MULTI-GROUP NEUTRON TRANSPORT THEORY IN PLANE GEOMETRY*

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I. INTRODUCTION

We wish to present a method for solving neutron transport problems involving the time-independent multi-group transport equations in plane geometry. The presentation is for systems exhibiting isotropic scattering, but the method can readily be extended for anisotropic scattering. It should be emphasized that the method is analytically exact and it provides numerically amenable expressions for the physical quantities of interest. Thus, the results can be used as a standard for evaluating approximate methods. Further, it is hoped that the analysis will shed more light on the general subject of neutron transport. The method is a generalization of the method of singular integral equations as reported by Bowden, McCrosson, and Rhodes (3); however, before the details of this method are presented we shall give a brief review of the problem.

The difficulty in solving neutron transport problems by an exact analysis arises from the fact that the general neutron transport equation is an integro-differential equation involving seven independent variables. Thus, at the outset, one is compelled to consider only highly idealized systems in order to reduce the number of independent variables. One of the simplest forms of the transport equation is the time-independent, one-speed equation in plane geometry. In this equation all the neutrons are treated as if they possess the same energy. There are only two independent variables: a spatial coordinate, and a direction cosine measured with respect to the axis of the spatial coordinate. Until a few years ago, very few exact solutions had been found even for this simple form of the transport equation, and these only for infinite systems. Then, in 1960, K. M. Case (4) presented a new approach for solving the equation. Since that time, numerous authors have solved various transport problems involving the one-speed equation. These investigations clearly indicate the versatility of Case's method.

Case's method consists of separating the independent variables of the transport equation and expressing the general solution as an eigenfunction expansion with arbitrary coefficients over the discrete and continuous spectrums of the separation parameter. Specific problems are then solved by applying appropriate boundary conditions to the general solution to determine the expansion coefficients. For semi-infinite systems, one typically obtains a singular integral equation for the expansion coefficient of the continuum. Fortunately, this singular integral equation can be solved using the standard methods presented by Muskhelishvili (11). In some cases the actual solving of the singular integral equations can be avoided by using orthogonality relations between the eigenfunctions.

Recently, another approach for solving one-speed transport problems was reported by Bowden, McCrosson, and Rhodes (3). This approach which is a modification of the transform method of Leonard and Mullikin (10), will be referred to in this thesis as the method of singular integral equations. In this approach, the one-speed transport equation is transformed into a singular integral equation with an arbitrary inhomogeneous term. This inhomogeneous term plays much the same role as the expansion coefficients of Case's method. Specific problems are solved by first determining the inhomogeneous term and then solving the singular integral equation for the neutron distribution. For semi-infinite systems the inhomogeneous term is generally expressed in terms of the emerging angular flux at the boundary. The emergent flux is, in turn, determined by a singular integral equation which is adjoint (11) to the one obtained for the continuum coefficient in Case's method. In the paper by Bowden, McCrosson and Rhodes (3), this duality between the two methods is further exhibited through equations which relate the inhomogeneous term and the expansion coefficients of Case's method. The two methods appear to be equally versatile and the degree of difficulty in solving specific problems is about the same in either case. Case's normal mode expansion of the neutron distribution has an advantage, however, in that it allows a direct comparison with the results of diffusion theory.

To include energy dependence, it is sometimes assumed that the neutrons can have any of several discrete energies, where the energy transfer from one energy group to another is described by an appropriate transfer matrix. In recent years numerous authors have attempted to extend Case's method to solve transport problems involving this multi-group transport equation. The generalization is quite straight forward in principle, but the results have not been altogether satisfactory. The reason is that in problems involving a semi-infinite medium one must solve a complicated system of singular integral equations. The general procedure for solving such systems has been developed by Muskhelishvili (11) and Vekua (17), but the theory is incomplete. Leonard and Ferziger (9) have attempted to further develop the theory to solve multi-group problems, but their results require the matrix solution of a certain analytic boundary value problem, and the equation which they derive for this matrix is extremely difficult to solve. In addition, the analysis of Leonard and Ferziger is restricted to systems which obey the principle of detailed balance and hence their results are not applicable to systems with fission sources.

In this thesis, we demonstrate that the method of singular integral equations provides an attractive alternative for solving multi-group problems. Again, as in Case's method, the procedure leads to a system of singular integral equations which must be solved. These equations can be treated formally using the analysis of Leonard and Ferziger

(9), but again the results are not amenable to numerical calculation. Fortunately, a procedure can be established for uncoupling the singular integral equations and the uncoupled form of the equations can be written down explicitly. This uncoupled system of equations can be solved using the conventional methods of Muskhelishvili (11) which were applicable in one-speed problems. As in one-speed problems, the relationship between Case's method and the method of singular integral equations can be exhibited. Hence, the expansion coefficients in Case's eigenfunction expansion can be determined using the method of singular integral equations.

An outline of the rest of the thesis is as follows: In Section II we shall present the method of Case. The procedure and notation are based largely on the work of Zelazny and Kuszell (18), who studied the problem in 1962. One important modification has been made, however, in that we have explicitly exhibited the degeneracy of the eigenfunctions of the continuum. This modification is in accordance with the recent work of Siewert and Zweifel (15) (16), and Siewert and Shieh (14). In Section III the method of singular integral equations is developed and in Section IV the relationship between this method and the method of Case is presented. In Sections V, VI, and VII we present applications of the method of singular integral equations. Specifically, the infinite medium Green's function for the multi-group equations is determined exactly in Section V. The Milne and critical problems are solved respectively in Sections VI and VII. The "solutions" to the latter two problems are expressed in terms of Fredholm equations of the second kind which are amenable to numerical calculation.

II. THE METHOD OF CASE; THE NORMAL MODE EXPANSION

Elementary Solutions

We present here the method of Case (4) for finding the general solution of the multi-group transport equations. Assuming isotropic scattering and plane geometry, the multi-group equations can be written in the form

$$(\mu \frac{\partial}{\partial x} + \sigma_i)\, \Psi_i(x,\mu) = \frac{1}{2} \sum_{j=1}^{N} c_{ij} \int_{-1}^{1} \Psi_j(x,\mu')\, d\mu' \,,$$

$$i = 1,2,\ldots,N, \quad -1 \leq \mu \leq 1 \,, \qquad (2.1)$$

where N represents the number of energy groups, σ_i is the total cross-section associated with energy group i, and the positive elements c_{ij} form what is commonly referred to as the transfer matrix. The transfer matrix takes into account all the processes that lead to the transfer of neutrons from all other energy groups to energy group i. With no loss in generality, we shall assume that the groups are labeled in such a way that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N \,.$$

In addition, we shall set

$$\sigma_N = 1 .$$

Thus, in the multi-group approximation, Eq. (2.1) can be interpreted as determining the group i angular distribution $\Psi_i(x,\mu)$, where the spatial coordinate x is measured in units of the maximum mean free path $(1/\sigma_N)$, and μ is the corresponding direction cosine. The elements σ_i and c_{ij} are now dimensionless quantities.

In accordance with the method of Case, the independent variables of Eq. (2.1) are separated by setting

$$\underset{\sim}{\Psi}(x,\mu) = e^{-x/\nu} \underset{\sim}{\Phi}(\nu,\mu) , \tag{2.2}$$

where

$$\underset{\sim}{\Psi}(x,\mu) = \begin{bmatrix} \Psi_1(x,\mu) \\ \Psi_2(x,\mu) \\ \cdot \\ \cdot \\ \cdot \\ \Psi_N(x,\mu) \end{bmatrix} \tag{2.3}$$

and

$$\underset{\sim}{\Phi}(\nu,\mu) = \begin{bmatrix} \varphi_1(\nu,\mu) \\ \varphi_2(\nu,\mu) \\ \cdot \\ \cdot \\ \cdot \\ \varphi_N(\nu,\mu) \end{bmatrix} . \tag{2.4}$$

The components of $\underset{\sim}{\Phi}(\nu,\mu)$ satisfy the following system of equations:

$$(\sigma_i\nu-\mu)\ \varphi_i(\nu,\mu) = \frac{\nu}{2}\sum_{j=1}^{N} c_{ij}\int_{-1}^{1}\varphi_j(\nu,\mu')\,d\mu',$$

$$i = 1,2,\ \ldots,\ N\ . \tag{2.5}$$

Clearly, the right-hand side of Eq. (2.5) depends only on ν. Let us then define the functions

$$g_i(\nu) \equiv \frac{1}{c}\sum_{j=1}^{N} c_{ij}\int_{-1}^{1}\varphi_j(\nu,\mu)\,d\mu\ , \tag{2.6}$$

where c represents the determinant of the transfer matrix, i.e.,

$$c \equiv \det\left\{c_{ij}\right\}\ . \tag{2.7}$$

The $g_i(\nu)$ are not well-defined if $c = 0$, and we shall therefore exclude this case in the following analysis. Siewert and Zweifel (15) (16) have already treated this particular case in detail.

The general solution to Eq. (2.5) can now be written in the form

$$\varphi_i(\nu,\mu) = \frac{c\nu}{2}\ P\ \frac{g_i(\nu)}{\sigma_i\nu-\mu} + \omega_i(\nu)\ \delta(\sigma_i\nu-\mu)\ , \tag{2.8}$$

$$i = 1,2,\ \ldots\ N\ ,$$

where P indicates that the Cauchy principal value is to be taken in integrals involving this term and δ represents the Dirac delta function. The functions $\omega_i(\nu)$, $i = 1,2, \ldots, N$, must be chosen such that Eq. (2.6) is satisfied. We shall consider the two cases $\nu \in (-1,1)$ and $\nu \notin (-1,1)$ separately.

<u>The Continuum Solution</u>

We first consider the case when the separation parameter ν lies on the interval $(-1,1)$. It is convenient to further divide this interval into subregions. Thus, we shall let region (n) be the union of the intervals $\left(-\frac{1}{\sigma_n}, -\frac{1}{\sigma_{n-1}}\right)$ and $\left(\frac{1}{\sigma_{n-1}}, \frac{1}{\sigma_n}\right)$, where $n = 2,3, \ldots, N$. In particular, region (1) is the interval $\left(-\frac{1}{\sigma_1}, \frac{1}{\sigma_1}\right)$. The eigensolutions for $\nu \in (n)$ will be denoted as follows:

$$\varphi_i^{(n)}(\nu,\mu) = \frac{c\nu}{2} P \frac{g_i^{(n)}(\nu)}{\sigma_i\nu-\mu} + \omega_i^{(n)}(\nu)\, \delta(\sigma_i\nu-\mu) \qquad (2.9)$$

$$i = 1,2, \ldots, N,$$

where the superscript (n) simply means that these functions (distributions) are defined for $\nu \in (n)$. Now, integrating Eq. (2.9) on μ from -1 to $+1$ and applying Eq. (2.6), we have

$$c \sum_{j=1}^{N} \left[\delta_{ij} - c_{ij} \frac{\nu}{2} P \int_{-1}^{1} \frac{du}{\sigma_j \nu - u} \right] g_j^{(n)}(\nu)$$

$$= \sum_{j=1}^{N} c_{ij} \omega_j^{(n)}(\nu) \int_{-1}^{1} \delta(\sigma_j \nu - u)\, du \ , \quad i = 1,2, \ldots, N. \qquad (2.10)$$

Since we are assuming a non-zero value for c, the transfer matrix has a unique inverse with elements

$$c_{ij}^{-1} = \frac{\gamma_{ij}}{c} \ , \qquad (2.11)$$

where γ_{ij} is the cofactor of c_{ji} in the transfer matrix. Hence, multiplying Eq. (2.10) by c_{ki}^{-1} and summing over i, we obtain

$$\sum_{j=1}^{N} \left[\gamma_{kj} - \delta_{kj}\, c\, \nu\, T\left(\frac{1}{\sigma_k \nu}\right) \right] g_j^{(n)}(\nu) = 0 \ , \quad k < n \qquad (2.12)$$

and

$$\sum_{j=1}^{N} \left[\gamma_{kj} - \delta_{kj}\, c\, \nu\, T(\sigma_k \nu) \right] g_j^{(n)}(\nu) = \omega_k^{(n)}(\nu) \ , \quad k \geq n, \qquad (2.13)$$

where

$$T(x) = \tanh^{-1} x \ . \qquad (2.14)$$

Let us define the elements

$$\omega_{kj}^{(n)}(\nu) = \begin{cases} \gamma_{kj} - \delta_{kj}\, c\, \nu\, T\left(\frac{1}{\sigma_k \nu}\right), & k < n \\ \gamma_{kj} - \delta_{kj}\, c\, \nu\, T(\sigma_k \nu), & k \geq n . \end{cases} \tag{2.15}$$

We write Eqs. (2.12) and (2.13) as follows:

$$\sum_{j=1}^{N} \omega_{kj}^{(n)}(\nu)\, g_j^{(n)}(\nu) = \begin{cases} 0, & k < n \\ \omega_k^{(n)}(\nu), & k \geq n . \end{cases} \tag{2.16}$$

Equation (2.16) represents a system of N equations for the N unknown $g_k^{(n)}(\nu)$ and the $N - n + 1$ unknowns $\omega_k^{(n)}(\nu)$. As was pointed out by Siewert and Zweifel (16), there is an $(N - n + 1)$-fold degeneracy in region (n), i.e., for each value of ν in region (n) there are $(N - n + 1)$ linearly independent eigensolutions.

We proceed by writing Eq. (2.16) in the form

$$\sum_{j=1}^{n-1} \omega_{kj}^{(n)}(\nu)\, g_j^{(n)}(\nu) = -\sum_{j=n}^{N} \gamma_{kj}\, g_j^{(n)}(\nu) , \quad k < n . \tag{2.17}$$

Now solving for $g_j^{(n)}(\nu)$, $j = 1, 2, \ldots, n - 1$, we find

$$g_j^{(n)}(\nu) = -\frac{1}{\Omega^{(n)}(\nu)} \sum_{i=1}^{n-1} \sum_{k=n}^{N} \Delta_{ji}^{(n)}(\nu)\, \gamma_{ik}\, g_k^{(n)}(\nu) , \quad j < n \tag{2.18}$$

where $\Omega^{(n)}(\nu)$ is the determinant of the matrix

$$\underset{\sim}{\Omega}^{(n)}(\nu) = \left\{\omega_{ij}^{(n)}(\nu)\right\}, \quad i,j=1,2,\ \ldots,n-1, \tag{2.19}$$

and $\Delta_{ij}^{(n)}(\nu)$ is the cofactor of $\omega_{ji}^{(n)}(\nu)$ in $\underset{\sim}{\Omega}^{(n)}(\nu)$. We shall assume that $\underset{\sim}{\Omega}^{(n)}(\nu)$ is non-singular for $\nu \in (n)$. In addition, we shall set

$$\Omega^{(1)}(\nu) \equiv 1 . \tag{2.20}$$

From Eqs. (2.16) and (2.18), it follows that

$$\omega_{l}^{(n)}(\nu) = \sum_{k=n}^{N} Q_{lk}^{(n)}(\nu)\ A_{k}^{(n)}(\nu)\ , \ l \geq n\ , \quad n=1,2,\ \ldots,\ N, \tag{2.21}$$

where $Q_{lk}^{(n)}(\nu)$ is the even function of ν,

$$Q_{lk}^{(n)}(\nu) = \Omega^{(n)}(\nu)\ \omega_{lk}^{(n)}(\nu) - \sum_{j=1}^{n-1}\sum_{i=1}^{n-1} \gamma_{lj}\ \Delta_{ji}^{(n)}(\nu)\ \gamma_{ik}, \tag{2.22}$$

and

$$A_{k}^{(n)}(\nu) = \frac{g_{k}^{(n)}(\nu)}{\Omega^{(n)}(\nu)} . \tag{2.23}$$

Using these results, we can write Eq. (2.9) in the form

$$\varphi_{i}^{(n)}(\nu,\mu) = A_{i}^{(n)}(\nu)\left\{\frac{c\nu}{2}\ P\ \frac{\Omega^{(n)}(\nu)}{\sigma_{i}\nu-\mu} + Q_{ii}^{(n)}(\nu)\ \delta(\sigma_{i}\nu-\mu)\right\}$$
$$+ \sum_{\substack{j=n\\ j\neq i}}^{N} Q_{ij}^{(n)}(\nu)\ A_{j}^{(n)}(\nu)\ \delta(\sigma_{i}\nu-\mu)\ , \tag{2.24}$$

where $i \geq n$. For $i < n$, we have

$$\varphi_i^{(n)}(\nu,\mu) = -\frac{c\nu}{2}\,\frac{\sum_{j=1}^{n-1}\sum_{k=n}^{N}\Delta_{ij}^{(n)}(\nu)\,\gamma_{jk}\,A_k^{(n)}(\nu)}{\sigma_i\,\nu-\mu}\;. \qquad (2.25)$$

Collecting terms containing $A_j^{(n)}(\nu)$ and using vector notation we can write Eqs. (2.24) and (2.25) in the form

$$\underset{\sim}{\Phi}^{(n)}(\nu,\mu) = \sum_{j=n}^{N} A_j^{(n)}(\nu)\,\underset{\sim}{\Phi}_j^{(n)}(\nu,\mu)\;, \qquad (2.26)$$

where

$$\underset{\sim}{\Phi}_j^{(n)}(\nu,\mu) = \begin{bmatrix} \varphi_{1,j}^{(n)}(\nu,\mu) \\ \varphi_{2,j}^{(n)}(\nu,\mu) \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{N,j}^{(n)}(\nu,\mu) \end{bmatrix} \qquad (2.27)$$

and

$$\varphi_{i,j}^{(n)}(\nu,\mu) = \begin{cases} -\dfrac{c\nu}{2}\,\dfrac{\sum_{k=1}^{n-1}\Delta_{ik}^{(n)}(\nu)\,\gamma_{kj}}{\sigma_i\nu-\mu}\;, & i=1,2,\ldots,n-1 \\ Q_{ij}^{(n)}(\nu)\,\delta(\sigma_i\nu-\mu), & i=n,n+1,\ldots,j-1 \\ \dfrac{c\nu}{2}\,P\,\dfrac{\Omega^{(n)}(\nu)}{\sigma_i\nu-\mu} + Q_{ii}^{(n)}(\nu)\,\delta(\sigma_i\nu-\mu), & i=j \\ Q_{ij}^{(n)}(\nu)\,\delta(\sigma_i\nu-\mu)\;, & i=j+1,j+2,\ldots,N. \end{cases} \qquad (2.28)$$

Since $\underset{\sim}{\Phi}^{(n)}(\nu,\mu)$ is a solution to Eq. (2.5) for completely arbitrary $A_j^{(n)}(\nu)$, $j \geq n$, the $\underset{\sim}{\Phi}_j^{(n)}(\nu,\mu)$ must themselves represent linearly independent solutions to Eq. (2.5).

The Discrete Solutions

If $\nu \notin (-1,1)$, the eigensolutions have the form

$$\varphi_i(\nu,\mu) = \frac{c\nu}{2} \frac{g_i(\nu)}{\sigma_i\nu-\mu} \quad . \tag{2.29}$$

The spectrum of ν can be determined by applying Eq. (2.6) to Eq. (2.29). Integrating Eq. (2.29) on μ from -1 to +1 and using Eq. (2.6), we have

$$\sum_{j=1}^{N} \left[\delta_{ij} - c_{ij}\, \nu\, T\left(\frac{1}{\sigma_j\nu}\right)\right] g_j(\nu) = 0 \quad . \tag{2.30}$$

Multiplying the last equation by c_{ki}^{-1} and summing over i, we obtain the following system of homogeneous equations for $g_j(\nu)$:

$$\sum_{j=1}^{N} \omega_{kj}(\nu)\, g_j(\nu) = 0 \;, \qquad k=1,2,\ldots,N, \tag{2.31}$$

where

$$\omega_{kj}(\nu) = \gamma_{kj} - \delta_{kj}\, c\, \nu\, T\left(\frac{1}{\sigma_k\nu}\right) \;, \quad \nu \notin (-1,1) \;. \tag{2.32}$$

Let us consider the coefficient matrix of Eq. (2.31):

$$\underset{\sim}{\Omega}(\nu) = \left\{\omega_{ij}(\nu)\right\}, \quad i,j=1,2,\ldots,N . \tag{2.33}$$

The nontrivial solutions of Eq. (2.31) occur at the zeros of $\Omega(\nu)$, where

$$\Omega(\nu) \equiv \det \underset{\sim}{\Omega}(\nu) . \tag{2.34}$$

Now, $\Omega(\nu)$ must be an even function of ν since the elements $\omega_{ij}(\nu)$ are even in ν. Thus, we shall denote the zeros of $\Omega(\nu)$ by $\pm\nu_s$, $s=0,1,\ldots,\alpha-1$, where α is the number of pairs of zeros. In Appendix A, the general procedure for determining α is outlined. We make the following assumptions:

(a) There are no multiple roots of $\Omega(\nu)$.

(b) The rank of $\underset{\sim}{\Omega}(\pm\nu_s)$ is $N-1$.

(c) $\Omega^{(N)}(\nu_s) \neq 0$.

Under these assumptions there is only one linearly independent solution of Eq. (2.31) corresponding to each zero $\pm\nu_s$ and this solution can be expressed in terms of $g_N(\pm\nu_s)$. Thus, we have

$$g_j(\pm\nu_s) = -a_{s\pm} \sum_{k=1}^{N-1} \Delta_{jk}^{(N)}(\nu_s)\, \gamma_{kN} \tag{2.35}$$

$$j=1,2,\ldots,N-1, \quad s=0,1,\ldots,\alpha-1 ,$$

where

$$a_{s\pm} = \frac{g_N(\pm\nu_s)}{\Omega^{(N)}(\nu_s)} \quad . \tag{2.36}$$

Using the above assumptions, we have found that there are 2α discrete eigensolutions of Eq. (2.5) which correspond to the case $\nu \notin (-1,1)$. These eigensolutions will be denoted by

$$\underset{\sim}{\Phi}_{s\pm}(\mu) = \begin{bmatrix} \varphi_{1,s\pm}(\mu) \\ \varphi_{2,s\pm}(\mu) \\ \vdots \\ \varphi_{N,s\pm}(\mu) \end{bmatrix} , \quad s=0,1,\ldots,\alpha-1 \tag{2.37}$$

where

$$\varphi_{i,s\pm}(\mu) = -\frac{c\nu_s}{2} \frac{\sum_{k=1}^{N-1} \Delta_{ik}^{(N)}(\nu_s)\, \gamma_{kN}}{\sigma_i\nu_s \mp \mu} , \quad i=1,2,\ldots,N-1, \tag{2.38}$$

and

$$\varphi_{N,s\pm}(\mu) = \frac{c\nu_s}{2} \frac{\Omega^{(N)}(\nu_s)}{\sigma_N\nu_s \mp \mu} \quad . \tag{2.39}$$

The above determination of the discrete and continuum eigensolutions closely parallels the work of Zelasny and Kuszell (<u>18</u>). In fact, the essential feature of these eigensolutions, that they form a complete set on the interval

(-1,1), has been known since 1962. We shall give a proof of this completeness property below. We have made an important deviation from the development of Zelazny and Kuszell, however, in that we have explicity represented the degenerate eigensolutions of the continuum. It is helpful to do this because in this representation some useful orthogonality relations between the eigensolutions can be obtained. This fact was first noted by Siewert and Zweifel (15) in considering radiative transfer problems involving the picket fence model. Later, Siewert and Shieh (14) allowed for the degeneracy of the eigensolutions in their studies of the two-group neutron transport equations.

Let us consider the following system of equations

$$(\sigma_1\nu-\mu)\,\varphi_i^{\dagger}(\nu,\mu) = \frac{\nu}{2}\sum_{j=1}^{N} c'_{ij}\int_{-1}^{1}\varphi_j^{\dagger}(\nu,\mu')\,d\mu' , \tag{2.40}$$

$$i=1,2,\dots,N\,,\quad -1 \leq \mu \leq 1\,,$$

where the c'_{ij} are the elements of the transposed transfer matrix. Equation (2.40) is said to be adjoint to Eq. (2.5) (14). From the analysis of Eq. (2.5), it is clear that the adjoint solutions $\underset{\sim}{\Phi}^{\dagger}(\nu,\mu)$ exhibit an (N-n+1)-fold degeneracy in region (n). These degenerate solutions have the form

$$\underset{\sim}{\Phi}_j^{(n)\dagger}(\nu,\mu) = \begin{bmatrix} \varphi_{1,j}^{(n)\dagger}(\nu,\mu) \\ \varphi_{2,j}^{(n)\dagger}(\nu,\mu) \\ \vdots \\ \varphi_{N,j}^{(n)\dagger}(\nu,\mu) \end{bmatrix}, \quad j=n,n+1,\ldots,N \qquad (2.41)$$

where

$$\varphi_{i,j}^{(n)\dagger}(\nu,\mu) = \begin{cases} -\dfrac{c\nu}{2} \dfrac{\sum\limits_{k=1}^{n-1} \gamma_{jk} \Delta_{ki}^{(n)}(\nu)}{\sigma_i\nu-\mu} & , \; i=1,2,\ldots,n-1 \\ Q_{ji}^{(n)}(\nu) \; \delta(\sigma_i\nu-\mu) & , \; i=n,n+1,\ldots,j-1 \\ \dfrac{c\nu}{2} \, P \, \dfrac{\Omega^{(n)}(\nu)}{\sigma_i\nu-\mu} + Q_{ii}^{(n)}(\nu) \; \delta(\sigma_i\nu-\mu) & , \; i=j \\ Q_{ji}^{(n)}(\nu) \; \delta(\sigma_i\nu-\mu) & , \; i=j+1,j+2,\ldots,N. \end{cases}$$

(2.42)

If we again make the assumptions listed under Eq. (2.34), there are 2α discrete eigensolutions of the adjoint system, and these have the form

$$\underset{\sim}{\Phi}_{s\pm}^{\dagger}(\mu) = \begin{bmatrix} \varphi_{1,s\pm}^{\dagger}(\mu) \\ \varphi_{2,s\pm}^{\dagger}(\mu) \\ \vdots \\ \varphi_{N,s\pm}^{\dagger}(\mu) \end{bmatrix}, \quad s=0,1,\ldots,\alpha-1 \qquad (2.43)$$

where

$$\varphi^{\dagger}_{i,s\pm}(\mu) = -\frac{c\nu_s}{2}\,\frac{\sum_{k=1}^{N-1} \gamma_{Nk}\,\bar{\Delta}^{(N)}_{ki}(\nu_s)}{\sigma_i\nu_s \mp \mu}\,, \quad i=1,2,\ldots,N-1 \tag{2.44}$$

and

$$\varphi^{\dagger}_{N,s\pm}(\mu) = \frac{c\nu_s}{2}\,\frac{\Omega^{(N)}(\nu_s)}{\sigma_N\nu_s \mp \mu}\;. \tag{2.45}$$

Orthogonality Relations

Let us arbitrarily choose two eigensolutions of Eq. (2.5) and (2.40), which we shall denote by $\underset{\sim}{\Phi}(\nu,\mu)$ and $\underset{\sim}{\Phi}^{\dagger}(\eta,\mu)$ respectively. We wish to prove the following important theorem.

Theorem (2.1). The eigensolutions $\underset{\sim}{\Phi}(\nu,\mu)$ and $\underset{\sim}{\Phi}^{\dagger}(\eta,\mu)$ are orthogonal on the interval $(-1 \leq \mu \leq 1)$ with respect to the weight function μ . That is

$$\int_{-1}^{1} \mu\left[\underset{\sim}{\Phi}^{\dagger}(\eta,\mu)\right]'\underset{\sim}{\Phi}(\nu,\mu)\,d\mu = 0 \quad , \quad \nu \neq \eta, \tag{2.46}$$

where the prime (') again denotes the transpose operation.

To prove this theorem, let us consider Eqs. (2.5) and (2.40):

$$\left(-\frac{\mu}{\nu}+\sigma_i\right)\varphi_i(\nu,\mu)=\frac{1}{2}\sum_{j=1}^{N}c_{ij}\int_{-1}^{1}\varphi_j(\nu,\mu')d\mu' \qquad (2.47)$$

$$\left(-\frac{\mu}{\eta}+\sigma_i\right)\varphi_i^{\dagger}(\eta,\mu)=\frac{1}{2}\sum_{j=1}^{N}c'_{ij}\int_{-1}^{1}\varphi_j^{\dagger}(\eta,\mu')d\mu' \quad , \qquad (2.48)$$

$$i=1,2,\ldots,N \; .$$

If we multiply Eq. (2.47) by $\varphi_i^{\dagger}(\eta,\mu)$ and Eq. (2.48) by $\varphi_i(\nu,\mu)$, integrate both on μ over (-1,1) and then sum over i, we obtain

$$\sum_{i=1}^{N}\int_{-1}^{1}\left(-\frac{\mu}{\nu}+\sigma_i\right)\varphi_i^{\dagger}(\eta,\mu)\,\varphi_i(\nu,\mu)\,d\mu$$

$$=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}c_{ij}\int_{-1}^{1}\varphi_j(\nu,\mu')\,d\mu'\int_{-1}^{1}\varphi_i^{\dagger}(\eta,\mu)\,d\mu \qquad (2.49)$$

$$\sum_{i=1}^{N}\int_{-1}^{1}\left(-\frac{\mu}{\eta}+\sigma_i\right)\varphi_i^{\dagger}(\eta,\mu)\,\varphi_i(\nu,\mu)\,d\mu$$

$$=\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}c'_{ij}\int_{-1}^{1}\varphi_i(\nu,\mu')\,d\mu'\int_{-1}^{1}\varphi_j^{\dagger}(\eta,\mu)\,d\mu \qquad (2.50)$$

Subtracting Eq. (2.49) from Eq. (2.50), we find

$$\left(\frac{1}{\nu} - \frac{1}{\eta}\right) \sum_{i=1}^{N} \int_{-1}^{1} \mu \, \varphi_i^{\dagger}(\eta,\mu) \, \varphi_i(\nu,\mu) \, d\mu = 0 \; ; \qquad (2.51)$$

thus

$$\int_{-1}^{1} \mu \left[\underset{\sim}{\Phi}^{\dagger}(\eta,\mu) \right]' \underset{\sim}{\Phi}(\nu,\mu) \, d\mu = 0 \quad , \text{ if } \nu \neq \eta \, . \qquad (2.52)$$

Since the degenerate solutions of the continuum can always be orthogonalized by using a Schmidt-type procedure, all the eigensolutions of Eq. (2.5) are orthogonal, or can be made so. We shall not carry out the orthogonalization procedure in this thesis, however, since we shall not be needing these results in our present analysis.

The normalization integrals, corresponding to the case when $\eta = \nu$ in Eq. (2.46), can be evaluated by using the explicit forms of the eigensolutions given by Eqs. (2.27), (2.37), (2.41), and (2.43). We find

$$\int_{-1}^{1} \mu \left[\underset{\sim}{\Phi}_{s\pm}^{\dagger}(\mu) \right]' \underset{\sim}{\Phi}_{s\pm}(\mu) \, d\mu = N_{s\pm} \; , \quad s=0,1,\ldots,\alpha-1 \qquad (2.53)$$

and

$$\int_{-1}^{1} \mu \left[\underset{\sim}{\Phi}_{k}^{(n)\dagger}(\eta,\mu) \right]' \underset{\sim}{\Phi}_{l}^{(n)}(\nu,\mu) d\mu = N_{kl}^{(n)}(\nu) \, \delta(\nu-\eta) \qquad (2.54)$$

$$k,l=n,n+1,\ldots,N, \quad n=1,2,\ldots,N,$$

where

$$N_{s\pm} = \pm \frac{c^2\nu_s^2}{2} \left\{ \sum_{i=1}^{N-1} \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \gamma_{Nk} \, \Delta_{ki}^{(N)}(\nu_s) \, \Delta_{il}^{(N)}(\nu_s) \, \gamma_{lN} \left[\frac{\sigma_i\nu_s}{(\sigma_i\nu_s)^2-1} - T\left(\frac{1}{\sigma_i\nu_s}\right)\right] + \left[\Omega^{(N)}(\nu_s)\right]^2 \left[\frac{\sigma_N\nu_s}{(\sigma_N\nu_s)^2-1} - T\left(\frac{1}{\sigma_N\nu_s}\right)\right]\right\} \quad (2.55)$$

and

$$N_{kl}^{(n)}(\nu) = \nu \left\{ \sum_{i=n}^{N} Q_{ki}^{(n)}(\nu) \, Q_{il}^{(n)}(n) + \delta_{kl} \frac{\pi^2c^2\nu^2}{4} \left[\Omega^{(n)}(\nu)\right]^2 \right\} \quad (2.56)$$

In the derivation of Eq. (2.54), we have used the partial fraction decomposition

$$\mu \, P\frac{1}{\sigma_k\eta-\mu} \, P\frac{1}{\sigma_k\nu-\mu} = \frac{1}{\eta-\nu}\left[P\frac{\nu}{\sigma_k\nu-\mu} - P\frac{\eta}{\sigma_k\eta-\mu}\right] + \mu\,\pi^2\,\delta(\sigma_k\nu-\mu)\,\delta(\sigma_k\eta-u), \quad (2.57)$$

which is a generalization of a similar expression given by Kuscer, McCormick and Summerfield (8).

Completeness Theorem

We now prove the most important property of the eigensolutions: completeness. The following theorem provides the exact statement of this property.

Theorem (2.2). The discrete eigensolutions $\underset{\sim}{\Phi}_{s\pm}(\mu)$ together with the eigensoltuions $\underset{\sim}{\Phi}_p^{(n)}(\nu,\mu)$ of the continuum form a complete set in the sense that any sufficiently well-behaved N-component vector $\underset{\sim}{V}(\mu)$, $-1 \leq \mu \leq 1$, can be represented in the form

$$\underset{\sim}{V}(\mu) = \sum_{s=0}^{\alpha-1} \left[a_{s+} \underset{\sim}{\Phi}_{s+}(\mu) + a_{s-} \underset{\sim}{\Phi}_{s-}(\mu) \right]$$

$$+ \sum_{n=1}^{N} \int_{<n>} \sum_{p=n}^{N} A_p^{(n)}(\nu) \, \underset{\sim}{\Phi}_p^{(n)}(\nu,\mu) \, d\nu \,, \qquad (2.58)$$

$$-1 \leq \mu \leq 1 \,.$$

In Eq. (2.58), the $a_{s\pm}$ and $A_p^{(n)}(\nu)$ are arbitrary expansion coefficients, and the range of integration, denoted by $<n>$, is over each of the intervals $\left(-\frac{1}{\sigma_n}, -\frac{1}{\sigma_{n-1}}\right)$ and $\left(\frac{1}{\sigma_{n-1}}, \frac{1}{\sigma_n}\right)$. A sufficient condition on $\underset{\sim}{V}(\mu)$ is that its components obey an H^* condition on $(-1,1)$. Specifically, a function $f(\mu)$ is said to obey an H^* condition of the interval (a,b) if there exists a constant β and a positive number ξ such that

$$\left| f(\mu) - f(\mu') \right| < \beta \left| \mu - \mu' \right|^{\xi} \qquad (2.59)$$

for $a < \mu, \mu' < b$, and if near the end points (a or b) $f(\mu)$

can be written in the form

$$f(\mu) = \frac{\tilde{f}(\mu)}{(\mu-d)^{\delta}} \;, \qquad 0 \leq \delta \leq 1 \;, \tag{2.60}$$

when d stands for either a or b and $\tilde{f}(\mu)$ satisfies Eq. (2.59) on the closed interval (a,b). It appears, however, that the necessary conditions on the components of $\underset{\sim}{V}(\mu)$ are even weaker than an H^* condition, and that Eq. (2.58) is satisfied by all N-component vectors of physical interest.

If the expansion given by Eq. (2.58) is truly a representation of $\underset{\sim}{V}(\mu)$, the orthogonality of the eigensolutions can be used to obtain the expansion coefficients. In particular, using Eq. (2.53), we find the discrete coefficients would have the form

$$a_{s\pm} = \frac{1}{N_{s\pm}} \int_{-1}^{1} \mu \left[\underset{\sim}{\Phi}^{\dagger}_{s\pm}(\mu)\right]' \underset{\sim}{V}(\mu)\, d\mu \;. \tag{2.61}$$

Thus, to prove Theorem (2.2), it is sufficient to demonstrate that the continuum coefficients $A_p^{(n)}(\nu)$, p=n,n+1, ...,N, can be determined from the equation

$$\underset{\sim}{V}^*(\mu) = \sum_{n=1}^{N} \int_{<n>} \sum_{p=n}^{N} A_p^{(n)}(\nu)\, \underset{\sim}{\Phi}_p^{(n)}(\nu,\mu)\, d\nu \;, \quad \mu \in (-1,1) \tag{2.62}$$

if $\underset{\sim}{V}^*(\mu)$ has the form

$$\underset{\sim}{V}^*(\mu) = \underset{\sim}{V}(\mu) - \sum_{s=0}^{\alpha-1} \left[a_{s+} \underset{\sim}{\Phi}_{s+}(\mu) + a_{s-} \underset{\sim}{\Phi}_{s-}(\mu) \right] , \quad \mu \in (-1,1) \tag{2.63}$$

with the $a_{s\pm}$ given by Eq. (2.61). To show this, we choose to write the components of $\sum_{p=n}^{N} A_p^{(n)}(\nu) \underset{\sim}{\Phi}_p^{(n)}(\nu,\mu)$ in the form

$$\sum_{p=n}^{N} A_p^{(n)}(\nu)\, \varphi_{i,p}^{(n)}(\nu,\mu) = \begin{cases} \dfrac{c\nu}{2} \dfrac{\Omega^{(n)}(\nu)\, A_i^{(n)}(\nu)}{\sigma_i\nu-\mu} , & i < n \\[2ex] \dfrac{c\nu}{2} P \dfrac{\Omega^{(n)}(\nu)\, A_i^{(n)}(\nu)}{\sigma_i\nu-\mu} + \Omega^{(n)}(\nu) \displaystyle\sum_{j=1}^{N} \omega_{ij}^{(n)}(\nu) A_j^{(n)}(\nu)\, \delta(\sigma_i\nu-\mu) , & i \geq n \end{cases} \tag{2.64}$$

Equation (2.64) was obtained from Eqs. (2.26), (2.9), (2.16), and (2.23). We must keep in mind, however, that once the coefficients $A_i^{(n)}(\nu)$, $i \geq n$, have been determined for region (n), the remaining coefficients $A_i^{(n)}(\nu)$, $i < n$, are specified by Eqs. (2.18) and (2.23).

Making the change of variables $\sigma_i\mu' = \mu$ and denoting the components of $\underset{\sim}{V}^*(\mu)$ by $V_i^*(\mu)$, $i=1,2,\ldots,N$, we substitute Eq. (2.64) into Eq. (2.62) to obtain

$$\sigma_i V_i^*(\sigma_i\mu') = \sum_{n=1}^{i} P \int_{<n>} \Big\{ \frac{c\nu}{2} \frac{\Omega^{(n)}(\nu) A_i^{(n)}(\nu)}{\nu-\mu'}$$

$$+ \Omega^{(n)}(\nu) \sum_{j=1}^{N} \omega_{ij}^{(n)}(\nu) A_j^{(n)}(\nu) \delta(\nu-\mu') \Big\} d\nu$$

$$+ \sum_{n=i+1}^{N} \int_{n} \frac{c\nu}{2} \frac{\Omega^{(n)}(\nu) A_i^{(n)}(\nu)}{\nu-\mu'} d\nu , \quad \mu' \epsilon \left(-\frac{1}{\sigma_i}, \frac{1}{\sigma_i}\right)$$

(2.65)

Let us now consider a subregion of $\left(-\frac{1}{\sigma_i}, \frac{1}{\sigma_i}\right)$. Following the convention which we introduced earlier, we shall denote this subregion by (k), when $k \leq i$. Then, performing the integrations over the Dirac delta functions, and dropping the primes, we write Eq. (2.65) in the form

$$\sigma_i V_i^*(\sigma_i\mu) = \Omega^{(k)}(\mu) \sum_{j=1}^{N} \omega_{ij}^{(k)}(\mu) A_j^{(k)}(\mu)$$

$$+ \sum_{n=1}^{N} P \int_{<n>} \frac{c\nu}{2} \frac{\Omega^{(n)}(\nu) A_i^{(n)}(\nu)}{\nu-\mu} d\nu , \qquad (2.66)$$

$$\mu \ \epsilon \ (k), \ k \leq i \quad i=1,2,\ldots,N .$$

Equation (2.66) represents a system of singular integral equations for the expansion coefficients $A_i^{(n)}(\nu)$. Our objective is to show that a solution to this system exists. To do this we use the methods developed by Muskhelishvili (11).

Initially, we assume that $A_i^{(n)}(\nu)$ exists for $\nu \in (n)$ and introduce the functions

$$N_i(z) = \frac{1}{2\pi i} \sum_{n=1}^{N} \int_{<n>} \frac{c\nu}{2} \frac{\Omega^{(n)}(\nu)\, A_i^{(n)}(\nu)}{\nu - z}\, d\nu \,, \qquad i=1,2,\ldots,N \,. \tag{2.67}$$

These functions have the following properties:

(a) $N_i(z)$ is analytic in the complex plane cut from -1 to +1.

(b) $N_i(z) \sim 1/z$ as $z \to \infty$.

We shall assume that the cut (-1,1) in the complex plane is directed from -1 to +1. Then, using Plemelj's formulas (11), we can find the limiting values of $N_i(z)$ as z approaches the cuts $\left(-\frac{1}{\sigma_k}, -\frac{1}{\sigma_{k-1}}\right)$ and $\left(\frac{1}{\sigma_{k-1}}, \frac{1}{\sigma_k}\right)$ from the left (+) and right (-):

$$N_i^{\pm}(\mu) = \frac{1}{2\pi i} \sum_{n=1}^{N} P \int_{<n>} \frac{c\nu}{2} \frac{\Omega^{(n)}(\nu)\, A_i^{(n)}(\nu)}{\nu - \mu}\, d\nu$$

$$\pm \frac{c\mu}{4}\, \Omega^{(k)}(\mu)\, A_i^{(k)}(\mu), \qquad \mu \in (k) \,. \tag{2.68}$$

Hence

$$N_i^+(\mu) - N_i^-(\mu) = \frac{c\mu}{2}\,\Omega^{(k)}(\mu)\,A_i^{(k)}(\mu)\,, \qquad \mu \in (k) \tag{2.69}$$

$$N_i^+(\mu) + N_i^-(\mu) = \frac{1}{i\pi}\sum_{n=1}^{N} P\int_{\langle n\rangle} \frac{c\nu}{2}\,\frac{\Omega^{(n)}(\nu)\,A_i^{(n)}(\nu)}{\nu-\mu}\,d\nu\,,$$

$$\mu \in \left(-\frac{1}{\sigma_i}\,,\,\frac{1}{\sigma_i}\right). \tag{2.70}$$

Similarly, using Eqs. (2.15) and (2.32), we note that the limiting values of $\omega_{ij}(z)$ as z approaches the cut (k) from the left and right are

$$\omega_{ij}^{\pm}(\mu) = \omega_{ij}^{(k)}(\mu) \pm \frac{i\pi c\mu}{2}\,\delta_{ij}\,, \qquad \mu \in (k)\,,\ i \geq k. \tag{2.71}$$

Therefore

$$\frac{1}{2}\left[\omega_{ij}^{+}(\mu)-\omega_{ij}^{-}(\mu)\right] = \frac{i\pi c\mu}{2}\,\delta_{ij}\,, \qquad \mu \in (k)\,,\ i \geq k \tag{2.72}$$

$$\frac{1}{2}\left[\omega_{ij}^{+}(\mu)+\omega_{ij}^{-}(\mu)\right] = \omega_{ij}^{(k)}(\mu)\,, \qquad \mu \in (k). \tag{2.73}$$

We can now write Eq. (2.66) in terms of $N_i^{\pm}$ and $\omega_{ij}^{\pm}$ to obtain the following boundary conditions:

$$\frac{c\mu}{2}\,\sigma_i V_i^*(\sigma_i\mu) = \sum_{j=1}^{N}\left[\omega_{ij}^+(\mu)\,N_j^+(\mu) - \omega_{ij}^-(\mu)\,N_j^-(\mu)\right],$$

$$\mu \in \left(-\frac{1}{\sigma_i}, \frac{1}{\sigma_i}\right), \quad i=1,2,\ldots,N. \tag{2.74}$$

Thus the problem is reduced to a non-homogeneous matrix Hilbert problem (11): to find the vector $\underset{\sim}{N}(z)$ with the properties listed under Eq. (2.67) which satisfies Eq. (2.74). If this Hilbert problem can be solved, the $A_i^{(n)}(\nu)$ exist in region (n) and can be determined from Eq. (2.69). To find the $N_i(z)$, we shall make a departure from the standard procedure of Muskhelishvili and use a trick which was used by Leonard and Ferziger (9) in solving a similar boundary value problem. This trick involves a generalization of the one used by Case (4) in proving the completeness theorem for the eigensolutions of the one-speed approximation.

Consider the functions

$$F_i(z) = \sum_{j=1}^{N} \omega_{ij}(z)\,N_j(z) - \frac{1}{2\pi i}\int_{-1}^{1} \frac{c\nu}{2}\,\frac{V_i^*(\nu)}{\nu-\sigma_i z}\,d\nu,$$

$$i=1,2,\ldots,N. \tag{2.75}$$

The functions $\omega_{ij}(z)$ and the integral in Eq. (2.75) are analytic in the plane cut from $-\frac{1}{\sigma_i}$ to $+\frac{1}{\sigma_i}$; $N_i(z)$ is analytic in the plane cut from -1 to +1. Thus, $F_i(z)$ is analytic everywhere in the complex plane, except possibly along a cut from -1 to +1. Moreover, $F_i(z)$ vanishes as $|z| \rightarrow \infty$. Let us, therefore, consider the limiting values of $F_i(z)$ as z approaches region (k). For values of $i \geq k$, we find

$$F_i^{\pm}(u) = \sum_{j=1}^{N} \omega_{ij}^{\pm}(u)\, N_j^{\pm}(u) - \frac{1}{2\pi i} P \int_{-1}^{1} \frac{c\nu}{2} \frac{V_i^*(\nu)}{\nu - \sigma_i u} d\nu$$

$$\mp \frac{cu}{4} \sigma_i V_i^*(u) \quad , \qquad u \in (k),\ i \geq k . \tag{2.76}$$

It follows from Eq. (2.74) that

$$F_i^+(u) - F_i^-(u) = 0 \ , \quad u \in (k) \ , \quad i \geq k. \tag{2.77}$$

Next consider the limiting values of $F_i(z)$ as z approaches region (k), where $i < k$. In this case we have

$$F_i^{\pm}(u) = \sum_{j=1}^{N} \omega_{ij}^{(k)}(u)\, N_j^{\pm}(u) - \frac{1}{2\pi i} \int_{-1}^{1} \frac{c\nu}{2} \frac{V_i^*(\nu)}{\nu - \sigma_i u} d\nu \ ,$$

$$u \in (k), \quad i < k \ ; \tag{2.78}$$

thus

$$F_i^+(u) - F_i^-(u) = \sum_{j=1}^{N} \omega_{ij}^{(k)}(u) \left[N_j^+(u) - N_j^-(u) \right],$$

$$u \in (k), \quad i < k . \tag{2.79}$$

From Eq. (2.69), we have

$$F_i^+(\mu) - F_i^-(\mu) = \frac{c\mu}{2} \sum_{j=1}^{N} \omega_{ij}^{(k)}(\mu)\, \Omega^{(k)}(\mu)\, A_i^{(k)}(\mu),$$

$$\mu \in (k)\,, \quad i < k\,. \qquad (2.80)$$

It follows from Eq. (2.16) that

$$F_i^+(\mu) - F_i^-(\mu) = 0, \qquad \mu \in (k),\ i < k. \qquad (2.81)$$

We have shown that $F_i(z)$ is continuous across the cut (-1,1). Therefore, $F_i(z)$ is an entire function which vanishes at infinity. We conclude, using Liouville's theorem, that

$$F_i(z) \equiv 0\,, \qquad i=1,2,\ldots,N\,. \qquad (2.82)$$

From Eq. (2.75), we obtain the following system of equations for $N_i(z)$, $i=1,2,\ldots,N$:

$$\sum_{j=1}^{N} \omega_{ij}(z)\, N_j(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{c\nu}{2} \frac{V_i^*(\nu)}{\nu - \sigma_i z}\, d\nu, \qquad (2.83)$$

$$i=1,2,\ldots,N\,.$$

It is clear that this system of equations has the solution

$$N_i(z) = \frac{1}{\Omega(z)} \sum_{j=1}^{N} \Delta_{ij}(z) \frac{1}{2\pi i} \int_{-1}^{1} \frac{c\nu}{2} \frac{V_j^*(\nu)}{\nu-\sigma_j z} d\nu , \tag{2.84}$$

$$i=1,2,\ldots,N ,$$

where $\Delta_{ij}(z)$ is the cofactor of $\omega_{ji}(z)$ in $\underset{\sim}{\Omega}(z)$. Since $\Delta_{ij}(z)$ and $\Omega(z)$ approach constant values as $|z| \to \infty$, this representation of $N_i(z)$ vanishes at infinity. Moreover, it obeys the boundary condition given by Eq. (2.74), and is analytic in the plane cut from -1 to +1, except perhaps at $z = \pm\nu_s$, $s=0,1,\ldots,\alpha-1$, when $\Omega(z)$ is equal to zero. Thus, all the required properties of $N_i(z)$ are satisfied if

$$\sum_{j=1}^{N} \Delta_{ij}(\nu_s) \int_{-1}^{1} \frac{c\nu}{z} \frac{V_i^*(\nu)}{\nu-\sigma_i\nu_s} d\nu = 0 \quad s=0,1,\ldots,\alpha-1 . \tag{2.85}$$

This condition is not satisfied in general, but we shall show that it does hold if $V_j^*(\nu)$ has the form given by Eq. (2.63).

First, we shall verify that Eq. (2.83) has a solution when $z = \pm\nu_s$, $s=0,1,\ldots,\alpha-1$. From the theory of linear equations, we know that a solution exists at the zeros of $\Omega(z)$ if

$$\sum_{i=1}^{N} Y_i(\pm\nu_s) \int_{-1}^{1} \frac{c\nu}{2} \frac{V_i^*(\nu)}{\nu-\sigma_i\nu_s} d\nu = 0, \quad s=0,1,\ldots,\alpha-1 , \tag{2.86}$$

where $Y_i(\pm\nu_s)$, i=1,2,...,N, is the solution to the following system of homogeneous equations:

$$\sum_{j=1}^{N} \omega'_{ij}(\nu_s)\; Y_j(\pm\nu_s) = 0\;, \quad i=1,2,\ldots,N. \qquad (2.87)$$

The coefficient matrix in Eq. (2.87) is the transpose of $\underset{\sim}{\Omega}(\nu_s)$.

To be consistent, we again make the assumptions listed under Eq. (2.34). The solution of Eq. (2.87) can then be written in the form

$$Y_i(\pm\nu_s) = -\frac{Y_N(\pm\nu_s)}{\Omega^{(N)}(\nu_s)} \sum_{k=1}^{N-1} \gamma_{Nk}\, \Delta_{ki}^{(N)}(\nu_s)\;, \quad i=1,2,\ldots,N-1, \qquad (2.88)$$

where $Y_N(\pm\nu_s)$ is unspecified. Comparing this solution with Eq. (2.43), we see that Eq. (2.86) is equivalent to the requirement that

$$\int_{-1}^{1} \mu \left[\underset{\sim}{\Phi}{}^{\dagger}_{\pm}(\mu)\right]' \underset{\sim}{V}^{*}(\mu)\; d\mu = 0\;, \quad s=0,1,\ldots,\alpha-1. \qquad (2.89)$$

It is easily verified, using the orthogonality relations between the discrete eigensolutions, that if $\underset{\sim}{V}^{*}(\mu)$ has the form given by Eq. (2.63), then Eq. (2.89) is satisfied. This fact ensures us that Eq. (2.85) is satisfied. Let us denote the components of a particular solution of Eq. (2.83) at $\pm\nu_s$ by $N_i^p(\pm\nu_s)$, $i=1,2,\ldots,N$. Then we can write

$$\sum_{j=1}^{N} \omega_{ij}(\nu_s)\, N_j^p(\pm\nu_s) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{c\nu}{2} \frac{V_i^*(\mu)}{\mu \mp \sigma_i \nu_s} d\mu , \tag{2.90}$$

$$i=1,2,\ldots,N .$$

Multiplying both sides of Eq. (2.90) by $\Delta_{ki}(\nu_s)$ and summing over i, we obtain

$$\sum_{j=1}^{N} \left[\sum_{i=1}^{N} \Delta_{ki}(\nu_s)\, \omega_{ij}(\nu_s) \right] N_j^p(\pm\nu_s)$$

$$= \frac{1}{2\pi i} \sum_{i=1}^{N} \Delta_{ki}(\nu_s) \int_{-1}^{1} \frac{c\mu}{2} \frac{V_i^*(\mu)}{\mu \mp \sigma_i \nu_s} d\mu , \quad k=1,2,\ldots,N . \tag{2.91}$$

Now, we note that $\underset{\sim}{\Omega}(\nu_s)$ has rank N-1, and that

$$\underset{\sim}{\Omega}_a(\nu_s) = \left\{ \Delta_{ij}(\nu_s) \right\}, \tag{2.92}$$

$$i,j=1,2,\ldots,N,$$

is the adjoint matrix of $\underset{\sim}{\Omega}(\nu_s)$. Thus, from the theory of matrices, we have

$$\sum_{i=1}^{N} \Delta_{ki}(\nu_s)\, \omega_{ij}(\nu_s) = 0 . \tag{2.93}$$

Equation (2.85) follows immediately from Eq. (2.91) and (2.93).

With Eq. (2.85) we can reason that $N_1(z)$ has removable singularities at $z = \pm\nu_s$, $s=0,1,\ldots,\alpha-1$. Thus $N_1(z)$, given by Eq. (2.84), satisfies all the requirements that it should, if $\underset{\sim}{V}^*(u)$ has the form shown in Eq. (2.63). This completes the proof of Theorem (2.2).

As a consequence of Theorem (2.2), we have the following theorem.

Theorem (2.3). The general solution of the multi-group equations can be written in the form

$$\underset{\sim}{\Psi}(x,u) = \sum_{s=0}^{\alpha-1} \left[a_{s+} \underset{\sim}{\Phi}_{s+}(u)\, e^{-x/\nu_s} + a_{s-} \underset{\sim}{\Phi}_{s-}(u)\, e^{x/\nu_s} \right]$$

$$+ \sum_{n=1}^{N} \int_{\langle n \rangle} \sum_{p=n}^{N} A_p^{(n)}(\nu)\, \underset{\sim}{\Phi}_p^{(n)}(\nu,\mu)\, e^{-x/\nu}\, d\nu \,, \quad -1 \leq \mu \leq 1 \,, \tag{2.94}$$

where $a_{s\pm}$ and $A_p^{(n)}(\nu)$ are arbitrary expansion coefficients.

To prove this theorem, we use the completeness property of the eigensolutions to write $\underset{\sim}{\Psi}(x,u)$ in the form

$$\underset{\sim}{\Psi}(x,\mu) = \sum_{s=0}^{\alpha-1} \left[b_{s+}(x)\, \underset{\sim}{\Phi}_{s+}(\mu) + b_{s-}(x)\, \underset{\sim}{\Phi}_{s-}(\mu) \right]$$

$$+ \sum_{n=1}^{N} \int_{<n>} \sum_{p=n}^{N} B_p^{(n)}(x,\nu)\, \Phi_p^{(n)}(\nu,\mu)\, d\nu , \tag{2.95}$$

where the expansion coefficients are functions of x. Substitution of Eq. (2.95) into Eq. (2.1) yields

$$\sum_{s=0}^{\alpha-1} \left\{ \left[\mu \frac{\partial b_{s+}(x)}{\partial x} + \sigma_i b_{s+}(x) \right] \varphi_{i,s+}(\mu) \right.$$

$$\left. + \left[\mu \frac{\partial b_{s-}(x)}{\partial x} + \sigma_i b_{s-}(x) \right] \varphi_{i,s-}(\mu) \right\}$$

$$+ \sum_{n=1}^{N} \int_{<n>} \sum_{p=n}^{N} \left[\mu \frac{\partial B_p^{(n)}(x,\nu)}{\partial x} + \sigma_i B_p^{(n)}(x,\nu) \right] \varphi_{i,p}^{(n)}(\nu,\mu)\, d\nu$$

$$- \frac{1}{2} \sum_{j=1}^{N} c_{ij} \int_{-1}^{1} \left\{ \sum_{s=0}^{\alpha-1} \left[b_{s+}(x)\, \varphi_{j,s+}(\mu') + b_{s-}(x)\, \varphi_{j,s-}(\mu') \right] \right.$$

$$\left. + \sum_{n=1}^{N} \int_{<n>} \sum_{p=n}^{N} B_p^{(n)}(x,\nu)\, \varphi_{j,p}^{(n)}(\nu,\mu')\, d\nu \right\} d\mu' = 0 , \tag{2.96}$$

$$i=1,2,\ldots,N , \quad -1 \leq \mu \leq 1 .$$

But recalling Eq. (2.5), we have

$$\frac{1}{2} \sum_{j=1}^{N} c_{ij} \int_{-1}^{1} \psi_j(\nu,\mu') \, d\mu' = (\sigma_i - \frac{\mu}{\nu}) \, \psi_i(\nu,\mu) \tag{2.97}$$

When the integrals over μ' in Eq. (2.96) are replaced by terms of the form given by the right-hand side of Eq. (2.97), we obtain

$$\sum_{s=0}^{\alpha-1} \left\{ \left[\frac{\partial b_{s+}(x)}{\partial x} + \frac{1}{\nu_s} b_{s+}(x) \right] \underset{\sim}{\Phi}_{s+}(\mu) + \left[\frac{\partial b_{s-}(x)}{\partial x} - \frac{1}{\nu_s} b_{s-}(x) \right] \underset{\sim}{\Phi}_{s-}(\mu) \right\}$$

$$+ \sum_{n=1}^{N} \int_{\langle n \rangle} \sum_{p=n}^{N} \left\{ \frac{\partial B_p^{(n)}(x,\nu)}{\partial x} + \frac{1}{\nu} B_p^{(n)}(x,\nu) \right\} \underset{\sim}{\Phi}_p^{(n)}(\nu,\mu) \, d\nu = 0 \, , \tag{2.98}$$

$$-1 \leq \mu \leq 1 \, .$$

This equation is valid for all μ on the interval from -1 to +1. Therefore, due to the linear independence of the eigensolutions, Eq. (2.98) implies that

$$b_{s\pm}(x) = a_{s\pm} \, e^{\mp x/\nu_s}, \quad s=0,1,\ldots,\alpha-1 \tag{2.99}$$

and

$$B_p^{(n)}(x,\nu) = A_p^{(n)}(\nu) \, e^{-x/\nu} \, , \; \nu \in (n) \, , \; p=n,n+1,\ldots,N, \tag{2.100}$$

where $a_{s\pm}$ and $A_p^{(n)}(\nu)$ are arbitrary. Theorem (2.3) follows immediately.

The form of the general solution in Eq. (2.94) is the obvious generalization of the two-group solution given by Siewert and Shieh (14).

Half-Range Problems

It may appear that we are now prepared to solve particular multi-group problems merely by applying the appropriate boundary conditions to the general solution and determining the expansion coefficients. We shall find, however, that although the expansion coefficients can usually be "determined" in the present formalism, only a limited class of problems can be solved, in the sense of being able to obtain numerically amenable expressions for the expansion coefficients. This follows from the mathematical complexity of the equations which are typically obtained for the expansion coefficients in problems involving a semi-infinite medium. To make these points clearer, let us briefly consider the Milne problem, which involves a semi-infinite medium.

We wish to determine the neutron distribution $\underset{\sim}{\Psi}^{(m)}(x,\mu)$ for a homogeneous, source-free medium occupying the half-space $x \geq 0$. The half-space $x < 0$ is assumed to be void; hence the neutrons must be provided by a source at $x = \infty$. In this case, we expect $\underset{\sim}{\Psi}^{(m)}(x,\mu)$ to have the form

$$\underset{\sim}{\Psi}^{(m)}(x,\mu) = \underset{\sim}{\Phi}_{o-}(\mu)\, e^{x/\nu_o} \tag{2.101}$$

for large x, where ν_o is assumed to be the largest positive zero of $\Omega(\nu)$, and the strength of the source has been suitably normalized. Since the neutrons which migrate from the half-space $x \geq 0$ into the vacuum will not return to the medium, we also know that

$$\underset{\sim}{\Psi}^{(m)}(0,\mu) = 0 \ , \quad \mu > 0 \ . \tag{2.102}$$

Equations (2.101) and (2.102) provide the necessary boundary conditions which must be applied to Eq. (2.94) to determine the neutron distribution of the Milne problem. By demanding that the solution have the form given by Eq. (2.101) as $x \longrightarrow \infty$, we find that we must set

$$a_{s-} = \begin{cases} 1, & s=0 \\ 0, & s=1,2,\ldots,\alpha-1 \end{cases} \tag{2.103}$$

and

$$A_p^{(n)}(\nu) = 0 \ , \quad -\frac{1}{\sigma_n} \lesssim \nu \lesssim -\frac{1}{\sigma_{n-1}} \ . \tag{2.104}$$

Thus, the solution has the form

$$\underset{\sim}{\Psi}^{(m)}(x,u) = \underset{\sim}{\Phi}_{o-}(u)e^{x/\nu_o} + \sum_{s=0}^{\alpha-1} a_{s+}\,\underset{\sim}{\Phi}_{s+}(u)e^{-x/\nu_s}$$

$$+ \sum_{n=1}^{N} \int_{n>} \sum_{p=n}^{N} A_p^{(n)}(\nu)\,\underset{\sim}{\Phi}_p^{(n)}(\nu,u)e^{-x/\nu}\,d\nu\ , \qquad (2.105)$$

where the notation n> means that the range of integration is over the positive segment of region (n). The remainder of the expansion coefficients are determined using the vacuum boundary condition given by Eq. (2.102). Imposing this condition on Eq. (2.105), we obtain

$$-\underset{\sim}{\Phi}_{o-}(u) = \sum_{s=0}^{\alpha-1} a_{s+}\,\underset{\sim}{\Phi}_{s+}(u) + \sum_{n=1}^{N} \int_{n>} \sum_{p=n}^{N} A_p^{(n)}(\nu)\,\underset{\sim}{\Phi}_p^{(n)}(\nu,u)\,d\nu\ ,$$

$$u > 0\ . \qquad (2.106)$$

The expansion coefficients in Eq. (2.106) could readily be determined if we had orthogonality relations over the half-range (0,1). That is, if a matrix $\underset{\sim}{W}(u)$ could be found such that

$$\int_0^1 \underset{\sim}{W}(u)\left[\underset{\sim}{\Phi}^{\dagger}(\eta,u)\right]'\,\underset{\sim}{\Phi}(\nu,u)\,du = 0\ , \quad \nu \neq \eta\ . \qquad (2.107)$$

Siewert and Zweifel (16) have determined the form of $\underset{\sim}{W}(\mu)$ for the special case where the determinant of the transfer matrix is zero, but the generalization to $c \neq 0$ has not been found. Thus, we are compelled to find the expansion coefficients by solving a system of singular integral equations similar in nature to those solved in the full-range completeness theorem ($-1 \leq \mu \leq 1$). In the present case, however, the solution is far more difficult because the basic interval is (0,1) and Leonard and Ferziger's (9) trick is not applicable.

In general terms, the problem can be characterized in the following way. We wish to determine the expansion coefficients a_{s+} and $A_p^{(n)}(\nu)$ in the expansion

$$\underset{\sim}{V}(\mu) = \sum_{s=0}^{\alpha-1} a_{s+} \underset{\sim}{\Phi}_{s+}(\mu) + \sum_{n=1}^{N} \int_{n>} \sum_{p=n}^{N} A_p^{(n)}(\nu)\, \underset{\sim}{\Phi}_p^{(n)}(\nu,\mu)\, d\nu \,,$$

$$\mu \in (0,1) \,, \tag{2.108}$$

where $\underset{\sim}{V}(\mu)$, $0 \leq \mu \leq 1$, is any sufficiently well-behaved N-component vector. In essence, the problem amounts to proving a half-range (0,1) completeness theorem for the eigenfunctions $\underset{\sim}{\Phi}_{s+}(\mu)$ and $\underset{\sim}{\Phi}_p^{(n)}(\nu,\mu)$. Due to the similarity of this problem and the full-range completeness proof, we can be somewhat brief.

In a fashion completely analogous to that used earlier we rewrite Eq. (2.108) in the form

$$\sigma_i V_i^*(\sigma_i\mu) = \Omega^{(k)}(\mu) \sum_{j=1}^{N} \omega_{ij}^{(k)}(\mu)\, A_j^{(k)}(\mu) + \sum_{n=1}^{N} P \int_{n>} \frac{c\nu}{2} \frac{\Omega^{(n)}(\nu)\, A_i^{(n)}(\nu)}{\nu-\mu}\, d\nu \qquad (2.109)$$

$$\mu \in \left(\frac{1}{\sigma_{k-1}}, \frac{1}{\sigma_k}\right) \qquad k \leq i ,$$

where

$$V_i^*(\sigma_i\mu) = V_i(\sigma_i\mu) - \sum_{s=0}^{\alpha-1} a_{s+}\, \varphi_{i,s+}(\sigma_i\mu) , \qquad (2.110)$$

$$i=1,2,\ldots,N .$$

Equation (2.109) can be interpreted as a system of singular integral equations for the coefficients $A_i^{(n)}(\nu)$, where $\nu \in \left(\frac{1}{\sigma_{n-1}}, \frac{1}{\sigma_n}\right)$. Again, solution of this system is facilitated by introducing the auxiliary functions

$$N_i(z) = \frac{1}{2\pi i} \sum_{n=1}^{N} \int_{n>} \frac{c\nu}{2} \frac{\Omega^{(n)}(\nu)\, A_i^{(n)}(\nu)}{\nu - z}\, d\nu , \qquad (2.111)$$

$$i=1,2,\ldots,N .$$

These functions are analytic in the complex plane cut from

0 to +1 and vanish as 1/z at infinity. From an application of the Plemelj formulas, we have

$$N_i^+(\mu) - N_i^-(\mu) = \frac{c\mu}{2}\,\Omega^{(k)}(\mu)\ A_i^{(k)}(\mu) \tag{2.112}$$

and

$$N_i^+(u) + N_i^-(u) = \frac{1}{i\pi}\sum_{n=1}^{N} P\int_{n>} \frac{c\nu}{2}\ \frac{\Omega^{(n)}(\nu)\ A_i^{(n)}(\nu)}{\nu-u}\,d\nu \tag{2.113}$$

where $u \in (\frac{1}{\sigma_{k-1}}, \frac{1}{\sigma_k})$. Writing Eq. (2.109) in terms of $N_i^{\pm}(u)$ and $\omega_{ij}^{\pm}(u)$, we obtain

$$\frac{cu}{2}\ \sigma_i V_i^*(\sigma_i u) = \sum_{j=1}^{N}\left[\omega_{ij}^+(u)\ N_j^+(u) - \omega_{ij}^-(u)\ N_j^-(u)\right] \tag{2.114}$$

$$u \in (\frac{-1}{\sigma_i}, \frac{1}{\sigma_i}),\quad i=1,2,\ldots,N.$$

We wish to solve this boundary value problem for $\underset{\sim}{N}(z)$, where $\underset{\sim}{N}(z)$ has the properties listed under Eq. (2.111). As we have noted, Leonard and Ferziger's trick (9), which was used to find $\underset{\sim}{N}(z)$ in the full-range completeness proof, is not applicable here. Indeed, we are unaware of a general procedure for solving such matrix Hilbert problems which have boundary conditions specified on open arcs. One approach for solving the problem is suggested, however, by the work of Muskhelishvili (11) and Vekua (17), who have presented a procedure for solving matrix Hilbert problems with boundary conditions specified over closed contours. In this particular procedure, the matrix Hilbert problem is solved in terms of a certain sectionally analytic matrix, called the fundamental matrix. Although this fundamental matrix is not generally expressible in closed form, Muskhelishvili and Vekua show how it can be constructed in terms of the solution of a certain class of Fredholm equations. A similar approach might be used to solve the matrix Hilbert problem described in Eq. (2.114), provided an appropriate fundamental matrix can be constructed. To illustrate the tenor of the approach, let us assume the existence of a fundamental matrix $\underset{\sim}{X}(z)$, with elements analytic in the cut plane, which satisfies the boundary condition

$$\left[\underset{\sim}{X}^{-}(\mu)\right]^{-1} \underset{\sim}{X}^{+}(\mu) = \left[\underset{\sim}{\Omega}^{-}(\mu)\right]^{-1} \underset{\sim}{\Omega}^{+}(\mu) \ , \quad \mu \in (0,1), \quad (2.115)$$

where

$$\underset{\sim}{\Omega}^{\pm}(\mu) = \left\{\omega_{ij}^{\pm}(\mu)\right\} , \qquad i,j=1,2,\ldots,N. \quad (2.116)$$

Furthermore, we require that the determinant of $\underset{\sim}{X}(z)$ be non-vanishing in the finite plane. The reason for this restriction will become apparent in the subsequent analysis. Now, let us assume that the i^{th} row vector of $\underset{\sim}{X}(z)$ behaves as z^{-K_i} at infinity. That is, all the elements in the i^{th}

row vector have degree less than, or equal to, $-K_i$ at infinity. In the spirit of Muskhelishvili, we shall stipulate that the determinant of the matrix $\underset{\sim}{P}(z)\ \underset{\sim}{X}(z)$ have a finite non-zero value at infinity, where $\underset{\sim}{P}(z)$ is a diagonal matrix with elements

$$P_{ij}(z) = Z^{K_i}\ \delta_{ij}. \qquad (2.117)$$

When we finally obtain a representation for $\underset{\sim}{N}(z)$, we will wish to investigate its behavior at infinity. This second requirement on the determinant of $\underset{\sim}{X}(z)$ will facilitate that investigation. It follows from this requirement that the degree of the determinant of $\underset{\sim}{X}(z)$ at infinity is

$$-\sum_{i=1}^{N} K_i \quad ,$$

and in Appendix B we show that this number is equal to $-\alpha$.

The construction of such a fundamental matrix appears to be extremely difficult. Leonard and Ferziger (9), in considering a matrix Hilbert problem completely analogous to that of Eq. (2.114), showed that the boundary conditions can be cast into a form involving closed contours so that the methods of Muskhelishvili and Vekua apply. By extending the theory for scalar Hilbert problems to the matrix case, they derive the explicit form of a Fredholm equation for construction of the fundamental matrix. However, the solution of this Fredholm equation is a matrix whose domain is defined over a set of closed contours which are specified, in part, by a limiting process. The analysis of such a Fredholm equation seems quite involved, and in particular, the form of the equation is not well-suited for numerical calculations. In addition, the analysis of Leonard and Ferziger is restricted to problems where the transfer matrix is symmetric. Thus, their results are not applicable for systems with fission sources. Nevertheless, we shall proceed as if $\underset{\sim}{X}(z)$ were known, and formally solve for $\underset{\sim}{N}(z)$

From Eqs. (2.32), (2.112) and (2.16), we note that

$$\sum_{j=1}^{N} \left[\omega_{ij}^{+}(\mu) \; N_j^{+}(\mu) - \omega_{ij}^{-}(\mu) \; N_j^{-}(\mu) \right] = 0 \; , \quad \mu \in \left(\frac{1}{\sigma_i} , 1\right) . \tag{2.118}$$

Thus, we can write Eq. (2.114) in the form

$$\frac{c\mu}{2} \, \epsilon_{in} \; \sigma_i V_i^{*}(\sigma_i\mu) = \sum_{j=1}^{N} \left[\omega_{ij}^{+}(\mu) \; N_j^{+}(\mu) - \omega_{ij}^{-}(\mu) \; N_j^{-}(\mu) \right] ,$$

$$\mu \in \left(\frac{1}{\sigma_{n-1}} , \frac{1}{\sigma_n}\right) , \tag{2.119}$$

where

$$\epsilon_{in} = \begin{cases} 0, & i < n \\ 1, & i \geqslant n . \end{cases} \tag{2.120}$$

Then, using Eq. (2.115), we have

$$\frac{c\mu}{2}\sum_{j=1}^{N} \xi_{ij}(\mu)\, \epsilon_{jn}\, \sigma_j\, V_j^{*}(\sigma_j\mu)$$

$$= \sum_{j=1}^{N} \left[X_{ij}^{+}(\mu)\, N_j^{+}(\mu) - X_{ij}^{-}(\mu)\, N_j^{-}(\mu) \right], \quad \mu \in \left(\frac{1}{\sigma_{n-1}}, \frac{1}{\sigma_n}\right), \tag{2.121}$$

where the elements $\xi_{ij}(\mu)$ are defined through the matrix equation

$$\underset{\sim}{X}^{-}(\mu)\left[\underset{\sim}{\Omega}^{-}(\mu)\right]^{-1} = \left\{\xi_{ij}(\mu)\right\}, \quad i,j=1,2,\ldots,N. \tag{2.122}$$

We can now solve for $N_i(z)$ in the usual way. We introduce the functions

$$F_i(z) = \sum_{j=1}^{N} X_{ij}(z)\, N_j(z) - \frac{1}{2\pi i}\sum_{j=1}^{N} \frac{c}{2}\int_0^{1/\sigma_j} \frac{\nu\, \xi_{ij}(\nu)\, \sigma_j\, V_j^{*}(\sigma_j\nu)}{\nu - z}\, d\nu\,, \tag{2.123}$$

$$i=1,2,\ldots,N\,.$$

Due to Eq. (2.121), and the properties of $\underset{\sim}{X}(z)$ and $\underset{\sim}{N}(z)$, the vector $\underset{\sim}{F}(z)$ is analytic everywhere in the finite plane. Moreover, from the analysis of Leonard and Ferziger, it appears that the $\underset{\sim}{X}$-matrix is bounded as $z \to \infty$. Hence

$\underset{\sim}{F}(z)$ vanishes at infinity. It follows from Liouville's theorem that

$$\sum_{j=1}^{N} X_{ij}(z)\, N_j(z) = \frac{1}{2\pi i} \sum_{j=1}^{N} \frac{c}{2} \int_0^{1/\sigma_j} \frac{\nu\, \xi_{ij}(\nu)\; \sigma_j\; V_j^*(\sigma_j \nu)}{\nu - z}\, d\nu, \tag{2.124}$$

$$i=1,2,\ldots,N$$

Solving Eq. (2.124) for $\underset{\sim}{N}(z)$, we find

$$N_i(z) = \frac{1}{2\pi i} \sum_{j=1}^{N} X_{ij}^{-1}(z) \sum_{k=1}^{N} \frac{c}{2} \int_0^{1/\sigma_k} \frac{\nu\, \xi_{jk}(\nu)\; \sigma_k\; V_k^*(\sigma_k \nu)}{\nu - z}\, d\nu, \tag{2.125}$$

$$i=1,2,\ldots,N\ .$$

Hence, $\underset{\sim}{N}(z)$ is analytic in the plane cut from 0 to +1, fulfilling one of the requirements listed under Eq. (2.111). Now, from the properties of $\underset{\sim}{X}(z)$, we know that the j^{th} column vector of $\underset{\sim}{X}^{-1}(z)$ has degree K_j at infinity. Thus, $\underset{\sim}{N}(z)$ will vanish at infinity if and only if $\underset{\sim}{V}^*(\mu)$ satisfies the α conditions

$$\sum_{k=1}^{N} \frac{c}{2} \int_0^{1/\sigma_k} \nu^l\, \xi_{jk}(\nu)\; \sigma_k\; V_k^*(\sigma_k \nu)\, d\nu = 0\ ,\quad j=1,2,\ldots,N \tag{2.126}$$

where $l=1,2,...,K_j$. These conditions are just sufficient to determine the α discrete expansion coefficients a_{s+}, which are contained in the expression for $\underset{\sim}{V}^*(u)$. The continuum expansion coefficients can be determined using Eq. (2.112).

Thus, the problem of determining the expansion coefficients for half-range problems, such as the Milne problem, hinges on our ability to find $\underset{\sim}{X}(z)$. To date, however, we are unaware of any computations of the $\underset{\sim}{X}$-matrix.

To summarize, we have seen that full-range problems, where the basic interval is (-1,1), are easily solved using the eigenfunction expansion. The expansion coefficients can be determined as in the full-range completeness proof, or more easily using the orthogonality relations. On the other hand, it is apparently very difficult using the eigenfunction expansion, to solve half-range problems, which are also of practical interest. It is apparent that this difficulty will be eliminated when a suitable method for computing the fundamental matrix has been established. However, in the next section, we shall present another method for solving multi-group problems. This second method provides equations which are amenable to numerical calculation, even for those problems which involve a semi-infinite medium.

III. THE METHOD OF SINGULAR INTEGRAL EQUATIONS

In this section we shall present an alternative approach for solving multi-group problems which is based on the method of singular integral equations (3). In this approach the neutron distribution is determined by transforming the multi-group equations into a system of singular integral equations which is solved using the methods of Muskhelishvili (11).

To obtain the system of singular integral equations for $\underset{\sim}{\Psi}(x,\mu)$, it is convenient to write Eq. (2.1) in the form

$$(\mu' \frac{\partial}{\partial x} + 1)\, \Psi_i(x,\sigma_i\mu') = \frac{1}{2\sigma_i} \sum_{j=1}^{N} c_{ij}\, \rho_j(x) \ , \quad i=1,2,\ldots,N, \tag{3.1}$$

where $\mu' \in (-\frac{1}{\sigma_i}, \frac{1}{\sigma_i})$ and

$$\rho_j(x) = \int_{-1}^{1} \Psi_j(x,\nu)\, d\nu \ . \tag{3.2}$$

We note that the change of variables $\mu = \sigma_k\mu'$, $\mu \in (-1,1)$, has been made in the k^{th} equation of Eq. (3.1). As in the one-speed approximation, we extend $\Psi_i(x,\sigma_i\mu')$ into the complex domain. Thus, replacing μ' in Eq. (3.1) with the complex variable z, we have

$$(z \frac{\partial}{\partial x} + 1)\, \Psi_i(x,\sigma_i z) = \frac{1}{2\sigma_i} \sum_{j=1}^{N} c_{ij}\, \rho_j(x)\ , \quad i=1,2,\ldots,N. \tag{3.3}$$

In the above equation, $\Psi_i(x,\sigma_i z)$ represents the analytic continuation of $\Psi_i(x,\sigma_i\mu')$ into the complex plane. Using Eq. (3.1), we note that

$$(z \frac{\partial}{\partial x} + 1)\nu\, \Psi_i(x,\sigma_i\nu) = \frac{z}{2\sigma_i} \sum_{j=1}^{N} c_{ij}\, \rho_j(x) + (\nu - z)\, \Psi_i(x,\sigma_i\nu). \tag{3.4}$$

We now introduce the functions

$$\Psi_i^{(o)}(x,\sigma_i z) = \frac{1}{\gamma_{ii}} \frac{c}{2} \int_{-1/\sigma_i}^{1/\sigma_i} \frac{\nu\, \Psi_i(x,\sigma_i\nu) - z\,\Psi_i(x,\sigma_i z)}{\nu - z}\, d\nu$$

$$- \frac{1}{\sigma_i\gamma_{ii}} \sum_{\substack{j=1\\ j\neq i}}^{N} \gamma_{ij}\, \sigma_j\, \Psi_j(x,\sigma_j z)\ , \quad i=1,2,\ldots,N\ , \tag{3.5}$$

where the elements γ_{ij} are defined in Eq. (2.11). When we operate on both sides of Eq. (3.5) with $(z \frac{\partial}{\partial x} + 1)$, and use Eqs. (3.2) through (3.4), we obtain

$$(z \frac{\partial}{\partial x} + 1)\, \Psi_i^{(o)}(x,\sigma_i z) = \frac{1}{2\sigma_i} \left\{ \frac{c}{\gamma_{ii}}\, \rho_i(x) - \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\gamma_{ij}}{\gamma_{ii}} \sum_{k=1}^{N} c_{jk}\, \rho_k(x) \right\} \tag{3.6}$$

It is easily verified, using Eq. (2.11), that the right-hand side of Eq. (3.6) is eqivalent to that of Eq. (3.3). Thus $\Psi_i^{(o)}(x,\sigma_i z)$ is a "particular solution" to Eq. (3.3). The "general solution" can be written in the form

$$\Psi_i(x,\sigma_i z) = \Psi_i^{(o)}(x,\sigma_i z) + \Psi_i^{(1)}(x,\sigma_i z), \quad i=1,2,\ldots,N \tag{3.7}$$

where $\Psi_i^{(o)}(x,\sigma_i z)$ is a solution to the homogeneous equation obtained by setting the right-hand side of Eq. (3.3) equal to zero, viz.,

$$\Psi_i^{(1)}(x,\sigma_i z) = -\frac{1}{\sigma_i\gamma_{ii}} f_i(z)\, e^{-x/z} \quad . \tag{3.8}$$

Here the function $f_i(z)$ is arbitrary; in specific transport problems, the form of $f_i(z)$ is determined by the boundary conditions. When the explicit expressions for $\Psi_i^{(o)}(x,\sigma_i z)$ and $\Psi_i^{(1)}(x,\sigma_i z)$ are substituted into Eq. (3.7), we find

$$\sum_{j=1}^{N} \gamma_{ij}\sigma_j\Psi_j(x,\sigma_j z) = \frac{c}{2}\int_{-1/\sigma_i}^{1/\sigma_i} \frac{\nu\sigma_i\,\Psi_i(x,\sigma_i\nu) - z\sigma_i\,\Psi_i(x,\sigma_i z)}{\nu - z}\,d\nu$$

$$- f_i(z)\, e^{-x/z}, \quad i=1,2,\ldots,N . \tag{3.9}$$

The integration over the second term of the integrand in

Eq. (3.9) can be performed explicitly to yield the following equations:

$$\sum_{j=1}^{N} \omega_{ij}(z)\ \sigma_j\ \Psi_j(x,\sigma_j z) = \frac{c}{2}\int_{-\frac{1}{\sigma_i}}^{\frac{1}{\sigma_i}} \frac{\nu\sigma_i\ \Psi_i(x,\sigma_i\nu)}{\nu - z}\, d\nu$$

$$- f_i(z)\ e^{-x/z}\ ,\quad z \in (-1,1) \qquad (3.10)$$

and

$$\sum_{j=1}^{N} \omega_{ij}^{(n)}(u)\ \sigma_j\ \Psi_j(x,\sigma_j u) = \frac{c}{2} P\int_{-1/\sigma_i}^{1/\sigma_i} \frac{\nu\sigma_i\ \Psi_i(x,\sigma_i\nu)}{\nu - u}\, d\nu$$

$$- f_i(u)\ e^{-x/u}\ ,\ u \in (n),\ n=1,2,\ldots,N, \qquad (3.11)$$

where $i=1,2,\ldots,N$, and $\omega_{ij}^{(n)}(u)$ and $\omega_{ij}(z)$ are defined in Eqs. (2.15) and (2.32) respectively. In Eq. (3.11), we note that the Cauchy principal value is used when $i \geq n$. The general solution of the multi-group equation can be found by solving the above system of singular integral equations for $\Psi_i(x,\sigma_i u)$ in terms of $f_i(u)$, $i=1,2,\ldots,N$. For specific transport problems, the functions $f_i(u)$, $i=1,2,\ldots,N$, are determined by applying the appropriate boundary conditions directly to Eq. (3.11). The details of this procedure for finding the $f_i(u)$ will be exemplified later, when we present the infinite-medium Green's function and the solutions to the Milne and critical problems. For the remainder of this section we shall be assuming that the functions $f_i(\mu)$, $i=1,2,\ldots,N$, are known.

Two different approaches can be used to solve Eq. (3.11). We have already seen one of these approaches in Section II. This approach consists of introducing an auxiliary vector, say $\underset{\sim}{N}(z)$, whose i^{th} component is analytic in the plane cut from $-\frac{1}{\sigma_i}$ to $+\frac{1}{\sigma_i}$. The system of singular integral equations is then reduced to a matrix Hilbert problem for $\underset{\sim}{N}(z)$. Although this method could certainly be used to solve Eq. (3.11), we choose to pursue a different approach. The reason would become apparent if we considered transport problems which involve a semi-infinite medium. When determining $f_i(\mu)$ for these problems, we would obtain a system of singular integral equations on the half-range (0,1) which is similar in form to the full-range equations given in Eq. (3.11). If the half-range singular integral equations are reduced to a matrix Hilbert problem, the $\underset{\sim}{X}$-matrix of Section II must be introduced to obtain the solution. Therefore, although this approach exhibits an interesting duality between the method of singular integral equations and Case's method, both methods suffer the same difficulty in calculating $\underset{\sim}{X}(z)$.

The second approach for solving Eq. (3.11) is more or less analogous to the Gauss algorithm of linear algebraic equations. The objective of such an approach is to reduce Eq. (3.11) to a system of equations with each equation containing only one unknown. One procedure for constructing such an algorithm is to consider the first equation of the system as a scalar singular integral equation for $\Psi_1(x,\sigma_1\mu)$, where the components $\Psi_i(x,\sigma_i\mu)$, $i=2,3,\ldots,N$, are treated as known functions. This singular integral equation can then be solved by the standard methods of Muskhelishvili (11), and the solution substituted into the remaining equations to obtain N-1 unknowns. Similiarly, the first equation of the latter system can be solved formally to obtain an expression for $\Psi_2(x,\sigma_2\mu)$. Substitution of this solution into the remaining equations yields N-2 coupled singular integral equations for N-2 unknowns. This procedure can be continued until an equation involving only $\Psi_N(x,\sigma_N\mu)$ is obtained. Then solving for $\Psi_N(x,\sigma_N\mu)$, we can go backwards in the elimination procedure to obtain the remaining $\Psi_i(x,\sigma_i\mu)$, $i=1,2,\ldots,N-1$. This is the tenor of the approach used by Zelazny and Kuszell (18) to demonstrate the completeness property of the eigensolutions in Case's method. The approach is attractive because only scalar singular integral equations have to be solved. On the other hand, the procedure of Zelazny and Kuszell is extremely arduous, even for a relatively small number of groups, and it does not exhibit all the equations in the general case.

Fortunately, in the present formalism, we can accomplish the same objective without using a laborious algorithmic scheme. Instead, we uncouple Eq. (3.11) by appealing to analytic function theory.

To obtain the uncoupled form of Eq. (3.11) we introduce the functions

$$Q_i(x,z) = \sum_{n=1}^{N} \frac{c}{2} \int_{<n>} \frac{R_n(\nu)\ \sigma_i\ \Psi_i(x,\sigma_i\nu)}{\nu-z}\, d\nu$$

$$+ \sum_{j=1}^{N} \sum_{n=1}^{N} \frac{c}{2} \int_{<n>} \frac{R_{ijn}(\nu)\ f_j(\nu)\ e^{-x/\nu}}{\nu-z}\, d\nu$$

$$- \sum_{j=1}^{N} \Delta_{ij}(z)\ \frac{c}{2} \int_{-1/\sigma_j}^{1/\sigma_j} \frac{\nu\ \sigma_j\ \Psi_j(x,\sigma_j\nu)}{\nu-z}\, d\nu\ ,$$

$$i=1,2,\ldots,N \qquad (3.12)$$

where $\Delta_{ij}(z)$ is defined in Eq. (2.84),

$$R_n(\nu) = \frac{1}{i\pi c}\left[\Omega^+(\nu) - \Omega^-(\nu)\right],\quad \nu\ \epsilon\ (n) \qquad (3.13)$$

and

$$R_{ijn}(\nu) = \frac{1}{i\pi c}\left[\Delta_{ij}^+(\nu) - \Delta_{ij}^-(\nu)\right],\quad \nu\ \epsilon\ (n). \qquad (3.14)$$

Here $\Omega^{\pm}(\nu)$ is the determinant of the matrix $\underset{\sim}{\Omega}^{\pm}(\nu)$, defined in Eq. (2.116), and $\Delta^{\pm}_{ij}(\nu)$ is the cofactor of $\omega^{\pm}_{ji}(\nu)$ in $\underset{\sim}{\Omega}^{\pm}(\nu)$. From Eq. (3.12), it is clear that the functions $Q_i(x,z)$, $i=1,2,\ldots,N$, are analytic everywhere in the complex plane except perhaps on the interval $(-1,1)$; they vanish as $1/z$ at infinity. Before we investigate the analytic properties of $Q_i(x,z)$ on the interval $(-1,1)$, we choose to write $Q_i(x,z)$ in a slightly different form. This is done by multiplying Eq. (3.10) by $\omega^{-1}_{li}(z)$ and summing on i to obtain

$$\Omega(z)\ \sigma_l\ \psi_l(x,\sigma_l z) = \sum_{i=1}^{N} \Delta_{li}(z)\ \frac{c}{2} \int_{-1/\sigma_i}^{1/\sigma_i} \frac{\nu\ \sigma_i\ \psi_i(x,\sigma_i\nu)}{\nu - z}\, d\nu$$

$$- \sum_{i=1}^{N} \Delta_{li}(z)\ f_i(z)\ e^{-x/z}, \quad l=1,2,\ldots,N\ . \tag{3.15}$$

Using the above equations, we can rewrite Eq. (3.12) as follows:

$$Q_i(x,z) = \sum_{n=1}^{N} \frac{c}{2} \int_{\langle n\rangle} \frac{R_n(\nu)\ \sigma_i\ \psi_i(x,\sigma_i\nu)}{\nu - z}\, d\nu - \Omega(z)\ \sigma_i\ \psi_i(x,\sigma_i z)$$

$$+ \sum_{j=1}^{N} \sum_{n=1}^{N} \frac{c}{2} \int_{\langle n\rangle} \frac{R_{ijn}(\nu)\ f_j(\nu)\ e^{-x/\nu}}{\nu - z}\, d\nu$$

$$- \sum_{j=1}^{N} \Delta_{ij}(z)\ f_j(z)\ e^{-x/z}\ . \tag{3.16}$$

Let us now consider the limiting values of $Q_i(x,z)$ as z approaches the cut (-1,1) from the left and right. Applying Plemelj's formulas to Eq. (3.16), we find

$$Q_i^+(x,\mu) - Q_i^-(x,\mu) = \left\{ i\pi c\ R_k(\mu) - \left[\Omega^+(\mu) - \Omega^-(\mu)\right] \right\} \sigma_i \Psi_i(x,\sigma_i\mu) + \sum_{j=1}^{N} \left\{ i\pi c\ R_{ijk}(\mu) - \left[\Delta_{ij}^+(\mu) - \Delta_{ij}^-(\mu)\right] \right\} f_j(\mu) e^{-x/\mu}, \tag{3.17}$$

where i=1,2,...,N, and $\mu \in (k)$, k=1,2,...,N. From Eqs. (3.13) and (3.14), it follows that $Q_i(x,z)$ is continuous across the cut (-1,1), i.e.,

$$Q_i^+(x,\mu) - Q_i^-(x,\mu) = 0 \ , \quad \mu \in (-1,1) \ . \tag{3.18}$$

Thus, the functions $Q_i(x,z)$, i=1,2,...,N, are analytic everywhere in the complex plane and they equal zero at infinity. We conclude from Liouville's theorem that

$$Q_i(x,z) \equiv 0 \ , \qquad i=1,2,\ldots,N \ . \tag{3.19}$$

When this result is applied to Eq. (3.16), we find that

$$\Omega(z)\sigma_i \Psi_i(x,\sigma_i z) = \sum_{n=1}^{N} \frac{c}{2} \int_{\langle n \rangle} \frac{R_n(\nu)\sigma_i \Psi_i(x,\sigma_i \nu)}{\nu - z} d\nu - g_i(x,z), \quad i=1,2,\ldots,N \tag{3.20}$$

where

$$g_i(x,z) = \sum_{j=1}^{N} \Delta_{ij}(z)\ f_j(z)\ e^{-x/z}$$

$$- \sum_{j=1}^{N} \sum_{n=1}^{N} \frac{c}{2} \int_{<n>} \frac{R_{ijn}(\nu)\ f_j(\nu)\ e^{-x/\nu}}{\nu - z}\, d\nu\ . \qquad (3.21)$$

The k^{th} equation of Eq. (3.20) determines $\Psi_k(x, \sigma_k z)$ everywhere in the complex plane, provided $\Psi_k(x, \sigma_k\nu)$, $\nu \in (-1,1)$, and $g_k(x,z)$ can be specified. In specific transport problems $g_k(x,z)$ is determined by the form of $f_j(z)$, $j=1,2,\ldots,N$, in Eq. (3.21). Now, let us consider the limiting forms of Eq. (3.20) as z approaches the directed cut (-1,1) from the left and right:

$$\Omega^{\pm}(u)\sigma_i\Psi_i(x, \sigma_i u) = \sum_{n=1}^{N} \frac{c}{2}\ P \int_{<n>} \frac{R_n(\nu)\ \ \sigma_i\Psi_i(x, \sigma_i\nu)}{\nu - u}\, d\nu$$

$$\pm \frac{i\pi c}{2} R_k(u)\ \ \sigma_i\Psi_i(x, \sigma_i u) - g_i^{\pm}(x,\mu) \qquad (3.22)$$

where $i=1,2,\ldots,N$, and $u \in (k)$, $k=1,2,\ldots,N$. From Eqs. (3.21) and (3.22), we obtain the following system of uncoupled singular integral equations for $\Psi_i(x, \sigma_i u)$, $u \in (-1,1)$, $i=1,2,\ldots,N$:

$$\Lambda(\mu)\sigma_i\Psi_i(x,\sigma_i\mu) = \sum_{n=1}^{N} \frac{c}{2} P\int_{<n>} \frac{R_n(\nu)\ \sigma_i\Psi_i(x,\sigma_i\nu)}{\nu-\mu}\,d\nu - g_i(x,\mu),$$

$$i=1,2,\ldots,N\ , \qquad \mu\ \varepsilon\ (-1,1)\ , \qquad (3.23)$$

where

$$\Lambda(\mu) = \frac{1}{2}\left[\Omega^+(\mu) + \Omega^-(\mu)\right] \qquad (3.24)$$

and

$$g_i(x,\mu) = \frac{1}{2}\sum_{j=1}^{N}\left[\Delta_{ij}^+(\mu) + \Delta_{ij}^-(\mu)\right] f_j(\mu)\ e^{-x/\mu}$$

$$- \sum_{j=1}^{N}\sum_{n=1}^{N} \frac{c}{2}\ P\int_{<n>} \frac{R_{ijn}(\nu)\ f_j(\nu)\ e^{-x/\nu}}{\nu-\mu}\,d\nu\ . \qquad (3.25)$$

The important feature of these equations is that $\Psi_k(x,\sigma_k\mu)$ can be represented in terms of the function $g_k(x,\mu)$, $\mu\ \varepsilon\ (-1,1)$, by solving the k^{th} scalar singular integral equation by the ordinary methods of Muskhelishvili. The shape of $g_k(x,\mu)$ is determined by the form of the functions $f_j(\mu)$, $j=1,2,\ldots,N$, $\mu\ \varepsilon\ (-1,1)$, in Eq. (3.25). Here, we shall illustrate the procedure for solving Eq. (3.23) by assuming the functions $g_i(x,\mu)$, $i=1,2,\ldots,N$, are known.

We begin by introducing auxiliary functions $M_i(x,z)$, $i=1,2,\ldots,N$, which are analytic in the complex plane cut from -1 to $+1$, and which vanish as $1/z$ at infinity, viz.,

$$M_i(x,z) = \frac{1}{2\pi i} \sum_{n=1}^{N} \frac{c}{2} \int_{<n>} \frac{R_n(\nu)\ \sigma_i \Psi_i(x,\sigma_i\nu)}{\nu - z} d\nu \ , \quad i=1,2,\ldots,N \ . \tag{3.26}$$

From Plemelj's formulas, we have

$$M_i^+(x,\mu) - M_i^-(x,\mu) = \frac{c}{2} R_k(\mu)\ \sigma_i \Psi_i(x,\sigma_i\mu) \tag{3.27}$$

$$M_i^+(x,\mu) + M_i^-(x,\mu) = \frac{1}{i\pi} \sum_{n=1}^{N} \frac{c}{2} P \int_{<n>} \frac{R_n(\nu)\ \sigma_i \Psi_i(x,\sigma_i\nu)}{\nu - \mu} d\nu \tag{3.28}$$

where $\mu \in (k)$, $k=1,2,\ldots,N$. Using the above equations, we can reduce Eq. (3.23) to a scalar Hilbert problem (11) for $M_i(x,z)$:

$$\frac{M_i^+(x,\mu)}{\Omega^+(\mu)} - \frac{M_i^-(x,\mu)}{\Omega^-(\mu)} = -\frac{c}{2} \frac{R_k(\mu)\ g_i(x,\mu)}{\Omega^+(\mu)\Omega^-(\mu)} \ , \quad \mu \in (k) \tag{3.29}$$

where $i=1,2,\ldots,N$ and $R_k(\mu)$ is given by Eq. (3.13). We wish to find a representation for $M_i(x,z)$ which satisfies the boundary conditions of Eq. (3.29). Moreover, this representation must be analytic in the complex plane cut from -1 to +1 and vanish as $1/z$ at infinity. Let us consider the functions

$$E_i(x,z) = \prod_{s=0}^{\alpha-1} (\nu_s^2 - z^2) \left\{ \frac{M_i(x,z)}{\Omega(z)} \right.$$

$$\left. + \frac{1}{2\pi i} \sum_{n=1}^{N} \frac{c}{2} \int_{<n>} \frac{R_n(\nu)\, g_i(x,\nu)}{\Omega^+(\nu)\Omega^-(\nu)(\nu - z)}\, d\nu \right\}, \qquad (3.30)$$

with $i=1,2,..,N$. Here, as in Section II, the complex numbers $\pm\nu_s$, $s=0,1,...,\alpha-1$, represent the zeros of $\Omega(z)$. Recalling the analytic properites of $\Omega(z)$ and $M_i(x,z)$, we conclude that $E_i(x,z)$ is analytic in the finite complex plane cut from -1 to +1. But in view of Eq. (3.29), we have

$$E_i^+(x,\mu) - E_i^-(x,\mu) = 0 \ , \ \mu \in (-1,1), \quad i=1,2,...,N. \quad (3.31)$$

Therefore, the functions $E_i(x,z)$, $i=1,2,...,N$, are analytic everywhere in the finite complex plane. Furthermore, using Eq. (2.32) through (2.34), we note that $\Omega(z)$ approaches a constant as $|z| \to \infty$. Hence, the terms within the braces of Eq. (3.30) vanish as $1/z$ at infinity. It follows from Eq. (3.30) that the behavior of $E_i(x,z)$ at infinity is that of a polynomial of degree $2\alpha - 1$, i.e.,

$$E_i(x,z) = \frac{1}{2\pi i} \sum_{j=0}^{2\alpha-1} b_j^{(i)}(x) z^j \ , \quad i=1,2,...,N \ , \quad (3.32)$$

where the coefficients $b_j^{(i)}(x)$ are arbitrary functions of

x. Substituting Eq. (3.32) into Eq. (3.30) and solving for $M_i(x,z)$, we find

$$M_i(x,z) = \frac{\Omega(z)}{2\pi i}\left\{\frac{\sum_{j=0}^{2\alpha-1} b_j^{(i)}(x) z^j}{\prod_{s=0}^{\alpha-1} (\nu_s^2 - z^2)} - \sum_{n=1}^{N} \frac{c}{2} \int_{\langle n\rangle} \frac{R_n(\nu)\, g_i(x,\nu)}{\Omega^+(\nu)\Omega^-(\nu)(\nu - z)}\, d\nu\right\} \tag{3.33}$$

with $i=1,2,\ldots,N$. We note that this expression for $M_i(x,z)$ meets all the requirements to be a solution of the scalar Hilbert problem, i.e., it is analytic in the plane cut from -1 to $+1$, satisfies Eq. (3.29), and vanishes as $1/z$ at infinity.

Now, recalling Eq. (3.26), we solve Eq. (3.20) for $\Psi_i(x,\sigma_i z)$:

$$\sigma_i \Psi_i(x,\sigma_i z) = \frac{2\pi i\, M_i(x,z) - g_i(x,z)}{\Omega(z)}, \quad i=1,2,\ldots,N, \tag{3.34}$$

where the appropriate representation for $M_i(x,z)$ is given in Eq. (3.33). It remains to determine the coefficients $b_j^{(i)}(x)$. In specific applications, the analytic behavior of $\Psi_i(x,\sigma_i z)$ in the complex plane can generally be reasoned by using the appropriate boundary conditions to formally solve Eq (3.1) for $\Psi_i(x,\sigma_i\mu)$, $\mu \in (-\frac{1}{\sigma_i}, \frac{1}{\sigma_i})$, in terms of $\rho_i(x)$; this solution can then be analytically continued into the complex domain. Thus, we can appeal to the analytic properties of $\Psi_i(x,\pm\sigma_i\nu_s)$, $s=0,1,\ldots,\alpha-1$, in Eq. (3.34) to obtain α N equations for the α N unknowns $b_j^{(i)}(x)$, $j=0,1,\ldots,\alpha-1$, $i=1,2,\ldots,N$. For example, if $\Psi_i(x,\sigma_i z)$, $i=1,2,\ldots,N$, is analytic at the zeros of $\Omega(z)$, it follows that the determining equations for $b_j^{(i)}(x)$ are

$$2\pi i\, M_i(x,\pm\nu_s) = g_i(x,\pm\nu_s) \tag{3.35}$$

where $i=1,2,\ldots,N$, $s=0,1,\ldots,\alpha-1$.

Finally, the neutron distribution $\Psi_i(x,\sigma_i u)$, $i=1,2,\ldots,N$, can be obtained by letting z tend to $\mu \in (-\frac{1}{\sigma_i}, \frac{1}{\sigma_i})$ in Eq. (3.34). We note from the above analysis that the neutron distribution is completely determined when the functions $f_i(z)$, $i=1,2,\ldots,N$ are specified at the zeros of $\Omega(z)$ and on the cut (-1,1). This fact alludes to a connection between these functions and the expansion coefficients of Eq. (2.94). We shall pursue this connection further in the next section.

IV. THE RELATIONSHIP BETWEEN THE TWO METHODS

We now derive some useful relations for expressing the expansion coefficients of Eq. (2.94) in terms of the functions $f_i(z)$, $i=1,2,\ldots,N$. These relations not only point to the dual roles played by the expansion coefficients and the functions $f_i(z)$, but also they show that the method of singular integral equations provides a new approach for determining the expansion coefficients.

We begin by casting Eq. (3.11) into the following form

$$- f_i(\mu)e^{-x/\mu} = \sum_{j=1}^{N} \omega_{ij}^{(n)}(\mu)\ \sigma_j \Psi_j(x,\sigma_j\mu) + \frac{c}{2} P \int_{-1}^{1} \frac{\nu \Psi_i(x,\nu)}{\sigma_i \mu - \nu} d\nu ,$$

$$i=1,2,\ldots,N, \quad \mu \in (n) . \qquad (4.1)$$

In the above system of equations we multiply the p^{th} equation ($p \geq n$) by $\mu\, \Omega^{(n)}(\mu)$; it is convenient to write the result as follows:

$$- \mu\, \Omega^{(n)}(\mu)\, f_p(\mu) e^{-x/\mu} = \mu\, \Omega^{(n)}(\mu) \sum_{j=n}^{N} \omega_{pj}^{(n)}(\mu)\, \sigma_j \Psi_j(x, \sigma_j \mu)$$

$$+ \mu\, \Omega^{(n)}(\mu) \sum_{k=1}^{n-1} \gamma_{pk}\, \sigma_k\, \Psi_k(x, \sigma_k \mu)$$

$$+ \frac{c\mu}{2} P \int_{-1}^{1} \frac{\nu\, \Omega^{(n)}(\mu)\, \Psi_p(x,\nu)}{\sigma_p \mu - \nu}\, d\nu \quad p \geq n \,. \tag{4.2}$$

Next, we multiply Eq. (4.1) by $-\mu \sum_{k=1}^{n-1} \gamma_{pk}\, \Delta_{ki}^{(n)}(\mu)$ and sum on i from 1 to $n-1$ to obtain

$$\mu \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \gamma_{pk}\, \Delta_{ki}^{(n)}(\mu)\, f_i(\mu)\, e^{-x/\mu}$$

$$= -\mu \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \gamma_{pk}\, \Delta_{ki}^{(n)}(\mu) \sum_{j=n}^{N} \gamma_{ij}\, \sigma_j\, \Psi_j(x, \sigma_j \mu)$$

$$- \mu \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \gamma_{pk}\, \Delta_{ki}^{(n)}(\mu) \sum_{j=1}^{n-1} \omega_{ij}^{(n)}(\mu)\, \sigma_j\, \Psi_j(x, \sigma_j \mu)$$

$$- \frac{c\mu}{2} \sum_{i=1}^{n-1} \int_{-1}^{1} \frac{\sum_{k=1}^{n-1} \gamma_{pk}\, \Delta_{ki}^{(n)}(\mu)\, \Psi_i(x,\nu)}{\sigma_i \mu - \nu}\, d\nu \,, \tag{4.3}$$

where $p \geq n$, $\mu \in (n)$. In addition, we note the identity

$$\sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \gamma_{pk} \Delta_{ki}^{(n)}(\mu) \sum_{j=1}^{n-1} \omega_{ij}^{(n)}(\mu) \sigma_j \Psi_j(x, \sigma_j \mu)$$

$$= \Omega^{(n)}(\mu) \sum_{k=1}^{n-1} \gamma_{pk} \sigma_k \Psi_k(x, \sigma_k \mu) \quad , \quad p \geq n, \ \mu \in (n), \tag{4.4}$$

which follows from the fact that

$$\frac{1}{\Omega^{(n)}(\mu)} \sum_{i=1}^{n-1} \Delta_{ki}^{(n)}(\mu) \, \omega_{ij}^{(n)}(\mu) = \delta_{kj}, \quad j,k=1,2,\ldots,n-1, \ \mu \in (n). \tag{4.5}$$

Adding Eqs. (4.2) and (4.3) and using Eq. (4.4), we find

$$- \mu \left[\Omega^{(n)}(\mu) \, f_p(\mu) - \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \gamma_{pk} \Delta_{ki}^{(n)}(\mu) \, f_i(\mu) \right] e^{-x/\mu}$$

$$= \mu \sum_{j=n}^{N} Q_{pj}^{(n)}(\mu) \, \sigma_j \Psi_j(x, \sigma_j \mu) + \frac{c\mu}{2} P \int_{-1}^{1} \frac{\nu \Omega^{(n)}(\mu) \, \Psi_p(x,\nu)}{\sigma_p \mu - \nu} d\nu$$

$$- \frac{c\mu}{2} \sum_{i=1}^{n-1} \int_{-1}^{1} \frac{\nu \sum_{k=1}^{n-1} \gamma_{pk} \Delta_{ki}^{(n)}(\mu) \, \Psi_i(x,\nu)}{\sigma_i \mu - \nu} d\nu \, , \quad p \geq n, \ \mu \in (n), \tag{4.6}$$

where $Q_{pj}^{(n)}(\mu)$ is defined in Eq. (2.22). From Eq. (2.41) we conclude that

$$- \mu \left[\Omega^{(n)}(\mu)\ f_p(\mu) - \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \gamma_{pk}\ \Delta_{ki}^{(n)}(\mu)\ f_i(\mu)\right] e^{-x/\mu}$$

$$= \int_{-1}^{1} \nu \left[\underset{\sim}{\Phi}^{(n)\dagger}(\mu,\nu)\right]' \underset{\sim}{\Psi}(x,\nu)\ d\nu, \qquad p \geq n\ ,\ \mu \in (n)\ . \tag{4.7}$$

In the above equation we replace $\underset{\sim}{\Psi}(x,\nu)$ by the eigenfunction expansion of Eq. (2.94), and employ the orthogonality relations between the eigenfunctions to obtain the desired result, viz.,

$$- \mu \left[\Omega^{(n)}(\mu)\ f_p(\mu) - \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \gamma_{pk}\ \Delta_{ki}^{(n)}(\mu)\ f_i(\mu)\right]$$

$$= \sum_{q=n}^{N} N_{pq}^{(n)}(\mu)\ A_p^{(n)}(\mu)\ ,\ p = n, n+1, \ldots, N,\ \mu \in (n), \tag{4.8}$$

where $N_{pq}^{(n)}(\mu)$ is given in Eq. (2.56). We note that these equations can be used to represent the continuum expansion coefficients $A_p^{(n)}(\mu)$, $p=n,n+1,\ldots,N$, in terms of the function, $f_i(\mu)$, $i=1,2,\ldots,N$, $\mu \in (n)$

To relate the discrete coefficients $a_{s\pm}$ to the functions $f_i(\pm\nu_s)$, we use Eq. (3.10) to write

$$\sum_{j=1}^{N} \omega_{ij}(z)\; z\; \sigma_j \Psi_j(x,\sigma_j z) = h_i(z) \;,\; i=1,2,\ldots,N, \tag{4.9}$$

where

$$h_i(z) = -\, z \left\{ \frac{c}{2} \int_{-1}^{1} \frac{\nu \Psi_i(x,\nu)}{\sigma_i z - \nu} + f_i(z)\, e^{-x/z} \right\}. \tag{4.10}$$

Viewing Eq. (4.9) as a system of linear algebraic equations for $z\sigma_i\, \Psi_i(x,\sigma_i z)$, i=1,2,...,N, we note that the coefficient matrix is zero at $\pm\nu_s$, s=0,1,...,α-1. Thus, $z\sigma_i\, \Psi_i(x,\sigma_i z)$ is bounded at $\pm\nu_s$ if and only if

$$\sum_{i=1}^{N} Y_i(\pm\nu_s)\; h_i(\pm\nu_s) = 0 \;,\quad s=0,1,\ldots,\alpha-1 \;, \tag{4.11}$$

where the functions $Y_i(\pm\nu_s)$ represent the solutions to Eq. (2.87). The form of the functions $Y_i(\pm\nu_s)$ is given in Eq. (2.88). We now substitute the explicit forms of $Y_i(\pm\nu_s)$ and $h_i(\pm\nu_s)$ into Eq. (4.11) and use Eq. (2.43) to obtain

$$\mp \nu_s \Big[\Omega^{(N)}(\nu_s)\; f_N(\pm\nu_s) - \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \gamma_{Nk}\, \Delta_{kj}^{(N)}(\nu_s)\; f_j(\pm\nu_s) \Big] e^{\mp x/\nu_s}$$
$$= \int_{-1}^{1} \nu \left[\underset{\sim}{\Phi}{}^{\dagger}_{s\pm}(\nu) \right]' \underset{\sim}{\Psi}(x,\nu)\, d\nu \;,\quad s=0,1,\ldots,\alpha-1. \tag{4.12}$$

Finally, inserting the eigenfunction expansion for $\underset{\sim}{\Psi}(x,\nu)$ into Eq. (4.12) and using the orthogonality relations between the eigenfunctions, we find

$$\mp \nu_s \left[\Omega^{(N)}(\nu_s)\, f_N(\pm\nu_s) - \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \gamma_{Nk}\, \Delta_{kj}^{(N)}(\nu_s)\, f_j(\pm\nu_s) \right]$$

$$= N_{s\pm}\, a_{s\pm} \quad , \quad s=0,1,\ldots,\alpha-1 \; , \qquad (4.13)$$

where $N_{s\pm}$ is defined in Eq. (2.55).

With Eqs. (4.8) and (4.13), we have two approaches for determining the neutron distribution using the functions $f_i(z)$. The first approach, which was developed in Section III, is independent of Case's method. It may be more convenient, however, to represent the neutron distribution by the eigenfunction expansion of Eq. (2.94). This is particularly true if one wishes to compare the results of multi-group transport theory with those of multi-group diffusion theory. In these cases, Eqs. (4.8) and (4.13) can be used to determine the expansion coefficients of the eigenfunction expansion.

In the subsequent sections we shall consider some specific transport problems to illustrate how the functions $f_i(z)$ are determined.

V. THE INFINITE MEDIUM GREEN'S FUNCTION

To demonstrate how the techniques of Section III can be employed to solve multi-group transport problems, let us first consider a simple problem involving an infinite, homogeneous, nonmultiplying medium. We wish to find the angular neutron distribution (infinite medium Green's function) corresponding to a plane source a $x = 0$ which emits monoenergetic neutrons in the direction $\mu = \mu_o$. In particular, here we shall determine the functions $f_i(z)$ which correspond to this source. The multi-group transport equation for this problem can be written in the form

$$\left(\mu \frac{\partial}{\partial x} + 1\right) \sigma_i \Psi_i^{(g)}(x,\sigma_i\mu) = \frac{1}{2} \sum_{j=1}^{N} c_{ij} \rho_j^{(g)}(x) + \delta_{iq} \delta(x) \delta(\sigma_i\mu-\mu_o) , \tag{5.1}$$

$$\mu \in \left(-\frac{1}{\sigma_i}, \frac{1}{\sigma_i}\right), \quad i=1,2,\ldots,N; \quad |\mu_o| < 1 ,$$

where

$$\rho_j^{(g)}(x) = \int_{-1}^{1} \Psi_j^{(g)}(x,\mu') \, d\mu' . \tag{5.2}$$

In Eq. (5.1), the Kronecker delta δ_{iq} signifies that the plane source emits neutrons into energy group q and

$\Psi_i^{(g)}(x,\sigma_i\mu)$ represents the Green's function corresponding to this particular source. We note that Eq. (5.1) can be used to define $\Psi_i^{(g)}(x,\sigma_i\mu)$ for <u>any</u> real value of μ, i.e., we can remove the restriction that $\mu \in (-\frac{1}{\sigma_i}, \frac{1}{\sigma_i})$.

Since the neutron distribution should vanish far from the source, we require that $\Psi_i^{(g)}(x,\sigma_i\mu) \longrightarrow 0$ as $|x| \longrightarrow \infty$, and use Eq. (3.11) to write

$$\sum_{j=1}^{N} \omega_{ij}^{(n)}(\mu)\ \sigma_j\ \Psi_j^{(g)}(x,\sigma_j\mu) - \frac{c}{2}\, P\int_{-\frac{1}{\sigma_i}}^{\frac{1}{\sigma_i}} \frac{\nu\sigma_i\Psi_i^{(g)}(x,\sigma_i\nu)}{\nu-\mu}\, d\nu = \begin{cases} -\ f_{i-}(\mu)\ e^{-x/\mu}\ , & \mu < 0 \\ 0\ , & \mu > 0 \end{cases} \tag{5.3}$$

where $x < 0$ and $\mu \in (n)$. Similarly, for $x > 0$ and $\mu \in (n)$, we have

$$\sum_{j=1}^{N} \omega_{ij}^{(n)}(\mu)\ \sigma_j\ \Psi_j^{(g)}(x,\sigma_j\mu) - \frac{c}{2}\, P\int_{-\frac{1}{\sigma_i}}^{\frac{1}{\sigma_i}} \frac{\nu\sigma_i\Psi_i^{(g)}(x,\sigma_i\nu)}{\nu-\mu}\, d\nu = \begin{cases} 0\ , & \mu < 0 \\ -\ f_{i+}(\mu)\ e^{-x/\mu}\ , & \mu > 0 \end{cases} \tag{5.4}$$

Let us investigate the behavior of $\Psi_i^{(g)}(x, \sigma_i\mu)$ in the neighborhood of the source. Integrating Eq. (5.1) on x over the interval $(-\epsilon, \epsilon)$, where ϵ is an arbitrarily small positive number, we obtain

$$\mu \sigma_i \Psi_i^{(g)}(x, \sigma_i\mu) \Big|_{-\epsilon}^{\epsilon} + \int_{-\epsilon}^{\epsilon} \left\{ \sigma_i \Psi_i^{(g)}(x, \sigma_i\mu) - \frac{1}{2} \sum_{j=1}^{N} c_{ij} \rho_j^{(g)}(x) \right\} dx$$

$$= \delta_{iq} \delta(\sigma_i \mu - \mu_o), \quad i=1,2,\ldots,N. \tag{5.5}$$

When $\epsilon \rightarrow 0$, Eq. (5.5) takes the form

$$\Psi_i^{(g)}(0^+, \sigma_i\mu) - \Psi_i^{(g)}(0^-, \sigma_i\mu) = \frac{\delta_{iq} \delta(\sigma_i\mu - \mu_o)}{\sigma_i\mu} ,$$

$$i=1,2,\ldots,N . \tag{5.6}$$

Equation (5.6) yields the so-called jump conditions for the Green's function corresponding to a plane source at the origin which emits neutrons into the q^{th} group. We note that $\Psi_i^{(g)}(x, \sigma_i\mu)$, $i \neq q$, is continuous across the origin of the spatial coordinate for all real values of μ; the q^{th} component is always continuous at $x = 0$ for real values of μ

outside the interval $(-\frac{1}{\sigma_q}, \frac{1}{\sigma_q})$.

The functions $f_{i+}(\mu)$, $\mu > 0$, are determined by applying Eq. (5.6) to Eqs. (5.3) and (5.4). When $|\mu| \leq 1/\sigma_q$, we find

$$\mu f_{i\pm}(\mu) = \mp\left\{\left[\gamma_{iq} - c\mu\delta_{iq}\, T(\sigma_q\mu)\right]\delta(\sigma_q\mu-\mu_o) + \frac{c\mu}{2} P \frac{\delta_{iq}}{\sigma_q\mu-\mu_o}\right\}, \qquad \mu \gtrless 0\ . \tag{5.8}$$

For $|\mu| > 1/\sigma_q$, we obtain

$$\mu f_{i\pm}(\mu) = \mp \frac{c\mu}{2}\frac{\delta_{iq}}{\sigma_q\mu-\mu_o}, \qquad \mu \gtrless 0\ . \tag{5.9}$$

Using Eq. (5.9), we analytically continue the functions $f_{i\pm}(\mu)$, $|\mu| > 1/\sigma_q$, into the complex domain as follows:

$$z\, f_{i\pm}(z) = \mp \frac{cz}{2}\frac{\delta_{iq}}{\sigma_q z-\mu_o}, \qquad \mathrm{Re}\ z \gtrless 0, \tag{5.10}$$

where $z \notin (-1,1)$. Thus, at the zeros of $\Omega(z)$, we have

$$f_{i\pm}(\pm\nu_s) = 0\ , \quad i \neq q\ , \quad s = 0,1,\ldots,\alpha-1 \tag{5.11}$$

and

$$\nu_s f_{q\pm}(\pm\nu_s) = -\frac{c\nu_s}{2}\frac{1}{\sigma_q\nu_s \mp \mu_o}, \quad s=0,1,\ldots,\alpha-1. \tag{5.12}$$

With the functions $f_{i\pm}$ determined, we can now solve Eqs. (5.3) and (5.4) for $\Psi_i^{(g)}(x,\mu)$, $x \lessgtr 0$, as in Section III, or we can use the relationships in Section IV to determine the expansion coefficients of Eq. (2.94).

It should be noted that the infinite medium Green's function can also be determined by working entirely within the framework of Section II, i.e., by Case's method. An outline of the procedure is as follows (cf. 14): Since the Green's function vanishes as $|x| \to \infty$, it is sufficient to expand $\underset{\sim}{\Psi}^{(g)}(x,\mu)$ in the form

$$\underset{\sim}{\Psi}^{(g)}(x,\mu) = -\sum_{s=0}^{\alpha-1} a_{s-}\,\underset{\sim}{\Phi}_{s-}(\mu)\,e^{x/\nu_s}$$
$$-\sum_{n=1}^{N}\int_{<n}\sum_{p=n}^{N} A_p^{(n)}(\nu)\,\underset{\sim}{\Phi}_p^{(n)}(\nu,\mu)\,e^{-x/\nu}d\nu,\quad x < 0 \tag{5.13}$$

or

$$\underset{\sim}{\Psi}^{(g)}(x,\mu) = \sum_{s=0}^{\alpha-1} a_{s+}\,\underset{\sim}{\Phi}_{s+}(\mu)\,e^{-x/\nu_s}$$
$$+\sum_{n=1}^{N}\int_{n>}\sum_{p=n}^{N} A_p^{(n)}(\nu)\,\underset{\sim}{\Phi}_p^{(n)}(\nu,\mu)\,e^{-x/\nu}d\nu,\quad x > 0, \tag{5.14}$$

where the notation <n signifies that the integration is over the negative segment of region (n); n> signifies integration over the positive segment of region (n). When the jump-condition for $\underset{\sim}{\Psi}^{(g)}(x,\mu)$ is applied to the above equations, we find

$$\delta_{iq} \frac{\delta(\mu-\mu_0)}{\mu} = \sum_{s=0}^{\alpha-1} \left[a_{s+} \varphi_{i,s+}(\mu) + a_{s-} \varphi_{i,s-}(\mu) \right]$$

$$+ \sum_{n=1}^{N} \int_{<n>} \sum_{p=n}^{N} A_p^{(n)}(\nu) \varphi_{i,p}^{(n)}(\nu,\mu)\, d\nu \,, \qquad (5.15)$$

$$i=1,2,\ldots,N \,, \qquad -1 \leq \mu \leq 1 \,.$$

Using Eq. (5.15), the expansion coefficients for the infinite medium Green's function can be determined as in the proof of the full-range completeness property of the eigenfunctions of Section II, or by using the orthogonality relations between these eigenfunctions.

We note that we have obtained a closed form solution for the functions $f_i(z)$ associated with the infinite medium Green's function. In the next two sections we shall consider the Milne and critical problems. In these problems the $f_i(z)$ are expressed in terms of the emerging neutron distribution at the boundary, and the emerging distribution is shown to be the solution of a coupled system of Fredholm equations.

VI. THE MILNE PROBLEM

Having demonstrated how the functions $f_i(z)$ are determined for a problem involving an infinite medium, we now turn to Milne's problem for an infinite half-space. The boundary conditions for this problem are given by Eqs. (2.101) and (2.102).

Before we can determine the $f_i(z)$ for the Milne problem, we must first demonstrate that $\underset{\sim}{\psi}^{(m)}(x,\mu)$, $-1 \leq \mu \leq 1$, can be analytically continued into the complex plane and ascertain the analytic properties of this continuation. We begin by noting that the neutron distribution satisfies Eq. (3.1) in the form

$$(\mu \frac{\partial}{\partial x} + 1)\sigma_i \Psi_i^{(m)}(x,\sigma_i\mu) = \frac{1}{2} \sum_{j=1}^{N} c_{ij} \int_{-1}^{1} \varphi_{j,o-}(\mu')e^{x/\nu_o}\, d\mu$$

$$+ \frac{1}{2} \sum_{j=1}^{N} c_{ij}\, \rho_j'(x) \quad , \qquad (6.1)$$

$$\mu \in (-\frac{1}{\sigma_i}, \frac{1}{\sigma_i}),\ i=1,2,\ldots,N ,$$

where the functions $\rho_j'(x)$, $j=1,2,\ldots,N$, are bounded and continuous for all real values of x. Let us consider Eq. (6.1) as a first-order differential equation in x and solve accordingly. In view of the boundary conditions on $\Psi_i^{(m)}$, the "solution" has the form

$$\sigma_i \Psi_i^{(m)}(x,\sigma_i\mu) = \sigma_i \varphi_{i,o-}(\sigma_i\mu)\, e^{x/\nu_o} - \frac{1}{2\mu}\sum_{j=1}^{N} c_{ij}\int_x^{\infty} \rho_j'(y)\, e^{(y-x)/\mu} dy, \qquad (6.2)$$

$$\mu \in (-\frac{1}{\sigma_i}, 0) ,$$

or

$$\sigma_i \Psi_i^{(m)}(x,\sigma_i\mu) = \sigma_i \varphi_{i,o-}(\sigma_i\mu)\left[e^{x/\nu_o} - e^{-x/\mu}\right] + \frac{1}{2\mu}\sum_{j=1}^{N} c_{ij}\int_0^{x} \rho_j'(y)\, e^{(y-x)/\mu} dy ,$$

$$\mu \in (0, \frac{1}{\sigma_i}) . \qquad (6.3)$$

Substituting the complex variable z for μ in Eqs. (6.2) and (6.3), we have

$$\sigma_i \Psi_i^{(m)}(x,\sigma_i z) = \sigma_i \varphi_{i,o-}(\sigma_i z) e^{x/\nu_o} - \frac{1}{2z}\sum_{j=1}^{N} c_{ij}\int_x^{\infty} \rho_j'(y)\, e^{(y-x)/z}\, dy, \quad \text{Re } z < 0, \qquad (6.4)$$

or

$$\sigma_i \Psi_i^{(m)}(x,\sigma_i z) = \sigma_i \varphi_{i,o-}(\sigma_i z) \left[e^{x/\nu_o} - e^{-x/z} \right]$$

$$+ \frac{1}{2z} \sum_{j=1}^{N} c_{ij} \int_o^x \rho_j'(y) e^{(y-x)/z} \, dy \ , \ \text{Re } z > 0 \ , \tag{6.5}$$

where the components $\varphi_{i,o-}(\sigma_i z)$ are defined by replacing μ with $\sigma_i z$ in Eqs. (2.38) and (2.39). It is clear that the function $\Psi_i^{(m)}(x,\sigma_i z)$, defined by Eqs. (6.4) and (6.5), is the unique continuation of $\Psi_i^{(m)}(x,\sigma_i \mu)$, $\mu \in (-1/\sigma_i, 1/\sigma_i)$, into the complex plane, and that this continuation is analytic everywhere in the complex plane cut along the imaginary axis, except at $z = \nu_o$ where it has a simple pole. Moreover, we note that

$$\lim_{x \to \infty} \Psi_i^{(m)}(x,\sigma_i z) = \varphi_{i,o-}(\sigma_i z) \, e^{x/\nu_o} \tag{6.6}$$

and

$$\Psi_i^{(m)}(0,\sigma_i z) = 0 \quad , \quad \text{Re } z > 0 \ . \tag{6.7}$$

Now consider Eq. (3.10) in the form

$$\sum_{j=1}^{N} \omega_{ij}(z)\ \sigma_j\ \Psi_j^{(m)}(x,\sigma_j z) - \frac{c}{2}\int_{-1/\sigma_i}^{1/\sigma_i} \frac{\nu\sigma_i\Psi_i^{(m)}(x,\sigma_i\nu)}{\nu - z}\, d\nu \tag{6.8}$$

$$= \begin{cases} -\, f_{i-}(z)e^{-x/z}, & \text{Re } z < 0 \\ -\, f_{i+}(z)e^{-x/z}, & \text{Re } z > 0, \end{cases}$$

$$i=1,2,\ldots,N .$$

From the analytic properties of $\Psi_i^{(m)}(x,\sigma_i z)$, it is easily deduced that $f_{i-}(z)$ is analytic everywhere in the half-plane Re $z < 0$, except perhaps on $(-1,0)$ and at $z = -\nu_o$ where $\Psi_i^{(m)}(x,\sigma_i z)$ has a simple pole. Similarly, $f_{i+}(z)$ is analytic everywhere except perhaps on $(0,1)$. But applying the Plemelj formulas (11) to Eq. (6.8), it is easily verified that $f_{i-}(z)$ and $f_{i+}(z)$ are continuous across the cuts $(-1,0)$ and $(0,1)$ respectively. Moreover, if we were to expand $\omega_{ii}(z)$ in a Taylor series about $-\nu_o$ in Eq. (6.8), we would find that $f_{i-}(z)$ is bounded at $-\nu_o$. Thus $f_{i-}(z)$ is analytic everywhere in the half-plane Re $z < 0$ with at most a removable singularity at $-\nu_o$ and $f_{i+}(z)$ is analytic everywhere in the half-plane Re $z > 0$.

In accordance with the analytic properties of $\Psi_i^{(m)}$ and $f_{i\pm}$, as z tends to $\mu \in (-1,1)$ in Eq. (6.8), we have

$$\sum_{j=1}^{N} \omega_{ij}^{(n)}(\mu)\ \sigma_j\ \psi_j^{(m)}(x,\sigma_j\mu) - \frac{c}{2}\, P \int_{-1/\sigma_i}^{1/\sigma_i} \frac{\nu\sigma_i\psi_i^{(m)}(x,\sigma_i\nu)}{\nu-\mu}\, d\nu$$

$$= \begin{cases} - f_{i-}(\mu)e^{-x/\mu} \ , & \mu \in (-\frac{1}{\sigma_n}, -\frac{1}{\sigma_{n-1}}) \\ \\ - f_{i+}(\mu)e^{-x/\mu}, & \mu \in (\frac{1}{\sigma_{n-1}}, \frac{1}{\sigma_n}), \end{cases}$$

$$i=1,2,\ldots,N\ ,\ n=1,2,\ldots,N\ . \tag{6.9}$$

Here we adopt the convention

$$1/\sigma_o \equiv 0 \quad , \tag{6.10}$$

so that the intervals $(-1/\sigma_n, -1/\sigma_{n-1})$ and $(1/\sigma_{n-1}, 1/\sigma_n)$ are properly defined when $n = 1$. To determine $f_{i-}(z)$, we note that the conponents $\psi_i^{(m)}$, $i=1,2,\ldots,N$, are subject to Eq. (6.6). Applying this condition to Eq. (6.9), we find that

$$f_{i-}(\mu) = 0\ ,\quad \mu \in (-1,0)\ ,\quad i=1,2,\ldots,N\ . \tag{6.11}$$

Since $f_{i-}(z)$ is analytic in the half-plane Re $z < 0$, it follows that

$$f_{i-}(z) = 0 \ , \ \text{Re } z < 0 \ , \quad i=1,2,\ldots,N \ . \qquad (6.12)$$

An expression for the $f_{i+}(z)$ can be obtained by setting $x = 0$ in Eq. (6.9) and applying Eq. (6.7). We find that

$$f_{i+}(\mu) = \frac{c}{2} \int_0^{1/\sigma_i} \frac{\nu\sigma_i\Psi_i^{(m)}(0,-\sigma_i\nu)}{\nu+\mu} \, d\nu \ , \qquad (6.13)$$

$$\mu \in (0,1), \quad i=1,2,\ldots,N \ ;$$

hence

$$f_{i+}(z) = \frac{c}{2} \int_0^{1/\sigma_i} \frac{\nu\sigma_i\Psi_i^{(m)}(0,-\sigma_i\nu)}{\nu+z} \, d\nu \ , \qquad (6.14)$$

$$\text{Re } z > 0 \ , \quad i=1,2,\ldots,N.$$

We note that the same expression for $f_{i+}(z)$ could have been obtained by applying Eq. (6.7) directly to Eq. (6.8). More important, however, is the fact that the functions $f_{i+}(z)$, and hence the neutron distribution $\underset{\sim}{\Psi}^{(m)}(x,\mu)$, are completely determined by the emerging neutron distribution at the vacuum interface. In view of Eq. (6.11), Eq. (6.9) yields the following system of homogeneous singular integral equations for $\Psi_i^{(m)}(0,-\sigma_i\mu)$:

$$\sum_{j=1}^{N} \omega_{ij}^{(n)}(\mu)\ \sigma_j\ \Psi_j^{(m)}(0,-\sigma_j\mu) - \frac{c}{2}\ P \int_0^{1/\sigma_i} \frac{\nu\sigma_i\Psi_i^{(m)}(0,-\sigma_i\nu)}{\nu-\mu} d\nu = 0,$$

$$\mu \in \left(\frac{1}{\sigma_{n-1}}, \frac{1}{\sigma_n}\right), \quad i,n=1,2,\ldots,N . \tag{6.15}$$

It is interesting to consider the form of the functions $f_{i+}(z)$ obtained by reducing Eq. (6.15) to a matrix Hilbert problem. In the present case the appropriate auxiliary vector $\underset{\sim}{N}(z)$ has components of the form

$$N_i(z) = \frac{1}{2\pi i}\ \frac{c}{2} \int_0^{1/\sigma_i} \frac{\nu\sigma_i\Psi_i^{(m)}(0,-\sigma_i\nu)}{\nu-z}\, d\nu$$

$$= \frac{1}{2\pi i} \sum_{j=1}^{N} \omega_{ij}(z)\ \sigma_j\ \Psi_j^{(m)}(0,-\sigma_j z)\ , \quad \mathrm{Re}\ z > 0\ . \tag{6.16}$$

Letting z approach the directed cut (0,1) from the left and right, we have

$$N_i^{+}(\mu) = \frac{1}{2\pi i} \sum_{j=1}^{N} \omega_{ij}^{+}(\mu)\ \sigma_j\ \Psi_j^{(m)}(0,-\sigma_j\mu)\ , \quad \mu \in (0,1) \tag{6.17}$$

and

$$N_i^{-}(\mu) = \frac{1}{2\pi i} \sum_{j=1}^{N} \omega_{ij}^{-}(\mu)\ \sigma_j\ \Psi_j^{(m)}(0,-\sigma_j\mu)\ , \quad \mu \in (0,1). \tag{6.18}$$

Solving each of the above equations for $\sigma_i \Psi_i^{(m)}(0,-\sigma_i\mu)$, $i=1,2,\ldots,N$, and equating the results we obtain the following Hilbert problem for $\underset{\sim}{N}(z)$:

$$\underset{\sim}{N}^{+}(\mu) = \underset{\sim}{\Omega}^{+}(\mu)\left[\underset{\sim}{\Omega}^{-}(\mu)\right]^{-1}\underset{\sim}{N}^{-}(\mu) = 0 \ , \quad \mu \in (0,1), \tag{6.19}$$

where $\underset{\sim}{\Omega}^{+}(\mu)$ and $\underset{\sim}{\Omega}^{-}(\mu)$ are defined in Eq. (2.116). In consideration of Eq. (6.16), we seek a solution of Eq. (6.19) which vanishes as 1/z at infinity. Let us assume that the transfer matrix $\underset{\sim}{c}$ is symmetric. Then $\underset{\sim}{\Omega}^{+}(\mu)$ and $\underset{\sim}{\Omega}^{-}(\mu)$ are symmetric and we can solve for $\underset{\sim}{N}(z)$ in terms of the transpose of the $\underset{\sim}{X}$-matrix of Section II. From Eq. (2.115), it follows that

$$\underset{\sim}{X}'^{+}(\mu)\left[\underset{\sim}{X}'^{-}(\mu)\right]^{-1} = \underset{\sim}{\Omega}^{+}(\mu)\left[\underset{\sim}{\Omega}^{-}(\mu)\right]^{-1} , \quad \mu \in (0,1) , \tag{6.20}$$

where the prime (') denotes the transpose operations; hence Eq. (6.19) can be rewritten as

$$\underset{\sim}{N}^{+}(\mu) = \underset{\sim}{X}'^{+}(\mu)\left[\underset{\sim}{X}'^{-}(\mu)\right]^{-1}\underset{\sim}{N}^{-}(\mu) , \quad \mu \in (0,1) . \tag{6.21}$$

It is now a simple matter to show that the components of $\underset{\sim}{N}(z)$ can be written in the form

$$N_i(z) = \frac{1}{2\pi i}\sum_{j=1}^{N} X'_{ij}(z)\, P_j(z), \quad i=1,2,\ldots,N, \tag{6.22}$$

where the $P_j(z)$, $j=1,2,\ldots,N$, are polynomials. Let us consider the degree of $P_j(z)$. We recall that the i^{th} row vector of $\underset{\sim}{X}(z)$ behaves as z^{-K_i} at infinity, where

$$\sum_{i=1}^{N} K_i = \alpha \quad . \tag{6.23}$$

Thus the i^{th} column vector of $\underset{\sim}{X}'(z)$ has degree $-K_i$ at infinity. Since the solution $\underset{\sim}{N}(z)$ should vanish as $1/z$ at infinity, we conclude that $P_j(z)$ is a polynomial of degree not greater than K_j-1 (P_j is identically equal to zero if $K_j \equiv 0$). Thus there are at most α undetermined constants in Eq. (6.22). Moreover, from Eq. (6.14) it follows that

$$f_{i+}(z) = \sum_{j=1}^{N} X'_{ij}(-z)\, P_j(-z) \ , \quad i=1,2,\ldots,N. \tag{6.24}$$

To determine the α unknowns in Eq. (6.24), we write Eq. (6.16) in the form

$$\sum_{j=1}^{N} \omega_{ij}(z)\ \sigma_j\ \psi_j^{(m)}(0,-\sigma_j z) = f_{i+}(-z) \ , \tag{6.25}$$

$$\text{Re } z > 0 \ , \quad i=1,2,\ldots,N.$$

Considering Eq. (6.25) as a system of linear algebraic equations for $\sigma_i\ \psi_i^{(m)}(0,-\sigma_i z)$, we note that $\sigma_i\ \psi_i^{(m)}(0,-\sigma_i z)$, $i=1,2,\ldots,N$, is bounded at $z = \nu_s$, $s=1,2,\ldots,\alpha-1$, only if

$$\sum_{i=1}^{N} Y_i(\nu_s)\, f_{i+}(-\nu_s) = 0 \quad , \quad s=1,2,\ldots,\alpha-1 \;, \tag{6.26}$$

where the $Y_i(\nu_s)$ are defined in Eq. (2.88). Thus

$$-\sum_{i=1}^{N-1} \sum_{k=1}^{N-1} \gamma_{Nk}\, \Delta_{ki}^{(N)}(\nu_s)\, f_{i+}(-\nu_s) + \Omega^{(N)}(\nu_s)\, f_{N+}(-\nu_s) = 0 \;,$$

$$s=1,2,\ldots,\alpha-1 \;. \tag{6.27}$$

In addition, if we expand $\omega_{ii}(z)$ in a Taylor series about $z = \nu_o$ in Eq. (6.25), we find that

$$-\sum_{i=1}^{N-1} \sum_{k=1}^{N-1} \gamma_{Nk}\, \Delta_{ki}^{(N)}(\nu_o)\, f_{i+}(-\nu_o) + \Omega^{(N)}(\nu_o)\, f_{N+}(-\nu_o) = \frac{1}{\nu_o} N_{o-} \;, \tag{6.28}$$

where N_{o-} is given by Eq. (2.55). Equations (6.27) and (6.28) constitute α constraints on $f_{i+}(z)$ which serve to determine the α unknown constants in Eq. (6.24). Although this procedure yields a very simple form for the functions $f_{i+}(z)$, it still suffers the difficulties in calculating $\underset{\sim}{X}'(z)$.

To obtain numerically amenable expressions for the $f_{i+}(z)$, we choose to determine the emerging distribution at the vacuum interface using the scalar method of Section III. In particular, setting $x = 0$ in Eq. (3.23) and using Eq. (6.11), we find that the emerging distribution also satisfies the following system of singular integral equations:

$$\Lambda(\mu)\ \sigma_i\ \Psi_i^{(m)}(0,-\sigma_i\mu) = \sum_{n=1}^{N} \frac{c}{2} P \int_{n>} \frac{R_n(\nu)\,\sigma_i \Psi_i^{(m)}(0,-\sigma_i\nu)}{\nu-\mu}\, d\nu - g_i(0,-\mu),$$

$$i=1,2,\ldots,N, \quad \mu \in (0,1), \tag{6.29}$$

where

$$g_i(0,\mu) = -\sum_{j=1}^{N}\sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_{ijn}(\nu)\ f_{j+}(\nu)}{\nu+\mu}\, d\nu\ . \tag{6.30}$$

In contrast to the situation in Section III, these singular integral equations are not uncoupled since the f_{i+}, $i=1,2,\ldots,N$, are known only in so far as they are given in terms of $\Psi_i^{(m)}(0,-\sigma_i\nu)$ in Eq. (6.13). When Eq. (6.13) is substituted into Eq. (6.30), we find that

$$g_i(0,-\mu) = \sum_{j=1}^{N} \frac{c}{2} \int_0^{1/\sigma_j} \nu\sigma_j \Psi_j^{(m)}(0,-\sigma_j\nu) \left[\frac{\tau_{ij}(\nu)-\tau_{ij}(\mu)}{\nu-\mu}\right] d\nu \tag{6.31}$$

where

$$\tau_{ij}(\mu) = \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_{ijn}(\eta)}{\eta+\mu}\, d\eta\ . \tag{6.32}$$

With the above form of $g_i(0,-\mu)$, the ordinary scalar methods of Muskhelishvili can be used to reduce Eq. (6.31) to an equivalent system of coupled Fredholm equations of the second kind.

Let us introduce the functions

$$M_i(z) = \frac{1}{2\pi i} \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_n(\nu)\ \sigma_i\ \Psi_i^{(m)}(0,-\sigma_i\nu)}{\nu - z}\, d\nu \ ,$$

$$i=1,2,\ldots,N, \qquad (6.33)$$

which are analytic in the plane cut from 0 to +1 and vanish as 1/z at infinity. Moreover,

$$M_i^+(\mu) + M_i^-(\mu) = \frac{1}{i\pi} \sum_{n=1}^{N} \frac{c}{2}\ P \int_{n>} \frac{R_n(\nu)\ \sigma_i\ \Psi_i^{(m)}(0,-\sigma_i\nu)}{\nu - \mu}\, d\nu \qquad (6.34)$$

and

$$M_i^+(\mu) - M_i^-(\mu) = \frac{c}{2}\, R_k(\mu)\ \sigma_i\ \Psi_i^{(m)}(0,-\sigma_i\mu)\ , \qquad (6.35)$$

$$\mu \in (1/\sigma_{k-1},\ 1/\sigma_k) \ , \quad i,k=1,2,\ldots,N\ .$$

Substituting the above expressions into Eq. (6.29), we find that $M_i(z)$ must be a solution, vanishing at infinity, of the non-homogeneous scalar Hilbert problem

$$M_1^+(\mu) - G(\mu)\, M_1^-(\mu) = -\frac{c}{2}\,\frac{R_k(\mu)\; g_i(0,-\mu)}{\Omega^-(\mu)} \;, \tag{6.36}$$

$$\mu \in (1/\sigma_{k-1} \,,\, 1/\sigma_k) \quad i,k=1,2,\ldots,N \;,$$

where

$$G(\mu) = \frac{\Omega^+(\mu)}{\Omega^-(\mu)} \;. \tag{6.37}$$

It can be shown that $G(\mu)$ is nonvanishing and obeys an H^* condition on $(0,1)$. In accordance with Muskhelishvili, then, we solve this Hilbert problem by considering the corresponding homogeneous problem: to find a scalar function $X(z)$, nonvanishing and analytic in the plane cut from 0 to +1, which

(a) satisfies the boundary condition

$$X^+(\mu) - G(\mu)\, X^-(\mu) = 0 \;,\; \mu \in (0,1), \tag{6.38}$$

(b) does not vanish as rapidly as $(z-d)$ or approach infinity as fast as $(z-d)^{-1}$ as $z \longrightarrow d$, where d stands for either of the endpoints 0 or 1.

In Appendix B, we show that an appropriate X-function is

$$X(z) = \frac{1}{(1-z)^{\alpha}}\, e^{\Gamma(z)} \;, \tag{6.39}$$

where

$$\Gamma(z) = \frac{1}{2\pi i} \int_0^1 \frac{\log G(\mu)}{\mu - z} \, d\mu \, . \qquad (6.40)$$

Using Eq. (6.38), we write Eq. (6.36) in the form

$$\frac{M_i^+(\mu)}{X^+(\mu)} - \frac{M_i^-(\mu)}{X^-(\mu)} = -\frac{c}{2} \, \frac{R_k(\mu) \; g_i(0,-\mu)}{X^+(\mu) \; \Omega^-(\mu)} \, , \qquad (6.41)$$

$$\mu \in (1/\sigma_{k-1} \, , \, 1/\sigma_k) \, , \quad i,k=1,2,\ldots,N.$$

Now consider the functions

$$E_i(z) = \frac{M_i(z)}{X(z)} + \frac{1}{2\pi i} \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_n(\nu) \; g_i(0,-\nu)}{X^+(\nu) \; \Omega^-(\nu) \; (\nu - z)} \, d\nu \, , \qquad (6.42)$$

$$i=1,2,\ldots,N \, .$$

In view of Eq. (6.41), these functions are analytic everywhere in the finite plane. At infinity they behave as a polynomial of degree $\alpha-1$. Thus we can write $E_i(z)$ in the form

$$E_i(z) = \frac{1}{2\pi i} \sum_{j=0}^{\alpha-1} b_j^{(i)} z^j \, , \qquad (6.43)$$

where the constants $b_j^{(i)}$ will be determined below using the analytic properties of $\Psi_i^{(m)}(0,-\sigma_i z)$. From Eq. (6.42) it follows that

$$M_i(z) = \frac{X(z)}{2\pi i}\left\{\sum_{j=0}^{\alpha-1} b_j^{(i)} z^j - \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_n(\nu)\; g_i(0,-\nu)}{X^+(\nu)\;\Omega^-(\nu)\;(\nu-z)}\, d\nu\right\} \tag{6.44}$$

$$i=1,2,\ldots,N\ .$$

Substituting the explicit form of $g_i(0,-\nu)$, given by Eq. (6.31), into the above equation and using Eq. (6.35), we obtain the following system of Fredholm equations for the emerging distribution:

$$\sigma_i\,\Psi_i^{(m)}(0,-\sigma_i u) = \frac{X^-(u)}{\Omega^-(u)}\left\{\sum_{j=0}^{\alpha-1} b_j^{(i)} u^j - \sum_{k=1}^{N}\int_0^{1/\sigma_k} K_{ik}^{(m)}(u,\eta)\;\sigma_k\Psi_k^{(m)}(0,-\sigma_k\eta)\,d\eta\right\} \tag{6.45}$$

$$u \in (0,\ 1/\sigma_i)\ ,\quad i=1,2,\ldots,N,$$

where

$$K_{ik}^{(m)}(\mu,\eta) = \frac{c\eta}{2}\left\{\sum_{n=1}^{N} \frac{c}{2} P \int_{n>} \frac{R_n(\nu)\left[\tau_{ik}(\eta) - \tau_{ik}(\nu)\right]}{X^{+}(\nu)\ \Omega^{-}(\nu)\ (\nu-\mu)\ (\eta-\nu)}\ d\nu\right.$$

$$+ \frac{1}{2}\ \frac{\left[\frac{1}{X^{+}(\mu)} + \frac{1}{X^{-}(\mu)}\right]\left[\tau_{ik}(\eta) - \tau_{ik}(\mu)\right]}{\eta - \mu} \quad . \tag{6.46}$$

The following identities involving X(z) are proven in Appendix B.

Identity (6.1).

$$X(z)\ X(-z) = \frac{\Omega(z)}{\Omega(\infty) \prod_{s=0}^{\alpha-1} (\nu_s^2 - z^2)} \quad , \tag{6.47}$$

where

$$\Omega(\infty) = \det\left\{\gamma_{ij} - \frac{c}{\sigma_i}\ \delta_{ij}\right\} \quad , \quad i,j=1,2,\dots,N, \tag{6.48}$$

denotes the value of $\Omega(z)$ at infinity.

Identity (6.2).

$$X(z) = \int_0^1 \frac{\gamma(\nu)}{\nu - z}\ d\nu \quad , \tag{6.49}$$

where

$$\gamma(\nu) = \frac{c}{2} \frac{R_n(\nu)}{\Omega(\infty)\; X(-\nu) \prod_{s=0}^{\alpha-1} (\nu_s^2 - \nu^2)}$$

$$\nu \in (1/\sigma_{n-1}\, ,\ 1/\sigma_n)\ ,\quad n=1,2,\ldots,N. \tag{6.50}$$

Identity (6.3).

$$\frac{1}{X(z)} = P(z) - \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_n(\nu)}{X^+(\nu)\; \Omega^-(\nu)\; (\nu - z)}\, d\nu\ , \tag{6.51}$$

where

$$P(z) = \sum_{k=0}^{\alpha} c_k\, z^k\ . \tag{6.52}$$

The constants c_k, $k=0,1,\ldots,\alpha$, are determined by the equations

$$\sum_{i=0}^{k} \Gamma_{\alpha+i-1}\, c_{\alpha+i-k} = 0\ ,\quad k=1,2,\ldots,\alpha \tag{6.53}$$

with

$$c_\alpha = (-1)^\alpha \tag{6.54}$$

and

$$\Gamma_i = \int_0^1 \nu^i\, \gamma(\nu)\, d\nu\ . \tag{6.55}$$

Identity (6.4).

$$\frac{\tau_{ij}(z)}{X(z)} = P_{ij}(z) + Q_{ij}(z) - \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{\tau_{ij}(\nu)\; R_n(\nu)}{X^+(\nu)\; \Omega^-(\nu)\; (\nu - z)}\, d\nu \tag{6.56}$$

$$i, j = 1, 2, \ldots, N ,$$

where

$$P_{ij}(z) = \sum_{k=0}^{\alpha-1} d_{ijk}\; z^{\alpha-k-1} \tag{6.57}$$

and

$$Q_{ij}(z) = \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_{ijn}(\nu)}{X(-\nu)\; (\nu + z)}\, d\nu \; . \tag{6.58}$$

In Eq. (6.57), the constants d_{ijk} are determined by the equations

$$d_{ijk} = \sum_{l=0}^{k} R_{ij}^{(l)}\; c_{\alpha+l-k} \tag{6.59}$$

where

$$R_{ij}^{(l)} = (-1)^l \frac{c}{2} \sum_{n=1}^{N} \int_{n>} \nu^l\; R_{ijn}(\nu)\, d\nu \; . \tag{6.60}$$

Using the above identities we can write Eq. (6.45) as

$$\sigma_i \, \Psi_i^{(m)}(0,-\sigma_i\mu) = \frac{1}{\Omega(\infty)\, X(-\mu) \prod_{s=0}^{\alpha-1} (\nu_s^2 - \mu^2)} \left\{ \sum_{j=0}^{\alpha-1} b_j^{(i)} \mu^j \right.$$

$$\left. - \sum_{k=1}^{N} \int_0^{1/\sigma_k} K_{ik}^{(m)}(\mu,\eta)\; \sigma_k \, \Psi_k^{(m)}(0,-\sigma_k\eta)\, d\eta \right\}$$

$$\mu \in (0,1/\sigma_i) \; , \quad i=1,2,\ldots,N \; ,$$

(6.61)

where the kernel is now free of terms involving Cauchy principal valued integrals:

$$K_{ik}^{(m)}(\mu,\eta) = \frac{c\eta}{2} \, \frac{1}{\eta-\mu} \left\{ \left[P_{ik}(\eta) + Q_{ik}(\eta) - \tau_{ik}(\eta)\, P(\eta) \right] \right.$$

$$\left. - \left[P_{ik}(\mu) + Q_{ik}(\mu) - \tau_{ik}(\eta)\, P(\mu) \right] \right\}. \tag{6.62}$$

Let us introduce the vectors

$$\underset{\sim}{\Psi}{}_{jk}^{(m)}(\mu) = \begin{bmatrix} \Psi_{1jk}^{(m)}(\mu) \\ \Psi_{2jk}^{(m)}(\mu) \\ \cdot \\ \cdot \\ \cdot \\ \Psi_{Njk}^{(m)}(\mu) \end{bmatrix} \tag{6.63}$$

$\mu \in (0,1)$, $j=0,1,\ldots,\alpha-1$, $k=1,2,\ldots,N$, which are determined by the following α N systems of Fredholm integral equations:

$$\sigma_i \Psi^{(m)}_{ijk}(\sigma_i\mu) = \frac{1}{X(-\mu)\,\Omega(\infty)\prod_{s=0}^{\alpha-1}(\nu_s^2 - \mu^2)} \Bigg\{ \delta_{ik}\,\mu^j$$

$$- \sum_{l=1}^{N} \int_0^{1/\sigma_l} K^{(m)}_{il}(\mu,\eta)\,\sigma_l\,\Psi^{(m)}_{ljk}(\sigma_l\eta)\,d\eta \Bigg\} ,$$

$$\mu \in (0,1/\sigma_i) \quad , \quad i=1,2,\ldots,N . \tag{6.64}$$

These equations can not be solved in closed form, but they can be numerically analysed by standard methods. Using Eqs. (6.61) and (6.64), we write the emerging distribution in terms of the elements Ψ_{ijk} as follows:

$$\Psi^{(m)}_i(0,-\sigma_i\mu) = \sum_{j=0}^{\alpha-1}\sum_{k=1}^{N} b^{(k)}_j\,\Psi^{(m)}_{ijk}(\sigma_i\mu) ,$$

$$\mu \in (0,1/\sigma_i) , \quad i=1,2,\ldots,N . \tag{6.65}$$

The constants $b^{(k)}_j$ are determined using the analytic properties of $\Psi^{(m)}_i(0,-\sigma_i z)$. We note that the analytic continuation of Eq. (6.61) into the complex domain Re $z > 0$ can be written in the form

$$\sigma_i \, \Psi_i^{(m)}(0,-\sigma_i z) = \frac{H_i(z)}{X(-z)\,\Omega(\infty) \prod_{s=0}^{\alpha-1} (\nu_s^2 - z^2)} \tag{6.66}$$

$$\operatorname{Re} z > 0 \,, \quad i=1,2,\ldots,N \,,$$

where

$$H_i(z) = \sum_{j=0}^{\alpha-1} \sum_{k=1}^{N} \left\{ \delta_{ik} z^j - \sum_{l=1}^{N} \int_0^{1/\sigma_l} K_{il}^{(m)}(z,\eta) \, \sigma_l \Psi_{ljk}^{(m)}(\sigma_l \eta) d\eta \right\} b_j^{(k)}. \tag{6.67}$$

In order that $\Psi_i^{(m)}(0,-\sigma_i z)$ be analytic at ν_s, $s=1,2,\ldots,\alpha-1$, we set

$$H_i(\nu_s) = 0 \,, \tag{6.68}$$

$$i=1,2,\ldots,N \,, \quad s=1,2,\ldots,\alpha-1 \,.$$

On the other hand, accounting for the fact that $\Psi_i^{(m)}(0,-\sigma_i z)$ has a simple pole at ν_o, we have

$$H_i(\nu_o) = - c \, \nu_o^2 \, X(-\nu_o) \, \Omega(\infty) \prod_{s=1}^{\alpha-1} (\nu_s^2 - \nu_o^2) \sum_{k=1}^{N-1} \Delta_{ik}^{(N)}(\nu_o) \, \gamma_{kN},$$

$$i=1,2,\ldots,N-1 \,, \tag{6.69}$$

and

$$H_N(\nu_o) = c\,\nu_o^2\, X(-\nu_o)\, \Omega(\infty) \prod_{s=1}^{\alpha-1} (\nu_s^2 - \nu_o^2)\, \Omega^{(N)}(\nu_o) \,. \qquad (6.70)$$

The αN constraints on the functions $H_i(z)$ in Eqs. (6.68) through (6.70) are just sufficient to determine the constants $b_j^{(k)}$. An explicit system of linear algebraic equations for the $b_j^{(k)}$ can be obtained by introducing the elements B_q, S_q and $\lambda_{p,q}$, $p,q=0,1,\ldots,\alpha N-1$, which are uniquely defined by the relations

$$B_{j+\alpha(k-1)} = b_j^{(k)} \qquad (6.71)$$

$$S_{j+\alpha(k-1)} = H_k(\nu_j) \qquad (6.72)$$

$$\lambda_{s+\alpha(i-1),\,j+\alpha(k-1)} = \delta_{ik}(\nu_s)^j$$

$$-\sum_{l=1}^{N} \int_0^{1/\sigma_l} K_{il}^{(m)}(\nu_s,\eta)\, \sigma_l\, \Psi_{ljk}^{(m)}(\sigma_l\eta)\, d\eta \,,$$

$$s,j=0,1,\ldots,\alpha-1 \,, \quad i,k=1,2,\ldots,N \,. \qquad (6.73)$$

Using these definitions, we can combine Eqs. (6.68) through (6.70) as follows:

$$\sum_{q=0}^{\alpha N-1} \lambda_{p,q} B_q = S_p , \qquad p=0,1,\ldots,\alpha N-1 . \tag{6.74}$$

The $b_j^{(k)}$ can therefore be determined by solving the above system of inhomogeneous equations for B_q, and using Eq. (6.71). With the $b_j^{(k)}$ so determined, Eq. (6.65) can be used to determine the emerging distribution at the vacuum interface. The functions $f_{i+}(z)$, $i=1,2,\ldots,N$, can then be obtained from Eq. (6.14).

VII. THE CRITICAL PROBLEM

The last application to be considered is the critical problem for an infinite slab bounded by vacuum. The slab, which extends from $x = -a$ to $x = a$, is assumed to be composed of a certain homogeneous multiplying material. We wish to determine the functions $f_i(z)$ and the half-thickness a, corresponding to the case when the system's neutron distribution $\underset{\sim}{\psi}^{(c)}(x,\mu)$ is self-sustaining. In this case, the distribution obeys the symmetry condition

$$\underset{\sim}{\psi}^{(c)}(x,\mu) = \underset{\sim}{\psi}^{(c)}(-x,-\mu) \ , \quad -1 \leq \mu \leq 1 \ . \qquad (7.1)$$

Furthermore, since there is no way for neutrons to enter the slab from the vacuum, we have the boundary conditions

$$\underset{\sim}{\psi}^{(c)}(\pm a,\mu) = 0, \quad \mu \lessgtr 0. \qquad (7.2)$$

We shall find that the functions $f_i(z)$ for the critical problem are determined by the emerging neutron distribution at either face of the slab, and that the singular integral equations obtained for the emerging distribution can be reduced to an equivalent system of coupled Fredholm integral equations. An exact statement of the criticality condition arises from this reduction.

Proceeding as in the Milne problem, it is easily demonstrated that the solution to the critical problem $\underset{\sim}{\Psi}^{(c)}(x,\mu)$ can be analytically continued into the complex domain. This continuation is analytic everywhere in the complex plane cut along the imaginary axis and has the property

$$\underset{\sim}{\Psi}^{(c)}(\pm a,z) = 0 \ , \qquad \text{Re } z \lessgtr 0 \ . \tag{7.3}$$

Since $\underset{\sim}{\Psi}^{(c)}(x,\mu)$, $\mu \in (-1,1)$, satisfies the multi-group transport equations, it follows that $\underset{\sim}{\Psi}^{(c)}(x,z)$ satisfies Eq. (3.10) in the form

$$\sum_{j=1}^{N} \omega_{ij}(z)\ \sigma_j\ \Psi_j^{(c)}(x,\sigma_j z) - \frac{c}{2}\int_{-1/\sigma_i}^{1/\sigma_i} \frac{\nu\ \sigma_i\ \Psi_i^{(c)}(x,\sigma_i\nu)}{\nu - z}\, d\nu$$

$$= \begin{cases} -\ f_{i-}(z)\ e^{-x/z} \ , & \text{Re } z < 0 \\ -\ f_{i+}(z)\ e^{-x/z} \ , & \text{Re } z > 0 \ , \end{cases} \tag{7.4}$$

$$i=1,2,\ldots,N \ .$$

When Eqs. (7.1) and (7.3) are applied to Eq. (7.4), we find

$$f_{i-}(z) = e^{a/z}\ \frac{c}{2}\int_{0}^{1/\sigma_i} \frac{\nu\ \sigma_i\ \Psi_i^{(c)}(a,\sigma_i\nu)}{\nu - z}\, d\nu \tag{7.5}$$

and

$$f_{i+}(z) = e^{-a/z} \frac{c}{2} \int_0^{1/\sigma_i} \frac{\nu \; \sigma_i \; \Psi_i^{(c)}(a, \sigma_i\nu)}{\nu + z} \, d\nu \; , \tag{7.6}$$

$$i=1,2,\ldots,N \; .$$

It is clear, from the symmetry condition in Eq. (7.1), that $f_{i+}(z)$ and $f_{i-}(z)$ can also be determined in terms of the emerging neutron distribution at $x = -a$. We note that a system of singular integral equations for the emerging distribution can be obtained by setting $x = a$ and letting z tend to $\mu \in (0,1)$ in Eq. (7.4). Such a system could be analysed, at least formally, by reducing it to a matrix Hilbert problem. In view of the difficulties in obtaining numerical results using the matrix method, we shall instead consider the singular integral equations which are obtained from Eq. (3.23). In particular, when $f_{i-}(z)$ and $f_{i+}(z)$ are substituted into Eq. (3.23) and x is set equal to a, we find

$$\Lambda(\mu) \; \sigma_i \; \Psi_i^{(c)}(a, \sigma_i\mu) = \sum_{n=1}^{N} \frac{c}{2} \, P \int_{n>} \frac{R_n(\nu) \; \sigma_i \; \Psi_i^{(c)}(a, \sigma_i\nu)}{\nu - \mu} \, d\nu - g_i(a,\mu)$$

$$\mu \in (0,1) \; , \quad i=1,2,\ldots,N \; . \tag{7.7}$$

Here $g_i(a,\mu)$ has the form

$$g_i(a,\mu) = \sum_{j=1}^{N} \frac{c}{2} \int_0^{1/\sigma_j} \frac{\nu\ \sigma_j\ \Psi_j^{(c)}(a,\sigma_j\nu)}{\nu - \mu} \left[\tau_{ij}(\nu) - \tau_{ij}(\mu)\right] d\nu$$

$$+ \sum_{j=1}^{N} \frac{c}{2} \int_0^{1/\sigma_j} \frac{\nu\ \beta_{ij}(\nu)\ \sigma_j\ \Psi_j^{(c)}(a,\sigma_j\nu)}{\nu + \mu}\, d\nu$$

$$+ \sum_{j=1}^{N} \theta_{ij}(\mu)\, \frac{c}{2} \int_0^{1/\sigma_j} \frac{\nu\ \sigma_j\ \Psi_j^{(c)}(a,\sigma_j\nu)}{\nu + \mu}\, d\nu\ , \tag{7.8}$$

where $\tau_{ij}(\mu)$ is defined by Eq. (6.32),

$$\theta_{ij}(\mu) = \frac{1}{2}\left[\Delta_{ij}^{+}(\mu) + \Delta_{ij}^{-}(\mu)\right] e^{-2a/\mu}$$

$$- \sum_{n=1}^{N} \frac{c}{2}\, P \int_{n>} \frac{R_{ijn}(\nu)\ e^{-2a/\nu}}{\nu - \mu}\, d\nu\ , \tag{7.9}$$

and

$$\beta_{ij}(\mu) = \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_{ijn}(\nu)\ e^{-2a/\nu}}{\nu + \mu}\, d\nu\ . \tag{7.10}$$

As in the Milne problem, we can not obtain a closed form solution for the emerging distribution, but we can reduce

(7.7) to a system of Fredholm integral equations which is amenable to numerical analysis. First, however, let us cast Eq. (7.9) into a form which is free of Cauchy principal valued integrals.

Consider the functions

$$D_{ij}(z) = \Delta_{ij}(z) - \Delta_{ij}(\infty) - \tau_{ij}(z) - \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_{ijn}(\nu)}{\nu - z}\, d\nu \tag{7.11}$$

$$i,j=1,2,\ldots,N ,$$

where $\Delta_{ij}(\infty)$ denotes the constant value of $\Delta_{ij}(z)$ at infinity. Obviously, $D_{ij}(z)$ is analytic everywhere in the complex plane cut along (-1,1) and vanishes at infinity. Furthermore, using the Plemelj formulas, we find

$$D^{+}_{ij}(\mu) - D^{-}_{ij}(\mu) = 0 \quad , \quad \mu \in (-1,1). \tag{7.12}$$

Hence

$$D_{ij}(z) \equiv 0 . \tag{7.13}$$

We conclude, from Eq. (7.11), that

$$\Delta_{ij}(z) = \Delta_{ij}(\infty) + \tau_{ij}(z) + \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_{ijn}(\nu)}{\nu - z}\, d\nu . \tag{7.14}$$

Using this representation for $\Delta_{ij}(z)$, we can rewrite Eq. (7.9) as

$$\theta_{ij}(\mu) = \left[\Delta_{ij}(\infty) + \tau_{ij}(\mu)\right] e^{-2a/\mu}$$

$$- \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_{ijn}(\nu)}{\nu - \mu} \left[e^{-2a/\nu} - e^{-2a/\mu}\right] d\nu. \qquad (7.15)$$

To reduce Eq. (7.7) to a system of Fredholm integral equations, we introduce the functions

$$M_i(z) = \frac{1}{2\pi i} \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_n(\nu)\ \sigma_i\ \Psi_i^{(c)}(a, \sigma_i\nu)}{\nu - z} d\nu, \quad i=1,2,\ldots,N. \qquad (7.16)$$

Following the procedure of Section VI, it can be shown that $M_i(z)$ can also be represented in the form

$$M_i(z) = \frac{X(z)}{2\pi i} \left\{ \sum_{j=0}^{\alpha-1} b_j^{(i)} z^j - \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_n(\nu)\ g_i(a,\nu)}{X^+(\nu)\ \Omega^-(\nu)\ (\nu - z)} d\nu \right\},$$

$$i=1,2,\ldots,N, \qquad (7.17)$$

where the constants $b_j^{(i)}$ are as yet undetermined and $X(z)$ is given by Eq. (6.39). Substituting the explicit expression for $g_i(a,\nu)$ into Eq. (7.17) and using the relations

$$M_i^+(\mu) - M_i^-(\mu) = \frac{c}{2}\; R_n(\mu)\; \sigma_i\; \Psi_i^{(c)}(a,\sigma_i\mu)\;,$$

$$\mu \in (1/\sigma_{n-1}\;,\; 1/\sigma_n)\;,\quad i,n=1,2,\ldots,N, \tag{7.18}$$

which follow from Eq. (7.16), we obtain the following system of coupled Fredholm integral equations for the emerging distribution:

$$\sigma_i\,\Psi_i^{(c)}(a,\sigma_i\mu) = \frac{1}{X(-\mu)\;\Omega(\infty)\prod_{s=0}^{\alpha-1}(\nu_s^2-\mu^2)}\left\{\sum_{j=0}^{\alpha-1} b_j^{(i)}\,\mu^j - \sum_{k=1}^{N}\int_0^{1/\sigma_i} K_{ik}^{(c)}(\mu,\eta)\;\sigma_k\,\Psi_k^{(c)}(a,\sigma_k\eta)\,d\eta\right\},$$

$$\mu \in (0,1/\sigma_i)\;,\qquad i=1,2,\ldots,N\;, \tag{7.19}$$

where

$$K_{ik}^{(c)}(\mu,\eta) = K_{ik}^{(m)}(\mu,\eta) + \frac{c\eta}{2}\left\{\sum_{n=1}^{N}\frac{c}{2}\,P\int_{n>}\frac{R_n(\nu)\left[\beta_{ik}(\eta)+\theta_{ik}(\nu)\right]}{X^+(\nu)\;\Omega^-(\nu)\,(\nu-\mu)\,(\eta+\nu)}\,d\nu - \frac{1}{2}\left[\frac{1}{X^+(\mu)}+\frac{1}{X^-(\mu)}\right]\left[\frac{\beta_{ik}(\eta)+\theta_{ik}(\mu)}{\eta+\mu}\right]\right\}. \tag{7.20}$$

We recall that the elements $K_{ik}^{(m)}(\mu,\eta)$ are specified by either Eq. (6.46) or (6.62). The task of calculating the

Cauchy principal valued integrals in Eq. (7.20) can be avoided by using Identities (6.1) and (6.3) to write $K_{ik}^{(c)}(\mu,\eta)$ in the form

$$K_{ik}^{(c)}(\mu,\eta) = K_{ik}^{(m)}(\mu,\eta) + \frac{cn}{2}\,\frac{1}{\eta+\mu}\Big\{ S_{ik}(\mu) - T_{ik}(\eta) + P(\mu)\left[\beta_{ik}(\eta) + \theta_{ik}(\mu)\right] + \left[\frac{1}{X(-\eta)} - P(-\eta)\right]\beta_{ik}(\eta)\Big\}, \tag{7.21}$$

where the polynomial $P(\mu)$ is defined in Eq. (6.52)

$$S_{ik}(\mu) = \sum_{n=1}^{N} \frac{c}{2}\int_{n>} \frac{R_n(\nu)X(-\nu)\Omega(\infty)\prod_{s=0}^{\alpha-1}(\nu_s^2-\nu^2)\left[\theta_{ik}(\nu)-\theta_{ik}(\mu)\right]}{\Omega^+(\nu)\,\Omega^-(\nu)\,(\nu-\mu)}\,d\nu, \tag{7.22}$$

and

$$T_{ik}(\eta) = \sum_{n=1}^{N} \frac{c}{2}\int_{n>} \frac{R_n(\nu)X(-\nu)\Omega(\infty)\prod_{s=0}^{\alpha-1}(\nu_s^2-\nu^2)\,\theta_{ik}(\nu)}{\Omega^+(\nu)\,\Omega^-(\nu)\,(\nu+\eta)}\,d\nu\,. \tag{7.23}$$

As in the Milne problem, we choose to expand the emerging distribution in terms of the vectors

$$\underset{\sim}{\Psi}{}_{jk}^{(c)}(\mu) = \begin{bmatrix} \Psi_{1jk}^{(c)}(\mu) \\ \Psi_{2jk}^{(c)}(\mu) \\ \vdots \\ \Psi_{Njk}^{(c)}(\mu) \end{bmatrix}$$

$$\mu \in (0,1),\ j=0,1,\ldots,\alpha-1,\ k=1,2,\ldots,N, \tag{7.24}$$

where $\underset{\sim}{\Psi}{}^{(c)}_{jk}(\mu)$ is the solution to the following system of Fredholm equations:

$$\sigma_i \Psi^{(c)}_{ijk}(\sigma_i\mu) = \frac{1}{X(-\mu)\Omega(\infty)\prod_{s=0}^{\alpha-1}(\nu_s^2-\mu^2)} \left\{ \delta_{ik}\,\mu^j - \sum_{l=1}^{N} \int_0^{1/\sigma_l} K^{(c)}_{il}(\mu,\eta)\, \sigma_l \Psi^{(c)}_{ljk}(\sigma_l\eta)d\eta \right\},$$

$$\mu \in (0,1/\sigma_i)\ ,\quad i=1,2,\ldots,N\ . \tag{7.25}$$

We again remark that such Fredholm integral equations can be analysed numerically by standard methods. With the vectors $\underset{\sim}{\Psi}{}^{(c)}_{jk}(\mu)$ so determined, we can write $\Psi^{(c)}_i(a,\sigma_i\mu)$ in the form

$$\Psi^{(c)}_i(a,\sigma_i\mu) = \sum_{j=0}^{\alpha-1}\sum_{k=1}^{N} b^{(k)}_j\, \Psi^{(c)}_{ijk}(\sigma_i\mu)$$

$$\mu \in (0,1/\sigma_i)\ ,\quad i=1,2,\ldots,N\ . \tag{7.26}$$

To determine the constants $b^{(k)}_j$ in Eq. (7.26), we write the analytic continuation of Eq. (7.19) into the complex domain Re $z > 0$ as follows:

$$\sigma_i \Psi_i^{(c)}(a, \sigma_i z) = \frac{L_i(z)}{X(-z)\,\Omega(\infty) \prod_{s=0}^{\alpha-1} (\nu_s^2 - z^2)}$$

$$\text{Re } z > 0, \quad i=1,2,\ldots,N \, , \tag{7.27}$$

where

$$L_i(z) = \sum_{j=0}^{\alpha-1} \sum_{k=1}^{N} b_j^{(k)} \left\{ \delta_{ik}\, z^j - \sum_{l=1}^{N} \int_0^{1/\sigma_l} K_{il}^{(c)}(z,\eta)\, \sigma_l \Psi_{ljk}^{(c)}(\sigma_l \eta)\, d\eta \right\}. \tag{7.28}$$

Since $\Psi_i^{(c)}(a, \sigma_i z)$ is analytic at ν_s, $s=0,1,\ldots,\alpha-1$, we deduce from Eq. (7.27) that

$$L_i(\nu_s) = 0 \, , \qquad s=0,1,\ldots,\alpha-1, \quad i=1,2,\ldots,N \, . \tag{7.29}$$

Now introduce the elements B_q and λ_{pq}, $p,q=0,1,\ldots,\alpha N-1$, which are defined below:

$$B_{j+\alpha(k-1)} = b_j^{(k)} \tag{7.30}$$

$$\lambda_{s+\alpha(i-1),\,j+\alpha(k-1)} = \delta_{ij}\,(\nu_s)^j - \sum_{l=1}^{N} \int_0^{1/\sigma_l} K_{il}^{(c)}(\nu_s,\eta)\; \sigma_l \psi_{ljk}^{(c)}(\sigma_l\eta)d\eta,$$

$$s,j=0,1,\ldots,\alpha-1\ , \quad i,j=1,2,\ldots,N\ . \tag{7.31}$$

It follows that we can write the αN constraints of Eq. (7.29) in the form

$$\sum_{q=0}^{\alpha N-1} \lambda_{p,q}\, B_q = 0,\ p=0,1,\ldots,\alpha-1\ . \tag{7.32}$$

This system of homogeneous equations has nontrivial solutions only if

$$\det\left\{\lambda_{p,q}\right\} = 0,\ p,q=0,1,\ldots,\alpha N-1\ . \tag{7.33}$$

Equation (7.33) is the criticality condition for the system since $\lambda_{p,q}$ depends only on the transfer matrix $\underset{\sim}{c}$, the cross sections $1/\sigma_i$, and the half-thickness a of the slab. Let us assume that the transfer matrix and cross sections are specified. After finding the smallest value of a which satisfies Eq. (7.33), we can substitute this critical half-thickness into Eq. (7.32) and solve for the B_q. The constants $b_j^{(k)}$ can in turn be obtained using Eq. (7.30). The functions $f_{i-}(z)$ and $f_{i+}(z)$ for the critical problem can then be computed by determining the emerging distribution from Eq. (7.26) and using Eqs. (7.5) and (7.6).

VIII. CONCLUSION

We have seen that the method of singular integral equations can be used to analyse a varied set of multi-group transport problems in plane geometry. The procedure consists of transforming the integro-differential form multi-group equations for $\underset{\sim}{\Psi}(x,\mu)$ into a system of functional equations for $\underset{\sim}{\Psi}(x,z)$, where the spatial coordinate enters only as a parameter and the variable z may be complex valued. Included in these equations are arbitrary functions $f_i(z)$, $i=1,2,\ldots,N$, which are determined in specific transport problems by the boundary conditions. When z is restricted to the interval (-1,1), these functional equations yield a system of coupled singular integral equations for the neutron distribution. An important feature of these singular integral equations is that they can be explicitly uncoupled with respect to the components of the neutron distribution using analytic function theory. In these "uncoupled" singular integral equations, the physical relationship between the various components of the neutron distribution is maintained by the persistent coupling of the functions f_i. In Section III, we chose to express the neutron distribution in terms of the f_i by solving the uncoupled (scalar) form of the singular integral equations. By virtue of their full-range character, the coupled (matrix) form of the singular integral equations could just as easily have been utilized to obtain the neutron distribution in terms of the functions f_i. Another way to obtain the neutron distribution from the $f_i(z)$ is to use Eqs. (4.8) and (4.13) to determine the expansion coefficients in the eigenfunction expansion given by Eq. (2.94).

In looking at specific applications, a closed form solution was obtained for the functions $f_i(z)$ associated with the infinite medium Green's function. In the Milne and critical problems the $f_i(z)$ were shown to be determined by the emerging distribution at the boundary. In each of these problems, the results of Section III were used to obtain singular integral equations for the emerging distribution in matrix and scalar form. In the Milne problem, we illustrated the power of the matrix approach by obtaining a closed form solution for the $f_i(z)$ in terms of the fundamental matrix $\underset{\sim}{X}(z)$. Due to the difficulties in calculating the $\underset{\sim}{X}$-matrix, however, we chose to determine the emerging distribution in the Milne and critical problems by reducing the scalar form of the singular integral equations to an equivalent system of Fredholm integral equations of the second kind. In the critical problem this reduction also yielded an exact statement of the criticality condition.

It appears that the above procedure for solving multi-group problems is amenable to numerical analysis using a high speed computer. Apparently, the major inconvenience in obtaining numerical results for the Milne and critical problems is the relatively large number of functions and parameters which must be calculated before the Fredholm integral equations for the emerging distribution can be analysed.

With regard to applications, we note that the above results may also be applied to problems involving the energy-dependent transport equation in the form

$$\left[\mu \frac{\partial}{\partial x} + \sigma(E)\right] \Psi(x,E,\mu) = \frac{1}{2} \int_{-1}^{1} d\mu' \int_{0}^{\infty} K(E',E)\, \Psi(x,E',\mu')\, dE', \qquad (8.1)$$

where the transfer kernel K(E',E) takes into account all processes which transfer neutrons of energy E' to energy E. Leonard and Ferziger (9) have shown that such problems can be reduced to a study of the multi-group equations by approximating $\Psi(x,E,\mu)$ by a finite expansion of Laguerre polynomials of order one:

$$\Psi(x,E,\mu) = M(E) \sum_{j=1}^{N} \chi_j(x,\mu)\, L_j^{(1)}(E) , \qquad (8.2)$$

where M(E) denotes the Maxwellian distribution and the components $\chi_j(x,\mu)$ are to be determined. Substituting the above representation for $\Psi(x,E,\mu)$ into Eq. (8.1) and using the orthogonality of the Laguerre polynomials, it is seen that the $\chi_j(x,\mu)$ are related in an elementary way to the solution of a system of integro-differential equations which have the same form, but not the same physical interpretation, as the multi-group equations. Using this approach, Leonard and Ferziger treated energy-dependent problems by analysing the resulting "multi-group" equations using Case's method. Of course, the "multi-group" equations obtained in this manner could also be analysed by the method of singular equations.

It should be noted that Eq. (8.1) can be analysed, at least formally, without resorting to approximations such as Eq. (8.2). This approach was pursued by Bednarz and Mika (1) using the method of Case. It would also be interesting to pursue this approach using the method of singular integral equations. In practical computations, however, some approximation is usually required, such as assuming a degenerate kernel or appealing to the multi-group approximation. Recently, Pahor (12) investigated the Milne and albedo problems for the energy-dependent transport equation by assuming an N-term degenerate kernel. Pahor's treatment of these half-space problems combines the methods of Chandrasekhar (6) and Case (4). Using the principles of invariance and reciprocity, Pahor shows that the emerging distribution for the Milne and albedo problems can be expressed in terms of a certain $\underset{\sim}{H}$-matrix and the Case eigenfunctions. The neutron distribution throughout the system can be determined from the distribution at the boundary using the completeness theorem of the Case eigenfunctions. Of particular interest is the fact that the H-matrix, which is a generalization of Chandrasekhar's H-function, is shown to satisfy a nonlinear matrix integral equation. Pahor discusses the possibilities for numerically solving this equation for the H-matrix by iteration. It is not inconceivable that the X-matrix introduced in Section II would also satisfy a similar type of nonlinear integral equation. This possibility should be investigated since such an equation could prove to be a useful tool for calculating $\underset{\sim}{X}(z)$.

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XI. APPENDICES

A. The Number of Zeros of $\Omega(z)$

We note from Eqs. (2.32) through (2.34) that $\Omega(z)$ is an even function of z which is analytic in the complex plane cut along (-1,1). The number of zeros 2α of $\Omega(z)$, $z \notin (-1,1)$, can therefore be calculated using the argument principle of complex variables. First we consider the functions $\Omega^+(\mu)$ and $\Omega^-(\mu)$, $\mu \in (-1,1)$, which were initially introduced in Section III, viz.,

$$\Omega^{\pm}(\mu) = \det \left\{ \omega_{ij}^{\pm}(\mu) \right\} , \qquad i,j=1,2,\ldots,N. \qquad (A.1)$$

When μ is an element of region (n), the elements $\omega_{ij}^{\pm}(\mu)$ have the form

$$\omega_{ij}^{\pm}(\mu) = \begin{cases} \gamma_{ij} - \delta_{ij}\, c\, \mu\, T\left(\frac{1}{\sigma_i \mu}\right) & , \; i < n \\ \gamma_{ij} - \delta_{ij}\, c\, \mu\, T(\sigma_i \mu) \pm \delta_{ij} \frac{i\,\pi\, c\, \mu}{2} & , \; i \geq n \end{cases}$$

(A.2)

Assuming the cut (-1,1) to be directed from -1 to +1, $\Omega^{\pm}(\mu)$ represent the limiting values of $\Omega(z)$ as z approaches (-1,1) from the left (+) and right (-). Of particular importance here is the fact that $\Omega^{\pm}(\mu)$ are continuous on (-1,1) except at the points $\pm 1/\sigma_i$, i=1,2,...,N, where they have logarithmic singularities. In this thesis we assume that $\Omega^+(\mu)$ and $\Omega^-(\mu)$ do not vanish on (-1,1). In addition, we note that as $|z|$ approaches infinity

$$\Omega(z) = \Omega(\infty) + O(1/z^2) \quad , \tag{A.3}$$

where

$$\Omega(\infty) = \det\left\{\gamma_{ij} - \frac{c}{\sigma_i} \quad \delta_{ij}\right\}, \qquad i,j=1,2,\ldots,N. \tag{A.4}$$

Using the above properties of $\Omega(z)$, we conclude that α is given by the argument principle in the form (cf. 2)

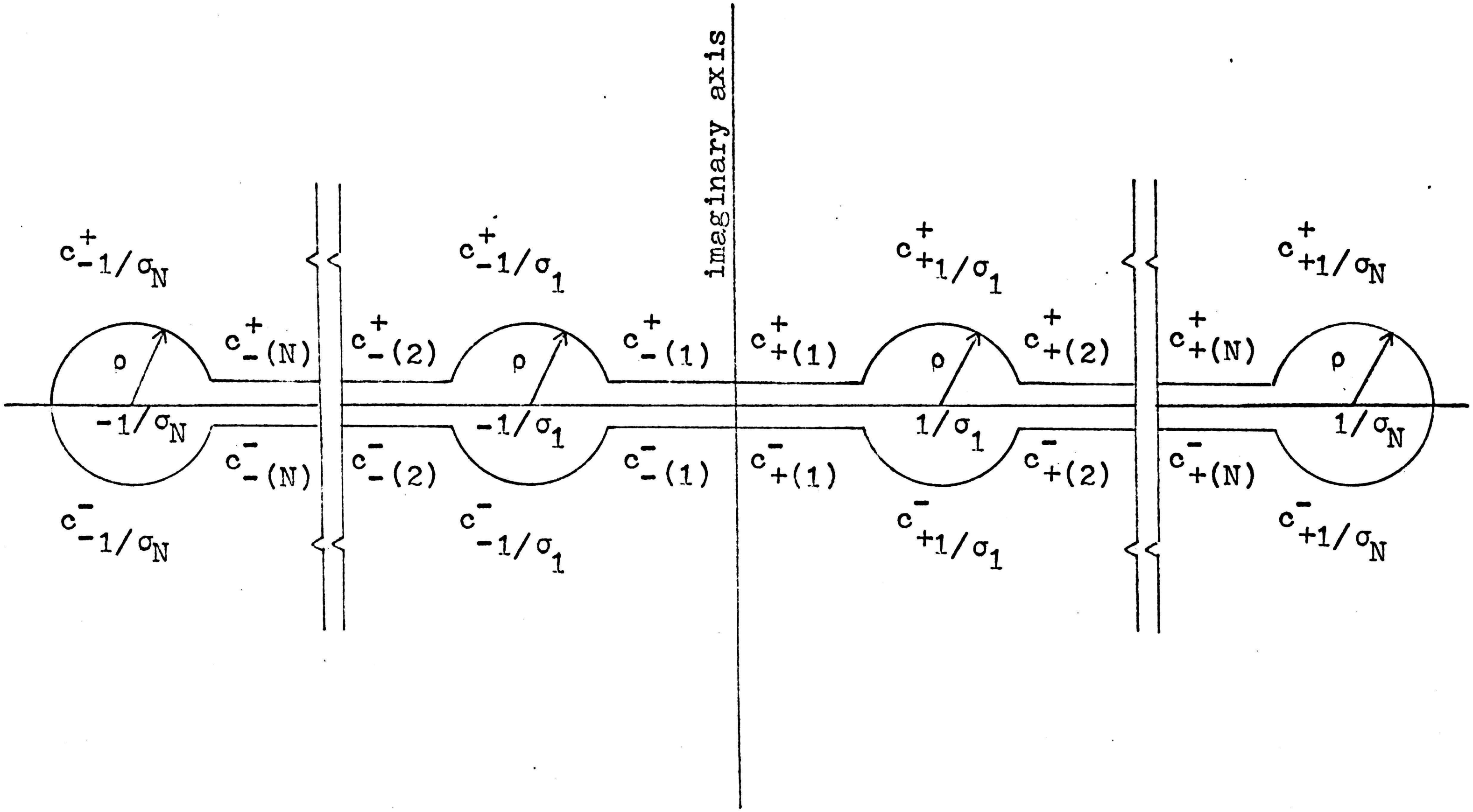
$$\alpha = \frac{1}{4\pi} \Delta_{c_o} \text{ Arg } \Omega(z) = \frac{1}{4\pi i} \int_{c_o} \frac{\Omega'(z)}{\Omega(z)} dz \ , \tag{A.5}$$

where

$$\Omega'(z) = \frac{d}{dz} \Omega(z) \tag{A.6}$$

and Δ_{c_o} Arg $\Omega(z)$ represents the change in the argument of $\Omega(z)$ around the contour c_o generated by letting $\rho \longrightarrow 0$ in Figure 1.

Let us calculate the change in the argument of $\Omega(z)$ as z traverses the circular arc $c^{\pm}_{\pm 1/\sigma_i}$. To do this we expand the determinant $\Omega(z)$ according to the elements of its



THE CONTOUR c_o

FIGURE 1

i^{th} row (or column) and write the result in the form

$$\Omega(z) = \log\,(z \mp 1/\sigma_i)\; W_1(z) + W_2(z) \quad , \tag{A.7}$$

where it can be shown that the functions $W_1(z)$ and $W_2(z)$, and their derivatives, have finite limits as $z \longrightarrow \pm\, 1/\sigma_i$ along any path; hence

$$(z \mp 1/\sigma_i)\,\frac{\Omega'(z)}{\Omega(z)}$$

$$= \frac{W_1(z) + (z\mp 1/\sigma_i)\,W_1'(z)\,\log(z\mp 1/\sigma_i) + (z\mp 1/\sigma_i)\,W_2'(z)}{\log\,(z\mp 1/\sigma_i)\,W_1(z) + W_2(z)} \tag{A.8}$$

tends to zero for all z on $c^{\pm}_{\pm 1/\sigma_i}$ as $\rho \longrightarrow 0$. It is now easy to deduce that

$$\lim_{\rho \to 0} \int_{c^{\pm}_{\pm 1/\sigma_i}} \frac{\Omega'(z)}{\Omega(z)}\,dz = 0\;. \tag{A.9}$$

Thus, the change in the argument of $\Omega(z)$ over the arc $c^{\pm}_{\pm\,1/\sigma_i}$ is zero. It follows that

$$\text{Arg}\;\Omega^+(\pm 1/\sigma_i - 0) = \text{Arg}\;\Omega^+(\pm 1/\sigma_i + 0) \tag{A.10}$$

and

$$\text{Arg}\ \Omega^-(\pm 1/\sigma_i - 0) = \text{Arg}\ \Omega^-(\pm 1/\sigma_i + 0) \quad , \tag{A.11}$$

where Arg $\Omega^{\pm}(\pm 1/\sigma_i - 0)$ denotes the limit of Arg $\Omega^{\pm}(\mu)$ as $\mu \rightarrow \pm 1/\sigma_i$ from the left, i.e., through real values less than $\pm 1/\sigma_i$; $\Omega^{\pm}(\pm 1/\sigma_i + 0)$ likewise denotes the limit of Arg $\Omega^{\pm}(\mu)$ as $\mu \rightarrow \pm 1/\sigma_i$ from the right.

We now may write α in terms of the changes in the argument of $\Omega(z)$ over the linear segments of the contour c_o:

$$\alpha = \frac{1}{4\pi} \sum_{i=1}^{N} \Big[\Delta_{c^+_-(i)} \text{Arg}\ \Omega^+(\mu) + \Delta_{c^+_+(i)} \text{Arg}\ \Omega^+(\mu)$$

$$+ \Delta_{c^-_+(i)} \text{Arg}\ \Omega^-(\mu) + \Delta_{c^-_-(i)} \text{Arg}\ \Omega^-(\mu) \Big] . \tag{A.12}$$

Recalling the notation $1/\sigma_o = 0$, we see that $\Delta_{c^+_-(i)}$ Arg $\Omega^+(\mu)$ ($\Delta_{c^+_+(i)}$ Arg $\Omega^+(\mu)$) represents the change in the argument of $\Omega^+(\mu)$ as μ increases from $-1/\sigma_i$ ($+1/\sigma_{i-1}$) to $-1/\sigma_{i-1}$ ($+1/\sigma_i$); $\Delta_{c^-_+(i)}$ Arg $\Omega^-(\mu)$ ($\Delta_{c^-_-(i)}$ Arg $\Omega^-(\mu)$) represents the change in Arg $\Omega^-(\mu)$ as μ decreases from $1/\sigma_i$ ($-1/\sigma_{i-1}$) to $1/\sigma_{i-1}$ ($-1/\sigma_i$). Equation (A.12) can be further simplified by noting from Eqs. (A.1) and (A.2) that

$$\Omega^{+}(\mu) = \Omega^{-}(-\mu) \tag{A.13}$$

and

$$\Omega^{+}(\mu) = \left[\Omega^{-}(\mu)\right]^{*} . \tag{A.14}$$

Here the asterisk (*) denotes complex conjugation. Using the above relations we find that

$$\mathrm{Arg}\,\Omega^{+}(0) = 0 \tag{A.15}$$

and

$$\alpha = \frac{1}{\pi}\,\mathrm{Arg}\,\Omega(1) , \tag{A.16}$$

where the argument of $\Omega(1)$ is uniquely defined in view of Eqs. (A.10) and (A.15).

B. The Scalar X-Function

It is clear that $X(z)$, as defined in Eq. (6.39), is analytic in the plane cut along (0,1). Moreover, applying the Plemelj formulas to Eq. (6.39) it is easily verified that $X(z)$ satisfies Eq. (6.38). Accordingly, in this appendix we shall demonstrate that $X(z)$ in Eq. (6.39) is indeed an appropriate X-function by showing it has the proper behavior at the endpoints 0 and 1. We shall then prove the identities involving $X(z)$ which were introduced in Section VI. In addition, we shall construct a nonlinear integral equation for calculating $X(z)$, and exhibit the relationship between $X(z)$ and the $\underset{\sim}{X}$-matrix of Section II.

The function $\Gamma(z)$ in Eq. (6.40) can be written in the form

$$\Gamma(z) = \frac{1}{2\pi i}\int_0^1 \frac{\log G(d_k)}{\nu - z}\, d\nu + \frac{1}{2\pi i}\int_0^1 \frac{\log G(\nu) - \log G(d_k)}{\nu - z}\, d\nu \tag{B.1}$$

$k=0,1$, where $d_0 = 0$ and $d_1 = 1$. It follows that

$$\Gamma(z) = \pm \frac{1}{2\pi i} \log G(d_k) \log (d_k - z) + \hat{\Gamma}_k(z) , \quad k=0,1, \tag{B.2}$$

where $\hat{\Gamma}_k(z)$ tends to a definite limit as $z \rightarrow d_k$; the plus sign goes with k=1, the minus sign with k=0. However, from Eqs. (6.37) and (A.14), we have

$$\frac{1}{2\pi i} \log G(\mu) = \frac{1}{\pi} \operatorname{Arg} \Omega^{+}(\mu), \quad \mu \in (0,1). \tag{B.3}$$

Recalling the values of Arg $\Omega^{+}(\mu)$ at 0 and 1, given by Eqs. (A.15) and (A.16), we conclude that

$$X_o(z) = e^{\Gamma(z)} \tag{B.4}$$

approaches a finite nonzero value as $z \to 0$, but as $z \to 1$

$$X_o(z) \sim (1-z)^{\alpha} \quad . \tag{B.5}$$

From Eq. (6.39), we have

$$X(z) = \frac{X_o(z)}{(1-z)^{\alpha}} \quad . \tag{B.6}$$

Thus X(z) is an appropriate X-function since it satisfies conditions (a) and (b) under Eq. (6.37).

Proof of Identity (6.1). Consider the function

$$R(z) = \frac{\Omega(z)}{\Omega(\infty) \prod_{s=0}^{\alpha-1} (\nu_s^2 - z^2)\, X(z)\, X(-z)} \quad . \tag{B.7}$$

This function is analytic everywhere in the complex plane, except perhaps for a cut along (-1,1), and approaches unity as $|z| \to \infty$. But using Plemelj's formulas and the boundary conditions given in Eq. (6.38), we find

$$\frac{R^+(\mu)}{R^-(\mu)} = 1 \, , \quad \mu \in (-1,1) \, . \tag{B.8}$$

Hence, R(z) is analytic everywhere, and from Liouville's theorem we have

$$R(z) \equiv 1 \quad . \tag{B.9}$$

Identity (6.1) follows immediately. Letting z tend to zero in Eq. (6.47) we obtain the following corollary

$$X^2(0) = \frac{c^{N-1}}{\Omega(\infty) \prod_{s=0}^{\alpha-1} \nu_s^2} \tag{B.10}$$

Proof of Identity (6.2). Since X(z) is analytic in the complex plane cut along the real axis from 0 to +1, and vanishes at infinity, we can use Cauchy's integral formula to write

$$X(z) = \frac{1}{2\pi i} \int_{c_1} \frac{X(z')}{z' - z} dz' \, , \tag{B.11}$$

where the closed contour c_1 has a clockwise orientation and encompasses the cut (0,1). Collapsing c_1 about this cut, we find

$$X(z) = \frac{1}{2\pi i} \int_0^1 \frac{X^+(\nu) - X^-(\nu)}{\nu - z} \, d\nu \; . \qquad (B.12)$$

But from the boundary conditions in Eq. (6.38) and Identity (6.1), we have

$$\frac{1}{2\pi i} \; X^+(\nu) - X^-(\nu) = \frac{c}{2} \; R_n(\nu) \frac{X^-(\nu)}{\Omega^-(\nu)}$$

$$= \frac{c}{2} \; \frac{R_n(\nu)}{\Omega(\infty) \, X(-\nu) \prod_{s=0}^{\alpha-1} (\nu_s^2 - \nu^2)} \; , \qquad (B.13)$$

$$\nu \in (1/\sigma_{n-1} \, , \, 1/\sigma_n) \; , \quad n=1,2,\ldots,N \; .$$

Identity (6.2) follows by virtue of Eq. (6.50).

For values of z outside the unit circle, Eq. (6.49) can be written in the form

$$X(z) = -\frac{1}{z} \int_0^1 \gamma(\nu) \left[1 + \frac{\nu}{z} + \frac{\nu^2}{z^2} + \ldots \right] d\nu \; . \qquad (B.14)$$

Thus

$$X(z) = -\sum_{i=0}^{\infty} \Gamma_i \, z^{-i-1} \; , \qquad |z| > 1, \qquad (B.15)$$

where Γ_i is defined in Eq. (6.65). We note from Eq. (6.39) that $X(z) \sim (-1/z)^{\alpha}$ as $|z| \to \infty$. Comparing this behavior with that of Eq. (B.15), we conclude

$$\Gamma_i = \begin{cases} 0, & i=0,1,\ldots,\alpha-2 \\ (-1)^{\alpha-1}, & i=\alpha-1 \,. \end{cases} \tag{B.16}$$

Proof of Identity (6.3). Since $1/X(z)$ is analytic for $|z| > 1$, and behaves as a polynomial of degree α at infinity, we can develop $1/X(z)$ in a Laurent series of the form

$$\frac{1}{X(z)} = \sum_{k=-\infty}^{\alpha} c_k z^k \,, \qquad |z| > 1 \,. \tag{B.17}$$

Specifically, we wish to determine the constants c_k, $k=0,1,\ldots,\alpha$, and thus determine the form of $1/X(z)$ at infinity. Recalling Eqs. (B.15) and (B.16), we write

$$\sum_{i=\alpha-1}^{\infty} \Gamma_i z^{-i-1} \sum_{k=-\infty}^{\alpha} c_k z^k = -1, \qquad |z| > 1 \,. \tag{B.18}$$

Equations (6.53) and (6.54) follow since the above equation is to hold for all z outside the unit circle. With the

c_k, $k=0,1,\ldots,\alpha$, determined by Eqs. (6.53) and (6.54), the polynomial $P(z)$ in Eq. (6.52) describes the behavior of $1/X(z)$ as $|z| \to \infty$. Since $\left[\frac{1}{X(z)} - P(z)\right]$ is analytic in the plane cut along (0,1) and vanishes at infinity, we can use Cuachy's integral formula to write

$$\frac{1}{X(z)} - P(z) = \frac{1}{2\pi i} \int_0^1 \frac{\frac{1}{X^+(\nu)} - \frac{1}{X^-(\nu)}}{\nu - z} \, d\nu \, . \qquad (B.19)$$

But using Eq. (6.38), we have

$$\frac{1}{2\pi i} \left[\frac{1}{X^+(\nu)} - \frac{1}{X^-(\nu)} \right] = - \frac{c}{2} \frac{R_n(\nu)}{X^+(\nu)\, \Omega^-(\nu)} \quad , \qquad (B.20)$$

$$\nu \in (1/\sigma_{n-1} \, , \, 1/\sigma_n) \, , \quad n=1,2,\ldots,N \, .$$

Inserting the above expression into Eq. (B.19), we obtain Identity (6.3).

Proof of Identity (6.4). For $|z| > 1$, the functions $\tau_{ij}(z)$ in Eq. (6.32) can be represented by the expansion

$$\tau_{ij}(z) = \sum_{l=0}^{\infty} R_{ij}^{(l)} \, z^{-l-1} \quad , \qquad |z| > 1 \, , \qquad (B.21)$$

where the coefficients $R_{ij}^{(l)}$ are given by Eq. (6.60).

Furthermore, using Eq. (B.17), we have

$$\frac{\tau_{ij}(z)}{X(z)} = \sum_{l=0}^{\infty} R_{ij}^{(l)} \, z^{-l-1} \sum_{k=-\infty}^{\alpha} c_k \, z^k \, , \qquad |z| > 1 \, . \tag{B.22}$$

Letting z tend to infinity in Eq. (B.22), we obtain

$$\frac{\tau_{ij}(z)}{X(z)} = \sum_{k=0}^{\alpha-1} d_{ijk} \, z^{\alpha-k-1} \, , \qquad |z| \to \infty \, , \tag{B.23}$$

where the constants d_{ijk} are defined in Eq. (6.59). Recalling the polynomials $P_{ij}(z)$ in Eq. (6.57), we conclude $\left[\frac{\tau_{ij}(z)}{X(z)} - P_{ij}(z)\right]$ is analytic in the complex plane cut from -1 to +1, and vanishes at infinity. We can again use Cauchy's integral formula to write

$$\frac{\tau_{ij}(z)}{X(z)} - P_{ij}(z) = \frac{1}{2\pi i} \int_{-1}^{0} \frac{\frac{1}{X(\nu)}\left[\tau_{ij}^{+}(\nu) - \tau_{ij}^{-}(\nu)\right]}{\nu - z} \, d\nu$$

$$+ \frac{1}{2\pi i} \int_{0}^{1} \frac{\tau_{ij}(\nu)\left[\frac{1}{X^{+}(\nu)} - \frac{1}{X^{-}(\nu)}\right]}{\nu - z} \, d\nu \tag{B.24}$$

From Eq. (6.32), we have

$$\frac{1}{2\pi i}\left[\tau^{+}_{ij}(\nu) - \tau^{-}_{ij}(\nu)\right] = \frac{c}{2} R_{ijn}(\nu), \tag{B.25}$$

$$\nu \in (1/\sigma_{n-1}\,,\ 1/\sigma_n)\,,\quad n=1,2,\ldots,N\,.$$

Substituting Eqs. (B.20) and (B.25) into Eq. (B.24), we obtain

$$\frac{\tau_{ij}(z)}{X(z)} - P_{ij}(z) = \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{R_{ijn}(\nu)}{X(-\nu)\ (\nu+z)}\, d\nu$$

$$- \sum_{n=1}^{N} \frac{c}{2} \int_{n>} \frac{\tau_{ij}(\nu)\ R_n(\nu)}{X^{+}(\nu)\ \Omega^{-}(\nu)\ (\nu-z)}\, d\nu\ , \tag{B.26}$$

which yields Identity (6.4) when Eq. (6.58) is used.

An iteration scheme, similar to that of Shure and Natelson (13), can be constructed for X(z). Let us introduce the function

$$\theta(z) = \left[\frac{1}{X(0)} + (-z)^{\alpha}\right] X(z) \tag{B.27}$$

which is analytic everywhere in the complex plane cut from 0 to +1, and tends to unity as z approaches either zero or infinity. From Cauchy's integral formula we have

$$\frac{\theta(z)-1}{z} = \frac{1}{2\pi i}\int_0^1 \frac{\theta^+(\nu)-\theta^-(\nu)}{\nu(\nu-z)}\,d\nu \quad . \tag{B.28}$$

However, using Identity (6.1) and Eq. (B.10), we obtain

$$\frac{1}{2\pi i}\left[\theta^+(\nu)-\theta^-(\nu)\right] = \frac{1}{2}\,\frac{\left[1+(-\nu)^\alpha X(0)\right]\left[1+(+\nu)^\alpha X(0)\right]R_n(\nu)}{c^{N-2}\prod_{s=0}^{\alpha-1}(1-\nu^2/\nu_s^2)\;\theta(-\nu)} , \tag{B.29}$$

$$\nu \in (1/\sigma_{n-1}\, ,\ 1/\sigma_n)\ ,\ n=1,2,\ldots,N.$$

Substituting this expression into Eq. (B.28), we find

$$\theta(z) = 1 - \frac{z}{2}\sum_{n=1}^{N}\int_{<n}\frac{\left[1+(-\nu)^\alpha X(0)\right]\left[1+(+\nu)^\alpha X(0)\right]R_n(\nu)}{c^{N-2}\prod_{s=0}^{\alpha-1}(1-\nu^2/\nu_s^2)\;\nu(\nu+z)\;\theta(\nu)}\,d\nu \; . \tag{B.30}$$

where the notation $<n$ implies that the integral is over the negative segment of region (n). It is seen that $\theta(z)$ is determined everywhere in the complex plane by its values on the interval (-1,0). Moreover, a nonlinear integral equation for $\theta(\mu)$, $-1 \leq \mu \leq 0$, is obtained by restricting z to the interval (-1,0) in Eq. (B.30). If $\theta(\mu)$, $-1 \leq \mu \leq 0$, is initially approximated by unity, it is expected that the iterative solution of this integral equation will converge rapidly. After determining $\theta(z)$, $X(z)$ can be obtained from Eq. (B.27). It should be noted, however, the numerical

integration scheme used in Eq. (B.30) must take into account the fact that the functions $R_n(\mu)$ generally have logarithmic singularities at the points $\pm 1/\sigma_i$, $i=1,2,\ldots,N$, and may oscillate quite rapidly on the interval $(-1,1)$. It may therefore prove more convenient to calculate $X(z)$ directly from Eq. (6.39).

Lastly, we mote that the scalar X-function is related to the determinant of the $\underset{\sim}{X}$-matrix introduced in Section II. Taking the determinant of both sides of Eq. (2.115), we obtain

$$\det \underset{\sim}{X}^{+}(\mu) - G(\mu) \cdot \det \underset{\sim}{X}^{-}(\mu) = 0 \ , \quad \mu \in (0,1). \tag{B.31}$$

But for $\underset{\sim}{X}(z)$ to be a suitable fundamental matrix, $\det \underset{\sim}{X}(z)$ must be a nonvanishing analytic function in the complex plane cut along $(0,1)$ which satisfies condition (b) under Eq. (6.38). Hence

$$\det \underset{\sim}{X}(z) = C\, X(z) \ , \tag{B.32}$$

where $X(z)$ is given in Eq. (6.39) and C is a nonzero constant. It follows that $\det \underset{\sim}{X}(z)$ vanishes as $1/z^{\alpha}$ at infinity.

MULTI-GROUP NEUTRON TRANSPORT THEORY IN PLANE GEOMETRY

F. Joseph McCrosson

Abstract

An exact analytical procedure is presented for solving multi-group neutron transport problems in plane geometry. The method consists of analysing a system of singular integral equations which is derived for the neutron distribution. Included in these singular integral equations are arbitrary inhomogeneous terms which depend only on the angle variable. Once these arbitrary functions are determined, the neutron distribution can be obtained from the singular integral equations by standard methods.

To illustrate the procedure the inhomogeneous terms associated with the infinite medium Green's function are determined. The Milne and critical problems are also considered. Here the inhomogeneous terms are expressed in terms of the emerging distribution at the boundary and the emerging distribution is given by a system of coupled Fredholm integral equations.

The relationship between this method and Case's eigenfunction method is investigated. This leads to an alternate procedure for determining the expansion coefficients in Case's method.