The Reflected Quasipotential: Characterization and Exploration

Kasie G. Farlow

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Martin V. Day, Chair
Peter E. Haskell
Shu-Ming Sun
Joseph A. Ball

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Kasie G. Farlow

(Abstract)

The Reflected Quasipotential $V(x)$ is the solution to a variational problem that arises in the study of reflective Brownian motion. Specifically, the stationary distributions of reflected Brownian motion satisfy a large deviation principle (with respect to a spatial scaling parameter) with $V(x)$ as the rate function. The Skorokhod Problem is an essential device in the construction and analysis of reflected Brownian motion and our value function $V(x)$. Here we characterize $V(x)$ as a solution to a partial differential equation $H(DV(x)) = 0$ in the positive n-dimensional orthant with appropriate boundary conditions. $H(p)$ is the Hamiltonian and $DV(x)$ is the gradient of $V(x)$. $V(x)$ is continuous but not differentiable in general. The characterization will need to be in terms of viscosity solutions. Solutions are not unique, thus additional qualifications will be needed for uniqueness. In order to prove our uniqueness result we consider a discounted version of $V(x)$ in a truncated region and pass to the limit. In addition to this characterization of $V(x)$ we explore the possibility of cyclic optimal paths in 3 dimensions.

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Chapter 1

Introduction

The Reflected Quasipotential is the solution to a variational problem that arises in the large deviation analysis of reflected Brownian motion. A reflected Brownian motion is a diffusion process which acts like an n-dimensional Brownian motion in the interior of the positive orthant while at the boundary a constraining force is applied to keep the Brownian motion in the positive orthant. The constraint mechanism is the Skorokhod Problem. The study of reflected Brownian motion is of interest because it appears in heavy traffic approximation of queueing networks; see [3], [23] and [16]. As the result of a large deviation theorem, the Reflected Quasipotential describes the exponential decay rate of the tails of the stationary distribution of the reflected Brownian motion. Here we provide a characterization of the Reflected Quasipotential in terms of viscosity solutions and explore the possibility of cyclic optimal paths in 3 dimensions.

The variational problem of interest is associated with parameters (b, A, D) (which will be defined later) and involves taking the infimum over functions constrained by the Skorokhod Problem. To motivate the study of the Reflected Quasipotential we will introduce the Skorokhod Problem and discuss how reflected Brownian motion and large deviation theory are related to the study of queueing networks. In Chapter 2, before defining the Reflected Quasipotential, we focus on the Skorokhod Problem in more detail and discuss the various conditions for existence and Lipschitz continuity of solutions to the Skorokhod Problem.

Our main task, which will be described in Chapter 4, is to identify $V(x)$ as a solution to a partial differential equation,

$$H(DV(x)) = 0$$

in the positive n-dimensional orthant,

$$\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_i \geq 0 \text{ all } i\}, \tag{1.1}$$

with appropriate boundary conditions. Here $H(p)$ is the Hamiltonian and $DV(x)$ is the gradient. $V(x)$ may not be differentiable. Thus the characterization will be in terms of viscosity solutions. Crandall and Lions introduced the notion of viscosity solutions as a class of generalized solutions to non-linear partial differential equations; see [9]. From [14] we know that often solutions to problems involving a Hamiltonian without u dependence, $p = Du$, are not unique without additional conditions. We will give 2-dimensional examples of non-uniqueness for our problem. With additional hypotheses we can obtain a uniqueness result in 2 dimensions. In order to do so we will need to consider a discounted version of our problem in a truncated domain. We then pass to the limit to obtain a uniqueness result for $V(x)$.

The simplicity of the geometry in 2 dimensions allows solutions to the variational problem and the associated minimizing trajectories to be calculated explicitly. In [3] Avram, Dai, and Hasenbein provide explicit analytical solutions to the variational problem in 2 dimensions. In higher dimensions things become more complicated. The literature mentions the possibility that in dimensions greater then two, minimizing trajectories (optimal paths) may cycle; see [16]. In Chapter 6 we will discuss a 3-dimensional search for optimal paths that cycle, or spiral, along the faces of the orthant. Liang and Hasenbein in [24] have explored the idea of cyclic optimal paths as well. They have shown that a specific type of cyclic path, known as an exotic spiral, is not optimal.

1.1 Motivation

1.1.1 Skorokhod Problem and Queueing

For a $\psi(t) \in \mathbb{R}^n$ that is right continuous and has limits from the left the Skorokhod Problem defines a constrained version $\phi(t)$ of $\psi(t)$. Associated with the Skorokhod Problem is a $n \times n$ constraint matrix D which has columns d_i. Starting from $\phi(0) = \psi(0)$ the constraint mechanism of the Skorokhod Problem acts along directions $d_i \in \mathbb{R}^n$, when $\phi_i(t) = 0$ to keep $\phi(t)$ in $\Omega \subset \mathbb{R}^n$. The vectors d_i are often called "directions of reflection" in the literature. We will restrict to the case where ψ and hence ϕ are absolutely continuous functions. In that case [15] gave the following heuristic description of the Skorokhod Problem. When $\phi(t)$ is in interior of Ω then ϕ behaves the same as ψ with $\dot{\phi}(t) = \dot{\psi}(t)$. If $\phi(t)$ is on the boundary of Ω then there are two possibilities. If the use of $\dot{\phi} = \dot{\psi}$ would allow ϕ to exit Ω then a correction term, η_D, is added so that $\dot{\phi}(t) = \dot{\psi}(t) + \dot{\eta}_D(t)$. η_D is refereed to as the "minimal push" needed to keep ϕ in Ω. Otherwise if the use of $\dot{\phi} = \dot{\psi}$ does not result in ϕ leaving Ω, no correction term is needed. When there is a unique solution, the Skorokhod Problem defines a deterministic mapping $\psi \mapsto \phi(t) = \Gamma(\psi(t))$. $\phi = \Gamma(\psi)$ is known as the Skorokhod Map. A precise description of the Skorokhod Problem will be given in Chapter 2.

Queueing networks are used as models in many important application areas such as manufacturing, telecommunications, computer networks and vehicle traffic. The behavior of a queue or network of queues is dependent on the ratio of service time to interarrival time, known as the traffic intensity. When the traffic intensity is less then 1, a queue attains an equilibrium as $t \to \infty$. In other words the stochastic process describing the queueing network has a stationary distribution. When the traffic intensity is greater then 1 the length of a queue grows. The term "heavy traffic" refers to situations in which the traffic intensity is less then but close to 1. See [21] for more details. The stochastic processes describing a queueing system typically move by discontinuous jumps occurring at random times. The rate functions are discontinuous when the queue is empty, making the stochastic model more difficult to analyze. The Skorokhod Problem provides a way to model such a system and deal with the discontinuities. In approximating queueing models, the boundaries are typically defined by the non-negative constraint on the queue size. If one were to pretend that the interior dynamics were still applicable (service still possible), and then correct for these "false" services, the resulting correction terms define the corresponding direction of constraint in the Skorokhod Problem. The idea is to construct a process $Y(t)$ with constant jump rates so that the desired process $X(t)$ is a solution to the Skorokhod Problem for $Y(t)$, $X(t) = \Gamma(Y(t))$. By analyzing $Y(t)$ we can obtain information about $X(t)$. Hence the Skorokhod Problem provides a convenient approach for the mathematical analysis of queueing systems and is used in the construction of the stochastic process which approximates a queueing system; see [17] and [2] for more details.

1.1.2 (b, A, D) Diffusion

Diffusion approximations have been an effective tool for simplifying or approximating physical or biological problems. Likewise, reflected diffusion processes are used to approximate queueing networks because they often simplify queueing network problems. Under certain conditions, one being heavy traffic, the stochastic process for the queue length converges in distribution to reflected Brownian motion; see [10], [23] and [16] for more details. Before defining reflected Brownian motion we state the following definitions, which can be found in [27] and [21]. The reader can refer to [27] and [21] for more details.

Definition 1. *A stochastic process is a parameterized collection of random variables* $\{X_t\}_{t\in T}$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$*, assuming values in* R^n*.*

T usually denotes $[0, \infty)$ and is often thought of as "time". For each fixed $t \in T$, $\omega \in \Omega$, $\omega \to X_t(\omega)$ is a random variable. For fixed $\omega \in \Omega$ $t \to X_t(\omega)$, is a sample path of X_t.

Definition 2. *A diffusion process is a stochastic process having the Strong Markov Property and whose sample paths are almost surely continuous.*

Definition 3. *Brownian Motion, $\{W_t\}$ is a real valued, Gaussian process with mean 0, independent increments and sample paths which are almost surely continuous.*

This process is known as standard Brownian motion if $W(0) = 0$ and $W(1)$ has covariance I. From [16] we have the following definition of reflected Brownian motion, expressed as a solution to a Skorokhod Problem.

Definition 4. *For given constant drift b, covariance matrix A and a constraint matrix D satisfying certain regularity conditions, $X = \Gamma(x_0 + bt + \sigma W_0)$ is a (b, A, D) reflected Brownian motion, where $\sigma\sigma^T = A$ and W_0 denotes standard Brownian motion.*

In the n-dimensional orthant $\mathbb{R}^n_+$, X behaves like an n-dimensional Brownian motion with covariance matrix A on the interior and is constrained at the boundary by being pushed along directions d_i; [16]. Reflected Brownian motions are often used to model the relation between arrivals and workloads in queueing systems or to approximate this relation in heavy traffic; see [10], [16] and [26]. Stationary distributions of these approximating reflected Brownian motion have been focused on in much of the literature because they serve as estimates for stationary distributions of the corresponding queueing networks. We will discuss the importance of the stationary distributions below. Next we will discuss the standard Quasipotential and see how the Reflected Quasipotential $V(x)$ comes into play in this situation.

1.1.3 Wentzell and Freidlin's Quasipotential

Wentzell and Freidlin's Quasipotential, $\overline{V}(x)$, is defined to be

$$\overline{V}(x) = \inf_{\phi \in C[-T,0], -T<0, \phi(-T)=0, \phi(0)=x} \int_{-T}^{0} L(\phi, \dot{\phi})dt.$$

ϕ is absolutely continuous and $L(x, v) = \frac{1}{2}\langle b(x) - v, A^{-1}(b(x) - v)\rangle$ is the Lagrangian. $\overline{V}(x)$, is a key component in the analysis of asymptotically small random perturbations of dynamical systems. In [20] they consider the Gaussian perturbation

$$dX^\epsilon(t) = b(X^\epsilon(t))dt + \epsilon\sigma(X^\epsilon(t))dW(t), \text{ (Itô sense)} \tag{1.2}$$

of the deterministic system

$$\dot{x} = b(x), \tag{1.3}$$

where $A = \sigma\sigma^T$ and $W(t)$ is Brownian motion. $\overline{V}(x)$ is the quasipotential for the deterministic system above. If b has potential, $b(x) = -\nabla U(x)$, where $U(x)$ is a continuously differentiable function, then $\overline{V}(x)$ differs from $U(x)$ by a multiplicative constant. Hence the name Quasipotential. From [20] it is well know that $\overline{V}(x)$ is Lipschitz. [20] states that it is easy to find examples such that $\overline{V}(x)$ is not differentiable. In [11] it was shown that $\overline{V}(x)$ is continuously differentiable in a neighborhood of the equilibrium point of the deterministic system above. We will soon see that the definition of our $V(x)$ is similar to that of the standard Quasipotential. However $V(x)$ involves taking the infimum over functions constrained by the Skorokhod Problem, which is why we call $V(x)$ the Reflected Quasipotential. The following theorem, (Theorem 4.3 from [20]) establishes a large deviation theorem which says that the behavior of the stationary distribution of X^ϵ as $\epsilon \to 0$ can be described in terms of $\overline{V}(x)$.

Theorem 1. *Let the point* 0 *be a unique stable equilibrium point of (1.3) and let the whole space* $\mathbb{R}^n$ *be attracted to* 0*. Furthermore assume that* X^ϵ *satisfies a stability condition that guarantees the existence of a stationary distribution. Then the process* X^ϵ *has a unique stationary distribution* μ^ϵ *for every* $\epsilon > 0$ *and we have*

$$\lim_{\epsilon\to 0} \epsilon^2 \ln(\mu^\epsilon(C)) = -\inf_{x\in C} \overline{V}(x) \tag{1.4}$$

for any compact C *which is the closure of its interior.*

1.1.4 Large Deviations and the Reflected Quasipotential

Large deviation theory is concerned with the asymptotic behavior of probability distributions and provides estimates for the probability of rare events. A rate function is used to quantify this probability. Theorem 1 above is a typical large deviations result, with $\bar{V}$ as the rate function. We now give the definition of a large deviation principle from Varadhan, which can be found in [30] and [16].

Definition 5. *A sequence of probability measures μ_n defined on a complete separable metric space $(\mathcal{X}, \mathcal{B})$ is said to satisfy the large deviation principle with rate function J, where $J : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is function with compact level sets, if for all $C \in \mathcal{B}$*

$$\limsup_{n\to\infty} \frac{1}{n} \log(\mu_n(C)) \leq -\inf_{x\in\bar{C}} J(x)$$

$$\liminf_{n\to\infty} \frac{1}{n} \log(\mu_n(C)) \geq -\inf_{x\in C^o} J(x).$$

$\bar{C}$ and C^o are the closure and interior of C respectively.

As mentioned above the stationary distributions of reflected Brownian motion can be analyzed with the help of a large deviation principle. Under certain conditions assumed on (b, A, D), $V(x)$ describes the exponential decay (tail behavior) of the stationary distributions; see Theorem 2 below. The appropriate stability condition is

$$D^{-1}b < 0; \tag{1.5}$$

[7]. This is a necessary and sufficient condition for the positive recurrence of reflected Brownian motion. If the reflected Brownian motion is positive recurrent then it has a stationary distribution; [16]. The stationary distribution provides information about the probability of rare events such as buffer overflow and large delays in the network. But in general an explicit description of the stationary distribution can be hard to obtain. One often has to resort to numerical approximations. The Reflected Quasipotential $V(x)$ is important because it plays the role of the rate function in a large deviation description of the tail of the stationary distribution. The knowledge of the tail behavior of the stationary distribution can lead to better convergence properties for numerical algorithms for approximating queueing networks proposed by Dai and Harrison; [16] and [3].

To be more precise we quote the following result from [16].

Theorem 2. *Suppose b, A and D satisfy our hypotheses stated in Section 2.2 of the next chapter. Let μ be the stationary distribution of the (b, A, D) reflected Brownian Motion X on the orthant $\mathbb{R}^n_+$ and $\mu_n(B) = \mu(nB)$. Then $\{\mu_n\}$ satisfies the large deviations principle with the rate function $V(x)$.*

Chapter 2

The Skorokhod Problem

The Skorokhod Problem is a useful tool in the study of queueing networks. Furthermore, the Skorokhod Map also arises in the analysis and construction of other constrained deterministic and stochastic processes. Since Skorokhod's introduction of the simplest version of the Skorokhod Problem in [28], it has been used in the analysis of a variety of processes such as stochastic differential equations with reflection (including reflecting Brownian motion), constrained ordinary differential equations, stochastic approximation algorithms and certain queueing and communication models. Skorokhod used Lipschitz continuity of the Skorokhod Map to construct and establish uniqueness of solutions to stochastic differential equations on $\mathbb{R}_+ = [0, \infty)$ with reflection on the origin. When the Skorokhod Map is Lipschitz continuous, the study of many constrained processes is greatly simplified; [17]. The domain of interest in [17] is a convex polyhedron with constant and possibly oblique constraint directions on each face. Here we are concerned with the specific domain $\Omega = \mathbb{R}^n_+$. We give a precise definition of the Skorokhod Problem for Ω below. For a more general definition see [18], [17] and [7]. In our setting, because of the boundary constraints, the Skorokhod Problem and Skorokhod Map are also referred to as the dynamic complementarity problem and reflection map respectively; [16].

2.1 Definition and Conditions

The nonnegative orthant is denoted $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_i \geq 0 \text{ all } i\}$. For $x \in \partial\Omega$ let $I(x) = \{i : x_i = 0\}$. (We will often us $I = I(x)$ when the meaning is clear.) For $J \subseteq \{1, 2, \ldots, n\}$, $\partial_J\Omega = \{x \in \partial\Omega : x_i = 0 \text{ for all } i \in J\}$ denotes the edges and faces of Ω. When $J = \{i\}$ we will use $\partial_i\Omega$ to denote the face where $x_i = 0$.

$$d_D(x) = \{\sum_{i \in I(x)} a_i d_i : a_i \geq 0, |\sum_{i \in I(x)} a_i d_i| = 1\}$$

is a set-valued function that described the set of directions of constraints allowed at each $x \in \partial\Omega$. $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. The Skorokhod Problem depends on the specification of a constraint matrix D, an invertible matrix whose columns are vectors we denote as d_i, $i \in \{1, 2 \ldots n\}$. The vectors d_i are often referred to as reflection vectors or "directions of reflection". We assume the d_i are normalized by $\langle d_i, n_i \rangle = -1$ where n_i are exterior normals $n_i = -e_i$. e_i is the ith standard basis vector for $\mathbb{R}^n$. If $x \in \partial\Omega$ and $I(x) = \{j\}$ then $d_D(x) = \{d_j\}$. Let $\mathcal{C}([0, \infty) : \mathbb{R}^n)$ denote the set of continuous functions mapping $[0, \infty)$ to $\mathbb{R}^n$. Likewise $\mathcal{C}([0, \infty) : \mathbb{R}^n_+)$ denotes the set of continuous functions mapping $[0, \infty)$ to $\mathbb{R}^n_+$. For $\eta_D \in \mathcal{C}([0, \infty) : \mathbb{R}^n)$ and $t \in [0, \infty)$, $|\eta_D|(t)$ is the total variation of η_D on $[0, t]$ with respect to the Euclidean norm.

Now we give the precise definition of the Skorokhod Problem, taken from [16].

Definition 6 (The Skorokhod Problem)**.** *Let $\psi \in \mathcal{C}([0, \infty) : \mathbb{R}^n)$, $\psi(0) \in \mathbb{R}^n_+$ be given. We say that (ϕ, η_D), $\phi \in \mathcal{C}([0, \infty) : \mathbb{R}^n_+)$ solves the Skorokhod Problem for ψ(with respect to $\mathbb{R}^n_+$ and the constraint matrix D) if $\phi(0) = \psi(0)$, and if for all $t \in [0, \infty)$*

1. $\phi(t) = \psi(t) + \eta_D(t)$,

2. $\phi(t) \in \mathbb{R}^n_+$,

3. $|\eta_D|(t) < \infty$,

4. $|\eta_D|(t) = \int_{[0,t]} 1_{\{\phi(s) \in \partial\mathbb{R}^n_+\}} d|\eta_D|(s)$,

5. *there exists a Borel measurable function* $\gamma_D : [0, \infty) \to \mathbb{R}^n$ *such that* $d|\eta_D|$*almost everywhere* $\gamma_D(t) \in d_D(\phi(t))$*, and* $\eta_D(t) = \int_{[0,t]} \gamma_D(s) d|\eta_D|(s)$.

When there is a unique solution to the Skorokhod Problem the mapping Γ defined by $\phi = \Gamma(\psi)$ is called the Skorokhod Map. In general the Skorokhod Problem can be defined in terms of $\psi,\ \phi,\ \eta_D \in \mathcal{D}([0, \infty) : \mathbb{R}^n)$, the set of functions mapping $[0, \infty)$ to $\mathbb{R}^n$ that are right continuous and have limits from the left; see [17] or [18]. Although we stated the definition above in terms of continuous functions we will be working only with absolutely continuous functions below.

2.1.1 Existence and Lipschitz Conditions for the Skorokhod Problem

In [18] Dupuis and Ishii developed conditions that guarantee that solutions to the Skorokhod Problem exist and that the Skorokhod Map $\phi = \Gamma(\psi)$ is Lipschitz continuous. (Dupuis and Ishii use n_i to represent inward normals contrary to our use of n_i for outward normals.) For $x \in \partial\Omega$ in general $n(x)$ denotes the set of outward normals at x. For $I(x) = \{i\}$ we have $n(x) = n_i$. For the definition of $n(x)$ when x is on a corner see [18]. Below we discuss their conditions for existence and Lipschitz continuity of solutions to the Skorokhod Problem.

Dupuis and Ishii's Lipschitz condition

Lipschitz continuity of the Skorokhod Map is proven in [18] by showing the existence of a convex set B which satisfies a set of conditions defined in terms of the Skorokhod Problem data. To our knowledge the existence of B is the weakest known sufficient condition for Lipschitz continuity. According to [18], Lipschitz continuity implies uniqueness of solutions to the Skorokhod Problem; see Theorem 6 below. Dupuis and Ishii's Assumption 2.1, which is a sufficient condition for Lipschitz condition, is stated below. We will refer to it by its given name in [18].

Assumption 2.1: There exists a compact, convex set B with $0 \in B^{\circ}$ (the interior of B), such that if $\nu(z)$ denotes the set of inward normals to B at $z \in \partial B$, then for $i = 1, 2, ..n$,

$$\begin{pmatrix} z \in \partial B \\ |\langle z, n_i \rangle| < 1 \end{pmatrix} \Rightarrow \langle \nu, d_i \rangle = 0 \text{ for all } \nu \in \nu(z).$$

See [18] for the definition of the set of inward normals. The following theorem (Theorem 2.1 from [18]) gives sufficient conditions for which Assumption 2.1 holds.

Theorem 3. *Assumption 2.1 holds in either of the two cases.*

1. $d_i = -n_i$

2. *There exists positive constants* a_i *such that*

$$a_i |\langle n_i, d_i \rangle| > \sum_{j \neq i} a_j |\langle n_i, d_j \rangle|$$

for all i.

Generalized Harrison-Reiman Condition

The Generalized Harrison-Reiman Condition (Condition 2.4 from [16]) is also a sufficient condition for Lipshitz continuity of solutions to the Skorokhod Problem. Let $Q = D - I$. The condition is that the matrix $|Q|$, with components $|q_{ij}|$, is an off diagonal matrix and has spectral radius less then 1.

Dupuis and Ishii's Existence Condition

The following assumption from Dupuis and Ishii is a sufficient condition under which solutions to the Skorokhod Problem exist. We use the name given in [18] and will refer to it as Assumption 3.1. In [18], they use a more general set $G \subset \mathbb{R}^n$. G is the closure of an open set possessing a smooth boundary. Assumption 3.1 associates the existences of solutions to the Skorokhod Problem with a projection onto Ω along directions d_i.

Assumption 3.1: There exists $\pi : \mathbb{R}^n \to \Omega$ such that if $y \in \Omega$, then $\pi(y) = y$, and if $y \notin \Omega$, then $\pi(y) \in \partial\Omega$, and $y - \pi(y) = \alpha\tilde{\gamma}$ for some $\alpha \leq 0$ and $\tilde{\gamma} \in d_D(\pi(y))$.

According to [18] the next theorem gives sufficient conditions for which Assumption 3.1 holds.

Theorem 4. *Assumption 3.1 holds in either of the following cases.*

1. $d_i = -n_i$
2. *Suppose the domain is a closed convex cone with vertex at the origin and that* $d_D(x) = -Dn(x)$ *for some* $n \times n$ *matrix* D*. Then we assume* $\langle n, Dn \rangle \geq a > 0$*, for all* $n \in n(0)$.

From [12] solutions to the Skorokhod Problem are associated with solving a linear complementarity problem, Definition 7 below. When Assumption 2.1 holds, Assumption 3.1 is equivalent to D being a P-matrix, defined below.

P-Matrix

From [8], D is a P-matrix if all the principal minors of D have positive determinants. D is a P-matrix if and only if solutions to the associated linear complementarity problem are unique. In other words the Skorokhod Problem has unique piecewise linear solutions. We will discuss the linear complementarity problem below; see Definition 7. The following theorem is a consequence of the results from [18].

Theorem 5. *When Assumption 2.1 and 3.1 hold solutions to the Skorokhod Problem exists and are Lipschitz continuous. (In this case we say that the Skorokhod Problem is well-posed.)*

Note that ϕ is absolutely continuous when ψ is. Under Assumption 2.1 and 3.1, when ψ is absolutely continuous, one can reformulate solutions to the Skorokhod Problem in terms of a constrained ordinary differential equation of the form

$$\dot{\phi}(t) = \pi(\phi(t), \dot{\psi}(t)).$$

$\pi(x, v)$ is the velocity projection of v defined in [7] in terms of a discrete projection $\pi : \mathbb{R}^n \to \Omega$:

$$\pi(x, v) = \lim_{h \downarrow 0} \frac{\pi(x + hv) - x}{h}. \tag{2.1}$$

$\pi(y) = \pi(0, y)$ is the discrete projection map of Assumption 3.1.

Theorem 6. *When ψ is absolutely continuous and Assumption 2.1 and 3.1 hold, there is a unique solution on any interval $[0, T]$ to*

$$\dot{\phi}(t) = \pi(\phi(t), \dot{\psi}(t)), \;\; \phi(0) \in \Omega$$

This follows from Theorems 5.1 in [18] with $\dot{\psi}(t) = b(t)$ and $\dot{\phi}(t) = \dot{x}(t)$.

Stability Condition

Stability means that for any starting point $x \in \Omega$ the path $\psi(t) = x + bt$ will, after constraint by the Skorokhod Problem, eventually make it back to the origin. In other words, $\phi(t) = \Gamma(x + bt) \to 0$ as $t \to \infty$. When the Skorokhod Map is Lipschitz continuous and D is an invertible matrix, [7] shows that

$$D^{-1}b < 0 \tag{2.2}$$

is a necessary and sufficient condition for $\phi(t) = \Gamma(x + bt) \to 0$ as $t \to \infty$ and for the associated reflected Brownian motion to be positive recurrent. This implies the existence of a stationary distribution. As mentioned in Chapter 1, this is important in the large deviation analysis of queueing networks. Note that $D^{-1}b < 0$ is equivalent to finding a vector $u > 0$ such that $-b = Du$.

2.2 Our Hypotheses

Throughout this dissertation we assume that Assumption 2.1, Assumption 3.1 (page 12 and page 13) and the Stability Condition (2.2) hold.

2.3 $\mathbb{R}^n$ Representation of the Velocity Projection

[13] shows that under Assumption 3.1 and Assumption 2.1 the associated velocity projection map $\pi(x, v)$ is identified with a collection of complementarity problems. From [7] and [13] we know in $\mathbb{R}^2$ $\pi(x, v) = v$ for $x \in \Omega^\circ$, and for $I(x) = \{i\}$,

$$\pi(x, v) = \begin{cases} v & \text{if } \langle v, n_i \rangle \leq 0 \\ v + \langle v, n_i \rangle d_i & \text{if } \langle v, n_i \rangle \geq 0 \end{cases}.$$

When $\langle v, n_i \rangle \geq 0$ the resulting velocity is tangent to the face where $x_i = 0$. Also note that in this case we can represent the projected velocity as $\pi(x, v) = P_i v$ where $P_i = I + d_i n_i^T$. We will use the associated complementarity problem to derive a representation of the form $\pi(x, v) = P_K v$, holding for $x \in \partial\Omega$ more generally.

Usually the velocity projection $\pi(x, v)$ is defined as (2.1) above, however in [13] it was shown to be equivalent to the following, which we take as our definition.

Definition 7. *The velocity projection of the Skorokhod Problem, $w = \pi(x, v)$ is characterized by the following linear complementarity problem: find w and a_i, $i \in I(x)$*

$$w = v + \sum_{i \in I(x)} a_i d_i,$$

such that for each $i \in I(x)$:

$$a_i \geq 0 \text{ for all } i \in I(x) \tag{2.3}$$

$$w_i = -\langle n_i, w\rangle \geq 0 \text{ for all } i \in I(x) \tag{2.4}$$

$$a_i w_i = 0 \text{ for all } i \in I(x). \tag{2.5}$$

For $J \subseteq \{1, 2, \ldots n\}$ let N_J be the matrix with column vectors n_j $j \in J$ and D_J be the matrix with columns d_j $j \in J$. For $K \subseteq I(x)$, define

$$B_K = -(N_K^T D_K)^{-1} N_K^T \tag{2.6}$$

and

$$P_K = I + D_K B_K. \tag{2.7}$$

If $K = \emptyset$ we take $P_K = I$ with $B_K = D_K = 0$. Also note that $N_K^T D_K$ is a $|K| \times |K|$ principal submatrix of D with components $-d_{ij}$. Since we know D is a P-matrix, the determinante of $N_K^T D_K$ is non-zero and $(N_K^T D_K)^{-1}$ exists. We will need this fact below.

The following is proven in [12].

Lemma 1. *Given* $x \in \Omega$ *the following are equivalent.*

1. $\pi(x, v) = w$
2. $w = P_K v$ *for* $K \subseteq I(x)$ *satisfying*
   (a) $B_K v \geq 0$ *when* $K \neq \emptyset$
   (b) $N_{I(x)\setminus K}^T P_K v \leq 0,$ *when* $K \neq I(x)$

Proof. First suppose $w = \pi(x, v)$ and let a_i, $i \in I(x)$ be as in the complementarity problem of Definition 7. Let $F = \{i \in I(x) | a_i > 0\}$ and $L = \{i \in I(x) | w_i = 0\}$. For any K with

$F \subseteq K \subseteq L$, we can solve for $a_i,\ i \in K$ using $w_i = 0$:

$$\langle n_i, v + \sum_{j \in K} a_j d_j \rangle = 0,\ i \in K$$

$$\Rightarrow N_K^T (v + D_K a_K) = 0$$

$$\Rightarrow a_K = \underbrace{-(N_K^T D_K)^{-1} N_K^T}_{=B_K} v$$

$$\Rightarrow w = \underbrace{(I + D_K B_K)}_{=P_K} v.$$

If $K = \emptyset$ it follows that $a_i = 0$ for all $i \in I(x)$ and

$$w = v = P_\emptyset v$$

because of our convention that $P_\emptyset = I$. So in this case $\pi(x,v) = P_K v$ for some $K \subseteq I(x)$. Observe that (2.3) implies 2(a). (2.4) means that $w_i \geq 0$ for $i \in I \backslash K$ so

$$N_{I \backslash K}^T P_K v \leq 0.$$

Hence 1 implies 2.

Now we want to show that if 2 holds then $\pi(x,v) = P_K v$. Suppose $K \subseteq I(x)$ satisfies 2(a) and 2(b). Define a_i, $i \in I(x)$ using $a_K = B_K v$ for $i \in K$ and $a_i = 0$ for $i \in I(x) \backslash K$. Then 2(a) gives (2.3). $w = P_K v = v + D_K B_K v$ gives $w = v + \sum_{i \in I(x)} a_i d_i$. Since

$$N_K^T w = N_K^T v + N_K^T D_K B_K v = N_K^T v - N_K^T D_K (N_K^T D_K)^{-1} N_K^T v = 0$$

we have $w_i = 0$ for $i \in K$. From 2(b) we have the rest of (2.4). Finally (2.5) holds since $w_i = 0$ for $i \in K$ and $a_i = 0$ for $i \in I(x) \backslash K$. □

Consider what the lemma says on a face where $I(x) = \{i\}$. For 2 to hold with $K = \{i\}$ we find

$$B_K = n_i^T,\ B_K v \geq 0 \iff \langle n_i, v \rangle \geq 0,$$

and

$$P_K = I + d_i n_i^T.$$

For 2 to hold with $K = \emptyset$ we find

$$P_K = I,\ N^T_{I(x)\setminus K} P_K v \leq 0 \iff \langle n_i, v\rangle \leq 0.$$

Thus we obtain the description of $\pi(x,v)$ on a face given at the beginning of this section.

We are interested in paths which end at a point x on $\partial\Omega$. We will need a description of these paths in terms of the Skorokhod dynamics later when we consider our viscosity solution boundary condition formulation. The final segment of these paths may be in the interior of Ω or may traverse the boundary of Ω. The paths we seek are of the form

$$\phi(t) = x + (T_2 - t)w \text{ with } \phi(t) \in \Omega \text{ and } w = \pi(\phi(t), v),\ \text{ for } T_2 - \delta < t < T_2$$

(some $\delta > 0$) with $\phi(T_2) = x$. In other words $\phi(t)$ is the final segment of $\phi = \Gamma(\psi)$ for some $\psi(t)$ with $\dot{\psi}(t) = v$ for $T_2 - \delta < t < T_2$ reaching x at $t = T_2$. Next we state a lemma describing such paths approaching a point $x \in \partial\Omega$.

Lemma 2. *Let $x \in \partial\Omega$, in order for $\phi(t) = x + (T_2 - t)w$ to satisfy $\phi(t) \in \Omega$ and $w = \pi(\phi(t), v)$, for $T_2 - \delta < t < T_2$ it is necessary and sufficient that there exits a $K \subseteq I(x)$ for which*

$$\begin{aligned} &w = P_K v \\ &B_K v \geq 0 \\ &N^T_{I(x)\setminus K} P_K v \geq 0. \end{aligned}$$

Proof. Suppose ϕ is such a path with $\dot{\phi} = w = \pi(\phi, v)$. Since $x \in \partial\Omega$ $I(x) \neq \emptyset$. Clearly we must have $w_i \leq 0$ for all $i \in I(x)$ since ϕ is a path approaching or on the edge or face that x is on. Defining

$$F = \{i \in I(x) : w_i = 0\}, \tag{2.8}$$

we see that $I(\phi(t)) = F$ for $T_2-\delta < t < T_2$. From the lemma above we see that $w = \pi(\phi(t), v)$ is equivalent to the existence of $K \subseteq F$ with

$$\begin{aligned} w &= P_K v \\ B_K v &\geq 0 \\ N^T_{F\setminus K} w &\leq 0. \end{aligned} \tag{2.9}$$

The condition $N^T_{F\setminus K} w \leq 0$ means $w_i \geq 0$ for all $i \in F \setminus K$. But we are assuming that $w_i \leq 0$ for all $i \in I(x)$, which means that in fact $w_i = 0$ for all $i \in F \setminus K$. Thus $w_i \leq 0$ for all $i \in I(x) \setminus K$. So we see that there exists $K \subseteq I(x)$ with

$$w = P_K v,\ B_K v \geq 0,\ \text{and}\ N^T_{I(x)\setminus K} P_K v \geq 0. \tag{2.10}$$

Conversely, suppose we have a v, w and $K \subseteq I(x)$ as in (2.10). Then $w_i = 0$ for $i \in K$ (because $N^T_K P_K v = 0$) and $w_i \leq 0$ for $i \in I(x) \setminus K$. Define F as in (2.8). Then $K \subseteq F$ and $N^T_{F\setminus K} w = 0$. So (2.9) holds for this F. Since $w_i < 0$ for $i \in I(x) \setminus F$ we have $I(\phi(t)) = F$ for $T_2 - \delta < t < T_2$, so that $w = \pi(\phi(t), v)$ for $T_2 - \delta < t < T_2$ as desired.

Thus (2.10) is equivalent to to the existence of a path approaching x as described, and (2.8) identifies the edge or face $\partial_F \Omega$ through which it approaches x. Note that this is not the same as saying $w = \pi(x, v)$; which is the velocity projection at a single point. That would involve the reverse inequality $N^T_{I(x)\setminus K} w \leq 0$. $\square$

Chapter 3

The Reflected Quasipotential

3.1 Definition

Next we define the Reflected Quasipotential $V(x)$. In the definition below $L(v) = \frac{1}{2}\langle v - b, A^{-1}(v-b)\rangle$ is the Lagrangian and A is a symmetric, positive definite matrix.

Definition 8 (The Reflected Quasipotential)**.** *For $x \in \Omega$*

$$V(x) = \inf_{\psi(t)\in AC, T_1 \le T_2, \phi(T_1)=0, \phi(T_2)=x} \int_{T_1}^{T_2} L(\dot{\psi}(t))dt. \tag{3.1}$$

The infimum is taken over absolutely continuous paths ψ, $\phi = \Gamma(\psi) \in \Omega$ such that $\psi(T_1) = \phi(T_1) = 0$ and $\phi(T_2) = x$.

Note that $V(x)$ depends on the parameters (b, A, D) which are subject to our hypotheses, Section 2.2 in Chapter 2. ϕ and ψ are defined on an arbitrary interval $[T_1, T_2]$. We could have defined them on $[0, T]$ as in [20] and [16] or $[-T, 0]$ as in [11] but chose the more general $[T_1, T_2]$ for convenience. Note that our definition of the Skorokhod Problem in the previous chapter is defined on $[0, t]$ with $\phi(0) = \psi(0)$, but we could have defined it on $[T_1, T_2]$ with $\phi(T_1) = \psi(T_1)$.

From the definition above we see that $V(x)$ is characterized as the minimum cost incurred by a path constrained by the Skorokhod Problem to reach a point $x \in \Omega$ starting from the origin. Here we do not state that the infimum in Definition 8 is achieved. However in Lemma 2 of [11] it was shown that for the standard Quasipotential the infimum is achieved. Therefore it is likely that the we could do the same for the Reflected Quasipotential.

Main Objective

Our main objective is to characterize $V(x)$ as a unique solution to a PDE, in Ω with the appropriate boundary conditions. For our uniqueness argument we will also need to consider a discounted, truncated version of $V(x)$, $V^{R,\gamma}(x)$. Moreover, we will show that $V^{R,\gamma}(x) \to V(x)$ as $\gamma \to 0$ and $R \to \infty$ (this is where the stability hypothesis is used). The definition of $V^{R,\gamma}(x)$ involves a function w, which will be explained latter. The truncated domain we will work with below is defined to be

$$\Omega^R = \{x \in \Omega \ : \ 0 < |x| < R\} \tag{3.2}$$

Definition 9. *Given a large positive truncation value $R > 0$, a discount rate $\gamma > 0$ and a continuous function $w(\cdot)$ on $\overline{\Omega^R}$, define the modified version $V^{R,\gamma}(x)$ to be the infimum of*

$$e^{-\gamma(T_2-T_1)} w(\phi(T_1)) + \int_{T_1}^{T_2} e^{-\gamma(T_2-t)} L(\dot{\psi}(t))\, dt.$$

over absolutely continuous ψ, $\phi(\cdot) = \Gamma(\psi(\cdot))$ subject to the restriction that $|\phi(t)| \leq R$ where $\phi(T_1)$ is allowed to be either 0 or $|\phi(T_1)| = R$, and $\phi(T_2) = x \in \overline{\Omega^R}$.

In other words we can start the path either at 0 as usual or at a point with $|\phi(T_1)| = R$ with "starting cost" $w(\phi(T_1))$.

3.2 Properties of the Reflected Quasipotential

Next we state several properties of $V(x)$ and $V^{R,\gamma}(x)$. The properties stated below for $V^{R,\gamma}(x)$ also hold for $V(x)$. The proofs of the properties for $V(x)$ follow from the proofs we state below for $V^{R,\gamma}(x)$ with $\gamma = 0$ and $R \to \infty$.

3.2.1 The Dynamic Programming Principle

Dynamic Programming arises in optimization, optimal control and differential games. The idea is that a value function v satisfies a functional equation called the Dynamic Programming Principle. This provides a characterization of the value function which can be used to derive other important results. In general when the value function is smooth enough it satisfies a Hamilton-Jacobi equation also know as the infinitesimal version of the Dynamic Programming Principle; [4]. According to [22], it was first discovered by Lions that for control problems the Dynamic Programming Principle implies that the value function is the viscosity solution of the Hamiltonian-Jacobi (Bellman) equation. See [4] for more details. For us the Dynamic Programming Principle will be useful when describing $V^{R,\gamma}(x)$ in terms of a viscosity solution in the next chapter.

Lemma 3 (Dynamic Programming Principle)**.** *For all $x \in \overline{\Omega^R}$, absolutely continuous $\phi = \Gamma(\psi)$ defined on $[T_1, T_2]$ with $\phi(T_2) = x$ and $T_1 \le T_2 - h < T_2$ it holds that*

$$V^{R,\gamma}(x) = \inf_{\phi}\{e^{-\gamma h}V(\phi(T_2 - h)) + \int_{T_2-h}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt\}. \tag{3.3}$$

From the lemma it follows that

$$V^{R,\gamma}(x) \le e^{-\gamma h}V^{R,\gamma}(\phi(T_2 - h)) + \int_{T_2-h}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))\,dt. \tag{3.4}$$

Proof. First consider any absolutely continuous $\phi = \Gamma(\psi)$ defined on $[T_1, T_2]$ where $\phi(T_1)$ is allowed to be either 0 or $|\phi(T_1)| = R$, $|\phi(t)| \le R$, $\phi(T_2) = x$ and $T_1 \le T_2 - h < T_2$. We know that

$$\begin{aligned}
&e^{-\gamma(T_2-T_1)}w(\phi(T_1)) + \int_{T_1}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt \\
&= e^{-\gamma(T_2-T_1)}w(\phi(T_1)) + \int_{T_1}^{T_2-h} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt + \int_{T_2-h}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt.
\end{aligned}$$

From the definition of $V^{R,\gamma}(\cdot)$,

$$V^{R,\gamma}(\phi(T_2-h)) \leq e^{-\gamma(T_2-h-T_1)}w(\phi(T_1)) + \int_{T_1}^{T_2-h} e^{-\gamma(T_2-h-t)}L(\dot{\psi}(t))dt.$$

Multiplying $V^{R,\gamma}(\phi(T_2-h))$ by $e^{-\gamma h}$ gives us

$$\begin{aligned}
&e^{-\gamma(T_2-T_1)}w(\phi(T_1)) + \int_{T_1}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt \\
&\geq e^{-\gamma h}V^{R,\gamma}(\phi(T_2-h)) + \int_{T_2-h}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt
\end{aligned}$$

and therefore inequality one way in (3.3);

$$V^{R,\gamma}(x) \geq \inf_{\phi}\{e^{-\gamma h}V^{R,\gamma}(\phi(T_2-h)) + \int_{T_2-h}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt\}.$$

Start with any $\phi = \Gamma(\psi)$ defined on $[T_2-h, T_2]$ with $\phi \in \overline{\Omega^R}$ and $\phi(T_2) = x$. Extend $\phi = \Gamma(\psi)$ to $[T_1, T_2-h]$ so that it is nearly optimal to $\phi(T_2-h)$ (with $\phi \in \overline{\Omega^R}$, $\phi(T_1) = 0$ or $|\phi(T_1)| = R$).

$$V(\phi(T_2-h)) + \epsilon > e^{-\gamma(T_2-h-T_1)}w(\phi(T_1)) + \int_{T_1}^{T_2-h} e^{-\gamma(T_2-h-t)}L(\dot{\psi}(t))dt.$$

From the definition of $V^{R,\gamma}(\cdot)$ and the previous line we have

$$\begin{aligned}
&V^{R,\gamma}(x) \leq e^{-\gamma(T_2-T_1)}w(\phi(T_1)) + \int_{T_1}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt \\
&= e^{-\gamma(T_2-T_1)}w(\phi(T_1)) + \int_{T_1}^{T_2-h} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt + \int_{T_2-h}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt \\
&= \int_{T_2-h}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt + e^{-\gamma h}\left(e^{-\gamma(T_2-h-T_1)}w(\phi(T_1)) + \int_{T_1}^{T_2-h} e^{-\gamma(T_2-h-t)}L(\dot{\psi}(t))dt\right) \\
&< \int_{T_2-h}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt + e^{-\gamma h}V(\phi(T_2-h)) + \epsilon e^{-\gamma h}.
\end{aligned}$$

Therefore

$$V^{R,\gamma}(x) < \inf_{\phi}\{\int_{T_2-h}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt + e^{-\gamma h}V(\phi(T_2-h)) + \epsilon e^{-\gamma h}\}.$$

Hence we can conclude

$$V^{R,\gamma}(x) = \inf_{\phi}\{e^{-\gamma h}V(\phi(T_2-h)) + \int_{T_2-h}^{T_2} e^{-\gamma(T_2-t)}L(\dot{\psi}(t))dt\}.$$

□

3.2.2 Scaling Lemma

The next lemma classifies our function $V(x)$ as a homogenous function. In other words $V(x)$ is a function with a multiplicative scaling factor. A similar lemma can be found in [3]. We state the Scaling Lemma below only for $V(x)$. Later we will use the Scaling Lemma in Chapter 6 where we are working with $V(x)$ and not $V^{R,\gamma}(x)$. Without more information about how $w(x)$ scales we can not completely formulate a version of the Scaling Lemma for $V^{R,\gamma}(x)$.

Lemma 4 (Scaling Lemma)**.** *For any positive k, and $x \in \Omega$, $V(kx) = kV(x)$.*

Proof. The Skorokhod problem scales in two ways. Suppose $k > 0$ and $\phi(\cdot) = \Gamma(\psi(\cdot))$ with $\phi(T_1) = 0$ and $\phi(T_2) = x$.

1. If $\tilde{\psi}(t) = \psi(T_1 + kt)$ then $\tilde{\phi}(t) = \phi(T_1 + kt)$ satisfies $\tilde{\phi}(t) = \Gamma(\tilde{\psi}(t))$.

2. If $\tilde{\psi}(t) = k\psi(T_1 + t)$ then $\tilde{\phi}(t) = k\phi(T_1 + t)$ satisfies $\tilde{\phi}(t) = \Gamma(\tilde{\psi}(t))$.

This depends on the fact that for $c > 0$, $x \in \mathbb{R}^n_+$ if and only if $cx \in \mathbb{R}^n_+$.

Now from above we see that $\tilde{\phi}(t) = k\phi(T_1 + \frac{t}{k})$ has $\tilde{\phi}(0) = 0$, $\tilde{\phi}(k(T_2 - T_1)) = kx$ and $\tilde{\phi}(\cdot) = \Gamma(\tilde{\psi}(\cdot))$ where $\tilde{\psi}(t) = k\psi(T_1 + \frac{t}{k})$. $\tilde{\psi}$ is absolutely continuous if and only if ψ is and

$$\dot{\tilde{\psi}}(s) = \frac{d}{ds}\tilde{\psi}(s) = k\frac{1}{k}\dot{\psi}(T_1 + \frac{s}{k}) = \dot{\psi}(T_1 + \frac{s}{k}),$$

so

$$\int_0^{(T_2-T_1)k} L(\dot{\tilde{\psi}}(s))ds = \int_0^{(T_2-T_1)k} L(\dot{\psi}(T_1+\frac{s}{k}))ds = k\int_{T_1}^{T_2} L(\dot{\psi}(t))dt.$$

Thus we have a one to one correspondence between the paths from 0 to x and the paths from 0 to kx. We also have that

$$\begin{aligned} V(kx) &= \inf_{\{\tilde{\psi}\in AC,(T_2-T_1)\geq 0,\tilde{\phi}((T_2-T_1)k)=kx,\tilde{\phi}(0)=0\}} \int_0^{(T_2-T_1)k} L(\dot{\tilde{\psi}}(t))dt \\ &= \inf_{\{\psi\in AC,(T_2-T_1)\geq 0,\phi(T_2)=x,\phi(T_1)=0\}} \int_{T_1}^{T_2} kL(\dot{\psi}(t))dt \\ &= k\{\inf_{\{\psi\in AC,(T_2-T_1)\geq 0,\phi(T_2)=x,\phi(T_1)=0\}} \int_{T_1}^{T_2} L(\dot{\psi}(t))dt\} = kV(x). \end{aligned}$$

□

3.2.3 Bounded Velocity Lemma

The next lemma tells us that we can produce a slowed down version of a path ψ, so that $\dot{\psi}$ is bounded, at a cost no more then that of the original path. From the lemma it follows that the velocity of an optimal path is bounded. This lemma will be particularly useful in our characterization of $V^{R,\gamma}(x)$ in terms of viscosity solutions. Let $||x|| = \langle x, A^{-1}x\rangle^{\frac{1}{2}}$. (When $A = I$ this reduces to the standard Euclidean norm $||x|| = |x|$.) We call this the A-norm.

Lemma 5 (Bounded Velocity Lemma). *Given a path $\psi(t)\in\mathbb{R}^n$, $\phi(t)=\Gamma(\psi(t))$,$|\phi(t)|\leq R$, $T_1\leq t\leq T_2$ (absolutely continuous), we can produce a "slowed down" version of ψ, $\bar{\psi}(\tau)$ with new time variable τ, $T_1\leq\tau\leq\bar{T}_2$, $\psi(t)=\bar{\psi}(\tau(t))$, and $T_2=\tau(\bar{T}_2)$, for which $\|\dot{\bar{\psi}}\|\leq\|b\|$ and (for any $C,\gamma\geq 0$)*

$$e^{-\gamma(\bar{T}_2-T_1)}C+\int_{T_1}^{\bar{T}_2} e^{-\gamma(\bar{T}_2-\tau)}L(\dot{\bar{\psi}}(\tau))\,d\tau \leq e^{-\gamma(T_2-T_1)}C+\int_{T_1}^{T_2} e^{-\gamma(T_2-\tau)}L(\dot{\psi}(t))\,dt. \quad (3.5)$$

Proof. First observe that for $\|v\| > 0$ the minimum of $\frac{1}{k}L(kv)$ over $k > 0$ occurs at $k = \frac{\|b\|}{\|v\|}$.

$$\begin{aligned}\frac{1}{k}L(kv) &= \frac{k}{2}\|v\|^2 - \langle v, A^{-1}b\rangle + \frac{1}{2k}\|b\|^2\\ &= \frac{1}{2k}(k\|v\| - \|b\|)^2 + \|v\|\|b\| - \langle v, A^{-1}b\rangle\\ &\geq \|v\|\|b\| - \langle v, A^{-1}b\rangle,\end{aligned}$$

with equality if (and only if) $k = \frac{\|b\|}{\|v\|}$.

Define $k(t) = \min(1, \|b\|/\|\dot{\psi}(t)\|)$ and the new time variable $\tau = \tau(t)$ by

$$\tau(t) = T_1 + \int_{T_1}^{t} \frac{1}{k(s)}ds,\ t \geq T_1.$$

When $\|\dot{\psi}(t)\| \leq \|b\|$ we have $k(t) = 1$ and the two time scales increase at the same rate. But when $\|\dot{\psi}(t)\| > \|b\|$ then $d\tau = \frac{1}{k(t)}\,dt > dt$ so that on the new time scale more time elapses, which will mean that the speed of ψ will be slower.

Before proceeding we should verify that $1/k(t)$ is integrable. Assuming that

$$\int_{T_1}^{T_2} L(\dot{\psi}(t))\,dt < \infty$$

(else (3.5) is trivial), we know that $\int_{T_1}^{T_2} \|\dot{\psi}(t)\|^2 dt < \infty$ and therefore

$$\int_{T_1}^{T_2} \|\dot{\psi}(t)\| dt < \infty.$$

Since $\frac{1}{k(t)} = \max(1, \|\dot{\psi}(t)\|/\|b\|)$ we have

$$\int_{T_1}^{T_2} \frac{1}{k(t)}\,dt \leq \int_{T_1}^{T_2} 1 + \frac{1}{\|b\|}\|\dot{\psi}(t)\|\,dt < \infty.$$

Thus $\tau(\cdot)$ is an absolutely continuous function. Since $1 \leq 1/k(t)$, $\tau(t)$ is strictly monotone and has a continuous inverse $t(\tau)$ defined for $T_1 \leq \tau \leq \bar{T}_2 \doteq \tau(T_2)$, also absolutely continuous with $dt = k(t(\tau))\,d\tau$. We define $\bar{\psi}$ on $[T_1, \bar{T}_2]$ by $\bar{\psi}(\tau) = \psi(t(\tau))$. It follows that $\bar{\psi}$ is absolutely continuous with

$$\dot{\bar{\psi}}(\tau) = \frac{d}{d\tau}\bar{\psi}(\tau) = k(t(\tau))\dot{\psi}(t(\tau)).$$

By our definition of $k(t)$ and our observation for $\frac{1}{k}L(kv)$ above it follows that $\frac{1}{k(t)}L(k(t)\dot{\psi}(t)) \leq L(\dot{\psi}(t))$. We now calculate that

$$\begin{aligned} e^{-\gamma(\bar{T}_2-T_1)}C + \int_{T_1}^{\bar{T}_2} e^{-\gamma(\bar{T}_2-\tau)} L(\dot{\bar{\psi}}(\tau))\,d\tau &= e^{-\gamma(\tau(T_2)-T_1)}C + \int_{T_1}^{T_2} e^{-\gamma(\tau(T_2)-\tau(t))} \frac{1}{k(t)} L(k(t)\dot{\psi}(t))\,dt \\ &\leq e^{-\gamma(\tau(T_2)-T_1)}C + \int_{T_1}^{T_2} e^{-\gamma(\tau(T_2)-\tau(t))} L(\dot{\psi}(t))\,dt. \end{aligned}$$

Now observe that $k(t) \leq 1$ so that $T_2 - t = \int_{\tau(t)}^{\tau(T_2)} k(t)\,d\tau \leq \tau(T_2) - \tau(t)$. In particular (taking $t = T_1$), $\tau(T_2) \geq T_2$. Thus $e^{-\gamma(\tau(T_2)-T_1)} \leq e^{-\gamma(T_2-T_1)}$ and $e^{-\gamma(\tau(T_2)-\tau(t))} \leq e^{-\gamma(T_2-t)}$, so we have

$$e^{-\gamma(\bar{T}_2-T_1)}C + \int_{T_1}^{\bar{T}_2} e^{-\gamma(\bar{T}_2-\tau)} L(\dot{\bar{\psi}}(\tau))\,d\tau \leq e^{-\gamma(T_2-T_1)}C + \int_{T_1}^{T_2} e^{-\gamma(T_2-t)} L(\dot{\psi}(t))\,dt.$$

□

Since A is a symmetric positive definite matrix then A^{-1} is also symmetric positive definite and has positive eigenvalues $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n > 0$. First note

$$||x||^2 = \langle x, A^{-1}x\rangle = x^T A^{-1} x.$$

Using the fact that A^{-1} has n linearly independent eigenvectors which form an orthonormal basis for $\mathbb{R}^n$ we have the following inequalities.

$$\lambda_n |x|^2 \leq ||x||^2 \leq \lambda_1 |x|^2;$$

[1].

If $\|\dot{\bar{\psi}}\| \leq \|b\|$ then there exists constants $c_1, c_2 > 0$ such that

$$c_2|\dot{\bar{\psi}}| \leq \|\dot{\bar{\psi}}\| \leq \|b\| \leq c_1|b|$$

so that

$$|\dot{\bar{\psi}}| \leq c|b|$$

for some $c > 0$. This leads us to the following corollary. Note that the only difference between the corollary and Lemma 5 is that in the corollary the bound on velocity is expresses in the Euclidean norm rather then the A-norm.

Corollary 1. *Let $c > 0$ be such that $|\dot{\bar{\psi}}| \leq c|b|$. Given a path $\psi(t) \in \mathbb{R}^n$, $\phi(t) = \Gamma(\psi(t))$, $|\phi| \leq R$, $T_1 \leq t \leq T_2$ (absolutely continuous), we can produce a "slowed down" version of ψ, $\bar{\psi}(\tau)$ with new time variable τ, $T_1 \leq \tau \leq \bar{T}_2$, $\psi(t) = \bar{\psi}(\tau(t))$, and $T_2 = \tau(\bar{T}_2)$, for which $|\dot{\bar{\psi}}| \leq c|b|$ and (for any $C, \gamma \geq 0$)*

$$e^{-\gamma(\bar{T}_2 - T_1)}C + \int_{T_1}^{\bar{T}_2} e^{-\gamma(\bar{T}_2-\tau)} L(\dot{\bar{\psi}}(\tau))\, d\tau \leq e^{-\gamma(T_2-T_1)}C + \int_{T_1}^{T_2} e^{-\gamma(T_2-\tau)} L(\dot{\psi}(t))\, dt. \tag{3.6}$$

3.2.4 Lipschitz Continuity of $V(x)$

It is well known in the literature that the standard Quasipotential function is Lipschitz continuous; see [20]. We can show the same for $V(x)$ and $V^{R,\gamma}(x)$. For any $x, y \in \Omega$ the Dynamic Programming Principle gives us $V(x) \leq V(\phi(T)) + \int_T^{T_2} L(\dot{\psi}(t))dt$, $0 \leq T \leq T_2$ where $\phi = \Gamma(\psi)$, $\phi(T_2) = x$ and $\phi(T) = y$. Consider the path $\psi(t) = y + \frac{x-y}{|x-y|}t$ so that and $\psi(|x-y|) = x$. Since $x, y \in \Omega$ it follows that $\phi = \Gamma(\phi)$. Taking $T = 0$ and $T_2 = |x-y|$ we see

$$\int_0^{T_2} L(\dot{\psi}(t))dt = \frac{1}{2}\langle \frac{x-y}{|x-y|} - b, A^{-1}(\frac{x-y}{|x-y|} - b)\rangle |x-y| \leq K|x-y|,$$

where K is a bound on $L(u)$ over unit vectors u. Then $|V(x) - V(y)| \leq K|x-y|$ and we can conclude $V(x)$ is Lipschitz continuous.

For $V^{R,\gamma}(x)$, take $x, y \in \overline{\Omega^R}$ and $0 < h \leq T_2$, then the Dynamic Programming Principle give us that

$$V^{R,\gamma}(x) \leq e^{-\gamma h} V^{R,\gamma}(\phi(T_2 - h)) + \int_{T_2-h}^{T_2} e^{-\gamma(T_2-t)} L(\dot{\psi}(t))\, dt.$$

Since $e^{-\gamma h}$ and $e^{-\gamma(T_2-t)}$ are bounded by 1 we have

$$V^{R,\gamma}(x) \leq V^{R,\gamma}(\phi(T_2 - h)) + \int_{T_2-h}^{T_2} L(\dot{\psi}(t))\, dt.$$

Proceeding as above with $V(x)$ it follows that $V^{R,\gamma}(x)$ is also Lipschitz continuous.

Chapter 4

Viscosity Solution

4.1 Definition

Our objective is to characterize $V(x)$ as the solution to a partial differential equation in Ω with appropriate boundary conditions. Since $V(x)$ is not differentiable in general the characterization will need to be in terms of viscosity solutions. The notion of viscosity solutions, a class of generalized solution for non-linear first-order partial differential equations, was introduced by Crandall and Lions. This has allowed a treatment of first-order Hamilton-Jacobi equations with non-smooth solutions. It has also had applications in deterministic optimal control and differential games, large deviations, asymptotic problems and various other problems where Hamilton-Jacobi equations arise naturally; see [5]. In [4] Bardi and Capuzzo-Dolcetta give an extensive introduction to the basic theory of viscosity solutions and its application to a variety of optimization and control problems. A fundamental part of the theory of viscosity solutions involves comparison and uniqueness results which identify the optimal value function as a unique viscosity solution to the appropriate Hamilton-Jacobi equation.

In [5] Barels and Lions considered problems of the type

$$H(x, u, Du) = 0, \text{ in } \Omega,$$

where Ω is a smooth, bounded, open set in $\mathbb{R}^n$. Here Du represents the gradient of u. This is the general form of the first-order Hamilton-Jacobi equation. They investigated existence and uniqueness of solutions satisfying non-linear boundary conditions of the form

$$F(x, u, Du) = 0, \text{ on } \partial\Omega.$$

The general idea in [5] was that at certain points of the boundary the boundary condition need not hold and such points behave like interior points. So one should impose the internal equation at those points. We will use the ideas from Barles and Lions in [5] to characterize $V(x)$ uniquely in terms of viscosity solutions in the 2-dimensional case. In $\mathbb{R}^n_+$ this would mean extending Barels and Lions formulation to domains with corners as well as extending their uniqueness argument to domains with corners. Some work has been done on this for particular cases in [19]; see [5].

4.1.1 Viscosity Solution: Interior Points

We are interested in a viscosity sense characterization of the Reflected Quasipotential $V(x)$ which is defined in Definition 8 above. We will see in the following chapter that additional conditions are needed for a unique characterization. Our uniqueness result Theorem 9 is involves the discounted truncated version $V^{R,\gamma}(x)$ in Definition 9. For that reason we discuss $V^{R,\gamma}$ below. The particular case of V is recovered by taking $\gamma = 0$ and $R \to \infty$. We begin by discussing $V^{R,\gamma}(x)$ as a viscosity solution of $\gamma V^{R,\gamma}(x) + H(DV^{R,\gamma}(x)) = 0$ for points in the interior of Ω^R and then present a viscosity-sense boundary condition formulation.

Recall that for our variational problem the Lagrangian is

$$L(v) = \frac{1}{2}\langle v - b, A^{-1}(v - b)\rangle. \tag{4.1}$$

(Recall for A and b our hypotheses in Section 2.2 .) The Hamiltonian is defined to be

$$H(p) = \sup_{v}\{\langle p, v\rangle - L(v)\} = \frac{1}{2}\langle p, Ap\rangle + \langle p, b\rangle. \tag{4.2}$$

Under Our Hypothese, Section 2.2 we show that $V^{R,\gamma}(x)$ is a viscosity solution of $\gamma V^{R,\gamma}(x) + H(DV^{R,\gamma}(x)) = 0$ in Ω^R plus boundary conditions on $\partial\Omega^R$. First we state in general the definition of a viscosity solution of $\gamma u(x) + H(Du(x)) = 0$ for interior points of a domain Ω. (In the definition below the reader can think of Ω as a general domain with $\Omega \subset \mathbb{R}^n$.)

Definition 10. *For interior points $x \in \Omega$, a continuous function $u(x)$ is a viscosity solution of $\gamma u(x) + H(Du(x)) = 0$ if*

$$\gamma u(x) + H(\xi) \leq 0 \ \forall \ \xi \in D^+_\Omega u(x) \ (\textit{subsolution condition}), \tag{4.3}$$

$$\gamma u(x) + H(\xi) \geq 0 \ \forall \ \xi \in D^-_\Omega u(x) (\textit{supersolution condition}), \tag{4.4}$$

where

$$D^+_\Omega u(x) = \{p : \nabla\Phi(x) = p \textit{ for some } \Phi \in \mathcal{C}^1(\Omega) \textit{ such that } u - \Phi \textit{ has a local max at } x\}$$
$$D^-_\Omega u(x) = \{p : \nabla\Phi(x) = p \textit{ for some } \Phi \in \mathcal{C}^1(\Omega) \textit{ such that } u - \Phi \textit{ has a local min at } x\}.$$

Here $\mathcal{C}^1(\Omega)$ denotes the set of differentiable functions defined on Ω whose derivative is continuous. $D^+_\Omega u(x)$ and $D^-_\Omega u(x)$ are called the superdifferential and subdifferential of u at x respectively. u is called a viscosity subsolution if it satisfies (4.3) and likewise a viscosity supersolution if it satisfies (4.4). If u is a subsolution and supersolution then it is a viscosity solution. Although the definition of $D^+_\Omega u(x)$ only requires $u - \Phi$ to have a local max at x. We can add $e^{c(y-x)^2}$ to $\Phi(y)$ for a large c to insure that the max is global. Moreover by adding a constant to Φ we can assume $u \leq \Phi$ with equality at x . Thus every $\xi \in D^+_\Omega u(x)$ occurs as $\xi = D\Phi(x)$ for such a Φ. Similar remarks apply for $\xi \in D^-_\Omega$. We use these simplifications in all that follows. When u is differentiable, u satisfies the equation above in the classical sense. We will show in Theorem 7 that $V^{R,\gamma}(x)$ is in fact a viscosity solution in Ω^R. See [4] for more information about viscosity solutions.

4.2 General Direct Boundary Condition Formulation

Here we introduce the viscosity sense boundary condition formulation. Before formally stating the boundary condition formulation we give a heuristic argument for our formulation in n dimensions. It turns out that only the "lateral" boundary of Ω^R is important for us. Thus for $I \subseteq \{1, 2, \ldots, n\}$ we define

$$\partial_I \Omega^R = \{x \in \partial\Omega^R : x_i = 0 \text{ for all } i \in I\}. \tag{4.5}$$

For $K \subseteq I$ define

$$H_K^-(p) = \sup_{B_K v \geq 0, N_{I\setminus K}^T P_K v \geq 0} \{\langle p, P_K v\rangle - L(v)\} \tag{4.6}$$

$$H_K(p) = \sup_{B_K v \geq 0} \{\langle p, P_K v\rangle - L(v)\}. \tag{4.7}$$

The matricies P_K and B_K are defined in Chapter 2. For $x \in \partial_I \Omega^R$ $(x \neq 0, R)$, we propose the following formulation,

$$\gamma V^{R,\gamma}(x) + \max_{K \subseteq I}(H_K^-(\xi)) \leq 0 \text{ for all } \xi \in D^+_{\Omega^R} V^{R,\gamma}(x) \text{ (subsolution)} \tag{4.8}$$

$$\gamma V^{R,\gamma}(x) + \max_{K \subseteq I}(H_K(\xi)) \geq 0 \text{ for all } \xi \in D^-_{\Omega^R} V^{R,\gamma}(x) \text{ (supersolution).} \tag{4.9}$$

We call this the *discounted general direct formulation*. The *general direct formulation* for $V(x)$ is recovered by taking $\gamma = 0$ and $R \to \infty$.

4.2.1 The Idea for $\mathbb{R}^n_+$

In this section we give heuristic arguments for the boundary conditions using the undiscounted case for clarity. Theorem 7 below will provide the formal proof in the general case of $V^{R,\gamma}$. For $n > 2$ there are edges and faces. So for a point x on the boundary we need to take into account all possible faces or edges a path could traverse as it approaches x. Hence, we consider all paths arising from the Skorokhod Problem that approach a point $x \in \partial\Omega$.

Subsolution Formulation

The definition of $V(x)$ implies that for any path $\phi(\cdot) = \Gamma(\psi(\cdot))$ to $x = \phi(T_2)$ from $0 = \phi(T_1)$ (some $0 \le T_1 < T_2$) we have

$$V(x) \le V(\phi(T_1)) + \int_{T_1}^{T_2} L(\dot{\psi}(t))dt.$$

If $\Phi \in \mathcal{C}^1$, $V \le \Phi$ with $V(x) = \Phi(x)$ (ie. $p = D\Phi(x) \in D_\Omega^+ V(x)$) and $0 < t \le T_2$ we have

$$\Phi(x) \le \Phi(\phi(T_2 - t)) + \int_{T_2-t}^{T_2} L(\dot{\psi}(t))dt$$

$$\frac{\Phi(\phi(T_2)) - \Phi(\phi(T_2 - t))}{t} - \frac{1}{t}\int_{T_2-t}^{T_2} L(\dot{\psi}(t))dt \le 0.$$

Let $t \to 0^+$, then

$$\langle D\Phi(\phi(T_2)), \dot{\phi}(T_2)\rangle - L(\dot{\psi}(T_2)) \le 0.$$

In this inequality we want to consider all paths $w = \dot{\phi}(T_2)$, $v = \dot{\psi}(T_2)$ which do occur for paths $\phi = \Gamma(\psi)$ reaching $x = \phi(T_2)$. If x is an interior point then $\dot{\phi}(T_2) = \dot{\psi}(T_2) = v$ is possible for any $v \in \mathbb{R}^n$. So we can conclude

$$H(p) = \sup_v \{\langle p, v\rangle - L(v)\} \le 0.$$

If $x \in \partial\Omega$, Lemma 2 in Chapter 2 says that piecewise linear paths with $w = \dot{\phi}(T_2)$ and $v = \dot{\psi}(T_2)$ do occur provided there exists $K \subseteq I(x)$ so that

$$w = P_K v$$

$$B_K v \ge 0$$

$$N_{I\setminus K}^T P_K v \ge 0.$$

So a subsolution necessary condition is

$$\max_{K\subseteq I}(H_K^-(p)) \le 0 \text{ for all } \xi \in D_\Omega^+ V(x) \tag{4.10}$$

where

$$H_K^-(p) = \sup_{B_K v \ge 0, N_{I\setminus K}^T P_K v \ge 0} \{\langle p, P_K v\rangle - L(v)\}. \tag{4.11}$$

Supersolution Formulation

This part of our heuristic derivation presumes that $\phi(\cdot) = \Gamma(\psi(\cdot))$ is optimal to $x = \phi(T_2)$, specifically for some $0 < t \leq T_2$

$$V(x) = V(\phi(T_2 - t)) + \int_{T_2-t}^{T_2} L(\dot{\psi})dt.$$

Suppose $\Phi \in \mathcal{C}^1$, $V \geq \Phi$ with $V(x) = \Phi(x)$ (ie. $p = D\Phi(x) \in D_\Omega^- V(x)$). It follows that

$$\frac{\Phi(\phi(T_2)) - \Phi(\phi(T_2 - t))}{t} - \frac{1}{t}\int_{T_2-t}^{T_2} L(\dot{\psi}(t))dt \geq 0.$$

Presuming continuity of $\dot{\phi}$ and $\dot{\psi}$ at T_2 we let $t \to 0^+$ to see that

$$\langle D\Phi(\phi(T_2)), \dot{\phi}(T_2)\rangle - L(\dot{\psi}(T_2)) \geq 0.$$

Unlike the subsolution argument this need not hold for any path we can construct to $\phi(T_2) = x$, but only for those which are optimal. We can not assume piecewise linearity as in Lemma 2, and thus must allow for a larger set of paths $w = \dot{\phi}(T_2)$, $v = \dot{\psi}(T_2)$ pairs. If x is an interior point then $\dot{\phi}(T_2) = \dot{\psi}(T_2)$ so we conclude

$$H(p) = \sup_v \{\langle p, v\rangle - L(v)\} \geq 0.$$

If $x \in \partial\Omega$, since $\phi(t) \to x$ as $t \to T_2$ and $\dot{\phi}(t) = \pi(\phi(t), \dot{\psi}(t))$ it seems that the most we could say is that $w = \pi(y, v)$ where $y \approx x$. If so then with $J = I(y) \subseteq I(x) = I$ and some $K \subseteq J \subseteq I$ we would have

$$w = P_K v,\ B_K v \geq 0,\ N_{J\setminus K}^T w \leq 0.$$

We must allow the possibility that $K = J$. So the possible v and w combinations are described by

$$w = P_K v,\ B_K v \geq 0,\ K \subseteq I.$$

This suggests that a supersolution necessary condition is

$$\max_{K\subseteq I}(H_K(p)) \geq 0 \text{ for all } \xi \in D_\Omega^- V(x) \tag{4.12}$$

where

$$H_K(p) = \sup_{B_K v \geq 0} \{\langle p, P_K v\rangle - L(v)\}. \tag{4.13}$$

The Proof of Theorem 7 will make this rigorous.

4.2.2 A General Representation for H_K and H_K^-

For a given $K \subseteq I(x)$ $H_K^-(p)$ and $H_K(p)$ are both instances of $H_C^o(P_K^T p)$, defined as follows for an arbitrary $m \times n$ matrix C

$$H_C^o(p) = \sup_{Cv \geq 0} \{\langle p, v\rangle - L(v)\}.$$

The following lemma describes an alternate formulation for H_C^o which will help us later when we consider an equivalent boundary condition formulation for two dimensions.

Lemma 6. *For a $m \times n$ matrix C and $\zeta \in \mathbb{R}^m$.*

$$H_C^o(p) = \inf_{\zeta \geq 0} H(p + C^T\zeta).$$

Proof. Let c_i be the rows of C, $i \in \{1, 2, ...m\}$ and $\mathcal{C} = \{v \in \mathbb{R}^n : \ Cv \geq 0\}$.

Since $H(p) \to \infty$ as $|p| \to \infty$, there does exist a $\zeta^o \geq 0$ which minimizes $H(p + C^T\zeta)$ over $\zeta \geq 0$ (see definition of H above, (4.2)). Let $p^o = p + C^T\zeta^o$ and $v^o = \nabla H(p^o)$. Thus

$$H(p^o) = \inf_{\zeta \geq 0} H(p + C^T\zeta).$$

Let $J = \{i : \zeta_i^o > 0\}$ and $J^c = \{i : \zeta_i^o = 0\}$. For $i \in J$ we have

$$
\begin{aligned}
0 = \frac{\partial}{\partial \zeta_i} H(p + C^T\zeta)|_{\zeta=\zeta^o} &= \langle c_i, \nabla H(p^o)\rangle \\
&= \langle c_i, v^o\rangle.
\end{aligned}
$$

For $i \in J^c$ we have

$$
\begin{aligned}
0 \leq \frac{\partial}{\partial \zeta_i} H(p + C^T\zeta)|_{\zeta=\zeta^o} &= \langle c_i, \nabla H(p^o)\rangle \\
&= \langle c_i, v^o\rangle.
\end{aligned}
$$

So $v^0 \in \mathcal{C}$.

Also we observe that

$$
\langle p - p^o, v^o\rangle = -\langle C^T\zeta^o, v^o\rangle = -\sum_{i\in J} \zeta_i^o \langle c_i, v^o\rangle = 0. \tag{4.14}
$$

For any $v \in C$ we have

$$
\langle p - p^o, v\rangle = -\sum_{i\in J^c} \zeta_i^o \langle c_i, v\rangle \leq 0.
$$

Therefore, for all $v \in C$ we have

$$
\begin{aligned}
\langle p, v\rangle - L(v) &= \langle p - p^o, v\rangle + \langle p^o, v\rangle - L(v) \\
&\leq \langle p^o, v\rangle - L(v) \\
&\leq H(p^o).
\end{aligned}
$$

and by (4.14) we have $\langle p, v^o\rangle - L(v^o) = H(p^o)$. Since $v^o \in \mathcal{C}$ we see that

$$
\sup_{v\in\mathcal{C}} \{\langle p, v\rangle - L(v)\} = H(p^o).
$$

Now we have

$$
H_C^o(p) = \sup_{v\in\mathcal{C}} \{\langle p, v\rangle - L(v)\} = H(p^o) = \inf_{\zeta\geq 0} H(p + C^T\zeta).
$$

$\square$

4.2.3 Discounted General Direct Formulation

Here we state and prove the *discounted general direct formulation* for the viscosity boundary conditions. If we take $\gamma = 0$ the proof for the *general direct formulation* for $V(x)$ in Ω follows easily from the proof of $V^{R,\gamma}(x)$ below.

Theorem 7. *For $x \in \Omega^R$, with $I = I(x)$, $V^{R,\gamma}(x)$ is a viscosity solution of $\gamma V^{R,\gamma}(x) + H(DV^{R,\gamma}(x)) = 0$ at interior points $x \in \Omega^R$:*

$$\gamma V^{R,\gamma}(x) + H(\xi) \leq 0 \text{ for all } \xi \in D^+_{\Omega^R} V^{R,\gamma}(x), \tag{4.15}$$

$$\gamma V^{R,\gamma}(x) + H(\xi) \geq 0 \text{ for all } \xi \in D^-_{\Omega^R} V^{R,\gamma}(x), \tag{4.16}$$

and when $I(x) \neq \emptyset$ $x \in \partial_I \Omega^R$,

$$\gamma V^{R,\gamma}(x) + \max_{K \subseteq I}(H^-_K(\xi)) \leq 0 \text{ for all } \xi \in D^+_{\Omega^R} V^{R,\gamma}(x), \tag{4.17}$$

$$\gamma V^{R,\gamma}(x) + \max_{K \subseteq I}(H_K(\xi)) \geq 0 \text{ for all } \xi \in D^-_{\Omega^R} V^{R,\gamma}(x). \tag{4.18}$$

Proof. We do not write out the argument for interior points separately. It follows from the following using $I = \emptyset$, the constraints $B_K v \geq 0$ and $N^T_{I \setminus K} P_K v \geq 0$ holding for all v, $P_\emptyset = I$, so that $H^-_\emptyset = H_\emptyset = H$.

Subsolution

By contradiction assume that (4.17) does not hold. For some $\xi \in D^+_{\Omega^R} V^{R,\gamma}(x)$,
and some K, $\emptyset \subseteq K \subseteq I$ $a > 0$,

$$\gamma V^{R,\gamma}(x) + \sup_{B_K v \geq 0, N^T_{I\setminus K} P_K v \geq 0} \{\langle \xi, P_K v\rangle - L(v)\} > a. \tag{4.19}$$

Let $D\Phi(x) = \xi$ where $\Phi \in \mathcal{C}^1$, $V^{R,\gamma}(x) = \Phi(x)$ and $V^{R,\gamma} \leq \Phi$ on $\overline{\Omega^R}$. Now let v be such that $w = P_K v$, $B_K v \geq 0$, $N^T_{I\setminus K} P_K v \geq 0$ and $\gamma\Phi(x) + \langle \xi, P_K v\rangle - L(v) \geq \frac{a}{2}$. We consider a linear path approaching x as in Lemma 2. Let $\psi(t)$ be such a path with $\dot\psi(t) = v$ for $t \in [T_2 - \delta_0, T_2]$, $\delta_0 > 0$, where $\phi(t)$ is a path with $\phi(T_2) = x$, $\dot\phi(t) = P_K v$, and for some $K \subseteq F \subseteq I$ $\phi(t) \in \partial_F \Omega^R$, $t \in [T_2 - \delta_0, T_2]$ with $N^T_{I\setminus F} P_K v > 0$. Using the continuity of Φ given $\tilde\epsilon = \frac{a}{4\gamma}$ there is $\delta_1 > 0$ such that $|\Phi(x) - \Phi(\phi(t))| \leq \tilde\epsilon$, $T_2 - \delta_1 \leq t \leq T_2$. Let $\tilde\delta = \min(\delta_0, \delta_1)$. Then for $t \in [T_2 - \tilde\delta, T_2]$, $\dot\psi(t) \equiv v$ and $\phi(t) = x + w(t - T_2)$ so that $\gamma\Phi(\phi(t)) + \langle \xi, \pi(\phi(t), \dot\psi(t))\rangle - L(\dot\psi(t)) > \frac{a}{4}$ for $t \in [T_2 - \tilde\delta, T_2]$. Using the continuity of $D\Phi(\phi(t))$, for $\epsilon \leq \frac{a}{4|P_K v|}$ with $\epsilon > 0$, there exists $\delta_2 > 0$ such that for $T_2 - \delta_2 \leq t \leq T_2$,

$$|D\Phi(\phi(T_2)) - D\Phi(\phi(t))| \leq \epsilon. \tag{4.20}$$

For $\delta = \min(\tilde\delta, \delta_2)$, we have

$$\begin{aligned}
\frac{d}{dt}(e^{\gamma t}\Phi(\phi(t))) &= \gamma e^{\gamma t}\Phi(\phi(t)) + e^{\gamma t}\langle D\Phi(\phi(t)), \pi(\phi(t), \dot\psi(t))\rangle \\
&= \gamma e^{\gamma t}\Phi(\phi(t)) + e^{\gamma t}\langle \xi, \pi(\phi(t), \dot\psi(t))\rangle - e^{\gamma t}\langle \xi - D\Phi(\phi(t)), \pi(\phi(t), \dot\psi(t))\rangle \\
&\geq \gamma e^{\gamma t}\Phi(\phi(t)) + e^{\gamma t}\langle \xi, \pi(\phi(t), \dot\psi(t))\rangle - e^{\gamma t}|\xi - D\Phi(\phi(t))||\pi(\phi(t), \dot\psi(t))| \\
&\geq e^{\gamma t}\left(\gamma\Phi(\phi(t) + \langle \xi, P_K v\rangle - \epsilon|P_K v|\right) \\
&\geq e^{\gamma t}\left(\gamma\Phi(\phi(t)) + \langle \xi, P_K v\rangle - \frac{a}{4}\right) > e^{\gamma t}L(\dot\psi(t))
\end{aligned}$$

So for $T_2 - \delta \leq T < T_2$, integrating both sides gives us

$$\begin{aligned}
e^{\gamma T_2}\Phi(\phi(T_2)) - e^{\gamma T}\Phi(\phi(T)) &> \int_T^{T_2} e^{\gamma t}L(\dot\psi(t))dt \\
e^{\gamma T_2}\Phi(\phi(T_2)) &> \int_T^{T_2} e^{\gamma t}L(\dot\psi(t))dt + e^{\gamma T}\Phi(\phi(T)) \\
V^{R,\gamma}(\phi(T_2)) &> \int_T^{T_2} e^{-\gamma(T_2 - t)}L(\dot\psi(t))dt + e^{-\gamma(T_2 - T)}V^{R,\gamma}(\phi(T)) \\
V^{R,\gamma}(x) &> \int_T^{T_2} e^{-\gamma(T_2 - t)}L(\dot\psi(t))dt + e^{-\gamma(T_2 - T)}V^{R,\gamma}(\phi(T))
\end{aligned}$$

which is a contradiction to the Dynamic Programming Principle, (3.3). This proves (4.17).

Supersolution

By contradiction assume that (4.18) does not hold. Then for some $\xi \in D^-_\Omega V^{R,\gamma}(x)$ and some $a > 0$ we have

$$\gamma V^{R,\gamma}(x) + \sup_{B_K v \geq 0} \{\langle \xi, P_K v\rangle - L(v)\} < -a. \tag{4.21}$$

for all $K \subseteq I$. Let $\Phi \in \mathcal{C}^1$ with $D\Phi(x) = \xi$, $V^{R,\gamma}(x) = \Phi(x)$ and $V^{R,\gamma} \geq \Phi$, on $\overline{\Omega^R}$. By virtue of Lemma 5 and Corollary 1 we can limit our considerations to paths $\phi(t) = \Gamma(\psi(t))$ $\phi(\bar{T}_2) = x$ with $|\dot{\psi}(t)| \leq c|b|$ where c is the constant from Corollary 1. Since $|\dot{\psi}(t)|$ is bounded, $|\dot{\phi}(t)|$ is also bounded. Note that $\dot{\phi}(t) = \dot{\psi}(t)$ or $\dot{\phi}(t) = P_K\dot{\psi}(t)$ depending on whether $\phi(t)$ is on the boundary and uses reflection. So from the bound on $|\dot{\psi}(t)|$, there is a bound $B > 0$ that applies to both $|\dot{\psi}(t)|$ and $|P_K\dot{\psi}(t)|$. Note that this B does not depend on $\psi(t)$ or $\phi(t)$ except for the fact that $|\dot{\psi}(t)| \leq c|b|$. For $x \in \partial_I\Omega^R$, we can find $\epsilon_o > 0$ so that if $|y - x| < \epsilon_o$ and $x_i > 0$ then $y_i > 0$. Therefore there exists $\delta_1 = \frac{\epsilon_0}{B}$, valid for any ψ with $|\dot{\psi}| \leq c|b|$, such that for all $t \in [\bar{T}_2 - \delta_1, \bar{T}_2]$, $|\phi(t) - \phi(\bar{T}_2)| < \epsilon_o$ and $I(\phi(t)) \subseteq I$.

Let $\tilde{\epsilon} = \frac{a}{4\gamma}$. Since $\Phi \in \mathcal{C}$ there exists δ_2 such that then for $\bar{T}_2 - \delta_2 \leq t \leq \bar{T}_2$,

$$|\Phi(\phi(t)) - \Phi(\phi(\bar{T}_2))| \leq \tilde{\epsilon}. \tag{4.22}$$

Let $\epsilon = \frac{a}{2B}$. Since $\Phi \in \mathcal{C}^1$there exists δ_3 such that for $\bar{T}_2 - \delta_3 \leq t \leq \bar{T}_2$,

$$|D\Phi(\phi(t)) - D\Phi(\phi(\bar{T}_2))| \leq \epsilon. \tag{4.23}$$

Define $\delta = \min(\delta_1, \delta_2, \delta_3)$.

Using the Dynamic Programming Principle we can find a nearly optimal, absolutely continuous path $\hat{\phi}(t) = \Gamma(\hat{\psi}(t))$ such that

$$V^{R,\gamma}(x) > \int_{T_2-\delta}^{T_2} e^{-\gamma(T_2-t)} L(\dot{\hat{\psi}}(t))dt + e^{-\gamma(\delta)} V^{R,\gamma}(\hat{\phi}(T_2 - \delta)) - e^{-\gamma\delta}\frac{a}{4}\delta. \tag{4.24}$$

Let $\hat{\epsilon} = e^{-\gamma\delta}\frac{a}{4}\delta$. This $\hat{\psi}(t)$ may not satisfy $|\dot{\hat{\psi}}(t)| \leq c|b|$. However by the Bounded Velocity Lemma we can slow down and rescale the function so that $|\dot{\psi}(t)| \leq c|b|$ where $\phi(t) = \Gamma(\psi(t))$ is the new slowed down version of $\hat{\phi}(t) = \Gamma(\hat{\psi}(t))$ defined on $[T_2 - \delta, \bar{T}_2]$, where $T_2 < \bar{T}_2$ since $|\dot{\psi}(t)| \leq |\dot{\hat{\psi}}(t)|$. (Otherwise $\phi = \Gamma(\psi)$ is optimal on $[T_2 - \delta, T_2]$ with the same $\hat{\epsilon}$ and we can skip to (4.27).) (4.24) then becomes

$$V^{R,\gamma}(x) > \int_{T_2-\delta}^{\bar{T}_2} e^{-\gamma(\bar{T}_2 - t)} L(\dot{\psi}(t))dt + e^{-\gamma(\bar{T}_2 - T_2 + \delta)} V^{R,\gamma}(\phi(T_2 - \delta))) - \hat{\epsilon}.$$

So by the Bounded Velocity Lemma $\phi = \Gamma(\psi)$ is a nearly optimal path on $[T_2 - \delta, \bar{T}_2]$ with $\phi(\bar{T}_2) = x$, specifically

$$V^{R\gamma}(\phi(T_2 - \delta)))e^{-\gamma(\bar{T}_2 - T_2 + \delta)} + \int_{T_2-\delta}^{\bar{T}_2} e^{-\gamma(\bar{T}_2 - t)} L(\dot{\psi}(t))dt < V^{R,\gamma}(x) + \hat{\epsilon}. \tag{4.25}$$

Consider $0 \leq T_2 - \delta < \bar{T}_2 - \delta$, by the Dynamic Programming Principle

$$V^{R,\gamma}(\phi(\bar{T}_2 - \delta)) \leq V^{R,\gamma}(\phi(T_2 - \delta))e^{-\gamma(\bar{T}_2 - T_2)} + \int_{T_2-\delta}^{\bar{T}_2-\delta} e^{-\gamma(\bar{T}_2 - \delta - t)} L(\dot{\psi}(t))dt. \tag{4.26}$$

Now we see that $\phi = \Gamma(\psi)$ is nearly optimal on $[\bar{T}_2 - \delta, \bar{T}_2]$ with the same $\hat{\epsilon}$:

$$\begin{aligned}
&V^{R,\gamma}(\phi(\bar{T}_2 - \delta))e^{-\gamma\delta} + \int_{\bar{T}_2-\delta}^{\bar{T}_2} e^{-\gamma(\bar{T}_2 - t)} L(\dot{\psi}(t))dt \\
&\leq V^{R,\gamma}(\phi(T_2 - \delta))e^{-\gamma(\bar{T}_2 - T_2 + \delta)} + \int_{T_2-\delta}^{\bar{T}_2-\delta} e^{-\gamma(\bar{T}_2 - t)} L(\dot{\psi}(t))dt + \int_{\bar{T}_2-\delta}^{\bar{T}_2} e^{-\gamma(\bar{T}_2 - t)} L(\dot{\psi}(t))dt \text{ by (4.26)} \\
&= V^{R,\gamma}(\phi(T_2 - \delta))e^{-\gamma(\bar{T}_2 - T_2 + \delta)} + \int_{T_2-\delta}^{\bar{T}_2} e^{-\gamma(\bar{T}_2 - t)} L(\dot{\psi}(t))dt \\
&< V^{R,\gamma}(x) + \hat{\epsilon} \text{ by (4.25).}
\end{aligned}$$

So we have

$$V^{R,\gamma}(\phi(\bar{T}_2 - \delta))e^{-\gamma\delta} + \int_{\bar{T}_2-\delta}^{\bar{T}_2} e^{-\gamma(\bar{T}_2 - t)} L(\dot{\psi}(t))dt < V^{R,\gamma}(x) + \hat{\epsilon}. \tag{4.27}$$

Since $\dot{\phi}(t) = P_K\dot{\psi}(t)$ for some $\emptyset \subseteq K \subseteq I$ we know from (4.21) and (4.22) that $\gamma\Phi(\phi(t)) + \langle \xi, \dot{\phi}(t) \rangle - L(\dot{\psi}(t)) < -\frac{3a}{4}$ for all $t \in [\bar{T}_2 - \delta, \bar{T}_2]$. Now for $\phi(t) = \Gamma(\psi(t))$ and $\bar{T}_2 - \delta \leq t \leq \bar{T}_2$

we have:

$$\begin{aligned}
\frac{d}{dt}(e^{\gamma t}\Phi(\phi(t))) - e^{\gamma t}L(\dot{\psi}(t)) &= \gamma e^{\gamma t}\Phi(\phi(t)) + e^{\gamma t}\langle D\Phi(\phi(t)), \dot{\phi}(t)\rangle - e^{\gamma t}L(\dot{\psi}(t)) \\
&= \gamma e^{\gamma t}\Phi(\phi(t)) + e^{\gamma t}\langle D\Phi(\phi(t)) - \xi, \dot{\phi}(t)\rangle + e^{\gamma t}\langle \xi, \dot{\phi}(t)\rangle - e^{\gamma t}L(\dot{\psi}(t)) \\
&\le e^{\gamma t}\left(|D\Phi(\phi(t)) - \xi||\dot{\phi}(t)| + \gamma\Phi(\phi(t)) + \langle \xi, \dot{\phi}(t)\rangle - L(\dot{\psi}(t))\right) \\
&\le e^{\gamma t}\left(\epsilon B + \gamma\Phi(\phi(t)) + \langle \xi, \dot{\phi}(t)\rangle - L(\dot{\psi}(t))\right) \\
&\le e^{\gamma t}\left(\frac{a}{2} + \gamma\Phi(\phi(t)) + \langle \xi, \dot{\phi}(t)\rangle - L(\dot{\psi}(t))\right) \\
&< -e^{\gamma t}\frac{a}{4}
\end{aligned}$$

Rearranging the terms above we have

$$\frac{d}{dt}(e^{\gamma t}\Phi(\phi(t))) < e^{\gamma t}L(\dot{\psi}(t)) - e^{\gamma t}\frac{a}{4}.$$

Integrating both sides :

$$\begin{aligned}
e^{\gamma \bar{T}_2}\Phi(\phi(\bar{T}_2)) - e^{\gamma(\bar{T}_2-\delta)}\Phi(\phi(\bar{T}_2-\delta)) &\le \int_{\bar{T}_2-\delta}^{\bar{T}_2} e^{\gamma t}L(\dot{\psi}(t))dt - \int_{\bar{T}_2-\delta}^{\bar{T}_2} e^{\gamma t}\frac{a}{4} \\
&\le \int_{\bar{T}_2-\delta}^{\bar{T}_2} e^{\gamma t}L(\dot{\psi}(t))dt - \frac{a}{4}\int_{\bar{T}_2-\delta}^{\bar{T}_2} e^{\gamma(\bar{T}_2-\delta)}dt \\
&= \int_{\bar{T}_2-\delta}^{\bar{T}_2} e^{\gamma t}L(\dot{\psi}(t))dt - \frac{a\delta}{4}e^{\gamma(\bar{T}_2-\delta)}.
\end{aligned}$$

Then

$$\Phi(\phi(\bar{T}_2)) \le e^{-\gamma\delta}\Phi(\phi(\bar{T}_2-\delta)) + \int_{\bar{T}_2-\delta}^{\bar{T}_2} e^{-\gamma(\bar{T}_2-t)}L(\dot{\psi}(t)) - \frac{a\delta}{4}e^{-\gamma\delta}.$$

So

$$V^{R,\gamma}(x) \le \int_{\bar{T}_2-\delta}^{\bar{T}_2} e^{-\gamma(\bar{T}_2-t)}L(\dot{\psi}(t))dt + V^{R,\gamma}(\phi(\bar{T}_2-\delta))e^{-\gamma\delta} - \hat{\epsilon}.$$

This gives us a contraction to (4.27). Therefore the supersolution condition holds as well.

$\square$

4.3 Barels-Lions Formulation

As mentioned above, Barles and Lions in [5] investigated the existence and uniqueness of the general form of the first-order Hamilton-Jacobi equation $H(x, u, \nabla u) = 0$, with a fully non-linear boundary condition of the form

$$F(x, u, \nabla u) = 0, \text{ on } \partial\Omega.$$

They established a uniqueness result based on the fact that the boundary function $F(x, t, p)$ is strictly increasing with respect to p in the normal direction. We exhibit a formulation of the viscosity-sense boundary conditions on the face $\partial_i\Omega$, $\Omega = \mathbb{R}^2_+$ in the general form considered in [5], which we will call *Barles-Lions formulation*: For $x \in \partial_i\Omega$

$$\min(H(\xi), F_i(\xi)) \leq 0 \text{ for all } \xi \in D^+_\Omega(x) \text{ (subsolution)}, \tag{4.28}$$

$$\max(H(\xi), F_i(\xi)) \geq 0 \text{ for all } \xi \in D^-_\Omega(x) \text{ (supersolution)}, \tag{4.29}$$

where the boundary function for $\partial_i\Omega$ is

$$\begin{aligned} F_i(\xi) &= \frac{1}{2}\left[\langle n_i, b + A\xi\rangle + \langle n_i, b + AP_i^T\xi\rangle\right] \qquad (4.30) \\ &= \langle n_i, b\rangle + \langle An_i + \frac{1}{2}\langle n_i, An_i\rangle d_i, \xi\rangle. \end{aligned}$$

Here the velocity projection matrix is $P_i = I + d_i n_i^T$. This is the P_K of Lemma 1 for $K = \{i\}$. The *Barles-Lions formulation* is equivalent to the *general direct formulation*, restated below for the special case for a face ($x_i = 0$ for only one coordinate i) where $\Omega = \mathbb{R}^2_+$:

$$\max_{K\subseteq I} H^-_K(\xi) \leq 0 \text{ for all } \xi \in D^+_\Omega V(x) \text{ (subsolution)},$$

$$\max_{K\subseteq I} H_K(\xi) \geq 0 \text{ for all } \xi \in D^-_\Omega V(x) \text{ (supersolution)}.$$

We will prove this equivalence for the discounted problem. For the discounted problem we have the following formulation of the viscosity-sense boundary conditions on the face $\partial_i\Omega^R$ in the general form considered in [5], which we will call the *discounted Barles-Lions* *formulation*:

$$\min(\gamma V^{R,\gamma}(x) + H(\xi), F_i(\xi)) \leq 0 \text{ for all } \xi \in D^+_{\Omega^R} V^{R,\gamma}(x) \text{ (subsolution)}, \tag{4.31}$$

$$\max(\gamma V^{R,\gamma}(x) + H(\xi), F_i(\xi)) \geq 0 \text{ for all } \xi \in D^-_{\Omega^R} V^{R,\gamma}(x) \text{ (supersolution)}. \tag{4.32}$$

with the same boundary function (4.30).

For $\mathbb{R}^2_+$ we know from Theorem 7 the following *discounted general direct formulation* of the viscosity solution properties does hold on $\partial_i \Omega^R$:

$$\max_{K\subseteq I} \left(\gamma V^{R,\gamma}(x) + H^-_K(\xi)\right) \leq 0 \text{ for all } \xi \in D^+_{\Omega^R} V^{R,\gamma}(x) \text{ (subsolution)}, \tag{4.33}$$

$$\max_{K\subseteq I} \left(\gamma V^{R,\gamma}(x) + H_K(\xi)\right) \geq 0 \text{ for all } \xi \in D^-_{\Omega^R} V^{R,\gamma}(x) \text{ (supersolution)}. \tag{4.34}$$

We state the equivalence in the following theorem.

Theorem 8. *In two dimensions the discounted Barles-Lions formulation is equivalent to the discounted general direct formulation.*

Proof. According to Lemma 6 we can rewrite the boundary functions in (4.33) and (4.34) as follows.

$$H^-_\emptyset(\xi) = \inf_{\zeta\geq 0} H(\xi + \zeta n_i), \tag{4.35}$$

$$H^-_{\{i\}}(\xi) = H_{\{i\}}(\xi) = \inf_{\zeta\geq 0} H(P_i^T \xi + \zeta n_i), \tag{4.36}$$

$$H(\xi) = H_\emptyset(\xi). \tag{4.37}$$

(The last is because of our convention that $B_\emptyset = 0$.) The connection between the *discounted Barels-Lions formulation* and the *discounted general direct formulation* is not apparent when considering just one ξ. It is only in considering *all* $\xi \in D^\pm_{\Omega^R} V^{R,\gamma}(x)$ that their relation emerges. It follows from Lemma 3 of Lions [25] that $D^\pm_{\Omega^R} V^{R,\gamma}(x)$ are made up of unions of sets of $\xi = p + \lambda n_i$ for λ in some unbounded interval. We will refer these sets of ξ as the *strands* of $D^\pm_{\Omega^R}$. To be more precise a strand of $D^+_{\Omega^R} V^{R,\gamma}(x)$ consists of all $\xi = p + \lambda n_i$ for some p and either all $\lambda \in \mathbb{R}$ (we will call this a *two-sided strand*) or all $\lambda \leq \lambda^+ < \infty$ with

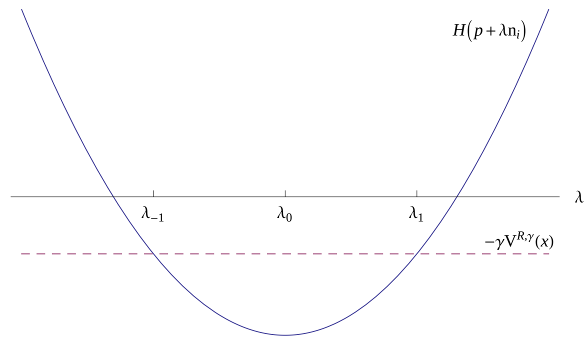

Figure 4.1: $\lambda \mapsto H(p + \lambda n_i)$

$\gamma V^{R,\gamma}(x) + H(p + \lambda^+ n_i) \leq 0$ (a *one-sided strand*). Similarly the strands of $D^-_{\Omega^R}$ consist of all $\xi = p + \lambda n_i$ for some p and either all $\lambda \in \mathbb{R}$ (two-sided strand) or all $\lambda \geq \lambda^- > -\infty$ with $\gamma V^{R,\gamma}(x) + H(p + \lambda^- n_i) \geq 0$ (one-sided strand). Lions' lemma shows that $D^{\pm}_{\Omega^R}$ are made up of unions of such strands. It is by comparing the two formulations of the boundary conditions for a full strand that their connection emerges.

The graph of $\lambda \mapsto H(p + \lambda n_i)$ is a parabola. The minimizing $\lambda = \lambda_0$ is determined by

$$0 = \langle n_i, \nabla H(p + \lambda_0 n_i) \rangle = \langle n_i, b + A(p + \lambda_0 n_i) \rangle,$$

which we easily solve to obtain

$$\lambda_0 = -\frac{\langle n_i, b + Ap \rangle}{\langle n_i, An_i \rangle}. \tag{4.38}$$

We know $H(p+\lambda n_i) \to \infty$ as $|\lambda| \to \infty$. For some p it is possible that $\gamma V^{R,\gamma}(x)+H(p+\lambda n_i) > 0$ for all λ. But otherwise there will be two roots, $\lambda_{-1} \leq \lambda_1$, of $\gamma V^{R,\gamma}(x) + H(p + \lambda n_i) = 0$. The minimum point $\lambda_0 = (\lambda_{-1} + \lambda_1)/2$ is their mean. Figure 4.1 is the typical graph of $\lambda \mapsto H(p + \lambda n_i)$.

Observe that for $\xi = p + \lambda n_i$ then we have

$$P_i^T \xi = p + \tilde{\lambda} n_i, \text{ where } \tilde{\lambda} = \langle d_i, p \rangle. \tag{4.39}$$

For $\xi = p + \lambda n_i$ we find that

$$\begin{aligned}
F_i(\xi) &= \frac{1}{2}[\langle n_i, b + A(p+\lambda n_i)\rangle + \langle n_i, b + A(p+\tilde{\lambda} n_i)\rangle] && (4.40)\\
&= \frac{1}{2}[2\langle n_i, b + Ap\rangle + \langle n_i, A\lambda n_i\rangle + \langle n_i, A\tilde{\lambda} n_i\rangle] && (4.41)\\
&= \frac{1}{2}[2\langle n_i, b + Ap\rangle \frac{\langle n_i, An_i\rangle}{\langle n_i, An_i\rangle} + \lambda\langle n_i, An_i\rangle + \tilde{\lambda}\langle n_i, An_i\rangle] && (4.42)\\
&= \frac{1}{2}\langle n_i, An_i\rangle(\lambda - 2\lambda_0 + \tilde{\lambda}) && (4.43)\\
&= \frac{1}{2}\langle n_i, An_i\rangle(\lambda - \lambda_{-1} - \lambda_1 + \tilde{\lambda}). && (4.44)
\end{aligned}$$

With these observations we compare the two boundary condition formulations within a strand.

Subsolutions

First consider a two-sided strand. Since $H(p + \lambda n_i) \to \infty$ it follows that

$$H_\emptyset^-(p + \lambda n_i) = \inf_{\zeta \geq 0} H(p + (\lambda + \zeta)n_i) \to \infty,$$

which means that the direct formulation of the subsolution fails. Since $F_i(p + \lambda n_i) \to \infty$ as well, the Barles-Lions formulation fails as well.

Now consider a one-sided strand. Since $\gamma V^{R,\gamma}(x) + H(p + \lambda^+ n_i) \leq 0$ we are in the case where $\gamma V^{R,\gamma}(x) + H(p + \lambda n_i) = 0$ has roots, so we have $\lambda_{-1} \leq \lambda^+ \leq \lambda_1$. In the direct formulation $\gamma V^{R,\gamma}(x) + H_\emptyset^-(\xi) \leq 0$ does hold for all ξ in the strand, since we can always take $\zeta = \lambda_1 - \lambda$. Using (4.39) and (4.36) we see that the direct formulation holds if and only if $\gamma V^{R,\gamma}(x) + \inf_{\zeta \geq 0} H(p + (\tilde{\lambda} + \zeta)n_i) \leq 0$, which we see to be equivalent to

$$\tilde{\lambda} \leq \lambda_1.$$

The Barles-Lions formulation says that $F_i(\xi) = \frac{1}{2}\langle n_i, An_i\rangle(\lambda - \lambda_{-1} - \lambda_1 + \tilde{\lambda}) \leq 0$ for those ξ in the strand with $\gamma V^{R,\gamma}(x) + H(\xi) > 0$, which corresponds to $\lambda < \lambda_{-1}$. We see that this is equivalent to $\lambda - \lambda_{-1} - \lambda_1 + \tilde{\lambda} \leq 0$ for all $\lambda < \lambda_{-1}$, which is also equivalent to $\tilde{\lambda} \leq \lambda_1$.

Supersolutions

If $\gamma V^{R,\gamma}(x) + H(\xi) \geq 0$ on the whole strand, then the Barles-Lions formulation is satisfied, and so is the direct formulation since $H_{\emptyset}(\xi) = H(\xi)$. This is always the case when $\gamma V^{R,\gamma}(x) + H(\xi) = 0$ has at most one root. So suppose $\gamma V^{R,\gamma}(x) + H(p + \lambda n_i) = 0$ has two distinct roots, $\lambda_{-1} < \lambda_1$, and the strand includes a point with $\gamma V^{R,\gamma}(x) + H(p + \lambda n_i) < 0$.

If the strand is one-sided, since it contains a point with $\gamma V^{R,\gamma}(x) + H(p + \lambda n_i) < 0$ we must have $\lambda^- \leq \lambda_{-1}$. So for either a one or two-sided strand the direct formulation says that $\gamma V^{R,\gamma}(x) + H_{\{i\}}(\xi) \geq 0$ for all $\lambda_{-1} < \lambda < \lambda_1$, which is simply that $\gamma V^{R,\gamma}(x) + \inf_{\zeta \geq 0}(H(p + (\tilde{\lambda} + \zeta)n_i)) \geq 0$. This is equivalent to

$$\lambda_1 \leq \tilde{\lambda}.$$

The Barles-Lions formulation says that $\lambda - \lambda_{-1} - \lambda_1 + \tilde{\lambda} \geq 0$ for all $\lambda_{-1} < \lambda < \lambda_1$, which is also equivalent to $\lambda_1 \leq \tilde{\lambda}$.

Thus in all cases the direct and Barles-Lions formulations coincide. $\square$

Chapter 5

Uniqueness

5.1 Examples of Non Uniqueness

Avram, Dai and Hasenbein in [3] have provided complete and explicit expressions for $V(x)$ in two dimensions. According to [3], an optimal path to a point $x \in \mathbb{R}^2_+$ is influenced by the boundary if it is contained in a certain cone associated with that boundary. The boundary influence is determined by two quantities, the "exit velocity" and the "entrance velocity." When x is not in one of these cones the optimal path is a direct linear path to x. When x is contained in one of these cones then the optimal path first travels along the boundary to a point w then leaves the boundary and enters the interior at a unique entrance angle to reach x. It is possible that one or both of these cones do not exist in which case the boundary or boundaries are not reflective and the constraint mechanism of the Skorokhod Problem has no influence on the optimal path.

By choosing specific values for the parameters (b, A, D) and using the formulas for $V(x)$ found in [3] we can produce explicit expressions for $V(x)$, which we then use to check our viscosity solution boundary condition formulation. We present two such examples here. In one of them, but not the other, we find that the zero function is also a solution. This tells us that solutions are not unique.

For the differentiable functions $V(x)$ below we check our boundary condition formulation on the two faces $\partial_1\Omega$ and $\partial_2\Omega$. The sub- and superdifferentials are of the form $\xi = p + \lambda n_i$ with $p = DV(x)$. When $\lambda \leq 0$, $\xi \in D^+_\Omega V(x)$ and $\lambda \geq 0$, $\xi \in D^-_\Omega V(x)$. Because of the Scaling Lemma it is enough to check one point on each boundary.

Example 1: ($\partial_1\Omega$ is reflective but $\partial_2\Omega$ is not reflective.)

Let $b = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. D is a P-matrix. Checking stability we see that $D^{-1}b = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$. Using techniques from [3] with $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we find the following explicit formula for $V(x)$. For x in a neighborhood of $\partial_1\Omega$,

$$V(x) = 3x_1 + x_2;$$

and for x in a neighborhood of $\partial_2\Omega$,

$$V(x) = \frac{\sqrt{5}}{||x||}(x_1^2 + x_2^2) + (2x_1 - x_2).$$

For $\xi = p + \lambda n_i$ we check the *Barles-Lions formulation* graphically by graphing $H(\xi)$ and $F_i(\xi)$ depicted in blue and green respectively in Figure 5.1. We see that V does satisfy our boundary condition formulation. For $\lambda \leq 0$ which correspond to $\xi \in D^+_\Omega V(x)$, we see that $\min(H(\xi), F_i(\xi)) \leq 0$. For $\lambda \geq 0$ which correspond to $\xi \in D^-_\Omega V(x)$, we see that $\max(H(\xi), F_i(\xi)) \geq 0$. Thus V satisfies our Barles-Lions boundary condition formulation in $\mathbb{R}^2_+$.

In Figure 5.2 we check to see whether the zero function $W(x) = 0$ is also a solution. By examining the figures we see that on $\partial_2\Omega$ for some $\lambda > 0$ $\max(H(\xi), F_i(\xi)) < 0$. So W fails the supersolution boundary condition on $\partial_2\Omega$.

Next we consider an example such that the zero function is also a viscosity solution.

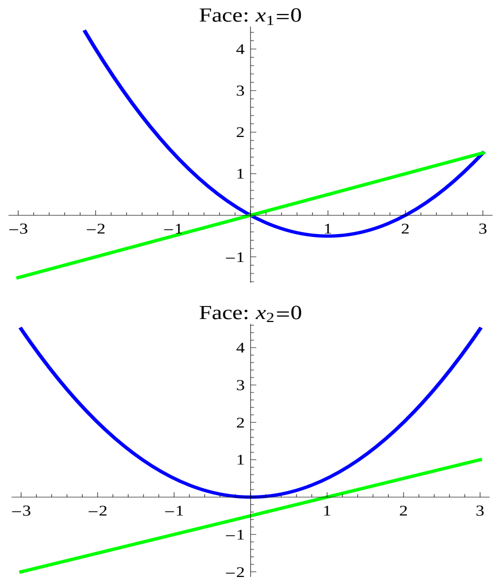

Figure 5.1: Example 1

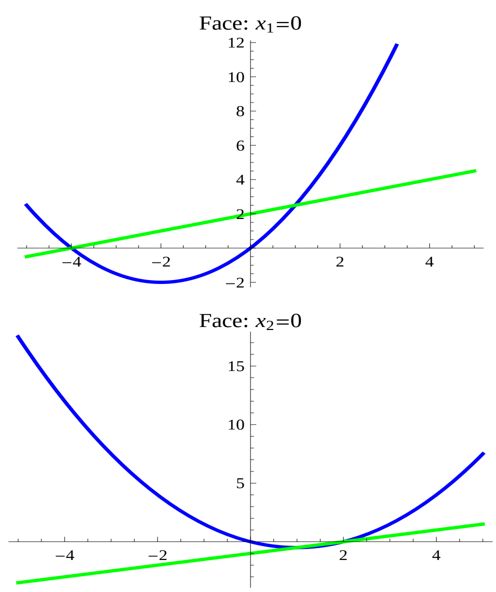

Figure 5.2: Example 1-zero function

Example 2: ($\partial_1\Omega$ and $\partial_2\Omega$ are not reflective)

Here we will consider $b = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Checking stability we see that $D^{-1}b = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$. By the considerations of [3] we find that for all $x \in \Omega$

$$V(x) = \frac{2}{||x||}(x_1^2 + x_2^2) + (2x_1).$$

Considering Figure 5.3 and Figure 5.4 we see that V and the $W = 0$ are both solutions. For $\lambda \leq 0$ which correspond to $\xi \in D_\Omega^+ V(x)$ and $\xi \in D_\Omega^+ W(x)$, we see that $\min(H(\xi), F_i(\xi)) \leq 0$. Likewise for $\lambda \geq 0$ which correspond to $\xi \in D_\Omega^- V(x)$ and $\xi \in D_\Omega^-$, we see that $\max(H(\xi), F_i(\xi)) \geq 0$. So V and W are both viscosity solutions with the Barles-Lions boundary conditions.

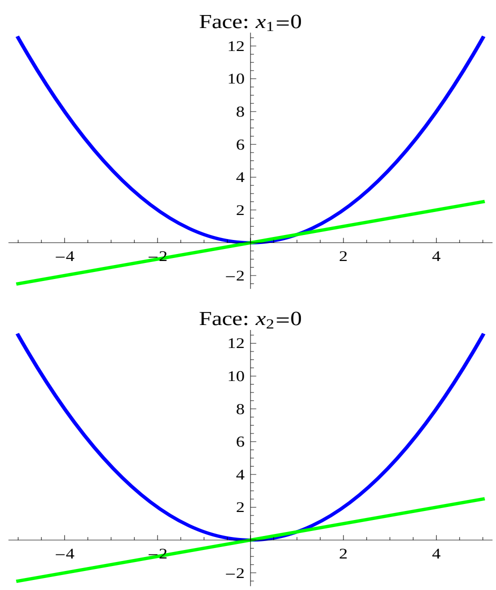

Figure 5.3: Example 2

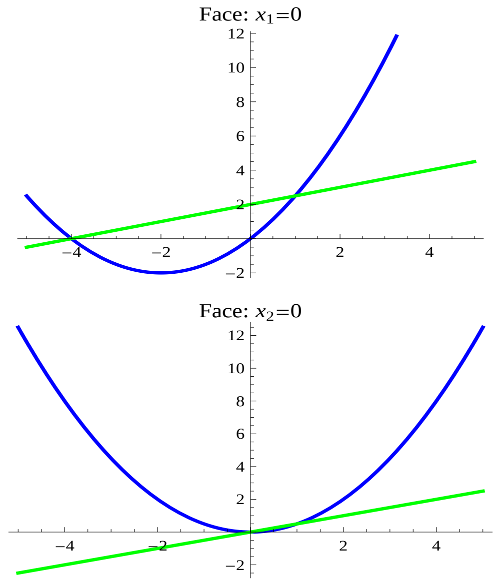

Figure 5.4: Example 2-zero function

5.2 When Zero is a Viscosity Solution

The zero function $W(x) = 0$ is always a solution of $H(DW(x)) = 0$ on the interior. As we saw in the previous examples it is a viscosity solution on the boundary of Ω. The lemma below provides a necessary and sufficient condition for when zero function is a viscosity solution on $\partial\Omega$ in the general case of $\Omega = \mathbb{R}^n_+$.

Lemma 7. *$W(x) = 0$ is a viscosity solution on $\partial\Omega$ if and only if for $x \in \partial\Omega$ $\langle n_i, b\rangle \geq 0$ for all $i \in I(x)$.*

Proof. **Subsolution**

Consider

$$H_K^-(\xi) = \sup_{B_k v \geq 0, N_{I\setminus K}^T P_K v \geq 0} \{\langle \xi, P_K v\rangle - L(v)\}$$

where as in Section 2.3 $B_K = -(N_K^T D_k)^{-1} N_K^T$ and $P_K = I + D_K B_K$. Since $DW(\cdot) = 0$ the superdifferentials are of the form $\xi = -N_I \lambda_I$, with $\lambda_i \geq 0$ for all $i \in I$. Note that the supremum above is taken over v such that $N_{I\setminus K}^T P_K v \geq 0$. This means that

$$-\lambda_{I\setminus K}^T N_{I\setminus K}^T P_K v \leq 0.$$

We also have

$$\begin{aligned}
\langle -N_K\lambda_K, P_K v\rangle &= \langle -N_K\lambda_K, v\rangle + \langle -N_K\lambda_K, D_K B_K v\rangle) \\
&= -\lambda_K^T N_K^T v + \lambda_K^T N_K^T D_K (N_K^T D_K)^{-1} N_K^T v) \\
&= -\lambda_K^T (N_K^T v - N_K^T D_K (N_K^T D_K)^{-1} N_K^T v) \\
&= -\lambda_K^T (N_K^T v - N_K^T v) = 0.
\end{aligned}$$

So $\langle \xi, P_K v\rangle \leq 0$, giving us

$$H_K^-(\xi) \leq \sup_{B_k v \geq 0, N_{I\setminus K}^T P_K v \geq 0} \{-L(v)\} \leq 0.$$

Therefore $\max_{K\subseteq I}(H_K^-(\xi)) \leq 0$ and $W = 0$ is always a subsolution.

Supersolution

We will show that for $x \in \partial\Omega$, $W(x) = 0$ is a supersolution if and only if $N_I^T b \geq 0$. First assume $N_I^T b \geq 0$ and consider

$$\max_{K\subseteq I}(H_K(\xi)) \geq 0 \text{ for all } \xi \in D^- V_\Omega(x).$$

We only need to show

$$H_K(\xi) = \sup_{B_K v \geq 0} \{\langle \xi, P_K v\rangle - L(v)\} \geq 0$$

for one K. The subdifferentials are of the form $\xi = N_I\lambda_I$, $\lambda_i \geq 0$ for all $i \in I$, . Take $K = \emptyset$ so that $H(\xi) = \langle \xi, Ab\rangle + \langle \xi, A\xi\rangle$, then

$$\begin{aligned} H(\xi) &= \langle N_I\lambda_I, b\rangle + \frac{1}{2}\langle N_I\lambda_I, AN_I\lambda_I\rangle \\ &\lambda_I^T N_I^T b + \frac{1}{2}\lambda_I^T N_I^T A\lambda_I N_I \\ &\geq 0, \end{aligned}$$

by our assumption and the fact that A is symmetric positive definite. So $W = 0$ is a supersolution if $N_I^T b \geq 0$. If we assume that W is a supersolution then we can consider each face $\partial_i\Omega$ separately and use our equivalent Barles-Lions formulation for 2 dimensions. For $x \in \partial_i\Omega$, the $\xi \in D_{\Omega}^{-}(x)$ consist of $\xi = \lambda_i n_i,\ \lambda_i \geq 0$. Assume by contradiction $\langle n_i, b\rangle < 0$ then

$$\lambda_0 = -\frac{\langle n_i, b\rangle}{\langle n_i, An_i\rangle} > 0.$$

Note that $H(\xi)$ has roots at $\lambda = 0$ and $\lambda = 2\lambda_0$. For $0 < \lambda < 2\lambda_0$ $H(\xi) < 0$ which means that $F_i(\xi) \geq 0$ since $W(x)$ is a supersolution, but by (4.30) we have

$$F_i(\xi) = \frac{1}{2}\langle n_i, An_i\rangle(\lambda - 2\lambda_0) < 0.$$

So we have a contradiction. Therefore If W is a supersolution then $\langle n_i, b\rangle \geq 0$ for all $i \in I$, and $N_I^T b \geq 0$.

So W is a supersolution if and only if $N_I^T b \geq 0$. □

5.3 Uniqueness in $\mathbb{R}_+^2$

Our next goal is to characterize $V(x)$ uniquely in terms of viscosity solutions. From the previous section we know that the zero function is often also a solution on the boundary of Ω, hence additional conditions are needed to characterize $V(x)$ uniquely. Barles and Lions in [5] established uniqueness based on the hypotheses that the boundary function F is strictly increasing in the direction of the outward normals n and that H is uniformly continuous. We are interested in

$$H(p) = \sup_v \{\langle p, v\rangle - L(v)\} = \frac{1}{2}\langle p, Ap\rangle + \langle p, b\rangle,$$
$$F_i(p) = \langle n_i, b\rangle + \langle An_i + \frac{1}{2}\langle n_i, An_i\rangle d_i, p\rangle.$$

For our boundary function F_i,

$$\frac{d}{d\lambda} F_i(p + \lambda n_i) = \frac{1}{2}\langle n_i, An_i\rangle > 0.$$

Therefore F_i is strictly increasing in the direction of n_i.

Another important property of F_i is Lipschitz continuity. Note that $|DF_i(p)| = |An_i + \frac{1}{2}\langle n_i, An_i\rangle d_i|$ for all p. Then

$$\begin{aligned} |F_i(p) - F_i(q)| &= |\langle An_i + \frac{1}{2}\langle n_i, An_i\rangle d_i, p\rangle - \langle An_i + \frac{1}{2}\langle n_i, An_i\rangle d_i, q\rangle| \\ &= |\langle An_i + \frac{1}{2}\langle n_i, An_i\rangle d_i, (p - q)\rangle| \\ &\leq |An_i + \frac{1}{2}\langle n_i, An_i\rangle d_i||p - q| \\ &= |DF_i||p - q|. \end{aligned}$$

Therefore

$$|F_i(p) - F_i(q)| \leq C_{F_i}|p - q|, \tag{5.1}$$

where $C_{F_i} = |An_i + \frac{1}{2}\langle n_i, An_i\rangle d_i|$.

However, unlike [5], our H is not uniformly continuous. Another difference is that they consider a domain which is a smooth, bounded open subset of $\mathbb{R}^n$. Consequently, we can not apply Barles and Lions' result directly to our uniqueness argument. By considering our discounted problem in a truncated region Ω^R and making adjustments to our Hamiltonian H we will be able to adapt their proof to deal with the boundary condition. We then show that we can pass to the limit ($\gamma \to 0$, $R \to \infty$) to obtain a uniqueness result for $V(x)$ for $\Omega = \mathbb{R}^2_+$, stated in the following theorem.

Theorem 9. *In $\mathbb{R}^2_+$ $V(x)$ is minimal among all nonnegative continuous supersolutions $w(x)$ of $H(DV(x)) = 0$ with boundary conditions $F_i(DV(x)) = 0$ (interpreted in the Barles-Lions sense) where $w(0) = 0$ and $w(x) \to +\infty$ as $|x| \to \infty$. In other words $V(x) \leq w(x)$ for any such supersolution.*

Here is an outline of the proof. To prove this we consider any $w(x)$ as in the theorem and use it in Definition 9 of $V^{R,\gamma}(x)$ in a truncated domain Ω^R.

1. In $\Omega^R = \{x \in \Omega :\ 0 < |x| < R\}$, $V^{R,\gamma}(x)$ is a viscosity solution of

   $\gamma u + H(Du) = 0$ with Barles-Lions type boundary conditions $F_i(Du(x)) = 0$ (on $\partial_i \Omega^R$).

2. $w(x)$ is a supersolution of the above discounted problem, *with* the γw term, and satisfies $V^{R,\gamma}(x) \leq w(x)$ for $|x| = R, 0$.

3. We apply the Barles and Lions proof in [5] to prove a comparison result, Theorem 10 below. This gives us $V^{R,\gamma}(x) \leq w(x)$ on $\overline{\Omega^R}$

4. Next we let $\gamma \to 0$ to show $V^{R,\gamma}(x) \to V^{R,0}(x)$ so that $V^{R,0}(x) \leq w(x)$ for all $x \in \overline{\Omega^R}$.

5. Finaly we argue that $V^{R,0}(x) \to V(x)$ as $R \to \infty$ so that $V(x) \leq w(x)$ in Ω, as desired.

Proof

Step 1 was already established in Theorem 7. Since $w(x)$ is a nonnegative supersolution of the undiscounted problem then for $\gamma > 0$, $w(x)$ is also a supersolution of the above discounted problem. From the definition of $V^{R,\gamma}(\cdot)$ it is clear that $V^{R,\gamma}(x) \leq w(x)$ for $x = 0$ and $|x| = R$. Therefore **step 2** is also established. For **step 3** we will prove the following theorem based on the argument of [5] to show $V^{R,\gamma}(x) \leq w(x)$ for all x in $\overline{\Omega^R}$.

Theorem 10. *Suppose $\gamma > 0$ and $u(x)$ and $w(x)$ are continuous on $\overline{\Omega^R}$ and are subsolution and supersolution (respectively) of*

$$\gamma v(x) + H(Dv(x)) = 0$$

in Ω^R with boundary conditions $F_i(Dv(x)) = 0$ on the faces $\partial_i\Omega^R$, with $u(x) \leq w(x)$ for $x = 0$ and $|x| = R$. Then $u(x) \leq w(x)$ on $\overline{\Omega^R}$.

It is understood that H and F_i refer specifically to

$$
\begin{aligned}
H(p) &= \frac{1}{2}\langle p, Ap\rangle + \langle p, b\rangle,\\
F_i(p) &= \langle n_i, b\rangle + \langle An_i + \frac{1}{2}\langle n_i, An_i\rangle d_i, p\rangle,
\end{aligned}
$$

and that the boundary conditions are interpreted in the sense of (4.31) and (4.32): for $x \in \partial_i\Omega^R$

$$\min(\gamma v(x) + H(p), F_i(p)) \leq 0 \text{ for all } p \in D^+_{\Omega^R}v(x) \text{ (subsolution)}, \tag{5.2}$$

$$\max(\gamma v(x) + H(p), F_i(p)) \geq 0 \text{ for all } p \in D^-_{\Omega^R}v(x) \text{ (supersolution)}. \tag{5.3}$$

Note that the boundary conditions are only for $0 < |x| < R$ with $x_i = 0$.

Most proofs of comparison results for viscosity solutions follow a common strategy: assume that $M = \sup(u(x) - w(x)) > 0$ and derive a contradiction. Intuitively, if $\bar{x}$ is an interior maximizing point for $u - w$ and u and w are smooth, then it would follow that $\xi = Du(\bar{x}) = Dw(\bar{x})$ belongs to both $D^+u(\bar{x})$ and $D^-w(\bar{x})$, so we get

$$\gamma u(\bar{x}) + H(\xi) \leq 0 \leq \gamma w(\bar{x}) + H(\xi),$$

which is not possible, since $u(\bar{x}) = M + w(\bar{x}) > w(\bar{x})$. Of course it is not this simple in general, because u and w may not be smooth, and it ignores the possibility that $\bar{x}$ could be a boundary point. To accommodate non-smooth functions the standard approach is to consider maximizing

$$\Phi(x, y) = u(x) - w(y) - \frac{|x-y|^2}{\epsilon^2}$$

over pairs (x, y). Simple estimates show that a maximizing pair $(\bar{x}^\epsilon, \bar{y}^\epsilon)$ must satisfy $|\bar{x}^\epsilon - \bar{y}^\epsilon|/\epsilon \to 0$ as $\epsilon \to 0$. $\Phi(x, y) \leq \Phi(\bar{x}^\epsilon, \bar{y}^\epsilon)$ implies that $\xi^\epsilon = 2(\bar{x}^\epsilon - \bar{y}^\epsilon)/\epsilon$ belongs to both $D^+u(\bar{x}^\epsilon)$ and $D^-w(\bar{y}^\epsilon)$. With appropriate continuity hypotheses, taking the limit in

$$\gamma u(\bar{x}^\epsilon) + H(\xi^\epsilon) \leq 0 \leq \gamma w(\bar{y}^\epsilon) + H(\xi^\epsilon), \tag{5.4}$$

leads to essentially the same contradiction as above. To deal with boundary conditions the idea of [5] is that by adding several additional terms to Φ it is possible to insure that the boundary conditions do not come into play at $\bar{x}^\epsilon$ and $\bar{y}^\epsilon$, so that we end up essentially back at (5.4) again. But the construction is delicate, involving several new parameters which must be carefully managed. The presentation in [5] leaves numerous details to the reader. Although we are not trying to be as general as they, we do need to pay attention to additional details because of the boundary at $x = 0$ and $|x| = R$. On the other hand our particular circumstances allow some simplifications. For these reasons we write out the proof of Theorem 10 in full.

Proof. We begin with the assumption that

$$M = \sup\{u(x) - w(x) \,:\, x \in \overline{\Omega^R}\} \tag{5.5}$$

is positive, $M > 0$, and will produce a contradiction.

Preliminaries

The argument below will use the hypothesis that H is uniformly continuous. As previously mentioned, our H does not satisfy this. However observe that if we replace H by a new Hamiltonian $\tilde{H}$ so that the signs of $\gamma v(x) + H(p)$ and $\gamma v(x) + \tilde{H}(p)$ agree, then $v(\cdot)$ is a viscosity solution for the H equation if and only if it is a solution for the $\tilde{H}$ equation. If B is a bound on both $|u|$ and $|w|$ on $\overline{\Omega^R}$, then altering H so that $H(p) = \tilde{H}(p)$ when $H(p) \leq \gamma B$ and $H(p) > \gamma B$ if and only if $\tilde{H}(p) > \gamma B$ will produce a new Hamiltonian for which our $u(\cdot)$ and $w(\cdot)$ are still solutions. Since $H(p) \to +\infty$ as $|p| \to \infty$ we can alter H outside a sufficiently large ball to produce such a substitute $\tilde{H}$. Moreover we can do this so that $\tilde{H}$ is smooth and constant outside a compact set. This makes $\tilde{H}$ uniformly continuous, while $u(\cdot)$ and $w(\cdot)$ remain solutions. By making this replacement we can assume that H is uniformly continuous.

We observe that F_i is linear and increasing in the n_i direction.

$$F_i(p + sn_i) = F_i(p) + s\nu_i \tag{5.6}$$

where

$$\nu_i = \langle An_i + \frac{1}{2}\langle n_i, An_i\rangle d_i, n_i\rangle = \frac{1}{2}\langle n_i, An_i\rangle > 0.$$

As we saw, F_i is Lipschitz with constant

$$C_{F_i} = |An_i + \frac{1}{2}\langle n_i, An_i\rangle d_i|.$$

Because $u(x) - w(x) \leq 0$ for $x = 0$ and $|x| = R$, and both functions are continuous, there is $\delta > 0$ so that

$$u(x) - w(y) < M/3 \text{ whenever } \max(|x|, |y|) < 3\delta \text{ or } R - 3\delta < \min(|x|, |y|).$$

We will take $0 \leq d(x) \leq 1$ to be a C^2 function on $\overline{\Omega^R}$, positive in the interior with $d(x) = 0$ on the boundary, and

$$Dd(x) = -n_i \text{ when } x \in \partial_i\Omega^R \text{ and } |x| > \delta.$$

(Such a function is easily constructed as $d(x) = \delta\phi(x_1/\delta)\phi(x_2/\delta)$ where $\phi : \mathbb{R} \to \mathbb{R}$ is C^2 with $\phi(x) = x$ for $x < 0$, $\phi(x) = 1$ for $x > 1$ and $0 < \phi(x)$ for $0 < x < 1$.) We define $n(x) = -Dd(x)$. Then both $d(x)$ and $n(x)$ will be Lipschitz in $\overline{\Omega^R}$. Let C_d, C_n be Lipschitz constants for d and n respectively.

We will use C (with no subscript) to denote a generic positive constant. Its value may change from one instance to the next.

First Perturbation: M^η

For $\eta > 0$ define

$$M^\eta = \sup_{\overline{\Omega^R}} \left(u(x) - w(x) + 2\eta d(x)\right). \tag{5.7}$$

Since $d \geq 0$ it is elementary that M^η is nondecreasing in η, and

$$M = \lim_{\eta \downarrow 0} M^\eta.$$

Second Perturbation: M_ϵ^η

For $\epsilon, \eta > 0$ define

$$M_\epsilon^\eta = \sup\left\{u(x) - w(y) - \frac{|x-y|^2}{\epsilon^2} + \eta d(x) + \eta d(y) \,:\, x, y \in \overline{\Omega^R},\ d(x) = d(y)\right\}. \tag{5.8}$$

Observe that this maximum is over pairs, but subject to the constraint that $d(x) = d(y)$. A maximizing pair will be denoted $(x^{\epsilon,\eta}, y^{\epsilon,\eta})$. For convenience, let

$$\Psi(x,y) = u(x) - w(y) - \frac{|x-y|^2}{\epsilon^2} + \eta d(x) + \eta d(y). \tag{5.9}$$

By considering $x = y$ we see that

$$0 < M \le M^\eta \le M_\epsilon^\eta.$$

In particular $0 < \Psi(x^{\epsilon,\eta}, y^{\epsilon,\eta})$, which implies that

$$\frac{|x^{\epsilon,\eta} - y^{\epsilon,\eta}|^2}{\epsilon^2} \le u(x^{\epsilon,\eta}) - w(y^{\epsilon,\eta}) + \eta d(x^{\epsilon,\eta}) + \eta d(y^{\epsilon,\eta}).$$

All the functions on the right are bounded (on $\overline{\Omega^R}$) so there is a constant C such that

$$|x^{\epsilon,\eta} - y^{\epsilon,\eta}| \le \epsilon C. \tag{5.10}$$

Third Perturbation: $\Phi(x,y)$

For $\epsilon, \eta, c_\epsilon, \alpha$ define

$$\begin{aligned}\Phi(x,y) &= u(x) - w(y) - \frac{|x-y|^2}{\epsilon^2} + \eta d(x) + \eta d(y) \\ &\qquad + c_\epsilon(d(x) - d(y)) - \frac{(d(x)-d(y))^2}{\alpha^2} - |x - x^{\epsilon,\eta}|^2 - |y - y^{\epsilon,\eta}|^2 \\ &= u(x) - w(y) - \Upsilon(x,y),\end{aligned}$$

where

$$\Upsilon(x,y) = \frac{|x-y|^2}{\epsilon^2} - \eta d(x) - \eta d(y) - c_\epsilon(d(x)-d(y)) + \frac{(d(x)-d(y))^2}{\alpha^2} + |x - x^{\epsilon,\eta}|^2 + |y - y^{\epsilon,\eta}|^2.$$

Let $(\bar{x}, \bar{y})$ be a maximizing pair for $\Phi(x,y)$ over $\overline{\Omega^R} \times \overline{\Omega^R}$ (dependence on η, ϵ, c_ϵ, α is suppressed in the notation). The parameters ϵ, η, α are assumed positive, but there is no presumption about the sign of c_ϵ.

Estimates for Maximizing Pairs

Notice that $\Psi(x,y) \leq u(x) - w(y) + 2\eta$. Thus if $\eta < M/3$ then $\max(|x|,|y|) < 3\delta$ will imply that $\Psi(x,y) < M$. Consequently either $3\delta < |x^{\epsilon,\eta}|$ or $3\delta < |y^{\epsilon,\eta}|$. By virtue of (5.10), for sufficiently small ϵ ($\epsilon < \frac{\delta}{C}$) it follows that both of these points have norms at least 2δ. Similar reasoning applies near $|x| = R$. Thus, for $\eta < \eta_0 = M/3$ and ϵ sufficiently small, we have

$$2\delta < |x^{\epsilon,\eta}| < R - 2\delta \text{ and } 2\delta < |y^{\epsilon,\eta}| < R - 2\delta. \tag{5.11}$$

The following lemma provides important estimates.

Lemma 8. *Suppose* $\epsilon|c_\epsilon| \leq \rho(\epsilon)$ *where* $\rho(\epsilon) \to 0$ *as* $\epsilon \to 0$. *Then there exists a function* $k(\epsilon)$, *independent of both* $0 < \alpha < 1$ *and* $0 < \eta < \eta_0$, *with* $k(\epsilon) \to 0$ *as* $\epsilon \to 0$ *and such that*

$$|\bar{x} - \bar{y}| \leq \epsilon k(\epsilon), \quad |\bar{x} - x^{\epsilon,\eta}| \leq k(\epsilon), \quad |\bar{y} - y^{\epsilon,\eta}| \leq k(\epsilon), \quad \textit{and } \frac{(d(\bar{x}) - d(\bar{y}))^2}{\alpha^2} \leq k(\epsilon).$$

Proof. For each $\epsilon > 0$ define $k(\epsilon)$ to be the supremum of

$$\frac{|\bar{x} - \bar{y}|}{\epsilon}, \ |\bar{x} - x^{\epsilon,\eta}|, \ |\bar{y} - y^{\epsilon,\eta}|, \text{ and } \frac{(d(\bar{x}) - d(\bar{y}))^2}{\alpha^2} \tag{5.12}$$

taken over all $0 < \eta < \eta_0$, all maximizing pairs $(x^{\epsilon,\eta}, y^{\epsilon,\eta})$ for M_ϵ^η, all $0 < \alpha \leq 1$ and all corresponding maximizing pairs $(\bar{x}, \bar{y})$ for Φ. Note that these are bounded respectively by $2R/\epsilon$, $2R$, $2R$, and $2B + 2(\eta_0 + |c_\epsilon|)$ (the latter stemming from $0 \leq \Phi(\bar{x}, \bar{y})$ with B an upper bound for $|u|$ and $|v|$). Thus $k(\epsilon) < \infty$. Our task is to show that $k(\epsilon) \to 0$ as $\epsilon \to 0$. If this were not so there would exist a sequence $\epsilon_n \downarrow 0$ and sequences η_n, α_n along which one of the four quantities in (5.12) has a positive $\limsup$. Since Ω^R is bounded we can assume that the corresponding $\bar{x}_n$ and $\bar{y}_n$ converge as well. We suppress the sequence index $\cdot_n$ in the following to simplify the notation, but all terms should be understood to be the n-th terms of the corresponding sequences.

Since $d(x^{\epsilon,\eta}) = d(y^{\epsilon,\eta})$ we have

$$M^\eta \leq M^\eta_\epsilon = \Psi(x^{\epsilon,\eta}, y^{\epsilon,\eta}) \leq \Phi(x^{\epsilon,\eta}, y^{\epsilon,\eta}) \leq \Phi(\bar{x}, \bar{y}),$$

so

$$\liminf_{n\to\infty} \Phi(\bar{x}, \bar{y}) \geq \liminf_{n\to\infty} M^\eta. \tag{5.13}$$

On the other hand from $0 \leq \Phi(\bar{x}, \bar{y})$ it follows that

$$\begin{aligned}\frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} &\leq \sup|u| + \sup|w| + 2\eta \sup|d| + |c_\epsilon||d(\bar{x}) - d(\bar{y})| \\ &\leq 2B + 2\eta_0 + C_d|c_\epsilon||\bar{x}-\bar{y}|.\end{aligned}$$

So for some constant C we have

$$\frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} \leq C \max(1, |c_\epsilon||\bar{x}-\bar{y}|).$$

If the maximum is 1 we have

$$|\bar{x}-\bar{y}|^2 \leq C\epsilon^2.$$

If the maximum is $|c_\epsilon||\bar{x}-\bar{y}|$ then $\frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} \leq C|c_\epsilon||\bar{x}-\bar{y}|$ so that

$$|\bar{x}-\bar{y}| \leq C|c_\epsilon|\epsilon^2.$$

Since $\epsilon|c_\epsilon| \leq \rho(\epsilon) \to 0$, we conclude in either case that for some constant C

$$|\bar{x}-\bar{y}| \leq C\epsilon. \tag{5.14}$$

By hypothesis $\lim_n \bar{x}$ and $\lim_n \bar{y}$ exist. Since $\epsilon \to 0$ the above implies that the limits of $\bar{x}$ and $\bar{y}$ agree. Let z be their common value.

$$\lim_n \bar{x} = z = \lim_n \bar{y}.$$

Now

$$\begin{aligned}\Phi(\bar{x}, \bar{y}) &\leq u(\bar{x}) - w(\bar{y}) + \eta d(\bar{x}) + \eta d(\bar{y}) - \frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} + C_d|c_\epsilon||\bar{x}-\bar{y}| \\ &\leq u(\bar{x}) - w(\bar{y}) + \eta d(\bar{x}) + \eta d(\bar{y}) + C|c_\epsilon|\epsilon.\end{aligned}$$

In the limit as $n \to \infty$ we can say

$$\begin{aligned}\limsup \Phi(\bar{x}, \bar{y}) &\le \lim\left(u(\bar{x}) - w(\bar{y}) + \eta d(\bar{x}) + \eta d(\bar{y})\right)\\ &= \lim\left(u(z) - w(z) + \eta d(z) + \eta d(z)\right) \;\; (\eta \text{ still depends on the sequence})\\ &\le \lim M^{\eta}\end{aligned}$$

But because of (5.13) it follows that both

$$\Phi(\bar{x}, \bar{y}) \to M^{\eta} \text{ and } u(\bar{x}) - w(\bar{y}) + \eta d(\bar{x}) + \eta d(\bar{y}) \to M^{\eta}.$$

The limit of the difference must thus be 0.

$$\begin{aligned}&u(\bar{x}) - w(\bar{y}) + \eta d(\bar{x}) + \eta d(\bar{y}) - \Phi(\bar{x}, \bar{y})\\ &\qquad= \left(\frac{|\bar{x} - \bar{y}|^2}{\epsilon^2} - c_\epsilon(d(\bar{x}) - d(\bar{y})) + \frac{(d(\bar{x}) - d(\bar{y}))^2}{\alpha^2} + |\bar{x} - x^{\epsilon,\eta}|^2 + |\bar{y} - y^{\epsilon,\eta}|^2\right) \to 0.\end{aligned}$$

Using (5.14) we know that $|c_\epsilon(d(\bar{x}) - d(\bar{y}))| \le |c_\epsilon| C_d C\epsilon \to 0$. Since all other terms are nonnegative they each converge to 0 individually. Thus none of the quantities in (5.12) can have a positive $\limsup$. This completes the proof of the lemma. □

Corollary 2. *Under the same hypotheses as the lemma, for all ϵ sufficiently small*

$$\delta < |\bar{x}| < R - \delta \text{ and } \delta < |\bar{y}| < R - \delta. \tag{5.15}$$

This follows easily from (5.11). Thus for all small ϵ we will have $n(\bar{x}) = n_i$ if $\bar{x} \in \partial_i \Omega^R$, and likewise for $\bar{y}$. Moreover, if $\bar{x}$ and $\bar{y}$ are on the boundary they must be on the same boundary.

Choosing c_ϵ and α to Avoid Boundary Conditions

The maximizing pair $(x^{\epsilon,\eta}, y^{\epsilon,\eta})$ does not depend on c_ϵ or α, but $(\bar{x}, \bar{y})$ does. We now discuss how it is possible to choose c_ϵ (with $\epsilon|c_\epsilon| \to 0$) and α to insure that the boundary conditions are *not satisfied* by F_i and therefore the Hamiltonian inequalities must hold in the sub- and supersolution conditions at $\bar{x}$ and $\bar{y}$. As we will see in the next subsection the sub- and superdifferentials we will be using at $\bar{x}$ and $\bar{y}$ are

$$\Upsilon_x(\bar{x},\bar{y}) \in D^+_{\Omega^R}u(\bar{x}), \quad -\Upsilon_y(\bar{x},\bar{y}) \in D^-_{\Omega^R}w(\bar{y}).$$

We want to *avoid* either

$$F_i(\Upsilon_x(\bar{x},\bar{y})) \leq 0 \text{ with } \bar{x} \in \partial_i\Omega^R, \tag{5.16}$$

or

$$F_i(-\Upsilon_y(\bar{x},\bar{y})) \geq 0 \text{ with } \bar{y} \in \partial_i\Omega^R. \tag{5.17}$$

The choice of c_ϵ and α will depend on the location of $x^{\epsilon,\eta}$ and $y^{\epsilon,\eta}$. Recall the constraint $d(x^{\epsilon,\eta}) = d(y^{\epsilon,\eta})$. By (5.10) we know $x^{\epsilon,\eta} - y^{\epsilon,\eta} \to 0$ as $\epsilon \to 0$ but by (5.11) $x^{\epsilon,\eta}$, $y^{\epsilon,\eta}$ are bounded away from 0 or $|x| = R$. Thus there are only three possibilities: $x^{\epsilon,\eta}$ and $y^{\epsilon,\eta}$ are both interior points, both on $\partial_1\Omega$ or both on $\partial_2\Omega$.

Suppose $d(x^{\epsilon,\eta}) = d(y^{\epsilon,\eta}) = 0$ with both on $\partial_i\Omega$. We take c_ϵ to be the solution of

$$F_i\left(2\frac{x^{\epsilon,\eta} - y^{\epsilon,\eta}}{\epsilon^2} + c_\epsilon n_i\right) = 0.$$

Observe from (5.6) that

$$F_i\left(2\frac{x^{\epsilon,\eta} - y^{\epsilon,\eta}}{\epsilon^2} + c_\epsilon n_i\right) = F_i\left(2\frac{x^{\epsilon,\eta} - y^{\epsilon,\eta}}{\epsilon^2}\right) + \nu_i c_\epsilon,$$

so that

$$c_\epsilon = -\frac{1}{\nu_i}F_i\left(2\frac{x^{\epsilon,\eta} - y^{\epsilon,\eta}}{\epsilon^2}\right). \tag{5.18}$$

We want to show $\epsilon c_\epsilon \to 0$ so that we can appeal to Lemma 8, but this requires a refined bound on $|x^{\epsilon,\eta} - y^{\epsilon,\eta}|$. We already have a preliminary bound from (5.10). The additional fact we have here is that $x^{\epsilon,\eta}$ and $y^{\epsilon,\eta}$ are both on $\partial_i\Omega$ so that $d(x^{\epsilon,\eta}) = 0 = d(y^{\epsilon,\eta})$. Thus the d-terms vanish in the supremum defining M^η_ϵ. From the fact that $\Psi(y^{\epsilon,\eta}, y^{\epsilon,\eta}) \leq \Psi(x^{\epsilon,\eta}, y^{\epsilon,\eta})$ we get the inequality

$$\frac{|x^{\epsilon,\eta} - y^{\epsilon,\eta}|^2}{\epsilon^2} \leq u(x^{\epsilon,\eta}) - u(y^{\epsilon,\eta}) + \eta(d(x^{\epsilon,\eta}) - d(y^{\epsilon,\eta})).$$

But this simplifies to
$$\frac{|x^{\epsilon,\eta}-y^{\epsilon,\eta}|^2}{\epsilon^2} \le u(x^{\epsilon,\eta}) - u(y^{\epsilon,\eta})$$
$u(\cdot)$ is uniformly continuous on $\overline{\Omega^R}$ so by virtue of (5.10), the right side is bounded by some $h(\epsilon) \to 0$ as $\epsilon \to 0$. It follows that

$$|x^{\epsilon,\eta}-y^{\epsilon,\eta}| \le \epsilon\sqrt{h(\epsilon)}.$$

Consequently
$$\frac{|x^{\epsilon,\eta}-y^{\epsilon,\eta}|}{\epsilon^2} \le \frac{\sqrt{h(\epsilon)}}{\epsilon}.$$
Using this in (5.18) and the fact that F_i is Lipschitz ($F_i(p) \le C(1+|p|)$), it follows that
$$|c_\epsilon| \le C\left(1+\frac{\sqrt{h(\epsilon)}}{\epsilon}\right),$$
so that
$$\epsilon|c_\epsilon| \le \rho(\epsilon) \doteq C\left(\epsilon+\sqrt{h(\epsilon)}\right) \to 0$$
as required.

For ϵ and η fixed, consider the sets

$$E^+(\epsilon,\eta) = \{0<\alpha<1 \,:\, \text{(5.16) holds for some maximizing pair } (\bar{x},\bar{y})\}$$
$$E^-(\epsilon,\eta) = \{0<\alpha<1 \,:\, \text{(5.17) holds for some maximizing pair } (\bar{x},\bar{y})\}$$

We claim that both $E^\pm(\epsilon,\eta)$ are either empty or have positive infima: $\alpha_0^\pm = \inf E^\pm(\epsilon,\eta)$. We write out the argument for E^-; the case of E^+ is analogous.

Consider $\alpha \in E^-(\epsilon,\eta)$. As worked out below,
$$-\Upsilon_y(\bar{x},\bar{y}) = 2\frac{\bar{x}-\bar{y}}{\epsilon^2} - \eta n(\bar{y}) + c_\epsilon n(\bar{y}) - 2\frac{d(\bar{x})-d(\bar{y})}{\alpha^2}n(\bar{y}) - 2(\bar{y}-y^{\epsilon,\eta}).$$

Since $\bar{y} \in \partial_i\Omega$ we have $d(\bar{y})=0$ and (by the corollary) $n(\bar{y})=n_i$. Thus $0 \le F_i(-\Upsilon_y(\bar{x},\bar{y}))$ becomes
$$0 \le F_i\left(2\frac{\bar{x}-\bar{y}}{\epsilon^2} - \eta n_i + c_\epsilon n_i - 2\frac{d(\bar{x})}{\alpha^2}n_i - 2(\bar{y}-y^{\epsilon,\eta})\right).$$

Since $F_i(p+sn_i)$ is monotone increasing in s we can drop the term $-2\frac{d(\bar{x})}{\alpha^2}n_i$ to obtain

$$0 \le F_i\left(2\frac{\bar{x}-\bar{y}}{\epsilon^2} - \eta n_i + c_\epsilon n_i - 2(\bar{y}-y^{\epsilon,\eta})\right). \tag{5.19}$$

If $E^-(\epsilon,\eta)$ were nonempty and $0 = \inf E^-(\epsilon,\eta)$, there would be a sequence $\alpha_k \downarrow 0$ in $E^-(\epsilon,\eta)$. By passing to a subsequence we can assume $\bar{x} \to \hat{x}$ and $\bar{y} \to \hat{y}$ along this sequence. From the lemma it follows that $d(\bar{x}) - d(\bar{y}) \to 0$ as well, so that $d(\hat{x}) = d(\hat{y})$. But then we have

$$M_\epsilon^\eta \le \lim_{\alpha_k \to 0} \Phi(\bar{x},\bar{y}) \le u(\hat{x}) - w(\hat{y}) - \frac{|\hat{x}-\hat{y}|^2}{\epsilon^2} + \eta d(\hat{x}) + \eta d(\hat{y}) - |\hat{x}-x^{\epsilon,\eta}|^2 - |\hat{y}-y^{\epsilon,\eta}|^2 \le M_\epsilon^\eta,$$

from which it follows that $\hat{x} = x^{\epsilon,\eta}$ and $\hat{y} = y^{\epsilon,\eta}$. We can pass to the limit in (5.19) along our sequence $\alpha_k \to 0$ to obtain

$$0 \le F_i\left(2\frac{x^{\epsilon,\eta}-y^{\epsilon,\eta}}{\epsilon^2} + c_\epsilon n_i - \eta n_i\right).$$

But this implies

$$0 < \nu_i \eta \le F_i\left(2\frac{x^{\epsilon,\eta}-y^{\epsilon,\eta}}{\epsilon^2} + c_\epsilon n_i\right),$$

contrary to the choose of c_ϵ. This proves that $E^-(\epsilon,\eta)$ is either empty or has positive infimum.

For any $\alpha \in E^-(\epsilon,\eta)$ we can improve the estimate on $(d(\bar{x})-d(\bar{y}))^2/\alpha^2$ of the lemma. Going back to $0 \le F_i(-\Upsilon_y(\bar{x},\bar{y}))$ and using the linearity of F_i in n_i we have

$$2\nu_i \frac{d(\bar{x})-d(\bar{y})}{\alpha^2} \le F_i(2\frac{\bar{x}-\bar{y}}{\epsilon^2} + c_\epsilon n(\bar{y}) - \eta n(\bar{y}) - 2(\bar{y}-y^{\epsilon,\eta})).$$

Since $\bar{y} \in \partial_i\Omega$ we know $d(\bar{y}) = 0$ and so $d(\bar{x}) - d(\bar{y}) = |d(\bar{x}) - d(\bar{y})|$. It follows from the Lipschitz continuity of F_i that for some constant C and all $\alpha \in E^-(\epsilon,\eta)$

$$\frac{|d(\bar{x})-d(\bar{y})|}{\alpha^2} \le C\left(1 + \frac{k(\epsilon)+\rho(\epsilon)}{\epsilon}\right).$$

It is easy to see that $E^-(\epsilon,\eta)$ is closed, so that when it is nonempty, the above holds for $\alpha_0^- = \inf E^-(\epsilon,\eta)$, which we know is positive. Take $\alpha^- = \alpha_0^-/2$. Then $\alpha^- \notin E^-(\epsilon,\eta)$ but α^- inherits the above bound. To see this observe that

$$\Phi_{\alpha^-}(\bar{x}_{\alpha_0},\bar{y}_{\alpha_0}) \le \Phi_{\alpha^-}(\bar{x}_{\alpha^-},\bar{y}_{\alpha^-}) \quad \text{and} \quad \Phi_{\alpha_0}(\bar{x}_{\alpha^-},\bar{y}_{\alpha^-}) \le \Phi_{\alpha_0}(\bar{x}_{\alpha_0},\bar{y}_{\alpha_0})$$

which implies that

$$\Phi_{\alpha_0}(\bar{x}_{\alpha^-}, \bar{y}_{\alpha^-}) - \Phi_{\alpha^-}(\bar{x}_{\alpha^-}, \bar{y}_{\alpha^-}) \leq \Phi_{\alpha_0}(\bar{x}_{\alpha_0}, \bar{y}_{\alpha_0}) - \Phi_{\alpha^-}(\bar{x}_{\alpha_0}, \bar{y}_{\alpha_0}).$$

But notice that $\Phi_{\alpha_0} - \Phi_{\alpha^-} = (\frac{1}{(\alpha^-)^2} - \frac{1}{\alpha_0^2})(d(x) - d(y))^2$. So we have

$$(d(x_{\alpha^-}) - d(y_{\alpha^-}))^2 \leq (d(x_{\alpha_0}) - d(y_{\alpha_0}))^2.$$

Consequently

$$\frac{1}{(\alpha^-)^2}|d(x_{\alpha^-}) - d(y_{\alpha^-})| \leq \frac{4}{\alpha_0^2}|d(x_{\alpha_0}) - d(y_{\alpha_0})|.$$

We have therefore for $\alpha = \alpha^-$ that $0 < \alpha \notin E^-(\epsilon, \eta)$ but satisfies

$$\frac{|d(\bar{x}) - d(\bar{y})|}{\alpha^2} \leq C\left(1 + \frac{k(\epsilon) + \rho(\epsilon)}{\epsilon}\right). \tag{5.20}$$

If $E^-(\epsilon, \eta) = \emptyset$ then we simply take $\alpha = 1$ for which the above bound also holds by the first inequality of the lemma.

Similar arguments apply to $E^+(\epsilon, \eta)$ to produce $0 < \alpha^+ \notin E^+(\epsilon, \eta)$. Finally we take α to be the smaller of these two. This positive α is in neither of $E^\pm(\epsilon, \eta)$ satisfies (5.20).

Next consider the case in which $x^{\epsilon,\eta}$ and $y^{\epsilon,\eta}$ are both interior points. In this case we simply take $c_\epsilon = 0$ and repeat the reasoning above. We must have $0 < \inf E^\pm(\epsilon, \eta)$ else there would be a sequence $\alpha_k \downarrow 0$ along which $\bar{x} \to \hat{x} = x^{\epsilon,\eta}$ and $\bar{y} \to \hat{y} = y^{\epsilon,\eta}$, but that is impossible since either $\bar{x}$ or $\bar{y}$ is a boundary point but $x^{\epsilon,\eta}$ and $y^{\epsilon,\eta}$ are both interior points. We take whichever of $E^\pm$ has the smaller infimum, and let α be half the infimum, giving $0 < \alpha \notin E^\pm(\epsilon, \eta)$ satisfying (5.20).

Coordinating the Parameters and the Final Contradiction

We now coordinate the choice of parameters as follows.

1. Choose a sequence $\eta_n \downarrow 0$ with $0 < \eta_n < \eta_0$.

2. Choose $\epsilon_n \downarrow 0$.

3. Choose c_{ϵ_n} as as in the preceding section (depending on the location of x^{ϵ_n,η_n} and y^{ϵ_n,η_n}).

4. Choose $0 < \alpha_n < 1$ so that $\alpha_n \notin E^{\pm}(\epsilon_n, \eta_n)$ and the inequality (5.20) holds.

These choices insure that, for each n, the maximizing pair $(\bar{x}, \bar{y}) = (\bar{x}_n, \bar{y}_n)$ of $\Phi(x, y)$ are points for which neither (5.16) nor (5.17) hold. The inequality

$$\Phi(x, \bar{y}) \leq \Phi(\bar{x}, \bar{y}),\ x \in \Omega^R$$

implies that

$$u(x) \leq \Upsilon(x, \bar{y}) + u(\bar{x}) - \Upsilon(\bar{x}, \bar{y})$$

with equality at $x = \bar{x}$. Thus

$$\xi_n^+ = \Upsilon_x(\bar{x}, \bar{y}) \in D^+ u(\bar{x}).$$

Likewise, $\Phi(\bar{x}, y) \leq \Phi(\bar{x}, \bar{y})$ implies that

$$\Upsilon(\bar{x}, \bar{y}) + w(\bar{y}) - \Upsilon(\bar{x}, y) \leq w(y)$$

with equality at $y = \bar{y}$. Thus

$$\xi_n^- = -\Upsilon_y(\bar{x}, \bar{y}) \in D^- w(\bar{y}).$$

Since neither (5.16) nor (5.17) hold by our careful choice of the parameters, it follows from the sub- and super solution properties that

$$\gamma u(\bar{x}_n) + H(\xi_n^+) \leq 0 \leq \gamma w(\bar{y}_n) + H(\xi_n^-)$$

and consequently

$$\gamma(u(\bar{x}_n) - w(\bar{y}_n)) \leq H(\xi_n^-) - H(\xi_n^+). \tag{5.21}$$

We now work out $\xi_n^{\pm}$ and use our various estimates to show that (5.21) leads to a contradiction.

$$\begin{aligned}
\xi_n^+ &= 2\frac{\bar{x}_n - \bar{y}_n}{\epsilon_n^2} + \eta_n n(\bar{x}_n) + c_{\epsilon_n} n(\bar{x}_n) - 2\frac{d(\bar{x}_n) - d(\bar{y}_n)}{\alpha_n^2} n(\bar{x}_n) + 2(\bar{x}_n - x^{\epsilon_n,\eta_n}) \\
\xi_n^- &= 2\frac{\bar{x}_n - \bar{y}_n}{\epsilon_n^2} - \eta_n n(\bar{y}_n) + c_{\epsilon_n} n(\bar{y}_n) - 2\frac{d(\bar{x}_n) - d(\bar{y}_n)}{\alpha_n^2} n(\bar{y}_n) - 2(\bar{y}_n - y^{\epsilon_n,\eta_n}).
\end{aligned}$$

Thus

$$|\xi_n^+ - \xi_n^-| \le \eta_n |n(\bar{x}_n) + n(\bar{y}_n)| + \left(|c_{\epsilon_n}| + 2\frac{|d(\bar{x}_n) - d(\bar{y}_n)|}{\alpha_n^2} \right) |n(\bar{x}_n) - n(\bar{y}_n)| + 2|\bar{x}_n - x^{\epsilon_n,\eta_n}| + 2|\bar{y}_n - y^{\epsilon_n,\eta_n}|.$$

Because from the lemma $|\bar{x}_n - \bar{y}_n| \le \epsilon_n k(\epsilon_n)$ and $n(\cdot)$ is Lipshitz, the bound (5.20) implies that the second term on the right vanishes in the limit as $n \to \infty$. The first vanishes because $n(\cdot)$ is bounded and $\eta_n \to 0$. The last two terms vanish by the lemma. Since $H(\cdot)$ is uniformly continuous, it follows that

$$H(\xi_n^-) - H(\xi_n^+) \to 0.$$

On the other side of (5.21) the lemma and uniform continuity of $u(\cdot)$ and $w(\cdot)$ imply that

$$\liminf[u(\bar{x}_n) - w(\bar{y}_n)] = \liminf[u(x^{\epsilon_n,\eta_n}) - w(y^{\epsilon_n,\eta_n})].$$

Since $\eta_n \to 0$ we have

$$\liminf[u(x^{\epsilon_n,\eta_n}) - w(y^{\epsilon_n,\eta_n})] = \liminf[u(x^{\epsilon_n,\eta_n}) - w(y^{\epsilon_n,\eta_n}) + \eta_n d(x^{\epsilon_n,\eta_n}) + \eta_n d(y^{\epsilon_n,\eta_n})].$$

Furthermore

$$\liminf[u(x^{\epsilon_n,\eta_n}) - w(y^{\epsilon_n,\eta_n}) + \eta_n d(x^{\epsilon_n,\eta_n}) + \eta_n d(y^{\epsilon_n,\eta_n})] \ge \liminf M_{\epsilon_n}^{\eta_n} \ge M,$$

giving us,

$$\liminf[u(\bar{x}_n) - w(\bar{y}_n)] \ge M.$$

Thus the limit inferior of both sides of (5.21) gives us

$$\gamma M \le 0,$$

which is contrary to our original hypothesis that $M > 0$ since $\gamma > 0$. Therefore $u(x) \le w(x)$ for all $x \in \overline{\Omega^R}$. □

Since $V^{R,\gamma}(\cdot) = u(\cdot)$ and $w(\cdot)$ satisfy the conditions of the previous Theorem 10 we can conclude $V^{R,\gamma}(x) \le w(x)$ for all $x \in \overline{\Omega^R}$ This completes **step** 3 of our argument for Theorem 9.

For **step** 4 we prove the following lemma.

Lemma 9. *$V^{R,\gamma}(x) \to V^{R,0}(x)$ as $\gamma \to 0$ for all $x \in \overline{\Omega^R}$.*

Proof. Recall the truncated, discounted version of our problem, where $V^{R,\gamma}$ is the infimum of

$$w(\phi(T_1))e^{-\gamma(T_2-T_1)} + \int_{T_1}^{T_2} e^{-\gamma(T_2-t)} L(\dot{\psi}(t))dt,$$

over $\phi(t) = \Gamma(\psi(\cdot))$ where $\phi(T_1)$ is allowed to be either 0 or $|\phi(T_1)| = R$ and subject to the restriction $|\phi(t)| \leq R$. This says that we can start a path in the usual way at 0 or at a point with $|\phi(T_1)| = R$ with "starting cost" $w(\phi(T_1))$. We will show that $V^{R,\gamma}(x) \to V^{R,0}(x)$ as $\gamma \to 0$ so that $V^{R,0} \leq w(x)$. $V^{R,\gamma} \leq V^{R,0}$ is obvious so what needs to be shown is $V^{R,0} \leq \liminf_{\gamma\to 0} V^{R,\gamma}$. The idea is to pass to the limit from a sequence of (nearly) optimal paths for $V^{R,\gamma}(x)$ as $\gamma \to 0$ to get a path $\phi(\cdot) = \Gamma(\psi(\cdot))$ which reaches $\phi(T) = x$ starting from some $t_0 < T$ with either $\phi(t_0) = 0$ or $|\phi(t_0)| = R$, and

$$w(\phi(t_0)) + \int_{t_0}^{T} L(\dot{\psi}(t))\, dt \leq \liminf_{\gamma\to 0} V^{R,\gamma}(x) + \epsilon.$$

We want to do this for an arbitrary $\epsilon > 0$ so that $V^{R,0}(x) \leq \liminf\limits_{\gamma\to 0} V^{R,\gamma}$.

Take $x \in \Omega^R$ and any $\epsilon > 0$. Let

$$B = \liminf_{\gamma\to 0} V^{R,\gamma}(x).$$

Our goal is to show

$$V^{R,0}(x) \leq B + \epsilon.$$

We know $V^{R,0}$ is continuous at 0 so given $\epsilon > 0$ there exists $\delta > 0$ such that $V^{R,0}(y) < \epsilon$ for all $|y| < \delta$. (This means $||y|| \leq c\delta$ for some constant c.) By our stability hypothesis and [7] we know there exits T_R with the property that for every $|y| \leq R$, the solution of the Skorokhod Problem on $[0, T_R]$ for $\tilde{\psi}(t) = y + bt$ has $\tilde{\phi}(T_R) = 0$. Using Lipschitz continuity of the Skorokhod Map we will show that there is $\delta_1 > 0$ so that any solution of the Skorokhod problem $\phi = \Gamma(\psi)$ with $\psi(0) = \phi(0) = y$, $|y| \leq R$ and $\int_0^{T_R} L(\dot{\psi}(t))dt < \delta_1$ will have

$|\phi(T_R)| \leq \delta$. To see this, assume $\int_0^{T_R} L(\dot{\psi})dt < \delta_1$. Using Hölder's inequality we have,

$$\begin{aligned}\int_0^{T_R} |\dot{\psi} - b|dt &\leq (T_R)^{\frac{1}{2}}(\int_0^{T_R} |\dot{\psi} - b|^2 dt)^{\frac{1}{2}} \\ &\leq (T_R)^{\frac{1}{2}}(c\int_0^{T_R} L(\dot{\psi})dt)^{\frac{1}{2}}.\end{aligned}$$

Note that $|\psi(t) - \tilde{\psi}(t)| \leq \int_0^{T_R} |\dot{\psi} - b|dt$ on $[0, T_R]$. Using the Lipschitz continuity of the Skorokhod Problem with Lipschitz constant K we have

$$|\phi(T_R) - \tilde{\phi}(T_R)| \leq K|\psi(T_R) - \tilde{\psi}(T_R)| \leq K(T_R(c\delta_1))^{\frac{1}{2}}.$$

Taking δ_1 such that $\delta = K(T_R(c\delta_1))^{\frac{1}{2}}$ it follows that $|\phi(T_R)| \leq \delta$.

Choose N so that $N\delta_1 > B + 1$ and take $\Delta T = NT_R$. Now if $\phi = \Gamma(\psi)$ and $\int_{T_1}^{T_2} L(\dot{\psi}(t))dt < B + 1$ over an interval $[T_1, T_2]$ with $T_2 \geq T_1 + \Delta T$, then over one of the subintervals $[T_1 + (k-1)T_R, T_1 + kT_R]$, $(k = 1, 2 \ldots N)$ we must have $\int L(\dot{\psi})dt < \delta_1$ and so $|\phi(T_1 + kT_R)| \leq \delta$. In other words, given the bound $B + 1$ on $\int L(\dot{\psi})dt$, if the time interval is at least as long as ΔT then the path must pass within δ of the origin at some point in the interval.

Since $B = \liminf_{\gamma \to 0} V^{R,\gamma}(x)$, we can choose a sequence $\gamma_n \to 0$ for which $V^{R,\gamma_n} \to B$. For each n take a nearly optimal path $\phi_n(\cdot) = \Gamma(\psi_n(\cdot))$ defined on $[0, T_n]$ with either $\phi_n(0) = 0$ or $|\phi_n(0)| = R$, $\phi_n(T_n) = x$ with

$$e^{-\gamma_n(T_n)}w(\phi_n(0)) + \int_0^{T_n} e^{-\gamma_n(T_n - t)}L(\dot{\psi}_n(t))dt \leq V^{R,\gamma_n}(x) + \frac{1}{n}. \tag{5.22}$$

Moreover by the Bounded Velocity Lemma we can assume $|\dot{\psi}_n| \leq c|b|$ for all n.

The idea is that we can extract a uniformly convergent subsequence of $\psi_n(\cdot)$ (and therefore of $\phi_n(\cdot)$ as well) and pass to the limit in (5.22) to produce a path which will show that $V^{R,0}(x) \leq B + \epsilon$. First we want to place the ψ_n in a common time interval, suppose that $T_n > \Delta T$. Then since $e^{-\gamma_n(T_n)}w(\phi_n(0)) \geq 0$,

$$\begin{aligned}\int_{T_n - \Delta T}^{T_n} L(\dot{\psi}_n(t))\,dt &\leq e^{\gamma_n \Delta T}\int_{T_n - \Delta T}^{T_n} e^{-\gamma_n(T_n - t)}L(\dot{\psi}_n(t))\,dt \\ &\leq e^{\gamma_n \Delta T}(V^{R,\gamma_n}(x) + \frac{1}{n}).\end{aligned}$$

For sufficiently large n, $e^{\gamma_n \Delta T}(V^{R,\gamma_n}(x) + \frac{1}{n}) < B + 1$ so we know there is a $t_0 > T_n - \Delta T$ at which $|\phi_n(t_0)| < \delta$. By shifting the time scale so that t_0 becomes 0 and T_n is replaced by $T_n - t_0$, we can assume $T_n \leq \Delta T$ and $|\phi_n(0)| < \delta$. Thus we can assume all $T_n \leq \Delta T$, at the expense of replacing $\phi_n(0) = 0$ by $|\phi_n(0)| < \delta$ for those paths with $|\phi_n(0)| \neq R$.

If $T_n < \Delta T$ we can extend ψ_n so to be defined on all of $[0, \Delta T]$ using $\psi_n(t) = \psi_n(T_n) + (t - T_n)b$ for $T_n < t$. Note that $L(\dot{\psi}_n(t)) = 0$ on the extension, so that $\int_0^{\Delta T} L(\dot{\psi}_n) = \int_0^{T_n} L(\dot{\psi}_n)$. Note that ϕ_n and ψ_n are continuous and pointwise bounded on $[0, \Delta T]$. For each n we have $|\dot{\psi}_n(t)| \leq c|b|$. Thus the ψ_n are equicontinuous. Since $[0, \Delta T]$ is compact ϕ_n are also uniformly bounded on $[0, \Delta T]$.

By the Arzela-Ascoli theorem since the ψ_n are uniformly bounded and equicontinuous we can pass to a uniformly convergent subsequence $\psi_n \to \psi$ on $[0, \Delta T]$. We can choose this subsequence so that either all $|\phi_n(0)| < \delta$ or all $|\phi_n(0)| = R$, and so that $T_n \to T_* \leq \Delta T$. Using the Lipschitz continuity of the Skorokhod Problem and the convergence of ψ_n, given $\epsilon > 0$ there exists $N > 0$ such that for all $n > N$, $|\psi_n(t) - \psi(t)| \leq \frac{\epsilon}{K}$ so that

$$|\phi_n(t) - \phi(t)| \leq K|\psi_n(t) - \psi(t)| \leq \epsilon.$$

Therefore convergence of ψ_n implies the uniform convergence of ϕ_n. It does not imply the convergence of $\dot{\psi}_n$. However the Lipschitz continuity of ψ_n implies that ψ is Lipschitz. Hence ψ is absolutely continuous.

The map $\psi \mapsto \int_0^{\Delta T} L(\dot{\psi}(t))\,dt$ is lower semicontinuous with respect to the topology of uniform convergence. This is a standard fact in the theory of large deviations; see Lemma 2.1 of [31]. Therefore we have

$$\int_0^{T_*} L(\dot{\psi}(t))dt \leq \liminf_{n\to\infty} \int_0^{T_*} L(\dot{\psi}_n(t))dt. \tag{5.23}$$

Since $e^{-\gamma_n(T_n - T_*)} \to 1$ and $T_n \to T_*$ as $n \to \infty$ using Fatou's lemma,

$$\begin{aligned}
\liminf_{n\to\infty} \int_0^{T_*} L(\dot{\psi}_n(t))dt &\leq \liminf_{n\to\infty} \int_0^{T_n} L(\dot{\psi}_n(t))dt \\
&\leq \liminf_{n\to\infty} \int_0^{T_n} e^{-\gamma_n(T_n - t)} L(\dot{\psi}_n(t))\,dt.
\end{aligned}$$

So we can say that

$$\int_0^{T_*} L(\dot{\psi}(t))\,dt \le \liminf_{n\to\infty} \int_0^{T_n} e^{-\gamma_n(T_n-t)} L(\dot{\psi}_n(t))\,dt. \tag{5.24}$$

Since $T_n \to T_*$ there exists $N_1 > 0$ such that for $n \ge N_1$, $|T_n - T_*| \le \delta$. Given an arbitrary $\epsilon > 0$ there exists $\delta > 0$ so that $|\psi_n(T_n) - \psi_n(T_*)| \le \frac{\epsilon}{2}$ whenever $|T_n - T_*| \le \delta$. There also exists $N_2 > 0$ such that for all $n > N_2$, $|\psi_n(T_*) - \psi(T_*)| \le \frac{\epsilon}{2}$. Therefore for $N = \max(N_1, N_2)$,

$$\begin{aligned} |\psi_n(T_n) - \psi(T_*)| &= |\psi_n(T_n) - \psi_n(T_*) + \psi_n(T_*) - \psi(T_*)| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This means $\psi_n(T_n) \to \psi(T_*)$, and so $\psi(T_*) = x$. Moreover, since $|\phi_n(t)| \le R$ (on $[0, T_n]$) it follows that $|\phi(t)| \le R$ on $[0, T_*]$.

Now we have two cases to consider: 1) $|\phi(0)| = R$, or 2) $|\phi(0)| \le \delta$.

Case 1:

We pass to the limit in (5.22) and use (5.24) to obtain

$$w(\phi(0)) + \int_0^{T_*} L(\dot{\psi}(t))\,dt \le B.$$

This implies $V^{R,0}(x) < B + \epsilon$.

Case 2:

We can drop the w term before passing to the limit in (5.22) and use (5.24) to obtain

$$\int_0^{T_*} L(\dot{\psi}(t))\,dt \le B.$$

Now since $|\phi(0)| \le \delta$ we tack on an straight-line segment joining $\phi(t_0) = 0$ (some $t_0 < 0$) to $\phi(0)$, staying in Ω^R with cost $\int_{t_0}^0 L(\dot{\psi})dt < \epsilon$. Putting the pieces together we have a path from $\phi(t_0) = 0$ to $\phi(T_*) = x$ staying in Ω^R with

$$\int_{t_0}^{T_*} L(\dot{\psi}(t))\,dt < B + \epsilon.$$

So in this case we also find that $V^{R,0}(x) < B + \epsilon$.

Therefore $V^{R,\gamma} \to V^{R,0}$ as $\gamma \to 0$.

$\square$

Finally for **step 5** we must show $\lim_{R\to\infty} V^{R,0}(x) = V(x)$. First note $V^{R,0}(x) \leq V(x)$. But if R is big enough so that $V(x) < \inf_{|y|=R} V(y)$, then all nearly optimal paths for $V(x)$ will *not* leave Ω_R. For R large enough $V(x) < \inf_{|y|=R} w(y)$ so that no path that starts with $|\phi(0)| = R$ can do better. Therefore $\lim_{R\to\infty} V^{R,0}(x) = V(x)$.

Chapter 6

3 Dimensional Cyclic Search

6.1 The Problem

The search for a cyclic optimal path for the Reflected Quasipotential in more than two dimensions is of importance in the literature. The literature mentions the possibility of such paths in general. They are known not to exist in some particular cases; see [16]. To our knowledge, no specific examples of cyclic optimal paths for a well-posed Skorokhod Problem have been given. Here we describe an interactive search for such a cyclic optimal path in the 3-dimensional positive orthant with a Skorokhod Problem which is well-posed. To accomplish this we assume "symmetry" on the faces of the orthant so that the cost along each axis has the same structure. This allows us to identify a cyclic path with a fixed point for a function based on the Skorokhod Problem for paths which cross a face. We then look for parameters associated with the Skorokhod Problem and the matrix A that produce a cyclic path and for which the Skorokhod Problem is well-posed. In particular we search for parameters in which the set B described in Assumption 2.1 exists. Recall that the existence of this set is a sufficient condition for the Lipschitz continuity of solutions to the Skorokhod Problem. For those parameters of the Skorokhod Problem which produce a cyclic path we show that no such set B can exist. Our results show that for $\alpha > 1$ or $\beta > 1$ (parameters described below) a cyclic path exists but Assumption 2.1 fails.

The set up

To simplify notation we will use $x(t), y(t)$ instead of $\phi(t), \psi(t)$. We will also use the interval $[0, T]$ instead of the more general $[T_1, T_2]$.

Our Goal

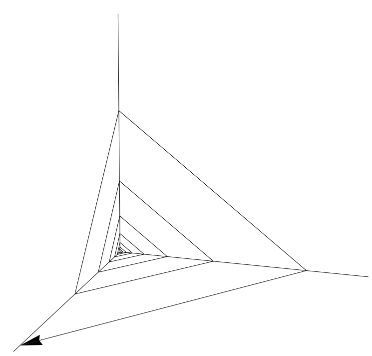

Figure 6.1: Cyclic Path

Our goal is to find examples (b, A, Skorokhod Problem parameters) in which some optimal paths cycle along the faces of the orthant as illustrated in Figure 6.1.

To search for examples we assume "symmetry" on the faces of the orthant with respect to rotations about the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ so that the cost along each axis has the same structure. This will allow us to identify a cyclic path with a fixed point for a function based on the Skorokhod Problem for paths which cross a face. We interactively adjust the parameters to search for an example with the desired properties. Once we have identified such a path we then check the conditions for existence, stability and Lipschitz continuity of solutions to the associated Skorokhod Problem.

To make the faces "symmetric" we define $b = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$, $d_1 = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}$, $d_2 = \begin{pmatrix} \beta \\ 1 \\ \alpha \end{pmatrix}$ and $d_3 = \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix}$. We let A be a matrix which has $b = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ as an eigenvector with eigenvalue 1, and two eigenvectors orthogonal to it with eigenvalue $\sigma > 0$. This works out to be

$$A = \begin{pmatrix} \frac{1}{3}(1+2\sigma) & -\frac{1}{3}(-1+\sigma) & -\frac{1}{3}(-1+\sigma) \\ -\frac{1}{3}(-1+\sigma) & \frac{1}{3}(1+2\sigma) & -\frac{1}{3}(-1+\sigma) \\ -\frac{1}{3}(-1+\sigma) & -\frac{1}{3}(-1+\sigma) & \frac{1}{3}(1+2\sigma) \end{pmatrix}.$$

This means that A^{-1} has eigenvalues $\frac{1}{\sigma}$ with eigenvectors orthogonal to b. When $\sigma = 1$, A is the identity matrix. So for $1 < \sigma$ we hoped that the optimal paths would stay on the coordinate faces pushing outward orthogonal to b while the Skorokhod mechanism moves them away from the origin with cost less than that of paths in the interior. To see why this would produce a cyclic path consider the motion on the face $\partial_3\Omega$ where $x_3(t) \equiv 0$ and the reflection vector is $d_3 = \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix}$. $\dot{x} = \pi(x, \dot{y})$ and we take $\dot{y} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix}$ with $\dot{y}_3 < 0$. Since $\dot{y}_3 < 0$ we have a path that is moving out of the positive orthant through $\partial_3\Omega$. The control problem charges us $L(\dot{y})$ but the Skorokhod Problem produces motion $\dot{x} = \dot{y} - \dot{y}_3 d_3 = \begin{pmatrix} \dot{y}_1 - \alpha\dot{y}_3 \\ \dot{y}_2 - \beta\dot{y}_3 \\ 0 \end{pmatrix}$.

It is possible that $L(\dot{y}) < L(\dot{x})$. Hence, we hoped that for the appropriate choices of α, β and σ perhaps the cheapest way to get away from the origin is to try to leave the positive orthant and let the Skorokhod Problem push you such that you move in the direction of $\dot{x} = \pi(x, \dot{y})$ with cost $L(\dot{y})$.

The Scaling Lemma (Lemma 4) gives us that:

$$V((0, x_2, 0)) = \rho x_2 \text{ and } V((x_1, 0, 0)) = \rho x_1.$$

Here $V(e_i) = \rho$ where e_i is the ith standard basis vector in $\mathbb{R}^3$. The ρ is the same for each coordinate axis because of symmetry. To find the cost function for our spiraling path assume a cyclic path exists for which the path cycles in the following way: $\partial_{\{2,3\}}\Omega \rightarrow \partial_{\{1,2\}}\Omega \rightarrow \partial_{\{1,3\}}\Omega \rightarrow \partial_{\{2,3\}}\Omega$. Recall that $\partial_{\{2,3\}}\Omega$ is the edge of Ω where $x_2 = x_3 = 0$, i.e. the x_1 -axis.

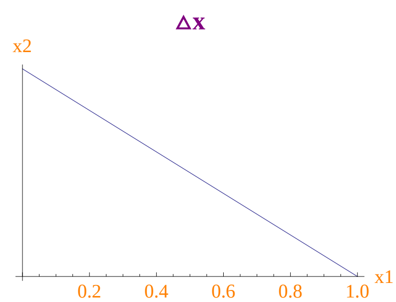

Figure 6.2: Path across $\partial_3\Omega$

Given a ρ we consider the problem of getting to the point $(1, 0, 0)$ on the x_1 axis by first getting to the point $(0, x_2, 0)$ on the x_2 axis with the cost ρx_2 and then move across the face $\partial_3\Omega$ in a straight line, taking into account all possible speeds and constrained dynamics. For the last leg of the path to the point $(1, 0, 0)$ (see Figure 6.2) we would have

$$V((1, 0, 0)) = \min_{x_2>0}\{V((0, x_2, 0)) + \text{Linecost}(x_2)\}$$

$$\rho = \min_{x_2>0}\{\rho x_2 + \text{Linecost}(x_2)\}.$$

Linecost is the minimal cost of direct paths from $(0, x_2, 0)$ to $(1, 0, 0)$, possibly using d_3. If $\psi(\rho) = \min_{x_2>0}\{\rho x_2 + \text{Linecost}(x_2)\}$ then ρ is a fixed point of $\psi(\rho)$. There also exists ρ_0 such that $\rho_0 x_i$ is the minimum cost of straight line motion from 0 to $x_i e_i$, considering all possible contained dynamics. We produce the plot of $\psi(\rho)$ with interactive controls for α, β and σ to search for cases in which $\rho_\star < \rho_0$, where $\rho_\star$ is the fixed point of $\psi(\rho)$. If there are cyclic optimal paths then $\psi(\rho)$ will have a fixed point with $\rho_\star < \rho_0$. In other words if a cyclic path existed, then the cost to get to $(1,0,0)$ by going to some point $(0,x_2,0)$ first and then crossing $\partial_3\Omega$ to $(1,0,0)$ would be less then the direct path along the x_1- axis from $(0,0,0)$ to $(1,0,0)$.

Details of Linecost

To find the cost function Linecost we minimized the function $\phi(t,a) = \frac{1}{t}L(t(\Delta x - ad_3))$ with respect to $t > 0$ and then with respect to $a \geq 0$ where $\Delta x = \begin{pmatrix} 1 \\ -x_2 \\ 0 \end{pmatrix}$. Here $\dot{x} = \Delta x$ and $\dot{y} = \Delta x - ad_3$ (see Figure 6.2). Using Mathematica we first calculated the minimum over $t > 0$ (explicitly), then found $a \geq 0$ which minimized the resulting expression and substituted the minimizing values into $\phi(t,a)$ to obtain the expression for Linecost.

Finding $\psi(\rho)$ requires minimizing $\{\rho x_2 + \text{Linecost}(x_2)\}$ with respect to x_2. Rather then doing this minimization we plotted $\psi(\rho)$ parametrically. Note that

$$\frac{\partial}{\partial x_2}(\{\rho x_2 + \text{Linecost}(x_2)\}) = 0$$

implies

$$\rho = -\frac{\partial}{\partial x_2}(\text{Linecost}(x_2)).$$

So we can plot $\psi(\rho)$ parametrically with x_2 as the parameter as long as we can calculate $\frac{\partial}{\partial x_2}(\text{Linecost}(x_2))$. This calculation was done using Mathematica.

We also have an alternative way to calculate Linecost in general for $\mathbb{R}^n$. We describe this method below.

Optimal Directional Rate with Reflection

Here we describe a method for calculating the minimal cost of a path which crosses a face. Suppose we have a velocity v with $\langle v, n_i \rangle = 0$ for all $i \in F \subseteq \{1,\ldots,n\}$. F corresponds to the set of reflection vectors d_i that could be used on a particular face. We want to calculate

$$\Phi_F(v) = \inf_{t>0,\ w\in\mathcal{C}_F} tL(t^{-1}v - w),$$

where $\mathcal{C}_F$ is the cone generated by the d_i:

$$\mathcal{C}_F = \left\{ \sum_{i\in F} a_i d_i :\ a_i \geq 0 \right\} \tag{6.1}$$

and $L(v) = \frac{1}{2}\langle v-b, A^{-1}(v-b)\rangle$, as usual. We would describe $\Phi_F(v)$ as the (minimal) cost of displacement v on the F-face, using an arbitrary time interval $[0,t]$ and taking advantage of reflection. It is easy to see that $\Phi_F(cv) = c\Phi_F(v)$ for any $c > 0$. Consequently we could write $\Phi_F(v) = \|v\|\rho_F(v)$, where $\rho_F(v) = \Phi_F(v/\|v\|)$ would be called the rate in direction v on face F.

We will focus on $\Phi_F(v)$ itself, and will assume in the following that $\{d_i :\ i \in F\}$ is linearly independent.

The Unconstrained K-Problem

For $K \subseteq F$ let

$$D_K = [d_i]_{i\in K}.$$

As a preliminary to the $\Phi_F(v)$ problem, consider the problem of finding

$$\inf_{t>0} \inf_{a} tL(t^{-1}v - D_K a)$$

where $a \in \mathbb{R}^{|K|}$ is *not* subject to the constraints $a_i \geq 0$.

First, given v, we minimize $L(v - D_K a)$ over $a \in \mathbb{R}^{|K|}$. This is a simple quadratic minimization problem in $\mathbb{R}^{|K|}$, with positive definite quadratic part $D_K^T A^{-1} D_K$, so it *is* minimized at a finite a^*. We find that

$$\frac{\partial}{\partial_{a_i}} L(v - D_K a) = -\langle d_i, A^{-1}(v - D_k a - b)\rangle.$$

Setting these equal to 0 for all $i \in K$ gives $0 = D_K^T A^{-1}(v - D_K a - b)$, which implies that the minimum occurs at

$$a^* = G_K(v - b),$$

where

$$G_K = (D_K^T A^{-1} D_K)^{-1} D_K^T A^{-1}. \tag{6.2}$$

Next we simplify the expression for $L(v - D_K a^*)$.

$$v - D_K a^* - b = (I - D_K G_K)(v - b),$$

so

$$\begin{aligned} L(v - D_K a^*) &= \frac{1}{2}\langle (I - D_K G_K)(v - b), A^{-1}(I - D_K G_K)(v - b)\rangle \\ &= \frac{1}{2}\langle v - b, Q_K(v - b)\rangle, \end{aligned}$$

where (after checking the matrix algebra) we have

$$Q_K = (I - D_K G_K)^T A^{-1}(I - D_K G_K) = \cdots = A^{-1}(I - D_K G_K). \tag{6.3}$$

In the case of $K = \emptyset$ there is no a-minimization; we just take $Q_K = A^{-1}$ in that case. Note that since A is symmetric positive definite Q_K is symmetric positive semi-definite in all cases. Thus in general we have

$$\inf_a L(v - D_K a) = \frac{1}{2}\langle v - b, Q_K(v - b)\rangle,$$

which has the same form as $L(v)$, just with Q_K in place of A^{-1}.

Next we consider the t minimization; we want the infimum over $t > 0$ of

$$\frac{t}{2}\langle (\frac{1}{t}v - b), Q_K(\frac{1}{t}v - b)\rangle = \frac{1}{2}\left[t^{-1}\langle v, Q_K v\rangle - 2\langle b, Q_K v\rangle + t\langle b, Q_K b\rangle\right]. \tag{6.4}$$

We can read off the minimum in several cases.

1. a. If $\langle v, Q_K v\rangle = 0$, the infimum occurs in the limit as $t \downarrow 0$, with a limiting value of $-\langle b, Q_K v\rangle$, which is 0 by the lemma below.

b. If $\langle b, Q_K b\rangle = 0$, the infimum occurs in the limit as $t \uparrow \infty$, with the limiting value of $-\langle b, Q_K v\rangle = 0$.

2. Otherwise there is a unique minimizing t^* which we easily find to be

$$t^* = \left(\frac{\langle v, Q_K v\rangle}{\langle b, Q_K b\rangle}\right)^{1/2},$$

and the value of the minimum is

$$\frac{1}{2t^*}\langle v, Q_K v\rangle - \langle b, Q_K v\rangle + \frac{t^*}{2}\langle b, Q_K b\rangle = \langle b, Q_K b\rangle^{1/2}\langle v, Q_K v\rangle^{1/2} - \langle b, Q_K v\rangle. \qquad (6.5)$$

Note from (6.4) that the infimum is achieved for a unique $t^* \in [0, \infty]$ unless both $\langle v, Q_K v\rangle = 0 = \langle b, Q_K b\rangle$, in which case *all* t achieve the minimum.

Here is the lemma we have used.

Lemma 10. *Suppose Q is symmetric, positive semi-definite. $\langle u, Qu\rangle = 0$ implies $Qu = 0$.*

Proof. Since Q is positive semi-definite we know that for all t and all w

$$\langle u + tw, Q(u + tw)\rangle = \langle u, Qu\rangle + 2t\langle w, Qu\rangle + t^2\langle u, Qu\rangle \geq 0.$$

If $\langle u, Qu\rangle = 0$ the only way this can be nonnegative for all t is for $\langle w, Qu\rangle = 0$ for all w as well forcing $Qu = 0$. $\square$

The $\Phi_F(v)$ Problem

$$\Phi_F(v) = \inf_{t>0,\ w\in\mathcal{C}_F} tL(t^{-1}v - w),$$

Recall the definition of C_F, (6.1) page 79. The following facts can be proven.

- If $v \in \mathcal{C}_F$ then $\Phi_F(v) = 0$.
- If $-b \in \mathcal{C}_F$ then $\Phi_F(v) = 0$.
- Otherwise $\inf_{w\in\mathcal{C}_F} tL(t^{-1}v - w) \to \infty$ both as $t \downarrow 0$ and as $t \uparrow \infty$.

The first case is equivalent to

$$\langle v, Q_F v\rangle = 0 \text{ and } G_F v \geq 0.$$

The second case is equivalent to

$$\langle b, Q_F b\rangle = 0 \text{ and } -G_F b \geq 0.$$

Consider the third case. The infimum over $t > 0$ can be limited to some compact interval: $0 < t_0 \leq t \leq t_1 < \infty$. Since $tL(t^{-1}v - w) \to \infty$ as $\|w\| \to \infty$, uniformly over $t \in [t_0, t_1]$, it follows that the infimum is achieved: for some $0 < t^* < \infty$ and $\bar{a}_i \geq 0$ we will have

$$\Phi_F(v) = t^* L(\frac{1}{t^*}v - D_F \bar{a}).$$

Let $\bar{K} = \{i \in F : \bar{a}_i > 0\}$. It follows that t^* and $a^* = (\bar{a}_i)_{i\in\bar{K}}$ are minimizers for the unconstrained $\bar{K}$-problem above, and so $\Phi_F(v)$ is given by (6.5) for $\bar{K}$. Moreover, since $\mathcal{C}_{\bar{K}} \subseteq \mathcal{C}_F$ it follows that $v \notin \mathcal{C}_{\bar{K}}$ and $-b \notin \mathcal{C}_{\bar{K}}$. So we have that

$$(\ \langle v, Q_{\bar{K}} v\rangle = 0 \text{ and } G_{\bar{K}} v \geq 0\ ) \text{ fails.} \tag{6.6}$$

and that

$$(\ \langle b, Q_{\bar{K}} b\rangle = 0 \text{ and } -G_{\bar{K}} b \geq 0\ ) \text{ fails.} \tag{6.7}$$

So in the third case ("Otherwise") above, we have both (6.6) and (6.7) as well as

$$a^* = G_{\bar{K}}(\frac{\langle b, Q_{\bar{K}} b\rangle^{1/2}}{\langle v, Q_{\bar{K}} v\rangle^{1/2}}v - b) \geq 0$$

with $\Phi_F(v)$ given by (6.5).

Next observe that for *any* $K \subseteq F$ the following hold.

- If $\langle v, Q_{\bar{K}} v\rangle = 0$ and $G_{\bar{K}} v \geq 0$ then $v \in \mathcal{C}_K \subseteq \mathcal{C}_F$ so that

$$\Phi_F(v) = 0.$$

- If $\langle b, Q_{\bar{K}} b\rangle = 0$ and $-G_{\bar{K}} b \geq 0$ then $-b \in \mathcal{C}_K \subseteq \mathcal{C}_F$ so that

$$\Phi_F(v) = 0.$$

- If neither of the above apply but $a^* = G_K(\frac{\langle b, Q_K b\rangle}{\langle v, Q_K v\rangle} v - b) \geq 0$, then we have from (6.5) that

$$\Phi_F(v) \leq \langle b, Q_K b\rangle^{1/2} \langle v, Q_K v\rangle^{1/2} - \langle b, Q_K v\rangle.$$

Algorithm

These observations imply that the following algorithm will produce $\Phi_F(v)$. Define r_K to be the minimal cost using reflections d_i $i \in K$. For each subset $K \subseteq F$ (including $K = \emptyset$) calculate r_K as follows:

a) if either ($\langle v, Q_K v\rangle = 0$ and $G_K v \geq 0$) or ($\langle b, Q_K b\rangle = 0$ and $-G_K b \geq 0$) take $r_K = 0$. (For $K = \emptyset$, use $Q_K = A^{-1}$ and $G_K = [0, \ldots 0]$.)

b) else if $G_K(\frac{\langle b, Q_K b\rangle^{1/2}}{\langle v, Q_K v\rangle^{1/2}} v - b) \geq 0$ take

$$r_K = \langle b, Q_K b\rangle^{1/2} \langle v, Q_K v\rangle^{1/2} - \langle b, Q_K v\rangle,$$

c) otherwise take $r_K = +\infty$.

Then

$$\Phi_F(v) = \min_{K \subseteq F} r_K.$$

The Optimal Directional Rate with Reflection provides us with a general and more efficient way to calculate Linecost and $\psi(\rho)$. To see how this connects to our previous description of Linecost we let $v = \begin{pmatrix} 1 \\ -x_2 \\ 0 \end{pmatrix}$. So we have that $F = \{3\}$, $K = \emptyset$ or $K = \{3\}$ and Linecost$(x_2) = \Phi_F(v)$. Also note here that Linecost$(x_2) = \min\{r_0(1, -x_2, 0), r_3(1, -x_2, 0)\}$. $r_0(1, -x_2, 0)$ is the minimal cost without reflection ($K = \emptyset$) across the face and $r_3(1, -x_2, 0)$ is the minimal cost using d_3 across the face. Here the minimization was done with respect to a first unlike our initial calculation for Linecost. We can also calculate ρ_0, the minimum direct cost to $(1, 0, 0)$. $\rho_0 = \min\{r_0(1, 0, 0), r_2(1, 0, 0), r_3(1, 0, 0), r_{\{2,3\}}(1, 0, 0)\}$.

6.2 Results for Fixed Point

The next step is to look for examples with $\rho_\star < \rho_0$. To do this we plot the graph of $\psi(\rho)$ and ρ and mark with a vertical line ρ_0 so that we can see if we can find the values of α, β and σ that give us a fixed point $\rho_\star$ less then ρ_0. See Figure 6.3 for an example of parameters that give us $\rho_\star < \rho$. In this example $\alpha = 1.4$, $\beta = -.5$ and $\sigma = 1$.

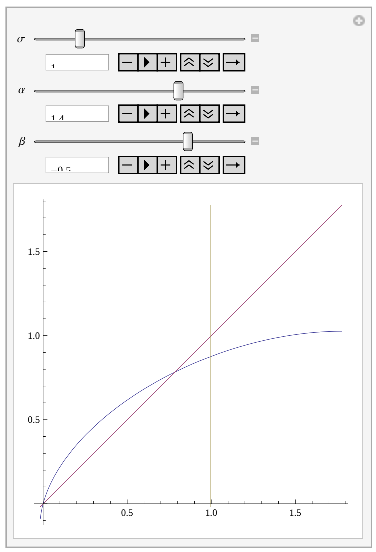

Figure 6.3: $\rho_\star < \rho_0$

By changing the parameters we see that a fixed point with $\rho_\star < \rho_0$ only occurs for $\alpha > 1$. In Figure 6.4 we see an example when $\alpha < 1$, $\rho_\star > \rho_0$. Changing β and σ does not seem to affect the presence of a fixed point but does affect the slope of the line segments of the cyclic paths. Now that we have found parameter values that give us a fixed point for $\psi(\rho)$ with $\rho_\star < \rho_0$ we need to check the stability, existence and Lipschitz continuity conditions of the Skorokhod Problem. Ultimately we wanted to find values of α and β so that solutions to the Skorokhod Problem are well-posed. Note that by changing the direction of the rotation of the cyclic path the roles of α and β switch. For example, we could have considered a path which crosses the face $\partial_1\Omega$ starting from the point $(0, x_2, 0)$ and ending at $(0, 0, 1)$ with the appropriate changes to Linecost and ρ_0. In this case $\rho_\star < \rho_0$ when $\beta > 1$. Hence, our

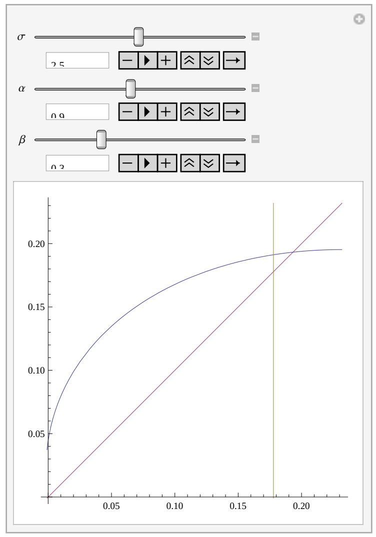

Figure 6.4: $\rho_\star > \rho_0$

results lead us to believe that $\alpha > 1$ or $\beta > 1$ are necessary for an optimal cyclic path in 3 dimensions.

Checking Conditions Related to the Skorokhod Problem

Recall our Hypotheses in Chapter 2 Section 2.2 and the various conditions we discussed related the the Skorokhod Problem. Here we explore the range of parameters α,β and σ for which our reflection matrix $D = \begin{pmatrix} 1 & \beta & \alpha \\ \alpha & 1 & \beta \\ \beta & \alpha & 1 \end{pmatrix}$ satisfies such conditions. By doing so we hope to find parameters such that we have a cyclic path ($\alpha > 1$ or $\beta > 1$) and satisfy conditions for the existence, stability and Lipschitz continuity of solutions to the Skorokhod Problem.

Independence of d_i

First we check that the reflection vectors d_i are linearly independent. Taking the determinante of D and setting it equal to 0 we have $\det(D) = 1 + \alpha^3 - 3\alpha\beta + \beta^3 = 0$, which can be factored as follows:

$$(1 + \alpha + \beta)((\alpha - 1)^2 - (\alpha - 1)(\beta - 1) + (\beta - 1)^2).$$

Since $x^2 - x + 1$ has no real roots the second factor is zero only when $\alpha = \beta = 1$ and is positive otherwise. The first term is zero when $\alpha + \beta = -1$. So we can conclude that $d_i, i \in \{1, 2, 3\}$ are linearly independent except when $\alpha = \beta = 1$ and $\alpha + \beta = -1$.

Stability Condition

From [7] and Condition 2.5 of [16],we know that stability for the Skorokhod Problem means that $D^{-1}b < 0$. This is equivalent to finding a vector $u > 0$ such that $-b = Du$. By symmetry we can assume $u = \begin{pmatrix} c \\ c \\ c \end{pmatrix}$, $c > 0$. Note that $\begin{pmatrix} 1 & \beta & \alpha \\ \alpha & 1 & \beta \\ \beta & \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = (1+\alpha+\beta, 1+\alpha+\beta, 1+\alpha+\beta)$.
So we can let $u = \frac{1}{1+\alpha+\beta} \begin{pmatrix} 1+\alpha+\beta \\ 1+\alpha+\beta \\ 1+\alpha+\beta \end{pmatrix}$. This implies that the stability condition for $b = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ holds if and only if $\alpha + \beta + 1 > 0$.

Lipschitz Condition

Assumption 2.1

Recall from Chapter 2 that the Lipschitz continuity of the Skorokhod Problem, Assumption 2.1, is equivalent to the existence of a special set B in $\mathbb{R}^n$. The existence of the set B described in Assumption 2.1 requires $\beta > 0$ when $\alpha > 1$. This is stated in the following lemma. (Similarly we can show that if $\beta > 1$ the existence of B requires $\alpha > 0$.) Soon we will conclude that in fact in both of these cases B can not exist.

Lemma 11. *The existence of the set B in Dupuis and Ishii's Condition 2.1 for* $D = \begin{pmatrix} 1 & \beta & \alpha \\ \alpha & 1 & \beta \\ \beta & \alpha & 1 \end{pmatrix}$ *and* $\alpha > 1$ *requires* $\beta > 0$.

Proof. Assume that B exists and satisfies Assumption 2.1. Let $z \in \partial B$ be a point near the x_1-axis where the second and third entry of z satisfy $z_2^2 + z_3^2 < 1$. By the definition B we have $\langle \nu, d_2 \rangle = \langle \nu, d_3 \rangle = 0$ for all $\nu \in \nu(z)$. So ν are scalar multiples of $d_2 \times d_3 = (1 - \alpha\beta, \alpha^2 - \beta, \beta^2 - \alpha) = V^{(1)}$.

Suppose by contradiction that $\beta \leq 0$. Let $b_1, b_2, b_3 > 0$ be the maximal points of B along the $\partial_{\{2,3\}}\Omega, \partial_{\{1,3\}}\Omega$ and $\partial_{\{1,2\}}\Omega$ axis respectively. Since $V^{(1)} \cdot \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix} = b_1(1 - \alpha\beta) > 0$ then

$$V^{(1)} \cdot \begin{pmatrix} 0 \\ b_2 \\ 0 \end{pmatrix} < V^{(1)} \cdot \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}.$$

This must be strict since otherwise $V^{(1)} \in \nu((0, b_2, 0))$, which means $V^{(1)}$ is parallel to $d_1 \times d_3$. (This is only true when $\beta = -(\alpha + 1)$; see note below). So we have

$$b_2(\alpha^2 - \beta) < b_1(1 - \alpha\beta)$$

$$b_2 < b_1 \frac{1-\alpha\beta}{\alpha^2-\beta}$$

where $(\alpha^2 - \beta) > 0$.

Using symmetry the same argument along the other axes where

$$V^{(2)} = d_3 \times d_1 \text{ and } V^{(3)} = d_1 \times d_3,$$

implies that

$$b_3 < b_2 \frac{1-\alpha\beta}{\alpha^2-\beta}$$

and

$$b_1 < b_3 \frac{1-\alpha\beta}{\alpha^2-\beta}.$$

Therefore

$$b_1 < b_1 (\frac{1-\alpha\beta}{\alpha^2-\beta})^3.$$

So

$$1 < \frac{1-\alpha\beta}{\alpha^2-\beta} \tag{6.8}$$

$$\iff \alpha^2 - \beta < 1 - \alpha\beta \tag{6.9}$$

$$\iff \alpha^2 - 1 < -\beta(\alpha - 1) \tag{6.10}$$

$$\iff \alpha + 1 < -\beta. \tag{6.11}$$

Then by squaring both sides we have $0 < (\alpha+1)^2 = \alpha^2+2\alpha+1 < \beta^2$ and so $0 < \alpha^2+\alpha+1 < \beta^2 - \alpha$.

Since $\beta^2 - \alpha > 0$ we can repeat the above starting from

$$V^{(1)} \cdot \begin{pmatrix} 0 \\ 0 \\ b_3 \end{pmatrix} < V^{(1)} \cdot \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}.$$

So

$$b_3(\beta^2 - \alpha) < b_1(1 - \alpha\beta) \text{ and we get } 1 < \frac{1-\alpha\beta}{\beta^2-\alpha}.$$

Then

$$\beta^2 - \alpha < 1 - \alpha\beta$$

$$\beta^2 - 1 < \alpha(1 - \beta)$$

$$-(\beta + 1) < \alpha \quad (\text{since } 1 - \beta > 0).$$

This says that $-\beta < \alpha + 1$, which contradicts (6.11). So we can conclude that $\beta > 0$.

Note: Consider the following to see that the inequality above is strict unless $\beta = -(\alpha + 1)$.

$$d_2 \times d_3 || d_1 \times d_3 \iff \begin{pmatrix} 1 - \alpha\beta \\ \alpha^2 - \beta \\ \beta^2 - \alpha \end{pmatrix} = c \begin{pmatrix} \beta^2 - \alpha \\ 1 - \alpha\beta \\ \alpha^2 - \beta \end{pmatrix}.$$

Then

$$1 - \alpha\beta = c(\beta^2 - \alpha) = c^2(\alpha^2 - \beta) = c^3(1 - \alpha\beta),$$

for some constant c. Which implies that $c = 1$ if and only

$$1 - \alpha\beta = \alpha^2 - \beta = \beta^2 - \alpha.$$

Then

$$\alpha^2 - \beta^2 = \beta - \alpha \text{ so } (\alpha + \beta) = -1 \ (\text{since}(\alpha - \beta) > 0).$$

This gives us $\beta = -1 - \alpha$.

But if $\beta = -(\alpha + 1)$, D does not satisfy the stability condition since $\alpha + \beta = -1$. So we can assume above that the inequalities are indeed strict.

□

Generalized Harrison-Reiman and Symmetry Conditions

The Generalized Harrison-Reiman Condition (Condition 2.4 from [16]) is a sufficient condition for Lipshitz continuity of solutions to the Skorokhod Problem. Let $Q = D - I$. The condition is that $|Q|$ has spectral radius less then 1 where $Q = \begin{pmatrix} 0 & \beta & \alpha \\ \alpha & 0 & \beta \\ \beta & \alpha & 0 \end{pmatrix}$. To find the eigenvalues of $|Q|$ we assume α and β are positive for simplification and replace the absolute values later. Here we have $\det(Q - \lambda I) = -\lambda^3 + 3\alpha\beta\lambda + \alpha^3 + \beta^3$. Setting $\det(Q - \lambda I) = 0$ it is not hard to see that $\alpha + \beta$ is a root of the characteristic equation. By polynomial division we see that

$$\lambda^3 - 3\alpha\beta\lambda - \alpha^3 - \beta^3 = (\lambda - (\alpha + \beta))(\lambda^2 + (\alpha + \beta)\lambda + (\alpha^2 - \alpha\beta + \beta^2)).$$

Setting the second factor equal to zero and solving for λ we get $\lambda = \frac{1}{2}(-(\alpha+\beta) \pm \sqrt{3}(\alpha+\beta)i)$. So the eigenvalues of Q are $\{(\alpha + \beta), \frac{1}{2}(-(\alpha + \beta) + \sqrt{3}(\alpha + \beta)i), \frac{1}{2}(-(\alpha + \beta) - \sqrt{3}(\alpha + \beta)i)\}$. Squaring the eigenvalues we have

$$\alpha^2 + 2\alpha\beta + \beta^2$$

from the first eigenvalue and

$$\alpha^2 - \alpha\beta + \beta^2$$

from remaining two eigenvalues. For nonnegative α and β the first equation above is clearly larger. So the spectral radius of $|Q|$ is $|\alpha| + |\beta|$. We conclude from this that in order for D to satisfy the Generalized Harrison-Reiman Condition we need $|\alpha| + |\beta| < 1$. Since we only had a fixed point with $\rho_\star < \rho_0$ when $\alpha > 1$ or $\beta > 1$, the parameter values we are interested in do not satisfy the Generalized Harrison Reiman Condition.

Existence

Assumption 3.1

Dupuis and Ishii's Assumption 3.1 in [18] for existence is satisfied if D is positive definite. We use the fact that D is positive definite if and only if $\frac{1}{2}(D + D^T)$ is and Sylvester's criterion in terms of determinants of principal submatrices for positive definiteness.

$$D + D^T = \begin{pmatrix} 2 & \alpha+\beta & \alpha+\beta \\ \alpha+\beta & 2 & \alpha+\beta \\ \alpha+\beta & \alpha+\beta & 2 \end{pmatrix},$$

and taking the determinante of $D + D^T$ we have

$$\det(D + D^T) = 8 - 6\alpha^2 + 2\alpha^3 - 12\alpha\beta + 6\alpha^2\beta - 6\beta^2 + 6\alpha\beta^2 + 2\beta^3.$$

This factors as

$$2(\alpha + \beta + 1)(\alpha + \beta - 2)^2.$$

Therefore D has positive determinante when $-1 < \alpha+\beta \neq 2$. The 2×2 principal minors have positive determinante when $(\alpha + \beta)^2 < 4$. So D is positive definite when $-1 < \alpha + \beta < 2$.

P-Matrix

Recalling the discussion of a P-matrix from Chapter 2 page 13, D has positive determinate when $\alpha + \beta + 1 > 0$ and $(\alpha, \beta) \neq (1, 1)$ which is necessary for D to be a P-Matrix. To see when D is a P-Matrix we must also find the parameter values when the determinante of the 2×2 principle minors are positive. This requires $\alpha\beta < 1$. We see that when Assumption 2.1 holds Assumption 3.1 holds when D is a P-matrix. So we are interested in when the parameters satisfy the following conditions $\alpha + \beta + 1 > 0$ and $\alpha\beta < 1$.

Search for Existence of the Set B

We know that $\alpha > 1$ or $\beta > 1$ is necessary to have a spiraling path. We want to find parameters such that Assumption 2.1 and 3.1 hold and $\alpha > 1$ or $\beta > 1$. It is not difficult to find parameters which satisfy the stability hypothesis and Assumption 3.1 with $\alpha > 1$ or $\beta > 1$. Finding parameters which also satisfy Assumption 2.1 is the challenge. Below we will offer two arguments which show that the set B does not exist for parameters which produce a cyclic path.

Axial Normal Argument

The following are what we call the "axial normals." These must be the normals to B at the points where it intersects the coordinate axes. Here we use the same idea as in Lemma 11. If in addition to $B = -B$ we presume that B is symmetric with respect to the rotations which interchange the coordinate axes then after rescaling we can assume that the points where B intersects the coordinate axes are $(\pm 1, 0, 0),\ (0, \pm 1, 0),\ (0, 0, \pm 1)$. Thus all of the following would need to be less then $1 - \alpha\beta = V^{(1)} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$:

$$V^{(1)} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \alpha^2 - \beta$$

$$V^{(1)} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = -\alpha^2 + \beta$$

$$V^{(1)} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \alpha + \beta^2$$

$$V^{(1)} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \alpha - \beta^2.$$

This requires that $|\alpha^2 - \beta| < 1 - \alpha\beta$ and $|\beta^2 - \alpha| < 1 - \alpha\beta$. By plotting this region we see that the parameter region in which the these inequalities are true does not contain $\alpha > 1$ or

$\beta > 1$ So the set B does not exist when $\alpha > 1$ or $\beta > 1$. See Figure 6.5. Here the values of α are on the horizontal axis and β on the vertical axis.

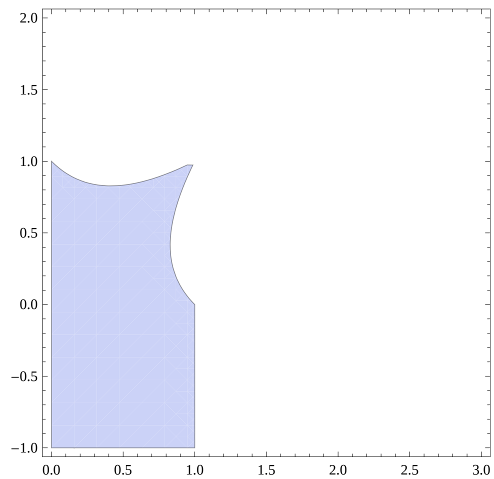

Figure 6.5: $|\alpha^2 - \beta| < 1 - \alpha\beta$ and $|\beta^2 - \alpha| < 1 - \alpha\beta$.

Note that although we presumed that B was symmetric with respect to the interchange of coordinate axes, the argument can be reformulated by arguing from intercept to intercept as in Lemma 11.

Composition of Operators L_i Argument

A second argument that the set B does not exist is based on Theorems 2.3 and 2.4 of Dupuis and Ramanan [17]. This argument shows that B does not exist for $\alpha > 1$. For $\beta > 1$ we have no conclusion about the set B using this argument.

From [17] we know that associated with the Skorokhod Problem is a collection of oblique projection operators $\{L_i, i = 1, 2, 3\}$ where L_i projects along the direction d_i on to the hyperplane $\{v : \langle v, n_i \rangle = 0\}$.

Definition 11. *The oblique projections associated with the Skorokhod Problem are given by*

$$L_i v = v + \langle v, n_i \rangle d_i.$$

Given a collection of linear operators $\mathcal{M}$ let $\mathcal{P}$ denote the set of all possible products of elements of $\mathcal{M}$. Next we state Dupuis and Ramanan's Theorem 2.3 in [17].

Theorem 11. *The following are equivalent:*

- *There exists and invariant set for the collection $\mathcal{M}$.*
- *The elements of $\mathcal{P}$ are uniformly bounded.*
- *There exists a norm $||\cdot||$ on R^n such that for all $M \in \mathcal{M}$ the operator norm with respect to this norm,*

$$||M|| = \sup_{x \neq 0} \frac{||Mx||}{||x||}$$

  is bounded by 1.

Theorem 2.4 from [17] tells us that the existence of B implies that every operator L_i is bounded by 1. So by Theorem 11 (Theorem 2.3 in [17]) we know that any sequence of compositions of operators L_i are uniformly bounded. This means that for any composition of operators L_i, call it P, the spectral radius of P is less then 1. First note that it is easy to see from the definition of L_i above that

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha & 1 & 0 \\ -\beta & 0 & 1 \end{pmatrix}, \; L_2 = \begin{pmatrix} 1 & -\beta & 0 \\ 0 & 0 & 0 \\ 0 & -\alpha & 1 \end{pmatrix}, \; L_3 = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider the composition $L_2 L_3 L_1$:

$$L_2 L_3 L_1 = \begin{pmatrix} \alpha\beta + \alpha\beta - \beta^3 & -\beta & \beta^2 - \alpha \\ 0 & 0 & 0 \\ \alpha^2 - \alpha\beta^2 & -\alpha & \alpha\beta \end{pmatrix}$$

$$L_2L_3L_1 - \lambda I = \begin{pmatrix} 2\alpha\beta - \beta^3 - \lambda & -\beta & \beta^2 - \alpha \\ 0 & -\lambda & 0 \\ \alpha^2 - \alpha\beta^2 & -\alpha & \alpha\beta - \lambda \end{pmatrix}.$$

$$\begin{aligned} \det(L_2L_3L_1 - \lambda I) &= -\lambda^3 + 3\alpha\beta\lambda^2 - \beta^3\lambda^2 - \alpha^3\lambda \\ &= \lambda(\lambda^2 + (3\alpha\beta - \beta^3)\lambda - \alpha^3). \end{aligned}$$

So we see that the set of eigenvalues is $\{0, \frac{1}{2}((3\alpha\beta - \beta^3) \pm \sqrt{-4\alpha^3 + (3\alpha\beta - \beta)^2})\}$. If $\alpha > 1$ then there is an eigenvalue greater then one. By graphing the region such that the maximal eigenvalue is less then 1 we see that no value of $\alpha > 1$ produces this region. See Figure 6.6.

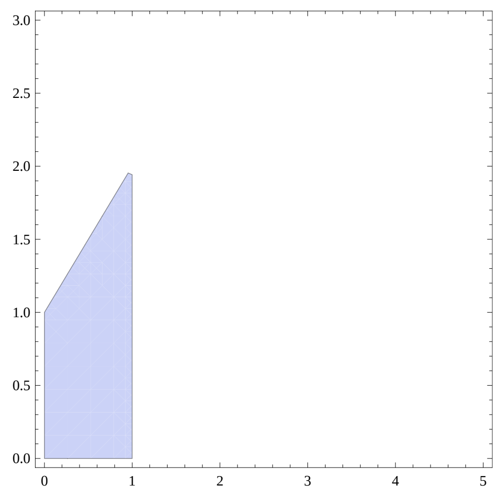

Figure 6.6: Region where spectral radius is less then 1

The values of α are on the horizontal axis and the values of β are on the vertical axis. So we can conclude that B does not exist for $\alpha > 1$.

6.3 Conclusion

From the discussion above we see that the parameters which give us the existence of a cyclic path do not give us the existence of the set B which is a sufficient condition for Lipschitz continuity of solutions to a Skorokhod Problem. Our work above leads us to believe that a cyclic optimal path in 3 dimensions for a Skorokhod Problem which is well-posed may not exist. Our search was restricted to a restricted class of "symmetric" parameters. It is possible that a cyclic optimal path for a well-posed Skorokhod Problem could exist for a different class of parameters. However, due to the symmetric nature of a cyclic path we believe that is unlikely.

6.4 Contributions of Others

Hasenbein and Liang are also concerned with the search for an optimal cyclic path. In [24] they looked at paths in 3 dimensions like our spiraling path which they call a "classic spiral" and another type of cyclic path called an "exotic spiral". They show that under certain conditions there always exists an optimal path which is a gradual path (a path which moves through the faces of strictly increasing dimensions) or a classic spiral. They also proved that "exotic spirals" are not optimal. Many of the observations made in [24] concerning existence and stability of solutions agree with our findings. The main difference between their work and ours is that they do not limit themselves to Skorokhod Problems which are Lipschitz continuous.

In Section 10 of [24] Hasenbein and Liang give an example of an optimal classic sprial where $r_1 = \alpha = \frac{3}{2}$, $r_2 = \beta = 0$ and $A = \Gamma = I$. Since $\alpha > 1$ the B-condition (Assumption 2.1) fails for this example which means that the Skorokhod Problem is not well-posed.

Chapter 7

Conclusion and Future Work

7.1 Characterization

As we know the Reflected Quasipotential arises in the study of Reflected Brownian motion and queueing networks. When certain conditions are satisfied, the stationary distribution of reflective Brownian motion satisfies a large deviation principle (with respect to a spatial scaling parameter) in which the Reflected Quasipotential provides information about the tail behavior of the stationary distribution. Here we provided a characterization of the Reflected Quasipotential in terms of viscosity solutions. Such a characterization has not been formulated in the literature previously. In dimensions greater then two, one needs to resort to numerical methods for finding $V(x)$. Under conditions where cyclic paths do not occur Majewski developed in [26] an algorithm to approximate $V(x)$ by a sequence of finite dimensional minimization problems. It is hoped that such a viscosity solution approach to the characterization of $V(x)$ can ease the difficulties faced when working with such a function. In particular, it may lead to more efficient ways to calculate $V(x)$ as well as to an analysis of the stationary distribution for approximating reflected Brownian motion.

To summarize, we described the Reflected Quasipotential $V(x)$, as a solution to

$$H(DV(x)) = 0$$

in $\Omega = \mathbb{R}^n_+ = \bigcap_{i=1,2,\ldots n}\{x : x_i \geq 0\}$ with appropriate boundary conditions. Since $V(x)$ may not be differentiable, the characterization is in terms of viscosity solutions. Although we were not able to describe $V(x)$ as a unique viscosity solution in n dimensions we were able to characterize $V(x)$ uniquely in 2 dimensions using our equivalent *Barles-Lions formulation.*

In the future we hope to extend our uniqueness proof to n dimensions. This will involve generalizing the Barles and Lions formulation to domains with corners as well as generalizing the Barles and Lions uniqueness proof to domains with corners.

The scaling lemma allows us to express $V(x)$ in the form,

$$V(x) = |x|w(\frac{x}{|x|}).$$

In 2 dimensions we can write $V(x)$ in terms of standard polar coordinates,

$$V(x) = rw(\theta).$$

This reduces the domain to $\theta \in [0, \frac{\pi}{2}]$. Perhaps by considering $V(x)$ in terms of polar coordinates where the geometry is much simpler and using our viscosity solution characterization in the Barles-Lions sense we can provide an alternative way to calculate explicit formulas for $V(x)$.

7.2 Exploration

In addition to our characterization of the Reflected Quasipotential we explored the possibility of cyclic optimal paths for $V(x)$ in 3 dimensions. As summarized above our search for a cyclic path where the Skorokhod Problem is well-posed was not successful. However we were able to determine parameters which produce a cyclic path. Our results were based on an interactive search using Mathematica. Our calculations show that for all $\alpha > 1$ or $\beta > 1$ there is a cyclic path which is optimal with respect to all straight line paths on the boundary, although we have not proven that it is optimal with respect to paths that pass through the interior at some point. A proof to verify our findings is something we hope to produce in the future.

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List of Notation:

- $D^{\pm}_{\Omega}u(x)$ pg 31
- H^{-}_{K} pg 32 equation (4.6)
- H_K pg 32 equation (4.7)
- $D^{\pm}_{\Omega^R}V^{R,\gamma}$ pg 37
- H^{o}_{C} pg 35 Lemma 6
- $F_i(p)$ pg 42 equation (4.30)
- λ_0 pg 44 equation (4.38)
- $\tilde{\lambda}$ pg 44 equation (4.39)
- $\lambda_{\pm 1}$ pg 44
- C_{F_i} pg 54
- $d(x)$ pg 58
- M pg 57 equation (5.5)
- M^{η} pg 58 equation (5.7)
- M^{η}_{ϵ} pg 59 equation (5.8)
- $(x^{\epsilon,\eta}, y^{\epsilon,\eta})$ pg 59
- Ψ pg 59 equation (5.9)
- $\Phi(x,y)$ pg 59
- $(\bar{x},\bar{y})$ pg 59
- $\Upsilon(x,y)$ pg 59
- $E^{\pm}(\epsilon,\eta)$ pg 64

- Linecost pg 78
- Q_K pg 80 equation (6.3)
- G_K pg 80 equation (6.2)
- Φ_F pg 81