INVESTIGATION OF A MATRIX SOLUTION OF VARIABLE CROSS-SECTION COLUMNS

by

ARTHUR E. ROWLAND

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

MASTER OF SCIENCE

in

Applied Mechanics

APPROVED:

Director of Graduate Studies

Dean of Engineering

APPROVED:

Head of Department

Major Professor

1951

Blacksburg, Virginia

155113

ABSTRACT

In 1949, W. T. Thompson(15) proposed a procedure for the solution of an n-section column using the matrix calculus.

Since the matrix method was rather long and unfamiliar to most engineers, it was felt that a further investigation was warranted in the hope that a further development of the Thompson method would result in a simplified procedure not involving matrices, thus placing a worthwhile tool in the hands of the practicing structural engineer.

From the theoretical point of view, at least, the investigation proved successful. However, the resulting equations could not be simplified as much as was originally hoped which meant that the practical value of the method was limited from the point of view of time required for the solution of a typical problem.

In an attempt to circumvent this difficulty, to some degree, a secondary investigation was undertaken. Two common types of columns were analyzed in detail and the results presented in graphical form in such a manner that the critical design load could be found, for the column types investigated, in a matter of minutes.

A bibliography of the more important column papers, dealing essentially with variable section columns, is inclu-

ded in the thesis to provide a convenient ready reference source of column information for future investigators.

TABLE OF CONTENTS

INTRODUCTION

The start of our present theory of elastic column action had its beginning a little over two centuries ago when Euler first published his classical column theory. Through the years that followed, additional knowledge of the subject was gained very slowly, since the problem was essentially an academic one; however, with the approach of the twentieth century, the increasing prevalence of steel in structures placed new and insistent demands on the structural engineer for better column design, which gave the development of column theory a new impetus.

It was natural that the problem of the variable section column should arise at an early stage, and although much time and effort has been expended in an attempt to solve the problem, relatively little headway has been made.

In 1949, W. T. Thompson(15) published a paper suggesting the use of matrix methods for the solution of a variable section elastic column. Although an excellent paper, it had three serious disadvantages.

First, an understanding of the method necessitated a knowledge of the matrix calculus and such knowledge is not possessed by the average practicing engineer.

Second, the method involved the process of successive matrix multiplications, ordinarily a tedious and time consuming operation.

Third, each series of multiplications was based on an initial approximation, which was actually a guess at the critical column load. A series of such guesses and their corresponding set of multiplications, would of course, eventually lead to the correct solution for critical column load, but the time and effort in most cases would be prohibitive.

It was felt, however, that further investigation along the same line would be well worth while with the hope that the matrix method could be replaced by simpler mathematical equations that would yield to quicker and easier solutions.

It was with this thought that the present investigation was begun.

REVIEW OF LITERATURE

Bairstow[1] and Stedman[1] presented a method of successive approximations whereby the critical load of any symmetrical column of varying cross section could be found. The basic Euler equation

$$EI \frac{d^2y}{dx^2} + Py = 0 \qquad (1)$$

was written first, then a critical load P was assumed. Graphs were prepared for the subsequent construction of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y versus x curves, the abscissa being divided into a convenient number of segments (the authors suggested ten for a half column length). A value of $\frac{d^2y}{dx^2}$ was assumed for the first segment, then plotted and mechanically integrated twice over the first segment, for $\frac{dy}{dx}$ and y, the latter being introduced into (1) to obtain a corrected value of $\frac{d^2y}{dx^2}$. The cycle was repeated until the desired accuracy was obtained for the first segment. (The authors stated that two cycles were ordinarily sufficient). The process was continued for each segment and if the resulting plot of $\frac{dy}{dx}$ versus x showed a zero value at the mid-point of the column, then the original assumption for critical load was correct, if not, then a new assumption for critical load had to be made and the entire process repeated until the correct load was found.

Morley[12] also suggested a method of successive approximations, one which is well known, and merely consists of assuming a simple deflection curve and substituting the expression for deflection in the basic column equation (i) to find, upon integration, a new deflection curve, the expression for which is again substituted in the basic equation and the process repeated until the desired accuracy is obtained. Morley suggested that this method could probably be used to advantage in graphical form and since the same process is discussed in Appendix II no further comment will be made here. Morley[13] later discussed the method at greater length. In a recent paper Miesse[11] has discussed essentially the same method.

Wilcken[20] investigated tapering columns whose cross-sections were either square or circular. In each case he investigated linear and a parabolic variation of the side or radius of the cross-section. He found an exact solution for the Euler equation (i) for each special case.

Dinnik[2] studied two types of variable section columns. The first was one whose cross-sectional moment of inertia varied in accordance with an exponential law of the form

$$I_x = Ie^{-\frac{mx}{L}} \qquad \text{(ii)}$$

where I was the maximum moment of inertia at the middle of the column and I_x was the moment of inertia at any point x measured from the mid-point. The length of the column was L. Dinnik wrote the governing equation as:

$$EI\,e^{-\frac{mx}{L}}\,y'' + Py = 0 \qquad \text{(iii)}$$

and expressed the solution as :

$$P = \frac{KEI}{L^2} \qquad \text{(iv)}$$

By assuming various values for m he obtained a set of K values which were listed in tabular form as a function of $\frac{i}{I}$ where i was taken as the moment of inertia at the ends of the column. A similar table was set up for fixed-ended columns. Dinnik followed the same procedure for the second case where the moment of inertia was varied in accordance with a power law of the following form:

$$I_x = I\left\{1-(1-k^2)\frac{x}{L}\right\}^m \qquad \text{(v)}$$

where

$$k^{2m} = \frac{i}{I} \qquad \text{(vi)}$$

The critical load was again expressed in the form of equation (iv) and solutions were worked out for different values of m. The results were tabulated as before as K versus $\frac{i}{I}$ for both pinned and fixed-ended columns. Dinnik[3] later extended this work by investigating a similar type of column but whose center portion had a constant moment of inertia.

The problem of the stepped column was also investigated by Dinnik[3] but for a column consisting of only three sections and of the type shown in Fig. 5 of this thesis. A comparison of his results and the results found herein is included in this thesis.

Gwinn and Miller[7] developed a method for solving a stepped column consisting of two sections. However, although the approach was different, the net result was the same as found in this thesis for a two section column.

Harris[8] solved the Euler equation (i) for three special cases: (1) for a column with constant $\frac{M}{EI}$, (2) constant $\frac{MC}{EI}$, (3) minimum use of material.

Turton[19] solved the problem of the column whose longitudinal profile tapered parabolically. His method was a direct solution of the Euler equation (i).

Templeton[14] used energy methods to solve the problem of a column tapering from the center. Letting I represent the moment of inertia at any station a distance x from the end of the column and letting I_o represent the moment of inertia at the ends, his taper was then described as:

$$\frac{I}{I_o} = A \neq B \left(\frac{x}{L}\right) \neq C \left(\frac{x}{L}\right)^2 \qquad \text{(vii)}$$

Thompson[16] developed a matrix solution for the variable section column. His method is discussed later in the thesis.

Timoshenko[18] has compiled a considerable amount of information concerning columns with varying cross-sections. It is interesting to note that he has pointed out that the effect of shear on solid columns is negligible. (See page 140 of Reference 16).

Other notable papers, not available to the author, have been included in the Bibliography.

LIST OF SYMBOLS

h_i	=	length of i^{th} section
E_i	=	modulus of elasticity of i^{th} section (assumed constant)
I_i	=	moment of inertia of i^{th} section
x_i	=	coordinate at end of i^{th} section
y_i	=	deflection at end of i^{th} section
y_i'	=	slope at end of i^{th} section
P	=	critical column load
b_i	=	$\sqrt{P/E_i I_i}$
L	=	total length of column
M_o, M_n	=	moments at ends of column
a_i	=	$b_i h_i$
R	=	$\frac{I_{i+1}}{I_i} \leq 1$ = moment of inertia taper ratio
n	=	number of sections in column
B	=	see page 22
C	=	see page 23
D	=	see page 24
F	=	see page 25
G	=	see page 26
H	=	see page 26

THE INVESTIGATION

Object of Investigation

It was the purpose of this thesis to develop a method whereby the critical load of an elastic column of variable cross section could be determined with accuracy in a reasonable length of time and employing the simplest mathematical methods possible.

Method of Procedure

Since this thesis was undertaken with the intention of using Thompson's[15] method as a starting point, it is necessary that a brief outline of his procedure be included.

Consider a variable section column such as shown in Figure 1a and the free body diagram of a typical element, Figure 1b.

A moment summation on the free body diagram results in the following differential equation:

$$E_i I_i \left(\frac{d^2 y_2}{dx^2}\right) = -P(y_i - y_{i-1}) + M_{i-1} \qquad (1)$$

Thompson wrote the solution for the deflection at i in terms of quantities at i-1 in the following form:

$$y_i = y_{i-1} + y_{i-1}'\left(\frac{1}{b_i}\right)\sin b_i h_i + \left(\frac{M_{i-1}}{E_i I_i b_i^2}\right)(1-\cos b_i h_i) \qquad (2)$$

He then differentiated twice for slope and moment and obtained:

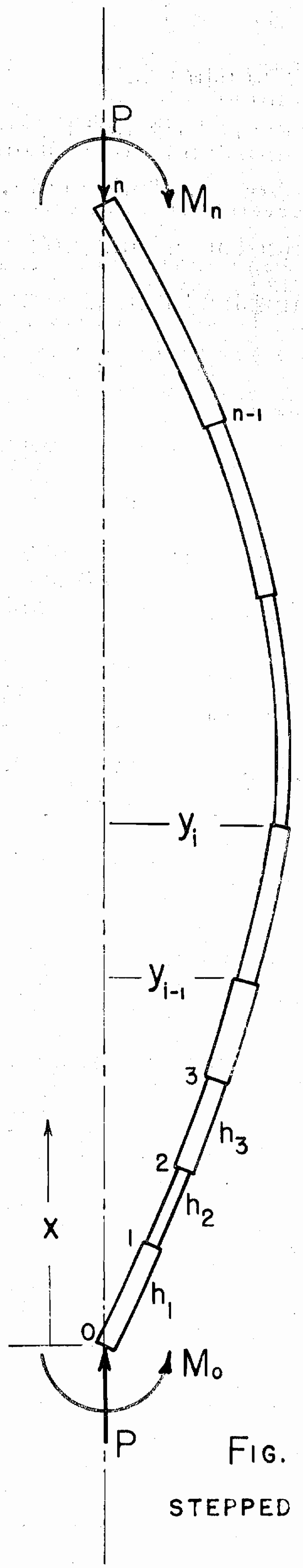

FIG. 1a

STEPPED COLUMN

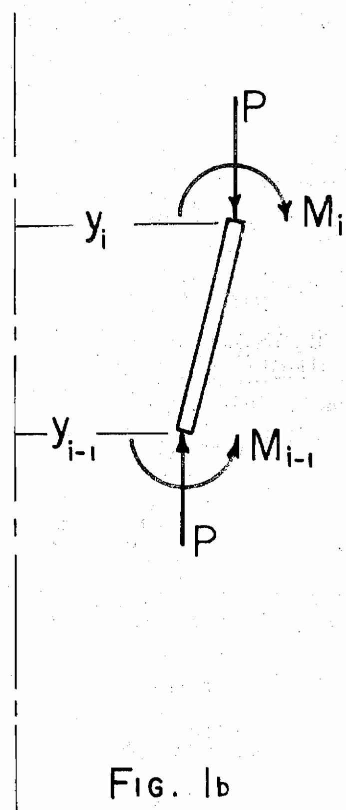

FIG. 1b

FREE BODY DIAGRAM

$$y_i' = y_{i-1}' \cos b_i h_i + \left(\frac{M_{i-1}}{E_i I_i b_i}\right) \sin b_i h_i \tag{3}$$

and

$$M_i = -y_{i-1}' E_i I_i b_i \sin b_i h_i + M_{i-1} \cos b_i h_i \tag{4}$$

Using equations (2), (3), and (4) he then wrote the following matrix relation which links the i^{th} section with the i-1 section:

$$\begin{bmatrix} y_i \\ y_i' \\ M_i \end{bmatrix} = \begin{bmatrix} 1 & (1/b_i)\sin b_i h_i & (1/E_i I_i b_i^2)(1-\cos b_i h_i) \\ 0 & \cos b_i h_i & (1/E_i I_i b_i)(\sin b_i h_i) \\ 0 & -E_i I_i b_i \sin b_i h_i & \cos b_i h_i \end{bmatrix} \begin{bmatrix} y_{i-1} \\ y_{i-1}' \\ M_{i-1} \end{bmatrix} \tag{5}$$

By using the relation established in equation (5) Thompson proceded to link the extreme ends of the column with the following equation:

$$\begin{bmatrix} y_n \\ y_n' \\ M_n \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} y_o \\ y_o' \\ M_o \end{bmatrix} \tag{6}$$

Then applying the boundary conditions of the problem he came to the following conclusions:

For a pin-ended column:

$y_n = y_o = 0$ and $M_n = M_o = 0$

hence $A_{12} = A_{32} = 0$ (7)

For fixed-end columns:

$y_n = y_o = 0$ and $y'_n = y_o' = 0$

hence $A_{13} = A_{23} = 0$ (8)

For a column with end 0 fixed and end n pinned or free:

$y_o = y_o' = 0$ and $M_n = 0$

hence $A_{33} = 0$ (9)

The above is an outline of Thompson's method and his results.

It was decided that an attempt should be made to develop expressions for the matrix elements A_{ij} in the hope that a simplified solution would result, circumventing matrix multiplication.

A rather obvious approach was taken:

For a one section column let $a_{ij} = A_{ij}$

For a two section column let $b_{ij} = A_{ij}$

For a three section column let $c_{ij} = A_{ij}$ etc. where A_{ij} represents any element of the square matrix in equation (6) and where:

$a_{ij} = A_{12}$ or A_{32} for a one section column with pinned ends. (See equation (7))

$a_{ij} = A_{13}$ or A_{23} for a one section column with fixed ends. (See equation (8))

$a_{ij} = A_{33}$ for a one section column with one end fixed and the other end pinned or free. (See equation (9))

Similarly:

$b_{ij} = b_{12}$ or b_{32} for a two section column with pinned ends. (See equation (7))

$b_{ij} = b_{13}$ or b_{23} for a two section column with fixed ends. (See equation (8))

$b_{ij} = b_{33}$ for a two section column with one end fixed and the other end pinned or free. (See equation (9))

Similarly with c_{ij} for a three section column, etc.

Now considering only pin-ended columns for illustration it is seen from equations (5) and (7) that the equation for a one section column would be:

$$a_{32} = -E_1 I_1 b_1 \sin b_1 h_1 = 0$$

or

$$a_{32} = -E_1 I_1 b_1 \sin a_1 = 0 \qquad (10)$$

For a column with two sections:

Multiplying successive matrices:

$$\begin{bmatrix} 1 & (1/b_2)\sin b_2 h_2 & (1/E_2 I_2 b_2^2)(1-\cos b_2 h_2) \\ 0 & \cos b_2 h_2 & (1/E_2 I_2 b_2)\sin b_2 h_2 \\ 0 & -E_2 I_2 b_2 \sin b_2 h_2 & \cos b_2 h_2 \end{bmatrix} \begin{bmatrix} 1 & (1/b_1)\sin b_1 h_1 & (1/E_1 I_1 b_1^2)(1-\cos b_1 h_1) \\ 0 & \cos b_1 h_1 & (1/E_1 I_1 b_1)\sin b_1 h_1 \\ 0 & -E_1 I_1 b_1 \sin b_1 h_1 & \cos b_1 h_1 \end{bmatrix}$$

$+ (1/b_2)\sin b_2h_2 \cos b_1h_1 - (E_1I_1b_1/E_2I_2b_2^2)(1-\cos b_2h_2)\sin b_1h_1$

$_1h_1 - (E_1I_1b_1/E_2I_2b_2)\sin b_2h_2 \sin b_1h_1$

$_2 \cos b_1h_1 - E_1I_1b_1 \cos b_2h_2 \sin b_1h_1$

$(1/E_1I_1b_1^2)(1-\cos b_1h_1) + (1/E_1I_1b_1b_2)\sin b_2h_2 \sin b_1h_1 + (1/E_2I_2b_2^2)(1-\cos b_2h_2)\cos b_1h_1$

$(1/E_1I_1b_1)\cos b_2h_2 \sin b_1h_1 + (1/E_2I_2b_2)\sin b_2h_2 \cos b_1h_1$

$-(E_2I_2b_2/E_1I_1b_1)\sin b_2h_2 \sin b_1h_1 + \cos b_2h_2 \cos b_1h_1$

$os\ b_1h_1 - E_1I_1b_1 \cos b_2h_2 \sin b_1h_1 = 0$

$a_1 - E_1I_1b_1 \cos a_2 \sin a_1 = 0$ (11)

sections:

matrix multiplications such as the

ld:

$a_2 \cos a_1 - E_2I_2b_2 \cos a_3 \sin a_2 \cos a_1$

$a_2 \sin a_1$

$_1I_1b_1/E_2I_2b_2)\sin a_3 \sin a_2 \sin a_1 = 0$ (12)

four sections:

$a_3 \cos a_2 \cos a_1$

$n\ a_3 \cos a_2 \cos a_1$

$a_3 \sin a_2 \cos a_1$

$$-E_1I_1b_1 \cos a_4 \cos a_3 \cos a_2 \sin a_1$$

$$+(E_4I_4b_4E_2I_2b_2/E_3I_3b_3) \sin a_4 \sin a_3 \sin a_2 \cos a_1$$

$$+(E_4I_4b_4E_1I_1b_1/E_2I_2b_2) \sin a_4 \cos a_3 \sin a_2 \sin a_1$$

$$+(E_3I_3b_3E_1I_1b_1/E_2I_2b_2) \cos a_4 \sin a_3 \sin a_2 \sin a_1$$

$$+(E_4I_4b_4E_1I_1b_1/E_3I_3b_3) \sin a_4 \sin a_3 \cos a_2 \sin a_1 = 0 \quad (13)$$

The repeated multiplications were continued for five and six section columns resulting in expressions for

$$e_{32} = 0 \quad (14)$$

$$f_{32} = 0 \quad (15)$$

Equations 10, 11, 12, 13, 14, and 15, were then simplified as much as possible and rewritten so that an attempt could be made to recognize common characteristics such as consistent similarities or differences in the terms. In other words, a basic pattern was sought so that a single equation could be written to represent A_{32} regardless of the number of sections in the column.

The same procedure was followed in an effort to find a similar pattern for A_{32} for the solution of the fixed-ended column and also for A_{33} for a column having one end fixed and the other pinned or free.

It should be mentioned that a detailed description of the procedure used in developing equations 16, 17, and 18, is impossible since it naturally involved much trial and error and some intuitive thought.

In determining the curves of Figure 6 and Figure 7, a large number of columns were worked out whose section lengths were equal (constant h_1) and whose moment of inertia varied by a constant proportion (constant R). These columns were of two common types as shown on the graphs. The critical loads so found were substituted in the Euler equation $P = \frac{\pi^2 EI}{L^2}$ and an effective moment of inertia computed. The ratio of effective moment of inertia to the maximum moment of inertia, was arbitrarily chosen as a convenient parameter and plotted against n, the number of sections in the column. These curves are shown in Figure 6 and Figure 7. The maximum moment of inertia is of course the largest cross-sectional moment of inertia to be found in the column. Columns with five sections or less were solved by equation 16. All others were solved by using the graphical method described in Appendix II.

A comparison of the results of the method developed herein with the graphical method and the method used by Dinnik[3] is included.

Results:

1. The attempts to find basic patterns and single equations to represent A_{32}, A_{23}, and A_{33} regardless of the number of sections, were successful in all three cases.

2. The equation for a pin-ended column having one or two sections is:

$$\sum_{i=1}^{n} (1/b_i) \tan a_i = 0 \qquad (16)$$

3. The equation for a pin-ended column having an odd number of sections equal to or greater than three is:

$$\sum_{i=1}^{n} (1/b_i) \tan a_i + B = 0 \qquad (17)$$

Where:

$$B = \sum_{k_n=1}^{n} \cdots\cdots \sum_{k_3=1}^{n} \sum_{k_2=1}^{n} \sum_{k_1=1}^{n} \left[\sum_{r=1}^{\frac{n-1}{2}} (-1)^r \frac{b_{k_2} b_{k_4} \ldots b_{k_{2r}}}{b_{k_1} b_{k_3} \ldots b_{k_{2r+1}}} \tan a_{k_1} \tan a_{k_2} \ldots \ldots \tan a_{k_{2r+1}} \right]$$

<u>Note</u>: All terms are dropped for which the following relation does not hold true.

$$1 \leqq k_1 < k_2 < k_3 < \ldots\ldots < k_n \leqq n$$

4. The equation for a pin-ended column having an even number of sections equal to or greater than four is:

$$\sum_{i=1}^{n} (1/b_i) \tan a_i + C = 0 \qquad (18)$$

Where:

$$C = \sum_{k_{n-1}=1}^{n} \cdots \sum_{k_3=1}^{n} \sum_{k_2=1}^{n} \sum_{k_1=1}^{n} \left[\sum_{r=1}^{\frac{n}{2}-1} (-1)^r \frac{b_{k_2} b_{k_4} \ldots b_{k_{2r}}}{b_{k_1} b_{k_3} \ldots b_{k_{2r+1}}} \tan a_{k_1} \tan a_{k_2} \ldots \tan a_{k_{2r+1}} \right]$$

<u>Note</u>: All terms are dropped for which the following relation does not hold true.

$$1 \leqq k_1 < k_2 < k_3 < \ldots\ldots < k_{n-1} \leqq n$$

5. The equation for a fixed-ended column having one or two sections is:

$$\sum_{i=1}^{n} b_i \tan a_i = 0 \qquad (19)$$

6. The equation for a fixed-ended column having an odd number of sections equal to or greater than three is:

$$\sum_{i=1}^{n} b_i \tan a_i + D = 0 \qquad (20)$$

Where:

$$D = \sum_{k_n=1}^{n} \cdots\cdots \sum_{k_3=1}^{n} \sum_{k_2=1}^{n} \sum_{k_1=1}^{n} \left[\sum_{r=1}^{\frac{n-1}{2}} (-1)^r \frac{b_{k_1} b_{k_3} \cdots b_{k_{2r+1}}}{b_{k_2} b_{k_4} \cdots b_{k_{2r}}} \tan a_{k_1} \tan a_{k_2} \cdots \cdots \tan a_{k_{2r+1}} \right]$$

<u>Note</u>: All terms are dropped for which the following relation does not hold true.

$$1 \leqq k_1 < k_2 < k_3 < \cdots\cdots < k_n \leqq n$$

7. The equation for a fixed-ended column having an even number of sections equal to or greater than four is:

$$\sum_{i=1}^{n} b_i \tan a_i + F = 0 \tag{21}$$

Where:

$$F = \sum_{k_{n-1}=1}^{n} \cdots\cdots \sum_{k_3=1}^{n} \sum_{k_2=1}^{n} \sum_{k_1=1}^{n} \left[\sum_{r=1}^{\frac{n}{2}-1} (-1)^r \frac{b_{k_1} b_{k_3} \cdots b_{k_{2r+1}}}{b_{k_2} b_{k_4} \cdots b_{k_{2r}}} \tan a_{k_1} \tan a_{k_2} \cdots \tan a_{k_{2r+1}} \right]$$

<u>Note</u>: All terms are dropped for which the following relation does not hold true.

$$1 \leqq k_1 < k_2 < k_3 < \cdots\cdots < k_{n-1} \leqq n$$

8. The equation for a column fixed at one end and pinned or free at the other and having an odd number of sections equal to or greater than three is:

$$1 + G = 0 \tag{22}$$

Where:

$$G = \sum_{k_{n-1}=1}^{n} \cdots \sum_{k_3=1}^{n} \sum_{k_2=1}^{n} \sum_{k_1=1}^{n} \left[\sum_{r=1}^{\frac{n-1}{2}} (-1)^r \frac{b_{k_1} b_{k_3} \cdots b_{k_{2r-1}}}{b_{k_2} b_{k_4} \cdots b_{k_{2r}}} \tan a_{k_1} \tan a_{k_2} \cdots \tan a_{k_{2r}} \right]$$

<u>Note</u>: All terms are dropped for which the following relation does not hold true.

$$1 \leqq k_1 < k_2 < k_3 < \cdots\cdots < k_{n-1} \leqq n$$

9. The equation for a column fixed at one end and pinned or free at the other and having an even number of sections equal to or greater than two is:

$$1 + H = 0 \qquad (23)$$

Where:

$$H = \sum_{k_n=1}^{n} \cdots \sum_{k_3=1}^{n} \sum_{k_2=1}^{n} \sum_{k_1=1}^{n} \left[\sum_{r=1}^{\frac{n}{2}} (-1)^r \frac{b_{k_1} b_{k_3} \cdots b_{k_{2r-1}}}{b_{k_2} b_{k_4} \cdots b_{k_{2r}}} \tan a_{k_1} \tan a_{k_2} \cdots \tan a_{k_{2r}} \right]$$

<u>Note</u>: All terms are dropped for which the following relation does not hold true.

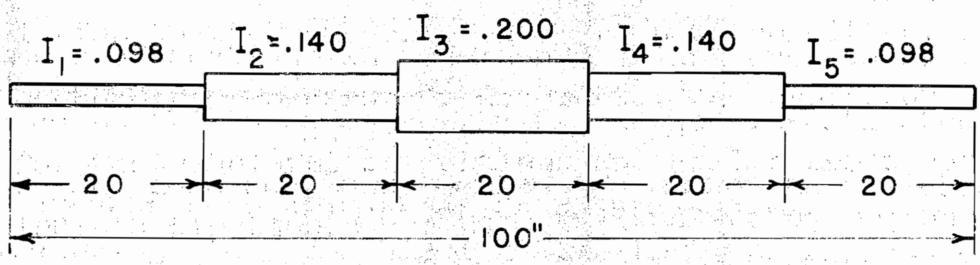

FIG. 2. SYMMETRICALLY TAPERED COLUMN WITH R = 0.7

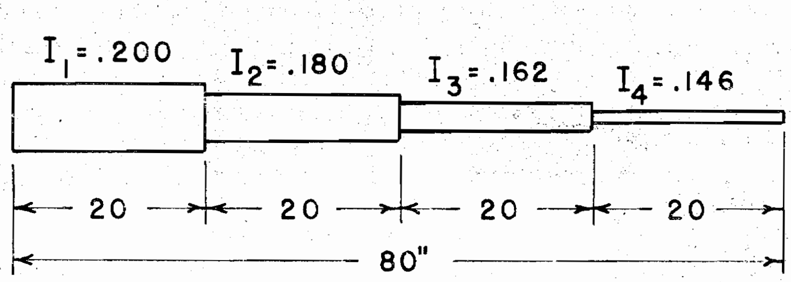

FIG. 3. UNSYMMETRICALLY TAPERED COLUMN WITH R = 0.9

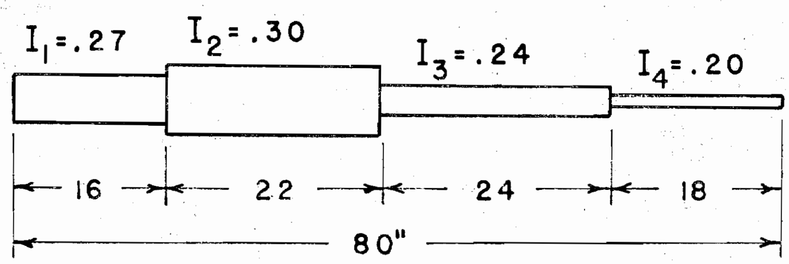

FIG. 4. NON-UNIFORMLY VARYING COLUMN

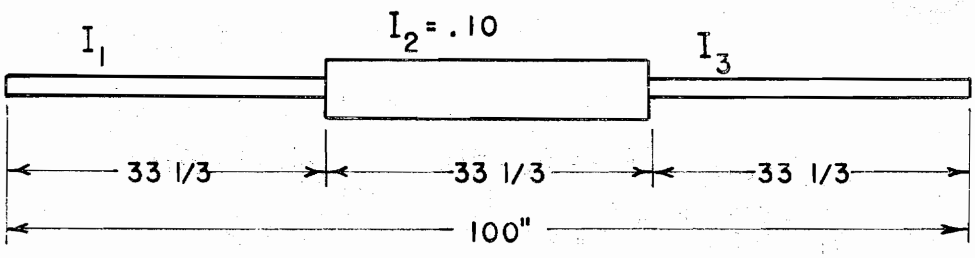

FIG. 5. SYMMETRICALLY TAPERED COLUMN

$$1 \leqq k_1 < k_2 < k_3 < \ldots\ldots\ldots < k_n \leqq n$$

10. The number of terms in each of equations 16 to 23 inclusive, will be $(2)^{n-1}$

11. The time required for each approximation using the above equations is $2.5(2)^{n-1}$ minutes.

12. Four to five approximations will ordinarily determine the critical load with reasonable accuracy.

13. The columns of Figures 2, 3, and 4 were solved by the graphical method described in Appendix II and by the method developed herein. A comparison of results is presented in Table 1.

TABLE 1

COMPARISON of GRAPHICAL METHOD and METHOD DEVELOPED HEREIN. ($E = 10^7$)

COLUMN	CRITICAL LOAD in POUNDS	
	GRAPHICAL METHOD	METHOD DEVELOPED HEREIN
Fig. 2	1500	1475
Fig. 3	2640	2640
Fig. 4	4060	4040

14. The column of Figure 5 was solved by Dinnik's(3)* method and by the method developed herein for four different moment of inertia ratios, namely R = 0.5, R = 0.6,

* See page 11 of thesis

R = 0.7, and R = 0.8. The results are shown in Table 2.

TABLE 2

COMPARISON of DINNIK'S[3] METHOD and

METHOD DEVELOPED HEREIN. ($E = 10^7$)

SEE FIG. 7	CRITICAL LOAD in POUNDS	
MOMENT of INERTIA RATIO R	DINNIK[3]	THESIS
0.5	690	706
0.6	770	775
0.7	830	839
0.8	890	898

15. For columns having more than five sections, the method outlined in Appendix II is recommended.

16. A complete solution (up to n = 10 and 9 respectively) is given in Figure 6 and Figure 7 for pinned columns with constant R and h_1 of the types shown.

17. The curves of Figure 6 give an immediate solution for any pin-ended two-section elastic column of the type shown on the graph.

18. The curves of Figure 7 give an immediate solution for any pin-ended symmetrical three-section elastic column of the type shown on the graph.

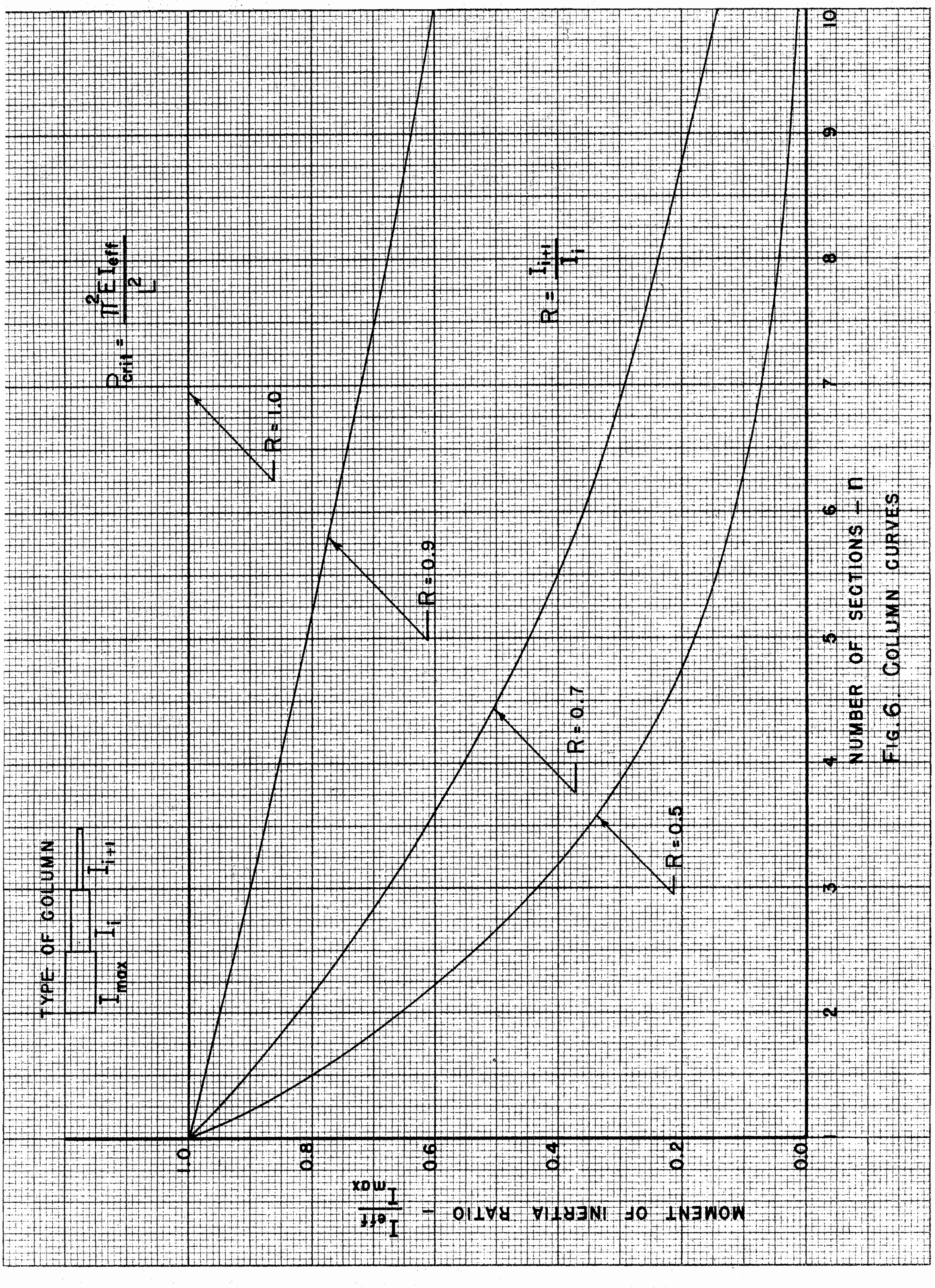

FIG. 6. COLUMN CURVES

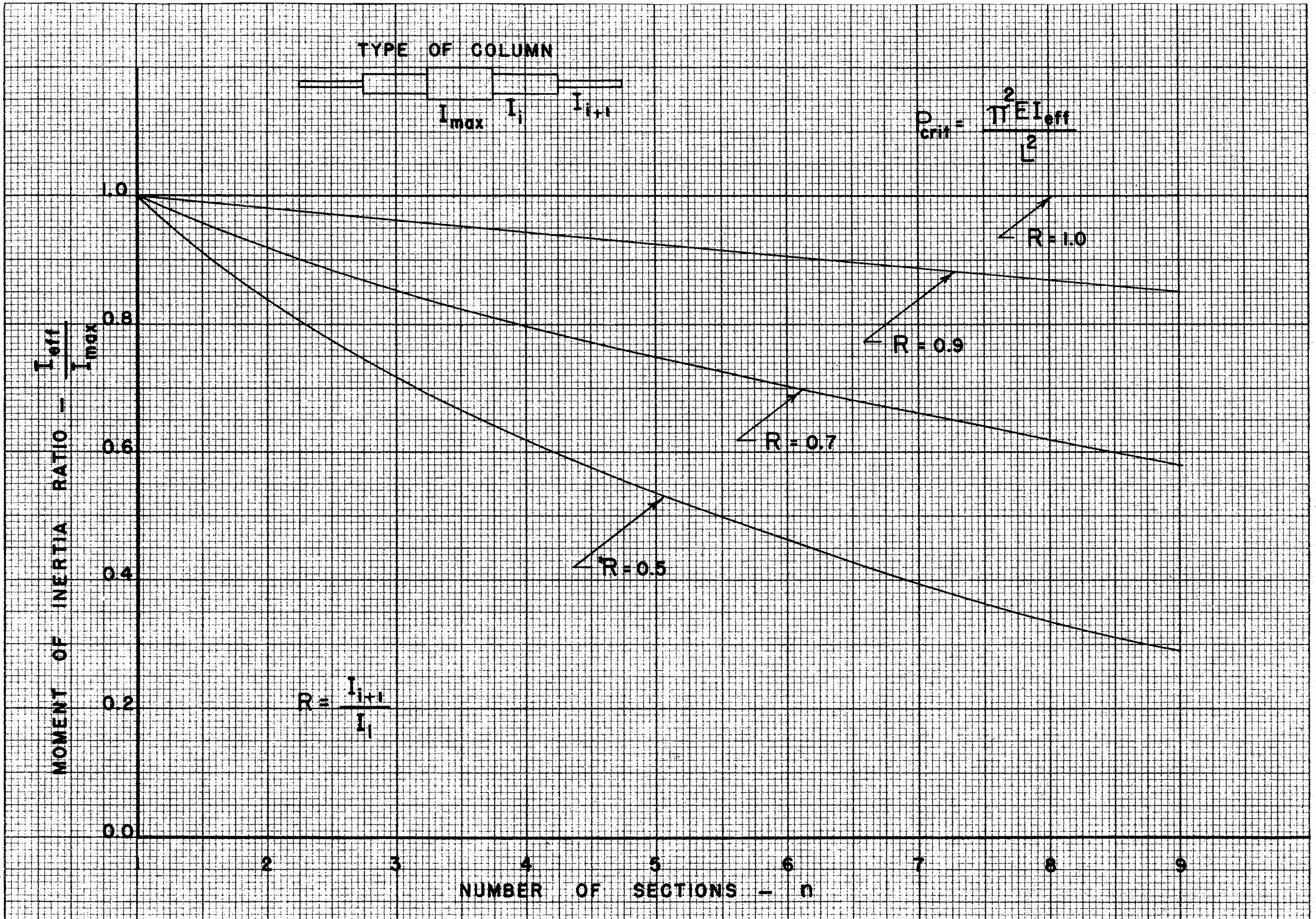

FIG. 7. COLUMN CURVES

19. It was found that for two and three section columns with constant h_i and whose moment of inertia did not vary greatly ($R \geqq 0.9$) the critical load could be closely approximated by an arithmetic mean of the critical column loads found by assuming each section as an individual column of length L.

DISCUSSION OF RESULTS

From equation (7) the solution for a pin-ended column could be expressed as

$$A_{12} = 0 \qquad \text{or} \qquad A_{32} = 0$$

A_{32} was developed merely because of the slight simplification offered by the zeros in the matrices.

A development of A_{12} and A_{32} will show that they differ only slightly in original form but when equated to zero they result in identical equations.

The same comments hold for the fixed-ended columns.

Equations 16 to 23 inclusive are believed to be the simplest forms for these equations although admittedly somewhat complicated in application. For this reason Appendix I has been included and shows the result of expanding equation 17 for a three and five section column and equation 18 for a four section column. (See equations 24, 25, and 26).

Table 3 is included in Appendix I as an added simplification. It covers columns having three to eight sections inclusive, and is a summary of all subscripts (in their proper order) required for the equation which defines the particular problem. It is believed that a better understanding and appreciation of Table 3 will exist if it is studied in conjunction with equations 24, 25, and 26.

It is noted from the results that the number of terms in equations 17, 18, 20, 21, 22, and 23 will increase very rapidly and the time required to work out a solution soon becomes prohibitive. For example, the equation for a five section column contains 16 terms and the time required for a single approximation is about 40 minutes. Assuming that four successive approximations are necessary then the total time required would be 2-2/3 hours.

Similarly a six section column would involve 32 terms and about 5-1/3 hours of steady work. Hence a five section column is considered to be the practical limit for a solution by the method developed in this thesis.

In applying equations 16 to 23 inclusive much work can be eliminated by making a good first approximation to the critical load. It was found that surprisingly good results were obtained by merely tabulating the critical column load for each section, assuming a column length of L. A careful examination of the tabulated results and the physical arrangement of the column will ordinarily lead to a very good first approximation for P.

In order to improve the practicability of the method set forth in this thesis, a secondary investigation was undertaken.

Two common types of columns were chosen and a large number of typical problems were worked out in each case. It was decided that the most efficient method of presenting the results, in generalized form, was through the medium of the Euler equation. This resulted in Figure 6 and Figure 7.

Hence, for a given column of either type shown, it is only necessary to pick off the value of moment of inertia ratio corresponding to the known number of sections n and the taper ratio R. Knowing the maximum moment of inertia, the effective moment of inertia is easily found and when substituted in the conventional Euler equation gives the critical column load P.

A comparison of the method developed and the graphical method described in Appendix II (see Table 1) showed very good agreement for three completely different columns. The results would appear to indicate that the method developed here is slightly conservative in comparison with the graphical method, however such a conclusion is not dependable unless a particularly large number of solutions are available.

A comparison with Dinnik's[3] method, (see Table 2) for a symmetrical three section column, showed excellent agreement with results, checking within about two percent.

CONCLUSIONS

As has been previously noted, the upper limit for the number of sections in a column is considered to be five if the column is to be solved by the method investigated herein. Any more sections require too much time and effort for a solution and a graphical method is then preferable.

Under these circumstances, the analytical method developed in this thesis is seriously limited in its range of applicability, nevertheless, there are many instances where columns of five sections or less can be found in actual structures, and in such cases, the method can be used to advantage.

In addition, columns of the type shown in Figure 6 and Figure 7 are quite often found in a variety of structures. It is not inconceivable that with this information available, columns could be deliberately designed to conform to the specific types shown in the graphs in order to save the time and cost of a lengthy analysis or test. Such a procedure is often practiced by designers.

A comparison with two other methods showed very good agreement, ordinarily within two percent.

SUMMARY

From the theoretical point of view, the investigation was successful. Equations were found for the solution of the pin-ended column, the fixed-ended column, and the column with one end fixed and the other pinned or free.

From the practical point of view, the success of the investigation is considered limited since the time required to solve columns with more than five sections is prohibitive.

Detailed solutions were worked out for two common types of columns. The results are presented in a convenient graphical form and give critical loads, for the types investigated, quickly and easily.

ACKNOWLEDGEMENTS

The author wishes to express his gratitude and appreciation to the members of his graduate committee, Dr. V. G. Szebehely, Professor D. H. Pletta, and Professor E. Q. Smith for their helpful suggestions and assistance throughout this investigation.

BIBLIOGRAPHY

1. Bairstow, L., and Stedman, E. W., "Critical Loads for Long Struts of Varying Sections", Engineering, Vol. 98, pp. 403-404, 1914.

2. Dinnik, A., "Design of Columns of Varying Cross Sections", Trans. ASME Vol. 51, 1929, paper APM-51-11.

3. Dinnik, A., "Design of Columns of Varying Cross Sections", Trans. ASME Vol. 54, 1932, paper APM-54-16.

4. Elwitz, E., "Die Lehre von der Knickfestigkeit", Vol. 1, p. 222, Dusseldorf, 1918.

5. Euler, L., see Ostwald's "Klassiker der exakten Wissenschaften", No. 175, Leipzig, 1910.

6. Franke, A. Z., Math. Physik, Vol. 49, 1901.

7. Gwinn, J. M., and Miller, Roy A., "A Method of Determining the Ultimate Strength of a Column Swaged at the Ends", Journal of the Aeronautical Sciences, Vol. 2, No. 3, pp. 94-96, May, 1935.

8. Harris, Charles O., "A Suggestion for Columns of Varying Cross Section", Journal of the Aeronautical Sciences, Vol. 9, No. 3, pp. 97-99, January, 1942.

9. Hoblit, Frederic M., "Buckling Load of a Stepped Column", Journal of the Aeronautical Sciences, Vol. 18, No. 2, pp. 124-126, February, 1951.

10. Lagrange, "Sur la figure des colonnes", Miscellanea Taurinensia, Vol. 5 which covers the years 1770-1773. Reprint in "Oeuvers de Lagrange", Vol. 2, pp. 125-170, Gautier Villars, Paris, 1868.

11. Miesse, C. C., "Determination of the Buckling Load for Columns of Variable Stiffness", ASME Trans., Vol. 71, Journal of Applied Mechanics, pp. 406-410, 1949.

12. Morley, Arthur, "Critical Loads for Ideal Long Columns", Engineering, Vol. 97, pp. 566-568, 1914.

13. Morley, Arthur, "Critical Loads for Long Tapering Struts", Engineering, Vol. 104, pp. 295-298, 1917.

14. Templeton, Hayden, "Approximate Solution for Tapered Pin-Ended Struts", Journal of the Royal Aeronautical Society, Vol. 48, pp. 6-11, 1944.

15. Thompson, William T., "Matrix Solution of the n-Section Column", Journal of the Aeronautical Sciences, Vol. 16, No. 10, pp. 623-624, October, 1949.

16. Thompson, William T., "Critical Load of Columns of Varying Cross Section", Journal of Applied Mechanics, Vol. 17, No. 2, pp. 132-134, June, 1950.

17. Timoshenko, S., Bulletin Polytechnic Institute, Kiev, 1908.

18. Timoshenko, S., "Theory of Elastic Stability", pp. 128-139, McGraw-Hill, New York, 1936.

19. Turton, F. J., "Pinned-Pinned Solid Struts with Parabolic Taper", Journal of the Royal Aeronautical Society, Vol. 46, pp. 146-151, 1942.

20. Wilcken, J. A., "The Bending of Columns of Varying Cross Section", Philosophical Magazine, Vol. 3, pp. 418-429, 1927.

21. Wilcken, J. A., "The Bending of Columns of Varying Cross Section - II", Philosophical Magazine, Vol. 3, pp. 1065-1072, 1927.

22. Young, Dana, "Inelastic Buckling of Variable Section Columns", Journal of Applied Mechanics, September, 1945.

VITA

1920 Born in Centerville, New Brunswick, Canada. Moved to Montreal and attended Model and Woodland elementary schools.

1938 Graduated from Verdun High School in Montreal. Found that Aeronautical Engineering was not taught in Canada so enrolled at the University of Detroit in Detroit, Michigan, where they had a five year cooperative engineering program.

1941 Became laboratory assistant in Physics Department at University of Detroit. Continued in that capacity until graduation. Duties were eventually almost entirely teaching.

1943 Graduated from University of Detroit with a Bachelor of Science Degree in Aeronautical Engineering. Took position with Aeronca Aircraft Corporation in Middletown, Ohio as a stress analyst. Was married in the same year.

1944 Studied advanced aircraft structural analysis at University of Cincinnati night school.

1945 Made Acting Chief of Structures at Aeronca Aircraft Corporation. Resigned to take position as stress analyst and structural design engineer at Southern Aircraft Corporation near Dallas, Texas.

1946 Airplane program cancelled at Southern Aircraft Corporation. Took position as Senior Structures Engineer at Beach Aircraft Corporation, Wichita, Kansas. Naturalized November 26.

1947 Left aircraft industry to teach. Took Assistant Professorship at Virginia Polytechnic Institute, Blacksburg, Virginia. Appointed Acting Head of Aeronautical Engineering Department in same year.

Present Associate Professor and Acting Head of Aeronautical Engineering Department, Virginia Polytechnic Institute.

Memberships:

Institute of the Aeronautical Sciences
The American Society for Engineering Education
Gamma Alpha Rho, National Honary Aeronautical Society

Arthur E. Rowland

APPENDIX I

SAMPLE EQUATIONS

From (17) the equation for a three section pin-ended column is:

$$(1/b_1) \tan a_1 + (1/b_2) \tan a_2 + (1/b_3) \tan a_3 - \frac{b_2}{b_1 b_3} \tan a_1 \tan a_2 \tan a_3 = 0 \qquad (24)$$

From (18) the equation for a four section pin-ended column is:

$$(1/b_1) \tan a_1 + (1/b_2) \tan a_2 + (1/b_3) \tan a_3 + (1/b_4) \tan a_4 - \frac{b_2}{b_1 b_3} \tan a_1 \tan a_2 \tan a_3 - \frac{b_2}{b_1 b_4} \tan a_1 \tan a_2 \tan a_4 - \frac{b_3}{b_1 b_4} \tan a_1 \tan a_3 \tan a_4 - \frac{b_3}{b_2 b_4} \tan a_2 \tan a_3 \tan a_4 \qquad (25)$$

From (17) the equation for a five section pin-ended column is:

$$(1/b_1) \tan a_1 + (1/b_2) \tan a_2 + (1/b_3) \tan a_3 + (1/b_4) \tan a_4 + (1/b_5) \tan a_5 - \frac{b_2}{b_1 b_3} \tan a_1 \tan a_2 \tan a_3 - \frac{b_2}{b_1 b_4} \tan a_1 \tan a_2 \tan a_4 - \frac{b_3}{b_1 b_4} \tan a_1$$

$$\tan a_3 \tan a_4 - \frac{b_3}{b_2 b_4} \tan a_2 \tan a_3 \tan a_4$$

$$-\frac{b_2}{b_1 b_5} \tan a_1 \tan a_2 \tan a_5 - \frac{b_3}{b_1 b_5} \tan a_1 \tan a_3 \tan a_5$$

$$-\frac{b_3}{b_2 b_5} \tan a_2 \tan a_3 \tan a_5 - \frac{b_4}{b_1 b_5} \tan a_1 \tan a_4 \tan a_5$$

$$-\frac{b_4}{b_2 b_5} \tan a_2 \tan a_4 \tan a_5 - \frac{b_4}{b_3 b_5} \tan a_3 \tan a_4 \tan a_5$$

$$+ \frac{b_2 b_4}{b_1 b_3 b_5} \tan a_1 \tan a_2 \tan a_3 \tan a_4 \tan a_5 \qquad (26)$$

Table 3 gives an ordered arrangement of subscripts for pin-ended columns having three to eight sections. This table should be self-explanatory when studied in conjunction with equations 24, 25, and 26.

Table 3

Subscripts for B and C Terms for Pin-Ended Column Equations

	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8
Subscripts for $\frac{b}{b\ b}$ terms	$\frac{2}{1,3}$	$\frac{2}{1,4}$ —	$\frac{2}{1,5}$ — —	$\frac{2}{1,6}$ — — —	$\frac{2}{1,7}$ — — — —	$\frac{2}{1,8}$ — — — — —
		$\frac{3}{1,4}$ $\frac{3}{2,4}$	$\frac{3}{1,5}$ $\frac{3}{2,5}$ —	$\frac{3}{1,6}$ $\frac{3}{2,6}$ — —	$\frac{3}{1,7}$ $\frac{3}{2,7}$ — — —	$\frac{3}{1,8}$ $\frac{3}{2,8}$ — — — —
			$\frac{4}{1,5}$ $\frac{4}{2,5}$ $\frac{4}{3,5}$	$\frac{4}{1,6}$ $\frac{4}{2,6}$ $\frac{4}{3,6}$ —	$\frac{4}{1,7}$ $\frac{4}{2,7}$ $\frac{4}{3,7}$ — —	$\frac{4}{1,8}$ $\frac{4}{2,8}$ $\frac{4}{3,8}$ — — —
				$\frac{5}{1,6}$ $\frac{5}{2,6}$ $\frac{5}{3,6}$ $\frac{5}{4,6}$	$\frac{5}{1,7}$ $\frac{5}{2,7}$ $\frac{5}{3,7}$ $\frac{5}{4,7}$ —	$\frac{5}{1,8}$ $\frac{5}{2,8}$ $\frac{5}{3,8}$ $\frac{5}{4,8}$ — —
					$\frac{6}{1,7}$ $\frac{6}{2,7}$ $\frac{6}{3,7}$ $\frac{6}{4,7}$ $\frac{6}{5,7}$	$\frac{6}{1,8}$ $\frac{6}{2,8}$ $\frac{6}{3,8}$ $\frac{6}{4,8}$ $\frac{6}{5,8}$ —
						$\frac{7}{1,8}$ $\frac{7}{2,8}$ $\frac{7}{3,8}$ $\frac{7}{4,8}$ $\frac{7}{5,8}$ $\frac{7}{6,8}$
Subscripts for $\frac{b\ b}{b\ b\ b}$ terms			$\frac{2,4}{1,3,5}$	$\frac{2,4}{1,3,6}$ — —	$\frac{2,4}{1,3,7}$ — — — — —	$\frac{2,4}{1,3,8}$ — — — — — — — — —
				$\frac{2,5}{1,3,6}$ $\frac{2,5}{1,4,6}$ —	$\frac{2,5}{1,3,7}$ $\frac{2,5}{1,4,7}$ — — — —	$\frac{2,5}{1,3,8}$ $\frac{2,5}{1,4,8}$ — — — — — — — —
				— $\frac{3,5}{1,4,6}$ $\frac{3,5}{2,4,6}$	$\frac{2,6}{1,3,7}$ $\frac{2,6}{1,4,7}$ $\frac{2,6}{1,5,7}$ — — —	$\frac{2,6}{1,3,8}$ $\frac{2,6}{1,4,8}$ $\frac{2,6}{1,5,8}$ — — — — — — —
					— $\frac{3,5}{1,4,7}$ — $\frac{3,5}{2,4,7}$ — —	$\frac{2,7}{1,3,8}$ $\frac{2,7}{1,4,8}$ $\frac{2,7}{1,5,8}$ $\frac{2,7}{1,6,8}$ — — — — — —
					— $\frac{3,6}{1,4,7}$ $\frac{3,6}{1,5,7}$ $\frac{3,6}{2,4,7}$ $\frac{3,6}{2,5,7}$ —	— $\frac{3,5}{1,4,8}$ — — $\frac{3,5}{2,4,8}$ — — — — —
					— — $\frac{4,6}{1,5,7}$ — $\frac{4,6}{2,5,7}$ $\frac{4,6}{3,5,7}$	— $\frac{3,6}{1,4,8}$ $\frac{3,6}{1,5,8}$ — $\frac{3,6}{2,4,8}$ $\frac{3,6}{2,5,8}$ — — — —
						— $\frac{3,7}{1,4,8}$ $\frac{3,7}{1,5,8}$ $\frac{3,7}{1,6,8}$ $\frac{3,7}{2,4,8}$ $\frac{3,7}{2,5,8}$ $\frac{3,7}{2,6,8}$ — — —
						— — $\frac{4,6}{1,5,8}$ — — $\frac{4,6}{2,5,8}$ — $\frac{4,6}{3,5,8}$ — —
						— — $\frac{4,7}{1,5,8}$ $\frac{4,7}{1,6,8}$ — $\frac{4,7}{2,5,8}$ $\frac{4,7}{2,6,8}$ $\frac{4,7}{3,5,8}$ $\frac{4,7}{3,6,8}$ —
						— — — $\frac{5,7}{1,6,8}$ — — $\frac{5,7}{2,6,8}$ — $\frac{5,7}{3,6,8}$ $\frac{5,7}{4,6,8}$
Subscripts for $\frac{b\ b\ b}{b\ b\ b\ b}$ terms					$\frac{2,4,6}{1,3,5,7}$	$\frac{2,4,6}{1,3,5,8}$ — — —
						$\frac{2,4,7}{1,3,5,8}$ $\frac{2,4,7}{1,3,6,8}$ — —
						— $\frac{2,5,7}{1,3,6,8}$ $\frac{2,5,7}{1,4,6,8}$ —
						— — $\frac{3,5,7}{1,4,6,8}$ $\frac{3,5,7}{2,4,6,8}$

APPENDIX II

A GRAPHICAL SOLUTION FOR VARIABLE SECTION ELASTIC COLUMNS[18]

For columns with more than five sections, the following iterative process is recommended:

The bending moment at any point in a column is $M = Py$ where P is the critical load and y is the deflection from the original straight position.

Since $P = M/y$ it is obvious that the trend or shape of the bending moment and deflection curves must be identical for a correct solution.

These curves would be sine waves for constant section columns; hence a sine wave should ordinarily be a very good first approximation of the moment curve of a variable section column.

If the corresponding M/EI curve is now plotted (using correct values of I) and integrated twice by any convenient graphical method, the deflection curve will result. If the bending moments at different points along the column are now divided by the deflections at corresponding points, a series of values for critical load will result.

If these values vary only slightly, a correct solution has been obtained. If the variation is great, then another similar approximation can be made, but is is now important to note that the deflection curve (obtained as described above) must more truly represent the true bending moment curve than the bending moment curve originally assumed; hence the deflection curve should be used as the bending moment curve for the next approximation.

It becomes obvious then that this is not a trial and error method, but it is an iterative process.

It has been found that only three iterations will almost always yield a solution within practical limits of accuracy.

The time required for a solution will run between two and three hours regardless of the number of sections in the column. The method is of course ideal for tapered columns.