and gum yields. The parameters of the beta distribution that approximated the diameter distribution were predicted from stand variables (age,site index, and density). The number of trees and volume per acre in each diameter class were then calculated, and per acre yield estimates were obtained by summing over the diameter classes of interest.

Following the same procedures, McGee and Della-Bianca (1967) successfully fitted the beta distribution to describe even-aged natural stands of yellow-poplar. From this diameter distribution information, Beck and Della-Bianca (1970) then developed reliable yield estimates for stands of even-aged unthinned yellow-poplar. A similar approach was used for loblolly pine plantations by Lenhart and Clutter (1971), Lenhart (1972), and Burkhart and Strub(1974). In each of these cases, the minimum and maximum diameters defining the limits of the distributions, as well as the pdf parameters were predicted from some function of stand characteristics.

Burkhart (1971) conducted an independent evaluation of the yield estimation technique presented by Bennett and Clutter (1968) for slash pine. He concluded that while variation of individual plots may be large, on the average, the technique gives accurate results.

Another distribution which is useful for describing diameter distributions is the Weibull. The pdf is flexible in shape, the parameters are reasonably easy to estimate, and the cdf exists in closed form, a major advantage over the beta pdf. The Weibull pdf exists in either a two or three parameter form, the three parameter pdf having the advantage of increased flexibility. First used as a diameter distribution model by Bailey(1972), the Weibull distribution has a wide range of applications. For example,it was used to construct models for loblolly pine plantations (Smalley and Bailey, 1974a, Feduccia et.al., 1979 and Schreuder and Swank, 1974),slash pine plantations (Clutter and Belcher, 1978, Dell et.al., 1979), shortleaf pine plantations (Smalley and Bailey, 1974b), longleaf pine plantations (Lohrey and Bailey, 1976) and white pine (Schreuder and Swank, 1974). Bailey and Dell(1973) concluded no other diameter distributions proposed exhibit as many desirable features as the Weibull.

Hafley and Schreuder(1977) compared six distributions(normal, lognormal, gamma, Weibull, beta, and S_B) in terms of flexibility of skewness and kurtosis, and for fitting the diameter distributions. They concluded that the S_B distribution was consistently better than the others, followed by the beta, Weibull, gamma, lognormal and normal distributions. However, for practical purposes, there were no real differences between the more theoretically and computationally complex S_B distribution and the beta and Weibull distributions.

Given a pdf and the parameter estimates, most published yield studies obtain volume yield on a per unit area basis in the following way.

1) Using the pdf, along with the number of surviving trees on the area, estimate the number of trees per unit area in each diameter class as:

$$N_i = Np(x_i)$$

where,

N_i = number of trees per unit area in diameter class, i,

N = total number of trees per unit area,

$p(x_i)$ = proportion of trees in diameter class, i,

$= \int_{dl}^{du} f(x)\,dx$, where f(x) is the pdf and dl and du are diameter limits such that dmin<dl<du<dmax.

2) Given a total height equation of the form, $H = f_1$(dbh, stand characteristics), and a total volume equation of the form, $V = f_2$(dbh,H), compute the volume per unit area of the midpoint tree of the i th diameter class by first estimating the tree's mean height and then using the total volume equation as follows:

$$v_i = f_2(dbh_i, H_i)$$

where,

v_i = volume per unit area of midpoint tree of ith diameter class,

dbh_i = dbh of midpoint tree of ith diameter class,

H_i = mean height of midpoint tree of ith diameter class obtained from f_1.

3) Compute the volume in the i th diameter class as follows:

$$V_i = N_i v_i$$

where,

V_i = total volume per unit area in ith class,

N_i = number of trees per unit area in diameter class i, as computed in step 1.

v_i = volume per unit area of midpoint tree of ith diameter class as computed in step 2.

(based on the assumption that tree diameters are uniformly distributed within the interval.)

Per unit yield estimates are obtained by summing over the diameter classes of interest. This method generally gives reliable yield estimates.

However, one shortcoming of the procedure outlined above is the class midpoint diameter is rarely the true mean for a diameter class, i.e. an incorrect assumption was made in Step 3. In addition, calculating volume per diameter class and summing to obtain a per unit area estimate involves unnecessary computations when only a single per unit area value is desired. Strub and Burkhart (1975) presented a class-interval-free method for obtaining yield estimates which eliminated the need for the assumption that diameters be uniformly distributed over an interval, as well as the dependency on fixed diameter class intervals to obtain yield estimates over specified diameter class limits. In addition, the class-interval-free method reduces the imprecision and bias inherent in using class midpoint diameters for volume estimates. The general equation form is given by,

$$TV = N \int_{L}^{U} g(D)\, f(D)\, dD \qquad (2.2.1)$$

where,

TV = expected stand volume per unit area,

N = number of trees per unit area,

D = dbh,

g(D) = individual tree volume equation,

f(d) = pdf for D,

L,U = lower and upper merchantable limits, respectively, for the product described by g(D).

In order to project the stand structure, and consequently the yield through time, the approach has generally been to predict the parameters of the diameter distribution at some future point in time. The ability to predict the parameter estimates for a given set of stand conditions is an essential feature in using pdf's to model diameter distributions.

One method of predicting the parameter estimates is to estimate the pdf parameters for each sample plot. Regression equations are then constructed to relate the parameters to stand characteristics such as age, site index, and number of trees. Given these equations, referred to as parameter prediction equations, and projected estimates of the stand characteristics (obtained from appropriate projection equations) the pdf parameters can be estimated, and thus the projected diameter distribution can be obtained. However, the parameter prediction equations typically have R^2 values ranging from 0.1 to 0.3, indicating poor model specification, or perhaps, that the parameters are not well related to varying stand characteristics.

As an alternative to the parameter prediction equations, Hyink (1980a, 1980b) introduced a method of solving for the parameters of a pdf approximating the diameter distribution using attributes from a whole stand model and the relationship given by the class-interval-free equation presented by

Strub and Burkhart (1975). The approach was to predict stand average attributes of interest for a specified set of stand conditions, and use these estimates as a basis to "recover" the parameters of the underlying diameter distribution using the method of moments. Hence it was called the "parameter-recovery" method.

When constructed independently, even from the same data set, stand average and diameter distribution models, which give different levels of resolution, do not necessarily produce the same estimates of stand yield for a given set of stand conditions (Daniels, et.al.,1979). The advantages of the procedure outlined by Hyink are mathematical compatibility between the whole stand and diameter distribution based yield models, ability to partition total yield by diameter class, and consistency among the various stand yield estimates.

Using this concept, Matney and Sullivan (1982) developed a model for thinned and unthinned loblolly pine plantations. Cao (1981) used a similar approach with a segmented Weibull cummulative distribution function to derive empirical diameter distributions from predicted stand attributes for thinned loblolly pine plantations.

Frazier (1981) also developed a method to approximate the diameter distribution of unthinned plantations of loblolly pine from whole stand predictions of stand attributes. The diameter distribution functions for estimating a stand attribute such as average diameter at breast height (dbh) or total volume per acre, were modelled using the beta pdf and the Weibull pdf. Given the stand attributes estimated from a whole stand equation, the parameters of the pdf were estimated.

Two types of parameter recovery systems were described by Frazier. The first used equations for the non-central moments of dbh, average diameter and average squared diameter being the first and second moments, respectively. The second type used volume, as a function of diameter, as one of the stand attributes used to solve for the parameters.

In unthinned loblolly pine plantations, the parameter models presented by Frazier (1981) represented a feasible alternative for predicting diameter distributions when compared to other conventional diameter distribution prediction methods.(Burkhart and Strub, 1974, Smalley and Bailey, 1974a). In addition to providing a model which can approximate the diameter distribution of stands, this method also insures numerical compatibility of the whole stand estimates of stand attributes and the diameter distribution estimates. Thus, given whole stand estimates of total basal area or total cubic-foot volume, basal area or cubic-foot volume by diameter class can be obtained. Because of the difficulty associated with specification of thinning effects on diameter distributions from stand and stock table projection, few diameter distribution models are available for thinned stands (Farrar, 1979).

Chapter III

DATA

Data for this study were collected by the U. S. Forest Service, Southeastern Forest Experiment Station from 141 circular, 1/4-acre plots established in the Appalachian mountains of North Carolina (93 plots), Virginia (31 plots), and Georgia (17 plots). The plots contained 75 percent or more yellow-poplar in the overstory, were free from insect and disease damage, and showed no evidence of past cutting (Beck and Della-Bianca, 1972).

Each plot was thinned (using low thinning) at the time of installation to obtain a range of basal areas for different site-age combinations. Site index at age 50 was determined for each plot with an equation published by Beck (1962). Volumes and basal areas were computed when the plots were thinned and again after five growing seasons. Heights were calculated by fitting a least squares equation relating height to diameter from measurements taken on every tenth tree. From the equation, heights were obtained for each tree in the plot. Then using existing equations (Beck 1963, 1964), a volume for each tree was computed. Plot volumes were then determined by summing the individual tree volumes. Table 1 shows a summary of the plot data before and after the first thinning (1), before and after the second thinning (2), five years after the second thinning (3) and 10 years after the second thinning (4). Basal area and cubic foot volume growth between the four measurement periods are presented in Table 2. The basal area, number of trees and cubic volume removed in each of the thinnings at measurements (1) and (2) are given in Table 3.

Table 1. Summary of stand characteristics at the time of the four plot measurements.

Growth Period	Variable	Minimum value	Mean	Maximum value	Standard Deviation
At time of first thinning (1)	Age	17.000	47.526	76.000	14.492
	Site	74.000	108.219	138.000	11.678
	NT	108.000	231.095	432.000	70.869
	BA	48.944	137.074	209.037	29.176
	CV	1336.300	5777.224	11170.700	1860.429
	BFV	490.700	18665.396	55032.400	11513.311
	RNT	32.000	103.737	340.000	61.551
	RBA	38.899	86.741	152.603	29.584
	RCV	1106.340	3856.974	8101.840	1575.150
	RBFV	329.000	14410.847	41106.000	8954.967
Five years after period (1) and at time of second thinning (2)	Age	22.000	52.201	81.000	14.638
	Site	74.000	108.219	138.000	11.678
	NT	32.000	102.849	320.000	58.539
	BA	37.993	97.131	163.998	30.609
	CV	1223.890	4579.897	9508.330	1768.553
	BFV	198.600	18279.735	49502.900	10209.791
	RNT	28.000	81.554	256.000	43.500
	RBA	21.809	85.567	150.057	29.255
	RCV	721.600	4093.961	8183.820	1694.037
	RBFV	198.000	16984.022	42459.000	9523.918
Five year after period (2) (no thinning) (3)	Age	27.000	57.071	86.000	14.666
	Site	74.000	107.721	138.000	11.827
	NT	28.000	81.886	256.000	43.468
	BA	33.376	97.773	163.760	30.395
	CV	1218.530	4864.039	9073.690	1823.227
	BFV	1905.500	21312.961	46473.000	10320.536
	RNT	28.000	81.400	252.000	42.931
	RBA	31.106	97.308	163.760	30.239
	RCV	1135.310	4841.826	9073.690	1817.256
	RBFV	1905.000	21219.950	46473.000	10287.229

Table 1. Continued.

Growth Period	Variable	Minimum value	Mean	Maximum value	Standard Deviation
Five years	Age	33.000	62.442	91.000	14.431
after	Site	74.000	107.551	138.000	11.819
period (3)	NT	28.000	81.217	252.000	43.191
(no thinning)	BA	39.530	110.632	177.485	32.161
(4)	CV	1567.450	5731.614	10052.500	1993.417
	BFV	3554.400	26352.014	51347.900	11194.138
	RNT	28.000	80.783	236.000	42.326
	RBA	39.530	110.267	177.485	31.978
	RCV	1567.450	5714.924	10052.500	1988.612
	RBFV	3554.000	26287.993	51347.000	11171.624

Where, Site = site index, base age 50 years
NT = number of trees/acre
BA = basal area/acre (sq.ft.)
CV = cubic-foot volume/acre
BFV = board-foot volume/acre
RNT = residual number of trees/acre
RBA = residual basal area/acre (sq.ft.)
RCV = residual cubic-foot volume/acre
RBFV = residual board-foot volume/acre

Table 2. Summary of basal area and cubic volume growth during the five-year periods between the four plot measurements.

Growth Period	Variable	Minimum value	Mean	Maximum value	Standard deviation	Mean annual growth
5-years after first thinning	BA1	25.245	85.350	152.603	30.313	
	BA2	37.993	97.388	171.009	31.253	
	BAGROWTH	4.623	11.988	32.969	5.208	2.398
	CV1	734.300	3775.866	8101.840	1624.174	
	CV2	1223.890	4570.519	9508.330	1780.950	
	CVGROWTH	317.930	794.652	1919.710	312.485	158.930
5-years after second thinning	BA1	21.809	85.245	150.057	29.398	
	BA2	33.376	97.773	163.760	30.395	
	BAGROWTH	4.455	12.528	32.364	4.458	2.506
	CV1	721.600	4073.356	8183.820	1705.448	
	CV2	1218.530	4864.039	9073.690	1823.227	
	CVGROWTH	260.050	790.682	2190.110	299.742	158.136
From 5 to 10 years after second thinning	BA1	31.106	97.741	163.760	30.223	
	BA2	39.530	110.632	177.485	32.161	
	BAGROWTH	-1.131	12.892	25.589	4.257	2.578
	CV1	1135.310	4874.815	9073.690	1808.376	
	CV2	1567.450	5731.614	10052.500	1993.417	
	CVGROWTH	-60.620	856.799	1739.520	322.622	171.360

where,
- BA1 = basal area/acre at beginning of growth period
- BA2 = basal area/acre at end of growth period
- BAGROWTH = BA2-BA1, i.e. 5 years growth
- CV1 = cubic-foot volume/acre at beginning of growth period
- CV2 = cubic-foot volume/acre at end of growth period
- CVGROWTH = CV2-CV1, i.e. 5 years growth

Table 3. Cubic-foot volume, basal area, and number of trees per acre removed in thinning at measurements one and two.

Variable	N	Mean	Standard Deviation	Variance	Minimum Value	Maximum Value
				Measure=1		
VR	141	1880.95042553	1254.94418494	1574884.90732	47.50000000	6275.0400000
VB	141	5656.81687943	1966.77358010	3868198.31539	1188.33000000	11170.7000000
BR	141	49.50071631	27.49815038	756.14827	1.79600000	136.7010000
BB	141	134.85079433	31.70053769	1004.92409	43.92200000	209.0370000
NR	141	126.78014184	56.63909455	3207.98703	12.00000000	312.0000000
NB	141	231.88652482	70.38497122	4954.04417	108.00000000	432.0000000
				Measure=2		
VR	124	547.18556452	275.39198717	75840.74660	51.74000000	1536.72000000
VB	124	4573.98185484	1806.18880742	3262318.00806	1223.89000000	9508.33000000
BR	124	13.05918548	7.14453527	51.04438	1.22700000	50.26800000
BB	124	97.82141935	31.05983264	964.71320	37.99300000	163.99800000
NR	124	24.35483871	24.45111052	597.85681	4.00000000	144.00000000
NB	124	106.00000000	59.14622915	3498.27642	32.00000000	320.00000000

Where, VR = volume removed in thinning
VB = volume before thinning
BR = basal area removed in thinning
BB = basal area before thinning
NR = number of trees removed in thinning
NB = number of trees before thinning

Chapter IV

METHODS AND PROCEDURES

4.1 SIMULTANEOUS GROWTH AND YIELD EQUATIONS

The first step in the analysis was to determine if the model forms and coefficients of the equations derived by Beck and Della-Bianca (1972, 1975) for predicting basal area growth, cubic-foot volume growth and yield, board-foot volume growth and yield and residual quadratic mean stand diameter growth were appropriate for stands thinned more than once. Using these equations and the initial stand characteristics at the 5-year remeasurement, the plots were projected to 5 and 10 years and the projected values were then compared to the observed data. In addition, the data from the 10-year remeasurement were projected to the 15-year point for comparison.

The mean difference between the observed and predicted values, standard deviation of the differences and mean of the absolute value of the differences were computed to check the bias, precision and average magnitude of the residuals, respectively. Also, the differences and absolute value of the differences were plotted over stand characteristics such as age, site index, and basal area, as well as the first order interaction terms and the terms in the prediction equations to check for trends or patterns indicating improper model specification or unaccounted for variation in the data. In addition, regression equations using the same model forms were fit for each measurement period. The R^2 (coefficient of determination) and residual values were calculated for each, and from this information, evaluations of the original model forms and coefficients were made.

These procedures were carried out for the basal area and cubic-foot volume growth equations of Beck and Della-Bianca (1972). The results indicated that the model forms were appropriate, but that different coefficients were needed for stands thinned more than once.

Based on these preliminary results, various options were considered. The first option was to use all the data to estimate a single set of coefficients for use over all growth periods. A second was to use the data immediately following initial thinning to estimate one set of coefficients (these would be identical to the coefficients of Beck and Della-Bianca, 1972), the data immediately following the second thinning to estimate a second set of coefficients, and the data following both thinnings to estimate a third set of coefficients. A final option was to determine if two of the growth periods could be combined together to simplify the prediction system. For example, combining periods one and two would produce a set of coefficients for prediction immediately after a thinning and period three would be used to fit an equation for prediction not immmediately after a thinning. An obvious problem with this type of system would be determining when to apply the equation for not immediately after a thinning. Combining the second and third periods would alleviate this problem. In this case, one would have two sets of coefficients, one for use after one thinning (based on period one) and another set for use after two thinnings (based on period two and three), regardless of how long it has been since a thinning.

With the options given above, there was a possibility that some reduced model form (i.e. same slope, different intercepts) was appropriate. For example, in the second option, separate coefficients could be estimated for each period while restricting the slope coefficients to be the same. All such possibilities were investigated.

To accomplish this part of the analysis, the data were sorted into three classes on the basis of number of thinnings. Measurements from the beginning and end of each of the three 5-year growth period were combined (i.e. growth and yield measurements were combined) to give a total of 282 observations per class.

The growth and yield measures must be combined in order to prevent a reduction in the original model form. As all the growth periods are fixed at five year intervals, exclusion of the yield measures, which represent a growth period of zero years, results in the following model reduction.

Given the model,

$$\ln Y = b_0 + b_1(1/S) + b_2(1/A_2) + b_3(A_1/A_2)(\ln B_1) + b_4(1-A_1/A_2) + b_5(S)(1-A_1/A_2)$$

the terms,

$$b_2(1/A_2) \quad \text{and} \quad b_4(1-(A_1/A_2)$$

can be rewritten as,

$$b_2(1/A_2) \quad \text{and} \quad b_4\left[(A_2-A_1)/A_2\right]$$

With the growth periods fixed at 5-year intervals, i.e. $A_2 - A_1 = 5$, the terms are,

$$b_2(1/A_2) \quad \text{and} \quad b_4(5/A_2)$$

As a result, the two terms are linear combinations of one another.

The consequences resulting from treating the observations from remeasured plots as if they are independent when in actuality they are not should not be too serious according to Sullivan and Clutter (1972).

The best option was determined on the basis of statistical analyses for determining optimal model forms, in particular, F-tests of full and reduced model forms, as well as on the basis of the predictive ability of the model forms. The F-tests were conducted using the sum of squared residuals from each equation form in terms of the logarithm of volume and basal area, as well as in cubic-foot volume (ft^3) and basal area (ft^2) terms. The F-tests in cubic- feet and square feet units would indicate the actual differences in volume and basal area fit due to the different model forms. Evaluations and comparisons of the predictive ability of the model forms were made according to predicted basal area and volume in terms of the mean residual, the mean absolute residual and the standard deviation of the residuals to check on the bias and precision.

Once the appropriate model forms were selected, the coefficients of the equations were estimated in two ways. First through ordinary least squares (OLS) procedures and then through a simultaneous fitting procedure. As the coefficients of the basal area projection equation are functions of those from the cubic volume projection equation (i.e. a1=b4/b3, a2=b5/b3) the ordinary least squares procedure for estimating coefficients of the volume equation can minimize the sum of squared residuals (SSE) for volume only. However, under the same circumstances, the simultaneous fitting procedure allows the minimization of the SSE for both volume and basal area.

The loss function to be minimized in the simultaneous fitting was defined such that equal weights were given to volume and basal area projection. In this case, the corresponding loss function was given by,

$$F = \frac{\sum_i (y_i - \hat{y}_i)^2}{\hat{\sigma}_y^2} + \frac{\sum_i (B_i - \hat{B}_i)^2}{\hat{\sigma}_B^2}$$

where,

y_i and $\hat{y}_i$ = the observed and predicted volume values, respectively,

B_i and $\hat{B}_i$ = the observed and predicted basal area values, respectively,

$\hat{\sigma}_y^2$ and $\hat{\sigma}_B^2$ = the estimates of the variance about the regression lines for volume and basal area, respectively, computed as the mean square error from ordinary least squares fits of equations 2.1.5 and 2.1.6.

Using the OLS coefficient estimates from the volume equation as starting values, the coefficients of the basal area equation, given by a1=b4/b3 and a2=b5/b3 were computed and the loss function, F, was evaluated. The coefficients were then adjusted through an iterative process until the loss function was minimized. The stopping criterion in the process was either a maximum number of iterations (1000) or no change in the coefficient estimates to six significant digits. It was expected that the simultaneous procedure would result in a slight sacrifice in volume fit for a greatly improved basal area fit. At the same time, the equations of the prediction system would remain compatible and numerically consistent.

Burkhart and Sprinz[1/] used this same procedure for projecting cubic volume and basal area growth of thinned old-field loblolly pine plantations using Sullivan and Clutter's (1972) simultaneous growth and yield equation forms. The simultaneous procedure greatly reduced the error in basal area projection while increasing the error in cubic volume projection only slightly. Reed (1982) also used this procedure to simultaneously estimate the parameters in tree taper and volume equations.

The two fitting procedures were evaluated and compared on the basis of cubic-foot volume and basal area prediction, as well as on the gains and losses in volume and basal area prediction due to the fitting procedures. Through this analysis, a consistent set of simultaneous growth and yield

[1/] Burkhart, H.E. and P.T. Sprinz. Cubic volume and basal area projection equations for thinned loblolly pine plantations. Submitted to Forest Science.

equations for thinned stands of yellow-poplar were obtained, and then incorporated into a stand-level computer simulation model that projects growth and yield of yellow-poplar stands given a set of initial conditions, a specified thinning regime, and a rotation age.

4.2 BOARD-FOOT VOLUME EQUATIONS

Previous studies involving board-foot volume prediction have generally produced equations with lower correlations and higher standard errors than similar cubic-foot volume equations. Brender and Clutter (1970) fitted a board-foot volume equation based on Sullivan and Clutter's (1972) model with a reduction in precision over cubic-foot volume. Also, when two seperate equations are fit, i.e. a cubic-foot and board-foot, using Sullivan and Clutter's model, illogical crossings of volume estimates may result. Beck and Della-Bianca (1975) also noted that this equation did not do well for board-foot volume prediction in thinned yellow-poplar stands, and that some measure of stand structure was needed.

For these reasons, equations similar to those fitted by Bennett (1970) relating board-foot volume to basal area and cubic-foot volume were fitted and evaluated. The equations would be used to express board-foot volume as a function of cubic volume predicted from the simultaneous growth and yield equation and either quadratic diameter or basal area, also predicted from the same set of equations. With this procedure, accuracy and precision should be increased, while preventing illogical crossings of board-foot and cubic-foot volume estimates associated with seperate prediction equations.

4.3 VOLUME REMOVED IN THINNING

To estimate the volume removed in thinning from below when the reduction in basal area or number of trees is known, equations presented by Field, et.al. (1978) were considered. They constructed the following equations on the basis of linear trends displayed in plots of proportion of volume removed versus proportional reduction in stand density.

$$-\log (Vr/Vb) / \sqrt{-\log(Br/Bb)} = \alpha\sqrt{-\log(Br/Bb)} \qquad (4.3.1)$$

$$-\log (Vr/Vb) / \sqrt{-\log(Nr/Nb)} = \beta\sqrt{-\log(Nr/Nb)} \qquad (4.3.2)$$

where,

Vr = cubic volume per acre removed,

Vb = cubic volume per acre before thinning,

Br = basal area per acre removed,

Bb = basal area per acre before thinning,

Nr = number of stems per acre removed,

Nb = number of stems per acre before thinning,

α, β = parameters to be estimated from the data.

These equation forms seem appropriate as they were derived from data taken from slash pine plantations thinned from below as the yellow poplar stands were. The equations were fitted using ordinary least squares regression techniques on the plot data to obtain estimates for α and β.

Then through algebraic manipulation, the following prediction equations were also specified.

$$\hat{V} = Vb(Br/Bb)^{\alpha} \qquad (4.3.3)$$

$$\hat{V} = Vb(Nr/Nb)^{\beta} \qquad (4.3.4)$$

where all variables are as previously defined.

The nonlinear equation forms 4.3.3 and 4.3.4 given above were then fit using nonlinear least squares procedures to obtain another set of coefficient estimates. The nonlinear fitting of these equations should reduce the transformation bias associated with the linear equation forms 4.3.1 and 4.3.2 when predicting the volume removed in thinning through the direct minimization of the SSE for the volume removed.

Both the linear and the nonlinear coefficient estimates were evaluated and then compared in terms of predicting volume removed in thinning based on the proportion of basal area or number of trees removed. For this analysis, volume removed was predicted with each of the four equations and then subtracted from the observed volume removed. The mean, the mean magnitude, and the standard deviations of these residual values were used as a basis for the evaluations and comparisons.

4.4 DIAMETER DISTRIBUTION PREDICTION

The parameter recovery procedure discussed by Frazier (1981) was used to estimate the parameters of the Weibull probability density function which was selected to describe the diameter distributions of yellow-poplar stands before and after thinning.

The Weibull probability density function exists in either a two or three parameter form. These two forms are defined as follows.

Three parameter Weibull density

$$f_X(x;a,b,c) = \begin{cases} (c/b)\left(\frac{x-a}{b}\right)^{c-1} \exp\left[-\left(\frac{x-a}{b}\right)^{c}\right] & a,b,c > 0 \\ & a < x < \infty \\ 0, \text{ otherwise} \end{cases}$$

Two parameter Weibull density

$$f_Y(y;b,c) = \begin{cases} (c/b)\left(\frac{y}{b}\right)^{c-1} \exp\left[-\left(\frac{y}{b}\right)^{c}\right] & y,b,c > 0 \\ 0, \text{ otherwise} \end{cases}$$

where $Y = X - a$.

With the general diameter distribution yield function,

$$Y_i = Nt\int_{D_L}^{D_U} g_i(x)f(x;\underline{\theta})dx ,$$

integration over the range of diameters, x, for any $g_i(x)$, gives the total per unit area value of the stand attribute defined by $g_i(x)$. Average diameter, basal area per acre, total cubic volume per acre and board-foot volume per acre are examples of such stand attributes. The number of stand attribute equations must equal the number of parameters to be estimated in order to solve the system of equations for the pdf parameters.

Frazier outlined two basic systems of equations for estimating the parameters. One consisted of the non-central moments of the random variable X, $E(X^i)$ and was called the moment-based parameter recovery system. The other system involved the use of one or more volume equations together with non-central moment equations, and was called the volume-based parameter recovery system. In this analysis, only the moment-based parameter recovery system was investigated.

As Frazier pointed out, the moment-based parameter recovery system is simply the method of moments technique of pdf parameter estimation (Mendenhall and Scheaffer, 1973), where the equation for the ith non-central moment of X is given by,

$$E(X^i) = \int_{\text{all } x} x^i f(x_i \; ; \; \underline{\theta}\,)dx \; = \; Y_i/N$$

where,

$$X^i = g_i(X)$$

The first non-central moment,

E(X) is estimated by $\dfrac{\sum_i X_i}{N} = \bar{x}$, the average diameter of the stand.

The second non-central moment,

$E(X^2)$ is estimated by $\dfrac{\text{basal area/acre}}{0.005454N} = \dfrac{\sum_i X_i^2}{N} = \overline{X^2}$

Although they have no practical forestry interpretations, the higher moments can be estimated in a similar manner.

i.e. $E(X^i)$ is estimated by,

$$\frac{\sum_k X_k^i}{N} = \overline{X^i}$$

Stand average estimates of the first k moments produce a system of k equations with k unknown parameters which can be used to obtain estimates of the pdf parameters while insuring compatibility between whole stand and diameter distribution estimates of the stand attributes described by the moment equations.

The moment-based system of equations for the three parameter Weibull distribution uses the first three non-central moments, $\overline{x}$, $\overline{x^2}$, $\overline{x^3}$. As this set of equations led to convergence problems, the three parameter Weibull pdf was reduced to the two parameter pdf form. Using the transformation Y=X-a, i.e. 'a' is set equal to a constant or predicted outside the system of equations, the three parameter Weibull system was simplified to a two parameter system. The two equations in the final system are,

$$\overline{x} = \int_0^\infty x f(x;b,c)\,dx = b\Gamma(1+1/c) \qquad (4.4.1)$$

$$\overline{x^2} = \int_0^\infty x^2 f(x;b,c)\,dx = b^2\Gamma(1+2/c) \qquad (4.4.2)$$

The estimated variance of the distribution is given by,

$$s^2 = \overline{x^2} - \overline{x}^2 = b^2[\Gamma(1+2/c) - \Gamma^2(1+1/c)] \qquad (4.4.3)$$

and the coefficient of variation is estimated by,

$$CV = s/\overline{x} = \frac{[\Gamma(1+2/c) - \Gamma^2(1+1/c)]^{\frac{1}{2}}}{\Gamma(1+1/c)} \qquad (4.4.4)$$

As the coefficeint of variation is a function of 'c' alone, given estimates of $\overline{x}$ and $\overline{x^2}$ it is possible to solve for 'c'. This then allows for the solution for 'b' from

$$\overline{x} = b\Gamma(1+1/c)$$

The whole stand and individual tree equations, developed from the plot and individual tree data sets, required by the moment-based parameter recovery system are given in Table 4.

The basal area equations are those presented earlier in the stand level projection equation section. A separate set of coefficients is used depending on the number of thinnings.

Initially, an independent equation was fit to predict $\overline{D}$, average stand diameter, as an estimate of the first non-central moment, and $\overline{D^2}$ = BA per acre/.005454N, where BA per acre was estimated from the stand level model, was used as an estimate of the second non-central moment. However, when the coefficient of variation for the Weibull distribution, given by,

$$\text{c.v.} = \frac{(\overline{D^2} - \overline{D}^2)^{\frac{1}{2}}}{\overline{D}}$$

was calculated, a negative variance, and thus a negative c.v. value sometimes resulted. Estimates of $\overline{D}$ and $\overline{D^2}$ from independent equations often produced illogical crossovers and hence negative variances (i.e. $(\overline{D^2}-\overline{D}^2)<0$). Frazier encountered similar difficulties when he predicted $\overline{D}$ and $\overline{D^2}$ independently. To condition the term $\overline{D^2}-\overline{D}^2$ to be greater than zero, Frazier predicted $\ln(\overline{D^2}-\overline{D}^2)$ and $\overline{D}$ and then solved for $\overline{D^2}$. For this analysis $\ln(\overline{D^2}-\overline{D}^2)$ was predicted and this

Table 4. Stand-level and individual tree equations developed from the plot and individual tree data sets.

(a) $\ln(BA_2) = (A_1/A_2) * \ln(BA_1) + (b_4/b_3) * (1 - A_1/A_2) + (b_5/b_3) * S * (1 - A_1/A_2)$ (4.4.5)

<u>For after 1 thinning</u>	<u>For after 2 thinnings</u>
$b_3 = 0.97473$	$b_3 = 0.98858$
$b_4 = 4.11893$	$b_4 = 5.84476$
$b_5 = 0.01293$	$b_5 = 0.00018$

(b) $\ln(\overline{D^2} - \bar{D}^2) = b_0 + b_1 \ln(BA) + b_2 \ln(HDOM) + b_3 (A_2 * NT/10000)$ (4.4.6)

<u>For before first thin</u>	<u>For all other measurements</u>
$b_0 = -13.40824$	$b_0 = -5.20164$
$b_1 = 0.45213$	$b_1 = 0.80773$
$b_2 = 3.05978$	$b_2 = 0.72383$
$b_3 = -0.20664$	$b_3 = -0.33560$

(c) $\bar{D} = [BA/(.005454 * N) - EXP(\ln(\overline{D^2} - \bar{D}^2))] ** 0.5$ (4.4.7)

(d) $\ln(Dmin) = 1.19439 + 0.05637 * ((BA/(NT * 0.005454)) ** 0.5)$ (4.4.8)

$+ 3.04022/(NT ** 0.5) - 394.07219/(A_2 * HDOM)$

(For all measures except before first thin, where Dmin is set equal to 5.0)

Table 4. Continued.

(e) $\ln(HDOM/H) = -0.09675 + (1/Dbh - 1/Dmax)*[3.70051 - 0.02828*\ln(BA) - 138.35633/AGE + 0.04010*S]$ (4.4.9)

(f) $TVOB = 0.01710 + 0.00248*Dbh^2*H$ (4.4.10)

(g) $\ln(BA) = b_0 + b_1(1/A) + b_2S + b_3(1/NT)$ (4.4.11)

	For before first thin	For after first thin	For after second thin
b_0	4.55808	4.16240	4.24861
b_1	-31.21173	-38.13602	-45.83883
b_2	0.01324	0.01606	0.01566
b_3	-77.35908	-47.19922	-37.7888

(h) $\ln(NT) = b_0 + b_1(1/A) + b_2S + b_3(1/BA)$ (4.4.12)

	For before first thin	For after first thin	For after second thin
b_0	6.43346	6.12444	6.12335
b_1	38.24834	59.93859	69.03772
b_2	-0.01309	-0.01911	-0.02083
b_3	-67.25874	-73.59987	-78.12201

(i) The Weibull parameter a = 0.5*Dmin

Table 4. Continued.

Where,

$\overline{D}$ = average tree dbh (inches)
$\overline{D^2}$ = average squared tree dbh ($inches^2$)
BA = basal area per acre (square feet)
BA_1 = basal area per acre at beginning of projection period
BA_2 = basal area per acre at end of projection period
A = stand age
A_1 = stand age at beginning of projection period
A_2 = stand age at end of projection period
Dbh = diameter at breast height (inches)
Dmin = minimum diameter (inches)
Dmax = maximum diameter (inches)
HDOM = average height of dominant and codominant trees (feet)
N = number of trees per acre
TVOB = total cubic-foot volume (outside bark) per tree
H = total height of tree (feet)
S = site index, base age 50

estimate, together with the basal area estimate from the stand level model, were used to solve for $\overline{D}$. Average diameter computed from the transformation gave fairly good results.

Several equations were fit for minimum diameter, Dmin, prediction. As there was so little variation in the minimum diameter of the stands before the first thinning (4.0 inches < Dmin < 7.0 inches), Dmin was set equal to 5.0 inches in this case. In all other cases, Dmin was predicted using the equation given in Table 4.

The total height equation is a slight modification of the one presented by Beck and Della-Bianca (1970). Number of trees was replaced by basal area in the original model form. The individual tree volume equation is of the same form presented by Beck (1963) and was also fitted using weighted least squares procedures.

Equations to predict number of trees from age, site index, and basal area and basal area from age, site, and number of trees were developed to increase the flexibility of the system. Seperate equations were fit for stands before the first thinning, after the first thinning, and after the second thinning.

The 'a' parameter of the Weibull distribution was calculated from Dmin as a=0.5(Dmin). Frazier tested several values for 'a' and found small differences in the final diameter distributions. However, a=0.5(Dmin) performed slightly better than the others. Preliminary tests using the yellow-poplar plot data produced similar results. Thus the equation a=0.5(Dmin) was used to estimate the parameter.

The computer solution routine written by Frazier in FORTRAN-Level G for loblolly pine stands was applied to the yellow-poplar data with certain modifications and revisions. After the appropriate equations, previously presented in Table 4, were entered into the computer routine, diameter distributions before the first thinning were predicted for the 141 plots. Specifically, observed basal area per acre, number of trees per acre, age, and average height of the dominants and codominants (calculated from the site index equation) were used to calculate the coefficient of variation

$$\text{c.v.} = \frac{(\overline{Y^2} - \overline{Y}^2)^{1/2}}{\overline{Y}}$$

where,

$\overline{Y} = \overline{D} - a$

$\overline{Y^2} = \overline{D^2} - 2\overline{D}a + a^2$

$\overline{D}$ = average diameter

$\overline{D^2}$ = average squared diameter

Using International Mathematical and Statistical Library (IMSL) subroutines for evaluating the gamma function (GAMMA) and the iterative solution of one equation in one unknown (ZBRENT), 'c' was solved in

$$c.v. = \frac{[\Gamma(1+2/c) - \Gamma^2(1+1/c)]^{\frac{1}{2}}}{\Gamma(1+1/c)}$$

Given the solution for 'c', 'b' was then calculated from

$$\overline{D} = b\Gamma(1+1/c)$$

Once the parameter estimates were obtained, number of trees, basal area, and cubic volume per acre by diameter class were calculated for each plot before the first thinning according to the procedures outlined earlier.

Following similar procedures, the diameter distributions of the plots immediately after the first thinning were predicted and then checked for logical consistencies which should exist between the unthinned and thinned diameter distributions, as well as for inconsistencies which may result from independent prediction of the two distributions.

First, the number of trees in each diameter class before and after thinning was checked to insure that the number in a given class did not increase with thinning. Other inconsistencies which could occur would be an increase in the maximum diameter, or a decrease in the minimum diameter after thinning.

An inspection of ten sample plot distributions predicted before and after thinning produced several inconsistencies. From this preliminary analysis it was apparent that the distribution predictions before and after thinning can not be performed independently, but must be conditioned such that the previously stated inconsistencies can not occur.

As an alternative to two independent predictions, first the diameter distribution prior to thinning was predicted as before, then a proportion of the basal area in each diameter class was removed to simulate the thinning. With this procedure it is impossible for the number of trees in a given class to increase as trees can only be removed from a class. Consequently minimum diameter can only increase and maximum diameter can only decrease, if they change at all.

To carry out the thinning algorithm, a function first had to be defined to specify the amount of basal area to be removed from each diameter class. The following equation relating the proportion of basal area removed in a diameter class to the ratio of the midpoint diameter of the class to the average squared diameter of the stand was used to "thin" the predicted stand table.

$$Pbar_i = \exp\left[b_1 \left(d_i^2 / \overline{d}^2 \right)^{b_2} \right] \qquad (4.4.14)$$

where,

$Pbar_i$ = proportion of basal area removed from diameter class, i,

d_i = midpoint diameter of class, i,

$\overline{d^2}$ = average squared diameter of stand,

b1,b2 = regression coefficients estimated from the data.

As the plot data were taken on stands thinned from below, the removal function "thins" heavily in the smaller diameter classes and proportionally less as the diameter classes increase in size. Seperate removal equations were fitted for stands after the first and second thinnings due to the obvious differences in the size-class distributions. Coefficient estimates and statistics of the fits are given in Table 5.

Once the basal area removal functions are defined, the thinning algorithm is as follows.

1) Predict the diameter distribution prior to thinning as initially described.
2) Starting with the smallest diameter class, remove the proportion of basal area specified by the removal function.
3) Proceed through the diameter classes until

Table 5. Coefficient estimates and fit statistics for fits of basal area removal function (4.4.14).

	For First Thinning	For Second Thinning
b_1	-0.70407	-2.61226
b_2	1.87666	2.00627
SSE	150.6588	82.2393
MSE	0.0843	0.0672
Sy.x	0.2902	0.2592
R^2	0.5614	0.4060

the desired level of basal area to be removed is attained.

4) If the required basal area removal is not obtained after the largest diameter class is reached, return to the smallest diameter class and remove the remaining basal area in that class. Proceed in this manner through the diameter classes until the desired level of basal area removal is attained.

Following these procedures, diameter distributions before and after the initial thinning were predicted for the 141 plots. Similarly, the distributions before and after the second thinning were predicted. To compare the observed and predicted diameter distributions, differences between the observed and predicted basal area per acre and total cubic-foot volume per acre were calculated. In addition, observed and predicted number of trees by diameter class were used to conduct a Chi-square (X^2) goodness-of-fit test statistic for each plot and for all plots combined. Evaluations of the parameter recovery model (for unthinned stands) and the thinning algorithm were made on the basis of the X^2 tail probabilities, or p-values.

As the basal area removal equations were fit using data from all 141 plots, one would expect plots with the proportion of basal area cut close to the mean removal to have the lower differences in observed and predicted distributions after thinning, and hence the smaller X^2 values. As the basal area removed deviates from the mean removal, the fit may become progressively worse. To determine what effects, if any, the amount of basal area removed in thinning has on the prediction of diameter distributions after thinning, three different methods of grouping the plots and their associated X^2 statistics were examined.

First, the plots were grouped according to the amount of basal area removed in thinning. Those having an amount removed in the range of the mean basal area removed, plus or minus one standard deviation represented one group. Those having amounts removed above and below the upper and lower bounds represented the other two groups.

Next, the plots were classified according to the proportion of basal area removed in thinning. In the manner described above, three groups were defined.

Finally, the plots were sorted on the basis of whether or not all the basal area required by the thinning was removed in one 'pass' through the diameter classes or if a second 'pass' was needed.

For each classification scheme, the average and sum of the X^2 statistics were calculated for each group to detect differences in fit among them. The results should indicate if fit and predictive ability are correlated with basal area removed, proportion of basal area removed, and/or number of 'passes' required to remove the specified level of basal area.

4.5 FLEXIBLE VOLUME EQUATIONS

The approach taken by Beck (1963) to obtain flexible volume tables for yellow-poplar in the southern Appalachians involved the fitting of four different fixed merchantable top limit equations. One drawback of this method is that merchantable volume can be estimated to only a limited number of top diameters. In the case of Beck (1963) the limits were four and eight inch top diameters (i.b. and o.b.). In addition, independent, unconstrained volume equations for various top limits often cross illogically within the range of the data.

The volume ratios approach presented by Burkhart (1977) seemed more appropriate in that it allows merchantable volume prediction to any top diameter limit. The procedure consists of three basic steps.

1) Predict total tree volume using a total volume equation (TOTALVOL)

2) Predict the ratio of merchantable volume to total volume as,

$$R = Vm/Vt = f(d,D)$$

where,

Vm = merchantable volume to top diameter, d,

Vt = total volume,

D = dbh,

f = function relating R to d and D.

3) Obtain merchantable volume to top diameter,d, as, TOTALVOL * R

Note: The ratio can be formulated for both inside and outside bark top diameter measures.

This method represents a relatively simple means for obtaining cubic volume to any top diameter limit. Through subtraction, volume between any two specified diameter limits can be estimated.

The first step in this procedure was to evaluate the total volume equation published by Beck (1963), where he used a combined variable equation weighted by $(1/D^2H)^2$, i.e.

$$TV/D^2H = b_0/D^2H + b_1$$

where,

TV = total volume

D = dbh

H = total height

which accounted for more than 98% of the total variation in volume.

Using the same individual tree data set as Beck used in his analysis, 100 observations were selected at random from the 337 total tree measurements and set aside for evaluation purposes. The remaining 237 observations were used to estimate the parameters in the following total volume equations

1) $TV = b_0 + b_1D^2H$

2) $TV/D^2H = b_0/D^2H + b_1$

3) $\ln(TV) = b_0 + b_1\ln(D) + b_2\ln(H)$

4) $TV = b_0 + b_1D + b_2DH + b_3D^2 + b_4H + b_5D^2H$

where all variables are as previously defined.

The fitted equations were then evaluated in terms of total volume prediction based on the analysis of the residual values representing observed minus predicted volumes . From this analysis, a total volume equation was selected.

The second step involved the definition of the function relating the ratio of merchantable volume over total volume to dbh and a merchantable top diameter, d. Burkhart (1977) fit the following nonlinear ratio equation for plantation and natural stand loblolly pine.

$$R = 1 + b1\left[t^{b2}/D^{b3}\right] \qquad (4.5.2)$$

where,

t = top diameter (o.b. or i.b.) in inches,

D = dbh in inches,

R = merchantable cubic-foot volume (o.b. or i.b.) to top diameter, t / total stem volume (o.b. or i.b.) in cubic feet,

$b1, b2, b3$ = regression coefficients.

With this form, the coefficient, b1, is less than zero, thus R is less than or equal to one, as it should be. Also, as t approaches zero, (i.e. as the top of the tree is approached), R approaches one. Using this volume ratios approach to merchantable volume prediction, plots of cubic-foot volume versus diameter squared times height for different top limits indicated different slope and intercepts for the various curves, but no illogical crossings often associated with independently fit fixed top limit equations. Thus with this ratio equation, it was possible to obtain logical and consistent cubic-foot volume estimates to any desired top limit. The volume ratios approach, when applied to the yellow-poplar data set, were evaluated on the basis of merchantable volume prediction. Analysis of the volume residual values in terms of the mean residual (observed minus predicted volume), the mean absolute residual, and the standard deviation of the residuals gave an indication of the bias and precision in prediction.

In a similar manner, Cao and Burkhart (1980) fit the following height ratio equation to obtain estimates of merchantable volume to any height limit.

$$R = 1 + b1\left[p^{b2}/H^{b3}\right] \tag{4.5.3}$$

where,

H = total tree height in feet,

p = distance from tip to height of interest, h,

= $H - h$ in feet,

R = as previously defined,

$b1,b2,b3$ = regression coefficients estimated from the data

Merchantable volume prediction with this height ratio involves the same basic steps as the diameter ratio, the only difference being the formulation in terms of height rather than diameter. Again, b1 is less than zero, restricting R to be less than or equal to one, and R is conditioned such that as p approaches zero, i.e. merchantable height, h, is approaching total height, H, R approaches one. As before, however, R is not conditioned at the lower end, so as p gets large and approaches H, R can become negative. With this ratio equation, cubic-foot volume can be estimated to any height limit, and through subtraction, volume between any two specified heights can also be estimated.

The height ratio equations were fit using nonlinear regression techniques, and then evaluated on the basis of merchantable volume prediction to various height limits. Again, this was done through analysis of the residuals corresponding to the observed minus the predicted volume values at the various height limits.

Given the height and diameter ratios presented by Cao and Burkhart (1980), implicit taper functions of the following forms could be obtained through a simple rearrangement of the equations.

$$t = f_1(D,H,p)$$

$$h = f_2(D,H,t)$$

where,

t = merchantable top diameter of interest,

h = merchantable height of interest,

D = dbh,

H = total height,

p = H - h.

Thus, in addition to merchantable volume prediction, height to a given diameter, and diameter to a given height could be estimated.

For example, let

$$R1 = Vm/Vt = 1 + b11\left[t^{b21}/D^{b31}\right]$$

and

$$R2 = Vm/Vt = 1 + b12\left[p^{b22}/H^{b32}\right]$$

Then,

$$t = \left[\frac{(R2 - 1)}{b11} D^{b31}\right]^{1/b21} = \left[\frac{b12\left[p^{b22}/H^{b32}\right]}{b11} D^{b31}\right]^{1/b21}$$

Similarly,

$$h = H - \left[\frac{b11\left[t^{b21}/D^{b31}\right]H^{b32}}{b12}\right]^{1/b22}$$

where all variables are as previously defined.

Clutter (1980) outlined an alternate method for obtaining implicit taper functions from the inside and outside bark diameter ratio equations presented by Burkhart (1977) and a total volume equation expressed as a function of dbh and total height. The procedure is as follows.

Using a variable-top merchantable volume equation of the following form,

$$Vm = Vt\left[1 - b_1 Dm^{b2} D^{b3}\right]$$

where,

Vm = merchantable o.b. stem volume to an o.b. top diameter, Dm,

Vt = total o.b. stem volume as given by a standard volume equation (i.e. V = g(D,H), where D = dbh and H = total height),

Dm = upper stem merchantability limit, o.b.,

b1,b2,b3 = regression coefficients.

and subsequent rearrangement of seperable differential equations, followed by integration led to the following implicit taper equations.

$$Dm = \left[kVt^{-1}b1^{-1}D^{-b3\left(\frac{b2-2}{b2}\right)}(H-M) \right]^{\frac{1}{b2-2}}$$

$$M = H - Dm^{b2-2}k^{-1}Vt\, b1\ D^{b3\left(\frac{b2}{b2-2}\right)}$$

$$Dm' = \left\{ k^{-1}(b2-2)^{-1}V'b1'D^{b3'}b2'\left[kVt^{-1}b1^{-1}D^{-b3\left(\frac{b2-2}{b2}\right)}\right]^{\left(\frac{b2'}{b2-2}\right)} \cdot (H - M)\left(\frac{b2' - b2 + 2}{b2-2}\right) \right\}^{½}$$

where,

T = distance from the top of the stem to Dm,

M = merchantable height, such that total height,

H = T + M

k = 0.005454

Dm' = upper stem merchantability limit, i.b.,

V' = total inside bark stem volume.

This same procedure was also used by Brister (1980) for site-prepared plantations of slash pine.

The set of implicit taper equations obtained through simple rearrangement of the volume ratios and the set derived according to the procedure described by Clutter (1980) were evaluated and compared on the basis of the prediction of diameter at a given height and height at a given diameter. The analysis was conducted on the basis of the mean residual, the mean magnitude of the residuals and the standard deviation of the residual values (observed minus predicted diameter at a given height and observed minus predicted height at a given diameter).

Cao, et.al. (1980) compared twelve ratio and taper equations in terms of taper and merchantable volume estimates to specified top diameters and height limits for loblolly pine. If a single equation is desired, they recommended a reliable taper equation that, when integrated, also provides reasonable merchantable volume estimates to either a specified merchantable height or diameter limit. They also concluded that the ratio equations presented by Cao and Burkhart (1980) produced good volume estimates and recommended them for predicting merchantable volume to various heights and/or top diameters. The volume ratios also have the advantage of being simple in form, producing good, relatively unbiased volume estimates, and being easily manipulated for the specification of implicit taper functions.

Chapter V

RESULTS AND DISCUSSION

5.1 SIMULTANEOUS GROWTH AND YIELD EQUATIONS

The first step in evaluating the model forms and coefficients of the equations derived by Beck and Della-Bianca (1972) for predicting basal area growth and cubic-foot volume growth and yield involved prediction over the additional measurement periods. Using the original model forms and coefficients, based on measurements taken five years after one thinning, the plots were projected to five and ten years and the projected values were compared to the observed data. In addition, the data from the 10-year remeasurement were projected to the fifteen year point for comparison. The results are given in Table 6.

While the initial set of coefficients predicted cubic-foot volume and basal area well over the first period, fit and predictive ability were somewhat less for the other periods. This was suggested by the residuals, representing observed minus predicted cubic volume and basal area values. The residuals also indicated that bias increases and precision decreases for cubic-foot volume and basal area prediction over the other three periods. In addition, the goodness of fit, measured by R^2, also decreases for the other

Table 6. Cubic-foot volume prediction over all periods using Beck and Della-Bianca's (1972) coefficients.

Period	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	Minimum value	Maximum value	Sum of squares	R^2
1	282	4.379	150.141	200.349	-529.294	759.876	11284651.83	0.9868
2	280	119.297	184.200	224.245	-678.595	816.547	18014590.35	0.9802
3	278	200.844	280.463	328.203	-667.458	1599.561	41051745.14	0.9612
4	138	440.366	555.566	521.613	-1116.924	1808.952	64036246.55	0.8824

where, Period 1 is 5 years after one thinning

Period 2 is 5 years after period 1, and after a second thinning

Period 3 is 5 years after period 2

Period 4 is 10 years after period 1

d is observed minus prediced cubic-foot volume measurement periods. In all cases, fit and predictive ability decrease as the projection age increases beyond the initial five year remeasurement.

Plots of residuals over stand characteristics such as age, site index, and basal area, as well as the first order interaction terms and the terms in the prediction equations indicated no obvious trends or relationships between the residual values and the stand characteristic terms. Aside from the bias and precision effects which may cause the residual values to not be centered around zero over the range of the independent variable, trends such as increasing variance with increasing magnitude of the independent variable were not present. The plots indicated that although the coefficients may not be appropriate over all the periods, the model forms are.

To further validate the model forms, separate regression equations were fitted for each time period using the same original model form. The R^2 and residual values were calculated for each of the periods. The results, given in Table 7, again indicate that the model forms are appropriate (high R^2, all variables significant at the alpha = 0.00001 level), and that only new coefficients are necessary (indicated by reduced bias and increased precision in prediction over that

Table 7. Cubic-foot volume prediction based on separate equations for each growth period.

Period	N	$\bar{d}$	$\|\bar{d}\|$	S_d	Minimum value	Maximum value	Sum of squares	R^2
1	282	7.075	149.000	198.777	-524.671	762.540	11117076.74	0.9870
2	280	4.673	138.674	193.339	-725.483	994.335	10435094.82	0.9885
3	278	6.491	176.599	257.812	-878.696	1216.140	18423134.52	0.9826
4	138	152.699	378.357	473.950	-1321.629	1458.583	33991912.49	0.9376

where, Period 1 is 5 years after one thinning

Period 2 is 5 years after period 1, and also after a second thinning

Period 3 is 5 years after period 2

Period 4 is 10 years after period 1

d is observed minus predicted cubic-foot volume associated with the original coefficient estimates). Again, residual plots indicated no trends over stand characteristics, first order interaction terms, or terms in the prediction equations.

Based on these results, it was concluded that the model forms were appropriate, but that different coefficients were necessary for stands thinned more than once.

Various options were analyzed regarding parameter estimation based on the number of thinnings. To accomplish this analysis, the data were first divided into three classes based on the number of thinnings as described earlier in the methods section. Given these three groups of measurements, four options were considered. First, all the data were combined to estimate a single set of coefficients. Second, the data immediately following the initial thinning were used to estimate one set of coefficients, the data immediately following the second thinning to estimate a second set of coefficients and finally, the data after the third growth period following both thinnings to estimate a third set of coefficients. Finally the third and fourth options involved the combination of two of the measurement periods.

In particular, the third option was to combine periods one and two to produce a set of coefficients for stands immediately after a thinning, and to use period three to estimate coefficients for stands not immediately after a thinning. The fourth and final option was to combine periods two and three to estimate a set of coefficients for stands after two thinnings and period one to obtain a set of coefficients for stands after one thinning.

For options one, three, and four, full and reduced model forms were also fitted. The full form estimated separate slope and intercept coefficients for each period, or group, (the full model for option one is in effect option two), whereas the reduced form estimated identical slope but different intercept coefficients for each period. For options three and four where two growth periods were being combined, full and reduced model forms, including a second reduced form which estimated identical slope and intercept coefficents for those two periods, were fitted to determine if the two periods could indeed be combined. A summary of how each of the model forms fit the data in terms of cubic-foot volume prediction is given in Table 8. As expected, the full model using three separate periods, option two, had the lowest sum of squared residuals and the reduced model form had the highest .

To determine whether or not the SSE between model forms were significantly different, which would then entail additional parameter estimation and hence additional complexity

Table 8. Summary of model form fits in terms of cubic-foot volume.

Model	SSE	d.f.	MSE	Sy.x	R^2
3 separate periods					
Full (option 2)	39975306.1	822	48631.8	220.526	0.9867
Reduced 1	42415638.2	832	50980.3	225.788	0.9859
Reduced 2 (option 1)	46285618.1	834	55498.3	235.581	0.9846
Periods 1 and 2 combined (option 3)					
Full	41950217.5	828	50664.5	225.088	0.9861
Reduced 1	44210974.9	833	53074.4	230.379	0.9853
Reduced 2	46285618.1	834	55498.3	235.581	0.9846
Periods 2 and 3 combined (option 4)					
Full	40998499.4	828	49515.1	222.520	0.9864
Reduced 1	42756589.4	833	51328.4	226.558	0.9858
Reduced 2	46285618.1	834	55498.3	235.581	0.9846

Where, Full indicates different slopes, different intercepts

Reduced 1 indicates same slopes, different intercepts

Reduced 2 indicates same slopes, same intercepts of the prediction system, F-tests for testing such differences were conducted on the SSE's. From the test results in Table 9, it appears that the differences in the SSE's for the model forms using the three separate periods are large enough to require a separate equation for each period. This would exclude option one, which involved combining all three periods for one set of coefficient estimates. The tests on the model forms involving periods two and three only indicate that one set of slope coefficients could be used for both periods. Similar tests involving periods one and two only indicate that the two periods could not be combined to estimate a single set of slope coefficients. This excluded option three. With periods two and three combined, the F-tests suggest that a separate set of coefficients is required for period one and the group containing periods two and three combined.

At this point, there were essentially two options to consider, option two-- a separate set of coefficients for each of the three periods and option four-- a set of coefficients for period one and a separate set of coefficients for periods two and three combined. An F-test to compare the SSE's of these two model forms gave borderline results (See Table 10). Thus other points had to be considered.

Table 9. Tests to determine significant differences in SSE's among model forms.

Model	SSE	d.f.		Calculated F	$F_{.01}$	$F_{.005}$
3 separate periods						
Full (Option 2)	39975306.1	822	Full vs. Reduced 1	5.018	2.32	2.52
Reduced 1	42415638.2	832	Full vs. Reduced 2	10.813	2.18	2.36
Reduced 2 (Option 1)	46285618.1	834	Reduced 1 vs. 2	37.956	4.61	5.30
Periods 1 and 2 Combined (Option 3)						
Full	41950217.5	828	Full vs. Reduced 1	8.924	3.02	3.35
Reduced 1	44210974.9	833	Full vs. Reduced 2	14.262	2.80	3.09
Reduced 2	46285618.1	834	Reduced 1 vs. 2	39.089	6.63	7.88
Periods 2 and 3 Combined (Option 4)						
Full	40998499.4	828	Full vs. Reduced 1	7.101	3.02	3.35
Reduced 1	42756589.4	833	Full vs. Reduced 2	17.796	2.80	3.09
Reduced 2	46285618.1	834	Reduced 1 vs. 2	68.836	6.63	7.88

Table 9. Continued.

Model	SSE	d.f.		Calculated F	$F_{.01}$	$F_{.005}$
Periods 1 and 2 (only)						
Full	21552171.6	550	Full vs. Reduced 1	0.275	3.02	3.35
Reduced 1	21606048.5	555	Full vs. Reduced 2	8.399	2.80	3.09
Reduced 2	23527083.0	556	Reduced 1 vs. 2	49.346	6.63	7.88
Periods 2 and 3 (only)						
Full	28858229.3	546	Full vs. Reduced 1	3.219	3.02	3.35
Reduced 1	29708828.3	551	Full vs. Reduced 2	3.226	2.80	3.09
Reduced 2	29881422.7	552	Reduced 1 vs. 2	3.201	6.63	7.88

Where, F vs. R is an F-test defined as follows,

$$F = \frac{(SSE_R - SSE_F)/(.d.f_R = d.f._F)}{SSE_F/d.f._F} \sim F_{(d.f._R - d.f._F),\ d.f._F}$$

Table 10. Test to determine significant differences in SSE's among options 2 and 4.

Model	SSE	d.f.		Calculated F	$F_{.01}$	$F_{.005}$
3 separate periods (option 2)	39975306.1	822				
			Option 2 versus Option 4	3.5066	2.802	3.091
Periods 2 and 3 combined (option 4)	40998499.4	828				

Where, Option 2 versus Option 4 is an F-test defined as follows:

$$F = \frac{(SSE_4 - SSE_2)/(d.f._4 - d.f._2)}{SSE_2/d.f._2} \sim F_{(d.f._4 - d.f._2),\ d.f._2}$$

First cubic-foot volume projection was evaluated using the full and two reduced model forms fitted on the combined measurements from periods two and three to determine what effects on fit and prediction selection of the reduced model form (same slope and intercept) had. From the results in Table 11 it appears that only slight sacrifices in fit and predictive ability were made when the reduced form for growth periods two and three was selected over the full model form. Therefore, it was concluded that grouping periods two and three had no practical effect on predictive ability.

Secondly, if separate equations were used for each period (option 2) how does one determine when to apply the equation applicable to prediction not immediately after a thinning? With option four there was no such confusion.

Finally, use of reduced model forms (option 4) decreases the number of equations in the prediction system and thus reduces the complexity of the system. Based on the slight losses in fit and predictive ability when periods two and three are combined, and the resulting simplification of the prediction system, option four was selected as the one to use in the final prediction system.

Once the appropriate option was selected, the coefficient estimates for the model forms were estimated in two ways.

Table 11. Cubic-foot volume prediction by full and reduced model forms for periods 2 and 3.

Model form	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	Minimum value	Maximum value	Sum of squares	R^2
Full	558	5.579	157.569	227.550	-878.696	1216.140	28858229.3	0.9860
Reduced 1	558	6.437	160.649	230.859	-816.263	1245.747	29708828.3	0.9856
Reduced 2	558	6.477	160.951	231.528	-807.367	1253.777	29881422.7	0.9855

where, Full indicates separate slopes, separate intercepts

Reduced 1 indicates same slopes, separate intercepts

Reduced 2 indicates same slopes, same intercepts

First using ordinary least squares (OLS), as was the case up to this point, and then using the simultaneous fitting procedure described earlier. As previously stated, this procedure minimizes the SSE of both cubic-foot volume and basal area, as opposed to the OLS procedure which minimizes the SSE with respect to cubic-foot volume only. The simultaneously fitted coefficients along with the OLS estimates are given in Table 12. The two fitting procedures were evaluated and compared in terms of cubic-foot volume and basal area prediction based on the average residual, the average absolute residual, and the standard deviation of the residual values to check on the bias, precision, and goodness-of-fit of each of the fitting methods. The summary statistics are given in Table 13.

With the simultaneous fitting procedure, one would expect slight losses in cubic-foot volume fit and significant gains in basal area fit. The cubic-foot volume results in Table 13 (combined periods) indicate a slight decrease rather than a slight increase in the SSE for volume. However, this is due to transformation bias as the coefficients were estimated through the fitting of the logarithm of volume. Overall, the simultaneous fitting procedure had little affect on cubic-foot volume fit or prediction as evidenced by only slight changes in the R^2, average residual, average absolute

Table 12. Simultaneous and ordinary least squares coefficient estimates for the cubic-foot volume equation.*

Coefficient	Ordinary Least Squares Estimates Period 1	Ordinary Least Squares Estimates Periods 2&3	Simultaneous Estimates Period 1	Simultaneous Estimates Periods 2&3
b_0	5.35197	5.33812	5.35740	5.33115
b_1	-101.90762	-99.08287	-102.45728	-97.95286
b_2	-21.95086	-25.14970	-21.95901	-25.19324
b_3	0.97489	0.98954	0.97473	0.98858
b_4	4.00752	6.05787	4.11893	5.84476
b_5	0.01385	-0.00204	0.01293	0.00018

*Equation: $$\ln(Y) = b_0 + b_1(1/S) + b_2(1/A_2) + b_3(A_1/A_2)\ln(BA_1) + b_4(1-A_1/A_2) + b_5(S)(1-A_1/A_2)$$

Where,
Y = cubic-foot volume
S = site index
A_2 = projected age
BA_1 = basal area at initial age, A_1

Table 13. Cubic-foot volume and basal area prediction using OLS and simultaneously estimated coefficients (all periods combined).

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	Minimum value	Maximum value	Sum of squares	R^2
Cubic-foot volume								
OLS	840	6.654	156.936	220.948	-807.533	1253.664	40995661.6	0.9864
Simultaneous	840	6.675	156.456	219.738	-808.913	1250.389	40548413.0	0.9865
Basal area								
OLS	419	1.039	2.970	3.720	-12.857	16.796	6237.131	0.9852
Simultaneous	419	0.782	2.899	3.685	-13.657	16.618	5932.760	0.9860

residual, and standard deviation of the residual values. On the other hand, clearer gains were obtained in basal area fit and prediction. The simultaneous fitting procedure reduced the SSE, the prediction bias, and the average magnitude of the residuals while also increasing precision in basal area fit and projection.

Table 14 presents cubic-foot volume and basal area projection and fit statistics over the separate growth periods. As was the case with cubic-foot volume fit and prediction over all the periods combined, the simultaneous fitting procedure had little affect over the individual growth period groupings. The simultaneous procedure also improved fit and prediction of basal area only slightly over the first period. Most of the decreases in bias and SSE were made in the second group consisting of growth periods two and three.

To further illustrate the effectiveness of the simultaneous fitting procedure , Table 15 presents three methods of basal area fit for the group containing periods two and three combined. The first method is based on an OLS fit of the basal area model form, independent of the cubic-foot volume fit. The second is based on an OLS fit of the volume equation with use of the ratios of the appropriate coefficients according to Beck and Della-Bianca (1972). The third also uses coefficient ratios, however the coefficient estimates are from the simultaneous fitting procedure.

Table 14. Cubic-foot volume and basal area prediction using OLS and simultaneously estimated coefficients (by period).

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	Minimum value	Maximum value	Sum of squares	R^2
<u>Cubic-foot volume (period 1)</u>								
OLS	282	7.075	149.000	198.777	-524.671	762.540	11117076.74	0.9870
Simultaneous	282	7.125	148.690	198.210	-525.337	760.710	11054019.02	0.9871
<u>Cubic-foot volume (periods 2 and 3)</u>								
OLS	558	6.477	160.951	231.528	-807.367	1253.777	29881422.7	0.9855
Simultaneous	558	6.447	160.380	230.023	-808.913	1250.389	29494394.0	0.9857
<u>Basal area (period 1)</u>								
OLS	141	0.724	2.821	3.560	-12.455	8.762	1848.39	0.9865
Simultaneous	141	0.613	2.802	3.556	-12.638	8.578	1823.20	0.9867
<u>Basal area (periods 2 and 3)</u>								
OLS	278	1.203	3.046	3.795	-12.848	16.798	4392.01	0.9844
Simultaneous	278	0.868	2.947	3.752	-13.657	16.618	4109.56	0.9854

Table 15. Statistics for basal area projection equations based on three different coefficient estimation methods for periods 2 and 3 combined.

Estimation Method	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	Minimum value	Maximum value	Sum of squares	R^2
OLS basal area fit	278	0.7531	2.9228	3.7446	-13.9398	16.5591	4041.83	0.9856
Basal area based on cubic-foot volume coefficient estimates	278	1.2028	3.0462	3.7952	-12.8485	16.7985	4392.01	0.9844
Basal area based on simultaneous fit cubic-foot volume coefficient estimates	278	0.8678	2.9473	3.7524	-13.6573	16.6183	4109.56	0.9854

As expected, the SSE associated with the direct OLS fit of the basal area equation is the lowest, and is thus used as a measure for comparison of the other two methods. Note the reduction in the SSE due to the simultaneous versus the OLS procedures using coefficient ratios. This was also expected.

Based on the improvement in fit and prediction, the simultaneous fitting procedure was used to estimate the coefficients of the final model forms selected. However, one final check on the set of equations forms was made. Cubic-foot volume and basal area were predicted with each of the two equations in the final system, given the same initial
conditions, to determine if there was any difference of practical significance between the two eqations fit on the basis of number of thinnings.

Given the results in Tables 16 and 17, it was concluded that the two equations produce values for both basal area and cubic-foot volume that are practically, as well as statistically significantly different. Thus it was decided to use the two separate equations in the prediction system. From this cubic-foot and basal area analysis, the following set of simultaneous growth and yield equations for thinned stands of yellow-poplar were selected for use in the final prediction system, (all variables as previously defined).

Table 16. Cubic volume prediction* using simultaneous fit coefficients given the same initial conditions.

		Basal area (sq.ft./acre)								
		After 1 thin			After 2 thin			Difference (2-1)		
	Age_2	70	90	110	70	90	110	70	90	110
		Cubic-foot volume/acre								
Site 90	20**	1425.3	1821.0	2214.4	1317.0	1688.4	2058.9	-108.3	-132.6	-155.5
	30	3006.9	3540.3	4033.4	3486.5	4114.5	4696.3	479.6	574.2	662.9
	40	4367.3	4936.3	5443.5	5672.7	6423.0	7092.8	1305.4	1486.7	1649.3
	50	5463.5	6025.9	6516.3	7596.8	8390.6	9083.5	2133.4	2364.7	2567.2
Site 110	20**	1753.1	2239.7	2723.6	1605.2	2057.9	2509.4	-147.9	-181.8	-214.2
	30	4031.3	4746.4	5407.5	4254.5	5020.9	5730.8	223.2	274.5	323.3
	40	6113.0	6909.5	7619.4	6926.5	7842.6	8660.4	813.4	933.1	1041.0
	50	7847.7	8655.6	9360.0	9279.2	10248.7	11095.1	1431.5	1593.1	1735.1
Site 130	20**	2023.2	2584.8	3143.3	1840.8	2360.0	2877.9	-182.4	-224.8	-265.4
	30	5071.2	5970.8	6802.5	4885.0	5765.0	6580.1	-186.2	-205.8	-222.3
	40	8028.6	9074.8	10007.1	7957.8	9010.3	9949.9	-70.9	-64.5	-57.3
	50	10577.0	11665.9	12615.2	10664.6	11778.9	12751.6	87.6	113.0	136.4

* Using equations 5.1.1 and 5.1.3, with initial age of 20 years.
** Yield prediction

Table 17. Basal area prediction* using simultaneous fit coefficients given the same initial conditions.

		Basal area (sq.ft./acre)								
		After 1 thin			After 2 thin			Difference(2-1)		
	Age_2	70	90	110	70	90	110	70	90	110
		Basal area (sq.ft./acre)								
Site 90	30	103.4	122.3	139.8	122.6	144.9	165.6	19.1	22.6	25.9
	40	125.7	142.6	157.6	162.2	183.9	203.3	36.4	41.3	45.7
	50	141.3	156.3	169.3	191.8	212.1	229.8	50.5	55.8	60.5
	60	152.8	166.2	177.7	214.6	233.3	249.5	61.8	67.1	71.8
Site 110	30	113.0	133.6	152.7	122.7	145.1	165.9	9.7	11.5	13.1
	40	143.6	162.8	180.0	162.5	184.2	203.6	18.9	21.4	23.7
	50	165.7	183.3	198.6	192.2	212.6	230.3	26.5	29.3	31.8
	60	182.4	198.3	212.0	215.1	233.9	250.1	32.7	35.6	38.0
Site 130	30	123.4	146.0	166.8	122.9	145.3	166.1	-0.6	-0.7	-0.8
	40	163.9	185.9	205.5	162.8	184.5	204.0	-1.2	-1.3	-1.5
	50	194.3	214.9	232.8	192.7	213.0	230.8	-1.7	-1.8	-2.0
	60	217.7	236.7	253.1	215.6	234.4	250.7	-2.1	-2.2	-2.4

*Using equations 5.1.2 and 5.1.4 , with an initial age of 20 years.

For stands thinned once,

$$\ln Y_2 = 5.35740 - 102.45728(1/S) - 21.95901(1/A_2) + 0.97473(A_1/A_2) \cdot(\ln B_1) + 4.11893(1-A_1/A_2) + 0.01293(S)(1-A_1/A_2) \quad (5.1.1)$$

$$\ln B_2 = (A_1/A_2)(\ln B_1) + (4.11893/0.97473)(1-A_1/A_2) + (0.01293/0.97473)(S)(1-A_1/A_2) \quad (5.1.2)$$

For stands thinned twice,

$$\ln Y_2 = 5.33115 - 99.95286(1/S) - 25.19324(1/A_2) + 0.98858(A_1/A_2) \cdot(\ln B_1) + 5.84476(1-A_1/A_2) + 0.00018(S)(1-A_1/A_2) \quad (5.1.3)$$

$$\ln B_2 = (A_1/A_2)(\ln B_1) + (5.84476/0.98858)(1-A_1/A_2) + (0.00018/0.98858)(S)(1-A_1/A_2) \quad (5.1.4)$$

5.2 BOARD-FOOT VOLUME EQUATIONS

Graphic trends indicated a strong linear relationship between board-foot volume and both basal area and cubic volume. Board-foot volume was also found to be linearly related to quadratic diameter. However, as quadratic diameter increased, so did the variance in volume.

Using basal area and quadratic diameter as measures of stand density or structure, six equations were fit to predict board-foot volume.

$$BFV = b_0 + b_1 BA + b_2 CFV$$

$$BFV = b_0 + b_1 (1/BA) + b_2 CFV$$

$$BFV = b_0 + b_1 (\ln BA) + b_2 CFV$$

$$BFV = b_0 + b_1 QD + b_2 CFV$$

$$BFV = b_0 + b_1 (1/QD) + b_2 CFV$$

$$BFV = b_0 + b_1 (\ln QD) + b_2 CFV$$

where,

BFV = board-foot volume per acre,

BA = basal area per acre,

QD = quadratic mean diameter,

CFV = cubic-foot volume per acre.

Fit and prediction statistics for each of these equations are given in Table 18.

Although the three equations containing quadratic diameter fit the data better than the three containing basal area, all three equations produced obvious trends in plots of the residuals, indicating improper model specification. On the other hand, no trends were apparent in the residual plots produced with the equations containing basal area. In addition, the three equations containing quadratic diameter tended to have the largest bias in prediction. Also, there

Table 18. Board-foot volume prediction based on six different equations.

Equation	$\bar{d}$	$\lvert\bar{d}\rvert$	Sd	SSE	Sy.x	R^2
1	0.01212	1539.38405	2062.67264	2357058607.	2068.2803	0.9664
2	0.02530	2984.12102	3665.95381	7445326394.	3675.9202	0.8939
3	-0.01795	2142.35766	2658.49463	3915446917.	2665.7221	0.9442
4	0.02356	1450.35447	1912.63380	2026626107.	1917.8335	0.9711
5	-1.30439	1330.72847	2003.41828	2223582333.	2008.8652	0.9683
6	1.39707	1260.78017	1773.29210	1742090030.	1778.1136	0.9752

Where, Equation 1 is $BFV = b_0 + b_1 BA + b_2 CFV$

Equation 2 is $BFV = b_0 + b_1 (1/BA) + b_2 CFV$

Equation 3 is $BFV = b_0 + b_1 \ln(BA) + b_2 CFV$

Equation 4 is $BFV = b_0 + b_1 QD + b_2 CFV$

Equation 5 is $BFV = b_0 + b_1 (1/QD) + b_2 CFV$

Equation 6 is $BFV = b_0 + b_1 \ln(QD) + b_2 CFV$

is currently a prediction equation for basal area, but not for quadratic diameter. For these reasons, only the three equations containing basal area were considered any further.

Of the three equations containing basal area, the one which used basal area with no transformation was best in terms of fit and bias and precision in prediction. For these reasons, this model form was selected to estimate board-foot volume from projected cubic volume and basal area from the growth and yield equations presented earlier. The board-foot volume equation is as follows,

$$BFV = 1363.09165 - 306.96647(BA) + 10.26187(CFV) \qquad (5.2.1)$$

5.3 VOLUME REMOVED IN THINNING

The equations presented by Field, et.al.(1978) for predicting volume removed in thinning as a function of the proportion of basal area or number of trees removed were fit using ordinary least squares procedures. The equations were then transformed to their nonlinear forms and refitted using nonlinear least squares estimation techniques. The coefficient estimates and fit statistics from both fitting procedures are given in Table 19. Only measures one and two of the plot data were used in the fittings as there were no thinnings at the time measures three and four were taken.

Table 19. Coefficient estimates and fit statistics for OLS and nonlinear least squares fits of Field et al. (1978) equations for predicting volume removed in thinning from below.

Equation	Coefficient estimate	SSE	MSE	Sy.x	R^2
Basal area model	α				
OLS fit	1.06614	1.43	0.01	0.073	0.9970
OLS fit*	1.06614	2593097.24	9822.34	99.108	0.9925
Nonlinear fit	1.10158	1855684.05	7029.11	83.840	0.9947
Number of trees model	β				
OLS fit	1.48118	66.09	0.25	0.500	0.9099
OLS fit*	1.48118	110229671.07	417536.63	646.171	0.6825
Nonlinear fit	2.10369	42767930.49	161999.74	402.492	0.8768

*Statistics in terms of volume removed

Note that the nonlinear fitting procedure reduced the SSE associated with the explanation of the variability in volume removed in thinning from below for both the number of trees and basal area equation forms. This is largely due to the elimination of tranformation bias associated with prediction of volume removed based on the linear forms of the equations, as well as the fact that the nonlinear form minimizes directly the SSE for volume removed. Also, both the linear and nonlinear equations containing proportion of basal area removed explained more variation in volume removed than the equations containing proportion of number of trees removed. This indicates that volume has a higher correlation with basal area than it does with number of trees alone.

With each set of coefficients, volume removed was predicted and subtracted from the observed volume removed. These residual values, which give an indication of the predictive ability of the equation forms, are summarized in Table 20. Although both fitting methods gave biased results, the bias associated with the nonlinear estimation procedure was less than half that associated with the OLS procedure in both the number of trees and basal area equation forms. Precision and the average magnitude of the residuals also improved considerably with the nonlinear fitting procedure.

Table 20. Prediction of volume removed in thinning from below for OLS and nonlinear least squares fits of Field *et al.* (1978) equation forms.

Equation	$\bar{d}$	$\lvert\bar{d}\rvert$	Sd	Minimum value(d)	Maximum value(d)	R^2
<u>Basal area model</u>						
OLS fit	-33.2758	71.2527	93.3320	-383.9596	167.1112	0.9925
Nonlinear fit	14.3899	67.0488	82.5909	-315.6250	267.4559	0.9947
<u>Number of trees model</u>						
OLS fit	-309.2161	458.5780	567.0625	-2167.0628	925.0515	0.6825
Nonlinear fit	144.3588	324.0752	375.6079	-1343.1039	1219.0966	0.8768

Using the nonlinear forms of the equations presented by Field, et.al. (1978), and then fitting with nonlinear least squares techniques produced equations which gave reliable estimates of volume removed in thinning from below when the proportion of basal area or number of trees removed was known. Results indicated that knowing the basal area removed will give better estimates of volume removed, in terms of prediction bias and precision than will knowing the number of trees removed.

5.4 DIAMETER DISTRIBUTION PREDICTION

With the appropriate equations and revisions, the computer solution routine written by Frazier (1981) in FORTRAN Level-G for loblolly pine stands was used to estimate the parameters of a Weibull distribution, and to subsequently produce a stand table before and after thinning for the 141 plots of the yellow-poplar data.

For each plot, total basal area and cubic-foot volume per acre were computed by summing across the diameter classes of the stand table. In each case, observed minus predicted basal area and cubic-foot volume per acre were calculated. The mean residual, mean absolute residual and the sum of the squared residuals, as well as an R^2 value were calculated. These values are given in Tables 21 and 22 for basal area

Table 21. Summary statistics for the residual values representing observed minus predicted basal area per acre.

	Period			
Criterion	Before first thin (1)	After first thin (2)	Before second thin (3)	After second thin (4)
$\bar{d}$	6.9185	6.9186	1.4521	1.4499
$\lvert\bar{d}\rvert$	6.9185	6.9186	1.4521	1.4499
SS	11146.36	11146.52	389.45	386.30
R^2	0.9208	0.9134	0.9972	0.9968

Where, $\bar{d}$ = mean residual value

$\lvert\bar{d}\rvert$ = mean absolute residual value

SS = sum of squared residual values

Table 22. Summary statistics for the residual values representing observed minus predicted total cubic-foot volume per acre.

Criterion	Period			
	Before first thin (1)	After first thin (2)	Before second thin (3)	After second thin (4)
$\bar{d}$	109.5497	154.3680	-173.3044	-153.2656
$\lvert\bar{d}\rvert$	211.9638	203.3688	180.8814	161.3829
SS	11499924.0	12439497.0	7053039.0	5672255.0
R^2	0.9788	0.9663	0.9841	0.9860

Where, $\bar{d}$ = mean residual value

$\lvert\bar{d}\rvert$ = mean absolute residual value

SS = sum of squared residual values

and cubic-foot volume, repectively. Each table presents these summary statistics for four periods: before the first thinning, after the first thinning, before the second thinning, and after the second thinning.

It is apparent that bias, represented by the mean residual, decreases and goodness-of-fit, represented by R^2, increases for both basal area and cubic volume for periods three and four versus periods one and two. Upon observation of the plot data, it appears this may be due to the fact that the diameter distributions of the stands become smoother and more unimodal after the thinnings. Before or immediately after the first thinning, the stands are generally irregular and often multimodal, making modeling with a Weibull distribution difficult. As the thinnings 'smoothed out' the distributions, the bias and goodness-of-fit improved for periods three and four. The smoothing effects of the thinnings are most noticeable with basal area as the parameter recovery solution procedure was conditioned on the basal area, and not on cubic volume.

In addition to evaluating the parameter recovery procedure and thinning algorithm at a whole stand level, they were also evaluated at a diameter distribution level. Using the plot data and the predicted number of trees obtained from the solution routines, the observed and predicted number of trees by diameter class were compared for each plot.

In particular, a Chi-square goodness-of-fit statistic was calculated for each plot before and after the first thinning, as well as before and after the second thinning. The statistic is defined as follows.

$$X^2 = \sum_{i=1}^{k} \frac{(E_i - O_i)^2}{E_i}$$

where,

$$E_i = \tfrac{1}{4}N \int_{DL_i}^{DU_i} f(x;\underline{\theta})\,dx$$, the expected frequency of trees in the ith dbh class,

O_i = observed frequency in the ith dbh class,

k = number of dbh classes.

The hypothesis to be tested is,

$$H_o : F_o(x) = H(x)$$

$$H_1 : F_o(x) \neq H(x)$$

at some significance level, α .

where,

$F_o(x)$ = hypothesized cumulative distribution function defined by the recovered parameters,

$H(x)$ = unknown population distribution function.

The IMSL subroutine MDCH was used to compute the p-values given by $Pr(X^2>x^2)$ for each plot, where x^2 is the computed X^2 value.

The Chi-square statistics were calculated on a plot basis (1/4-acre) rather than on a per acre basis to avoid the error associated with multiplying the observed number of trees per diameter class on a plot basis by four to obtain per acre values. Instead, the predicted number of trees per acre in each class was divided by four.

Table 23 presents a summary of the calculated Chi-square statistics and correspondind p-values before and after the first and second thinnings for the 141 plots. Trends similar to those found earlier in bias and R^2 values are also present here. The goodness-of-fit, measured by the Chi-square statistic, improves as the time from the initial measurement and number of thinnings increase. The associated p-values indicate that the hypothesized and unknown population distribution functions are not different at the alpha=0.2573 significance level (for the worst case).

To further evaluate the thinning algorithm, the Chi-square statistics were analyzed in greater detail. First, to determine if there was any relationship between the goodness-of-fit and the amount of basal area removed in thinning, the Chi-square values were sorted into three classes as follows,

Table 23. Summary statistics for the Chi-square statistic, X^2, and the computed p-values[1] for evaluation of predicted diameter distributions before and after thinning.

Period	X^2 Minimum	X^2 Mean	X^2 Maximum	p-value Minimum	p-value Mean	p-value Maximum	N
(1) Before first thin	8.5400	26.5744	64.6900	0.0000	0.2573	0.9796	141
(2) After first thin	1.2900	19.4372	106.8800	0.0000	0.3813	0.9957	141
(3) Before second thin	2.2300	11.8167	158.1000	0.0000	0.5335	0.9874	140
(4) After second thin	1.3900	8.3786	27.3000	0.0005	0.5443	0.9879	140

[1] p-value = $p\ (X^2 \geq x^2)$ where $X^2 = \sum_{i=1}^{k} \frac{(Oi-Ei)^2}{Ei}$

Σ = summation over all k diameter classes where predicted number of trees per class, Ei, is greater than 0.25 trees per quarter acre

Oi = observed number of trees per quarter acre plot in the ith diameter class

Class = 1 if BAREM < $\overline{BAREM}$ - SD

Class = 2 if $\overline{BAREM}$ - SD < BAREM < $\overline{BAREM}$ + SD

Class = 3 if BAREM > $\overline{BAREM}$ + SD

where,

BAREM = basal area removed in thinning,

$\overline{BAREM}$ = mean BAREM for all plots,

SD = standard deviation of BAREM for all plots.

The results, summarized in Table 24 indicate that fit is improved as the amount of basal area removed in thinning is increased. Again note the obvious differences in the Chi-square values for the two thinning periods.

The Chi-square values were also grouped according to the proportion of basal area removed in thinning. The classes were defined as before with proportion of basal area removed in place of basal area removed. The results, given in Table 25 are almost identical to those in Table 24 where the sort was based on basal area removed.

Finally, the Chi-square values were sorted according to the number of 'passes' through the diameter classes required to reach the specified level of residual basal area. The classes were defined as follows,

Table 24. Summary statistics for the Chi-square statistic, X^2, and the computed p-values for evaluation of predicted diameter distributions after thinning, sorted according to basal area removed in thinning.

	X^2			p-value			
Period	Minimum	Mean	Maximum	Minimum	Mean	Maximum	N
			Class = 1				
After first thin	6.2900	40.7547	105.6000	0.0000	0.1502	0.9586	19
After second thin	2.2300	10.0550	27.3000	0.0043	0.5579	0.9749	20
			Class = 2				
After first thin	1.2900	17.5301	106.8800	0.0000	0.3878	0.9957	98
After second thin	1.3900	8.1681	24.1700	0.0005	0.5501	0.6990	101
			Class = 3				
After first thin	2.8000	10.3479	63.2100	0.0000	0.5378	0.9027	24
After second thin	2.8300	7.7332	21.1400	0.0120	0.5000	0.9705	19

Where, BREM = basal area removed in thinning
Class = 1 if BREM < mean BREM minus 1 standard deviation
Class = 2 if mean BREM minus 1 std. dev. $\leq$ BREM $\leq$ mean BREM plus 1 std. dev.
Class = 3 if BREM > mean BREM plus 1 std. dev.

Table 25. Summary statistics for the Chi-square statistic, X^2, and the computed p-values for evaluation of predicted diameter distributions after thinning, sorted according to proportion of basal area removed in thinning.

Period	X^2 Minimum	X^2 Mean	X^2 Maximum	p-value Minimum	p-value Mean	p-value Maximum	N
Class = 1							
After first thin	6.2900	33.4938	106.8800	0.0000	0.2374	0.9586	24
After second thin	2.2300	10.0000	27.3000	0.0043	0.5825	0.9749	17
Class = 2							
After first thin	1.2900	18.5500	105.6000	0.0000	0.3910	0.9957	90
After second thin	1.3900	8.2620	24.1700	0.0040	0.5615	0.9879	107
Class = 3							
After first thin	2.8000	9.8996	63.2100	0.0000	0.4770	0.9027	27
After second thin	2.9200	7.4363	24.1000	0.0005	0.3890	0.8193	16

Where, PROPBA = proportion of basal area removed in thinning
Class = 1 if PROPBA < mean PROPBA minus 1 standard deviation
= 2 if mean PROPBA minus 1 std. dev. ≤ PROPBA ≤ mean PROPBA plus 1 std. dev.
= 3 if PROPBA > mean PROPBA plus 1 std. dev.

Cycle = 1 if required basal area removal is obtained in one pass through the diameter classes,

Cycle = 2 if specified basal area removal requires second pass through diameter classes.

As expected, the summary statistics in Table 26 indicate an improvement in fit with plots requiring an additional "pass" through the diameter classes. The results are in agreement with those associated with the sorts based on basal area and proportion of basal area removed in thinning, i.e., as basal area removal is increased, bias and goodness-of-fit are improved. Again, the differences in Chi-square values between the two thinning periods are present.

From the results in Tables 24, 25, and 26 it is obvious a relationship exists between the goodness-of-fit and the amount of basal area removed in thinning. Stand tables were produced from the plot data before and after thinnings from plots that were thinned both lightly and heavily to find possible reasons or explanations for the relationship. It was noted that in all thinnings, light or heavy, the trees in the smaller diameter classes were, for the most part, completely removed. The thinning algorithm, which removes a proportion of basal area from each class tends to leave few

Table 26. Summary statistics for the Chi-square statistic, X^2, and the computed p-values for evaluation of predicted diameter distributions after thinning, sorted according to the number of cycles through the diameter classes.

	X^2			p-value			
Period	Minimum	Mean	Maximum	Minimum	Mean	Maximum	N
			Cycle = 1				
After first thin	6.2900	27.5480	106.8800	0.0000	0.2932	0.9586	71
After second thin	2.2000	9.9875	27.3000	0.0040	0.5403	0.9879	55
			Cycle = 2				
After first thin	1.2900	11.2104	63.2100	0.0000	0.4707	0.9957	70
After second thin	1.3900	7.3376	24.1000	0.0005	0.5470	0.9773	85

Where, Cycle = 1 if required basal area removed in thin is obtained before or at the time the largest diameter class is reached.

Cycle = 2 if required basal area removed in thin is not obtained at time largest diameter class is reached necessitating a second "pass" through the diameter classes.

trees in the lower classes. When a thinning is light, only one pass through the diameter classes is required, and some trees remain in the lower classes. However, when a thinning is moderate to heavy, requiring a second pass by the thinning algorithm, all trees in the lower classes are removed, until the desired level of basal area is obtained. As a result, when the thinning algorithm is required to make a second pass through the diameter classes, thus eliminating all trees in the lower classes, a stand table more closely approximating the actual thinned stand table should be produced.

However, while this may account for some of the differences in goodness-of-fit, for the most part, there seem to be no general trends or relationships to explain the correlation between basal area removed and goodness-of-fit. For example, two plots with similar initial stand characteristics and stand structure before thinning were both thinned lightly. For the stand tables after thinning, one had a very high Chi-square value and the other a very low value. Similar differences were found for stands that were thinned heavily. At this point, the relationship between the Chi-square values and the basal area removed in thinning can not be adequately explained.

Overall, the parameter recovery method for estimating the parameters of the Weibull distribution for stands before thinning gave reasonable estimates of number of trees per acre, basal area per acre and cubic-foot volume per acre by diameter class. In addition, the thinning algorithm produced stand and stock tables with reliable estimates of these stand characteristics consistent with the stand and stock tables generated before thinning.

5.5 FLEXIBLE VOLUME EQUATIONS

The evaluations of various total volume equations indicated that the weighted combined variable equation form used by Beck (1963) performed just as well, if not better, than any of the other model forms for the yellow-poplar data set in terms of fit and prediction. The decision was based on R^2 values which measured fit according to the amount of variation in volume explained by the regressions and also on the bias and precision of prediction. Burkhart (1977) also found the weighted combined variable form (Spurr, 1952) to produce good results after analysis of three total volume equations for loblolly pine.

Thus, the total volume equations used in the remainder of the taper analysis are,

$$TVOB = 0.010309 + 0.002399*D^2H \qquad (5.5.1)$$

$$TVIB = 0.000109 + 0.001908*D^2H \qquad (5.5.2)$$

where,

TVOB = total cubic-foot volume outside bark,

TVIB = total cubic-foot volume outside bark,

D = dbh in inches,

H = total height in feet.

The nonlinear ratio equation presented by Burkhart (1977) for estimating merchantable volume inside or outside bark to a given top diameter was fitted using nonlinear least squares with the yellow-poplar individual tree data which is summarized in Table 27. The coefficient estimates and fit statistics are presented in Table 28.

Analysis of the predictive ability of this diameter ratio equation form was based on the results presented in Table 29. At each diameter measure along the tree's bole, the merchantable volume up to that diameter point was also known. The residual values in Table 29 represent the observed minus the predicted merchantable volume values for all the observations along the length of the tree combined. The high R^2 values for merchantable volume inside and outside bark indicate the equations explain a high percentage of the variation in merchantable volume. At the same time, however, all three ratio equations produce slightly negatively biased volume estimates.

Table 27. Merchantable volume (cu.ft.) data summary.

Volume measure	Minimum	Mean	Maximum	Standard deviation	Number of observations
All observations combined					
Outside bark	0.02	47.45	259.80	47.42	6328
Inside bark	0.01	38.97	219.34	39.63	6328
To a specified diameter limit					
Outside bark					
4-inch top	0.32	42.01	240.76	46.25	489
6-inch top	0.82	41.48	239.76	47.98	516
8-inch top	1.45	40.58	256.96	42.49	509
Inside bark					
4-inch top	0.32	34.22	219.10	39.99	489
6-inch top	0.70	32.00	218.06	35.53	516
8-inch top	1.36	33.59	201.75	35.10	509
To a specified height limit					
Outside bark					
17 feet	0.09	18.38	75.52	16.07	331
33 feet	1.39	34.50	136.54	28.39	310
49 feet	5.43	49.25	188.85	38.41	287
Inside bark					
17 feet	0.07	15.10	64.46	13.61	331
33 feet	0.90	28.35	116.44	24.02	310
49 feet	4.54	40.42	161.08	32.45	287

Table 28. Coefficient estimates from nonlinear least squares fit of Burkhart's (1977) volume ratio equation form.*

	Outside bark volume	Inside bark volume (t is o.b.)	Inside bark volume (t is i.b.)
b_1	-0.40075	-0.41905	-0.57082
b_2	2.09311	2.08760	1.95847
b_3	1.88125	1.89466	1.81287
SSE	94.603	97.666	107.578
MSE	0.015	0.015	0.017
Sy.x	0.122	0.122	0.130
R^2	0.8066	0.8007	0.7805

*Equation form:

$$R = Vm/Vt = 1 + b_1\ [t^{b2}/D^{b3}]$$

Where, Vm = merchantable volume (i.b. or o.b.) in cubic feet
Vt = total volume (i.b. or o.b.) in cubic feet
t = merchantable top diameter in inches (i.b. or o.b.)
D = dbh in inches

Table 29. Merchantable volume prediction based on the diameter ration equation, Burkhart (1977), to a given top diameter (all N = 6328 observations combined).

Equation	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
MVOB	1.40172	6.22895	9.65217	601884.224	0.9577
MVIB[1]	2.16361	5.75858	8.96177	537764.916	0.9459
MVIB[2]	2.17625	6.00825	9.34375	582352.323	0.9414

Where, [1]top diameter is outside bark

[2]top diameter is inside bark

MVOB = merchantable volume outside bark

MVIB = merchantable volume inside bark

d = observed minus predicted merchantable volume

Table 30 presents similar evaluation criteria for the three ratio equations for merchantable volume prediction to a 4-, 6-, and 8- inch top diameter. Note that all three ratios improve in fit and predictive ability as the top diameter becomes smaller and merchantable volume approaches total volume. And again, all three ratio equations produce negatively biased merchantable volume estimates, with the bias decreasing as merchantable top diameter approaches zero.

In a similar manner, the nonlinear height ratio equation presented by Cao and Burkhart (1980) for estimating merchantable volume inside or outside bark to a given height limit was fitted using nonlinear regression procedures. The coefficient estimates and fit statistics are given in Table 31, and the summary statistics for analysis and evaluation of the predictive ability of the equations are presented in Table 32. At each height measure along the length of the tree, the merchantable volume, inside and outside bark, up to that point is known. The residuals, given by d, in Table 32 represent the observed minus the predicted merchantable volume values at a particular height, h, for all the observations combined. The high R^2values reflect a high percentage of the variation in merchantable volume accounted for by the equations. The fit, as well as the bias and preci-

Table 30. Merchantable volume prediction to a 4-, 6-, and 8-inch top limit using the diameter ratio equation presented by Burkhart (1977).

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
Merchantable volume o.b.						
8-inch top	509	2.10234	3.90888	5.10721	15500.128	0.9831
6-inch top	516	1.25972	3.11091	4.76089	12491.879	0.9895
4-inch top	489	0.09658	2.56909	4.64228	10521.356	0.9899
Merchantable volume i.b.						
8-inch top, i.b.	519	2.90512	4.48502	5.47775	19923.176	0.9688
8-inch top, o.b.	509	2.48484	3.75793	4.96252	15653.087	0.9752
6-inch top, i.b.	554	2.05127	3.35234	4.44993	13281.500	0.9810
6-inch top, o.b.	516	1.82932	3.02325	4.50788	12192.054	0.9851
4-inch top, i.b.	522	1.14939	2.76966	4.52253	11345.772	0.9864
4-inch top, o.b.	489	0.79220	2.41311	4.01081	8157.132	0.9886

Where, d is observed minus predicted merchantable volume.

Table 31. Coefficient estimates from nonlinear least squares fit of Cao and Burkhart's (1980) volume ratio equation form.*

	Outside bark volume	Inside bark volume
b_1	-1.06843	-1.23140
b_2	2.52423	2.55120
b_3	2.53181	2.58930
SSE	3.669	4.256
MSE	0.00058	0.00067
Sy.x	0.024	0.026
R^2	0.9925	0.9913

*Equation form: $R = Vm/Vt = 1 + b_1(p^{b2}/H^{b3})$

Where, p = distance in feet from the tree tip to the limit of utilization

H = total tree height (from the ground) in feet

Vm = merchantable cubic foot volume (o.b. or i.b.) from the stump to the utilization limit, specified by p

Vt = total cubic foot volume (o.b. or i.b.) above the stump

b_1, b_2, b_3 = regression coefficients estimated from the data

Table 32. Merchantable volume prediction to a given height based on the height ratio equation presented by Cao and Burkhart (1980).

Equation	N	$\lvert\bar{d}\rvert$	$\bar{d}$	S_d	SSE	R^2
Merchantable volume, o.b.						
Over all observed heights	6328	-1.00085	2.25964	4.06321	110795.407	0.9922
17 feet (top of first log)	331	-0.53045	0.68196	1.04061	450.485	0.9947
33 feet (top of second log)	310	-1.29387	1.63730	2.47759	2415.755	0.9903
49 feet (top of third log)	287	-1.17989	2.18018	3.45873	3820.916	0.9909
Merchantable volume, i.b.						
Over all observed heights	6328	0.26957	2.14446	3.85834	94697.243	0.9905
17 feet (top of first log)	331	-0.11840	0.62181	1.03354	357.183	0.9942
33 feet (top of second log)	310	-0.34332	1.30311	2.05143	1336.927	0.9925
49 feet (top of third log)	287	0.10424	2.04679	3.23860	3002.834	0.9900

sion in prediction, reflected by the mean, mean absolute and standard deviation of the residuals, are noticeably better than those associated with the diameter ratio equations.

Again, in addition to checking fit and prediction for all the observations combined, merchantable volume fit and prediction were also checked at certain specified heights. Specifically, the height ratio equations were evaluated at the top of the first log (at 17 feet), and at the approximate top of the second and third logs, i.e. at 33 and 49 feet, respectively. The summary statistics, given in Table 32 show merchantable volume fit and predictive ability to be highest for the first log, and somewhat less for the second and third logs.

Overall the volume ratio equations predict merchantable volume to a specified diameter or height limit reasonably well and represent an alternative to fitting seperate fixed top limit equations.

While the height and diameter ratio equations increase flexibility in terms of merchantable volume prediction, they also allow the derivation of implicit taper functions. Given the following height and diameter ratios presented by Cao and Burkhart (1980), implicit taper equations were obtained through algebraic manipulation.

$$R = Vm/Vt = 1 + b_{11} (t^{b_{21}}/D^{b_{31}})$$

$$R = Vm/Vt = 1 + b_{12} (p^{b_{22}}/H^{b_{32}})$$

where all variables are as previously defined.

At each height measurement point, the diameter i.b. and o.b. was also recorded. With the implicit taper equations, given in Table 33, diameter to a given height, and height to a given diameter were predicted for evaluation purposes. Residuals repesenting observed minus predicted height at a given diameter and predicted diameter at a given height are summarized in Table 34 for all the observations combined. It appears as though prediction and fit for outside bark measures are slightly better than those for inside bark measures. However, all the equations appear to fit the data reasonably well, while tending to give negatively biased taper estimates.

In addition to evaluating the fit and predictive ability of the four taper equations over the entire stem profile, they were also evaluated over various portions of the trees. The set of measurements from each tree were separated into three groups. The first group contained all the observations from stump height to one third of the tree's total height. The second consisted of the measurements corres-

Table 33. Implicit taper equations obtained through algebraic rearrangement of the diameter and height ratios presented by Cao and Burkhart (1980).*

For predicting diameter at a given height

$$t = 1.59758\ D^{0.89878}\ H^{-1.20959}\ (H - h)^{1.20597} \quad (5.5.5)$$

$$t' = 1.48078\ D^{0.92566}\ H^{-1.32210}\ (H - h)^{1.30265} \quad (5.5.6)$$

For predicting height at a given diameter

$$h = H - 0.67809\ t^{0.82921}\ D^{-0.74528}\ H^{1.00300} \quad (5.5.7)$$

$$h = H - 0.73981\ t'^{\,0.76767}\ D^{-0.71059}\ H^{1.01493} \quad (5.5.8)$$

*Where,
D = dbh in inches
H = total height in feet
h = merchantable height in feet above stump height
t = merchantable diameter, o.b. at height, h
t' = merchantable diameter, i.b. at height, h

Table 34. Prediction of diameter at a given height and height to a given diameter based on the implicit taper functions in Table 31. (All N = 6328 observations combined).

Equation	$\bar{d}$	$\lvert\bar{d}\rvert$	Sd	SSE	R^2
Outside bark diameter	0.88457	1.48454	1.59488	21045.131	0.8865
Inside bark diameter	0.85556	1.49658	1.60989	21029.893	0.8658
Height at an o.b. top diameter	4.90256	7.98558	8.14681	572019.969	0.8739
Height at an i.b. top diameter	5.30175	8.61867	8.73125	660207.970	0.8545

ponding to the middle third of the tree, and the third group was made up of the observations from the top third of the tree. This grouping was done to determine if the equations fit and/or predicted better over different portions of the trees. The results are summarized in Table 35.

Note that merchantable diameter at a given height, inside or outside bark, fit and prediction are best in the lower one third of the trees in terms of bias and R^2. Merchantable height prediction at a given top diameter , inside or outside bark, is also best in terms of bias and precision in the bottom third of the trees. However, the height prediction equations seem to fit the top third slightly better than the bottom third of the trees, based on the R^2 values.

Again, the tree measurements were divided into three different groups. The first consisted of those observations from stump height up to six feet, The second from six feet to two thirds of the tree's total height, and the third from two thirds to total tree height. This grouping was done to determine if the butt section was being fit and predicted differently than the other tree sections. The results are given in Table 36.

Merchantable diameter fit, inside and outside bark, appears to be highest in the butt section. However, the middle section has the lowest bias in prediction of all three

Table 35. Taper prediction over various portions of the trees.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
o.b. diameter						
Portion 1	2280	-0.61735	1.01020	1.07567	3505.945	0.9465
Portion 2	2596	1.85296	1.86941	1.24831	12957.019	0.7124
Portion 3	1452	1.51159	1.54126	0.93351	4582.167	0.2320
i.b. diameter						
Portion 1	2280	-0.67768	1.06985	1.13744	3995.587	0.9290
Portion 2	2596	1.83738	1.85404	1.21923	12621.513	0.6631
Portion 3	1452	1.50776	1.52754	0.87539	4412.792	0.0527
Height at an o.b. diameter						
Portion 1	2280	-3.31337	5.04987	5.20927	86874.968	0.5690
Portion 2	2596	9.59341	9.68065	5.43168	315479.441	0.4199
Portion 3	1452	9.41693	9.56482	5.30947	169665.559	0.6009
Height at an i.b. diameter						
Portion 1	2280	-3.60502	5.42049	5.53810	99529.529	0.5062
Portion 2	2596	10.17123	10.26772	5.69254	352657.236	0.3516
Portion 3	1452	10.58156	10.69231	5.59619	208021.205	0.5107

Where, Portion = 1 if RELHT = $^h/H \leq 0.33$
= 2 if $0.33 <$ RELHT ≤ 0.67
= 3 if $0.67 <$ RELHT ≤ 1.00

Table 36. Taper prediction over various portions of the trees.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
o.b. diameter						
Portion 1	337	-1.63642	1.69054	0.77932	1106.505	0.9211
Portion 2	4539	0.87116	1.45110	1.62015	15356.459	0.8650
Portion 3	1452	1.51159	1.54126	0.93351	4582.167	0.2320
i.b. diameter						
Portion 1	337	-1.82039	1.87360	0.83741	1352.386	0.8877
Portion 2	4539	0.84560	1.45868	1.62744	15264.714	0.8416
Portion 3	1452	1.50776	1.52754	0.87539	4412.792	0.0527
Height at an o.b. diameter						
Portion 1	337	-8.33322	8.70409	4.58839	30476.077	---
Portion 2	4539	4.44113	7.42705	7.88794	371878.332	0.7711
Portion 3	1452	9.41693	9.56482	5.30947	169665.559	0.6009
Height at an i.b. diameter						
Portion 1	337	-9.35405	9.71362	4.58569	36552.506	---
Portion 2	4539	4.70090	7.87403	8.33585	415634.259	0.7442
Portion 3	1452	10.58156	10.69231	5.59619	208021.205	0.5107

Where Portion = 1 if merchantable height $\leq$ 6.0
= 2 if 6.0 < merchantable height and $^h/H \leq 0.67$
= 3 if $0.67 < {}^h/H \leq 1.00$

sections. As was the case with the other grouping scheme, diameters in the bottom portion tended to be over-predicted while those in the upper two portions tended to be under-predicted. Like merchantable diameter prediction, merchantable height prediction in the middle section had the lowest bias. As only one height measure in the bottom section could be taken, no R^2value could be computed to evaluate the fit. As was the case with the other grouping method, merchantable heights in the bottom portion tended to be over-predicted, while those in the upper two portions were underpredicted.

As an alternative to the taper function derivation given by Cao and Burkhart (1980), the method oultlined by Clutter (1980) was also used for evaluation and comparison purposes. The taper functions derived according to the procedures described previously are given in Table 37.

When these taper equations were evaluated, illogical results were obtained. This was due to the numerical values of the coefficients obtained from the fitting of the ratio equations from Burkhart (1977), in particular, b2, from the outside bark diameter ratio equation. As b2 (2.09311) is close to 2.00, the term, 1/(b2 - 2.0), in Clutter's formulation is exceedingly large, causing illogical taper results.

Table 37. Implicit taper equations obtained according to the method described by Clutter (1980).

For predicting diameter at a given height

$$t = (17.02948 + 3.96293\ D^2H)^{-10.74024} D^{20.20511} (H-h)^{10.74024}$$

$$t' = 0.22637\ (0.0001085 + 0.00191\ D^2H)^{0.5} (17.02953 + 3.96288\ D^2H)^{-11.21067} D^{21.09010} \cdot (h-H)^{11.21067}$$

For predicting height at a given diameter

$$h = H - t^{0.09311} (17.02948 + 3.96293\ D^2H)\ D^{-1.881253}$$

$$h = H - 1.41695\ t'^{\,0.08920}\ (0.0001085 + 0.00191\ D^2H)^{-0.04460} (17.02953 + 3.96288\ D^2H) \cdot D^{-1.88125}$$

*Where,
D = dbh in inches
H = total height in feet
h = merchantable height in feet above stump height
t = merchantable diameter, o.b., at height, h
t' = merchantable diameter, i.b., at height, h

In an attempt to alleviate this problem, the diameter ratio coefficients were reestimated according to the following equation form. For comparison purposes, the height ratio coefficients were also reestimated in a similar manner.

$$Vm = Vt(1 + b_{11} (t^{b_{21}}/D^{b_{31}})) \tag{5.5.3}$$

$$Vm = Vt(1 + b_{21}' (p^{b_{22}}/H^{b_{32}})) \tag{5.5.4}$$

where all variable are as previously defined.

The coefficient estimates obtained from the modified equation forms as well as the original coefficients are given in Tables 38 and 39. Note that the new estimates are larger than the old ones. In particular, the estimate for b2 in the diameter ratio equation is greater than 3.00. As pointed out earlier, this will prevent the term, 1/(b2 - 2.00) from becom+ng too large, and should improve taper prediction.

The additional sets of coefficients obtained through the modified fitting of the original equation forms to improve taper prediction were also used to estimate merchantable volume to a given top diameter and merchantable volume to a specified height limit. The results were evaluated and compared to those obtained from the original equation forms. In addition, new implicit taper functions obtained through algebraic rearrangement of the two modified ratio equations,

Table 38. Coefficient estimates from nonlinear least squares fit of Burkhart's (1977) volume ratio equation form.*

	Outside bark volume		Inside bark volume (t is o.b.)		Inside bark volume (t is i.b.)	
	Equation (5.5.9)	Equation (5.5.10)	Equation (5.5.9)	Equation (5.5.10)	Equation (5.5.9)	Equation (5.5.10)
b_1	-0.40075	-0.51817	-0.41905	-0.49534	-0.57082	-0.84085
b_2	2.09311	3.36235	2.08760	3.38736	1.95847	3.19319
b_3	1.88125	3.18701	1.89466	3.20048	1.81287	3.09032
SSE	94.603	198,282.71	97.666	142,990.14	107.578	167,374.13
MSE	0.015	31.349	0.015	22.607	0.017	26.462
Sy.x	0.122	5.599	0.122	4.755	0.130	5.144
R^2	0.8066	0.9861	0.8007	0.9856	0.7805	0.9832

*Equation forms:

$R = Vm/Vt = 1 + b_1[t^{b2}/D^{b3}]$ (5.5.9) and $Vm = Vt*[1 + b_1(t^{b2}/D^{b3})]$ (5.5.10)

Where, Vm = merchantable volume (i.b. or o.b.) in cubic feet
Vt = total volume (i.b. or o.b.) in cubic feet
t = merchantable top diameter in inches (i.b. or o.b.)
D = dbh in inches

Table 39. Coefficient estimates from nonlinear least squares fit of Cao and Burkhart (1980) volume ratio equation.*

	Outside bark volume		Inside bark volume	
	Equation (5.5.11)	Equation (5.5.12)	Equation (5.5.11)	Equation (5.5.12)
b_1	-1.06843	-1.08119	-1.23140	-1.21089
b_2	2.52423	2.58566	2.55120	2.63739
b_3	2.53181	2.59535	2.58930	2.67073
SSE	3.669	24494.622	4.256	17237.382
MSE	0.00058	3.873	0.00067	2.725
Sy.x	0.024	1.968	0.026	1.651
R^2	0.9925	0.9983	0.9913	0.9983

*Equation Forms:

$R = Vm/Vt = 1 + b_1[p^{b2}/H^{b3}]$ (5.5.11) and $Vm = Vt*[1 + b_1(p^{b2}/H^{b3})]$ (5.5.12)

Where, p = distance in feet from the tree tip to the limit of utilization
H = total tree height (from the ground) in feet
Vm = merchantable cubic-foot volume (o.b. or i.b.) from the stump to the utilization limit, specified by p
Vt = total cubic-foot volume (o.b. or i.b.) above the stump
b_1, b_2, b_3 = regression coefficients estimated from the data

as well as those derived by Clutter's method with the new coefficients, were evaluated and compared against the first set of taper equations based on the original set of coefficients.

First, an evaluation and comparison of the various equations for merchantable volume prediction will be made. Then the taper results will be discussed.

The results of merchantable volume prediction to a given top diameter, (all observations combined) for both sets of diameter ratio coefficients are given in Table 40. In terms of merchantable volume outside bark, the modified coefficients improved the fit, (a significant decrease in the SSE), but simultaneously increased the bias in prediction slightly. At the same time, the precision improved and the average magnitude of the residuals decreased. The modified coefficients also greatly improved the fit and prediction of merchantable volume inside bark to both outside- and inside- bark top diameters. In both cases, the SSE values were reduced by approximately one half. The bias in prediction was reduced by over 85%, while the precision increased in both volume estimates. Therefore, it was concluded that the modified coefficient estimates perform better than the original estimates in terms of merchantable volume fit and prediction inside or outside bark to a given top diameter.

Table 40. Merchantable volume prediction based on two methods of fitting the diameter ratio equation, to a given top diameter (All N = 6328 observations combined).

Equation	$\bar{d}$	$\lvert\bar{d}\rvert$	Sd	SSE	R^2
MVOB					
Equation 1	1.40172	6.22895	9.65217	601884.224	0.9577
Equation 2	-1.54804	4.48247	7.29263	351650.159	0.9753
MVIB[1]					
Equation 1	2.16361	5.75858	8.96177	537764.916	0.9459
Equation 2	-0.25495	3.98416	6.55031	271881.489	0.9726
MVIB[2]					
Equation 1	2.17625	6.00825	9.34375	582352.323	0.9414
Equation 2	-0.28474	4.34226	7.03084	313273.772	0.9685

Where, [1]top diameter is outside bark

[2]top diameter is inside bark

MVOB = merchantable volume outside bark
MVIB = merchantable volume inside bark
d = observed minus predicted merchantable volume

Equation 1 = $R = 1 + b_1(t^{b2}/D^{b3})$

Equation 2 = $Vm = Vt^* [1 + b_1(t^{b2}/D^{b3})]$

Further comparisons of the two sets of diameter ratio coefficients were made through merchantable volume predictions to specified top diameters. Table 41 presents the prediction results at 4-, 6-, and 8- inch top diameters. Beck's (1963) equations were also included (for 4- and 8-inch top diameters, o.b.) for comparison purposes.

The modified coefficients decrease bias slightly in outside bark volume prediction to an 8- and 6- inch top, but increase it for a 4-inch top. While precision is increased for an 8-inch top, it is decreased for the 6- and 4- inch tops. Both sets of coefficients are similar in terms of fitting the data, i.e. explaining variation in merchantable volume outside bark. Beck's equations consistently exhibited the largest bias, but fell between the two ratio equations in terms of precision and fit. As was noted earlier, the modified estimates greatly improved inside bark volume prediction to an inside or outside bark top diamter limit. For all three top diameters, inside or outside bark, the ratios using the modified coefficients produced volume estimates with lower bias and higher precision (except for the 4-inch top, i.b.) in prediction than either the original ratio equation estimates or Beck's equations while also explaining more of the variation in volume.

Table 41. Merchantable volume preduction to a 4-, 6-, and 8-inch top limit based on 3 prediction equations.*

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
Merchantable volume o.b.						
8-inch top o.b.	509					
Equation 1		2.10234	3.90888	5.10721	15500.128	0.9831
Equation 2		-1.05654	2.96050	4.76933	12123.430	0.9868
Equation 3		-2.53708	3.95566	4.81892	15073.110	0.9836
6-inch top o.b.	516					
Equation 1		1.25972	3.11091	4.76089	12491.879	0.9895
Equation 2		-1.11099	2.67133	5.02786	13655.798	0.9885
4-inch top o.b.	489					
Equation 1		0.09658	2.56909	4.64228	10521.356	0.9899
Equation 2		-1.28068	2.54411	4.90397	12537.913	0.9880
Equation 3		-1.40226	2.58560	4.82053	12301.440	0.9882
Merchantable volume i.b.						
8-inch top i.b.	519					
Equation 1		2.90512	4.48502	5.47775	19923.176	0.9688
Equation 2		0.06343	2.94949	4.21562	9207.717	0.9856
8-inch top o.b.	509					
Equation 1		2.48484	3.75793	4.96252	15653.087	0.9752
Equation 2		-0.15512	2.59928	4.15806	8795.289	0.9861
Equation 3		-2.15635	3.42463	4.29972	11758.440	0.9814

Table 41. Continued.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
6-inch top i.b.	554					
Equation 1		2.05127	3.35234	4.44993	13281.500	0.9810
Equation 2		-0.22669	2.52741	4.10874	9364.055	0.9866
6-inch top o.b.	516					
Equation 1		1.82932	3.02325	4.50788	12192.054	0.9851
Equation 2		-0.12830	2.36898	4.27159	9405.538	0.9885
4-inch top i.b.	522					
Equation 1		1.14939	2.76966	4.52253	11345.772	0.9864
Equation 2		-0.30748	2.40685	4.53810	10779.008	0.9871
4-inch top o.b.	489					
Equation 1		0.79219	2.41311	4.01081	8157.132	0.9886
Equation 2		-0.33214	2.25545	4.04066	8021.508	0.9888
Equation 3		-1.91482	2.80181	4.72238	12675.780	0.9823

*Where,

Equation 1 is $R = 1 + b_1(t^{b2}/D^{b3})$

Equation 2 is $V_M = V_T * [1 + b_1(t^{b2}/D^{b3})]$

Equation 3 is Beck's (1963) weighted combined variable equation.

Analysis of merchantable volume prediction to a given height for both sets of coefficients was based on the results in Table 42 for all the observations combined. While the original coefficients gave better outside bark merchantable volume estimates to a given height in terms of fit and prediction, the new estimates were better for inside bark merchantable volume prediction to a specified height limit. However, the actual differences in bias, precision, and fit were small in both cases. Therefore, a closer comparison of volume prediction to specific heights was made.

Results for the evaluation of merchantable volume prediction, inside and outside bark, to the approximate tops of the first, second, and third logs are given in Table 43. The original set of estimates performed consistently better in terms of fit and prediction than the modified set of coefficients in outside bark volume prediction to the three height limits. Except for a slightly higher precision at the 17 and 49 foot points, the same held true for inside bark volume fit and prediction.

While the modified coefficients improved merchantable volume fit and prediction to a specified top diameter, some losses were incurred in volume fit and prediction to a specified height limit. However, it was decided that the large gains in merchantable volume prediction to a specified top

Table 42. Merchantable volume prediction to a given height based on the two methods of fitting the height ratio equation presented by Cao and Burkhart (1980). (All N = 6328 observations combined).*

Equation	$\bar{d}$	$\lvert\bar{d}\rvert$	Sd	SSE	R^2
Merchantable volume o.b.					
Equation 1	-1.00085	2.25964	4.06321	110795.407	0.9922
Equation 2	-1.36810	2.45102	4.19888	123392.821	0.9913
Merchantable volume i.b.					
Equation 1	0.26957	2.14446	3.85934	94697.243	0.9905
Equation 2	-0.08121	2.17206	3.85766	94197.438	0.9905

*Where, Equation 1 = $R = 1 + b_1\ (p^{b2}/H^{b3})$

Equation 2 = $Vm = Vt*(1 + b_1(p^{b2}/H^{b3})$

Table 43. Merchantable volume prediction to a given height based on two methods of fitting the height ratio equation.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
Merchantable volume o.b.						
17 feet (top of first log)	331					
Equation 1		-0.53045	0.68196	1.04061	450.485	0.9947
Equation 2		-0.85447	0.98536	1.21318	727.365	0.9915
33 feet (top of second log)	310					
Equation 1		-1.29387	1.63730	2.47759	2415.755	0.9903
Equation 2		-1.79349	2.00326	2.78831	3399.514	0.9863
49 feet (top of third log)	287					
Equation 1		-1.17989	2.18018	3.45873	3820.916	0.9909
Equation 2		-1.68112	2.43506	3.70158	4729.784	0.9888

Table 43. Continued.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
Merchantable volume i.b.						
17 feet (top of first log)	331					
Equation 1		-0.11840	0.62181	1.03354	357.183	0.9942
Equation 2		-0.39221	0.68904	1.02907	400.385	0.9935
33 feet (top of second log)	310					
Equation 1		-0.34332	1.30311	2.05143	1336.927	0.9925
Equation 2		-0.83523	1.42800	2.11356	1596.604	0.9910
49 feet (top of third log	287					
Equation 1		0.10424	2.04679	3.23860	3002.834	0.9900
Equation 2		-0.40650	2.05448	3.21730	3007.807	0.9900

Where, Equation 1 = $R = 1 + b_1(p^{b2}/H^{b3})$

Equation 2 = $Vm = Vt\,[1 + b_1(p^{b2}/H^{b3})]$

diameter outweighed the slight losses in volume fit and prediction to a specified height limit.

As for taper prediction, there were three sets of equations for comparison. The first obtained from the rearrangement of the original volume ratios, the second from the rearrangement of the modified volume ratios, and the final set from Clutter's procedure using the modified ratio coefficients. The new sets of taper equations based on the modified set of coefficients and the set derived by Clutter's procedure using the same modified set of coefficients are given in Tables 44 and 45. The three equation sets of taper prediction equations were used to predict inside and outside bark diameters at specified heights and heights at specified inside or outside bark diameters. Residual values equal to the observed minus predicted heights and diameters were computed and used for evaluation and comparison of the three sets of equations. The results of these predictions for all the observations combined are given in Table 46.

With all the observations combined, the set of taper equations based on the modified volume ratio equation coefficients produced consistently better taper estimates than the other two sets in terms of fit, bias, and precision of prediction for estimation of both diameter at a given height and height at a given diameter. It should also be noted

Table 44. Implicit taper equations obtained through algebraic rearrangement of the diameter and height ratios presented by Cao and Burkhart (1980) with modified coefficient estimation.

For predicting diameter at a given height

$$t = 1.24452\ D^{0.94785} H^{-0.77189} (H-h)^{0.76900} \qquad (5.5.13)$$

$$t' = 1.12099\ D^{0.96779} H^{-0.83638} (H-h)^{0.82594} \qquad (5.5.14)$$

For predicting height at a given diameter

$$h = H - 0.75242\ t^{1.30038} D^{-1.23257} H^{1.00375} \qquad (5.5.15)$$

$$h = H - 0.87085\ t'^{\,1.21074} D^{-1.17173} H^{1.01264} \qquad (5.5.16)$$

*Where,
D = dbh in inches
H = total height in feet
h = merchantable height in feet above stump height
t = merchantable diameter, o.b., at height h
t' = merchantable diameter, i.b., at height h

Table 45. Implicit taper equations obtained according to the method described by Clutter (1980) with modified coefficient estimates.

For predicting diameter at a given height

$$t = (2.41739 + 0.56253D^2H)^{-0.73403} D^{2.33934} (H-h)^{0.73403} \quad (5.5.17)$$

$$t' = 15.02728\ (0.00010854 + 0.00190800D^2H)^{0.5} (2.41739 + 0.56255D^2H)^{-1.24321} \cdot D^{2.36186} (H-h)^{0.74321} \quad (5.5.18)$$

For predicting height at a given diameter

$$h = H - t^{1.36235} (2.41739 + 0.56253D^2H) D^{-3.18701} \quad (5.5.19)$$

$$H = H - 0.02609\ t'^{\,1.34552} (0.00010854 + 0.00190800D^2H)^{-0.67276} \cdot (2.41739 + 0.56255D^2H)^{1.67276} D^{-3.17794} \quad (5.5.20)$$

Where, D = dbh in inches
H = total height in feet
h = merchantable height in feet above stump height
t = merchantable diameter, o.b., at height h
t' = merchantable diameter, i.b., at height h

Table 46. Summary of taper prediction by 3 sets of equations (all observations combined).

Equation	$\bar{d}$	$\lvert\bar{d}\rvert$	s_d	SSE	R^2
Outside bark diameter					
Set 1	0.88457	1.48454	1.59488	21045.131	0.8865
Set 2	-0.12047	0.60165	0.87244	4907.677	0.9735
Set 3	-0.34798	0.71177	0.91226	6031.751	0.9675
Inside bark diameter					
Set 1	0.85556	1.49658	1.60989	21029.893	0.8658
Set 2	-0.07297	0.61908	0.85654	4675.620	0.9702
Set 3	-0.20894	0.68846	0.93446	5801.132	0.9630
Height at an o.b. diameter					
Set 1	4.90256	7.98558	8.14681	572019.969	0.8739
Set 2	-0.58478	3.90039	5.37075	184665.670	0.9593
Set 3	-2.49951	4.84380	5.76270	249645.694	0.9450
Height at an i.b. diameter					
Set 1	5.30175	8.61867	8.73125	660207.970	0.8545
Set 2	-0.33152	4.39079	5.85233	217393.924	0.9521
Set 3	-1.61358	5.13666	6.56985	289567.438	0.9362

Where, Set 1 = taper equations based on ordinary volume ratio fits
Set 2 = taper equations based on modified volume ratio fits
Set 3 = taper equations based on Clutter's procedure using modified ratio coefficients

that the taper equations derived by Clutter's method performed consistently better than those based on the rearrangement of the original volume ratio equations and coefficients for prediction of both height at a given diameter and diameter at a given height.

To determine how well the three sets of taper equations performed over various portions of the trees, the observations from each tree were divided into three groups according to the two methods described earlier. That is, first the tree measures were divided (based on relative height) as those in the bottom third, the middle third, and the top third of the trees. Second the tree measures were divided as those from stump height up to six feet, from six feet to two thirds tree height, and from two thirds to total tree height.

Taper prediction results from the first grouping method are given in Tables 47 and 48. In all cases, the equations obtained from rearrangement of the modified coefficient ratios explained the most variation in taper, as evidenced by the lowest SSE values. The precision of the modified coefficient set was also greatest for both merchantable diameter and height prediction over all portions except for the top one where the equations derived according to Clutter's procedure had slightly greater precision in height and diam-

Table 47. Prediction of diameter at a given height over various portions of the tree.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
Outside bark diameter						
Portion 1	2280					
Set 1		-0.61735	1.01020	1.07567	3505.945	0.9465
Set 2		-0.26640	0.44411	0.54944	849.812	0.9870
Set 3		-0.41333	0.63886	0.65320	1361.897	0.9792
Portion 2	2596					
Set 1		1.85296	1.86941	1.24831	12957.019	0.7124
Set 2		0.29491	0.57815	0.78333	1818.121	0.9596
Set 3		0.03056	0.58961	0.84752	1866.366	0.9586
Portion 3	1452					
Set 1		1.51159	1.54126	0.93351	4582.167	0.2320
Set 2		-0.63399	0.89103	1.06835	2239.744	0.6246
Set 3		-0.92217	1.04467	1.03977	2803.488	0.5301
Inside bark diameter						
Portion 1	2280					
Set 1		-0.67768	1.06985	1.13744	3995.587	0.9290
Set 2		-0.33190	0.52522	0.60730	1091.685	0.9806
Set 3		-0.17564	0.60053	0.75359	1354.511	0.9757

Table 47. Continued.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
Portion 2	2596					
Set 1		1.83738	1.85404	1.21923	12621.513	0.6631
Set 2		0.35124	0.61458	0.79130	1945.140	0.9481
Set 3		0.11782	0.60575	0.87328	2015.046	0.9462
Portion 3	1452					
Set 1		1.50776	1.52754	0.87539	4412.792	0.0527
Set 2		-0.42486	0.77450	0.97406	1638.795	0.6482
Set 3		-0.84544	0.97440	0.89785	2421.515	0.4802

Where, Portion = 1 if RELHT = $h/H \leq 0.33$
= 2 if $0.33 <$ RELHT ≤ 0.67
= 3 if $0.67 <$ RELHT ≤ 1.00

Set 1 = taper equations based on ordinary volume ratio fits
Set 2 = taper equations based on modified volume ratio fits
Set 3 = taper equations based on Clutter's procedure using modified ratio coefficients.

Table 48. Prediction of height at a given diameter over various portions of the tree.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
To an outside bark diameter						
Portion 1	2280					
Set 1		-3.31337	5.04987	5.20927	86874.968	0.5690
Set 2		-2.29687	3.58933	5.02446	69562.083	0.6549
Set 3		-4.07952	5.58031	5.96290	118977.361	0.4097
Portion 2	2596					
Set 1		9.59341	9.68065	5.43168	315479.441	0.4199
Set 2		2.07478	3.86366	4.71955	68976.453	0.8732
Set 3		-0.08413	4.03193	5.32209	73520.862	0.8648
Portion 3	1452					
Set 1		9.41693	9.56482	5.30947	169665.559	0.6009
Set 2		-2.65135	4.45451	4.97548	46127.134	0.8915
Set 3		-4.33689	5.13885	4.53467	57147.470	0.8656
To an inside bark diameter						
Portion 1	2280					
Set 1		-3.60502	5.42049	5.53810	99529.529	0.5062
Set 2		-2.87199	5.33865	5.43674	86169.056	0.5725
Set 3		-2.44705	5.61670	7.15746	130404.203	0.3530

Table 48. Continued.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
Portion 2	2596					
Set 1		10.17123	10.26772	5.69254	352657.236	0.3516
Set 2		2.66574	4.40802	5.09749	85877.090	0.8421
Set 3		0.67791	4.58244	6.06917	96779.358	0.8221
Portion 3	1452					
Set 1		10.58156	10.69231	5.59619	208021.205	0.5107
Set 2		-1.70110	4.44184	5.32513	45347.777	0.8933
Set 3		-4.40172	5.37376	4.85852	62383.876	0.8533

Where, Portion = 1 if $h/H \leq 0.33$

= 2 if $0.33 < h/H \leq 0.67$

= 3 if $0.67 < h/H \leq 1.000$

Sets are defined as before

eter prediction. Bias in prediction was sometimes better with set (2) and sometimes better with set (3). Generally, taper equation sets (2) and (3) were markedly better than the set based on the original ratio equation coefficients (1). From this grouping scheme, the modified volume ratio taper set (2) for the most part, produced the least biased and most precise estimates of height to a specified diameter and diameter to a specified height.

A summary of the taper prediction residual values for the second grouping method are given in Tables 49 and 50. The same trends in taper fit, and prediction that were present in the first grouping method were also observed in this second grouping method, i.e. overall, set (2) was found to give better taper estimates than either set (1) or (3). A final observation made regarding the two grouping methods was that taper fit and predictive ability in the first portion decreased for all three sets when it included only the butt section measures (less than or equal to 6.00 feet) This would seem reasonable as taper prediction is generally poorest in this portion of the tree.

Based on the merchantable volume and taper prediction results, the modified ratio equations were selected as the best forms for coefficient estimation. The modified ratio equations produced coefficient estimates which explained

Table 49. Prediction of diameter at a given height over various portions of the tree.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
Outside bark diameter						
Portion 1	337					
Set 1		-1.63642	1.69054	0.77932	1106.505	0.9211
Set 2		-0.35882	0.42864	0.58855	159.778	0.9886
Set 3		-0.51208	0.68382	0.71247	258.931	0.9815
Portion 2	4539					
Set 1		0.87116	1.45110	1.62015	15356.459	0.8650
Set 2		0.06150	0.52192	0.74089	2508.154	0.9779
Set 3		-0.15212	0.60735	0.79447	2969.332	0.9739
Portion 3	1452					
Set 1		1.51159	1.54126	0.93351	4582.167	0.2320
Set 2		-0.63399	0.89103	1.06835	2239.744	0.6246
Set 3		-0.92217	1.04467	1.03977	2803.488	0.5301
Inside bark diameter						
Portion 1	337					
Set 1		-1.82039	1.87360	0.83741	1352.386	0.8877
		-0.55458	0.61984	0.62541	235.070	0.9805
		-0.33032	0.65389	0.82856	267.437	0.9778

Table 49. Continued.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
Portion 2	4539					
Set 1		0.84560	1.45868	1.62744	15264.714	0.8416
Set 2		0.07535	0.56930	0.78213	2801.755	0.9709
Set 3		0.00368	0.59955	0.82812	3112.181	0.9677
Portion 3	1452					
Set 1		1.50776	1.52754	0.87539	4412.792	0.0527
Set 2		-0.42486	0.77450	0.97406	1638.795	0.6482
Set 3		-0.84544	0.97440	0.89785	2421.515	0.4802

Where, Portion = 1 if merchantable height (MERCHT) $\leq$ 6.0 feet
= 2 if 6.0 < MERCHT and h/H $\leq$ 0.67
= 3 if 0.67 < h/H $\leq$ 1.00

Set 1 = taper equations based on ordinary volume ratio fits
Set 2 = taper equations based on modified volume ratio fits
Set 3 = taper equations based on Clutter's procedure using modified ratio coefficients

Table 50. Prediction of height at a given diameter over various portions of the tree.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
To an outside bark diameter						
Portion 1	337					
Set 1		-8.33322	8.70409	4.58839	30476.077	---
Set 2		-3.07600	4.11720	8.91784	29910.003	---
Set 3		-5.11568	6.72382	9.29593	37854.558	---
Portion 2	4539					
Set 1		4.41130	7.42705	7.88794	371878.332	0.7711
Set 2		0.26126	3.70704	4.88562	108628.533	0.9331
Set 3		-1.71750	4.60984	5.57916	154643.666	0.9048
Portion 3	1452					
Set 1		9.41693	9.56482	5.30947	169665.559	0.6009
Set 2		-2.65135	4.45451	4.97548	46127.134	0.8915
Set 3		-4.33689	5.13885	4.53467	57147.470	0.8656
To an inside bark diameter						
Portion 1	337					
Set 1		-9.35405	9.71362	4.58569	36552.506	---
Set 2		-4.87515	5.85985	8.95085	34929.057	---
Set 3		-4.05687	6.99161	10.93534	45725.850	---

Table 50. Continued.

Equation	N	$\bar{d}$	$\lvert\bar{d}\rvert$	S_d	SSE	R^2
Portion 2	4539					
Set 1		4.70090	7.87403	8.33585	415634.259	0.7442
Set 2		0.44394	4.26538	5.47889	137117.090	0.9156
Set 3		-0.54026	4.92309	6.30034	181457.711	0.8883
Portion 3	1452					
Set 1		10.58156	10.69231	5.59619	208021.205	0.5107
Set 2		-1.70110	4.44184	5.32513	45347.777	0.8933
Set 3		-4.40172	5.37376	4.85852	62383.876	0.8533

Where, Portion = 1 if merchantable height (MERCHT) $\leq$ 6.0 feet
= 2 if 6.0 < MERCHT and $h/H \leq 0.67$
= 3 if $0.67 < h/H \leq 1.00$

Sets are defined as before.

more than 96% of the variation in merchantable volume inside or outside bark to a specified top diameter and over 99% of the variation in volume prediction inside or outside bark to a specified height limit. In both cases, the bias in prediction was less than 1.55 cubic feet.

The implicit taper equations obtained through rearrangement of the modified ratio equations accounted for more than 97% of the variability in diameter (inside or outside bark) at a given height and greater than 95% of the variability in height at a given diameter inside or outside bark. Merchantable diameter prediction bias, i.b. or o.b., at a given height and merchantable height prediction bias at an i.b. or o.b. diameter limit were less than 0.125 inches and 0.600 feet, respectively.

Thus, with the diameter and height ratio equation forms presented by Burkhart (1977) and Cao and Burkhart (1980) reliable estimates of merchantable volume, i.b. or o.b., can be easily obtained to either a specified diameter or height limit. Volume between any two diameter or height limits can be obtained through subtraction. Also, through rearrangement of the ratio equations, implicit taper functions to predict height at a given diameter or diameter at a given height were specified.

Chapter VI

APPLYING THE MODELS

This section outlines the steps required to obtain stand and diameter distribution level estimates of number of trees, basal area, and cubic-foot volume per acre for a given set of initial conditions, thinning regime, and rotation age.

6.1 STAND-LEVEL ESTIMATES

Stand-level estimates of number of trees, basal area and cubic volume at some projected age when site index, initial age, and basal area are given are obtained as follows.

1) Compute number of trees per acre from,

$$\ln(NT) = b_0 + b_1(1/A) + b2(S) + b3(1/BA) \qquad (4.4.12)$$

2) Calculate basal area per acre as,

$$\ln(BA) = b_0 + b_1(1/A) + b_2(S) + b_3(1/NT) \qquad (4.4.11)$$

3) Estimate cubic-foot volume per acre by,

$$\ln(Y) = b_0 + b_1(1/S) + b_2(1/A_2) + b_3(A_1/A_2)(\ln B_1) + b_4(1-A_1/A_2) + b_5(S)(1-A_1/A_2) \qquad (5.1.1)$$

where, the coefficients in the above equations depend on the thinning regime (i.e. whether after the first or the second thinning).

6.2 DIAMETER DISTRIBUTION ESTIMATES

Diameter distribution estimates of number of trees, basal area and cubic-foot volume by diameter class when site index, initial age, and basal area are given are obtained by first specifying the following inputs for for use in a computer solution routine written to carry out the parameter recovery computations described in section 4.4.

1) Initial age,
2) projected age,
3) initial basal area and/or number of trees,
4) site index,
5) number of previous thinnings,
6) basal area removed in thinning, if a thinning is desired (set equal to zero otherwise).

Given these inputs, the computer solution routine estimates the parameters of a Weibull distribution and subsequently produces a stand and stock table at the projected age. To obtain a dtand table at the present age, projected age is set equal to initial age (1 and 2 above). If a thinning is specified, a second table containing number of trees, basal area, and cubic-foot volume by diameter class after thinning is also given. The stand table after thinning is produced according to the procedures described in section 4.4 on page 53, in which a thinning algorithm removes a specified proportion of basal area from each diameter class of the corresponding stand table generated before thinning.

6.3 NUMERICAL EXAMPLE

To compare the estimates of number of trees, basal area, and cubic-foot volume per acre from the stand level and diameter distribution models, the following set of initial conditions and thinning options were inputed into the appropriate stand-level equations and parameter recovery solution routine.

Initial conditions: basal area = 132 sq.ft.
site index = 100 feet
age = 35 years

Thinning options: Thin to 80 sq.ft. at age 35,
project to age 50,
thin to 90 sq.ft. at age 50,
project to age 70.

The stand level and diameter distribution estimates obtained at each step are presented in Table 51.

Table 51. Stand level and diameter distribution estimates of number of trees, basal area and cubic-foot volume per acre.

Age (years)	Before thinning: Number of trees	Before thinning: Basal area (sq.ft.)	Before thinning: Total volume(o.b.) (cu.ft.)	After thinning: Number of trees	After thinning: Basal area (sq.ft.)	After thinning: Total volume(o.b.) (cu.ft.)
Stand level estimates						
35	301	132	4748	149	80	2912
50	149	114	4950	95	90	4009
70	95	135	6934			
Diameter distribution estimates						
35	301	131	4756	124	79	3040
50	123	113	5136	82	89	4140
70	81	131	6954			

Chapter VII

SUMMARY AND CONCLUSIONS

Analysis and evaluation of the equations presented by Beck and Della-Bianca (1972) for predicting basal area growth and cubic-foot volume growth and yield in yellow-poplar stands after a single thinning to various levels of basal area indicated that the same equation forms could be used for stands thinned more than once. However, seperate parameter estimates were required for stands thinned more than once. The coefficients in the final equations were estimated using a simultaneous fitting procedure. The process of simultaneously fitting the basal area and cubic-foot volume equations produces a system of equations that are compatible and numerically consistent. The procedure is also more statistically efficient in that the basal area growth information is used in the fitting procedure. As a result, the fit and prediction of basal area were improved, while affecting the accuracy and precision of volume projection very little. Given estimates of basal area and cubic-foot volume from these equations, board-foot volumes can also be calculated.

Stand tables were then derived from the whole stand attributes by solving for the parameters of a two parameter

Weibull distribution according to the parameter recovery method. When applying the system, the same stand level basal area equation is applied when deriving diameter distributions as when estimating overall stand basal area in order to ensure compatibility between the two levels of stand detail.

Overall, the parameter recovery procedure for estimating the parameters of the diameter distributions of the stands before thinning gave reasonable estimates of number of trees, basal area, and cubic-foot volume per acre by diameter class. The thinning algorithm which removed a proportion of the basal area from each class, to simulate a thinning from below, produced stand and stock tables after thinning that were consistent with those generated before thinning.

Finally, the modified fitting of the diameter and height ratio equations presented by Burkhart (1977) and Cao and Burkhart (1980) produced reliable estimates of merchantable volume, i.b. or o.b., to either a specified diameter or height limit, where volume between any two diameter or height limits can be obtained through subtraction. Through rearrangement of the ratio equations, implicit taper functions were specified to predict height at a given diameter and diameter at a given height.

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A GROWTH AND YIELD PREDICTION MODEL FOR
THINNED STANDS OF YELLOW-POPLAR

by

BRUCE R. KNOEBEL

(ABSTRACT)

Analysis and evaluation of the simultaneous growth and yield equations presented by Beck and Della-Bianca (1972) for predicting basal area growth and cubic-foot volume growth and yield in yellow-poplar stands after a single thinning indicated that a separate set of coefficients was required for stands thinned twice. A joint loss function involving both volume and basal area was used to estimate the coefficients in the system of equations. The estimates obtained were analytically compatible, invariant for projection length, and numerically equivalent with alternative applications of the equations. Given estimates of basal area and cubic-foot volume from these equations, board-foot volumes can also be calculated.

As an adjunct to the stand level equations, compatible stand tables were derived by solving for the parameters of the Weibull distribution from attributes predicted with the stand-level equations. This procedure for estimating the parameters of the diameter distributions of the stands before thinning gave reasonable estimates of number of trees,

basal area, and cubic-foot volume per acre by diameter class. The thinning algorithm removes a proportion of the basal area from each diameter class and produces stand and stock tables after thinning from below that are consistent with those generated before thinning.

Finally, volume ratio equations were fitted to provide estimates of merchantable volume, i.b. or o.b., to either a specified diameter or height limit, where volume between any two diameter or height limits can be obtained through subtraction. Through rearrangement of the ratio equations, implicit taper functions were specified to predict height at a given diameter and diameter at a given height.