AN ANALYSIS OF THRESHOLD CHARACTERISTICS OF QUASI-LINEARIZED PHASE-LOCKED LOOP DEMODULATION FOR WIDEBAND FREQUENCY-MODULATED SIGNALS

by

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A	Amplitude of sinusoid in root-mean-square
A_e	Equivalent amplifier gain
B_{if}	IF filter bandwidth
f	Frequency
$F(s)$	Transfer function of loop filter
f_m	Frequency of the modulating signal
Δf	Peak frequency devation
G_m	Carrier signal phase power spectal density
G_n	Noise power spectral density
G_{if}	Normalized noise power spectral density by IF bandwidth
$H(s)$	Closed-loop transfer function
K_d	Phase detector gain
K_f	Loop filter gain
K_o	Open-loop gain
K_v	VCO gain
n_i	Input noise
N_i	Input noise power
N_o	Output noise power of the loop
P	Probability density function
S_i	Input signal power
S_{if}	Signal power over IF bandwidth
$(S/N)_i$	Input signal power-to-noise power ratio
$(S/N)_{i,th.}$	Threshold input-signal power-to-noise power ratio

v_i	Input signal
v_d	Output of phase detector
v_f	Output of loop filter
v_o	Output of VCO
ω	Radian frequency of VCO
ω_c	Radian frequency of received signal
W_i	IF bandwidth-to-baseband bandwidth ratio
ω_n	Loop natural frequency
ω_{no}	Relative loop natural frequency
Z	Ratio of $(S/N)_i$-to-$(S/N)_o$
θ	Phase error
ϕ_i	Input signal phase
ϕ_o	Phase of VCO output
Φ_i	Input signal phase spectrum
Φ_o	VCO output phase spectrum
β	Frequency modulation index
ζ	Loop damping factor
ζ_o	Relative loop damping factor
σ^2	Variance
$\overline{\phi^2}$	Mean-square phase error

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I. INTRODUCTION

Discussed herein is the theoretical analysis of the threshold characteristics of quasi-linearized phase-locked loop demodulation for wideband frequency-modulated signals.

In the performance of frequency demodulation the signal reception deteriorates very rapidly below a certain critical value, called the threshold, of the input carrier power-to-noise power ratio, and improve linearly above this value. Therefore, the threshold characteristics is one of the essential elements for optimization of the minimum input power communications systems design.

Using phase-locked loop demodulation techniques considerable improvement in the threshold can be obtained compared to that of a conventional frequency demodulation system. It is also well suited for tracking narrow-band signals emitted by moving space vehicles. Therefore, in present day, phase-locked loop demodulation has become the standard technique in deep-space communication applications.

Much of the earlier work on the threshold behavior of phase-locked demodulations for optimum reception has been discussed in several papers[3,6,9], which are based on different criterion of threshold behavior.

This thesis presents an analysis of the performance of phase-locked-loop behavior based on Viterbi's application of the Fokker-Plank technique and Booton's Quasi-Linearization technique. The analysis in this thesis for the threshold criterion follows very closely the work done by Develet[3] in a study of threshold criterion for phase-locked demodulation for arbitrary information and noise spectral densities.

The main purpose of this thesis is to analyze the threshold characteristics and to observe the effects of the input bandwidth ratio, the frequency modulation index, and the phase loop parameters on phase-locked demodulation.

In Chapter 2, the basic loop is described and its fundamental equations and transfer functions are derived. Chapter 3 develops a quasi-linearized loop demodulator model in the presence of noise. Under the assumption of the quasi-linearized model, the mean-square phase error is derived in Chapter 4, and finally, the threshold criterion is discussed in Chapter 5.

The threshold characteristics will be investigated under the following assumptions: (1) the phase loop is statistically locked-in, (2) the phase of the signal is a zero-mean Gaussian process, (3) the incoming signal and the additive white Gaussian noise are statistically independent, and (4) the input signal is wideband frequency-modulated by sinusoidal message.

II. FORMULATION OF THE BASIC LOOP EQUATIONS OF A HIGH-GAIN SECOND-ORDER PHASE-LOCKED LOOP IN THE ABSENCE OF NOISE

This chapter presents some basic analytical expressions governing the phase-locked loop as shown in Figure 2-1, such as, the basic loop equations, the transfer functions, and the loop parameters.

The description in this chapter will be based on the following assumptions: (1) the loop has a high gain, (2) the closed-loop transfer function has two poles, (3) the loop is locked-in, and (4) there is no noise present in the loop.

Suppose that the input signal to the loop of Figure 2-1 is a sinusoidal waveform as

$$v_i(t) = \sqrt{2}\, A \sin[\omega_c t + \phi_i(t)] \tag{2.1}$$

where A denotes rms amplitude of the sinusoid.
The output of the voltage controlled oscillator (VCO) is assumed to be a sinusoid whose oscillating frequency is equal to the input frequency of the loop. This output can be expressed as

$$v_o(t) = \sqrt{2} \cos[\omega_c t + \phi_o(t)] \tag{2.2}$$

The rms amplitude of the VCO output is assumed to be unity.

The output of the phase detector (multiplier), which is product of the incoming signal $v_i(t)$ and the VCO output $v_o(t)$, can be written as

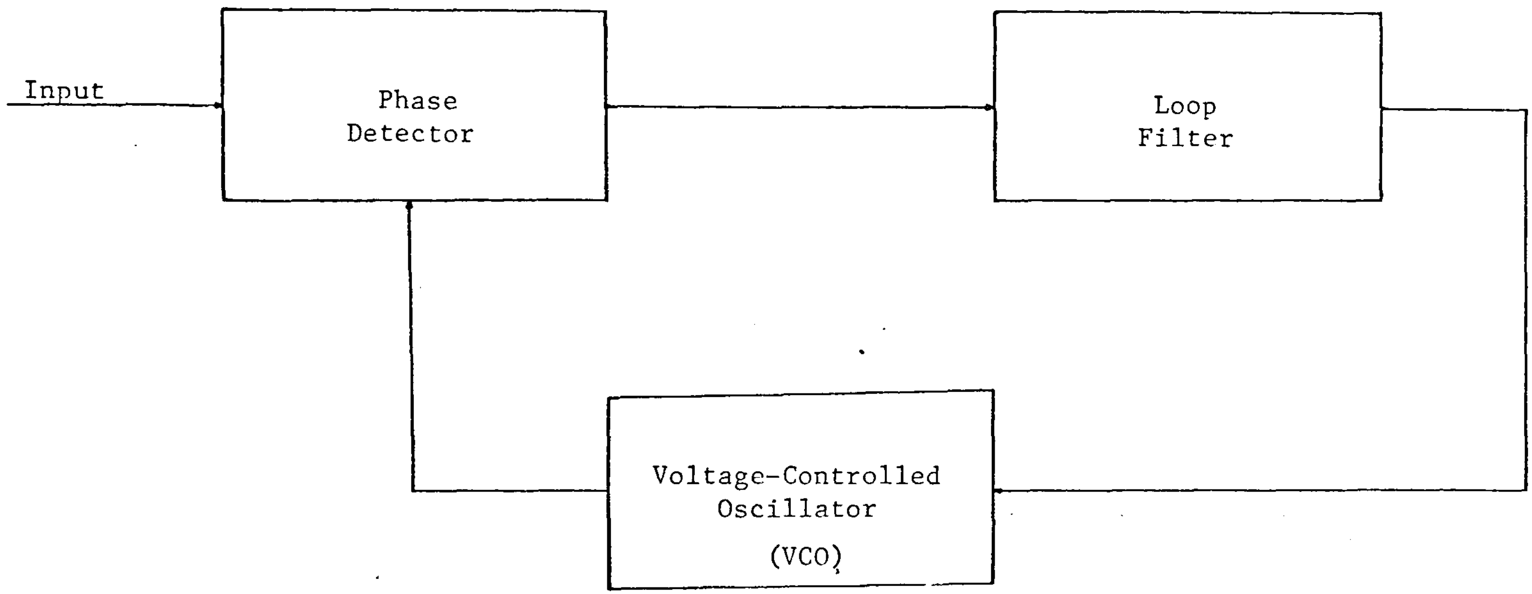

Figure 2-1. Basic phase-locked loop block diagram.

$$v_d(t) = K_d v_i(t) \cdot v_o(t)$$

$$= AK_dK_v\{\sin[\phi_i(t)-\phi_o(t)]+\sin[2\omega_c t+\phi_i(t)+\phi_o(t)]\}$$

where K_d is a d.c gain of the phase detector, and K_v is a d.c gain of the VCO. The multiplier output $v_d(t)$ is fed into the loop filter which removes the second harmonic frequency component. Hence, the input waveform to the VCO can be expressed as

$$v_f(t) = AK_dK_fK_v L^{-1}\{L[\sin(\phi_i(t)-\phi_o(t))]F(s)\}$$

where K_f is a d.c gain and F(s) is the transfer function of the loop filter, respectively.

The frequency of the VCO output is controlled by the input voltage $v_f(t)$, and the relation between the output phase $\phi_o(t)$ and the loop filter output is given by

$$\frac{d\phi_o(t)}{dt} = K_v v_f(t) \tag{2.4}$$

By taking the Laplace transform of Equation (2.4), the phase of the VCO output can be expressed as

$$\Phi_o(s) = K_v \frac{V_f(s)}{s} \tag{2.5}$$

Thus, it is understood from Equation (2.5) that the VCO output phase is proportional to the integral of the input voltage v_f. Here, it must be noted that the steady state behavior of the loop will be only considered through this study.

Taking the Laplace transform of Equation (2.3) and substituting into Equation (2.5)

$$\Phi_o(s) = AK_o L\{\sin\theta(t)\}\frac{F(s)}{s} \tag{2.6}$$

where, $K_o = K_d K_f K_v$ is the open-loop gain,

$\theta = \phi_i - \phi_o$ is the phase error.

Since it is assumed that the loop is capable of reducing the phase error to a small enough value $|\theta| < \frac{\pi}{6}$, the loop is completely phase-locked, the phase of the VCO output tracks the phase of the input signal closely. Under the locked-in condition, the output of the phase detector will be linearly proportional to the phase error, i.e., $v_d \propto \theta$. Thus, the phase error can be expressed in the frequency domain as

$$\theta(s) = \Phi_i(s) - AK_o[\Phi_i(s) - \Phi_o(s)]\frac{F(s)}{s} \tag{2.7}$$

This is the fundamental equation for the phase error which specifies the behavior of the phase-locked loop in the absence of noise. The mathematical model of the loop may be portrayed by the block diagram shown in Figure 2-2.

For the analysis of the loop response, it is necessary to find the relationship between input and output signals by means of the loop transfer functions. From the mathematical model for the linearized phase-locked loop (Figure 2-2), the closed-loop transfer function is obtained as

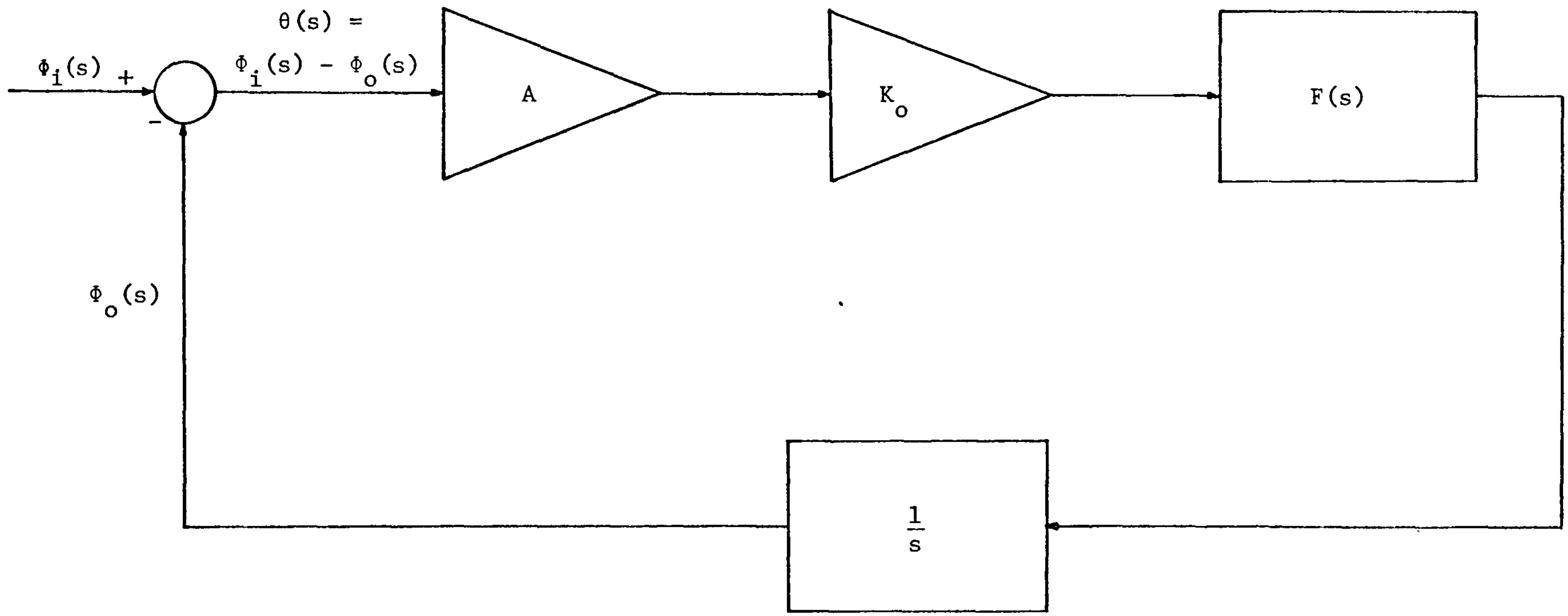

Figure 2-2. Mathematical model of linearized phase-locked loop.

$$\frac{\Phi_o(s)}{\Phi_i(s)} = H(s)$$

$$= \frac{AK_oF(s)}{S + AK_oF(s)} \tag{2.8}$$

The relation of the phase error to the input, i.e., the phase-error transfer function can be obtained as follows:

$$\frac{\theta(s)}{\Phi_i(s)} = 1 - H(s)$$

$$= \frac{S}{S + AK_oF(s)} \tag{2.9}$$

A lag-lead filter will be employed as the loop filter as shown in Figure (2-3), and the transfer function of the filter gives

$$F(s) = \frac{1 + \tau_1 S}{1 + \tau_2 S} \tag{2.10}$$

where,

$$\tau_1 = R_2C_2,$$

$$\tau_2 = (R_1 + R_2)C_2.$$

For this transfer function, the closed-loop transfer function becomes

$$H(s) = \frac{(AK_o\tau_1/\tau_2)S + AK_o/\tau_2}{S^2 + (1+AK_o\tau_1)S/\tau_1 + AK_o/\tau_2} \tag{2.11}$$

By letting

$$\omega_n^2 = \frac{AK_o}{\tau_2},$$

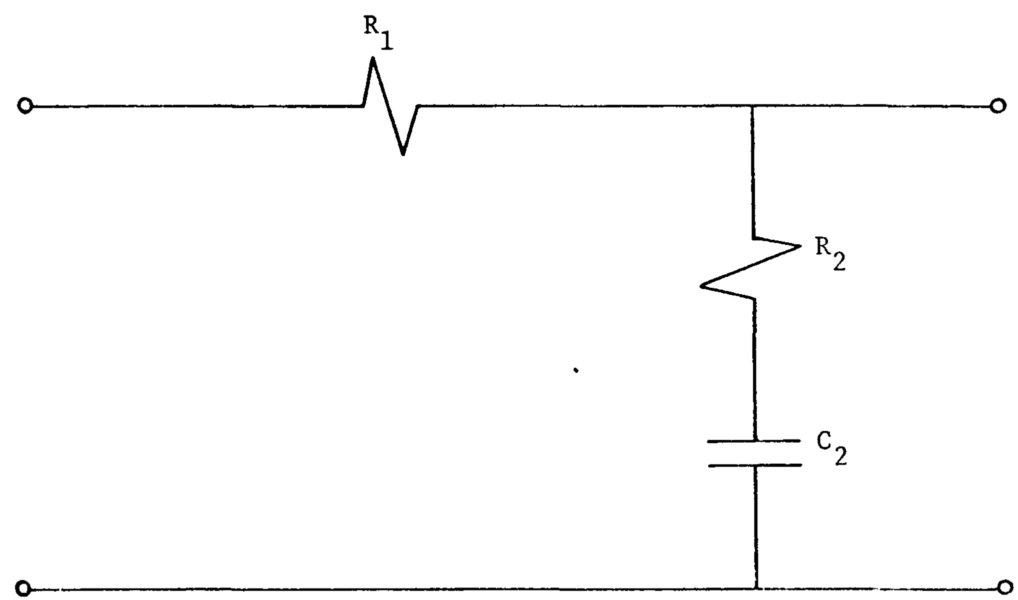

Figure 2-3. Lag-lead filter.

and

$$2\zeta\omega_n = \frac{1 + AK_o \tau_1}{\tau_2} ,$$

Equation (2.11) can be expressed as

$$H(s) = \frac{(2\zeta\omega_n - \omega_n^2/AK_o)S + \omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} \tag{2.12}$$

where ζ denotes the loop damping factor, and ω_n is the loop natural frequency. Considering that the loop has a high gain such that $\omega_n^2/AK_o << 2\zeta\omega_n$, Equation (2.12) will be reduced to

$$H(s) = \frac{2\zeta\omega_n S + \omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} \tag{2.13}$$

Equation (2.13) implies that the phase-locked loop is a second-order system. Since the loop gain is assumed high and its transfer function has two poles, the phase loop may be defined as a high-gain second-order phase-locked loop.

The magnitude of the frequency response of this high-gain second-order loop for several values of damping factor is illustrated in Figure 2-4. From this figure it can be seen that the phase-locked loop performs as a low-pass filter for the input phase. Phase error response is plotted in Figure 2-5 for the high-gain second-order loop with $\zeta = 0.707$. This demonstrates to us that the phase error approaches zero as the normalized frequency ω_n/ω is increased. Furthermore, we notice that the loop tracks reasonably well in the low-frequency region, but it will fail at the high frequencies.

In the following chapter an analysis is presented of a quasi-linearized loop in the presence of noise. This provides a more general situation of the phase-locked loop performance.

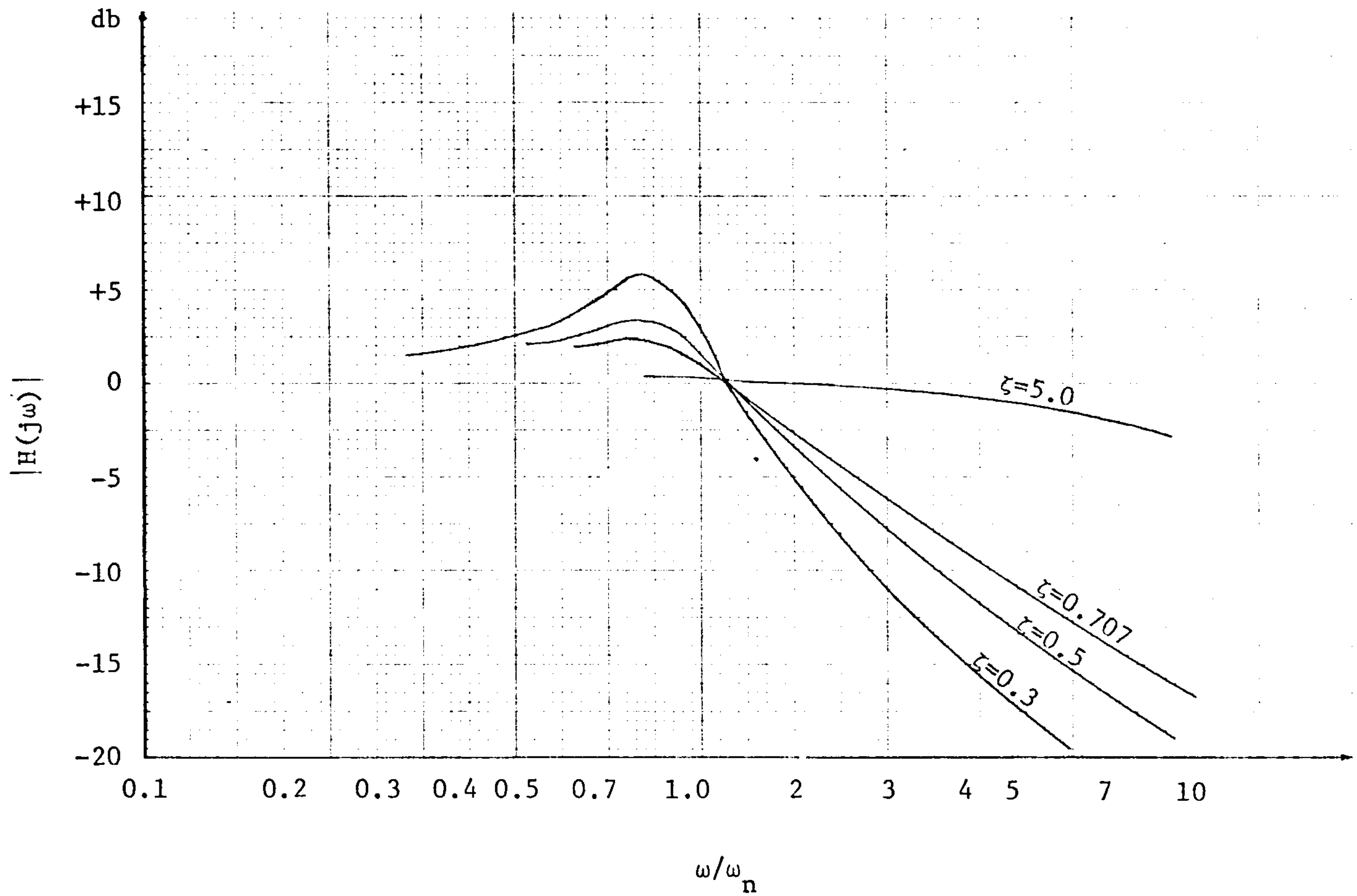

Figure 2-4. Frequency response of high-gain second-order loop.

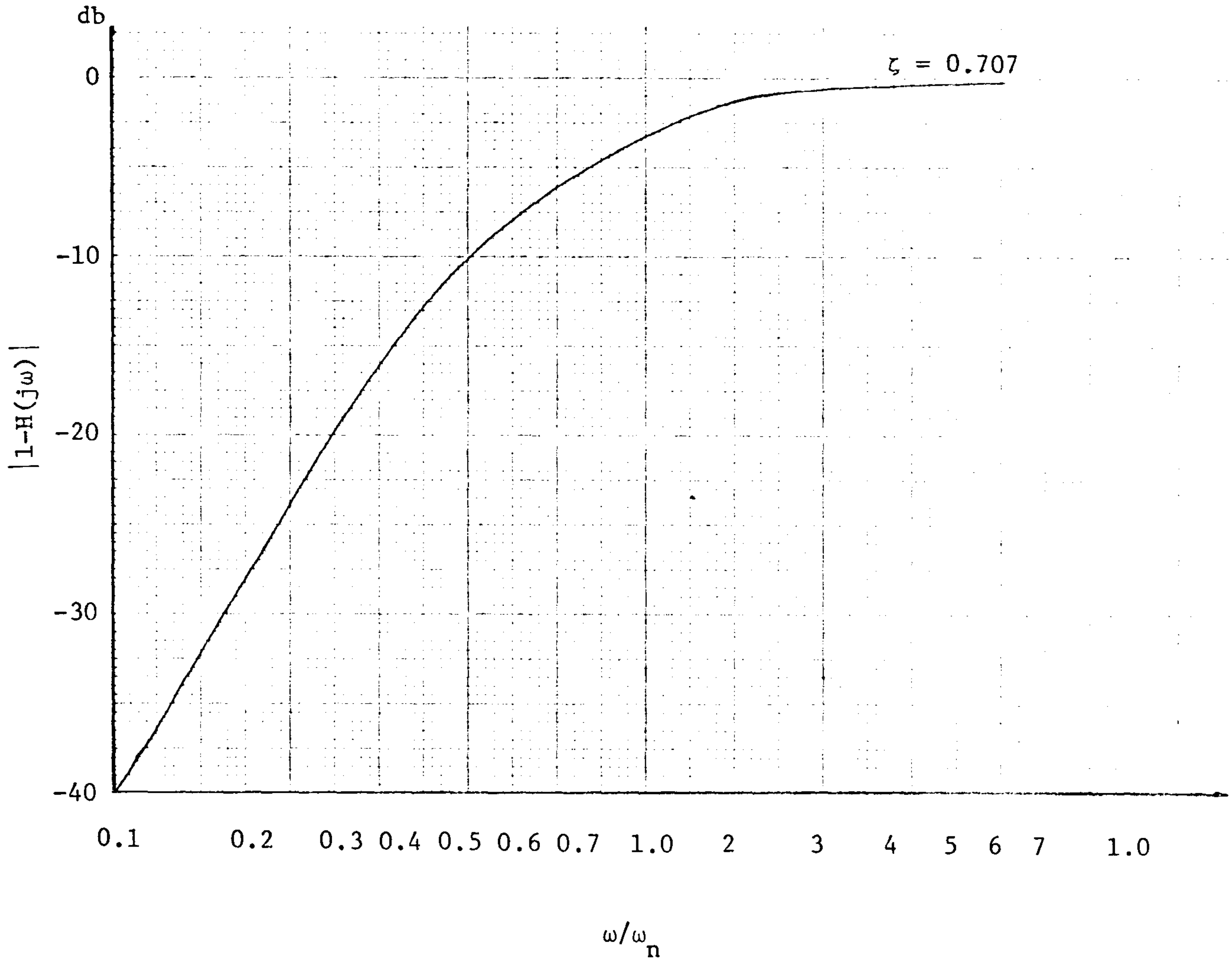

Figure 2-5. Ratio of phase-error to carrier signal phase response of high-gain second-order loop at damping factor $\zeta = 0.707$.

III. DEVELOPMENT OF A QUASI-LINEARIZED PHASE-LOCKED LOOP DEMODULATOR MODEL

For a simple analysis of the fundamental behavior of the phase-locked loop, we have assumed that, in Chapter II: (1) the phase error is suffiently small such that $\sin\theta \approx \theta$, (2) the loop operates in the absence of noise. Under these conditions the precise behavior of the loop performance may be predicted by the linearized loop model as shown in Figure 2-2.

However, in practical cases like a deep-space communications systems, it may not be expected that the phase error is always small such as $|\theta| < \pi/6$. In addition, there exists a significant additive noise through the transmission channel plus noise caused internally in the loop. When the phase error is considerable, the loop can no longer be regarded as a linearized loop, and it will have a sinusoidal nonlinear phase gain.

The exact solution for the second-order phase-locked loop with nonlinearity is extremely difficult to obtain. Therefore, it is desirable that the nonlinear element, $A\sin\theta$, in Equation (2.6) should be converted to an equivalent linear element. One way to convert the nonlinear gain to an equivalent linear gain is by Booton's quasi-linearization technique[3]. Thus, by using this technique, a mathematical model of phase-locked loop in the presence of noise will be established, and then the corresponding equation and transfer function will be derived by the same procedure as explained in the previous chapter.

A basic phase-locked demodulator, which contains an IF band-pass filter followed by the phase-locked loop (Figure 2-1) is considered. It is assumed that the received signal contains the message component plus additive white Gaussian noise. The noise density spectrum is uniform over the bandwidth of the IF filter.

Under these operating conditions, an analytical block diagram of the phase-locked loop demodulator may be represented as shown in Figure 3-1. Consider the output of the IF filter as

$$v_i(t) = \sqrt{2}\,A\sin[\omega_c t+\phi_i(t)] + n(t) \tag{3.1}$$

where, $n(t) = \sqrt{2}\,x(t)\cos\omega_c t + \sqrt{2}\,y(t)\sin\omega_c t$ is white Gaussian noise. The quadrature components of noise, $x(t)$ and $y(t)$ are statistically independent, and stationary Gaussian processes. Their probability density functions are:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{x^2}{2\sigma_x^2}}$$

$$p(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}}\, e^{-\frac{y^2}{2\sigma_y^2}} \tag{3.2}$$

in which σ_x^2 or σ_y^2 is the variance of noise component.

As mentioned in the previous chapter the VCO output is

$$v_o(t) = \sqrt{2}\cos[\omega_o t + \phi_o(t)] \tag{3.3}$$

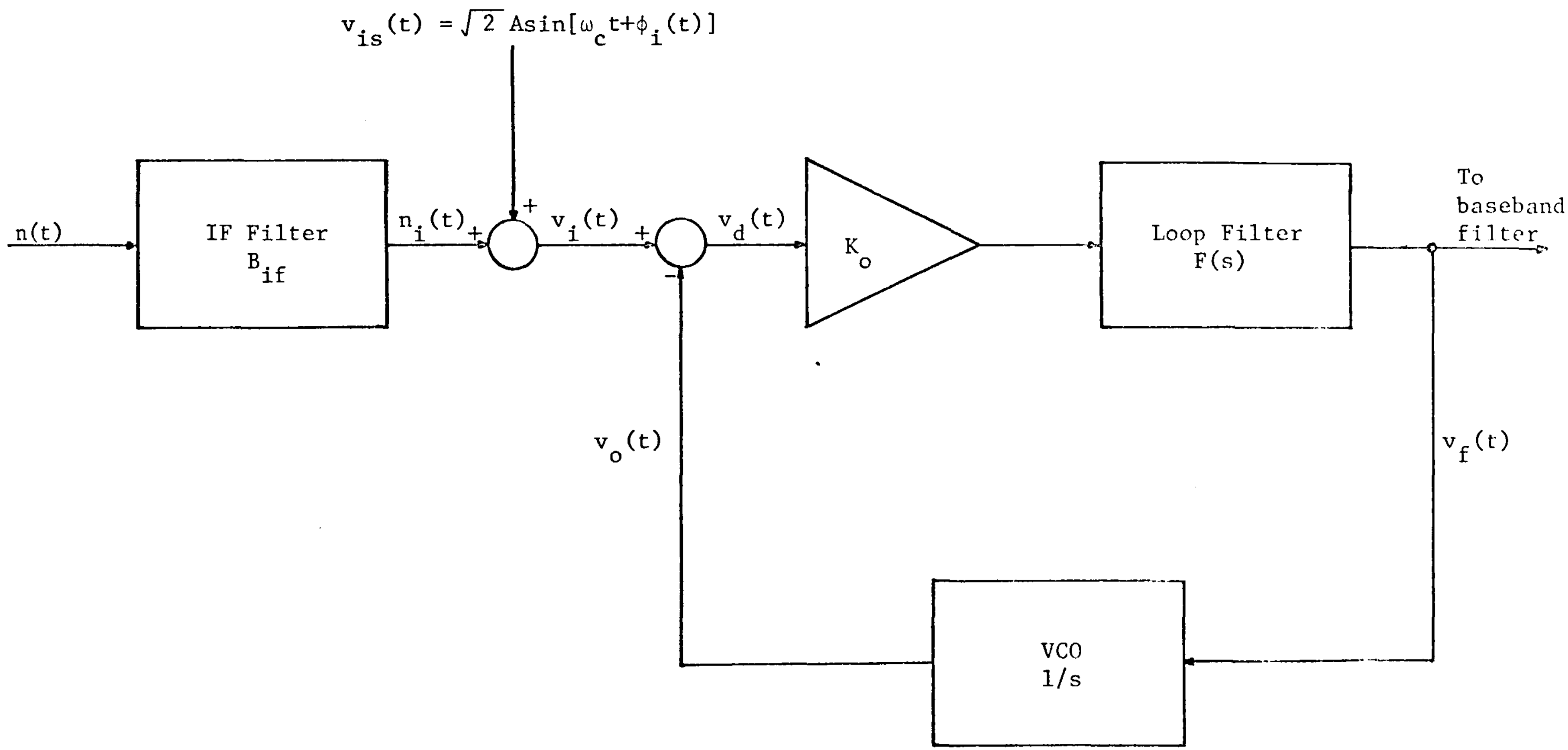

Figure 3-1. Basic diagram of phase-locked loop demodulator in the presence of an additive noise.

where ω_o is radian frequency of the VCO output. The output waveform of the phase detector can be shown as

$$v_d(t) = 2K_d\{A\sin[\omega_c t+\phi_i(t)]+n(t)\} \cdot \cos[\omega_o t+\phi_o(t)] \qquad (3.4)$$

If we assume that the incoming signal to the loop is tracked-in, say, the carrier frequency is initially tuned at the VCO frequency, then $\omega_c = \omega_o$. The waveform to the loop filter can be represented by

$$\begin{aligned} v_d(t) &= AK_d\{\sin[\phi_i(t)-\phi_o(t)]+\sin[2\omega_c t+\phi_i(t)+\phi_o(t)]\} \\ &\quad + K_d x(t)\{\cos\phi_o(t)+\cos[2\omega_c t+\phi_o(t)]\} \\ &\quad + K_d y(t)\{-\sin\phi_o(t)+\sin[2\omega_c t+\phi_o(t)]\} \qquad (3.5) \end{aligned}$$

After suppressing the double frequency components, the output of the loop filter becomes

$$\begin{aligned} v_f(t) &= K_d K_f L^{-1}\{L[A\sin(\phi_i(t)-\phi_o(t)] \\ &\quad + x(t)\cos\phi_o(t)-y(t)\sin\phi_o(t)]F(s)\} \qquad (3.6) \end{aligned}$$

Letting $n_i(t) = x(t)\cos\phi_o(t)+y(t)\sin\phi_o(t)$, $v_f(t)$ can be simplified as

$$v_f(t) = K_d K_f L^{-1}\{L[A\sin\theta(t)+n_i(t)]F(s)\} \qquad (3.7)$$

where $\theta(t) = \phi_i(t) - \phi_o(t)$ is the phase error. It is noticed that Equation (3.6) is nonlinear due to the presence of the sinusoidal term. The nonlinearity should be converted to the equivalent linear form so that a quasi-linearization technique may be applied for obtaining an approximate solution.

Before using this technique, we will try to obtain the probability density of the phase error. It is not wise to come to the conclusion that the probability distribution of the phase error is Gaussian with zero mean. However, according to Viterbi's work by using Fokker-Plank techniques[12] and Charles-Lindseys' experimental results[2], it is known that the probability distribution of the phase error of the second-order loop is essentially Gaussian with zero mean for large signal-to-noise ratio in the loop bandwidth.

Thus, under these operating conditions, the nonlinear element $A\sin\theta(t)$ of Equation (3.7) can be replaced by the average gain[3]:

$$A_e = \int_{-\infty}^{\infty} \dot{f}(\theta)p(\theta)d\theta$$

$$= \frac{A}{\sqrt{2\pi\ \sigma^2}} \int_{-\infty}^{\infty} \cos\theta \cdot e^{-\frac{\theta^2}{2\sigma^2}} d\theta \qquad (3.8a)$$

where

A_e = equivalent element gain,

σ^2 = Variance of phase error,

$f(\theta) = A\cos\theta$,

$$p(\theta) = \frac{1}{\sqrt{2\pi\ \sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}}$$

From Equation (3.8) we have

$$A_e = Ae^{-\frac{\sigma^2}{2}} \qquad (3.8b)$$

Substituting Equation (3.8b) into (3.7), we get

$$v_f(t) = K_d K_f L^{-1}\{L[A_e + n_i(t)]F(s)\} \tag{3.9}$$

Hence, the loop equation for the phase error of a quasi-linearized phase locked-loop is obtained as

$$\theta(s) = \Phi_i(s) - A_e K_o[1+N(s)]\frac{F(s)}{S} \tag{3.10}$$

where N(s) denotes the density spectrum of the normalized noise $\frac{n_i(t)}{A_e}$. This equation shows that, under the equivalent linearized conditions, the steady state loop behavior of phase error can be analyzed.

Thus, an alternative mathematical model of the quasi-linearized phase-locked loop is portrayed in Figure 3-2. In this model $\Phi_i(s)$ is the input phase spectrum produced by the modulating signal, and $\Phi_n(s)$ denotes the noise phase spectrum of the VCO phase jitter $\phi_n(t)$.

The closed-loop transfer function of Figure 3-2 can be expressed as

$$H(s) = \frac{A_e K_o F(s)}{S + A_e K_o F(s)} \tag{3.11}$$

Therefore, the closed-loop transfer function of the high gain loop employing the filter of Figure 2-3 may be written as

$$H(s) = \frac{2\zeta\omega_n S + \omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} \tag{3.12}$$

where,

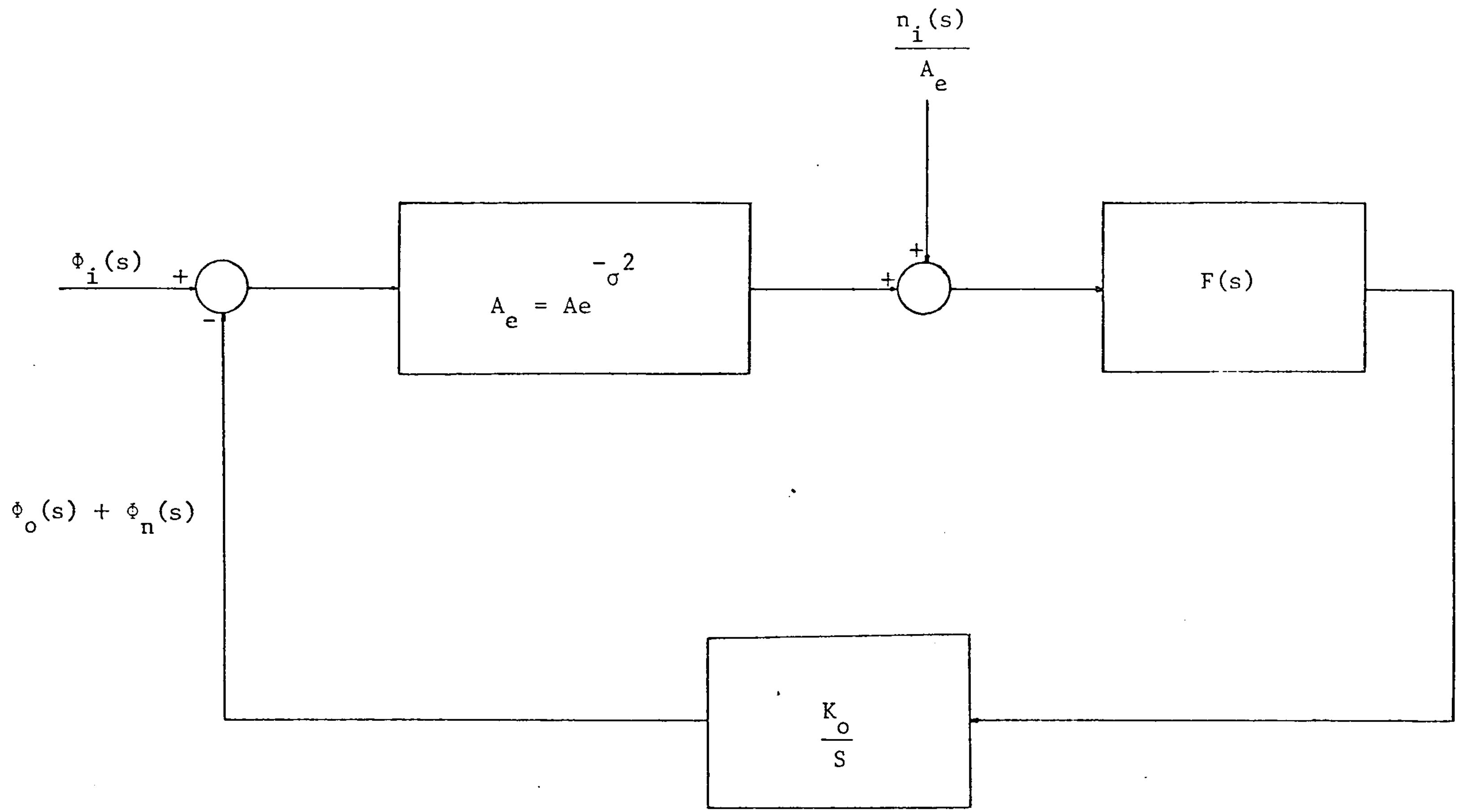

Figure 3-2. Mathematical model of a quasi-linearized phase-locked loop.

$$\omega_n = \frac{AK_o}{\tau_2} e^{-\frac{\sigma^2}{4}} \text{ is loop natural frequency,}$$

$$\zeta = \frac{1}{2\omega_n} \quad 1 + AK_o\,{}_1 e^{-\frac{\sigma^2}{2}} \text{ is loop damping factor.}$$

Letting $\omega_{no} = \dfrac{AK_o}{\tau_2}$ is relative loop natural frequency,

and

$$\zeta_o = \frac{\tau_1}{} \frac{AK_o}{\tau_2} \text{ is relative loop damping factor,}$$

then the loop natural frequency and damping can be written as

$$\omega_n = \omega_{no}\, e^{-\frac{\sigma^2}{4}},$$

and

$$\zeta = \zeta_o\, e^{-\frac{\sigma^2}{4}}, \text{ respectively.}$$

IV. DERIVATION OF MEAN-SQUARE PHASE ERROR OF QUASI-LINEARIZED PHASE-LOCKED LOOP DEMODULATOR

This chapter is devoted to obtain the mean-square phase error of the quasi-linearized phase-locked loop under the operating conditions mentioned in the previous chapter.

Throughout this thesis, when the power spectral density involving bandwidth is considered, it must be noted that the single-sided spectrum is employed. The single-sided spectrum is related to the double-sided spectrum which is uniformly distributed over the bandwidth $(-f_o, f_o)$. That is

$$G_1(f) = \begin{cases} 2G_{11}(f) & f \geq 0 \\ 0 & f < 0 \end{cases}$$

where,

$G_1(f)$ = single-sided spectrum over the bandwidth $0 \leq f \leq f_o$,

$G_{11}(f)$ = double-sided spectrum over the bandwidth $-f_o \leq f \leq f_o$.

As shown in Figure 3-2, the total phase error produced at the output of the phase detector is

$$\phi_e(t) = \phi_i(t) - \phi_o(t) - \phi_n(t) \tag{4.1}$$

Since the signal and noise are statistically independent processes, the total mean-square phase error can be obtained by using

Parseval's theorem as

$$\overline{\phi_e^2(t)} = \frac{1}{2\pi j} \int_o^{+j\infty} |\Phi_i(s)-\Phi_o(s)|^2 ds + \frac{1}{2\pi j} \int_o^{+j\infty} |\Phi_n(s)|^2 ds \qquad (4.2)$$

From Equation (4.2), it is observed that the first term represents the mean-square phase error due to modulation and the second the mean-square phase error due to noise.

In order to analyze the random behavior of the loop, it is worthwhile to express Equation (4.2) in terms of the closed-loop transfer function, H(s). Then the mean-square phase error due to modulation can be written as

$$\overline{\phi_m^2(t)} = \frac{1}{2\pi j} \int_o^{+j\infty} G_n(s)\,|1-H(s)|^2 ds \qquad (4.3)$$

where,

$$G_m(s) = |\Phi_i(s)|^2,$$

i.e., magnitude square of the input phase transform function is the same as the power spectral density of the input signal phase.

Before attempting the actual computation of Equation (4.4), the following assumptions will be made to obtain the meaningful results: (1) the power spectral density of the input signal phase is uniformly distributed over the base-band f_m, (2) the loop natural frequency is so large that $\omega_m/\omega_n << 1$.

In these assumptions, the first one is usually the case in practice. The second assumption restricts the validity of computations to more interesting region of the high output signal-to-noise ratio of the demodulator.

Hence, for the closed-loop transfer function given by Equation (3.12) we obtain

$$\overline{\phi_n^2(t)} = \frac{1}{2\pi j}\int_o^{+j\infty} G_m|1-H(s)|^2 ds$$

$$\simeq \frac{G_m}{2\pi j}\int_o^{f_m} \frac{\omega^4}{\omega_n^4}\, df(2\pi j)$$

$$= \frac{G_m f_m \omega_m^4 e^{\sigma^2}}{5\omega_{no}^4} \tag{4.4}$$

Since the noise spectrum density was assumed to be uniformly distributed with the normalized amplitude over the bandwidth of IF filter, the mean-square phase error due to noise through the loop filter H(s) will be given by

$$\overline{\phi_n^2(t)} = \frac{G_n}{2\pi j}\int_o^{+j\infty}|H(s)|^2 ds$$

$$= \frac{G_n}{2\pi j}\int_o^{+j\infty}\left|\frac{2\zeta\omega_n + \omega_n^2}{S^2 + 2\zeta\omega_n + \omega_n^2}\right|^2 ds$$

$$= \frac{G_n\omega_n(1 + 4\zeta^2)}{8\zeta} \tag{4.5}$$

where,

$$G_n = \frac{G_{if}\, e^{\sigma^2}}{S_{if}} \tag{4.6}$$

In Equation (4.5) S_{if} and G_{if} denote the signal and noise power spectra over the IF filter bandwidth. The signal and noise processes are assumed to be zero-mean Gaussian, and the mean-square is given by the variance. Therefore, from Equation (4.4), (4.5), and (4.6) the total mean-square phase error $\overline{\phi_e^2(t)}$ of Equation (4.3) can be given as

$$\sigma^2 = \frac{f_m G_m \omega_m^4 e^{\sigma^2}}{5\ \omega_{no}^4} + \frac{G_{if}\omega_{no}(1+4\zeta_o^2 e^{-\frac{\sigma^2}{2}})e^{\sigma^2}}{8\zeta_o S_{if}} \tag{4.7}$$

V. INVESTIGATION OF MAXIMUM DEMODULATING SENSITIVITY LIMIT WITH HIGH STABILITY

This chapter is intended to determine the limitation of the maximum demodulating sensitivity with high stability. In this thesis, the maximum demodulating sensitivity limit is defined as the threshold of the demodulator.

It has been recognized that the phase-locked loop performs a band-pass filtering operation on phase inputs produced by modulating message and noise. Hence, one of the important criteria for high demodulating sensitivity is minimization of the mean-square phase error, σ^2. Inspection of Equation (4.7) shows that it is also affected by such significant factors as the relative loop natural frequency ω_{no} and damping factor ζ_o. Therefore, it is obvious that these three factors, i.e., ω_{no}, ζ_o, and σ play important roles to improve high demodulating sensitivity.

Since the signal power spectrum over IF filter bandwidth S_{if} can be assumed to be equal to the input signal power S_i of the demodulator, from Equation (4.7), the input signal power of the demodulator can be represented as:

$$S_i = \frac{G_{if}[1+4\zeta_o^2 e^{-\sigma^2/2}]e^{\sigma^2}\omega_{no}}{8\zeta_o[\sigma^2 - f_m\omega_m^4 G_m e^{\sigma^2}/5\omega_{no}^4]} \tag{5.1}$$

Consequently, it is suggested that the maximum demodulating sensitivity can be derived by minimizing the input signal power.

Assuming the parameters: σ, ζ_o, G_{if}, G_m, and ω_m are as constants, we may find an expression for $\omega_{no}]_{min}$ to give a minimum S_i by taking the first derivative of Equation (5.1) with respect to ω_{no} and equating the result to zero. That is,

$$\omega_{no}\Big|_{min} = \left(\frac{f_m \omega_m^4 e^{\sigma^2}}{\sigma^2}\right)^{\frac{1}{4}}$$

Substituting this into Equation (5.1), we have

$$S_i\Big|_{min} = \frac{5\pi[1+4\zeta_o^2 e^{-\frac{\sigma^2}{2}}]}{32\zeta_o} \left(\frac{e^{\sigma^2}}{\sigma^2}\right)^{\frac{5}{4}} (G_m f_m)^{\frac{1}{4}} (2G_{if} f_m \beta) \tag{5.2}$$

Thus, one of the criteria for maximum demodulating sensitivity is achieved.

Consider the wideband frequency-modulated carrier to the input of the IF filter as:

$$v_{is} = 2\,A\sin[\omega_c t + \beta\cos\omega_m t] \tag{5.3}$$

where,

$\beta = \dfrac{\Delta f}{f_m}$ is frequency modulation index,

Δf = peak frequency deviation,

$f_m = \dfrac{\omega_m}{2\pi}$ is maximum frequency of sinusoidal message.

An approximation of the requirement bandwidth of the IF filter for distortionless transmission is

$$B_{if} = 2\beta f_m \tag{5.4}$$

If the IF bandwidth is assumed to be $2\beta f_m$ the single-sided power spectral density of the input signal phase in the loop is given by

$$G_m = \frac{\beta^2}{f_m} \tag{5.5}$$

where $W_i = \beta_{if}/f_m$ is the input bandwidth ratio.

Since the input noise power to the loop can be given as $N_i = 2G_{if}f_m\beta$ the input signal-to-noise power ratio is

$$(S/N)_i = \frac{S_i}{2G_{if}f_m\beta} \tag{5.6}$$

From inspection of Figure 3-2, G_mf_m is the mean-square signal power and G_nf_m the noise power in the demodulator output, respectively. Therefore, the output signal-to-noise power ratio of the demodulator can be expressed by Equation (4.6), (5.5), and (5.6) as:

$$(\frac{S}{N})_o = \frac{2\beta^3}{e^{\sigma^2}} (\frac{S}{N})_i \tag{5.7}$$

Thus, using Equation (5.2), (5.5), (5.6), and (5.7), we can write as follows:

$$(S/N)_i = \frac{e^{\frac{6}{5}\sigma^2}}{2\beta\sigma^2} \frac{5\ [1+4\zeta_o^2 e^{-\frac{\sigma^2}{2}}]}{16\zeta_o}^{\frac{4}{5}} (\frac{S}{N})_o^{\frac{1}{5}} \tag{5.8}$$

As was done in Equation (5.1) for ω_{no}, $(S/N)_i$ may be minimized from Equation (5.8) with respect to the mean-square phase error σ^2.

Assuming that $(S/N)_o$ is fixed, a minimum value of $(S/N)_i$ will be obtained for a particular value of damping factor. From the curves of the frequency response of the loop shown in Figure 2-4, it can be seen that the suitable range for the loop stability may be from 0.3 to 1.0. Within this range, as shown in Figure 5-1, the damping factor for minimum $(S/N)_i$ is approximately 0.707 and hence, maximum demodulating sensitivity occurs at $\sigma^2 = 1.0201$ radians. Substitution of this value into Equation (5.8) yields the following threshold relation:

$$(S/N)_{i,th.}\Big|_{\substack{\zeta_o = 0.707 \\ \beta = 1}} = 6.1 + (1/5)(S/N)_o(\text{db}) \tag{5.9}$$

where $(S/N)_{i,th.}\Big|_{\zeta_o = 0.707}$ denotes the input threshold in the value of decibels for the damping factor 0.707. The threshold characteristics are depicted in Figure 5-2 for damping factors: 0.3, 0.5, 0.707, and 1.0.

From Equation (5.7), by putting $Z = (S/N)_i/(S/N)_o$, we have

$$\sigma^2 = \ell_n[2\beta^3 Z]$$

Substituting this into Equation (5.8), the input signal-to-noise ratio can be obtained by

$$\left(\frac{S}{N}\right)_i = \frac{[W_i^3 Z/4]^{\frac{3}{2}}\left\{\frac{5\pi}{16}\left[1 + \frac{4\zeta_o^2}{(W_i^3 Z/4)^{1/2}}\right]\right\}}{Z^{1/4}\left\{W_i \ell n[W_i^3 Z/4]\right\}^{5/4}} \tag{5.10}$$

Derivation of Equation (5.10) from (5.1) is shown in Appendix A.

Graphs $(S/N)_o$-versus-$(S/N)_i$ are shown in Figure 5-3, 5-4, and 5-5 for damping factors 0.3, 0.5, 0.707, and 1.0, respectively. By the digital computer IBM 7040 the curves of threshold [Equation (5.9)], and $(S/N)_o$-versus-$(S/N)_i$ [Equation (5.10)] for various ζ_o, W_i, and β are plotted in Figure 5-2 to 5-5. The FORTRAN program is shown in Appendix B.

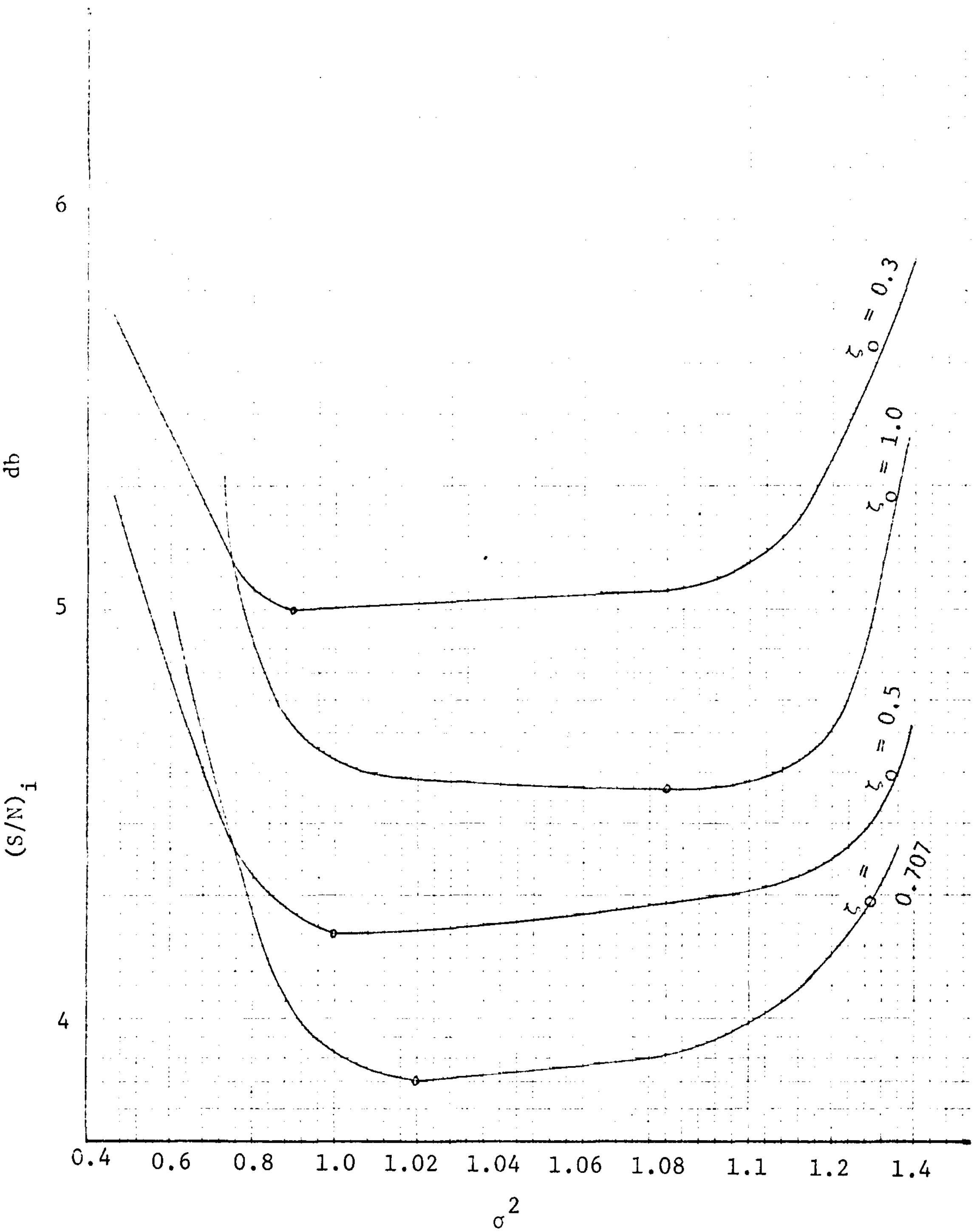

Figure 5-1. Input signal-to-noise ratio versus mean-square phase error for various damping factors.

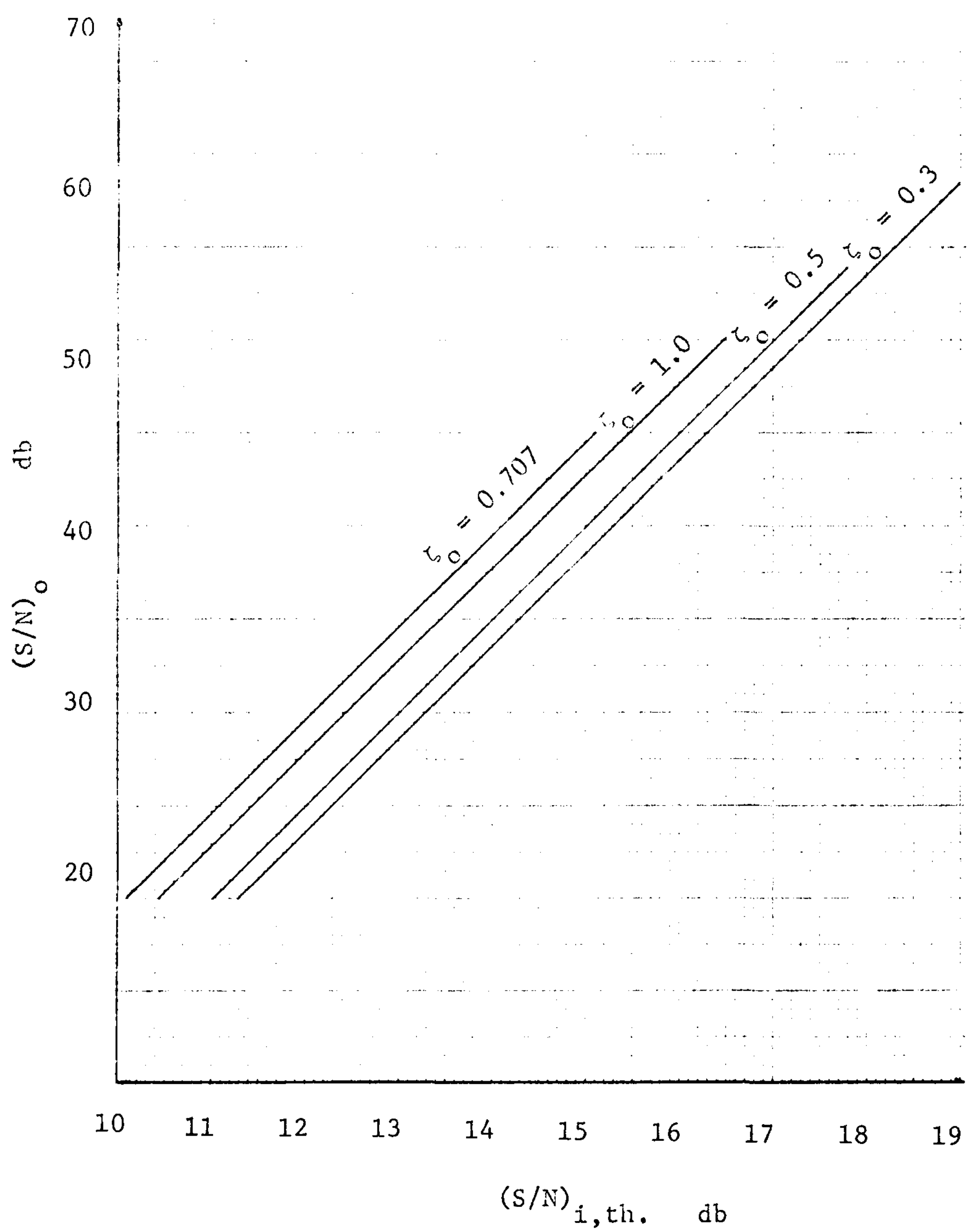

Figure 5-2. Threshold characteristics for various damping factors.

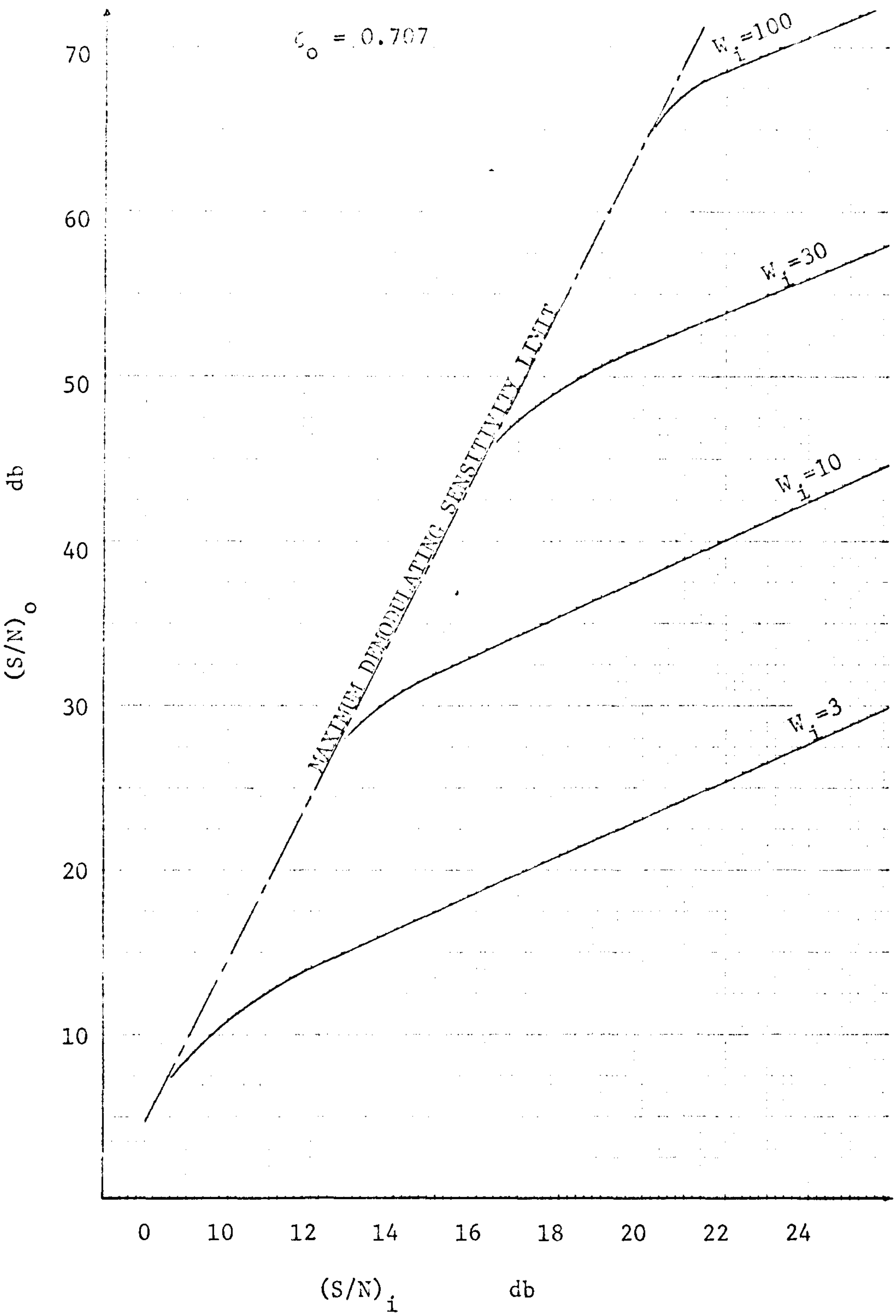

Figure 5-3. Output signal-to-noise ratio versus input-signal-to-noise ratio for various input bandwidth ratio at $\zeta_o = 0.707$.

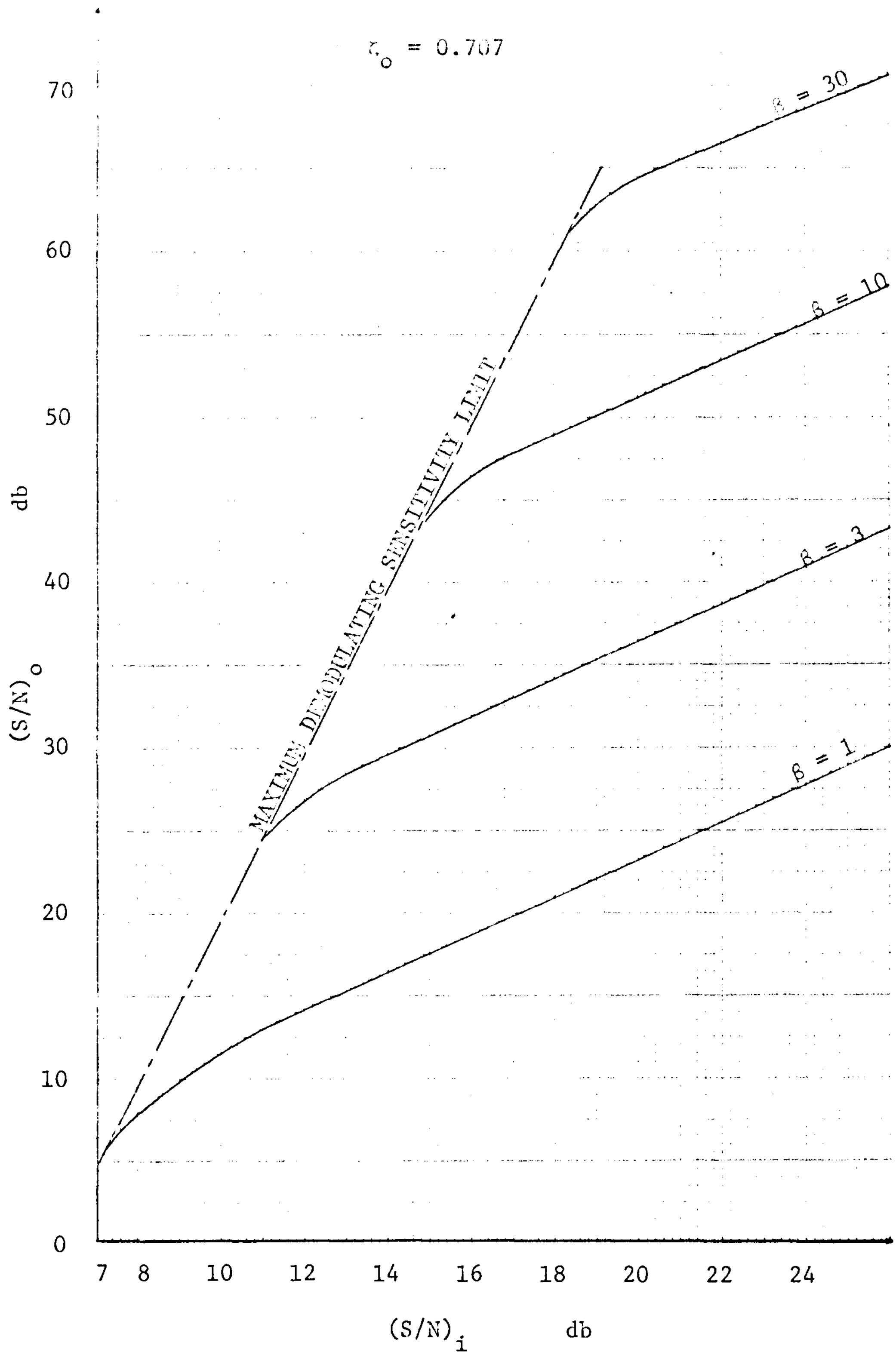

Figure 5-4. Output signal-to-noise ratio versus input signal-to-noise ratio for various modulation indices at $\zeta_o = 0.707$.

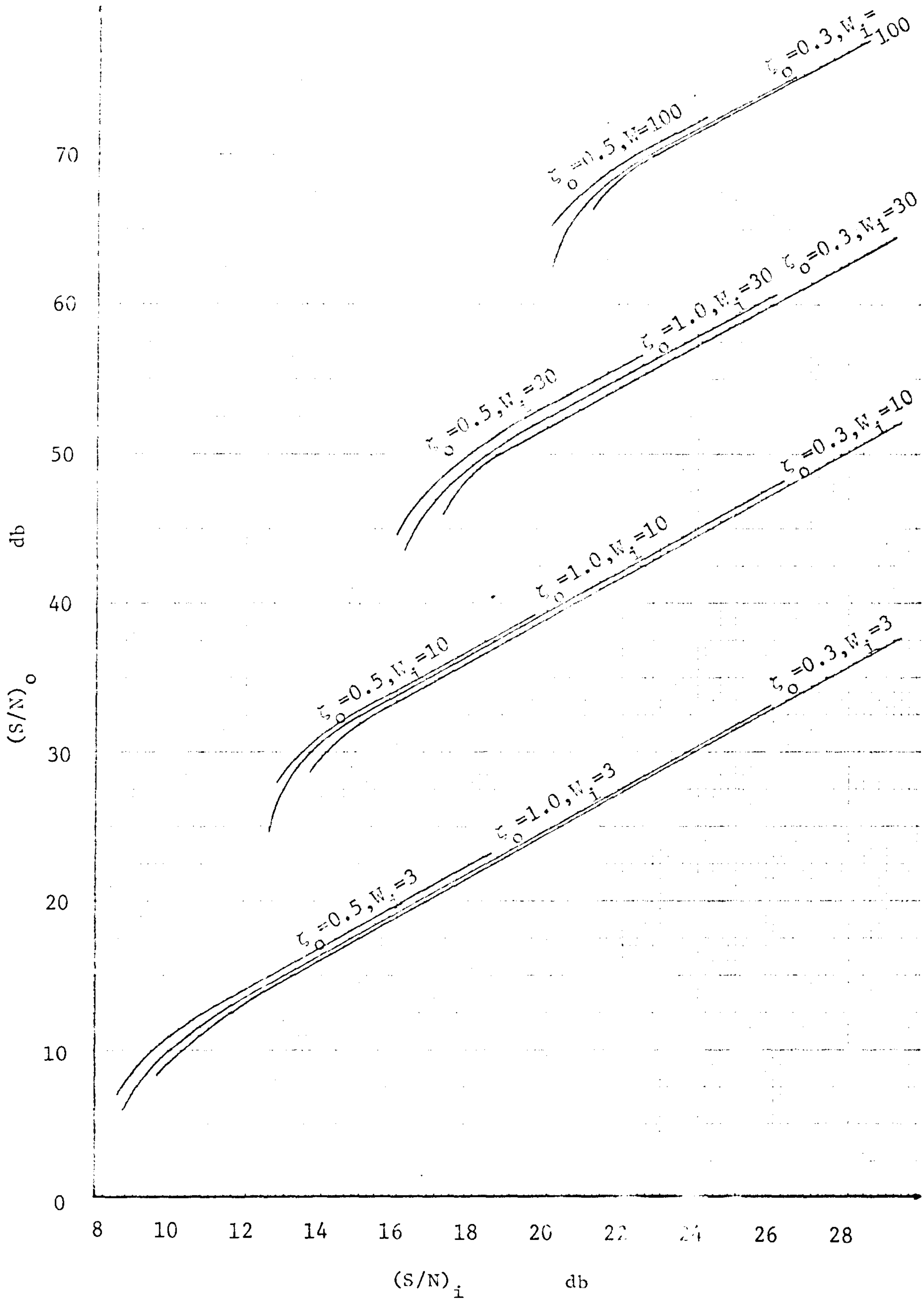

Figure 5-5. Output signal-to-noise ratio versus input signal-to-noise ratio for various input bandwidth and damping factors.

VI. CONCLUSIONS

Utilization of Figure 2-4 makes it possible to predict that a proper range of damping factor for a suitable response of the high-gain second-order loop is between 0.3 and 1.0. For this range, the curves in Figure 5-1 shows that the mean-square phase error never exceeds its minimum value by more than 8.0 percent.

From Figure 5-4, the minimum threshold is bounded at $(S/N)_i$ = +7.2 db. It is also seen from Figure 5-2 that the difference of the input threshold is approximately 1.2 db for $0.3 \leq \zeta_o \leq 1.0$. Hence, we can conclude that for suitable performance the damping factor may be chosen in any of the above range.

In the above threshold, it can be seen from Figure 5-3 to 5-5 that $(S/N)_0$ versus $(S/N)_i$ approach the characteristics of a conventional FM demodulation asymptotically. Also, as seen from these figures, near threshold, there is a risk of the demodulating sensitivity being degraded rapidly. Therefore, to avoid this possibility of degradation, the input threshold should be chosen about +2 db above threshold for linear demodulation. It is also noted that the input bandwidth ratio or modulation index does not affect linearly on $(S/N)_o$. If a minimum power optimization is desirable, the lower input bandwidth ratio or modulation index is suitable.

In conclusion it must be mentioned that by utilizing quasi-linearization techniques, the satisfactory performance of the high-gain second-order phase-locked loop demodulation may be expected only for value of output signal-to-noise ratio above +5 db for damping factor 0.707.

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X. APPENDICES

APPENDIX A

Derivation of Equation (5.10) from (5.1)

From Equation (5.1) we have

$$\omega_{no}\Big|_{min} = \left(\frac{G_m f_m \omega_m^4 e^{\sigma^2}}{\sigma^2}\right)^{\frac{1}{4}}$$

Substituting $\omega_{no}\big|_{min}$ into Equation (5.1),

$$S_i\Big|_{min} = \frac{G_{if} e^{\sigma^2}\left[1+4\zeta_o^2 e^{-\frac{\sigma^2}{2}}\right]\left(G_m f_m \omega_m^4 e^{\sigma^2}/\sigma^2\right)^{\frac{1}{4}}}{8\zeta_o\left[\sigma^2 - G_m f_m \omega_m^4 e^{\sigma^2}/5\left(G_m f_m \omega_m^4 e^{\sigma^2}/\sigma^2\right)\right]}$$

$$= \frac{5\pi\left[1+4\zeta_o^2 e^{-\frac{\sigma^2}{2}}\right]}{16\zeta_o}\left(\frac{e^{\sigma^2}}{\sigma^2}\right)^{\frac{5}{4}}\left(G_{if} f_m\right)\left(G_m f_m\right)^{\frac{1}{4}}$$

$$= \frac{5\pi\left[1+4\zeta_o^2 e^{-\frac{\sigma^2}{2}}\right]}{32\zeta_o}\left(\frac{e^{\sigma^2}}{\sigma^2}\right)\left(G_m f_m\right)^{\frac{1}{4}}\left(2G_{if} f_m \beta\right) \qquad (5.2)$$

Since $\left(\frac{S}{N}\right)_i = \frac{S_{if}}{2\beta G_{if} f_m}$

and $\left(\frac{S}{N}\right)_o = \frac{2G_m f_m}{2G_n f_m} = \frac{2\beta G_m f_m}{e^{\sigma^2}}\left(\frac{S_{if}}{2G_{if} f_m \beta}\right) = \frac{2\beta G_m f_m}{e^{\sigma^2}}\left(\frac{S}{N}\right)_i,$

Then $$G_m f_m = \frac{e^{\sigma^2}}{2\beta} \frac{(\frac{S}{N})_o}{(\frac{S}{N})_i} \tag{5.7a}$$

Therefore, $(\frac{S}{N})_i$ can be expressed as

$$(\frac{S}{N})_i = \frac{e^{6/5\sigma^2}}{2\beta\sigma^2} \left[\frac{5\pi[1 + 4\zeta_o^2 e^{-\sigma^2/2}]}{16\zeta_o}\right]^{4/5} (\frac{S}{N})_o^{1/5} \tag{5.8}$$

From Equation (5.3) and (5.4),

$$G_m = \frac{2\overline{\phi_i^2(t)}}{f_m} = \frac{2\overline{[\beta\cos\omega_m t]^2}}{f_m} = \frac{\beta^2}{f_m} \tag{5.5}$$

Thus, substituting this into (5.7a), we have

$$(\frac{S}{N})_o = \frac{2\beta^3}{e^{\sigma^2}} (\frac{S}{N})_i \tag{5.7}$$

Substituting Equation (5.5) into (5.7a), by putting $Z = (S/N)_i/(S/N)_o$, we have

$$\sigma^2 = \ln[2\beta^3 Z]$$

Then, substituting this into Equation (5.8), $(\frac{S}{N})_i$ can be expressed as

$$(\frac{S}{N})_i = \frac{[2\beta^3 Z]}{Z^{1/4}\{2\beta\ln[2\beta^3 Z]\}^{5/4}} \left\{\frac{5\pi}{16\zeta_o} [1 + \frac{4\zeta_o^2}{(2\beta^3 Z)^{1/2}}]\right\}$$

$$= \frac{[W_i^3 Z/4]^{3/2}}{Z^{1/4}\{W_i\ln[W_i^3 Z/4]\}^{5/4}} \left\{\frac{5\pi}{16\zeta_o} [1 + \frac{4\zeta_o^2}{(W_i^3 Z/4)^{1/2}}]\right\} \tag{5.10}$$

APPENDIX B

Fortran Program Variables Names

DAMP	Relative damping factor, ζ_o
MØD	Frequency modulation index, β
WIDTH	Input bandwidth ratio, W_i
ISNR	Input signal-to-noise power ratio, $(S/N)_i$
ØSNR	Output signal-to-noise power ratio, $(S/N)_o$
IØSNR	Input-to-output signal-to-noise power ratio, $(S/N)_i/(S/N)_o$
ISNRDB	Input signal-to-noise power ratio in decibels, $(S/N)_i$ (db)
ØSNRDB	Output signal-to-noise power ratio in decibels, $(S/N)_o$ (db)
ITHDB	Input threshold, in decibels, $(S/N)_{i,th.}$ (db)
ØTHDB	Output threshold in decibels, $(S/N)_{o,th.}$ (db)

Fortran IV Digital Computer Program

```
      DIMENSIØN      DAMP(7), WIDTH(9), MØD(9), IØSNR(75),
     C               ISNR(75), ØSNR(75), ISNRDB(75), ØSNRDB(75),
     C               ITHDB(75), ØTHDB(75)
      READ(5, 10)    (DAMP(I), I=1, 4)
   10 FØRMAT (4F10.5)
      READ (5, 20)   (WIDTH(J), J=1, 9)
   20 FØRMAT (9F8.5)
      READ (5,30)    (IØSNR(K), K=1, 75)
   30 FØRMAT (5F10.7)
      WRITE (6, 1)
    1 FØRMAT (1H1, 11X, 4HDAMP, 13X, 5HWIDTH, 13X, 3HMØD, 5HIØSNR,
     C        15X, 6HISNRDB, 15X, 6HØSNRDB, 15X, 5HØTHDB, ///)
      DØ  100    I=1, 4
      DØ  200    J=1, 9
      DØ  300    K=1, 75
      S = DAMP(I)
      W = WIDTH(J)
      B = MØD(J)
      Z = IØSNR(K)
      A1 = W**3
      R1 = 0.25*A1*Z
      IF (R1 .LE. 1.0) GØ TØ  300
      R2 = Z*0.25
```

```
A2 = ALØG (R1)

R3 = W*A2

A3 = 16.*S

R4 = 5.*3.14159/A3

A4 = S**2

R5 = 4.*A4

A51 = R1**0.5

A52 = R1**1.5

R6 = 1. + R5/A51

A6 = R3*1.25

AR = A51/R2*A6

BR = R4*R6

R = ISNR(K)

RR = ØSNR(K)

RDB = ISNRDB(K)

RRDB = ØSNRDB(K)

MØD(J) = W/2.

ISNR(K) = AR*BR

ØSNR(K) = R/RR

RLØG = ALØG10(R)

RDB = 10.*RBØG

RRLØG = ALØG10(RR)

ØSNRDB(K) = 10.*RRLØG

T1 = ITHDB(K)

T2 = ØTHDB(K)
```

```
      ITHDB = ISNRDB
      T11 = T1 - 6.97481
      T12 = T1 - 6.19258
      T13 = T1 - 6.10048
      T14 = T1 - 6.40822
      IF (S .EQ. 0.3) ØTHDB(K) = T11*5.
      IF (S .EQ. 0.5) ØTHDB(K) = T12*5.
      IF (S .EQ. 0.707) ØTHDB(K) = T13*5.
      IF (S .EQ. 1.0) ØTHDB(K) = T14*5.
      WRITE (6, 2) DAMP(I), WIDTH(J), MØD(J), IØSNR(K),
     C             ISNRDB(K), ØSNRDB(K), ITHDB(K), ØTHDB(K)
    2 FØRMAT (1X, 7F16.7)
  300 CØNTINUE
  200 CØNTINUE
  100 CØNTINUE
      STØP
      END
```

ABSTRACT

AN ANALYSIS OF THRESHOLD CHARACTERISTICS OF QUASI-LINEARIZED PHASE-LOCKED LOOP DEMODULATION FOR WIDEBAND FREQUENCY-MODULATED SIGNALS

An analytical threshold criterion in approximation has been developed for the basic phase-locked loop demodulator utilizing quasi-linearization technique. The analysis is based on assumptions that the loop is excited by an input FM signal and additive white Gaussian noise. This paper defines the threshold criterion by the characteristics of maximum demodulating sensitivity limit. Finally, the effects of the modulation indecies and loop parameters on the threshold characteristics are discussed from a theoretical and practical point of view.