A Model of the Formation of Multilayer Networks*

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Abstract

We study the formation of multilayer networks where payoffs are determined by the degrees of players in each network. We begin by imposing either concavity or convexity in degree on the payoff function of the players. We then explore distinct network relationships that result from inter- and intra-network spillovers captured by the properties of supermodularity/submodularity and strategic complementarity respectively. We show the existence of equilibria and characterize them. Additionally, we establish both necessary and sufficient conditions for an equilibrium to occur. We also highlight the connection, in equilibrium, between inter-network externalities and the identity of linked players in one network given the identity of linked players in the other network. Furthermore, we analyze efficient multilayer networks. Finally, we extend our models to contexts with more than two layers, and scenarios where agents receive a bonus for being connected to the same individuals in both networks.

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1 Introduction

Individuals, firms, and nations are typically connected with each other bilaterally in multiple networks. Yet we know very little about the formation of multiple networks, especially about how one network may impact the formation of the other networks. For instance, an individual's social network can easily affect their professional network since firms routinely examine a job applicant's social media profile before hiring. Belonging to a peer-to-peer marketing network like *Amway* might negatively impact other people's decisions to extend a friendship link while belonging to a baking club might positively impact this decision. Similarly, membership in certain golf clubs or being part of an alumni network can have impacts on a person's business network.

The purpose of this paper is to develop a model of the simultaneous formation of multiple networks within the same universe of agents. This allows us to focus on who will form links with whom, based on the nature of spillovers across networks. We call such networks *multilayer networks*. The extensive literature on single layer network formation can be seen as a special case similar to the partial equilibrium analysis of markets: it is assumed that the other networks do not play a significant role. In fact, to extend this market analogy further, the formation of bilateral links across several networks is akin to engaging in general equilibrium analysis where multiple markets affect each other.

To illustrate why the study of multilayer network formation is important relative to single network formation consider the following scenario. Suppose that the network under scrutiny is a dominant group network, *i.e.*, a network where agents with links are all linked together while the remaining agents have no links. Moreover, suppose that the researcher knows that externalities obtained by agents from links in which they are not involved are very low. When considering only one network (only externalities between the links formed by players in the network), based on the existing literature (Goyal and Joshi, 2006), the researcher may infer that the player's payoff function is not concave in the number of links formed by her. However, as we will show, the same architecture can also arise when the payoff function of a given player is concave in the player's own links, and there are externalities between the links formed by this player in distinct networks. In other words, ignoring one network may lead the researcher to make incorrect inferences about equilibrium networks. Hence, it is important to be aware of situations where individuals are members of multiple networks, especially in the presence of externalities across these networks.

To address such issues, our paper takes the first step towards incorporating cross network effects by studying the simultaneous formation of two different networks using the standard toolkit of microeconomic theory. Our objective is to provide a modeling framework for such networks and also identify the insights and challenges that might arise in going from single layer to multilayer networks.[1] Apart from showing that equilibrium multilayer networks always exist, the questions addressed in the paper can be summarized as follows: *How do the incentives and patterns of connections differ depending on the nature of the externalities across different network layers? How do externalities within each layer alter these outcomes? How does inequality in links across agents in one network generate or undermine inequality in the other?*

We follow Jackson (2014) who suggests that an ideal model of network formation should be link based by using agents' degrees as the components of the payoff function. Moreover, an individual's degree turns out to be critical in many situations like the labor market, for bargaining or risk sharing, and for firms it is well known that they matter for collaborative research. In order to disentangle the impact of different types of externalities, we introduce two different models of multilayer network formation. In the first model, called the *Multilayer Degree* model (or Degree model), we examine situations where every player's payoff function has two arguments: the number of links or degrees he has in networks G and H respectively. In other words, a player's payoff depends on the number of his own links in the two networks. We then study network formation using very standard properties. So, we first consider convexity and concavity in the two arguments of the payoff function. We also examine the impact of externalities across networks by imposing supermodularity or submodularity on the arguments of the payoff function, *i.e.*, we assume that links in one network affect the marginal profitability of forming links in the other network. The strategic aspect of network formation derives from the fact that link formation requires mutual consent.

The second model is called the *Multilayer Externality* model (or Externality model) where we augment the Multilayer Degree model by adding two components to the payoff function of a given player: the total number of links formed by all the *other players* in networks G and H respectively. We allow for strategic complementarity between the links formed by player i in a given network and the links formed by all the other players in this network. Hence, in this model, we have both inter- and intra-network spillovers. At first glance, Externality model might seem like a robustness check for the

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[1] Note that hypergraphs are not ideally suited to tackle these issues since the typical hyperedge connects multiple vertices making it hard to study the incentives for players to link with each other. Apart from making it hard to interpret results, it poses additional challenges. For instance, we typically assume that a link requires mutual consent of the pair of involved agents. The possibility of having multiple agents involved in the same link will require different notions of consent.

Degree model. However, when we examine the formal structure of these two models it is easy to see that it is important to analyze both. At the least while the Degree model only has inter-network externalities, the Externality model has intra-network externalities as well and can help us understand the trade-off between these two types of externalities. Both these models differ from single layer network models in terms of the properties they incorporate. For instance, it is easy to see that concavity and convexity play an important role in determining the number of links that will be formed in both single and multiple layer networks. However, along with supermodularity and submodularity (properties that capture inter-network spillovers and do not occur in single layer networks) these properties also help us determine which agents will connect to whom. In fact, it is hard to explain why different interaction patterns arise in single layer networks between ex ante identical agents. However, if we realize that these agents might be involved in link formation in multiple layers, these inter- and intra-network externalities can be used to explain the networks that form. Finally, as already mentioned, the strategic complementarity property (which is commonly assumed in single layer networks) helps us understand the trade-off between these two types of externalities in the multilayer model.

We can find several examples of these types of situations in the economics literature. Firms often establish bilateral R&D collaboration links concurrently to lower production costs, as in Goyal and Joshi (2003). These collaborations may involve sharing productive assets for higher capacity utilization by forming links in one network and collaborating in R&D for process innovation by forming links in another network. Consider agents in a community that seek information/resources to achieve their goals. For example, information can be obtained by participating in sports or political discussions with other community members. As a result, an agent can form two types of links to access information. At the macro level, these could be links between firms in the financial sector and real sector to obtain resources as shown by the propagation of shocks in Tzavellas (2023).

In terms of results, for both models, we first establish the existence of equilibrium networks. Given that players have discrete strategies consisting of a $(n-1)$ dimensional vector for each network, showing that multilayer equilibrium networks always exist in pure strategies involves techniques that are different from the ones we use in the study of single networks. In single layer network models, in equilibrium every player needs to play the best response to other players. One way to think about this in a multilayer network is that players need to be playing best responses to all the others in both layers, but these best responses need to take inter-network spillovers into account. This implies that the existence of equilibrium in the Externality model requires a different approach from the Degree model.

Next, we examine the properties of equilibrium networks. Under convexity we find that as in single layer networks, equilibrium multilayer networks consist of dominant group networks. However, once we allow for inter-network spillovers, in the form of supermodularity or submodularity we are able to determine whether players form links with the same players or different players in both networks. Although multilayer equilibrium networks are harder to characterize under concavity (also true in single layer networks), we identify what drives link formation under both supermodularity and submodularity, allowing us to narrow down the set of possible equilibria. Specifically, regular networks where all nodes have the same number of links are always a possibility under a specific condition. As for irregular networks, we provide conditions such that under supermodularity players who have not formed the same number of links in G and H can be divided into two groups: those that are linked together in G and those that are linked together in H. In case of submodularity, we provide additional conditions on the magnitude of the different properties (concavity, submodularity, and strategic complementarity) of the payoff functions which allow us to obtain the same type of characterization. It is worth emphasizing that we provide necessary and sufficient conditions for equilibrium networks, except for the Externality model under concavity, which can be seen as one of the key contributions of our paper. In general, providing necessary conditions for equilibrium networks is difficult and this becomes an even harder problem in the context of multilayer networks where there are more degrees of freedom. Throughout the paper, we also discuss similarities and differences with single layer networks. For the sake of illustration, we identify two such instances here. Under convexity, we highlight situations where the mechanisms driving the proofs of equilibrium networks are similar for single layer and multiple layer networks. In case of concavity, at equilibrium each network may contain unlinked players with fewer links than other players in the same network. This is possible because of inter-network externalities that may reduce the marginal payoff associated with an additional link. Obviously, this cannot happen in the absence of inter-network spillovers, in particular in single layer models.

The type of equilibrium networks we obtain can be found in several different economic environments. For instance, convexity implies high interconnectedness among a small group of players. Ahuja (2000) reports such patterns in R&D collaborations as do Goyal and Joshi (2003) and Billand et al. (2016). In multilayer networks such things are quite likely to occur in other areas as well. For instance, social media platforms like LinkedIn and Facebook may easily exhibit such groups. Some people have interacted with the same individuals in their social and professional networks, while others may interact with different sets on each. And there might also be individuals who interact only on one of these networks. Concavity can lead to relatively equal degree distributions among players. Such outcomes can occur when there is a need for risk sharing as in Bramoulle and Kranton (2007). Other instances for the need for regular networks can occur when blockages in the network are possible due to either congestion effects or links not being fully reliable (Roy and Sarangi, 2009).

Next, we provide results about the efficiency of multilayer networks. To the best of our knowledge, ours is the first paper to do so. Specifically, under convexity, we show that intra-network and inter-network externalities give rise to efficient multilayer networks where the neighborhoods of agents are nested in each network. Under concavity and in the absence of intra-network externalities, efficient multilayer networks contain regular networks. Additionally, efficient multilayer networks coincide with equilibrium networks.

Finally, we explore two extensions of the Degree model. In the first, we examine what happens to multilayer equilibrium networks when we allow for more than two networks. In the second, we incorporate the identity of players into the Degree model by assuming that common neighborhoods in the two networks, provide additional payoffs.

The literature on the formation of multiple networks is relatively thin. A few recent papers that examine this phenomenon are Joshi, Mahmud, and Sarangi (2020), Joshi, Mahmud and Tzavellas (2020), and Joshi, Mahmud, Sarangi, and Tzavellas (2021). The first paper examines how one network is formed as a function of the other in a two-stage setting. In this paper, agents form network G, given a specific network H in stage 1 and engage in an effort game on both networks in stage 2. In the second paper, this analysis is extended to the case of co-evolution of multiple networks. There is a seed network that is non-empty while other networks are empty. The network formation happens *sequentially* across networks. In the third paper, the framework of multigraph formation is applied in the context of an overlapping generations model to analyze intergenerational mobility. Individuals in each generation inherit their parental network, which subsequently influences the network formation when they reach adulthood. Similar to our model, these papers show that the incentive to form links in each network depends both on intra- and inter-layer externalities generated by the effort game on the networks. One common feature in all these papers is that they use the linear-quadratic payoff function introduced by Ballester et al. (2006) and therefore are able to use Bonacich centrality and obtain precise results about network architectures. Our model differs from these papers in important ways. First, we do not rely on a linear quadratic payoff function; rather we use a general payoff function. Second, we do not model costs as is often done in the network formation literature as a linear and additive function. In our models, both costs and benefits are subsumed in one function and can be quite general. Third, we study the simultaneous formation of multiple layers relying on standard properties used in microeconomic theory. Walsh (2021) also examines network formation, but differs from these papers as it adapts Jackson and Wolinsky's notion of pairwise stability to multilayer networks in the context of the symmetric connections models (Jackson and Wolinsky, 1996).

An early attempt to model the simultaneous formation of links in two networks can be found in Billand et al. (2016). They develop a model of competitive intelligence gathering among oligopoly firms where directed links denote which firms are spied on and therefore require no consent. These links affect a firm's demand within each market, but the cross-market spillovers occur through the production costs and not through links between firms. Billand et al. (2014) examine collaborative R&D between firms that produce two different products. Again, the cost of production is what links these two activities.

A related set of papers allows players to be engaged in multiple activities but assumes the network structure is fixed. Chen, Zenou, and Zhou (2018) assume a fixed network where individuals exert effort in two types of activities that are interdependent. Cheng, Huang, and Xing (2021) allow individuals to form multiple relationships with other individuals and each relationship is captured by a repeated prisoner's dilemma. While their basic model uses a fixed network, they also consider what happens if an individual is allowed to form only one additional link and ask whether this will be a multiplex link or not.[2] Walsh (2019) studies two different activities (the formation of different local public goods) in two separate networks that are given. However, he allows for the possibility of synergies in effort across the two networks. In a recent paper, Tzavellas (2023) studies the capacity of an arbitrary multilayer network to transmit agent specific shocks across the economy. Similar to our inter- and intra-network spillover mechanism, he shows that idiosyncratic shocks transmit in the economy according to

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[2] Jackson and Nei (2015) extend these models of multiple activities because they do not assume that the network is fixed. They study the formation of a single network which defines both trade relationships and the possibility of wars between countries.

an inter- and an intra-layer cascade margin. Interestingly, the paper identifies volatility levels that are larger than economies consisting of a single layer network.

The rest of the paper is organized as follows. In Section 2 we develop the model setup. In Section 3 we provide results for equilibrium multilayer networks under convexity, and in Section 4 we study these networks under concavity. In Section 5 we provide a brief overview of two extensions to our model. In Section 6 we conclude. All proofs are provided in Section 7 (Appendix).

2 Model Setup

We denote by $[\![a, b]\!]$ the set $\{a, \ldots b\}$, with $a, b \in \mathbb{N}$ and $a < b$. Let the set of nodes or players be denoted by $N = [\![1, n]\!]$, with $n \geq 4$, and n even.[3] We use E to denote the set of *unordered* pairs of N, *i.e.*, $E = \{(i, j) \in N \times N, i \neq j\}$. Throughout the paper, an unordered pair $(i, j) \in E$ is called a *(pairwise) link* and is denoted by ij. A *network* is a pair (N, E).

Each player is a member of two networks, $G = (N, E_G)$ and $H = (N, E_H)$. If players i and j are linked in G, then $ij \in E_G$. The number of links in which player i is involved in G, *i.e.*, the *degree* of i, is denoted by g^i. These definitions apply to network H as well. The *multilayer (ML) network*, $M = (G, H)$, is a formal description of players and the links between them in networks G and H.

Network definitions. A network is *complete* if each player is linked with every other player, and is *empty* if each player is linked with no other player. In a *dominant group network*, there exists a group of players that are all linked with each other, while the remaining players have no links. Thus, complete and empty networks are polar cases of dominant group networks. Network G is *regular* if every player has the same degree in G. An induced subnetwork (or subnetwork) $G[N']$ of G is another network, formed on the subset of players $N' \subseteq N$ and all of the links from G connecting pairs of players in that subset. We denote by $\mathcal{N}_G^c = \{N' \subseteq N : G[N'] \text{ is complete}\}$ the set of subsets (or groups) of players that induce a complete subnetwork in G. The same properties define the corresponding networks and groups of players for network H.

Strategies and payoff functions of players. Every player announces her intended

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[3] When n is odd the problem is less easily tractable, but the results are qualitatively similar.

links: $s_G^{i,j} = 1$ implies that player i intends to form a link with player $j \neq i$ in G, and $s_G^{i,j} = 0$ otherwise. We assume that consent is necessary (and sufficient) for a link to form, that is $ij \in E_G$ if and only if $s_G^{i,j} = s_G^{j,i} = 1$. The same applies to links in network H. Players only use pure strategies. With a slight abuse of notation, a strategy of player i is denoted by $s^i = (s_G^i, s_H^i)$, where $s_G^i = (s_G^{i,j})_{j \in N \setminus \{i\}}$. Let s_G^{-i} and s_H^{-i} denote the strategies of all players except player i in networks G and H respectively.
A strategy profile $(s_G, s_H) = ((s_G^1, \ldots, s_G^n), (s_H^1, \ldots, s_H^n))$ therefore induces/supports a multilayer (ML) network $M(s_G, s_H) = (G[s_G], H[s_H])$ – when there is no confusion, we write $M = (G, H) = M(s_G, s_H)$. Sometimes we also write $(s_G, s_H) = (s_G^i, s_H^i, s_G^{-i}, s_H^{-i})$. Let $U^i = U^i(s_G, s_H)$ be the payoff function of player i. Finally, let s_{G+ij} be the strategy profile identical to s_G except that $s_{G+ij}^{i,j} \times s_{G+ij}^{j,i} = 1$ while $s_G^{i,j} \times s_G^{j,i} = 0$. In other words, players i and j are linked in $G[s_{G+ij}]$ but not in $G[s_G]$. The profile s_{H+ij} is similarly defined for network H.

Equilibrium strategies. Let $B^i(s_G^{-i}, s_H^{-i}) = \{(s_G^i,\ s_H^i) \in [\![0,1]\!]^{n-1} \times [\![0,1]\!]^{n-1} : U^i(s_G^i, s_H^i, s_G^{-i}, s_H^{-i}) \geq U^i(\hat{s}_G^i, \hat{s}_H^i, s_G^{-i}, s_H^{-i})$, for all $(\hat{s}_G^i, \hat{s}_H^i) \in [\![0,1]\!]^{n-1} \times [\![0,1]\!]^{n-1}\}$ denote the set of best responses of player i when she faces (s_G^{-i}, s_H^{-i}). If for some profile of strategies, (s_G, s_H), we have $(s_G^i, s_H^i) \in B^i(s_G^{-i}, s_H^{-i})$, for all players $i \in N$, then the resulting profile (s_G, s_H) constitutes a *Nash equilibrium.*

Definition 1 *A ML network $M = (G, H)$ is a multilayer pairwise equilibrium (MLPE) network if the two following conditions hold:*

1. *There exists a Nash equilibrium strategy profile (s_G, s_H) that supports M.*
2. *For $ij \notin E_G$, $U^i(s_{G+ij}, s_H) - U^i(s_G, s_H) \geq 0 \Rightarrow U^j(s_{G+ij}, s_H) - U^j(s_G, s_H) < 0$, for $ij \notin E_H$, $U^i(s_G, s_{H+ij}) - U^i(s_G, s_H) \geq 0 \Rightarrow U^j(s_G, s_{H+ij}) - U^j(s_G, s_H) < 0$.*

Observe that an MLPE network is a refinement of Nash equilibrium: it is a Nash equilibrium network where there does not exist a pair of players with an incentive to form a link in G or in H. It takes into account mutual consent in link formation in both networks G and H.

ML Efficient Networks. An *efficient* ML network is one that maximizes $\sum_{i \in N} U^i(s_G, s_H)$. To narrow down the number of potential efficient networks, we assume that the second-difference conditions on ψ are strict.

2.1 The Multilayer Degree Model

In this model, the payoff function of each player depends only on the number of links she has formed in each network. However, due to the mutual consent requirement for link formation, equilibrium networks do involve strategic considerations. This simple model provides a benchmark scenario to study the formation of multilayer networks. Let $\psi : [\![0, n-1]\!] \times [\![0, n-1]\!] \to \mathbb{R}$ be a symmetric function, *i.e.*, $\psi(x, y) = \psi(y, x)$. For realized networks G and H, we denote by U^i_ψ the payoff function of each player i:

$$U^i_\psi(s_G, s_H) = \psi(g^i, h^i). \tag{1}$$

The symmetry of the function ψ eliminates the identity of the two networks and allows us to study network formation without giving *a priori* importance to one network over the other.

For all admissible (g^i, h^i), we denote by $\psi_G(g^i, h^i) = \psi(g^i, h^i) - \psi(g^i - 1, h^i)$ the marginal payoff of i from an additional link in G. Moreover, $\psi_{GG}(g^i, h^i) = \psi_G(g^i, h^i) - \psi_G(g^i - 1, h^i)$ measures the impact of i's existing links in G on the marginal payoff of i from establishing additional links in G. These definitions apply to network H as well.

Definition 2 *The function ψ is convex in its first and second argument if for all admissible (g^i, h^i), we have $\psi_{GG}(g^i, h^i) \geq 0$ and $\psi_{HH}(g^i, h^i) \geq 0$ respectively. Similarly, concavity of ψ is obtained by replacing '$\geq$' with '$\leq$' in the above inequalities.*

Therefore, when ψ is convex (respectively concave), the marginal payoff of player i from forming additional links in a given network is increasing (respectively decreasing) in the number of links formed by i in this network. We have strict convexity (or concavity) when the inequality is strict. Additionally, for technical reasons, we sometimes extend the concavity of ψ to $\psi(-1, \cdot)$ and $\psi(\cdot, -1)$. When we use this property, we specify it.

The next property on ψ characterizes the impact of links formed by player i in a given network on the marginal payoff of player i from forming additional links in the other network. For all admissible (g^i, h^i), let $\psi_{GH}(g^i, h^i) = \psi_G(g^i, h^i) - \psi_G(g^i, h^i - 1)$. The corresponding object for ψ_{HG} can be defined similarly.

Definition 3 *The payoff function ψ is supermodular if for all admissible (g^i, h^i), we have $\psi_{GH}(g^i, h^i) \geq 0$ and $\psi_{HG}(g^i, h^i) \geq 0$. Submodularity of ψ is obtained by replacing '$\geq$' by '$\leq$' in the above inequalities.*

In other words, when ψ is supermodular (respectively submodular), the marginal payoff of player i from forming additional links in a given network is increasing (respectively decreasing) in the number of links formed by i in the other network.

We now present two applications of our setting: the first where ψ is convex in its two arguments (Application 1.a) and the second where ψ is concave in its two arguments (Application 2).

Application 1 Links formation between monopolies. Consider n monopoly firms with the inverse demand function of firm i given by $p_i(q_i) = \alpha - q_i$. Firms can collaborate in order to decrease their marginal production costs. Suppose these collaborations are of two types, like the sharing of productive assets (plants for instance) for higher capacity utilization, and collaborations in R&D in order to obtain process innovation. The first type of collaborations result in network G and the second type in network H. Let the marginal production costs of a firm i be given by $c^i(g^i, h^i) = \kappa - f\left(g^i, h^i\right)$, with $f_1', f_2' > 0$, and $\kappa > f(n-1, n-1)$. The cost of forming links is captured by an increasing function $\sigma(g^i, h^i)$ in its two arguments. The reduced payoff function is then given by:

$$\psi(g^i, h^i) = \left(\frac{\alpha - \kappa + f(g^i, h^i)}{2}\right)^2 - s(g^i, h^i). \tag{2}$$

Note that ψ is convex when σ is a linear function. Here, we assume that $f(g^i, h^i) = g^i + h^i$, $\sigma(g^i, h^i) = \frac{1}{8}(1 + (g^i)^2 + (h^i)^2 + 5g^i h^i)$, and $\alpha = \kappa + 1$. Consequently, ψ is convex in its two arguments and submodular.

Application 2. Links formation between information seekers. Consider n players who live in a community, and seek information to satisfy some goals. Each player owns some piece of information that other players can access by interacting with her through different activities. For instance, information can be accessed while participating in sporting activities with other members of the community, or while participating in political discussions with other members of the community. Therefore, a player can form two types of links allowing her to access to information. As a result, the benefits of player i are given by the following Cobb-Douglas type function: $A((g^i+1)(h^i+1))^\alpha$ with $A, \alpha > 0$. The payoffs of player i is given by $\psi(g^i, h^i) = A((g^i+1)(h^i+1))^\alpha - b(g^i + h^i)$, where $b > 0$ is the cost associated with each link. It can be checked that ψ satisfies concavity in its two arguments and supermodularity.

2.2 The Multilayer Externality model

In this section, we study an augmented version of the Multilayer Degree model where, in addition to her degrees in networks G and H, player i also benefits from the total number of links formed by the other players in both networks. This model allows us to go beyond the Degree model in three ways. First, by allowing for both inter- and intra-network externalities, we can now study the trade-off between these two different types of externalities. Second, this model incorporates the links of all the other players within a network. So the externalities are generated by links a player cannot form and must take as given. This has different implications for strategic behavior relative to the Degree model. Finally, in this model, since different players can have different numbers of links within a network, the payoff function no longer satisfies symmetry.

Let G^{-i} and H^{-i} denote the number of links in G and H respectively in which player i is not involved, and let $\theta : [\![0, n-1]\!] \times [\![0, (n-1)(n-2)/2]\!] \to \mathbb{R}$ be a function. Let x be the number of links formed by player i in the considered network, and X be the number of links formed between the other players in this network. For all admissible (x, X), we denote the marginal payoff of player i from adding a link in one network by $\theta_1(x, X) = \theta(x, X) - \theta(x - 1, X)$. Moreover, let $\theta_{11}(x, X) = \theta_1(x, X) - \theta_1(x - 1, X)$. Convexity and concavity of θ in its first argument are defined in the same way as for ψ. In the Multilayer Externality model, we assume that θ satisfies strategic complementarity, *i.e.*, intra-network externalities are positive.

Definition 4 *The function θ satisfies strategic complementarity when for all admissible (x, X), $\theta_1(x, X)$ is increasing in X.*

We define two additional payoff functions. The first one, U^i_θ, is inspired by the *Playing the Field model* of Goyal and Joshi (GJ, 2006). We have for every $i \in N$,

$$U^i_\theta(s_G) = \theta(g^i, G^{-i}), \text{ and } U^i_\theta(s_H) = \theta(h^i, H^{-i}).$$

Note here that each of these utility functions does not allow externalities across networks. The payoff function for the Multilayer Externality model, U^i, is a convex combination of U^i_ψ, defined above, and U^i_θ, where U^i_θ is associated both with network G and network H. Let $t \in (0, 1)$, we have

$$\begin{aligned} U^i(s_G, s_H) &= \mathcal{U}^i(g^i, h^i, G^{-i}, H^{-i}) \\ &= t\psi(g^i, h^i) + (1 - t)[\theta(g^i, G^{-i}) + \theta(h^i, H^{-i})]. \end{aligned} \tag{3}$$

This class of payoff functions allows us to incorporate into the model the role of the links between all the other players in a tractable manner.

We illustrate the Multilayer Externality model through two applications. The first one is a modification of Application 1, where ψ is convex in its two arguments and θ is convex in its first argument. In Application 3, ψ is concave in its two arguments and θ is concave in its first argument.

Application 1'. Consider Application 1.a, replace function σ by $\sigma(g^i, h^i) = \frac{1}{16}(2 + (g^i)^2 + (h^i)^2 + 10g^ih^i)$, and subtract the two following functions $\sigma_1(g^i, G^{-i}) = \frac{1}{16}((g^i)^2 - \frac{1}{8}g^iG^{-i})$, and $\sigma_2(h^i, H^{-i}) = \frac{1}{16}((h^i)^2 - \frac{1}{8}h^iH^{-i})$. Both latter functions are associated with θ and reflect the fact that the cost of forming a link is influenced by the intra-network externalities arising from the total number of links formed by other firms. Note that the difference between the profit of firm i in Application 1 and 1' is $\frac{1}{128}(g^iG^{-i} + h^iH^{-i})$. Clearly, ψ is convex in its two arguments, submodular, and θ is convex in its first argument and satisfies strategic complementarity.

Application 3. Production of two goods. We consider a situation where n firms potentially collaborate with each other and produce two different goods. Each firm i chooses effort levels, x_1^i and x_2^i, to produce two goods 1 and 2 respectively. The quantity of good $\ell \in \{1, 2\}$ produced by firm i is given by $q_\ell^i = 2\zeta_\ell^i(x_\ell^i)^{\frac{1}{2}}$, where ζ_ℓ^i is a productivity parameter that depends on the collaborative R&D links formed by firms for increasing the efficiency of effort. We assume that firm i's number of collaborations in network G, g^i, and network H, h^i, allow for knowledge production and increase firm i's absorptive capacity. Moreover, this absorptive capacity allows the firm to benefit from knowledge externalities resulting from collaborations between other firms. Let $\zeta_1^i = \zeta(g^i, |G^{-i}|) = \big((g^i + \kappa)(|G^{-i}| + K)\big)^{1/2}$ and $\zeta_2^i = \zeta(h^i, |H^{-i}|) = \big((h^i + \kappa)(|H^{-i}| + K)\big)^{1/2}$, with $\kappa, K > 0$. Effort costs are assumed to be linear, and the profit (gross of the cost of forming links) of firm i is then given by: $q_1^i + q_2^i - x_1^i - x_2^i$.

Each firm i chooses effort levels that maximize its gross profit, that is $x_1^i = (\zeta_1^i)^2$ and $x_2^i = (\zeta_2^i)^2$. The cost of forming links for firm i is given by $C^i(g^i, h^i) = (g^i + h^i)^2$. This captures the fact that forming links requires resources, say time, and the opportunity cost of these resources is increasing. The profit to a firm i in a multilayer network can be written as: $t(\psi(g^i, h^i)) + (1 - t)(\theta(g^i, |G^{-i}|) + \theta(h^i, |H^{-i}|))$, with $t = 1/2$, $\psi(g^i, h^i) = -(g^i)^2 - (h^i)^2 - 4g^ih^i$, $\theta(g^i, |G^{-i}|) = -(g^i)^2 + 2(g^i + \kappa)(|G^{-i}| + K)$, and $\theta(h^i, |H^{-i}|) = -(h^i)^2 + 2(h^i + \kappa)(|H^{-i}| + K)$. Clearly, ψ is concave in its two arguments and submodular, θ is concave in its first argument and satisfies strategic complementarity.

3 The Multilayer Model under Convexity

This section examines the two multilayer models $-$ the Degree and the Externality Model under convexity of the payoff functions and points out the differences between the two models.

3.1 The Multilayer Degree Model

We start with the existence and the characterization of MLPE networks in this model. It turns out that while in an MLPE network (G, H), both G and H are dominant group networks, they need not be identical. Let $\mathcal{D}^G$ and $\mathcal{D}^H$ be the sets of players who belong to the completely connected group in the dominant group networks G and H respectively, and by $|\mathcal{D}^G|$ and $|\mathcal{D}^H|$ be the cardinality of these sets.[4]

Theorem 1 *Suppose the payoff function is given by* (1) *and satisfies convexity.*

1. *An MLPE network always exists. Moreover, in an MLPE network $M = (G, H)$, both G and H are dominant group networks.*
2. *Suppose that $M = (G, H)$, where G and H are group dominant networks of size $g + 1$ and $h + 1$ respectively with $h \geq g$, $g < n - 1$, and $h > 0$.*
   - i. *Let ψ be supermodular. (a) When $\mathcal{D}^G \setminus \mathcal{D}^H \neq \emptyset$ and $\mathcal{D}^H \setminus \mathcal{D}^G \neq \emptyset$, M is an MLPE if and only if $\psi(g, 0) - \psi(0, 0) \geq 0$ and $\psi(1, h) - \psi(0, h) < 0$. (b) When $\mathcal{D}^G = \mathcal{D}^H$, M is an MLPE if and only if $\psi(g, g) - \psi(0, g) \geq 0$ and $\psi(1, 0) - \psi(0, 0) < 0$.*
   - ii. *Let ψ be submodular. (a) When $\mathcal{D}^G \cap \mathcal{D}^H \neq \emptyset$ and $N \setminus (\mathcal{D}^G \cup \mathcal{D}^H) \neq \emptyset$, M is an MLPE if and only if* $\min\{\psi(g, h) - \psi(0, h), \psi(g, h) - \psi(g, 0)\} \geq 0$, *and $\psi(1, 0) - \psi(0, 0) < 0$. (b) When $\mathcal{D}^G = N \setminus \mathcal{D}^H$, M is an MLPE if and only if $\psi(g, 0) - \psi(0, 0) \geq 0$ and $\psi(g, 1) - \psi(g, 0) < 0$.*

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[4] In the next result, we do not provide the necessary and sufficient conditions for a ML network $M = (G, H)$ to be an MLPE when both G and H are either complete or empty. In the former case, the condition is $\psi(n-1, n-1) \geq \max\{\psi(0, n-1), \psi(0, 0)\}$ when ψ is supermodular, and $\psi(n-1, n-1) \geq \psi(0, n-1)$ when ψ is submodular. In the latter case the condition is $\psi(0, 0) < \psi(1, 0)$.

Let us now provide intuition for Theorem 1. Intuitively, increasing marginal returns (convexity of ψ) means that players either want as many links as possible or no links at all in each network. Hence for each network we can divide the players into two groups. Moreover, it is easy to establish the existence of MLPE, when there are only inter-network externalities. Indeed, because of convexity, the maximum of ψ is obtained at $(g^i, h^i) \in \{0, n-1\}^2$. For the ease of exposition, let the maximum of ψ be reached at $(n-1, 0)$. Then, in a ML network where G is complete and H is empty is clearly an MLPE. Next, it is easy to see what the necessary and sufficient conditions do in Theorem 1: they prevent the most incentivized players from deleting links when they belong to the dominant group and from forming a link when they do not. We will now illustrate how Theorem 1 can be used to find MLPE networks.

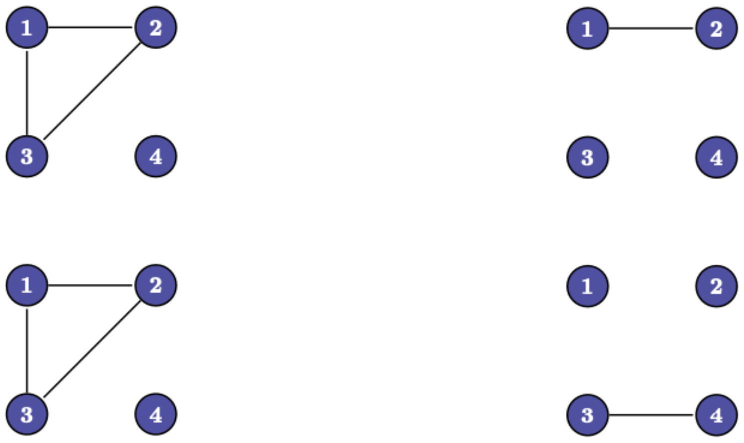
Application 1 (revisited). Suppose that $N = [\![1, 14]\!]$. We have $\psi(1,0) - \psi(0,0) = 5/8$, and $\psi(g,0) - \psi(0,0) = (g/8)(g+4)$. Therefore, $M = (G, H)$, where $\mathcal{D}^G = N \setminus \mathcal{D}^H$, and $|\mathcal{D}^H| \geq |\mathcal{D}^G|$, is an MLPE if and only if $|\mathcal{D}^G| > 5$, that is $|\mathcal{D}^G| = |\mathcal{D}^H| = 7$ or $|\mathcal{D}^G| = 6 < |\mathcal{D}^H| = 8$.

From Theorem 1, we know that the only non-empty equilibrium networks are dominant group networks. So now we ask the following question: how do the MLPE networks vary with inter-network externalities? In the next theorem, we examine how a specific number of links in one network can affect link formation in the other network. This result provides starkly different outcomes: in the first part of Theorem 2, where ψ is supermodular, we find that the same set of players form links in networks G and H – as in Figure (1.a), while in the second part of Theorem 2, where ψ is submodular, we show that there does not exist a single player with links in both networks – as in Figure (1.b).

Theorem 2 *Let $M = (G, H)$ be an MLPE network.*

1. *Suppose ψ is convex and supermodular. Moreover, suppose that there exists $k \in [\![1, n-1]\!]$, such that $\psi_G(1, k) \geq 0$. If $|\mathcal{D}^G| \geq k+1$ or $|\mathcal{D}^H| \geq k+1$, then $\mathcal{D}^G = \mathcal{D}^H$.*
2. *Suppose ψ is convex and submodular. Moreover, suppose that there exists $k \in [\![1, n-1]\!]$, such that $\psi_G(n-1,\ k) < 0$. If $|\mathcal{D}^H| \geq k+1$ or $|\mathcal{D}^G| \geq k+1$, then $\mathcal{D}^G \cap \mathcal{D}^H = \emptyset$.*

Two things follow immediately. Since $|\mathcal{D}^G| \geq 2$, by replacing k with 1, we get: (1) If ψ is convex and supermodular, then $[\psi_G(1,1) \geq 0] \Rightarrow [\mathcal{D}^G = \mathcal{D}^H]$. (2) If ψ is convex and



(a) $(\hat{G}, \hat{H})$ (b) $(\bar{G}, \bar{H})$

Figure 1: Examples of multilayer networks

submodular, then $[\psi_G(n-1,\ 1) < 0] \Rightarrow [\mathcal{D}^G \cap \mathcal{D}^H = \emptyset]$. We now provide intuition behind the two parts of the theorem.

1. Suppose player i belongs to a dominant group in $\mathcal{D}^G$ with $|\mathcal{D}^G| \geq k+1$, *i.e.*, player i has formed $g^i = |\mathcal{D}^G| - 1 \geq k$ links in G. The symmetry and supermodularity of ψ allows us to make cross-network comparisons, that is $\psi_H\left(g^i, 1\right) = \psi_G\left(1, g^i\right) \geq \psi_G(1, k)$. Moreover, since $\psi_G(1, k) \geq 0$, player i has an incentive to form an additional link in H. Since i was chosen arbitrarily, this holds for all players in $\mathcal{D}^G$. Thus, the $\psi_G(1, k) \geq 0$ condition is the driving force that leads players to form additional links in G (and in H by symmetry of ψ).

2. Suppose that player $i \in \mathcal{D}^G \cap \mathcal{D}^H$ in an MLPE. Then, we should have $\psi_G(n-1, k) \geq \psi_G(n-1, |\mathcal{D}^H| - 1) \geq \psi_G(|\mathcal{D}^G| - 1, |\mathcal{D}^H| - 1) \geq 0$. The first inequality follows from the submodularity of ψ, the second one from the convexity of ψ, and the last one is due to the fact that M is an MLPE. In other words, player i should have an incentive to form an additional link in G in a ML network where i is linked with $n-2$ players in G and k players in H. But the $\psi_G(n-1, k) < 0$ condition limits the formation of additional links.

Theorem 2 is important in that it highlights the role played by inter-network externalities in our setting. Supermodularity or submodularity of ψ, captured by ψ_{GH}, bring out the nuanced role of inter-network spillovers in ML networks. Having shown how they matter, our next result focuses on the magnitude of these spillovers relative to convexity. We show that if the supermodularity of ψ is sufficiently high relative to its convexity, players who belong to the dominant group in one network also belong to the dominant group in the other network. Similarly, if the submodularity of ψ is sufficiently high, then there does not exist players who simultaneously belong to the dominant group in both G and H. To summarize, the next result highlights the connection between inter-network externalities and the identity of linked players in one network given the identity of linked players in the other network.

Corollary 1 *Let $M = (G, H)$ be an MLPE. Let $k \in [\![2, n-2]\!]$.*

1. *Suppose that ψ is convex and supermodular, $\psi_G(k, 0) \geq 0$, and $\sum_{\ell=0}^{k-1} \psi_{GH}(1, k-\ell) \geq \sum_{\ell=0}^{k-2} \psi_{GG}(k-\ell, 0)$. If $|\mathcal{D}^G| \geq k+1$ or $|\mathcal{D}^H| \geq k+1$, then $\mathcal{D}^G = \mathcal{D}^H$.*
2. *Suppose that ψ is convex and submodular, $\psi(n-1-k, 0) - \psi(n-2-k, 0) < 0$, and $\sum_{\ell=1}^{k} |\psi_{GH}(n-1-k, \ell)| \geq \sum_{\ell=0}^{k-1} \psi_{GG}(n-1-\ell, k)$. If $|\mathcal{D}^G| \geq k+1$ or $|\mathcal{D}^H| \geq k+1$, then $\mathcal{D}^G \cap \mathcal{D}^H = \emptyset$.*

Let us provide the intuition for the first part of the corollary. First, note that the condition $\psi_G(k, 0) \geq 0$ means that ψ is increasing in its first argument at k when H has no links. Second, we know from Theorem 2 that the result is obtained if $\psi_G(1, k) \geq 0$. Clearly, the difference between $\psi_G(1, k)$ and $\psi_G(k, 0)$ is positively driven by the supermodularity of ψ since $k > 0$ (specifically by $\sum_{\ell=0}^{k-1} \psi_{GH}(1, k-\ell)$), and negatively driven by the convexity of ψ in its first argument since $1 < k$ (specifically by $\sum_{\ell=0}^{k} \psi_{GG}(k-\ell, 0)$). Therefore, if the supermodularity is sufficiently high compared to the convexity, we obtain $\psi_G(1, k) \geq \psi_G(k, 0)$. The intuition behind the second part of the corollary is similar.

This corollary emphasizes the significance of supermodularity and the strength of the interaction between the two networks in determining the type of MLPE networks. Indeed, if we impose $\psi_{GH} = 0$, then the marginal payoff of player i from an additional link in G is independent of his number of links in H. In our framework, this coincides with a situation where the two layers are independent and hence the dominant groups in G and H may not be related.

We now introduce results concerning efficient ML networks. In the Degree model, $M = (G, H)$, is efficient if it maximizes $W_\psi(G, H) = \sum_{i \in N} \psi(g^i, h^i)$. Our first lemma emphasizes the crucial role of supermodularity of ψ in characterizing efficient ML networks.

Lemma 1 *Let ψ be strictly supermodular. Then, in an efficient ML network $M = (G, H)$, we have for every $i \in N$, $g^i = h^i$.*

Observe that in the context of the Degree model, we have $\max_{(g^i,h^i)} \sum_{(g^i,h^i)} \psi(g^i, h^i) = \sum_{(g^i,h^i)} \max_{(g^i,h^i)} \psi(g^i, h^i)$. Therefore, our goal is to determine the pair (g^i, h^i) that maximizes ψ. If $g^i \neq h^i$, then either $g^i > h^i$, or $g^i < h^i$. Let us consider the case where $g^i > h^i$. In this case, $\psi(g^i, h^i)$ is not a maximum because, by symmetry and strict supermodularity of ψ, we have $\psi(g^i, g^i) - \psi(g^i, h^i) = \psi(g^i, g^i) - \psi(h^i, g^i) > \psi(g^i, h^i) - \psi(h^i, h^i)$. We now examine the implications of the strict convexity property of ψ on efficient ML networks.

Proposition 1 *Let ψ be strictly convex in each of its arguments. Then, an efficient ML network $M = (G, H)$ is such that G is either empty or complete and H is either empty or complete. Moreover, if in addition ψ is strictly supermodular, then the networks G and H are both empty or both complete.*

The reasoning behind Proposition 1 is quite simple. Due to the strict convexity of ψ, its maximum is reached at a point where g^i and h^i belong to $\{0, n-1\}$. It is possible to achieve the maximum at $\psi(0, n-1)$ when ψ is sufficiently submodular relative to its convexity. However, this is not possible if ψ is supermodular, as shown by Lemma 1. Furthermore, since there are no externalities between players, efficient ML networks are MLPE.

3.2 The Multilayer Externality Model

We now augment the Multilayer Degree model to allow for situations where players in a given network also incur benefits from the links between other players in that network. In this section, the payoff function satisfies Equation (3), ψ is symmetric, and θ exhibits strategic complementarity. Recall that ψ relies on the players' own degrees while θ captures the externalities of others' total degrees. We establish the existence of MLPE networks in the following theorem and show that under convexity, only dominant group networks can be equilibria. This result is driven by convexity in the same way as in Theorem 1. We also provide a result analogous to Theorem 2 that identifies the pattern of interaction between players in a multilayer network. The reasoning is similar to that of Theorem 2, but we need additional conditions to account for intra-network externalities. When ψ is supermodular, we add a condition that provides players an incentive to add links when intra-network externalities are at their highest. For the sake of brievety in Appendix A2, we include Theorem 3' which provides the necessary and sufficient conditions for obtaining a given MLPE network $M = (G, H)$, where G and H are group dominant networks, as in the Degree model.

Conversely, when ψ is submodular, we add a condition which ensures that players will not add links when intra-network externalities are at their lowest.

Theorem 3 *Suppose ψ and θ are convex in their first argument. Then, an MLPE network always exists, and in an MLPE network $M = (G, H)$, both G and H are dominant group networks. Moreover,*

1. *Suppose ψ is supermodular, $\theta_1(1,0) > 0$, and there exists $k \in [\![1, n-1]\!]$, $\psi_G(1,k) > 0$. If $|\mathcal{D}^G| \geq k+1$ or $|\mathcal{D}^H| \geq k+1$, then $\mathcal{D}^G = \mathcal{D}^H$.*
2. *Suppose ψ is submodular and $t\psi_G(n-1,1) + (1-t)\theta_1\left(n-1, \frac{(n-1)(n-2)}{2}\right) < 0$. Then, $\mathcal{D}^G \cap \mathcal{D}^H = \emptyset$.*

Recall that in the Playing the Field model (GJ, 2006), as well as in the mutilayer Degree model, equilibrium networks are dominant group networks. Theorem 3 shows that the result is preserved under externalities both within and accross networks since the result is driven by increasing marginal returns in links. Essentially, in this context the inter-network externalities (supermodularity and submodularity of ψ) allow for a selection of MLPE networks.

The study of multilayer networks provides another important insight that we may not see when examining networks G and H in isolation. We know that in the Playing the Field model, in non-empty and non-complete equilibrium networks, players can be in asymmetric positions: some players will have a non-zero number of links while other players will have formed no links. In the multilayer network framework, such asymmetries between players may be mitigated by the fact that the group of players who have formed links in one network, say G, may not coincide with the group of players that have formed links in network H. In other words, there will exist situations where despite significant asymmetries between players within each network, overall players may have the same total number of links and therefore obtain the same payoffs (see $M = (\bar{G}, \bar{H})$ drawn in Figure (1.b)).

There are similarities between the results obtained in the Degree model when ψ is convex and in the multilayer Externality model when both ψ and θ are convex.[5] This could suggest that the introduction of θ is only used to check the robustness, which is not the case. In fact, we establish that the strategic complementarity of θ determines which ML networks can be MLPE networks. For the sake of simplicity,

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[5] It is easy that see that there exists a version of Corollary 1 that also incorporates intra-network externalities.

we focus on MLPE networks $M = (G, H)$ as described in Parts 1 and 2 of Theorem 3. Specifically, we consider ML networks where $\mathcal{D}^G = \mathcal{D}^H$ when ψ is supermodular, and $D^G \cap D^H = \emptyset$ when ψ is submodular. In the latter case, as an illustration, we assume that $|D^G| = |N \setminus D^H|$. Also, let $t = 1/2$ and $\eta(x) = \frac{x(x-1)}{2}$. In the following result, we show that the strategic complementarity property plays a crucial role in determining which ML networks can be MLPE networks. Specifically, when the strategic complementarity of θ is sufficiently high, certain ML networks are not MLPE.

Proposition 2 *Suppose ψ and θ are convex in their first argument.*

1. *Let ψ be supermodular, and $M = (G, H)$ be a ML network where $\mathcal{D}^G = \mathcal{D}^H$, with $|\mathcal{D}^G| = g + 1$. Suppose that $\psi(g, g) + \theta(g, \eta(g)) \geq \psi(0, g) + \theta(0, \eta(g))$, $\psi(g, g) + 2\theta(g, \eta(g)) \geq \psi(0, 0) + 2\theta(0, \eta(g))$, and $\psi_G(1, 0) + \theta_1(1, 0) < 0$. M is an MLPE network if and only if $\sum_{\ell=0}^{\eta(g+1)-1} \theta_{12}(1, \eta(g + 1) - \ell) < |\psi_G(1, 0) + \theta_1(1, 0)|$.*
2. *Let ψ be submodular, and $M = (G, H)$ be a ML network where $\mathcal{D}^G = N \setminus \mathcal{D}^H$, and $|\mathcal{D}^H| \geq |\mathcal{D}^G|$. Suppose that $\psi(g, 0) + \theta(g, \eta(g)) \geq \psi(0, 0) + \theta(0, \eta(g))$, and $\psi_H(g, 1) + \theta_1(1, 0) < 0$. M is an MLPE network if and only if $\sum_{\ell=0}^{\eta(h+1)-1} \theta_{12}(1, \eta(h + 1) - \ell) < |\psi_H(g, 1) + \theta_1(1, 0)|$.*

Why are the ML networks $M = (G, H)$, such that $\mathcal{D}^G = \mathcal{D}^H$, not MLPE when strategic complementarity is sufficiently high? Note that M is an MLPE only if player i who forms no link in G and H has no incentive to form a link in either of these networks. It is clear that an isolated player in a network may obtain positive externalities from the links formed by the others. The higher these externalities are, the greater the incentive for i to form a link. Application 1' illustrates the potential impact of introducing intra-network externalities on MLPE networks when ψ is submodular.

Application 1' (revisited). Let $N = [\![1, 14]\!]$. From Theorem 3', $M = (G, H)$, where $\mathcal{D}^G = N \setminus \mathcal{D}^H$, and $|\mathcal{D}^H| \geq |\mathcal{D}^G|$, is an MLPE if and only if $\psi(g, 0) + \theta(g, (g-1)g/2) \geq \psi(0, 0) + \theta(0, (g-1)g/2)$, and $\psi(g, 1) + \theta(1, h(h+1)/2) < \psi(g, 0) + \theta(1, h(h+1)/2)$. These two properties are satisfied when $g \geq 7$ (the second condition does not hold for $g \leq 6$). Consequently, M is an MLPE if and only if $\mathcal{D}^G = \mathcal{D}^H = 7$.

We now examine efficient ML networks in the presence of both intra- and inter-network externalities. In an efficient ML network, $M = (G, H)$, $W_{\psi+\theta}(G, H) = \sum_{i \in N} \psi(g^i, h^i) + \theta(g^i, G^{-i}) + \theta(h^i, H^{-i})$ is maximal. The study of efficient networks requires to take into account for each player the role of additional links formed by others on his marginal payoff. Here we assume that θ is *strictly convex in its second argument, i.e.*, $\theta_{22}(x, X+1) = \theta(x, X+1) - \theta(x, X) - (\theta(x, X) - \theta(x, X-1)) > 0$.

In the following proposition, we establish that the neighborhoods of players in efficient ML networks are *nested* when both ψ and θ are strictly convex in their two arguments. Moreover, we show that strict supermodularity of ψ implies that if $M = (G, H)$ is an efficient ML network, then G and H are both complete, or both empty. We denote by $g(i;j) = \{\ell : i\ell \in E_G, \ell \neq j\}$, the set of players that are neighbors of player i in network G, except possibly player j – definition for H is similar.

Proposition 3 *Suppose that ψ and θ are strictly convex in their two arguments and θ satisfies strict strategic complementarity. Let $M = (G, H)$ be an efficient ML network. Then, for players i and j, we have $g(i;j) \subseteq g(j;i)$ or $g(j;i) \subseteq g(i;j)$, and $h(i;j) \subseteq h(j;i)$ or $h(j;i) \subseteq h(i;j)$. Moreover, if ψ is strictly supermodular, then G and H are both empty or both complete.*

When ψ is strictly supermodular, all marginal payoffs are increasing, so the link formation process continues until all links are formed. As a result, in efficient ML networks, both G and H are either empty or complete. Let us examine the mechanisms underlying the first part of the proposition where the neighborhoods of players i and j are nested. Suppose not. Then, there would exist a player i' (respectively j') that belongs to the neighborhood of player i (respectively j) in G, but not to that of player j (respectively i) in G. Due to the convexity of ψ and θ in their first argument, the incremental payoff obtained by players i, i', j, and j' when the links ij' and ji' are added is higher than the incremental payoff when the links ii' and jj' were added. Furthermore, due to the convexity of θ in its second argument, the other players obtain a higher incremental payoff when links ij' and ji' are added in G than when links ii' and jj' were added in G.

It is important to note that the results obtained in Propositions 1 and 3 are similar when ψ is supermodular: both networks G and H are either empty or complete. However, the underlying reasons for these results are quite different. In the absence of intra-network externalities, the symmetry of ψ plays a crucial role in obtaining the result. When intra-network externalities are introduced, the player's payoff function is no longer symmetric and the result is now obtained through the convexity and strategic complementarity properties of θ. Moreover, because of intra-network externalities, we cannot rule out the possibility that efficient ML networks may not be MLPE networks.

4 The Multilayer Model under Concavity

In this section we examine the two multilayer network models when the functions ψ and θ are both concave.

4.1 The Multilayer Degree Model

We begin with a result concerning the existence and properties of MLPE networks. We characterize the groups of players that are linked together in G and H in an MLPE network $M = (G, H)$ under the supermodularity or submodularity of ψ. For supermodularity, we define two sets of players associated with the ML network M, for whom there exists another player who has formed strictly more links in one network and no more links in the other network: $N^1_{sup}(M) = \{i \in N : \exists j \in N, g^j > g^i \text{ and } h^j \leq h^i\}$, and $N^2_{sup}(M) = \{i \in N : \exists j \in N, g^j \leq g^i \text{ and } h^j > h^i\}$. Similarly, for submodularity, we define $N^1_{sub}(M) = \{i \in N : \exists j \in N, g^j > g^i \text{ and } h^j \geq h^i\}$ and $N^2_{sub}(M) = \{i \in N : \exists j \in N, g^j \geq g^i \text{ and } h^j > h^i\}$. Finally, for M, we define three subsets of players based on the number of links formed in G relative to the number of links formed in H: $N_g(M) = \{i \in N : g^i < h^i\}$, $N_h(M) = \{i \in N : g^i > h^i\}$, and $N_{gh}(M) = \{i \in N : g^i = h^i\}$. For $M = (G, H)$, recall that $\mathcal{N}^c_G(M)$ and $\mathcal{N}^c_H(M)$ contain the subsets of players that induce a complete subnetwork in G and H, respectively. Next result states that some players are linked together in G or in H when ψ is either supermodular or submodular.

Theorem 4 *Suppose the payoff function is given by* (1) *and satisfies concavity. An MLPE network, $M = (G, H)$, always exists.*

1. *Suppose ψ satisfies supermodularity. Then, there exist $N_G \in \mathcal{N}^c_G(M)$ and $N_H \in \mathcal{N}^c_H(M)$ such that $N_G = N^1_{sup}(M) \cup N_g(M)$ and $N_H = N^2_{sup}(M) \cup N_h(M)$.*
2. *Suppose ψ satisfies submodularity. Then, players in $N^1_{sub}(M)$ are linked together in G, and players in $N^2_{sub}(M)$ are linked together in H.*

The reasoning behind point 2 is quite simple. Consider player $i \in N^1_{sub}(M)$, such that there exists player j with $g^i < g^j$ and $h^i \leq h^j$. It is clear that player i's payoff if he forms an additional link in G will be higher than player j's payoff from one of his links in G, due to the concavity and submodularity of ψ. Point 1 uses the same logic on $N^1_{sup}(M)$. Furthermore, when $h^i > g^i$, player i obtains a higher marginal payoff if he forms an additional link in G relative to the marginal payoff he obtains from a link in

H. Indeed, because of the symmetry, the concavity and the supermodularity of ψ, we have $\psi_G(g^i+1,h^i) \geq \psi_G(g^i+1,g^i) \geq \psi_G(h^i,g^i) = \psi_H(g^i,h^i)$.
Example 1 illustrates the role played by inter-network externalities when ψ satisfies concavity in its two arguments.

Example 1. Suppose $N = [\![1,4]\!]$, $\psi(1,1)-\psi(0,1) = \varepsilon$, $\varepsilon > 0$, and $\psi(1,1)-\psi(0,0) > 0$. Let $M = (G,H)$ be such that players 1 and 2 are linked in both networks, and players 3 and 4 have formed no link in both networks. Then, M is an MLPE if and only if $\psi_{GH}(1,1) = \psi(1,1) - \psi(0,1) - (\psi(1,0) - \psi(0,0)) = \varepsilon' > \varepsilon$.

The example highlights the fact that if the inter-network externalities are sufficiently high, then each network in an MLPE may contain unlinked players who have strictly fewer links than the other players in this network. This cannot happen in an MLPE when inter-networks externalities are sufficiently low because of the concavity of ψ within each network. More precisely, in Example 1, both networks G and H are *dominant group* despite the *concavity* of ψ. This result is not possible in scenarios with only one layer (see Proposition 3.2, page 328 in GJ, 2006).

We now examine the necessary and sufficient conditions for obtaining a result analogous to Theorem 1. Let $\varphi(y) = \max\{\arg\max_{x \in [\![0,n-1]\!]} \psi(x,y)\}$. Moreover, let $Q_G(M)$ (respectively $Q_H(M)$) denote the set of players in M such that $g^j < \varphi(h^j)$ (respectively $h^j < \varphi(g^j)$). The necessary and sufficient conditions require that ψ satisfies a specific property called (P1). For $z \leq \varphi(x)$, let $d(x;b) = \psi(x,z) - \psi(x,z-b)$.

(P1) For $x \leq \varphi(y)$ and $y \leq \varphi(x)$, we have $\sum_{\ell'=0}^{b-1} \sum_{\ell=0}^{a-1} \psi_{GH}(x-\ell, y-\ell') \leq d(y;a) + d(x;b)$.

When (P1) holds, for $x \leq \varphi(y)$ and $y \leq \varphi(x)$, we have $\psi(x,y) - \psi(x-a,y-b) \geq 0$. In other words, if players have no incentive to remove a links in network G and b links in network H, then they also have no incentive to simultaneously remove a links in G and b links in H. This property holds when ψ is submodular, or the supermodularity of ψ is bounded above on pairs (x,y), where $x \leq \varphi(y)$ and $y \leq \varphi(x)$.

Proposition 4 *Suppose that the payoff function is given by* (1)*, satisfies concavity and (P1). Let $M = (G,H)$. M is an MLPE network if and only if (a) for every $i \in N$, $g^i \leq \varphi(h^i)$ and $h^i \leq \varphi(g^i)$, (b) all players in $Q_G(M)$ are linked together in G, and (c) all players in $Q_H(M)$ are linked together in H.*

The intuition behind this result follows from Theorem 4. We know that players who have formed fewer links in G than others may have an incentive to form additional links. In Proposition 4, we provide the condition for this using φ. Specifically, because of the concavity of ψ, if $g^i < \varphi(h^i)$, then player i has an incentive to form an additional link in G, while if $g^i > \varphi(h^i)$, then player i has an incentive to remove links in G. We now illustrate how φ can be used to find MLPE networks.

Application 2 (revisited). Let $N = [\![1, 4]\!]$, and suppose that $A = 5/4$, $\alpha = 3/10$, and $b = 3/10$. Then, $\varphi(0) = 0$, and $\varphi(x) = 1$, for $x \in [\![1, 3]\!]$. Moreover, $\varphi(1, 1) > \varphi(0, 0)$. In that case, three types of ML networks, $M = (G, H)$, are MLPE: in M, both G and H are empty, or G and H are regular networks where each player is involved in one link, or two players are involved in 1 link in G and H and the two other players have formed no links in G and H.

Recall that in Theorem 4 when ψ satisfies submodularity the result is not as strong as when ψ satisfies supermodularity. In fact, for more precise results under submodularity, we will need to assume that submodularity is greater than concavity, *i.e.*, $|\psi_{GG}(x+1, y)| < |\psi_{GH}(x+1, y)|$. From this, it follows that $\psi_G(x+1, y-1) > \psi_G(x, y)$. We deduce that if $h^j - h^i > g^i - g^j > 0$, then the marginal payoff obtained by player i from an additional link in G is greater than the payoff that player j obtains from one of his links in H. Consider for example that $h^j - h^i = 2$ and $g^i - g^j = 1$. We have $\psi_G(g^i + 1, h^i) = \psi_G(g^j + 2, h^j - 2) > \psi_G(g^j + 1, h^j - 1) > \psi_G(g^j, h^j)$. In other words, if player i has more links in G and fewer links in H compared to player j and $h^j - h^i > g^i - g^j$, then player i has an incentive to form an additional link in G.[6] Finally, we assume that $\psi_G(1, n-1) > 0$ to ensure that players always have an incentive to form at least one link.

We show that, under these assumptions, players who do not have the maximal sum of degrees in a ML network, say $M = (G, H)$, belong to $N_G \in \mathcal{N}^c_G(M)$ or $N_H \in \mathcal{N}^c_H(M)$. Let $N^{\max}(M) = \{j \in N : \ \forall \ell \in N, g^j + h^j \geq g^\ell + h^\ell\}$ be the set of players who have the maximal sum of degrees in M.

Proposition 5 *Suppose ψ satisfies strict concavity in each of its arguments and strict submodularity. Let $|\psi_{GG}(x+1, y)| < |\psi_{GH}(x+1, y)|$, and $|\psi_{HH}(x, y+1)| < |\psi_{HG}(x, y+1)|$. Moreover, $\psi_G(1, n-1) > 0$. Then, in an MLPE network $M = (G, H)$, there exist $N_G \in \mathcal{N}^c_G(M)$ and $N_H \in \mathcal{N}^c_H(M)$, such that $N \setminus N^{\max}(M) = N_G \cup N_H$.*

What underlies this result? Clearly, when $i \notin N^{\max}(M)$ there is a player $j \in N^{\max}(M)$ such that $g^i - g^j < h^j - h^i$ (or $g^i - g^j > h^j - h^i$). There are two possibilities. (a)

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[6] In the next result, we also use the extension for the concavity of ψ to $\psi(-1, \cdot)$ and $\psi(\cdot, -1)$.

Suppose $g^i \leq g^j$ and $h^i < h^j$ (or $g^i < g^j$ and $h^i \leq h^j$). Then, by Theorem 4.2 player i has an incentive to form an additional link in G or in H. (b) Suppose $g^i > g^j$ and $h^i < h^j$ (or $g^i < g^j$ and $h^i > h^j$). Since $g^i - g^j < h^j - h^i$, we can use the condition $|\psi_{GG}(x+1, y)| < |\psi_{GH}(x+1, y)|$ to establish that player i has an incentive to form an additional link in G (or in H) giving us the result.

Next, we examine efficient networks when ψ is strictly concave. We know by Lemma 1, that when in addition ψ is strictly supermodular, all players have formed the same number of links in G and H.

Proposition 6 *Suppose that ψ is strictly concave. Then, in an efficient ML network $M = (G, H)$, G and H are regular networks. Moreover, if ψ is supermodular, then all players have the same number of links in both networks.*

The intuition for this result is identical to the one for Proposition 1. Not surprisingly, since there are no externalities between players, an efficient ML network is also an MLPE.

4.2 The Multilayer Externality Model

We first deal with the existence and characterization of MLPE networks. The existence of MLPE is difficult to obtain when ψ is supermodular. Basically, unlike a single layer network problem where we would have to worry about deletion of links in either G or H, we now have to worry about the simultaneous deletion of links in both G and H. Therefore, we introduce a specific property, called (P2).[7] For $a \in [\![0, x-1]\!]$, let $\delta(x, a; y) = t(\psi(x, y) - \psi(x-a, y)) + (1-t)\left(\theta\left(x, \frac{(n-2)x}{2}\right) - \theta\left(x-a, \frac{(n-2)x}{2}\right)\right)$. We now state the required property.

(P2) If $\delta(x, a; y) \geq 0$ and $\delta(y, b; x) \geq 0$, for every $a \in [\![1, x-1]\!]$, $b \in [\![1, y-1]\!]$, then $t\sum_{\ell'=0}^{b-1}\sum_{\ell=0}^{a-1}\psi_{GH}(x-\ell, y-\ell') \leq \delta(x, a; y) + \delta(y, b; x)$.

First, note that (P2) is a local property as it only considers situations where all players form x and y links in G and H, respectively. Second, $\delta(x, a; y)$ measures the loss of player i when deviating from the regular network by removing a links. Therefore, conditions $\delta(x, a; y) \geq 0$ and $\delta(y, b; x) \geq 0$ imply that we only consider regular networks with x links (and with y links) where no player has an incentive to remove any number

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[7] Unlike a single layer network model where we would have to worry about deletion of link in either G or H we now have to worry about the simultaneous deletion of links in both G and H.

of links in each network (due to the symmetry of ψ) taking care of the concern raised above. Third, (P2) states that in such regular networks the supermodularity, *i.e.*, $\sum_{\ell'=0}^{b-1}\sum_{\ell=0}^{a-1}\psi_{GH}(x-\ell,y-\ell') = \psi(x,y)-\psi(x-a,y)-(\psi(x,y-b)-\psi(x-a,y-b))$, is bounded above by the sum of the losses associated with the deviations from the regular network by removing either a links in G, or b links in H, *i.e.*, $\delta(x,a;y)+\delta(y,b;x)$.

The characterization of MLPE networks and its underlying intuition is similar to that given in Theorem 4, except that we do not have very precise results under supermodularity. This is because the payoff function is no longer symmetric when intra-network externalities are introduced.

Theorem 5 *Let the payoff function be given by* (3)*. Moreover, suppose that ψ and θ are concave and θ satisfies strict strategic complementarity. If ψ is supermodular and satisfies (P2), or ψ is submodular, then an MLPE network always exists. Let $M=(G,H)$ be an MLPE network.*

1. *Suppose ψ satisfies supermodularity. Then, players in $N^1_{sup}(M)$ are linked together in G, and players in $N^2_{sup}(M)$ are linked together in H.*
2. *Suppose ψ satisfies submodularity. Then, players in $N^1_{sub}(M)$ are linked together in G, and players in $N^2_{sub}(M)$ are linked together in H.*

To obtain more precise characterization results, we need additional assumptions regarding the relative magnitude of the supermodularity or submodularity of ψ compared to the concavity of ψ and θ, as well as the strategic complementarity of θ. Specifically, we assume that the former property is significantly greater than the latter properties. To present the result, we need the following definition. Let $N^{\min}(M)=\{j\in N: g^j+h^j\le g^\ell+h^\ell,\forall \ell\in N\}$ be the set of players who have the lowest sum of degrees, and $D_a(M)=\{j\in N: g^j-h^j=a\}$ be the set of players for whom the difference between the links they are involved in G and H is a. When ψ is supermodular, there are two possibilities for an MLPE network M. (i) Some players in $N^{\min}(M)$ do not have the same difference in degrees between networks G and H. In that case, all players who are not in $N^{\min}(M)$ have an incentive to form links in G or H, and as a result, these players are linked together in G or H. (ii) All players j in $N^{\min}(M)$ have the same difference in degrees. In that case, all players i such that $g^i-h^i\neq g^j-h^j$ have an incentive to form links in G or in H, and as a result, these players are linked together G or in H. When ψ is submodular, the result we obtain is consistent with that stated in Proposition 5.[8] To obtain sharper results for MLPE networks in the presence of both inter- and intra-network externalities, additional conditions on the externalities are again necessary.

Proposition 7 *Suppose that ψ satisfies strict concavity in each of its arguments, and θ satisfies strict concavity in its first argument and strict strategic complement. Moreover, $|\psi_{GH}(x+1,y)| - \theta_{12}(x+1, X-x+1) > |\psi_{GG}(x+1,y)| + |\theta_{11}(x+1, X-x+1)|$ and $|\psi_{GH}(x,y+1)| - \theta_{12}(y+1, Y-y+1) > |\psi_{HH}(x,y+1)| + |\theta_{11}(y+1, Y-y+1)|$. Let $M = (G,H)$ be an MLPE network.*

1. *Suppose that ψ satisfies strict supermodularity, and $\psi(1,0) + \theta(1,0) \geq \psi(0,0) + \theta(0,0)$. There are two possibilities:*
   (a) *There are two players $j, j' \in N^{\min}(M)$ such that $g^j - h^j \neq g^{j'} - h^{j'}$. Then, there exist $N_G \in \mathcal{N}_G^c(M)$ and $N_H \in \mathcal{N}_H^c(M)$ such that $N \setminus N^{\min}(M) = N_G \cup N_H$.*
   (b) *For all players $j \in N^{\min}(M)$, $g^j - h^j = a$. Then, there are $N_G \in \mathcal{N}_G^c(M)$ and $N_H \in \mathcal{N}_H^c(M)$ such that $N \setminus D_a(M) = N_G \cup N_H$.*
2. *Suppose that ψ satisfies strict submodularity, and $\psi(1,n-1) + \theta(1,0) - (\psi(0,n-1) + \theta(0,0)) \geq 0$. In an MLPE network $M = (G,H)$, there exist $N_G \in \mathcal{N}_G^c(M)$ and $N_H \in \mathcal{N}_H^c(M)$, such that $N \setminus N^{\max}(M) = N_G \cup N_H$.*

The underlying intuition for the second part of the proposition is similar to that given for Proposition 5. The first part of the proposition, however, is more involved. Consider a player $i \notin N^{\min}(M)$. Then, there is a player $j \in N^{\min}(M)$ such that $g^i + h^i > g^j + h^j$. There are three possible cases based on these inequalities. The reasoning behind the supermodularity outcomes is similar to the type of reasoning used in Proposition 5, albeit with more steps.

While it may initially appear that the introduction of intra-network externalities has no impact on the properties of MLPE networks, we illustrate through an example that these externalities do play a significant role in determining equilibrium networks when their magnitude is sufficiently high.

Example 1 (revisited). Suppose $\theta_{12}(1,1) > \varepsilon'$. Then, $\psi(1,0) + \theta(1,1) - (\psi(0,0) + \theta(0,1)) > \psi(1,1) + \theta(1,0) - (\psi(0,1) + \theta(0,0))$. Consequently, M given in Example 1 cannot be an MLPE.

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[8] Note that in the next result, we extend the concavity of ψ to $\psi(-1,\cdot)$ and $\psi(\cdot,-1)$, and that of θ in its first argument to $\theta(-1,\cdot)$.

Let us explain the mechanisms at play in this example. Suppose that $g^i = g^j - 1$, and $h^i = h^j$. Then, $G^{-i} = G^{-j} + 1$. Thus, the difference between the marginal payoffs obtained by players i and j from their last link in G is $|\psi_{11}(g^i + 1, h^i)| + |\theta_{GG}(g^i + 1, G^{-i})| + \theta_{GH}(g^i + 1, G^{-i})$. Clearly, the first two terms capture the impact of the concavity of ψ and θ and the last term captures the impact of the strategic complementarity of θ. The strategic complementarity plays a similar role to the concavity in that it drives players with the fewest links in G to form links with one another. When ψ is submodular, the strategic complementarity of θ can result in MLPE networks where players in each network have significantly different connections. Recall that the strategic complementarity increases the incentive for each player to form an additional link in networks where others have already formed many links. This phenomenon is illustrated in Application 3.[9]

Application 3 (revisited). Suppose $N = [\![1, 4]\!]$, $\kappa = 1$, $K = 2$. Then $M = (G, H)$ where G is complete and H is empty is an MLPE.

We now turn our attention to efficient ML networks. Characterizing these networks becomes more difficult when intra-network externalities are introduced. For instance, let ψ be supermodular, and suppose a ML network with two players i and j such that $g^i < g^j - 1$ and $h^i \geq h^j$. Hence, in G, j has a neighbor, say j', not linked to i. At first glance, due to the concavity of ψ and θ, one might think that replacing the link jj' with ij' in G would increase the total payoff. However, it affects the payoffs of players i and j through both functions ψ and θ. First, due to the concavity and supermodularity of ψ, we have $\psi(g_i + 1, h_i) + \psi(g_j - 1, h_j) \geq \psi(g_i, h_i) - \psi(g_j, h_j)$. Second, θ plays a more complex role. While the concavity of θ in its first argument favors the replacement of the link, strategic complementarity of θ and its convexity in its second argument (as stated in Proposition 3) do not since $G^{-i} > G^{-j} + 1$. As a result, when the concavity of ψ and θ and the supermodularity of ψ are sufficiently high relative to the strategic complementarity of θ and its convexity in its second argument, then replacing the link jj' with the link ij' would increase the total utility of the multilayer network.

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[9] Note that if the terms $g^i \times G^{-i}$ and $h^i \times H^{-i}$ are removed from the payoff function used in Application 3, the previous result no longer holds.

5 Extensions

We now present two extensions keeping in mind that a complete examination of all the different possibilities in these cases is outside the scope of the current paper. First, we relax the assumption regarding the number of layers and consider scenarios with more than two layers. Second, we examine situations where players receive a bonus from being linked with the same players in both networks.

Going Beyond Two Layers. Let us consider ML networks, M, with K layers, where $M = (G_k)_{k \in \llbracket 1,K \rrbracket}$. To illustrate, let us provide a natural setting that generalizes the Degree model framework in this direction. We define the payoff function as $\psi((g^i_k)_{k \in \llbracket 1,K \rrbracket})$, and we assume that $\psi((g^i_k)_{k \in \llbracket 1,K \rrbracket}) = \psi(\pi((g^i_k)_{k \in \llbracket 1,K \rrbracket}))$, where $\pi((g^i_k)_{k \in \llbracket 1,K \rrbracket})$ is a permutation of $(g^i_k)_{k \in \llbracket 1,K \rrbracket}$. This assumption is the analog of the *symmetry* property when M contains two layers. For K layers, ψ is said supermodular if $\psi(g^i_\ell + 1, g^i_{\ell'}, (g^i_m)_{m \neq \ell,\ell'}) - \psi(g^i_\ell, g^i_{\ell'}, (g^i_m)_{m \neq \ell,\ell'}) \geq \psi(g^i_\ell + 1, g^i_{\ell'} - 1, (g^i_m)_{m \neq \ell,\ell'}) - \psi(g^i_\ell, g^i_{\ell'} - 1, (g^i_m)_{m \neq \ell,\ell'})$ for any two arbitrary layers ℓ and ℓ'. Assuming that ψ is convex in each of its arguments, we have $\psi(g^i_\ell + 1, (g^i_\ell)_{k \neq \ell}) - \psi(g^i_\ell, (g^i_\ell)_{k \neq \ell}) \geq \psi(g^i_\ell, (g^i_\ell)_{k \neq \ell}) - \psi(g^i_\ell - 1, (g^i_\ell)_{k \neq \ell})$. It follows that if player i has formed a link in G_k, then he has an incentive to form an additional link in this network. Hence, the result obtained in Theorem 1 will be preserved. Suppose now that ψ is concave in each of its arguments and supermodular, and that M is an MLPE with $g^i_\ell < g^i_{\ell'}$. Then, because of the symmetry of ψ, we have $\psi(g^i_\ell + 1, g^i_{\ell'}, (g^i_m)_{m \neq \ell,\ell'}) - \psi(g^i_\ell, g^i_{\ell'}, (g^i_m)_{m \neq \ell,\ell'}) \geq \psi(g^i_{\ell'}, g^i_{\ell'}, (g^i_m)_{m \neq \ell,\ell'}) - \psi(g^i_{\ell'} - 1, g^i_{\ell'}, (g^i_m)_{m \neq \ell,\ell'}) \geq \psi(g^i_{\ell'}, g^i_\ell, (g^i_m)_{m \neq \ell,\ell'}) - \psi(g^i_{\ell'} - 1, g^i_\ell, (g^i_m)_{m \neq \ell,\ell'})$. Therefore, player i has an incentive to form an additional link in network G_ℓ. This result is consistent with Theorem 4, Part 1. To sum up, the possibility of more than two layers opens up a richer set of possibilities for inter-network spillovers. However, if these spillovers are not in opposing directions then we obtain results similar to what we find in two layers.

The role of common neighbors. We now assume that in the Degree model, every player obtains a bonus for each link he forms with a player who is part of his neighborhood in both networks. This is a simple way to incorporate the identity of players into the payoff function. The marginal payoff obtained by player i from forming a link with player j in network G, is now given by $\psi(g^i + 1, h^i) - \psi(g^i, h^i) + \delta_{ij}$, with $\delta_{ij} = a$, if $ij \in E_H$, and $\delta_{ij} = 0$ otherwise. A similar definition applies for network H. Let us present some results for the case where ψ satisfies supermodularity. Suppose ψ is convex and let $M = (G, H)$ be an MLPE. Then, (a) players who have formed a link in G with a player they are not linked to in H are linked together in G, and (b) players who have formed links in G only with players they are linked to in H, are linked in G with all players who have formed links in G, and are their neighbors in H. Part (a) follows from the fact that if player i has formed a link with a player who is not his neighbor in G, then i has an incentive to form a link with all players in G because of the convexity property. Part (b) is also a consequence of the convexity property but limited to i's neighbors – if i has formed a link in G with one of his neighbors j in H, he has an incentive to form a link in G with all his neighbors in H. Indeed, by forming an additional link in G with one of his neighbors in H, player i obtains a marginal payoff that is at least equal to the marginal payoff he obtains from link ij. Suppose now that ψ is concave and let $M = (G, H)$ be an MLPE. Consider player $i \in N_g(M)$, *i.e.*, $g^i < h^i$. By construction, there exists a player j who is a neighbor of player i in network H, but not in network G. As a result, $\psi(g^i, h^i) - \psi(g^i, h^i - 1) \geq 0$. Furthermore, due to the symmetry, concavity, and supermodularity of ψ, we have $\psi(g^i + 1, h^i) - \psi(g^i, h^i) \geq \psi(g^i, h^i) - \psi(g^i, h^i - 1)$. In other words, player i has an incentive to form an additional link in G. This means that all players in $N_g(M)$ are connected in network G. Similarly, players in $N_h(M)$ are all connected in network H. In summary, the introduction of a bonus for common neighbors would alter some results. However, the main mechanisms described in the Degree model would remain valid.

6 Conclusion

In this paper we examine the simultaneous formation of two networks using basic principles of microeconomic theory. The number of links that players have in each network form the components of their payoff function. We then impose different properties on this function to capture different types of externalities across the networks. Our focus has been on identifying the incentives for players to establish links with others under these types of externalities. Jackson (2014) sets out to explain the role of networks for understanding human behavior. He suggests that ideally a model of network formation should do three things. (1) It should allow for dependencies and network effects at the link level. This feature is clearly present in our model. (2) It allows for the endogenous formation of network relationships. In our model externalities determine the incentives to form links and mutual consent is necessary for establishing the links. Hence, the links are truly endogenous. (3) An ideal model of network formation should be tractable enough to test it with data. Here we believe that more research is needed.

Since our payoff function does not utilize a specific functional form and only imposes properties on the arguments of the payoff functions it may be harder to take it directly to the data. However, we believe that our paper provides important insights about the trade-offs between inter and intra-network externalities and the role they play for equilibrium and efficient networks. This allows us to highlight the importance of studying multilayer networks by demonstrating that the study of single layer networks may not provide the correct insights. For instance, if we find that players i and j have not formed a link it might be due to the fact that they have formed a link in another network and there are negative inter-network spillovers. In that sense we believe that our paper provides a tractable framework for further analysis of the formation of multilayer networks.

7 Appendix

Appendix A. MLPE networks under convexity

Appendix A1. ML Degree Model

Graphical sequences of degrees. Every player is interested in the formation of a number of links (her degree) in each network. However all n-uples $\vec{g} = (g^1, \ldots, g^n)$ and $\vec{h} = (h^1, \ldots, h^n)$ cannot be used to represent a network. To avoid this problem, we need the following graph-theoretic definition.

Definition 5 *A sequence* $(x^1, x^2, \ldots, x^n)$ *is graphical if there exists a network whose nodes have degrees* x^1, x^2, …, x^n.

We first present a lemma that is used in the proof of Theorem 1 for the existence part.

Lemma 2 *Let* n *be even. Then, for every* $x \in [\![0, n-1]\!]$, *the* n*-uple* $(x, x, \ldots, x)$ *is graphical.*

The proof of Lemma 2 is based on a theorem from Erdös and Gallai (1960).

Theorem ***(Erdös and Gallai, 1960).*** *A ntuple* $(x^1, x^2, \ldots, x^n)$ *of non-negative integers, such that* $x^1 \geq x^2 \geq \ldots \geq x^n$, *and whose sum is even is graphical if and only* *if*

$$\sum_{i=1}^{r} x^i \leq r(r-1) + \sum_{i=r+1}^{n} \min\{x^i, r\}, \text{ for every } r,\ 1 \leq r < n.^{10} \tag{4}$$

Proof of Lemma 2 First, since n is even, the sum of the n-uple $(x, x, \dots, x)$ is even. Then Equation (4) can be written as

$$rx \leq r(r-1) + \sum_{i=r+1}^{n} \min\{x, r\}, \text{ for every } r,\ 1 \leq r < n. \tag{5}$$

There are two cases. Suppose $r \leq x$. Then equation (5) is satisfied if $rx \leq r(r-1) + (n-r)r \Rightarrow x \leq (r-1)+(n-r) \Rightarrow x \leq n-1$. This equation is always satisfied. Suppose $r > x$. Then equation (5) is $rx \leq r(r-1)+(n-r)x$ If $x = n-1$, then $(n-1, \dots, n-1)$ is graphical since the complete network supports this sequence. Similarly, if $x = 0$, then $(0, \dots, 0)$ is graphical since the empty network supports this sequence. We now deal with x, $0 < x < n-1$. We have $r(r-1)+(n-r)x-rx = r(r-x-1)+x(n-r) \geq 0$, for $r \in [\![x+1, n]\!]$. □

Proof of Theorem 1

Existence. First, since $[\![0, n-1]\!]^2$ is finite, $\psi(x, y)$ admits a maximum over $[\![0, n-1]\!]^2$. Let $(g^\star, h^\star) \in \arg\max_{(g,h)\in[\![0,n-1]\!]^2} \psi(g, h)$. Consider the following degrees sequences: $(g^\star, \dots, g^\star)$ and $(h^\star, \dots, h^\star)$. These two sequences are graphical by Lemma 2. Let $G^\star$ and $H^\star$ be networks where each player is involved in $g^\star$ and $h^\star$ links respectively. Networks $G^\star$ and $H^\star$ are MLPE by construction.

Characterization. Let $M = (G, H)$ be an MLPE network. To introduce a contradiction, suppose wlog that $ij \notin E_G$ and $g^i, g^j > 0$. Then, we have for $\ell \in \{i, j\}$, $(g^\ell, h^\ell) \in [\![1, n-2]\!] \times [\![0, n-1]\!]$, $\psi_G(g^\ell + 1, h^\ell) \geq \psi_G(g^\ell, h^\ell) \geq 0$. The first inequality comes from convexity and the second one follows from the fact that M is an MLPE network. Therefore, players i and j have an incentive to form a link together, a contradiction.

We now show successively the two parts of the proposition.

1. a. **Necessary condition.** Suppose that some players who belong to the dominant group in G (H) do not belong to the dominant group in H (G). Let $\Delta_\ell = \psi(g-\ell, 0) - \psi(g-(\ell+1), 0)$. Suppose that $\psi(g, 0) - \psi(0, 0) \geq 0$. For $k \in [\![1, g]\!]$, we establish that $\psi(g, 0) - \psi(g-k, 0) \geq 0$. To introduce a contradiction suppose that there is $k \in [\![1, g]\!]$ such that $\psi(g, 0) - \psi(g-k, 0) < 0$. Note that

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[10] This theorem can also be found in Harary, 1969, Chapter 6 pp. 59-62 and the statement here is based on his presentation.

$\psi(g,0)-\psi(g-k,0)=\sum_{\ell=0}^{k-1}\Delta_\ell$. Since Δ_ℓ is decreasing in ℓ by the convexity of ψ, Δ_{k-1} is the lowest term in the sum $\sum_{\ell=0}^{k-1}\Delta_\ell$. Moreover, we know that $\sum_{\ell=0}^{k-1}\Delta_\ell<0$, it follows that $\Delta_{k-1}<0$. Note that since Δ_ℓ is decreasing in ℓ and $\Delta_{k-1}<0$, we have $\sum_{\ell=k}^{g-1}\Delta_\ell<0$. Therefore, $\psi(g,0)-\psi(0,0)=\sum_{\ell=0}^{g-1}\Delta_\ell<0$, a contradiction. Consequently, player i who has formed no links in H has no incentive to remove links in G. Moreover, by using the same argument as in the previous point, by symmetry and convexity of ψ, and the fact that $h\geq g$ and $\psi(g,0)-\psi(0,0)\geq 0$, we have $\psi(0,h)-\psi(0,0)=\psi(h,0)-\psi(0,0)\geq\psi(g,0)-\psi(0,0)$. Consequently, player i who has formed no link in G has no incentive to remove links in H. Furthermore, by supermodularity of ψ, $\psi(g,h)-\psi(0,h)\geq\psi(g,0)-\psi(0,0)$, and player i who has formed h links in H, has no incentive to remove links in G. Similarly, $\psi(g,h)-\psi(g,0)\geq\psi(0,h)-\psi(0,0)$, and player i who has formed g links in G has no incentive to remove links in H. Note that $\psi(g,h)-\psi(0,0)=\psi(g,h)-\psi(g,0)+\psi(g,0)-\psi(0,0)\geq 0$. Thus, player i who has formed g links in G and h links in H has no incentive to remove links in both networks. Let us establish that players who have formed no links in one network have no incentive to form a link in this network. Players who have formed no links in G and have formed h links in H have no incentive to form a link in G since $\psi(1,h)-\psi(0,h)<0$. Players who have no links in H and has formed g links in G has no incentive to form a link in H since $\psi(g,1)-\psi(g,0)=\psi(1,g)-\psi(0,g)\leq\psi(1,h)-\psi(0,h)$. The equality comes from the symmetry of ψ and the inequality follows from the supermodularity of ψ. Finally, players who have formed no links in G and H have no incentive to form a link in one network by using the same argument: $\psi(0,1)-\psi(0,0)=\psi(1,0)-\psi(0,0)\leq\psi(1,h)-\psi(0,h)$. We conclude that $M=(G,H)$ is an MLPE network.

Sufficient condition. Let $M=(G,H)$ be an MLPE network. Then, players who have formed no links in H have no incentive to remove all their links in G, *i.e.*, $\psi(g,0)-\psi(0,0)\geq 0$. Note that players who have formed links in G have an incentive to form an additional link in G by convexity of ψ in its first argument. Consequently, players who have formed h links in H and no links in G should have no incentive to form a link in G, *i.e.*, $\psi(1,h)-\psi(0,h)<0$.

b. The proof of this part is similar to the previous proof, except that players in the dominant group in G and in H have formed g links in both networks and players who do not belong to the dominant group in G or in H have formed no links in the other network.

2. a. **Necessary condition.** Since $\min(\psi(g,h)-\psi(0,h),\psi(g,h)-\psi(g,0))\geq 0$, players who have formed links in G and H have no incentive to remove links in one of these networks because of the convexity of ψ by using the same arguments as in point 1. Moreover, we have $\psi(g,h)-\psi(0,0)=\psi(g,h)-\psi(0,h)+\psi(0,h)-\psi(0,0)\geq 0$. Indeed, $\psi(g,h)-\psi(0,h)\geq 0$, and by submodularity of ψ, $\psi(0,h)-\psi(0,0)\geq \psi(g,h)-\psi(g,0)\geq 0$. Hence players who have formed links in G and H have no incentive to remove links in both networks. Players who have formed links in G and no links in H have no incentive to remove links since $\psi(g,0)-\psi(0,0)\geq \psi(g,h)-\psi(0,h)$ by submodularity of ψ. By symmetry of ψ, players who have formed no links in G and links in H have no incentive to remove links. We now establish that no player who has formed no links in at least one network has an incentive to add links. Consider successively players who have formed no links in G and links in H, and players who have formed no links in H and links in G. By submodularity of ψ, we have for the former $\psi(1,h)-\psi(0,h)\leq\psi(1,0)-\psi(0,0)<0$, and for the latter $\psi(g,1)-\psi(g,0)=\psi(1,g)-\psi(0,g)\leq\psi(1,0)-\psi(0,0)<0$ where the equality follows from the symmetry of ψ. Moreover, players who have formed no links in G and in H have no incentive to form a link in G or in H since $\psi(0,1)-\psi(0,0)=\psi(1,0)-\psi(0,0)<0$. We conclude that M is a PEML network.

   Sufficient condition. Suppose that M is an MLPE network. Players who have formed links in both networks do not have an incentive to remove their links in one of the two networks: $\psi(g,h)-\psi(g,0)\geq 0$ and $\psi(g,h)-\psi(0,h)\geq 0$. Finally, players who have formed no links in any network have no incentive to form a link in G or H, *i.e.*, $\psi(0,1)-\psi(0,0)=\psi(1,0)-\psi(0,0)<0$.
   b. The proof of the part b is similar to the previous proof, except that players in the dominant group in G (H) have formed no links in H (G).

□

Proof of Theorem 2 We prove successively the two parts of the theorem.

1. To introduce a contradiction, suppose that there exists an MLPE network, say (G,H) where player i is such that $i\in\mathcal{D}^G$, $i\notin\mathcal{D}^H$. We assume wlog that $|\mathcal{D}^G|\geq k+1$. By Theorem 1, player i has formed $|\mathcal{D}^G|-1$ links in G. It follows that $g^i\geq k$. Moreover, by Theorem 1, $i\notin\mathcal{D}^H$ only if $h^i=0$. We have $\psi_H\left(g^i,1\right)=$

$\psi_G\left(1, g^i\right) \geq \psi_G\left(1, k\right) \geq 0$. Equality follows from the symmetry of ψ, the first inequality follows from supermodularity of ψ, and the last inequality from the assumption in the statement of the proposition. Then, player i has an incentive to form a link in H. Moreover, by inspecting the proof of Theorem 1, we know that every player in $\mathcal{D}^H$ has an incentive to form a link with i, a contradiction. Therefore, we obtain $\mathcal{D}^G \subseteq \mathcal{D}^H$. If $\mathcal{D}^G \subseteq \mathcal{D}^H$, then $|\mathcal{D}^H| \geq |\mathcal{D}^G| \geq k+1$. By using the same arguments, we obtain $\mathcal{D}^H \subseteq \mathcal{D}^G$. It follows that $\mathcal{D}^H = \mathcal{D}^G$.

2. To introduce a contradiction, suppose wlog that there exists an MLPE network, say (G, H), such that $|\mathcal{D}^H| \geq k+1$, and $i \in \mathcal{D}^G \cap \mathcal{D}^H$. We have $\psi_G(g^i, h^i) \geq 0$ for $g^i, h^i > 0$. Player i has formed $|\mathcal{D}^H| - 1$ links in H by Theorem 1. Hence, $\psi_G(g^i, |\mathcal{D}^H| - 1) \geq 0$. We have $0 > \psi_G(n-1, k) \geq \psi_G(n-1, |\mathcal{D}^H| - 1) \geq \psi_G(g^i, |\mathcal{D}^H| - 1) \geq 0$. The first inequality follows from the assumption in the statement of the proposition, the second inequality from the fact that $|\mathcal{D}^H| - 1 \geq k$ and ψ is submodular, and the third inequality from the fact that ψ is convex. We obtain a contradiction. Therefore, we have $\mathcal{D}^G \cap \mathcal{D}^H = \emptyset$.

$\square$

Proof of Corollary 1 We show the two parts of the result successively. First, we have $\sum_{\ell=0}^{k-1} \psi_{GH}(1, k-\ell) = \psi(1,k) - \psi(0,k) - (\psi(1,0) - \psi(0,0))$ and $\sum_{\ell=0}^{k-2} \psi_{GG}(k-\ell, 0) = \psi(k,0) - \psi(k-1,0) - (\psi(1,0) - \psi(0,0))$. Since $\sum_{\ell=0}^{k-1} \psi_{GH}(1, k-\ell) \geq \sum_{\ell=0}^{k-2} \psi_{GG}(k-\ell, 0)$, we have $\psi(1,k) - \psi(0,k) \geq \psi(k,0) - \psi(k-1,0)$. Therefore, $\psi(1,k) - \psi(0,k) \geq 0$, and by Theorem 2.1, we obtain the result.
Second, we have $\sum_{\ell=1}^{k} |\psi_{GH}(n-1-k, \ell)| = \psi(n-1-k, 0) - \psi(n-2-k, 0) - (\psi(n-1-k, k) - \psi(n-2-k, k))$ and $\sum_{\ell=0}^{k-1} \psi_{GG}(n-1-\ell, k) = \psi(n-1, k) - \psi(n-2, k) - (\psi(n-1-k, k) - \psi(n-2-k, k))$. Since $\sum_{\ell=1}^{k} |\psi_{GH}(n-1-k, \ell)| \geq \sum_{\ell=0}^{k-1} \psi_{GG}(n-1-\ell, k)$, we have $\psi(n-1-k, 0) - \psi(n-2-k, 0) \geq \psi(n-1, k) - \psi(n-2, k)$. Therefore, $\psi(n-1,k) - \psi(n-2,k) < 0$, and by Theorem 2.2, we obtain the result. $\square$

Proof of Lemma 1 Let $M = (G, H)$ be an efficient ML network. Let $(g^i, h^i) \in [\![0, n-1]\!]^2$ be such that $\psi(g^i, h^i) \geq \psi(\bar{g}^i, \bar{h}^i)$ for all $(\bar{g}^i, \bar{h}^i) \in [\![0, n-1]\!]^2$. To introduce a contradiction, we suppose wlog that $g^i > h^i$. Then, we have $\psi(g^i, g^i) - \psi(g^i, h^i) = \psi(g^i, g^i) - \psi(h^i, g^i) > \psi(g^i, h^i) - \psi(h^i, h^i) \geq 0$, a contradiction. The equality follows from the symmetry of ψ, the first strict inequality follows from the strict supermodularity of ψ, and the second inequality follows from the assumption that $\psi(g^i, h^i)$ is maximal. The result follows from the fact that $\max_{(g^i,h^i)}\{\sum_{i \in N} \psi(g^i, h^i)\} = \sum_{i \in N} \max_{(g^i,h^i)}\{\psi(g^i, h^i)\}$.

$\square$

Proof of Proposition 1 Let $M = (G, H)$ be an efficient ML network. To introduce a contradiction, suppose that $g^i \in [\![1, n-2]\!]$ satisfies $\psi(g^i, h^i) \geq \psi(\bar{g}^i, h^i)$ for all $\bar{g}^i \in [\![0, n-1]\!]$. Then, $\psi(g^i+1, h^i) - \psi(g^i, h^i) > \psi(g^i, h^i) - \psi(g^i-1, h^i) \geq 0$, a contradiction. The first inequality follows from the strict convexity of ψ in its first argument, and the second one follows from the assumption that $\psi(g^i, h^i)$ is maximal. Consequently, we have $g^i \in \{0, n-1\}$. By using the same argument for h^i, we conclude that the candidates for being the maximum of ψ are $(0,0)$, $(0, n-1)$, $(n-1, 0)$, $(n-1, n-1)$. The result follows from the fact that $\max_{(g^i,h^i)}\{\sum_{i\in N} \psi(g^i, h^i)\} = \sum_{i\in N} \max_{(g^i,h^i)}\{\psi(g^i, h^i)\}$. Suppose that ψ is in addition strictly supermodular. Then, we know, by Lemma 1, that $g^i = h^i$ for every i.

$\square$

Appendix A2. ML Externality Model

For the next two results, we introduce $\eta(x) = \frac{(x-1)x}{2}$ to simplify the presentation.

Proof of Theorem 3

[Existence part.] If $t\psi(0,0) + (1-t)\theta(0,0) > t\psi(n-1,0) + (1-t)\theta(n-1,0)$, then $M = (G, H)$, where G and H are empty, is an MLPE. Indeed, $t\psi(0,0) + (1-t)\theta(0,0) > t\psi(1,0) + (1-t)\theta(1,0)$ since $t\psi(\cdot,0) + (1-t)\theta(\cdot,0)$ is convex as the sum of convex functions, and $\max\{t\psi(\cdot,0) + (1-t)\theta(\cdot,0)\} \in \{0, n-1\}$. Suppose that $t\psi(n-1,0) + (1-t)\theta(n-1,0) \geq t\psi(0,0) + (1-t)\theta(0,0)$ and let $\xi = \eta(n-1)$. Then, by strategic complementarity of θ, $\Delta_1 = t\psi(n-1,0) + (1-t)\theta(n-1,\xi) - (t\psi(0,0) + (1-t)\theta(0,\xi)) \geq t\psi(n-1,0) + (1-t)\theta(n-1,0) - (t\psi(0,0) + (1-t)\theta(0,0))$, and no player has an incentive to remove links in G when G is the complete network and H is the empty network. There are two possibilities. (a) Suppose that $t\psi(n-1,n-1) + (1-t)\theta(n-1,0) < t\psi(n-1,0) + (1-t)\theta(0,0)$ (this is not possible if ψ is supermodular). Then, $M = (G, H)$, where G is complete and H is empty, is an MLPE. Indeed, $t\psi(n-1,0) + (1-t)\theta(0,0) > t\psi(n-1,1) + (1-t)\theta(1,0)$ since $\max\{t\psi(n-1,\cdot) + (1-t)\theta(\cdot,0)\} \in \{0, n-1\}$ by convexity of $t\psi(n-1,\cdot) + (1-t)\theta(\cdot,0)$. (b) Suppose that $t\psi(n-1,n-1) + (1-t)\theta(n-1,0) \geq t\psi(n-1,0) + (1-t)\theta(0,0)$. By strategic complementarity of θ, $\Delta_2 = t\psi(n-1,n-1) + (1-t)\theta(n-1,\xi) - (t\psi(n-1,0) + (1-t)\theta(0,\xi)) \geq t\psi(n-1,n-1) + (1-t)\theta(n-1,0) - (t\psi(n-1,0) + (1-t)\theta(0,0))$. We show that $M = (G, H)$, where G and H are complete, is an MLPE. First, no player has an incentive to remove links in one network $[t\psi(n-1,n-1) + (1-t)\theta(n-1,\xi) \geq t\psi(n-1,0) + (1-t)\theta(0,\xi)]$. Second, no player has an incentive to remove links in both networks since Δ_1, $\Delta_2 \geq 0$, *i.e.*, $t\psi(n-1,n-1) + 2(1-t)\theta(n-1,\xi) - (t\psi(0,0) + 2(1-t)\theta(0,\xi)) \geq 0$.

Characterization part. To introduce a contradiction consider an MLPE network (G, H) such that there exist two unlinked players i and j, with $g^i, g^j \in [\![1, n-2]\!]$. Since ψ satisfies convexity in g^i, we have: $\psi(g^i+1, h^i) - \psi(g^i, h^i) \geq \psi(g^i, h^i) - \psi(g^i-1, h^i)$. Likewise, since θ satisfies convexity in g^i, we have: $\theta(g^i+1, G^{-i}) - \theta(g^i, G^{-i}) \geq \theta(g^i, G^{-i}) - \theta(g^i-1, G^{-i})$. It follows that we have $\mathcal{U}^i(g^i+1, h^i, G^{-i}, H^{-i}) \geq \mathcal{U}^i(g^i, h^i, G^{-i}, H^{-i})$, and by similar arguments $\mathcal{U}^j(g^j+1, h^i, G^{-j}, H^{-j}) \geq \mathcal{U}^j(g^j, h^j, G^{-j}, H^{-j})$, a contradiction. We obtain the same contradiction when $h^i, h^j \in [\![1, n-2]\!]$. The result follows.

Parts 1 and 2 of the theorem.

1. To introduce a contradiction, suppose that there exists an MLPE network, say (G, H), where a player i is such that $i \in \mathcal{D}^G$ and $i \notin \mathcal{D}^H$. We also suppose wlog that $|\mathcal{D}^G| \geq k+1$. Player i has formed $|\mathcal{D}^G| - 1$ links in G by Theorem 3. It follows that $g^i \geq k$. By inspecting the proof of Proposition 2, we know that $\psi_H\left(g^i, 1\right) > 0$. Moreover, we have $\theta_1(1, H^{-i}) \geq \theta_1(1, 0) > 0$. The first inequality follows from the strict strategic complementarity of θ. The second inequality follows from the assumption in the statement of the proposition. Consequently, $t\psi_H\left(g^i, 1\right) + (1-t)\theta_1(1, H^{-i}) > 0$, and player i has an incentive to form a link in H. By the characterization part, every player in $\mathcal{D}^H$ has an incentive to form a link with i, a contradiction. Therefore, we obtain $\mathcal{D}^G \subseteq \mathcal{D}^H$. If $\mathcal{D}^G \subseteq \mathcal{D}^H$, then $|\mathcal{D}^H| \geq |\mathcal{D}^G| \geq k+1$. By using the same arguments, we obtain $\mathcal{D}^H \subseteq \mathcal{D}^G$. It follows that $\mathcal{D}^H = \mathcal{D}^G$.

2. To introduce a contradiction, suppose that there exists an MLPE network, say (G, H), where a player i is such that $i \in \mathcal{D}^G \cap \mathcal{D}^H$. Then, $g^i, h^i > 0$. We have $0 > t\psi_G(n-1, 1) + (1-t)\theta_1\left(n-1, \frac{(n-1)(n-2)}{2}\right) \geq t\psi_G(n-1, 1) + (1-t)\theta_1(n-1, G^{-i}) \geq t\psi_G(n-1, h^i) + (1-t)\theta_1\left(n-1, G^{-i}\right) \geq t\psi_G(g^i, h^i) + (1-t)\theta_1(g^i, G^{-i}) \geq 0$. The first inequality follows from the assumption in the statement of the proposition. The second inequality follows from strict strategic complement of θ. The third inequality follows from submodularity of ψ. The fourth inequality follows from convexity of ψ and θ. The last inequality follows from the fact that (G, H) is an MLPE. We obtain a contradiction. The result follows.

$\square$

To simplify the presentation of the next theorem, we let $t = 1/2$.

Theorem 3' *Suppose ψ and θ satisfy convexity, ψ is supermodular or submodular, and θ satisfies strict strategic complementarity. Moreover, suppose that $M = (G, H)$, where G and H are group dominant networks of size $g+1$ and $h+1$ respectively with*

$h \geq g$, $g < n-1$, and $h > 0$.

1. *Let ψ be supermodular.*
   (a) *When $\mathcal{D}^G \setminus \mathcal{D}^H \neq \emptyset$ and $\mathcal{D}^H \setminus \mathcal{D}^G \neq \emptyset$, M is an MLPE if and only if (i) $\psi(g,0)+\theta(g,\eta(g)) \geq \psi(0,0)+\theta(0,\eta(g))$, (ii) $\psi(g,h)+\theta(g,\eta(g))+\theta(h,\eta(h)) \geq \psi(0,0)+\theta(0,\eta(g))+\theta(0,\eta(h))$, and (iii) $\max\{\psi_G(1,h)+\theta_1(1,\eta(g+1)), \psi_H(g,1)+\theta_1(1,\eta(h+1))\} < 0$.*
   (b) *When $\mathcal{D}^G = \mathcal{D}^H$, M is a MLPE if and only if $\psi(g,g)+\theta(g,\eta(g)) \geq \psi(0,g)+\theta(0,\eta(g))$, $\psi(g,g)+2\theta(g,\eta(g)) \geq \psi(0,0)+2\theta(0,\eta(g))$, and $\psi_G(1,0)+\theta_1(1,\eta(g+1)) < 0$.*
2. *Let ψ be submodular.*
   (a) *When $\mathcal{D}^G \cap \mathcal{D}^H \neq \emptyset$, and $N \setminus (\mathcal{D}^G \cup \mathcal{D}^H) \neq \emptyset$, M is an MLPE if and only if $\min\{\psi(g,h)+\theta(g,\eta(g)) - [\psi(0,h)+\theta(0,\eta(g))], \psi(g,h)+\theta(h,\eta(h))-[\psi(g,0)+\theta(0,\eta(h))]\} \geq 0$, and $\psi_H(0,1)+\theta_1(1,\eta(h+1)) < 0$.*
   (b) *When $\mathcal{D}^G = N \setminus \mathcal{D}^H$, M is an MLPE if and only if $\psi(g,0)+\theta(g,\eta(g)) \geq \psi(0,0)+\theta(0,\eta(g))$, and $\psi_H(g,1)+\theta_1(1,\eta(h+1)) < 0$.*

Proof Since the proofs of the two parts of Theorem 3' use similar arguments, we only present the proof of part 1. Note that in part 2, players who have no incentive to remove links in each network have no incentive to remove links in both networks since by submodularity $\psi(g,h)-\psi(g,0) < \psi(0,h)-\psi(0,0)$.

(Part a.) **Necessary condition.** Suppose that $\psi(g,0)+(\theta(g,\eta(g)) - [\psi(0,0)+\theta(0,\eta(g))] \geq 0$. Let $\Delta_\ell = \psi(g-\ell,0)+\theta(g-\ell,\eta(g))-[\psi(g-(\ell+1),0)+\theta(g-(\ell+1),\eta(g))]$. For $k \in [\![1,g]\!]$, we establish that $\psi(g,0)+\theta(g,\eta(g))-[\psi(g-k,0)+\theta(g-k,\eta(g))] \geq 0$. To introduce a contradiction suppose that there is $k \in [\![1,g]\!]$ such that $\psi(g,0)+\theta(g,\eta(g))-[\psi(g-k,0)+\theta(g-k,\eta(g))] < 0$. We have $\psi(g,0)+\theta(g,\eta(g))-[\psi(g-k,0)+\theta(g-k,\eta(g))] = \sum_{\ell=0}^{k-1}\Delta_\ell$. Since Δ_ℓ is decreasing in ℓ by the convexity of ψ and θ, Δ_{k-1} is the lowest term in the sum $\sum_{\ell=0}^{k-1}\Delta_\ell$. Since $\sum_{\ell=0}^{k-1}\Delta_\ell < 0$, it follows that $\Delta_{k-1} < 0$. Note that since Δ_ℓ is decreasing in ℓ and $\Delta_{k-1} < 0$, we have $\sum_{\ell=k}^{g-1}\Delta_\ell < 0$. It follows that $\psi(g,0)+\theta(g,\eta(g))-[\psi(0,0)+\theta(0,\eta(g))] = \sum_{\ell=0}^{g-1}\Delta_\ell < 0$, a contradiction. Consequently, player i who has formed no links in H has no incentive to remove links in G. Moreover, by using the same argument as in the previous point, by convexity of ψ and θ, symmetry of ψ and strategic complementarity of θ, and since $h \geq g$, we have $\psi(0,h)+\theta(h,\eta(h))-[\psi(0,0)+\theta(0,\eta(h))] = \psi(h,0)+\theta(h,\eta(h))-(\psi(0,0)+\theta(0,\eta(h))) \geq \psi(g,0)+\theta(g,\eta(g))-[\psi(0,0)+\theta(0,\eta(g))]$. Consequently, player i who has formed no link in G has no incentive to remove links in H. Furthermore, by supermodularity of

ψ, we have $\psi(g,h)+\theta(g,\eta(g))-[\psi(0,h)+\theta(0,\eta(g))] \geq \psi(g,0)+\theta(g,\eta(g))-[\psi(0,0)+\theta(0,\eta(g))]$, and player i who has formed h links in H, has no incentive to remove links in G. Similarly, player i who has formed g links in G has no incentive to remove links in H. Moreover, by (ii) player i who has formed g links in G and h links in H has no incentive to remove links in both networks. Let us establish that players who have no links in one network have no incentive to form a link in this network. Players who have formed no links in G (H) and has formed h (g) links in H (G) have no incentive to form a link in G (H) since by (iii) $\psi_1(1,h)+\theta_G(1,\eta(g+1)) < 0$ and $\psi_H(g,1)+\theta_1(1,\eta(h+1)) < 0$. Finally, players who have formed no links in G and H have no incentive to form a link in one network by using the same argument: $\psi_G(1,0)+\theta_1(1,\eta(g+1)) \leq \psi_G(1,g)+\theta_1(1,\eta(g+1))$ and $\psi_H(0,1)+\theta_1(1,\eta(h+1)) \leq \psi_H(g,1)+\theta_1(1,\eta(h+1))$ by supermodularity of ψ.

Sufficient condition. Let $M=(G,H)$ be an MLPE network. Then, players who have formed no links in H have no incentive to remove all their links in G, *i.e.*, $\psi(g,0)+\theta(g,\eta(g))-[\psi(0,0)+\theta(0,\eta(g))] \geq 0$. Moreover, players who belong to a group dominant network in both networks have no incentive to remove all their links, hence $\psi(g,h)+\theta(g,\eta(g))+\theta(h,\eta(h)) \geq \psi(0,0)+\theta(0,\eta(g))+\theta(0,\eta(h))$. Note that players who have formed links in g have an incentive to form an additional link in G by convexity of ψ and θ in their first argument. Consequently, players who have formed h links in H and no links in G should have no incentive to form a link in G, *i.e.*, $\psi_G(1,h)+\theta(1,\eta_1(g+1)) < 0$ and players who have formed g links in G and no links in H have no incentive to form a link in H, *i.e.*, $\psi_H(g,1)+\theta_1(1,\eta(h+1)) < 0$.

(Part b.) The proof of part b is similar to the previous proof, except that players in the dominant group in G and in H have formed g links in both networks and players who do not belong to the dominant group in G or in H have formed no links in both networks.

□

Proof of Proposition 2 We show only the first part of the proposition since the second part is established with similar arguments. By Theorem 3', we know that M is an MLPE if and only if $\psi(1,0)+\theta(1,\eta(g+1))-\psi(0,0)-\theta(0,\eta(g+1)) < 0$. Note that $\psi(1,0)+\theta(1,\eta(g+1))-\psi(0,0)-\theta(0,\eta(g+1))-(\psi(1,0)+\theta(1,0)-\psi(0,0)-\theta(0,0)) = \sum_{\ell=0}^{\eta(g+1)-1}\theta_{12}(1,\eta(g+1)-\ell)$. The result follows. □

Proof of Proposition 3 Wlog, we let $t=1/2$ to simplify the presentation. Let $M=(G,H)$ be a ML-efficient network. First, we establish that for every $i,j \in N$, (a) $g(i;j) \subseteq g(j;i)$ or $g(j;i) \subseteq g(i;j)$, and (b) $h(i;j) \subseteq h(j;i)$ or $h(j;i) \subseteq h(i;j)$. We restrict our attention to case (a) since cases (a) and (b) are symmetric. To introduce a contradiction, suppose that there are two players i and j such that $g(i;j) \nsubseteq g(j;i)$ and $g(j;i) \nsubseteq g(i;j)$. Then, there are two players, say i' and j' such that $i' \in g(i;j) \setminus g(j;i)$, and $j' \in g(j;i) \setminus g(i;j)$. We denote by S the set of players $\{i, i', j, j'\}$. Let G' be such that $E'_G = E_G \setminus \{ii', jj'\}$. Since $M = (G,H)$ is an efficient ML network, we have $W_{\psi+\theta}(G,H) - W_{\psi+\theta}(G',H) \geq 0$, i.e.,

$$\begin{array}{rl} & \sum_{\ell\in S}(\psi(g^\ell,h^\ell)+\theta(g^\ell,G^{-i})) + \sum_{\ell\in N\setminus S}(\psi(g^\ell,h^\ell)+\theta(g^\ell,G^{-i})) \\ \geq & \sum_{\ell\in S}(\psi(g^\ell-1,h^\ell)+\theta(g^\ell-1,G^{-\ell}-1)) + \sum_{\ell\in N\setminus S}(\psi(g^\ell,h^\ell)+\theta(g^\ell,G^{-\ell}-2)) \end{array}$$

Let $\bar{G}$ be such that $E(\bar{G}) = E_G \cup \{ij', ji'\}$. Then, we have

$$\begin{array}{rl} & \sum_{\ell\in S}\theta(\bar{g}^\ell,\bar{G}^{-\ell}) - \sum_{\ell\in S}\theta(g^\ell,G^{-\ell}) \\ = & \sum_{\ell\in S}\theta(g^\ell+1,G^{-\ell}+1) - \sum_{\ell\in S}\theta(g^\ell,G^{-\ell}) \\ = & \sum_{\ell\in S}\left[\theta(g^\ell+1,G^{-\ell}+1) - \theta(g^\ell,G^{-\ell}+1)\right] \\ & +\sum_{\ell\in S}\left[\theta(g^\ell,G^{-\ell}+1) - \theta(g^\ell,G^{-\ell})\right] \\ > & \sum_{\ell\in S}\left[\theta(g^\ell,G^{-\ell}+1) - \theta(g^\ell-1,G^{-\ell}+1)\right] \\ & +\sum_{\ell\in S}\left[\theta(g^\ell-1,G^{-\ell}+1) - \theta(g^\ell-1,G^{-\ell})\right] \\ > & \sum_{\ell\in S}\left[\theta(g^\ell,G^{-\ell}) - \theta(g^\ell-1,G^{-\ell})\right] \\ & +\sum_{\ell\in S}\left[\theta(g^\ell-1,G^{-\ell}) - \theta(g^\ell-1,G^{-\ell}-1)\right]. \end{array}$$

The first strict inequality follows from the strict convexity of θ. The second one follows from the strict strategic complementarity of θ and its strict convexity in its second argument. Moreover, by the strict convexity of ψ in its first argument, we have $\sum_{\ell\in S}\psi(g^\ell+1,h^\ell) - \sum_{\ell\in S}\psi(g^\ell,h^\ell) > \sum_{\ell\in S}\psi(g^\ell,h^\ell) - \sum_{\ell\in S}\psi(g^\ell-1,h^\ell)$. Finally, by the strict convexity of θ in its second argument, we have $\sum_{\ell\in N\setminus S}[\theta(g^\ell,G^{-\ell}+2) - \theta(g^\ell,G^{-\ell})] > \sum_{\ell\in N\setminus S}[\theta(g^\ell,G^{-\ell}) - \theta(g^\ell,G^{-\ell}-2)]$. We conclude that $W_{\psi+\theta}(\bar{G},H) - W_{\psi+\theta}(G,H) > W_{\psi+\theta}(G,H) - W_{\psi+\theta}(G',H) > 0$, a contradiction.

Suppose in addition that ψ is strictly supermodular. We start by showing that in an ML efficient network M, we have for every pair of players (i,j), $g(i;j) = g(j;i)$ and $h(i;j) = h(j;i)$. We know that for every two players i and j, we have $g(i;j) \subseteq g(j;i)$ or $g(j;i) \subseteq g(i;j)$ and $h(i;j) \subseteq h(j;i)$ or $h(j;i) \subseteq h(i;j)$. There are two possibilities. The neighborhood of one player, say i, contains the neighborhood of the other, say j, in both networks, or not. We examine these two possibilities successively.

1. Suppose that $g(i;j) \supseteq g(j;i)$ and $h(i;j) \supseteq h(j;i)$, where at least one of these order relation is strict, say the former one. Let $\Delta_G = g(i;j) \setminus g(j;i)$, and $\Delta_H = h(i;j) \setminus h(j;i)$, with $\delta_G = |\Delta_G|$, and $\delta_H = |\Delta_H|$. We denote by Λ_G (Λ_H) the set

of players who do not belong to $\Delta_G \cup \{i, j\}$ ($\Delta_H \cup \{i, j\}$). Consider network G' (H') identical to G (H) except that player j forms a link with each player in Δ_G (Δ_H), and network $\bar{G}$ ($\bar{H}$) identical to G (H) except that player i does not form links with players in Λ_G (Λ_H). Note that since $g^i > g^j$, we have $G^{-i} < G^{-j}$. Clearly, $\theta(g^j + \delta_G, G^{-j}) - \theta(g^j, G^{-j}) = \theta(g^i, G^{-j}) - \theta(g^i - \delta_G, G^{-j}) > \theta(g^i, G^{-i}) - \theta(g^i - \delta_G, G^{-i})$. The equality follows from $g^j + \delta_G = g^i$. The inequality follows from $G^{-i} < G^{-j}$ and the strict supermodularity of θ. Moreover, $\theta(g^i, G^{-i} + \delta_G) - \theta(g^i, G^{-i}) = \theta(g^i, G^{-j}) - \theta(g^i, G^{-j} - \delta_G) > \theta(g^j, G^{-j}) - \theta(g^j, G^{-j} - \delta_G)$. The equality follows from $G^{-i} + \delta_G = G^{-j}$. The inequality follows from the strict supermodularity of θ and $g^i > g^j$. Note that $\psi(g^j + \delta_G, h^j + \delta_H) - \psi(g^j, h^j) = \psi(g^i, h^i) - \psi(g^i - \delta_G, h^i - \delta_H)$.

We now establish that $\sum_{\ell \in \Delta_G \cap \Lambda_H} [\theta(g^\ell + 1, G^{-\ell} + \delta_G - 1) - \theta(g^\ell, G^{-\ell})] > \sum_{\ell \in \Delta_G \cap \Lambda_H} [\theta(g^\ell, G^{-\ell}) - \theta(g^\ell - 1, G^{-\ell} - (\delta_G - 1))]$. This inequality can be rewritten as follows.

$$\begin{aligned} & \textstyle\sum_{\ell \in \Delta_G \cap \Lambda_H} [\theta(g^\ell + 1, G^{-\ell} + \delta_G - 1) - \theta(g^\ell, G^{-\ell} + \delta_G - 1)] \\ & + \textstyle\sum_{\ell \in \Delta_G \cap \Lambda_H} [\theta(g^\ell, G^{-\ell} + \delta_G - 1) - \theta(g^\ell, G^{-\ell})] \\ > \; & \textstyle\sum_{\ell \in \Delta_G \cap \Lambda_H} [\theta(g^\ell, G^{-\ell}) - \theta(g^\ell - 1, G^{-\ell})] \\ & + \textstyle\sum_{\ell \in \Delta_G \cap \Lambda_H} [\theta(g^\ell - 1, G^{-\ell}) - \theta(g^\ell - 1, G^{-\ell} - (\delta_G - 1))]. \end{aligned}$$

This inequality holds because of the following arguments. First, note that for each $\ell \in \Delta_G \cap \Lambda_H$, $\theta(g^\ell + 1, G^{-\ell} + \delta_G - 1) - \theta(g^\ell, G^{-\ell} + \delta_G - 1) \geq \theta(g^\ell + 1, G^{-\ell}) - \theta(g^\ell, G^{-\ell}) > \theta(g^\ell, G^{-\ell}) - \theta(g^\ell - 1, G^{-\ell})$. The first inequality follows from the supermodularity of θ and $\delta_G \geq 1$. The second inequality follows from the strict convexity of θ in its first argument. Second, we have for each $\ell \in \Delta_G \cap \Lambda_H$, $\theta(g^\ell, G^{-\ell} + \delta_G - 1) - \theta(g^\ell, G^{-\ell}) > \theta(g^\ell - 1, G^{-\ell} + \delta_G - 1) - \theta(g^\ell - 1, G^{-\ell}) \geq \theta(g^\ell - 1, G^{-\ell}) - \theta(g^\ell - 1, G^{-\ell} - (\delta_G - 1))$. The first inequality follows from the strict strategic complementarity of θ and $\delta_G \geq 1$. The second inequality follows from the strict convexity of θ in its second argument. Moreover, by strict convexity of ψ in its first argument, for $\ell \in \Delta_G \cap \Lambda_H$, $\psi(g^\ell + 1, h^\ell) - \psi(g^\ell, h^\ell) > \psi(g^\ell, h^\ell) - \psi(g^\ell - 1, h^\ell)$.

Finally, from the strict convexity of θ in its second argument, we have $\sum_{\ell \in \Lambda_G} \theta(g^\ell, G^{-\ell} + \delta_G) - \theta(g^\ell, G^{-\ell}) > \sum_{\ell \in \Lambda_G} \theta(g^\ell, G^{-\ell}) - \theta(g^\ell, G^{-\ell} - \delta_G)$. All the results obtained in G are true for network H when player j forms links with players in $h(i; j) \setminus h(j; i)$. For players $\ell \in \Delta_{GH} = \Delta_G \cap \Delta_H$, we now establish that $\psi(g^\ell + 1, h^\ell + 1) - \psi(g^\ell, h^\ell) > \psi(g^\ell, h^\ell) - \psi(g^\ell - 1, h^\ell - 1)$. Let us rewrite the previous inequality as follows:

$$\psi(g^\ell+1,h^\ell+1)-\psi(g^\ell,h^\ell+1)+\psi(g^\ell,h^\ell+1)-\psi(g^\ell,h^\ell)$$
$$-(\psi(g^\ell,h^\ell)-\psi(g^\ell-1,h^\ell)+\psi(g^\ell-1,h^\ell)-\psi(g^\ell-1,h^\ell-1))>0.$$

First, we have $\psi(g^\ell+1,h^\ell+1)-\psi(g^\ell,h^\ell+1)>\psi(g^\ell,h^\ell+1)-\psi(g^\ell-1,h^\ell+1)$ $>\psi(g^\ell,h^\ell)-\psi(g^\ell-1,h^\ell)$. The first inequality follows from the strict convexity of ψ in its first argument, and the second one follows from the strict supermodularity of ψ. Second, by symmetric arguments, we obtain $\psi(g^\ell,h^\ell+1)-\psi(g^\ell,h^\ell)>$ $\psi(g^\ell-1,h^\ell)-\psi(g^\ell-1,h^\ell-1)$.

It follows that $W_{\psi+\theta}(G',H')-W_{\psi+\theta}(G,H)>W_{\psi+\theta}(G,H)-W_{\psi+\theta}(\bar{G},\bar{H})\geq 0$, a contradiction.

2. Suppose that $g(i;j)\supseteq g(j;i)$ and $h(i;j)\subseteq h(j;i)$, where at least one of these order relation is strict, say the former one. Obviously, the case where $g(i;j)\subseteq g(j;i)$ and $h(i;j)\supseteq h(j;i)$ is symmetric. Let $\Delta'_H=h(j;i)\setminus h(i;j)$, and $\delta'_H=|\Delta'_H|$. We consider network G' (H') identical to G (H) except that player j (i) forms links with players in Δ_G (Δ_H), and network $\bar{G}$ ($\bar{H}$) identical to G (H) except that player i (j) removes links with players in Δ_G (Δ_H). By using similar arguments as in point 2, we obtain that $W_{\psi+\theta}(G',H')-W_{\psi+\theta}(G,H)>W_{\psi+\theta}(G,H)-W_{\psi+\theta}(\bar{G},\bar{H})\geq 0$, a contradiction.

From points 1 and 2, we know that for each pair of players (i,j), we have $g(i;j)=g(j;i)$ and $h(i;j)=h(j;i)$. Hence, G and H are empty or complete.

We now establish that G and H are both empty or both complete. To introduce a contradiction, suppose that G is complete and H is empty. Then, we have for each player $\psi(n-1,n-1)+2\theta\left(n-1,\frac{(n-1)^2}{2}\right)\leq\psi(n-1,0)+\theta(0,0)+\theta\left(n-1,\frac{(n-1)^2}{2}\right)$ and $\psi(n-1,0)+\theta(0,0)+\theta\left(n-1,\frac{(n-1)^2}{2}\right)\geq\psi(0,0)+2\theta(0,0)$. It follows that $\psi(n-1,n-1)-\psi(n-1,0)\leq\psi(n-1,0)-\psi(0,0)=\psi(0,n-1)-\psi(0,0)$, which contradicts the strict supermodularity of ψ. The result follows. □

Appendix B. MLPE Networks in the ML Degree Model under Concavity

Proof of Theorem 4

1. **Existence of MLPE.** The arguments that show the existence of MLPE networks are similar to those given in Theorem 1.

Since $\psi(g,h) \in \max_{\ell \in [\![0,g]\!], \ell' \in [\![0,h]\!]}\{\psi(g-\ell, h-\ell')\}$, players do not have a strict incentive to remove links in G, H or in both networks. Moreover since $\psi(g,h) > \min\{\psi(g+1,h), \psi(g,h+1)\}$, no pair of players have an incentive to form an additional link in G or in H.

2. **Properties of MLPE.** Suppose that $M = (G,H)$ is an MLPE network. We successively show the two parts of the proposition.

1. It is sufficient to establish that all players in $N^1_{\text{sup}}(M) \cup N_g(M)$ are linked together in G since the arguments for showing $N^2_{sup}(M) \cup N_h(M)$ are similar.
   (a) Consider player $i \in N^1_{sup}(M)$, and $g^i < n-1$. Then, there is a player $\ell \in N$ such that $g^\ell \geq g^i + 1$, and $h^\ell \leq h^i$. We have $\psi(g^i+1,\ h^i) - \psi(g^i, h^i) \geq \psi(g^i+1, h^\ell) - \psi(g^i, h^\ell) \geq \psi(g^\ell, h^\ell) - \psi(g^\ell - 1, h^\ell) \geq 0$. The first inequality follows from $h^i \geq h^\ell$ and the supermodularity of ψ. The second inequality follows from $g^\ell \geq g^i + 1$ and the strict concavity in its first argument of ψ. The last inequality follows from the fact that $M = (G,\ H)$ is an MLPE network. Therefore, player i has an incentive to add a link in G.

   (b) Consider player $i \in N_g(M)$, and $g^i < n-1$. Then, we have $0 \leq \psi(g^i, h^i) - \psi(g^i, h^i - 1) = \psi(h^i, g^i) - \psi(h^i - 1, g^i) \leq \psi(g^i+1, g^i) - \psi(g^i, g^i) \leq \psi(g^i + 1, h^i) - \psi(g^i, h^i)$. The first inequality follows from the fact that M is an MLPE network. The equality follows from the symmetry of ψ. The second inequality follows from the concavity in its first argument of ψ and $g^i + 1 \leq h^i + 1$. The last inequality follows from the supermodularity of ψ and $g^i < h^i$. Therefore, player i has an incentive to add a link in G.

   The result follows from points (a) and (b).

2. Again, we only show the first part of point 2 of the proposition. We show that all players in $N^1_{\text{sub}}(M)$ are linked together. To introduce a contradiction, suppose players $i, j \in N^1_{\text{sub}}(M)$ are not linked in G. There is a player $\ell \in N$ such that $g^\ell > g^i$, $h^\ell \geq h^i$. Then, $\psi(g^i+1, h^i) - \psi(g^i, h^i) \geq \psi(g^i+1, h^\ell) - \psi(g^i, h^\ell) \geq \psi(g^\ell, h^\ell) - \psi(g^\ell - 1, h^\ell) \geq 0$. The first inequality follows from $h^i \geq h^\ell$ and the submodularity of ψ. The second inequality follows from $g^\ell \geq g^i + 1$ and the concavity of ψ. The last inequality follows from the fact that $M = (G,H)$ is an MLPE. Therefore, player i has an incentive to add a link in G. By using the same reasoning for player j, we conclude that players i and j are linked in G, a contradiction.

□

Proof of Proposition 4 Suppose that (a), (b) and (c) hold. Let $\Delta_\ell = \psi(\varphi(h^i) - \ell, h^i) - \psi(\varphi(h^i) - (\ell+1), h^i)$. First, player $i \notin (Q_G(M) \cup Q_H(M))$ satisfies $g^i = \varphi(h^i)$ and $h^i = \varphi(g^i)$, and $g^i + 1 \neq \varphi(h^i)$, $h^i + 1 \neq \varphi(g^i)$, by (a), (b), and (c). Hence, i has no incentive to add a link in G or H, and has no incentive to remove links in one network or simultaneously in both networks because of (P1). Second, consider player $i \in Q_G(M) \setminus Q_H(M)$. Note that $\psi(\varphi(h^i), h^i) \geq \psi(a, h^i)$ for every $a < \varphi(h^i)$. We have $\psi(\varphi(h^i), h^i) - \psi(\varphi(h^i) - a, h^i) = \sum_{\ell=0}^{a-1} \Delta_\ell$. Since ψ is concave, Δ_ℓ is increasing in ℓ, $\Delta_\ell \geq \Delta_0 \geq 0$. It follows that if player i has formed $g^i < \varphi(h^i)$ links, then i has an incentive to form an additional link in G. Since $i \notin Q_H(M)$, we have $h^i = \varphi(g^i)$, and $h^i + 1 \neq \varphi(g^i)$ by (a), hence player i has no incentive to form an additional link in H. Moreover, player i has no incentive to remove links in one network or simultaneously in both networks because of (P1). We obtain a similar result for $i \in Q_H(M) \setminus Q_G(M)$. Finally, player $i \in Q_G(M) \cap Q_H(M)$ has an incentive to form an additional link in both networks by using similar arguments. It follows that M is an MLPE.

Suppose $M = (G, H)$ is an MLPE. There is no player i such that $g^i > \varphi(h^i)$ or $h^i > \varphi(g^i)$, since player i would remove links in G or in H. Clearly, players in $Q_G(M)$ are linked together in G since each of them has an incentive to form an additional link in G. Likewise, players in $Q_H(M)$ are linked together in H. In particular, if $\hat{g} \in \arg\max_{g \in [\![0, n-2]\!]}\{\psi(g, h^i)\}$ and $\hat{g} < \varphi(h^i)$, then $\psi_G(\hat{g}+1, h^i) \geq 0$ since ψ is concave.

$\square$

The proof of the following lemma is straightforward and omitted.

Lemma 3 *Suppose that ψ is concave in each of its arguments and submodular. Let $|\psi_{GG}(x+1, y)| < |\psi_{GH}(x+1, y)|$. Then, $\psi(x+1, y-1) - \psi(x, y-1) > \psi(x, y) - \psi(x-1, y)$. Similarly, let $|\psi_{HH}(x, y+1)| < |\psi_{HG}(x, y+1)|$. Then, $\psi(x-1, y+1) - \psi(x-1, y) > \psi(x, y) - \psi(x, y-1)$.*

Proof of Proposition 5 Let $M = (G, H)$ be an equilibrium. Consider player $i \in N \setminus N^{\max}(M)$. Hence, there exists player j such that $g^j + h^j > g^i + h^i$. There are three possibilities.

1. First, $g^j \geq g^i$ and $h^j > h^i$ or $g^j > g^i$ and $h^j \geq h^i$. From Theorem 4.2, we know that player i has an incentive to form an additional link in G or in H.
2. Second, $g^j < g^i$ and $h^j > h^i$. Since $g^j + h^j > g^i + h^i$, we have $g^i - g^j + 1 \leq h^j - h^i$. We establish that player i has an incentive to form a link in G. Let $g^j > 0$. We

have

$$
\begin{aligned}
& \psi(g^i+1,h^i)-\psi(g^i,h^i) \\
= \;& \psi(g^i+1,h^j-(h^j-h^i))-\psi(g^i,h^j-(h^j-h^i)) \\
\geq \;& \psi(g^i+1,h^j-(g^i-g^j+1))-\psi(g^i,h^j-(g^i-g^j+1)) \\
= \;& \psi(g^j+1+g^i-g^j,h^j-(g^i-g^j)-1)-\psi(g^j+g^i-g^j,h^j-(g^i-g^j)-1) \\
> \;& \psi(g^j,h^j)-\psi(g^j-1,h^j)\geq 0.
\end{aligned}
$$

The first inequality follows from $g^i-g^j+1 \leq h^j-h^i$, and ψ is submodular. The second inequality follows from $\psi(x+1,y-1)-\psi(x,y-1) > \psi(x,y)-\psi(x-1,y)$. When $g^j>0$ does not hold, we use arguments similar to those of point 1. Second, suppose that $g^j=0$. Then, we extend $\psi(\cdot,\cdot)$ to $\psi(-1,\cdot)$ and preserve the strict concavity in its first argument. From the previous inequalities, we have $\psi(g^i+1,h^i)-\psi(g^i,h^i) > \psi(0,h^j)-\psi(-1,h^j)$. Moreover, we have $\psi(0,h^j)-\psi(-1,h^j) > \psi(1,h^j)-\psi(0,h^j) \geq \psi(1,n-1)-\psi(0,n-1) > 0$, the result follows.

3. Third, $g^j > g^i$ and $h^j < h^i$. Since $g^j+h^j > g^i+h^i$, we have $h^i-h^j+1 \leq g^j-g^i$. We establish that player i has an incentive to form a link in H by using similar arguments as in point 2.

Each player who is not in $N^{\max}(M)$ has an incentive to form an additional link in G or in H, the result follows. $\square$

Proof of Proposition 6 Since $[\![0,n-1]\!]^2$ is finite, $\Gamma = \arg\max_{(x,y)\in[\![0,n-1]\!]^2}\{\psi(x,y)\}$ is non-empty. Let $(g,h)\in\Gamma$. We have $\max\left\{\sum_{i\in N}\psi(g^i,h^i)\right\} = \sum_{i\in N}\max\{\psi(g^i,h^i\} = \sum_{i\in N}\psi(g,h)$. By Lemma 1, if ψ is supermodular, then $g=h$ in an efficient network. The result follows.

$\square$

Appendix C. MLPE Networks in the ML Externality Model under Concavity

Appendix C1. Theorem 5

Proof of Theorem 5

[Existence of MLPE]. Let $\iota(x) = \frac{(n-2)x}{2}$, and for $y\in[\![0,n-1]\!]$, $\varphi_y(x) = \max\{\arg\max_{a\in[\![0,n-1]\!]}\{t\psi(a,y)+(1-t)\theta(a,\iota(x))\}\}$, and $\Xi_y = \{x\in[\![0,n-1]\!] : \varphi_y(x)\geq x\}$. Clearly, Ξ_y is non-empty since $\varphi_y(0)\geq 0$, and finite. Consequently, Ξ_y admits a maximal element, say $\bar{x}_y$. By construction, $\varphi_y(\bar{x}_y) \geq \bar{x}_y$, and we establish that $\varphi_y(\bar{x}_y) = \bar{x}_y$. To introduce a contradiction suppose that $\xi = \varphi_y(\bar{x}_y) > \bar{x}_y$. Let us consider $\varphi_y(\xi)$. By strategic complementarity and $\iota(\xi) > \iota(\bar{x})$, for every $a < \xi$, $t\psi(\xi, y) + (1-t)\theta(\xi, \iota(\xi)) - (t\psi(a, y) + (1-t)\theta(a, \iota(\xi))) \geq t\psi(\xi, y) + (1-t)\theta(\xi, \iota(\bar{x}_y)) - (t\psi(a, y) + (1-t)\theta(a, \iota(\bar{x}_y))) \geq 0$. It follows that $\xi \in \Xi_y$, a contradiction. Consequently, for every $y \in [\![0, n-1]\!]$ – the number of links formed by each player in the regular network H, there exists a MLPE network G that is regular, and by symmetry of ψ, there also exists a MLPE network H that is regular for every $y \in [\![0, n-1]\!]$. Let $\hat{g}_y$ and $\hat{h}_x$ be the maximal element of Ξ_y and Ξ_x respectively, and suppose that ψ is submodular. We now show that $\hat{g}_y \geq \hat{g}_{y+1}$, i.e., $\hat{g}_y$ is decreasing in y. To introduce a contradiction, suppose that $\hat{g}_y < \hat{g}_{y+1}$. By construction, there is $a < \hat{g}_{y+1}$, $t\psi(a, y) + (1-t)\theta(a, \iota(\hat{g}_{y+1})) > t\psi(\hat{g}_{y+1}, y) + (1-t)\theta(\hat{g}_{y+1}, \iota(\hat{g}_{y+1}))$ and $t\psi(a, y+1) + (1-t)\theta(a, \iota(\hat{g}_{y+1})) \leq \psi(\hat{g}_{y+1}, y+1) + (1-t)\theta(\hat{g}_{y+1}, \iota(\hat{g}_{y+1}))$. Hence $\psi(\hat{g}_{y+1}, y) - \psi(a, y) < (\hat{g}_{y+1}, y+1) - \psi(a, y+1)$, contradicting the submodularity of ψ. By using the same arguments, we establish that $\hat{g}_y \leq \hat{g}_{y+1}$, i.e., $\hat{g}_y$ is increasing in y when ψ is supermodular. Finally, we use a process to establish the existence result, this process converges to an MLPE network where all players form the same number of links in each network.

1. Suppose that ψ satisfies submodularity. Let $((G_0, H_0), (G_1, H_0), (G_1, H_1), (G_2, H_1), (G_2, H_2), \ldots)$ be a sequence of ML networks, where $G_0, G_1, G_2, \ldots$, and $H_0, H_1, H_2, \ldots$ are regular networks. Also G_0 is the empty network and H_0 is the complete network. Moreover, let $h_k = \hat{h}_{g_k}$ and $g_k = \hat{g}_{h_{k-1}}$ for $k > 0$. We know that $g_{k+1} \geq g_k$ and $h_{k+1} \leq h_k$ for every $k \geq 0$. We establish that there always exists $k \geq 0$, such that $(G_k, H_k) = (G_{k+1}, H_{k+1})$. Indeed, because the number of links in which players are involved is bounded by 0 and $n-1$, $(g_\ell)_{\ell \in \mathbb{N}}$ is increasing and $(h_\ell)_{\ell \in \mathbb{N}}$ is decreasing, it is not possible that $g_{k+1} > g_k$ or $h_{k+1} < h_k$ for every $k \geq 0$. Let $M^\star = (G^\star, H^\star) = (G_k, H_k)$, with $(G_k, H_k) = (G_{k+1}, H_{k+1})$. Clearly, no pair of unlinked players has an incentive to form a link, and no player has an incentive to remove links in $G^\star$ or $H^\star$. Finally, no player has an incentive to remove any of her links in both networks simultaneously. Indeed, we have $t\psi(g^\star_{h^\star}, h^\star_{g^\star}) + (1-t)\theta(g^\star_{h^\star}, \iota(g^\star_{h^\star})) > t\psi(g^\star_{h^\star} - \ell, h^\star_{g^\star}) + (1-t)\theta(g^\star_{h^\star} - \ell, \iota(g^\star_{h^\star}))$, for $\ell \in [\![1, g^\star_{h^\star}]\!]$, and $t\psi(g^\star_{h^\star}, h^\star_{g^\star}) + (1-t)\theta(h^\star_{g^\star}, \iota(h^\star_{g^\star})) > t\psi(g^\star_{h^\star}, h^\star_{g^\star} - \ell') + (1-t)\theta(h^\star_{g^\star} - \ell', \iota(h^\star_{g^\star}))$, for $\ell' \in [\![1, h^\star_{g^\star}]\!]$. Moreover, $\psi(g^\star_{h^\star}, h^\star_{g^\star}) - \psi(g^\star_{h^\star} - \ell, h^\star_{g^\star} - \ell') = \psi(g^\star_{h^\star}, h^\star_{g^\star}) - \psi(g^\star_{h^\star} - \ell, h^\star_{g^\star}) + \psi(g^\star_{h^\star} - \ell, h^\star_{g^\star}) - \psi(g^\star_{h^\star} - \ell, h^\star_{g^\star} - \ell') \geq \psi(g^\star_{h^\star}, h^\star_{g^\star}) - \psi(g^\star_{h^\star} - \ell, h^\star_{g^\star}) + \psi(g^\star_{h^\star}, h^\star_{g^\star}) - \psi(g^\star_{h^\star}, h^\star_{g^\star} - \ell')$ since $\psi(g^\star_{h^\star} - \ell, h^\star_{g^\star}) - \psi(g^\star_{h^\star} -$

$\ell, h^\star_{g^\star} - \ell') \geq \psi(g^\star_{h^\star}, h^\star_{g^\star}) - \psi(g^\star_{h^\star}, h^\star_{g^\star} - \ell')$ by submodularity of ψ. It follows that $t\psi(g^\star_{h^\star}, h^\star_{g^\star}) + (1-t)\theta(g^\star_{h^\star}, \iota(g^\star_{h^\star})) - t\psi(g^\star_{h^\star} - \ell, h^\star_{g^\star} - \ell') - (1-t)(\theta(g^\star_{h^\star}, \iota(g^\star_{h^\star})) + \theta(h^\star_{g^\star} - \ell', \iota(h^\star_{g^\star}))) \geq 0$. Consequently, $M^\star$ is an MLPE network.

2. Suppose that ψ satisfies supermodularity. Let $((G_0, H_0), (G_1, H_0), (G_1, H_1), (G_1, H_2), \ldots)$ be a sequence of ML networks, where $G_0, G_1, G_2, \ldots$, and $H_0, H_1, H_2, \ldots$ are regular networks, and both G_0 and H_0 are empty networks. Moreover, $h_k = \hat{h}_{g_k}$ for $k \geq 0$ and $g_k = \hat{g}_{h_k}$ for $k > 0$. We know that $g_{k+1} \geq g_k$ and $h_{k+1} \geq h_k$ for every $k \geq 0$ since ψ is supermodular. We establish that there always exists $k \geq 0$, such that $(G_k, H_k) = (G_{k+1}, H_{k+1})$. Indeed, because the number of links in which players are involved is bounded by 0 and $n-1$, $(g_\ell)_{\ell \in \mathbb{N}}$ and $(h_\ell)_{\ell \in \mathbb{N}}$ are increasing, it is not possible that $g_{k+1} > g_k$ or $h_{k+1} > h_k$ for every $k \geq 0$. Let $M^\star = (G^\star, H^\star) = (G_k, H_k)$, with $(G_k, H_k) = (G_{k+1}, H_{k+1})$. Clearly, no pair of unlinked players has an incentive to form a link, and no player has an incentive to remove links in $G^\star$ or $H^\star$. Finally, no player has an incentive to remove any of her links in both networks simultaneously. Indeed, we have $\delta(g^\star_{h^\star}, \ell; h^\star_{g^\star}) > 0$, and by symmetry of ψ, $\delta(h^\star_{g^\star}, \ell'; g^\star_{h^\star}) > 0$. Due to (P2), we have $\delta(g^\star_{h^\star}, \ell; h^\star_{g^\star}) + \delta(h^\star_{g^\star}, \ell'; g^\star_{h^\star}) > t(\psi(g^\star_{h^\star}, h^\star_{g^\star}) - \psi(g^\star_{h^\star} - \ell, h^\star_{g^\star}) - (\psi(g^\star_{h^\star} - \ell, h^\star_{g^\star}) - \psi(g^\star_{h^\star} - \ell, h^\star_{g^\star} - \ell')))$. Consequently, $t\psi(g^\star_{h^\star}, h^\star_{g^\star}) + (1-t)(\theta(g^\star_{h^\star}, \iota(g^\star_{h^\star})) + \theta(h^\star_{g^\star}, \iota(h^\star_{g^\star}))) \geq t\psi(g^\star_{h^\star} - \ell, h^\star_{g^\star} - \ell') + (1-t)(\theta(g^\star_{h^\star}, \iota(g^\star_{h^\star})) + \theta(h^\star_{g^\star} - \ell', \iota(h^\star_{g^\star})))$. We conclude that $M^\star$ is an MLPE network.

[Characterization of MLPE]. Suppose that $M = (G, H)$ is an MLPE network. We successively show the two parts of the proposition.

1. We only establish that $N^1_{sup}(M) \in \mathcal{N}^c_G(M)$ since the arguments for showing $N^1_{sup}(M) \in \mathcal{N}^c_H(M)$ are similar. To introduce a contradiction, suppose that players i and j belong to $N^1_{sup}(M)$ and are not linked in G. Consider player i. There is player $\ell \in N$ such that $g^\ell \geq g^i + 1$ and $h^\ell \leq h^i$. Thus $G^{-\ell} \leq G^{-i} - 1$. Consequently, $\psi(g^i + 1, h^i) + \theta(g^i + 1, G^{-i}) - (\psi(g^i, h^i) + \theta(g^i, G^{-i})) \geq \psi(g^i + 1, h^\ell) - \psi(g^i, h^\ell) + \theta(g^i + 1, G^{-i}) - \theta(g^i, G^{-i}) \geq \psi(g^\ell, h^\ell) - \psi(g^\ell - 1, h^\ell) + \theta(g^\ell, G^{-i}) - \theta(g^\ell - 1, G^{-i}) > \psi(g^\ell, h^\ell) - \psi(g^\ell - 1, h^\ell) + \theta(g^\ell, G^{-\ell}) - \theta(g^\ell - 1, G^{-\ell}) \geq 0$. The first inequality follows from $h^i \geq h^\ell$ and the supermodularity of ψ. The second inequality follows from $g^\ell \geq g^i + 1$ and the concavity in the first argument of ψ and θ. The third inequality follows from $G^{-\ell} + 1 \leq G^{-i}$ and the strict strategic complement of θ. The last inequality follows from the fact that $M = (G, H)$ is an MLPE network. Therefore, player i has an incentive to add a link in G. By using the same arguments for player j, we conclude that players i and j are linked in

G, a contradiction.

2. The proof for the case where ψ is submodular is obtained by using arguments similar to those given in point 1.

$\square$

The proof of Lemma 4 is straightforward, and therefore omitted.

Appendix C2. Proposition 7

Lemma 4 *Suppose that ψ is strictly concave in each of its argument and θ is strictly concave and satisfies strict strategic complement.*

1. *Let ψ be strictly supermodular. Suppose that $\psi_{GH}(x+1,y+1)-\theta_{12}(x+1,X-x+1) > |\psi_{GG}(x+1,y)|+|\theta_{11}(x+1,X-x+1)|$ and $\psi_{HG}(x+1,y+1)-\theta_{12}(y+1,Y-y+1) > |\psi_{HH}(x,y+1)|+|\theta_{11}(y+1,Y-y+1)|$. Then,*

$$\begin{aligned}&\psi(x+1,y+1)-\psi(x,y+1)+\theta(x+1,X-x)-\theta(x,X-x)\\ >\ &\psi(x,y)-\psi(x-1,y)+\theta(x,X-x+1)-\theta(x-1,X-x+1),\ \textit{and}\end{aligned}$$

$$\begin{aligned}&\psi(x+1,y+1)-\psi(x+1,y)+\theta(y+1,Y-y)-\theta(y,Y-y)\\ >\ &\psi(x,y)-\psi(x,y-1)+\theta(y,Y-y+1)-\theta(y-1,Y-y+1).\end{aligned}$$

2. *Let ψ be strictly submodular. Suppose that $|\psi_{GH}(x+1,y)|-\theta_{12}(x+1,X-x+1) > |\psi_{GG}(x+1,y)|+|\theta_{11}(x+1,X-x+1)|$ and $|\psi_{GH}(x,y+1)|-\theta_{12}(y+1,Y-y+1) > |\psi_{HH}(x,y+1)|+|\theta_{11}(y+1,Y-y+1)|$. Then,*

$$\begin{aligned}&\psi(x+1,y-1)-\psi(x,y-1)+\theta(x+1,X-x)-\theta(x,X-x)\\ >\ &\psi(x,y)-\psi(x-1,y)+\theta(x,X-x+1)-\theta(x-1,X-x+1),\ \textit{and}\end{aligned}$$

$$\begin{aligned}&\psi(x-1,y+1)-\psi(x-1,y)+\theta(y+1,Y-y)-\theta(y,Y-y)\\ >\ &\psi(x,y)-\psi(x,y-1)+\theta(y,Y-y+1)-\theta(y-1,Y-y+1).\end{aligned}$$

Proof of Proposition 7 We prove successively the two parts of the proposition. Let $M=(G,H)$ be an equilibrium.

A. Suppose that ψ is supermodular. Consider player $i \notin N^{\min}(M)$. Hence, there exists player j such that $g^j+h^j < g^i+h^i$. There are four possibilities. When $g^i < g^j$ and $h^i > h^j$ or $g^i > g^j$ and $h^i < h^j$, we know by Theorem 5.2.(a) that player i has an incentive to add a link in G or in H. The last two possibilities are symmetric: $g^i \geq g^j$

and $h^i > h^j$ or $g^i > g^j$ and $h^i \geq h^j$. Because of the symmetry, we restrict our attention to the case $g^i \geq g^j$ and $h^i > h^j$. We divide our analysis into three points.

1. Suppose that $h^i - h^j > g^i - g^j$. We have $h^i - h^j \geq g^i - g^j + 1 = \delta + 1$ First, we assume that $g^j > 0$, we have

$$\begin{aligned}
& \psi(g^i+1,h^i) + \theta(g^i+1,G^{-i}) - (\psi(g^i,h^i) + \theta(g^i,G^{-i})) \\
= \; & \psi(g^i+1,h^j+(h^i-h^j)) + \theta(g^i+1,G^{-i}) - \psi(g^i,h^j+(h^i-h^j)) - \theta(g^i,G^{-i}) \\
\geq \; & \psi(g^i+1,h^j+\delta+1) + \theta(g^i+1,G^{-i}) - \psi(g^i,h^j+\delta+1) - \theta(g^i,G^{-i}) \\
= \; & \psi(g^j+1+\delta,h^j+\delta+1) + \theta(g^j+1+\delta,G^{-j}-\delta) \\
& -\psi(g^j+\delta,h^j+\delta+1) - \theta(g^j+\delta,G^{-j}-\delta) \\
\geq \; & \psi(g^j,h^j) + \theta(g^j,G^{-j}+1) - \psi(g^j-1,h^j) - \theta(g^j-1,G^{-j}+1) \\
\geq \; & \psi(g^j,h^j) + \theta(g^j,G^{-j}) - \psi(g^j-1,h^j) - \theta(g^j-1,G^{-j}) \geq 0.
\end{aligned}$$

The first inequality follows from the fact that $h^i - h^j \geq g^i - g^j + 1$ and ψ is strictly supermodular. The second inequality follows from $g^i \geq g^j$ and Lemma 1, i.e., $\psi(x+1,y+1) - \psi(x,y+1) + \theta(x+1,X-x) - \theta(x,X-x) > \psi(x,y) - \psi(x-1,y) + \theta(x,X-x+1) - \theta(x-1,X-x+1)$. The third inequality follows from the strategic complement property of θ. The last inequality follows from the fact $M = (G,H)$ is an MLPE network.

Second, we assume that $g^i > g^j = 0$. We extend $\psi(\cdot,h^j) + \theta(\cdot,\cdot)$ to $\psi(-1,h^j) + \theta(-1,X)$, with $X \geq 0$, and preserve the concavity of ψ and θ in their first argument. Because of the previous inequalities, we have $\psi(g^i+1,h^i) + \theta(g^i+1,G^{-i}) - (\psi(g^i,h^i) + \theta(g^i,G^{-i})) \geq \psi(0,h^j) + \theta(0,G^{-i}+g^i) - (\psi(-1,h^j) + \theta(-1,G^{-i}+g^i))$. We have

$$\begin{aligned}
& \psi(0,h^j) + \theta(0,X) - (\psi(-1,h^j) + \theta(-1,X)) \\
\geq \; & \psi(1,h^j) + \theta(1,X) - (\psi(0,h^j) + \theta(0,X)) \\
\geq \; & \psi(1,0) + \theta(1,X) - (\psi(0,0) + \theta(0,X)) \\
\geq \; & \psi(1,0) + \theta(1,0) - (\psi(0,0) + \theta(0,0)) \geq 0.
\end{aligned}$$

The first inequality comes from $\psi(0,h^j) - \psi(-1,h^j) \geq \psi(1,h^j) - \psi(0,h^j)$ by concavity of ψ and θ in their first argument. The second inequality follows from supermodularity of ψ. The third inequality follows from the strategic complement of θ. The last inequality follows from our assumption.

Third, we assume that $g^i = g^j = 0$. Then, $\psi(1,h^i) + \theta(1,X) - (\psi(0,h^i) + \theta(0,X)) \geq \psi(1,0) + \theta(1,X) - (\psi(0,0) + \theta(0,X)) \geq \psi(1,0) + \theta(1,0) - (\psi(0,0) + \theta(0,0)) > 0$. The first inequality follows from the fact that ψ is supermodular.

The second inequality follows from the strategic complement of θ. The last inequality follows from our assumption. We conclude that player i has an incentive to form an additional link in G.

2. Suppose that $h^i - h^j < g^i - g^j$. Then, we use the same arguments as in point 1 to conclude that player i has an incentive to form an additional link in H.

3. Suppose $h^i - h^j = g^i - g^j$. Moreover, suppose that there exists $j, j' \in N^{\min}(M)$ such that $g^j - g^{j'} \neq h^{j'} - h^j$. In that case, we also have $h^i - h^{j'} \neq g^i - g^{j'}$. Consequently, by points 1 and 2, all players in $N \setminus N^{\min}(M)$ have an incentive to form an additional link in G or in H by points 1 and 2. Consequently, if there exists player $i \notin N^{\min}(M)$ who does not have an incentive to form an additional link in G or in H, then $g^j - g^{j'} = h^{j'} - h^j$, for $j, j' \in N^{\min}(M)$. It follows that $i \in D_a(M)$, with $a = g^j - h^j$.

B. Suppose that ψ is submodular. Let $i \in N \setminus N^{\max}(M)$. Then, there is player j such that $g^j + h^j > g^i + h^i$. There are three possibilities.

1. First, $g^j \geq g^i$ and $h^j > h^i$ or $g^j > g^i$ and $h^j \geq h^i$. From Theorem 5.2, we know that player i has an incentive to form an additional link in G or in H.

2. Second, $g^j < g^i$ and $h^j > h^i$. Let $g^i = g^j + \delta$, $\delta \geq 1$. Since $g^j + h^j > g^i + h^i$, we have $g^i - g^j + 1 = \delta + 1 \leq h^j - h^i$. We establish that player i has an incentive to form an additional link in G. Let $g^j > 0$. We have

$$\begin{aligned}
& \psi(g^i+1, h^i) + \theta(g^i+1, G^{-i}) - (\psi(g^i, h^i) + \theta(g^i, G^{-i})) \\
= \ & \psi(g^i+1, h^j - (h^j - h^i)) + \theta(g^i+1, G^{-i}) - (\psi(g^i, h^j - (h^j - h^i)) + \theta(g^i, G^{-i})) \\
\geq \ & \psi(g^i+1, h^j - (\delta+1) + \theta(g^i+1, G^{-i}) - (\psi(g^i, h^j - (\delta+1) + \theta(g^i, G^{-i})) \\
= \ & \psi(g^j+1+\delta, h^j - (\delta+1)) + \theta(g^j+1+\delta, G^{-j} - \delta) \\
& -\psi(g^j+\delta, h^j - (\delta+1)) - \theta(g^j+\delta, G^{-j} - \delta) \\
\geq \ & \psi(g^j, h^j) + \theta(g^j, G^{-j}+1) - (\psi(g^j-1, h^j) + \theta(g^j-1, G^{-j}+1)) \\
\geq \ & \psi(g^j, h^j) + \theta(g^j, G^{-j}) - (\psi(g^j-1, h^j) + \theta(g^j-1, G^{-j})) \geq 0.
\end{aligned}$$

The first inequality follows from the fact that ψ is submodular and $g^i - g^j + 1 \leq h^j - h^i$. The second inequality follows from Lemma 4, $\psi(x+1, y-1) - \psi(x, y-1) + \theta(x+1, X-x) - \theta(x, X-x) > \psi(x, y) - \psi(x-1, y) + \theta(x, X-x+1) - \theta(x-1, X-x+1)$. The third inequality follows from the strategic complement property of θ. The last inequality follows from the fact that M is an MLPE network. Second, suppose that $g^i > g^j = 0$. Then, we extend ψ and θ to $\psi(-1, \cdot)$ and $\theta(-1, \cdot)$ and preserve the concavity of these functions in their first argument. Because of the previous inequalities, we have $\psi(g^i+1,h^i)+\theta(g^i+1,G^{-i})-(\psi(g^i,h^i)+\theta(g^i,G^{-i})) \geq \psi(0,h^j)+\theta(0,G^{-i}+g^i)-(\psi(-1,h^j)+\theta(-1,G^{-i}+g^i))$. Moreover, we also have $\psi(0,h^j)+\theta(0,G^{-i}+g^i)-(\psi(-1,h^j)+\theta(-1,G^{-i}+g^i)) \geq \psi(1,h^j)+\theta(1,G^{-i}+g^i)-(\psi(0,h^j)+\theta(0,G^{-i}+g^i)) \geq \psi(1,n-1)+\theta(1,0)-(\psi(0,n-1)+\theta(0,0)) \geq 0$. The result follows.

3. Third, $g^j > g^i$ and $h^j < h^i$. Since $g^j+h^j > g^i+h^i$, we have $g^j-g^i \geq h^i-h^j+1$. We establish that player i has an incentive to form a link in H by using similar arguments to those given in Point 2.

□

References

[1] G. Ahuja. Collaboration networks, structural holes, and innovation: A longitudinal study. *Administrative Science Quarterly*, 45(3):425–455, 2000.

[2] C. Ballester, A. Calvó-Armengol, and Y. Zenou. Who's who in networks. wanted: The key player. *Econometrica*, 74(5):1403–1417, 2006.

[3] P. Billand, C. Bravard, S. Chakrabarti, and S. Sarangi. Business intelligence and multimarket competition. *Journal of Public Economic Theory*, 18(2):248–267, 2016.

[4] P. Billand, C. Bravard, S. Chakrabarti, and S. Sarangi. A note on networks of collaboration in multi-market oligopolies. *The Annals of Regional Science*, 53(2):325–335, Sep 2014.

[5] Y. Bramoullé and R. Kranton. Risk-sharing networks. *Journal of Economic Behavior and Organization*, 64(3):275–294, 2007.

[6] Y.-J. Chen, Y. Zenou, and J. Zhou. Multiple activities in networks. *American Economic Journal: Microeconomics*, 10(3):34–85, August 2018.

[7] C. Cheng, W. Huang, and Y. Xing. A theory of multiplexity: Sustaining cooperation with multiple relations. *SSRN Electronic Journal*, 01 2021.

[8] P. Erdős and T. Gallai. Gráfok előírt fokszámú pontokkal. *Matematikai Lapok*, 11:264–274, 1960.

[9] S. Goyal and S. Joshi. Networks of collaboration in oligopoly. *Games and Economic Behavior*, 43(1):57–85, 2003.

[10] S. Goyal and S. Joshi. Unequal connections. *International Journal of Game Theory*, 34(3):319–349, Oct 2006.

[11] M. O. Jackson. Networks in the understanding of economic behaviors. *Journal of Economic Perspectives*, 28(4):3–22, November 2014.

[12] M. O. Jackson and S. Nei. Networks of military alliances, wars, and international trade. *Proceedings of the National Academy of Sciences*, 112(50):15277–15284, 2015.

[13] M. O. Jackson and A. Wolinsky. A strategic model of social and economic networks. *Journal of Economic Theory*, 71(1):44–74, 1996.

[14] S. Joshi, A. S. Mahmud, and S. Sarangi. Network formation with multigraphs and strategic complementarities. *Journal of Economic Theory*, 188:105033, 2020.

[15] S. Joshi, A. S. Mahmud, S. Sarangi, and H. Tzavellas. Inheritance and evolution of networks across generation. *Working Paper*, 2021.

[16] S. Joshi, A. S. Mahmud, and H. Tzavellas. The tangled webs we weave: Multiplex formation. *Working Paper*, 2020.

[17] A. Roy and S. Sarangi. Revisiting Friendship Networks. *Economics Bulletin*, 29(4):2640–2647, 2009.

[18] H. Tzavellas. A multilayer view of systemic importance and aggregate fluctuations. *Forthcoming in International Economic Review*, 2023.

[19] A. M. Walsh. Games on multi-layer networks. *Working Paper*, 2019.

[20] E. W. Walsh. An equilibrium concept for multi-network formation games. *Working Paper*, 2021.