A CONTROL PROBLEM FOR BURGERS' EQUATION

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(ABSTRACT)

Burgers' equation is a one-dimensional simple model for convection-diffusion phenomena such as shock waves, supersonic flow about airfoils, traffic flows, acoustic transmission, etc. For high Reynolds number, the open-loop system (no control) produces steep gradients due to the nonlinear nature of the convection.

The steep gradients are stabilized by feedback control laws. In this phase, sufficient conditions for the control input functions and the location of sensors are obtained. Also, explicit exponential decay rates for open-loop and closed-loop systems are obtained.

Numerical experiments are given to illustrate some of typical results on convergence and stability.

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Chapter I. Introduction

1.1. Introduction.

In this chapter, we present several examples to motivate the control problem for Burgers' equation. In Section 1.2, we consider some physical examples which can be represented by Burgers' equation. A brief history of the control problem related to Burgers' equation will be discussed in Section 1.3. In Section 1.4, we present the basic notation.

1.2. Physical Models.

Many physical problems can be modelled, approximately or exactly, by Burgers' equation. In this section we consider some of the more illuminating examples that are available in standard references. Other examples involving acoustic transmission and turbulence in hydrodynamic flows can be found in [**14**] and the references given there. The following three examples are taken almost directly from [**14**].

Example 1.2.1. (Shock Waves).

An impulsively-started piston moving at a constant velocity into a tube containing a compressible fluid initially at rest creates compression waves. The compression waves eventually coalesce, due to the nonlinear nature of the convection, to form a single shock wave. The one-dimensional unsteady motion of the fluid is governed by the continuity equation

$$\frac{\partial}{\partial t}\rho(t,x) + \rho(t,x)\frac{\partial}{\partial x}v(t,x) + v(t,x)\frac{\partial}{\partial x}\rho(t,x) = 0 \tag{1.2.1}$$

and the x-momentum equation

$$\frac{\partial}{\partial t}v(t,x) + v(t,x)\frac{\partial}{\partial x}v(t,x) + (\frac{\partial}{\partial x}p(t,x))/\rho(t,x) = \delta\frac{\partial^2}{\partial x^2}v(t,x), \tag{1.2.2}$$

where ρ is the density, v is the velocity, p is the pressure and δ is the "diffusivity of sound". It is convenient to replace the density by the sound speed, $a = a(t,x)$, via $a(t,x)/a_0 = (\rho(t,x)/\rho_0)^{\frac{\gamma-1}{2}}$, where $\gamma > 1$ is the specific heats ratio and the subscript 0 refers to the undisturbed values, ([**27**]). Equations (1.2.1) and (1.2.2) become

$$\frac{\partial}{\partial t}a(t,x) + v(t,x)\frac{\partial}{\partial x}a(t,x) + \frac{\gamma-1}{2}a(t,x)\frac{\partial}{\partial x}v(t,x) = 0 \tag{1.2.3}$$

and

$$\frac{\partial}{\partial t}v(t,x) + v(t,x)\frac{\partial}{\partial x}v(t,x) + \frac{2}{\gamma-1}a(t,x)\frac{\partial}{\partial x}a(t,x) = \delta\frac{\partial^2}{\partial x^2}v(t,x), \tag{1.2.4}$$

where δ is a function of the undisturbed (to the right of the shock) viscosity, density, specific heat and thermal conductivity of the medium. Equations (1.2.3) and (1.2.4) can be simplified by introducing the Riemann invariants,

$$r(t,x) = \frac{a(t,x)}{\gamma-1} + \frac{v(t,x)}{2}, \qquad s(t,x) = \frac{a(t,x)}{\gamma-1} - \frac{v(t,x)}{2} \tag{1.2.5}$$

to give

$$\frac{\partial}{\partial t}r(t,x) + (a(t,x) + v(t,x))\frac{\partial}{\partial x}r(t,x) = \frac{\delta}{2}\frac{\partial^2}{\partial x^2}(r(t,x) - s(t,x)) \tag{1.2.6}$$

and

$$\frac{\partial}{\partial t}s(t,x) - (a(t,x) - v(t,x))\frac{\partial}{\partial x}s(t,x) = \frac{\delta}{2}\frac{\partial^2}{\partial x^2}(s(t,x) - r(t,x)). \qquad (1.2.7)$$

Consider the propagation of a disturbance into an initially undisturbed region, $s = s_0$ where $s_0 = \frac{a_0}{\gamma-1}$. Then the problem is governed by equation (1.2.6). But from equation (1.2.5),

$$a(t,x) + v(t,x) = \frac{\gamma+1}{2}r(t,x) + \frac{\gamma-3}{2}s_0, \qquad (1.2.8)$$

thus equation (1.2.6) becomes

$$\frac{\partial}{\partial t}r(t,x) + (\frac{\gamma+1}{2}r(t,x) + \frac{\gamma-3}{2}s_0)\frac{\partial}{\partial x}r(t,x) = \frac{\delta}{2}\frac{\partial^2}{\partial x^2}r(t,x). \qquad (1.2.9)$$

As the final step we introduce the change of variables

$$u(t,x) = \frac{\gamma+1}{2}(r(t,x) - r_0), \qquad \xi = x - a_0 t\,, \qquad (1.2.10)$$

to give Burgers' equation

$$\frac{\partial}{\partial t}u(t,\xi) + u(t,\xi)\frac{\partial}{\partial \xi}u(t,\xi) = \frac{\delta}{2}\frac{\partial^2}{\partial \xi^2}u(t,\xi). \qquad (1.2.11)$$

From equations (1.2.8) and (1.2.10) we have $u(t,\xi) = \{\, a(t,\xi) + v(t,\xi)\,\} - \{\, v_0 + a_0\,\}$, where $v_0 = 0$, i.e., $u(t,\xi)$ is the excess wavelet velocity (the difference between propagation speeds of disturbance in stagnation and nonstagnation conditions). The coordinate ξ is measured relative to a frame of reference moving with the undisturbed speed of sound a_0.

Example 1.2.2. (Traffic Flow).

One-dimensional continuum model of highway traffic flow problem is governed by the continuity equation

$$\frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}f(t,x) = 0, \tag{1.2.12}$$

where $u(t,x)$ is the density of cars and $f(t,x)$ is the flux of cars, ([**41**]). If drivers look ahead and modify their speed accordingly, then $f(t,x)$ can be represented by

$$f(t,x) = u(t,x)v(t,x) - D\frac{\partial}{\partial x}u(t,x), \tag{1.2.13}$$

where $v(t,x) = v_0(1 - u(t,x)/u_s)$ is the local mean velocity, v_0 is the speed when $u(t,x) = 0$, u_s is the saturation density at $v(t,x) = 0$, and $D > 0$ is the diffusion coefficient given by

$$D = \tau v_r^2, \tag{1.2.14}$$

where v_r is a random velocity of cars and τ is the mean collision time for cars, ([**30**]). Combining equations (1.2.12), (1.2.13) and (1.2.14) gives

$$(\frac{\partial}{\partial t}u(t,x) + v_0\frac{\partial}{\partial x}u(t,x)) - 2(\frac{u(t,x)}{u_s})v_0\frac{\partial}{\partial x}u(t,x) - D\frac{\partial^2}{\partial x^2}u(t,x) = 0. \tag{1.2.15}$$

By introducing the moving reference frame

$$\xi = -x + v_0 t \tag{1.2.16}$$

we obtain

$$\frac{\partial}{\partial t}u(t,\xi) + 2(\frac{v_0}{u_s})u(t,\xi)\frac{\partial}{\partial \xi}u(t,\xi) - D\frac{\partial^2}{\partial \xi^2}u(t,\xi) = 0. \tag{1.2.17}$$

Let $w(t,\xi) = u(t,\xi)/(\frac{u_s}{2})$. Then the equation (1.2.17) becomes

$$\frac{\partial}{\partial t}w(t,\xi) + v_0 w(t,\xi)\frac{\partial}{\partial \xi}w(t,\xi) - D\frac{\partial^2}{\partial \xi^2}w(t,\xi) = 0. \tag{1.2.18}$$

Finally, we introduce the normalizations

$$s = \frac{t}{t_0} \quad \text{and} \quad \eta = \frac{\xi}{x_0} \tag{1.2.19}$$

to obtain Burgers' equation

$$\frac{\partial}{\partial s}w(s,\eta) + w(s,\eta)\frac{\partial}{\partial \eta}w(s,\eta) = \frac{1}{\mathrm{R}}\frac{\partial^2}{\partial \eta^2}w(s,\eta), \tag{1.2.20}$$

where $x_0 = v_0 t_0$, t_0 is the mean time between successive cars passing a stationary observer and R is the dimensionless constant defined by

$$\mathrm{R} = (\frac{v_0}{v_r})^2\frac{t_0}{\tau}. \tag{1.2.21}$$

Note that the normalized reference frame η is given by $\eta = -\frac{x}{x_0} + \frac{t}{t_0}$ and that $u(\frac{t}{t_0}, \frac{\xi}{x_0})$ and $w(s,\eta) = u(\frac{t}{t_0}, \frac{\xi}{x_0})/(\frac{u_s}{2})$ represent the relative car density and the normalized car density, respectively.

Example 1.2.3. (Supersonic Flow about Airfoils).

For slender wings or airfoils subjected to a steady-state supersonic inviscid flow, it is convenient to think of the wing as a perturbation of a uniform free stream. If the governing partial differential equation is expressed as a velocity potential $\phi(x,y)$ it is possible to develop a hierarchy of solutions, the lowest order of which is linear.

In fact, the governing equation for the first-order problem is given by

$$(1 - M_\infty^2)\frac{\partial^2}{\partial x^2}\phi(x,y) + \frac{\partial^2}{\partial y^2}\phi(x,y) = 0, \qquad (1.2.22)$$

where M_∞ is the freestream Mach number, x is the streamwise coordinate and y is the stream-normal coordinate, ([**14**, p161]).

Although solutions to equation (1.2.22) give an accurate description of the flow at an intermediate distance from the body, in the farfield a nonlinear second-order correction must be added to account for the merging of the characteristics. In this case, equation (1.2.22) is replaced by

$$(1 - M_\infty^2 - (\gamma+1)M_\infty^4\frac{\partial}{\partial x}\phi(x,y))\frac{\partial^2}{\partial x^2}\phi(x,y) + \frac{\partial^2}{\partial y^2}\phi(x,y) = 0, \qquad (1.2.23)$$

where $\gamma > 1$ is the adiabatic gas constant, e.g., for air, $\gamma \approx 1.4$.

However, the description (1.2.23) is inappropriate locally when characteristics merge the velocity potential, since a shock wave will occur. One way to permit shocks to appear in the solution is to add a diffusion term to equation (1.2.23), ([**9**]). In this case, equation (1.2.23) is replaced by

$$\delta\frac{\partial^3}{\partial x^3}\phi(x,y) + (1 - M_\infty^2 - (\gamma+1)M_\infty^4\frac{\partial}{\partial x}\phi(x,y))\frac{\partial^2}{\partial x^2}\phi(x,y) + \frac{\partial^2}{\partial y^2}\phi(x,y) = 0, \quad (1.2.24)$$

where $\delta > 0$ is a small diffusion coefficient. The term $\delta\frac{\partial^3}{\partial x^3}\phi(x,y)$ will only contribute to the solution in regions where shocks occur. In the limit $\delta \to 0$ the inviscid solution is recovered. The effect of including the term $\delta\frac{\partial^3}{\partial x^3}\phi(x,y)$ permits a continuous solution to be obtained in the vicinity of the shock so that the gross effect of the shock on, say, the pressure distribution on the airfoil can be obtained.

For simplicity, consider a symmetric airfoil and introduce the following transformation of coordinates. Two directions appear in the problem, that of the freestream and that of the outgoing Mach lines. The airfoil lies nearly along the former and the wave pattern nearly along the latter. Thus the solution can be expressed more naturally in terms of the oblique coordinates

$$\xi = x - \sqrt{M_\infty^2 - 1}\, y \quad \text{and} \quad \eta = \sqrt{M_\infty^2 - 1}\, y. \tag{1.2.25}$$

Then equation (1.2.24) becomes

$$\delta \frac{\partial^3}{\partial \xi^3}\phi(\xi,\eta) - (\gamma+1)M_\infty^4 \frac{\partial}{\partial \xi}\phi(\xi,\eta)\frac{\partial^2}{\partial \xi^2}\phi(\xi,\eta) - 2(M_\infty^2 - 1)\frac{\partial^2}{\partial \eta \partial \xi}\phi(\xi,\eta) + (M_\infty^2 - 1)\frac{\partial^2}{\partial \eta^2}\phi(\xi,\eta) = 0. \tag{1.2.26}$$

With only single wave family oriented along the planes ξ = constant, the term $(M_\infty^2 - 1)\frac{\partial^2}{\partial \eta^2}\phi(\xi,\eta)$ is of higher order and may be dropped, ([20]). Hence, in what follows we consider

$$\delta \frac{\partial^3}{\partial \xi^3}\phi(\xi,\eta) - (\gamma+1)M_\infty^4 \frac{\partial}{\partial \xi}\phi(\xi,\eta)\frac{\partial^2}{\partial \xi^2}\phi(\xi,\eta) - 2(M_\infty^2 - 1)\frac{\partial^2}{\partial \eta \partial \xi}\phi(\xi,\eta) = 0. \tag{1.2.27}$$

Finally, by the additional transformations

$$u(\xi,\eta) = \frac{(\gamma+1)M_\infty^4}{\sqrt{2(M_\infty^2 - 1)}}\frac{\partial}{\partial \xi}\phi(\xi,\eta) \quad \text{and} \quad \zeta = \sqrt{2(M_\infty^2 - 1)}\, \xi, \tag{1.2.28}$$

we obtain Burgers' equation

$$\frac{\partial}{\partial\eta}u(\zeta,\eta) + u(\zeta,\eta)\frac{\partial}{\partial\zeta}u(\zeta,\eta) = \delta\frac{\partial^2}{\partial\zeta^2}u(\zeta,\eta). \tag{1.2.29}$$

Here, $u(\zeta,\eta) = \frac{(\gamma+1)M_\infty^4}{\sqrt{2(M_\infty^2-1)}}\frac{\partial}{\partial\xi}\phi(\sqrt{2(M_\infty^2-1)}\,\xi, \sqrt{M_\infty^2-1}\,y)$ is the modified velocity component in the outgoing Mach line direction.

1.3. Previous Work Related to the Problem.

The Burgers' equation

$$\frac{\partial}{\partial t}u(t,x) + u(t,x)\frac{\partial}{\partial x}u(t,x) = \epsilon\frac{\partial^2}{\partial x^2}u(t,x) \tag{1.3.1}$$

was introduced by Burgers, ([**5,6,7**]), as a simple model for turbulence, where $\epsilon > 0$ is a viscosity coefficient. Since then many researchers have considered the conservation law

$$\frac{\partial}{\partial t}u(t,x) + u(t,x)\frac{\partial}{\partial x}u(t,x) = 0 \tag{1.3.2}$$

and the "viscosity solution"

$$u(t,x) = \lim_{\epsilon\downarrow 0} u^\epsilon(t,x), \tag{1.3.3}$$

where $u^\epsilon(t,x)$ satisfies equation (1.3.1), ([**10,19,22,26,28,29,31**]).

Oleinik, ([**31**]), proved that for any L^∞-initial data, there is a unique viscosity solution for equation (1.3.2) and the solution satisfies the "entropy condition"

$$\frac{u(t,x+a) - u(t,x)}{a} < \frac{E}{t} \tag{1.3.4}$$

for all $t > 0$, $a > 0$, $-\infty < x < \infty$ and for some constant $E > 0$. A complete discussion of these results may be found in [**37**].

Almost no results exist for the control problem associated with Burgers' equation. Chen, Wang and Weerakoon, ([**8,40**]), considered an optimal control problem for equation (1.3.2) with $-\infty < x < \infty$. The problem was to select an initial function to minimize a specific cost functional J. They obtained sufficient conditions for the differentiability of J with respect to the initial function and explicit expression of the entropy solution of (1.3.2) in terms of initial data.

In this paper we consider a control problem for Burgers' equation (1.3.1) defined on a finite interval. Specifically, we will find several feedback laws stabilizing the nonlinear system (1.3.1) with a certain exponential decay rate. The feedback laws will be obtained from the linearized equation. From this point of view, Curtain, ([**12**]), considered a stabilization problem for semilinear evolution equations. Using Kiehöfer's stability results for semilinear evolution equations, ([**24**]), she showed that, under certain conditions, there exists a finite dimensional compensator which produces a stable closed-loop system. These finite dimensional compensators are obtained from the linearized control system. Applying her results to Burgers' equation (1.3.1) with, for example, Dirichlet boundary conditions, one can obtain stabilizability results of the closed-loop system which are very similar with ours. However, in this case, there is a restriction on the action of output operators. The domain of the output operator should be taken as a certain proper subspace of L^2 which contains the Sobolev space H_0^1.

Throughout this paper, feedback laws will be chosen as "optimal", instead of finite dimensional compensators, in the sense that they minimize certain energy functionals.

Well-posedness and stability results for the open-loop system are obtained in Chapter II. In Chapter III, a "shifted" linear control problem ((LCP)^) is introduced. Under appropriate selection of input and output operators, the (LCP)^ is stabilizable and detectable. The feedback control law obtained from (LCP)^ produces the desired degree of stability for the closed-loop nonlinear system (Theorem 3.2.10). Finally, in Chapter IV, a numerical scheme for computing the "feedback functional gains" is developed and several numerical experiments are performed.

1.4. Notation.

If $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ are normed linear spaces, then $\mathcal{L}(X,Y)$ will denote the space of all bounded linear operators from X to Y. For any $A \in \mathcal{L}(X,Y)$, $||A||$ or $||A||_{\mathcal{L}(X,Y)}$ will denote the operator norm on the space $\mathcal{L}(X,Y)$. In the event that $X = Y$ we denote $\mathcal{L}(X,Y)$ by $\mathcal{L}(X)$. From time to time we will use $||\cdot||$ without any subindex for vector or operator norm. In all such cases the appropriate index for $||\cdot||$ will be understood from the context. For a Hilbert space X, we denote the inner product on $X \times X$ by $< \cdot, \cdot >_X$. Given a linear operator A from X into itself, we denote its domain, spectrum, resolvent and adjoint by $\mathcal{D}(A)$, $\sigma(A)$, $\rho(A)$ and A^*, respectively. For real numbers a, b with $a < b$, $L^p(a,b;X)$, $1 < p < \infty$, will be the space of all Lebesgue measurable functions f from (a,b) to X such that $||f||_{L^p(a,b)} = (\int_a^b |f(x)|^p dx)^{\frac{1}{p}} < \infty$. The spaces $H^k(a,b)$ and $H_0^k(a,b)$ are the standard Sobolev spaces defined by $H^k(a,b) = \{f \in L^2(a,b) | f^{(j)} \in L^2(a,b), j = 0,1,...,k\}$ and $H_0^k(a,b) = \{f \in H^k(a,b) \,|\, f^{(j)}(a) = f^{(j)}(b) = 0, j = 0,1,...,k-1\}$, respectively. The dual space $H^{-k}(a,b)$ of $H_0^k(a,b)$ is the space of all continuous linear functionals on $H_0^k(a,b)$ represented by the inner product $< \cdot, \cdot >_{L^2(a,b)}$.

Chapter II. Well-Posedness and Stability of Burgers' Equation

In this chapter, we consider well-posedness and stability properties of the solution for Burgers' equation with Dirichlet boundary condition. These results will be needed in the analyses of our control problems in Chapter III.

2.1. Basic Definitions and Preliminaries.

Consider the initial value problem

$$\begin{aligned} \frac{d}{dt}z(t) + \mathcal{A}z(t) &= f(t, z(t)), \qquad t > t_0 \\ z(t_0) &= z_0 \end{aligned} \tag{2.1.1}$$

on a Hilbert space X, where $-\mathcal{A}$ is the infinitesimal generator of an analytic semigroup $S(t)$ satisfying $||S(t)||_{\mathcal{L}(X)} \leq Me^{\omega t}$, $t \geq t_0$, for some constants $M = M(\omega) \geq 1$ and $\omega \geq 0$. Since $S(t)$ is analytic, the fractional powers of $\mathcal{A}_1 = \mathcal{A} + aI$ are well-defined for any $a > \omega$, ([**32**, Chapters 1 and 2]). Since $0 \in \rho(\mathcal{A}_1)$, the resolvent of $\mathcal{A}_1$, $\mathcal{A}_1^{\mu}$ is invertible for all $0 \leq \mu \leq 1$. Therefore, the graph norm $||z|| + ||\mathcal{A}_1^{\mu}z||$ on the domain $\mathcal{D}(\mathcal{A}_1^{\mu})$ of $\mathcal{A}_1^{\mu}$ is equivalent to the norm $||z||_{\mu} = ||\mathcal{A}_1^{\mu}z||$. We denote the Hilbert space $\mathcal{D}(\mathcal{A}_1^{\mu})$ with the norm $||z|| + ||\mathcal{A}_1^{\mu}z||$ or $||\mathcal{A}_1^{\mu}z||$ by $(X_{\mu}, ||\cdot||_{\mu})$.

We shall make the following assumption, (see [**21,32**]).

HYPOTHESIS (F): Let U be an open subset of $[t_0, \infty) \times X_{\mu}$. The function $f : U \longrightarrow X$ satisfies the hypothesis (F) if for every $(t, z) \in U$ there is a neighborhood

$V \subset U$ and constants $L \geq 0$, $0 < \theta \leq 1$ such that

$$\|f(t_1, z_1) - f(t_2, z_2)\|_X \leq L(|t_1 - t_2|^\theta + \|z_1 - z_2\|_\mu) \tag{2.1.2}$$

for all $(t_i, z_i) \in V$, $i = 1, 2$, i.e., f is locally Hölder continuous in t, locally Lipschitzian in z, on U.

Now we are ready to state the local existence theorem for the solution of equation (2.1.1). The following result appears as Theorem 3.3.3 in [**21**].

THEOREM 2.1.1. *Let $\mathcal{A}$ be as before and f satisfy hypothesis (F). Then for any $(t_0, z_0) \in U \subset \mathbf{R}^+ \times X_\mu$, there exists $T = T(t_0, z_0) > 0$ such that equation (2.1.1) has a unique (strong) solution $z(t)$ on $[t_0, t_0 + T)$ with initial value $z(t_0) = z_0$.*

2.2. Well-Posedness and Stability of Burgers' Equation.

In this section we consider the well-posedness and stability properties of Burgers' equation, with Dirichlet boundary condition, on a finite interval $[0, \ell]$ given by

$$\begin{aligned}
&\frac{\partial}{\partial t} z(t,x) = \epsilon \frac{\partial^2}{\partial x^2} z(t,x) - z(t,x) \frac{\partial}{\partial x} z(t,x), \qquad 0 < x < \ell, \quad t > 0 \\
&\quad z(t,0) = z(t,\ell) = 0, \\
&\quad z(0,x) = z_0(x),
\end{aligned} \tag{2.2.1}$$

where $\epsilon = \frac{1}{\mathrm{Re}} > 0$, Re is the Reynolds number. In order to place the system (2.2.1) into a semigroup framework let $z(t)(\cdot) = z(t, \cdot)$, $z_0(\cdot) = z(0, \cdot)$ and $H = L^2(0, \ell)$.

Define an operator A by

$$A\phi = \epsilon\,\phi'' \tag{2.2.2}$$

for all $\phi \in \mathcal{D}(A) = H^2(0,\ell) \cap H_0^1(0,\ell)$. Then the system (2.2.1) can be written as an initial value problem

$$\begin{aligned} \frac{d}{dt}z(t) &= Az(t) + f(t, z(t)), \qquad t > 0 \\ z(0) &= z_0 \end{aligned} \tag{2.2.3}$$

on the space H, where $f(t,z) = -zz'$ is defined on the space $H_0^1(0,\ell)$. It is well-known, (see **[21,32,39]**), that A generates an analytic semigroup $S(t)$ on H.

We summarize the basic properties of the infinitesimal generator A and its semigroup $S(t)$, $t \geq 0$, in the following remark.

REMARK 2.2.1. (i) The spectrum $\sigma(A)$ of A consists of all eigenvalues $\lambda_n = -\epsilon n^2\pi^2/\ell^2$, $n = 1, 2, \ldots$, and for each eigenvalue λ_n the corresponding eigenfunction ϕ_n is given by

$$\phi_n(x) = \sqrt{2}\,\sin\frac{n\pi}{\ell}x, \qquad 0 < x < \ell. \tag{2.2.4}$$

(ii) The operator A is self-adjoint, i.e., $A = A^*$, and the semigroup $S(t)$ can be represented by the following formula

$$S(t)z = \sum_{n=1}^{\infty} e^{-(\epsilon n^2\pi^2/\ell^2)t} < z, \phi_n > \phi_n \tag{2.2.5}$$

for all $z \in H$, where ϕ_n's are defined by equation (2.2.4). Moreover, from equation

(2.2.5), it is easy to see that $S(t)$ has the stability property

$$||S(t)||_{\mathcal{L}(H)} \leq e^{-(\epsilon\pi^2/\ell^2)t}, \qquad t \geq 0. \tag{2.2.6}$$

A simple application of Schwartz inequality gives the following first Poincaré inequality, ([**42**, p116]).

LEMMA 2.2.2. *For any* $z \in H_0^1(0,\ell)$,

$$||z||_H \leq \ell ||z'||_H, \tag{2.2.7}$$

where $H = L^2(0,\ell)$.

PROOF: For any $z \in H_0^1(0,\ell)$, z is absolutely continuous and is given by $z(x) = \int_0^x z'(s)ds$, $0 \leq x \leq \ell$, and hence, by Schwartz inequality,

$$\begin{aligned} |z(x)| &\leq \int_0^x |z'(s)|\, ds \\ &\leq (\int_0^x ds\,)^{\frac{1}{2}} (\int_0^x |z'(s)|^2\, ds)^{\frac{1}{2}} \\ &\leq \sqrt{\ell} \int_0^\ell |z'(s)|^2\, ds = \sqrt{\ell}\, ||z'||_H. \end{aligned}$$

Therefore,

$$\begin{aligned} ||z||_H &= (\int_0^\ell |z(s)|^2\, ds)^{\frac{1}{2}} \\ &\leq \sqrt{\ell}\, ||z'||_H (\int_0^\ell ds)^{\frac{1}{2}} = \ell\, ||z'||_H. \end{aligned}$$

∎

REMARK 2.2.3. (i) The above lemma gives an equivalent norm $||z||_{H_0^1} \equiv ||z'||_{L^2}$ on the space $H_0^1(0,\ell)$.

(ii) It is well-known that $\mathcal{D}((-A)^{\frac{1}{2}}) = H_0^1(0,\ell)$, ([**21**, p29],[**25**, p326]).

LEMMA 2.2.4. *For any $z \in H_0^1(0,\ell) = \mathcal{D}((-A)^{\frac{1}{2}})$, the following inequalities hold.*

$$(i) \quad ||S(t)z||_{H_0^1} \leq e^{-\gamma t}\,||z||_{H_0^1}, \qquad (t \geq 0), \tag{2.2.8}$$

$$(ii) \quad ||S(t)z||_{H_0^1} \leq \Big(\frac{1+\ell}{\sqrt{2\epsilon}}\frac{1}{\sqrt{t}} + \frac{\pi(1+\ell)}{\ell}\Big)\, e^{-\gamma t}\,||z||_{L^2(0,\ell)}, \qquad (t > 0), \tag{2.2.9}$$

where $\gamma = \epsilon\pi^2/\ell^2$.

PROOF: For any $z \in H_0^1(0,\ell)$, we know that $(-A)^{\frac{1}{2}}S(t)z = S(t)(-A)^{\frac{1}{2}}z$. Hence, by Remark 2.2.1,

$$\begin{aligned} ||S(t)z||_{H_0^1} &= ||S(t)z||_H + ||(-A)^{\frac{1}{2}}S(t)z||_H \\ &\leq ||S(t)||\,(\,||z|| + ||(-A)^{\frac{1}{2}}z||\,) \\ &\leq e^{-\gamma t}\,||z||_{H_0^1}, \end{aligned}$$

where $H = L^2(0,\ell)$. The inequality (ii) follows from Remark 2.2.1, Lemma 2.2.2 and the estimate

$$\begin{aligned} ||S(t)z||_{H_0^1} &\leq (1+\ell)\,||(-A)^{\frac{1}{2}}S(t)z||_H \\ &= (1+\ell)\,||(-A)^{\frac{1}{2}}\sum_{n=1}^{\infty} e^{\lambda_n t} < z, \phi_n > \phi_n|| \\ &= (1+\ell)\,||\sum_{n=1}^{\infty} \frac{n\pi}{\ell} e^{\lambda_n t} < z, \phi_n > \sqrt{2}\cos\frac{n\pi}{\ell}x|| \\ &\leq (1+\ell)\,(\sup\{\frac{n\pi}{\ell} e^{(\lambda_n+\gamma)t} : n = 1, 2, \dots\})e^{-\gamma t}||z||_H \end{aligned}$$

and

$$\sup\{\frac{n\pi}{\ell}e^{(\lambda_n+\gamma)t} : n = 1,2,\dots\} \le \begin{cases} \frac{1}{\sqrt{2\epsilon}}\frac{1}{\sqrt{t}}, & 0 < t \le \frac{\ell^2}{2\epsilon\pi^2} \\ \frac{\pi}{\ell}, & t \ge \frac{\ell^2}{2\epsilon\pi^2}, \end{cases}$$

where $\lambda_n = -\epsilon n^2\pi^2/\ell^2$ and $\phi_n(x) = \sqrt{2}\sin\frac{n\pi}{\ell}x$, $n = 1,2,\cdots$. ∎

REMARK 2.2.5. The inequality (2.2.9) holds for every $z \in H = L^2(0,\ell)$, since the semigroup $S(t)$ is analytic.

Now we have the well-posedness and stability properties of Burgers' equation (2.2.1) on the space $H_0^1(0,\ell)$. The following theorem is an application of Theorem 5.1.1 in **[21]**.

THEOREM 2.2.6. *For any given $\beta > 0$, $0 < \beta < \gamma = \epsilon\pi^2/\ell^2$, there is a $\rho = \rho(\ell,\epsilon,\beta) > 0$ such that for any initial data $z_0 \in H_0^1(0,\ell)$, with $||z_0||_{H_0^1} \le \frac{\rho}{2}$, there is a unique solution $z(t) = z(t,0;z_0) \in H_0^1(0,\ell)$ of equation (2.2.1). Moreover, the solution satisfies the inequality*

$$||z(t,0;z_0)||_{H_0^1} \le 2e^{-\beta t}||z_0||_{H_0^1} \qquad (t \ge 0) \tag{2.2.10}$$

and $\rho = \rho(\ell,\epsilon,\beta) > 0$ can be chosen to satisfy

$$0 < \rho < \frac{\sqrt{\epsilon}\,\ell(\gamma-\beta)}{\sqrt{2\pi}\,(1+\ell)(\ell\sqrt{\gamma-\beta}+\sqrt{2\pi\epsilon})}. \tag{2.2.11}$$

PROOF: Note that 0 is an equilibrium point for the system (2.2.3). Since $\mathcal{D}((-A)^{\frac{1}{2}}) = H_0^1$, if the nonlinear term $f(z) = -zz'$ satisfies the hypothesis (F) in Section 2.1

with index $\mu = \frac{1}{2}$, then, by Theorem 2.1.1, we have a unique local solution $z(t,0;z_0)$ on the space H_0^1. It is easy to see that

$$||f(z_1) - f(z_2)||_{L^2} \le (||z_1||_{H_0^1} + ||z_2||_{H_0^1})\,||z_1 - z_2||_{H_0^1}$$

for all $z_1, z_2 \in H_0^1$, uniformly in $t > 0$. Hence, f satisfies the hypothesis (F).

For the global existence and uniqueness of the solution $z(t,0;z_0) \in H_0^1$, let z_0 be any initial data in H_0^1 with $||z_0||_{H_0^1} \le \frac{\rho}{2}$, where $\rho = \rho(\ell, \epsilon, \beta)$ satisfies the condition (2.2.11). It follows that

$$\rho\{\frac{1+\ell}{\sqrt{2\epsilon}}\int_0^\infty \frac{1}{\sqrt{s}}e^{-(\gamma-\beta)s}\,ds + \frac{(1+\ell)\pi}{\ell(\gamma-\beta)}\} < \frac{1}{2}, \tag{2.2.12}$$

since $\int_0^\infty \frac{1}{\sqrt{s}}e^{-(\gamma-\beta)s}\,ds = \frac{1}{\sqrt{\gamma-\beta}}\frac{1}{\Gamma(\frac{1}{2})}\frac{\pi}{\sin\frac{\pi}{2}} = \frac{\sqrt{\pi}}{\sqrt{\gamma-\beta}}$, where Γ is the Gamma function and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Let $||z_0||_{H_0^1} \le \frac{\rho}{2}$. Then, by the local existence property, there is a unique solution $z(t,0;z_0) \in H_0^1$ satisfying the inequality $||z(t,0;z_0)||_{H_0^1} < \rho$ on an interval $[0,t_1)$ for some $t_1 > 0$, where t_1 is chosen as large as possible. We will show that t_1 must be infinity. Suppose that t_1 is finite. Then we must have $||z(t_1)||_{H_0^1} \ge \rho$. Note that, on the interval $[0,t_1)$,

$$||f(z(t))||_H = ||-z(t)z'(t)||_H \le (||z(t)||_{H_0^1})^2 \le \rho^2, \tag{2.2.13}$$

where $' = \frac{d}{dx}$. Lemma 2.2.4, Remark 2.2.5 together with inequalities (2.2.12), (2.2.13) yield

$$\begin{aligned}
||z(t_1)||_{H_0^1} &= ||S(t_1)z_0 + \int_0^{t_1} S(t_1 - s)f(z(s))ds\,||_{H_0^1} \\
&\le ||S(t_1)z_0||_{H_0^1} + \int_0^{t_1} ||S(t_1 - s)f(z(s))||_{H_0^1}\,ds \\
&\le e^{-\gamma t_1}||z_0||_{H_0^1} + \rho^2 \int_0^{t_1} \{ \frac{1+\ell}{\sqrt{2\epsilon}} \frac{1}{\sqrt{t_1 - s}} + \frac{(1+\ell)\pi}{\ell} \}\, e^{-\gamma(t_1 - s)}\,ds \\
&\le \frac{\rho}{2} + \rho^2 \{ \frac{1+\ell}{\sqrt{2\epsilon}} \int_0^\infty \frac{1}{\sqrt{s}} e^{-\gamma s}\,ds + \frac{(1+\ell)\pi}{\ell} \int_0^\infty e^{-\gamma s}\,ds \} \\
&= \frac{\rho}{2} + \rho \{ \rho(\frac{1+\ell}{\sqrt{2\epsilon}} \sqrt{\frac{\pi}{\gamma}} + \frac{(1+\ell)\pi}{\ell\gamma}) \} \\
&< \frac{\rho}{2} + \frac{\rho}{2} \\
&= \rho\,.
\end{aligned}$$

This is a contradiction. Therefore, the unique global solution $z(t, 0; z_0)$ exists. Moreover, from the above estimate, we know that if $||z_0||_{H_0^1} \le \frac{\rho}{2}$ then $||z(t, 0; z_0)||_{H_0^1} < \rho$ for all $t \in [0, \infty)$.

Finally, we will derive the stability result (2.2.10). Let $w(t) = \sup\{\, ||z(s)||_{H_0^1} e^{\beta s} \,|\, 0 \le s \le t \,\}$. We then have

$$\begin{aligned}
&||z(t)||_{H_0^1} e^{\beta t} \\
&\le e^{\beta t} (\, ||S(t)z_0||_{H_0^1} + \int_0^t ||S(t-s)f(z(s))||_{H_0^1} ds\,) \\
&\le e^{\beta t} (e^{-\gamma t}||z_0||_{H_0^1} + \int_0^t \{ \frac{1+\ell}{\sqrt{2\epsilon}} \frac{1}{\sqrt{t-s}} + \frac{(1+\ell)\pi}{\ell} \}\, e^{-\gamma(t-s)}\, ||z(s)||^2_{H_0^1}\, ds\,) \\
&\le e^{-(\gamma-\beta)t}||z_0||_{H_0^1} + \rho \int_0^t (\frac{1+\ell}{\sqrt{2\epsilon}} \frac{1}{\sqrt{t-s}} + \frac{(1+\ell)\pi}{\ell})\, e^{-(\gamma-\beta)(t-s)}\, ||z(s)||_{H_0^1} e^{\beta s}\, ds
\end{aligned}$$

$$\leq ||z_0||_{H_0^1} + \frac{1+\ell}{\sqrt{2\epsilon}}\rho\,(\int_0^t \frac{1}{\sqrt{t-s}}e^{-(\gamma-\beta)(t-s)}\,ds\,)\,w(t) + \frac{(1+\ell)\pi}{\ell}\rho\,(\int_0^t e^{-(\gamma-\beta)s}\,ds\,)\,w(t)$$
$$\leq ||z_0||_{H_0^1} + \rho\{\,\frac{1+\ell}{\sqrt{2\epsilon}}\int_0^\infty \frac{1}{\sqrt{s}}e^{-(\gamma-\beta)s}\,ds + \frac{(1+\ell)\pi}{\ell(\gamma-\beta)}\,\}\,w(t)$$
$$\leq ||z_0||_{H_0^1} + \frac{1}{2}w(t)\,.$$

Therefore, $w(t) \leq 2||z_0||_{H_0^1}$ and $||z(t)||_{H_0^1} \leq 2e^{-\beta t}||z_0||_{H_0^1}$. ∎

REMARK 2.2.7. Rankin, ([**34**]), considered well-posedness properties for a certain type of semilinear evolution equations where the nonlinear terms are in divergence form. According to his results, we can see that equation (2.2.1) has a unique (strong) solution for initial data in $L^p(0,\ell)$, $p \geq 4$. To get this result, he used the analyticity of the semigroup $S(t)$ and the fact that the differential operator $\frac{d}{dx}$ on $H_0^1(0,\ell)$ can be represented by

$$\sqrt{\epsilon}\,\frac{d}{dx} = (-A)^{\frac{1}{2}}\,B \tag{2.2.14}$$

for some bounded operator $B \in \mathcal{L}(H)$, where $H = L^2(0,\ell)$. In general, it is not true that $\sqrt{\epsilon}\,\frac{d}{dx} = (-A)^{\frac{1}{2}}$. This result can be used to analyze the stability property of the solution of Burgers' equation with initial data in $L^p(0,\ell)$, $p \geq 4$.

Chapter III. Linear Control Problem and Exponential Decay Rate of the Controlled Burgers' Equation

In this chapter we consider a distributed control problem for Burgers' equation. As we noted in Chapter II, the open-loop solution of Burgers' equation decays exponentially in the topology of an energy space (see Theorem 2.2.6). However, the decay rate depends on the viscosity $\epsilon > 0$. We now explore the possibility of obtaining an exponential decay rate independent of viscosity by feedback laws.

In Section 3.1 an abstract framework for a distributed parameter control problem will be given. The results in Section 3.1 will be applied to the Burgers' equation in Section 3.2. For the distributed parameter control problem we introduce two approaches. First, after stabilizing the solution using bounded output we let the viscosity take care of any possible "steep" gradient nature of the solution. Our second approach is to stabilize both the solution and its gradient through an unbounded observation. Since for both cases the abstract frameworks are similar we will consider the first case as a corollary of the second one.

3.1. Distributed Parameter Control Problem.

The basic model is governed by

$$\frac{d}{dt}z(t) = Az(t) + Bu(t), \qquad z(0) = z_0 \in H,$$
$$y(t) = Cz(t), \qquad t \geq 0, \tag{3.1.1}$$

where U and Y are Hilbert spaces, $u(\cdot) \in L^2(0,\infty;U)$, $y(\cdot) \in L^2(0,\infty;Y)$, and A is the infinitesimal generator of an analytic semigroup $S(t)$ on H. Assume that $B \in \mathcal{L}(U,H)$ and A is self-adjoint with compact resolvent. In order to allow for possible unboundedness of the operator C, we assume that $C \in \mathcal{L}(W,Y)$, where W is a Hilbert space such that

$$\mathcal{D}(A) \subset W \subset H \tag{3.1.2}$$

with continuous dense injections. Of course, the system (3.1.1) is understood in the mild form

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)ds$$
$$y(t) = CS(t)z_0 + C\int_0^t S(t-s)Bu(s)ds, \qquad t \geq 0. \tag{3.1.3}$$

REMARK 3.1.1. The output $y(t)$ defined by equation (3.1.3) is well-defined for all $t > 0$ and all $z \in H$ because the semigroup $S(t)$ is analytic. But, $y(0) = Cz_0$ is defined only for $z_0 \in W$.

Since the ouput operator C may be unbounded, we need the following hypothesis.

(H1). For each $T > 0$, there is a constant $c_T > 0$ such that

$$\int_0^T ||CS(t)z||_Y^2 \, dt \leq c_T^2 \, ||z||_H^2 \tag{3.1.4}$$

for all $z \in W$.

REMARK 3.1.2. (i) If the observation operator C is bounded on the state space H, then W can be chosen as $W = H$. In this case, the hypothesis (H1) is automatically satisfied.

(ii) If (H1) is satisfied we understand $y(t)$ in (3.1.3) as a continuous extension to H of the mapping $z \in W \mapsto y(\cdot) \in L^2(0,T;Y)$.

We now consider the performance index

$$J(u) = \int_0^\infty \{||y(t)||_Y^2 + < u(t), Ru(t) >_U\} dt, \tag{3.1.5}$$

where $y(t)$ is given by equation (3.1.3), and $R \in \mathcal{L}(U)$ is positive definite satisfying the inequality $< u, Ru >_U \geq \gamma ||u||_U^2$ for some $\gamma > 0$ and for every $u \in U$. The linear quadratic regulator problem is :

(LQR) : Find $u(\cdot) \in L^2(0, \infty; U)$ minimizing the cost functional given by equation (3.1.5) subject to the system (3.1.3).

For the existence of an admissible control u such that $J(u) < \infty$ and for the exponential stability of the closed-loop system we need the following two hypotheses.

(H2). The system (3.1.1) is *stabilizable* in the sense that there is a feedback operator $K \in \mathcal{L}(H, U)$ such that the closed loop semigroup $S_K(t) \in \mathcal{L}(H)$ given by

$$S_K(t)z = S(t)z + \int_0^t S(t-s)BKS_K(s)z\, ds \tag{3.1.6}$$

for all $t \geq 0$ and $z \in H$ decays exponentially.

(H3). The system (3.1.1) is *detectable* in the sense that there exists an operator $F \in \mathcal{L}(Y, H)$ such that the output injection semigroup $S_F(t) \in \mathcal{L}(H)$ given by

$$S_F(t)z = S(t)z + \int_0^t S_F(t-s)FCS(s)z\, ds \tag{3.1.7}$$

for all $t \geq 0$ and $z \in W$ decays exponentially.

REMARK 3.1.3. (i) If (H1) and (H2) are satisfied, then for any $z_0 \in H$, there is an admissible control $u_{z_0}(\cdot) \in L^2(0, \infty; U)$ such that $J(u_{z_0}) < \infty$. This finiteness of $J(u_{z_0})$ follows from the observation that, by hypotheses (H1) and (H2), there is a $T > 0$, a constant $c_T > 0$ and a feedback operator $K \in \mathcal{L}(H, U)$ such that

$$||S_K(T)||_{\mathcal{L}(H)} < 1 \quad \text{and} \quad \int_0^T ||CS_K(t)z||_Y^2\, dt < c_T^2 ||z||_H^2 \tag{3.1.8}$$

for all $z \in W$. We now have

$$\int_0^\infty ||CS_K(t)z||_Y^2\, dt \leq c_T^2 \left(\frac{1}{1 - ||S_K(T)||_{\mathcal{L}(H)}^2}\right) ||z||_H^2. \tag{3.1.9}$$

Let $u_{z_0}(t) = KS_K(t)z_0$. Then it is easy to see that $J(u_{z_0}) < \infty$.

(ii) Let (H1) and (H3) be satisfied. Then for any $z_0 \in H$ and $u(\cdot) \in L^2(0, \infty; U)$ with $J(u) < \infty$, $z(t)$ defined by equation (3.1.3) is in $L^2(0, \infty; H)$, ([**33**, p134-135]).

The following result which is obtained using the theory developed by Pritchard and Salamon, ([**33**, Theorems 3.3,3.4]). Since the basic arguments are similar to those in [**33**] we only give a brief sketch of the proof.

THEOREM 3.1.4. *Let hypotheses (H1), (H2) and (H3) be satisfied. Then there is a unique optimal control* $\bar{u}(\cdot) \in L^2(0,\infty;U)$ *for the linear quadratic regulator problem (LQR) and* $\bar{u}(\cdot)$ *is given by the feedback law*

$$\bar{u}(t) = -R^{-1}B^*\Pi\, z(t), \qquad t \geq 0 \tag{3.1.10}$$

where $\Pi \in \mathcal{L}(H)$ *is the unique nonnegative self-adjoint solution of the algebraic Riccati equation*

$$A^*\Pi\, z + \Pi A\, z - \Pi B R^{-1} B^* \Pi\, z + C^*C\, z = 0 \tag{3.1.11}$$

in W^* *for every* $z \in W$*. Moreover, the closed-loop semigroup* $S_\Pi(t) \in \mathcal{L}(H)$ *decays exponentially.*

SKETCH OF THE PROOF: The existence of a nonnegative self-adjoint solution $\Pi \in \mathcal{L}(H)$ satisfying equation (3.1.11) for every $z \in \mathcal{D}(A)$ and the uniqueness of an optimal control $\bar{u}(\cdot)$ given by the formula (3.1.10) follow from the hypotheses (H1), (H2), and Remark 3.1.3.(i). On the other hand, the right hand side of the equation

$$\Pi A\, z = -A^*\Pi\, z + \Pi B R^{-1} B^* \Pi\, z - C^*C\, z \tag{3.1.12}$$

is well-defined for all $z \in W$. Thus, we can extend the left hand side of equation (3.1.12) to $z \in W$ continuously, since $\mathcal{D}(A)$ is densely embedded in W. Finally, the uniqueness of a nonnegative self-adjoint operator Π and the exponential stability property of the closed-loop semigroup $S_\Pi(t)$ follow from (H1), (H3), Remark 3.1.3.(ii) and work of Datko, ([**13**, Theorem 1]). ∎

Corollary 3.1.5 (Bounded Output). *Let (H2) and (H3) be satisfied. If the ouput operator C is bounded, i.e., $C \in \mathcal{L}(W, Y)$ and $W = H$, then all conclusions of Theorem 3.1.4 hold under the condition $W = H$*

3.2. Applications to Burgers' Equation.

Now consider the linearization of Burgers' equation given by

$$\begin{aligned} &\frac{\partial}{\partial t} v(t,x) = \epsilon \frac{\partial^2}{\partial x^2} v(t,x), \qquad t > 0, \qquad 0 < x < \ell \\ &v(t,0) = v(t,\ell) = 0 \\ &v(0,x) = v_0(x), \end{aligned} \tag{3.2.1}$$

where $\epsilon = \frac{1}{\mathrm{Re}}$, Re is the Reynolds number. Introducing $z(t)(\cdot) = v(t,\cdot)$ as in Section 2.2 we can reformulate equation (3.2.1) into an abstract Cauchy problem

$$\frac{d}{dt} z(t) = Az(t), \quad z(0) = z_0 \tag{3.2.2}$$

on the state space $H = L^2(0,\ell)$, where $A\phi = \epsilon\,\phi''$ for all $\phi \in \mathcal{D}(A) = H^2(0,\ell) \cap H_0^1(0,\ell)$, (see Section 2.2).

For the control input operator B and the observation output operator C we consider the Hilbert spaces $U = \mathbf{R}$, $W = H_0^1(0,\ell) = \mathcal{D}((-A)^{\frac{1}{2}})$, and $Y = \mathbf{R}^{k+m}$.

Assume that $B \in \mathcal{L}(U,H)$ and $C \in \mathcal{L}(W,Y)$ are defined by

$$Bu = b(\cdot)u, \qquad b(\cdot) \in H, \qquad u \in U \tag{3.2.3}$$

and

$$Cz = (\tilde{z}(\bar{x}_1), \ldots, \tilde{z}(\bar{x}_k), \tilde{z}'(\bar{y}_1), \ldots, \tilde{z}'(\bar{y}_m)), \tag{3.2.4}$$

where $\bar{x}_i, \bar{y}_j \in (0,\ell)$, $1 \leq i \leq k$, $1 \leq j \leq m$, and

$$\tilde{z}(\bar{x}_i) = \frac{1}{2\delta}\int_{\bar{x}_i-\delta}^{\bar{x}_i+\delta} z(x)dx, \qquad \tilde{z}'(\bar{y}_j) = \frac{1}{2\delta}\int_{\bar{y}_j-\delta}^{\bar{y}_j+\delta} z'(x)dx. \tag{3.2.5}$$

In equation (3.2.5), $\delta > 0$ is chosen so that $(\bar{x}_i-\delta, \bar{x}_i+\delta) \subset (0,\ell)$ and $(\bar{y}_j-\delta, \bar{y}_j+\delta) \subset (0,\ell)$ for all $1 \leq i \leq k$, $1 \leq j \leq m$.

Now consider the following linear control problem

(LCP): Find $\bar{u}(\cdot) \in L^2(0,\infty;U)$ minimizing the performance index

$$J(u) = \int_0^\infty \{||y(t)||_Y^2 + < u(t), Ru(t) >_U\} e^{2\alpha t} dt, \qquad \alpha > 0 \tag{3.2.6}$$

subject to the governing equations

$$\begin{aligned} \frac{d}{dt}z(t) &= Az(t) + Bu(t), \qquad z(0) = z_0 \\ y(t) &= Cz(t), \qquad t \geq 0, \end{aligned} \tag{3.2.7}$$

where A, B, C are as above and $R \in \mathcal{L}(U)$ is positive definte.

REMARK 3.2.1. (i) If the output operator $C \in \mathcal{L}(W,Y)$ is given by

$$Cz = (\tilde{z}(\bar{x}_1), \ldots, \tilde{z}(\bar{x}_k)) \tag{3.2.8}$$

then C is a bounded operator on H. In this case, W and Y can be chosen as $W = H$ and $Y = \mathbf{R}^k$.

(ii) For each i, $1 \le i \le k$, $\tilde{z}(\bar{x}_i)$ given by equation (3.2.5) represents an average value of $z(x)$ over a small neighborhood of $\bar{x}_i$. We can regard each $\bar{x}_i$, $1 \le i \le k$, as the location of a "sensor".

Similarly, $\bar{y}_j$, $1 \le j \le m$, can be considered as the location of a sensor measuring the average of the gradient of $z(x)$ in the δ-neighborhood of $\bar{y}_j$.

(iii) The weight function $e^{2\alpha t}$ in the definition of the cost functional J will play an important role in the exponential decay rate (see Theorems 3.2.8 and 3.2.11). It gives rise to a question of existence of an admissible control $u(\cdot)$ such that $J(u) < \infty$.

For the control problem (LCP), we introduce an "α-shifted" control system, ([**18**]). Let $\hat{z}(t) = z(t)e^{\alpha t}$, $\hat{u}(t) = u(t)e^{\alpha t}$ and $\hat{y}(t) = y(t)e^{\alpha t}$. We then have a modified linear control problem

(LCP)^ : Find $\bar{\hat{u}} \in L^2(0, \infty; U)$ minimizing the cost functional

$$\hat{J}(\hat{u}) = \int_0^\infty \{\, ||\hat{y}(t)||_Y^2 + < \hat{u}(t), R\hat{u}(t) >_U \} \, dt \tag{3.2.9}$$

subject to

$$\begin{aligned} &\frac{d}{dt}\hat{z}(t) = (A + \alpha I)\,\hat{z}(t) + B\hat{u}(t), \qquad \hat{z}(0) = z_0 \\ &\hat{y}(t) = C\hat{z}(t), \qquad t \ge 0. \end{aligned} \tag{3.2.10}$$

Of course, the solutions for (3.2.10) are taken as mild solutions. If we solve the problem (LCP)$\hat{}$ and apply

$$\bar{u}_\alpha(t) = e^{-\alpha t}\,\bar{\hat{u}}(t) \qquad (t \geq 0) \tag{3.2.11}$$

to the original control system (3.2.7), then the resulting optimal trajectory $\bar{z}_\alpha(t)$ will satisfy the inequality

$$\|\,\bar{z}_\alpha(t)\,\|_H \leq M e^{-\alpha t}\|z_0\|_H, \tag{3.2.12}$$

where $M \geq 1$ is a constant and $\alpha > 0$ is the desired degree of stability.

REMARK 3.2.2. A discussion of the "α-shifted" problem for finite dimensional systems first appeared in [**1**]. Anderson and Moore showed that, for finite dimensional systems, the control problem (LCP) is "equivalent" to (LCP)$\hat{}$ in the following senses :

(i) The minimum value of J defined by equation (3.2.6) is the same as the minimum value of $\hat{J}$ given by equation (3.2.9).

(ii) If $\bar{\hat{u}}(t) = g(\hat{z}(t))$ is the optimal control for (LCP)$\hat{}$ for some function g, then $\bar{u}(t) = e^{-\alpha t} g(z(t)e^{\alpha t})$ is the optimal control for (LCP) and conversely.

Now, we apply the results in Section 3.1 to the problem (LCP)$\hat{}$ in order to obtain an optimal control $\bar{\hat{u}} \in L^2(0,\infty;U)$ for (LCP)$\hat{}$.

REMARK 3.2.3. (i) From Remark 2.2.1, the spectrum $\sigma(A+\alpha I)$ of the infinitesimal generator $A + \alpha I$ consists of all eigenvalues $\lambda_{\alpha,n}$, $n = 1, 2, \cdots$, given by

$$\lambda_{\alpha,n} = \alpha - \frac{\epsilon n^2 \pi^2}{\ell^2} \tag{3.2.13}$$

and for each n, $n = 1, 2, \cdots$, the eigenvector $\phi_{\alpha,n}$ corresponding to $\lambda_{\alpha,n}$ is given by

$$\phi_{\alpha,n}(x) = \sqrt{2} \sin \frac{n\pi}{\ell} x. \tag{3.2.14}$$

(ii) We are interested in the stabilization problem for the system (3.2.1) with small viscosity $\epsilon = \frac{1}{\mathrm{Re}} > 0$, i.e., high Reynolds number Re. Thus, we assume that for any given $\alpha > 0$ there is a positive eigenvalue $\lambda_{\alpha,n}$.

Let $\alpha > 0$ be given and let

$$n_\alpha = \max \{ n \in \mathbf{N} : \lambda_{\alpha,n} = \alpha - \frac{\epsilon n^2 \pi^2}{\ell^2} \geq 0 \}. \tag{3.2.15}$$

Since A is self-adjoint (see Remark 2.2.1) and the set $\{ \phi_{\alpha,n} : n = 1, 2, \cdots \}$ is a basis for $H = L^2(0, \ell)$, we can express $W = H_0^1(0, \ell)$ as

$$W = \{ z \in H : \sum_{n=1}^{\infty} n^2 \, | < z, \phi_{\alpha,n} > |^2 < \infty \} \tag{3.2.16}$$

and identify $z \in H$ with the sequence $\{< z, \phi_{\alpha,n} >\}_{n \in \mathbf{N}}$. Assume that $b_n \in U$ and $c_n \in Y$ satisfy

$$Bu = \{< b_n, u >\}_{n \in \mathbf{N}} \qquad \text{and} \qquad Cz = \sum_{n=1}^{\infty} c_n < z, \phi_{\alpha,n} > \tag{3.2.17}$$

and that

$$\sum_{n=1}^{\infty} ||b_n||_U^2 < \infty, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} ||c_n||_Y^2 < \infty,$$

(see [**33**, p137-143]).

Now, we consider hypothesis (H1) for the α-shifted system (LCP)^.

LEMMA 3.2.4. *For each $T > 0$, there is a constant $\hat{c}_T > 0$ such that*

$$\int_0^T ||C\hat{S}(t)z||_Y^2 \, dt \le \hat{c}_T^2 \, ||z||_H^2 \tag{3.2.18}$$

for all $z \in W = H_0^1(0,\ell)$, where $H = L^2(0,\ell)$, the output operator C is defined by equation (3.2.4) and $\hat{S}(t)$ is the analytic semigroup generated by $A + \alpha I$.

PROOF: Since the infinitesimal generator $A + \alpha I$ of $\hat{S}(t)$ is self-adjoint, the semigroup $\hat{S}(t)$ can be represented by

$$\hat{S}(t)\, z = \sum_{n=1}^{\infty} e^{\lambda_{\alpha,n} t} < z, \phi_{\alpha,n} >_H \phi_{\alpha,n}$$

for all $z \in H$, where $\phi_{\alpha,n}(x) = \sqrt{2}\sin\frac{n\pi}{\ell}x$ is the eigenfunction corresponding to the eigenvalue $\lambda_{\alpha,n} = \alpha - \epsilon n^2\pi^2/\ell^2$ of $A+\alpha I$, $n = 1, 2, \cdots$. Hence, for any $z \in W \subset H$, we have

$$\begin{aligned} ||\, C\hat{S}(t)z \,||_Y^2 &= ||\, C(\sum_{n=1}^{\infty} e^{\lambda_{\alpha,n} t} < z, \phi_{\alpha,n} >_H \phi_{\alpha,n}) \,||_Y^2 \\ &= ||\, \sum_{n=1}^{\infty} e^{\lambda_{\alpha,n} t} < z, \phi_{\alpha,n} >_H C(\phi_{\alpha,n}) \,||_Y^2 \\ &\le \sum_{n=1}^{\infty} e^{2\lambda_{\alpha,n} t} \,| < z, \phi_{\alpha,n} >_H |^2 \,||\, C(\phi_{\alpha,n}) \,||_Y^2 \,. \end{aligned}$$

By the expression (3.2.17) of the ouput operator C, it is easy to see that, for each

n, $n = 1, 2, \cdots$,

$$C(\phi_{\alpha,n}) = \sum_{j=1}^{\infty} c_j < \phi_{\alpha,n}, \phi_{\alpha,j} >= c_n. \tag{3.2.19}$$

Let $c_n = (c_{n,1}, \cdots, c_{n,k}, c_{n,k+1}, \cdots, c_{n,k+m}) \in \mathbf{R}^{k+m}$. Then it is easy to see that, by the definition (3.2.4) of the operator C,

$$c_{n,i} = \frac{\sqrt{2}\ell}{n\pi\delta} \sin\frac{n\pi\bar{x}_i}{\ell} \sin\frac{n\pi\delta}{\ell} \quad \text{and} \quad c_{n,k+j} = \frac{\sqrt{2}}{\delta} \cos\frac{n\pi\bar{y}_j}{\ell} \sin\frac{n\pi\delta}{\ell} \tag{3.2.20}$$

for $1 \le i \le k$, $1 \le j \le m$. Thus, $|c_{n,i}| \le \frac{\sqrt{2}\ell}{\pi\delta}$ and $|c_{n,k+j}| \le \frac{\sqrt{2}}{\delta}$ for all $n = 1, 2, \cdots$, $1 \le i \le k$, $1 \le j \le m$. Therefore, for any $z \in W$,

$$\begin{aligned} \| C\hat{S}(t)z \|_Y^2 &\le \sum_{n=1}^{\infty} e^{2\lambda_{\alpha,n} t} | < z, \phi_{\alpha,n} >_H |^2 \| C(\phi_{\alpha,n}) \|_Y^2 \\ &= \sum_{n=1}^{\infty} e^{2\lambda_{\alpha,n} t} | < z, \phi_{\alpha,n} >_H |^2 (\sum_{i=1}^{k+m} |c_{n,i}|^2) \\ &\le \sum_{n=1}^{\infty} e^{2\lambda_{\alpha,n} t} | < z, \phi_{\alpha,n} >_H |^2 (\frac{2k\ell^2}{\pi^2\delta^2} + \frac{2m}{\delta^2}) \\ &= (\frac{2k\ell^2}{\pi^2\delta^2} + \frac{2m}{\delta^2}) \sum_{n=1}^{\infty} e^{2\lambda_{\alpha,n} t} | < z, \phi_{\alpha,n} >_H |^2 . \end{aligned}$$

Note that, from definition (3.2.15) of n_α, $\lambda_{\alpha,n} = \alpha - \epsilon n^2\pi^2/\ell^2 < 0$ for all $n \ge n_\alpha + 1$. here, without loss of generality, we assume that $\lambda_{\alpha,n} > 0$ for all $n = 1, 2, \cdots, n_\alpha$.

Let $M = (\frac{2k\ell^2}{\pi^2\delta^2} + \frac{2m}{\delta^2})$. Then

$$\begin{aligned}
&\int_0^T ||C\hat{S}(t)z||_Y^2\, dt \\
&\leq M \sum_{n=1}^{\infty} (\int_0^T e^{2\lambda_{\alpha,n} t}\, dt)\, |< z, \phi_{\alpha,n} >|^2 \\
&= M[\sum_{n=1}^{n_\alpha} (\int_0^T e^{2\lambda_{\alpha,n} t}\, dt)\, |< z, \phi_{\alpha,n} >|^2 + \sum_{n=n_{\alpha+1}}^{\infty} (\int_0^T e^{2\lambda_{\alpha,n} t}\, dt)\, |< z, \phi_{\alpha,n} >|^2] \\
&\leq M[\sum_{n=1}^{n_\alpha} \frac{1}{2\lambda_{\alpha,n}} (e^{2\lambda_{\alpha,n} T} - 1)\, |< z, \phi_{\alpha,n} >|^2 + \sum_{n=n_{\alpha+1}}^{\infty} T\, |< z, \phi_{\alpha,n} >|^2] \\
&\leq M[\frac{1}{2\lambda_{\alpha,n_\alpha}} (e^{2\lambda_{\alpha,1} T} - 1) \sum_{n=1}^{n_\alpha} |< z, \phi_{\alpha,n} >|^2 + T \sum_{n=n_{\alpha+1}}^{\infty} |< z, \phi_{\alpha,n} >|^2] \\
&\leq M[\frac{1}{2\lambda_{\alpha,n_\alpha}} (e^{2\lambda_{\alpha,1} T} - 1) + T]\, ||z||_H^2,
\end{aligned}$$

where we have used the facts that $0 < \lambda_{\alpha,n_\alpha} = \alpha - \epsilon n_\alpha^2\pi^2/\ell^2 \leq \lambda_{\alpha,n} = \alpha - \epsilon n^2\pi^2/\ell^2$ for all $n = 1, 2, \cdots, n_\alpha + 1$. By letting $\hat{c}_T^2 = M[\frac{1}{2\lambda_{\alpha,n_\alpha}} (e^{2\lambda_{\alpha,1} T} - 1) + T] > 0$, we have proved the lemma. ∎

The following lemma is an application of stabilizability and detectability results of Pritchard and Salamon, ([**33**, Section 4.2]).

Lemma 3.2.5. *For each $n = 1, 2, \cdots, n_\alpha$, let*

$$X_{\alpha,n} = \{\frac{i\ell}{n} : i = 1, 2, \cdots, n-1\} \quad \text{and} \quad Y_{\alpha,n} = \{\frac{(2j+1)\ell}{2n} : j = 0, 1, \cdots, n-1\}. \tag{3.2.21}$$

Then the following statements hold.

(a) $b_n = < b(\cdot), \phi_{\alpha,n} > \neq 0$ for all $n = 1, 2, \cdots, n_\alpha$ if and only if the system (3.2.10) is stabilizable in H.

(b) Choose $\delta > 0$ so small that $\delta < \frac{\ell}{2n_\alpha}$. Then for each $n = 1, 2, \cdots, n_\alpha$, there exists at least one $\bar{x}_i$ or $\bar{y}_j$, $1 \leq i \leq k$, $1 \leq j \leq m$, such that

$$\bar{x}_i \notin X_{\alpha,n}, \qquad \bar{y}_j \notin Y_{\alpha,n}$$

if and only if the system (3.2.10) is detectable through $C \in \mathcal{L}(W, Y)$.

PROOF: (a) From Remark 3.2.3, we know that the spectrum $\sigma(A + \alpha I)$ of $A + \alpha I$ consists of all eigenvalues $\lambda_{\alpha,n} = \alpha - \epsilon n^2\pi^2/\ell^2 < 0$, $n = 1, 2, \cdots$. Thus, for all n, $n \geq n_\alpha + 1$, we know that $\lambda_{\alpha,n} < 0$. Let H_u be the linear span of eigenfunctions $\phi_{\alpha,n}, \cdots, \phi_{n_\alpha}$. Then the dimension of H_u is n_α and hence the system (3.2.10) is stabilizable if and only if the projected system of (3.2.10) onto H_u is controllable if and only if $b_n = < b(\cdot), \phi_{\alpha,n} > \neq 0$ for all n, $n = 1, 2, \cdots, n_\alpha$.

For (b), let $c_n = (c_{n,1}, \cdots, c_{n,k}, c_{n,k+1}, \cdots, c_{n,k+m})$, $n = 1, 2, \cdots, n_\alpha$, be defined by equation (3.2.19). Then, by equation (3.2.20), we have

$$c_{n,i} = \frac{\sqrt{2}\ell}{n\pi\delta} \sin\frac{n\pi\bar{x}_i}{\ell} \sin\frac{n\pi\delta}{\ell} \quad \text{and} \quad c_{n,k+j} = \frac{\sqrt{2}}{\delta} \cos\frac{n\pi\bar{y}_j}{\ell} \sin\frac{n\pi\delta}{\ell} \tag{3.2.22}$$

for $1 \leq i \leq k$, $1 \leq j \leq m$. By the dual statements of (a), (3.2.10) is detectable through $C \in \mathcal{L}(W, Y)$ if and only if $c_n \neq 0$ for all $n = 1, 2, \cdots, n_\alpha$. Hence, (b) holds. ∎

REMARK 3.2.6. If $n_\alpha = 1$, then $X_{\alpha,1}$ is the empty set.

We now return to the original control problem (LCP). The following theorem is the main result for our control problem (LCP).

THEOREM 3.2.7. *Let $\alpha > 0$ be given. Suppose that $b(\cdot) \in H = L^2(0,\ell)$, $\delta > 0$, $\bar{x}_i$, $1 \le i \le k$, and $\bar{y}_j$, $1 \le j \le m$, satisfy the conditions (a) and (b) of Lemma 3.2.5. Then there is a unique optimal control $\bar{u}_\alpha(\cdot) \in L^2(0,\infty;\mathbf{R})$ for the problem (LCP) such that*

$$\bar{u}_\alpha(t) = -R^{-1}B^*\Pi_\alpha\,\bar{z}_\alpha(t), \qquad t \ge 0 \tag{3.2.23}$$

where $\bar{z}_\alpha(t)$ is the corresponding optimal trajectory and $\Pi_\alpha \in \mathcal{L}(H)$ is the unique nonnegative self-adjoint operator satisfying the algebraic Riccati equation

$$(A+\alpha I)^*\Pi_\alpha z + \Pi_\alpha(A+\alpha I)z - \Pi_\alpha BR^{-1}B^*\Pi_\alpha z + C^*Cz = 0 \tag{3.2.24}$$

in $W^* = H^{-1}(0,\ell)$ for every $z \in W = H_0^1(0,\ell)$. Moreover, the closed loop semigroup $S_{\Pi_\alpha}(t) \in \mathcal{L}(H)$ satisfies the following stability property

$$\|\, S_{\Pi_\alpha}(t)\,\|_{\mathcal{L}(H)} \le Me^{-(\alpha+\omega)t} \tag{3.2.25}$$

for some constants $M = M(\alpha,\epsilon) \ge 1$ and $\omega = \omega(\alpha,\epsilon) > 0$.

PROOF: By Lemmas 3.2.4 and 3.2.5, we know that the α-shifted control system (3.2.10) satisfies all hypotheses (H1), (H2) and (H3) in Section 3.1 (with $z(t)$, $y(t)$, $u(t)$, A, $S(t)$ and J replaced by $\hat{z}(t)$, $\hat{y}(t)$, $\hat{u}(t)$, $A+\alpha I$, $\hat{S}(t)$ and $\hat{J}$, respectively). Hence, by Theorem 3.1.4, there is a unique optimal control $\bar{\hat{u}}(t)$ for (LCP)^ and the corresponding closed-loop semigroup $\hat{S}(t)$ decays exponentially, i.e.,

$$\|\hat{S}(t)\|_{\mathcal{L}(H)} \le \hat{M}e^{-\omega t} \qquad t \ge 0 \tag{3.2.26}$$

for some constants $\hat{M} = \hat{M}(\alpha, \epsilon) \ge 1$ and $\omega = \omega(\alpha, \epsilon) > 0$. Moreover, $\bar{\hat{u}}(t)$ is given by

$$\bar{\hat{u}}(t) = -R^{-1}B^*\Pi_\alpha \bar{\hat{z}}(t) \tag{3.2.27}$$

where $\bar{\hat{z}}(t)$ is the resulting optimal trajectory for the α-shifted system (3.2.10) and Π_α is the unique nonnegative self-adjoint solution of equation (3.2.24). Since the semigroup $\hat{S}(t)$ is generated by $A + \alpha I - BR^{-1}B^*\Pi_\alpha$, the infinitesimal generator of the closed-loop semigroup $S_{\Pi_\alpha}(t)$ for the original system (3.2.7) is $A - BR^{-1}B^*\Pi_\alpha$. Hence, $S_{\Pi_\alpha}(t) = \hat{S}(t)e^{-\alpha t}$ and, by the relation (3.2.26), $S_{\Pi_\alpha}(t)$ satisfies the inequality (3.2.25) with $M = \hat{M}$. Moreover, the optimal control $\bar{u}_\alpha(t)$ for (LCP) is given by the formula (3.2.23), since $\bar{u}_\alpha(t) = \bar{\hat{u}}(t)e^{-\alpha t} = -R^{-1}B^*\Pi_\alpha\bar{\hat{z}}(t)e^{-\alpha t} = -R^{-1}B^*\Pi_\alpha \bar{z}_\alpha(t)$, where $\bar{z}_\alpha(t) = \bar{\hat{z}}(t)e^{-\alpha t}$ is the corresponding optimal trajectory for the original system (3.2.7). This completes the proof. ∎

The optimal control $\bar{u}_\alpha(\cdot) \in L^2(0,\infty;\mathbf{R})$ obtained in Theorem 3.2.7 is given by the feedback law (3.2.23). Define the feedback operator $K_\alpha \in \mathcal{L}(H, U)$ by

$$K_\alpha = -R^{-1}B^*\Pi_\alpha. \tag{3.2.28}$$

Then the optimal control $\bar{u}_\alpha(t)$, $t \ge 0$, is given by

$$\bar{u}_\alpha(t) = K_\alpha \, \bar{z}_\alpha(t) \tag{3.2.29}$$

and the infinitesimal generator for the closed-loop semigroup $S_{\Pi_\alpha}(t)$ is

$$A + BK_\alpha = A - BR^{-1}B^*\Pi_\alpha. \tag{3.2.30}$$

Recall that $H = L^2(0,\ell)$ and $U = \mathbf{R}$. Thus, by Riesz representation theorem, (see, e.g., [**11**, p13]), there is a unique *feedback gain function* $k_\alpha(\cdot) \in L^2(0,\ell)$ such that

$$K_\alpha z = \int_0^\ell k_\alpha(s) z(s)\, ds \tag{3.2.31}$$

for all $z \in L^2(0,\ell)$.

We are now at a point where we can obtain a stability result for the controlled heat equation.

Corollary 3.2.8. *Let $\alpha > 0$ be given. Suppose that $b(\cdot) \in L^2(0,\ell)$, $\delta > 0$, $\bar{x}_i$, $1 \le i \le k$, and $\bar{y}_j$, $1 \le j \le m$, satisfy the conditions (a) and (b) of Lemma 3.2.5. Let the feedback gain function $k_\alpha(\cdot) \in L^2(0,\ell)$ be given by the formula (3.2.31). Then for any initial data $z_0 \in L^2(0,\ell)$, there is a unique (strong) solution of the controlled heat equation*

$$\begin{aligned} \frac{\partial}{\partial t} z(t,x) &= \epsilon \frac{\partial^2}{\partial x^2} z(t,x) + b(x) \int_0^\ell k_\alpha(s) z(t,s)\, ds \\ z(t,0) &= z(t,\ell) = 0 \\ z(0,x) &= z_0(x) \in L^2(0,\ell) \end{aligned} \tag{3.2.32}$$

and the solution $z(t)(\cdot) = z(t,\cdot)$ satisfies the following inequality

$$\|\, z(t)(\cdot)\, \|_{L^2(0,\ell)} \le M_\alpha\, e^{-(\alpha+\omega)t}\, \|z_0\|_{L^2(0,\ell)} \tag{3.2.33}$$

for some constants $M_\alpha = M_\alpha(\alpha,\epsilon) \geq 1$ *and* $\omega = \omega(\alpha,\epsilon) > 0$.

PROOF: The existence and uniqueness of a strong solution $z(t)(\cdot) = z(t,\cdot)$ with initial data $z_0 \in L^2(0,\ell)$ follow from the fact that the closed-loop operator $A+BK_\alpha$ defined by equation (3.2.30) generates an analytic semigroup $S_{\Pi_\alpha}(t)$. The stability result (3.2.33) follows from the inequality (3.2.25). ∎

REMARK 3.2.9. From Remark 2.2.1, the open-loop solution $z(t)(\cdot) = z(t,\cdot)$ for heat equation (3.2.32), with $k_\alpha \equiv 0$, satisfies the stability result

$$|| \, z(t) \, ||_{L^2(0,\ell)} \leq e^{-(\epsilon\pi^2/\ell^2)t} \, ||z_0||_{L^2(0,\ell)} \tag{3.2.34}$$

for all $z_0 \in L^2(0,\ell)$.

Now we are ready to state the main result of this section.

THEOREM 3.2.10. *Let* $\alpha > 0$ *be given. Suppose that* $b(\cdot) \in H = L^2(0,\ell)$, $\delta > 0$, $\bar{x}_i$, $1 \leq i \leq k$, *and* $\bar{y}_j$, $1 \leq j \leq m$, *satisfy the conditions (a) and (b) of Lemma 3.2.6. Let* $k_\alpha(\cdot) \in H$ *be the linear feedback gain function defined by the formula (3.2.31). Then there exist constants* $\rho = \rho(\alpha,\epsilon) > 0$ *and* $M = M(\alpha,\epsilon) \geq 1$ *such that for any initial data* $v_0(\cdot) \in H_0^1(0,\ell)$, *with* $||v_0||_{H_0^1} \leq \frac{\rho}{2M}$, *the controlled Burgers'* *equation*

$$\frac{\partial}{\partial t}v(t,x) = \epsilon\frac{\partial^2}{\partial x^2}v(t,x) - v(t,x)\frac{\partial}{\partial x}v(t,x) + b(x)\int_0^\ell k_\alpha(s)v(t,s)\,ds$$

$$v(t,0) = v(t,\ell) = 0, \tag{3.2.35}$$

$$v(0,x) = v_0(x) \in H_0^1(0,\ell)$$

has a unique (strong) solution and the solution $v(t)(\cdot) = v(t,\cdot)$ *satisfies the following stability property*

$$\|\,v(t)\,\|_{H_0^1} \leq 2Me^{-\alpha t}\,\|v_0(\cdot)\|_{H_0^1}. \tag{3.2.36}$$

PROOF: Let the operators A, B, C and K_α be as in this section, i.e., defined by equations (3.2.2), (3.2.3), (3.2.4) and (3.2.28), respectively. Define the nonlinear function $f : H_0^1(0,\ell) \to L^2(0,\ell)$ by

$$f(z) = BK_\alpha z - zz' \tag{3.2.37}$$

where $' = \frac{d}{dx}$. Then, the map f satisfies the hypothesis (F) in Section 2.1, since for any $z_1, z_2 \in H_0^1(0,\ell)$,

$$\|\,f(z_1) - f(z_2)\,\|_{L^2(0,\ell)} \leq (\,\|\,BK_\alpha\|_{\mathcal{L}(H)} + \|z_1\|_{H_0^1} + \|z_2\|_{H_0^1}\,)\,\|z_1 - z_2\|_{H_0^1}. \tag{3.2.38}$$

Note that the operator BK_α is bounded on the state space $H = L^2(0,\ell)$. Thus, by Theorem 2.1.1, we have a unique local (strong) solution of equation (3.2.35).

Let $S_{K_\alpha}(t)$, $t \geq 0$, be the analytic semigroup on H generated by the operator $A + BK_\alpha$. Then, by Corollary 3.2.8, $S_{K_\alpha}(t)$ satisfies the inequality

$$\|S_{K_\alpha}(t)\,z\|_H \leq M_\alpha\,e^{-(\alpha+\omega)t}\,\|z\|_H \tag{3.2.39}$$

for all $z \in H$ and for some constants $M_\alpha = M_\alpha(\alpha, \epsilon) \geq 1$ and $\omega = \omega(\alpha, \epsilon) > 0$. Since $S_{K_\alpha}(t)$ is analytic, for any β with $\alpha < \beta < \alpha + \omega$, there is a constant $\tilde{M}_\alpha = \tilde{M}_\alpha(\alpha, \epsilon, \beta) \geq 1$ such that

$$|| S_{K_\alpha}(t)\, z||_{H_0^1} \leq M_\alpha\, e^{-\beta t}\, ||z||_{H_0^1} \tag{3.2.40}$$

$$|| S_{K_\alpha}(t)\, z||_{H_0^1} \leq \tilde{M}_\alpha\, \frac{1}{\sqrt{t}}\, e^{-\beta t}\, ||z||_H \tag{3.2.41}$$

for all $z \in H_0^1$. Let

$$M = \max\{\, M_\alpha,\ \tilde{M}_\alpha \,\} \tag{3.2.42}$$

and choose $\rho > 0$ with

$$0 < \rho < \frac{\sqrt{\beta - \alpha}}{2\sqrt{\pi} M}. \tag{3.2.43}$$

Then it is easy to see that

$$\rho\, M \int_0^\infty \frac{1}{\sqrt{s}} e^{-(\beta-\alpha)s}\, ds < \frac{1}{2}. \tag{3.2.44}$$

Thus, by arguments similar to those in the proof of Theorem 2.2.6 together with inequalities (3.2.40)-(3.2.44) and the expression

$$v(t) = S_{K_\alpha}(t) v_0 + \int_0^t S_{K_\alpha}(t-s) g(v(s)) ds, \tag{3.2.45}$$

the unique global solution $v(t)(\cdot) = v(t, \cdot)$ for the controlled Burgers' equation (3.2.35) exists and satisfies the inequality (3.2.36), where $g(v(t)) = -v(t)v'(t)$. ∎

Chapter IV. Approximation and Numerical Results

In Chapter III, we considered a linear control problem. Under certain conditions on the input and the output operators there is a unique optimal control which guarantees the desired degree of stability for the closed-loop system (see Lemma 3.2.5, Theorem 3.2.7, Corollary 3.2.8, and Theorem 3.2.10).

In this chapter we consider various approximation schemes for the control problems considered in Chapter III. Abstract approximation schemes and its applications will be discussed in Section 4.1. In Section 4.2, three feedback schemes are introduced. Numerical results will be shown in Section 4.3. Finally, the conclusion of this paper will be given in Section 4.4.

Throughout this chapter, we use superscript N in the designation of subspaces, operators and matrices in the N-th approximating system and corresponding control problem, like H^N, A^N, B^N, etc. Hence the superscript N indicates the order of approximation.

4.1. Approximation Schemes for Linear Regulator Problem.

In this section we consider an approximation scheme for the abstract linear regulator problem with bounded input and output operators and then apply the scheme to our control problem. The following approximation scheme is based on the results from [**4,23**]. For more approximation schemes for linear regulator problems, see [2,3,17] and refences given there.

We suppose that H, U and Y are separable Hilbert spaces, that $A : \mathcal{D}(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $S(t)$ on H and that

$$B \in \mathcal{L}(U,H) \qquad \text{and} \qquad C \in \mathcal{L}(H,Y). \tag{4.1.1}$$

Consider a control problem given by

$$\begin{aligned} &\frac{d}{dt}z(t) = Az(t) + Bu(t), \qquad z(0) = z_0 \in H \\ &y(t) = Cz(t) \end{aligned} \tag{4.1.2}$$

and an associated performance index

$$J(z_0,u) = \int_0^\infty \{||y(t)||_Y^2 + < Ru(t), u(t) >_U\}dt, \tag{4.1.3}$$

where $R \in \mathcal{L}(U)$ is self-adjoint and strictly positive. We interprete the control system (4.1.2) in the mild sense:

$$\begin{aligned} &z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)ds \\ &y(t) = Cz(t) \end{aligned} \tag{4.1.4}$$

The abstract linear optimal regulator problem is stated as

$(\mathcal{R})$: Minimize $J(z_0,u)$ over $u \in L^2(0,\infty;U)$ subject to the system (4.1.4).

The following theorem is well-known, (see [4,16,43]), and can be viewed as a revised version of Theorem 3.1.4.

THEOREM 4.1.1. *Assume that (A,B) is stabilizable and (A,C) is detectable. Then there is a unique nonnegative self-adjoint solution Π of the algebraic Riccati equation*

$$A^*\Pi + \Pi A - \Pi BR^{-1}B^*\Pi + C^*C = 0 \tag{4.1.5}$$

and the unique optimal control $\bar{u}(\cdot) \in L^2(0,\infty;U)$ is given by

$$\bar{u}(t) = -R^{-1}B^*\Pi S_\Pi(t)z_0, \tag{4.1.6}$$

where $S_\Pi(t)$ is the C_0-semigroup generated by $A - BR^{-1}B^*\Pi$. Moreover, $S_\Pi(t)$ is exponentially stable.

We next formulate a sequence of approximate regulator problems and present a convergence result for the corresponding Riccati operators.

Let H^N, $N = 1,2,\cdots$, be a sequence of finite dimensional linear subspaces of H and $P^N : H \to H^N$ be the canonical orthogonal projections. Assume that $S^N(t)$ is a sequence of C_0-semigroups on H^N with infinitesimal generators $A^N \in \mathcal{L}(H^N)$. Let $B^N \in \mathcal{L}(U,H^N)$ and $C^N \in \mathcal{L}(H^N,Y)$. We then consider the family of regulator problems:

$(\mathcal{R}^N)$: Minimize $J^N(z_0^N,u)$ over $u \in L^2(0,\infty;U)$ subject to the control system

$$z^N(t) = S^N(t)z_0^N + \int_0^t S^N(t-s)B^Nu(s)ds, \quad z^N(0) = z_0^N \equiv P^Nz_0 \tag{4.1.7}$$

$$y^N(t) = C^Nz^N(t), \tag{4.1.8}$$

where

$$J^N(z_0^N, u) = \int_0^\infty \{||y^N(t)||_Y^2 + < Ru(t), u(t) >_U\} dt. \tag{4.1.9}$$

REMARK 4.1.2. If, for each N, (A^N, B^N) is stabilizable and (A^N, C^N) is detectable, then, by Theorem 4.1.1, there is a unique optimal control $\bar{u}^N(t)$ for the finite dimensional problem $(\mathcal{R}^N)$ and it is given by

$$\bar{u}^N(t) = -R^{-1}(B^N)^*\Pi^N S_\Pi^N(t) z_0^N, \tag{4.1.10}$$

where $S_\Pi^N(t)$ is the C_0-semigroup on H^N generated by $A^N - B^N R^{-1}(B^N)^*\Pi^N$ and $\Pi^N \in \mathcal{L}(H^N)$ is the unique nonnegative self-adjoint solution of

$$(A^N)^*\Pi^N + \Pi^N A^N - \Pi^N B^N R^{-1}(B^N)^*\Pi^N + (C^N)^*C^N = 0. \tag{4.1.11}$$

For the finite dimensional approximation systems, it is not clear that (A^N, B^N) is stabilizable even if the original system (A, B) is stabilizable. Similarly, it is not clear that the detectability property of (A, C) is preserved under the finite dimensional projections. Another question we have to consider is the convergence of approximates Π^N and $\bar{u}^N(t)$ to the infinite dimensional solutions Π and $\bar{u}(t)$, respectively. For these reasons, we need the following assumptions.

Let $S^N(t) = e^{A^N t}$, $t \geq 0$.

$(A1)^N$: For each $z \in H$ we have

(i) $S^N(t)P^N z \longrightarrow S(t)z$, and

(ii) $S^N(t)^* P^N z \longrightarrow S(t)^* z$,

where the convergences are uniform in t on bounded subsets of $[0, \infty)$.

$(A2)^N$: (i) For each $u \in U$, $B^N u \longrightarrow Bu$ and for each $z \in H$, $(B^N)^* P^N z \longrightarrow B^* z$.

(ii) For each $z \in H$, $C^N P^N z \longrightarrow Cz$ and for each $y \in Y$, $(C^N)^* y \longrightarrow C^* y$.

$(A3)^N$: (i) The family of the pairs (A^N, B^N) is *uniformly stabilizable*, i.e., there exists a sequence of operators $K^N \in \mathcal{L}(H^N, U)$ such that $\sup ||K^N|| < \infty$ and

$$||e^{(A^N + B^N K^N)t} P^N||_{\mathcal{L}(H)} \leq M_1 e^{-\omega_1 t}, \qquad t \geq 0, \tag{4.1.12}$$

for some positive constants $M_1 \geq 1$ and $\omega_1 > 0$ which are independent of N.

(ii) The family of the pairs (A^N, C^N) is *uniformly detectable*, i.e., there exists a sequence of operators $F^N \in \mathcal{L}(Y, H^N)$ such that $\sup ||F^N|| < \infty$ and

$$||e^{(A^N + F^N C^N)t} P^N||_{\mathcal{L}(H)} \leq M_2 e^{-\omega_2 t}, \qquad t \geq 0, \tag{4.1.13}$$

for some constants $M_2 \geq 1$ and $\omega_2 > 0$ which are independent of N.

REMARK 4.1.3. (i) The condition $(A3)^N$(ii) is a relaxation of the coercivity assumption in [4], (see [**23**, p3]).

(ii) Suppose that $B^N = P^N B$ and $C^N = CP^N$. Then $(A2)^N$ holds, since it follows from $(A1)^N$ that $P^N z \to z$ for all $z \in H$.

By simple modification of results from [**23**, Theorem 2.1] and [**4**, Theorem 2.2], we have the following fundamental convergence results.

THEOREM 4.1.4. *Let (A,B) be stabilizable and (A,C) be detectable. Suppose that $(A1)^N-(A3)^N$ are satisfied. Then, for each N, the finite dimensional algebraic Riccati Equation (4.1.11) admits a unique nonnegative self-adjoint solution Π^N such that*

$$sup\{||\Pi^N||_{\mathcal{L}(H^N)} : N=1,2,\cdots\}<\infty \tag{4.1.14}$$

and

$$\Pi^N P^N z \longrightarrow \Pi z \tag{4.1.15}$$

for every $z\in H$. Moreover, there exist positive constants $M_3\geq 1$ and ω_3 (independent of N) such that

$$||e^{(A^N-B^NR^{-1}(B^N)^*\Pi^N)t}P^N||_{\mathcal{L}(H)}\leq M_3e^{-\omega_3 t}, \qquad t\geq 0. \tag{4.1.16}$$

Next, we will apply the previous approximation results to our control problem for the linearized Burgers' equation. Throughout the rest of this section we assume that $R=I$ and $\ell=1$ for convenience. The governing equation for our control problem is given by

$$\begin{aligned}&\frac{\partial}{\partial t}z(t,x)=\epsilon\frac{\partial^2}{\partial x^2}z(t,x)+b(x)u(t), \qquad t>0, \quad 0<x<1\\ &z(t,0)=z(t,1)=0\\ &z(0,x)=z_0(x),\end{aligned} \tag{4.1.17}$$

where $b(\cdot)\in L^2(0,1)$. Let $H=L^2(0,1)$, $Y=\mathbf{R}^k$ and $U=\mathbf{R}$. Let $Az=\epsilon z''$ for

$z \in \mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1)$ and $B \in \mathcal{L}(U,H)$, $C \in \mathcal{L}(H,Y)$ be defined by

$$Bu = b(\cdot)u, \qquad u \in U \tag{4.1.18}$$

$$C(z) = (\tilde{z}(\bar{x}_1), \tilde{z}(\bar{x}_2), \cdots, \tilde{z}(\bar{x}_k)), \qquad z \in H \tag{4.1.19}$$

where $\tilde{z}(\bar{x}_i) = \frac{1}{2\delta}\int_{\bar{x}_i-\delta}^{\bar{x}_i+\delta} z(s)ds$, $\bar{x}_i$'s denote the location of the sensors (see Remark 3.2.1), and $\delta > 0$ is chosen small enough that $(\bar{x}_i - \delta, \bar{x}_i + \delta) \subset (0,1)$ for all i, $1 \leq i \leq k$. Then our linear regulator problem can be stated as

$(\mathcal{R}_\alpha)$: Minimize $J_\alpha(z_0, u_\alpha)$ over $u_\alpha(\cdot) \in L^2(0,\infty;U)$ subject to

$$z_\alpha(t) = S_\alpha(t)z_0 + \int_0^t S_\alpha(t-s)Bu_\alpha(s)ds, \tag{4.1.20}$$

$$y_\alpha(t) = Cz_\alpha(t), \tag{4.1.21}$$

where $\alpha > 0$, $z_\alpha(t) = z(t)e^{\alpha t}$, $y_\alpha(t) = y(t)e^{\alpha t}$, $u_\alpha(t) = u(t)e^{\alpha t}$, $S_\alpha(t)$ is the analytic semigroup generated by $A + \alpha I$, and

$$J_\alpha(z_0, u_\alpha) = \int_0^\infty \{||y_\alpha(t)||_Y^2 + ||u_\alpha(t)||_U^2\}dt. \tag{4.1.22}$$

Assume that $(A+\alpha I, B)$ is stabilizable and that $(A+\alpha I, C)$ is detectable. The necessary and sufficient conditions to stabilizability and detectability can be found in Lemma 3.2.5. Then, from Theorem 4.1.1, the corresponding Riccati equation and the optimal control $\bar{u}_\alpha(t)$ for problem $(\mathcal{R}_\alpha)$ are given by

$$(A+\alpha I)^*\Pi_\alpha + \Pi_\alpha(A+\alpha I) - \Pi_\alpha BB^*\Pi_\alpha + C^*C = 0 \tag{4.1.23}$$

and

$$\bar{u}_\alpha(t) = -B^*\Pi_\alpha \bar{z}_\alpha(t), \tag{4.1.24}$$

where $\bar{z}_\alpha(t)$ is the corresponding closed loop trajectory.

For the uniform stabilizability and detectability assumptions $(A3)^N$ we introduce a sesquilinear form $a(\cdot,\cdot) : V \times V \to \mathbf{C}$ defined by

$$a(z,w) = \int_0^1 \epsilon z'(x)\bar{w}'(x)dx, \qquad z,w \in V \tag{4.1.25}$$

where $V = H_0^1(0,1)$. Note that, to allow the use of the theory of sectorial operators and sesquilinear forms in discussing the spectra of various operators, we assume in defining $a(\cdot,\cdot)$ that the functions in V are complex valued. It is easy to see that the sesquilinear form $a(\cdot,\cdot)$ is V-coercive, ([**42**, p274]), i.e.,

$$|a(z,w)| \le \epsilon \, ||z||_V \, ||w||_V \quad \text{(continuity)} \tag{4.1.26}$$

and

$$\mathcal{R}e\, a(z,z) + \gamma ||z||_H^2 \ge \epsilon \, ||z||_V^2 \quad \text{(Gårding's inequality)} \tag{4.1.27}$$

for all $z, w \in V$ and $\gamma \ge \epsilon > 0$. Furthermore, it follows from the bounds (4.1.26) and (4.1.27) that there exists, in a unique manner, an operator $\tilde{A} \in \mathcal{L}(V, V^*)$ such that

$$a(z,w) = < -\tilde{A}z, w >_{V^*,V} \tag{4.1.28}$$

and

$$\overline{a(z,w)} = < -\tilde{A}^* w, z >_{V^*,V} \tag{4.1.29}$$

for all $z, w \in V$, (see, e.g., [**42**, pp271-275]).

REMARK 4.1.5. (i) Since $Az = \epsilon z''$ for all $z \in \mathcal{D}(A) = H^2 \cap H_0^1$ and $\mathcal{D}(A)$ is dense in V, the operator $\tilde{A}$ is nothing but the unique extension of A from $\mathcal{D}(A)$ to V. For this reason, we will denote $\tilde{A}$ by A. Hence, throughout this section, the operator A will be understood from the context.

(ii) Since $A = A^*$, it is easy to see that $\tilde{A} = \tilde{A}^*$.

REMARK 4.1.6. From Gårding's inequality (4.1.27), we know that for any $\lambda \in \sigma(A)$,

$$\mathcal{R}e\,\lambda \leq -\gamma \leq -\epsilon. \tag{4.1.30}$$

Turning next to approximations for $(\mathcal{R}_\alpha)$, we divide the unit interval [0,1] into $N+1$ equal subintervals to get $[x_i, x_{i+1}]$, $x_i = \frac{i}{N+1}$, $i = 0, 1, \cdots, N$. For each i, $1 \leq i \leq N$, let $h_i^N(x)$ denote the linear spline basis function defined by

$$h_i^N(x) = \begin{cases} (N+1)(x - x_{i-1}), & x_{i-1} \leq x \leq x_i \\ -(N+1)(x - x_{i+1}), & x_i \leq x \leq x_{i+1} \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.31}$$

Let H^N be the N-dimensional finite element space given by

$$H^N = \{ \sum_{i=1}^{N} z_i h_i^N(x) : z_i \in \mathbf{R},\ i = 1, 2, \cdots, N \}. \tag{4.1.32}$$

Then we have a sequence of finite dimensional (real) subspaces $H^N \subset V$, $n = 1, 2, \cdots$. Moreover, it is well-known, ([**36**],[**23**, p15]), that the family of H^N satisfies the following approximation condition:

(APP) : For each $z \in V$, there exists an element $z^N \in H^N$ such that

$$||z - z^N||_V \leq \varepsilon(N), \text{ where } \varepsilon(N) \to \infty.$$

Let $P : H \to H^N$ be the canonical orthogonal projection onto H^N. Then, from the approximation property (APP), it is a trivial matter to see that

$$P^N z \longrightarrow z \qquad \text{as } N \to \infty, \quad \text{for } z \in H. \tag{4.1.33}$$

For the finite demensional regulator problem $(\mathcal{R}_\alpha^N)$ we choose

$$B^N = P^N B \qquad \text{and} \qquad C^N = CP^N. \tag{4.1.34}$$

Then conditions $(A2)^N$(i),(ii) follow from Remark 4.1.3.

To obtain the finite dimensional representation A^N of A, consider the restriction of the sesquilinear form $a(\cdot,\cdot)$ to $H^N \times H^N$. We then have a representation A^N of A satisfying

$$a(z,w) = < -A^N z, w > \tag{4.1.35}$$

and

$$a(z,w) = < -(A^N)^* w, z > \tag{4.1.36}$$

for all $z, w \in H^N$. Equation (4.1.36) follows from the fact that H^N is a real Hilbert space. We know also that $A^N = (A^N)^*$, since $A = A^*$.

REMARK 4.1.7. Since $H^N \subset H$, by equations (4.1.35) and (4.1.36), it is easy to see that for any $\lambda \in \sigma(A^N)$,

$$\mathcal{R}e\,\lambda \leq -\gamma \leq -\epsilon. \tag{4.1.37}$$

Let $S^N(t)$ be the C_0-semigroup generated by A^N. Then the conditions $(A1)^N$(i),(ii) follow from the results of Banks and Kunish, ([**4**, Lemma 3.2]). Note that $S^N(t) = (S^N(t))^*$. Moreover, for each fixed time $t > 0$, the convergence rates follow from the next theorem, ([**15**, Theorem 4.1]).

THEOREM 4.1.8. *We have the following estimates:*

$$(i)\quad ||S(t)z - S^N(t)P^N z||_V \leq \tilde{c}\,(\frac{1}{N+1})(\frac{1}{t})||z||_H, \qquad (t > 0) \tag{4.1.38}$$

$$(ii)\quad ||S(t)z - S^N(t)P^N z||_H \leq \tilde{c}\,(\frac{1}{N+1})^2(\frac{1}{t})||z||_H \qquad (t > 0) \tag{4.1.39}$$

for some positive constant $\tilde{c}$ which is independent of N and t.

For the condition $(A3)^N$(i) we need a certain *preservation of exponential stabilizability under approximation*, (see (POES) in [4]). The following result is taken from [**4**, Lemma 3.3].

THEOREM 4.1.9. *Let (A, B) be (exponentially) stabilizable. Suppose that the approximation condition (APP) holds. Then the approximations defined through equations (4.1.34)-(4.1.36) satisfy the condition $(A3)^N$(i), i.e., the family of pairs (A^N, B^N) is uniformly stabilizable.*

By the dual arguments of Theorem 4.1.9 we can see that the condition $(A3)^N$ (ii) holds under the assumption that (A, C) is detectable. We summarize our discussion up to this point as the following theorem.

THEOREM 4.1.10. *Let $(A+\alpha I, B)$ be stabilizable and $(A+\alpha I, C)$ be detectable. Let A^N, B^N, C^N be defined as in equations (4.1.34) and (4.1.35). Then we have*

$$\Pi_\alpha^N P^N z \longrightarrow \Pi_\alpha z, \qquad z \in H \tag{4.1.40}$$

and

$$S^N(t) P^N z \longrightarrow S(t) z, \qquad z \in H \tag{4.1.41}$$

where the convergence is uniform in t on bounded subsets of $[0, \infty)$, P^N is the orthogonal projection onto H^N, and Π_α^N satisfies

$$(A^N + \alpha I^N)^* \Pi_\alpha^N + \Pi_\alpha^N (A^N + \alpha I^N) - \Pi_\alpha^N B^N (B^N)^* \Pi_\alpha^N + (C^N)^* C^N = 0. \tag{4.1.42}$$

REMARK 4.1.11. Note that $S(t) = \hat{S}(t) e^{-\alpha t}$, $S^N(t) = \hat{S}^N(t) e^{-\alpha t}$, where $S(t)$, $S^N(t)$, $\hat{S}(t)$ and $\hat{S}^N(t)$ are semigroups generated by A, A^N, $A + \alpha I$ and $A^N + \alpha I^N$, respectively.

Next, consider the matrix representations of operators on the space H^N. Let the approximate solution $z^N(t, x)$ of $z(t, x)$ on H^N be given by

$$z^N(t, x) = \sum_{i=1}^{N} z_i^N(t) h_i^N(x) \tag{4.1.43}$$

for some $z_i^N(t) \in \mathbf{R}$, $i = 1, \cdots, N$, where $z(t, x)$ is the open loop solution (with $u \equiv 0$) for equation (4.1.17). Then, from equations (4.1.34) and (4.1.35), we have a finite dimensional ODE system

$$[G^N]\frac{d}{dt}\{z^N(t)\} = [\tilde{A}^N]\{z^N(t)\} + \{\tilde{B}^N\}u(t), \qquad (4.1.44)$$

where $\{z^N(t)\} = [z_1^N(t), \cdots, z_N^N(t)]^T$,

$$[G^N] = [< h_j^N, h_i^N >]_{N\times N}$$
$$= \frac{1}{6(N+1)}\begin{bmatrix} 4 & 1 & 0 & 0 & & \dots & 0 \\ 1 & 4 & 1 & 0 & & \dots & 0 \\ 0 & 1 & 4 & 1 & & \dots & 0 \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & 1 & 4 & 1 \\ 0 & & \dots & & 0 & 1 & 4 \end{bmatrix}_{N\times N}, \qquad (4.1.45)$$

$$[\tilde{A}^N] = \epsilon(N+1)\begin{bmatrix} -2 & 1 & 0 & 0 & & \dots & 0 \\ 1 & -2 & 1 & 0 & & \dots & 0 \\ 0 & 1 & -2 & 1 & & \dots & 0 \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & 1 & -2 & 1 \\ 0 & & \dots & & 0 & 1 & -2 \end{bmatrix}_{N\times N}, \qquad (4.1.46)$$

$$\{\tilde{B}^N\} = [< b, h_1^N >, < b, h_2^N >, \cdots, < b, h_N^N >]^T, \qquad (4.1.47)$$

where $< b, h_j^N >= \int_0^1 b(x)h_j^N(x)dx$, $1 \le j \le N$. Since $[G^N]$ is invertible, by multiplying $[G^N]^{-1}$ to both sides of (4.1.44), we get

$$\frac{d}{dt}\{z^N(t)\} = [A^N]\{z^N(t)\} + \{B^N\}u(t),$$
$$\{z^N(0)\} = \{z_0^N\}, \qquad (4.1.48)$$

where

$$[A^N] = [G^N]^{-1}[\tilde{A}^N], \quad \{B^N\} = [G^N]^{-1}\{\tilde{B}^N\} \qquad (4.1.49)$$

and $\{z_0^N\} = [G^N]^{-1}[< z_0, h_1^N >, \cdots, < z_0, h_N^N >]^T$.

Next, consider a representation C^N of the operator C on H^N. It is easy to see that $C^N : H^N \to \mathbf{R}^k$ is given by

$$[C^N] = [\tilde{h}_j^N(\bar{x}_i)]_{k\times N}, \tag{4.1.50}$$

where $\tilde{h}_j^N(\bar{x}_i) = \frac{1}{2\delta}\int_{\bar{x}_i-\delta}^{\bar{x}_i+\delta} h_j^N(x)dx$, $1 \le i \le k$, $1 \le j \le N$.

Finally, we have a finite dimensional Riccati equation

$$(A^N + \alpha I^N)^*\Pi_\alpha^N + \Pi_\alpha^N(A^N + \alpha I^N) - \Pi_\alpha^N B^N(B^N)^*\Pi_\alpha^N + (C^N)^*C^N = 0 \tag{4.1.51}$$

and the corresponding feedback gain operator K_α^N given by

$$K_\alpha^N = -(B^N)^*\Pi_\alpha^N. \tag{4.1.52}$$

Therefore, the closed loop system (4.1.18) can be represented by

$$\begin{aligned} \frac{d}{dt}\{z^N(t)\} &= (A^N + B^N K_\alpha^N)\{z^N(t)\} \\ \{z^N(0)\} &= \{z_0^N\}. \end{aligned} \tag{4.1.53}$$

REMARK 4.1.12. Consider an unbounded observation operator $\tilde{C} \in \mathcal{L}(H_0^1(0,1);\mathbf{R}^{k+m})$ given by equation (3.2.4) in Section 3.2, i.e.,

$$\tilde{C}z = (\tilde{z}(\bar{x}_1), \cdots, \tilde{z}(\bar{x}_k), \tilde{z}'(\bar{y}_1), \ldots, \tilde{z}'(\bar{y}_m)), \tag{4.1.54}$$

where $\tilde{z}(\bar{x}_i) = \frac{1}{2\delta}\int_{\bar{x}_i-\delta}^{\bar{x}_i+\delta} z(x)dx$, $1 \le i \le k$, $\tilde{z}'(\bar{y}_j) = \frac{1}{2\delta}\int_{\bar{y}_j-\delta}^{\bar{y}_j+\delta} z'(x)dx$, $1 \le j \le m$, are defined by equation (3.2.5). Then the finite dimensional approximation $\tilde{C}^N$ of $\tilde{C}$ on H^N can be represented by

$$\tilde{C}^N = \begin{pmatrix} \tilde{C}_1^N \\ \tilde{C}_2^N \end{pmatrix}, \tag{4.1.55}$$

where $\tilde{C}_1^N : H^N \to \mathbf{R}^k$ and $\tilde{C}_2^N : H^N \to \mathbf{R}^m$ are given by

$$[\tilde{C}_1^N] = [(\tilde{C}_1^N)_{ij}]_{k\times N}, \quad (\tilde{C}_1^N)_{ij} = \tilde{h}_j^N(\bar{x}_i), \quad (1 \le i \le k, 1 \le j \le N) \tag{4.1.56}$$

$$[\tilde{C}_2^N] = [(\tilde{C}_2^N)_{ij}]_{m\times N}, \quad (\tilde{C}_2^N)_{ij} = (N+1)\tilde{\psi}_j^N(\bar{y}_i), \quad (1 \le i \le m, 1 \le j \le N) \tag{4.1.57}$$

where h_j^N, $1 \le j \le N$, is the linear spline function given by equation (4.1.31) and ψ_j^N, $1 \le j \le N$, is defined by

$$\psi_j^N(x) = \begin{cases} 1, & x_{j-1} < x < x_j \\ -1, & x_j < x < x_{j+1} \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.58}$$

Thus $[\tilde{C}^N]$ is a $(k+m)\times N$ matrix, and the corresponding finite dimensional Riccati equation and feedback gain operator K_α^N are given by equations (4.1.51) and (4.1.52) with $C^N = \tilde{C}^N$.

Finally, we discuss an algorithm for finding the unique nonnegative self-adjoint Riccati solution for equation (4.1.51). We employ the Potter's method, (see, e.g., [**35**]), to obtain Π_α^N. The first step in Potter's method is to form $2N \times 2N$ matrix

$$M^N = \begin{bmatrix} (A^N + \alpha I^N)^* & (C^N)^* C^N \\ B^N (B^N)^* & -(A^N + \alpha I^N) \end{bmatrix}. \tag{4.1.59}$$

Next, find all eigenvalues and eigenvectors of M^N and form the matrix

$$Z^N = \begin{bmatrix} Q_1^N \\ Q_2^N \end{bmatrix}, \tag{4.1.60}$$

where the columns of Z^N are the eigenvectors of M^N corresponding to the eigenvalues with positive real part. When eigenvalues occur in complex conjugate pairs, so do the eigenvectors. In this case, the real and imaginary part of the eigenvector each forms a column of Z^N. Finally, the solution to the Riccati equation (4.1.51) is given by the formula $\Pi_\alpha^N = Q_1^N (Q_2^N)^{-1}$.

REMARK 4.1.13. From the numerical results we found that the Riccati solution operators $\Pi_\alpha^N = \Pi_\alpha^N(\epsilon)$ blow up when the viscosity $\epsilon > 0$ goes to 0 for fixed $\alpha > 0$. Also, when α goes to infinity with ϵ fixed the same phenomenon has been observed.

4.2. Feedback Schemes.

From Sections 3.1, 3.2 and 4.1, we know that the optimal control $u(\cdot) \in L^2(0, \infty; U)$ for our linear regulator problem is given by a feedback form

$$u(t) = Kz(t) \tag{4.2.1}$$

and the feedback operator $K \in \mathcal{L}(H, U)$ can be represented by

$$Kz(t) = \int_0^1 k(s) z(t,s) ds \tag{4.2.2}$$

for some $k(\cdot) \in H$, where $H = L^2(0,1)$, $U = \mathbf{R}$, $z(t)(\cdot) = z(t,\cdot)$ is the corresponding optimal trajectory, (see Theorems 3.1.4, 3.2.7 and 4.1.1). Therefore, the infinite dimensional closed loop system for Burgers' equation (3.2.35) is given by

$$\begin{aligned} \frac{d}{dt} z(t) &= (A + BK) z(t) + f(z(t)), \qquad t > 0 \\ z(0) &= z_0 \end{aligned} \tag{4.2.3}$$

on $H = L^2(0,1)$, where $Az = \epsilon z''$ for $z \in \mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1)$, $B \in \mathcal{L}(U,H)$, $Bu = b(\cdot)u$ for $u \in U$, $b(\cdot) \in H$, and $f(z) = -zz'$, $' = \frac{d}{dx}$, and $z_0 \in H$ or H_0^1 depending on the observation operator C.

In this section we introduce three schemes for feedback law; feedback by the optimal L^2-gain function $k(\cdot)$ in equation (4.2.2), the step function averaging $k(\cdot)$ on each interval $I_j = (\bar{y}_{j-1}, \bar{y}_j)$, and by the average value of $k(\cdot)$ in a small neighborhood of each center point $\hat{y}_j$ of I_j, where $\bar{y}_0 = 0$, $\bar{y}_{m+1} = 1$, $\bar{y}_j$'s, $1 \leq j \leq m$, are defined as in equation (3.2.4).

Let the finite element space H^N be defined by equation (4.1.32) and the approximations $z^N(t,x)$, K^N and $k^N(x)$ on H^N for $z(t,x)$, K and $k(x)$, respectively, be given by

$$z^N(t,x) = \sum_{i=1}^{N} z_i^N(t) h_i^N(x), \qquad k^N(x) = \sum_{i=1}^{N} k_i^N h_i^N(x) \tag{4.2.4}$$

and

$$K^N = [\gamma_1^N, \gamma_2^N, \cdots, \gamma_N^N]. \tag{4.2.5}$$

Then the coefficients k_i^N's in equation (4.2.4) are determined by the formula

$$[k_1^N, k_2^N, \cdots, k_N^N]^T = [G^N]^{-1} [\gamma_1^N, \gamma_2^N, \cdots, \gamma_N^N]^T, \tag{4.2.6}$$

where $[G^N]$ is as in equation (4.1.45). Hence, on H^N, the approximate control $u^N(t)$ becomes

$$u^N(t) = K^N z^N(t) = \sum_{i=1}^{N} \gamma_i^N z_i^N(t) \tag{4.2.7}$$

and the finite dimensional system for the closed-loop system (4.2.3) is

$$\frac{d}{dt}\{z^N(t)\} = ([A^N] + \{B^N\}[K^N])\{z^N(t)\} + f^N(\{z^N(t)\}), \qquad t > 0$$
$$\{z^N(0)\} = \{z_0^N\}, \tag{4.2.8}$$

where $\{z^N(t)\} = [z_1^N(t), \cdots, z_N^N(t)]^T$, $[A^N]$, $\{B^N\}$, $\{z_0^N\}$ are defined as in equations (4.1.44) - (4.1.49), $[K^N]$ is as in equation (4.2.6), and

$$f^N(\{z^N(t)\}) = [G^N]^{-1}\tilde{f}^N(\{z^N(t)\}), \tag{4.2.9}$$

$$\tilde{f}^N(\{z^N(t)\}) = \frac{-1}{6}\begin{bmatrix} z_1^N(t)z_2^N(t) + (z_2^N(t))^2 \\ -(z_1^N(t))^2 - (z_1^N(t))(z_2^N(t)) + (z_2^N(t))(z_3^N(t)) + (z_3^N(t))^2 \\ \vdots \\ -(z_{N-2}^N(t))^2 - (z_{N-2}^N(t))(z_{N-1}^N(t)) + (z_{N-1}^N(t))(z_N^N(t)) + (z_N^N(t))^2 \\ -(z_{N-1}^N(t))^2 - (z_{N-1}^N(t))(z_N^N(t)) \end{bmatrix}.$$

To solve the nonlinear ODE system (4.2.8) we use the 4-th order Runge-Kutta method, (see, e.g., [**38**]).

REMARK 4.2.1. For the representation (4.2.5) of K^N, we used the fact that the input operator $B \in \mathcal{L}(U, H)$ is 1-dimensional.

Now, consider the following approximation schemes for various feedback laws.

Scheme 1. First, we consider a feedback form (4.2.2), say K_o, corresponding to the optimal control $u_o(t)$. Then, on H^N, K_o^N is determined by

$$\begin{aligned} K_o^N &= [\gamma_1^N, \gamma_2^N, \cdots, \gamma_N^N]_{1\times N} \\ &= -\{B^N\}^T[\Pi^N] \\ &= -[b_1^N, b_2^N, \cdots, b_N^N][\Pi^N] \end{aligned} \tag{4.2.10}$$

and hence

$$u_o^N(t) = K_o^N z^N(t) = \sum_{i=1}^{N} \gamma_i^N z_i^N(t), \tag{4.2.11}$$

where $b_i^N =< b(\cdot), h_i^N >_H$, $1 \leq i \leq N$, are as in equation (4.1.47) and $[\Pi^N]$ is the Riccati matrix solution for equation (4.1.42). The subscript o indicates the "optimal" feedback.

REMARK 4.2.2. For the control problem with bounded observation operator we use only Scheme 1. Numerical results are presented in Figures 4.3.2-4.3.9. Scheme 2 and Scheme 3 are designed for the problem with unbounded output operators. Of course, those schemes can be applied to the first case with intervals $[\bar{x}_{i-1}, \bar{x}_i]$ and center points $\hat{x}_i$, $1 \leq i \leq k$.

Scheme 2. Consider the following feedback form K_a:

$$K_a z = \int_0^1 (\sum_{j=1}^{m+1} a_j \chi_{I_j}(s)) z(t,s) ds, \tag{4.2.12}$$

where $\bar{y}_0 = 0, \bar{y}_{m+1} = 1$ and for $j = 1, \cdots, m+1$,

$$\begin{aligned} a_j &= \frac{1}{\bar{y}_j - \bar{y}_{j-1}} \int_{I_j} k(s) ds \\ &= \text{the average of } k(x) \text{ over the interval } I_j = (\bar{y}_{j-1}, \bar{y}_j), \end{aligned}$$

$$\chi_{I_j}(x) = \begin{cases} 1, & x \in I_j \\ 0, & \text{otherwise.} \end{cases}$$

and for each j, $1 \leq j \leq m$, $\bar{y}_j$ is the location of sensor measuring $\frac{\partial}{\partial x} z(t,x)$. Then on H^N, the control function $u_{\mathbf{a}}^N(t)$ becomes

$$
\begin{aligned}
u_{\mathbf{a}}^N(t) &= K_{\mathbf{a}}^N z^N(t) \\
&= \int_0^1 (\sum_{j=1}^{m+1} a_j^N \chi_{I_j}(s)) z^N(t,s) ds \\
&= \sum_{i=1}^{N} u_i^N z_i^N(t),
\end{aligned}
\tag{4.2.13}
$$

where

$$
u_i^N = \sum_{j=1}^{m+1} a_j^N \int_{I_j} h_i^N(x) dx \quad , \quad a_j^N = \frac{1}{\bar{y}_j - \bar{y}_{j-1}} \sum_{n=1}^{N} k_n^N (\int_{I_j} h_n^N(x) dx). \tag{4.2.14}
$$

Thus the feedback operator $K_{\mathbf{a}}^N$ is represented by

$$
K_{\mathbf{a}}^N = [u_1^N, u_2^N, \cdots, u_N^N]. \tag{4.2.15}
$$

Scheme 3. Consider the feedback form

$$
K_{\mathbf{ap}} z = \sum_{j=1}^{m+1} a_j \tilde{z}(t, \hat{y}_j), \tag{4.2.16}
$$

where a_j's are as in Scheme 2 and for each j, $1 \leq j \leq m$, $\hat{y}_j$ is the center point of the interval $I_j = (\bar{y}_{j-1}, \bar{y}_j)$ and $\tilde{z}(t, \hat{y}_j) = \frac{1}{2\delta} \int_{\hat{y}_j - \delta}^{\hat{y}_j + \delta} z(t,s) ds$, $0 < \delta << 1$. It follows that on H^N,

$$
\begin{aligned}
\tilde{z}^N(t, \hat{y}_j) &= \frac{1}{2\delta} \int_{\hat{y}_j - \delta}^{\hat{y}_j + \delta} (\sum_{i=1}^{N} z_i^N(t) h_i^N(s)) ds \\
&= \sum_{i=1}^{N} z_i^N(t) (\frac{1}{2\delta} \int_{\hat{y}_j - \delta}^{\hat{y}_j + \delta} h_i^N(s) ds),
\end{aligned}
\tag{4.2.17}
$$

and

$$
\begin{aligned}
u_{\mathrm{ap}}^N(t) &= K_{\mathrm{ap}}^N z^N(t) \\
&= \sum_{j=1}^{m+1} a_j^N \tilde{z}(t, \hat{y}_j) \\
&= \sum_{i=1}^{N} z_i^N(t) \sum_{j=1}^{m+1} a_j^N \Big(\frac{1}{2\delta} \int_{\hat{y}_j-\delta}^{\hat{y}_j+\delta} h_i^N(s)ds\Big) \\
&= \sum_{i=1}^{N} v_i^N z_i^N(t), \qquad v_i^N = \sum_{j=1}^{m+1} a_j^N \Big(\frac{1}{2\delta} \int_{\hat{y}_j-\delta}^{\hat{y}_j+\delta} h_i^N(s)ds\Big).
\end{aligned} \tag{4.2.18}
$$

Hence, the feedback operator K_{ap}^N is given by

$$
K_{\mathrm{ap}}^N = [v_1^N, v_2^N, \cdots, v_N^N]. \tag{4.2.19}
$$

4.3. Numerical Results.

In this section, we discuss how our results work for relaxation of "steep" gradient of the solution for Burgers' equation, with Dirichlet boundary condition, through numerical experiments. The computer codes described in this section were implemented on a VAX 8800.

For Examples 4.3.1-4.3.3, the length ℓ for space domain, Reynolds nummber Re, initial function $z_0(\cdot) \in H_0^1(0,1)$ and control input function $b(\cdot) \in L^2(0,1)$ will be chosen as 1, 60, $\sin \pi x$ and e^x, respectively. Thus the governing equation is given by

$$\frac{\partial}{\partial t}z(t,x) = \frac{1}{60}\frac{\partial^2}{\partial x^2}z(t,x) - z(t,x)\frac{\partial}{\partial x}z(t,x) + e^x \int_0^1 k_\alpha(s)z(t,s)\,ds$$

$$z(t,0) = z(t,1) = 0 \tag{4.3.1}$$

$$z(0,x) = \sin \pi x,$$

where the feedback gain function $k_\alpha(\cdot) \in L^2(0,1)$ will be determined by the desired degree $\alpha > 0$ of stability and the action of output operator C.

The robustness of the feedback controller exhibited, for example, in Figure 4.3.2 will be discussed in Example 4.3.4. For this particular example, Reynolds numbers 60, 80, 100 and 120 are chosen.

REMARK 4.3.1. (i) From the numerical experiments, we found that if Reynolds number Re is less than 60, then the diffusion phenomena dominate convection phenomena. In this case, the formation of steep gradient due to convection term $-z(t,x)\frac{\partial}{\partial x}z(t,x)$ of the open-loop solution, i.e., $k_\alpha(\cdot) \equiv 0$ in equation (4.3.1), is not clear. But, for Reynolds number greater than 60, the open-loop solution creates "sharp" gradient in finite time (see Figure 4.3.1). Of course, the solution dies out eventually, because of the diffusion term $\frac{1}{\mathrm{Re}}\frac{\partial^2}{\partial x^2}z(t,x)$.

(ii) The control input function $b(x) = e^x$ is defined for all $x \in [0,1]$. Thus, the feedback control acts on the whole domain $[0,1]$. But, one can choose any L^2-function $b(\cdot) \in L^2(0,1)$ satisfying the stabilizability condition in Lemma 3.2.5. In fact, $b(x) = e^x$ satisfies the stabilizability condition for any desired degree of stability

$\alpha > 0$, since the coefficients b_n, $n = 1, 2, \ldots$, representing the input function $b(\cdot)$ are not zero, i.e., $b_n = < b(\cdot), \sin \pi x >_{L^2(0,1)} = \int_0^1 e^x \sin \pi x \, dx \neq 0$ for all $n = 1, 2, \ldots$ (see Lemma 3.2.5).

(iii) The initial function $z_0(x) = \sin \pi x$ is chosen for our numerical experiments. Other typical H_0^1-functions such as the "hat function" defined by

$$z_0(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}] \\ -2x + 2, & x \in [\frac{1}{2}, 1] \end{cases} \tag{4.3.2}$$

can be used for initial data. But, we found that the solution of Burgers' equation (4.3.1) with initial data $z_0(x)$ replaced by the hat function has almost similar phenomena, such as the creation or relaxation of steep gradient, as those of solution with initial data $z_0(x) = \sin \pi x$.

To show trajectories of open-loop and closed-loop solutions, the order N of approximation is chosen as $N = 32$ for both cases. And the corresponding trajectories from time $t = 0.0$ to $t = 1.0$ are shown. The convergence of the feedback L^2-gain functions $k_\alpha(\cdot) \in L^2(0,1)$ are shown for $N = 8, 16, 32, 64$ and 128.

EXAMPLE 4.3.1. (Bounded Observation)

The observation operator $C \in \mathcal{L}(L^2(0,1), \mathbf{R}^3)$ for this example is given by

$$C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75)), \tag{4.3.3}$$

where $\tilde{z}(\bar{x})$ is the average value of $z(\cdot) \in L^2(0,1)$ in a small neighborhood of $\bar{x}$, $\bar{x} =$

0.3, 0.5, 0.75, and defined by equation (3.2.5)

$$\tilde{z}(\bar{x}) = \frac{1}{2\delta} \int_{\bar{x}-\delta}^{\bar{x}+\delta} z(s)\, ds. \tag{4.3.4}$$

Here, $\delta > 0$ is chosen so small that each open interval $(\bar{x} - \delta, \bar{x} + \delta)$ is contained in the whole domain $(0, 1)$. The desired degree α of stability are chosen 0.3 and 0.6 for Figures 4.3.3 and 4.3.5, respectively. For both cases, $n_\alpha = \max\{ n \in \mathbf{N} : \alpha - \frac{1}{60}n^2\pi^2 \geq 0 \} = 1$ and hence the set $X_{\alpha,1}$ defined in Lemma 3.2.5 is empty. Thus, all assumptions in Theorem 3.2.7 are satisfied.

The feedback L^2-gain functions $k_\alpha(\cdot)$ are given in Figures 4.3.2 and 4.3.4. From these plots, it is easy to see that control action is concentrated on the location of sensors. This phenomenon is natural, since the optimal control is obtained to minimize the cost functional J defined by equation (3.2.9) whose first term $||\hat{y}(t)||_Y^2 = ||C\hat{z}(t)||_{\mathbf{R}^3}^2 = \sum_{i=1}^3 |\tilde{\hat{z}}(\bar{x}_i)|^2$, where $\bar{x}_i = 0.3$, 0.5 and 0.75 for $i = 1, 2$ and 3, respectively.

The corresponding closed-loop trajectories are shown in Figure 4.3.3 (for $\alpha = 0.3$) and Figure 4.3.5 (for $\alpha = 0.6$). From Figures 4.3.3 and 4.3.5, we can see how the controllers contribute to stabilization of the steep gradient as well as the solution itself. Scheme 1 described in Section 4.2 was used for both cases.

EXAMPLE 4.3.2. (Identity Output Operator)

For this example, we take the identity operator I on $L^2(0, 1)$ for output operator C. In this case, the output space Y is $L^2(0, 1)$. The convergence of gain functions and corresponding closed-loop trajectories for $\alpha = 0.3$ and 0.6 are shown in Figures 4.3.6-4.3.7 and 4.3.8-4.3.9, respectively. Since the observation operator is the identity, this example gives the information about maximal control action. We note the following observation concerning the convergence rate of the gain function. Theoretically, the rate is $O(\frac{1}{N})$, ([**23**, p15]). But, in this example, the rate seems to be faster than $O(\frac{1}{N})$ (see Figures 4.3.2 and 4.3.4). Another observation is about the location of maximal control action. The location moves to the left portion of domain as the degree of stability $\alpha > 0$ increases. In other words, we should put more action on the front part of domain to get a higher exponential decay rate $\alpha > 0$. (See also Figures 4.3.2 and 4.3.4).

EXAMPLE 4.3.3. (Unbounded Observation)

Although we haven't provided proof for the convergence of our approximation schemes for the control problem with unbounded output operators, we give the numerical evidence for the convergence of the feedback gain functions through examples. The convergence of the approximation scheme for the control problem with unbounded input/output operators is an open problem.

Figures 4.3.10-4.3.17 are concerned with the control problems with unbounded observation operator $C \in \mathcal{L}(H_0^1(0,1), \mathbf{R}^6)$. Figures 4.3.10-4.3.12 show how control action is related to the location of sensors. For these three cases, the number 0.3 is chosen for the desired degree of stability $\alpha > 0$. Thus, $n_\alpha = 1$, $X_{\alpha,1}$ is an empty set and $Y_{\alpha,1} = \{\frac{1}{2}\}$, (see Lemma 3.2.5). The output operators C for Figures 4.3.10, 4.3.11 and 4.3.12 are given by

$$C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75), \tilde{z}'(0.2), \tilde{z}'(0.4), \tilde{z}'(0.6)), \tag{4.3.5}$$

$$C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75), \tilde{z}'(0.4), \tilde{z}'(0.6), \tilde{z}'(0.88)) \tag{4.3.6}$$

and

$$C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75), \tilde{z}'(0.3), \tilde{z}'(0.5), \tilde{z}'(0.75)), \tag{4.3.7}$$

respectively, where the $\tilde{z}(\bar{x}_i)$'s are defined by equation (4.3.4) and $\tilde{z}'(\bar{y}_j)$, $j = 1, 2, 3$, is defined by

$$\tilde{z}'(\bar{y}_j) = \frac{1}{2\delta} \int_{\bar{y}_j - \delta}^{\bar{y}_j + \delta} \frac{d}{ds} z(s)\, ds, \qquad 0 < \delta << 1. \tag{4.3.8}$$

From these examples, it is easy to observe that the control action depends strongly on the location of sensors measuring the gradient of the solution instead of the solution itself. Similar arguments in Example 4.3.1 can be used to explain this phenomena. But, at this time, the first term in the performance index J consists of two components:

$$\| Cz(t) \|_Y^2 = \sum_{i=1}^{3} |\tilde{z}(\bar{x}_i)|^2 + \sum_{i=1}^{3} |\tilde{z}'(\bar{y}_j)|^2, \tag{4.3.9}$$

where $Y = \mathbf{R}^6$. From equation (3.2.22) in Section 3.2, it is easy to see that the second term $\sum_{i=1}^{3} |\tilde{z}'(\bar{y}_j)|^2$ dominates the first term.

To see the control effects on the closed-loop solution (Figure 4.3.13), the gain function in Figure 4.3.12 was chosen. For this case, we used Scheme 1 described in Section 3.2.

Figures 4.3.14, 4.3.15 show the average of the gain function in Figure 4.3.12 and the corresponding closed-loop solution (Scheme 2). The average values were estimated on each interval $(\bar{y}_{j-1}, \bar{y}_j)$, where $\bar{y}_j = 0$, 0.3, 0.5, 0.75, 1.0 for $j = 0, 1, 2, 3, 4$, respectively. Scheme 3 was applied to obtain Figure 4.3.16. From Figures 4.3.13, 4.3.15 and 4.3.16, one can observe that Schemes 2 and 3 work very well compared with Scheme 1 for relaxation of the steep gradient of the solution. From a practical point of view, Scheme 3 is the most "efficient" in the sense that we need only data from sensors to calculate the control function $u(t)$, yet produce similar effects (see Section 4.2).

EXAMPLE 4.3.1. (Robustness)

In this example we show the robustness of the feedback controller showed in Figure 4.3.2. The feedback controller is obtained from the control system (4.1.17) with Re=60, $\alpha = 0.3$, $b(x) = e^x$ and the ouput operator C defined by equation (4.3.3).

Figures 4.3.1, 4.3.17, 4.3.19 and 4.3.21 show open-loop trajectories for Reynolds numbers 60, 80, 100 and 120, respectively. The corresponding closed-loop trajectories are shown in Figures 4.3.3, 4.3.18, 4.3.20 and 4.3.22, respectively. The order N of approximation is chosen as $N = 32$ for Re=60, 80, 100 and $N = 64$ for Re=120. From these examples, it is easy to see that the feedback controller obtained for Re=60 stabilizes the steep gradient of the solution for Burgers' equation with various Reynolds numbers. However, we see that the sharp gradient is relaxed slowly as Reynolds number increases (see Figures 4.3.3, 4.3.18, 4.3.20 and 4.3.22). Note that, from Theorem 3.2.10, the closed-loop nonlinear system $v(t)$ given by equation (3.2.35) satisfies the stability property

$$||\, v(t)\, ||_{H_0^1} \leq 2Me^{-\alpha t}\, ||v_0(\cdot)||_{H_0^1}\,. \tag{4.3.10}$$

Although the exponential decay rate α is independent of Reynolds number, the constant $M = M(\alpha, \mathrm{Re})$ depends strongly on the Reynolds number, Re.

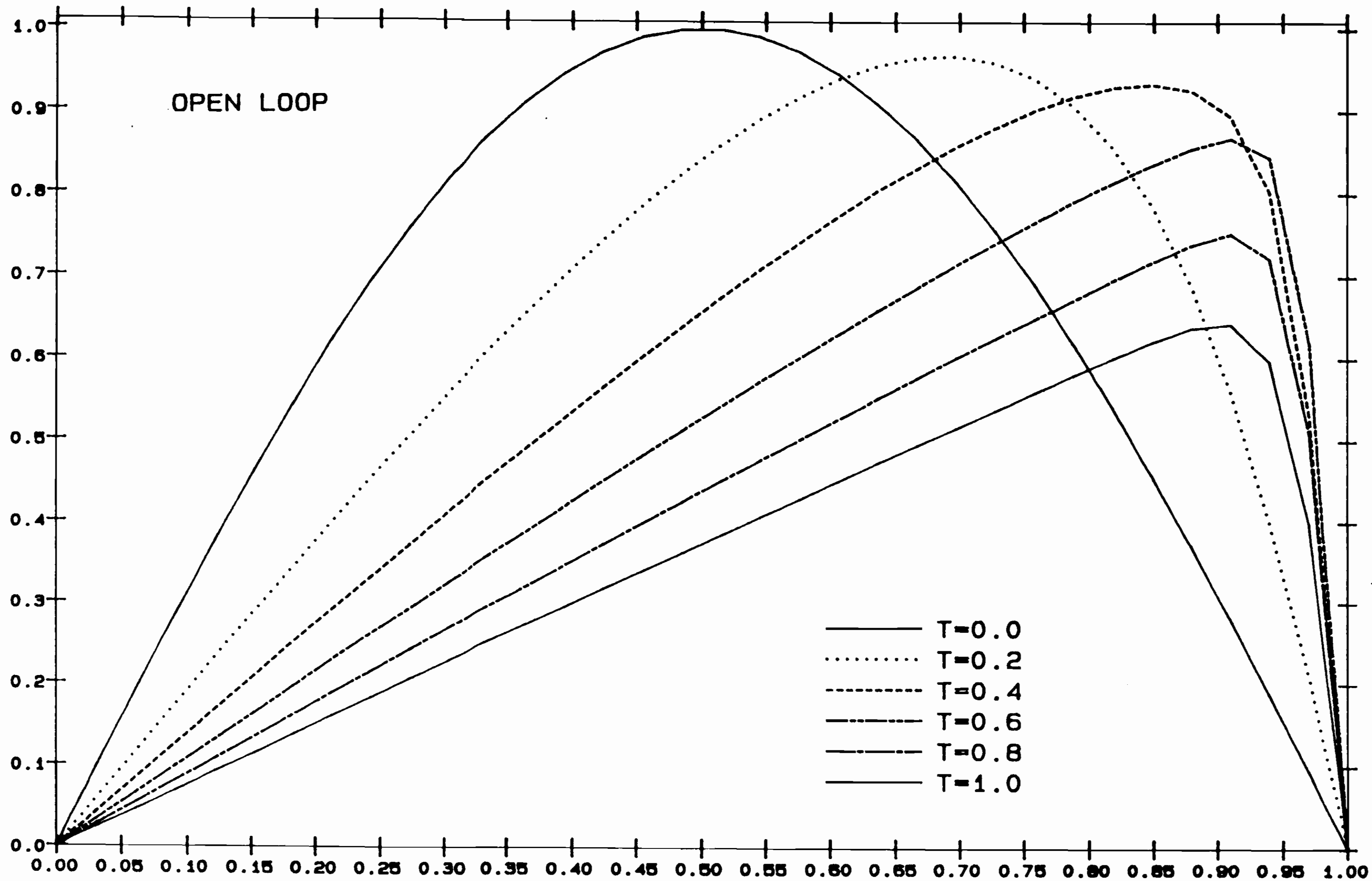

Figure 4.3.1. Open Loop (Re=60, $z_0(x) = \sin \pi x$)

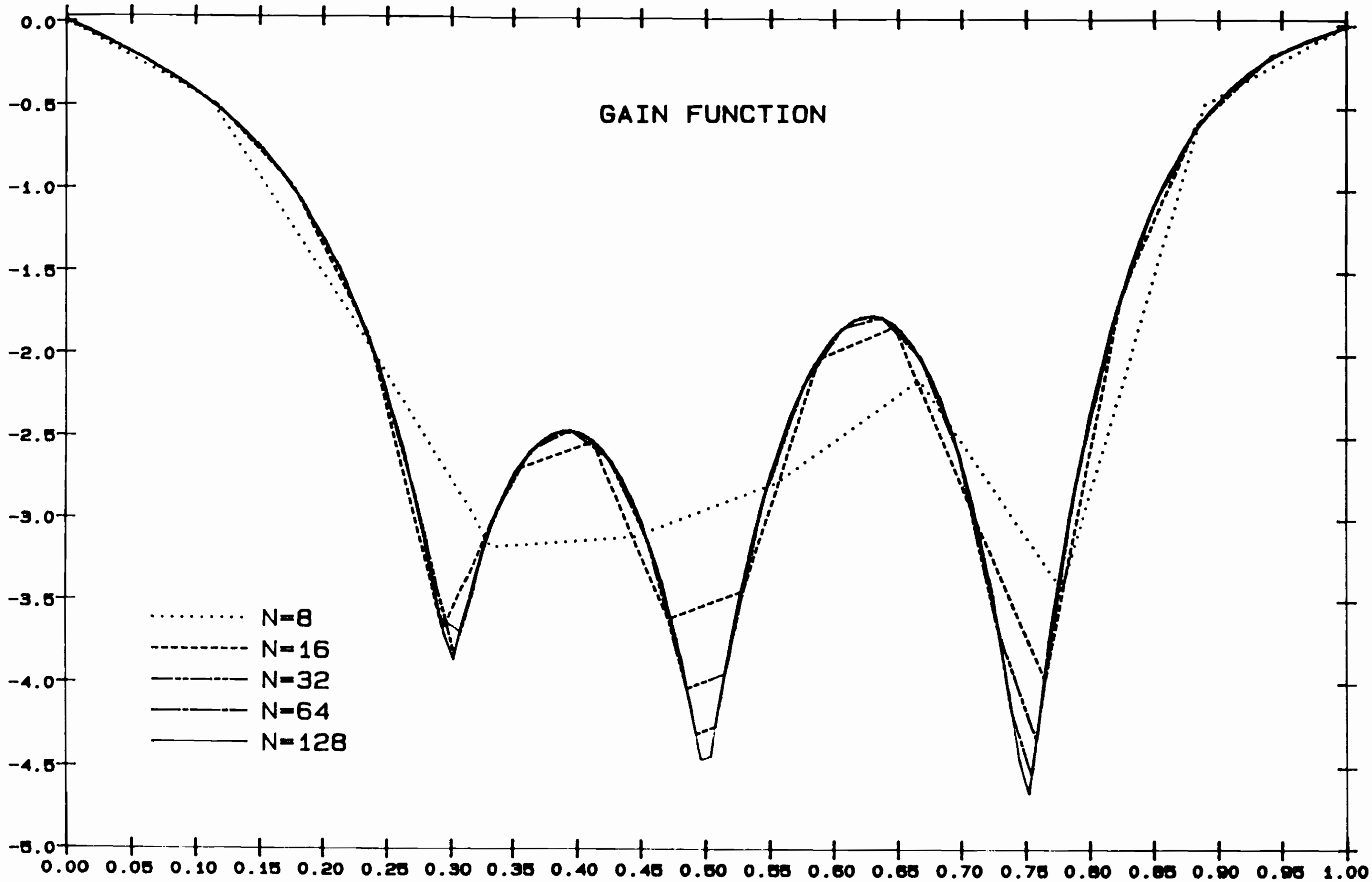

Figure 4.3.2. Gain Function $k_{\alpha,1}(\cdot)$ ($\alpha = 0.3$, $C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75))$)

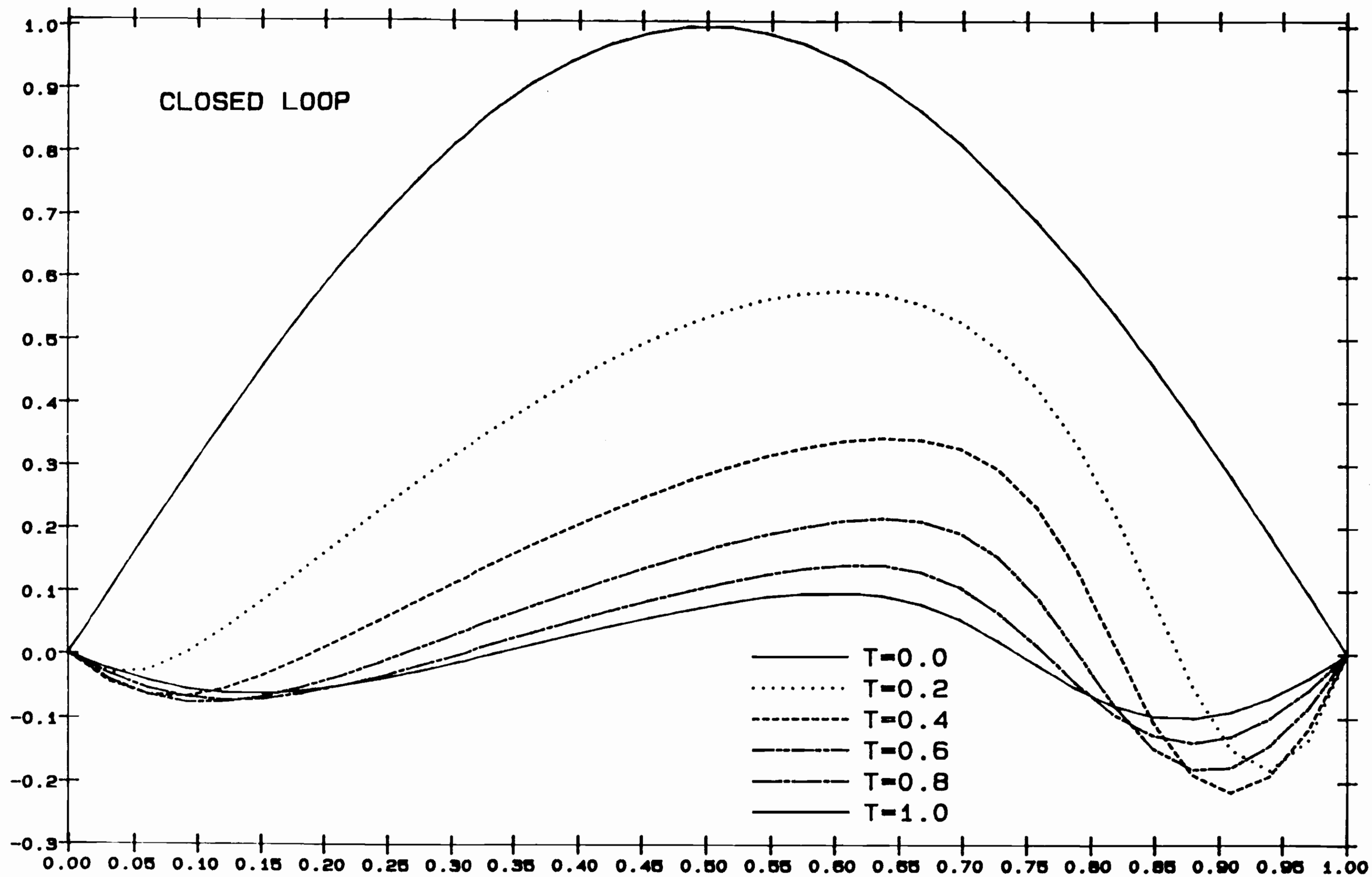

Figure 4.3.3. Closed Loop (feedback by $k_{\alpha,1}(\cdot)$)

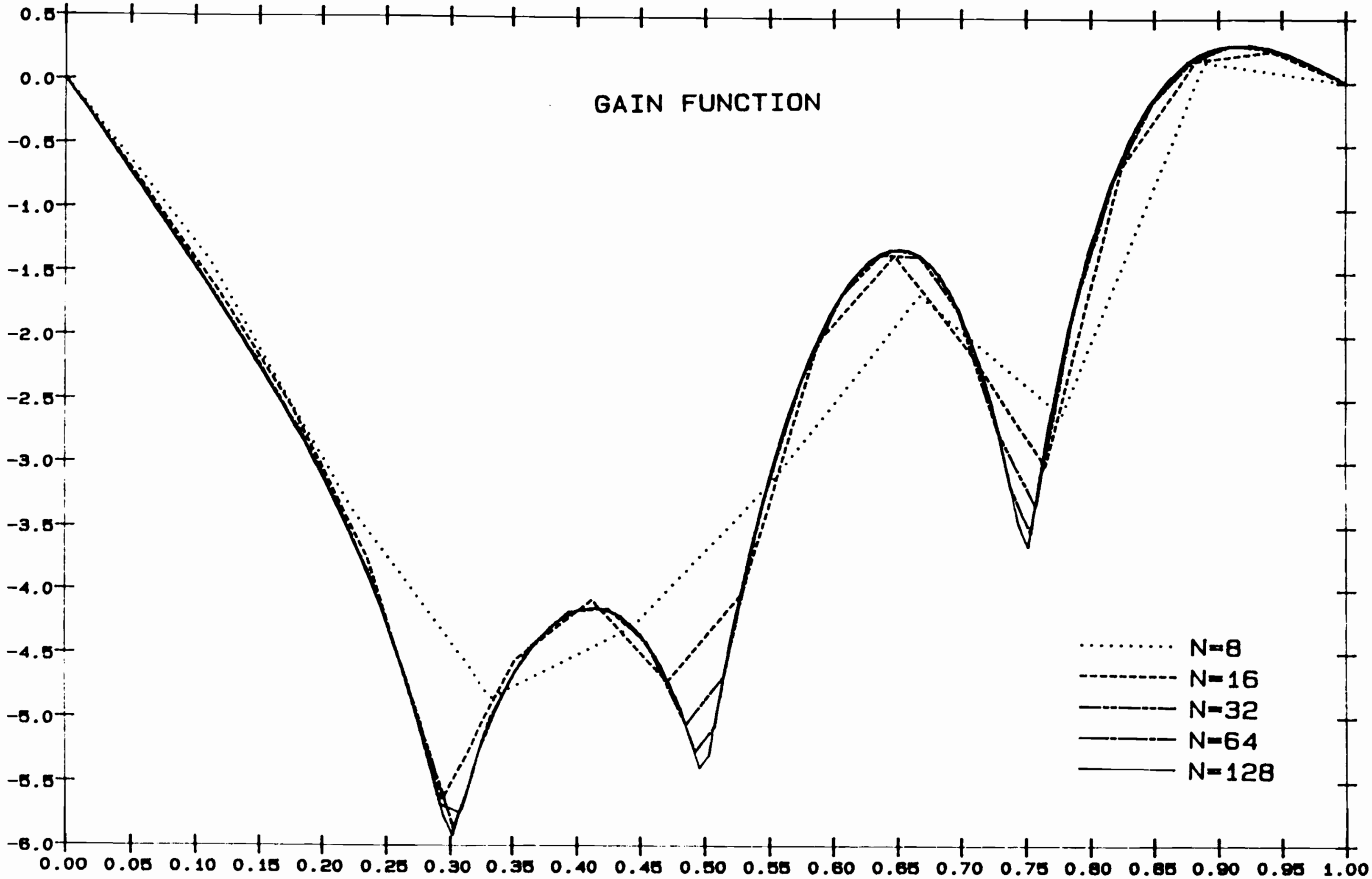

Figure 4.3.4. Gain Function $k_{\alpha,2}(\cdot)$ ($\alpha = 0.6$, $C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75))$)

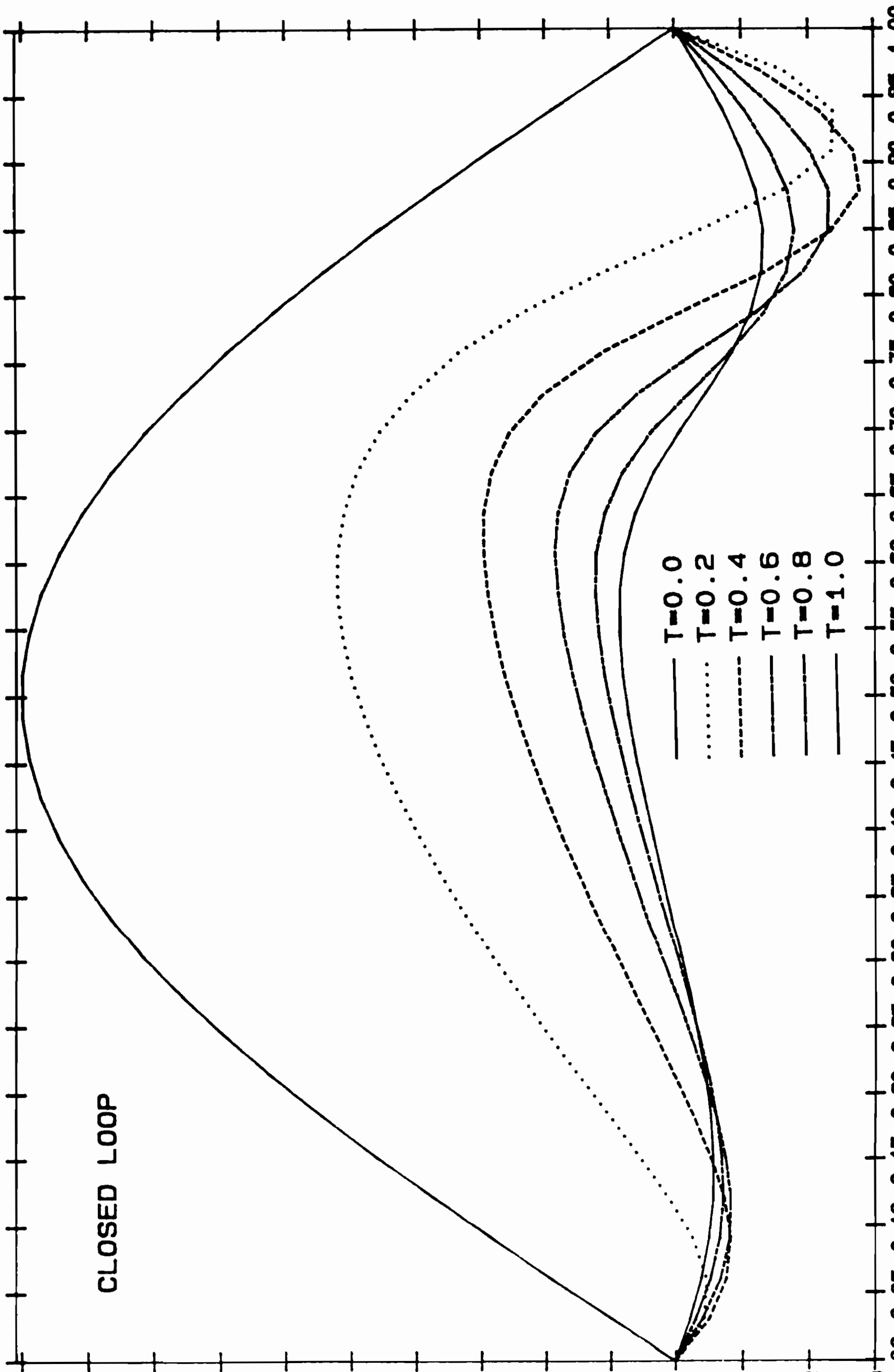

Figure 4.3.5. Closed Loop (feedback by $k_{\alpha,2}(\cdot)$)

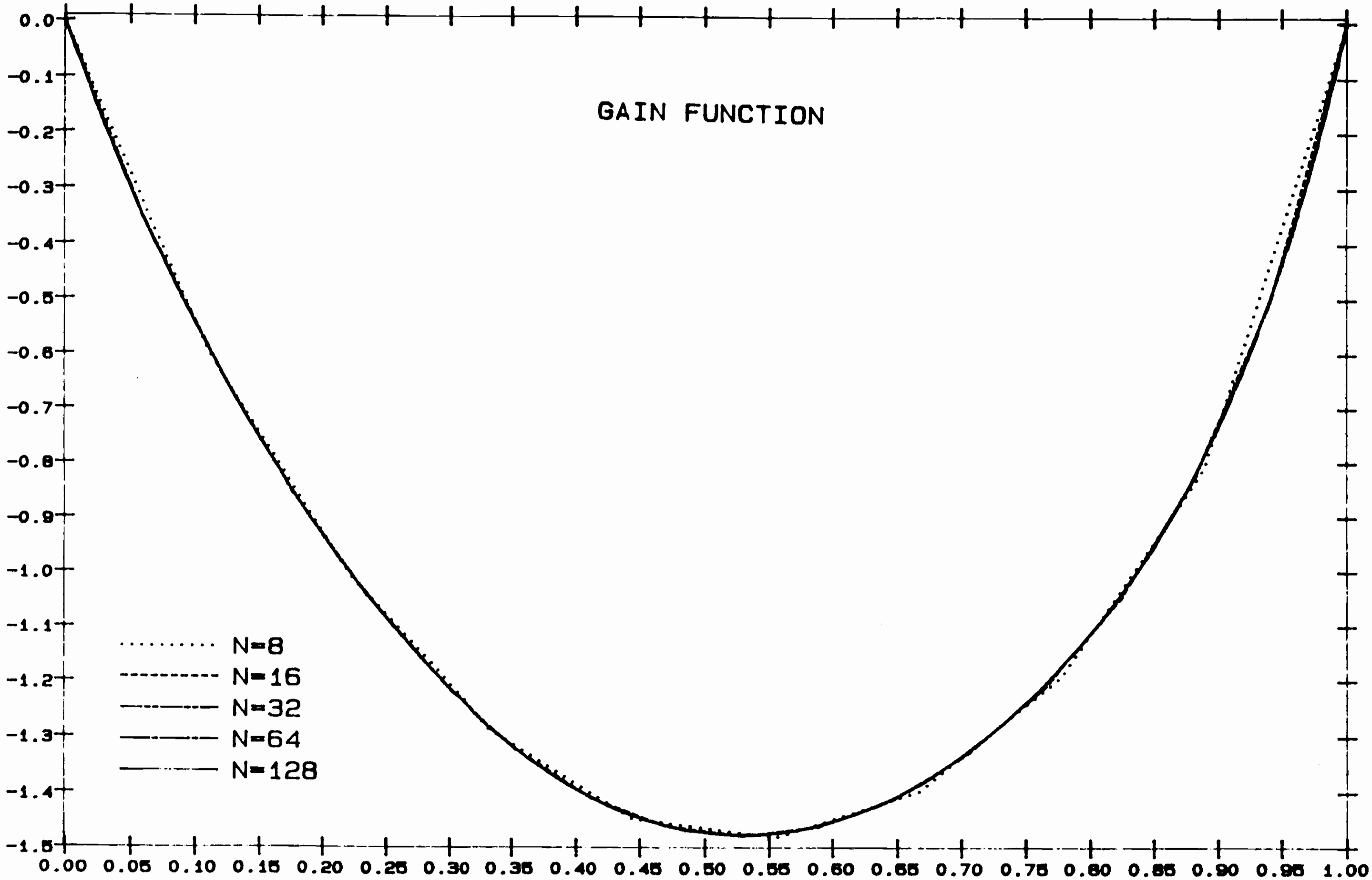

Figure 4.3.6. Gain Function $k_{\alpha,3}(\cdot)$ ($\alpha = 0.3$, $C = I$)

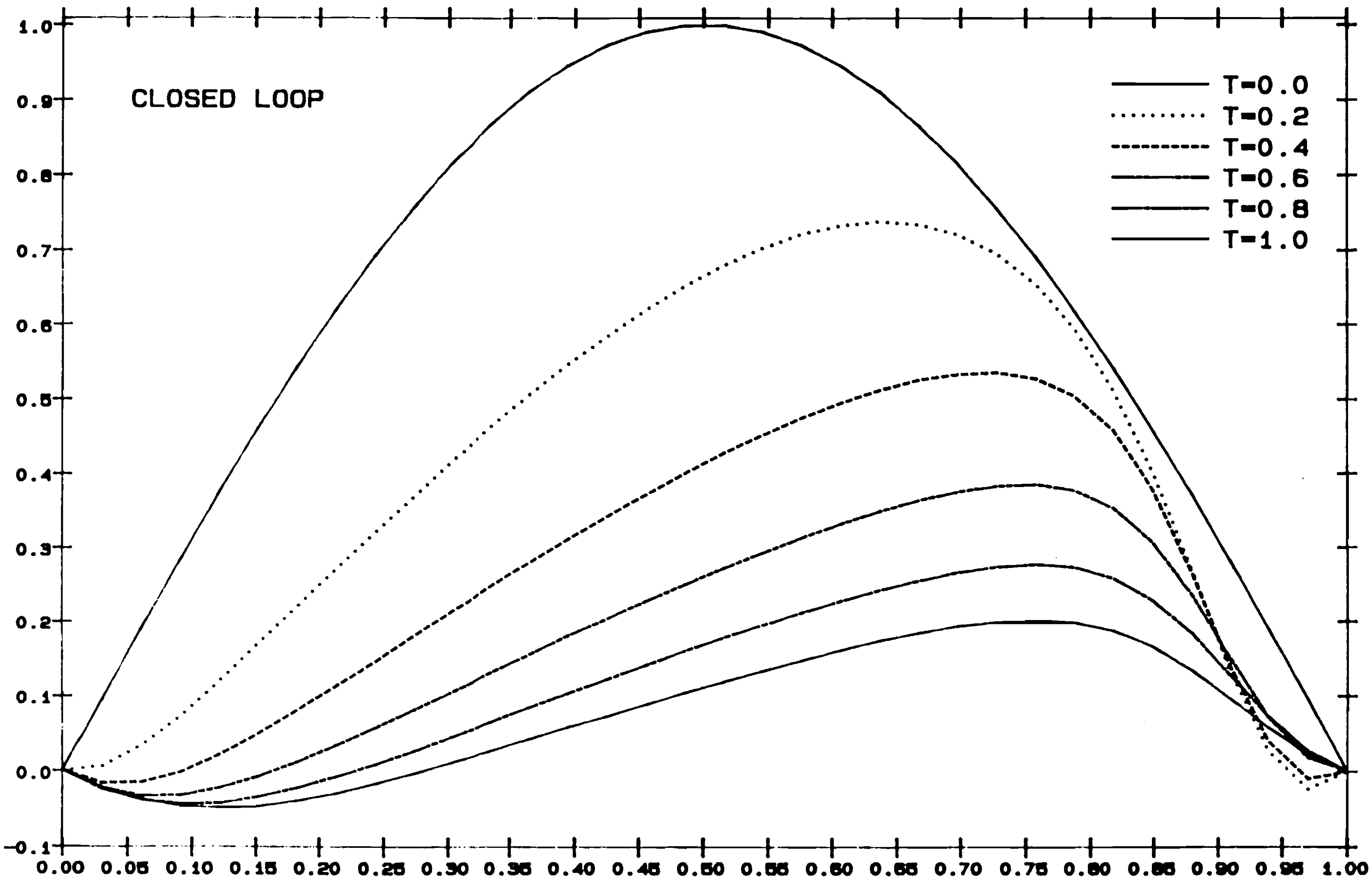

Figure 4.3.7. Closed Loop (feedback by $k_{\alpha,3}(\cdot)$)

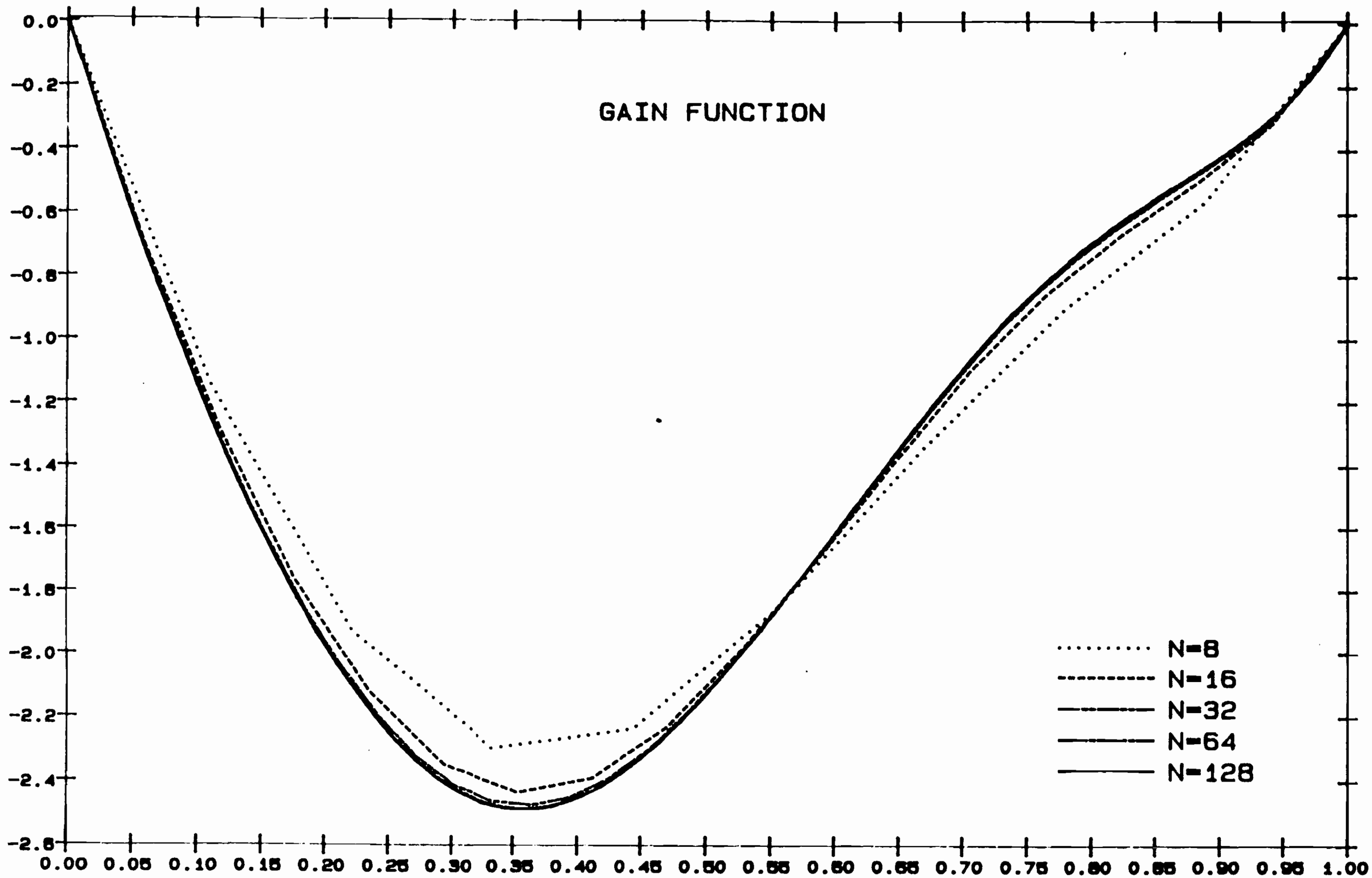

Figure 4.3.8. Gain Function $k_{\alpha,4}(\cdot)\,(\alpha = 0.6,\ C = I)$

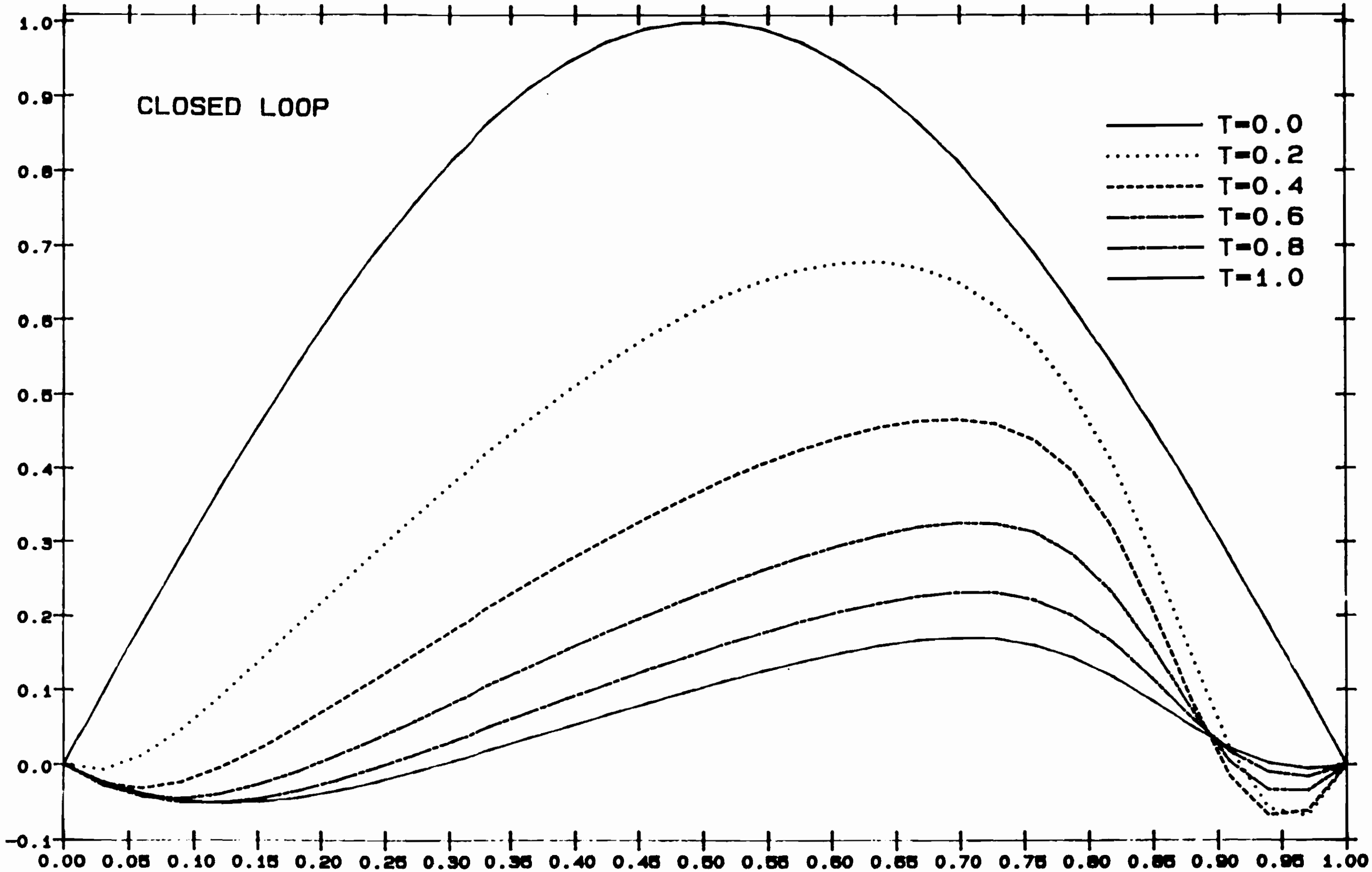

Figure 4.3.9. Closed Loop (feedback by $k_{\alpha,4}(\cdot)$)

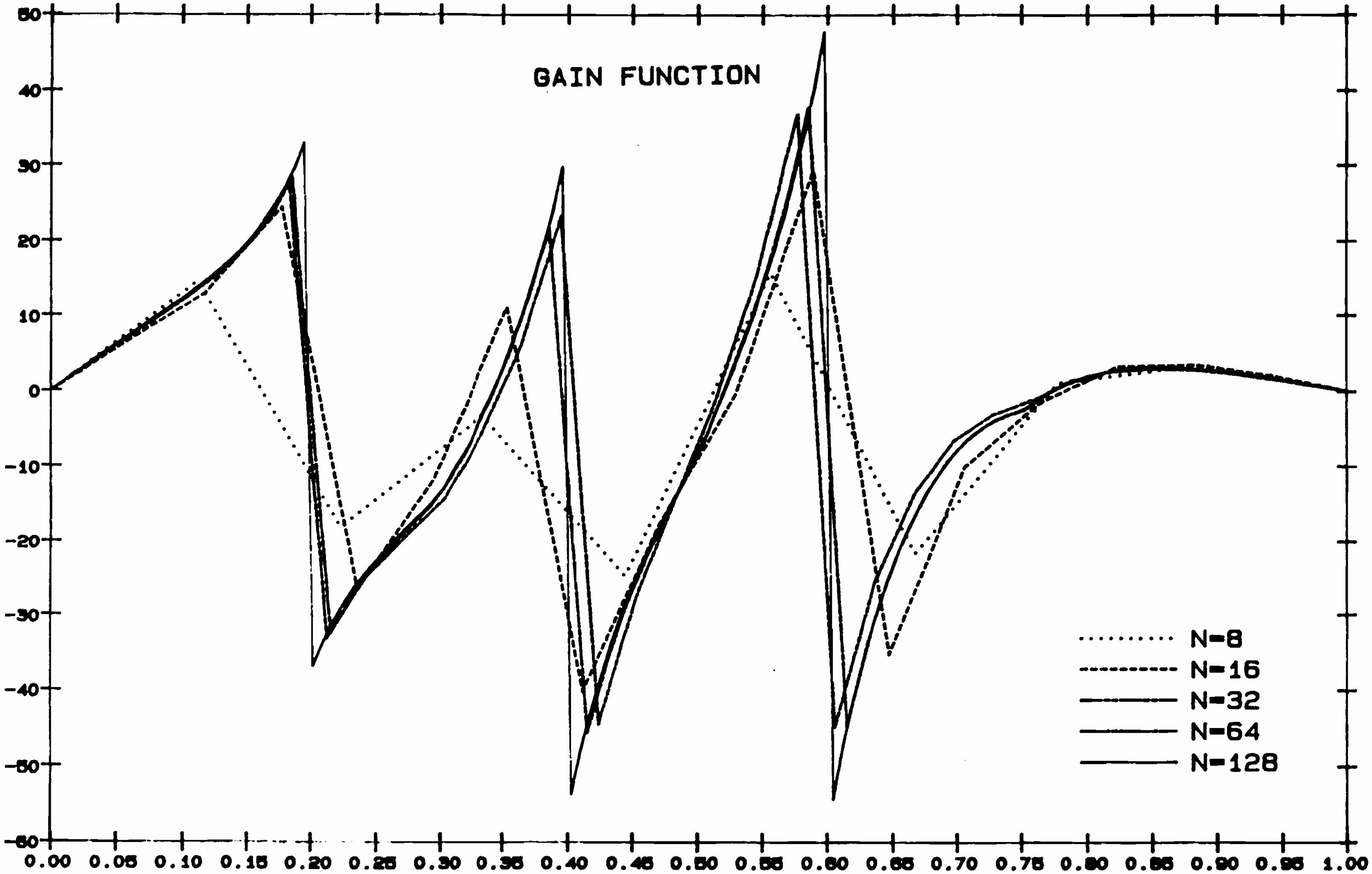

Figure 4.3.10. Gain Function $k_{\alpha,5}(\cdot)$

$(\alpha = 0.3,\ C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75), \tilde{z}'(0.2), \tilde{z}'(0.4), \tilde{z}'(0.6))\ )$

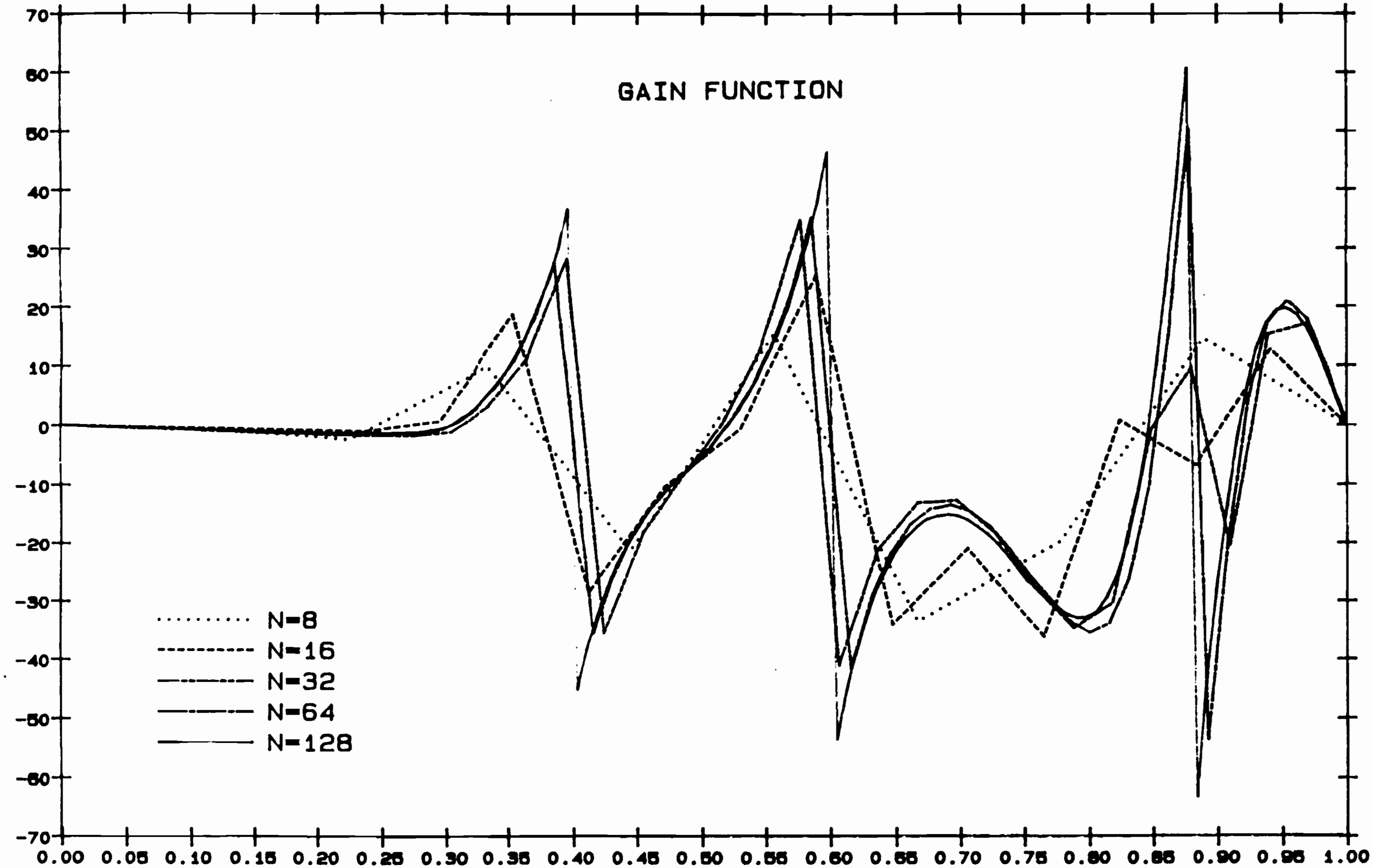

Figure 4.3.11. Gain Function $k_{\alpha,6}(\cdot)$

$(\alpha = 0.3,\ C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75), \tilde{z}'(0.4), \tilde{z}'(0.6), \tilde{z}'(0.88))\ )$

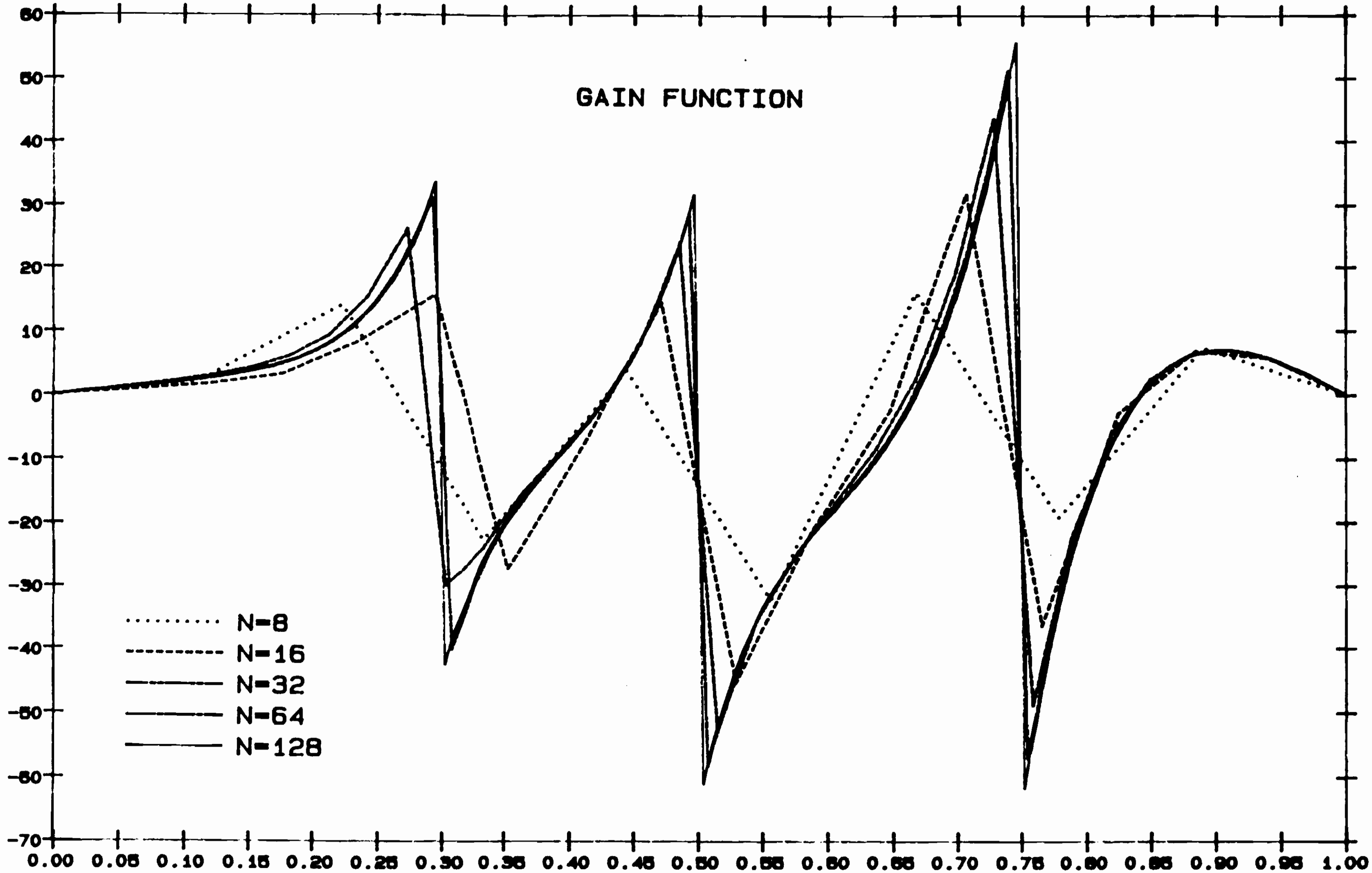

Figure 4.3.12. Gain Function $k_{\alpha,7}(\cdot)$

$(\alpha = 0.3,\ C(z) = (\tilde{z}(0.3), \tilde{z}(0.5), \tilde{z}(0.75), \tilde{z}'(0.3), \tilde{z}'(0.5), \tilde{z}'(0.75))\ )$

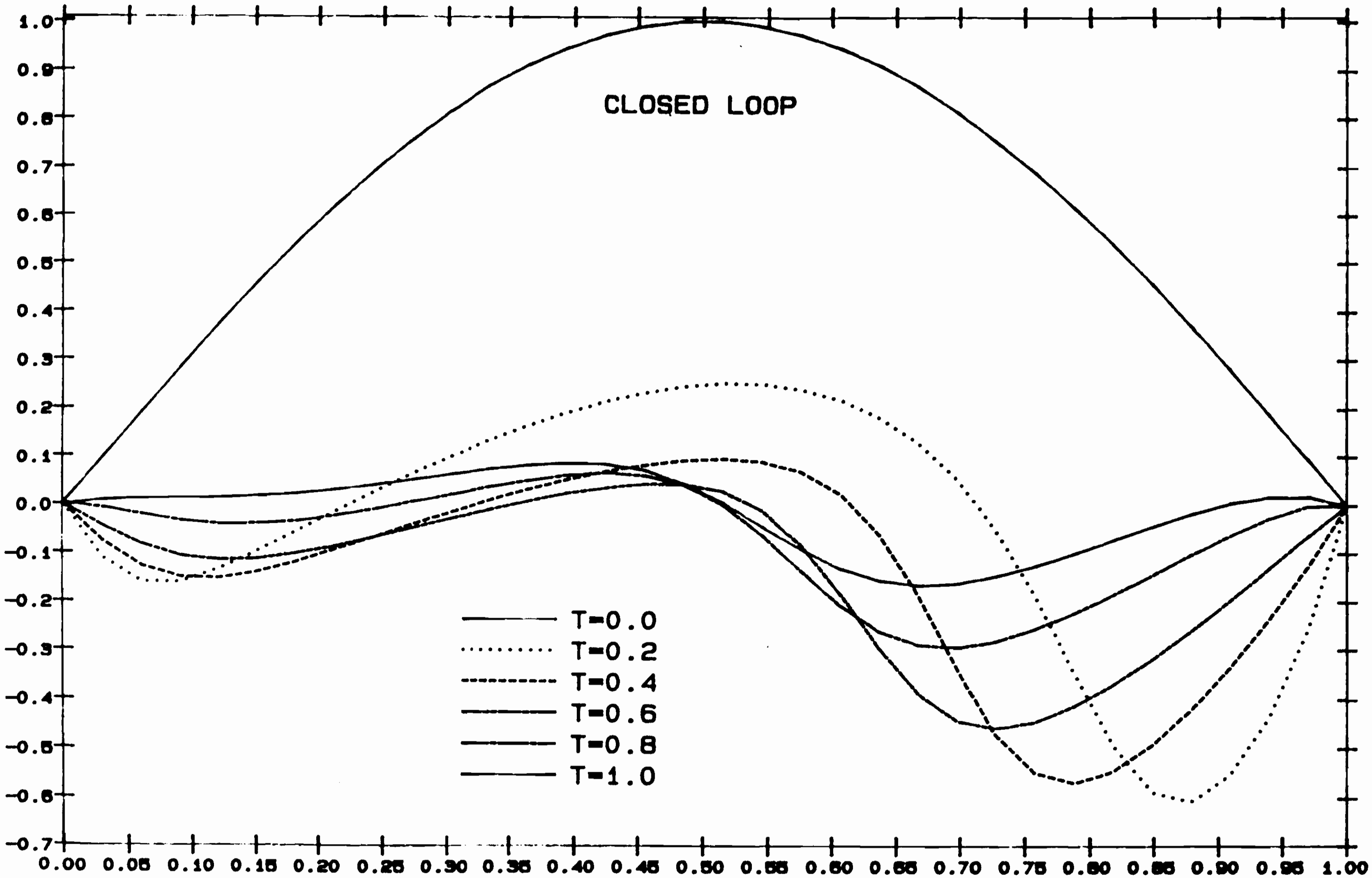

Figure 4.3.13. Closed Loop (feedback by $k_{\alpha,7}(\cdot)$)

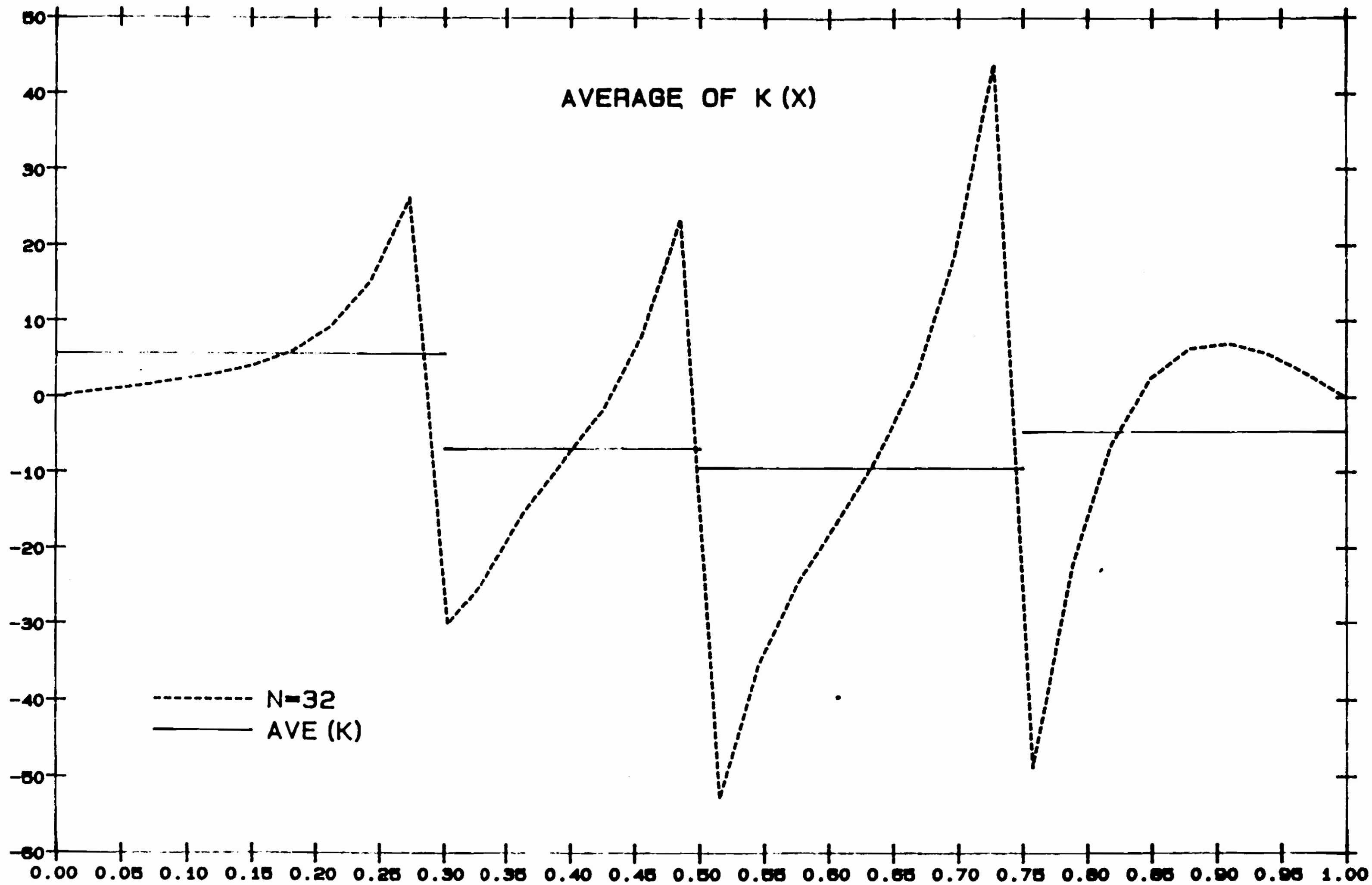

Figure 4.3.14. Average Step Function $\tilde{k}_{\alpha,7}(\cdot)$ of $k_{\alpha,7}(\cdot)$ (Scheme 2)

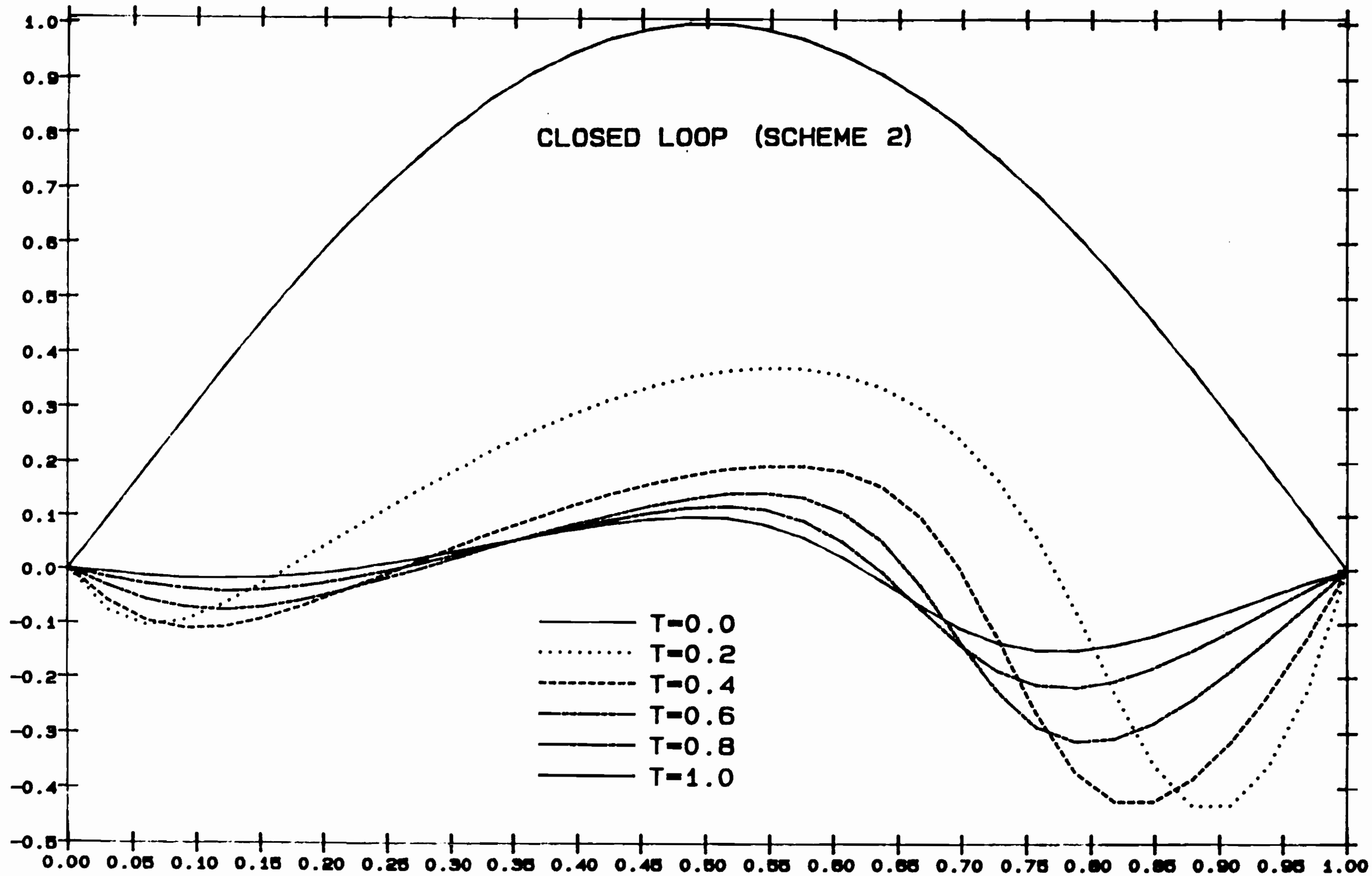

Figure 4.3.15. Closed Loop (feedback by $\tilde{k}_{\alpha,7}(\cdot)$)

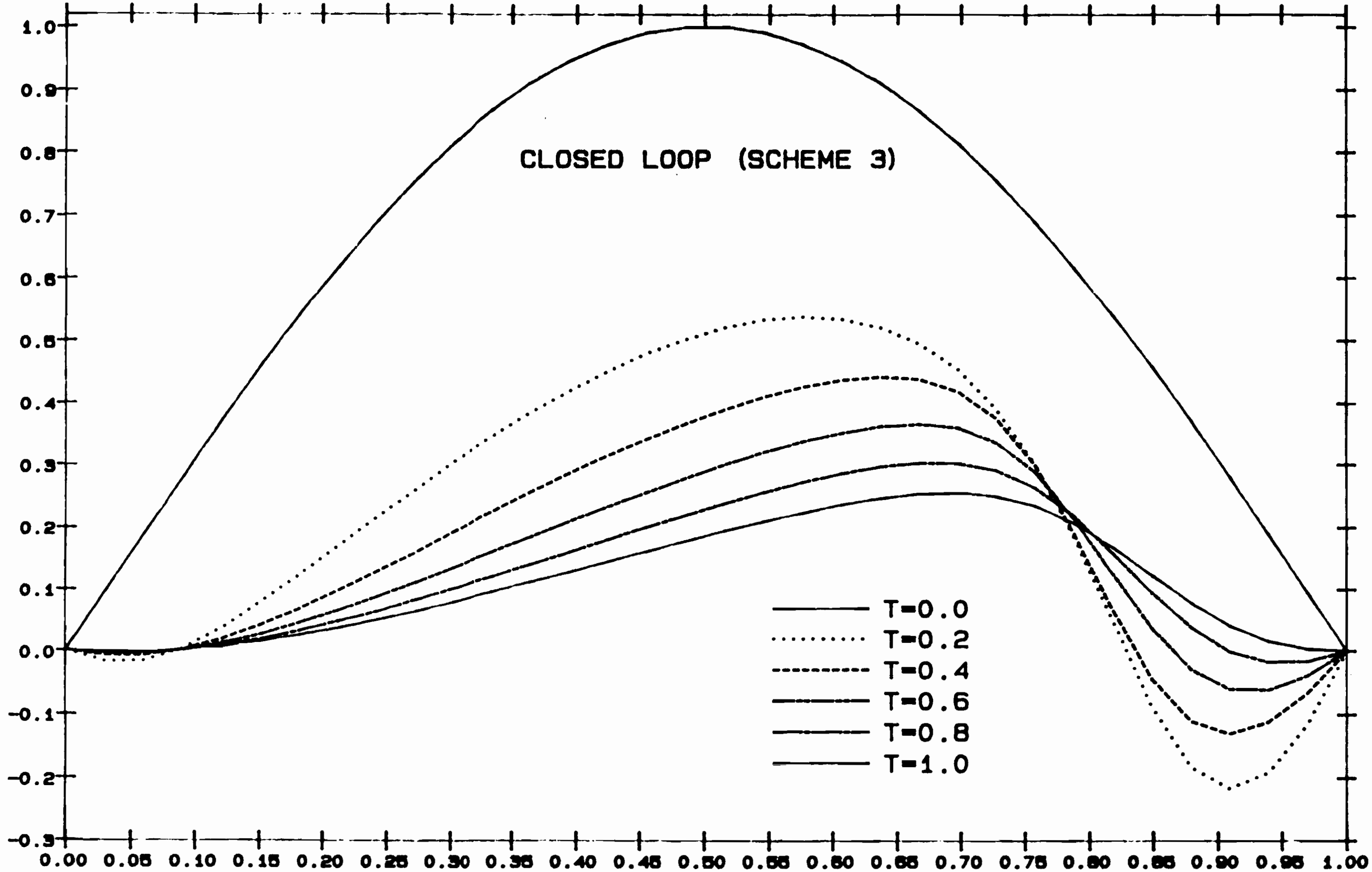

Figure 4.3.16. Closed Loop (feedback by Scheme 3)

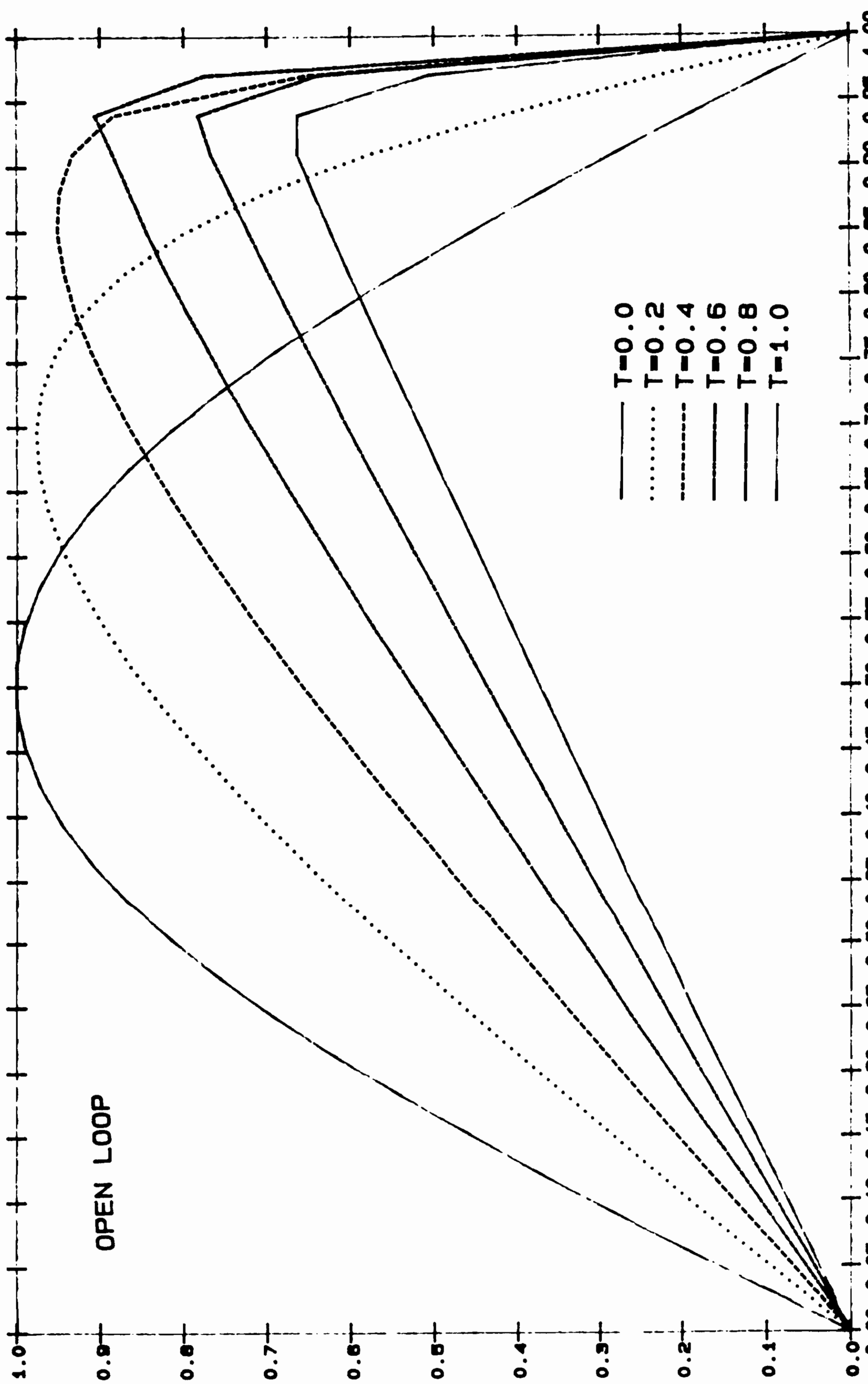

Figure 4.3.17. Open Loop (Re=80, $z_0(x) = \sin \pi x$)

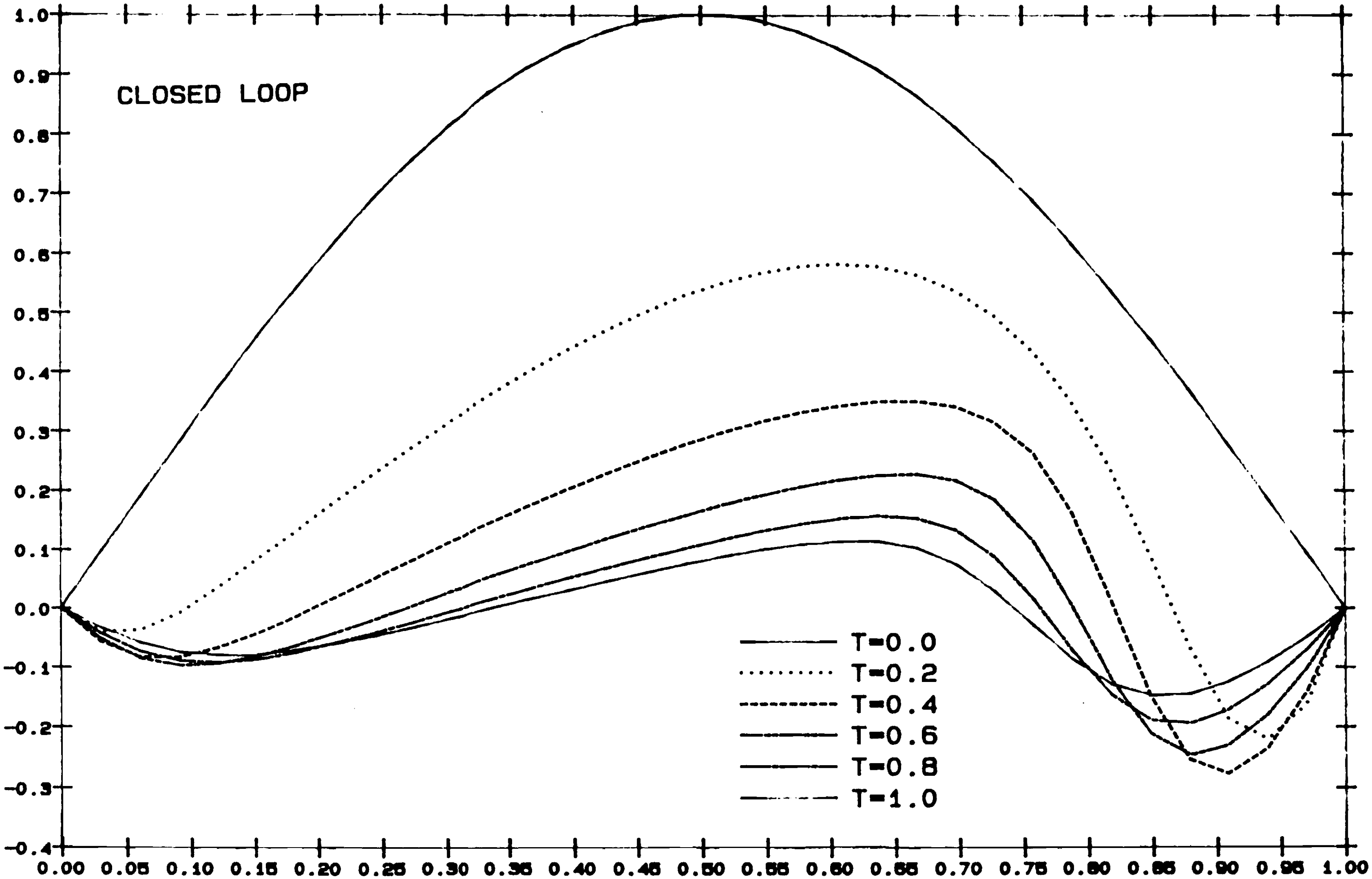

Figure 4.3.18. Closed Loop (Re=80, feedback by $k_{\alpha,1}(\cdot)$)

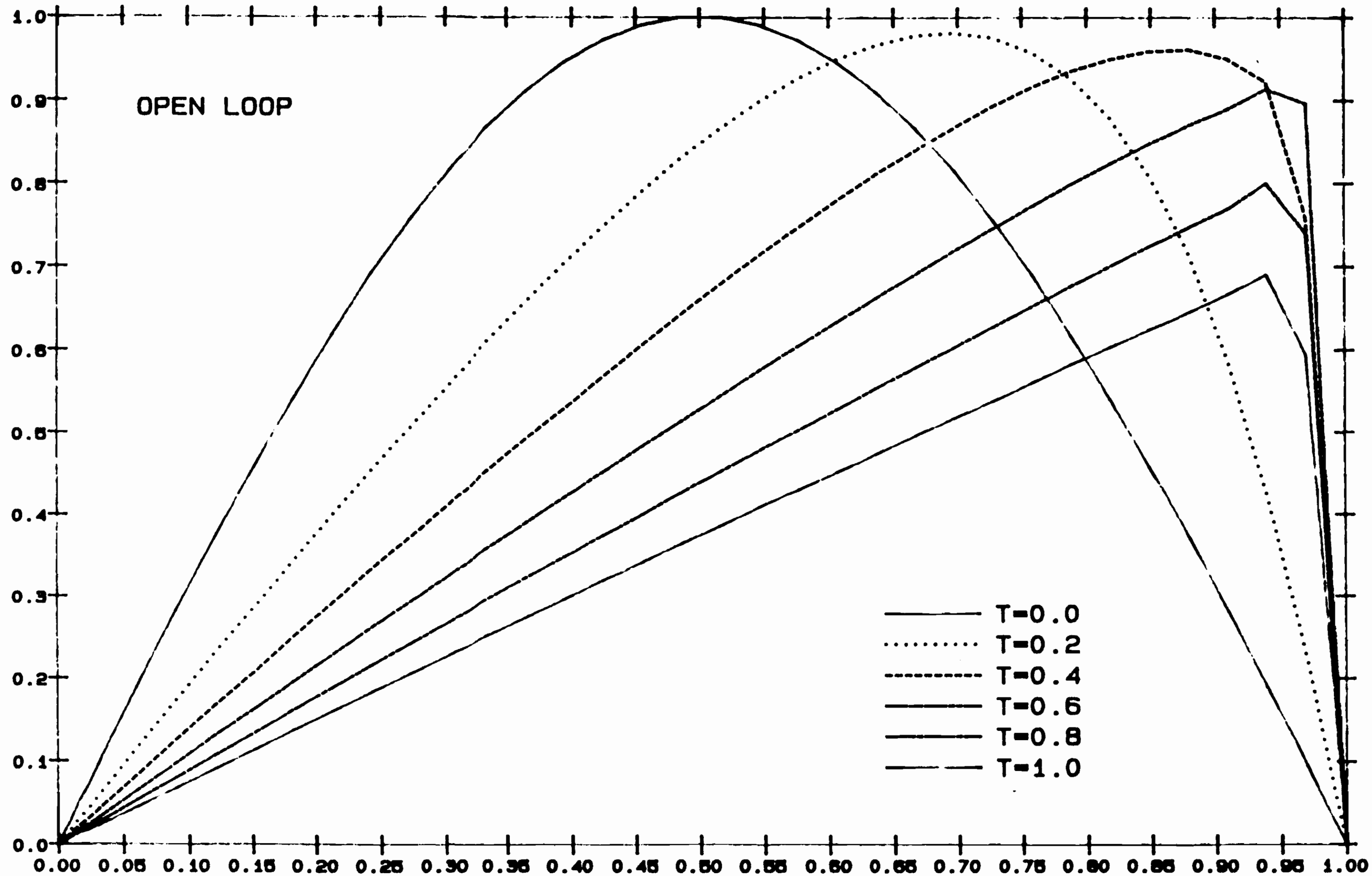

Figure 4.3.19. Open Loop (Re=100, $z_0(x) = \sin \pi x$)

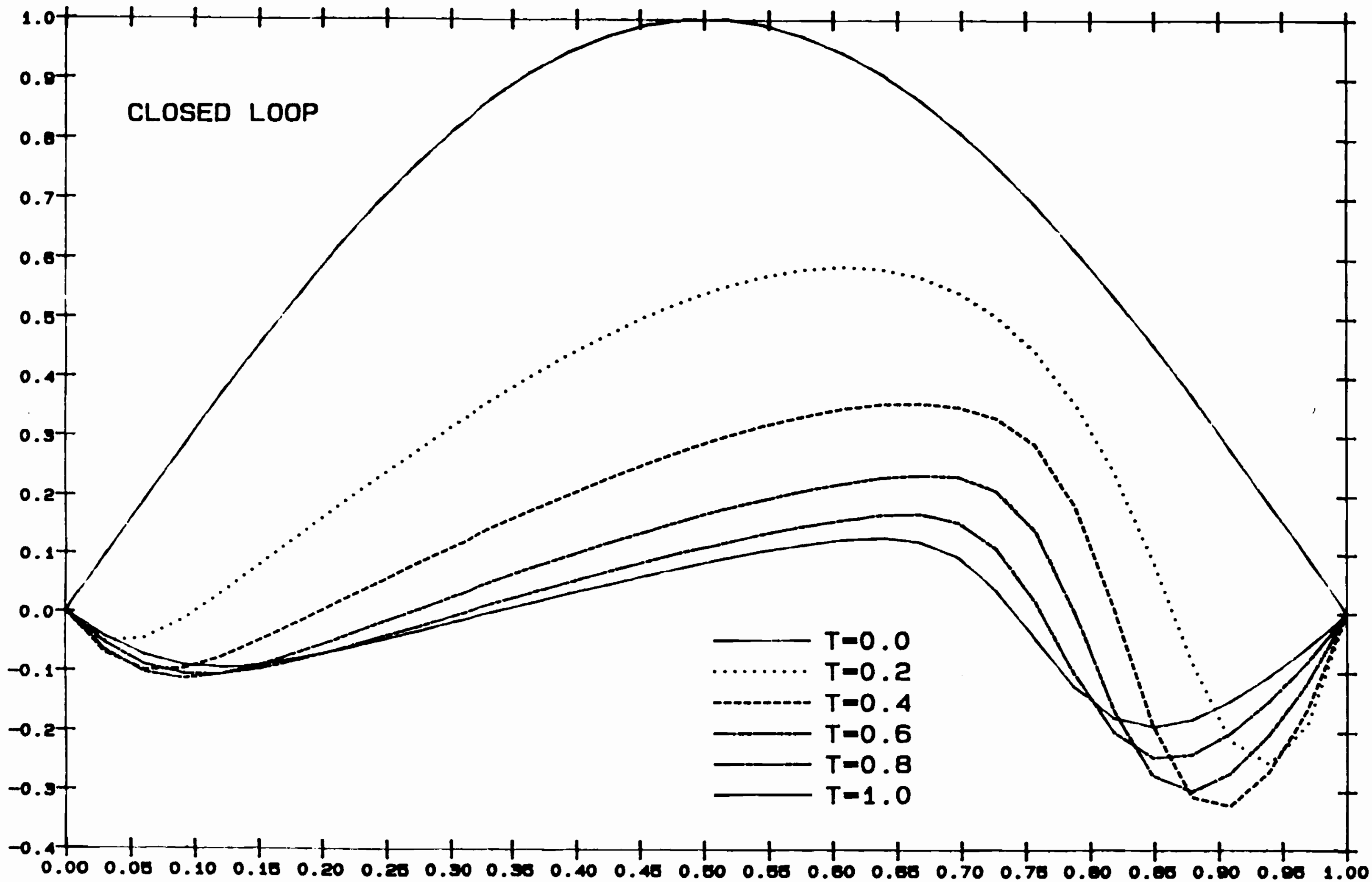

Figure 4.3.20. Closed Loop (Re=100, feedback by $k_{\alpha,1}(\cdot)$)

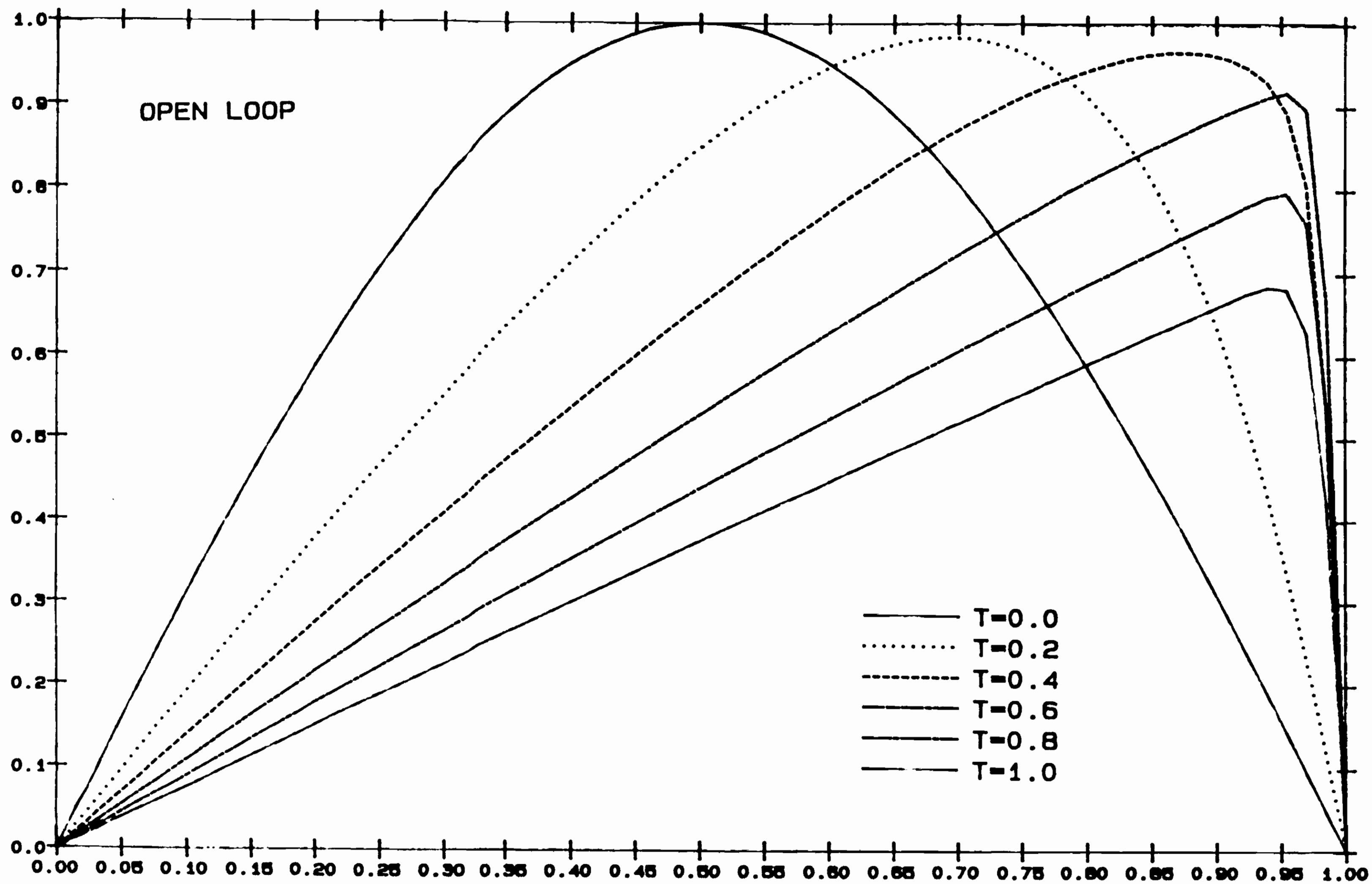

Figure 4.3.21. Open Loop (Re=120, $N = 64$, $z_0(x) = \sin \pi x$)

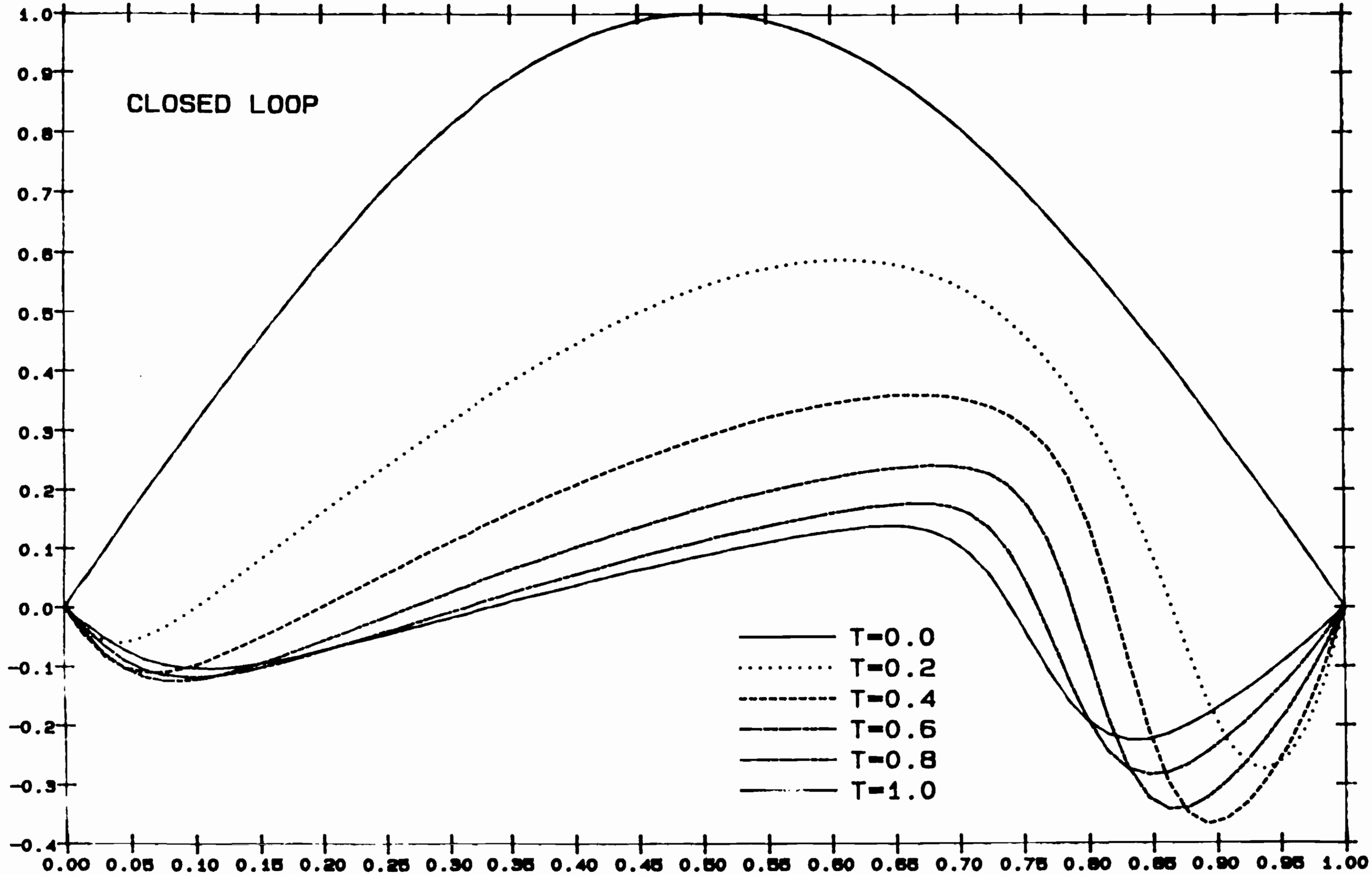

Figure 4.3.22. Closed Loop (Re=120, $N = 64$, feedback by $k_{\alpha,1}(\cdot)$)

4.4. Summary.

In this paper, we looked at a feedback control problem for a nonlinear equation, in particular, Burgers' equation. The method consists of linearization of the nonlinear equation. We used the linear quadratic regulator (LQR) problem to find optimal feedback gains. The linearized equation is the heat equation. It was also proved that, under appropriate selection of the input functions and "sensors", the LQR problem for the linearized problem is detectable and stabilizable. We then analysed a "shifted quadratic cost" to construct gains which produce a fixed decay rate. In particular, we showed that the closed-loop system satisfies the inequality

$$||\, z(t,0;z_0)\,||_{H_0^1} \leq M(\epsilon)e^{-\alpha t}\,||z_0||_{H_0^1},$$

where $\alpha > 0$ does not depend on the Reynolds number, but $M(\epsilon)$ does (Theorem 3.2.10).

We also developed a numerical scheme for computing the feedback functional gains. For bounded inputs and bounded outputs, the convergence follows from well-known results. Convergence for unbounded input and output is an open question.

Several numerical experiments were performed and the following observations were made:

1) The functional gains depend strongly on the "sensor location" and their type.

For example, if the output operator C is given by

$$C(z) = (\tilde{z}(.3), \tilde{z}(.5), \tilde{z}(.75)),$$

then the gain function is concentrated on the location of sensors, 0.3, 0.5 and 0.75 (Figures 4.3.2 and 4.3.4). On the other hand, for the unbounded output operator C given by

$$C(z) = (\tilde{z}(.3), \tilde{z}(.5), \tilde{z}(.75), \tilde{z}'(.4), \tilde{z}'(.6), \tilde{z}'(.88))$$

the control action is concentrated on the points 0.4, 0.6 and 0.88 which are the locations of sensors measuring the gradient (Figure 4.3.11).

2) Even for an unbounded output operator, the numerical convergence is observed (Figures 4.3.10, 4.3.11 and 4.3.12). Thus it appears that the question of convergence for such systems could be answered theoretically.

3) The closed-loop nonlinear system is stabilized (as predicted) by linear feedback laws. Moreover, the steep gradients (for $\epsilon \approx 0$) are smoothed out by feedback.

4) To test the "robustness" of the feedback control law, two experiments were performed. First, we replaced the functional gain by its average over some intervals and noticed that the performance was almost the same. Second, we obtained the functional gain $k(\cdot)$ from the control system at the Reynolds number, Re=60, and applied it to the closed-loop system

$$\frac{\partial}{\partial t}z(t,x) = \frac{1}{\mathrm{Re}}\frac{\partial^2}{\partial x^2}z(t,x) - z(t,x)\frac{\partial}{\partial x}z(t,x) - e^x \int_0^1 k(s)z(t,s)\,ds \qquad (4.4.1)$$

at Re=80, 100 and 120 (Figures 4.3.18, 4.3.20 and 4.3.22). Although the performance was decreased, the system (4.4.1) was still stabilized and smoothed out.

These results provided some insight into the possibility of using linear feedback laws for nonlinear distributed parameter systems. On the other hand, they raised several interesting questions that need further study. We plan to:

(i) develop approximation schemes for linear regulator problems with unbounded input and ouput operators;

(ii) investigate the "shock" stabilization problems for conservation laws (zero viscosity) by boundary feedback controls.

REFERENCES

[1] B. D. O. Anderson and J. B. Moore, "Linear System Optimisation with Prescribed Degree of Stability." *Proc. IEEE*, Vol. 116, No. 12, (1969), 2083-2087.

[2] H. T. Banks and J. A. Burns, "Hereditary Control Problems: Numerical Methods based on Averaging approximation", *SIAM J. Control and Optim.*, Vol. 16, No. 2, (1978), 169-208.

[3] H. T. Banks, J. A. Burns and E. M. Cliff, "Parameter Estimation and Identification for Systems with Delays", *SIAM J. Control and Optim.*, Vol. 19, No. 6, (1981), 791-828.

[4] H. T. Banks and K. Kunish, "The Linear Regulator Problem for Parabolic Systems." *SIAM J. Control and Optim.*, Vol. 22, No. 5, (1984), 684-698.

[5] J. M. Burgers, "Mathematical Examples Illustrating Relations Occuring in the Theory of Turbulent Fluid Motion." *Trans. Roy. Neth. Acad. Sci.*, Amsterdam, Vol. 17, (1939), 1-53.

[6] J. M. Burgers, "A Mathematical Model Illustrating the Theory of Turbulence", *Adv. in Appl. Mech.*, Vol. 1, (1948), 171-199.

[7] J. M. Burgers, "Statistical Problems Connected with Asymptotic Solution of One-dimensional Nonlinear Diffusion Equation", in *Statistical Models and Turbulence* (eds., M. Rosenblatt and C. van Atta), (1972), Springer, Berlin, 41-60.

[8] G. Chen, H. K. Wang and S. Weerakoon, "An Initial Value Control Problem for Burgers' Equation", in *Distributed Parameter Systems*, (ed. F. Kappel, K. Kunish and W. Schappacher), Lecture Notes in Control and Information Sciences, Vol. 75, (1985), 52-76.

[9] W. C. Chin, "Pseudo-Transonic Equation with a Diffusion Term", *AIAA Journal*, Vol. 16, (1978), 87-88.

[10] J. D. Cole, "On a Quasi-Linear Parabolic Equation Occuring in Aerodynamics." *Quart. Appl. Math.*, Vol. IX, No. 3, (1951), 225-236.

[11] J. B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1985.

[12] R. F. Curtain, "Stability of Semilinear Evolution Equations in Hilbert Space", *J. Math. Pures et Appl*, Vol. 63, (1984), 121-128.

[13] R. Datko, "Uniform Asymptotic Stability of Evolutionary Process in a Banach Space." *SIAM J. Math. Anal.*, Vol. 3, No. 3, (1972), 428-445.

[14] C. A. J. Fletcher, "Burgers' Equation : A Model for All Reasons." *Numerical Solution of Partial Differential Equations,* J. Noye ed., North-Holland Publ. Co., (1982), 139-225.

[15] H. Fujita and A. Mizutani, "On the Finite Element Method for Parabolic Equations I: Approximation of Holomorphic Semi-Groups", *J. Math. Soc. Japan*, Vol. 28, (1976), 749-771.

[16] J. S. Gibson, "The Riccati Integral Equations for Optimal Control Problems on Hilbert Spaces", *SIAM J. Control and Optim.*, Vol. 17, (1979), 537-565.

[17] J. S. Gibson, "Linear-Quadratic Optimal Control of Hereditary Differential Systems: Infinite Dimensional Riccati Equations and Numerical Approximations", *SIAM J. Control and Optim.*, Vol. 21, No. 1, (1983), 95-139.

[18] J. S. Gibson and I. G. Rosen, "Shifting the Closed-Loop Spectrum in the Optimal Linear Quadratic Regulator Problem for Hereditary Systems", Institute for Computer Applications for Science and Engineering, NASA Langley Research Center, Hampton, VA, ICASE Report 86-16, 1986.

[19] J. Glimm and P. Lax, *Decay of Solutions of Systems of Nonlinear Hyperbolic Conservation Laws*, Amer. Math. Soc. Memoir, No. 101, A.M.S., Providence, 1970.

[20] W. D. Hayes, "Pseudo Transonic Similitude and First-Order Wave Structure", *Journal of the Aeronautical Sciences*, Vol. 21, (1954), 721-730.

[21] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, New York, 1981.

[22] E. Hopf, "The Partial Differential Equation $u_t + uu_x = \mu u_{xx}$." *Comm. Pure and Appl. Math*, Vol. 3, (1950), 201-230.

[23] K. Ito, "Strong Convergence and Convergence Rates of Approximating Solutions for Algebraic Riccati Equations in Hilbert Spaces", LCDS/CCS Report #87-15, Brown University, 1987.

[24] H. Kielhöfer, "Stability and Semilinear Evolution Equations in Hilbert Space", *Arch. Rat. Mech. Anal.*, Vol. 57, (1974), 150-165.

[25] I. Lasiecka, "Unified Theory for Abstract Parabolic Boundary Problems - A Semigroup Approach", *Appl. Math. Optim*, Vol. 6, (1980), 287-333.

[26] P. D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Schock Waves*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 11, SIAM, 1973.

[27] M. J. Lighthill, "Viscosity Effects in Sound Waves of Finite Amplitude", in *Surveys in Mechanics* (ed. G. K. Bachelor and R. M. Davies), Cambridge University Press, Cambridge, (1956), 250-351.

[28] V. P. Maslov, "On a New Principle of Superposition for Optimization Problems", *Uspekhi Mat. Nauk*, Vol. 42, No. 3, (1987), 39-48.

[29] V. P. Maslov, "A New Approach to Generalized Solutions of Nonlinear Systems", *Soviet Math. Dokl*, Vol. 35, No. 1, (1987), 29-33.

[30] T. Musha and H. Higuchi, "Traffic Current Fluctuation and the Burgers' Equation." *Japanese J. of Applied Physics*, Vol. 17, No. 5, (1978), 811-816.

[31] O. A. Oleĭnik, "Discontinuous Solutions of Nonlinear Differential Equations." *Uspsekhi Math. Nauk*, Vol. 12, (1957), 3-73, (English Translation *Amer. Math. Soc. Trans.* , Series 2, 26, 95-172).

[32] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[33] A. J. Pritchard and D. Salamom, "The Linear Quadratic Control Problem for Infinite Dimensional Systems with Unbounded Input and Output Operators." *SIAM J. Control and Optimization*, Vol. 25, No. 1, (1987), 121-144.

[34] S. M. Rankin, "Semilinear Evolution Equations in Banach Spaces with Application to Parabolic Partial Differential Equations", Preprint, 1989.

[35] D. L. Russell, *Mathematics of Finite-Dimensional Control Systems: Theory and Design*, Marcel Dekker Inc., 1979.

[36] M. H. Schultz, *Spline Analysis*, Prentice-Hall, N.J., 1973.

[37] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, 1983.

[38] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, Springer-Verlag, New York, 1980

[39] J. A. Walker, *Dynamical Systems and Evolution Equations, Theory and Applications*, Plenum Press, New York, 1980.

[40] S. Weerakoon, *An Initial Value Control Problem for the Burgers' Equation*, Ph.D. Thesis, Department of Mathematics, Pennsylvania State University, December, 1984.

[41] G. B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1974.

[42] J. Wloka, *Partial Differential Equations*, Cambridge University Press, Cambridge, 1987.

[43] J. Zabczyk, "Remark on the Algebraic Riccati Equation in Hilbert Spaces", *Applied Math. Optim.*, Vol. 2, (1976), 251-258.

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