Lie Derivations on Rings of Differential Operators

by

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(ABSTRACT)

Derivations on rings of differential operators are studied. In particular, we ask whether Lie derivations are forced to be associative derivations. This is established for the Weyl algebras, which provides the details of a theorem of A. Joseph.

The ideas are extended to localizations of Weyl algebras. As a corollary, the implication is verified for the universal enveloping algebras of nilpotent Lie algebras.

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Table of Contents

Abstract ii

Acknowledgements iii

Introduction **1**

Chapter 1 **4**

Rings of Differential Operators

Chapter 2 **15**

Associative Derivations

Chapter 3 **24**

Lie Derivations I

Chapter 4 **35**

Lie Derivations II

References **43**

Vita 44

Introduction

The main objective of this study is to understand derivations on rings of differential operators. Assume that K is a field of characteristic zero. If E is a not necessarily associative K-algebra then a *K-derivation* $D : E \to E$ is a K-linear map from E to itself which satisfies the "Leibniz" or "product " rule,

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$$

for all $a, b \in E$. The model for derivations is the familiar derivative from calculus: consider E to be the polynomial $K[X]$ and let $D = \frac{d}{dX}$. On the other hand, E might be a Lie algebra, in which case the Leibniz rule takes the form

$$D([ab]) = [D(a)b] + [aD(b)].$$

Some ambiguity may arise when E is an associative algebra because E also has the structure of a Lie algebra under the commutator bracket $[a, b] = ab - ba$ for $a, b \in E$. We will refer to *associative* derivations when the relevant product is the associative product and *Lie* derivations when the product is the commutator bracket. The interplay between these two types of derivations is the concern of this thesis. At this point we only mention that an associative derivation is always a Lie derivation. Let D be an associative derivation of E. For a, b in E,

$$\begin{aligned} D([a, b]) &= D(ab - ba) \\ &= D(a)b + aD(b) - D(b)a - bD(a) \\ &= [D(a), b] + [a, D(b)]. \end{aligned}$$

Thus D is a Lie derivation.

A deeper relationship between associative and Lie derivations is first discussed in a survey article of I. N. Herstein ([4]) in 1961. He reports I. Kaplansky's observation that every Lie K-derivation of the full algebra of $n \times n$ matrices over K is a scalar multiple of the trace plus an associative derivation. In general, if E is an associative algebra then there is a K-linear map $\tau : E \to K$ with $\tau([E, E]) = 0$ which sends $a \in E$ to $\tau(a)\cdot 1$. For trivial reasons, the function must be a Lie derivation of E. Herstein asks to what extent Kaplansky's observation holds for simple, or even primitive, algebras. In particular, if $[E, E] = E$ and E is a well-behaved primitive algebra must every Lie derivation be associative?

After some initial work of Martindale ([6]), the issue lay dormant. It was resurrected in a paper of Farkas and Letzter ([3]) concerned with noncommutative differential geometry. Following the suggestion of some physicists they describe algebras which might be considered as the coordinate rings of noncommutative symplectic manifolds. Associated to these algebras is a differential 2-form which satisfies a cocycle condition. It turned out that the 2-form is not a coboundary when Lie derivations are associative.

The quintessential noncommutative symplectic algebra is the Weyl algebra $A_1(K)$. Formally, it is the K-algebra with generators p and q and the single relation $[q, p] = 1$. It is realizable as the subalgebra of K-endomorphisms of the polynomial ring $K[X]$ generated by X as left multiplication and $\frac{d}{dX}$ by sending p to X and q to $\frac{d}{dX}$. The associative derivations are well known to be inner ([2]). This means that if D is an associative K-derivation of $A_1(K)$ then there exists some $b \in A_1(K)$ with $D(t) = [b, t]$ for all $t \in A_1(K)$. It is customary to borrow notation from the theory of Lie algebras and write $D = ad\ b$. In a paper which is not very well known ([5]), Joseph indicates that the inner derivations also account for all Lie derivations.

The Weyl algebra is the most prominant member of an entire family of algebras known as rings of differential operators. Assume that A is a commutative regular K-affine domain (e.g., the polynomial algebra in finitely many variables). The subalgebra of $\mathrm{End}_K(A)$ generated by A (considered as left multipliers) and $\mathrm{Der}_K(A)$, the A-module of K-derivations on A, is the ring $\mathcal{D}(A)$ of differential operators on A. (See [7] for an extensive discussion.) It is reasonable to ask whether the result for $A_1(K)$

generalizes: is every Lie derivation of $\mathcal{D}(A)$ automatically an associative derivation? After all, $\mathcal{D}(A)$ is a Noetherian simple algebra with many desirable finiteness properties ([7]). The gap between Joseph's theorem and this question seems to be very large. We settle a more modest question, although we attempt to prove as much as we can for $\mathcal{D}(A)$ in general.

As might be expected from a construction beginning with regular algebras, $\mathcal{D}(A)$ is very well-behaved locally. Indeed, if $\mathcal{M}$ is a maximal ideal of A then $\mathcal{D}(A_{\mathcal{M}})$ is a localization of $\mathcal{D}(A)$ and is isomorphic to some localization of a Weyl algebra. (To be more precise, we need the Weyl algebra $A_n(K)$ which is the ring of differential operators on the polynomial algebra $K[X_1, \ldots, X_n]$.) For this reason we look at a "local" version of the general question. We show that if B is a localization of an integral extension of $K[X_1, \ldots, X_n]$ at the powers of a special single element then Lie derivations of $\mathcal{D}(B)$ which vanish on the identity must be associative. In particular, we provide details for Joseph's theorem in the case of $A_n(K)$.

The special result on $A_n(K)$ has recently surfaced in a paper of De Wilde and Van Hauten ([1]). They regard $A_n(K)$ as a factor algebra of the universal enveloping algebra $U(L)$ with L a Heisenberg algebra. For example, if L is the Lie algebra of strictly upper triangular 3×3 matrices and

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then h is central in L and $U(L)/(h-1)U(L) \simeq A_1$. (Indeed,

$$\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right] = h.)$$

A description of Lie derivations of $U(L)$ along the lines of Kaplansky's description for matrices is presented. Thus, in some sense, the theorems found in the following thesis may also be regarded as results about Lie derivations for enveloping algebra of nilpotent Lie algebras.

Chapter 1. Rings of Differential Operators

In this chapter, we study the ring of differential operators on a commutative regular affine domain over a field.

Let A be a commutative affine domain over K, a field of characteristic zero. This notation will be kept fixed throughout unless otherwise specified. A K-linear transformation $\delta : A \to A$ is called a *K-derivation* of A if $\delta(ab) = a\delta(b) + b\delta(a)$ for all $a, b \in A$. The set of all such derivations is denoted by $\mathrm{Der}_K A$. $\mathrm{Der}_K A$ is an A-module by defining bX to be the derivation which sends a to $bX(a)$ for $X \in \mathrm{Der}_K A$. Note that it is a Lie algebra over K under the commutator bracket. Clearly $\mathrm{Der}_K A \subseteq \mathrm{End}_K(A)$. Now we identify A with left multiplications in $\mathrm{End}_K(A)$.

Definition 1.1. *The ring of differential operators* on A, denoted by $\mathcal{D}(A)$, is the K-subalgebra of $\mathrm{End}_K A$ generated by A and $\mathrm{Der}_K A$.

We note that A is a $\mathcal{D}(A)$-module via $f * a = f(a)$ for $f \in \mathcal{D}(A), a \in A$. In particular, $X * a = [X, a]$ for $X \in \mathrm{Der}_K A$.

Example 1.1. $A_n(K)$ denotes K-algebra with $2n$ generators $p_1, \ldots, p_n, q_1, \ldots, q_n$ and relations

$$[q_i, p_j] = \delta_{i,j} \text{ (the Kronecker delta)}$$

and

$$[p_i, p_j] = 0 = [q_i, q_j].$$

It is called the nth Weyl algebra over K. Let $K[X_1, \ldots, X_n]$ be the polynomial algebra in n indeterminates. Since $\mathrm{Der}_K(K[X_1, \ldots, X_n])$ is the free $K[X_1, \ldots, X_n]$-module on $\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}$ ([7]), the ring of differential operators $\mathcal{D}(K[X_1, \ldots, X_n])$ is generated as K-algebra by $X_1, \ldots, X_n$ and $\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}$. Since $\frac{\partial}{\partial X_i}(X_j) = \delta_{ij}$,

$$\Big[\frac{\partial}{\partial X_i}, X_j\Big] = \delta_{ij}.$$

The reader can easily show that $\left[\frac{\partial}{\partial X_i}, \frac{\partial}{\partial X_j}\right] = 0$. It follows that $A_n(K)$ maps onto $\mathcal{D}(K[X_1, \ldots, X_n])$ as an algebra. Since $A_n(K)$ is simple ([2]) the kernel of this map is zero. Thus $A_n(K)$ is isomorphic to the ring of differential operators over the commutative polynomial K-algebra $K[X_1, \ldots, X_n]$.

Notice that $p_1, \ldots, p_n$ in $A_n(K)$ are algebraically independent over K.

Definition 1.2. Let F be the free A-module on symbols da, $a \in A$. $\Omega_K(A) = F/N$ is *the module of Kähler differentials* of A where N is the submodule of F generated by $d\lambda, d(a+b) - da - db$, $d(ab) - a(db) - b(da)$ for $\lambda \in K, a, b \in A$. The derivation $d : A \rightarrow \Omega_K(A)$ given by $d(a) = da$ is the universal derivation of A.

There is a classical connection between $\mathrm{Der}_K A$ and $\Omega_K(A)^*$ where $-^*$ denotes the dual $\mathrm{Hom}_A(-, A)$.

Let $\Psi : \mathrm{Hom}_A(\Omega_K(A), A) \rightarrow \mathrm{Der}_K A$ be the map given by $\Psi(\theta) = \theta \circ d$ for $\theta \in \mathrm{Hom}_A(\Omega_K(A), A)$. Equivalently, $\Psi(\theta)(a) = \theta(da)$ for all $a \in A$. If $\theta_1 = \theta_2$ in $\mathrm{Hom}_A(\Omega_K(A), A)$, then $\theta_1 - \theta_2 = 0$, so $(\theta_1 - \theta_2) \circ d = 0$. Thus we have $\theta_1 \circ d = \theta_2 \circ d$. Hence Ψ is well-defined. To prove that Ψ is an A-homomorphism, let θ_1, θ_2 be in $\mathrm{Hom}_A(\Omega_K(A), A)$ and $u_1, u_2 \in A$. Then, for all $a \in A$,

$$\begin{aligned} \Psi(u_1\theta_1 + u_2\theta_2)(a) &= (u_1\theta_1 + u_2\theta_2)(da) \\ &= u_1\theta_1(da) + u_2\theta_2(da). \end{aligned}$$

It follows that $\Psi(u_1\theta_1 + u_2\theta_2) = u_1\Psi(\theta_1) + u_2\Psi(\theta_2)$.

Now let $\phi : \mathrm{Der}_K A \rightarrow \mathrm{Hom}_A(\Omega_K(A), A)$ defined by $\phi(X)(udv) = u(Xv)$ for $X \in \mathrm{Der}_K A$, $u, v \in A$. The universality of Ω shows that given any derivation $X \in \mathrm{Der}_K A$ there exists a unique $\phi(X) \in \mathrm{Hom}_A(\Omega_K(A), A)$ such that $X = \phi(X) \circ d$. This shows that ϕ is well defined. For $X_1, X_2 \in \mathrm{Der}_K A$ and $a_1, a_2 \in A$,

$$\begin{aligned} \phi(a_1X_1 + a_2X_2)(udv) &= u(a_1X_1 + a_2X_2)v \\ &= a_1uX_1v + a_2uX_2v \end{aligned}$$

$$\begin{aligned} &= a_1\phi(X_1)(udv) + a_2\phi(X_2)(udv) \\ &= (a_1\phi(X_1) + a_2\phi(X_2))(udv). \end{aligned}$$

Thus ϕ is an A-homomorphism. For $X \in \mathrm{Der}_K A$, $a \in A$,

$$\begin{aligned} \Psi \circ \phi(X)(a) &= \Psi(\phi(X))(a) = \phi(X)(da) = X(a) \text{ and} \\ \phi \circ \Psi(\theta)(udv) &= \phi(\Psi(\theta))(udv) = u(\Psi(\theta))(v) \\ &= u\theta(dv) = \theta(udv) \end{aligned}$$

for $udv \in \Omega_K(A)$. Thus $\Psi \circ \phi$ and $\phi \circ \Psi$ are identity functions of $\mathrm{Der}_K A$ and $\mathrm{Hom}_A(\Omega_K(A), A)$, respectively. It follows that Ψ is an isomorphism of A-modules. This explicitly establishes $\mathrm{Der}_K A \simeq \Omega_K(A)^*$ ([7]).

We get more information for the case when A is regular. Regularity of A implies that $\Omega_K(A)$ is a finitely generated projective A-module; in fact, ([7]) indicates that A is regular if and only if $\Omega_K(A)$ is projective. Under the assumption of regularity, we study a projective basis for $\Omega_K(A)$ and for $\mathrm{Der}_K A$. In general, if P is a finitely generated projective A-module then P has a projective basis $\{(p_j, f_j) \mid j = 1, \ldots, m\}$ where $p_j \in P$ and $f_j \in \mathrm{Hom}_A(P, A)$. That is, $x = \sum f_j(x)p_j$ for all $x \in P$. Let $Q(A)$ be the field of fractions of A. Then the rank of P is the dimension of $Q(A) \otimes_A P$ as a vector space over $Q(A)$. By the trace, we get $\sum f_j(p_j) = \mathrm{rank}(P) \cdot 1$. This follows from the fact that for a free module M, $f_j(p_i) = \delta_{ij}$ and $\sum_{j=1}^m f_j(p_j) = 1 + \cdots + 1 = m \cdot 1$, where $\{p_1, \ldots, p_m\}$ is a basis for M.

Suppose $\Omega_K(A)$ has a projective basis $\{(\sum_i u_i^{(t)} dv_i^{(t)}, f_t)\}$ for $t = 1, \ldots, m$. If $x \in \Omega_K(A)$, then $x = \sum_t \sum_i f_t(x) u_i^{(t)} dv_i^{(t)}$ for $u_i^{(t)}, v_i^{(t)} \in A$. Note that if $\{(p_1, f_1), \ldots, (p_{j-1}, f_{j-1}), (p_j + ap'_j, f_j), (p_{j+1}, f_{j+1}), \ldots, (p_m, f_m)\}$ is a projective basis for P then $\{(p_1, f_1), \ldots, (p_{j-1}, f_{j-1}), (p_j, f_j), (p'_j, af_j), (p_{j+1}, f_{j+1}), \ldots, (p_m, f_m)\}$ is also a projective basis of P. By stretching out those $u_i^{(t)} dv_i^{(t)}$, we take a basis $\{(u_i^{(t)} dv_i^{(t)}, f_t)\}, t = 1, \ldots, m$. Rewriting, we may assume that there is a projective basis of the form $\{(u_i dv_i, g_i)\}$ for $\Omega_K(A)$. Now we get a projective basis $\{(dv_i, h_i)\}$ for $\Omega_K(A)$ where

$h_i = u_i g_i, h_i \in \Omega_K(A)^*$. From $\Omega_K(A)^* \overset{\Psi}{\simeq} \mathrm{Der}_K A$, we may choose X_i in $\mathrm{Der}_K A$ with $X_i = \Psi(h_i)$. To get a projective basis for $\Omega_K(A)^* = \mathrm{Der}_K A$, we use the next lemma.

Lemma 1.1. *Let P be a projective R-module and set $P^* = Hom_R(P, R)$. If $\{(X_i, f_i)\}$ is a projective basis for P then P^* is a projective module with basis $\{(f_i, X_i)\}$ where we identify P with P^{**} in usual way : if $X \in P$ and $f^* \in P^*$ then X sends f^* to $f^*(X)$.*

Suppose that $\{(X_i, (dv_i)^{**})\}$ is a projective basis for $\mathrm{Der}_K A$. By Lemma 1.1, for each $v \in A$, dv can be identified with a member of $(\mathrm{Der}_K A\ )^*$. We may assume that $\{(X_i, dv_i)\}$ is a projective basis for $\mathrm{Der}_K A$. In fact, for any $Y \in \mathrm{Der}_K A$,

$$\begin{aligned} Y &= \sum dv_i(Y)(X_i) \\ &= \sum Y(dv_i)X_i \\ &= \sum Y(v_i)X_i. \end{aligned}$$

Instead of $\{(X_i, dv_i)\}$, we use a basis $\{(X_i, v_i)\}$ for $\mathrm{Der}_K A$ where $Y = \sum Y(v_i)X_i$ for all $Y \in \mathrm{Der}_K A$. We call this basis an *elite basis* for $\mathrm{Der}_K A$. The rank formula says that $\sum X_i(v_i) = \mathrm{rank}(\mathrm{Der}_K A)$. In $\mathcal{D}(A)$, this says $\sum [X_i, v_i] = \mathrm{rank}(\mathrm{Der}_K A)$. If A is the coordinate ring of a curve with rank 1 then $\sum [X_i, v_i] = 1$. This is the correct generalization of $[q, p] = 1$ in A_1.

Definition 1.3. A *Weyl basis* for $\mathrm{Der}_K A$ is an elite basis $\{(X_i, a_i)\}$ for $\mathrm{Der}_K A$ such that $X_i(a_j) = \delta_{ij}$.

We have the following three examples of the rings of differential operators with Weyl basis for the module of derivations. Let $S = K[p_1, \ldots, p_n]$ in $A_n(K)$.

Example 1.2. Recall that $A_n(K)$ is isomorphic to the ring of differential operators on S. $\mathrm{Der}_K S$ has a Weyl basis $\{(\frac{\partial}{\partial p_i}, p_i)\}$. By considernig q_i acting as $\frac{\partial}{\partial p_i}$, we see that $q_i(p_i) = 1$ and $q_i(p_j) = 0$, if $i \neq j$.

Let A be a regular K-affine domain and let $\mathcal{M}$ be a multiplicatively closed subset of A. Denote by $A_{\mathcal{M}}$ the localization of A at $\mathcal{M}$. The next examples show that we may choose an elite basis $\{(\delta_i, a_i)\}_{i=1}^n$ for $\mathrm{Der}_K(A_{\mathcal{M}})$ which is also one for $\mathrm{Der}_K(A)$.

Example 1.3. Consider the localization of Weyl algebra by a single element c in S. $A_n(K)_c = \mathcal{D}(S)_c$ equals $\mathcal{D}(S_c)$. Thus $\mathcal{D}(S_c)$ is generated by $\mathrm{Der}_K S$ and S_c. $\mathrm{Der}_K(S_c)$ has a Weyl basis as in example 1.2.

Example 1.4. Take any maximal ideal P of S. Let S_P be the localization of S at P. Note that S is a commutative Noetherian integral domain. Regularity of S shows that S_P is regular. Thus $\Omega_K(S_P)$ is free of rank n over S_P with a basis $dp_1, \ldots, dp_n$. By duality, $\mathrm{Der}_K(S_P)$ is free on $\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}$.

Next example exhibits an elite basis for derivations of an elliptic curve.

Example 1.5. Let E be the coordinate ring of the elliptic curve $y^2 = x^3 - x$; $E = \mathbf{C}[x,y]/(y^2 - x^3 + x)$ where $\mathbf{C}$ is the field of complex numbers. Let $\delta = y\frac{\partial}{\partial x} + \frac{1}{2}(3x^2 - 1)\frac{\partial}{\partial y}$. Note that $\delta(y^2 - x^3 + x) = 0$ so δ induces an endomorphism of E. It can be shown that $\mathrm{Der}_{\mathbf{C}}(E)$ is free E-module on δ ([7]).

We show that $\{(-\frac{9}{2}xy\delta, \bar{x}), ((3x^2-2)\delta, \bar{y})\}$ is an elite basis for $\mathrm{Der}_{\mathbf{C}}(E)$ where $\bar{x}, \bar{y}$ are the images of x, y under the epimorphism $\phi : \mathbf{C}[x,y] \to E$, respectively. Notice that $\delta(x) = y, \delta(y) = \frac{1}{2}(3x^2 - 1)$. Take any derivation D on E, i.e., $D = a\delta$ for some $a \in E$. Then

$$\begin{aligned}
& (Dx)(-\frac{9}{2}xy\delta) + (Dy)(3x^2 - 2)\delta \\
= \; & (a\cdot y)(-\frac{9}{2}\delta) + a\cdot\frac{1}{2}(3x^2 - 1)(3x^2 - 2)\delta \\
= \; & -\frac{9}{2}ax(x^3 - x) + \frac{1}{2}a(9x^4 - 9x^2 + 2)\delta \\
= \; & a\delta = D.
\end{aligned}$$

Thus $\{(-\frac{9}{2}xy\delta, \bar{x}), ((3x^2-2)\delta, \bar{y})\}$ is an elite basis for $\mathrm{Der}_{\mathbf{C}}(E)$. The ring of differential operators on E is isomorphic to $E[u;\delta]$ with the relations $[u,x] = y$ and $[u,y] = \frac{1}{2}(3x^2-1)$.

The next proposition shows that given an elite basis $\{(\delta_i, a_i)\}$, we can generalize the projective formula for $\mathcal{D}(A)$. If A is an affine domain, $\mathcal{D}(A)$ has a filtration $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \cdots$ where

$$\begin{aligned}
\mathcal{D}_0 &= A \\
\mathcal{D}_1 &= \mathcal{D}_0 + \mathrm{Der}_K A \\
\mathcal{D}_2 &= \mathcal{D}_1 + (\mathrm{Der}_K A)(\mathrm{Der}_K A) \\
&\vdots \\
\mathcal{D}_s &= \mathcal{D}_{s-1} + (\mathrm{Der}_K A)^s \\
&\vdots
\end{aligned}$$

Notice that $[\mathcal{D}_j, \mathcal{D}_1] \subseteq \mathcal{D}_j$ by induction on j. If $j = 0$, the assertion is trivial. Assume that the result is true for all $k < j$. For every $a \in A$, $[\mathcal{D}_j, \mathcal{D}_1](a) = [\mathcal{D}_j(a), \mathcal{D}_1] + [\mathcal{D}_j, \mathcal{D}_1(a)]$. By induction, $[\mathcal{D}_j, \mathcal{D}_1](a) \subseteq \mathcal{D}_{j-1}$ for all a in A. Thus we conclude that $[\mathcal{D}_j, \mathcal{D}_1] \subseteq \mathcal{D}_j$.

Proposition 1.1 ([3]). *Let A be a commutative regular K-affine domain. Given an elite basis $\{(\delta_i, a_i)\}$, we have $\sum[\omega, a_i]\delta_i \equiv s\omega \pmod{\mathcal{D}_{s-1}}$ whenever $\omega \in \mathcal{D}_s$.*

Proof. We show the above result by induction on s. It is trivial if $s = 0$. Suppose it is true for all $k < s$. If we let $\omega \in \mathcal{D}_s$, then $\omega = W + X$ where $W \in \mathcal{D}_{s-1}$ and $X \in (\mathrm{Der}_K A)^s$.

$$\begin{aligned}
\sum_i [\omega, a_i]\delta_i &= \sum [W + X, a_i]\delta_i \\
&= \sum [W, a_i]\delta_i + \sum [X, a_i]\delta_i \\
&\equiv (s-1)W + \sum [X, a_i]\delta_i \pmod{\mathcal{D}_{s-2}} \\
&\equiv sW + \sum [X, a_i]\delta_i \pmod{\mathcal{D}_{s-1}},
\end{aligned}$$

by induction. Since X is in $(\mathrm{Der}_K A)^s$, we may write $X = \sum YZ$ for $Y \in \mathrm{Der}_K A$ and

$Z \in (\mathrm{Der}_K A)^{s-1}$. Then

$$\begin{aligned}\sum_i [YZ, a_i]\delta_i &= \sum_i Y[Z, a_i]\delta_i + \sum_i [Y, a_i] Z\delta_i \\ &= Y\sum [Z, a_i]\delta_i + \sum [Y, a_i](\delta_i Z + [Z, \delta_i]) \\ &\equiv Y \cdot (s-1)Z + YZ + \sum [Y, a_i][Z, \delta_i] \pmod{\mathcal{D}_{s-2}},\end{aligned}$$

by induction and $\sum [Y, a_i]\delta_i = Y$ for $Y \in \mathrm{Der}_K A$. Since $[\mathcal{D}_j, \mathcal{D}_1] \subseteq \mathcal{D}_j$, we obtain $[Z, \delta_i] \in \mathcal{D}_{s-1}$ for $Z \in (\mathrm{Der}_K A)^{s-1}$. Hence

$$\begin{aligned}\sum_i [YZ, a_i]\delta_i &\equiv Y \cdot (s-1)Z + YZ \pmod{\mathcal{D}_{s-1}} \\ &\equiv sYZ \pmod{\mathcal{D}_{s-1}}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_i [X, a_i]\delta_i &= \sum \sum_i [YZ, a_i]\delta_i \\ &\equiv \sum sYZ \equiv sX \pmod{\mathcal{D}_{s-1}}.\end{aligned}$$

Therefore $\sum [\omega, a_i]\delta_i \equiv s\omega \pmod{\mathcal{D}_{s-1}}$ as required. □

By using this projective formula, we have the following result.

Lemma 1.2. *Let A be a commutative regular affine algebra over a field K of characteristic zero. Let $(\delta_1, a_1), \ldots, (\delta_n, a_n)$ be an elite basis for $\mathrm{Der}_K(A)$. If $x \in \mathcal{D}(A)$ commutes with all a_i then $x \in A$.*

Proof. Suppose $x \in \mathcal{D}_s \backslash \mathcal{D}_{s-1}$. Then

$$\begin{aligned}sx &\equiv \sum [x, a_i]\delta_i \pmod{\mathcal{D}_{s-1}} \\ &\equiv 0 \pmod{\mathcal{D}_{s-1}}.\end{aligned}$$

Thus $sx \in \mathcal{D}_{s-1}$ which is impossible. Therefore $s = 0$, so $x \in \mathcal{D}_0 = A$. □

Note that we may choose an elite basis $\{(\delta_i, a_i)\}_{i=1}^n$ for $\mathrm{Der}_K(A_{\mathcal{M}})$ which is also one for $\mathrm{Der}_K(A)$. Thus A in lemma 1.2 can be replaced by $A_{\mathcal{M}}$.

We have a similar result to lemma 1.2 for the case when A is an integral extension of a commutative K-affine domain.

Lemma 1.3. *Let A be an integral extension of a commutative K-affine domain B. If x in $\mathcal{D}(A)$ commutes with B, then $x \in A$.*

Proof. Let $E = \{y \in \mathcal{D}_m(A) \mid [y, B] = 0\}$ We proceed by induction on $m \geq 1$ that $E \cap \mathcal{D}_m(A) \subseteq A$. First suppose $m = 1$. If $x \in E \cap \mathcal{D}_1(A)$ we may decompose $x = a + D$ where $a \in A$ and $D \in \mathrm{Der}_K A$. It suffices to show that $D = 0$. So suppose D is in E. Choose an arbitrary element ω in A. Let $f(t) = t^k + b_{k-1}t^{k-1} + \cdots + b_0$ be the minimal polynomial for $\omega \in A$ over B, so $f(\omega) = 0$. Then

$$\begin{aligned} 0 &= D(f(\omega)) = k\omega^{k-1}D(\omega) + b_{k-1}(k-1)\omega^{k-2}D(\omega) + \cdots + b_1 D(\omega) \\ &= (k\omega^{k-1} + (k-1)b_{k-1}\omega^{k-2} + \cdots + b_1)D(\omega), \end{aligned}$$

since $[D, b] = 0$ for all $b \in B$. By the minimality of f, we conclude that $D(\omega) = 0$ and so $D = 0$. Thus $x \in A$.

Now assume that the result is true for all $1 < k < m$. Let $x \in \mathcal{D}_m(A)$ for $m > 1$ and $a \in A$. Then $[x, a] \in \mathcal{D}_{m-1}(A)$ and $[x, B] = 0$. But

$$\Big[[x, a], B\Big] = \Big[[x, B], a\Big] + \Big[x, [a, B]\Big] = 0,$$

since $[x, B] = 0$ and $[a, B] = 0$. By induction, we get $[x, a] \in A$, i.e., $[x, A] \subseteq A$ for $x \in \mathcal{D}_m(A), m > 1$. But the projective formula tells us that if $x \in \mathcal{D}(A)$ then $mx = \sum [x, a_i]\delta_i + \nu$ where $\nu \in \mathcal{D}_{m-1}$ and $\{(\delta_i, a_i)\}$ is an elite basis for $\mathrm{Der}_K A$. Since $[x, a_i] \in A, mx \in \mathcal{D}_1 + \mathcal{D}_{m-1} = \mathcal{D}_{m-1}$ and hence, $x \in \mathcal{D}_{m-1}$ for $m > 1$. By induction, we have $x \in A$ if $x \in \mathcal{D}_m$, for $m > 1$. □

Let D be an associative algebra derivation on an algebra F. The following lemma gives us a sufficient condition for D to be a derivation on a subalgebra E.

Lemma 1.4. *Suppose E is a subalgebra of F. Let D be an associative algebra derivation on F such that $D(e_i) \in E$ for a set of algebra generators $\{e_i \mid i \in I\}$ of E. Then D is a derivation on E.*

Proof. It suffices to show that $D(a) \in E$ for $a = {e_1}^{i_1} \cdots {e_s}^{i_s}$. We argue by induction on $|a| = \sum_{j=1}^{s} i_j$. If $|a| = 1$, it is trivial. Choose the smallest t such that $i_t \neq 0$. Then

$$D(a) = D(a_t b) = D(a_t)b + a_t D(b) \in E$$

by induction. Thus $D(a) \in E$ for all $a \in E$. We have shown that $D(E) \subseteq E$, hence D is a derivation on E. □

There is a similar result from elementary calculus, called the existence of potentials for gradients.

Theorem 1.1. *Let $D_1, \ldots, D_n$ be commuting derivations on a ring R. Assume that $E_i, \ldots, E_n$ are endomorphisms of R such that $D_i E_i = id$ for all i and $D_k E_l = E_l D_k$ for $k \neq l$. If $f_1, \ldots, f_n$ are elements in R which satisfy $D_i(f_j) = D_j(f_i)$ for $1 \leq i, j \leq n$, then there exists an element h in R with $D_i(h) = f_i$.*

Proof. We proceed by induction on n to show that there exists an element h_n in R with $D_i(h_n) = f_i$ for $1 \leq i \leq n$. Let

$$h_1 = E_1(f_1) \text{ and } h_n = \sum_{j=1}^{n} E_j(f_j) - \sum_{l=2}^{n} E_l D_l(h_{l-1}).$$

If $n = 1$ then $D_1(h_1) = f_1$, so we are done. Assume that $D_i(h_{n-1}) = f_i$ for $1 \leq i \leq n-1$. For $1 \leq i \leq n$,

$$\begin{aligned} D_i(h_n) &= D_i\Big(\sum_{j=1}^{n-1} E_j(f_j) + E_n(f_n) - \sum_{l=2}^{n-1} E_l D_l(h_{l-1}) - E_n D_n(h_{n-1})\Big) \\ &= D_i(h_{n-1}) + D_i E_n(f_n) - D_i E_n D_n(h_{n-1}). \end{aligned}$$

If $i < n$ then

$$\begin{aligned} D_i(h_n) &= f_i + E_n D_i(f_n) - E_n D_n(f_i) \\ &= f_i, \end{aligned}$$

by induction and $D_i(f_n) = D_n(f_i)$. If $i = n$ then

$$\begin{aligned} D_n(h_n) &= D_n(h_{n-1}) + f_n - D_n(h_{n-1}) \\ &= f_n, \end{aligned}$$

since $D_n E_n = $ id. Therefore there exists $h \in R$ such that $D_i(h) = f_i$ for $i = 1, \ldots, n$.
□

For the existence of potentials for gradient from calculus, we take D_i and E_i be the formal differentiation and integration with respect to X_i, respectively, in a commutative polynomial ring with n variables $X_1, \ldots, X_n$.

Example 1.6. Let $\{(q_i, p_i)\}_{i=1}^n$ be a Weyl basis for $\mathrm{Der}_K A$ and consider $R = \mathcal{D}(A)$. We can also write as $R = A[q_1, \ldots, q_n]$. Let $f = \sum_k f_k$ be an element in R where all f_k are monomials in $q_1, \ldots, q_n$. Define, for $i = 1, \ldots, n$

$$\begin{aligned} D_i(f) &= [f, p_i] \text{ and} \\ E_i(f) &= \sum_k \frac{1}{(n_{ki}+1)(n_{ki}+2)}[f_k q_i^2, p_i] \end{aligned}$$

where n_{ki} is the power of q_i in the monomial f_k. Clearly D_i's are commuting derivations on R and E_i's are linear on A. Extend E_i linearly.

$$\begin{aligned} D_i E_i(f) &= D_i\Big(\sum_k \frac{1}{(n_{ki}+1)(n_{ki}+2)}[f_k q_i^2, p_i]\Big) \\ &= \sum_k \frac{1}{(n_{ki}+1)(n_{ki}+2)}\Big[[f_k q_i^2, p_i], p_i\Big]. \end{aligned}$$

But

$$\begin{aligned} \Big[[f_k q_i^2, p_i], p_i\Big] &= \Big[(f_k[q_i^2, p_i] + [f_k, p_i]q_i^2), p_i\Big] \\ &= \Big[(f_k \cdot 2q_i + n_{ki} f_k q_i), p_i\Big] \\ &= (n_{ki}+2)[f_k q_i, p_i]. \end{aligned}$$

Therefore

$$\begin{aligned} D_i E_i(f) &= \sum_k \frac{1}{n_{ki}+1}[f_k q_i, p_i] \\ &= \sum_k \frac{1}{n_{ki}+1}(f_k + n_{ki} f_k) \\ &= \sum_k f_k = f, \end{aligned}$$

showing that $D_iE_i =$ id. For $i \neq j$,

$$D_iE_j(f) = \sum_k \frac{1}{(n_{kj}+1)(n_{kj}+2)}\Big[[f_kq_j^2, p_j], p_i\Big].$$

From the Jacobian identity,

$$\begin{aligned}\Big[[f_kq_j^2, p_j], p_i\Big] &= \Big[[f_kq_j^2, p_i], p_j\Big] \\ &= \Big[[f_k, p_i]q_j^2, p_j\Big].\end{aligned}$$

Thus for $i \neq j$,

$$D_iE_j(f) = \sum_k \frac{1}{(n_{kj}+1)(n_{kj}+2)}\Big[[f_k, p_i]q_j^2, p_j\Big].$$

On the other hand,

$$\begin{aligned}E_jD_i(f) &= E_j(\sum_k [f_k, p_i]) \\ &= \sum_k \frac{1}{(n_{kj}+1)(n_{kj}+2)}\Big[[f_k, p_i]q_j^2, p_j\Big].\end{aligned}$$

Thus we obtain that $D_iE_j = E_jD_i$ for $i \neq j$.

Let T be a Lie derivation on $\mathcal{D}(A)$. Then

$$0 = T([p_i, p_j]) = [T(p_i), p_j] + [p_i, T(p_j)],$$

i.e., $D_j(T(p_i)) = D_i(T(p_j))$. It follows from theorem 1.1 that there exists an element h in $\mathcal{D}(A)$ such that $D_i(h) = T(p_i)$, i.e., $[h, p_i] = T(p_i)$ for all i.

Chapter 2. Associative Derivations

In this chapter, we study an associative derivation on a K-algebra $\mathcal{E}$. A K-linear mapping D of $\mathcal{E}$ into itself such that $D(xy) = xD(y) + D(x)y$ for all $x, y \in \mathcal{E}$ is called an *associative derivation* of $\mathcal{E}$. Throughout, we denote $K[p_1, \ldots, p_n, q_1, \ldots, q_n]$ by the nth Weyl algebra and $K(p_1, \ldots, p_n)[q_1, \ldots, q_n]$ by $B_n(K)$, a localization of Weyl algebra.

We shall prove that if α_i, β_i, $i = 1, \ldots, n$ are the elements in $B_n(K)$ which satisfy the conditions that $[\alpha_i, p_j] + [p_i, \alpha_j] = 0$, $[\alpha_i, q_j] + [p_i, \beta_j] = 0$, and $[\beta_i, q_j] + [q_i, \beta_j] = 0$ for $1 \leq i, j \leq n$, then there exists an associative algebra derivation D on $B_n(K)$ with $D(p_i) = \alpha_i$ and $D(q_i) = \beta_i$ for $1 \leq i \leq n$. Let A be a regular commutative K-affine domain. Furthermore, we show that if $\mathrm{Der}_K A$ has a Weyl basis $\{(q_i, p_i)\}_{i=1}^{n}$ and $\beta_i \in \mathcal{D}(A)$ with $[\beta_i, q_j] + [q_i, \beta_j] = 0$ for $1 \leq i, j \leq n$, there exists an associative derivation D on $\mathcal{D}(A)$ such that D vanishes on A and $D(q_i) = \beta_i$.

We assume throughout that an associative derivation on K-algebra $\mathcal{E}$ vanishes on its center. It is true that the center of the ring $\mathcal{D}(A)$ of differential operators of A is, in fact, a finite field extension of the base field K ([3]). Also, a derivation vanishes on finite field extensions of K.

McConnell-Robson indicates that any derivation on A can be uniquely extended to a derivation of any localization of A (15.1.23). We begin with the following lemmas.

Lemma 2.1. *If $\{(\delta_i, a_i)\}$ is an elite basis for $\mathrm{Der}_K A$ then it is one for $\mathrm{Der}_K Q(A)$ with $Q(A)$ being the field of fractions of A.*

Proof. Let $X \in \mathrm{Der}_K Q(A)$. Then there is a nonzero element $a \in A$ such that $aX \in \mathrm{Der}_K A$. Thus it follows that $aX = \sum aX(a_i)\delta_i$. By multiplying the both sides by a^{-1} on the left, $X = \sum X(a_i)\delta_i$, showing that $\{(\delta_i, a_i)\}$ is an elite basis for $\mathrm{Der}_K Q(A)$. □

The next lemma shows that $Q(A)$ is a finite field extension of $K(a_1, \ldots, a_n)$ where

$\{(\delta_i, a_i)\}$ is an elite basis for $\mathrm{Der}_K A$.

Lemma 2.2. *$Q(A)|K(a_1,\ldots,a_n)$ is a finite field extension.*

Proof. Suppose that $Q(A)|K(a_1,\ldots,a_n)$ is not algebraic. Then there exist $u_1,\ldots,u_t$, $t \geq 1$ where $K(a_1,\ldots,a_n)(u_1,\ldots,u_t)$ is purely transcendental over $K(a_1,\ldots,a_n)$ and $Q(A)$ is a finite field of $K(a_1,\ldots,a_n)(u_1,\ldots,u_t)$. Let $D = \frac{\partial}{\partial u_1}$. By this we mean the unique derivation on $Q(A)$ such that

$$\begin{aligned} D(K(a_1,\ldots,a_n)) &= 0 \\ D(u_1) &= 1 \\ D(u_r) &= 0 \text{ for } r > 1. \end{aligned}$$

Clearly $D \in \mathrm{Der}_K Q(A)$. By lemma 2.1, $D = \sum D(a_i)\delta_i = 0$, which is a contradiction. Thus $Q(A)$ is algebraic, equivalently, a finite field extension over $K(a_1,\ldots,a_n)$. □

A *Lie derivation* T of a Lie algebra $\mathcal{L}$ is a K-linear map from $\mathcal{L}$ to itself such that $T([x,y]) = [T(x),y] + [x,T(y)]$ for $x,y \in \mathcal{L}$. We also assume throughout that a Lie derivation on $\mathcal{L}$ is linear on its center.

The following lemma provides conditions on generators and relations for $\mathcal{D}_1$ which ensure that $D : \mathcal{D}_1 \to \mathcal{D}(A)$ is a left $\mathcal{D}_0$-module derivation and a Lie derivation. Suppose that $D : \mathcal{D}_1 \to \mathcal{D}(A)$ is an associative derivation. If $\{(\delta_i, a_i)\}_{i=1}^n$ is an elite basis for $\mathrm{Der}_K A$, we may write $[\delta_i, a_j] = h_{ij}$ and $[\delta_i, \delta_j] = \sum_t f_{ij}^t \delta_t$ with $h_{ij}, f_{ij}^t \in A$. Then

$$\begin{aligned} D([\delta_i, a_j]) &= [D(\delta_i), a_j] + [\delta_i, D(a_j)], \\ D(\delta_k) &= \sum_i \Big(D(h_{ki})\delta_i + h_{ki} D(\delta_i) \Big) \text{ and} \\ \sum_t D(f_{ij}^t)\delta_t + \sum_t f_{ij}^t D(\delta_t) &= [D(\delta_i), \delta_j] + [\delta_i, D(\delta_j)]. \end{aligned}$$

Lemma 2.3. *Let $\{(\delta_i, a_i)\}_{i=1}^n$ be an elite basis for $\mathrm{Der}_K A$. Let $D : A \to \mathcal{D}(A)$ be an A-bimodule derivation. If $\beta_i \in \mathcal{D}(A)$, $i = 1,\ldots,n$ are such that*

(*i*) $D([\delta_i, a_j]) = [\beta_i, a_j] + [\delta_i, D(a_j)]$,

$$(ii) \quad \beta_k = \sum_i \Big(D(h_{ki})\delta_i + h_{ki}\beta_i \Big) \text{ and}$$

$$(iii) \quad \sum_t D(f_{ij}^t)\delta_t + \sum_t f_{ij}^t \beta_t = [\beta_i, \delta_j] + [\delta_i, \beta_j]$$

$$\text{where } [\delta_i, \delta_j] = \sum_t f_{ij}^t \delta_t \text{ and } [\delta_i, a_j] = h_{ij} \text{ for } f_{ij}^t, h_{ij} \in A,$$

then there exists a unique extension $D : \mathcal{D}_1 \to \mathcal{D}(A)$ *which is a left* $\mathcal{D}_0$*-module derivation and a Lie derivation such that* $D(\delta_i) = \beta_i$.

Proof. Let $D : A \to \mathcal{D}(A)$ be an A-bimodule derivation and let β_i be the elements in $\mathcal{D}(A)$ with conditions as mentioned above. Given $X \in \mathcal{D}_1$ we may write $X = \sum_i [X, a_i]\delta_i + r$ for $r \in A$. Define

$$D(X) = \sum_i D([X, a_i])\delta_i + \sum [X, a_i]\beta_i + D(r).$$

First we show that D is well-defined on $\mathcal{D}_1$. If $X = Y$ in $\mathcal{D}_1$ where

$$X = \sum [X, a_i]\delta_i + r \text{ and } Y = \sum [Y, a_i]\delta_i + s$$

for $r, s \in \mathcal{D}_0$, then $X - Y = \sum [X - Y, a_i]\delta_i + (r - s)$. Hence

$$\begin{aligned} 0 &= D(X - Y) = \sum D([X - Y, a_i])\delta_i + \sum [X - Y, a_i]\beta_i + D(r - s) \\ &= \sum \Big(D([X, a_i]) - D([Y, a_i]) \Big)\delta_i + \sum ([X, a_i] - [Y, a_i])\beta_i + D(r) - D(s) \\ &= D(X) - D(Y), \end{aligned}$$

since D is an A-bimodule derivation. Thus $D(X) = D(Y)$. If $X \in \mathcal{D}_0$, then $[X, a_i] = 0$, and so $D(X) = D(r)$. Thus D is well-defined on $\mathcal{D}_1$. Additivity of D follows from the definition of D and the fact that D is an A-bimodule derivation of A.

To show that D is a left $\mathcal{D}_0$-module derivation, it suffices to show that $D(f\delta_k) = D(f)\delta_k + f\beta_k$ for $f \in A$, since D is an A-bimodule derivation of A into $\mathcal{D}(A)$. From $\delta_k = \sum [\delta_k, a_i]\delta_i$, we have $f\delta_k = \sum_i [f\delta_k, a_i]\delta_i$. The definition of D implies that

$$\begin{aligned} D(f\delta_k) &= \sum_i \Big(D([f\delta_k, a_i])\delta_i + [f\delta_k, a_i]\beta_i \Big) \\ &= \sum \Big(D(f[\delta_k, a_i])\delta_i + f[\delta_k, a_i]\beta_i \Big). \end{aligned}$$

Since D is a derivation of A,

$$\begin{aligned} D(f\delta_k) &= \sum\Big(D(f)[\delta_k, a_i]\delta_i + fD([\delta_k, a_i])\delta_i + f[\delta_k, a_i]\beta_i\Big) \\ &= D(f)\delta_k + f\sum\Big(D([\delta_k, a_i])\delta_i + [\delta_k, a_i]\beta_i\Big). \end{aligned}$$

By the definition of $D(\delta_k)$, $D(f\delta_k) = D(f)\delta_k + f\beta_k$.

To prove that D is a Lie derivation on $\mathcal{D}_1$, we first show that $D([\delta_k, f]) = [\beta_k, f] + [\delta_k, D(f)]$ for any $f \in A$. Let $D' = [D, ad(\delta_k)] - ad(D(\delta_k))$ where $ad(\ )$ is a derivation on A defined by $ad(x) = [x, *]$. It is enough to show that $A \subseteq M = \{f \in Q(A) : D'(f) = 0\}$ where $Q(A)$ is the field of fraction of A. Clearly, D' is a derivation on A, hence on $Q(A)$, so for any $f, g \in M$, $D'(f+g) = 0$ and $D'(fg) = 0$. Thus M is a ring. Furthermore, M is a field. For, if $f \in M$, then $D'(f^{-1}) = -f^{-1}(D'f)f^{-1} = 0$. By the hypotheses (i), we conclude that the a_i's are in M. Thus $K[a_1, \ldots, a_n]$ is a subset of M, hence M contains $K(a_1, \ldots, a_n)$. Since $Q(A)$ is a finite field extension of $K(a_1, \ldots, a_n)$ from lemma 2.2, D' vanishes on $Q(A)$. This result comes from the fact that a derivation vanishes on any finite field extension of a field on which it already vanishes. Therefore $A \subseteq M$, so $D([\delta_k, f]) = [\delta_k, D(f)] + [D(\delta_k), f]$ for any $f \in A$. It easily follows that $D([X, f]) = [D(X), f] + [X, D(f)]$ for $X \in \mathcal{D}_1, f \in A$.

Next we show that $D([\delta_i, f\delta_k]) = [\beta_i, f\delta_k] + [\delta_i, D(f\delta_k)]$.

$$\begin{aligned} D([\delta_i, f\delta_k]) &= D([\delta_i, f]\delta_k + f[\delta_i, \delta_k]) \\ &= D([\delta_i, f])\delta_k + [\delta_i, f]\beta_k + D(f)[\delta_i, \delta_k] + fD([\delta_i, \delta_k]), \end{aligned}$$

since D is a left $\mathcal{D}_0$-module derivation of $\mathcal{D}_1$. But (iii) and the previous result stating that $D([\delta_i, f]) = [\beta_i, f] + [\delta_i, D(f)]$ implies that

$$\begin{aligned} D([\delta_i, f\delta_k]) &= ([\beta_i, f] + [\delta_i, D(f)])\delta_k + [\delta_i, f]\beta_k + D(f)[\delta_i, \delta_k] + f([\beta_i, \delta_k] \\ &\quad +[\delta_i, \beta_k]) \\ &= [\beta_i, f\delta_k] + [\delta_i, D(f)\delta_k] + [\delta_i, f\beta_k] \\ &= [\beta_i, f\delta_k] + [\delta_i, D(f\delta_k)]. \end{aligned}$$

Hence we proved that $D([\delta_i, f\delta_k]) = [\beta_i, f\delta_k] + [q_i, D(f\delta_k)]$ and so $D([\delta_i, X]) = [\beta_i, X] + [\delta_i, D(X)]$ for all $X \in \mathcal{D}_1$. By this result and the fact that D is a left $\mathcal{D}_0$-module derivation of $\mathcal{D}_1$, we conclude that $D([X,Y]) = [D(X), Y] + [X, D(Y)]$ for $X, Y \in \mathcal{D}_1$. Therefore D is a Lie derivation from $\mathcal{D}_1$ to $\mathcal{D}(A)$. □

In fact, D described as in lemma 2.3 can be extended to an associative derivation on $\mathcal{D}(A)$. We prove the result.

Theorem 2.1. *Assume that $Der_K A$ has an elite basis $\{(\delta_i, a_i)\}_{i=1}^n$. Let $D : A \to \mathcal{D}(A)$ be an A-bimodule derivation. If $\beta_i \in \mathcal{D}(A)$, $i = 1, \ldots, n$ are such that*

$$(i) \quad D([\delta_i, a_j]) = [\beta_i, a_j] + [\delta_i, D(a_j)],$$

$$(ii) \quad \beta_k = \sum_i \left(D(h_{ki})\delta_i + h_{ki}\beta_i\right) \text{ and}$$

$$(iii) \quad \sum_t D(f_{ij}^t)\delta_t + \sum_t f_{ij}^t \beta_t = [\beta_i, \delta_j] + [\delta_i, \beta_j]$$

$$\text{where } [\delta_i, \delta_j] = \sum_t f_{ij}^t \delta_t \text{ and } [\delta_i, a_j] = h_{ij} \text{ for } f_{ij}^t, h_{ij} \in A,$$

then D can be extended to an associative derivation on $\mathcal{D}(A)$ such that $D(\delta_i) = \beta_i$.

Proof. By lemma 2.3, we may assume that $D : \mathcal{D}_1 \to \mathcal{D}(A)$ is a left $\mathcal{D}_0$-module derivation and a Lie derivation such that $D(\delta_i) = \beta_i$. For any Z in $\mathcal{D}_m$, we define

$$(*) \qquad \xi_m(Z) = mZ - \sum_i [Z, a_i]\delta_i$$

and note that $\xi_m(Z) \in \mathcal{D}_{m-1}$.

Assume that D is an additive function on $\mathcal{D}_{m-1}$ for $m > 1$. We want to extend D to an additive function on $\mathcal{D}_m$. For $Z \in \mathcal{D}_m$, define

$$mD(Z) = \sum_i D([Z, a_i])\delta_i + \sum [Z, a_i]\beta_i + D(\xi_m(Z)).$$

To show that D is well-defined on $\mathcal{D}_m$, let $Z \in \mathcal{D}_{m-1}$. Then $\xi_{m-1}(Z) = (m-1)Z - \sum_i [Z, a_i]\delta_i = \xi_m(Z) - Z$ by $(*)$. The definition of D on $\mathcal{D}_{m-1}$ shows that

$$(m-1)D(Z) = \sum D([Z, a_i])\delta_i + \sum [Z, a_i]\beta_i + D(\xi_{m-1}(Z)).$$

But $D(\xi_{m-1}(Z)) = D(\xi_m(Z)) - D(Z)$. Thus we get $mD(Z) = \sum_i D([Z,a_i])\delta_i + \sum[Z,a_i]\beta_i + D(\xi_m(Z))$. It follows that D is well-defined on $\mathcal{D}_m$. Since D is additive in $\mathcal{D}_{m-1}$ and $\xi_m(X+Y) = \xi_m(X) + \xi_m(Y)$, it follows that D is additive in $\mathcal{D}_m$, and so D is an additive function on $\mathcal{D}(A)$.

Next we show that $D(fY) = D(f)Y + fD(Y)$ for $Y \in \mathcal{D}_m$, $f \in A$. We may assume that $m > 0$ since D is a derivation of A into $\mathcal{D}(A)$. We proceed by induction on m. For the case of $m = 1$, it is trivial by the hypothesis. Now we assume that the result is true for all $1 < k < m$. For any $Y \in \mathcal{D}_m$, define $\xi_m(Y) = mY - \sum[Y,a_i]\delta_i$. Then $\xi_m(fY) = mfY - \sum[fY,a_i]\delta_i = f\xi_m(Y)$ and

$$\begin{aligned} mD(fY) &= \sum\Big(D([fY,a_i])\delta_i + [fY,a_i]\beta_i\Big) + D(f\xi_m(Y)) \\ &= \sum\Big(D(f[Y,a_i])\delta_i + f[Y,a_i]\beta_i\Big) + D(f\xi_m(Y)). \end{aligned}$$

By induction, we have

$$\begin{aligned} mD(fY) &= \sum\Big(D(f)[Y,a_i]\delta_i + fD([Y,a_i])\delta_i + f[Y,a_i]\beta_i\Big) \\ &\quad + D(f)\xi_m(Y) + fD(\xi_m(Y)) \\ &= D(f)\cdot mY + f\cdot mD(Y). \end{aligned}$$

Thus $D(fY) = D(f)Y + fD(Y)$ for $Y \in \mathcal{D}_m$, $f \in A$.

Next we claim that $D(XY) = D(X)Y + XD(Y)$ for $X \in \mathcal{D}_m$, $Y \in \mathcal{D}_s$ and $m > 0, s \geq 0$. We proceed by induction on $m+s$. If $m = 1$ and $s = 0$ then for $X \in \mathcal{D}_1, h \in A$, $D(Xh) = D(hX) - D([h,X])$ and so $D(Xh) = D(X)h + XD(h)$ from the hypotheses. Now assume that the result is true for all $1 < k < m+s$ and $m > 0, s \geq 0$. For $X \in \mathcal{D}_m$, $Y \in \mathcal{D}_s$, we define

$$\xi_m(X) = mX - \sum[X,a_i]\delta_i \text{ and } \xi_s(Y) = sY - \sum[Y,a_i]\delta_i.$$

Notice that $\xi_m(X) \in \mathcal{D}_{m-1}$ and $\xi_s(Y) \in \mathcal{D}_{s-1}$. Then

$$\xi_m(X)Y = mXY - \sum[X,a_i]\delta_iY \text{ and } X\xi_s(Y) = sXY - \sum X[Y,a_i]\delta_i.$$

On the other hand, $XY \in \mathcal{D}_{m+s}$, so

$$\begin{aligned}
\xi_{m+s}(XY) &= (m+s)XY - \sum [XY, a_i]\delta_i \\
&= (m+s)XY - \sum\Big([X,a_i]Y\delta_i + X[Y,a_i]\delta_i\Big) \\
&= (m+s)XY - \sum\Big([X,a_i]\delta_i Y + X[Y,a_i]\delta_i\Big) - \sum [X,a_i][Y,\delta_i],
\end{aligned}$$

for $\xi_{m+s}(XY) \in \mathcal{D}_{m+s-1}$. If we solve for $\xi_{m+s}(XY)$, we have $\xi_{m+s}(XY) = \xi_m(X)Y + X\xi_s(Y) - \sum [X,a_i][Y,\delta_i]$. From the definition of D,

$$\begin{aligned}
(m+s)D(XY) &= \sum\Big(D([XY,a_i])\delta_i + [XY,a_i]\beta_i\Big) + D(\xi_{m+s}(XY)) \\
&= \sum\Big(D(X[Y,a_i]) + D([X,a_i]Y)\Big)\delta_i + \sum (X[Y,a_i] + [X,a_i]Y)\beta_i \\
&\quad + D(\xi_m(X)Y) + D(X\xi_s(Y)) - \sum D([X,a_i][Y,\delta_i]).
\end{aligned}$$

By induction on $m+s$, we get

$$\begin{aligned}
(m+s)D(XY) &= \sum\Big(D(X)[Y,a_i] + XD([Y,a_i]) + D([X,a_i])Y + [X,a_i]D(Y)\Big)\delta_i \\
&\quad + \sum (X[Y,a_i] + [X,a_i]Y)\beta_i + D(\xi_m(X))Y + \xi_m(X)D(Y) \\
&\quad + D(X)\xi_s(Y) + XD(\xi_s(Y)) - D([X,a_i])[Y,\delta_i] - [X,a_i]D([Y,\delta_i]) \\
&= D(X)\cdot sY + X\cdot sD(Y) + \sum D([X,a_i])\delta_i Y \\
&\quad + \sum [X,a_i]\Big(D(Y)\delta_i + Y\beta_i - D([Y,\delta_i])\Big) \\
&\quad + D(\xi_m(X))Y + \xi_m(X)D(Y).
\end{aligned}$$

If $D([Y,\delta_i]) = [D(Y),\delta_i] + [Y,\beta_i]$, then

$$\begin{aligned}
(m+s)D(XY) &= s(D(X)\cdot Y + XD(Y)) + \sum D([X,a_i])\delta_i Y \\
&\quad + \sum [X,a_i](\delta_i D(Y) + \beta_i Y) + D(\xi_m(X))Y + \xi_m(X)D(Y) \\
&= s(D(X)\cdot Y + XD(Y)) + m(D(X)\cdot Y + XD(Y)) \\
&= (m+s)(D(X)\cdot Y + XD(Y)).
\end{aligned}$$

Now we must show that $D([Y,\delta_i]) = [D(Y),\delta_i] + [Y,\beta_i]$ for $Y \in \mathcal{D}_s$. If $s = 0$, we are done by the hypotheses. For the case of $s > 0$, let $Y = Y'\delta_j$ for $Y' \in \mathcal{D}_{s-1}$. Then

$$D([Y,\delta_i]) = D\Big([Y'\delta_j, \delta_i]\Big)$$

$$
\begin{aligned}
&= D\Big(Y'[\delta_j,\delta_i] + [Y',\delta_i]\delta_j\Big) \\
&= D(Y')[\delta_j,\delta_i] + Y'D([\delta_j,\delta_i]) + D([Y',\delta_i])\delta_j + [Y',\delta_i]\beta_j \\
&= D(Y')[\delta_j,\delta_i] + Y'D([\delta_j,\delta_i]) + \Big([D(Y'),\delta_i] + [Y',\beta_i]\Big)\delta_j + [Y',\delta_i]\beta_j,
\end{aligned}
$$

by induction on $m+s$. Since $D([\delta_j,\delta_i]) = [\beta_j,\delta_i] + [\delta_j,\beta_i]$,

$$
\begin{aligned}
D([Y,\delta_i]) &= [D(Y')\delta_j,\delta_i] + [Y'\delta_j,\beta_i] + [Y'\beta_j,\delta_i] \\
&= [D(Y'\delta_j),\delta_i] + [Y'\delta_j,\beta_i] \\
&= [D(Y),\delta_i] + [Y,\beta_i].
\end{aligned}
$$

Thus $D([Y,\delta_i]) = [D(Y),\delta_i] + [Y,\beta_i]$ for $Y\in\mathcal{D}_s$. It follows that $D(XY) = D(X)Y + XD(Y)$ for $X\in\mathcal{D}_m$, $Y\in\mathcal{D}_s$. □

If $\mathrm{Der}_K A$ has a Weyl basis $\{(q_i,p_i)\}_{i=1}^n$, then conditions on generators and relations for $\mathcal{D}_1$ in the above theorem are $[D(p_i),q_j]+[p_i,\beta_j] = 0$, and $[\beta_i,q_j]+[q_i,\beta_j] = 0$ if $D(1) = 0$. We prove the existence of an associative derivation on $B_n(K)$.

Theorem 2.2. *If $\alpha_i,\beta_i\in K(p_1,\ldots,p_n)[q_1,\ldots,q_n]$, $i = 1,\ldots,n$ are such that $[\alpha_i,p_j]+[p_i,\alpha_j] = 0$, $[\alpha_i,q_j] + [p_i,\beta_j] = 0$, and $[\beta_i,q_j] + [q_i,\beta_j] = 0$ for $1\le i,j\le n$, then there exists an associative algebra derivation D on $K(p_1,\ldots,p_n)[q_1,\ldots,q_n]$ with $D(p_i) = \alpha_i$ and $D(q_i) = \beta_i$ for $1\le i\le n$.*

Proof. Let $D(p_i) = \alpha_i, D(q_i) = \beta_i$ and $D(1) = 0$ for $i = 1,\ldots,n$. First we extend D to a derivation of $K[p_1,\ldots,p_n]$ into $K(p_1,\ldots,p_n)[q_1,\ldots,q_n]$. From the antisymmetry of p_i and q_i, we may write

$$mX = \sum[q_i,X]p_i + \zeta$$

where X and ζ are elements of $K[p_1,\ldots,p_n]$ with total degrees $\le m$ and $\le m-1$, respectively. We use the same method as in the proof of theorem 2.1 after switching p_i with q_i and vice versa. But in this case there is no term with q_i's and we need condition $[\alpha_i,p_j]+[p_i,\alpha_j] = 0$. Thus D can be extended to a derivation of $K[p_1,\ldots,p_n]$ into $K(p_1,\ldots,p_n)[q_1,\ldots,q_n]$. If we let $S = K[p_1,\ldots,p_n]\setminus\{0\}$ then S is an Ore set in $K[p_1,\ldots,p_n]$ and $K(p_1,\ldots,p_n) = K[p_1,\ldots,p_n]\ S^{-1}$. D can be extended to a derivation of $K(p_1,\ldots,p_n)$ into $K(p_1,\ldots,p_n)[q_1,\ldots,q_n]$ ([7]). From theorem 2.1 for $A = K(p_1,\ldots,p_n)$, the result follows. □

Theorem 2.3. *Let A be a commutative regular K-affine domain. If $\mathrm{Der}_K A$ has a Weyl basis $\{(q_i,p_i)\}_{i=1}^n$ and $\beta_i \in \mathcal{D}(A)$ with $[\beta_i,q_j]+[q_i,\beta_j]=0$ for $1 \le i,j \le n$, then there exists an associative derivation D on $\mathcal{D}(A)$ such that D vanishes on A and $D(q_i)=\beta_i$.*

Proof. Let $D(A)=0$, $D(q_i)=\beta_i$ and $D(1)=0$ for all i. Then D is a derivation of A into $\mathcal{D}(A)$. Theorem 2.1 for the case of $\alpha_i = 0$ proves the result. □

Chapter 3. Lie Derivations I

In this chapter, we show that a Lie derivation on any localization of nth Weyl algebra over a field is associative and vanishes on identity. Furthermore, we prove that if $U(\mathcal{L})$ is the universal enveloping algebra of $\mathcal{L}$ over the scalar field K where $\mathcal{L}$ is a finite-dimensional nilpotent Lie algebra, then a Lie derivation T on $U(\mathcal{L})$ which is linear over the center of $U(\mathcal{L})$ is associative.

Let K be a field of characteristic zero. $A_n(K)$ will denote the nth Weyl algebra $K[p_1, \ldots, p_n, q_1, \ldots, q_n]$ over K with $2n$ generators $p_1, \ldots, p_n, q_1, \ldots, q_n$ and relations as in example 1.1. Recall that a *Lie derivation* T of a Lie algebra $\mathcal{L}$ is a K-linear map from $\mathcal{L}$ to $\mathcal{L}$ such that $T([x,y]) = [T(x), y] + [x, T(y)]$ for $x, y \in \mathcal{L}$. We assume throughout that a Lie derivation on a Lie algebra is linear over its center.

Recall that $B_n(K) = K(p_1, \ldots, p_n)[q_1, \ldots, q_n]$. The following lemma will be used to show that there is no Lie derivation T from $A_n(K)$ to $B_n(K)$ such that $T(1) = 1$, $T(p_i) = \frac{T(1)}{2} p_i$ and $T(q_i) = \frac{T(1)}{2} q_i$ for each i.

Lemma 3.1. *Let $T : A_n(K) \to B_n(K)$ be a Lie derivation such that $T(p_i) = \frac{T(1)}{2} p_i$ and $T(q_i) = \frac{T(1)}{2} q_i$ for $T(1)$ in the center of $B_n(K)$. Then $T(p_i^k) = (1 - \frac{k}{2})T(1)p_i^k$ and $T(q_i^k) = (1 - \frac{k}{2})T(1)q_i^k$.*

Proof. Let T be a Lie derivation from $A_n(K)$ to $B_n(K)$ satisfying $T(p_i) = \frac{T(1)}{2} p_i$ and $T(q_i) = \frac{T(1)}{2} q_i$ for $T(1) \in \mathbf{Z}(B_n(K))$, the center of $B_n(K)$. We first argue that $T(p_i q_i)$ is in $\mathbf{Z}(B_n(K))$ for $1 \le i \le n$. Since

$$\begin{aligned} \frac{T(1)}{2} q_i &= T(q_i) = T([q_i, p_i q_i]) \\ &= [\frac{T(1)}{2} q_i, p_i q_i] + [q_i, T(p_i q_i)], \end{aligned}$$

it follows that $[q_i, T(p_i q_i)] = \frac{T(1)}{2} q_i - \frac{T(1)}{2}[q_i, p_i q_i] = 0$. Similarly, we get $[T(p_i q_i), p_i] = 0$

from $\frac{T(1)}{2}p_i = T([p_iq_i, p_i])$. But for all $s \neq i$,

$$\begin{aligned} 0 &= T([p_iq_i, p_s]) = [T(p_iq_i), p_s] + [p_iq_i, \frac{T(1)}{2}p_s] \\ &= [T(p_iq_i), p_s] \text{ and} \\ 0 &= T([q_s, p_iq_i]) = [q_s, T(p_iq_i)]. \end{aligned}$$

Thus $[T(p_iq_i), p_k] = 0$ and $[q_k, T(p_iq_i)] = 0$ for all $1 \leq k \leq n$. Hence $T(p_iq_i)$ commutes with all p_i's and q_i's. Thus $T(p_iq_i)$ must be in $\mathbf{Z}(B_n(K))$ for all i.

Second, we argue by induction on k that $T(p_i^k) = (1 - \frac{k}{2})T(1)p_i^k$ and $T(q_i^k) = (1 - \frac{k}{2})T(1)q_i^k$. Note that by induction,

$$\begin{aligned} T([q_i, p_i^k]) &= kT(p_i^{k-1}) \\ &= k \cdot (1 - \frac{(k-1)}{2})T(1)p_i^{k-1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} T([q_i, p_i^k]) &= [\frac{T(1)}{2}q_i, p_i^k] + [q_i, T(p_i^k)] \\ &= \frac{T(1)}{2} \cdot kp_i^{k-1} + [q_i, T(p_i^k)]. \end{aligned}$$

By comparing ones on the right hand sides of equations for $T([q_i, p_i^k])$, we obtain $[q_i, T(p_i^k)] = k(1 - \frac{k}{2})T(1)p_i^{k-1}$. But for all s,

$$\begin{aligned} 0 &= T([p_i^k, p_s]) = [T(p_i^k), p_s] + [p_i^k, \frac{T(1)}{2}p_s] \\ &= [T(p_i^k), p_s]. \end{aligned}$$

Therefore $T(p_i^k)$ commutes with all p_j's. Since $T(p_iq_i)$ is in $\mathbf{Z}(B_n(K))$,

$$\begin{aligned} kT(p_i^k) &= T([p_iq_i, p_i^k]) = [p_iq_i, T(p_i^k)] \\ &= p_i[q_i, T(p_i^k)] = k(1 - \frac{k}{2})T(1)p_i^k. \end{aligned}$$

Hence we have $T(p_i^k) = (1 - \frac{k}{2})T(1)p_i^k$. Similarly, we can show that $T(q_i^k) = (1 - \frac{k}{2})T(1)q_i^k$. □

Let $\mathcal{M}$ be a multiplicatively closed subset of $K[p_1, \dots, p_n]$. Denote by $A_n(K)_{\mathcal{M}}$ the localization of $A_n(K)$. The following lemma is very crucial to show that a Lie derivation T on $A_n(K)_{\mathcal{M}}$ has the property that $T(1) = 0$.

Lemma 3.2. *There is no Lie derivation $T : A_n(K) \rightarrow B_n(K)$ with $T(1) = 1$, $T(p_i) = \frac{1}{2}p_i$ and $T(q_i) = \frac{1}{2}q_i$ for each i.*

Proof. Suppose that there exists a Lie derivation $T : A_n(K) \rightarrow B_n(K)$ such that $T(1) = 1, T(p_i) = \frac{1}{2}p_i$ and $T(q_i) = \frac{1}{2}q_i$. From lemma 3.1, $T(p_i^k) = (1 - \frac{k}{2})T(1)p_i^k$ and $T(q_i^k) = (1-\frac{k}{2})T(1)q_i^k$ for all i. Now let $p = p_i$ and $q = q_i$. We calculate that $\Big[[q^3,p^5],[q^2,p^3]\Big] = -360p^3$. Hence $T\Big(\Big[[q^3,p^5],[q^2,p^3]\Big]\Big) = -360T(p^3) = 180p^3$. On the other hand,

$$
\begin{aligned}
T([q^3,p^5]) &= [-\frac{1}{2}q^3,p^5] + [q^3,-\frac{3}{2}p^5] = -2[q^3,p^5] \text{ and} \\
T([q^2,p^3]) &= [q^2,-\frac{1}{2}p^3] = -\frac{1}{2}[q^2,p^3].
\end{aligned}
$$

In fact, $T\Big(\Big[[q^3,p^5],[q^2,p^3]\Big]\Big) = -\frac{5}{2}\Big[[q^3,p^5],[q^2,p^3]\Big] = 900p^3$. But this is impossible, since $p^3 \neq 0$. Therefore there is no Lie derivation T on $A_n(K)$ with $T(1) = 1, T(p_i) = \frac{1}{2}p_i$ and $T(q_i) = \frac{1}{2}q_i$. □

Let T be a Lie derivation on $A_n(K)_{\mathcal{M}}$ with $T(1) = 0$. The next result guarantees the existence of an associative derivation which agrees with T at p_i's and q_i's. Let $M = A_n(K)_{\mathcal{M}}$.

Lemma 3.3. *Let T be a Lie derivation on $A_n(K)_{\mathcal{M}}$ with $T(1) = 0$. Then there exists an associative derivation S with $(T-S)(p_i) = 0$ and $(T-S)(q_i) = 0$ for $1 \leq i \leq n$.*

Proof. Let $\alpha_i = T(p_i)$ and $\beta_i = T(q_i)$. Then $0 = T([q_i,q_j]) = [\beta_i,q_j] + [q_i,\beta_j]$. Also, $0 = T([p_i,p_j]) = [\alpha_i,p_j] + [p_i,\alpha_j]$ and $0 = T([p_i,q_j]) = [\alpha_i,q_j] + [p_i,\beta_j]$ for all $1 \leq i,j \leq n$. By theorem 2.2, there exists an associative derivation S on $B_n(K)$ with $S(p_i) = \alpha_i$ and $S(q_i) = \beta_i$. Let $f \in \mathcal{M}$. From

$$
\begin{aligned}
0 &= S([f \cdot f^{-1}]) \\
&= S(f) \cdot f^{-1} + f \cdot S(f^{-1}),
\end{aligned}
$$

we conclude that $S(f^{-1}) = -f^{-1}S(f)f^{-1}$ is an element of M, since S agrees with T at each f. This can be easily shown by induction and the fact that $(S-T)(p_iq_i)$ is in the center of $B_n(K)$. Lemma 1.4 tells us that S is an associative derivation on M with $S(p_i) = \alpha_i$ and $S(q_i) = \beta_i$, i.e., $(T-S)(p_i) = 0 = (T-S)(q_i)$. □

Lemma 3.4. *Let A be a subalgebra of $K(p_1, \ldots, p_n)$ closed under differentiation and let T be a Lie derivation on $A[q_1, \ldots, q_n]$. If $j \geq 1$ is such that $T(Aq^l) = 0$ for all $l < j$ and $T(q_k^2) = 0$ for all $j_k \geq 1$ where $q^j = q_1^{j_1} \cdots q_n^{j_n}$ and $j = j_1 + \cdots + j_n$, then $T(\frac{\partial}{\partial p_k} hq^j) = 0$, i.e., $T([q_k, hq^j]) = 0$ for all $h \in A$ and $j_k \geq 1$.*

Proof. Let $q^j = q_1^{j_1} \cdots q_n^{j_n}$ with $j = j_1 + \cdots + j_n$. Let k be an element with $j_k \geq 1$ and $h \in A$. If h has no p_k terms, then $\frac{\partial}{\partial p_k} hq^j = 0$, so $T(\frac{\partial}{\partial p_k} hq^j) = 0$. Assume that h has a p_k term. Let $q' = q_1^{j_1} \cdots q_k^{j_k - 1} \cdots q_n^{j_n}$. Then

$$\begin{aligned} T([q_k^2, hq']) &= [T(q_k^2), hq'] + [q_k^2, T(hq')] \\ &= 0, \end{aligned}$$

by assumption. But

$$\begin{aligned} [q_k^2, hq'] &= q_k[q_k, hq'] + [q_k, hq']q_k \\ &= q_k[q_k, h]q' + [q_k, h]q'q_k \\ &= q_k(\frac{\partial}{\partial p_k}h)q' + (\frac{\partial}{\partial p_k}h)g^j \\ &= 2(\frac{\partial}{\partial p_k}h)g^j + (\frac{\partial^2}{\partial p_k^2}h)q'. \end{aligned}$$

By applying T, $T([q_k^2, hq']) = 2T((\frac{\partial}{\partial p_k}h)g^j)$. The result follows. □

The following proposition shows that if a Lie derivation T on $\mathcal{D}(A_{\mathcal{M}})$ vanishes on $\mathcal{D}(A)$ then T vanishes on $A_{\mathcal{M}}$.

Proposition 3.1. *Let A be a commutative regular affine domain and $\mathcal{M}$ be a multiplicatively closed subset of A. If $T : \mathcal{D}(A_{\mathcal{M}}) \to \mathcal{D}(A_{\mathcal{M}})$ is a Lie derivation such that T restricted to $\mathcal{D}(A)$ is zero then $T(A_{\mathcal{M}}) = 0$.*

Proof. Let $F \in A_{\mathcal{M}}$. We first argue that $T(F) \in A_{\mathcal{M}}$. If $a \in A$, then $[a, F] = 0$, and so

$$\begin{aligned} 0 &= T([a,F]) = [T(a), F] + [a, T(F)] \\ &= [a, T(F)], \end{aligned}$$

since $T(a) = 0$. Thus $[a, T(F)] = 0$ for all $a \in A$, i.e., $T(F)$ commutes with A. It follows from lemma 1.2 that $T(F) \in A_{\mathcal{M}}$.

Next we prove $T(a\delta(F)) = 0$ for all $a \in A$, $\delta \in \mathrm{Der}_K A$. Consider arbitrary elements $a \in A$ and $\delta \in \mathrm{Der}_K A$. Then

$$\begin{aligned} T([a\delta, F]) &= T(a[\delta, F] + [a, F]\delta) \\ &= T(a\delta(F)), \end{aligned}$$

since $[a, F] = 0$. On the other hand,

$$\begin{aligned} T([a\delta, F]) &= [T(a\delta), F] + [a\delta, T(F)] \\ &= a\delta(T(F)) \end{aligned}$$

by the hypothesis that $T(a\delta) = 0$. Thus $T(a\delta(F)) = a\delta(T(F))$ for $F \in A_{\mathcal{M}}$. Since $\delta(F)$ is in $\mathcal{D}(A_{\mathcal{M}})$, we can find a particular nonzero element a in A such that $a\delta(F) \in \mathcal{D}(A)$. For this $a \neq 0$, $T(a\delta(F)) = 0$, and so $a\delta(T(F)) = 0$. Since $A_{\mathcal{M}}$ is a domain and $a \neq 0$, $\delta(T(F)) = 0$. Thus $T(F)$ in $A_{\mathcal{M}}$ commutes with all of derivations δ of A. Since $\mathcal{D}(A_{\mathcal{M}})$ is generated by $A_{\mathcal{M}}$ and $\mathrm{Der}_K A$, we see that $T(F)$ is in the center of $\mathcal{D}(A_{\mathcal{M}})$. From $T(a\delta(F)) = a\delta(T(F))$, we obtain $T(a\delta(F)) = 0$ for all $a \in A, \delta \in \mathrm{Der}_K A$.

Finally we show that $T(F) = 0$ for all $F \in A_{\mathcal{M}}$. Let ρ be the rank of $\mathrm{Der}_K(A)$. Since A is regular, we can choose $\delta_1, \ldots, \delta_n \in \mathrm{Der}_K A$ and $a_1, \ldots, a_n \in A$ such that $\sum \delta_i(a_i) = 1$. This is the rank formula when we replace a_i with $\frac{1}{\rho} a_i$. Now, we calculate $T(\sum_i [\delta_i, a_i F])$.

$$\begin{aligned} \sum_i [\delta_i, a_i F] &= \sum_i ([\delta_i, a_i]F + a_i[\delta_i, F]) \\ &= (\sum \delta_i(a_i))F + \sum a_i \delta_i(F) \\ &= F + \sum a_i \delta_i(F) \end{aligned}$$

from $\sum \delta_i(a_i) = 1$. By applying T, we obtain $T(\sum_i [\delta_i, a_i F]) = T(F)$, since $T(\sum a_i \delta_i(F))=0$. But

$$\begin{aligned} T(\sum_i [\delta_i, a_i F]) &= \sum_i [T(\delta_i), a_i F] + \sum_i [\delta_i, T(a_i F)] \\ &= 0, \end{aligned}$$

since $T(\delta_i) = 0$ and $T(a_i F)$ is in the center of $\mathcal{D}(A_{\mathcal{M}})$. Therefore $T(F) = 0$ for all $F \in A_{\mathcal{M}}$. □

The following proposition says that if T is a Lie derivation as in proposition 3.1 then T vanishes on $\mathcal{D}(A_{\mathcal{M}})$.

Proposition 3.2. *Let A be a commutative regular affine domain. If $T : \mathcal{D}(A_{\mathcal{M}}) \to \mathcal{D}(A_{\mathcal{M}})$ is a Lie derivation such that $T(\mathcal{D}(A)) = 0$, then $T(\mathcal{D}(A_{\mathcal{M}})) = 0$.*

Proof. Let $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \cdots \subseteq \mathcal{D}_s \cdots$ be the standard filtration of $\mathcal{D}(A_{\mathcal{M}})$. We argue by induction on s that $T(\mathcal{D}_s) = 0$. If $s = 0$, the result is true from the previous proposition. Assume that $T(\mathcal{D}_k) = 0$ for all $0 < k < s$. We want to claim that $T(F) \in A_{\mathcal{M}}$ for $F \in \mathcal{D}_s$. Let $F \in \mathcal{D}_s$ and $a \in A$. Then $[a, F] \in \mathcal{D}_{s-1}$, so by induction, we have $T([a, F]) = 0$. As in the proof of proposition 3.1, $T(F)$ commutes with A. It follows from lemma 1.2 that $T(F) \in A_{\mathcal{M}}$.

Next we want to show that $T([a\delta, F]) = 0$ for all $a \in A$ and $\delta \in \mathrm{Der}_K A$. Let $a \in A$ and $\delta \in \mathrm{Der}_K A$ be arbitrary. Then we have

$$\begin{aligned} T([a\delta, F]) &= [a\delta, T(F)] = a\delta(T(F)) \text{ and} \\ T([a\delta, F]) &= T(a[\delta, F] + [a, F]\delta). \end{aligned}$$

From $\mathcal{D}(A_{\mathcal{M}}) = \mathcal{D}(A)_{\mathcal{M}}$, note that if $\gamma \in \mathcal{D}(A_{\mathcal{M}})$ then we can write $\gamma = a_0^{-1}\gamma_0$ or $a_0\gamma = \gamma_0$ for some nonzero $\gamma_0 \in \mathcal{D}(A)$ and $a_0 \in A$. Thus for $F \in \mathcal{D}_s$ we may choose a nonzero element b in A such that $bF \in \mathcal{D}(A)$. But

$$\begin{aligned} b^2[\delta, F] &= b([\delta, bF] - [\delta, b]F) \\ &= b[\delta, bF] - [\delta, b]bF \in \mathcal{D}(A), \end{aligned}$$

since $bF\in\mathcal{D}(A)$. Hence there exists a nonzero element b^2 in A with $b^2[\delta,F]\in\mathcal{D}(A)$. Next we want to show that there exists a nonzero a' in A such that $[a',F]\delta\in\mathcal{D}(A)+\mathcal{D}_{s-1}$ for $\delta\in\text{Der}_K A$. Note that $bF\in\mathcal{D}(A)$ and $[b,F]\in\mathcal{D}_{s-1}$. But

$$\begin{aligned}[b^2,F] &= b[b,F]+[b,F]b \\ &= [b,bF]+b[b,F]-[b,[b,F]] \\ &= 2[b,bF]-[b,[b,F]].\end{aligned}$$

Since $[b,bF]\in\mathcal{D}(A)$ and $[b,[b,F]]\in\mathcal{D}_{s-2}$, $[b^2,F]$ must be in $\mathcal{D}(A)+\mathcal{D}_{s-2}$. Thus $[b^2,F]\delta\in\mathcal{D}(A)+\mathcal{D}_{s-1}$ for $\delta\in\text{Der}_K A$. For special nonzero b^2 in A, $b^2[\delta,F]\in\mathcal{D}(A)$ and $[b^2,F]\delta\in\mathcal{D}(A)+\mathcal{D}_{s-1}$. By the hypothesis and induction, $T(a[\delta,F])=0=T([a,F]\delta)$, and so $T([a\delta,F])=T(a[\delta,F]+[a,F]\delta)=0$ for $a{=}b^2$. From $T([a\delta,F])=a\delta(T(F))$, we get $a\delta(T(F))=0$ for special nonzero $a{=}b^2$ in A. It follows that $\delta(T(F))=0$ for all $\delta\in\text{Der}_K A$. We conclude that $T(F)$ is in the center of $\mathcal{D}(A_{\mathcal{M}})$, and so $T([c\delta,F])=0$ for all $c\in A$ and $\delta\in\text{Der}_K A$.

Let $\{(\delta_i,a_i)\}$ be an elite basis for $\text{Der}_K A$ and let ρ=rank $(\text{Der}_K(A))$. Then

$$T(\sum[\delta_i,a_iF])=\sum[T(\delta_i),a_iF]+\sum[\delta_i,T(a_iF)]=0,$$

since $T(\delta_i)=0$ and $T(a_iF)$ is in the center of $\mathcal{D}(A_{\mathcal{M}})$. On the other hand,

$$\begin{aligned}T(\sum[\delta_i,a_iF]) &= T(\sum[\delta_i,a_i]F+\sum a_i[\delta_i,F]) \\ &= T(\rho\cdot F+\sum[a_i\delta_i,F]-\sum[a_i,F]\delta_i) \\ &= T(\rho\cdot F-\sum[a_i,F]\delta_i),\end{aligned}$$

since $\sum_i[\delta_i,a_i]=\rho$ and $T([a_i\delta_i,F])=0$ for all i. Note that $\{(\delta_i,a_i)\}$ is also an elite basis for $\text{Der}_K(A_{\mathcal{M}})$, since $\text{Der}(A_{\mathcal{M}})=A_{\mathcal{M}}\cdot\text{Der}(A)$. From the projective basis formula,

$$\begin{aligned}T(-\sum[a_i,F]\delta_i) &= T(sF+\text{ element in }\mathcal{D}_{s-1}) \\ &= T(sF)\end{aligned}$$

by induction. Thus we have $T(\sum[\delta_i,a_iF])=(s+\rho)T(F)$. Since $T(\sum[\delta_i,a_iF])=0$ from the previous argument, $(s+\rho)T(F)=0$, so $T(F)=0$ for $F\in\mathcal{D}_s$. Thus we proved that $T(\mathcal{D}(A_{\mathcal{M}}))=0$. □

This proposition is very useful in proving one of our main theorems, which states that a Lie derivation T on $\mathcal{D}(A_{\mathcal{M}})$ is associative. To show this, we construct an associative derivation S which agrees with T on $\mathcal{D}(A)$, i.e., $S - T$ is a Lie derivation which vanishes in $\mathcal{D}(A)$. Then we apply proposition 3.2.

Theorem 3.1. *If T is a Lie derivation on $A_n(K)_{\mathcal{M}}$, then T is an associative algebra derivation with $T(1) = 0$.*

Proof. First we want to prove $T(1) = 0$. Let $M = A_n(K)_{\mathcal{M}}$. If $a \in M$, then $0 = T([a,1]) = [a, T(1)]$. In particular, $T(1)$ commutes with all p_i's, hence by lemma 1.2, $T(1) \in K[p_1, \dots, p_n]_{\mathcal{M}}$. But $T(1)$ also commutes with all q_i's, so $T(1)$ must be in the center of M. If $T(1) \neq 0$, set $S = \frac{1}{T(1)}T$. Then S is a Lie derivation with $S(1) = 1$. Let $\alpha_i = \frac{1}{2}p_i - S(p_i)$ and $\beta_i = \frac{1}{2}q_i - S(q_i)$. Then

$$[\alpha_i, p_j] + [p_i, \alpha_j] = [-S(p_i), p_j] + [p_i, -S(p_j)] = -S([p_i, p_j]) = 0.$$

Also,

$$\begin{aligned} [\alpha_i, q_j] + [p_i, \beta_j] &= [\frac{1}{2}p_i - S(p_i), q_j] + [p_i, \frac{1}{2}q_j - S(q_j)] \\ &= \begin{cases} -1 - S([p_i, q_j]) = 0 & \text{for } i = j \\ -S([p_i, q_j]) = 0 & \text{for } i \neq j. \end{cases} \end{aligned}$$

Similarly $[\beta_i, q_j] + [q_i, \beta_j] = 0$ for $1 \leq i, j \leq n$. By theorem 2.2, there exists an associative derivation D on $B_n(K)$ where $D(p_i) = \alpha_i$ and $D(q_i) = \beta_i$ for $1 \leq i \leq n$. Define $S^* : A_n(K) \rightarrow B_n(K)$ by $S^*(u) = S(u) + D(u)$. Then S^* is a Lie derivation such that $S^*(1) = 1, S^*(p_i) = \frac{1}{2}p_i$ and $S^*(q_i) = \frac{1}{2}q_i$. From lemma 3.2, there is no such S^*. Therefore $T(1) = 0$. By lemma 3.3, we replace T by $T - S'$ where S' is an associative derivation on M with $(T - S')(p_i) = 0$ and $(T - S')(q_i) = 0$ for all i. We may assume without any loss of generality that $T(p_i) = 0 = T(q_i)$. From lemma 3.1 and $T(1) = 0$, we conclude that $T(p_i^k) = 0 = T(q_i^k)$ for all $k > 0$ and $1 \leq i \leq n$.

Let $a = p_1^{t_1} \cdots p_n^{t_n} \in K[p_1, \dots, p_n]$. Next we argue by induction on $t = t_1 + \cdots + t_n$ that $T(a) = 0$. It is trivial for $t = 1$. Assume that the result is true for all $1 < k < t$.

Since $T(p_i^2) = 0 = T(q_i^2), T(p_iq_i) = 0$ from $0 = T([q_i^2, p_i^2]) = 4T(p_iq_i) + T(2)$. But for each i, $[T(a), p_i] = T([a, p_i]) = 0$ and $[q_i, T(a)] = T([q_i, a]) = 0$, by induction. Thus $T(a)$ is in the center of M. But

$$\begin{aligned} 0 &= T([\sum_i p_iq_i, a]) = \sum T(p_i[q_i, a]) \\ &= \sum_i t_iT(a) = tT(a), \end{aligned}$$

so $T(a) = 0$. Thus we conclude that $T(a) = 0$ for $a \in K[p_1, \dots, p_n]$.

Third, we want to prove by induction on $j = j_1 + \cdots + j_n$ that $T(Aq^j) = 0$ for $A \in K[p_1, \dots, p_n], q^j = q_1^{j_1} \cdots q_n^{j_n}$. If $j = 0$, the result follows from the previous argument. Suppose $T(Aq^l) = 0$ for all $l < j$. From lemma 3.4, $T(\frac{\partial}{\partial p_k} hq^j) = 0$ for $j_k > 0$, all $h \in A$. Every polynomial in $K[p_1, \dots, p_n]$ has a polynomial antiderivative with respect to p_k. Thus $T(hq^j) = 0$ for all $h \in K[p_1, \dots, p_n]$. It follows from proposition 3.2 that $T(a) = 0$ for every $a \in M$. Therefore T is an associative derivation on $A_n(K)_{\mathcal{M}}$ with $T(1) = 0$. □

Let $\mathcal{L}$ be a finite-dimensional nilpotent Lie algebra and let $U(\mathcal{L})$ be the universal enveloping algebra of $\mathcal{L}$ over the scalar field k. The following theorem shows that a Lie derivation on $U(\mathcal{L})$ which is linear over the center of $U(\mathcal{L})$ is associative. We first prove the next lemma.

Lemma 3.5. *Let $U(\mathcal{L})$ be the universal enveloping algebra described as above. Assume $T : U(\mathcal{L}) \to U(\mathcal{L})$ is a Lie derivation which is linear over the center $\mathbf{Z} = \mathbf{Z}(U(\mathcal{L}))$ of $U(\mathcal{L})$. If $C = \mathbf{Z} \backslash \{0\}$ then T extends to a Lie derivation from $C^{-1}U(\mathcal{L})$ to itself that is linear over $K = C^{-1}\mathbf{Z}$.*

Proof. Since $U(\mathcal{L})$ is an integral domain from the Poincaré-Birkhoff-Witt theorem, we may invert elements of C. Let $T : U(\mathcal{L}) \to U(\mathcal{L})$ be a Lie derivation which is linear over $\mathbf{Z}$. Define $T : C^{-1}U(\mathcal{L}) \to C^{-1}U(\mathcal{L})$ by $T(a^{-1}\omega) = a^{-1}T(\omega)$ for $a \in C$ and $\omega \in U(\mathcal{L})$. This obviously extends the original function. Let $a_1, a_2 \in C$ and $\omega_1, \omega_2 \in U(\mathcal{L})$. If $a_1^{-1}\omega_1 = a_2^{-1}\omega_2$ then $a_2\omega_1 = a_1\omega_2$. Thus $T(a_2\omega_1) = T(a_1\omega_2)$ and so

$a_2T(\omega_1) = a_1T(\omega_2)$. Hence $a_1^{-1}T(\omega_1) = a_2^{-1}T(\omega_2)$ and it follows that T is well- defined on $C^{-1}U(\mathcal{L})$.

To show that T is a Lie derivation on $C^{-1}U(\mathcal{L})$, we first show that T is linear over $K = C^{-1}\mathbf{Z}$.

$$\begin{aligned} T(a_1^{-1}\omega_1 + a_2^{-1}\omega_2) &= T\Big(a_1^{-1}a_2^{-1}(a_2\omega_1 + a_1\omega_2)\Big) \\ &= a_1^{-1}a_2^{-1}T(a_2\omega_1 + a_1\omega_2) \\ &= a_1^{-1}a_2^{-1}\Big(T(a_2\omega_1) + T(a_1\omega_2)\Big) \\ &= T(a_1^{-1}\omega_1) + T(a_2^{-1}\omega_2). \end{aligned}$$

For $\gamma = z_0^{-1}z_1 \in K$,

$$\begin{aligned} T(\gamma a_1^{-1}\omega_1) &= T(z_0^{-1}a_1^{-1}z_1\omega_1) = z_0^{-1}a_1^{-1}T(z_1\omega_1) \\ &= z_0^{-1}a_1^{-1}z_1T(\omega_1) = \gamma T(a_1^{-1}\omega_1). \end{aligned}$$

Therefore T is linear over K. Since the commutator bracket is bilinear over the center and T is a Lie derivation on $U(\mathcal{L})$,

$$\begin{aligned} T([a_1^{-1}\omega_1, a_2^{-1}\omega_2]) &= T(a_1^{-1}a_2^{-1}[\omega_1, \omega_2]) \\ &= a_1^{-1}a_2^{-1}T([\omega_1, \omega_2]) \\ &= a_1^{-1}a_2^{-1}\Big([T(\omega_1), \omega_2] + [\omega_1, T(\omega_2)]\Big) \\ &= [a_1^{-1}T(\omega_1), a_2^{-1}\omega_2] + [a_1^{-1}\omega_1, a_2^{-1}T(\omega_2)] \\ &= [T(a_1^{-1}\omega_1), a_2^{-1}\omega_2] + [a_1^{-1}\omega_1, T(a_2^{-1}\omega_2)]. \end{aligned}$$

Thus we have shown that T is a Lie derivation on $C^{-1}U(\mathcal{L})$. □

Theorem 3.2. *Let $U(\mathcal{L})$ be the universal enveloping algebra of $\mathcal{L}$ over the scalar field k where $\mathcal{L}$ is a finite-dimensional nilpotent Lie algebra. If $T : U(\mathcal{L}) \to U(\mathcal{L})$ is a Lie derivation which is linear over the center $\mathbf{Z} = \mathbf{Z}(U(\mathcal{L}))$ of $U(\mathcal{L})$ then T is associative.*

Proof. Lemma 3.5 shows that T can be extended to a Lie derivation on $C^{-1}U(\mathcal{L})$ that is linear over $K = C^{-1}\mathbf{Z}$. A theorem of McConnell and Joseph ([7], 14.6.9) states that $C^{-1}U(\mathcal{L}) \cong A_n(K)$ for some n. Since a Lie derivation on $A_n(K)$ is associative by our result, theorem 3.1, $T : U(\mathcal{L}) \to U(\mathcal{L})$ is associative. $\square$

Chapter 4. Lie Derivations II

In chapter 3, we proved that a Lie derivation T on any localization of Weyl algebra is associative.

Let A be an integral extension of a commutative polynomial ring $B = K[a_1, \ldots, a_n]$ over K , a field of characteristic zero. In this section, we shall prove that a Lie derivation T on $\mathcal{D}(A_u)$ with $T(1) = 0$ for special element u in B, in fact, is associative. This theorem was motivated by the following example, the coordinate ring of circle.

Example 4.1. Let A be the coordinate ring of S^1; $A = \mathbf{R}[x,y]/(x^2+y^2-1)$ where $\mathbf{R}$ is the field of real numbers. $A \simeq \mathbf{R}[p, \sqrt{1-p^2}]$. We see that A is an integral extension of $\mathbf{R}[p]$, where $f(t) = t^2 + p^2 - 1$ is a minimal polynomial in $\mathbf{R}[p]$. Let $\delta = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Then $\mathrm{Der}_{\mathbf{R}}(A)$ is free A-module on δ.

We show that $\{(x\delta, \bar{y}), (-y\delta, \bar{x})\}$ is an elite basis for $\mathrm{Der}_{\mathbf{R}}(A)$ where $\bar{x}, \bar{y}$ are the respective images of x and y under the endomorphism $\phi : \mathbf{R}[x,y] \to A$. Clearly $\delta(x^2+y^2-1) = 0$. From $\delta(x) = -y, \delta(y) = x$, we get $x\delta(y) - y\delta(x) = 1$, which satisfies the rank formula. Take any derivation D on A, i.e., $D = a\delta$ for some $a \in A$. Then $(Dx)(-y\delta)+(Dy)x\delta= (-ay)(-y\delta)+(ax)x\delta = a\delta = D$. Thus $\{(x\delta, \bar{y}), (-y\delta, \bar{x})\}$ is an elite basis for $\mathrm{Der}_{\mathbf{R}}(A)$.

The ring of differential operators on A is isomorphic to $U(\mathcal{L})/(x^2+y^2-1)$ with $\mathcal{L}$ being the three-dimensional solvable real Lie algebra on generators x, y, u with relations $[x,y] = 0, [u,x] = -y$ and $[u,y] = x$. Here $U(\mathcal{L})$ denotes the enveloping algebra of $\mathcal{L}$. Note that if D is a Lie derivation for $\mathcal{D}(A)$ then $1 \otimes D$ is a Lie derivation for $\mathbf{C} \otimes_{\mathbf{R}} \mathcal{D}(A)$ where $\mathbf{C}$ is the field of complex numbers. Let $[q,p] = 1$. Since $\mathbf{C} \otimes_{\mathbf{R}} \mathcal{D}(A) \simeq \mathbf{C}[p, p^{-1}, q]$, a Lie derivation $1 \otimes D$ on $\mathbf{C} \otimes_{\mathbf{R}} \mathcal{D}(A)$ must be associative, from the theorem in chapter 3.

Let $\{\delta_1, \ldots, \delta_n\}$ be a basis for $\mathrm{Der}_K(B)$ with the property that $\delta_i(a_i) = 1, \delta_i(a_j) =$

0 if $i \neq j$.

McConnell-Robson shows that there exists a nonzero element $u \in B \subseteq A$ such that $\Omega(A_u)$, the module of Kähler differentials of A_u, is a free A_u-module with $da_1, \cdots, da_n$ and $\mathrm{Der}_K A_u$ is free of finite rank over A_u with basis $\delta_1, \ldots, \delta_n$. To see this, let L and Q be the fields of fractions of A and B, respectively. By duality, it suffices to show that for some $u \in B$, $\Omega(A_u)$ is free over A_u on $da_1, \cdots, da_n$.

Since QA is finite dimensional over Q, QA is Artinian. Therefore QA is a field and so is equal to L. Thus for each $a \in A$, we write $a^{-1} = u^{-1}c$ for some $u \in B, c \in A$ and so $a^{-1} \in A_u$.

Let N be the A-module generated by $da_1, \cdots, da_n$. Clearly $N \subseteq \Omega_K(A)$. From $\Omega_K(A_{\mathcal{M}}) = A_{\mathcal{M}} \otimes_A \Omega_K(A)$, $\Omega_K(L) = L \otimes_A \Omega_K(A)$. Since L is a finite field extension of Q, $\Omega_K(L)$ is a free L-module with basis $da_1, \cdots, da_n$. Thus $\Omega_K(L) = L \otimes_A N$ and so $L \otimes_A (\Omega_K(A)/N) = 0$. Since $\Omega_K(A)$ is finitely generated, there exists $a \in A$ such that $A_a \otimes_A (\Omega_K(A)/N) = 0$. We may choose $a \in B$, from the result above. It follows that $\Omega_K(A_a)$ is free over A_a on $da_1, \cdots, da_n$. Then duality shows that $\mathrm{Der}_K A_a$ is free on $\delta_1, \ldots, \delta_n$.

A basis $\{c_i\}$ for L over Q can be chosen from A. Then, for each K-algebra generator x of A, $x = b_x^{-1} \sum_i b_{x,i} c_i$ for some $b_x, b_{x,i} \in B$. By taking $b = \prod_x b_x$, we can arrange that every K-algebra generator x is a B_b-linear combination of that basis. Thus $\{c_i\}$ is a basis for A_b over B_b. Let $u = ab$. Thus we have shown that there exists $u \in B$ such that $\mathrm{Der}_K A_u$ is a free A_u-module with basis $\delta_1, \ldots, \delta_n$. Also the derivation ring of A_u is equal to $A_u[\delta_1, \ldots, \delta_n]$. Thus all δ_i of $\mathrm{Der}_K B$ extends to A_u.

We assume throughout that a Lie derivation on $\mathcal{D}(A_u)$ is linear over the center of $\mathcal{D}(A_u)$ which is a finite field extension of the base field K. We begin with some preliminary lemmas.

Lemma 4.1. *Let A be an integral extension of a commutative polynomial B. Suppose that T is a Lie derivation on $\mathcal{D}(A_u)$ where T is associative when restricted to*

A_u. *If $T(a_i) = 0$ for algebra generators a_i of B, then T vanishes on A_u.*

Proof. Let $a_1, \ldots, a_n$ be the algebra generators of B. Assume that $T(a_i) = 0$ for each i. First we show that T vanishes on B. Let $b = a_1^{i_1} \cdots a_n^{i_n}$. Induct on $i = i_1 + \cdots + i_n$. It is trivial for the case of $i = 1$. Assume that the result is true for all $j < i$. Choose the smallest s such that $i_s > 0$. Then $T(b) = T(a_s \cdot a_s^{i_s - 1} \cdots a_n^{i_n}) = a_s T(a_s^{i_s-1} \cdots a_n^{i_n}) = 0$ by induction.

Finally we claim that $T(F) = 0$ for all $F \in A_u$. Let ν be an element in A. If $f(t) = t^k + b_{k-1}t^{k-1} + \cdots + b_0$ is its minimal polynomial over B, then $f(\nu) = 0$. Clearly $T(ba) = (T(b))a + bT(a) = bT(a)$ for all $b \in B, a \in A$. By induction, it is easily shown that $T(\nu^m) = m\nu^{m-1}T(\nu)$ for any positive integers m. Then

$$\begin{aligned} 0 = T(f(\nu)) &= T(\nu^k) + b_{k-1}T(\nu^{k-1}) + \cdots + b_1T(\nu) + T(b_0) \\ &= (k\nu^{k-1} + (k-1)b_{k-1}\nu^{k-2} + \cdots + b_1)T(\nu) \\ &= k(\nu^{k-1} + (k - 1/k)b_{k-1}\nu^{k-2} + \cdots + (1/k)b_1)T(\nu). \end{aligned}$$

From the minimality of f, $T(\nu) = 0$. Thus T vanishes on A. If $F \in A_u$ then $b_0 F \in A$ for some nonzero b_0 in B. Hence $0 = T(b_0F) = b_0T(F) + (T(b_0))F = b_0T(F)$. It follows that $T(F) = 0$, since $b_0 \neq 0$. □

Let $\{(\delta_i, a_i)\}$ be a Weyl basis of $\mathrm{Der}_K(B)$ where $B = K[a_1, \ldots, a_n]$. Recall that all $\delta_1, \ldots, \delta_n$ in $\mathrm{Der}_K B$ can be extended to $\mathrm{Der}_K(A_u)$. We next show that $\{(\delta_i, a_i)\}_{i=1}^n$ is a Wely basis for $\mathrm{Der}_K(A_u)$. It follows from lemma 4.1 that if X is a derivation of A_u with $X(a_i) = 0$ for each i, then $X = 0$.

Now say $Y \in \mathrm{Der}_K(A_u)$. Set $X = Y - \sum Y(a_i)\tilde{\delta}_i$ where each $\tilde{\delta}_i$ is extended by δ_i from $\mathrm{Der}_K(A_u) = A_u \cdot \mathrm{Der}_K(B)$. Then $X \in \mathrm{Der}_K(A_u)$. For each k, $X(a_k) = Y(a_k) - Y(a_k) = 0$, since $\delta_k(a_k) = 1, \delta_i(a_k) = 0$ if $i \neq k$. By the previous paragraph, $X = 0$, so $Y = \sum Y(a_i)\delta_i$. Thus $\{(\delta_i, a_i)\}_{i=1}^n$ is a Weyl basis for $\mathrm{Der}_K(A_u)$.

From now on we will denote a Weyl basis of $\mathrm{Der}_K(A_u)$ by $\{(\delta_i, a_i)\}_{i=1}^n$.

Lemma 4.2. *Let T be a Lie derivation on $\mathcal{D}(A_u)$ with $T(1) = 0$ and $T(a_i) = 0$ for all i. Then*

$$(i) \quad T(A_u) \subseteq A_u$$
$$(ii) \quad T(F\delta_k) - (T(F))\delta_k \in A_u$$

for any $F \in A_u$ and $1 \leq k \leq n$.

Proof. (i) Let $F \in A_u$. Then $0 = T([F, a_i]) = [T(F), a_i]$ for each i. Hence $T(F) \in A_u$, according to lemma 1.3. Thus $T(A_u) \subseteq A_u$.
(ii) Let $F \in A_u$. Note that

$$[T(F\delta_k), a_i] = T([F\delta_k, a_i]) = \begin{cases} T(F) = [(T(F))\delta_k, a_k] & i = k \\ 0 = [(T(F))\delta_k, a_i] & i \neq k. \end{cases}$$

Thus we get $[T(F\delta_k) - (T(F))\delta_k, a_i] = 0$ for all i and , hence $T(F\delta_k) - (T(F))\delta_k \in A_u$ from lemma 1.3. □

The following lemma shows that there exists an associative derivation on $\mathcal{D}(A_u)$ which agrees with T on A_u.

Lemma 4.3. *Let A be an integral extension of $B = K[a_1, \ldots, a_n]$ and let u be the element in B as described above. If T is a Lie derivation on $\mathcal{D}(A_u)$ with $T(1) = 0$, then there exists an associative derivation D on $\mathcal{D}(A_u)$ such that $(T - D)$ vanishes on A_u. In particular, T has the Leibniz property when restricted to A_u.*

Proof. Let T be a Lie derivation on $\mathcal{D}(A_u)$ with $T(1) = 0$. Since $[a_i, a_j] = 0$, $0 = T([a_i, a_j]) = [T(a_i), a_j] + [a_i, T(a_j)]$. Thus $[T(a_i), a_j] = [T(a_j), a_i]$. By example 1.6, there is a t in $\mathcal{D}(A_u)$ such that $[t, a_i] = T(a_i)$ for all i. Let $D(x) = [t, x]$. Then $(T - D)(a_i) = 0$ for all i. Thus we may assume that $T(a_i) = 0$ for all i and $T(1) = 0$.

We show that $T(rc) = (T(r))c + rT(c)$ for $r, c \in A_u$. Let $r, c \in A_u$. If c is in the center of $\mathcal{D}(A_u)$, then $T(rc) = cT(r)$ and $T(c) = cT(1) = 0$. Assume that c is not in the center of $\mathcal{D}(A_u)$. Since $\mathcal{D}(A_u)$ is generated by A_u and $\delta_1, \ldots, \delta_n$, we can choose

k with $[\delta_k, c] \neq 0$. Let $\delta = \delta_k$ for this c. Then

$$\begin{aligned} T([r\delta, c]) &= [T(r\delta), c] + [r\delta, T(c)] \\ &= [(T(r))\delta, c] + r[\delta, T(c)] \\ &= (T(r))[\delta, c] + r[\delta, T(c)], \end{aligned}$$

since $T(c) \in A_u, T(r\delta) - (T(r))\delta \in A_u$ from lemma 4.2. But $T([r\delta, c]) = T(r[\delta, c])$ for all $r \in A_u$, hence we have $T(r[\delta, c]) = r[\delta, T(c)] + (T(r))[\delta, c]$ for all $r \in A_u$. In particular, if $r = 1$, then $[\delta, T(c)] = T([\delta, c])$. In fact, we have

$$(*) \qquad T(r\omega) = rT(\omega) + (T(r))\omega$$

where $r \in A_u$ and $\omega = [\delta, c]$. By substituting c^2 for c in $(*)$, it follows that $T(r \cdot 2c[\delta, c]) = rT(2c[\delta, c]) + T(r) \cdot 2c[\delta, c]$, equivalently,

$$(**) \qquad T(rc\omega) = rT(c\omega) + (T(r))c\omega.$$

If we let $r = c$ in $(*)$, $T(c\omega) = cT(\omega) + (T(c))\omega$. Substitute this equality in $(**)$ to get $T(rc\omega) = rcT(\omega) + r(T(c))\omega + (T(r))c\omega$. If we replace r with rc in $(*)$, we obtain $T(rc\omega) = rcT(\omega) + (T(rc))\omega$. By comparing this equation with the one in the previous line, we get

$$(T(rc))\omega = r(T(c))\omega + (T(r))c\omega.$$

Since $\omega \neq 0$, $T(rc) = rT(c) + (T(r))c$ for all $r \in A_u$. Thus we have shown that $T(rc) = rT(c) + (T(r))c$ for all $r, c \in A_u$. From lemma 4.1 it follows that T vanishes on A_u. □

The next lemma gives us the sufficient condition on a Lie derivation T to obtain an associative derivation S where S vanishes on A_u and S agrees with T on each δ_k.

Lemma 4.4. *Let T be a Lie derivation on $\mathcal{D}(A_u)$ with $T(A_u) = 0$ and $T(1) = 0$. Then there exists an associative derivation S with $(T - S)(A_u) = 0$ and $(T - S)(\delta_i) = 0$.*

Proof. For each i, $0 = T([\delta_k, a_i]) = [T(\delta_k), a_i]$. Thus $T(\delta_k) \in A_u$ for each k from lemma 1.3. Let $\beta_k = T(\delta_k)$ for each k where $\beta_k \in \mathcal{D}(A_u)$. Then clearly $0 = T([\delta_i, \delta_j]) = [\beta_i, \delta_j] + [\delta_i, \beta_j]$. By theorem 2.3, there exists an associative derivation S on $\mathcal{D}(A_u)$ with $S(A_u) = 0$ and $S(\delta_i) = \beta_i$, i.e., $(T - S)(A_u) = 0 = (T - S)(\delta_i)$ for each i. □

Lemma 4.5. *Assume that T is a Lie derivation on $\mathcal{D}(A_u)$. Let $\varepsilon, \delta \in \mathcal{D}(A_u)$ such that $\varepsilon \in Der_K(A_u)$and $[\varepsilon, \delta] = 0$. If $T(A_u\delta) = 0$ and $T(\varepsilon^2) = 0$ then $T([\varepsilon, A_u]\delta\varepsilon) = 0$.*

Proof. Let $F \in A_u$. Then $T([\varepsilon^2, F\delta]) = [T(\varepsilon^2), F\delta] + [\varepsilon^2, T(F\delta)] = 0$. On the other hand,

$$\begin{aligned}[\varepsilon^2, F\delta] &= \varepsilon[\varepsilon, F\delta] + [\varepsilon, F\delta]\varepsilon \\ &= \varepsilon[\varepsilon, F]\delta + [\varepsilon, F]\delta\varepsilon \\ &= 2[\varepsilon, F]\delta\varepsilon + [\varepsilon, [\varepsilon, F]]\delta.\end{aligned}$$

Thus $0 = T([\varepsilon^2, F\delta]) = 2T([\varepsilon, F]\delta\varepsilon)$, since $T([\varepsilon, [\varepsilon, F]]\delta) = 0$. □

We find a simple condition for a Lie derivation T on $\mathcal{D}(A_u)$ to vanish everywhere.

Lemma 4.6. *If T is a Lie derivation on $\mathcal{D}(A_u)$ such that T vanishes on A_u and $T(\delta_i) = 0 = T(1)$ for each i, then T vanishes on $\mathcal{D}(A_u)$.*

Proof. Let T be a Lie derivation on $\mathcal{D}(A_u)$ with $T(A_u) = 0, T(\delta_i) = 0$ and $T(1) = 0$. For each t, we proceed by induction on k to show $T(\delta_t^k) = 0$ for each k. From

$$T([\delta_t^k, a_j]) = \begin{cases} kT(\delta_t^{k-1}) & \text{if } j = t \\ 0 & \text{if } j \neq t, \end{cases}$$

we get $[T(\delta_t^k), a_j] = 0$ for all j. On the other hand, $[T(\delta_t^k), \delta_j] = T([\delta_t^k, \delta_j]) = 0$ for all j. Thus $T(\delta_t^k)$ is in the center of $\mathcal{D}(A_u)$. From $[T(a_t\delta_t), a_j] = T([a_t\delta_t, a_j]) = 0$ and $[T(a_t\delta_t), \delta_j] = T([a_t\delta_t, \delta_j]) = 0$, it follows that $T(a_t\delta_t)$ is in the center of $\mathcal{D}(A_u)$. Since $T(\delta_t^k)$ and $T(a_t\delta_t)$ are in the center of $\mathcal{D}(A_u)$,

$$\begin{aligned}0 &= T([\delta_t^k, a_t\delta_t]) = T([\delta_t^k, a_t]\delta_t) \\ &= kT(\delta_t^k),\end{aligned}$$

and so $T(\delta_t^k) = 0$ for $1 \leq t \leq n$.

Next we argue by induction on i that $T(F\delta^i) = 0$ for all $F \in A_u$ and $\delta^i = \delta_1^{i_1} \cdots \delta_n^{i_n}$ where $i = i_1 + \cdots + i_n$. It is trivial for the case of $i = 0$. Assume that $T(F\delta^k) = 0$ for all $k < i$. Let s be an element with $i_s > 0$. Since $T(\delta_s^2) = 0$, from lemma 4.5, $T([\delta_s, F]\delta^i) = 0$ for $i_s \geq 1$. By induction,

$$[T(F\delta^i), a_j] = T([F\delta^i, a_j]) = 0$$

for each j and

$$[\delta_j, T(F\delta^i)] = T([\delta_j, F\delta^i]) = T([\delta_j, F]\delta^i) = 0$$

for each j with $i_j \geq 1$. Hence

$$\begin{aligned} T([a_s\delta_s, F\delta^i]) &= [T(a_s\delta_s), F\delta^i] + [a_s\delta_s, T(F\delta^i)] \\ &= 0 \text{ and} \\ T([a_s\delta_s, F\delta^i]) &= T(a_s[\delta_s, F\delta^i] + [a_s, F\delta^i]\delta_s) \\ &= T(a_s[\delta_s, F]\delta^i) - i_s T(F\delta^i). \end{aligned}$$

Thus we have $i_s T(F\delta^i) = T(a_s[\delta_s, F]\delta^i)$ where i_s is the power of δ_s. But from the previous lemma, by replacing F with $a_s F$, we obtain

$$\begin{aligned} 0 &= T([\delta_s, a_s F]\delta^i) \\ &= T(F\delta^i) + T(a_s[\delta_s, F]\delta^i) \end{aligned}$$

for all $F \in A_u$. Therefore $i_s T(F\delta^i) = -T(F\delta^i)$, and hence $T(F\delta^i) = 0$ for all $F \in A_u$.□

Now we shall prove one of our main theorems.

Theorem 4.1. *Let A be an integral extension of a polynomial algebra B and let u be the element of B such that $\Omega(A_u)$, the module of Kähler differentials of A_u, is a free A_u-module with $da_1, \cdots, da_n$ and $\mathrm{Der}_K(A_u)$ is free of finite rank over A_u with basis $\delta_1 \ldots, \delta_n$ where $\{(\delta_i, a_i)\}_{i=1}^n$ is a Weyl basis of $\mathrm{Der}_K(B)$. If T is a Lie derivation on $\mathcal{D}(A_u)$ with $T(1) = 0$ then T is an associative derivation on $\mathcal{D}(A_u)$.*

Proof. According to lemma 4.3, we may subtract an associative derivation from T and assume that $T(A_u) = 0$. Lemma 4.4 shows that there exists an associative derivation S with $(T - S)(A_u) = 0$ and $(T - S)(\delta_i) = 0$. But lemma 4.6 assures that $(T - S)$ vanishes on $\mathcal{D}(A_u)$. Therefore T is an associative derivation on $\mathcal{D}(A_u)$. □

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