ANALYTICAL DETERMINATION OF AUTOCORRELATION AND NOISE POWER DENSITY SPECTRUM OF RANDOMLY MODULATED PULSE WIDTH SQUARE WAVES

by

Hen Suh Park

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APPROVED: ______________________________
Chairman, Dr. Man Y. Rhee

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Walton W. Cannon Charles A. Holt

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Ralph R. Wright

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TABLE OF CONTENTS

I. INTRODUCTION

This chapter presents Average Area Method of evaluating the autocorrelation and Wiener's Power Density Spectrum pertaining to randomly modulated square waves and introduce briefly the Gamma and Beta-Distribution Density Function.

1.1 Randomly Modulated Pulse Width with Uniform Density

I consider the square wave that is a sequence of square pulses of height E with their leading edges at 2b aparts and the maximum pulse width b shown in Fig. 1.

Let x and $P_1(x)$ be the random pulse duration and its probability density function, respectively. For the present problems we assume that (1) the random pulse durations are stochastically independent of one another and (2) a series of pulses is stationary ergodic random process, and (3) the probability density function for the random pulse durations of being found in the interval $0<t<b$ is uniform density.

The problem can be simplified by changing the b-time period scale to unit time period scale shown in Fig. 2.

We compute first the autocorrelation function over the unit interval. As we will notice from the definition

$$R_{11}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_1(t)f_1(t+\tau)dt, \qquad (1.1.1)$$

the problems of finding the autocorrelation function needs the displacement, the multiplication, and the time average of $f_1(t)f_1(t+\tau)$ over the unit interval. Without loss of generality a sample function chosen from the ensemble of random time function with randomly modulated pulse width square waves is shown in Fig. 3.

According to the stationary ergodic assumption at the first place, the time averaging, over infinite time interval, of the shaded rectangle areas can be replaced by the ensemble averaging with uniform density function so that the expected value of the shaded squares is

$$A_T(\tau) = \int_{\tau}^{1} E^2 (x-\tau) P_1(x)\,dx \tag{1.1.2}$$

in which case

$$P_1(x) = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$$

It is convenient to introduce here the relative autocorrelation function defined by equation $A_T(\tau) = \frac{A_T(\tau)}{2E^2}$.

Therefore, the relative autocorrelation function is

$$\begin{aligned} A_T(\tau) &= \int_{\tau}^{1} (x-\tau) P_1(x)\,dx \\ &= \frac{1}{4}(1-\tau)^2 \qquad (1.1.3) \\ &\qquad \text{for } 0<\tau<1. \end{aligned}$$

A further observation on the properties of the eq. (1.1.1) states that the relative autocorrelation function for the interval $-1<\tau<0$ is

$$R_{11}(\tau) = R_{11}(-\tau) = A_T(\tau) = \frac{1}{4}(1-|\tau|)^2 \quad \text{for } -1<\tau<0\ ,\ 0<\tau<1.$$

Let us consider next the autocorrelation for $2<\tau<3$. When $2<\tau<3$, the displaced and folded pulse trains are shown in Fig. 4.

Keeping in mind that in a practical situations we are concerned with the stationary ergodic random pulse width which are stochastically independent of each other, it is found for the ensemble average area of partially and completely overlapping rectangle indicated by the shaded line in Fig. 4 to be

$$A_P(\tau') = \int_{\tau'}^{1}\int_{y-\tau'}^{1} E^2(y-\tau')P_{12}(x,y)\,dxdy + \int_{\tau'}^{1}\int_{0}^{y-\tau'} E^2 x P_{12}(x,y)\,dxdy$$

$$= \frac{E^2}{6}\ [(2+\tau')(1-\tau')^2] \qquad (1.1.4)$$

Therefore the relative autocorrelation function for the interval $2<\tau<3$ is

$$A_P(\tau') = \frac{A_P(\tau')}{2E^2} = \frac{1}{12}(2+\tau')(1-\tau')^2 \qquad (1.1.5)$$

Now we are in a sufficient position to be able to obtain the relative autocorrelation function for the other interval $-2<\tau<-1$ and then determine the entire relative autocorrelation. To do this, we begin to notice the eq. (1.1.1). When $\tau = 2+\tau'$, the autocorrelation for the stationary ergodic process and stochastically independent pulse trains is

$$R_{11}(2+\tau') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x_1)f_2(x_2)P_{12}(x_1,x_2\ ;\ \tau = 2+\tau')dx_1dx_2$$

or

$$R_{11}(2+\tau') = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(t)f_1(t+2+\tau')dt \qquad (1.1.6)$$

in which x_1 and x_2 stand for the random pulse widths at $t=t_1$ and $t=t_1+2+\tau'$, respectively. The displacement of $f_1(t)$ to the right by $\tau=2-\tau'$ leads the autocorrelation function $R_{11}(\tau)$ to

$$R_{11}(\tau) = R_{11}(-2+\tau') = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(t)f_1(t-2+\tau')dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(x_1)F_2(x_2)P_{12}(x_1,x_2\ ;\ \tau=-2+\tau')dx_1dx_2$$

(1.1.7)

It follows from the ergodicity, stationarity, and stochastical independence that

$$P_{12}(x_1,x_2;\tau=2+\tau') = P_1(x_1)P_2(x_2;\tau=\tau'+2) = P_1(x_1)P_2(x_2;\tau=\tau')$$

$$P_{12}(x_1,x_2;\tau=-2+\tau') = P_1(x_1)P_2(x_2;\tau=-2+\tau') = P_1(x_1)P_2(x_2;\tau=\tau')$$

and then

$$P_{12}(x_1,x_2;\tau=2+\tau') = P_{12}(x_1x_2;\tau=-2+\tau') \qquad (1.1.8)$$

Imposing the equation (1.1.8) into the equations (1.1.6) and (1.1.7), one obtains

$$R_{11}(2+\tau') = R_{11}(-2+\tau')$$

so that knowledges of $R_{11}(2+\tau')$ gives $R_{11}(-2+\tau')$. Finally the properties of autocorrelation make us to determine the complete curve in the infinite range on τ-axis. The complete curve is shown in Fig. 5.

1.2 Wiener's Theorem for the Power Density Spectrum

Let $f_p(t)$ be the periodic message and $f_n(t)$ be the noise signal with zero mean so that the input $f_i(t)$ to the systems is generally an additive mixture of a message and noise. These input function $f_i(t)$ are members of their stationary ensembles.

Since the variable $f_p(t)$ and random variable $f_n(t)$ are stochastically independent, it follows that the autocorrelation function $R_{ii}(\tau)$ of the continuous input function $f_i(t)$ is

$$R_{ii}(\tau) = R_{pp}(\tau) + R_{nn}(\tau) \qquad (1.2.1)$$

where

$$R_{pp}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} f_p(t) f_p(t+\tau) dt \qquad (1.2.2)$$

$$R_{nn}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_n(t) f_n(t+\tau) dt \qquad (1.2.3)$$

in which case $R_{pp}(\tau)$ becomes a periodic function with a periodicity 2T. Note that $R_{ii}(\tau)$ is composed of two components, one periodic and the other nonperiodic.

By the Wiener's Theorem for autocorrelation the power density spectrum of the messages and noise are given as

$$\Phi_{pp}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{pp}(\tau) \cos \omega\tau d\tau \qquad (1.2.4)$$

and

$$\Phi_{nn}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{nn}(\tau) \cos \omega\tau d\tau \ , \qquad (1.2.5)$$

respectively, where ω is the angular frequency.

In going through the present problems for the power density spectrum of the randomly modulated pulse width square wave with uniform probability density, first of all there must be a special point that needs to decompose the autocorrelation function into two components, one periodic and the other aperiodic which is random component. From the eq.s (1.1.3) and (1.1.5) we know that a periodic and aperiodic components are

$$A_p(\tau) = \frac{1}{12}(2+\tau)(1-\tau)^2$$

and

$$A_n(\tau) = A_T(\tau)[u(\tau+1)-u(\tau-1)] - A_p(\tau) \qquad \text{for} \quad 1<\tau<1,$$

respectively.

By using the Wiener's Theorem we observe that the noise power density spectrum for this random component is

$$\Phi_{nn}(\omega) = \frac{1}{4\pi}\,\frac{1}{\omega^2}\left[1-\left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^2\right]$$

1.3 <u>The Need and Purpose of Using Gamma and Beta-Distribution</u>

Up to now we have been concerned with the assumption that the probability density distributions of given randomly modulated pulse width were assumed to be known a priori and that the problems was discussed to determine the power spectrum density and autocorrelation function from it. Even if we can make a set of physical measurements of the power density spectrum function, it is not known which power spectrum was produced by which probability density that might be no longer uniform density. Therefore we have to start from the most generally reasonable assumption that the density function is Beta-Distribution Function over the interval $0<x<1$. Furthermore we often face a quite different problem. As some of statistical properties of random variables or process can be changed by varying the various parameters of the assumed Beta-Distribution Function, we may ask how the variation of the power density spectrum and autocorrelation function with variance, medium, and mean is. For example, we might wish to know the variation of noise power density spectrum with the variance for fixed medium. From the variation of noise power density spectrum with the variance and mode it can be established to make the probabilistic design criterion of pulse generator.

Another purpose is to determine the most fitted Beta-distribution to any randomly modulated pulse width signal which will have an important factor in Square Wave Frequency Modulation.

1.4 The Gamma-Distribution Function

The Gamma-Distribution function can be defined by the equation

$$\Gamma(x) = \frac{1}{\Gamma(\alpha+1)\beta^{\alpha+1}} x^{\alpha} e^{-\frac{x}{\beta}} \qquad x>0$$

$$= 0 \qquad x<0$$

This is a two parameter family of distributions, the parameters being α and β. β must be positive and α must be greater than minus one. The function is shown in Fig. 6 for $\beta = 1$ and several values of α which will play an important role in autocorrelation function.

Let the mean and variance be μ and σ, respectively. It follows from the gamma-distribution density function that

$$\mu = \beta(\alpha+1)$$

$$\sigma^2 = \beta^2(\alpha+1).$$

1.5 <u>The Beta-Distribution Density Function</u>

The Density Function

$$\beta(x) = (n+1)\binom{n}{n_1}(1-x)^{n-n_1}x^n \quad , \quad (n-n \ , \ n_1>-1) \text{ for } 0<x<1$$

$$= 0 \qquad \text{elsewhere}$$

is called the Beta-Distribution Density Function. This function represents a two-parameter family of distributions and the various curves for the representative values of n and n_1 which play a very important factor for the randomly modulated pulse trains, are plotted in Fig. 7.

It is notified from Fig. 7 that the uniform density function, over the unit interval, becomes a special case of the Beta-Distribution density when $n = n_1 = 0$.

Let the mean, variance, and mode be denoted by a, s, and X_m, respectively. From the Beta-Distribution density we find

$$a = \frac{X_m + \frac{1}{n}}{1 + \frac{2}{n}}$$

$$s^2 = \frac{(X_m + \frac{1}{n})(1 - X_m + \frac{1}{n})}{(1 + \frac{2}{n})^2 (n+3)}$$

$$X_m = \frac{n_1}{n}$$

II. DEVELOPMENT OF AUTOCORRELATIONS AND POWER SPECTRUMS

In this chapter the autocorrelation function and power density spectrum of a series of square waves with random amplitude and random pulse width will be presented. Final expressions for the autocorrelations and power density spectrums is developed to meet Fortran Program.

2.1 Autocorrelation and Power Spectrum of Randomly Modulated Pulse Width with Beta-Distribution Only.

When the displacement time τ lies between 0 and 1, the area of the overlapping parts can be expressed as

$$E^2(x-\tau)$$

which is also another random variable defined by random pulse width ξ. The probability for this amount of area being occurred is our presumed Beta-Distribution Function

$$B_\xi(x) = (n+1)\binom{n}{n_1}(1-x)^{n-n_1}x^{n_1} \qquad (2.1.1)$$

where n, n_1 will be determined by considering the noise component power density spectrum. Therefore the expected value of $E^2(x-\tau)$ yields

$$A_T(\tau) = \int_\tau^1 E^2(x-\tau)B_\xi(x)dx \qquad (2.1.2)$$

This equation gives the relative autocorrelation function

$$A_T(\tau) = \frac{A_T(\tau)}{2E^2} = \frac{1}{2}\int_{\tau}^{1}(x-\tau)B_{\xi}(x)dx \qquad (2.1.3)$$

Let us consider nest the autocorrelation function over the interval $2<\tau<3$. If the square waves with random pulse width is displaced by an amount of $\tau = 2+\tau'(0<\tau'<1)$ seconds and folded, we have two cases which are the partially and completely overlapping random sequence as shown in Fig. 3. When $\tau'<y$, from observation on Fig. 3 it follows that the average area of partially and completely overlapping parts indicated by the shaded line on the time axis is

$$A_p(\tau') = E^2\int_{\tau'}^{1}\int_{y-\tau'}^{1}(y-\tau')B_{\xi n}(x,y)dxdy +$$

$$E^2\int_{\tau'}^{1}\int_{0}^{y-\tau'}xB_{\xi n}(x,y)dxdy \qquad (2.1.4)$$

which brings the relative autocorrelation function

$$A_p(\tau') = \frac{A_p(\tau')}{2E^2} = \frac{1}{2}\int_{\tau'}^{1}\int_{y-\tau'}^{1}(y-\tau')B_{\xi n}(x,y)dxdy +$$

$$\frac{1}{2}\int_{\tau'}^{1}\int_{0}^{y-\tau'}xB_{\xi n}(x,y)dxdy \qquad (2.1.5)$$

Now let us stand $B_\eta(y)$ for the probability that the succeeding pulse width length y is occurred after the time elapse by 2 seconds and apply the statistical independent property and ergodic stationary process presumption to the joint probability density function of two random variables ξ and η in which ξ represents the length of one pulse in a random series and the other η the succeeding random pulse width variable, respectively.

The probability density of occurrence of ξ at $t=t_1$ and η at $t=t_1+2$ can be simply expressed as

$$P_{\xi\eta}(x,y) = B_\xi(x)B_\eta(y) \tag{2.1.6}$$

which is a simple product of each probabilities.

From the points discussed above it is observed that

$$A_p(\tau') = \frac{1}{2}\int_{\tau'}^{1} (y-\tau')B_\eta(y)dy \int_{y-\tau'}^{1} B_\xi(x)dx + \frac{1}{2}\int_{\tau'}^{1} B_\eta(y)dy \int_{0}^{y-\tau} xB_\xi(x)dx \tag{2.1.7}$$

Replacing the argument τ' by the new argument τ, it represents a periodic component of the autocorrelation function of the random process with randomly modulated pulse width square waves.

By the Wiener Theory the random noise component of the autocorrelation function caused by random pulse width modulation can be formed by subtracting $A_p(\tau)$ from $A_T(\tau)$. Thus

$$A_n(\tau) = \frac{1}{2}\int_{\tau}^{1} [(y-\tau)B_\eta(y)\int_{0}^{y-\tau} B_\xi(x)dx - B_\xi(y)\int_{0}^{y-\tau} xB_\xi(x)dx]dy$$

therefore the Wiener's Power Density Spectrum of the random component is

$$\Phi_{nn}(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} R_{11}(\tau)\ \mathrm{Cos}\,\omega\tau d\tau.$$

$$= \frac{1}{2}\int_{0}^{1}\mathrm{Cos}\,\omega\tau d\tau\int_{\tau}^{1} [(y-\tau)B_\eta(y)\int_{0}^{y-\tau} B_\xi(x)dx - B_\eta(y)\int_{0}^{y-\tau} xB_\xi(x)dx]dy \qquad (2.1.8)$$

where $B_\xi(x) = B_\eta(y) = (n+1)\binom{n}{n_1}(1-x)^{n-n_1}x^{n_1}$.

2.2 Square Waves Series with Random Pulse Width and Amplitude.

In the case that a train of square waves has the random amplitude and pulse width with Gamma and Beta-Distribution density, respectively, the visualization of forming the several overlapping parts for the computation of autocorrelation function over the intervals $0<\tau<1$, and $2<\tau<3$ is shown in Fig. 8 and 9.

The first thing to be computed is the autocorrelation function over the unit interval. Taking into account of that the amplitude to be a random variable, we find the autocorrelation

$$A_T(\tau) = \frac{1}{2}\int_0^\infty {}^2\Gamma(u)\,du\int_\tau^1 (x-\tau)B_\xi(x)\,dx$$

$$= \frac{1}{2}(\alpha+1)(\alpha+2)\beta^2\int_\tau^1 (\alpha-\tau)B_\xi(x)\,dx \qquad (2.2.1)$$

Let us take next the computation over the interval $2<\tau<3$. The shaded parts on the time axis in Fig. 9 shows the area of partially and completely overlapping parts. From the observation on Fig. 9 it is computed that

$$A_p(\tau) = \frac{1}{2}\int_0^\infty \Gamma(u)\,du\int_0^\infty \nu\Gamma(\nu)\,d\nu\,[\int_\tau^1 (y-\tau)B_\eta(y)\,dy\int_{y-\tau}^1 B_\xi(x)\,dx -$$

$$\int_\tau^1 B_\eta(y)\,dy\int_0^{y-\tau} xB_\xi(x)\,dx]$$

$$= \frac{1}{2}(\alpha+1)^2\beta^2[\int_\tau^1 (y-\tau)B_\eta(y)\,dy\int_{y-\tau}^1 B_\xi(x)\,dx-\int_\tau^1 B_\eta(y)\,dy\int_0^{y-\tau} xB_\xi(x)\,dx]$$

$$(2.2.2)$$

From Equation (2.2.1) and (2.2.2) it follows that the random component of autocorrelation function is

$$A_{nn}(\tau) = [u(\tau+1)- (\tau-1)]\frac{1}{2}(\alpha+1)\beta^2\int_{\tau}^{1}\{(y-\tau)B_{\eta}(y) +$$

$$(\alpha+1)B_{\eta}(y)[(y-\tau)\int_{0}^{y-\tau}B_{\xi}(x)dx-\int_{0}^{y-\tau}xB_{\xi}(x)dx]\}dy$$

We define the new relative autocorrelation function as

$$\overline{A}_{nn}(\tau) = \frac{A_{\tau}(\tau)}{\frac{1}{2}(\alpha+1)\beta^2} = \int_{\tau}^{1}\{(y-\tau)B_{\eta}(y)+(\alpha+1)B_{\eta}(y)$$

$$[(y-\tau)\int_{0}^{y-\tau}B_{\xi}(x)dx-\int_{0}^{y-\tau}xB_{\xi}(x)dx]\}dy \qquad (2.2.3)$$

The above equation (2.2.3) tells us that the relative autocorrelation is independent of the parameter β.

By the use of Wiener-Khinchin's Theorem we observe the power density spectrum as

$$\pi\Phi_{nn}(\omega) = \frac{1}{2}\int_{-\infty}^{\infty}\overline{A}_{nn}(\tau)\mathrm{Cos}\,\omega\tau d\tau$$

$$= \int_{0}^{1}\mathrm{Cos}\,\omega\tau d\tau\int_{\tau}^{1}\{(y-\tau)B_{\eta}(y)+(\alpha+1)B_{\xi}(y)[(y-\tau)\int_{0}^{y-\tau}B_{\xi}(x)dx -$$

$$\int_{0}^{y-\tau}xB_{\xi}(x)dx]\}dy$$

By the computer 7040 the various curves of the autocorrelation function and power density spectrum for the very important parameters n_1, n is plotted in Fig. 10 to 18 and will be discussed in the next chapter.

III. DISCUSSION OF THE INFLUENCE OF GAMMA AND BETA-DISTRIBUTION ON THE AUTOCORRELATION AND POWER DENSITY SPECTRUM.

In this chapter we discuss the variation of autocorrelation and power density spectrum with α, n, and n_1 of Gamma and Beta-distributions, and further examine the physical interpretation on the density function.

3.1 The Influence of Gamma-Distribution on the Autocorrelation and Power Density Spectrum.

The autocorrelation and power density spectrum are plotted in Fig. 10 and 11 for $\beta = 1$ and the typical values 0, 1, and 4 of α which give the significant density curves as well as variances and means. From the discussion in section 1.4 the increasing value of the variance and mean of Gamma-distribution function for the fixed value β, will be increased by increasing the parameter α.

As shown in Fig. 10 and 11, the increasing α merely changes the values of autocorrelation at the origin on the displacement τ-axis, and consequently the power density spectrum become proportionally broader, as the variance as well as mean become greater. In a practical situation it cannot be further improved by adjusting the variance and mean of the random amplitude with Gamma-distribution in order to make a narrow noise power density spectrum, because the modulating square wave generator has a limiting and desired value of amplitude which has random variation around the mean value.

In the next section we will see that the broadness of noise power density spectrum depends predominantly upon the Beta-distribution function of randomly modulated pulse width.

3.2 The Influence of Beta-Distribution on the Autocorrelation and Power Density Spectrum.

As shown in Fig. 12 and 13, for the fixed mode at $x = 0.5$ we observe that the noise power density spectrum broadens with the decreasing variance. The smaller variance gives the more flat bandwidth characteristics of noise power density spectrum than of noise power density spectrum with the larger variance. In the events that the occurrence of ending edges of the pulses is being found around the mode $x_m = 0.5$, the smaller variance means that the ending edges of pulses can occur around the mode $x_m = 0.5$ more likely than that of the larger variance. This will give us a physical interpretation that the square pulse generator producing less random pulse width supplies the more flat noise power density spectrum.

Let us consider next the behavior at the mode $x_m = 0.1$ and 0.2, respectively. From Fig. 14 and 15 we know that the power density spectrum and the autocorrelation at the fixed medium x_m = 0.1 and 0.2 broaden with decreasing variance. The pulse width having less randomness at the smaller mode than one half gives the broader random power density spectrum because the randomness of some of the pulse trains is predominantly more effective in producing an autocorrelation than one which the quasi-periodicity of the pulse trains does.

When the mode x_m = 0.8 and 0.9, the autocorrelation and power density spectrum are plotted in Fig. 16, 17 and 18. For the decreasing variance at the mode x_m = 0.8, we observe the power density spectrum become narrower as shown in Fig. 16 and 17. Whereas, the less randomness of pulse trains having the smaller mode than one half supplies the broader power density spectrum. The less randomness of pulse width which has the larger mode than one half gives the narrower power density spectrum. The contribution of quasi-periodicity to the power density spectrum having the larger mode is much more effective in producing an autocorrelation than that of the randomness of the pulse width. This result tells us the physical meaning that the smaller standard deviation at the ending edges of pulses presents more quasi-periodic pulse signal which will give the narrower power density spectrum.

IV. CONCLUSIONS

In Chapter III we have seen that a Beta-distribution function of random-width square pulses plays a very important role in improving the narrowness of the noise power density spectrum, but Gamma-distribution of randomly modulated amplitude of square waves has little effect on the power density spectrum of a fixed value of x_m of random amplitude of square pulses. It is very interesting to mention that the noise power density spectrum at the smaller mode, say $x_m = 0.1$, becomes broader with decreasing value of variance. The pulses occurring around $x_m = 0.9$ give much longer mean pulse widths than at the mode around $x_m = 0.1$. Since the pulse width is a major factor in determining the magnitude of the autocorrelation function, at large values of x_m, large variances provide for more short pulses and, thus, decrease the magnitude of the autocorrelation function. The noise power density spectrum broadens not only with decreasing variance around the mode $x_m = 0.9$ but also with the mode approaching to $x_m = 1.0$.

We could find all possible optimum values of parameters of Beta-distribution which produces the noise power density spectrum measured by an electronic correlator, and consequently determine the most probable Beta-distribution function.

If the square wave frequency modulation signal with a Beta-distribution of pulse width is applied to the input of Wide Band FM Discriminator that has nearly linear characteristics in the considerable frequency range, the noise voltage is developed at the output terminals by the random pulse width at the input terminals. We can expect that the noise power density spectrum of the output voltage could depend upon the parameters of the Beta-distribution function and the Discriminator.

V. LIST OF ILLUSTRATIONS

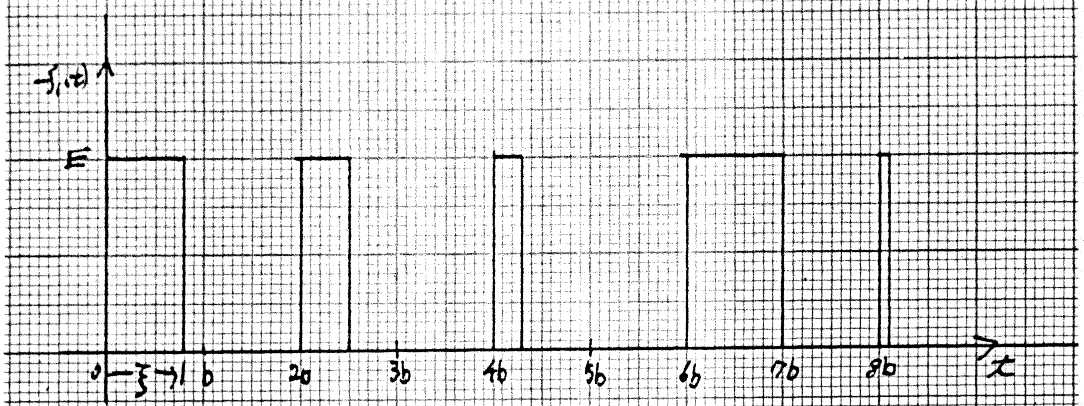

Fig. 1. Pulse Width Modulated Square Waves

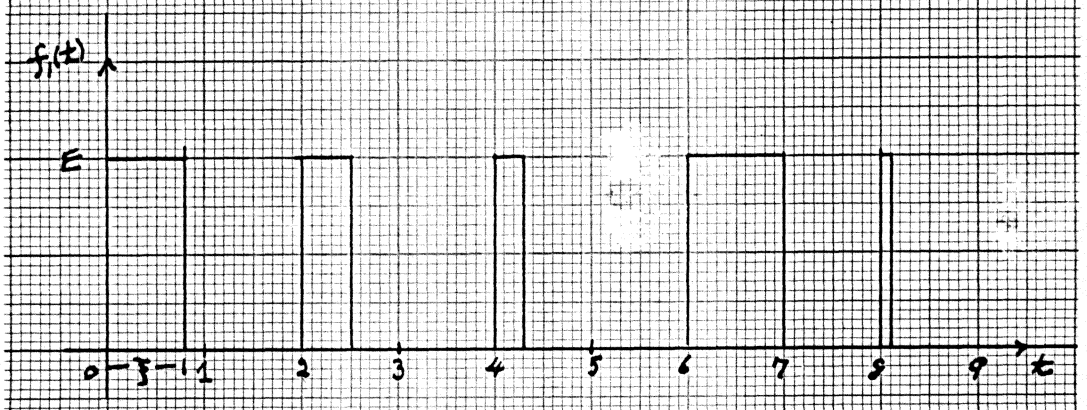

Fig. 2. Pulse Width Modulated Wave with Unit Period Scale

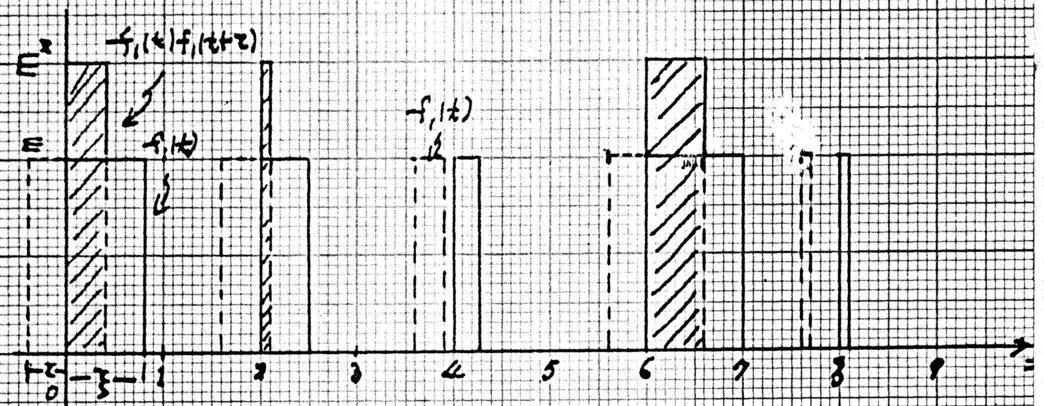

Fig. 3. The displaced and folded random time function

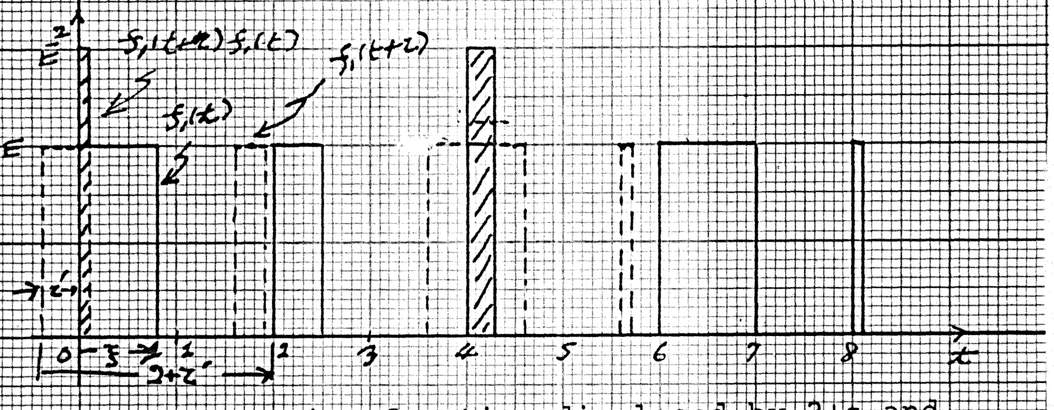

Fig. 4. Random time function displaced by $2+\tau$ and folded

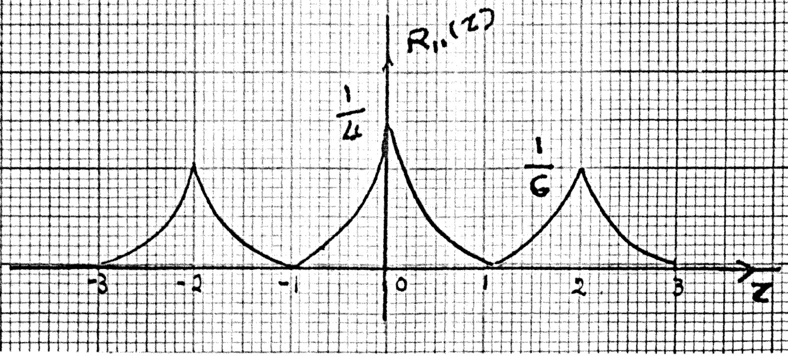

Fig. 5. The Entire Autocorrelation Function

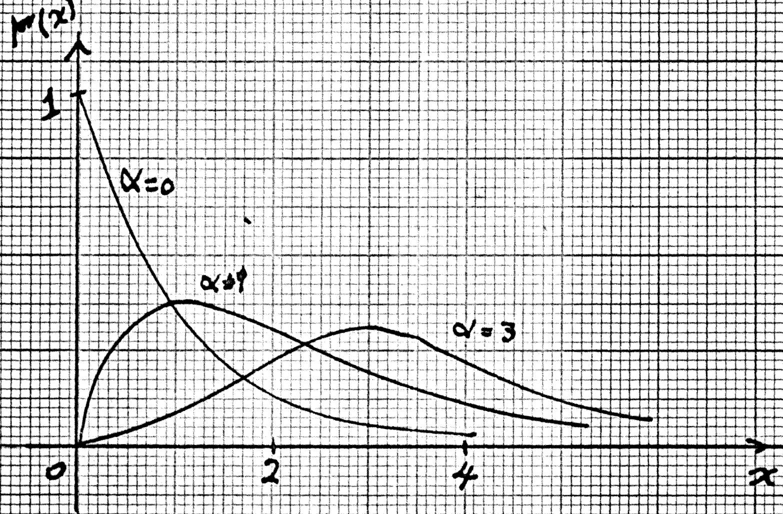

Fig. 6. Gamma-Distribution Density Function

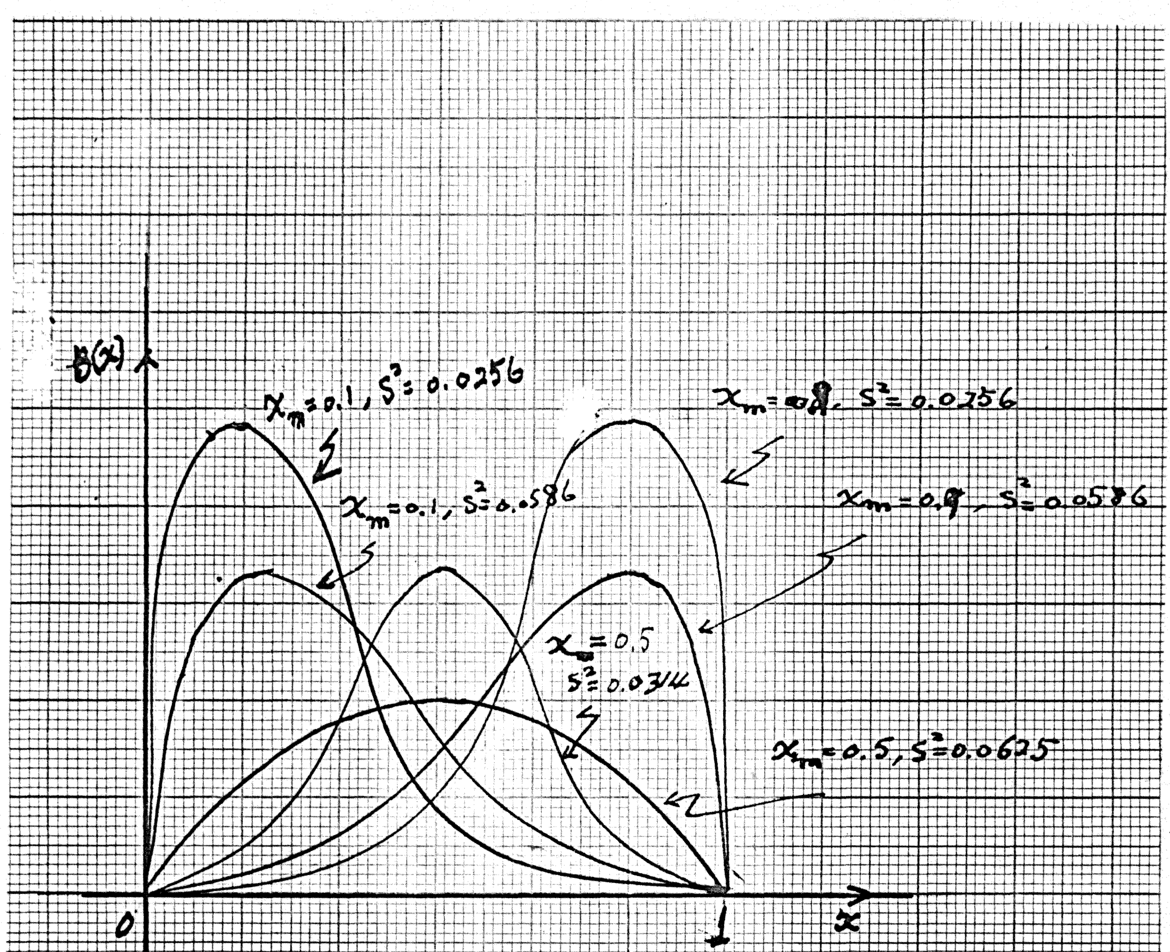

Fig. 7. Beta-Distribution Density Function

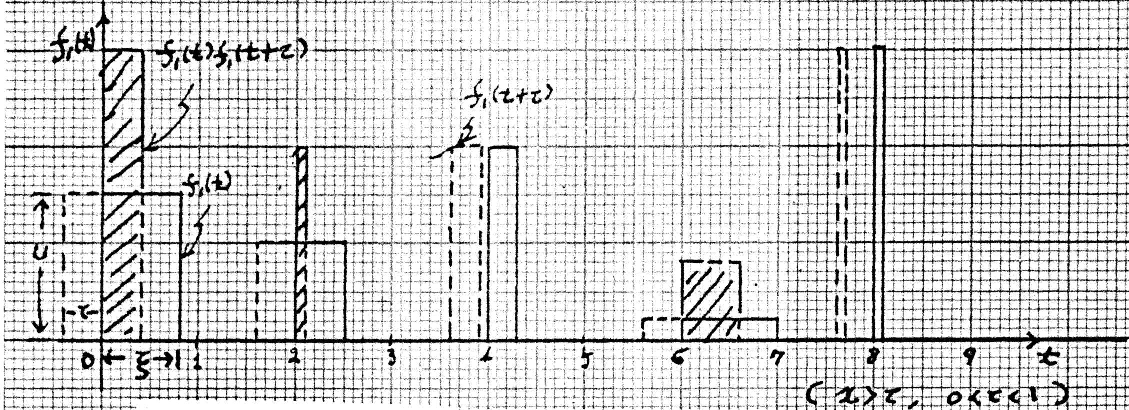

Fig. 8. The Displaced and Folded Randomly Modulated Function with Gamm and Beta-Distribution

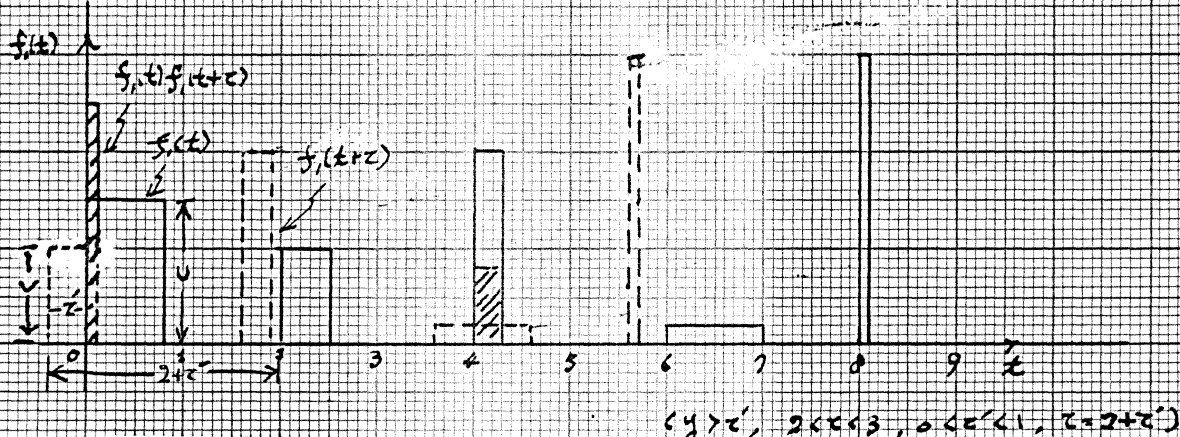

Fig. 9. The Displaced and Folded Random Time Function

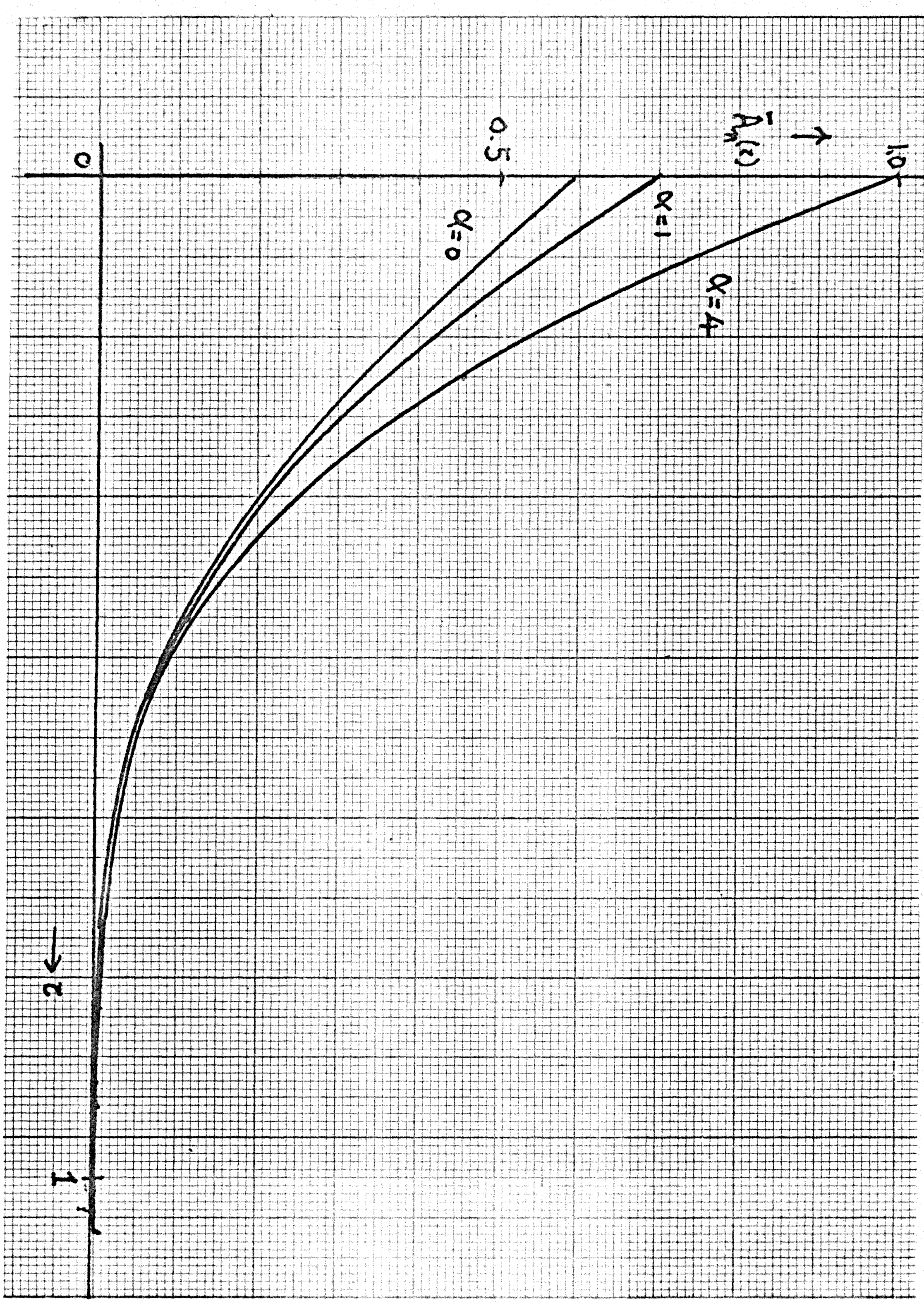

Fig. 10. Autocorrelation for $\alpha = 0, 1, 4$, $\beta = 1$, and $x_m = 0.5$

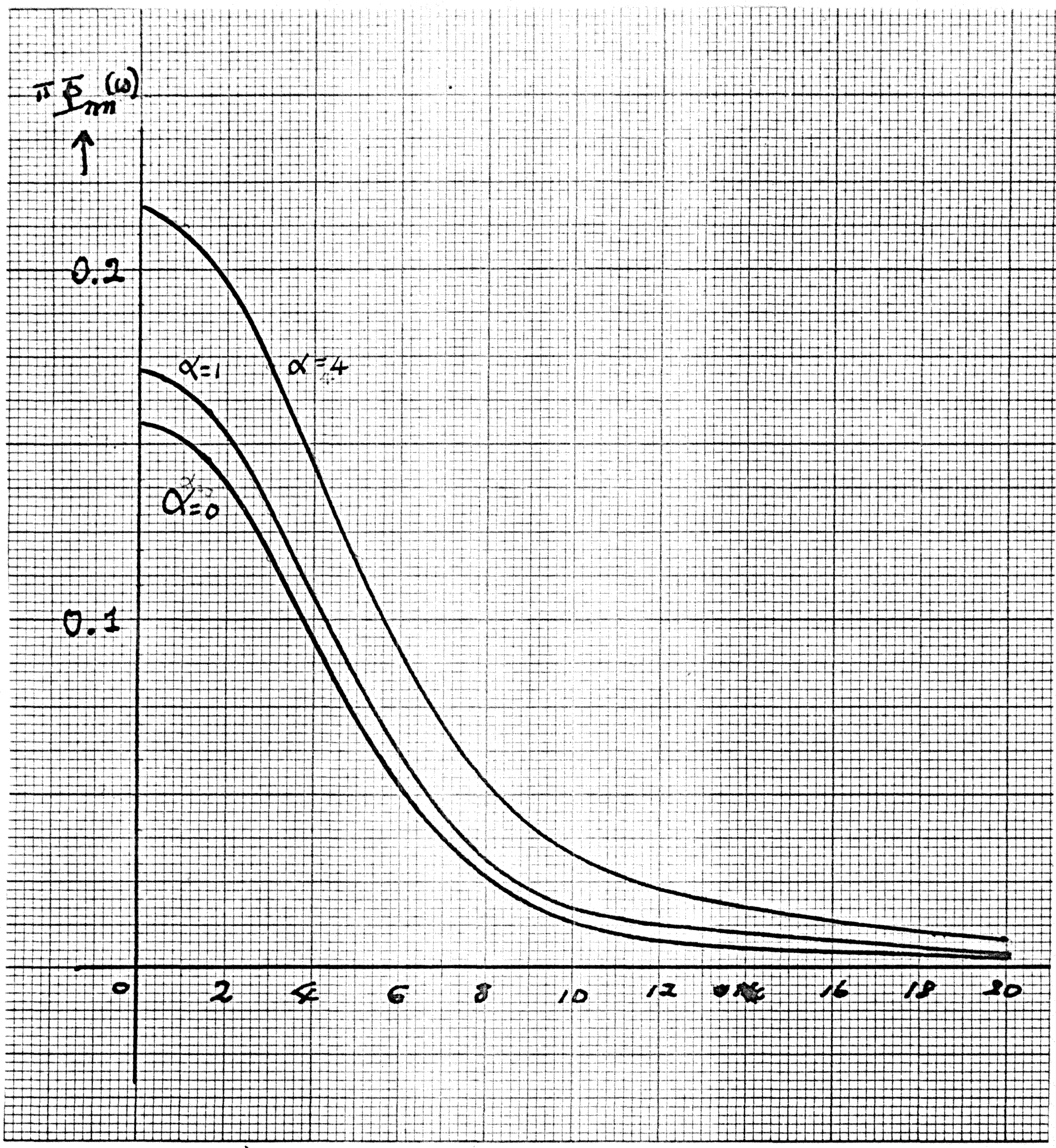

Fig. 11. Noise Power Density Spectrum for Fig. 10

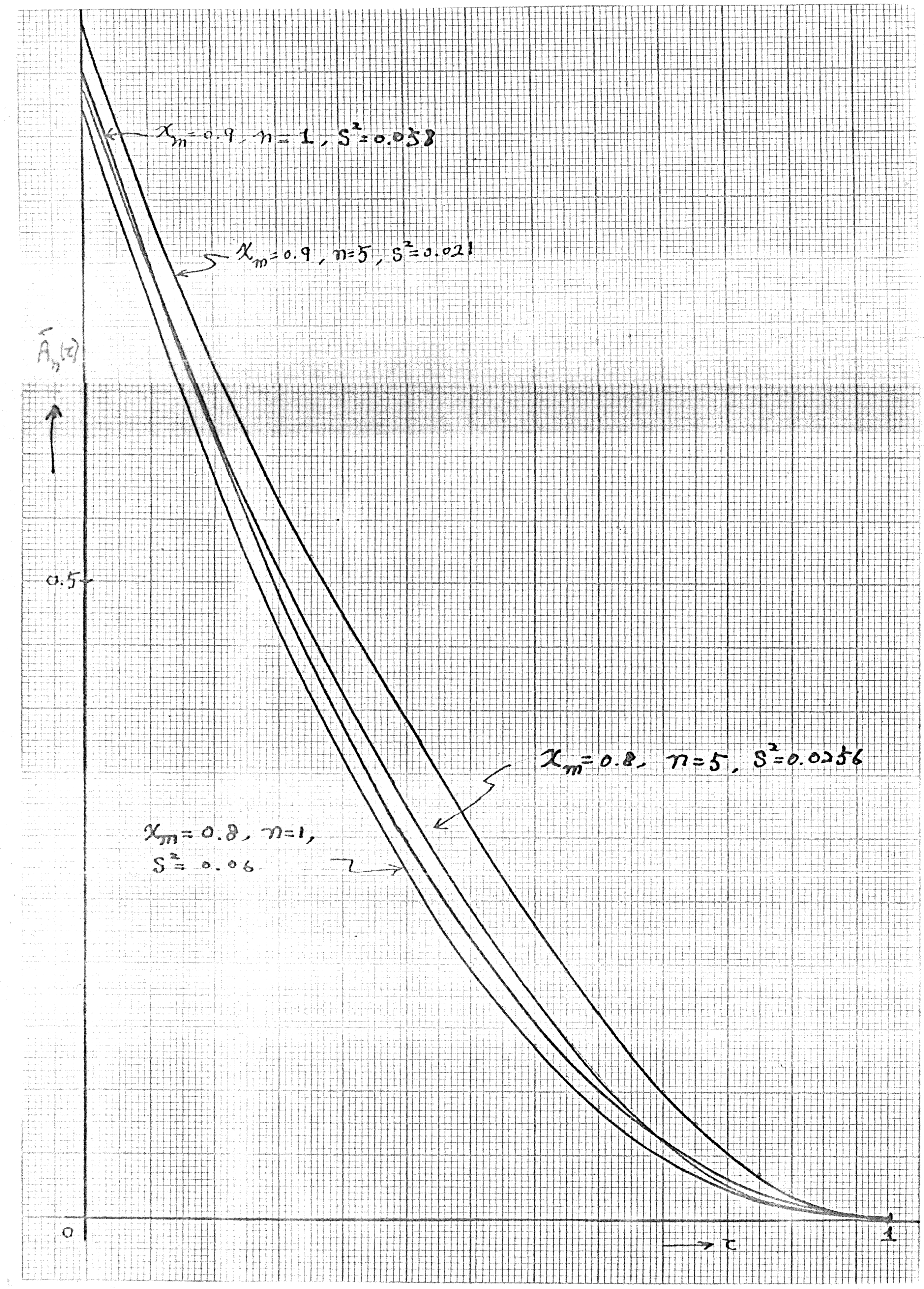

Fig. 16. Autocorrelation for x_m = 0.8 and 0.9

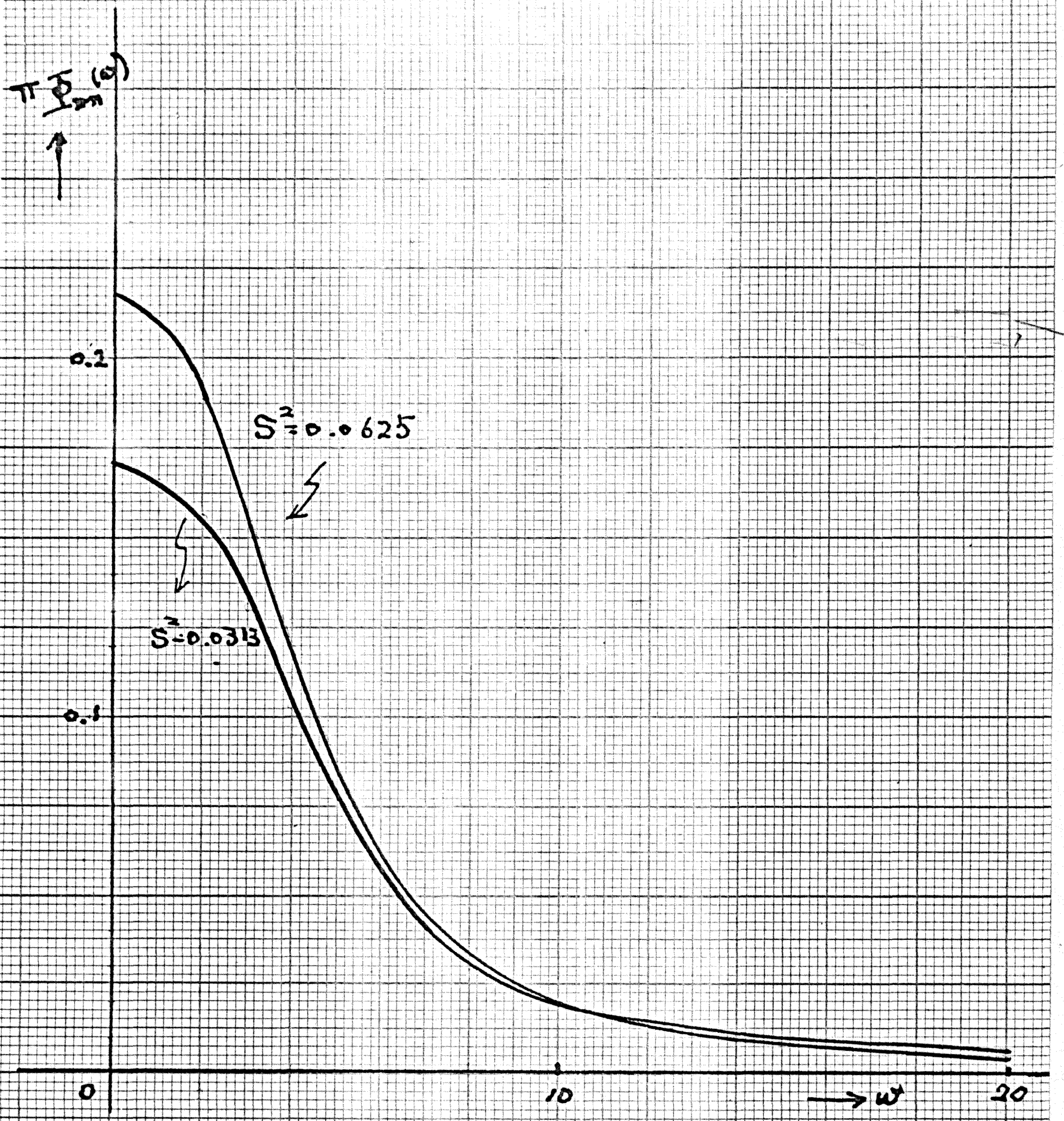

Fig. 13. Noise Power Density Spectrum for Fig. 12

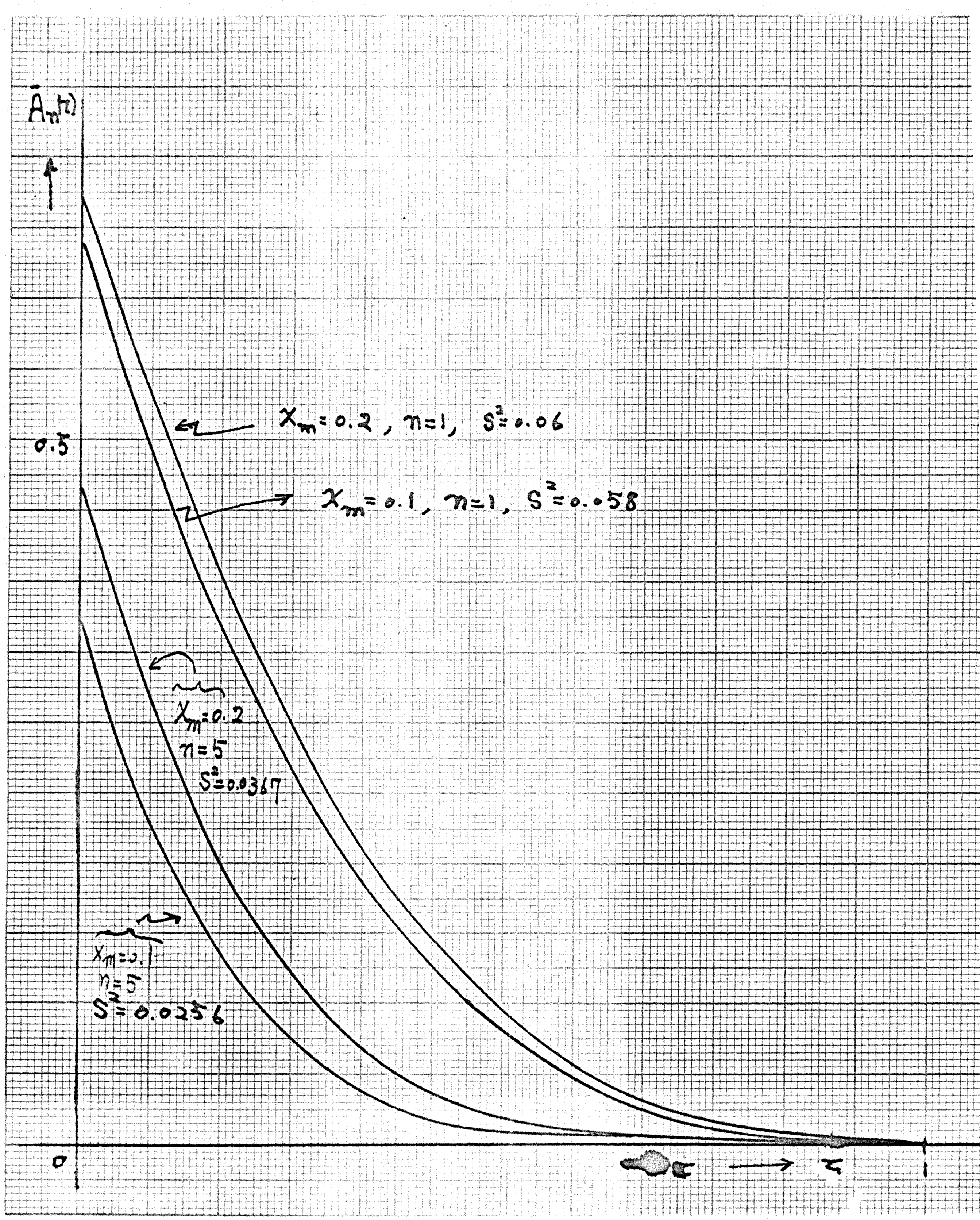

Fig. 14. Autocorrelation for x_m = 0.1 and 0.2

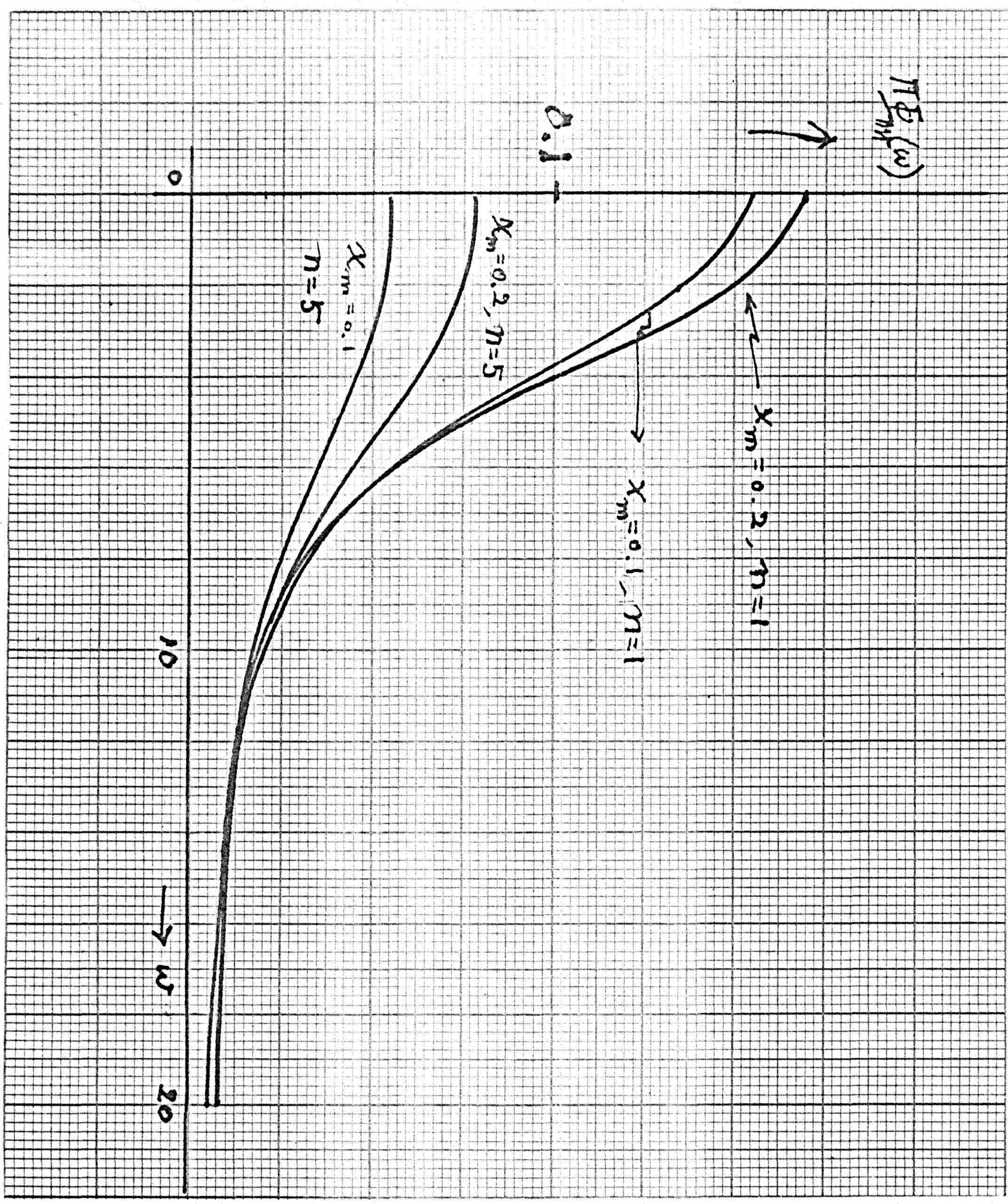

Fig. 15. Noise Power Density Spectrum for Fig. 14

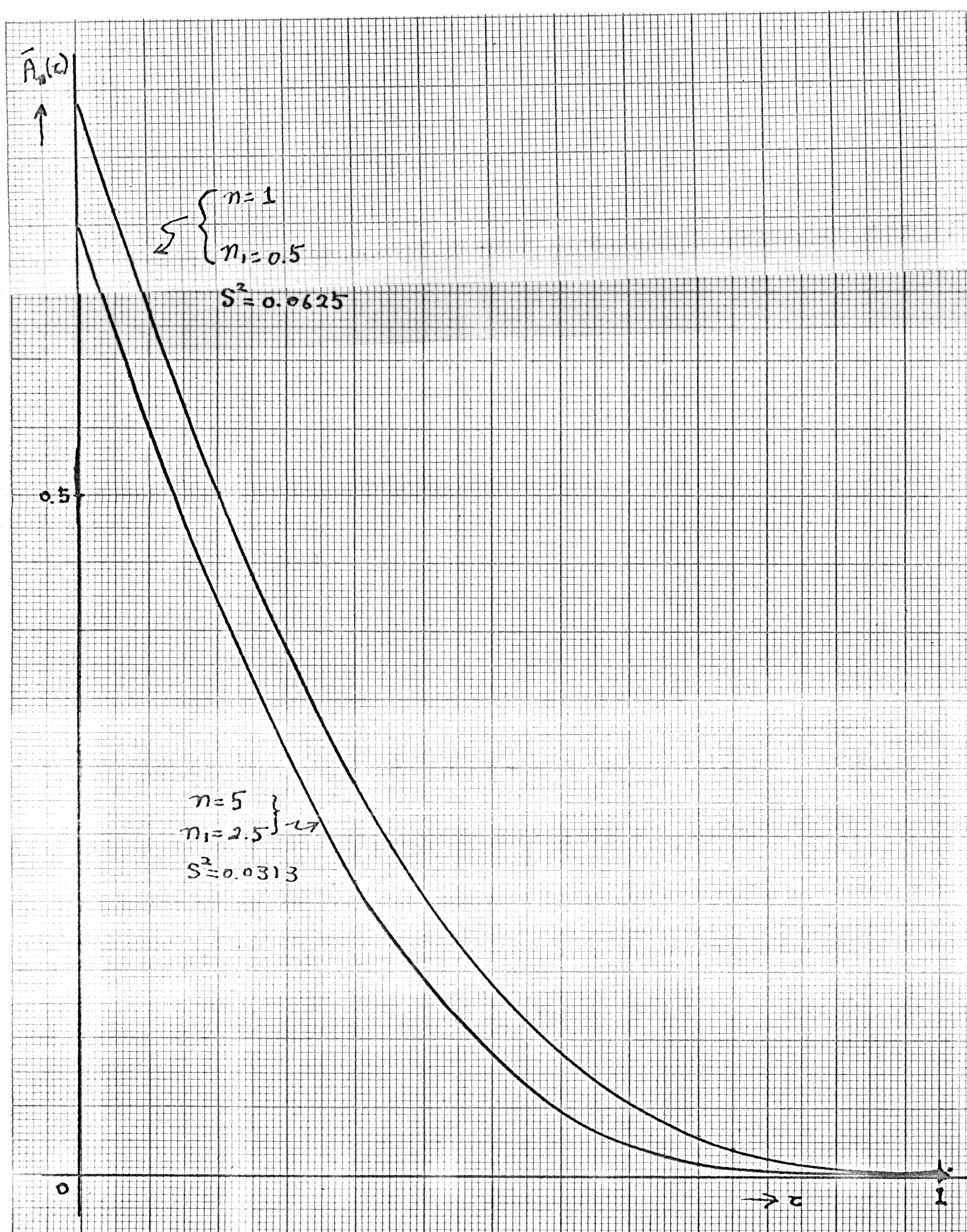

Fig. 12. Autocorrelation for $x_m = 0.5$

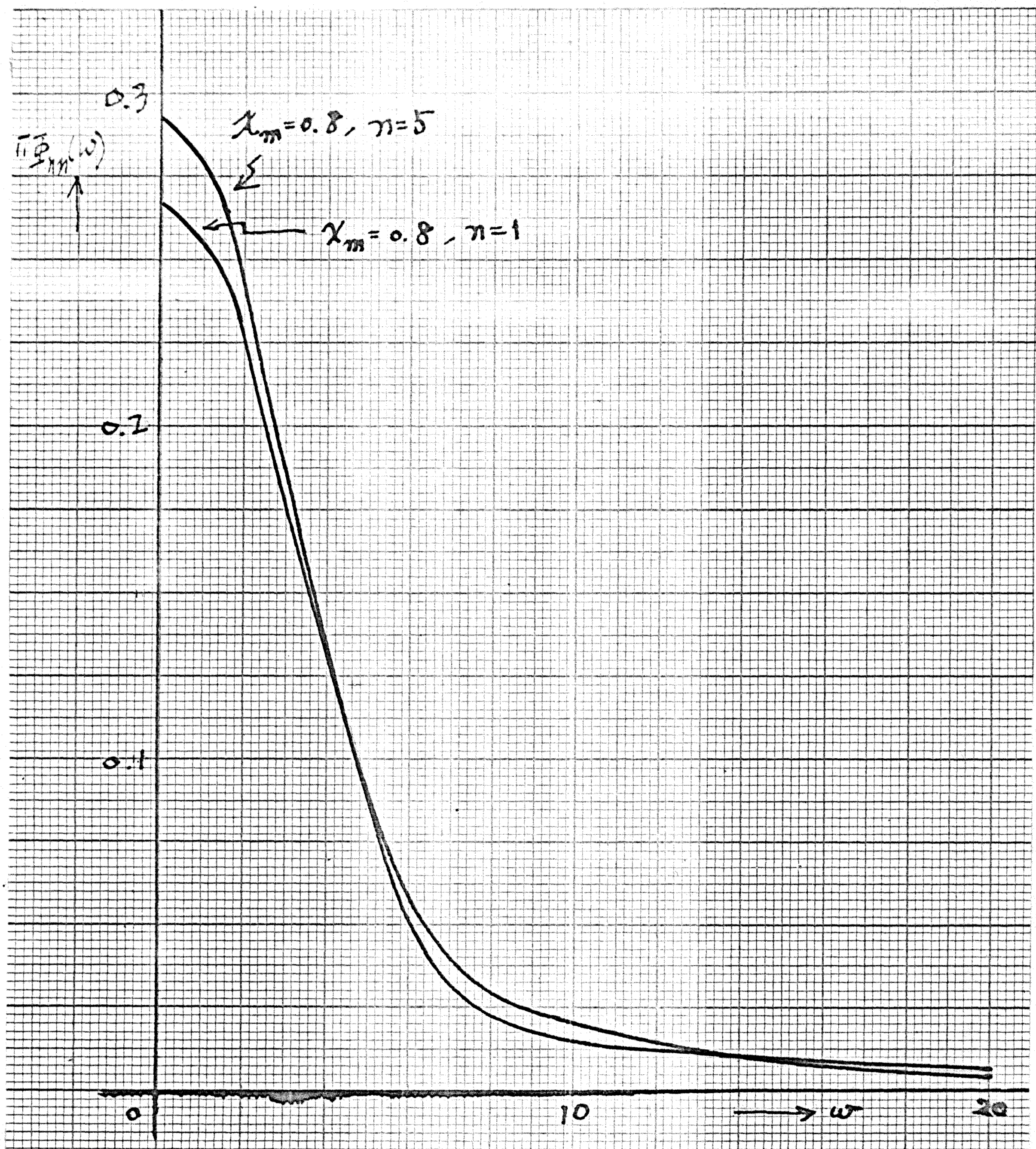

Fig. 17. Noise Power Density Spectrum for $x_m = 0.8$

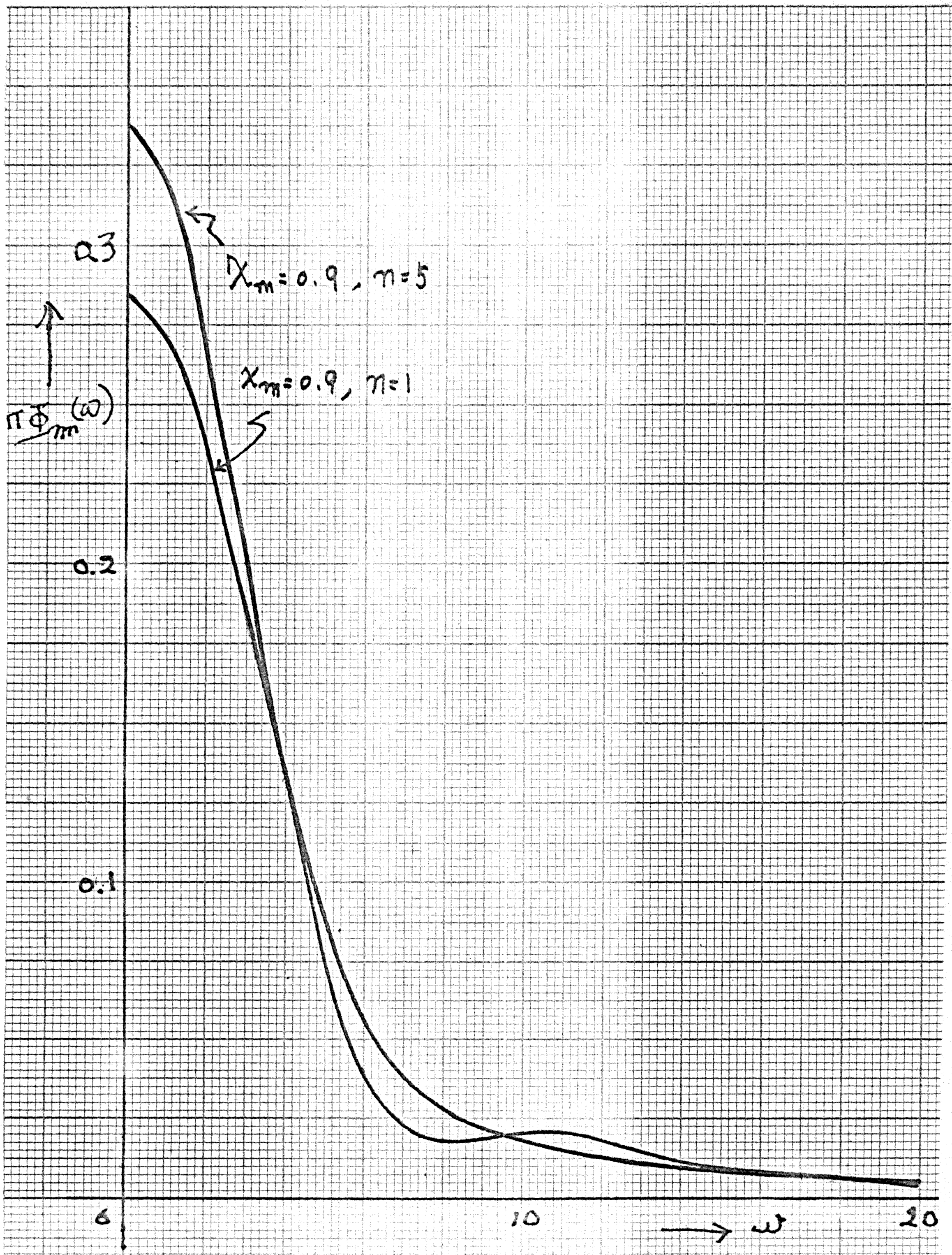

Fig. 18. Noise Power Density Spectrum for x = 0.9

VI. BIBLIOGRAPHY

Lee, Y. W. (1960), Statistical Theory of Communication, J. Wiley and Sons, Inc., New York.

Lee, Y. W. (1954), On Wiener Filters and Predictors, Proceedings of the Symposium on Information Theory, New York, N. Y., Polytechnic Press of Polytechnic Institute of Brooklyn, Brooklyn, N. Y.

Pearson, Karl (1934), Tables of Incomplete Beta-Function, Cambridge University Press, London.

Rhee, M. Y. (1965), Statistical Determination of Correlation Functions and Power Density Spectra, Technical Report, Engineering Experiment Station, Virginia Polytechnic Institute, Blacksburg, Virginia.

Richard Von Mixes (1964), Mathematical Theory of Probability and Statistics, Academic Press, New York, N. Y.

Wiener, Norbert (1949), Extrapolation, Interpolation, and Smoothing of Stationary Time Series, M. I. T. Press, John Wiley & Sons, Inc., N. Y.

Wiener, Norbert (1958), Nonlinear Problems in Random Theory, M. I. T. Press and J. Wiley & Sons, Inc., N. Y.

Wilk, Samuel S. (1962), Mathematical Statistics, J. Wiley & Sons, Inc., N. Y.

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APPENDIX A

Fortran Program of Integral

```
      Dimension A(10), B(10), C(10), U(50), AL(3)
      Read (5,1) M,N
 1    Format (2I3)
      Read (5,2) (A(I), I = 1,M), (B(I), I=1,M), (C(I), I=1,M)
 2    Format (8F10.0)
      Read (5,2) (AL(I), I=1,N)
      DO 10 I=1,N
      ALPH = AL(I)
      DO 20 J=1,M
      AA=A(J)
      BB=B(J)
      CC=C(J)
      D=BB-CC
      Write (6,3) ALPH,BB,CC,AA
 3    Format (1H1,1X,6H ALPH=,F10.3,10X,3H N=,F10.3,10X,4H
      N1=,F10.3, 110X,3H A=,F10.3)
      T1=0.
      DO 30 I1=1,25
      Y=T1
      V2=0.
      V3=0.
 25   Y=Y+0.02
      K=1
      GO TO 14
 22   V2=V2+4.*Q1
      Y=Y+0.02
      EY=1.-Y
      IF(ABS(EY)-0.0001) 26,26,27
 27   K=2
      GO TO 14
 23   V3=V3+2.*Q1
      GO TO 25
 26   K=3
      GO TO 14
 28   V4=Q1
      IF(ABS(T1)-0.96) 29,24,24
```

```
24   UU=0.02*(V2+V4)*AA/3.
     GO TO 50

29   UU=0.02*(V2+V3+V4)*AA/3.
     GO TO 50

14   Z=Y-T1
     G1=0.
     G2=0.
     H1=0.
     H2=0.
     X=0.

15   X=X+0.01
     A2=((1.-X)**D)*(X**CC)
     H1=H1+4.*A2
     G1=G1+4.*X*A2
     X=X+0.01
     EX=Z-X
     IF(ABS(EX)-0.0001) 16,16,17

17   A3=((1.-X)**D)*(X**CC)
     H2=H2+2.*A3
     G2=G2+2.*X*A3
     GO TO 15

16   A4=((1.-X)**D)*(X**CC)
     B4=X*A4
     IF(ABS(Z)-0.02) 18,18,19

18   H=0.01*(H1+A4)/3.
     G=0.01*(G1+B4)/3.
     GO TO 60

19   H=0.01*(H1+H2+A4)/3.
     Write (6,51) Z,H

51   Format (1X,3H Z=,E15.8,10X,3H H=,E15.8)
     G=0.01*(G1+G2+B4)/3.

60   H4=Z*H-G
     Q=((1.-Y)**D)*(Y**CC)
     Q1=Z*Q+(ALPH+1.)*Q*H4*AA
     GO TO (22,23,28),K

50   U(I1)=UU
     Write (6,4) T1,UU

 4   Format (1X,3H T=,E15.8,10X 3H U=,E15.8)
```

```
30   T1=T1+0.04
     W=0.
     DO 40 I2=1,101
     T=0.
     I3=1
     S2=0.
     S3=0.
     S1=COS(W*T)*U(I3)

35   T=T+0.04
     I3=I3+1
     S2=S2+4.*COS(W*T)*U(I3)
     T=T+0.04
     I3=I3+1
     ET=0.96-T
     IF (ABS(ET)-0.0001) 36,36,37

37   S3=S3+2.*COS(W*T)*U(I3)
     GO TO 35

36   S4=COS(W*T)*U(I3)
     F=0.04*(S1+S2+S3+3.*S4)/3.
     Write (6,5) W,F

 5   Format (1X,3H W=,E15.8,10X,3H F=,E15.8)

40   W=W+0.2

20   Continue

10   Continue

     Stop

     End
```

ABSTRACT

The Wiener theory of the minimum mean square error criterion is well furnished by knowing the autocorrelation function of the imput to the linear system. This input signal is generally an additive mixture of a piecewise continuous message and a noise.

The problem considered in this paper is the determination of the autocorrelation function and also their power density spectrum of the noise component for the random base and height modulated square wave whose leading edges are periodic functions of time.

We note that the adopted probability density function for heights of random square wave have Gamma-Distribution Density Function. In addition to this distribution function, we consider the rectangular and Beta-density function on the base of square waves. In fact, the leading edges of most periodic-random base function can be simply described by using the rectangular and Beta-Density function.

Another matter under consideration is the visualization of the variations noise power density spectrum immersed in the masked signal (mixtured signal) with respect to the variance σ^2 and S^2 of Gamma and Beta-distribution, respectively.