Appendix A
Estimation of Parameters in Weibull Proportional Intensity Model

For continuous intensity nonhomogeneous Poisson processes, Cox and Lewis (1966) defined the probability, starting from time t_j, given that there has been no failure from time t_j to time t_{j+1}, of the next failure in the interval (t_{j+1}, $t_{j+1} + \Delta t$) as

$$\lambda(t_{j+1})\exp\left\{-\int_{t_j}^{t_{j+1}}\lambda(u)du\right\}\Delta t + O(\Delta t)$$

where $l(t)$ is the intensity function of the process.

Thus, the total likelihood is expressed as:

$$L = \left\{\prod_{j=1}^{n}\lambda(t_j)\exp\left[-\int_{t_{j-1}}^{t_j}\lambda(u)du\right]\right\}\exp\left\{-\int_{t_n}^{t_{n+1}}\lambda(u)du\right\} \tag{A.1}$$

where t_j is the jth failure. The last term in this expression represents the likelihood of no failure from the time of the last repair, at time t_n, until censoring time t_{n+1}. Therefore, the likelihood function is expressed as

$$L = \prod_{j=1}^{n}\lambda(t_j)\exp\left[-\sum_{j=1}^{n+1}\int_{t_{j-1}}^{t_j}\lambda(u)du\right] \tag{A.2}$$

Considering the structure of proportional intensity model the likelihood becomes

$$L = \left[\prod_{j=1}^{n}\lambda_0(t_j)\right]\exp(\gamma^T z_{sum})\times\exp\left[-\sum_{j=1}^{n+1}\exp(\gamma^T z_j)\int_{t_{j-1}}^{t_j}\lambda_0(u)du\right] \tag{A.3}$$

where

$$z_{sum} = \sum_{j=1}^{n} z_j$$

Thus, the log-likelihood function is

$$l = \sum_{j=1}^{n}\log(\lambda_0(t_j)) + \gamma^T z_{sum} - \sum_{j=1}^{n+1}\exp(\gamma^T z_j)\int_{t_{j-1}}^{t_j}\lambda_0(u)du \tag{A.4}$$

Notice that

$$\int_a^b \lambda_0(u)du = \left(\frac{b}{\eta}\right)^{\beta} - \left(\frac{a}{\eta}\right)^{\beta}$$

and denote

$$w_j = \int_{t_{j-1}}^{t_j}\lambda_0(u)du = \left(\frac{t_j}{\eta}\right)^{\beta} - \left(\frac{t_{j-1}}{\eta}\right)^{\beta}$$

Using the relationship above, we obtain

$$l(\beta,\eta,\gamma,\delta) = n\log\left(\frac{\beta}{\eta}\right) + (\beta-1)\sum_{j=1}^{n}\log\left(\frac{t_j}{\eta}\right) + \gamma^T z_{sum} - \sum_{j=1}^{n+1}\exp(\gamma^T z_j) w_j \qquad \text{(A.5)}$$

where n is the total number of failures observed.

Optimal values of β, η and the covariate coefficients **γ** can be found using the Newton-Raphson algorithm, in which the (m+1)th iteration is given by

$$\begin{bmatrix}\beta^{(m+1)}\\ \eta^{(m+1)}\\ \gamma^{(m+1)}\end{bmatrix} = \begin{bmatrix}\beta^{(m)}\\ \eta^{(m)}\\ \gamma^{(m)}\end{bmatrix} - \left[G^{(m)}\right]^{-1} g^{(m)}, \; m = 1,2,\ldots$$

where $G^{(m)}$ represents the matrix of the second-order partial derivatives of $l(\beta,\eta,\gamma)$ with respect to β, η and **γ**. $g^{(m)}$ is the vector of first-order partial derivatives of $l(\beta,\eta,\gamma)$ with respect to β,η and **γ**.

The first-order derivatives of l are, noting

$$\frac{d}{d\chi}(\alpha^{\chi}) = \alpha^{\chi}\log(\alpha)$$

as follows:

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{j=1}^{n}\log\left(\frac{t_j}{\eta}\right) - \sum_{j=1}^{n}\exp(\gamma^T z_{sum}) \left[\left(\frac{t_j}{\eta}\right)^{\beta}\log\left(\frac{t_j}{\eta}\right) - \left(\frac{t_j}{\eta}\right)^{\beta}\log\left(\frac{t_j}{\eta}\right)\right]$$

$$\frac{\partial l}{\partial \eta} = -\frac{\beta}{\eta}\left\{n - \sum_{j=1}^{n+1}\exp(\gamma^T z_{sum})\left[\left(\frac{t_j}{\eta}\right)^{\beta} - \left(\frac{t_{j-1}}{\eta}\right)^{\beta}\right]\right\}$$

$$\frac{\partial l}{\partial \gamma_k} = z_{sum_k} - \sum_{j=1}^{n+1} z_{jk}\exp(\gamma^T z_{sum})\left[\left(\frac{t_j}{\eta}\right)^{\beta} - \left(\frac{t_{j-1}}{\eta}\right)^{\beta}\right]$$

k = 1, 2, … , M

where z_{jk} is the value of the *k*th covariate observed at *j*th failure.

The second-order derivatives are derived as follows:

$$\frac{\partial^2 l}{\partial \beta^2} = -\frac{n}{\beta^2} - \sum_{j=1}^{n+1}\exp(\gamma^T z_{sum})\left[\left(\frac{t_j}{\eta}\right)^{\beta}\log\left(\frac{t_j}{\eta}\right)^{2} - \left(\frac{t_{j-1}}{\eta}\right)^{\beta}\log\left(\frac{t_{j-1}}{\eta}\right)^{2}\right]$$

$$\frac{\partial^2 l}{\partial\beta\partial\eta} = -\frac{1}{\eta}\left\{ n - \sum_{j=1}^{n+1}\exp(\gamma^T z_j)\left[\left(\frac{t_j}{\eta}\right)^\beta - \left(\frac{t_{j-1}}{\eta}\right)^\beta\right]\right\}$$

$$+\frac{\beta}{\eta}\sum_{j=1}^{n+1}\exp(\gamma^T z_j)\left[\left(\frac{t_j}{\eta}\right)^\beta \log\left(\frac{t_j}{\eta}\right) - \left(\frac{t_{j-1}}{\eta}\right)^\beta \log\left(\frac{t_{j-1}}{\eta}\right)\right]$$

$$\frac{\partial^2 l}{\partial\beta\partial\gamma_k} = -\sum_{j=1}^{n+1} z_{jk}\exp(\gamma^T z_j)$$

$$\left[\left(\frac{t_j}{\eta}\right)^\beta \log\left(\frac{t_j}{\eta}\right) - \left(\frac{t_{j-1}}{\eta}\right)^\beta \log\left(\frac{t_{j-1}}{\eta}\right)\right]$$

k = 1, 2, … , M

$$\frac{\partial^2 l}{\partial\eta^2} = \frac{\beta}{\eta^2} n - \frac{\beta(\beta+1)}{\eta^2}\sum_{j=1}^{n+1}\exp(\gamma^T z_j)\left[\left(\frac{t_j}{\eta}\right)^\beta - \left(\frac{t_{j-1}}{\eta}\right)^\beta\right]$$

$$\frac{\partial^2 l}{\partial\eta\partial\gamma_k} = \frac{\beta}{\eta}\sum_{j=1}^{n+1} z_{jk}\exp(\gamma^T z_j)\left[\left(\frac{t_j}{\eta}\right)^\beta - \left(\frac{t_{j-1}}{\eta}\right)^\beta\right],$$

k = 1, 2, … , M

$$\frac{\partial^2 l}{\partial\gamma_{k_1}\partial\gamma_{k_2}} = \sum_{j=1}^{n} z_{jk_1} z_{jk_2}\exp(\gamma^T z_j)\left[\left(\frac{t_j}{\eta}\right)^\beta - \left(\frac{t_{j-1}}{\eta}\right)^\beta\right],$$

k_1, k_2 *= 1, 2, … , M*