FREE VIBRATION OF A ROTATING BEAM-CONNECTED SPACE STATION

by

James LaMar Milner

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APPROVED:

Chairman Dr. Ricardo Chicurel

Dr. Daniel Frederick

Professor W. E. Pace

Professor F. J. Maher

Professor J. H. Sword

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LIST OF SYMBOLS

$\vec{I}, \vec{J}, \vec{K}$	Inertia space axes with origin at the center of the earth
$\vec{i}, \vec{j}, \vec{k}$	Rotational axes of the space station
$\vec{e}_\xi, \vec{e}_\eta, \vec{e}_\rho$	
$\vec{e}_{a_i}, \vec{e}_{b_i}, \vec{e}_{c_i}$	Rotational axes of the ith stage
$\vec{e}_{p_i}, \vec{e}_{s_i}, \vec{e}_{t_i}$	
$\vec{e}_{1_i}, \vec{e}_{2_i}, \vec{e}_{3_i}$	Principal axes of the ith stage
$\vec{e}_v, \vec{e}_{p_1 p_2}, \vec{e}_w$	Unit vectors of the beam orientation
$\vec{d}_1$	Vector from the center of mass of the 1st stage to the beam-attachment point on the 1st stage
$\vec{d}_2$	Vector from the center of mass of the 2nd stage to the beam-attachment point on the 2nd stage
$\vec{d}_{12}$	Vector from the center of mass of Stage 1 to the center of mass of Stage 2

$\vec{d}_{p_1p_2}$	Vector from P_1 to P_2
$\vec{R}_c(r,t)$	Position vector of the point r on the beam at time t
$\vec{R}_i$	Position vector of the ith stage
$\vec{R}_{p_i}$	Position vector of the beam-attachment point on the ith stage
$\vec{\omega}_i$	Rotation vector of the ith stage
a_1 - a_7	Arguments of trigonometric terms in the beam deflection equations
a_{ij}	Elements of the characteristic determinant
c_1 - c_4	Arbitrary constants of the beam deflection
e_{ij}	Elements of the characteristic determinant for the special case of $A_1B_2 - A_2B_1 \equiv 0$
m	Mass per unit length of the beam

p	Frequency function for the vibration
r	Position coordinate along the beam longitudinal axis
t	Time
$v(r, t)$	Deflection of the beam in the plane perpendicular to the plane of rotation
$w(r, t)$	Deflection of the beam in the plane of rotation
x_i, y_i, z_i	Rectangular position coordinates of the ith stage
A_1 - A_6	Frequency dependent terms identified in the equations for C_{w1} and C_{w2}
$\widetilde{A}_1$ - $\widetilde{A}_4$	Frequency dependent terms relating C_{w1} to c_1 - c_4
B_1 - B_6	Frequency dependent terms identified in the equations for C_{w1} and C_{w2}
$\widetilde{B}_1$ - $\widetilde{B}_4$	Frequency dependent terms relating C_{w2} to c_1 - c_4

C_1 - C_{33}	Algebraic combinations of the physical constants of the system
$\tilde{C}_1$ - $\tilde{C}_{22}$	Algebraic combinations of the physical constants of the system
C_{w1}, C_{w2}	Frequency dependent terms of the particular solution for the beam deflection in the plane of the orbit
D_1 - D_{28}	Algebraic combinations of the physical constants of the system appearing in the boundary condition equations
$\tilde{D}_1$ - $\tilde{D}_8$	Frequency dependent terms identified in the nonhomogeneous boundary condition equations
$D(p^2)$	Characteristic determinant
EI	Flexural rigidity of the beam
$(IA)_1$, $(IA)_2$, $(IA)_3$	Principal moments of inertia of Stage 1
$(IB)_1$, $(IB)_2$, $(IB)_3$	Principal moments of inertia of Stage 2
L	Length of the beam

$\tilde{L}$	Length, $\tilde{L} = L + d_1 + d_2$
L_D	Length of the deflected beam, $L_D = L - \frac{1}{2} \int_o^L \left[\left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right] dr$
M_A	Mass of Stage 1
M_B	Mass of Stage 2
M_T	Total mass of the space station, $M_T = M_A + M_B + mL$
P_i	Connection point where the end of the beam is attached to the ith stage
$R(r)$	Position dependent deflection of the beam in the plane of rotation
α_i	Yaw angle of the ith stage
$\bar{\alpha}$	Angle of the constraint equation
β_i	Pitch angle of the ith stage
$\bar{\beta}$	Angle of the constraint equation
γ_i	Roll angle of the ith stage

$\eta(t)$	Time function of the beam deflection in the plane of rotation
η_o	Constant
θ	Attitude angle of the space station locating a moving frame of reference
$\lambda_1, \lambda_2, \lambda_3$	Angles made by $\vec{d}_{12}$ with the inertia space axes
μ	Angle made by $\vec{e}_{p_1p_2}$ with the $\vec{I}$ axis
$\mu_{v_i}, \mu_{r_i}, \mu_{w_i}$	Angles made by $\vec{e}_{2_i}$ with the beam orientation vectors
μ_{vr_i}, μ_{wr_i}	Angles of the boundary conditions
$\tau(t)$	Time dependent perturbation of the spin rate
ψ	Elevation angle of the space station from the plane of the orbit
Ω	Spin rate of the space station

I. INTRODUCTION

The problem of determining the natural frequencies of the free vibration of a rotating, beam-connected space station is considered by developing a mathematical model of the system which represents the general three-dimensional motion of the various components of the space station. The configuration studied is composed of two space modules connected by a flexible beam where the system is made to spin in the plane of its orbit in order to produce an artificial gravity environment within the space modules. The orbital configuration of the space station is shown in Figure 1.

The kinetic energy and potential energy of the space station are formulated in terms of a set of generalized coordinates, and a Lagrangian function is developed for the system. Hamilton's principle is applied to determine the governing equations for the motion of the rotating space station. Boundary conditions representing the clamped-clamped attachment of the beam to each space module are applied to the ends of the beam. Thus the mixed problem of a continuous beam with two large end masses is reduced to the problem of a continuous beam with nonhomogeneous boundary conditions.
The vibration of the space station is considered to be limited to small angular and linear displacements from the motion corresponding to steady rotation as a rigid body. Within the limits of this small deflection approximation, the

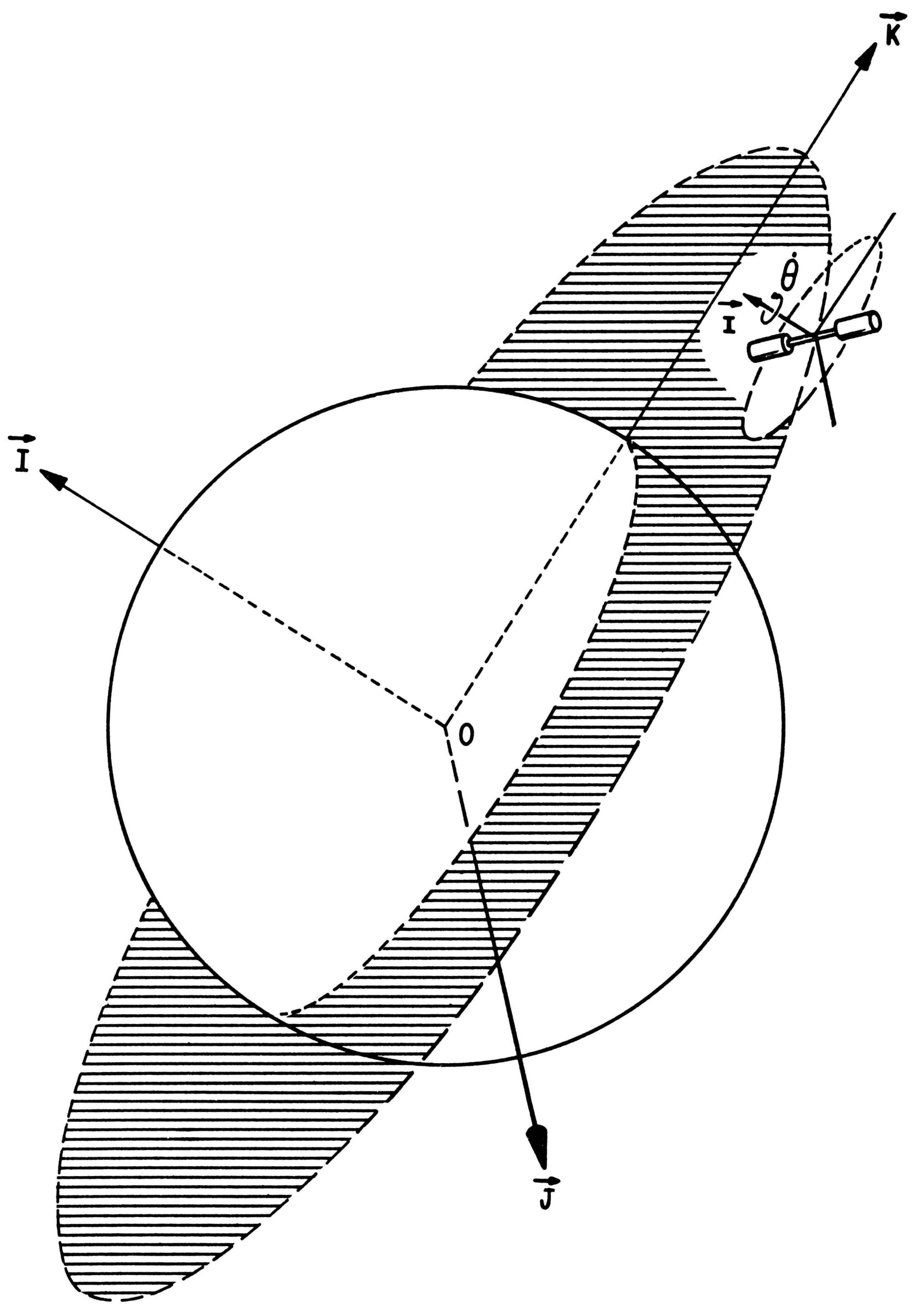

FIG. 1. ORBITAL CONFIGURATION OF THE SPACE STATION

motion of the space station in the plane of rotation is shown to be uncoupled from the motion of the space station out of the plane of rotation.

An exact solution is obtained for the beam deflection in the plane of rotation. This solution is substituted into the appropriate boundary equations, and a characteristic determinant is developed. In addition, the behavior of the characteristic determinant as the frequency parameter takes on negative values is investigated with five special cases in which the form of the exact solution for the beam deflection is modified.

A procedure to solve for the zeros of the characteristic determinant is programmed for digital solution on the IBM 7094. Numerical results of the analysis for a given space station configuration are presented in Appendix III.

II. REVIEW OF THE LITERATURE

During the last decade a considerable amount of emphasis has been placed upon creating an artificial gravity environment within a rotating space station. Suddath [1][1], Kurzhals and Keckler [2], Krause [3], Polstorff [4], and others have studied various single body problems, while such authors as Chobotov [5], Fowler [6], Pengelley [7], Tai and Loh [8], and Targoff [9] have discussed the problem of rotation of cable-connected space stations.

The concept of compression-member-connected compartments has been examined by Tai, Andrew, Loh, and Kamrath [10] in a paper in which the stability and response of thirteen rotating space station configurations was investigated. The configurations studied included single-cable-connected compartments, multiple-cable-connected compartments, and compartments connected by compression members to a central hub.

A recent paper by Liu [11] presents an analysis of two cable-connected space stations rotating about an axis normal to their orbital plane. By using a concept of concentrated fictitious masses and a Galerkin approach, a solution was obtained for the free vibration of the rotating system.

[1] [ ] refers to references on page 74

III. DEVELOPMENT OF THE MATHEMATICAL MODEL

1. Motion of the Stages

In order to study the motion of the rotating space station, the motion of each of the two stages of the station is described independently. The position of the center of mass of the ith stage at any time is given by

$$\vec{R}_i = x_i\vec{I} + y_i\vec{J} + z_i\vec{K} \qquad (i = 1, 2) \qquad \text{------ (1)}$$

as shown in Figure 2. The motion of each stage is considered to have three translation components such that

$$\dot{\vec{R}}_i = \dot{x}_i\vec{I} + \dot{y}_i\vec{J} + \dot{z}_i\vec{K} \qquad (i = 1, 2) \qquad \text{------ (2)}$$

The rotational motion of each stage is represented by the vector sum of five independent angular velocities shown in Figure 3.

$$\vec{\omega}_i = \dot{\theta}\vec{I} + \dot{\psi}\vec{k} + \dot{\beta}_i\vec{e}_{\xi} + \dot{\alpha}_i\vec{e}_{c_i} + \dot{\gamma}_i\vec{e}_{s_i} \qquad (i = 1, 2) \qquad \text{------ (3)}$$

where $\dot{\theta}$ represents the rate of change of the attitude angle of the space station in the plane of the orbit

$\dot{\psi}$ represents the rate of change of the elevation angle of the space station from the plane of the orbit

$\dot{\beta}_i$ is the rate of pitch of the ith stage about its center of mass

$\dot{\alpha}_i$ is the rate of yaw of the ith stage about its center of mass

$\dot{\gamma}_i$ is the rate of roll of the ith stage about its center of mass

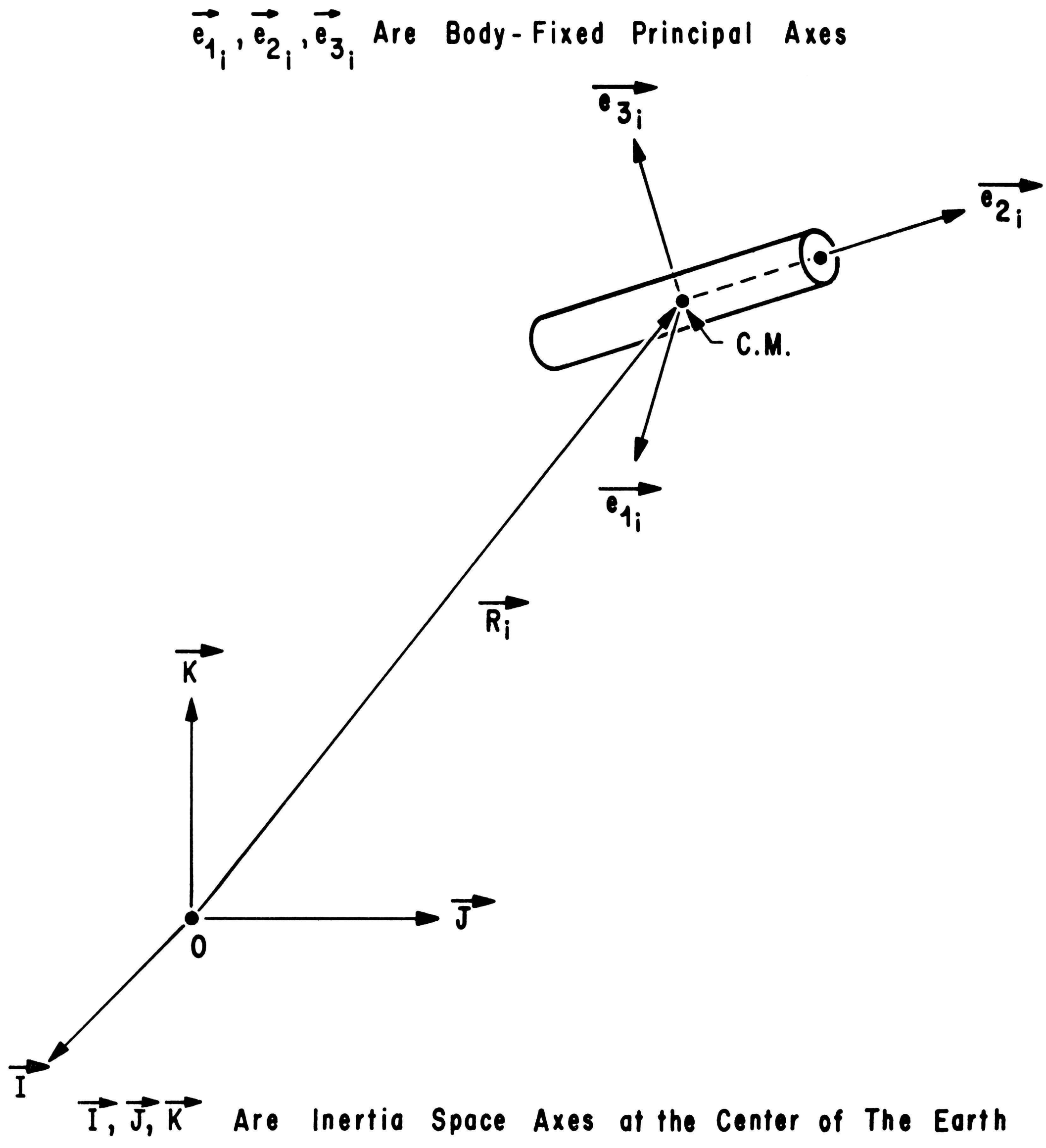

FIG. 2. POSITION OF THE i th STAGE

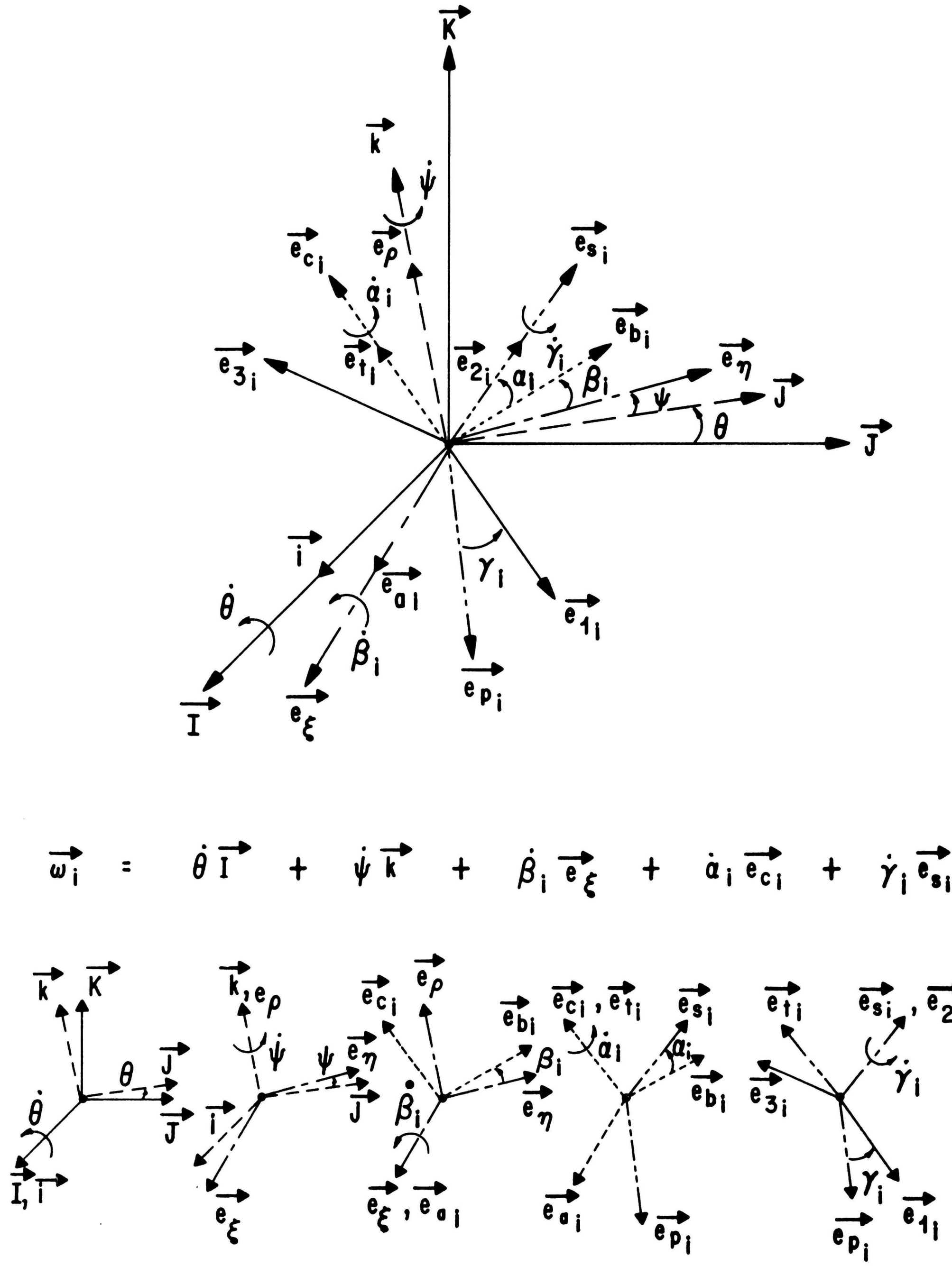

FIG. 3. ROTATIONAL MOTION OF THE ith STAGE

Transformations from each of the five sets of axes to inertia space axes are given by

$$\begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = [T_1] \begin{bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{bmatrix}$$

$$\begin{bmatrix} \vec{e}_\xi \\ \vec{e}_\eta \\ \vec{e}_\rho \end{bmatrix} = [T_2] \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = [T_{21}] \begin{bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{bmatrix}$$

$$\begin{bmatrix} \vec{e}_a \\ \vec{e}_b \\ \vec{e}_c \end{bmatrix}_i = [T_3]_i \begin{bmatrix} \vec{e}_\xi \\ \vec{e}_\eta \\ \vec{e}_\rho \end{bmatrix} = [T_{321}]_i \begin{bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{bmatrix} \qquad (i = 1, 2)$$

$$\begin{bmatrix} \vec{e}_p \\ \vec{e}_s \\ \vec{e}_t \end{bmatrix}_i = [T_4]_i \begin{bmatrix} \vec{e}_a \\ \vec{e}_b \\ \vec{e}_c \end{bmatrix}_i = [T_{4321}]_i \begin{bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{bmatrix} \qquad (i = 1, 2)$$

$$\begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}_i = [T_5]_i \begin{bmatrix} \vec{e}_p \\ \vec{e}_s \\ \vec{e}_t \end{bmatrix}_i = [T_{54321}]_i \begin{bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{bmatrix} \qquad (i = 1, 2)$$

where

$$[T_1] = \begin{bmatrix} 1 , & 0 & , 0 \\ 0 , & \cos\theta & , \sin\theta \\ 0 , & -\sin\theta & , \cos\theta \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} \cos\psi & , \sin\psi & , 0 \\ -\sin\psi & , \cos\psi & , 0 \\ 0 & , 0 & , 1 \end{bmatrix}$$

$$[T_3]_i = \begin{bmatrix} 1 , & 0 & , 0 \\ 0 , & \cos\beta_i & , \sin\beta_i \\ 0 , & -\sin\beta_i & , \cos\beta_i \end{bmatrix} \qquad (i = 1, 2)$$

$$[T_4]_i = \begin{bmatrix} \cos\alpha_i & , \sin\alpha_i & , 0 \\ -\sin\alpha_i & , \cos\alpha_i & , 0 \\ 0 & , 0 & , 1 \end{bmatrix} \qquad (i = 1, 2)$$

$$[T_5]_i = \begin{bmatrix} \cos\gamma_i & , 0 , & -\sin\gamma_i \\ 0 & , 1 , & 0 \\ \sin\gamma_i & , 0 , & \cos\gamma_i \end{bmatrix} \qquad (i = 1, 2)$$

and

$$[T_{21}] = \begin{bmatrix} (\cos\psi) & , (\cos\theta\sin\psi) & , (\sin\theta\sin\psi) \\ (-\sin\psi) & , (\cos\theta\cos\psi) & , (\sin\theta\cos\psi) \\ (0) & , (-\sin\theta) & , (\cos\theta) \end{bmatrix}$$

$$[T_{321}]_i = \begin{bmatrix} (\cos\psi) & (\cos\theta\sin\psi) & (\sin\theta\sin\psi) \\ (-\sin\psi\cos\beta_i) & (\cos\theta\cos\psi\cos\beta_i - \sin\theta\sin\beta_i) & (\sin\theta\cos\psi\cos\beta_i + \cos\theta\sin\beta_i) \\ (\sin\psi\sin\beta_i) & (-\cos\theta\cos\psi\sin\beta_i - \sin\theta\cos\beta_i) & (-\sin\theta\cos\psi\sin\beta_i + \cos\theta\cos\beta_i) \end{bmatrix}$$

(i = 1, 2)

$$[T_{4321}]_i = \begin{bmatrix} (\cos\psi\cos\alpha_i - \sin\psi\cos\beta_i\sin\alpha_i) & \begin{pmatrix} \cos\theta\sin\psi\cos\alpha_i + \cos\theta\cos\psi\cos\beta_i\sin\alpha_i \\ -\sin\theta\sin\beta_i\sin\alpha_i \end{pmatrix} & \begin{pmatrix} \sin\theta\sin\psi\cos\alpha_i + \sin\theta\cos\psi\cos\beta_i\sin\alpha_i \\ +\cos\theta\sin\beta_i\sin\alpha_i \end{pmatrix} \\ (-\cos\psi\sin\alpha_i - \sin\psi\cos\beta_i\cos\alpha_i) & \begin{pmatrix} -\cos\theta\sin\psi\sin\alpha_i + \cos\theta\cos\psi\cos\beta_i\cos\alpha_i \\ -\sin\theta\sin\beta_i\cos\alpha_i \end{pmatrix} & \begin{pmatrix} -\sin\theta\sin\psi\sin\alpha_i + \sin\theta\cos\psi\cos\beta_i\cos\alpha_i \\ +\cos\theta\sin\beta_i\cos\alpha_i \end{pmatrix} \\ (\sin\psi\sin\beta_i) & (-\cos\theta\cos\psi\sin\beta_i - \sin\theta\cos\beta_i) & (-\sin\theta\cos\psi\sin\beta_i + \cos\theta\cos\beta_i) \end{bmatrix}$$

(i = 1, 2)

$$
[T_{54321}]_i = \begin{bmatrix}
(\cos\psi\cos\alpha_i\cos\gamma_i - \sin\psi\cos\beta_i\sin\alpha_i\cos\gamma_i - \sin\psi\sin\beta_i\sin\gamma_i), \ldots \\
(-\cos\psi\sin\alpha_i - \sin\psi\cos\beta_i\cos\alpha_i) \qquad , \ldots \\
(\cos\psi\cos\alpha_i\sin\gamma_i - \sin\psi\cos\beta_i\sin\alpha_i\sin\gamma_i + \sin\psi\sin\beta_i\cos\gamma_i), \ldots \\
\ldots, \begin{pmatrix} \cos\theta\sin\psi\cos\alpha_i\cos\gamma_i + \cos\theta\cos\psi\cos\beta_i\sin\alpha_i\cos\gamma_i \\ -\sin\theta\sin\beta_i\sin\alpha_i\cos\gamma_i + \cos\theta\cos\psi\sin\beta_i\sin\gamma_i \\ +\sin\theta\cos\beta_i\sin\gamma_i \end{pmatrix}, \ldots\ldots \\
\ldots, \begin{pmatrix} -\cos\theta\sin\psi\sin\alpha_i + \cos\theta\cos\psi\cos\beta_i\cos\alpha_i \\ -\sin\theta\sin\beta_i\cos\alpha_i \end{pmatrix} \qquad , \ldots\ldots \\
\ldots, \begin{pmatrix} \cos\theta\sin\psi\cos\alpha_i\sin\gamma_i + \cos\theta\cos\psi\cos\beta_i\sin\alpha_i\sin\gamma_i \\ -\sin\theta\sin\beta_i\sin\alpha_i\sin\gamma_i - \cos\theta\cos\psi\sin\beta_i\cos\gamma_i \\ -\sin\theta\cos\beta_i\cos\gamma_i \end{pmatrix}, \ldots\ldots \\
\ldots, \begin{pmatrix} \sin\theta\sin\psi\cos\alpha_i\cos\gamma_i + \sin\theta\cos\psi\cos\beta_i\sin\alpha_i\cos\gamma_i \\ +\cos\theta\sin\beta_i\sin\alpha_i\cos\gamma_i + \sin\theta\cos\psi\sin\beta_i\sin\gamma_i \\ -\cos\theta\cos\beta_i\sin\gamma_i \end{pmatrix} \\
\ldots, \begin{pmatrix} -\sin\theta\sin\psi\sin\alpha_i + \sin\theta\cos\psi\cos\beta_i\cos\alpha_i \\ +\cos\theta\sin\beta_i\cos\alpha_i \end{pmatrix} \\
\ldots, \begin{pmatrix} \sin\theta\sin\psi\cos\alpha_i\sin\gamma_i + \sin\theta\cos\psi\cos\beta_i\sin\alpha_i\sin\gamma_i \\ +\cos\theta\sin\beta_i\sin\alpha_i\sin\gamma_i - \sin\theta\cos\psi\sin\beta_i\cos\gamma_i \\ +\cos\theta\cos\beta_i\cos\gamma_i \end{pmatrix}
\end{bmatrix}
$$

$$(i = 1, 2)$$

The components of the angular velocity about a set of body-fixes principal axes of each stage $(\vec{e}_{1_i}, \vec{e}_{2_i}, \vec{e}_{3_i})$ are determined from

$$\vec{\omega}_i = \dot{\theta}\vec{I} + \dot{\psi}\vec{k} + \dot{\beta}_i\vec{e}_\xi + \dot{\alpha}_i\vec{e}_{c_i} + \dot{\gamma}_i\vec{e}_{s_i} = \omega_{1_i}\vec{e}_{1_i} + \omega_{2_i}\vec{e}_{2_i} + \omega_{3_i}\vec{e}_{3_i}$$

$$(i = 1, 2) \qquad \text{------} \ (4)$$

Using the transformation equations to write $\vec{k}, \vec{e}_\xi, \vec{e}_{c_i}, \vec{e}_{s_i}, \vec{e}_{1_i}, \vec{e}_{2_i}, \vec{e}_{3_i}$ in terms of the inertia space axes $\vec{I}, \vec{J}, \vec{K}$, we solve equation (4) to obtain

$$\omega_{1_i} = \dot{\theta}(\cos\psi\cos\alpha_i\cos\gamma_i - \sin\psi\cos\beta_i\sin\alpha_i\cos\gamma_i - \sin\psi\sin\beta_i\sin\gamma_i) + \dot{\psi}(\sin\beta_i\sin\alpha_i\cos\gamma_i - \cos\beta_i\sin\gamma_i) + \dot{\beta}_i\cos\alpha_i\cos\gamma_i - \dot{\alpha}_i\sin\gamma_i$$

$$(i = 1, 2) \qquad \text{------} \ (5)$$

$$\omega_{2_i} = -\dot{\theta}(\sin\psi\cos\beta_i\cos\gamma_i + \cos\psi\sin\alpha_i) + \dot{\psi}\sin\beta_i\cos\alpha_i - \dot{\beta}_i\sin\alpha_i + \dot{\gamma}_i$$

$$(i = 1, 2) \qquad \text{------} \ (6)$$

$$\omega_{3_i} = \dot{\theta}(\sin\psi\sin\beta_i\cos\gamma_i - \sin\psi\cos\beta_i\sin\alpha_i\sin\gamma_i + \cos\psi\cos\alpha_i\sin\gamma_i) + \dot{\psi}(\cos\beta_i\cos\gamma_i + \sin\beta_i\sin\alpha_i\sin\gamma_i) + \dot{\beta}_i\cos\alpha_i\sin\gamma_i + \dot{\alpha}_i\cos\gamma_i$$

$$(i = 1, 2) \qquad \text{------} \ (7)$$

2. Configuration of the Space Station

The position of the space station at a given time is shown in Figure 4, where the origin of the inertia space axes is fixed at the center of the earth. Points 1 and 2 represent the centers of mass of the stages while points P_1 and P_2 represent the ends of the beam connecting the two stations. Vector $\vec{d}_1$ is the directed distance from the center of mass of Stage 1 to the connection point

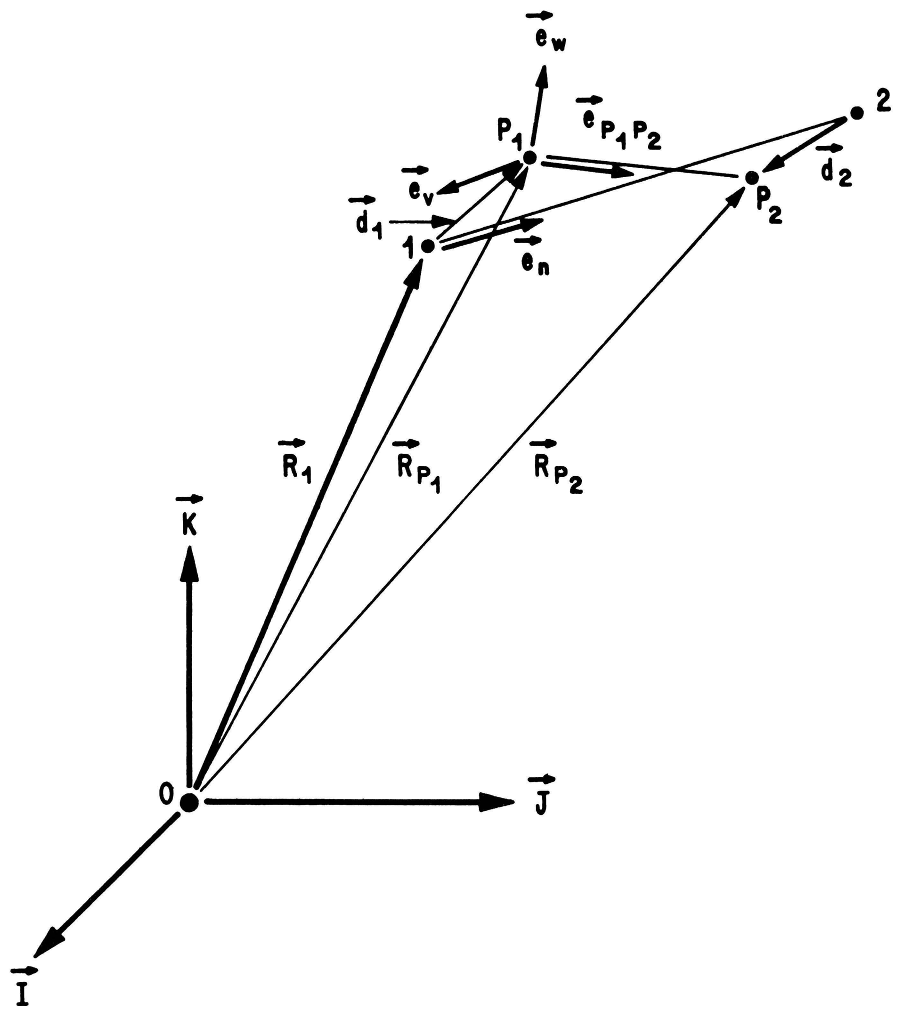

FIG. 4. POSITION OF THE SPACE STATION

of the beam and vector $\vec{d}_2$ is the directed distance from the center of mass of Stage 2 to the point where the beam is connected to Stage 2. The unit vector $\vec{e}_\eta$ is along the line from the center of mass of Stage 1 to the center of mass of Stage 2. The unit vector $\vec{e}_{p_1p_2}$ is along the line drawn from P_1 to P_2. Thus we have

$$\vec{R}_{p_1} = \vec{R}_1 + \vec{d}_1 = (x_1 - d_1\cos\psi\sin\alpha_1 - d_1\sin\psi\cos\beta_1\cos\alpha_1)\,\vec{I}$$
$$+\begin{pmatrix} y_1 - d_1\cos\theta\sin\psi\sin\alpha_1 + d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 \\ - d_1\sin\theta\sin\beta_1\cos\alpha_1 \end{pmatrix}\vec{J}$$
$$+\begin{pmatrix} z_1 - d_1\sin\theta\sin\psi\sin\alpha_1 + d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 \\ + d_1\cos\theta\sin\beta_1\cos\alpha_1 \end{pmatrix}\vec{K}$$

$$\vec{R}_{p_2} = \vec{R}_2 + \vec{d}_2 = (x_2 + d_2\cos\psi\sin\alpha_2 + d_2\sin\psi\cos\beta_2\cos\alpha_2)\,\vec{I}$$
$$+\begin{pmatrix} y_2 + d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ + d_2\sin\theta\sin\beta_2\cos\alpha_2 \end{pmatrix}\vec{J}$$
$$+\begin{pmatrix} z_2 + d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ - d_2\cos\theta\sin\beta_2\cos\alpha_2 \end{pmatrix}\vec{K}$$

$$\vec{d}_{12} = d_{12}\vec{e}_\eta = \vec{R}_2 - \vec{R}_1 = (x_2 - x_1)\,\vec{I} + (y_2 - y_1)\,\vec{J} + (z_2 - z_1)\,\vec{K}$$

$$\vec{d}_{p_1p_2} = d_{p_1p_2}\,\vec{e}_{p_1p_2} = \vec{R}_{p_2} - \vec{R}_{p_1} = \vec{d}_{12} + \vec{d}_2 - \vec{d}_1$$
$$= \begin{pmatrix} x_2 - x_1 + d_2\cos\psi\sin\alpha_2 + d_2\sin\psi\cos\beta_2\cos\alpha_2 \\ + d_1\cos\psi\sin\alpha_1 + d_1\sin\psi\cos\beta_1\cos\alpha_1 \end{pmatrix}\vec{I}$$
$$+\begin{pmatrix} y_2 - y_1 + d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ + d_2\sin\theta\sin\beta_2\cos\alpha_2 + d_1\cos\theta\sin\psi\sin\alpha_1 \\ - d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 + d_1\sin\theta\sin\beta_1\cos\alpha_1 \end{pmatrix}\vec{J}$$

$$+\begin{pmatrix} z_2 - z_1 + d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ - d_2\cos\theta\sin\beta_2\cos\alpha_2 + d_1\sin\theta\sin\psi\sin\alpha_1 \\ - d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 - d_1\cos\theta\sin\beta_1\cos\alpha_1 \end{pmatrix}\vec{K}$$

The unit vectors $(\vec{e}_v, \vec{e}_{p_1p_2}, \vec{e}_w)$ describe the beam orientation. Vector $\vec{e}_w$ lies in the plane of the orbit and defines the direction of the beam deflection $w(r,t)$, while vector $\vec{e}_v$ is orthogonal to $\vec{e}_{p_1p_2}$ and $\vec{e}_w$ and defines the direction of the beam deflection $v(r,t)$ out of the orbit plane. These vectors are derived in Appendix I, from which we write

$$\vec{e}_v = \vec{I}$$

$$-\frac{\vec{J}}{L^2}\left\{\begin{bmatrix} x_2 - x_1 + d_2\cos\psi\sin\alpha_2 + d_2\sin\psi\cos\beta_2\cos\alpha_2 \\ + d_1\cos\psi\sin\alpha_1 + d_1\sin\psi\cos\beta_1\cos\alpha_1 \end{bmatrix} \cdot \begin{bmatrix} y_2 - y_1 + d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ + d_2\sin\theta\sin\beta_2\cos\alpha_2 + d_1\cos\theta\sin\psi\sin\alpha_1 \\ - d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 + d_1\sin\theta\sin\beta_1\cos\alpha_1 \end{bmatrix}\right\}$$

$$-\frac{\vec{K}}{L^2}\left\{\begin{bmatrix} x_2 - x_1 + d_2\cos\psi\sin\alpha_2 + d_2\sin\psi\cos\beta_2\cos\alpha_2 \\ + d_1\cos\psi\sin\alpha_1 + d_1\sin\psi\cos\beta_1\cos\alpha_1 \end{bmatrix} \cdot \begin{bmatrix} z_2 - z_1 + d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ - d_2\cos\theta\sin\beta_2\cos\alpha_2 + d_1\sin\theta\sin\psi\sin\alpha_1 \\ - d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 - d_1\cos\theta\sin\beta_1\cos\alpha_1 \end{bmatrix}\right\}$$

$$\vec{e}_{p_1p_2} = \frac{\vec{I}}{L}\begin{bmatrix} x_2 - x_1 + d_2\cos\psi\sin\alpha_2 + d_2\sin\psi\cos\beta_2\cos\alpha_2 \\ + d_1\cos\psi\sin\alpha_1 + d_1\sin\psi\cos\beta_1\cos\alpha_1 \end{bmatrix}$$

$$+ \frac{\vec{J}}{L}\begin{bmatrix} y_2 - y_1 + d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ + d_2\sin\theta\sin\beta_2\cos\alpha_2 + d_1\cos\theta\sin\psi\sin\alpha_1 \\ - d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 + d_1\sin\theta\sin\beta_1\cos\alpha_1 \end{bmatrix}$$

$$+ \frac{\vec{K}}{L}\begin{bmatrix} z_2 - z_1 + d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ - d_2\cos\theta\sin\beta_2\cos\alpha_2 + d_1\sin\theta\sin\psi\sin\alpha_1 \\ - d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 - d_1\cos\theta\sin\beta_1\cos\alpha_1 \end{bmatrix}$$

$$\vec{e}_w = -\frac{\vec{J}}{L}\begin{bmatrix} z_2 - z_1 + d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ - d_2\cos\theta\sin\beta_2\cos\alpha_2 + d_1\sin\theta\sin\psi\sin\alpha_1 \\ - d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 - d_1\cos\theta\sin\beta_1\cos\alpha_1 \end{bmatrix}$$

$$+ \frac{\vec{K}}{L}\begin{bmatrix} y_2 - y_1 + d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ + d_2\sin\theta\sin\beta_2\cos\alpha_2 + d_1\cos\theta\sin\psi\sin\alpha_1 \\ - d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 + d_1\sin\theta\sin\beta_1\cos\alpha_1 \end{bmatrix}$$

3. Motion of the Beam

The position of a point on the beam at any time is given by

$$\vec{R}_c(r,t) = \vec{R}_1 + \vec{d}_1 + r\vec{e}_{p_1p_2} + w(r,t)\,\vec{e}_w + v(r,t)\,\vec{e}_v \qquad \text{------ (8)}$$

where r is the position along the beam longitudinal axes measured from P_1

$w(r,t)$ is the deflection of the beam in the direction of $\vec{e}_w$, measured from the vector $\vec{e}_{p_1p_2}$

$v(r,t)$ is the deflection of the beam in the direction of $\vec{e}_v$, measured from the vector $\vec{e}_{p_1p_2}$

Using the vector identities of the previous article, we write

$$\vec{R}_c = \Big\{ x_1 + d_1(-\cos\psi\sin\alpha_1 - \sin\psi\cos\beta_1\cos\alpha_1) + \frac{r}{L}\Big[x_2 - x_1 + d_2\cos\psi\sin\alpha_2 + d_2\sin\psi\cos\beta_2\cos\alpha_2 + d_1\cos\psi\sin\alpha_1 + d_1\sin\psi\cos\beta_1\cos\alpha_1\Big] + v \Big\}\vec{I}$$

$$+ \Big\{ y_1 + d_1\Big(-\cos\theta\sin\psi\sin\alpha_1 + \cos\theta\cos\psi\cos\beta_1\cos\alpha_1 - \sin\theta\sin\beta_1\cos\alpha_1\Big)$$

$$+ \frac{r}{L}\Big[y_2 - y_1 + d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 + d_2\sin\theta\sin\beta_2\cos\alpha_2 + d_1\cos\theta\sin\psi\sin\alpha_1 - d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 + d_1\sin\theta\sin\beta_1\cos\alpha_1\Big]$$

$$- \frac{w}{L}\Big[z_2 - z_1 + d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 - d_2\cos\theta\sin\beta_2\cos\alpha_2 + d_1\sin\theta\sin\psi\sin\alpha_1 - d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 - d_1\cos\theta\sin\beta_1\cos\alpha_1\Big]$$

$$- \frac{v}{L^2}\Big[x_2 - x_1 + d_2\cos\psi\sin\alpha_2 + d_2\sin\psi\cos\beta_2\cos\alpha_2 + d_1\cos\psi\sin\alpha_1 + d_1\sin\psi\cos\beta_1\cos\alpha_1\Big]$$

$$\cdot\Big[y_2 - y_1 + d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 + d_2\sin\theta\sin\beta_2\cos\alpha_2 + d_1\cos\theta\sin\psi\sin\alpha_1 - d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 + d_1\sin\theta\sin\beta_1\cos\alpha_1\Big]\Big\}\vec{J}$$

$$
\begin{aligned}
&+\Bigg(z_1 + d_1\Big(-\sin\theta\sin\psi\sin\alpha_1 + \sin\theta\cos\psi\cos\beta_1\cos\alpha_1 \\
&\qquad + \cos\theta\sin\beta_1\cos\alpha_1 \Big) \\
&+ \frac{r}{L}\Big[z_2 - z_1 + d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 \\
&\qquad - d_2\cos\theta\sin\beta_2\cos\alpha_2 + d_1\sin\theta\sin\psi\sin\alpha_1 \\
&\qquad - d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 - d_1\cos\theta\sin\beta_1\cos\alpha_1 \Big] \\
&+ \frac{w}{L}\Big[y_2 - y_1 + d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 \\
&\qquad + d_2\sin\theta\sin\beta_2\cos\alpha_2 + d_1\cos\theta\sin\psi\sin\alpha_1 \\
&\qquad - d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 + d_1\sin\theta\sin\beta_1\cos\alpha_1 \Big] \\
&- \frac{v}{L^2}\Big[x_2 - x_1 + d_2\cos\psi\sin\alpha_2 + d_2\sin\psi\cos\beta_2\cos\alpha_2 \\
&\qquad + d_1\cos\psi\sin\alpha_1 + d_1\sin\psi\cos\beta_1\cos\alpha_1 \Big] \\
&\quad \cdot \Big[z_2 - z_1 + d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 \\
&\qquad - d_2\cos\theta\sin\beta_2\cos\alpha_2 + d_1\sin\theta\sin\psi\sin\alpha_1 \\
&\qquad - d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 - d_1\cos\theta\sin\beta_1\cos\alpha_1 \Big] \Bigg) \vec{K}
\end{aligned}
$$

Using small motion approximations for ψ, β_1, β_2, α_1, α_2, v, w and neglecting 3rd order and higher terms, the position vector $\vec{R}_c(r,t)$ becomes

$$
\begin{aligned}
\vec{R}_c = &\Bigg\{ x_1 - \psi\left(d_1 + d_{12}\frac{r}{L} - d_1\frac{r}{L} - d_2\frac{r}{L}\right) \\
&+ \alpha_1\left(d_1\frac{r}{L} - d_1\right) + \alpha_2\, d_2\frac{r}{L} + v \Bigg\}\vec{I} \\
&+\Bigg\{ y_1 + d_1\cos\theta\left(1 - \frac{1}{2}\psi^2 - \frac{1}{2}\beta_1^2 - \frac{1}{2}\alpha_1^2 - \psi\alpha_1\right)\left(1 - \frac{r}{L}\right) \\
&- d_2\cos\theta\left(1 - \frac{1}{2}\psi^2 - \frac{1}{2}\beta_2^2 - \frac{1}{2}\alpha_2^2 - \psi\alpha_2\right)\frac{r}{L} \\
&+ d_{12}\cos\theta\frac{r}{L} - d_1\beta_1\sin\theta\left(1 - \frac{r}{L}\right) + d_2\beta_2\sin\theta\,\frac{r}{L} \Bigg\}\vec{J}
\end{aligned}
$$

$$
\begin{aligned}
&\left\{ -\frac{w}{L}\left[(d_{12} - d_1 - d_2)\sin\theta - (d_1\beta_1 + d_2\beta_2)\cos\theta\right] \right. \\
&\left. -\frac{v}{L^2}\left[-\psi(d_{12} - d_1 - d_2) + (d_1\alpha_1 + d_2\alpha_2)\right](d_{12} - d_1 - d_2)\cos\theta \right\} \\
&+\left\{ z_1 + d_1\sin\theta\left(1 - \frac{1}{2}\psi^2 - \frac{1}{2}\beta_1^2 - \frac{1}{2}\alpha_1^2 - \psi\alpha_1\right)\left(1 - \frac{r}{L}\right) \right. \\
&\quad - d_2\sin\theta\left(1 - \frac{1}{2}\psi^2 - \frac{1}{2}\beta_2^2 - \frac{1}{2}\alpha_2^2 - \psi\alpha_2\right)\frac{r}{L} + d_{12}\sin\theta\frac{r}{L} \\
&\quad + d_1\beta_1\cos\theta\left(1 - \frac{r}{L}\right) - d_2\beta_2\cos\theta\frac{r}{L} \\
&\quad + \frac{w}{L}\left[(d_{12} - d_1 - d_2)\cos\theta + (d_1\beta_1 + d_2\beta_2)\sin\theta\right] \\
&\quad \left. - \frac{v}{L^2}\left[-\psi(d_{12} - d_1 - d_2) + (d_1\alpha_1 + d_2\alpha_2)\right](d_{12} - d_1 - d_2)\sin\theta \right\}\vec{K}
\end{aligned}
$$

------ (9)

Thus the vector velocity of a point on the beam is

$$
\begin{aligned}
\dot{\vec{R}}_c(r,t) = &\left\{ \dot{x}_1 - \dot{\psi}\left(d_1 + d_{12}\frac{r}{L} - d_1\frac{r}{L} - d_2\frac{r}{L}\right) - \psi\dot{d}_{12}\frac{r}{L} + d_1\dot{\alpha}_1\left(\frac{r}{L} - 1\right) \right. \\
&\left. + d_2\dot{\alpha}_2\frac{r}{L} + \dot{v} \right\}\vec{I} \\
&+\left\{ \dot{y}_1 + d_1\cos\theta(-\psi\dot{\psi} - \beta_1\dot{\beta}_1 - \alpha_1\dot{\alpha}_1 - \psi\dot{\alpha}_1 - \alpha_1\dot{\psi})\left(1 - \frac{r}{L}\right) \right. \\
&\quad - d_1\dot{\theta}\sin\theta\left(1 - \frac{1}{2}\psi^2 - \frac{1}{2}\beta_1^2 - \frac{1}{2}\alpha_1^2 - \psi\alpha_1\right)\left(1 - \frac{r}{L}\right) \\
&\quad - d_2\cos\theta(-\psi\dot{\psi} - \beta_2\dot{\beta}_2 - \alpha_2\dot{\alpha}_2 - \psi\dot{\alpha}_2 - \alpha_2\dot{\psi})\frac{r}{L} \\
&\quad + d_2\dot{\theta}\sin\theta\left(1 - \frac{1}{2}\psi^2 - \frac{1}{2}\beta_2^2 - \frac{1}{2}\alpha_2^2 - \psi\alpha_2\right)\frac{r}{L} - d_{12}\dot{\theta}\sin\theta\frac{r}{L} \\
&\quad + \dot{d}_{12}\cos\theta\frac{r}{L} - d_1\beta_1\dot{\theta}\cos\theta\left(1 - \frac{r}{L}\right) - d_1\dot{\beta}_1\sin\theta\left(1 - \frac{r}{L}\right) \\
&\quad \left. + d_2\beta_2\dot{\theta}\cos\theta\frac{r}{L} + d_2\dot{\beta}_2\sin\theta\frac{r}{L} \right\}\vec{J}
\end{aligned}
$$

$$
\begin{aligned}
\Bigg\{ & -\frac{w}{L}\Big[(d_{12}-d_1-d_2)\dot{\theta}\cos\theta + \dot{d}_{12}\sin\theta + (d_1\beta_1 + d_2\beta_2)\dot{\theta}\sin\theta \\
& \qquad -(d_1\dot{\beta}_1 + d_2\dot{\beta}_2)\cos\theta\Big] \\
& -\frac{\dot{w}}{L}\Big[(d_{12}-d_1-d_2)\sin\theta - (d_1\beta_1 + d_2\beta_2)\cos\theta\Big] \\
& -\frac{v}{L^2}\Big[-\psi(d_{12}-d_1-d_2) + (d_1\alpha_1 + d_2\alpha_2)\Big] \\
& \cdot\Big[-(d_{12}-d_1-d_2)\dot{\theta}\sin\theta + \dot{d}_{12}\cos\theta\Big] \\
& -\frac{v}{L^2}\Big[-\dot{d}_{12}\psi - \dot{\psi}(d_{12}-d_1-d_2) + (d_1\dot{\alpha}_1 + d_2\dot{\alpha}_2)\Big] \\
& \cdot\Big[(d_{12}-d_1-d_2)\cos\theta\Big] \\
& -\frac{\dot{v}}{L^2}\Big[-\psi(d_{12}-d_1-d_2) + (d_1\alpha_1 + d_2\alpha_2)\Big](d_{12}-d_1-d_2)\cos\theta \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
+\Bigg\{ & \dot{z}_1 + d_1\sin\theta(-\psi\dot{\psi} - \beta_1\dot{\beta}_1 - \alpha_1\dot{\alpha}_1 - \psi\dot{\alpha}_1 - \alpha_1\dot{\psi})\left(1-\frac{r}{L}\right) \\
& + d_1\dot{\theta}\cos\theta\left(1 - \frac{1}{2}\psi^2 - \frac{1}{2}\beta_1^2 - \frac{1}{2}\alpha_1^2 - \psi\alpha_1\right)\left(1-\frac{r}{L}\right) \\
& - d_2\sin\theta(-\psi\dot{\psi} - \beta_2\dot{\beta}_2 - \alpha_2\dot{\alpha}_2 - \psi\dot{\alpha}_2 - \alpha_2\dot{\psi})\frac{r}{L} \\
& - d_2\dot{\theta}\cos\theta\left(1 - \frac{1}{2}\psi^2 - \frac{1}{2}\beta_2^2 - \frac{1}{2}\alpha_2^2 - \psi\alpha_2\right)\frac{r}{L} \\
& + d_{12}\dot{\theta}\cos\theta\frac{r}{L} + \dot{d}_{12}\sin\theta\frac{r}{L} - d_1\beta_1\dot{\theta}\sin\theta\left(1-\frac{r}{L}\right) \\
& + d_1\dot{\beta}_1\cos\theta\left(1-\frac{r}{L}\right) + d_2\beta_2\dot{\theta}\sin\theta\frac{r}{L} - d_2\dot{\beta}_2\cos\theta\frac{r}{L} \\
& + \frac{w}{L}\Big[-(d_{12}-d_1-d_2)\dot{\theta}\sin\theta + \dot{d}_{12}\cos\theta \\
& \qquad + (d_1\beta_1 + d_2\beta_2)\dot{\theta}\cos\theta + (d_1\dot{\beta}_1 + d_2\dot{\beta}_2)\sin\theta\Big] \\
& + \frac{\dot{w}}{L}\Big[(d_{12}-d_1-d_2)\cos\theta + (d_1\beta_1 + d_2\beta_2)\sin\theta\Big] \Bigg\} \vec{K}
\end{aligned}
$$

$$\left\{\begin{array}{l} -\frac{v}{L^2}\left[-\psi(d_{12} - d_1 - d_2) + (d_1\alpha_1 + d_2\alpha_2)\right] \\ \cdot\left[(d_{12} - d_1 - d_2)\dot{\theta}\cos\theta + \dot{d}_{12}\sin\theta\right] \\ -\frac{v}{L^2}\left[-\dot{d}_{12}\psi - \dot{\psi}(d_{12} - d_1 - d_2) + (d_1\dot{\alpha}_1 + d_2\dot{\alpha}_2)\right] \\ \cdot\left[(d_{12} - d_1 - d_2)\sin\theta\right] \\ -\frac{\dot{v}}{L^2}\left[-\psi(d_{12} - d_1 - d_2) + (d_1\alpha_1 + d_2\alpha_2)\right](d_{12} - d_1 - d_2)\sin\theta \end{array}\right\}$$

------ (10)

4. Constraint Equation for the Configuration

The vector from the center of mass of Stage 1 to the center of mass of Stage 2 has been previously identified as $\vec{d}_{12}$. Consider the plane formed by $\vec{d}_{12}$ and $\vec{k}$ as shown in Figure 5.

The angle between the beam longitudinal axis and the plane of $\vec{d}_{12}$, $\vec{k}$ is $\bar{\alpha}$, where

$$\sin\bar{\alpha} \simeq \bar{\alpha} = \frac{d_1\sin\alpha_1 + d_2\sin\alpha_2}{d_{p_1p_2}} \simeq \frac{d_1\alpha_1 + d_2\alpha_2}{L}$$

and $\cos\bar{\alpha} \simeq 1 - \frac{1}{2}\bar{\alpha}^2 = 1 - \frac{1}{2L^2}(d_1\alpha_1 + d_2\alpha_2)^2$

so the projection of the beam onto the $\vec{d}_{12}$, $\vec{k}$ plane is $L_D\cos\bar{\alpha}$. L_D represents the deflected length of the beam.

Similarly, the angle formed by the line of length $L_D\cos\bar{\alpha}$ with the line drawn between points 1 and 2 is $\bar{\beta}$, where

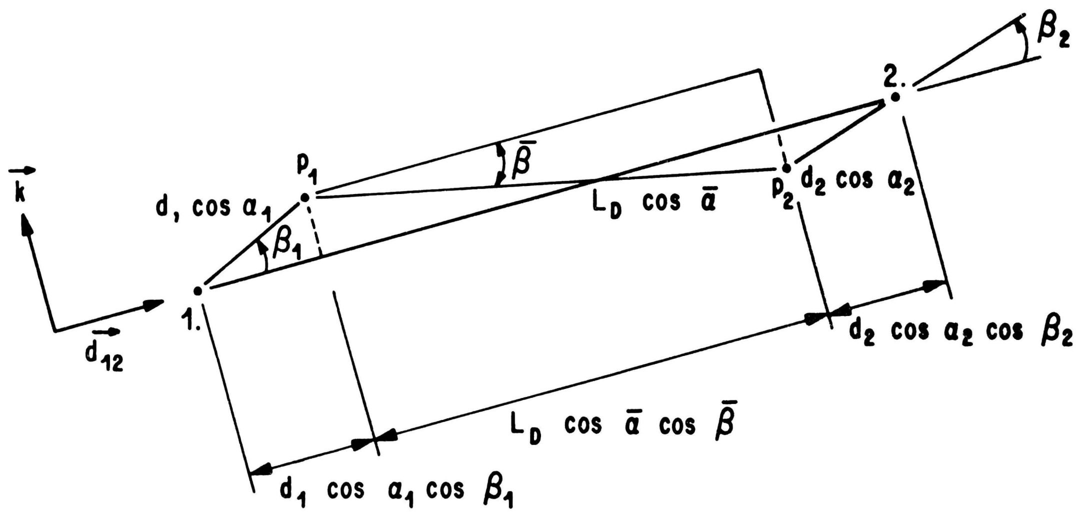

FIG. 5. THE PLANE FORMED BY $\vec{d}_{12}$, $\vec{k}$ SHOWING THE PROJECTED LENGTHS OF THE CONFIGURATION

$$\sin\bar{\beta} \simeq \bar{\beta} = \frac{d_1\cos\alpha_1\sin\beta_1 + d_2\cos\alpha_2\sin\beta_2}{L_D\cos\bar{\alpha}}$$

$$\simeq \frac{d_1\beta_1\left(1 - \frac{1}{2}\alpha_1^2\right) + d_2\beta_2\left(1 - \frac{1}{2}\alpha_2^2\right)}{L_D\left[1 - \frac{1}{2L^2}(d_1\alpha_1 + d_2\alpha_2)^2\right]} \simeq \frac{d_1\beta_1 + d_2\beta_2}{L}$$

(neglecting higher order terms)

and $\cos\bar{\beta} \simeq 1 - \frac{1}{2}\bar{\beta}^2 = 1 - \frac{1}{2L^2}(d_1\beta_1 + d_2\beta_2)^2$

Thus we have

$$d_{12}(t) = d_1\cos\alpha_1\cos\beta_1 + L_D\cos\bar{\alpha}\cos\bar{\beta} + d_2\cos\alpha_2\cos\beta_2$$

$$\simeq d_1\left(1 - \frac{1}{2}\alpha_1^2 - \frac{1}{2}\beta_1^2\right)$$

$$+ L\left[1 - \frac{1}{2L^2}(d_1\alpha_1 + d_2\alpha_2)^2 - \frac{1}{2L^2}(d_1\beta_1 + d_2\beta_2)^2\right]$$

$$- \frac{1}{2}\int_0^L\left[\left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial w}{\partial r}\right)^2\right]dr + d_2\left(1 - \frac{1}{2}\alpha_2^2 - \frac{1}{2}\beta_2^2\right) \quad \text{-----} \ (11)$$

which is the constraint equation of the configuration.

5. Kinetic Energy of the Space Station

The total kinetic energy of the space station is composed of the kinetic energies of Stages 1 and 2 plus the kinetic energy of the beam. Therefore

$$T_{Total} = \frac{1}{2}M_A\dot{\vec{R}}_1^2 + \frac{1}{2}(IA)_1\omega_{1_1}^2 + \frac{1}{2}(IA)_2\omega_{2_1}^2 + \frac{1}{2}(IA)_3\omega_{3_1}^2$$

$$+ \frac{1}{2}M_B\dot{\vec{R}}_2^2 + \frac{1}{2}(IB)_1\omega_{1_2}^2 + \frac{1}{2}(IB)_2\omega_{2_2}^2 + \frac{1}{2}(IB)_3\omega_{3_2}^2$$

$$+ \frac{1}{2}\int_0^L m\dot{\vec{R}}_c^2\,dr \qquad \text{------ (12)}$$

where $\dot{\vec{R}}_1$ and $\dot{\vec{R}}_2$ are given in equation (2)

ω_{1_i}, ω_{2_i}, ω_{3_i} are given in equations (5), (6) and (7)

$\dot{\vec{R}}_c$ is given in equation (10).

6. Potential Energy of the Space Station

For an orbiting space station the gravity field of the earth is associated with the centripetal acceleration of the mass center of the space station in its rotation about the center of the earth. We wish to study only the motion of the space station superimposed on the translation of its center of mass and neglect the small effect of the gravity gradient. Therefore we consider that the potential energy of the space station is the internal bending energy of the connecting beam, written as

$$U = \frac{1}{2}\int_0^L EI_w\left(\frac{\partial^2 w}{\partial r^2}\right)^2 dr + \frac{1}{2}\int_0^L EI_v\left(\frac{\partial^2 v}{\partial r^2}\right)^2 dr \qquad \text{------ (13)}$$

and for the beam used in the analysis $I_w = I_v = I$.

IV. DERIVATION OF THE GOVERNING EQUATIONS

1. Development of the Lagrangian Function

We now have the kinetic energy and potential energy of the space station given in equations (12) and (13). We write the Lagrangian K = T - V to obtain

$$K = \frac{1}{2}M_A\dot{\vec{R}}_1^2 + \frac{1}{2}(IA)_1\,\omega_{1_1}^2 + \frac{1}{2}(IA)_2\,\omega_{2_1}^2 + \frac{1}{2}(IA)_3\,\omega_{3_1}^2$$

$$+ \frac{1}{2}M_B\dot{\vec{R}}_2^2 + \frac{1}{2}(IB)_1\,\omega_{1_2}^2 + \frac{1}{2}(IB)_2\,\omega_{2_2}^2 + \frac{1}{2}(IB)_3\,\omega_{3_2}^2$$

$$+ \frac{1}{2}\int_0^L m\dot{\vec{R}}_c^2\,dr$$

$$- \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial r^2}\right)^2 dr - \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 v}{\partial r^2}\right)^2 dr \qquad \text{------ (14)}$$

2. Hamilton's Principle

By application of Hamilton's Principle in accordance with The Calculus of Variations we obtain, after substituting the constraint equation and neglecting second order terms,

$$\delta \int_{t_1}^{t_2} K\,dt = 0$$

$$= \int_{t_1}^{t_2} \left\{ \begin{array}{l} -M_A\ddot{x}_1 - M_B(\ddot{x}_1 - \tilde{L}\ddot{\psi}) \\ -mL\left[\ddot{x}_1 - \ddot{\psi}(d_1 + \frac{1}{2}L) - \frac{1}{2}d_1\ddot{\alpha}_1 + \frac{1}{2}d_2\ddot{\alpha}_2\right] + \int_0^L -m\ddot{v}dr \end{array} \right\} \delta x_1 dt$$

$$+ \int_{t_1}^{t_2} \left(\begin{array}{l} -M_A\ddot{y}_1 - M_B(\ddot{y}_1 - \tilde{L}\dot{\theta}^2\cos\theta - \tilde{L}\ddot{\theta}\sin\theta) \\ -mL\left[\begin{array}{l} \ddot{y}_1 - \frac{1}{2}d_1\dot{\theta}^2\cos\theta - \frac{1}{2}d_1\ddot{\theta}\sin\theta + \frac{1}{2}d_2\dot{\theta}^2\cos\theta + \frac{1}{2}d_2\ddot{\theta}\sin\theta \\ - \frac{1}{2}\tilde{L}\dot{\theta}^2\cos\theta - \frac{1}{2}\tilde{L}\ddot{\theta}\sin\theta + \frac{1}{2}d_1\beta_1\dot{\theta}^2\sin\theta - \frac{1}{2}d_1\beta_1\ddot{\theta}\cos\theta \\ - d_1\dot{\beta}_1\dot{\theta}\cos\theta - \frac{1}{2}d_1\ddot{\beta}_1\sin\theta - \frac{1}{2}d_2\beta_2\dot{\theta}^2\sin\theta \\ + \frac{1}{2}d_2\beta_2\ddot{\theta}\cos\theta + d_2\dot{\beta}_2\dot{\theta}\cos\theta + \frac{1}{2}d_2\ddot{\beta}_2\sin\theta \end{array} \right] \\ + \int_0^L -m(w\dot{\theta}^2\sin\theta - w\ddot{\theta}\cos\theta - 2\dot{w}\dot{\theta}\cos\theta - \ddot{w}\sin\theta)\,dr \end{array} \right) \delta y_1 dt$$

$$+ \int_{t_1}^{t_2} \left(\begin{array}{l} -M_A\ddot{z}_1 - M_B(\ddot{z}_1 - \tilde{L}\dot{\theta}^2\sin\theta + \tilde{L}\ddot{\theta}\cos\theta) \\ -mL\left[\begin{array}{l} \ddot{z}_1 - \frac{1}{2}d_1\dot{\theta}^2\sin\theta + \frac{1}{2}d_1\ddot{\theta}\cos\theta + \frac{1}{2}d_2\dot{\theta}^2\sin\theta - \frac{1}{2}d_2\ddot{\theta}\cos\theta \\ - \frac{1}{2}\tilde{L}\dot{\theta}^2\sin\theta + \frac{1}{2}\tilde{L}\ddot{\theta}\cos\theta - \frac{1}{2}d_1\beta_1\dot{\theta}^2\cos\theta - \frac{1}{2}d_1\beta_1\ddot{\theta}\sin\theta \\ - d_1\dot{\beta}_1\dot{\theta}\sin\theta + \frac{1}{2}d_1\ddot{\beta}_1\cos\theta + \frac{1}{2}d_2\beta_2\dot{\theta}^2\cos\theta \\ + \frac{1}{2}d_2\beta_2\ddot{\theta}\sin\theta + d_2\dot{\beta}_2\dot{\theta}\sin\theta - \frac{1}{2}d_2\ddot{\beta}_2\cos\theta \end{array} \right] \\ + \int_0^L -m(-w\dot{\theta}^2\cos\theta - w\ddot{\theta}\sin\theta - 2\dot{w}\dot{\theta}\sin\theta + \ddot{w}\cos\theta)\,dr \end{array} \right) \delta z_1 dt$$

$$+\int_{t_1}^{t_2}\left\{\begin{array}{l} -M_B\tilde{L}\left(\ddot{z}_1\cos\theta-\ddot{y}_1\sin\theta+L\ddot{\theta}\right)-(IA)_1(\ddot{\theta}+\ddot{\beta}_1) \\ -(IB)_1(\ddot{\theta}+\ddot{\beta}_2) \\ -mL\left[\begin{array}{l} \frac{1}{3}d_1^2\ddot{\theta}+\frac{1}{3}d_2^2\ddot{\theta}+\frac{1}{3}\tilde{L}^2\ddot{\theta}-\frac{1}{2}\ddot{y}_1d_1\sin\theta+\frac{1}{2}\ddot{y}_1d_2\sin\theta \\ -\frac{1}{2}\ddot{y}_1\tilde{L}\sin\theta-\frac{1}{2}\ddot{y}_1d_1\beta_1\cos\theta+\frac{1}{2}\ddot{y}_1d_2\beta_2\cos\theta \\ +\frac{1}{2}\ddot{z}_1d_1\cos\theta-\frac{1}{2}\ddot{z}_1d_2\cos\theta+\frac{1}{2}\ddot{z}_1\tilde{L}\cos\theta \\ -\frac{1}{2}\ddot{z}_1d_1\beta_1\sin\theta+\frac{1}{2}\ddot{z}_1d_2\beta_2\sin\theta-\frac{1}{3}d_1d_2\ddot{\theta}+\frac{1}{3}d_1\tilde{L}\ddot{\theta} \\ +\frac{1}{3}d_1^2\ddot{\beta}_1-\frac{1}{6}d_1d_2\ddot{\beta}_2-\frac{2}{3}d_2\tilde{L}\ddot{\theta}-\frac{1}{6}d_1d_2\ddot{\beta}_1+\frac{1}{3}d_2^2\ddot{\beta}_2 \\ +\frac{1}{6}d_1\tilde{L}\ddot{\beta}_1-\frac{1}{3}d_2\tilde{L}\ddot{\beta}_2 \end{array}\right] \\ +\int_0^L -m\left[\begin{array}{l} -\ddot{y}_1w\cos\theta-\ddot{z}_1w\sin\theta-d_1\ddot{w}\left(\frac{r}{L}-1\right) \\ -d_2\ddot{w}\frac{r}{L}+\tilde{L}\ddot{w}\frac{r}{L} \end{array}\right]dr \end{array}\right\}\delta\theta\,dt$$

$$+\int_{t_1}^{t_2}\left\{\begin{array}{l} M_B\tilde{L}(\ddot{x}_1-\tilde{L}\ddot{\psi})+(IA)_1(\dot{\theta}\dot{\gamma}_1+\ddot{\theta}\gamma_1-\dot{\theta}^2\psi-\dot{\theta}^2\alpha_1) \\ +(IA)_2\dot{\theta}(\dot{\theta}\psi+\dot{\theta}\alpha_1-\dot{\gamma}_1)-(IA)_3(\dot{\theta}\dot{\gamma}_1+\ddot{\theta}\gamma_1+\ddot{\alpha}_1+\ddot{\psi}) \\ +(IB)_1(\dot{\theta}\dot{\gamma}_2+\ddot{\theta}\gamma_2-\dot{\theta}^2\psi-\dot{\theta}^2\alpha_2)+(IB)_2\dot{\theta}(\dot{\theta}\psi+\dot{\theta}\alpha_2-\dot{\gamma}_2) \\ -(IB)_3(\dot{\theta}\dot{\gamma}_2+\ddot{\theta}\gamma_2+\ddot{\alpha}_2+\ddot{\psi}) \end{array}\right\}\delta\psi\,dt$$

$$
\Bigg\{ -mL\Bigg[\ddot{\psi}\left(d_1^2 + d_1L + \frac{1}{3}L^2\right) - \ddot{x}_1\left(d_1 + \frac{1}{2}L\right) + \frac{1}{2}d_1\ddot{\alpha}_1\left(d_1 + \frac{1}{3}L\right)
$$

$$
- d_2\ddot{\alpha}_2\left(\frac{1}{2}d_1 + \frac{1}{3}L\right) - \frac{1}{2}\ddot{y}_1 d_1\cos\theta\,(\psi + \alpha_1)
$$

$$
+ \frac{1}{2}\ddot{y}_1 d_2\cos\theta\,(\psi + \alpha_2) - \frac{1}{2}\ddot{z}_1 d_1\sin\theta\,(\psi + \alpha_1)
$$

$$
+ \frac{1}{2}\ddot{z}_1 d_2\sin\theta\,(\psi + \alpha_2) + \frac{1}{3}d_1^2\dot{\theta}^2(\psi + \alpha_1) + \frac{1}{3}d_2^2\dot{\theta}^2(\psi + \alpha_2)
$$

$$
- \frac{1}{6}d_1 d_2\dot{\theta}^2(2\psi + \alpha_1 + \alpha_2) + \frac{1}{6}d_1\tilde{L}\dot{\theta}^2(\psi + \alpha_1)
$$

$$
- \frac{1}{3}d_2\tilde{L}\dot{\theta}^2(\psi + \alpha_2)\Bigg]
$$

$$
+ \int_0^L -m\Bigg[-(d_1 + r)\ddot{v} + \ddot{y}_1\cos\theta\, v + \ddot{z}_1\sin\theta\, v + d_1\dot{\theta}^2\left(\frac{r}{L} - 1\right)v
$$

$$
d_2\dot{\theta}^2\frac{r}{L}v - \tilde{L}\dot{\theta}^2\frac{r}{L}v\Bigg]dr\Bigg\}
$$

$$
+ \int_{t_1}^{t_2}\Bigg[\left\{M_B d_1\left[\left(1 + \frac{d_1}{L}\right)\beta_1 + \frac{d_2}{L}\beta_2\right] + \frac{1}{2}mLd_1\beta_1\right\}(\ddot{y}_1\cos\theta + \ddot{z}_1\sin\theta)\,\delta\beta_1 dt
$$

$$
+ \frac{1}{2}mLd_1\left[\left(1 + \frac{d_1}{L}\right)\beta_1 + \frac{d_2}{L}\beta_2\right]
$$

$$
- m\frac{d_1}{L}\int_0^L w\,dr
$$

$$
- \frac{1}{2}mLd_1(\ddot{z}_1\cos\theta - \ddot{y}_1\sin\theta)
$$

$$
- \left[M_B d_1\tilde{L}\dot{\theta}^2 + \frac{1}{3}mLd_1\tilde{L}\dot{\theta}^2 + \frac{1}{6}mLd_1^2\dot{\theta}^2 - \frac{1}{3}mLd_1 d_2\dot{\theta}^2\right]
$$

$$
\cdot\left[\left(1 + \frac{d_1}{L}\right)\beta_1 + \frac{d_2}{L}\beta_2\right] - (IA)_1(\ddot{\theta} + \ddot{\beta}_1) - \frac{1}{3}mLd_1^2\ddot{\beta}_1
$$

$$
\left. + \frac{1}{6}mLd_1d_2\ddot{\beta}_2 - mL\left(\frac{1}{2}d_1^2 + \frac{1}{6}d_1L\right)\ddot{\theta} - \frac{1}{6}mLd_1(d_1 + L)\dot{\theta}^2\beta_1 - \frac{1}{6}mLd_1d_2\dot{\theta}^2\beta_2 - m\frac{d_1}{L}\int_0^L\left[-(r - L)\ddot{w} - d_1\dot{\theta}^2 w\right]dr \right]
$$

$$
+ \int_{t_1}^{t_2} \left[\left\{ \begin{array}{l} M_B d_2\left[\left(1 + \frac{d_2}{L}\right)\beta_2 + \frac{d_1}{L}\beta_1\right] - \frac{1}{2}mLd_2\beta_2 \\ + \frac{1}{2}mLd_2\left[\left(1 + \frac{d_2}{L}\right)\beta_2 + \frac{d_1}{L}\beta_1\right] \\ - m\frac{d_2}{L}\int_0^L w\,dr \end{array} \right\} (\ddot{y}_1\cos\theta + \ddot{z}_1\sin\theta) \right.
$$

$$
+ \frac{1}{2}mLd_2(\ddot{z}_1\cos\theta - \ddot{y}_1\sin\theta) - (IB)_1(\ddot{\theta} + \ddot{\beta}_2)
$$

$$
- \left[M_B d_2\tilde{L}\dot{\theta}^2 + \frac{1}{3}mLd_2\tilde{L}\dot{\theta}^2 + \frac{1}{6}mLd_1d_2\dot{\theta}^2 - \frac{1}{3}mLd_2^2\dot{\theta}^2\right]
$$

$$
\cdot\left[\left(1 + \frac{d_2}{L}\right)\beta_2 + \frac{d_1}{L}\beta_1\right] - \frac{1}{3}mLd_2^2\ddot{\beta}_2 + \frac{1}{6}mLd_1d_2\ddot{\beta}_1
$$

$$
+ mL\left(\frac{1}{2}d_1d_2 + \frac{1}{3}d_2L\right)\ddot{\theta} + mL\left(\frac{1}{2}d_1d_2 + \frac{1}{3}d_2L + \frac{1}{3}d_2^2\right)\dot{\theta}^2\beta_2
$$

$$
\left. - \frac{1}{6}mLd_1d_2\dot{\theta}^2\beta_1 - m\frac{d_2}{L}\int_0^L(-r\ddot{w} - d_1\dot{\theta}^2 w)\,dr \right] \delta\beta_2 dt
$$

$$
+ \int_{t_1}^{t_2} \left[\left\{ \begin{array}{l} M_B d_1\left[\left(1 + \frac{d_1}{L}\right)\alpha_1 + \frac{d_2}{L}\alpha_2\right] \\ + \frac{1}{2}mLd_1\left[\left(1 + \frac{d_1}{L}\right)\alpha_1 + \frac{d_2}{L}\alpha_2\right] \\ + m\frac{d_1}{L}\int_0^L v\,dr + \frac{1}{2}mLd_1(\psi + \alpha_1) \end{array} \right\} (\ddot{y}_1\cos\theta + \ddot{z}_1\sin\theta) \right. \quad \delta\alpha_1 dt
$$

$$- \left[M_B d_1 \tilde{L} \dot{\theta}^2 + \frac{1}{3} m L d_1 \tilde{L} \dot{\theta}^2 - \frac{1}{3} m L d_1 d_2 \dot{\theta}^2 + \frac{1}{6} m L d_1^2 \dot{\theta}^2 \right]$$

$$\cdot \left[\left(1 + \frac{d_1}{L} \right) \alpha_1 + \frac{d_2}{L} \alpha_2 \right] + (IA)_1 (\dot{\theta} \dot{\gamma}_1 + \ddot{\theta} \gamma_1 - \dot{\theta}^2 \psi - \dot{\theta}^2 \alpha_1)$$

$$+ (IA)_2 (\dot{\theta}^2 \psi + \dot{\theta}^2 \alpha_1 - \dot{\theta} \dot{\gamma}_1) - (IA)_3 (\dot{\theta} \dot{\gamma}_1 + \ddot{\theta} \gamma_1 + \ddot{\psi} + \ddot{\alpha}_1)$$

$$- \frac{1}{3} m L d_1^2 \ddot{\alpha}_1 + \frac{1}{2} m L d_1 \ddot{x}_1 - \frac{1}{2} m L d_1 \ddot{\psi} \left(d_1 + \frac{1}{3} L \right) + \frac{1}{6} m L d_1 d_2 \ddot{\alpha}_2$$

$$- m L d_1 \left(\frac{1}{2} d_1 + \frac{1}{6} L \right) (\psi + \alpha_1) \dot{\theta}^2$$

$$- m \frac{d_1}{L} \int_0^L \left[(r - L) \ddot{v} + d_1 \dot{\theta}^2 v + r \dot{\theta}^2 v \right] dr \Bigg]$$

$$+ \int_{t_1}^{t_2} \Bigg[\Bigg\{ M_B d_2 \left[\left(1 + \frac{d_2}{L} \right) \alpha_2 + \frac{d_1}{L} \alpha_1 \right]$$

$$- \frac{1}{2} m L d_2 (\psi + \alpha_2)$$

$$+ \frac{1}{2} m L d_2 \left[\left(1 + \frac{d_2}{L} \right) \alpha_2 + \frac{d_1}{L} \alpha_1 \right]$$

$$+ m \frac{d_2}{L} \int_0^L v \, dr \Bigg\} (\ddot{y}_1 \cos\theta + \ddot{z}_1 \sin\theta)$$

$$- \left[M_B d_2 \tilde{L} \dot{\theta}^2 + \frac{1}{3} m L d_2 \tilde{L} \dot{\theta}^2 + \frac{1}{6} m L d_1 d_2 \dot{\theta}^2 - \frac{1}{3} m L d_2^2 \dot{\theta}^2 \right]$$

$$\cdot \left[\left(1 + \frac{d_2}{L} \right) \alpha_2 + \frac{d_1}{L} \alpha_1 \right] + (IB)_1 (\dot{\theta} \dot{\gamma}_2 + \ddot{\theta} \gamma_2 - \dot{\theta}^2 \psi - \dot{\theta}^2 \alpha_2)$$

$$+ (IB)_2 (\dot{\theta}^2 \psi + \dot{\theta}^2 \alpha_2 - \dot{\theta} \dot{\gamma}_2) - (IB)_3 (\dot{\theta} \dot{\gamma}_2 + \ddot{\theta} \gamma_2 + \ddot{\psi} + \ddot{\alpha}_2)$$

$$- \frac{1}{3} m L d_2^2 \ddot{\alpha}_2 - \frac{1}{3} m L d_2 \ddot{x}_1 + m L d_2 \ddot{\psi} \left(\frac{1}{2} d_1 + \frac{1}{3} L \right)$$

$$\delta \alpha_2 \, dt$$

$$
\left[+ \frac{1}{6} m L d_1 d_2 \ddot{\alpha}_1 + m L d_2 \left(\frac{1}{2} d_1 + \frac{1}{3} L \right) (\psi + \alpha_2)\, \dot{\theta}^2 \right.
$$

$$
\left. - m \frac{d_2}{L} \int_0^L (r\ddot{v} + d_1 \dot{\theta}^2 v + r \dot{\theta}^2 v)\, dr \right]
$$

$$
+ \int_{t_1}^{t_2} \left[\begin{array}{l} -(IA)_1(\dot{\theta}^2 \gamma_1 + \dot{\theta}\dot{\psi} + \dot{\theta}\dot{\alpha}_1) + (IA)_2(\theta\ddot{\psi} + \theta\ddot{\alpha}_1 + \ddot{\theta}\psi + \ddot{\theta}\alpha_1 - \ddot{\gamma}_1) \\ +(IA)_3(\dot{\theta}^2 \gamma_1 + \dot{\theta}\dot{\psi} + \dot{\theta}\dot{\alpha}_1) \end{array} \right] \delta\gamma_1 dt
$$

$$
+ \int_{t_1}^{t_2} \left[\begin{array}{l} -(IB)_1(\dot{\theta}^2 \gamma_2 + \dot{\theta}\dot{\psi} + \dot{\theta}\dot{\alpha}_2) + (IB)_2(\theta\ddot{\psi} + \theta\ddot{\alpha}_2 + \ddot{\theta}\psi + \ddot{\theta}\alpha_2 - \ddot{\gamma}_2) \\ +(IB)_3(\dot{\theta}^2 \gamma_2 + \dot{\theta}\dot{\psi} + \dot{\theta}\dot{\alpha}_2) \end{array} \right] \delta\gamma_2 dt
$$

$$
+ \int_{t_1}^{t_2} \int_0^L \left\{ \begin{array}{l} - \left[M_B \frac{\partial^2 w}{\partial r^2} + m \left(\frac{d_1}{L} \beta_1 + \frac{d_2}{L} \beta_2 \right) + \frac{1}{2} m L \frac{\partial^2 w}{\partial r^2} \right] \\ \cdot \left[\ddot{y}_1 \cos\theta + \ddot{z}_1 \sin\theta \right] - m(\ddot{z}_1 \cos\theta - \ddot{y}_1 \sin\theta) \\ + \left[M_B \tilde{L} + m L \left(\frac{1}{2} d_1 + \frac{1}{3} L \right) \right] \dot{\theta}^2 \frac{\partial^2 w}{\partial r^2} - m(\ddot{w} - w\dot{\theta}^2) \\ - m(d_1 + r)\ddot{\theta} + m d_1 \ddot{\beta}_1 \left(\frac{r}{L} - 1 \right) + m d_2 \ddot{\beta}_2 \frac{r}{L} \\ + m d_1 \left(1 + \frac{d_1}{L} \right) \dot{\theta}^2 \beta_1 + m d_1 \frac{d_2}{L} \dot{\theta}^2 \beta_2 - EI \frac{\partial^4 w}{\partial r^4} \end{array} \right\} \delta w\, dr\, dt
$$

$$
+ \int_{t_1}^{t_2} \left[-EI \frac{\partial^2 w}{\partial r^2} \delta \left(\frac{\partial w}{\partial r} \right) \right] \Bigg|_{r = 0}^{L} dt
$$

$$+\int_{t_1}^{t_2}\int_0^L\left\{\left[-M_B\frac{\partial^2 v}{\partial r^2}+m\left(-\psi+\frac{d_1}{L}\alpha_1+\frac{d_2}{L}\alpha_2\right)-\frac{1}{2}mL\frac{\partial^2 v}{\partial r^2}\right]\right.$$
$$\cdot\left[\ddot{y}_1\cos\theta+\ddot{z}_1\sin\theta\right]$$
$$+\left[M_B\tilde{L}+mL\left(\frac{1}{2}d_1+\frac{1}{3}L\right)\right]\dot{\theta}^2\frac{\partial^2 v}{\partial r^2}-m\ddot{v}-m\ddot{x}_1$$
$$+m\ddot{\psi}(d_1+r)-md_1\ddot{\alpha}_1\left(\frac{r}{L}-1\right)-md_2\ddot{\alpha}_2\frac{r}{L}$$
$$\left.-m\dot{\theta}^2\left(-\psi+\frac{d_1}{L}\alpha_1+\frac{d_2}{L}\alpha_2\right)(d_1+r)-EI\frac{\partial^4 v}{\partial r^4}\right\}\delta v\,dr\,dt$$

$$+\int_{t_1}^{t_2}\left[-EI\frac{\partial^2 v}{\partial r^2}\delta\left(\frac{\partial v}{\partial r}\right)\right]\Bigg|_{r=0}^{L}dt \qquad \text{------ (15)}$$

Now we note that x_1, y_1, z_1 are independent generalized coordinates, so we have three equations associated with the coordinates, written as

$$\ddot{x}_1=\left[\frac{M_B\tilde{L}+mL\left(d_1+\frac{1}{2}L\right)}{M_T}\right]\ddot{\psi}+\frac{mLd_1}{2M_T}\ddot{\alpha}_1-\frac{mLd_2}{2M_T}\ddot{\alpha}_2$$
$$-\frac{m}{M_T}\int_0^L\ddot{v}\,dr \qquad \text{------ (16)}$$

$$\ddot{y}_1=\frac{M_B\tilde{L}}{M_T}\dot{\theta}^2\cos\theta+\frac{M_B}{M_T}\tilde{L}\ddot{\theta}\sin\theta$$
$$-\frac{m}{M_T}\int_0^L\left(w\dot{\theta}^2\sin\theta-w\ddot{\theta}\cos\theta-2\dot{w}\dot{\theta}\cos\theta-\ddot{w}\sin\theta\right)dr$$

$$- \frac{mL}{2M_T}\left[(L + 2d_1)\,\dot\theta^2\cos\theta + (L + 2d_1)\ddot\theta\,\sin\theta + (d_2\beta_2 - d_1\beta_1)\,\dot\theta^2\sin\theta \right.$$
$$- (d_2\beta_2 - d_1\beta_1)\,\ddot\theta\cos\theta + (d_1\ddot\beta_1 - d_2\ddot\beta_2)\,\sin\theta$$
$$\left. + 2(d_1\dot\beta_1 - d_2\dot\beta_2)\,\dot\theta\cos\theta\right]$$

------ (17)

$$\ddot z_1 = \frac{M_B\tilde L}{M_T}\dot\theta^2\sin\theta - \frac{M_B\tilde L}{M_T}\ddot\theta\cos\theta$$

$$- \frac{m}{M_T}\int_0^L(-w\dot\theta^2\cos\theta - w\ddot\theta\sin\theta - 2\dot w\dot\theta\sin\theta + \ddot w\cos\theta)\,dr$$

$$+ \frac{mL}{2M_T}\left[(L + 2d_1)\,\dot\theta^2\sin\theta - (L + 2d_1)\,\ddot\theta\cos\theta - (d_2\beta_2 - d_1\beta_1)\dot\theta^2\cos\theta \right.$$
$$- (d_2\beta_2 - d_1\beta_1)\,\ddot\theta\sin\theta - (d_1\ddot\beta_1 - d_2\ddot\beta_2)\,\cos\theta$$
$$\left. + 2(d_1\dot\beta_1 - d_2\dot\beta_2)\,\dot\theta\sin\theta\right]$$

------ (18)

Combining equations (17) and (18), we may write

$$\ddot z_1\cos\theta - \ddot y_1\sin\theta = -\left[\frac{M_B\tilde L}{M_T} + \frac{mL}{2M_T}(L + 2d_1)\right]\ddot\theta - \frac{mL}{2M_T}(d_2\beta_2 - d_1\beta_1)\dot\theta^2$$

$$- \frac{mL}{2M_T}(d_1\ddot\beta_1 - d_2\ddot\beta_2) + \frac{m}{M_T}\int_0^L(w\dot\theta^2 - \ddot w)\,dr$$

------ (19)

and

$$\ddot{z}_1\sin\theta + \ddot{y}_1\cos\theta = \left[\frac{M_B\tilde{L}}{M_T} + \frac{mL}{2M_T}(L + 2d_1)\right]\dot{\theta}^2 + \frac{mL}{2M_T}(d_1\beta_1 - d_2\beta_2)\ddot{\theta}$$

$$+ \frac{mL}{M_T}(d_1\dot{\beta}_1 - d_2\dot{\beta}_2)\dot{\theta} + \frac{m}{M_T}\int_0^L (w\ddot{\theta} + 2\dot{w}\dot{\theta})\,dr$$

------ (20)

We also have θ, ψ, γ_1, γ_2 as independent generalized coordinates with associated equations given by

$$\left[M_B\tilde{L}L + (IA)_1 + (IB)_1 + mL\left(d_1^2 + \frac{1}{3}L^2 + d_1L\right)\right]\ddot{\theta}$$

$$+ \left[M_B\tilde{L} + \frac{1}{2}mL(L + 2d_1)\right](\ddot{z}_1\cos\theta - \ddot{y}_1\sin\theta)$$

$$- \left[\frac{1}{2}mL(d_1\beta_1 - d_2\beta_2) + m\int_0^L w\,dr\right](\ddot{y}_1\cos\theta + \ddot{z}_1\sin\theta) + m\int_0^L (d_1 + r)\ddot{w}\,dr$$

$$+ \left[(IA)_1 + mLd_1\left(\frac{1}{2}d_1 + \frac{1}{6}L\right)\right]\ddot{\beta}_1 + \left[(IB)_1 - mLd_2\left(\frac{1}{2}d_1 + \frac{1}{3}L\right)\right]\ddot{\beta}_2 = 0$$

------ (21)

$$M_B\tilde{L}(\ddot{x}_1 - \tilde{L}\ddot{\psi}) + (IA)_1(\dot{\theta}\dot{\gamma}_1 + \ddot{\theta}\gamma_1 - \dot{\theta}^2\psi - \dot{\theta}^2\alpha_1)$$

$$+ (IA)_2\dot{\theta}(\dot{\theta}\psi + \dot{\theta}\alpha_1 - \dot{\gamma}_1) - (IA)_3(\dot{\theta}\dot{\gamma}_1 + \ddot{\theta}\gamma_1 + \ddot{\psi} + \ddot{\alpha}_1)$$

$$+ (IB)_1(\dot{\theta}\dot{\gamma}_2 + \ddot{\theta}\gamma_2 - \dot{\theta}^2\psi - \dot{\theta}^2\alpha_2) + (IB)_2\dot{\theta}(\dot{\theta}\psi + \dot{\theta}\alpha_2 - \dot{\gamma}_2)$$

$$- (IB)_3(\dot{\theta}\dot{\gamma}_2 + \ddot{\theta}\gamma_2 + \ddot{\psi} + \ddot{\alpha}_2)$$

$$- mL\left[\ddot{\psi}\left(d_1^2 + d_1L + \frac{1}{3}L^2\right) - \ddot{x}_1\left(d_1 + \frac{1}{2}L\right) + \frac{1}{2}d_1\ddot{\alpha}_1\left(d_1 + \frac{1}{3}L\right) - d_2\ddot{\alpha}_2\left(\frac{1}{2}d_1 + \frac{1}{3}L\right) + \dot{\theta}^2\psi\left(\frac{1}{2}d_1^2 - \frac{1}{2}d_1d_2 + \frac{1}{6}d_1L - \frac{1}{3}d_2L\right) + d_1\dot{\theta}^2\alpha_1\left(\frac{1}{2}d_1 + \frac{1}{6}L\right) - d_2\dot{\theta}^2\alpha_2\left(\frac{1}{2}d_1 + \frac{1}{3}L\right)\right]$$

$$+ \left(\frac{1}{2}mL[d_1(\psi + \alpha_1) - d_2(\psi + \alpha_2)] - m\int_0^L v\,dr\right)(\ddot{y}_1\cos\theta + \ddot{z}_1\sin\theta)$$

$$+ \int_0^L m(d_1 + r)(\ddot{v} + \dot{\theta}^2 v)\,dr = 0 \qquad \text{------ (22)}$$

$$- (IA)_2\ddot{\gamma}_1 + (IA)_2\ddot{\theta}\psi + (IA)_2\ddot{\theta}\alpha_1 + [-(IA)_1 + (IA)_2 + (IA)_3]\,\dot{\theta}\,(\dot{\psi} + \dot{\alpha}_1)$$

$$+ [-(IA)_1 + (IA)_3]\,\dot{\theta}^2\gamma_1 = 0 \qquad \text{------ (23)}$$

$$- (IB)_2\ddot{\gamma}_2 + (IB)_2\ddot{\theta}\psi + (IB)_2\ddot{\theta}\alpha_2 + [-(IB)_1 + (IB)_2 + (IB)_3]\,\dot{\theta}\,(\dot{\psi} + \dot{\alpha}_2)$$

$$+ [-(IB)_1 + (IB)_3]\dot{\theta}^2\gamma_2 = 0 \qquad \text{------ (24)}$$

3. Boundary Conditions of the Beam

The boundary conditions represent a fixed attachment point at each end of the beam. That is, the beam deflection and slope at each end are considered to be consistent with the motion of the respective stages. The derivation of these boundary conditions is shown in Appendix II, from which we write

$$\beta_1 = \frac{1}{\tilde{L}}\,[-d_2 w_{,1}(L,t) + (L + d_2)\,w_{,1}(0,t)] \qquad \text{------ (25)}$$

$$\beta_2 = \frac{1}{\tilde{L}} [(L + d_1)\, w,_1(L,t) - d_1 w,_1(0,t)] \qquad \text{------ (26)}$$

$$\alpha_1 = \frac{1}{\tilde{L}} [d_2 v,_1(L,t) - (L + d_2)\, v,_1(0,t)] \qquad \text{------ (27)}$$

$$\alpha_2 = \frac{1}{\tilde{L}} [-(L + d_1)\, v,_1(L,t) + d_1 v,_1(0,t)] \qquad \text{------ (28)}$$

4. The Governing Equations

By substituting equations (16), (19), (20), (25), (26), (27), (28) into equations (21), (22), (23), (24) and the remaining terms in equation (15) and by neglecting second order terms, we obtain a set of six independent coordinates (θ, ψ, γ_1, γ_2, w, v) which yield six governing equations and eight associated boundary conditions. The equations are shown below as

$$\tilde{C}_1\ddot{\theta} + \tilde{C}_2\ddot{w},_1(0,t) + \tilde{C}_3\ddot{w},_1(L,t) + \tilde{C}_7 \int_0^L \ddot{w}dr + m \int_0^L r\ddot{w}dr = 0 \qquad \text{------ (29)}$$

$$C_9\ddot{\psi} - C_5\ddot{\theta}\gamma_1 - C_6\ddot{\theta}\gamma_2 - \tilde{C}_4\dot{\theta}^2\psi - \left[\frac{\tilde{C}_5 d_2}{\tilde{L}} - \frac{\tilde{C}_6(d_1 + L)}{\tilde{L}}\right]\dot{\theta}^2 v,_1(L,t)$$

$$- \left[\frac{-\tilde{C}_5(d_2 + L)}{\tilde{L}} + \frac{\tilde{C}_6 d_1}{\tilde{L}}\right]\dot{\theta}^2 v,_1(0,t) - \left[\frac{-C_{10} d_2}{\tilde{L}} + \frac{C_{11}(d_1 + L)}{\tilde{L}}\right]\ddot{v},_1(L,t)$$

$$- \left[\frac{C_{10}(d_2 + L)}{\tilde{L}} - \frac{C_{11} d_1}{\tilde{L}}\right]\ddot{v},_1(0,t) - C_7\dot{\theta}\dot{\gamma}_1 - C_8\dot{\theta}\dot{\gamma}_2 - C_{12}\int_0^L \ddot{v}dr$$

$$- m \int_0^L r\ddot{v}dr - \tilde{C}_7\dot{\theta}^2 \int_0^L vdr - m\dot{\theta}^2 \int_0^L rvdr = 0 \qquad \text{------ (30)}$$

$$(IA)_2\ddot{\gamma}_1 - (IA)_2\ddot{\theta}\psi - \frac{(IA)_2d_2}{\tilde{L}}\ddot{\theta}v,_1(L,t) + \frac{(IA)_2(L+d_2)}{\tilde{L}}\ddot{\theta}v,_1(0,t)$$

$$+ C_7\dot{\theta}\dot{\psi} + \frac{C_7d_2}{\tilde{L}}\dot{\theta}\dot{v},_1(L,t) - \frac{C_7(L+d_2)}{\tilde{L}}\dot{\theta}\dot{v},_1(0,t) + C_5\dot{\theta}^2\gamma_1 = 0$$

$$\text{------ (31)}$$

$$(IB)_2\ddot{\gamma}_2 - (IB)_2\ddot{\theta}\psi - \frac{(IB)_2d_1}{\tilde{L}}\ddot{\theta}v,_1(0,t) + \frac{(IB)_2(L+d_1)}{\tilde{L}}\ddot{\theta}v,_1(L,t)$$

$$+ C_8\dot{\theta}\dot{\psi} + \frac{C_8d_1}{\tilde{L}}\dot{\theta}\dot{v},_1(0,t) - \frac{C_8(L+d_1)}{\tilde{L}}\dot{\theta}\dot{v},_1(L,t) + C_6\dot{\theta}^2\gamma_2 = 0$$

$$\text{------ (32)}$$

$$m\ddot{w} + EIw,_{1111} - m\dot{\theta}^2w - \tilde{C}_{24}\dot{\theta}^2w,_{11} + \frac{m^2\dot{\theta}^2}{M_T}\int_0^L wdr - \frac{m^2}{M_T}\int_0^L \ddot{w}dr$$

$$+ (\tilde{C}_7 + mr)\ddot{\theta} - \frac{1}{\tilde{L}}\left[C_{20}(L+d_2) + \frac{m^2Ld_1d_2}{2M_T} + md_1r\right]\ddot{w},_1(0,t)$$

$$- \frac{1}{\tilde{L}}\left[-C_{20}d_2 - \frac{m^2Ld_2}{2M_T}(L+d_1) + md_2r\right]\ddot{w},_1(L,t)$$

$$- \frac{1}{\tilde{L}}[\tilde{C}_{12}(L+d_2) + \tilde{C}_{23}d_1]\,\dot{\theta}^2w,_1(0,t)$$

$$+ \frac{1}{\tilde{L}}[\tilde{C}_{12}d_2 + \tilde{C}_{23}(L+d_1)]\,\dot{\theta}^2w,_1(L,t) = 0 \qquad \text{------ (33)}$$

$$m\ddot{v} + EIv_{,1111} - \tilde{C}_{24}\dot{\theta}^2 v_{,11} - \frac{m^2}{M_T}\int_0^L \ddot{v}dr - (C_{12} + mr)\ddot{\psi}$$

$$- (\tilde{C}_7 + mr)\,\dot{\theta}^2\psi - \frac{1}{\tilde{L}}\left[C_{20}(L + d_2) + \frac{m^2 L d_1 d_2}{2M_T} + md_1 r\right]\ddot{v}_{,1}(0,t)$$

$$+ \frac{1}{\tilde{L}}\left[C_{20}d_2 + \frac{m^2 L d_2}{2M_T}(L + d_1) - md_2 r\right]\ddot{v}_{,1}(L,t) - \frac{d_1}{\tilde{L}}(\tilde{C}_7 + mr)\,\dot{\theta}^2 v_{,1}(0,t)$$

$$- \frac{d_2}{\tilde{L}}(\tilde{C}_7 + mr)\,\dot{\theta}^2 v_{,1}(L,t) = 0 \qquad \text{------ (34)}$$

and $w(0,t) = 0$ ------ (35)

$$w(L,t) = 0 \qquad \text{------ (36)}$$

$$EIw_{,11}(0,t) + D_1\ddot{\theta} + D_2\ddot{w}_{,1}(0,t) + D_3\ddot{w}_{,1}(L,t) + D_4\dot{\theta}^2 w_{,1}(0,t)$$

$$+ D_5\dot{\theta}^2 w_{,1}(L,t) + D_6\dot{\theta}^2\int_0^L w\,dr + D_7\int_0^L \ddot{w}\,dr + \frac{md_1}{\tilde{L}}\int_0^L r\ddot{w}\,dr = 0 \qquad \text{------ (37)}$$

$$EIw_{,11}(L,t) + D_8\ddot{\theta} - D_3\ddot{w}_{,1}(0,t) + D_{10}\ddot{w}_{,1}(L,t) - D_5\dot{\theta}^2 w_{,1}(0,t)$$

$$+ D_{12}\dot{\theta}^2 w_{,1}(L,t) + D_{13}\dot{\theta}^2\int_0^L w\,dr + D_{14}\int_0^L \ddot{w}\,dr - \frac{md_2}{\tilde{L}}\int_0^L r\ddot{w}\,dr = 0 \qquad \text{------ (38)}$$

$$v(0,t) = 0 \qquad \text{------ (39)}$$

$$v(L,t) = 0 \qquad \text{------ (40)}$$

$$EIv_{,11}(0,t) - \frac{(L+d_2)}{\tilde{L}} C_5\ddot{\theta}\gamma_1 + \frac{d_1}{\tilde{L}}C_6\ddot{\theta}\gamma_2 - \frac{(L+d_2)}{\tilde{L}}C_7\ddot{\theta}\gamma_2$$

$$+ \frac{d_1}{\tilde{L}}C_8\ddot{\theta}\gamma_2 - D_{15}\ddot{\psi} - D_{16}\dot{\theta}^2\psi + D_{17}\ddot{v}_{,1}(0,t) + D_{18}\ddot{v}_{,1}(L,t)$$

$$+ D_{19}\dot{\theta}^2 v_{,1}(0,t) + D_{20}\dot{\theta}^2 v_{,1}(L,t) + \frac{d_1}{\tilde{L}}\tilde{C}_7\dot{\theta}^2 \int_0^L vdr + \frac{d_1}{\tilde{L}}m\dot{\theta}^2 \int_0^L rvdr$$

$$+ D_{21}\int_0^L \ddot{v}dr + \frac{d_1}{\tilde{L}}m\int_0^L r\ddot{v}dr = 0 \qquad \text{------ (41)}$$

$$EIv_{,11}(L,t) - \frac{d_2}{\tilde{L}}C_5\ddot{\theta}\gamma_1 + \frac{(L+d_1)}{\tilde{L}}C_6\ddot{\theta}\gamma_2 - \frac{d_2}{\tilde{L}}C_7\ddot{\theta}\gamma_1 + \frac{(L+d_1)}{\tilde{L}}C_8\ddot{\theta}\gamma_2$$

$$- D_{22}\ddot{\psi} - D_{23}\dot{\theta}^2\psi - D_{18}\ddot{v}_{,1}(0,t) - D_{25}\ddot{v}_{,1}(L,t) - D_{20}\dot{\theta}^2 v_{,1}(0,t)$$

$$- D_{27}\dot{\theta}^2 v_{,1}(L,t) - \frac{d_2}{\tilde{L}}\tilde{C}_7\dot{\theta}^2\int_0^L vdr - \frac{d_2}{\tilde{L}}m\dot{\theta}^2\int_0^L rvdr + D_{28}\int_0^L \ddot{v}dr$$

$$- \frac{d_2}{\tilde{L}}m\int_0^L r\ddot{v}dr = 0 \qquad \text{------ (42)}$$

where the C's, $\tilde{C}$'s and D's are combinations of physical constants given below

$$C_1 = M_B\tilde{L} + \frac{1}{2}mL(L + 2d_1)$$

$$C_2 = M_B\tilde{L}L + (IA)_1 + (IB)_1 + mL(d_1^2 + \frac{1}{3}L^2 + d_1L)$$

$$C_3 = (IA)_1 + mLd_1(\frac{1}{2}d_1 + \frac{1}{6}L)$$

$$C_4 = (IB)_1 - mLd_2\left(\frac{1}{2}d_1 + \frac{1}{3}L\right)$$

$$C_5 = (IA)_1 - (IA)_3$$

$$C_6 = (IB)_1 - (IB)_3$$

$$C_7 = (IA)_1 - (IA)_2 - (IA)_3$$

$$C_8 = (IB)_1 - (IB)_2 - (IB)_3$$

$$C_9 = (IA)_3 + (IB)_3 + mL(d_1^2 + d_1L + \frac{1}{3}L^2) + M_B\tilde{L}^2 - \frac{C_1^2}{M_T}$$

$$C_{10} = (IA)_3 + \frac{1}{2}mLd_1(d_1 + \frac{1}{3}L) - \frac{mLd_1C_1}{2M_T}$$

$$C_{11} = (IB)_3 - mLd_2\left(\frac{1}{2}d_1 + \frac{1}{3}L\right) + \frac{mLd_2C_1}{2M_T}$$

$$C_{12} = m\left[d_1 - \frac{C_1}{M_T}\right]$$

$$C_{13} = d_1\left[M_B\left(1 + \frac{d_1}{L}\right) + \frac{1}{2}mL\left(2 + \frac{d_1}{L}\right)\right]$$

$$C_{14} = d_1(M_B + \frac{1}{2}mL)\frac{d_2}{L}$$

$$C_{15} = \left[M_B\tilde{L} + mL\left(\frac{1}{2}d_1 + \frac{1}{3}L\right)\right]d_1\left(1 + \frac{d_1}{L}\right)$$

$$C_{16} = \left[M_B\tilde{L} + mL\left(\frac{1}{2}d_1 + \frac{1}{3}L\right)\right]d_1\frac{d_2}{L}$$

$$C_{17} = (IA)_1 + \frac{1}{3}mLd_1^2$$

$$C_{18} = \frac{1}{6}mLd_1d_2$$

$$C_{19} = \frac{1}{6} m L d_1 (d_1 + L)$$

$$C_{20} = \frac{m d_1}{M_T}\left(\frac{1}{2} m L - M_T\right)$$

$$C_{21} = \left[M_B \left(1 + \frac{d_2}{L}\right) + \frac{1}{2} m d_2 \right] d_2$$

$$C_{22} = \left[M_B \widetilde{L} + m L \left(\frac{1}{2} d_1 + \frac{1}{3} L \right) \right] d_2 \left(1 + \frac{d_2}{L}\right)$$

$$C_{23} = (IB)_1 + \frac{1}{3} m L d_2^2$$

$$C_{24} = m L d_2 \left(\frac{1}{2} d_1 + \frac{1}{3} L + \frac{1}{3} d_2 \right)$$

$$C_{25} = m d_2 \left[\frac{m L}{2 M_T} + \frac{d_1}{L} - \frac{C_1}{M_T L} \right]$$

$$C_{26} = \frac{m^2 L d_2}{2 M_T}$$

$$C_{27} = (IA)_3 + \frac{1}{2} m L d_1 \left(d_1 + \frac{1}{3} L \right)$$

$$C_{28} = (IA)_3 + \frac{1}{3} m L d_1^2$$

$$C_{29} = (IA)_1 - (IA)_2 + m L d_1 \left(\frac{1}{2} d_1 + \frac{1}{6} L \right)$$

$$C_{30} = (IB)_3 - m L d_2 \left(\frac{1}{2} d_1 + \frac{1}{3} L \right)$$

$$C_{31} = (IB)_3 + \frac{1}{3}mLd_2^2$$

$$C_{32} = (IB)_1 - (IB)_2 - mLd_2\left(\frac{1}{2}d_1 + \frac{1}{3}L\right)$$

$$C_{33} = M_B\tilde{L} + mL\left(\frac{1}{2}d_1 + \frac{1}{3}L\right)$$

$$\tilde{C}_1 = C_2 - \frac{C_1^2}{M_T}$$

$$\tilde{C}_2 = \frac{C_3(L + d_2)}{\tilde{L}} - \frac{C_1 mLd_1}{2M_T\tilde{L}}(L + 2d_2) - \frac{C_4 d_1}{\tilde{L}}$$

$$\tilde{C}_3 = \frac{-C_3 d_2}{\tilde{L}} + \frac{C_1 mLd_2}{2M_T\tilde{L}}(L + 2d_1) + \frac{C_4(L + d_1)}{\tilde{L}}$$

$$\tilde{C}_4 = -(IA)_1 + (IA)_2 - (IB)_1 + (IB)_2 - mL\left(\frac{1}{2}d_1^2 - \frac{1}{2}d_1 d_2 + \frac{1}{6}d_1 L - \frac{1}{3}d_2 L\right)$$

$$+ \frac{mLC_1}{2M_T}(d_1 - d_2)$$

$$\tilde{C}_5 = -C_3 + (IA)_2 + \frac{mLd_1C_1}{2M_T}$$

$$\tilde{C}_6 = -C_4 + (IB)_2 - \frac{mLd_2C_1}{2M_T}$$

$$\tilde{C}_7 = md_1 - \frac{mC_1}{M_T}$$

$$\tilde{C}_8 = \frac{m^2L^2d_1^2}{4M_T} - C_{17}$$

$$\tilde{C}_9 = C_{18} - \frac{m^2L^2d_1d_2}{4M_T}$$

$$\tilde{C}_{10} = \frac{C_1C_{13}}{M_T} - \frac{m^2L^2d_1^2}{4M_T} - C_{15} - C_{19}$$

$$\tilde{C}_{11} = \frac{C_1C_{14}}{M_T} + \frac{m^2L^2d_1d_2}{4M_T} - C_{16} - C_{18}$$

$$\tilde{C}_{12} = md_1\left[-\frac{C_1}{M_TL} - \frac{mL}{2M_T} + \frac{(d_1 + L)}{L}\right]$$

$$\tilde{C}_{13} = \frac{m^2L^2d_2^2}{4M_T} - C_{23}$$

$$\tilde{C}_{14} = \frac{C_1C_{21}}{M_T} - \frac{m^2L^2d_2^2}{4M_T} - C_{22} + C_{24}$$

$$\tilde{C}_{15} = \frac{mLd_1C_1}{2M_T} - C_{27}$$

$$\tilde{C}_{16} = \frac{mLd_1C_1}{2M_T} - C_{29}$$

$$\tilde{C}_{17} = \frac{m^2L^2d_1^2}{4M_T} - C_{28}$$

$$\tilde{C}_{18} = \frac{C_1C_{13}}{M_T} - C_{15} - C_{29}$$

$$\tilde{C}_{19} = \frac{mLd_2C_1}{2M_T} + C_{30}$$

$$\tilde{C}_{20} = \frac{mLd_2C_1}{2M_T} + C_{32}$$

$$\tilde{C}_{21} = \frac{m^2L^2d_2^2}{4M_T} - C_{31}$$

$$\tilde{C}_{22} = \frac{C_1C_{21}}{M_T} - C_{22} - C_{32}$$

$$\tilde{C}_{23} = md_2\left[\frac{C_1}{M_TL} - \frac{mL}{2M_T} - \frac{d_1}{L}\right]$$

$$\tilde{C}_{24} = C_{33} - \frac{C_1}{M_T}\left(M_B + \frac{1}{2}mL\right)$$

$$D_1 = \frac{1}{\tilde{L}}[\tilde{C}_5(L + d_2) - (IA)_2(L + d_2) - \tilde{C}_6d_1 + (IB)_2d_1]$$

$$D_2 = \frac{1}{\tilde{L}^2}[\tilde{C}_8(L + d_2)^2 - 2\tilde{C}_9d_1(L + d_2) + \tilde{C}_{13}d_1^2]$$

$$D_3 = \frac{1}{\tilde{L}^2}[-\tilde{C}_8(L + d_2)d_2 + \tilde{C}_9(L^2 + d_1L + d_2L + 2d_1d_2) - \tilde{C}_{13}(L + d_1)d_1]$$

$$D_4 = \frac{1}{\tilde{L}^2}[\tilde{C}_{10}(L + d_2)^2 - 2\tilde{C}_{11}(L + d_2)d_1 + \tilde{C}_{14}d_1^2]$$

$$D_5 = \frac{1}{\tilde{L}^2}[-\tilde{C}_{10}(L + d_2)d_2 + \tilde{C}_{11}(L^2 + d_1L + d_2L + 2d_1d_2) - \tilde{C}_{14}(L + d_1)d_1]$$

$$D_6 = \frac{1}{\tilde{L}} [\tilde{C}_{12}(L + d_2) - C_{25}d_1]$$

$$D_7 = \frac{1}{\tilde{L}} [C_{20}(L + d_2) + C_{26}d_1]$$

$$D_8 = \frac{1}{\tilde{L}} [\tilde{C}_5 d_2 - (IA)_2 d_2 - \tilde{C}_6(L + d_1) + (IB)_2(L + d_1)]$$

$$D_{10} = \frac{1}{\tilde{L}^2} [-\tilde{C}_8 d_2^2 + 2\tilde{C}_9(L + d_1) d_2 - \tilde{C}_{13}(L + d_1)^2]$$

$$D_{12} = \frac{1}{\tilde{L}^2} [-\tilde{C}_{10} d_2^2 + 2\tilde{C}_{11}(L + d_1) d_2 - \tilde{C}_{14}(L + d_1)^2]$$

$$D_{13} = \frac{1}{\tilde{L}} [\tilde{C}_{12} d_2 - C_{25}(L + d_1)]$$

$$D_{14} = \frac{1}{\tilde{L}} [C_{20} d_2 + C_{26}(L + d_1)]$$

$$D_{15} = \frac{1}{\tilde{L}} [\tilde{C}_{15}(L + d_2) + \tilde{C}_{19} d_1]$$

$$D_{16} = \frac{1}{\tilde{L}} [\tilde{C}_{16}(L + d_2) + \tilde{C}_{20} d_1]$$

$$D_{17} = \frac{1}{\tilde{L}^2} [\tilde{C}_{17}(L + d_2)^2 - 2\tilde{C}_9(L + d_2) d_1 + \tilde{C}_{21} d_1^2]$$

$$D_{18} = \frac{1}{\tilde{L}^2} [-\tilde{C}_{17}(L + d_2) d_2 + \tilde{C}_9(L^2 + d_1 L + d_2 L + 2d_1 d_2) - \tilde{C}_{21}(L + d_1) d_1]$$

$$D_{19} = \frac{1}{\tilde{L}^2}[\tilde{C}_{18}(L + d_2)^2 - 2(\tilde{C}_{11} + \tilde{C}_9)(L + d_2)\,d_1 + \tilde{C}_{22}d_1^2]$$

$$D_{20} = \frac{1}{\tilde{L}^2}\left[\begin{array}{l} -\tilde{C}_{18}(L + d_2)\,d_2 + (\tilde{C}_{11} + \tilde{C}_9)(L^2 + d_1L + d_2L + 2d_1d_2) \\ -\tilde{C}_{22}(L + d_1)\,d_1 \end{array}\right]$$

$$D_{21} = \frac{1}{\tilde{L}}[\tilde{C}_{20}(L + d_2) + \tilde{C}_{26}d_1]$$

$$D_{22} = \frac{1}{\tilde{L}}[\tilde{C}_{15}d_2 + \tilde{C}_{19}(L + d_1)]$$

$$D_{23} = \frac{1}{\tilde{L}}[\tilde{C}_{16}d_2 + \tilde{C}_{20}(L + d_1)]$$

$$D_{25} = \frac{1}{\tilde{L}^2}[\tilde{C}_{17}d_2^2 - 2\tilde{C}_9(L + d_1)\,d_2 + \tilde{C}_{21}(L + d_1)^2]$$

$$D_{27} = \frac{1}{\tilde{L}^2}[\tilde{C}_{18}d_2^2 - 2(\tilde{C}_{11} + \tilde{C}_9)(L + d_1)\,d_2 + \tilde{C}_{22}(L + d_1)^2]$$

$$D_{28} = \frac{1}{\tilde{L}}[C_{20}d_2 + C_{26}(L + d_1)]$$

V. SOLUTION OF THE GOVERNING EQUATIONS FOR THE MOTION OF THE SPACE STATION IN THE PLANE OF ROTATION

1. The Uncoupled Motion of the Space Station

The governing equations derived in the preceding chapter are seen to be uncoupled into two types of motion. Equations (29) and (33) represent the in-plane motion of the station with boundary conditions given by equations (35) - (38). Equations (30), (31), (32), (34) describe the out-of-plane motion of the station with associated boundary conditions given by equations (39) - (42). We may now treat the in-plane motion of the space station independently from the out-of-plane motion and determine a solution for the bending motion of the beam in the plane of rotation.

2. The Beam Deflection Equation

We assume that θ, the angle of rotation of the space station in the plane of the orbit, is given by

$$\theta(t) = \Omega t + \tau(t)$$

where Ω is a constant and $\tau(t)$ is a small quantity representing the time dependent perturbation of θ. Then the time derivatives are

$$\dot{\theta} = \Omega + \dot{\tau}(t)$$

$$\ddot{\theta} = \ddot{\tau}(t)$$

Substituting the above identities into equations (24), (33), (35) - (38) and neglecting non-linear terms, we obtain

$$\tilde{C}_1\ddot{\tau} + \tilde{C}_2\ddot{w},_1(0,t) + \tilde{C}_3\ddot{w},_1(L,t) + \tilde{C}_7\int_0^L \ddot{w}dr + m\int_0^L r\ddot{w}dr = 0 \qquad \text{------ (43)}$$

$$m\ddot{w} + EIw,_{1111} - m\Omega^2 w - \tilde{C}_{24}\Omega^2 w,_{11} + \frac{m^2\Omega^2}{M_T}\int_0^L wdr - \frac{m^2}{M_T}\int_0^L \ddot{w}dr$$

$$+ (\tilde{C}_7 + mr)\ddot{\tau} - \frac{1}{\tilde{L}}\left[C_{20}(L + d_2) + \frac{m^2Ld_1d_2}{2M_T} + md_1r\right]\ddot{w},_1(0,t)$$

$$- \frac{1}{\tilde{L}}\left[-C_{20}d_2 - \frac{m^2Ld_2}{2M_T}(L + d_1) + md_2r\right]\ddot{w},_1(L,t)$$

$$- \frac{1}{\tilde{L}}[\tilde{C}_{12}(L + d_2) + \tilde{C}_{23}d_1]\,\Omega^2 w,_1(0,t)$$

$$+ \frac{1}{\tilde{L}}[\tilde{C}_{12}d_2 + \tilde{C}_{23}(L + d_1)]\,\Omega^2 w,_1(L,t) = 0 \qquad \text{------ (44)}$$

$$w(0,t) = 0 \qquad \text{------ (45)}$$

$$w(L,t) = 0 \qquad \text{------ (46)}$$

$$EIw,_{11}(0,t) + D_1\ddot{\tau} + D_2\ddot{w},_1(0,t) + D_3\ddot{w},_1(L,t) + D_4\Omega^2 w,_1(0,t)$$

$$+ D_5\Omega^2 w,_1(L,t) + D_6\Omega^2\int_0^L wdr + D_7\int_0^L \ddot{w}dr + \frac{md_1}{\tilde{L}}\int_0^L r\ddot{w}dr = 0 \qquad \text{------ (47)}$$

$$EIw_{,11}(L,t) + D_8\ddot{\tau} - D_3\ddot{w}_{,1}(0,t) + D_{10}\ddot{w}_{,1}(L,t) - D_5\Omega^2 w_{,1}(0,t)$$

$$+ D_{12}\Omega^2 w_{,1}(L,t) + D_{13}\Omega^2 \int_0^L w\,dr + D_{14}\int_0^L \ddot{w}\,dr - \frac{md_2}{\tilde{L}}\int_0^L r\ddot{w}\,dr = 0 \qquad \text{------ (48)}$$

By solving equation (43) for $\ddot{\tau}$ and substituting into equation (44), we have the governing equation of the beam deflection in the plane of rotation, given by

$$m\ddot{w} + EIw_{,1111} - m\Omega^2 w - \tilde{C}_{24}\Omega^2 w_{,11} + \frac{m^2\Omega^2}{M_T}\int_0^L w\,dr$$

$$- \left[\frac{m^2}{M_T} + \frac{\tilde{C}_7}{\tilde{C}_1}(\tilde{C}_7 + mr)\right]\int_0^L \ddot{w}\,dr - \frac{m}{\tilde{C}_1}(\tilde{C}_7 + mr)\int_0^L r\ddot{w}\,dr$$

$$- \left[\frac{C_{20}}{\tilde{L}}(L + d_2) + \frac{m^2 L d_1 d_2}{2M_T\tilde{L}} + \frac{md_1 r}{\tilde{L}} + \frac{\tilde{C}_2}{\tilde{C}_1}(\tilde{C}_7 + mr)\right]\ddot{w}_{,1}(0,t)$$

$$- \left[\frac{-C_{20}d_2}{\tilde{L}} - \frac{m^2 L d_2}{2M_T\tilde{L}}(L + d_1) + \frac{md_2 r}{\tilde{L}} + \frac{\tilde{C}_3}{\tilde{C}_1}(\tilde{C}_7 + mr)\right]\ddot{w}_{,1}(L,t)$$

$$- \frac{1}{\tilde{L}}[\tilde{C}_{12}(L + d_2) + \tilde{C}_{23}d_1]\Omega^2 w_{,1}(0,t)$$

$$+ \frac{1}{\tilde{L}}[\tilde{C}_{12}d_2 + \tilde{C}_{23}(L + d_1)]\Omega^2 w_{,1}(L,t) = 0 \qquad \text{------ (49)}$$

With the assumption $w(r,t) = \eta(t)R(r)$, the governing equation becomes

$$m\ddot{\eta}R(r) + EI\eta R^{IV}(r) - m\Omega^2\eta R(r) - \tilde{C}_{24}\Omega^2\eta R''(r) + \frac{m^2\Omega^2\eta}{M_T}\int_0^L R(r)\,dr$$

$$- \left[\frac{m^2}{M_T} + \frac{\tilde{C}_7^2}{\tilde{C}_1}\right]\ddot{\eta}\int_0^L R(r)\,dr - \frac{\tilde{C}_7}{\tilde{C}_1} mr\ddot{\eta}\int_0^L R(r)\,dr - \frac{m\tilde{C}_7}{\tilde{C}_1}\ddot{\eta}\int_0^L rR(r)\,dr$$

$$- \frac{m^2}{\tilde{C}_1} r\ddot{\eta}\int_0^L rR(r)\,dr - \left[\frac{C_{20}}{\tilde{L}}(L + d_2) + \frac{m^2 L d_1 d_2}{2M_T\tilde{L}} + \frac{\tilde{C}_2\tilde{C}_7}{\tilde{C}_1}\right]\ddot{\eta}R'(0)$$

$$- \left[\frac{md_1}{\tilde{L}} + \frac{m\tilde{C}_2}{\tilde{C}_1}\right] r\ddot{\eta}R'(0) - \left[-\frac{C_{20}d_2}{\tilde{L}} - \frac{m^2 L d_2}{2M_T\tilde{L}}(L + d_1) + \frac{\tilde{C}_3\tilde{C}_7}{\tilde{C}_1}\right]\ddot{\eta}R'(L)$$

$$- \left[\frac{md_2}{\tilde{L}} + \frac{m\tilde{C}_3}{\tilde{C}_1}\right] r\ddot{\eta}R'(L) - \frac{1}{\tilde{L}}[\tilde{C}_{12}(L + d_2) + \tilde{C}_{23}d_1]\,\Omega^2\eta R'(0)\,]$$

$$+ \frac{1}{\tilde{L}}[\tilde{C}_{12}d_2 + \tilde{C}_{23}(L + d_1)]\,\Omega^2\eta R'(L) = 0 \qquad \text{------ (50)}$$

Dividing by η and rearranging terms, we write

$$\frac{\ddot{\eta}}{\eta}\left\{\begin{array}{l} mR(r) - \left[\dfrac{m^2}{M_T} + \dfrac{\tilde{C}_7^2}{\tilde{C}_1}\right]\displaystyle\int_0^L R(r)\,dr - \dfrac{\tilde{C}_7}{\tilde{C}_1} mr\displaystyle\int_0^L R(r)\,dr \\ - m\dfrac{\tilde{C}_7}{\tilde{C}_1}\displaystyle\int_0^L rR(r)\,dr - \dfrac{m^2 r}{\tilde{C}_1}\displaystyle\int_0^L rR(r)\,dr \\ - \left[\dfrac{C_{20}}{\tilde{L}}(L + d_2) + \dfrac{m^2 L d_1 d_2}{2M_T\tilde{L}} + \dfrac{\tilde{C}_2\tilde{C}_7}{\tilde{C}_1}\right] R'(0) \end{array}\right\}$$

$$\left\{ - \left[-\frac{C_{20}d_2}{\tilde{L}} - \frac{m^2Ld_2}{2M_T\tilde{L}}(L + d_1) + \frac{\tilde{C}_3\tilde{C}_7}{\tilde{C}_1}\right] R'(L) - \left[\frac{md_1}{\tilde{L}} + \frac{m\tilde{C}_2}{\tilde{C}_1}\right] rR'(0) - \left[\frac{md_2}{\tilde{L}} + \frac{m\tilde{C}_3}{\tilde{C}_1}\right] rR'(L) \right\}$$

$$= -\frac{m^2\Omega^2}{M_T}\int_0^L R(r)\,dr + \frac{1}{\tilde{L}}[\tilde{C}_{12}(L + d_2) + \tilde{C}_{23}d_1]\,\Omega^2R'(0)$$

$$- \frac{1}{\tilde{L}}[\tilde{C}_{12}d_2 + \tilde{C}_{23}(L + d_1)]\,\Omega^2R'(L) - EIR^{IV}(r) + m\Omega^2R(r)$$

$$+ \tilde{C}_{24}\,\Omega^2R''(r) \qquad \text{------ (51)}$$

We now assume $\eta(t) = \eta_o e^{ipt}$ and take two time derivatives to obtain $\ddot{\eta}(t) = -p^2\eta_o e^{ipt} = -p^2\eta(t)$. Substituting this value, we remove the time dependence from the governing equation and write

$$EIR^{IV}(r) - \tilde{C}_{24}\Omega^2R''(r) - m(\Omega^2 + p^2)R(r)$$

$$+ rmp^2\left\{\frac{\tilde{C}_7}{\tilde{C}_1}\int_0^L R(r)\,dr + \frac{m}{\tilde{C}_1}\int_0^L rR(r)\,dr + \left[\frac{d_1}{\tilde{L}} + \frac{\tilde{C}_2}{\tilde{C}_1}\right]R'(0) + \left[\frac{d_2}{\tilde{L}} + \frac{\tilde{C}_3}{\tilde{C}_1}\right]R'(L)\right\}$$

$$+ \left[\frac{p^2C_{20}}{\tilde{L}}(L + d_2) + \frac{m^2p^2Ld_1d_2}{2M_T\tilde{L}} + \frac{p^2\tilde{C}_2\tilde{C}_7}{\tilde{C}_1} - \frac{\Omega^2\tilde{C}_{12}}{\tilde{L}}(L + d_2) - \frac{\Omega^2\tilde{C}_{23}d_1}{\tilde{L}}\right] R'(0)$$

$$+ \left[- \frac{p^2 C_{20} d_2}{\tilde{L}} - \frac{m^2 p^2 L d_2}{2 M_T \tilde{L}} (L + d_1) + \frac{p^2 \tilde{C}_3 \tilde{C}_7}{\tilde{C}_1} + \frac{\Omega^2 \tilde{C}_{12} d_2}{\tilde{L}} + \frac{\Omega^2 \tilde{C}_{23}}{\tilde{L}} (L + d_1) \right] R'(L)$$

$$+ \left[\frac{m^2 \Omega^2}{M_T} + \frac{m^2 p^2}{M_T} + \frac{p^2 \tilde{C}_7^2}{\tilde{C}_1} \right] \int_0^L R(r)\,dr + mp^2 \frac{\tilde{C}_7}{\tilde{C}_1} \int_0^L rR(r)\,dr = 0 \qquad \text{------ (52)}$$

To simplify the algebra we make two identities, where

$$C_{w1} = mp^2 \left\{ \frac{\tilde{C}_7}{\tilde{C}_1} \int_0^L rR(r)\,dr + \frac{m}{\tilde{C}_1} \int_0^L rR(r)\,dr + \left[\frac{d_2}{\tilde{L}} + \frac{\tilde{C}_3}{\tilde{C}_1} \right] R'(L) + \left[\frac{d_1}{\tilde{L}} + \frac{\tilde{C}_2}{\tilde{C}_1} \right] R'(0) \right\} \qquad \text{------ (53)}$$

$$C_{w2} = \left[\frac{p^2 C_{20}}{\tilde{L}} (L + d_2) + \frac{m^2 p^2 L d_1 d_2}{2 M_T \tilde{L}} + \frac{p^2 \tilde{C}_2 \tilde{C}_7}{\tilde{C}_1} - \frac{\Omega^2 \tilde{C}_{12}}{\tilde{L}} (L + d_2) - \frac{\Omega^2 \tilde{C}_{23} d_1}{\tilde{L}} \right] R'(0)$$

$$+ \left[- \frac{p^2 C_{20} d_2}{\tilde{L}} - \frac{m^2 p^2 L d_2}{2 M_T \tilde{L}} (L + d_1) + \frac{p^2 \tilde{C}_3 \tilde{C}_7}{\tilde{C}_1} + \frac{\Omega^2 \tilde{C}_{12} d_2}{\tilde{L}} + \frac{\Omega^2 \tilde{C}_{23}}{\tilde{L}} (L + d_1) \right] R'(L)$$

$$+ \left[\frac{m^2\Omega^2}{M_T} + \frac{m^2p^2}{M_T} + \frac{p^2\tilde{C}}{\tilde{C}_1} \right] \int_0^L R(r)\,dr + mp^2\frac{\tilde{C}_7}{\tilde{C}_1} \int_0^L rR(r)\,dr \qquad \text{------ (54)}$$

Thus equation (52) may be written

$$EIR^{IV}(r) - \tilde{C}_{24}\Omega^2R''(r) - m(\Omega^2 + p^2)\,R(r) + C_{w1}r + C_{w2} = 0 \qquad \text{------ (55)}$$

An exact solution to this differential equation is

$$R(r) = c_1 \sinh a_1 r + c_2 \cosh a_1 r + c_3 \sin a_2 r + c_4 \cos a_2 r + \frac{C_{w1} r}{m(\Omega^2 + p^2)} + \frac{C_{w2}}{m(\Omega^2 + p^2)} \qquad \text{------ (56)}$$

where

$$a_1 = \left[\frac{\tilde{C}_{24}\,\Omega^2 + \sqrt{\tilde{C}_{24}^2\,\Omega^4 + 4EIm(\Omega^2 + p^2)}}{2EI} \right]^{\frac{1}{2}} \qquad \text{------ (57)}$$

$$a_2 = \left[\frac{-\tilde{C}_{24}\Omega^2 + \sqrt{\tilde{C}_{24}^2\,\Omega^4 + 4EIm(\Omega^2 + p^2)}}{2EI} \right]^{\frac{1}{2}} \qquad \text{------ (58)}$$

The constants C_{w1} and C_{w2} are now expressed in terms of the arbitrary constants c_j ($j = 1 - 4$) of the solution by substituting equation (56) into equations (53) and (54). Carrying out this substitution, we obtain

$$A_1C_{w1} - A_2C_{w2} = A_3c_1 + A_4c_2 + A_5c_3 + A_6c_4 \qquad \text{------ (59)}$$

and

$$-B_1C_{w1} + B_2C_{w2} = B_3c_1 + B_4c_2 + B_5c_3 + B_6c_4 \qquad \text{------ (60)}$$

where

$$A_1 = \frac{1}{\tilde{L}(\Omega^2 + p^2)}\left(\tilde{C}_1\tilde{L}\Omega^2 + p^2\left[\tilde{C}_1\tilde{L} - \frac{1}{2}\tilde{C}_7L^2\tilde{L} - \frac{1}{3}mL^3\tilde{L} - d_1\tilde{C}_1 - d_2\tilde{C}_1 - \tilde{L}\tilde{C}_2 - \tilde{L}\tilde{C}_3\right]\right)$$

$$A_2 = \frac{mp^2L}{(\Omega^2 + p^2)}\left[\frac{\tilde{C}_7}{m} + \frac{L}{2}\right]$$

$$A_3 = \frac{mp^2}{\tilde{L}}\left[\frac{1}{a_1}\tilde{L}\tilde{C}_7(\cosh a_1L - 1) + \frac{1}{a_1}m\tilde{L}\left(L\cosh a_1L - \frac{1}{a_1}\sinh a_1L\right) + (d_1\tilde{C}_1 + \tilde{L}\tilde{C}_2)\,a_1 + (d_2\tilde{C}_1 + \tilde{L}\tilde{C}_3)\,a_1\cosh a_1L\right]$$

$$A_4 = \frac{mp^2}{\tilde{L}}\left[\frac{1}{a_1}\tilde{L}\tilde{C}_7\sinh a_1L + \frac{1}{a_1}m\tilde{L}\left(L\sinh a_1L - \frac{1}{a_1}\cosh a_1L + \frac{1}{a_1}\right) + (d_2\tilde{C}_1 + \tilde{L}\tilde{C}_3)\,a_1\sinh a_1L\right]$$

$$A_5 = \frac{mp^2}{\tilde{L}}\left[-\frac{1}{a_2}\tilde{L}\tilde{C}_7(\cos a_2L - 1) + \frac{1}{a_2}m\tilde{L}\left(\frac{1}{a}\sin a_2L - L\cos a_2L\right) + (d_1\tilde{C}_1 + \tilde{L}\tilde{C}_2)\,a_2 + (d_2\tilde{C}_1 + \tilde{L}\tilde{C}_3)\,a_2\cos a_2L\right]$$

$$A_6 = \frac{mp^2}{\tilde{L}}\left[\frac{1}{a_2}\tilde{L}\tilde{C}_7\sin a_2L + \frac{1}{a_2}m\tilde{L}\left(\frac{1}{a_2}\cos a_2L - \frac{1}{a_2} + L\sin a_2L\right) - (d_2\tilde{C}_1 + \tilde{L}\tilde{C}_3)\,a_2\sin a_2L\right]$$

$$B_1 = \frac{1}{m(\Omega^2 + p^2)} \left[\frac{p^2 C_{20} L}{\tilde{L}} - \frac{m^2 p^2 L^2 d_2}{2M_T \tilde{L}} + \frac{p^2 \tilde{C}_7}{\tilde{C}_1} (\tilde{C}_2 + \tilde{C}_3) - \frac{\Omega^2 \tilde{C}_{12} L}{\tilde{L}} + \frac{\Omega^2 \tilde{C}_{23} L}{\tilde{L}} + \frac{m^2 L^2}{2M_T} (\Omega^2 + p^2) + \frac{p^2 \tilde{C}_7^2 L^2}{2\tilde{C}_1} + \frac{m p^2 \tilde{C}_7 L^3}{3\tilde{C}_1} \right]$$

$$B_2 = \frac{1}{(\Omega^2 + p^2)} \left[(\Omega^2 + p^2) - \frac{mL}{M_T} (\Omega^2 + p^2) - \frac{p^2 \tilde{C}_7^2 L}{\tilde{C}_1 m} - \frac{p^2 \tilde{C}_7 L^2}{2\tilde{C}_1} \right]$$

$$B_3 = \left\{ \frac{a_1 p^2 C_{20}}{\tilde{L}} (L + d_2) + \frac{a_1 m^2 p^2 L d_1 d_2}{2M_T \tilde{L}} + a_1 p^2 \frac{\tilde{C}_2 \tilde{C}_7}{\tilde{C}_1} - \frac{a_1 \Omega^2 \tilde{C}_{12}}{\tilde{L}} (L + d_2) - \frac{a_1 \Omega^2 \tilde{C}_{23} d_1}{\tilde{L}} - \frac{m^2 (\Omega^2 + p^2)}{a_1 M_T} - \frac{p^2 \tilde{C}_7^2}{a_1 \tilde{C}_1} - \sinh a_1 L \, \frac{m p^2 \tilde{C}_7}{a_1^2 \tilde{C}_1} + \cosh a_1 L \left[- \frac{a_1 p^2 C_{20} d_2}{\tilde{L}} - \frac{a_1 m^2 p^2 L d_2}{2M_T \tilde{L}} (L + d_1) + \frac{a_1 p^2 \tilde{C}_3 \tilde{C}_7}{\tilde{C}_1} + \frac{a_1 \Omega^2 \tilde{C}_{12} d_2}{\tilde{L}} + \frac{m L p^2 \tilde{C}_7}{a_1 \tilde{C}_1} + \frac{a_1 \Omega^2 \tilde{C}_{23}}{\tilde{L}} (L + d_1) + \frac{m^2 (\Omega^2 + p^2)}{a_1 M_T} + \frac{p^2 \tilde{C}_7^2}{a_1 \tilde{C}_1} \right] \right\}$$

$$B_4 = \left\{ \sinh a_1 L \left[- \frac{a_1 p^2 C_{20} d_2}{\widetilde{L}} - \frac{a_1 m^2 p^2 L d_2}{2 M_T \widetilde{L}} (L + d_1) - \frac{a_1 p^2 \widetilde{C}_3 \widetilde{C}_7}{\widetilde{C}_1} + \frac{a_1 \Omega^2 \widetilde{C}_{12} d_2}{\widetilde{L}} + \frac{m L p^2 \widetilde{C}_7}{a_1 \widetilde{C}_1} + \frac{a_1 \Omega^2 \widetilde{C}_{23}}{\widetilde{L}} (L + d_1) + \frac{m^2 (\Omega^2 + p^2)}{a_1 M_T} + \frac{p^2 \widetilde{C}_7^2}{a_1 \widetilde{C}_1} \right] - \cosh a_1 L \frac{m p^2 \widetilde{C}_7}{a_1^2 \widetilde{C}_1} + \frac{m p^2 \widetilde{C}_7}{a_1^2 \widetilde{C}_1} \right\}$$

$$B_5 = \left\{ \frac{a_2 p^2 C_{20}}{\widetilde{L}} (L + d_2) + \frac{a_2 m^2 p^2 L d_1 d_2}{2 M_T \widetilde{L}} + \frac{a_2 p^2 \widetilde{C}_2 \widetilde{C}_7}{\widetilde{C}_1} - \frac{a_2 \Omega^2 \widetilde{C}_{12}}{\widetilde{L}} (L + d_2) - \frac{a_2 \Omega^2 \widetilde{C}_{23} d_1}{\widetilde{L}} + \frac{p^2 \widetilde{C}_7^2}{a_2 \widetilde{C}_1} + \frac{m^2 (\Omega^2 + p^2)}{a_2 M_T} + \sin a_2 L \frac{m p^2 \widetilde{C}_7}{a_2^2 \widetilde{C}_1} + \cos a_2 L \left[- \frac{a_2 p^2 C_{20} d_2}{\widetilde{L}} - \frac{a_2 m^2 p^2 L d_2}{2 M_T \widetilde{L}} (L + d_1) + \frac{a_2 p^2 \widetilde{C}_3 \widetilde{C}_7}{\widetilde{C}_1} + \frac{a_2 \Omega^2 \widetilde{C}_{12} d_2}{\widetilde{L}} - \frac{m L p^2 \widetilde{C}_7}{a_2 \widetilde{C}_1} + \frac{a_2 \Omega^2 \widetilde{C}_{23}}{\widetilde{L}} (L + d_1) - \frac{m^2 (\Omega^2 + p^2)}{a_2 M_T} - \frac{p^2 \widetilde{C}_7^2}{a_2 \widetilde{C}_1} \right] \right\}$$

$$B_6 = \left\{ \sin a_2 L \left[\frac{a_2 p^2 C_{20} d_2}{\tilde{L}} + \frac{a_2 m^2 p^2 L d_2}{2 M_T \tilde{L}} (L + d_1) - \frac{a_2 p^2 \tilde{C}_3 \tilde{C}_7}{\tilde{C}_1} - \frac{a_2 \Omega^2 \tilde{C}_{12} d_2}{\tilde{L}} + \frac{m L p^2 \tilde{C}_7}{a_2 \tilde{C}_1} - \frac{a_2 \Omega^2 \tilde{C}_{23}}{\tilde{L}} (L + d_1) + \frac{m^2 (\Omega^2 + p^2)}{a_2 M_T} + \frac{p^2 \tilde{C}_7^2}{a_2 \tilde{C}_1} \right] + \cos a_2 L \frac{m p^2 \tilde{C}_7}{a_2^2 \tilde{C}_1} - \frac{m p^2 \tilde{C}_7}{a_2^2 \tilde{C}_1} \right\}$$

Solving equations (59) and (60) simultaneously, we write

$$C_{w1} = \tilde{A}_1 c_1 + \tilde{A}_2 c_2 + \tilde{A}_3 c_3 + \tilde{A}_4 c_4 \qquad \text{------ (61)}$$

and

$$C_{w2} = \tilde{B}_1 c_1 + \tilde{B}_2 c_2 + \tilde{B}_3 c_3 + \tilde{B}_4 c_4 \qquad \text{------ (62)}$$

where $\tilde{A}_1 = \dfrac{A_3 B_2 + A_2 B_3}{A_1 B_2 - A_2 B_1}$

$$\tilde{A}_2 = \frac{A_4 B_2 + A_2 B_4}{A_1 B_2 - A_2 B_1}$$

$$\tilde{A}_3 = \frac{A_5 B_2 + A_2 B_5}{A_1 B_2 - A_2 B_1}$$

$$\tilde{A}_4 = \frac{A_6 B_2 + A_2 B_6}{A_1 B_2 - A_2 B_1}$$

$$\tilde{B}_1 = \frac{B_3 + B_1 \tilde{A}_1}{B_2}$$

$$\widetilde{B}_2 = \frac{B_4 + B_1\widetilde{A}_2}{B_2}$$

$$\widetilde{B}_3 = \frac{B_5 + B_1\widetilde{A}_3}{B_2}$$

$$\widetilde{B}_4 = \frac{B_6 + B_1\widetilde{A}_4}{B_2}$$

3. Development of the Characteristic Determinant

By making the substitutions for $\theta = \Omega t + \tau(t)$ and $w = \eta_o e^{ipt} R(r)$ and substituting $\ddot{\tau}$ from equation (43) into the boundary condition equations given by equations (45) - (48), we have, after algebraic simplification and neglecting non-linear terms,

$$R(0) = 0 \qquad \text{------ (63)}$$

$$R(L) = 0 \qquad \text{------ (64)}$$

$$EIR''(0) + \widetilde{D}_1 R'(0) + \widetilde{D}_2 R'(L) + \widetilde{D}_3 \int_0^L R(r)\,dr - \widetilde{D}_4 \int_0^L rR(r)\,dr = 0 \qquad \text{------ (65)}$$

$$EIR''(L) + \widetilde{D}_5 R'(0) + \widetilde{D}_6 R'(L) + \widetilde{D}_7 \int_0^L R(r)\,dr + \widetilde{D}_8 \int_0^L rR(r)\,dr = 0 \qquad \text{------ (66)}$$

The identities $\widetilde{D}_1$ through $\widetilde{D}_8$ are given below as

$$\widetilde{D}_1 = D_4\Omega^2 - p^2 \left[D_2 - \frac{D_1\widetilde{C}_2}{\widetilde{C}_1} \right]$$

$$\tilde{D}_2 = D_5\Omega^2 - p^2 \left[D_3 - \frac{D_1\tilde{C}_3}{\tilde{C}_1} \right]$$

$$\tilde{D}_3 = D_6\Omega^2 - p^2 \left[D_7 - \frac{D_1\tilde{C}_7}{\tilde{C}_1} \right]$$

$$\tilde{D}_4 = mp^2 \left[\frac{d_1}{\tilde{L}} - \frac{D_1}{\tilde{C}_1} \right]$$

$$\tilde{D}_5 = -D_5\Omega^2 - p^2 \left[-D_3 - \frac{D_8\tilde{C}_2}{\tilde{C}_1} \right]$$

$$\tilde{D}_6 = D_{12}\Omega^2 - p^2 \left[D_{10} - \frac{D_8\tilde{C}_3}{\tilde{C}_1} \right]$$

$$\tilde{D}_7 = D_{13}\Omega^2 - p^2 \left[D_{14} - \frac{D_8\tilde{C}_7}{\tilde{C}_1} \right]$$

$$\tilde{D}_8 = mp^2 \left[\frac{d_2}{\tilde{L}} + \frac{D_8}{\tilde{C}_1} \right]$$

The solution for $R(r)$ given in equations (56), (61), (62) is substituted into equations (63) - (66) to give a set of four simultaneous homogeneous equations in the unknown p^2, written as

$$a_{11}c_1 + a_{12}c_2 + a_{13}c_3 + a_{14}c_4 = 0$$

$$a_{21}c_1 + a_{22}c_2 + a_{23}c_3 + a_{24}c_4 = 0$$

$$a_{31}c_1 + a_{32}c_2 + a_{33}c_3 + a_{34}c_4 = 0$$

$$a_{41}c_1 + a_{42}c_2 + a_{43}c_3 + a_{44}c_4 = 0$$

from which we obtain an associated characteristic determinant

$$D(p^2) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

The elements a_{ij} of the determinant are defined as

$$a_{11} = \frac{\widetilde{B}_1}{m(\Omega^2 + p^2)}$$

$$a_{12} = \frac{\widetilde{B}_2}{m(\Omega^2 + p^2)} + 1$$

$$a_{13} = \frac{\widetilde{B}_3}{m(\Omega^2 + p^2)}$$

$$a_{14} = \frac{\widetilde{B}_4}{m(\Omega^2 + p^2)} + 1$$

$$a_{21} = \sinh a_1 L + \frac{L\widetilde{A}_1 + \widetilde{B}_1}{m(\Omega^2 + p^2)}$$

$$a_{22} = \cosh a_1 L + \frac{L\widetilde{A}_2 + \widetilde{B}_2}{m(\Omega^2 + p^2)}$$

$$a_{23} = \sin a_2 L + \frac{L\widetilde{A}_3 + \widetilde{B}_3}{m(\Omega^2 + p^2)}$$

$$a_{24} = \cos a_2 L + \frac{L\tilde{A}_4 + \tilde{B}_4}{m(\Omega^2 + p^2)}$$

$$a_{31} = \left[\tilde{D}_1 a_1 + \tilde{D}_2 a_1 \cosh a_1 L + \frac{\tilde{D}_3}{a_1}(\cosh a_1 L - 1) + \frac{\tilde{B}_1 L(2\tilde{D}_3 - \tilde{D}_4 L)}{2m(\Omega^2 + p^2)} \right.$$
$$\left. - \frac{\tilde{D}_4}{a_1}\left(L\cosh a_1 L - \frac{1}{a_1}\sinh a_1 L \right) + \frac{\tilde{A}_1(6\tilde{D}_1 + 6\tilde{D}_2 + 3\tilde{D}_3 L^2 - 2\tilde{D}_4 L^3)}{6m(\Omega^2 + p^2)} \right]$$

$$a_{32} = \left[EIa_1^2 + \tilde{D}_2 a_1 \sinh a_1 L + \frac{\tilde{D}_3}{a_1}\sinh a_1 L \right.$$
$$- \frac{\tilde{D}_4}{a_1}\left(L\sinh a_1 L - \frac{1}{a_1}\cosh a_1 L + \frac{1}{a_1} \right) + \frac{\tilde{B}_2 L(2\tilde{D}_3 - \tilde{D}_4 L)}{2m(\Omega^2 + p^2)}$$
$$\left. + \frac{\tilde{A}_2(6\tilde{D}_1 + 6\tilde{D}_2 + 3\tilde{D}_3 L^2 - 2\tilde{D}_4 L^3)}{6m(\Omega^2 + p^2)} \right]$$

$$a_{33} = \left[\tilde{D}_1 a_2 + \tilde{D}_2 a_2 \cos a_2 L - \frac{\tilde{D}_3}{a_2}(\cos a_2 L - 1) + \frac{\tilde{B}_3 L(2\tilde{D}_3 - \tilde{D}_4 L)}{2m(\Omega^2 + p^2)} \right.$$
$$\left. - \frac{\tilde{D}_4}{a_2}\left(\frac{1}{a_2}\sin a_2 L - L\cos a_2 L \right) + \frac{\tilde{A}_3(6\tilde{D}_1 + 6\tilde{D}_2 + 3\tilde{D}_3 L^2 - 2\tilde{D}_4 L^3)}{6m(\Omega^2 + p^2)} \right]$$

$$a_{34} = \left[- EIa_2^2 - \tilde{D}_2 a_2 \sin a_2 L + \frac{\tilde{D}_3}{a_2}\sin a_2 L \right.$$
$$- \frac{\tilde{D}_4}{a_2}\left(\frac{1}{a_2}\cos a_2 L - \frac{1}{a_2} + L\sin a_2 L \right) + \frac{\tilde{B}_4 L(2\tilde{D}_3 - \tilde{D}_4 L)}{2m(\Omega^2 + p^2)}$$
$$\left. + \frac{\tilde{A}_4(6\tilde{D}_1 + 6\tilde{D}_2 + 3\tilde{D}_3 L^2 - 2\tilde{D}_4 L^3)}{6m(\Omega^2 + p^2)} \right]$$

$$a_{41} = \left[EIa_1^2 \sinh a_1 L + \tilde{D}_5 a_1 + \tilde{D}_6 a_1 \cosh a_1 L + \frac{\tilde{D}_7}{a_1}(\cosh a_1 L - 1) + \frac{\tilde{D}_8}{a_1}\left(L\cosh a_1 L - \frac{1}{a_1}\sinh a_1 L\right) + \frac{\tilde{B}_1 L(2\tilde{D}_7 + \tilde{D}_8 L)}{2m(\Omega^2 + p^2)} + \frac{\tilde{A}_1(6\tilde{D}_5 + 6\tilde{D}_6 + 3\tilde{D}_7 L^2 + 2\tilde{D}_8 L^3)}{6m(\Omega^2 + p^2)} \right]$$

$$a_{42} = \left[EIa_1^2 \cosh a_1 L + \tilde{D}_6 a_1 \sinh a_1 L + \frac{\tilde{D}_7}{a_1}\sinh a_1 L + \frac{\tilde{D}_8}{a_1}\left(L\sinh a_1 L - \frac{1}{a_1}\cosh a_1 L + \frac{1}{a_1}\right) + \frac{\tilde{B}_2 L(2\tilde{D}_7 + \tilde{D}_8 L)}{2m(\Omega^2 + p^2)} + \frac{\tilde{A}_2(6\tilde{D}_5 + 6\tilde{D}_6 + 3\tilde{D}_7 L^2 + 2\tilde{D}_8 L^3)}{6m(\Omega^2 + p^2)} \right]$$

$$a_{43} = \left[-EIa_2^2 \sin a_2 L + \tilde{D}_5 a_2 + \tilde{D}_6 a_2 \cos a_2 L - \frac{\tilde{D}_7}{a_2}(\cos a_2 L - 1) + \frac{\tilde{D}_8}{a_2}\left(\frac{1}{a_2}\sin a_2 L - L\cos a_2 L\right) + \frac{\tilde{B}_3 L(2\tilde{D}_7 + \tilde{D}_8 L)}{2m(\Omega^2 + p^2)} + \frac{\tilde{A}_3(6\tilde{D}_5 + 6\tilde{D}_6 + 3\tilde{D}_7 L^2 + 2\tilde{D}_8 L^3)}{6m(\Omega^2 + p^2)} \right]$$

$$a_{44} = \left[-EIa_2^2 \cos a_2 L - \tilde{D}_6 a_2 \sin a_2 L + \frac{\tilde{D}_7}{a_2}\sin a_2 L + \frac{\tilde{D}_8}{a_2}\left(\frac{1}{a_2}\cos a_2 L - \frac{1}{a_2} + L\sin a_2 L\right) + \frac{\tilde{B}_4 L(2\tilde{D}_7 + \tilde{D}_8 L)}{2m(\Omega^2 + p^2)} + \frac{\tilde{A}_4(6\tilde{D}_5 + 6\tilde{D}_6 + 3\tilde{D}_7 L^2 + 2\tilde{D}_8 L^3)}{6m(\Omega^2 + p^2)} \right]$$

The characteristic determinant has been programmed for solution on the IBM 7094 digital computer. The values of p^2 for which $D(p^2) = 0$ are the natural frequencies of the in-plane motion of the space station, with associated mode shapes. Numerical results of this analysis for a particular space station design are presented in Appendix III, Section 1.

4. Special Cases for Values of $p^2 \leq -\Omega^2$

In order to investigate the behavior of the characteristic determinant when p^2 takes on negative values of magnitude exceeding Ω^2, we recognize five regions where the general solution obtained in the previous article is not valid. Discussion of these regions follows.

a. The Case Where p^2 Is Between $p^2 = -\Omega^2$ and $p^2 = \frac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$

Within this region the governing equation is unchanged from equation (55), but the solution is written more conveniently as:

$$R(r) = c_1 \sinh a_1 r + c_2 \cosh a_1 r + c_3 \sinh a_3 r + c_4 \cosh a_3 r + \frac{C_{w1} r}{m(\Omega^2 + p^2)} + \frac{C_{w2}}{m(\Omega^2 + p^2)} \qquad \text{------ (67)}$$

where a_1 is given by equation (57) and

$$a_3 = \left[\frac{\tilde{C}_{24}\Omega^2 - \sqrt{\tilde{C}_{24}^2\Omega^4 + 4EIm(\Omega^2 + p^2)}}{2EI} \right]^{\frac{1}{2}} \qquad \text{------ (68)}$$

The procedure of Section 3 again leads to a characteristic determinant valid within the stated region. Numerical results of the analysis of this case are presented in Section 2 of Appendix III.

b. The Case Where $p^2 \equiv -\Omega^2$

For this case the governing equation becomes

$$EIR^{IV}(r) - \tilde{C}_{24}\Omega^2 R''(r) + C_{w1} r + C_{w2} = 0 \qquad \text{------ (69)}$$

for which the solution is

$$R(r) = c_1 \sinh a_4 r + c_2 \cosh a_4 r + c_3 r + c_4 + \frac{C_{w1} r^3}{6\tilde{C}_{24}\Omega^2} + \frac{C_{w2} r^2}{2\tilde{C}_{24}\Omega^2} \qquad \text{------ (70)}$$

where

$$a_4 = \left[\frac{\tilde{C}_{24}\Omega^2}{EI}\right]^{\frac{1}{2}} \qquad \text{------ (71)}$$

Following the procedure of Section 3 we obtain a characteristic determinant valid for the case $p^2 \equiv -\Omega^2$. Numerical results of the analysis for this case are presented in Section 3 of Appendix III.

c. The Case Where p^2 Takes on Negative Values Larger than $p^2 = \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$

The governing equation for this region is given by equation (55), but the solution will now be written as:

$$R(r) = c_1 \sinh a_5 r \sin a_6 r + c_2 \sinh a_5 r \cos a_6 r + c_3 \cosh a_5 r \sin a_6 r + c_4 \cosh a_5 r \cos a_6 r + \frac{C_{w1} r}{m(\Omega^2 + p^2)} + \frac{C_{w2}}{m(\Omega^2 + p^2)} \quad \text{------ (72)}$$

where

$$a_5 = \left[\frac{\tilde{C}_{24}\Omega^2}{4EI} + \sqrt{\frac{-m(\Omega^2 + p^2)}{4EI}} \right]^{\frac{1}{2}} \quad \text{------ (73)}$$

$$a_6 = \left[\frac{-\tilde{C}_{24}\Omega^2}{4EI} + \sqrt{\frac{-m(\Omega^2 + p^2)}{4EI}} \right]^{\frac{1}{2}} \quad \text{------ (74)}$$

The procedure of Section 3 leads to a characteristic determinant valid within the stated region. Numerical results of the analysis for this case are presented in Section 4 of Appendix III.

d. <u>The Case Where $p^2 \equiv \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$</u>

The governing equation for this case is

$$EIR^{IV}(r) - \tilde{C}_{24}\Omega^2 R''(r) + \frac{\tilde{C}_{24}^2\Omega^4}{4EI} R(r) + C_{w1} r + C_{w2} = 0 \quad \text{------ (75)}$$

for which the solution is

$$R(r) = c_1 \sinh a_7 r + c_2 \cosh a_7 r + c_3 r \sinh a_7 r + c_4 r \cosh a_7 r - \frac{4EIC_{w1} r}{\tilde{C}_{24}^2\Omega^4} - \frac{4EIC_{w2}}{\tilde{C}_{24}^2\Omega^4} \quad \text{------ (76)}$$

where $a_7 = \left[\frac{\tilde{C}_{24}\Omega^2}{2EI}\right]^{\frac{1}{2}}$ ------ (77)

Following the procedure of Section 3 we obtain a characteristic determinant valid for the case $p^2 \equiv \frac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$. Numerical results of the analysis for this case are presented in Section 5 of Appendix III.

e. The Region Where $A_1B_2 - A_2B_1$ Is Identically Zero

This region gives rise to a singularity in calculating the value of the characteristic determinant; therefore the simultaneous solution of equations (59) and (60) must be modified. In order to simplify the discussion of this point of singularity, the physical constants of the space station design given in Appendix III are used as an example.

For the given data $A_2 \equiv 0$. Therefore the equation $A_1B_2 - A_2B_1 \equiv 0$ becomes $A_1B_2 \equiv 0$, and since $B \neq 0$ we must have $A_1 = 0$. From the definition of A_1 on page 54 we solve for the p^2 for which $A_1 = 0$. Thus

$$p_s^2 = \frac{-\tilde{C}_1\tilde{L}\Omega^2}{\tilde{C}_1\tilde{L} - \frac{1}{2}\tilde{C}_7L^2\tilde{L} - \frac{1}{3}mL^3\tilde{L} - d_1\tilde{C}_1 - d_2\tilde{C}_1 - \tilde{L}\tilde{C}_2 - \tilde{L}\tilde{C}_3}$$

which falls within the region discussed in paragraph a of this section. For this value of p^2 equations (59) and (60) become

$$0 = A_3c_1 + A_4c_2 + A_5c_3 + A_6c_4 \qquad \text{------ (78)}$$

and
$$C_{w2} = \frac{B_3c_1 + B_4c_2 + B_5c_3 + B_6c_4 + B_1C_{w1}}{B_2} \qquad \text{------ (79)}$$

The identity of equation (79) is used with the solution for $R(r)$ of equation (67) and substituted into the boundary equations given by equations (63) - (66) to obtain four simultaneous equations, written as

$$e_{11}c_1 + e_{12}c_2 + e_{13}c_3 + e_{14}c_4 + e_{15}\frac{C_{w1}}{m(\Omega^2 + p^2)} = 0$$

$$e_{21}c_1 + e_{22}c_2 + e_{23}c_3 + e_{24}c_4 + e_{25}\frac{C_{w1}}{m(\Omega^2 + p^2)} = 0$$

$$e_{31}c_1 + e_{32}c_2 + e_{33}c_3 + e_{34}c_4 + e_{35}\frac{C_{w1}}{m(\Omega^2 + p^2)} = 0$$

$$e_{41}c_1 + e_{42}c_2 + e_{43}c_3 + e_{44}c_4 + e_{45}\frac{C_{w1}}{m(\Omega^2 + p^2)} = 0$$

and a fifth equation is obtained from equation (78), rewritten as

$$e_{51}c_1 + e_{52}c_2 + e_{53}c_3 + e_{54}c_4 + e_{55}\frac{C_{w1}}{m(\Omega^2 + p^2)} = 0$$

Thus the characteristic determinant for this case becomes

$$D(p_s^2) = \begin{vmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} \\ e_{41} & e_{42} & e_{43} & e_{44} & e_{45} \\ e_{51} & e_{52} & e_{53} & e_{54} & e_{55} \end{vmatrix}$$

where the elements e_{ij} of the determinant are defined as

$$e_{11} = \frac{B_3}{m(\Omega^2 + p_s^2) B_2}$$

$$e_{12} = \frac{B_4}{m(\Omega^2 + p_s^2) B_2} + 1$$

$$e_{13} = \frac{B_5}{m(\Omega^2 + p_s^2) B_2}$$

$$e_{14} = \frac{B_6}{m(\Omega^2 + p_s^2) B_2} + 1$$

$$e_{15} = \frac{B_1}{B_2}$$

$$e_{21} = \sinh a_1 L + \frac{B_3}{m(\Omega^2 + p_s^2) B_2}$$

$$e_{22} = \cosh a_1 L + \frac{B_4}{m(\Omega^2 + p_s^2) B_2}$$

$$e_{23} = \sinh a_4 L + \frac{B_5}{m(\Omega^2 + p_s^2) B_2}$$

$$e_{24} = \cosh a_4 L + \frac{B_6}{m(\Omega^2 + p_s^2) B_2}$$

$$e_{25} = L + \frac{B_1}{B_2}$$

$$e_{31} = \left[\tilde{D}_1 a_1 + \tilde{D}_2 a_1 \cosh a_1 L + \frac{\tilde{D}_3}{a_1}(\cosh a_1 L - 1) + \frac{\tilde{D}_3 B_3 L}{m(\Omega^2 + p_s^2) B_2} - \frac{\tilde{D}_4}{a_1}\left(L\cosh a_1 L - \frac{1}{a_1}\sinh a_1 L \right) - \frac{\tilde{D}_4 B_3 L^2}{2m(\Omega^2 + p_s^2) B_2} \right]$$

$$e_{32} = \left[EIa_1^2 + \tilde{D}_2 a_1 \sinh a_1 L + \frac{\tilde{D}_3}{a_1} \sinh a_1 L + \frac{\tilde{D}_3 B_4 L}{m(\Omega^2 + p_s^2) B_2} - \frac{\tilde{D}_4}{a_1}\left(L \sinh a_1 L - \frac{1}{a_1}\cosh a_1 L + \frac{1}{a_1} \right) - \frac{\tilde{D}_4 B_4 L^2}{2m(\Omega^2 + p_s^2) B_2} \right]$$

$$e_{33} = \left[\tilde{D}_1 a_4 + \tilde{D}_2 a_4 \cosh a_4 L + \frac{\tilde{D}_3}{a_4} (\cosh a_4 L - 1) + \frac{\tilde{D}_3 B_5 L}{m(\Omega^2 + p_s^2) B_2} - \frac{\tilde{D}_4}{a_4}\left(L \cosh a_4 L - \frac{1}{a_4}\sinh a_4 L \right) - \frac{\tilde{D}_4 B_4 L^2}{2m(\Omega^2 + p_s^2) B_2} \right]$$

$$e_{34} = \left[EIa_4^2 + \tilde{D}_2 a_4 \sinh a_4 L + \frac{\tilde{D}_3}{a_4} \sinh a_4 L + \frac{\tilde{D}_3 B_6 L}{m(\Omega^2 + p_s^2) B_2} - \frac{\tilde{D}_4}{a_4}\left(L \sinh a_4 L - \frac{1}{a_4}\cosh a_4 L + \frac{1}{a_4} \right) - \frac{\tilde{D}_4 B_6 L^2}{2m(\Omega^2 + p_s^2) B_2} \right]$$

$$e_{35} = \left[\tilde{D}_1 + \tilde{D}_2 + \frac{\tilde{D}_3 L^2}{2} - \frac{\tilde{D}_4 L^3}{3} + \frac{B_1 L}{B_2}\left(\tilde{D}_3 - \frac{\tilde{D}_4 L}{2} \right) \right]$$

$$e_{41} = \left[EIa_1^2 \sinh a_1 L + \tilde{D}_5 a_1 + \tilde{D}_6 a_1 \cosh a_1 L + \frac{\tilde{D}_7}{a_1}(\cosh a_1 L - 1) + \frac{\tilde{D}_7 B_3 L}{m(\Omega^2 + p_s^2) B_2} + \frac{\tilde{D}_8}{a_1}\left(L \cosh a_1 L - \frac{1}{a_1} \sinh a_1 L \right) + \frac{\tilde{D}_8 B_3 L^2}{2m(\Omega^2 + p_s^2) B_2} \right]$$

$$e_{42} = \left[EIa_1^2\cosh a_1L + \tilde{D}_6 a_1 \sinh a_1 L + \frac{\tilde{D}_7}{a_1}\sinh a_1 L + \frac{\tilde{D}_7 B_4 L}{m(\Omega^2 + p_s^2)B_2} + \frac{\tilde{D}_8}{a_1}\left(L\sinh a_1 L - \frac{1}{a_1}\cosh a_1 L + \frac{1}{a_1}\right) + \frac{\tilde{D}_8 B_4 L^2}{2m(\Omega^2 + p_s^2)B_2} \right]$$

$$e_{43} = \left[EIa_4^2\sinh a_4 L + \tilde{D}_5 a_4 + \tilde{D}_6 a_4 \cosh a_4 L + \frac{\tilde{D}_7}{a_4}(\cosh a_4 L - 1) + \frac{\tilde{D}_7 B_5 L}{m(\Omega^2 + p_s^2)B_2} + \frac{\tilde{D}_8}{a_4}\left(L\cosh a_4 L - \frac{1}{a_4}\sinh a_4 L\right) + \frac{\tilde{D}_8 B_5 L^2}{2m(\Omega^2 + p_s^2)B_2} \right]$$

$$e_{44} = \left[EIa_4^2\cosh a_4 L + \tilde{D}_6 a_4 \sinh a_4 L + \frac{\tilde{D}_7}{a_4}\sinh a_4 L + \frac{\tilde{D}_7 B_6 L}{m(\Omega^2 + p_s^2)B_2} + \frac{\tilde{D}_8}{a_4}\left(L\sinh a_4 L - \frac{1}{a_4}\cosh a_4 L + \frac{1}{a_4}\right) + \frac{\tilde{D}_8 B_6 L^2}{2m(\Omega^2 + p_s^2)B_2} \right]$$

$$e_{45} = \left[\tilde{D}_5 + \tilde{D}_6 + \frac{\tilde{D}_7 L^2}{2} + \frac{\tilde{D}_8 L^3}{3} + \frac{B_1 L}{B_2}\left(\tilde{D}_7 + \frac{\tilde{D}_8 L}{2}\right) \right]$$

$$e_{51} = A_3$$

$$e_{52} = A_4$$

$$e_{53} = A_5$$

$$e_{54} = A_6$$

$$e_{55} = 0$$

The characteristic determinant has been programmed for solution on the IBM 7094 digital computer. Numerical results of the analysis are presented in Section 6 of Appendix III.

VI. DISCUSSION

The results of this analysis of the free vibration of a rotating beam-connected space station have shown that the motion of the system can be considered to be divided into motion in the plane of rotation which is uncoupled from the motion of the system in the plane perpendicular to the plane of rotation. For motion in the plane of rotation an exact solution for the beam deflection has been obtained, and a set of nonhomogeneous boundary conditions representing a fixed-fixed attachment of the beam to the space modules has been used to give a characteristic determinant. Computer programs have been developed to solve for the zeros of the characteristic determinant and to calculate the natural modes of vibration of the rotating system.

An exact solution for the beam deflection for motion of the system in the plane perpendicular to the plane of rotation can be obtained using similar methods, but was not included in this study because of the length of the algebraic forms and computer programs.

A particular space station made up of two manned space modules connected by a flexible beam has been studied to provide an example giving numerical results of the analysis.

The natural frequencies and mode shapes for the six lowest modes have been calculated for spin rates varying from 0 to 3.5 rev/min. The effect of the spin rate has been shown to be an increase in the natural frequencies corresponding to an "apparent" increase in beam stiffness due to rotation.

As discussed in Section 4, Chapter V, analyses have been made to investigate the behavior of the characteristic determinant when p^2 is less than or equal to $-\Omega^2$. The numerical results for these special cases demonstrate the existence of two rigid body modes with non-zero frequencies $p^2 \equiv -\Omega^2$ and $p^2 \equiv \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$ respectively. For these frequencies the value of the characteristic determinant has been shown to approach zero as $p^2 \to -\Omega^2$ and as $p^2 \to \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$, while $D(p^2 \equiv -\Omega^2) \neq 0$ and $D\left(p^2 \equiv \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}\right) \neq 0$. Therefore for each of these frequencies we have the rigid body mode $R(r) \equiv 0$ since the arbitrary constants c_j $(j = 1 - 4)$ of the solution must vanish in order to satisfy the boundary condition equations.

Frequency sweeps have been conducted for negative values of p^2, and no instabilities for the motion of the rotating system in the plane of the orbit have been found.

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APPENDIX I

DERIVATION OF THE BEAM ORIENTATION VECTORS

In order to describe the motion of the point on the beam, we identify an orthogonal set of unit vectors $(\vec{e}_v, \vec{e}_{p_1p_2}, \vec{e}_w)$ as shown in Figure 4. We define the unit vector $\vec{e}_w$ from the vector equation

$$\vec{I} \times \vec{d}_{p_1p_2} = d_{p_1p_2} \sin\mu \vec{e}_w$$

$$= -\vec{J}\begin{bmatrix} z_2 - z_1 + d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ - d_2\cos\theta\sin\beta_2\cos\alpha_2 + d_1\sin\theta\sin\psi\sin\alpha_1 \\ - d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 - d_1\cos\theta\sin\beta_1\cos\alpha_1 \end{bmatrix}$$

$$+\vec{K}\begin{bmatrix} y_2 - y_1 + d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ + d_2\sin\theta\sin\beta_2\cos\alpha_2 + d_1\cos\theta\sin\psi\sin\alpha_1 \\ - d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 + d_1\sin\theta\sin\beta_1\cos\alpha_1 \end{bmatrix}$$

where the angle μ is shown in Figure 6. Taking the dot product of the vector with itself, we obtain the scalar equation

$$(d_{p_1p_2}\sin\mu)^2 = (z_2 - z_1)^2 + (y_2 - y_1)^2 + d_2^2\sin^2\psi\sin^2\alpha_2$$
$$+ d_2^2\cos^2\psi\cos^2\beta_2\cos^2\alpha_2 + d_2^2\sin^2\beta_2\cos^2\alpha_2$$
$$+ d_1^2\sin^2\psi\sin^2\alpha_1 + d_1^2\cos^2\psi\cos^2\beta_1\cos^2\alpha_1$$
$$+ d_1^2\sin^2\beta_1\cos^2\alpha_1$$

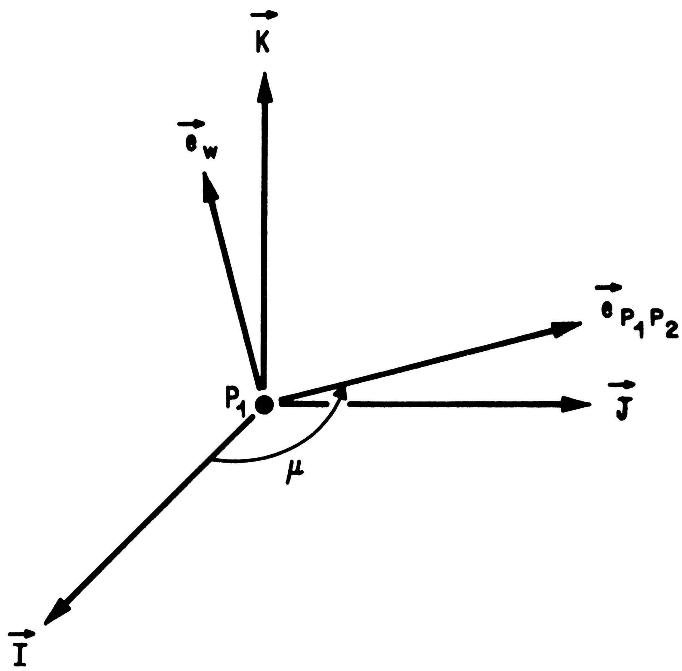


FIG. 6. THE ANGLE μ BETWEEN $\vec{e}_{P_1P_2}$ AND $\vec{I}$

$$+ 2(z_2 - z_1)\left[\begin{array}{l} d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ - d_2\cos\theta\sin\beta_2\cos\alpha_2 + d_1\sin\theta\sin\psi\sin\alpha_1 \\ - d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 \\ - d_1\cos\theta\sin\beta_1\cos\alpha_1 \end{array}\right]$$

$$+ 2(y_2 - y_1)\left[\begin{array}{l} d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ + d_2\sin\theta\sin\beta_2\cos\alpha_2 + d_1\cos\theta\sin\psi\sin\alpha_1 \\ - d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 \\ + d_1\sin\theta\sin\beta_1\cos\alpha_1 \end{array}\right]$$

$$- 2d_2^2\sin\psi\cos\psi\cos\beta_2\sin\alpha_2\cos\alpha_2 + 2d_1d_2\sin^2\psi\sin\alpha_1\sin\alpha_2$$

$$- 2d_1d_2\sin\psi\cos\psi\cos\beta_1\cos\alpha_1\sin\alpha_2$$

$$- 2d_1d_2\sin\psi\cos\psi\cos\beta_2\sin\alpha_1\cos\alpha_2$$

$$+ 2d_1d_2\cos^2\psi\cos\beta_1\cos\beta_2\cos\alpha_1\cos\alpha_2$$

$$+ 2d_1d_2\sin\beta_1\sin\beta_2\cos\alpha_1\cos\alpha_2$$

$$- 2d_1^2\sin\psi\cos\psi\cos\beta_1\sin\alpha_1\cos\alpha_1$$

We now identify the angles λ_k $(k = 1,\ 2,\ 3)$ which the vector $\vec{d}_{12}$ makes with the inertia space axes shown in Figure 7a. From Figure 7b we see that

$$\lambda_1 = \frac{\pi}{2} + \psi$$

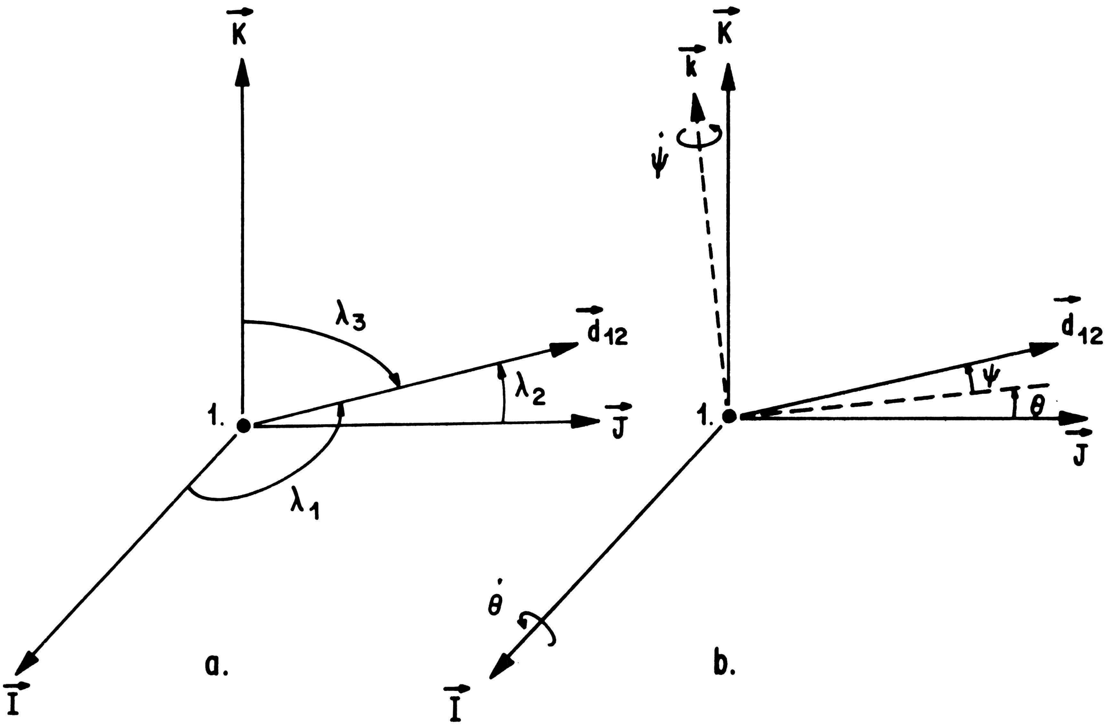


FIG. 7. ORIENTATION OF $\vec{d}_{12}$ IN INERTIA SPACE

and

$\lambda_2 \simeq \theta$ when ψ is assumed to be a small angle

From the sum of the direction cosines we have, neglecting higher order terms,

$$\lambda_3 = \frac{\pi}{2} - \theta$$

Taking the dot product of $\vec{d}_{12}$ with each of the inertia axes, we write

$$\vec{d}_{12} \cdot \vec{I} = d_{12}\cos\lambda_1 = x_2 - x_1 \simeq -d_{12}\psi$$

$$\vec{d}_{12} \cdot \vec{J} = d_{12}\cos\lambda_2 = y_2 - y_1 \simeq d_{12}\cos\theta$$

$$\vec{d}_{12} \cdot \vec{K} = d_{12}\cos\lambda_3 = z_2 - z_1 \simeq d_{12}\sin\theta$$

Substituting these identities into the equation for $(d_{p_1p_2}\sin\mu)^2$, we have

$$\begin{aligned}(d_{p_1p_2}\sin\mu)^2 = {} & d_{12}^2 + d_2^2\psi^2\sin^2\alpha_2 + d_2^2\cos^2\alpha_2 + d_1^2\psi^2\sin^2\alpha_1 + d_1^2\cos^2\alpha_1 \\ & + 2d_{12}\begin{bmatrix} d_2\psi\sin\alpha_2 - d_2\cos\beta_2\cos\alpha_2 + d_1\psi\sin\alpha_1 \\ - d_1\cos\beta_1\cos\alpha_1 \end{bmatrix} \\ & - 2d_2^2\psi\cos\beta_2\sin\alpha_2\cos\alpha_2 + 2d_1d_2\psi^2\sin\alpha_1\sin\alpha_2 \\ & - 2d_1d_2\psi\cos\beta_1\cos\alpha_1\sin\alpha_2 - 2d_1d_2\psi\cos\beta_2\sin\alpha_1\cos\alpha_2 \\ & + 2d_1d_2\cos\alpha_1\cos\alpha_2\cos(\beta_1 - \beta_2) - 2d_1^2\psi\cos\beta_1\sin\alpha_1\cos\alpha_1\end{aligned}$$

We assume α_1, α_2, β_1, β_2 are small angles and neglect higher order terms to obtain

$$(d_{p_1p_2}\sin\mu)^2 \simeq d_{12}^2 + d_2^2 + d_1^2 - 2d_{12}d_2 - 2d_{12}d_1 + 2d_1d_2 = (d_{12} - d_1 - d_2)^2$$

Thus $d_{p_1p_2}\sin\mu = (d_{12} - d_1 - d_2) \simeq L$ for small deflection theory. We note that a similar procedure can be used to show that

$$\vec{d}_{p_1p_2} \cdot \vec{d}_{p_1p_2} = d^2_{p_1p_2} \simeq L^2 \text{ for small deflection theory.}$$

The equations for $\vec{e}_w$ and $\vec{e}_{p_1p_2}$ may now be written as

$$\vec{e}_w = -\frac{\vec{J}}{L}\begin{bmatrix} z_2 - z_1 + d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ - d_2\cos\theta\sin\beta_2\cos\alpha_2 + d_1\sin\theta\sin\psi\sin\alpha_1 \\ - d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 - d_1\cos\theta\sin\beta_1\cos\alpha_1 \end{bmatrix}$$
$$+\frac{\vec{K}}{L}\begin{bmatrix} y_2 - y_1 + d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ + d_2\sin\theta\sin\beta_2\cos\alpha_2 + d_1\cos\theta\sin\psi\sin\alpha_1 \\ - d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 + d_1\sin\theta\sin\beta_1\cos\alpha_1 \end{bmatrix}$$

$$\vec{e}_{p_1p_2} = \frac{\vec{I}}{L}\begin{bmatrix} x_2 - x_1 + d_2\cos\psi\sin\alpha_2 + d_2\sin\psi\cos\beta_2\cos\alpha_2 \\ + d_1\cos\psi\sin\alpha_1 + d_1\sin\psi\cos\beta_1\cos\alpha_1 \end{bmatrix}$$
$$+\frac{\vec{J}}{L}\begin{bmatrix} y_2 - y_1 + d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ + d_2\sin\theta\sin\beta_2\cos\alpha_2 + d_1\cos\theta\sin\psi\sin\alpha_1 \\ - d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 + d_1\sin\theta\sin\beta_1\cos\alpha_1 \end{bmatrix}$$

$$+\frac{\vec{K}}{L}\left[\begin{array}{l} z_2 - z_1 + d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ - d_2\cos\theta\sin\beta_2\cos\alpha_2 + d_1\sin\theta\sin\psi\sin\alpha_1 \\ - d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 - d_1\cos\theta\sin\beta_1\cos\alpha_1 \end{array}\right]$$

From the orthogonality of the unit vectors it can be shown by algebraic manipulation that the third vector of the triad is given by

$$\vec{e}_v = \vec{I}$$

$$-\frac{\vec{J}}{L^2}\left\{\left[\begin{array}{l} x_2 - x_1 + d_2\cos\psi\sin\alpha_2 + d_2\sin\psi\cos\beta_2\cos\alpha_2 \\ + d_1\cos\psi\sin\alpha_1 + d_1\sin\psi\cos\beta_1\cos\alpha_1 \end{array}\right] \cdot \left[\begin{array}{l} y_2 - y_1 + d_2\cos\theta\sin\psi\sin\alpha_2 - d_2\cos\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ + d_2\sin\theta\sin\beta_2\cos\alpha_2 + d_1\cos\theta\sin\psi\sin\alpha_1 \\ - d_1\cos\theta\cos\psi\cos\beta_1\cos\alpha_1 + d_1\sin\theta\sin\beta_1\cos\alpha_1 \end{array}\right]\right\}$$

$$-\frac{\vec{K}}{L^2}\left\{\left[\begin{array}{l} x_2 - x_1 + d_2\cos\psi\sin\alpha_2 + d_2\sin\psi\cos\beta_2\cos\alpha_2 \\ + d_1\cos\psi\sin\alpha_1 + d_1\sin\psi\cos\beta_1\cos\alpha_1 \end{array}\right] \cdot \left[\begin{array}{l} z_2 - z_1 + d_2\sin\theta\sin\psi\sin\alpha_2 - d_2\sin\theta\cos\psi\cos\beta_2\cos\alpha_2 \\ - d_2\cos\theta\sin\beta_2\cos\alpha_2 + d_1\sin\theta\sin\psi\sin\alpha_1 \\ - d_1\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 - d_1\cos\theta\sin\beta_1\cos\alpha_1 \end{array}\right]\right\}$$

APPENDIX II

DERIVATION OF THE BOUNDARY CONDITIONS

The relationship between the slope of the beam at the end points and the rotation angles of the respective stages is established for the fixed-fixed boundary conditions by the following derivation:

The orthogonal set of unit vectors $\vec{e}_v$, $\vec{e}_{p_1p_2}$, $\vec{e}_w$ has been described in Appendix I. From Figure 8 we identify the angles μ_{v_1}, μ_{r_1}, μ_{w_1} made by the vector $\vec{e}_{2_1}$ with $\vec{e}_v$, $\vec{e}_{p_1p_2}$, $\vec{e}_w$ respectively, where

$$
\begin{aligned}
\vec{e}_{2_1} \cdot \vec{e}_v = \cos\mu_{v_1} = & \ (-\cos\psi\sin\alpha_1 - \sin\psi\cos\beta_1\cos\alpha_1) \\
& - \frac{1}{L^2}\begin{bmatrix} x_2 - x_1 + d_2\cos\psi\sin\alpha_2 + d_2\sin\psi\cos\beta_2\cos\alpha_2 \\ + d_1\cos\psi\sin\alpha_1 + d_1\sin\psi\cos\beta_1\cos\alpha_1 \end{bmatrix} \\
& \cdot \left\{ \begin{aligned} & (y_2 - y_1)\begin{bmatrix} -\cos\theta\sin\psi\sin\alpha_1 + \cos\theta\cos\psi\cos\beta_1\cos\alpha_1 \\ -\sin\theta\sin\beta_1\cos\alpha_1 \end{bmatrix} \\ & + (z_2 - z_1)\begin{bmatrix} -\sin\theta\sin\psi\sin\alpha_1 + \cos\theta\sin\beta_1\cos\alpha_1 \\ + \sin\theta\cos\psi\cos\beta_1\cos\alpha_1 \end{bmatrix} \\ & - d_2\sin^2\psi\sin\alpha_1\sin\alpha_2 + d_2\sin\psi\cos\psi\cos\beta_2\sin\alpha_1\cos\alpha_2 \\ & - d_1\sin^2\psi\sin^2\alpha_1 + d_1\sin\psi\cos\psi\cos\beta_1\sin\alpha_1\cos\alpha_1 \end{aligned} \right\}
\end{aligned}
$$

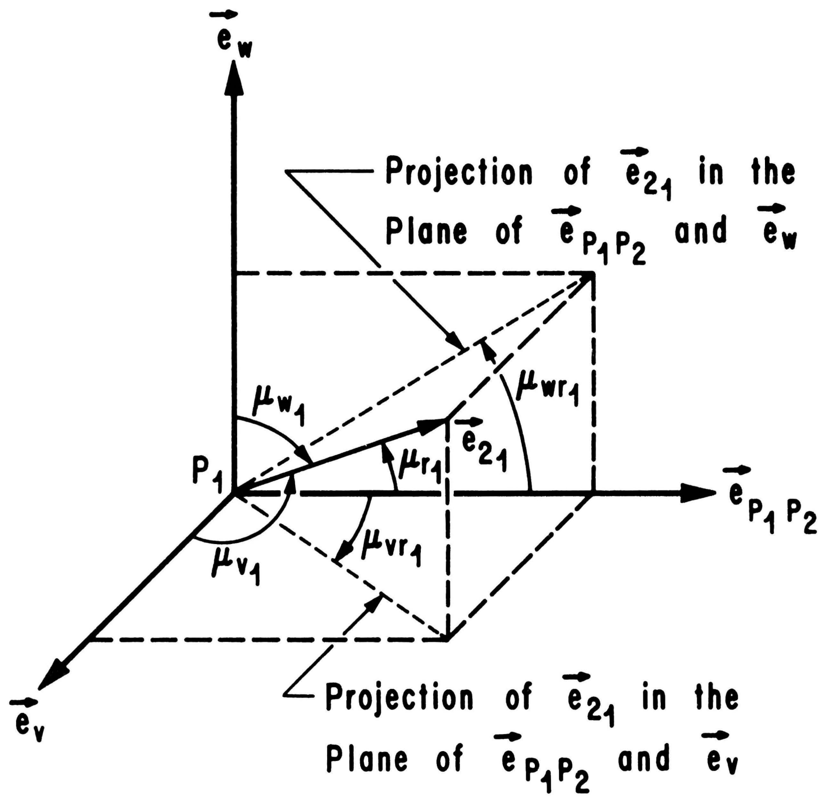


FIG. 8. ORIENTATION OF $\vec{e}_{2_1}$ WITH THE TRIAD $\vec{e}_v$, $\vec{e}_{P_1P_2}$, $\vec{e}_w$

$$\left\{\begin{array}{l} + d_2\sin\psi\cos\psi\cos\beta_1\cos\alpha_1\sin\alpha_2 - d_1\sin^2\beta_1\cos^2\alpha_1 \\ - d_2\sin\beta_1\sin\beta_2\cos\alpha_1\cos\alpha_2 - d_1\cos^2\psi\cos^2\beta_1\cos^2\alpha_1 \\ - d_2\cos^2\psi\cos\beta_1\cos\beta_2\cos\alpha_1\cos\alpha_2 \\ + d_1\sin\psi\cos\psi\cos\beta_1\sin\alpha_1\cos\alpha_1 \end{array}\right\}$$

$$\vec{e}_{2_1} \cdot \vec{e}_{p_1p_2} = \cos\mu_{r_1} = \frac{(x_2 - x_1)}{L}\left(-\cos\psi\sin\alpha_1 - \sin\psi\cos\beta_1\cos\alpha_1\right)$$

$$+ \frac{(y_2 - y_1)}{L}\left[\begin{array}{l} -\cos\theta\sin\psi\sin\alpha_1 + \cos\theta\cos\psi\cos\beta_1\cos\alpha_1 \\ -\sin\theta\sin\beta_1\cos\alpha_1 \end{array}\right]$$

$$+ \frac{(z_2 - z_1)}{L}\left[\begin{array}{l} -\sin\theta\sin\psi\sin\alpha_1 + \sin\theta\cos\psi\cos\beta_1\cos\alpha_1 \\ +\cos\theta\sin\beta_1\cos\alpha_1 \end{array}\right]$$

$$+ \frac{1}{L}\left[\begin{array}{l} - d_2\sin\alpha_1\sin\alpha_2 - d_1 \\ - d_2\cos\alpha_1\cos\alpha_2\cos(\beta_1 - \beta_2) \end{array}\right]$$

$$\vec{e}_{2_1} \cdot \vec{e}_{w} = \cos\mu_{w_1} = \frac{(y_2 - y_1)}{L}\left[\begin{array}{l} -\sin\theta\sin\psi\sin\alpha_1 + \cos\theta\sin\beta_1\cos\alpha_1 \\ +\sin\theta\cos\psi\cos\beta_1\cos\alpha_1 \end{array}\right]$$

$$+ \frac{(z_2 - z_1)}{L}\left[\begin{array}{l} \cos\theta\sin\psi\sin\alpha_1 - \cos\theta\cos\psi\cos\beta_1\cos\alpha_1 \\ +\sin\theta\sin\beta_1\cos\alpha_1 \end{array}\right]$$

$$+ \frac{d_2}{L}\sin\psi\left(\sin\beta_1\cos\alpha_1\sin\alpha_2 - \sin\beta_2\sin\alpha_1\cos\alpha_2\right)$$

$$+ \frac{d_2}{L}\cos\psi\cos\alpha_1\cos\alpha_2\sin(\beta_2 - \beta_1)$$

Using small deflection theory and neglecting higher order terms, we write

$$\cos\mu_{v_1} \simeq -\alpha_1\left(1+\frac{d_1}{L}\right) - \alpha_2\left(\frac{d_2}{L}\right)$$

$$\cos\mu_{r_1} \simeq 1$$

$$\cos\mu_{w_1} \simeq \beta_1\left(1+\frac{d_1}{L}\right) + \beta_2\left(\frac{d_2}{L}\right)$$

Now the angle μ_{vr_1} made by the projection of $\vec{e}_{2_1}$ on the plane formed by $\vec{e}_v$, $\vec{e}_{p_1p_2}$ is related to $v,_1(0,t)$ with the equation

$$\tan\left[\mu_{vr_1}\right] = \frac{\cos\mu_{v_1}}{\cos\mu_{r_1}} = \frac{-\alpha_1\left(1+\frac{d_1}{L}\right) - \alpha_2\left(\frac{d_2}{L}\right)}{1} = v,_1(0,t)$$

so $$v,_1(0,t) = -\alpha_1\left(1+\frac{d_1}{L}\right) - \alpha_2\left(\frac{d_2}{L}\right)$$

and similarly we have for the other end of the beam

$$v,_1(L,t) = -\alpha_2\left(1+\frac{d_2}{L}\right) - \alpha_1\left(\frac{d_1}{L}\right)$$

Also the angle μ_{wr_1} made by the projection of $\vec{e}_{2_1}$ on the plane formed by $\vec{e}_{p_1p_2}$, $\vec{e}_w$ is related to $w,_1(0,t)$ with the equation

$$\tan[\mu_{wr_1}] = \frac{\cos\mu_{w_1}}{\cos\mu_{r_1}} = \frac{\beta_1\left(1+\frac{d_1}{L}\right) + \beta_2\left(\frac{d_2}{L}\right)}{1} = w,_1(0,t)$$

so $\quad w,_1(0,t) = \beta_1\left(1+\frac{d_1}{L}\right) + \beta_2\left(\frac{d_2}{L}\right)$

and similarly for the other end of the beam

$$w,_1(L,t) = \beta_2\left(1+\frac{d_2}{L}\right) + \beta_1\left(\frac{d_1}{L}\right)$$

From the equations for $w,_1(0,t)$ and $w,_1(L,t)$ we solve for β_1 and β_2, written as

$$\beta_1 = \frac{1}{\tilde{L}}\left[-d_2 w,_1(L,t) + (L+d_2)\,w,_1(0,t)\right]$$

$$\beta_2 = \frac{1}{\tilde{L}}\left[(L+d_1)\,w,_1(L,t) - d_1 w,_1(0,t)\right]$$

and from the equations for $v,_1(0,t)$ and $v,_1(L,t)$ we write

$$\alpha_1 = \frac{1}{\tilde{L}}\left[d_2 v,_1(L,t) - (L+d_2)\,v,_1(0,t)\right]$$

$$\alpha_2 = \frac{1}{\tilde{L}}\left[-(L+d_1)\,v,_1(L,t) + d_1 v,_1(0,t)\right]$$

APPENDIX III

NUMERICAL RESULTS OF THE ANALYSIS

1. The Basic Solution

A program has been written in Fortran IV for the IBM 7094 digital computer to solve for the zeros of the characteristic determinant described in Section 3, Chapter V. This program utilizes an iteration procedure to conduct a frequency sweep for incremental values of p^2. When the value of the determinant changes sign, the increment is progressively decreased to converge to the value of p^2 for which $D(p^2) \simeq 0$. This value of p^2 is listed as p_j^2 $(j = 1, 2, \ldots N)$, the j th natural frequency of the system, and an associated eigenvector representing the mode shape of the j th mode is calculated. A subroutine is then utilized to calculate the value of $R_j(r)$ for incremental values of r from 0 to L, and the results are plotted to give the j th mode shape of the beam.

The data given in Table I are used for input data to the computer program. These data represent two manned space modules launched by Saturn-type launch vehicles and then connected by a flexible tunnel. The tunnel is 4 feet in diameter with thin wall construction of a steel wire grid sealed with a soft polymer membrane. The grid provides bending stiffness

Table I. Physical Constants of the Space Station

Component	Mass	Principal Moments of Inertia (slug - ft^2)			Associated Distance	EI (lb - ft^2)
Stage 1	M_A = 1552.8 slugs	$(IA)_1 = 9(10^5)$	$(IA)_2 = 1(10^5)$	$(IA)_3 = 9(10^5)$	d_1 = 46 ft	----
Stage 2	M_B = 1552.8 slugs	$(IB)_1 = 9(10^5)$	$(IB)_2 = 1(10^5)$	$(IB)_3 = 9(10^5)$	d_2 = 46 ft	----
Beam	m = .1242 slug/ft	-----	-----	-----	L = 47 ft	$.184(10^9)$

and astronaut protective structure while the membrane serves as a micro-meteorite shield and as closure for a shirt-sleeve atmosphere.

A spin rate Ω varying from 0 to 3.5 rev/min has been studied. The maximum Ω of 3.5 rev/min is such that the gravity level is approximately .1 g at the tunnel end (approximately .3 g in the crew quarters). For each spin rate the natural frequencies of the six lowest modes are tabulated in Table II. Mode shapes for these frequencies are shown in Graphs 1 - 6. The effect of the spin rate is seen to be a slight increase in the natural frequencies corresponding to an "apparent" increase in stiffness of the beam due to rotation, reference Hurty and Rubinstein [12].

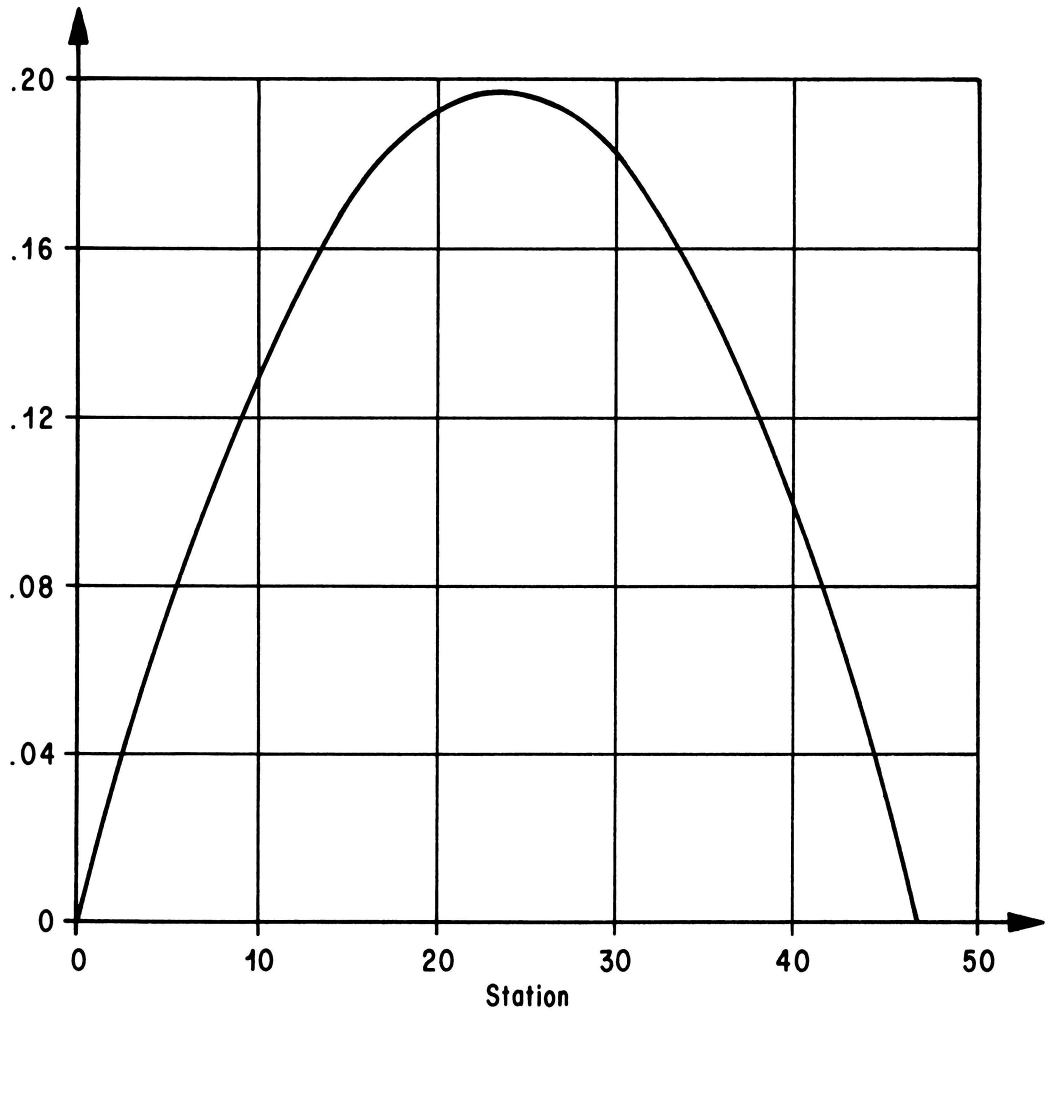
In addition, the value of the characteristic determinant has been shown to approach zero as p^2 approaches $-\Omega^2$. Since the basic solution is not valid for values of $p^2 \leq -\Omega^2$ due to changes in the governing equation or the solution, as discussed in Section 4 of Chapter V, we consider the special cases in the following sections.

2. p^2 Between $p^2 = -\Omega^2$ and $p^2 = \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$

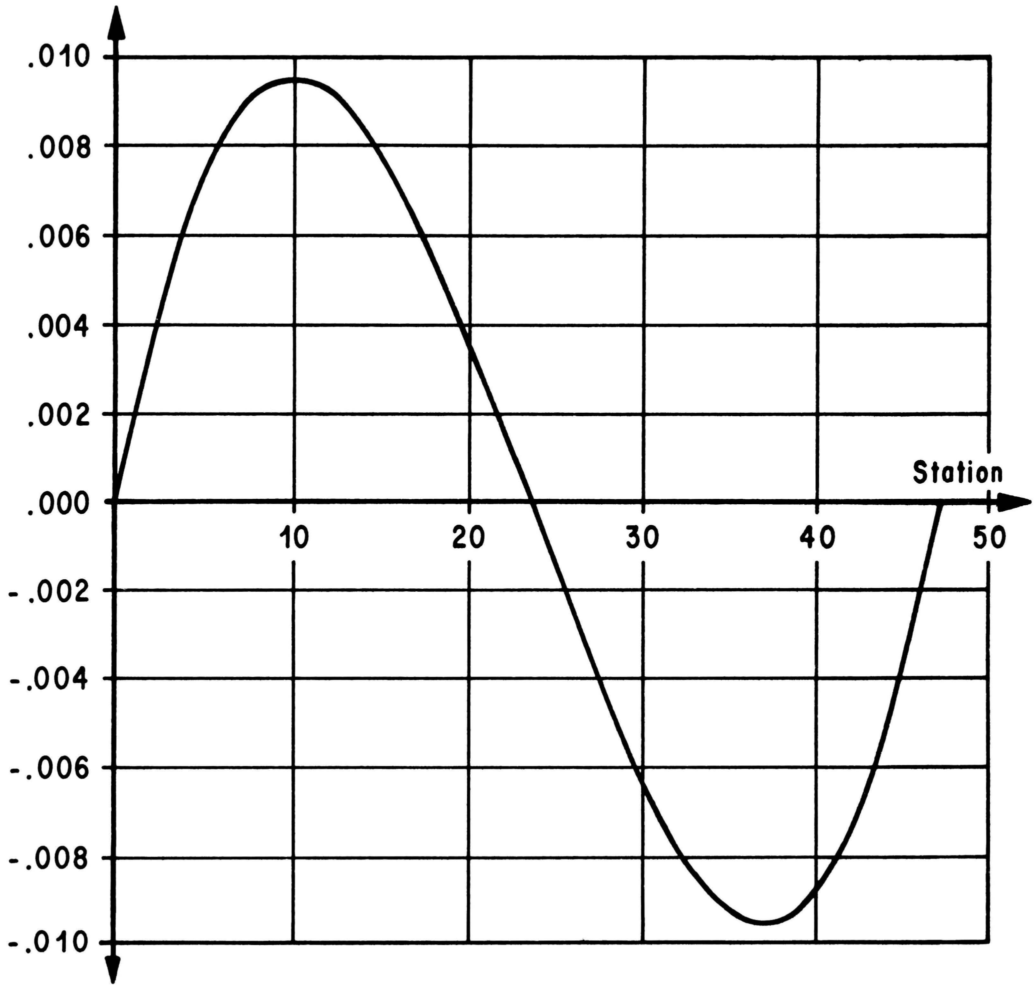
As shown in paragraph a, Section 4, Chapter V, the solution is given by equation (67). The characteristic determinant for this solution has been programmed, and a frequency sweep for incremental values of p^2 within

Table II. Natural Frequencies of the System

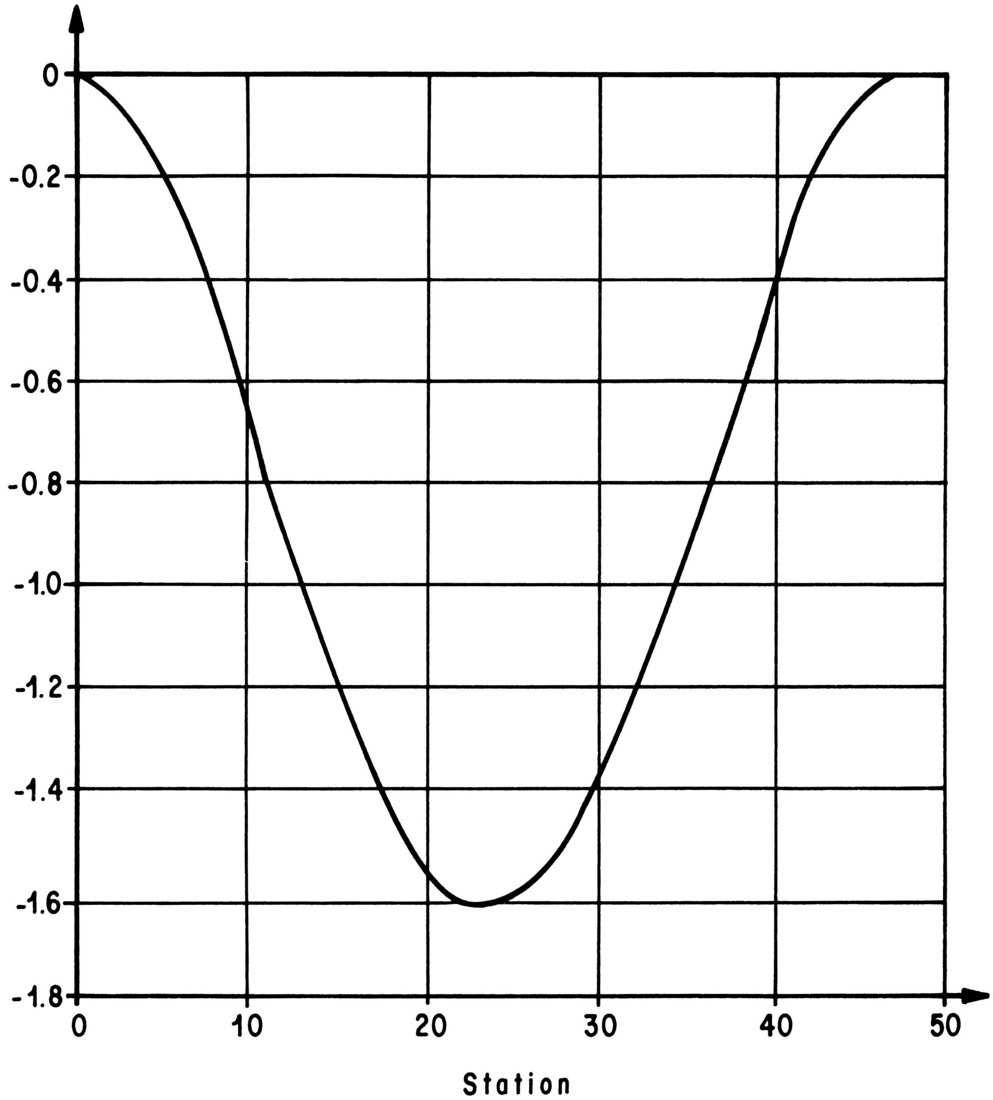
Mode	p_i^2 (rad/sec)^2	$\Omega^2 = 0$ (rad/sec)^2	$\Omega^2 = .01343$ (rad/sec)^2	$\Omega^2 = .02686$ (rad/sec)^2	$\Omega^2 = .04029$ (rad/sec)^2	$\Omega^2 = .05372$ (rad/sec)^2	$\Omega^2 = .06715$ (rad/sec)^2	$\Omega^2 = .08058$ (rad/sec)^2	$\Omega^2 = .09401$ (rad/sec)^2	$\Omega^2 = .10744$ (rad/sec)^2	$\Omega^2 = .12087$ (rad/sec)^2	$\Omega^2 = .13430$ (rad/sec)^2
1.	p_1^2	.861857(10^1)	.870433(10^1)	.879009(10^1)	.887584(10^1)	.896159(10^1)	.904732(10^1)	.913305(10^1)	.921878(10^1)	.930449(10^1)	.939020(10^1)	.947590(10^1)
2.	p_2^2	.143452(10^3)	.143631(10^3)	.143810(10^3)	.143990(10^3)	.144169(10^3)	.144348(10^3)	.144528(10^3)	.144707(10^3)	.144886(10^3)	.145065(10^3)	.145244(10^3)
3.	p_3^2	.153237(10^6)	.153302(10^6)	.153368(10^6)	.153433(10^6)	.153499(10^6)	.153564(10^6)	.153630(10^6)	.153695(10^6)	.153760(10^6)	.153826(10^6)	.153891(10^6)
4.	p_4^2	.115565(10^7)	.115590(10^7)	.115614(10^7)	.115638(10^7)	.115662(10^7)	.115687(10^7)	.115711(10^7)	.115735(10^7)	.115759(10^7)	.115784(10^7)	.115808(10^7)
5.	p_5^2	.444411(10^7)	.444435(10^7)	.444487(10^7)	.444539(10^7)	.444631(10^7)	.444644(10^7)	.444699(10^7)	.444749(10^7)	.444802(10^7)	.444853(10^7)	.444906(10^7)
6.	p_6^2	.121290(10^8)	.121300(10^8)	.121309(10^8)	.121318(10^8)	.121327(10^8)	.121336(10^8)	.121345(10^8)	.121354(10^8)	.121363(10^8)	.121372(10^8)	.121381(10^8)



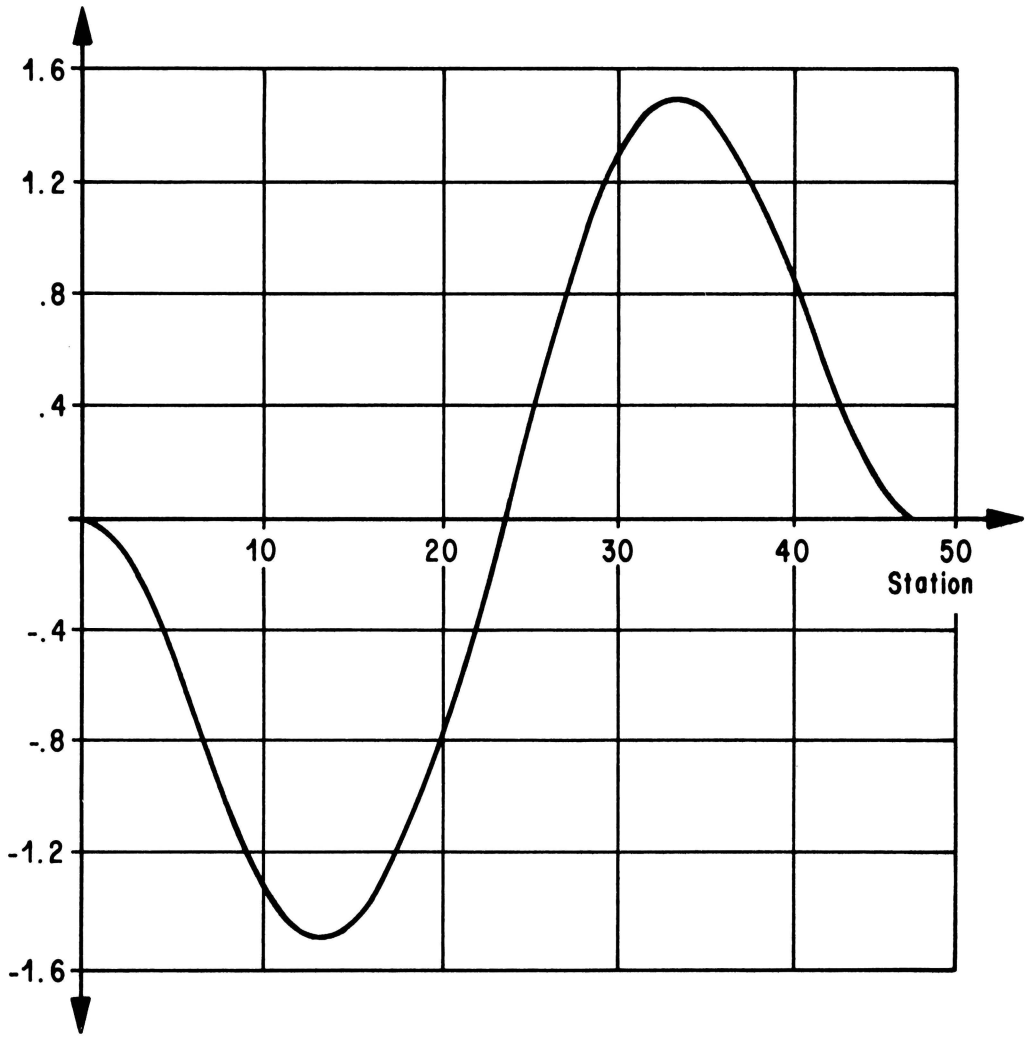
GRAPH 1. NORMALIZED MODE SHAPE CORRESPONDING TO 1st NATURAL FREQUENCY OF THE ROTATING SYSTEM



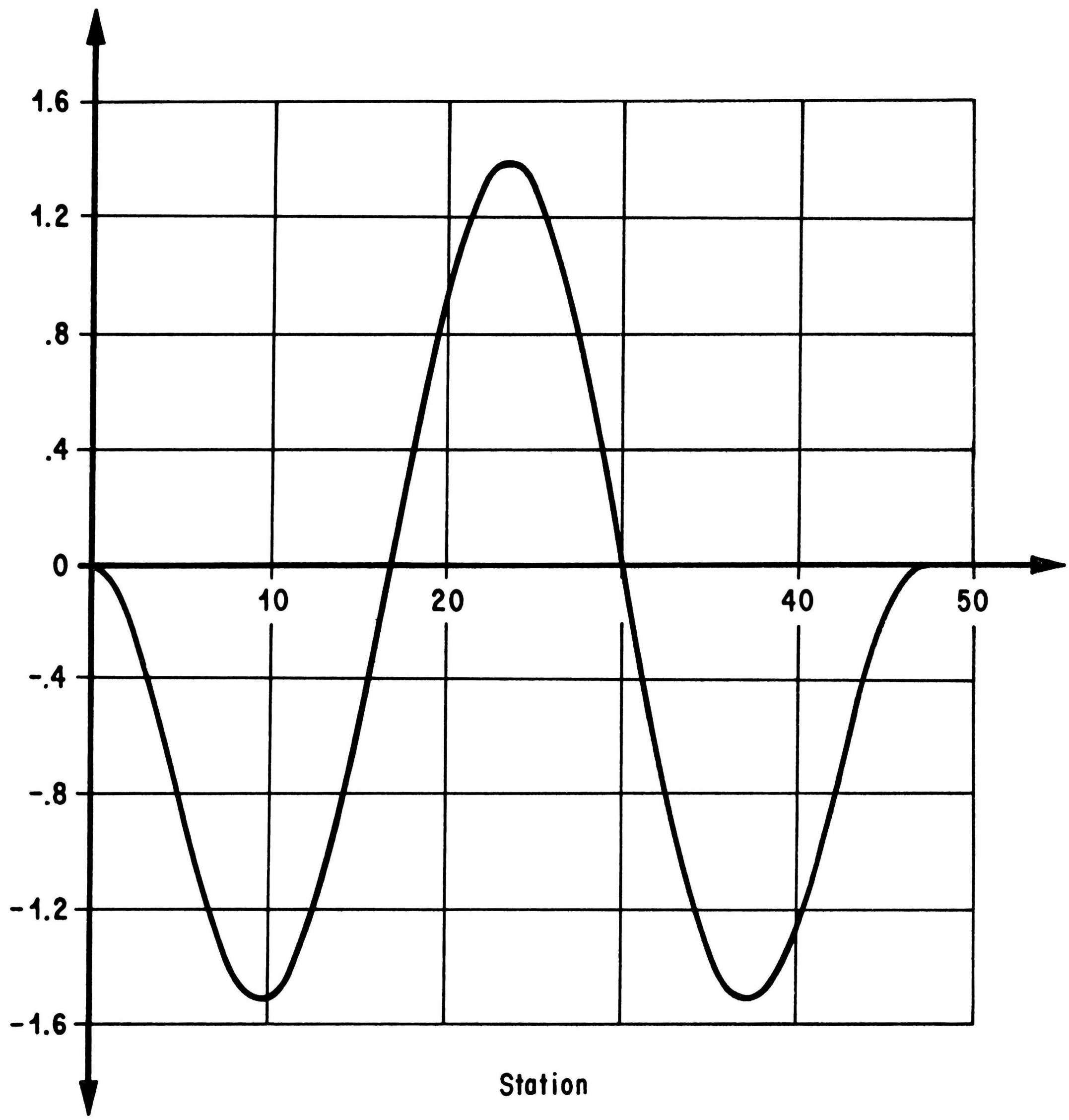
GRAPH 2. NORMALIZED MODE SHAPE CORRESPONDING TO 2nd NATURAL FREQUENCY OF THE ROTATING SYSTEM



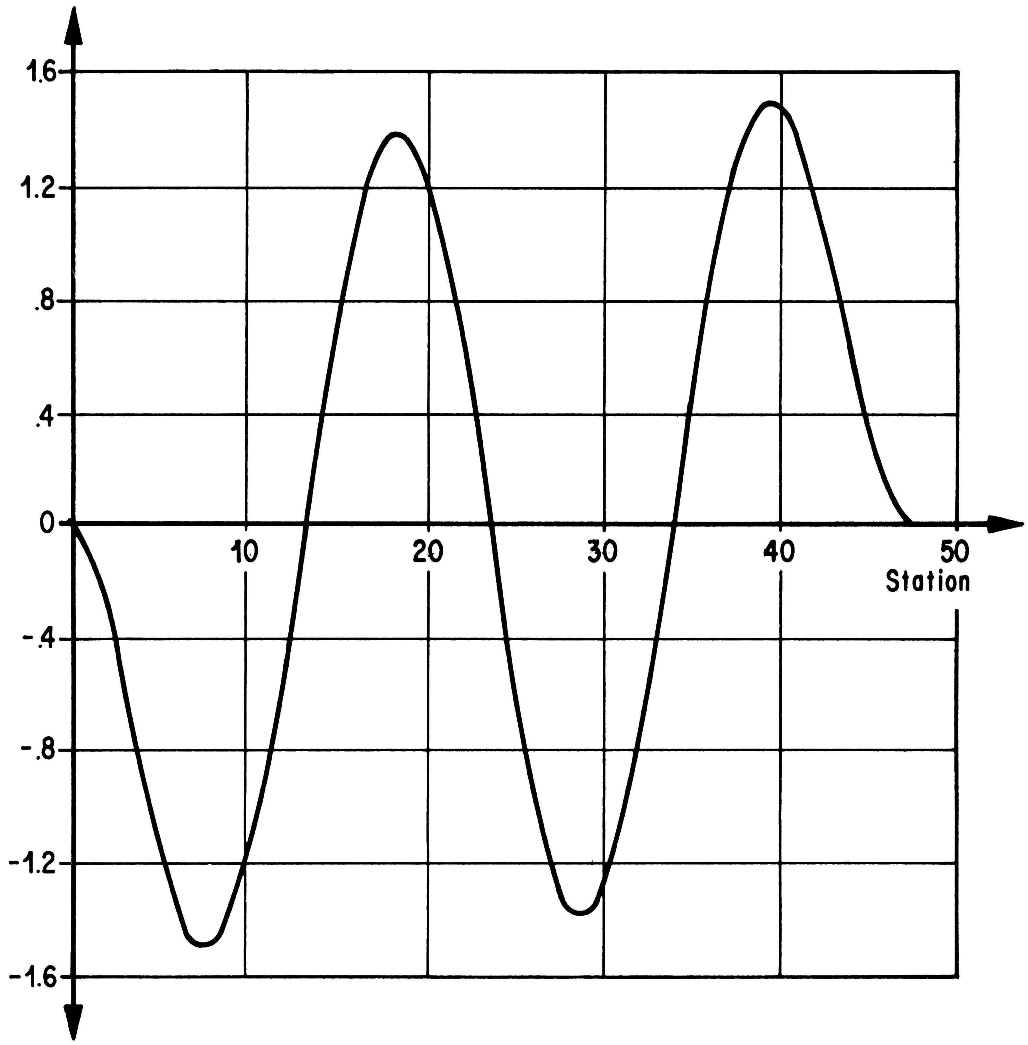
GRAPH 3. NORMALIZED MODE SHAPE CORRESPONDING TO 3rd NATURAL FREQUENCY OF THE ROTATING SYSTEM



GRAPH 4. NORMALIZED MODE SHAPE CORRESPONDING TO 4th NATURAL FREQUENCY OF THE ROTATING SYSTEM



GRAPH 5. NORMALIZED MODE SHAPE CORRESPONDING TO 5th NATURAL FREQUENCY OF THE ROTATING SYSTEM



GRAPH 6. NORMALIZED MODE SHAPE CORRESPONDING TO 6th NATURAL FREQUENCY OF THE ROTATING SYSTEM

this region has been conducted. The results indicate that $D(p^2) \rightarrow 0$ at both end points and that there are no values of p^2 within this region for which $D(p^2) = 0$. However, a singularity exists within this region, as discussed in paragraph e, Section 4, Chapter V. The value of $D(p^2)$ becomes large on both sides of this singularity, which is evaluated in Section 6 of this chapter.

3. $p^2 \equiv -\Omega^2$

As shown in paragraph b, Section 4, Chapter V, the governing equation and solution are given by equations (69) and (70) when $p^2 \equiv -\Omega^2$. The characteristic determinant for this case has been programmed and the value of $D(p^2 \equiv -\Omega^2)$ found to be non-zero. However, the numerical results obtained from the basic solution and the numerical results of the analysis for the special case of p^2 between $-\Omega^2$ and $\dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$ indicate that there is a natural mode at $p^2 \equiv -\Omega^2$. Therefore since $D(p^2 \equiv -\Omega^2) \neq 0$ we must have $R(r) \equiv 0$ for all r, which is the rigid body mode with associated frequency $p^2 \equiv -\Omega^2$. Thus we find that the rigid body frequency is not the usual $p^2 = 0$ for a non-rotating body, but rather a finite number equal to the negative spin rate.

4. p^2 With Negative Values Larger than $p^2 = \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$

As shown in paragraph c, Section 4, Chapter V, the solution for p^2 within this region is given by equation (72). The characteristic determinant for this region has been programmed, and a frequency sweep for incremental values of p^2 has been conducted. The results indicate that $D(p^2) \rightarrow 0$ as $p^2 \rightarrow \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$ and that there are no values of p^2 within this region for which $D(p^2) = 0$.

5. $p^2 \equiv \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$

The governing equation and solution for this point are given by equations (75) and (76). The characteristic determinant for this case has been programmed and the value of $D\left(p^2 \equiv \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}\right)$ found to be non-zero. However, the numerical results in Sections 2 and 4 of this chapter indicate that there is a natural mode at $p^2 \equiv \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$. Therefore, since $D\left(p^2 \equiv \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}\right) \neq 0$, we must have $R(r) \equiv 0$ for all r. Thus we have another rigid body mode with associated frequency $p^2 \equiv \dfrac{-\tilde{C}_{24}^2\Omega^4 - 4EIm\Omega^2}{4EIm}$ as in Section 3 of this chapter.

6. The Singularity Where $A_1B_2 - A_2B_1 \equiv 0$

As shown in paragraph c, Section 4, Chapter V, a singularity exists in the calculation of the characteristic determinant. A computer program has been written to study this singularity, and the results indicate that $D(p_s^2) \neq 0$. Thus, since the numerical results of the analysis in the neighborhood of the singularity do not indicate a natural mode at the singularity, we conclude that p_s^2 is not a natural frequency of the system.

FREE VIBRATION OF A ROTATING BEAM-CONNECTED SPACE STATION

James LaMar Milner

Abstract

The free vibration of a rotating beam-connected space station is analyzed with a mathematical model of the space station which represents the general three-dimensional motion of the various components of the system. The space station is composed of two space modules connected by a flexible beam, and the system is caused to spin in the plane of its orbit in order to produce an artificial gravity environment within the space modules.

The kinetic energy and potential energy of the space station are used to develop a Lagrangian function of the system. Hamilton's principle is used to determine a set of governing equations, and a set of boundary conditions representing a clamped-clamped attachment of the beam to each space module is applied to the ends of the beam. Within the limits of small deflection theory the motion of the space station is shown to be uncoupled into two separate types of motion, one in the plane of rotation and the other perpendicular to the plane of rotation.

An exact solution is obtained for the beam deflection in the plane of rotation. The application of the nonhomogeneous boundary conditions leads

to a set of simultaneous equations in the frequency p^2, from which a characteristic determinant is developed. A procedure to solve for the zeros of the characteristic determinant is programmed for digital solution on the IBM 7094.

Results of the analysis for a given space station design are presented in the form of tables showing the natural frequencies of free vibration of the space station for various spin rates. The effect of the spin rate is shown to be an "apparent" increase in the stiffness of the beam. Mode shapes showing the normalized bending deflection of the beam in its six lowest modes of vibration are presented. By analysis of five special cases of negative values of p^2 the existence of two rigid body modes with non-zero values of p^2 is demonstrated, and it is shown that the configuration studied has no instabilities for motion in the plane of the orbit.