

STATIC AND SEQUENTIAL LOCATION-ALLOCATION PROBLEMS
ON NETWORKS AND AREAS WITH PROBABILISTIC DEMANDS

by

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(ABSTRACT)

Location-allocation problems arise in many practical settings and may be generically stated as follows: Given the location or distribution of a set of customers and their associated demands, simultaneously determine an optimal location of a number of supply facilities and their allocation of products or services to the customers, so as to minimize total location and transportation costs. This study is concerned with the development, convergence analysis, and testing of exact and heuristic algorithms for location-allocation problems in which demands can occur continuously over regions according to some probability density functions.

In this context, minimum location problems on undirected networks are considered in which demands can occur on links with uniform probability distributions. Three types of networks are considered. The first type is a chain graph. It is shown that except for the 1-median case, the problem is generally nonconvex. However, for the p -median case, it is shown that all local and global minima to the problem may be discovered by solving a series of linear programming problems. This analysis forms the basis for similar problems on trees and graphs with isolated cycles.

These problems are then extended to multiperiod versions in which demands may change dynamically over time periods and at most one facility can be located per time period. Chain and tree graphs are considered in conjunction with three optimization strategies: myopic, long-range, and discounted present worth. It is hoped that the exact methods developed for these special networks will lead to at least effective heuristics for problems on more general networks.

Finally, location-allocation problems are considered in which the region to be served is a convex polygon having a uniform demand distribution. Both single and multifacility formulations are considered. For the single facility problem, an algorithm is developed which is shown to converge to a global optimal solution. This analysis is extended to the nonconvex multifacility case, and although optimality is not guaranteed, an algorithm is presented for finding a good starting solution which increases the likelihood of finding an optimal solution. Extensions of the above problems to include discrete demand points and computational experience are also provided.

To my wife, Victoria, and our children, Ian and Sierra

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CHAPTER I

INTRODUCTION

1.1 Problem Descriptions

A problem which frequently arises in many practical settings can be stated generically as follows: Given the location or distribution of a set of customers and their associated demands for a given product or service, determine the optimum location of a number of service facilities and the allocation of products or services from facilities to customers, so as to minimize total location and transportation costs. This problem is commonly known as the Location-Allocation Problem.

This class of problems has numerous applications among which are the location of warehouses, distribution centers, service and production facilities, and emergency service facilities (see Francis and White [1974], and Handler and Mirchandani [1979]).

The Location-Allocation problem was originally formulated by Cooper [1963] and is still an area of active research. Since its origination, several special cases and extensions have been studied by various researchers. Examples of these include the rectilinear distance location-allocation problem, the Euclidean and squared-Euclidean distance location-allocation problem, the discrete location-allocation or fixed charge location problem, and location-allocation on networks. Some extensions which have been considered include stochastic location-allocation problems, dynamic location-allocation problems, and location-allocation problems with area destinations. These problems, together with additional background information are discussed in Chapter II.

In this study, we examine single and multifacility location-allocation problems in which demands are described on regions according to some probability density function instead of being necessarily fixed at discrete locations. Extensions of these problems, namely, sequential location decisions and dynamic demand variations are also considered. Below we discuss the basic problems to be considered and mention the extensions of these problems which will be investigated.

We begin, in Chapter III, by studying the simplest version of this problem type. Namely, we consider a location-allocation problem on a chain graph in which the demand on each link is described by a uniform probability density function. The problem is to locate p facilities on this chain, and to ascertain what portion of the chain is to be served by each facility so as to minimize the total expected transportation costs. Subsequently we address similar problems on more general networks. In particular, we consider the problems of locating two facilities on a tree graph and one facility on a graph with isolated cycles.

In Chapter IV, we extend the aforementioned problems to multiperiod versions in which demands may change dynamically over time periods. We assume that at most one facility can be built per time period and give special attention to the optimization strategy for evaluating siting decisions. In particular, a comparison is made between myopic, long-range, and discounted present worth strategies.

Finally, in Chapter V, we consider location-allocation problems in which the region to be served is a convex polygon having a demand defined by a uniform probability density function. The problem is to

locate n facilities and determine the subregions which each facility must serve, based on Euclidean distance measure, so as to minimize total expected transportation costs. Extensions to include discrete demand points are also provided.

1.2 Research Objectives

The objectives of this study are to develop exact and/or heuristic algorithms for the nonconvex location-allocation problems introduced in Section 1.1, and to study convergence properties of and provide computational experience with such algorithms. Optimization techniques proposed to be used include classical optimization and calculus methods, iterative direct search techniques, enumerative and graph theoretic procedures.

CHAPTER II

LITERATURE REVIEW

2.1 Introduction

In this chapter, we provide a brief review of the vast literature that exists in location theory as relevant to this study. Since pure location problems constitute important subproblems in the study on location-allocation problems, these are addressed first. Thereafter, we discuss location and location-allocation problems on networks, the rectilinear, Euclidean, squared Euclidean and fixed charge discrete location-allocation problems, and variants such as stochastic, area destination, sequential and stochastic location-allocation problems.

2.2 Pure Location Problems

The (discrete demand) multifacility location problem considers the location of m new facilities to serve n existing facilities in order to minimize total interactive transportation costs. Denoting the location of existing facility j in the plane by $P_j = (a_j, b_j)$, $j=1, \dots, n$ and the location of new facility i by $X_i = (x_i, y_i)$, $i=1, \dots, m$ (these coordinates being the decision variables), and letting $d(Z_i, Z_j)$ represent a distance measure between new or existing facility locations Z_i and Z_j , the multifacility location (MFLOC) problem may be generically stated as

$$(\text{MFLOC}) \text{ minimize } \sum_{i=1}^m \sum_{j=1}^n w_{ij} d(X_i, P_j) + \sum_{i=1}^{n-1} \sum_{k=i+1}^n v_{ik} d(X_i, X_k)$$

where w_{ij} is an annual cost incurred by separating new facility i and existing facility j by a unit distance, and v_{ik} is a similar cost of separating new facilities i and k by a unit distance.

In the special case when $m = 1$, the problem reduces to one of locating a single facility with respect to n existing facilities and is known as a single facility location problem (SFLOC). Variants of problems SFLOC, MFLOC are obtained by selecting different distance measures, by restricting new facility locations to specific sites, and by characterizing demand as being either stochastic or distributed over areas.

2.2.1 Rectilinear Distance Location Problems

The rectilinear distance location problem is one in which the distance norm is defined by $d(X,P) = |x-a| + |y-b|$, where $X = (x,y)$ and $P = (a,b)$. The objective function in this case is separable and the determination of the x and y new facility coordinates can be treated as two separate optimization problems. Francis [1963] is credited with first solving the rectilinear SFLOC problem. The rectilinear MFLOC problem was originally proposed by Francis [1963], who subsequently solved it in the special case of equal weightings (see Francis [1964]). Cabot, Francis, and Stary [1970] solved the general rectilinear MFLOC problem by decomposing it into two independent subproblems, each of which is equivalent to a linear program which is the dual of a minimum cost network flow problem. Fulkerson's out-of-kilter algorithm was then used to obtain an optimal solution. Wesolowsky and Love [1971b] also solved the general rectilinear MFLOC problem by using a linear programming formulation. A nonlinear approximation to the rectilinear MFLOC problem was developed by Wesolowsky and Love [1972] which was then solved using a gradient search procedure.

A direct search algorithm for the rectilinear MFLOC problem was developed by Pritsker and Ghare [1970]. However, as demonstrated by Rao [1973], this was essentially a primal simplex based linear programming approach, and the stated optimality conditions were not sufficient in the presence of degeneracy. A complete set of necessary and sufficient optimality conditions have been derived by Juel and Love [1976].

An alternative, efficient, primal simplex based algorithm for this problem was developed by Sherali and Shetty [1978]. Perhaps the most efficient algorithm available for this problem is due to Picard and Ratliff [1978] who showed that the problem can be solved via at most $(n-1)$ minimum cut problems on derived networks containing at most $(m-2)$ vertices. Later, Kolen [1981] showed that the methods of Sherali and Shetty, and Picard and Ratliff are essentially equivalent, and differ only in the efficiency of computational implementation.

2.2.2 Squared-Euclidean Distance Problems

If $d(X,P) = (x-a)^2 + (y-b)^2$, then the resulting problem is called the squared-Euclidean distance problem, and is again separable in the x and y variables. In this case, both the SFLOC and MFLOC problems can be solved using standard calculus techniques. Determining the optimal solution in the MFLOC problem involves the solution of two systems of n linear equations in n variables.

2.2.3 Euclidean Distance Problems

For Euclidean distance problems, the distance norm is defined by $d(X,P) = ((x-a)^2 + (y-b)^2)^{1/2}$. Unlike the squared-Euclidean case there exist points of nondifferentiability in this problem, and even where the

extremal equations are defined, they are non-linear and an exact closed-form solution cannot be obtained. However, a fixed-point iterative scheme was used by Weiszfeld [1937], Cooper [1963], and Kuhn and Kuenne [1962] to solve the SFLOC problem by finding a solution to the extremal equations when it exists. The scheme is commonly labeled the Weiszfeld procedure, and issues concerning its convergence have been investigated by Katz [1969], Kuhn [1973], Ostresh [1978], and Morris [1981].

An extension of the Weiszfeld procedure used in solving the Euclidean distance problem was developed by Eyster, White, and Wierwille [1973] for SFLOC and was extended to MFLOC. This procedure is based on approximating the objective with hyperboloids to eliminate indeterminacies in the extremal equations. The technique has been labeled the hyperboloid approximation procedure (HAP). Recently, Charalambous [1982] has developed a set of optimality conditions for the problem which lead to a far more efficient algorithm, devoid of the ill-conditioning of the Hessian matrix which exists in the HAP method as the degree of accuracy is increased.

Among other contributions for this problem are the convex programming approach of Love [1969] and the necessary optimality conditions due to Francis and Cabot [1972]. Extensions of the Euclidean distance problem results and algorithms for general ℓ_p distances have been proposed by Drezner and Wesolowsky [1978], Morris and Verdini [1979] and Juel and Love [1981], among others.

2.2.4 Variants and Extensions

Several variants and extensions of the problems discussed above have been addressed in the literature. One type of variant involves the addition of metric side constraints restricting the separation of facilities, as in Schaefer and Hurter [1974]. A second variant involves the consideration of stochastic components in the location problem. For example, Cooper [1974] and Katz and Cooper [1974, 1976] have considered Euclidean distance SFLOC problems in which demand points are randomly distributed, and Seppala [1975] has considered a version of this problem in which the transportation costs are random variables. A noteworthy paper in this area by Aly and White [1978] treats the multifacility Euclidean distance problem in which the demand points and the interactions between facilities are random variables. Both unconstrained and chance constrained formulations have been addressed. An analogous stochastic version for the single facility rectilinear distance problem has been considered by Wesolowsky [1977].

A third variant involves models which permit the relocation of facilities over a multiperiod horizon, which become necessary due to changes in costs and demand patterns. Contributions in this area include the dynamic programming approach of Wesolowsky [1973a] for the rectilinear distance SFLOC problem, and an extension of this study to the multifacility case by Wesolowsky and Truscott [1976].

A fourth variant permits destinations to be either single demand points or be uniformly distributed over regions. Wesolowsky and Love [1971a] considered such a rectilinear distance MFLOC problem for which a gradient search algorithm was developed. An analogous extension of the

Euclidean SFLOC problem was solved by Love [1972] using a nonlinear optimization technique. Bennett and Mirakhor [1974] solved an approximation of the same problem by replacing the areas with their centers of gravity. Drezner and Wesolowsky [1980] addressed a similar problem with ℓ_p distances and more general demand area shapes. A two-stage iteration procedure based on the Weiszfeld procedure was developed for the SFLOC problem.

Finally, other variants of the location problems include the use of different distance norms, such as that proposed by Ward and Wendell [1980] (which leads to linear programming formulations) or involve some special considerations, as for example in Odoni and Sadiq [1982] in which the demand is continuously distributed over a region, parts of which can be traversed by a high-speed corridor.

2.3 Location and Location-Allocation Problems on Networks

Let $G(N,A)$ be an n -vertex undirected network with weights (demands) h_i attached to its nodes $V_i \in N$ and weights (distances) w_j associated with its links $a_j \in A$. Let $X_p = \{x_1, \dots, x_p\}$ represent the decision variables, namely, the locations of the p facilities which are to be located on G , and let $d(V_i, X_p) = \min \{d(V_i, x_1), \dots, d(V_i, x_p)\}$ measure the shortest path distances in G .

2.3.1 Minisum (Absolute Median) Problems

The objective of the (absolute) p -median problem is to locate p facilities on the network G so as to minimize the sum of the weighted distances to each node from its nearest facility, that is, to

$$\underset{X_p \text{ on } G}{\text{minimize}} f(X_p) = \sum_{i=1}^n h_i d(V_i, X_p)$$

Note that when $p \geq 2$, this problem may be viewed as a location-allocation problem since the location of facilities determines the allocation of its services to meet nodal demands. Hakimi [1964] has shown that one may restrict attention to the vertices of the graph in determining the optimal locations, and has proposed an enumerative graph theoretic solution procedure for the problem. Another class of solution procedures for the problem have stemmed from the reformulation of the p -median problem as an integer programming problem by Reville and Swain [1970]. For example, Jarvinen, et al [1972] have proposed a branch and bound algorithm for the problem based on this reformulation. The NP hardness of the problem has prompted the development of various heuristic solution procedures, including those due to Maranzana [1964] and Teitz and Bart [1968]. One noteworthy feature of this class of location-allocation problems is that for situations in which G is a tree network, the problem simplifies considerably because there exists a unique path between any two points in the network and because special collapsing, disconnecting, aggregation and decomposition strategies can be employed with such a graph. The reader is referred to Kariv and Hakimi [1979b], Minieka [1978] and Handler and Mirchandani [1979] for excellent tutorials on the subject.

Several variants of this problem have received attention, including cases with capacity restrictions on arcs, maximum distance constraints, probabilistic vertex weights, and probabilistic travel times on arcs. Two variants of particular interest to our work are those considered by

Minieka [1977] and Chiu [1982] for the case $p = 1$. Minieka [1977] studied the problem in which the single facility to be located serves points on the edges as well as the vertices, with the relevant distance for serving an arc being given by the distance from the facility to the most distance point on the arc. Chiu [1982] has formulated and solved a single facility problem with discrete nodal demands as well as continuous link demands. Here, distances on arcs are measured exactly. An efficient specialization for the tree network problem has also been considered.

2.3.2 Minimax (Absolute Center) Problems

The objective of the (absolute) p -center problem is to locate p facilities on G so as to minimize the maximum weighted distance between a node and its nearest facility. That is, to

$$\begin{array}{l} \text{minimize } f(X_p) = \max_{1 \leq i \leq n} h_i d(v_i, X_p). \\ X_p \text{ on } G \end{array}$$

(The corresponding problem when X_p is restricted to N is known simply as the p -center problem. We briefly address only the absolute p -center problem and refer the reader to Minieka [1978], Halpern and Maimon [1982], Kariv and Hakimi [1979a] and Handler and Mirchandani [1979] for further details.)

Although the absolute p -center problem is related to the median location problem, its solution methods are quite different. Hakimi [1964] first proposed a very simple enumerative approach for $p = 1$ which determines a local center on each link and thereby solves the problem. A more efficient implementation of this method which restricts attention to only subsets of links has subsequently been developed (see

Christofides [1975]). However, this technique does not readily extend to the general p -center problem. Computationally efficient exact algorithms for this problem have been developed by Christofides and Viola [1971] and by Garfinkel, et al [1977]. Heuristics, such as that due to Singer [1968], also exist for the problem. The reader is referred to the books by Christofides [1975], Handler and Mirchandani [1979], and Minieka [1978] for further discussions and for extensions of this problem.

2.4 Continuous Location-Allocation Problems with Discrete Demand Points

Continuous location-allocation problems address the problem of determining the location of m supply centers in a plane to serve n customers with fixed locations, and to simultaneously determine the allocation of services or products from the supply centers to the customers in order to minimize total transportation costs.

Let $d(X_i, P_j)$ represent the distance between the locations of supply center i and customer j . Then the continuous location-allocation problem for m supply centers and n customers can be stated as follows:

$$\begin{aligned}
 \text{(LAP) minimize} \quad & \sum_{i=1}^m \sum_{j=1}^n t_{ij} w_{ij} d(X_i, P_j) \\
 \text{subject to} \quad & \sum_{j=1}^n w_{ij} \leq s_i, \quad i=1, \dots, m \\
 & \sum_{i=1}^m w_{ij} = d_j, \quad j=1, \dots, n \\
 & w_{ij} \geq 0 \quad i=1, \dots, m; \quad j=1, \dots, n
 \end{aligned}$$

where the decision variables are:

w_{ij} = annual number of units transported from supply center i to customer j .

$X_i = (x_i, y_i) \equiv$ location of facility i , $i=1, \dots, m$

and where the data is symbolized by:

t_{ij} = transportation cost per unit shipped per unit distance from
supply center i to customer j , $i=1, \dots, m$; $j=1, \dots, n$

s_i = annual capacity of supply center $i=1, \dots, m$

d_j = annual demand of customer $j=1, \dots, n$.

It is important to notice that the location-allocation problem is comprised of two basic problems. That is, if the locations of the supply centers are known, then the problem becomes the ordinary transportation problem. While if the transportation flows are known, then the problem becomes a pure location problem.

The above formulation is generally referred to as the capacitated version of problem LAP since each supply center has a capacity restriction s_i . In the case when $s_i = \infty$ for all i , the problem is referred to as uncapacitated. It should be noted that in the uncapacitated case, the total demand of any given customer is entirely satisfied by a single facility. The interpretation of $s_i = \infty$ is that the capacity of supply center i is effectively determined by its total allocation in an optimal solution. Furthermore, note that a more general statement of the problem could consider multiple commodities along with interactions between new facilities as well. Such a version has been addressed by Shetty and Sherali [1977].

2.4.1 Rectilinear Distance Location-Allocation Problems

In the case of the rectilinear norm, Problem LAP becomes a nonconvex bilinear programming problem (see Vaish [1974]). Love and Morris [1975] have developed an exact solution procedure for the uncapacitated version of this problem. The procedure consists of using a set reduction algorithm to reduce the possible solution set, and the problem is shown to be equivalent to the p-median problem on a weighted connected graph. Subsequently, Sherali and Shetty [1977] developed a convergent cutting plane algorithm to solve the capacitated version of LAP. This algorithm was further improved upon by Shetty and Sherali [1977] who proposed deeper (negative-edge extension reverse polar) cutting planes, and provided a more efficient computational implementation. This procedure was actually developed for a more general problem which permitted interactions between new facilities and involved multiple commodities.

2.4.2 Euclidean and Squared Euclidean Distance Location-Allocation Problems

The uncapacitated version of problem LAP with Euclidean distances was originally formulated by Cooper [1963], who proposed an enumerative procedure for its solution. However the method proved to be impractical on problems of any reasonable size. Several heuristic solution procedures were also proposed by Cooper [1964, 1967]. Subsequently, Eilon, et al [1971] developed a computationally tractable iterative solution procedure for this problem. An alternative heuristic as well as an exact branch and bound algorithm for this problem was later developed by Kuenne and Soland [1972]. For the special case when $m = 2$,

Ostresh [1975] provided an improved version of this algorithm. A variant of the uncapacitated version of LAP which uses ℓ_p distances has been considered by Love and Juel [1982]. This problem has been shown to be equivalent to a concave minimization problem, and several perturbation type of strategies have been applied to the problem.

The capacitated version of the location-allocation problem was first considered by Cooper [1972] who provided an exact enumerative, combinatorial approach, and also developed two heuristic methods for the problem. Perhaps the most tractable, though not effective solution method which exists for the capacitated Euclidean distance location-allocation problem is the cutting plane algorithm due to Selim [1979]. As far as a heuristic strategy for this problem is concerned, the method of Murtagh and Niwattisyawong [1982] which employs the commercial nonlinear programming package MINOS along with a reasonable starting solution, appears to be the most promising.

The squared Euclidean distance location-allocation problem has surprisingly not received any significant attention in the literature. The principal reason being that it shares a common structure with the Euclidean distance problem and algorithms developed for the latter problem may be readily extended to this problem. In other words, no apparent simplification of methodology has been as yet ascertained for this class of problems.

2.5 Fixed-Charge Location/Discrete Location-Allocation Problems

A variation of the above class of location-allocation problems, which has received considerable attention, is the fixed-charge location problem or the discrete location-allocation problem. These problems

differ from the previously discussed problems by restricting the location of supply centers to a fixed finite set of potential sites. The "fixed charge" is the cost associated with opening a supply center at a given location.

Effroymsen and Ray [1966] introduced a branch and bound algorithm for the solution of the uncapacitated version of the fixed charge location problem. Subsequently, a dynamic programming based decomposition algorithm for this problem was developed by Curry and Skeith [1969]. Recently, Erlenkotter [1978] has developed a dual ascent enumeration algorithm for this problem. As has been pointed out by Guignard and Spielberg [1979] and by Sherali and Adams [1982], algorithms employing dual ascent, and performing logical tests on cutting planes in an enumerative framework, are most effective for this class of problems.

The capacitated version of the fixed charge location problem is computationally somewhat harder to solve because of the larger LP-IP gap that it presents. The algorithms proposed for this problem are either implicit enumeration strategies which differ in the computation of lower bounds and tree exploration techniques, or use Benders' decomposition algorithm. Among the former techniques are those of Sa [1969], Davis and Ray [1969], Ellwein and Gray [1971], Akinc and Khumawala [1977], (a method later improved by Nauss [1978]), and Bitran, et al [1981], whereas Balinski and Wolfe [1963] have employed the latter strategy.

Several variants of the fixed charge problem have been considered. A particularly important case considered by Sherali and Adams [1982] seeks the location of facilities on a set of potential sites in a

one-to-one fashion, and the subsequent allocation of products to customers so as to minimize total production, location and transportation costs. This problem is shown to be transformable to a fixed-charge location problem with assignment side constraints. An efficient Lagrangian relaxation, enumeration and decomposition composite algorithm was developed for the problem. The approaches that exist for handling more general side constraints include those by Spielberg [1966] for the uncapacitated problem, and by Ellwein and Gray [1971], Guignard and Spielberg [1979] and Geoffrion and McBride [1979] for the capacitated problem. Among other variants for the fixed-charge location problem are the consideration of customer locations alone for potential new facility sites by El-Shaieb [1973] in the uncapacitated case, the development of a heuristic for a case in which the construction and operation cost is a general continuous concave function by Feldman, et al [1966], and the formulation and solution of a multistage version of the capacitated fixed charge problem by Nagelhout and Thompson [1981].

2.6 Stochastic Location-Allocation Problems

Similar to the stochastic versions of network location-allocation problems (see Handler and Mirchandani [1979]), stochastic versions of both rectilinear and Euclidean distance location-allocation problems have been considered. Maruchek and Aly [1981] have made a noteworthy contribution to the rectilinear distance case. They consider uncapacitated situations in which customer demands are characterized by bivariate uniform and normal probability density functions over rectangular regions. A branch and bound algorithm has been developed to minimize an expected weighted distance objective function.

A stochastic version of problem LAP with Euclidean distances has been considered by Selim [1979]. Here, customer demands are taken to be independent random variables with known probability distribution. Two approaches have been presented to obtain a deterministic equivalent to this problem. One is a chance constrained approach while the other is a two stage approach with recourse. Cutting plane algorithms have been developed to solve both problems.

2.7 Location-Allocation Problems with Area Destinations

Aside from the mention of the extension to the multifacility case of the algorithm in Drezner and Wesolowsky [1980], the only explicit consideration of an area destination location-allocation problem is due to Leamer [1968]. Leamer considers a multifacility, uncapacitated problem using Euclidean distance norms, in which the demand points are uniformly distributed over an area which is either a square, an equilateral triangle or a circle. A heuristic, perturbation algorithm was implemented to find solutions which could not be improved upon by perturbing any one single facility location. The principal observation was that as the number of facilities increased, the areas served by each facility tended to be hexagonal, except for distorting effects on the boundaries of the region.

2.8 Sequential and Dynamic Location-Allocation Problems

This class of location-allocation problems involves time dependent components, information and decisions. The variations with respect to time may be continuous or the time horizon may be partitioned into a finite number of discrete periods. Typically, the demand varies in a prescribed fashion with respect to time, and facility siting decisions

(and hence allocations) need to be made sequentially by the addition of extra facilities and/or relocating previously constructed facilities.

A good example of the sequential types of decisions involved is presented by Scott [1971]. Here, an uncapacitated Euclidean distance location-allocation problem is considered in which one new facility must be located in every period over a given horizon. Two strategies are investigated. One, a myopic strategy in which a new facility is located in every period to minimize present period costs. A second approach is a long range strategy in which the facilities must ultimately be located in an optimal multifacility solution, and one must then decide which of these optimal location sites receive a new facility in which period. Note that as a third strategy, one may attempt to minimize total discounted costs.

A related problem has been considered by Fong and Srinivasan [1981a,b]. Here, the problem is to determine a schedule of capacity expansions for the supply centers and a schedule of shipments to the customers so as to meet the market demands over a finite, discrete planning horizon at minimum discounted capacity expansion and shipment costs. A heuristic algorithm was developed for this problem.

By far, a good majority of dynamic location-allocation problems have been considered in the setting of discrete or fixed charge plant location (and allocation) problems. Both capacitated and uncapacitated versions in which the demand is prescribed over a finite, discrete horizon have been addressed. Khachaturov and Astakhov [1976] present a combinatorial heuristic for such problems, and Truscott [1980] examines

an (exact) implicit enumeration algorithm on a zero-one linear integer programming formulation of the problem.

More complicated versions of the same problem permit relocation of facilities in the sense that previously established facilities may be closed concurrently with the opening of new facilities at other sites. Sweeney and Taltham [1976] develop a composite mixed-integer approach for the static single period problem along with a dynamic programming adjustment routine for finding the optimal sequence of configurations over multiple periods. An excellent paper in this context is due to Van Roy and Erlenkotter [1982], which is an extension of the method in Erlenkotter [1978]. A computationally efficient dual ascent branch and bound algorithm has been developed for the uncapacitated version of the problem. Several extensions, which include the capacitated version, are discussed.

The general subject of facility location models in which the relocation of service centers is a principal feature forms an important class of problems in the context of siting emergency and public service facilities. Rosenthal, et al [1978] discuss one such application in which relocations must be effected concomitantly with the change in demand locations, which is described in this case by a stationary Markov chain. Another related application is described in Chrissis, et al [1978] and in Chrissis [1980]. Here, a dynamic set covering problem is addressed in which the solution (operating facility locations) must change from period to period along with changes in the set covering matrix coefficients, which reflect alterations in the set of customer locations which can be served from each potential site.

Dynamic location-allocation problems with continuous time variations have also been studied, although to a lesser extent. Tapiero [1971] considers a capacitated version of problem LAP with Euclidean distances in which demands and production rates are functions of time. A variational calculus problem is formulated and solved analytically using Euler-Lagrange type of necessary optimality conditions. Another version of this class of problems has been studied by Rao and Rutenberg [1977]. Here, a discrete site location-allocation problem is considered in which demands can grow at varying rates in different markets over a continuous time interval. The model also considers variations in plant sizing which may result due to prescribed economies of scale. A two-step algorithm is proposed in which the timing of location sitings are determined first, and then plant capacities are optimized using a nonlinear programming routine.

For further references to problems in this area and for a comparative study on various proposed heuristic methods, the interested reader is referred to Erlenkotter [1981].

2.9 Conclusions

A survey of the literature indicates that the proposed location-allocation problems either have not been explicitly considered or have no existing solution procedure which is effective.

CHAPTER III

NETWORK LOCATION PROBLEMS WITH PROBABILISTIC LINK DEMANDS

3.1 Introduction

Minisum network location problems are a special class of location-allocation problems in which supply centers, demand points and the flow of products between these points are all confined to a graph with certain designated nodes and links. In the earlier formulations originating with the work of Hakimi [1964,1965], demand was restricted to occur only at the nodes of the network. However, as pointed out by Chiu [1982], Handler and Mirchandani [1979], and Minieka [1977] problems involving the location of emergency or public service or utility repair stations, one must allow for demands to occur continuously on links as well. Hence, more recent contributions have attempted to accommodate this feature.

In order to satisfy a given demand pattern, some p supply centers must be located in the network. In some cases, these supply centers are restricted to discrete nodal points in the network, but in most cases, they are generally permitted to lie anywhere within the network. The problem then becomes one of locating the supply facilities or centers and determining the flow of products from these supply centers to the customer locations, so as to minimize the total transportation cost which is directly proportional to the amount of product flow and the distance traversed. Since the facilities are usually treated as being uncapacitated and the network as being undirected, each customer point gets effectively served by the supply facility which is physically nearest to it in the network. This chapter is concerned with minisum

location problems on some special undirected networks with demands occurring continuously on links according to some uniform probability distributions, and in which uncapacitated supply facilities may be located anywhere within the network.

Several versions of and approaches to continuous link demand minimsum location problems on networks have been considered recently. Handler and Mirchandani [1979] formulate such a problem and then approximate it by replacing the continuous demand on each link with a concentrated centroidal demand point. Alternatively, one may obtain a closer approximation by using several concentrated demand points to replace the continuous demand on each link.

The first non-discretized approach to this problem is due to Minieka [1977]. Here however, the demand on a link is said to be served by travelling to the furthest end of the link. Hence, the objective is to locate a facility which minimizes the sum of distances from the facility to the furthest point on each link. Such a location is called a general absolute median of the network.

A similar type of problem has been considered by Slater [1981]. Here, a tree network is specified in which the demand is characterized by a collection of subtrees. A facility is said to serve a demand subtree by travelling in the network to the closest point in that subtree. The objective is therefore to find a set of single facility locations, each element of which minimizes the total sum of distances from the closest points in the demand subtrees. Slater calls such a set of points the branch centroid of the tree.

Perhaps the first contribution in which continuous link demands are actually served pointwise appears in a recent paper by Chiu [1982]. Here, a single facility is considered and the average distance function to be minimized is characterized as a function of the location of this facility. This characterization leads to exact as well as heuristic 1-median location algorithms on general networks and on trees. We note that our methodology subsumes Chiu's 1-median location algorithm on a tree network (considering uniform link demand densities).

Analogous to the class of problems addressed in this paper are minimax or center location problems on networks with continuous demands, and minisum location problems in planes with demands occurring continuously over designated areas. In the latter type of problems, the distance from supply facilities to demand points may be measured using, for example, rectilinear or Euclidean distance norms. We refer the interested reader to Tamir and Zemel [1982] for a treatment of the p-center location problem on trees with continuous link demands, and to Wesolowsky and Love [1971] and Drezner and Wesolowsky [1980] for a treatment of location problems with area demands. For other excellent tutorials on the subject of network location problems, the reader may refer to Handler and Mirchandani [1979], Kariv and Hakimi [1979a, 1979b], and Minieka [1978].

This chapter is organized as follows. We begin by first formulating and solving a p-median location problem on a chain graph with continuous link demands. It is shown that while the problem is generally nonconvex, one may readily identify through a series of trivial linear programming problems, a discrete set of candidate

locations for local and global optimal solutions. This analysis provides a capability of addressing more general networks. In particular, it is shown that it leads naturally to a method of locating one facility on a tree, which turns out to be identical to a recent algorithm of Chiu [1982], and that it permits one to extend this method to locating 2-medians on a tree. Finally, we address the problem of locating one facility on a graph with isolated cycles, i.e., a graph in which every link is included in at most one cycle. This type of a graph has been previously considered by Goldman [1971] in the context of nodal demands. The presentation is illustrated with numerical examples and some computational experience is also provided. We remark that although location problems on special networks are considered herein, as noted in Handler and Mirchandani [1979], this often forms the basis for obtaining solutions for problems on more general networks.

3.2 p-median Location Problem on a Chain

Consider a chain graph with $m + 1$ distinct nodes designated $1, 2, \dots, m+1$ and m links $(j, j+1)$, $j=1, \dots, m$. Assume that the chain graph is laid out as an interval $[0, b]$ and let a_j be the distance of node j from node 1, $j = 1, \dots, m+1$. In particular, note that $a_1 = 0$ and $a_{m+1} = b$. Further, let the demand for a given service or product on link $(j, j+1)$ be $w_j(a_{j+1} - a_j)$ while the actual location of this demand on link $(j, j+1)$ is a random variable with a uniform probability density function. Then the expected transportation cost of serving an infinitesimal length dx of link $(j, j+1)$ from a unit distance away is given by

$$w_j(a_{j+1}-a_j)[dx/(a_{j+1}-a_j)] = w_j dx.$$

We will henceforth refer to w_j as the demand density on link $(j, j+1)$ and denote the total piecewise continuous demand density function on $[0, b]$ as

$$f(x) = w_j > 0, a_j \leq x < a_{j+1}, j=1, \dots, m \quad (3.1)$$

with $f(b) \equiv w_m$. The quantity $\int_0^b f(x)dx$ will be referred to as the total demand.

The problem is to locate p facilities on the real line, and to ascertain what portion I_i of the interval $[0, b]$ is to be served by each facility $i=1, \dots, p$, so as to minimize the total expected transportation cost. Mathematically, the problem may be stated as

$$\begin{aligned} \text{CP: minimize} \quad & \sum_{i=1}^p \int_{I_i} f(x)d(x, y_i)dx \\ \text{subject to} \quad & \bigcup_{i=1}^p I_i = [0, b] \end{aligned}$$

where the decision variables $y_1, \dots, y_p$ denote the locations of the p facilities and where in general, $d(x, y_i)$ measures the shortest distance between points x and y_i in the network. In the above context, $d(x, y_i) = |x - y_i|$.

Some remarks are in order at this point. Firstly, if there exist any concentrated (nodal) demands, say h_j at node j , then we assume that this node is split into two nodes separated by a (small) distance ϵ , with a uniform demand density of h_j/ϵ on this interval. By making ϵ sufficiently small, as close an approximation as desired may be

obtained. We will refer to this as the ϵ -splitting of a nodal demand. We remark that this will not eventually be necessary in the case $p = 1$ for the problems we consider. (Computationally, a value of $\epsilon = 0.0001$ is quite reasonable.) Secondly, note that (3.1) implies that

$$F(t) = \int_0^t f(x)dx, \quad 0 \leq t \leq b \quad (3.2)$$

is an increasing function. Algorithmically, we find this a convenient assumption. In case the graph contains segments of zero demand densities, then these segments may be given a demand density of $\Delta > 0$, where Δ is sufficiently small so that the total extra demand created multiplied by b is less than the desired tolerance on the optimal objective value. Again, this will not be necessary when $p = 1$. Finally note that we are assuming uniform demand densities on links. This assumption leads to trivial subproblems in the sequel. A parallel, though less efficient, development is possible using any piecewise continuous demand density function $f(\cdot)$.

To begin with, let us establish an important necessary optimality condition for Problem CP. For notational simplicity, we denote

$$z_i = (y_i + y_{i+1})/2 \text{ for } i = 1, \dots, p-1, \quad (3.3)$$

and accordingly denote $z_i^* = (y_i^* + y_{i+1}^*)/2$ for $i = 1, \dots, p-1$.

Theorem 3.1. Let $Y^* = \{y_1^*, \dots, y_p^*\}$ be an optimal solution to Problem CP

where $y_1^* \leq \dots \leq y_p^*$. Then it is necessary that $0 < y_1^* < \dots < y_p^* < b$,

$I_1 \equiv [0, z_1^*]$, $I_i \equiv [z_{i-1}^*, z_i^*]$ for $i=2, \dots, p-1$ and $I_p \equiv [z_{p-1}^*, b]$, and

furthermore, that

$$\int_0^{y_1^*} f(x)dx = \int_{y_1^*}^{z_1^*} f(x)dx, \quad (3.4)$$

$$\int_{z_{i-1}^*}^{y_i^*} f(x)dx = \int_{y_i^*}^{z_i^*} f(x)dx \quad \text{for } i=2, \dots, p-1, \quad (3.5)$$

and

$$\int_{z_{p-1}^*}^{y_p^*} f(x)dx = \int_{y_p^*}^b f(x)dx. \quad (3.6)$$

Proof. It should be clear that each $y_i^* \in [0, b]$ and each I_i is as defined by the theorem. Furthermore, no two facilities will be coincident since this essentially makes the two facilities equivalent to a single facility, and total cost may be reduced by moving one of the facilities to a distinct point. Finally, in order to verify that $y_1^* > 0$, $y_p^* < b$, and that conditions (3.4), (3.5), and (3.6) hold, let us rewrite the total cost (TC) for any solution $Y = (y_1, \dots, y_p)$ with $0 \leq y_1 \leq \dots \leq y_p \leq b$ as follows by noting that demands get served by the closest facility:

$$\begin{aligned} TC = & \left[\int_0^{y_1} f(x)(y_1 - x)dx + \int_{y_1}^{z_1} f(x)(x - y_1)dx \right] \\ & + \sum_{i=2}^{p-1} \left[\int_{z_{i-1}}^{y_i} f(x)(y_i - x)dx + \int_{y_i}^{z_i} f(x)(x - y_i)dx \right] \\ & + \left[\int_{z_{p-1}}^{y_p} f(x)(y_p - x)dx + \int_{y_p}^b f(x)(x - y_p)dx \right]. \end{aligned} \quad (3.7)$$

Now, let us obtain the right hand partial derivative $\partial^+ TC / \partial y_1$ of TC with respect to y_1 . Denoting $f^+(t)$ as the limit of $f(\cdot)$ at t from the right if t increases with y_1 and from the left otherwise,

$$\begin{aligned} \frac{\partial^+ TC}{\partial y_1} &= \int_0^{y_1} f(x)dx + \frac{1}{2} \left(\frac{y_2 - y_1}{2} \right) f^+(z_1) - \int_{y_1}^{z_1} f(x)dx \\ &\quad - \frac{1}{2} \left(\frac{y_2 - y_1}{2} \right) f^+(z_1) = \int_0^{y_1} f(x)dx - \int_{y_1}^{z_1} f(x)dx. \end{aligned}$$

In a similar manner, the left hand partial derivative $\partial^- TC / \partial y_1$ is given by the same expression. This may be verified for $i=2, \dots, p$ as well. Hence, TC is differentiable, and in fact,

$$\frac{\partial TC}{\partial y_1} = \int_0^{y_1} f(x)dx - \int_{y_1}^{z_1} f(x)dx \quad (3.8)$$

$$\frac{\partial TC}{\partial y_i} = \int_{z_{i-1}}^{y_i} f(x)dx - \int_{y_i}^{z_i} f(x)dx \quad i=2, \dots, p-1 \quad (3.9)$$

$$\frac{\partial TC}{\partial y_p} = \int_{z_{p-1}}^{y_p} f(x)dx - \int_{y_p}^b f(x)dx. \quad (3.10)$$

Since $\partial TC / \partial y_i = 0$, $i=1, \dots, p$ is a necessary optimality condition, the conditions (3.4), (3.5), (3.6) are respectively obtained from (3.8), (3.9), (3.10). Finally, the above conditions also imply that $y_1^* > 0$,

$y_p^* < b$ and the proof is complete.

Corollary 3.2. The conditions embodied by Theorem 3.1 are also sufficient if $p = 1$.

Proof. If $p = 1$, then $g(y_1) \equiv dTC/dy_1 = \int_0^{y_1} f(x)dx - \int_{y_1}^b f(x)dx$, is

continuous, and the right hand derivative $\frac{d^+}{dy_1} g(y_1) = 2f^+(y_1) \geq 0$.

Hence, TC is convex and so $g(y_1^*) = 0$ is both necessary and sufficient for optimality. This completes the proof.

Observe that the conditions (3.4), (3.5) and (3.6) assert that at an optimal solution Y^* , for each individual facility, the demand served by this facility on either side is the same. Secondly, note that the ϵ -splitting of nodal demands preserves the differentiability of $TC(\cdot)$. Thirdly, in the case $p = 1$, this ϵ -splitting is not necessary. One may simply locate a point at which no more than half the total demand lies on either side of it, and then (letting $\epsilon \rightarrow 0$), this clearly represents an optimal solution by Corollary 3.2. Finally, when $p > 1$, Problem CP is generally nonconvex. For example, consider the simple case in which $p = 2$, $b = 1$, and $f(x) = 4$ if $0 \leq x < 1/2$ and $f(x) = 1$ if $1/2 \leq x \leq 1$. Then $(y_1, y_2) = (5/32, 15/32)$ and $(y_1, y_2) = (1/4, 1/2)$ both satisfy the conditions of Theorem 3.1. However, the total cost for the first solution is $37/128$ while that of the second solution is $40/128$.

We will now proceed to use Theorem 3.1 in order to efficiently determine all solutions Y which satisfy the first order optimality conditions. From among these, the total cost will be evaluated at candidates for a global optimum, and hence, Problem CP will be solved.

Toward this end, observe that by specifying y_1 in conditions (3.4), (3.5), (3.6), since $f(x) > 0$ on $[0, b]$, one sequentially obtains

$z_1, y_2, z_2, \dots, z_{p-1}, y_p$ in a unique fashion so long as each of these points do not exceed b . Since we cannot guarantee that this will hold for any selected y_1 , we create a new artificial link $(m+1, m+2)$ beyond b , with $[a_{m+1}, a_{m+2}] \equiv [b, M]$, where M is some large number, and extend the definition of $f(\cdot)$ as $f(x) = w_m$ for $b < x \leq M$. It is also clear that Problem CP remains equivalent with this new definition of $f(x)$, $0 \leq x \leq M$. Let us now define a point z_p which is such that

$$\int_{z_{p-1}}^{y_p} f(x)dx = \int_{y_p}^{z_p} f(x)dx. \quad (3.11)$$

In other words, given y_1 , we use (3.4), (3.5), (and (3.3)) to sequentially determine $z_1, y_2, \dots, z_{p-1}, y_p$ and then, finally use (3.11) in order to determine z_p . If it so turns out that $z_p = b$, then we will have found a $Y = (y_1, \dots, y_p)$ which satisfies conditions (3.4), (3.5), (3.6). Such a pair (Y, Z) , where $Z \equiv (z_1, \dots, z_p)$, will be called a feasible completion. On the other hand, if $z_p \neq b$, then we will refer to (Y, Z) as an infeasible completion. In either case, the pair (Y, Z) will be called a completion. Now, consider the following result which will help us determine feasible completions via a series of trivial linear programming problems.

Theorem 3.3. Let $(\bar{Y}, \bar{Z})$ be any completion. Then, it is possible to associate with each y_i a link $(r(i), r(i)+1)$, and with each z_i a link $(s(i), s(i)+1)$, such that as y_1 increases from $\bar{y}_1$ to $\bar{y}_1 + \epsilon$ for some

$\xi > 0$, all resulting completions (Y, Z) satisfy $a_{r(i)} \leq y_i \leq a_{r(i)+1}$, and $a_{s(i)} \leq z_i \leq a_{s(i)+1}$ for $i=1, \dots, p$. Moreover, for all such completions, y_i and z_i may be obtained as linear functions of y_1 , $i=1, \dots, p$, as say,

$$y_i = \theta_i y_1 + \phi_i, \quad i=1, \dots, p \quad (3.12)$$

$$z_i = \alpha_i y_1 + \beta_i, \quad i=1, \dots, p \quad (3.13)$$

Proof. Using induction, let us first verify this result for y_1 and z_1 .

Let k be such that $a_k \leq \bar{y}_1 < a_{k+1}$. Then we may let $r(1) = k$, $\theta_1 = 1$, $\phi_1 = 0$. Next, consider z_1 . Noting (3.2) and (3.4), we know that $F(z_1) = 2F(y_1)$. Taking right hand derivatives with respect to y_1 , we obtain

$$f^+(z_1) \frac{d^+ z_1}{dy_1} = 2f^+(y_1) \quad (3.14)$$

or $d^+ z_1 / dy_1 > 0$. Hence, selecting k such that $a_k \leq \bar{z}_1 < a_{k+1}$, we may let $s(1) = k$. Furthermore, since z_1 is a continuous function of y_1 , and since $f^+(z_1) = w_{s(1)}$, $f^+(y_1) = w_{r(1)}$, we obtain from (3.14) that

$$z_1 = \left(\frac{2w_{r(1)}}{w_{s(1)}} \right) (y_1 - \bar{y}_1) + \bar{z}_1 \quad (3.15)$$

which is of the form $z_1 = \alpha_1 y_1 + \beta_1$.

Now, suppose that for $i=1, \dots, n-1$, we have designated links or intervals within which y_i and z_i lie for all completions for which

$\bar{y}_1 \leq y_1 \leq \bar{y}_1 + \xi$ for some $\xi > 0$, and moreover, for $i=1, \dots, n-1$, y_i and z_i are known linear functions of y_1 of the form (3.12), (3.13) for these completions. Let us show that this implies that the above is true for y_n and z_n as well. First, note that by (3.3),

$$y_n = 2z_{n-1} - y_{n-1} \quad (3.16)$$

which means that $d^+y_n/dy_1 = 2\alpha_{n-1} - \theta_{n-1}$ for $\xi > 0$ small enough. Now,

if $a_k < \bar{y}_n < a_{k+1}$ for some k , we may let $r(n) = k$. Otherwise, if

$\bar{y}_n = a_k$ for some k , then we may let $r(n) = k$ if $2\alpha_{n-1} - \theta_{n-1} \geq 0$ and

let $r(n) = k-1$ otherwise. Furthermore, using the linear forms for z_{n-1}

and y_{n-1} in (3.16) gives $y_n = \theta_n y_1 + \phi_n$.

Next, observe from (3.5) or (3.11) that

$$F(z_n) = 2F(y_n) - F(z_{n-1}). \quad (3.17)$$

Taking right hand derivatives with respect to y_1 , since the completions determine each y_i and z_i as continuous functions of y_1 ,

$$\begin{aligned} f^+(z_n) \left(\frac{d^+z_n}{dy_1} \right) &= 2f^+(y_n) \frac{d^+y_n}{dy_1} - f^+(z_{n-1}) \frac{d^+z_{n-1}}{dy_1} \\ &= 2w_{r(n)}\theta_n - w_{s(n-1)}\alpha_{n-1}. \end{aligned} \quad (3.18)$$

Hence, if

$a_k < \bar{z}_n < a_{k+1}$ for some k , then we may let $s(n) = k$. Otherwise, if

$\bar{z}_n = a_k$ for some k , then we may let $s(n) = k$ if

$2w_{r(n)}\theta_n - w_{s(n-1)}\alpha_{n-1} \geq 0$ and let $q(n) = k-1$ otherwise. Furthermore,

in this case, $f^+(z_n) = w_{s(n)}$ and we obtain from (3.18) that

$$z_n = \left(\frac{2w_{r(n)}\theta_n - w_{s(n-1)}\alpha_{n-1}}{w_{s(n)}} \right) (y_1 - \bar{y}_1) + \bar{z}_n \quad (3.19)$$

which is of the form $z_n = \alpha_n y_1 + \beta_n$ and the proof is complete.

Now let us determine lower and upper bounds, LB and UB respectively, on the value of y_1 for any feasible completion (Y, Z) . Then we will use Theorem 3.3 to obtain specific values of y_1 in the interval $[LB, UB]$ which give candidates for global optima.

First consider LB. Let $L_0 = (a_2 - a_1)/2p \equiv a_2/2p$ be the length of the first link divided into $2p$ parts. Then clearly, $y_i = (2i-1)L_0$, $i=1, \dots, p$ and $z_i = 2iL_0$, $i=1, \dots, p$ is a completion. Further, it is feasible if $m = 1$ (single link), and no feasible completion can have a smaller value of y_1 . Thus $LB = L_0$.

Next, consider UB. Let $\hat{y}_1$ be such that $2F(\hat{y}_1) = F(a_m)$. So from (3.4), if $y_1 = \hat{y}_1$, then $z_1 = a_m$. This would give $y_2 = \hat{y}_2 \equiv 2a_m - \hat{y}_1$. Let $\ell = \hat{y}_2 - a_m = (a_m - \hat{y}_1)$. Since the demand density is uniformly equal to w_m beyond a_m , we would obtain $z_p = a_m + 2(p-1)\ell$. If $z_p \geq b$, then clearly $UB = \hat{y}_1$. Hence, letting L_f be the length of the final link, i.e., $L_f = (b - a_m)$, then if $(a_m - \hat{y}_1) \geq L_f/2(p-1)$, we may let

$UB = \hat{y}_1$. Otherwise, as for LB, we may clearly let $UB = a_m + L_f/2p$.

Note that if there is only one link, i.e., $m = 1$, then $LB = UB$ since $a_m = 0$ and $L_f/2p \equiv L_0$.

Now, let us define some critical completions called breakpoints.

Let the completion which determines LB above be called the first breakpoint $(\bar{Y}, \bar{Z})$. For this completion, determine $r(i)$ and $s(i)$ using Theorem 3.2 and obtain the associated linear functions (3.12), (3.13). Consider the linear programming problem

$$\underline{P}: \text{maximize } y_1$$

$$\text{subject to } a_{r(i)} < \theta_i y_1 + \phi_i < a_{r(i)+1}, \quad i=1, \dots, p \quad (3.20)$$

$$a_{s(i)} \leq \alpha_i y_1 + \beta_i \leq a_{s(i)+1}, \quad i=1, \dots, p. \quad (3.21)$$

Note that P is solved trivially by a minimum ratio test which uses either the left or right inequality in (3.20), (3.21), depending on the sign of the coefficient of y_1 . Let $y_{1\max}$ solve P. Note by Theorem 3.3

that $y_{1\max} > \bar{Y}_1$. Hence, the next breakpoint is defined as the

completion $y_1 = y_{1\max}$, $y_i = \theta_i y_{1\max} + \phi_i$, $i=2, \dots, p$ and

$z_i = \alpha_i y_{1\max} + \beta_i$, $i=1, \dots, p$. Thereafter, again using Theorem 3.3, the

$r(i)$ and $s(i)$ associated with this breakpoint completion are identified along with the linear functions (3.12), (3.13) and Problem P is solved.

This gives the third breakpoint. The process is continued until a breakpoint is obtained for which $y_1 \geq UB$. The following result

establishes that it is sufficient to consider only breakpoints for determining an optimal solution to Problem CP. Moreover, it identifies

some breakpoints as being noncandidates for optimal solutions; hence the total cost need not be evaluated at such breakpoints.

Theorem 3.4. It is sufficient to consider only breakpoints in determining an optimal solution to Problem CP, and in particular, only those breakpoints for which $z_p = b$, and moreover, which result from solving Problem P in which both θ_p and α_p have the same sign.

Proof. By Theorem 3.1, it is sufficient to consider only feasible completions for which $LB \leq y_1 \leq UB$. The breakpoints determine all such completions except those for which y_1 lies between $\bar{y}_1$ and $\tilde{y}_1$, where $(\bar{Y}, \bar{Z})$ and $(\tilde{Y}, \tilde{Z})$ are two consecutive breakpoints, both of which are feasible completions, with $\alpha_p \equiv 0$ in Problem P formulated at $(\bar{Y}, \bar{Z})$.

However for such a case, viewing the total cost function TC of Equation (3.7) as a function of y_1 , with z_i and y_i given by (3.12), (3.13) for $i=1, \dots, p$, it follows that

$$\frac{d^+TC}{dy_1} = \sum_{i=1}^p \frac{\partial TC}{\partial y_i} \frac{\partial^+ y_i}{\partial y_1} = \sum_{i=1}^p \frac{\partial TC}{\partial y_i} \theta_i = 0 \quad (3.22)$$

for $\bar{y}_1 \leq y_1 \leq \tilde{y}_1$, since for all such y_1 values, $\partial TC / \partial y_i \equiv 0$ for each $i=1, \dots, p$ from (3.8), (3.9), (3.10). Hence the total cost function is constant between these breakpoints. Consequently, it is sufficient to consider only breakpoints for which $z_p = b$.

To complete the proof, suppose that a breakpoint $(\bar{Y}, \bar{Z})$ which is not a feasible completion leads to a feasible completion breakpoint $(\tilde{Y}, \tilde{Z})$

through the solution of Problem P. For $\bar{y}_1 \leq y_1 < \tilde{y}_1$, we have as in (3.22),

$$\frac{d^+ TC}{dy_1} = \frac{\partial TC}{\partial y_p} \theta_p \quad (3.23)$$

because from (3.8), (3.9), $\partial TC / \partial y_i = 0$ for $i=1, \dots, p-1$. Now, if $\alpha_p > 0$, then from (3.10) and (3.11)

$$\begin{aligned} \frac{\partial TC}{\partial y_p} &= \int_{z_{p-1}}^{y_p} f(x) dx - \int_{y_p}^b f(x) dx = \int_{y_p}^{z_p} f(x) dx - \int_{y_p}^b f(x) dx \\ &= - \int_{z_p}^b f(x) dx < 0. \end{aligned}$$

Consequently, if $\theta_p < 0$, then TC is increasing between these breakpoints

and so $(\tilde{Y}, \tilde{Z})$ could not be optimal. In a similar manner, $\alpha_p < 0$ implies

$$\frac{\partial TC}{\partial y_p} = \int_b^{z_p} f(x) dx > 0, \text{ and so if } \theta_p > 0, \text{ then again } (\tilde{Y}, \tilde{Z}) \text{ could not be}$$

optimal. This completes the proof.

We are now ready to formally state the algorithm to solve Problem CP.

Algorithm CP

Initialization: Determine lower and upper bounds LB and UB on y_1 .

Obtain the completion which determines LB as the first breakpoint $(\bar{Y}, \bar{Z})$.

Step 1. Given a breakpoint, determine $r(i)$ and $s(i)$ $i=1, \dots, p$ as in Theorem 3.3 along with the linear functional expression (3.12), (3.13) for y_i and z_i , $i=1, \dots, p$. Solve Problem P and let $\tilde{y}_1$ be the optimal

solution. Use (3.12) and (3.13) to determine $(\tilde{Y}, \tilde{Z})$ by substituting

$$y_1 = \tilde{y}_1.$$

Step 2. If $\tilde{z}_p \neq b$, or $\bar{z}_p = b$ or if θ_p and α_p are of opposite signs, go to Step 3. Otherwise, compute the total cost for $(\tilde{Y}, \tilde{Z})$ and update the incumbent solution if necessary.

Step 3. If $\tilde{y}_1 \geq UB$, stop with the incumbent as an optimal solution.

Otherwise, set $(\bar{Y}, \bar{Z}) = (\tilde{Y}, \tilde{Z})$ and return to Step 1.

As an expedient in computing total costs, it may be verified that if a segment $[r, s] \subseteq [0, b]$ with some demand density w is served by a point at $x < r$, then the associated cost is $\frac{w}{2} (s-r)^2 + w(s-r)(r-x)$, and if $x > r$, then the associated cost is $\frac{w}{2} (s-r)^2 + w(s-r)(x-s)$, and if $r \leq x \leq s$, then the associated cost is $\frac{w}{2} [(x-r)^2 + (s-x)^2]$.

To illustrate an iteration of the solution procedure, consider a case $p = 2$, $b = 24$, with $f(x) = w_1 = 1$ for $0 \leq x < 9$, $f(x) = w_2 = 3$ for $9 \leq x < 13$, $f(x) = w_3 = 1$ for $13 \leq x < 17$ and $f(x) = w_4 = 3$ for $17 \leq x \leq 24$. (The extension of $f(\cdot)$ would therefore have $f(x) = w_5 = 3$ for $24 \leq x \leq M$.) Suppose we currently have a breakpoint $\bar{y}_1 = 4.5$, $\bar{z}_1 = 9$, $\bar{y}_2 = 13.5$ and $\bar{z}_2 = 20$.

From Theorem 3.3 we obtain the linear functions (3.12) and (3.13) as follows. First, $r(1) = 1$ and $s(1) = 2$. From (3.15), this gives $z_1 = [2(1)/3] (y_1 - 4.5) + 9 = (2/3) y_1 + 6$. Next from (3.16),

$y_2 = 2z_1 - y_1 = (1/3) y_1 + 12$, with $r(2) = 3$. Further, $s(2) = 4$.

Hence, from (3.19), we get

$$z_2 = \{[2(1)(1/3) - 3(2/3)]/3\}(y_1 - 4.5) + 20 = -(4/9) y_1 + 22. \text{ Problem}$$

P may then be formulated as $\max \{y_1: y_1 \leq 9, (2/3) y_1 + 6 \leq 13,$

$$(1/3) y_1 + 12 \leq 17, 17 \leq -(4/9) y_1 + 22\} \text{ which gives } \tilde{y}_1 = 9.$$

This means $\tilde{z}_1 = 12$, $\tilde{y}_2 = 15$ and $\tilde{z}_2 = 18$. This is an infeasible

completion, and hence we now repeat the above step. (There turn out to be a total of 11 breakpoints, with y_1 ranging from 2.25 to 10.1667.

The only feasible completion, and hence the optimal solution, is obtained with $y_1 = 9.8333$, $z_1 = 15$, $y_2 = 20.1667$, and $z_2 = 24$.)

3.3 2-median Location Problem on a Tree

In this section, we consider the 1-median and 2-median problems on an undirected tree $T \equiv T(N,A)$ with discrete demands h_v associated with the nodes or vertices $v \in N$, and with uniform demand densities w_ℓ associated with the links $\ell \in A$. Using the development of the foregoing section, the 1-median case admits an efficient Goldman [1971] type of algorithm which turns out to be identical with the method obtained by Chiu [1982]. For the 2-median case, the algorithm turns out to be somewhat more involved, though also quite effective.

Let us begin by developing a method to locate a single median.

Toward this end, consider a link $\bar{\ell} \equiv (r,s) \in A$. Disconnecting (r,s) from T results in two subtrees T_r and T_s which respectively contain the

nodes r and s . For $k = r, s$, let the subtree T_k have node set N_k and arc set A_k , and let L_ℓ denote the length of any link ℓ . Furthermore, for $k = r, s$, let D_v^k and D_ℓ^k respectively denote the distances from node k to node $v \in N_k$ and from node k to the closest point (node) on link $\ell \in A_k$. Then, for $k = r, s$, the total cost of serving subtree T from a point y on link (r, s) is given by

$$C(y, T_k) = \sum_{v \in N_k} h_v (D_v^k + d(y, k)) + \sum_{\ell \in A_k} \int_0^{L_\ell} w_\ell [x + D_\ell^k + d(y, k)] dx \quad (3.24)$$

i.e.,

$$C(y, T_k) = C(k, T_k) + d(y, k) H_k \quad (3.25)$$

where $C(k, T_k)$ is the cost of serving T_k from k and H_k is the total demand associated with subtree T_k .

What this means is that if one were constrained to locate the best facility on link (r, s) , then the equivalent problem is to consider the single link (r, s) with demand weights H_r at r , H_s at s and $w_{\bar{\ell}}$ on the link $\bar{\ell} \equiv (r, s)$. It is easy to show that a point is optimal if and only if it is an optimal (constrained) location on all links containing that point. In other words, if an interior point on a link has exactly half the total demand weight on either side of it, then it is an optimal location on T , and if a node in T is such that if the subtrees obtained by disconnecting this node have demand weights no more than half the total demand, then this node is an optimal location in T .

Algorithmically, therefore one can proceed in a Goldman [1971]-Chiu

[1982] like fashion by picking a node with the greatest degree say and testing the optimality condition. If it doesn't hold at this node, then all subtrees of demand weight no more than half the total demand may be collapsed at this node, i.e., such subtrees may be deleted with their total demands being aggregated with the demand at this node. This proceeds until either an optimal node results or a single link remains. In the latter case, the optimal facility is obtained by locating a 1-median on this remaining link.

Now, consider the 2-median case. Since both facilities must lie on the path connecting some two end nodes, a rudimentary algorithm would proceed as follows. Let $E = \{e_1, \dots, e_g\}$ be the set of end nodes of T and let $P(e_i, e_j)$ be the unique path connecting some pair of end nodes $e_i, e_j \in E$. Consider the (constrained) problem of locating 2 facilities on this path. Let $a_1 = e_i, a_2, \dots, a_m, a_{m+1} = e_j$ be the nodes on this path. Suppose that some node a_k on the path $P(e_i, e_j)$ has degree greater than two. In other words, there exists some subtree, say T_k , which results from disconnecting T at a_k and which does not share any link with $P(e_i, e_j)$. Then the cost of serving T_k from a point y on $P(e_i, e_j)$ is given by $C(y, T_k) = C(a_k, T_k) + d(y, a_k)H_k$ in the spirit of (3.24), (3.25), where the quantities are defined therein. What this means is that all such subtrees T_k (there could be more than one at a_k) may be deleted, and their demand weights may be aggregated with a_k . This results in a problem of the type in the foregoing section and one may optimally

locate two facilities on this chain and evaluate its total cost. Note that one would have to perform the ϵ -splitting of nodes with positive demand and create artificially small demand densities on links of zero density. Further, if the two facilities turn out to share the collapsed demand at a node, then this is not a problem since if only one facility is to serve this node (and its associated subtrees), then the difference in total cost is inconsequential when ϵ is small enough. In this manner, by solving $|E|(|E|-1)/2$ such path problems, and picking the best solution obtained, one can obtain the optimal solution to the 2-median problem.

Fortunately, one need not solve so many path problems. As demonstrated in the sequel, it is possible to pre-process the problem in order to reduce the number of such path subproblems that need to be solved. Toward this end consider the following two definitions. Let r, s be two points in T and for some $t \in P(r, s)$, let T_t be a subtree (if one exists) which results when T is disconnected at t and which is such that it shares no links with $P(r, s)$. Then T_t will be said to be rooted at t on $P(r, s)$. Secondly, suppose T_t is a subtree of T which results when T is disconnected at t . Then, the operation of deleting T_t and adding the total demand weight associated with T_t to the demand at point t will be referred to as collapsing T_t into t .

Now, let $\bar{y}$ denote an optimal 1-median location on T and suppose that subtrees T_i , $i \in \Lambda$ are rooted at $\bar{y}$. Then the following result

localizes the optimal 2-median locations y_1^*, y_2^* on T .

Theorem 3.5. $\bar{y} \in P(y_1^*, y_2^*)$. Moreover, one of the 2-median lies either at $\bar{y}$ or in some subtree T_n , $n \in \Lambda$, with the largest demand weight.

Proof. The proof is similar to the discrete nodal case in Mirchandani and Oudjit [1980], and is therefore omitted.

Observe that Theorem 3.5 permits us to examine only those path subproblems associated with end nodes $e_i, e_j \in E$, such that e_i belongs to some maximum weight subtree rooted at $\bar{y}$, and e_j belongs to some other subtree rooted at $\bar{y}$. The following results yield further reductions in the number of subproblems that need to be solved by collapsing portions of T_i , $i \in \Lambda$.

Lemma 3.6. Let $\tau(N, A)$ be some tree and let $\hat{y}$ denote an optimal single facility location on τ . Consider an end e of τ and let the demand on node e be increased from its present value. Then, the new optimal single facility location $y' \in P(\hat{y}, e)$ in τ .

Proof. Consider the subtrees which result when τ is disconnected at $\hat{y}$, and let τ' be that subtree which contains e . Since each resulting subtree in this disconnection contains no more than half the total original demand weight, it follows that $y' \in \tau'$. Hence, in order to (optimally) locate y' on τ , it is sufficient to (optimally) locate y' on τ' with all the other subtrees obtained upon disconnecting τ at $\hat{y}$ collapsed into $\hat{y}$. Now, if on the contrary, $y' \notin P(\hat{y}, e)$, then y' must

lie on a subtree rooted at a node $a_k \in P(\hat{y}, e)$, $y' \neq a_k$. Since excluding a_k , this subtree contains strictly less than half the new total demand weight, this is impossible, and the proof is complete.

We now are ready to state our reduction theorem. But before this, one can trivially delete all end links of T which have zero total demand.

Theorem 3.7. Let $\bar{y}$ denote an optimal single facility location on T and let $T_0(N_0, A_0)$ be a subtree obtained by disconnecting T at $\bar{y}$. Treat $\bar{y}$ as an end node of T_0 . Consider any point $r \in T_0$ such that r is not an end node of T_0 . Let T_r be some subtree of T_0 not containing $\bar{y}$ which results when T_0 is disconnected at r . Treat r as a node of T_0 and hence an end node of T_r . Find a node $e(r)$ in T_r which is furthest away from the node r . Let M denote the midpoint of the path $P(\bar{y}, e(r))$ in T_0 , and denote by T_M that subtree of T_0 containing $f(r)$ which is obtained by disconnecting T_0 at M . Treat M as an end node of T_M . By neglecting $T_0 - T_M$, locate optimally a single facility on T_M , and let y_M denote this location. Treat y_M as a node of T_M . Then, an equivalent 2-median problem on T results by performing the following reductions, designated $R_1(r, T_r)$ on T_0 :

(a) If $y_M \notin T_r$, then collapse T_r into r .

(b) If $y_M \in T_r$, then for each node $v \in P(y_M, M)$, including y_M , if a subtree T_v of T_r is rooted at v on $P(y_M, M)$, then collapse T_v into v .

Proof. First of all, note that T_0 can receive at most one of the two facilities, say y_1^* , at a point other than $\bar{y}$. Now, consider case (a).

It is sufficient to show that if T_0 receives y_1^* , it cannot lie on $T_r - \{r\}$. First of all, note that since $y_m \notin T_r$, then $M \notin T_r$ and so, $M \in P(\bar{y}, r)$. Now, if $y_1^* \in T_r - \{r\}$, then for any $x \in T_M$, we get

$$\begin{aligned} d(\bar{y}, x) &= d(\bar{y}, M) + d(M, x) = d(M, r) + d(r, e(r)) + d(M, x) \\ &\geq d(M, r) + d(r, y_1^*) + d(M, x) = d(y_1^*, M) + d(M, x) \geq d(y_1^*, x) \end{aligned} \quad (3.26)$$

and so y_1^* would serve at least the tree T_M . But by Lemma 3.6, this means that y_1^* must lie on $P(y_M, M)$, a contradiction.

Next, if $y_M \in T_r$, consider two cases, namely $M \in T_r$ and $M \notin T_r$.

Suppose $M \in T_r$, so that T_M is a subtree of T_r . Now, either $y_1^* \in T_M$, or $y_1^* \notin T_M$. If $y_1^* \notin T_M$, then since case (b) is only collapsing portions of $T_M \subseteq T_r$, it is valid. If $y_1^* \in T_M$, then as above, it must serve at least T_M , and so by Lemma 3.6, $y_1^* \in P(y_M, M)$. Consequently, again case (b) is valid.

Finally, suppose $y_M \in T_r$ and $M \notin T_r$. Now, either facility $y_1^* \in T_r$ or $y_1^* \notin T_r$. If $y_1^* \notin T_r$, case (b) is again valid. If $y_1^* \in T_r$, then as in (3.26), it must serve at least T_r , and so $y_1^* \in P(y_M, r) \subseteq P(y_M, M)$. Consequently, one may collapse the subtrees of T_r rooted at nodes on this path as stated in case (b), and obtain an equivalent problem. This completes the proof.

Theorem 3.7 gives a general reduction strategy. The benefits derived from its usage depend on the way it is implemented. Below we state a method which we recommend. Note that after solving the reduced problem, the solution to the original problem is recovered by noting the distance and orientation of y_1^* and y_2^* from $\bar{y}$.

Reduction Algorithm 1

Initialization: Locate $\bar{y}$ and perform this algorithm for each T_0

identified by Theorem 3.7. From $\bar{y}$, determine the furthest node e in T_0 and let r be the mid-point of $P(\bar{y}, e)$. Determine a subtree T_r of T_0 , not containing $\bar{y}$, which is obtained by disconnecting T_0 at r .

Step 1. Perform the reduction $R_1(r, T_r)$ of Theorem 3.7. Designate a point ξ as being either the point r or the midpoint M of the path from $\bar{y}$ to $e(r)$ defined in Theorem 5, whichever is further from $\bar{y}$.

Step 2. Find an original node k of T_0 closest to ξ on the path $P(\bar{y}, \xi)$, which may be ξ itself, such that when T_0 is disconnected at k , a subtree T_k of T_0 results which does not contain any reduced portion of T_0 . If no such node k exists, stop. Otherwise, set $r = k$, $T_r \equiv T_k$, and go to Step 1.

As an example, consider the tree depicted in Figure 3.1(a), where the quantity shown against each link ℓ is $\{L_\ell, w_\ell\}$ and the quantity shown at each node v is (h_v) . Using the single facility algorithm, $\bar{y}$ gets located as shown in Figure 3.1(a). Hence, there are two candidates for T_0 . Considering the T_0 with the single link, the reduction algorithm merely shrinks this link as shown in Figure 3.1(b). Now, consider the other subtree T_0 .

Reduction Algorithm 1 first locates $r = M$ as a point on the link (2,3) at a distance of 2.5 from node 2, with $e(r) = 4$. The single facility on T_M gets located at node 3. Hence, according to case (b) in Theorem 3.5, nodes 4,5 and the adjoining links get collapsed into node 3 as shown in Figure 3.1(b). Next, we backtrack to node 2 and let $r = 2$, and denote by T_r the subtree containing nodes 2,6,7,8. Now, $e(2) = 7$ and M gets located on the link (2,6) at a distance of 0.5 from node 2. The single facility location problem on T_M gives node 6 as y_M . Consequently, by case (b) of Theorem 3.7, Figure 3.1(b) results.

Now, we need to solve two subproblems, namely, on the path joining nodes 1' and 3 and on the path joining nodes 1' and 6. The first

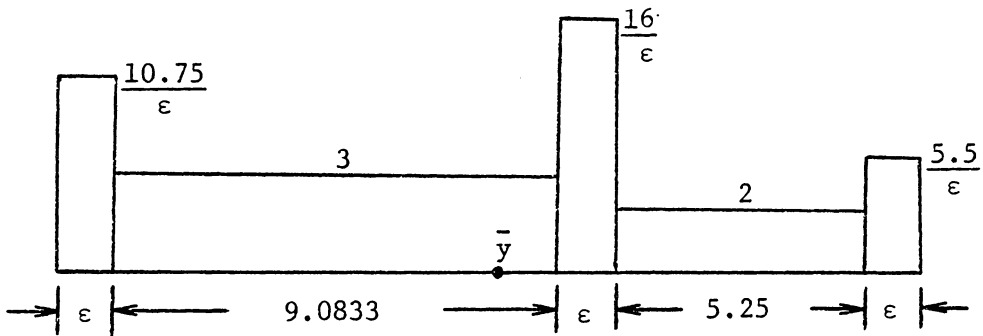
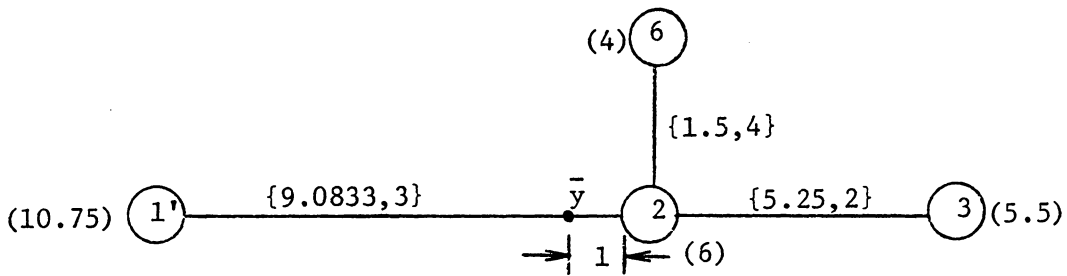
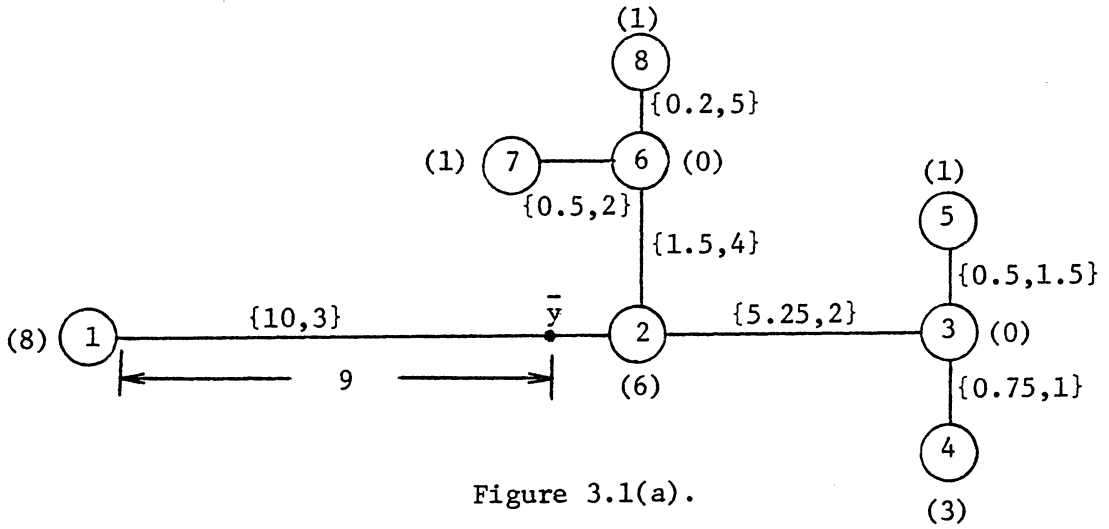


Figure 3.1. Illustrative Example

subproblem chain is shown in Figure 3.1(c). Using $\varepsilon = 0.00001$, we obtain y_1^* at a distance of 7.444 from $\bar{y}$ toward node 1' (or node 1 in the original problem), and y_2^* gets located at node 2 (within 3 decimals of accuracy). The total cost of this solution is 140.762. Similarly, solving the other subproblem, we discover that the first solution is a global optimum to the 2-median problem.

In concluding this section, we mention that through an example, one may show that the following conjecture is false: Let y_1^* and y_2^* be the optimal 2-median locations on a path $P(e_i, e_j)$, $e_i, e_j \in E$, such that y_1^* and y_2^* each lie on the interior of some links. Then y_1^* and y_2^* are optimal locations for T . Further, except for Reduction Algorithm 1, the above technique can be used to locate p -medians provided they are constrained to lie on some path $P(e_i, e_j)$, $e_i, e_j \in E$.

3.4 1-median Location Problem on a Graph with Isolated Cycles

In this section, we will consider the location of a single facility on an undirected graph $G(N, A)$ which is such that each link is involved in at most one cycle. We will call such a graph as being one with isolated cycles. The location of a single facility on such a graph having only nodal demands h_v for $v \in N$ has been considered by Goldman [1971]. In addition, we will permit herein nonnegative uniform demand densities w_ℓ on links $\ell \in A$ as well, and derive an alternative algorithm.

To begin with, let us introduce some terminology and notation. Consider a link $\ell \in A$ and denote its length by L_ℓ . A link (or node) will be called cycleless if it is not involved in any cycle. Let C be a cycle and let y be a point in $G - C$. Then the leading node $y(C)$ of C is that node on C which is closest to y . (Note that for our graph G , all points in C are accessed from y through $y(C)$). Next, consider any cycle C and let $2P_C$ denote the perimeter of the cycle, namely, $2P_C = \sum_{\ell \in C} L_\ell$. For each node $v \in C$, define the antipode of v as that unique point $a(v)$ on C such that $d(v, a(v)) = P_C$. If $a(v)$ is not already a node in C , create a dummy node at $a(v)$ with zero demand weight. Hence, each node of C has a nodal antipode. Similarly, for each link $(r, s) \in C$, denote the link $(a(r), a(s))$ as its antipodal link.

Now, if the single facility to be located is constrained to lie on a cycleless link (r, s) , then an equivalent problem is obtained by examining the two disjoint subgraphs G_r and G_s obtained by disconnecting link (r, s) , with $r \in G_r$, and $s \in G_s$. For each cycle C in G_r , one may locate $r(C)$ and split the cycle at the antipode $a(r(C))$. In this way, an equivalent tree results. Then as in (3.24), (3.25), the equivalent problem is to simply aggregate the total demand weight of G_r at node r and the total demand weight of G_s at node s . In this manner, the best location on a cycleless link may be found. We use this concept later to devise an efficient algorithm.

Next, consider the constrained problem in which the single facility is restricted to optimally lie on some cycle C . Since each link is

involved in only one cycle, the demand on any point in $G - C$ can be satisfied along a path which departs from C at a unique node in C . Consequently, with each node in C , one can associate such a subgraph of G served through this node, and as above, convert this subgraph into a tree. Then using (3.24), (3.25), one may equivalently delete this subgraph, and add its total demand to the demand at the associated node. Consequently, the cycle constrained problem may be reduced to that of locating a facility on a simple circuit graph. This is considered next.

Hence, let C be a circuit graph with nodes $1, 2, \dots, g$, say, oriented in a counterclockwise fashion. (The "orientation" is pictorial; the links are, of course, undirected. Further, for notational convenience, node $g + 1$ will refer to node 1, and node 0 will refer to node g .) Note that g must be even by our creation of antipodes. Let $\ell = 1, \dots, g$ be the links of C , with link ℓ being $(\ell, \ell + 1)$ of length L_ℓ , say. Let h_v be the demand weight on node v , and w_ℓ the density on link ℓ . Let $d_1, \dots, d_{g+1}$ be the distances of nodes $1, \dots, g+1$ from node 1 measured in a counterclockwise direction from node 1. Next, for any point x in the cycle, let $W^+(x) = [\text{total demand weight on the counterclockwise chain from } x \text{ to } a(x), \text{ excluding the demand at } x]$ and let $W^-(x) = [\text{total demand weight on } C] - W^+(x)$. Define $TC(y)$ as the total cost of serving the circuit from a point at a distance y from node 1 measured counterclockwise, $0 \leq y \leq 2P_C$. The following result characterizes this function.

Theorem 3.8. $TC(y)$, $0 \leq y \leq 2P_C$ is a continuous function. Furthermore,

$TC(\cdot)$ has the following properties:

- (a) For each pair of antipodal links (r,s) and $(a(r),a(s))$, $TC(y)$ is convex for $d_r \leq y \leq d_s$ and concave for $d_{a(r)} \leq y \leq d_{a(s)}$ if $w_r \geq w_{a(r)}$ and the convexity is vice versa if $w_r \leq w_{a(r)}$.
- (b) For any point $x \in C$, denoting D_x^+ and D_x^- as respectively the right and left hand derivatives of $TC(\cdot)$ with respect to y evaluated at x ,

$$D_x^+ = W^-(x) - W^+(x) = -D_{a(x)}^+ \quad (3.27)$$

$$D_x^- = D_x^+ - 2(h_x - h_{a(x)}) = -D_{a(x)}^- \quad (3.28)$$

where $h_x = h_{a(x)} = 0$ if x is not a nodal point in C .

Proof. The continuity of $TC(y)$ is clear since a small perturbation in y results in only a small perturbation in the distances over which points in C are served. By the nature of the circuit, in order to prove (a) and (b), there is no loss in generality in considering $r = 1$, $s = 2$, and taking x as a point on the link $(1,2)$. Now, the total cost function $TC(y)$ for $0 \leq y \leq d_2$ is readily obtained as

$$\begin{aligned} TC(y) = & \sum_{j=2}^{a(1)} h_j(d_j - y) + \sum_{j=a(2)}^{g+1} h_j(y + 2P_C - d_j) + \sum_{\ell=2}^{a(1)-1} w_\ell [L_\ell^2/2 + L_\ell(d_\ell - y)] \\ & + \sum_{\ell=a(2)}^g w_\ell [L_\ell^2/2 + L_\ell(y + 2P_C - d_{\ell+1})] \\ & + \frac{w_1}{2} [y^2 + (L_1 - y)^2] + w_{a(1)} [y^2/2 + y(d_{a(1)} - y)] \\ & + w_{a(1)} [(L_{a(1)} - y)^2/2 + (L_{a(1)} - y)(y + 2P_C - d_{a(2)})]. \end{aligned} \quad (3.29)$$

Differentiating (3.29) with respect to y and noting that

$P_C = d_{a(1)} = d_{a(2)} - L_{a(1)}$, we get,

$$\begin{aligned} \left. \frac{dTC}{dy} \right|_x &= \left[\sum_{j=a(2)}^{g+1} h_j + \sum_{\ell=a(2)}^g w_\ell L_\ell + w_1 y + w_{a(1)} \{L_{a(1)} - y\} \right] \\ &\quad - \left[\sum_{j=2}^{a(1)} h_j + \sum_{\ell=2}^{a(1)-1} w_\ell L_\ell + w_1 (L_1 - y) + w_{a(1)} y \right] \end{aligned}$$

which from the definitions of $W^+(x)$ and $W^-(x)$ is simply,

$$\left. \frac{dTC}{dy} \right|_x = W^-(x) - W^+(x). \quad (3.30)$$

Note that $d^2TC/dy^2 \Big|_x = 2(w_1 - w_{a(1)})$. By treating r or $a(r)$ as node 1, this proves part (a). Furthermore, since $a(x)$ is antipodal to x , $W^+(a(x)) = W^-(x)$ and $W^-(a(x)) = W^+(x)$. Using this fact and (3.30) establishes (3.27). By using a mirror image argument, in which clockwise and counterclockwise directions get interchanged, we obtain (3.28) and the proof is complete.

Based on Theorem 3.8, we now characterize an optimal solution. Theorem 3.9 below proves that an optimal solution belongs to $S_1 \cup S_2$, where

$$S_0 = \{\text{nodes } v: D_v^+ = 0, D_v^- = 0, w_v = w_{a(v)}, w_{(v-1)} = w_{a(v-1)}\}$$

$$S_1 = \{\text{nodes } v: D_v^+ \geq 0 \text{ and } D_v^- \leq 0, \text{ and if } D_v^+ = 0 \text{ then}$$

$$w_v \geq w_{a(v)}, \text{ and if } D_v^- = 0 \text{ then } w_{(v-1)} \geq w_{a(v-1)}\} - S_0 \quad (3.31)$$

$$S_2 = \{x: x \in \ell \equiv (r, s) \text{ such that } w_r > w_{a(r)}, D_r^+ < 0, D_s^- > 0$$

$$\text{and } \left. \frac{dTC}{dy} \right|_x = 0\}. \quad (3.32)$$

Theorem 3.9. The set $S_1 \cup S_2$ defined by (3.31), (3.32) contains an optimal single facility location on C .

Proof. By Theorem 3.8, note that $TC(y)$, $0 \leq y \leq 2P_C$, is continuous, $TC(0) = TC(2P_C)$, and on each link it is smooth and convex or concave. Hence, the only nodes v to be considered as candidates for an optimal solution are those for which $D_v^+ \geq 0$ and $D_v^- \leq 0$. Furthermore, if either D_v^+ or D_v^- equals zero, this must be due to the derivative of a convex function vanishing. Thirdly, if $TC(\cdot)$ is constant on both links incident at v , then v may be disregarded. This characterizes set S_1 . Similarly, the only candidates for an optimal solution on the interior of links correspond to such points at which the derivative of a strictly convex function vanishes. Such points are characterized by S_2 . Clearly, it is impossible for $S_1 = \phi$ and $S_2 = \phi$ if the circuit has positive total demand and this completes the proof.

Corollary 3.10. If some node $v \in C$ has demand weight at least half the total demand weight, then v is optimal on C .

Proof. Follows from the fact that $D_x^+ \geq 0$ for all x on the counterclockwise chain from v to $a(v)$, not including $a(v)$, and $D_x^+ \leq 0$ for all x on the counterclockwise chain from $a(v)$ to v , not including v . (Alternatively, in this case, $S_1 = \{v\}$ and $S_2 = \phi$.)

The foregoing results yield the following algorithm to locate an optimal facility on a circuit C .

Algorithm C

Initialization. For each node $v=1, \dots, g$, compute D_v^+ and D_v^- . As an expedient, note from (3.27) and (3.28) that only half of these need to be calculated. Further, if D_r^+ is known and s is the next node after r in the counterclockwise direction, then

$$D_s^+ = D_r^+ + 2\{(h_s + w_r L_r) - (h_{a(s)} + w_{a(r)} L_{a(r)})\}. \quad (3.33)$$

Thirdly, the $g/2$ quantities D_v^- that need to be computed, may be found from (3.28).

Step 1. Determine the sets S_1 and S_2 as in (3.31), (3.32) and compute the total cost associated with each point in $S_1 \cup S_2$. Select the best solution as optimal to the 1-median problem on the circuit C .

Finally, let us piece together the results dealing with the location of a single facility constrained to lie on a cycleless link and on a cycle. Toward this end, let v be any node in G . Because of the nature of G , there may be incident at v some cycleless links, and if v is not cycleless, then there are incident at v pairs of links, each pair being associated with some cycle. Disconnecting G at v results in a set of disconnected subgraphs, one associated with every cycleless link and one with every pair of links of some cycle incident at v . Let $G(v)$ denote the set of these subgraphs and consider the following reduction result.

Theorem 3.11. Let v be a node in G and let $G(v) = \{G_1, \dots, G_r\}$ be as defined above. If any $G_i \in G(v)$ has demand weight (not counting that at

node v) no more than half the total demand weight of G , then an equivalent problem results by deleting G_i and adding its demand to that at node v .

Proof. Let y^* be an optimal facility location in G and suppose

$y^* \in G_i - \{v\}$ for some $G_i \in G(v)$. Note that an optimal constrained

location on any cycle C in G is, by Corollary 3.10, on the leading node $v(C)$ of C , and similarly, a constrained optimum on a cycleless link in G is on that end of the link which is closer to v . Now, consider any path

$P(y^*, v)$. If this path contains only cycleless links, then by the

foregoing argument, v is at least as good as y^* . Similarly, if this path contains a link involved in some cycle C , then the leading node

$v(C)$ of C on $P(y^*, v)$ is at least as good as y^* , and continuing in this

fashion, we again conclude that v is at least as good as y^* . This completes the proof.

We can now state the algorithm to locate a median on G .

Step 1. Pick a node v of maximum degree. If the degree of v is less than or equal to unity, locate an optimal 1-median on this remaining node or single link.

Step 2. Disconnect G at v and examine the demand weight on each $G_i \in G(v)$. If the demand on any $G_i \in G(v)$ is no more than half the total demand on G , collapse G_i into v . If the remaining subgraph is a cycle, go to Step 3. Otherwise, return to Step 1.

Step 3. Locate an optimal 1-median on the remaining circuit graph.

To illustrate the nonconvex step in the above procedure, consider the 1-median problem on the circuit graph of Figure 3.2. For simplicity, let each link be of length unity, and let the quantity shown in Figure 3.2 against each node v be (h_v) and that against each link ℓ be (w_ℓ) . The antipodal nodes are shown connected by dotted lines in Figure 3.2.

Using (3.27), we compute $D_1^+ = 38 - 32 = 6$. Next, using (3.33), we compute $D_2^+ = 6 + 2(7-11) = -2$, $D_3^+ = -14$ and $D_4^+ = -10$. By (3.27), the respective antipodal nodes of 1,2,3,4 then have $D_5^+ = -6$, $D_6^+ = 2$, $D_7^+ = 14$ and $D_8^+ = 10$. Now, using (3.28), we find $D_1^- = 6 - 2(5-7) = 10$, $D_2^- = 10$, $D_3^- = 0$ and $D_4^- = -8$. Again for the antipodal nodes by (3.28), we obtain, $D_5^- = -10$, $D_6^- = -10$, $D_7^- = 0$, and $D_8^- = 8$. From (3.31), (3.32), we obtain $S_1 = \{6\}$, $S_2 = \emptyset$. (Note in formulating S_1 that $D_7^+ > 0$, $D_7^- = 0$ but $w_6 = 2 < w_{a(6)} \equiv w_2 = 3$.) Consequently by Theorem 3.9, node 6 is an optimal 1-median location.

3.5 Computational Experience

In this section we present some computational experience for the principal problem of locating p -medians on a chain. The Algorithm CP of Section 3.2 was coded in Fortran IV and run on an IBM 3081 Series D24 computer using the Fortran G1 compiler with the double precision option, under normal load batch operation. Table 3.1 summarizes the results obtained for three randomly generated demand patterns and for various

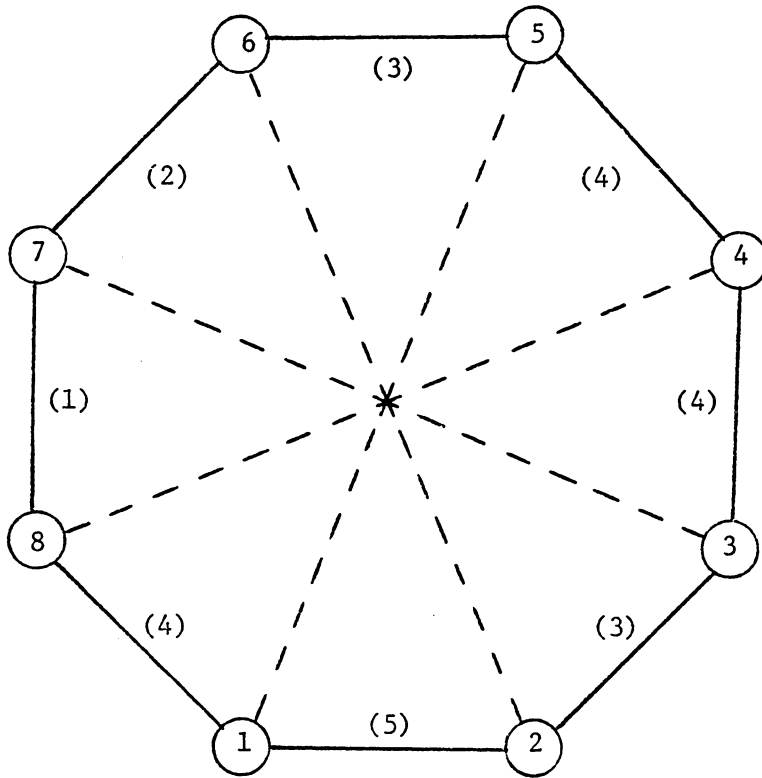


Figure 3.2. Locating 1-median on a Cycle

number of facilities. The three demand patterns involve a randomly generated uniform demand density on each of the 10, 30 and 50 links respectively. It may be observed that for reasonably sized problems, the effort is quite moderate even with the derivative of the total cost function vanishing at several points.

Table 3.1. Computational Experience with Algorithm CP

Number of Links (m)	Number of Facilities (p)	Number of Breakpoints	Number of Feasible Completions	Execution Time CPU (secs)
10	2	34	1	< 0.01
	4	82	3	0.01
	6	176	7	0.04
	8	352	11	0.12
	10	681	27	0.31
30	2	106	1	< 0.01
	4	332	3	0.05
	6	884	15	0.23
	8	2198	47	0.79
	10	5678	103	2.55
50	2	176	1	0.01
	4	598	1	0.09
	6	2030	13	0.53
	8	7476	63	2.63
	10	24974	223	11.05

CHAPTER IV

SEQUENTIAL LOCATION-ALLOCATION PROBLEMS ON CHAINS AND TREES WITH PROBABILISTIC LINK DEMANDS

4.1 Introduction

This chapter is concerned with multiperiod minimum location problems on trees with probabilistic link demands in which multiple facilities need to be located, but due to budgetary constraints or some other factor, at most one facility can be built per time period. That is, the facilities must be located in a sequential fashion. In addition, it is possible that demands may change dynamically over the time periods.

As discussed in Section 2.8, a good example of the sequential types of decisions involved is presented by Scott [1971], which incidentally is the viewpoint adopted herein. In addition to the myopic and long-range strategies considered by Scott [1971], we consider a third strategy, namely, one of minimizing the total discounted costs.

Note that a central problem that arises in the context of sequential locations is that of locating a single facility on a network in the presence of existing service facilities. One such problem was considered by Minieka [1980]. It was shown that the myopic versions of both the absolute median and general absolute median problems admit to previous solution techniques if the distances in the network are appropriately redefined. For further references to problems in this area, see Section 2.8.

This chapter is organized as follows. We begin by formulating and solving sequential siting decision problems on chain graphs, in which

the location of demands on links may occur probabilistically according to some uniform continuous density function. The demand may change dynamically over time and a single facility must be located at each prescribed point in time. Three strategies are considered. The first is a myopic strategy for locating p facilities, the second is a long-range strategy for locating p facilities, and the third strategy minimizes a discounted present worth cost function in locating two facilities over some finite planning horizon. We remark that although these problems are generally nonconvex, one may quite readily identify all local minima and hence obtain a global optimal solution. We also remark that the problems on a chain provide not only insights, but also viable methodologies for similar problems on more complex networks. In particular, we extend this analysis herein to solve exactly the 3-facility myopic problem and the 2-facility long-range and discounted cost problems on a tree graph. It is hoped that these exact algorithms on specialized networks will lead to at least effective heuristics on more general networks.

4.2 A Myopic Strategy for the Multiperiod p -facility Problem on a Chain

Consider a chain graph with $m + 1$ distinct nodes designated $1, 2, \dots, m+1$ and m links $(j, j+1)$, $j = 1, \dots, m$. Assume that the chain graph is laid out as an interval $[0, b]$. Accordingly, let a_j be the distance of node j from node 1, $j = 1, \dots, m+1$. In particular, note that $a_1 = 0$ and $a_{m+1} = b$. Further, let the demand for a given service or product on link $(j, j+1)$ in time period t be $w_j^t(a_{j+1} - a_j)$, $t = 1, \dots, \tau$, while the actual location of this demand on link $(j, j+1)$ is a random

variable with a uniform probability density function. Then the expected transportation cost of serving an infinitesimal length dx of link $(j, j+1)$ in period t from a unit distance away is given by

$$w_j^t(a_{j+1} - a_j) \cdot [dx / (a_{j+1} - a_j)] = w_j^t dx.$$

We will henceforth refer to w_j^t as the demand density on link $(j, j+1)$ in period t and denote the total piecewise continuous demand density function on $[0, b]$ in period t , $t = 1, \dots, \tau$ as

$$f^t(x) = w_j^t > 0, \quad a_j \leq x < a_{j+1}, \quad j = 1, \dots, m \quad (4.1)$$

with $f^t(b) = w_m^t$. The quantity $\int_0^b f^t(x) dx$ will be referred to as the total demand in period t .

The problem is to use a myopic strategy in locating a single facility in each of p designated time periods t_i , $i = 1, \dots, p$ where $0 \leq t_1 < t_2 < \dots < t_p \leq \tau$. For each time period t_q , $q \in \{1, \dots, p\}$, one assumes that $q - 1$ of the p planned facilities have been located at points, say, $\bar{y}_1, \dots, \bar{y}_{q-1}$ on $[0, b]$ at times $t_1, \dots, t_{q-1}$ respectively. The problem at time t_q is then to locate a single new facility on the real line and ascertain what portion I_i of the interval $[0, b]$ is to be served by each facility $i = 1, \dots, q$, so as to minimize the expected present period transportation costs. Mathematically the period t_q problem may be stated as

$$\begin{aligned}
\text{MCP}(q): \quad & \text{minimize} \quad \sum_{i=1}^{q-1} \int_{I_i} f^t{}^q(x) d(x, \bar{y}_i) dx \\
& + \int_{I_q} f^t{}^q(x) d(x, y_q) dx \\
& \text{subject to} \quad \bigcup_{i=1}^q I_i = [0, b]
\end{aligned}$$

where the decision variable y_q denotes the location of the single new facility.

We will assume that concentrated nodal demands and segments of zero demand density are handled as in Section 3.2.

Notice that since the single new facility in Problem MCP(q) must be located among $q - 1$ existing facilities on the real line, then at optimality, the new facility location, y_q , must be adjacent to either one or two existing facilities. For convenience, let us refer to Problem $P_{\ell r}$ as that of locating a new facility on an interval

$[\bar{y}_\ell, \bar{y}_r] \subseteq [0, b]$, where $\bar{y}_\ell$ and $\bar{y}_r$ represent adjacent existing facility locations. Similarly, P_ℓ and P_r will denote the problems of locating a new facility on intervals $[\bar{y}_\ell, b]$ and $[0, \bar{y}_r]$ respectively.

The following theorem establishes a necessary optimality condition for Problem $P_{\ell r}$. Similar conditions are then stated for Problems P_ℓ and P_r . (Note that $f^t{}^q(x)$ is written as $f(x)$ for notational convenience below).

Theorem 4.1. Let y^* be an optimal facility location on $[\bar{y}_\ell, \bar{y}_r]$ and let

$z_\ell^* = (\bar{y}_\ell + y^*)/2$, $z_r^* = (y^* + \bar{y}_r)/2$. Then it is necessary that

$\bar{y}_\ell < y^* < \bar{y}_r$, and furthermore, that

$$\int_{z_\ell^*}^{y^*} f(x)dx = \int_y^{z_r^*} f(x)dx \quad (4.2)$$

Proof. It should be clear that $\bar{y}_\ell < y^* < \bar{y}_r$ and that each facility serves the set of points on $[0, b]$ closest to it. Accordingly, to verify that (4.2) holds, let us rewrite the expected total cost $TC(y)$ for serving $[\bar{y}_\ell, \bar{y}_r]$ given any solution y on $(\bar{y}_\ell, \bar{y}_r)$ as follows:

$$\begin{aligned} TC(y) = & \int_{\bar{y}_\ell}^{z_\ell} f(x)(x - \bar{y}_\ell)dx + \int_{z_\ell}^y f(x)(y - x)dx \\ & + \int_y^{z_r} f(x)(x - y)dx + \int_{z_r}^{\bar{y}_r} f(x)(\bar{y}_r - x)dx \end{aligned} \quad (4.3)$$

where $z_\ell = (\bar{y}_\ell + y)/2$ and $z_r = (y + \bar{y}_r)/2$.

By taking right and left hand derivatives of $TC(y)$ with respect to y , it may be readily verified that $TC(\cdot)$ is differentiable, and in fact,

$$D(y) = \frac{dTC}{dy} = \int_{z_\ell}^y f(x)dx - \int_y^{z_r} f(x)dx \quad (4.4)$$

Noting that $dTC/dy = 0$ is a necessary optimality condition completes the proof.

Corollary 4.2. Let y^* be an optimal solution to Problem P_r . Then

$$\int_0^{y^*} f(x)dx = \int_y^{z_r^*} f(x)dx \quad (4.5)$$

Corollary 4.3. Let y^* be an optimal solution to Problem P_ℓ . Then

$$\int_{z_\ell}^{y^*} f(x)dx = \int_y^{b^*} f(x)dx \quad (4.6)$$

Observe that conditions (4.2), (4.5), and (4.6) assert that at optimality, the demand served by the new facility on either side of it is the same. Despite this simple condition, it should be pointed out that Problems $P_{\ell r}$, P_r , and P_ℓ are in general nonconvex. Consider, for example, Problem P_r with $[0, \bar{y}_r] = [0, 12]$, and suppose that $f(x)$ is 2.5 if $0 \leq x \leq 2$, is 1 if $2 \leq x < 7$, is 2 if $7 \leq x < 8$ and is 6 if $8 \leq x \leq 12$. Then $y = 2$ and $y = 8$ both satisfy the conditions of Corollary 4.2. However, the total cost for the first solution is 74.5 while that of the second is 77.5.

Now let us develop a solution procedure for Problem $P_{\ell r}$.

(Problems P_r and P_ℓ may be solved in an analogous manner). The idea is to find all zeroes of $D(y)$ defined in (4.4) on the interval $[\bar{y}_\ell, \bar{y}_r]$.

Again for notational convenience, we treat the interval $[\bar{y}_\ell, \bar{y}_r]$ as

$[0, c]$ where $c = \bar{y}_r - \bar{y}_\ell$, and let the demand on $[0, c]$ be given by

$f(x) = \omega_j > 0$, $\alpha_j \leq x < \alpha_{j+1}$, $j = 1, \dots, \mu$, and $f(c) = \omega_\mu$ where $\alpha_1 = 0$

and $\alpha_{\mu+1} = c$. (Note that this function may have redefined demands at $x = 0$ and $x = c$ without consequence).

First, observe that $y_{\min} = \alpha_2/2$ and $y_{\max} = (\alpha_\mu + \alpha_{\mu+1})/2$ are respectively lower and upper bounds on the zeroes of $D(y)$. Hence,

starting with $y = y_{\min}$, we will increase y to $y_{\max}$, while detecting zeroes of $D(y)$ along the way.

Toward this end, noting that $z_\ell = (0+y)/2$ and $z_r = (y+c)/2$ in (4.4), observe that the right hand derivative of $D(y)$ with respect to y is given by

$$G(y) = \frac{d^+ D(y)}{dy} = 2f^+(y) - \frac{1}{2} [f^+(z_\ell) + f^+(z_r)] \quad (4.7)$$

where $f^+(\cdot)$ is the right hand limit of $f(\cdot)$. The following result lays the groundwork for solving $P_{\ell r}$.

Theorem 4.4. Given $y = \bar{y} \in [y_{\min}, y_{\max}]$, denote $\bar{z}_\ell = \bar{y}/2$ and

$\bar{z}_r = (\bar{y}+c)/2$, and let α_y , α_ℓ , and α_r respectively be the smallest α_j

strictly greater than $\bar{y}$, $\bar{z}_\ell$, $\bar{z}_r$. Define

$$\tilde{y} = \min\{\alpha_y, 2\alpha_\ell, 2\alpha_r - c\}. \quad (4.8)$$

Then $\tilde{y} > \bar{y}$ and moreover, $G(y)$ is a constant for $\bar{y} \leq y < \tilde{y}$.

Proof. From the definitions of α_y , α_ℓ , α_r and from (4.8), it is clear

that $\tilde{y} > \bar{y}$. Furthermore, for $\bar{y} \leq y < \tilde{y}$, since $y < \alpha_y$, $y/2 < \alpha_\ell$ and

$(y+c)/2 < \alpha_r$ from (4.8), it follows that $f^+(y) = \omega_y$, $f^+(z_\ell) = \omega_\ell$, and

$f^+(z_r) = \omega_r$. Noting (4.7), the proof is complete.

Observe that Theorem 4.4 gives us an effective means of determining the nondifferentiable points of the continuous piecewise linear function $D(\cdot)$. Hence, we may seek the zeroes of $D(y)$ by noting

that given $\bar{y}$,

$$D(y) = D(\bar{y}) + G(\bar{y})(y - \bar{y}) \quad \text{for } \bar{y} \leq y < \tilde{y}$$

where $\tilde{y}$ is given by (4.8). Thus, to see if $D(y) = 0$ in this interval, we may compute

$$\hat{y} = \begin{cases} \bar{y} - \frac{D(\bar{y})}{G(\bar{y})} & \text{if } D^+(\bar{y}) \neq 0 \\ +\infty & \text{otherwise} \end{cases} \quad (4.9)$$

and if $\bar{y} \leq \hat{y} \leq \tilde{y}$, then $y = \hat{y}$ is a zero of $D(y)$. Otherwise, we may replace $\bar{y}$ by $\tilde{y}$ and repeat. As an expedient, note that if at some zero $\hat{y}$ of $D(y)$, it turns out that $G(\hat{y}) < 0$, then clearly $\hat{y}$ is not a candidate for a minimizing point of $TC(y)$, and hence $TC(\hat{y})$ need not be evaluated. The algorithm is formalized below.

Algorithm A_{lr}

Initialization. Determine $y_{\min}$ and $y_{\max}$ and let $\bar{y} = y_{\min}$.

Step 1. Determine $\tilde{y}$ as in Theorem 4.4 and find $\hat{y}$ from (4.9).

Step 2. If $\bar{y} < \hat{y} < \tilde{y}$ and $G(\hat{y}) > 0$, then compute $TC(\hat{y})$ and update the incumbent solution if necessary. Otherwise go to Step 3.

Step 3. If $\tilde{y} \geq y_{\max}$, stop with the incumbent as an optimal solution.

Otherwise, set $\bar{y} = \tilde{y}$ and go to Step 1.

Using Algorithm A_{lr} , a global optimal solution to the nonconvex Problem P_{lr} can be determined quite effectively. In a parallel fashion,

it is possible to solve Problems P_ℓ and P_r . Now, we need to turn our attention to the problem of choosing candidate subintervals, $[0, \bar{y}_r]$, $[\bar{y}_\ell, \bar{y}_r]$, or $[\bar{y}_\ell, b]$, to receive the new facility. Obviously, since there are assumed to be $q - 1$ existing facilities, a global optimal solution to problem MCP(q) could be found by determining the optimal solution on each of the q subintervals and choosing one with the minimum total cost. However, a more efficient approach is characterized by the following theorem which determines an upper bound on the saving in cost obtained by locating a new facility in a subinterval.

Theorem 4.5. Let $[0, c]$ be an interval associated with Problem $P_{\ell r}$ and let $\Delta TC(y)$ be the decrease in total cost realized by locating an additional facility at $y \in (0, c)$. Let y_0 be any point in $(0, c)$. Then

$$\Delta TC(y) \leq B = \frac{c}{2} \cdot \max \left\{ \int_{\frac{y_{\min}}{2}}^{\frac{y_0+c}{2}} f(x) dx, \int_{\frac{y_0}{2}}^{\frac{y_{\max}+c}{2}} f(x) dx \right\} \quad \text{for all } y \in (0, c) \quad (4.10)$$

Proof. Let $y \in (0, c)$ and let $I = [z_\ell, z_r]$. Then $\Delta TC(y)$ = old cost of serving I - new cost of serving I , is no greater than the difference computed by assuming that all of the demand on I is concentrated at y . Thus

$$\Delta TC(y) \leq \min\{y, c-y\} \int_{z_\ell}^{z_r} f(x) dx \leq \frac{c}{2} \int_{z_\ell}^{z_r} f(x) dx \quad (4.11)$$

First, note that $z_\ell \geq y_{\min}/2$ and $z_r \leq (y_{\max}+c)/2$. Next, either $y \leq y_0$

or $y > y_0$. If $y \leq y_0$, then $z_r \leq (y_0+c)/2$ and thus

$$\int_{z_l}^{z_r} f(x)dx \leq \int_{y_{\min}}^{\frac{y_0+c}{2}} f(x)dx.$$

Similarly,

$$\int_{z_l}^{z_r} f(x)dx \leq \int_{\frac{y_0}{2}}^{\frac{y_{\max}+c}{2}} f(x)dx \text{ if } y > y_0.$$

Using this in (4.11), we obtain (4.10) and the proof is complete.

Noting the form of (4.10), in order to compute a tight bound, a good choice of y_0 appears to be the 1-median location on $[0,c]$ ignoring the existence of facilities at the end points. Hence, the demand weight on either side of y_0 is the same. Furthermore, if y_0 is chosen in this manner, the following corollaries for Problems P_r and P_ℓ follow, where $y_{\min}$ and $y_{\max}$ are appropriately defined as before.

Corollary 4.6. Let $[0,c]$ be an interval associated with Problem P_r .

Then

$$\Delta TC(y) \leq B \leq (c-y_{\min}) \int_0^{\frac{y_0+c}{2}} f(x)dx \quad \text{for all } y \in [0,c). \quad (4.12)$$

Corollary 4.7. Let $[0,c]$ be an interval associated with Problem P_ℓ .

Then

$$\Delta TC(y) \leq B = y_{\max} \int_{\frac{y_0}{2}}^c f(x)dx \quad \text{for all } y \in (0,c] \quad (4.13)$$

Now, for a given subinterval j , $j = 1, \dots, q$, let ΔTC_j represent the

actual decrease in total cost which would be realized by an optimal location on that subinterval, and let B_j represent the appropriate bound computed from (4.10), (4.12), or (4.13). Then an algorithm for solving Problem MCP(q) is as follows.

Algorithm MCP(q)

Step 1. If $q = 1$, then compute the optimal 1-median on $[0, b]$ and stop. Otherwise, use the $q - 1$ existing facility locations to subdivide $[0, b]$ into q subintervals and compute the bound B_j for each subinterval,

$j = 1, \dots, q$. Let $S = \{B_1, \dots, B_q\}$.

Step 2. Let $B_k = \max_{B_j \in S} \{B_j\}$ and compute the optimal location on

subinterval k along with ΔTC_k by solving either Problem P_{lr} , P_l , or P_r .

Update the incumbent solution and the best savings ΔTC^* obtained thus far, if necessary. Replace S by $S - \{B_k\} \cup \{B_j : B_j \leq \Delta TC^*\}$.

Step 3. If $S = \phi$, stop with incumbent as an optimal solution.

Otherwise, return to Step 2.

4.3 A Long-Range Strategy for the Multiperiod p-facility Problem on a Chain

In this section, we again assume a sequential multiperiod formulation of the minisum p-facility problem on a chain. However, the strategy used for optimization is the opposite extreme of the myopic case. That is, the long-range strategy is a preemptive approach which assumes that aggregate long-range costs incurred after the total number

of facilities becomes stationary, outweighs the costs incurred during the initial sequential location of the facilities.

For this reason, the final facility locations can be computed by solving the static p -median problem with respect to the projected long-range demand. This can be done using the method of Section 3.2. Hence, the problem becomes one of optimally sequencing the opening of facilities from among these predetermined locations.

Using an approach similar to Scott [1971], let us assume that facility i is to be constructed at the beginning of prescribed time period t_i , $i = 1, \dots, p$, with the system becoming stationary after time period t_p . Hence, the configuration at time period t_q continues to exist through periods $t_{q+1}, \dots, t_{q+1}$.

Certainly one method of sequencing the opening of the facilities would be to use a myopic approach as in Section 4.2. However, another method which is probably more consistent with the philosophy of long-range planning is to pick that sequence of openings which minimizes the cumulative discounted cost incurred from the start to the final location of the p facilities. Note that there are $p!$ such sequences. The following forward recursive dynamic programming strategy may be used to determine an optimal sequence.

Let us first define the stages, states, and decisions. There are as many stages as there are facilities to be located, namely p , and at each stage q , some q out of the p facilities are assumed to have been located. Hence, the state variables at stage q are all combinations of q out of p facilities; hence, there are $p!/q!(p-q)!$ state variables

at stage q . The optimal decision made for each state variable at stage q determines which of the facilities in this combination should be constructed last. Next, let us construct the forward recursive equation. Toward this end, let S_q denote a state variable at stage q , i.e., S_q is a set of some q out of the p facilities. Define $TC_q(S_q)$ as the minimal total discounted cost up till time period $t_{q+1}-1$ associated with having constructed the facilities in S_q (in an optimal sequence) by time t_q . Defining $t_{p+1} = t_p + 1$, it follows that $TC_p(\{1, \dots, p\})$ is the optimal total objective function value. Note that the accumulated discounted cost incurred in periods $t_q, t_q+1, \dots, t_{q+1}-1$ given S_q is independent of the sequence in which the facilities were opened. Denoting this cost by $C_q(S_q)$, it follows that this cost is the present worth of the following cost incurred in each of the aforementioned periods, that is,

$$C_q(S_q) = \sum_{t=t_q}^{t_{q+1}-1} \left(\frac{1}{1+d}\right)^{t-t_q} \sum_{i \in S_q} \int_{I_i} f^t(x) d(y_i, x) dx$$

where d = discount rate, I_i = optimal subinterval of $[0, b]$ served by facility i in the configuration S_q , y_i is the (long-range) location of facility $i \in S_q$, and as before, $f^t(\cdot)$ characterizes the demand at time t . Hence, the forward recursion formula is given as

$$TC_q(S_q) = \min_{k \in S_q} [C_q(S_q) + TC_{q-1}(S_q - \{k\})],$$

with $TC_0(\phi) = 0$.

Observe that in essence, therefore, one needs to compute the total cost associated with every distinct subset of facilities. Because of the combinatorial nature of the problem, a little reflection will reveal that it is futile to attempt any state reduction techniques in this context.

Before closing this section, let us note that by virtue of Section 3.3, the long-range 2-facility problem on a tree can be solved as a trivial extension of the above procedure.

4.4 A Discounted Present Worth Strategy for the Multiperiod 2-facility Problem on a Chain

In this section, we consider a compromise between the myopic and long-range strategies of the previous sections. Namely, we consider a strategy which minimizes a discounted present worth cost function over some finite planning horizon τ . Let the chain graph be laid out as in Section 4.2 with the piecewise continuous uniform demand density function on $[0, b]$ in period t defined by $f^t(\cdot)$, $t = 1, \dots, \tau$. Without loss of generality, let us assume that facility 1 is located in time period $t_1 = 1$ and facility 2 is located in prescribed time period t_2 where $t_1 = 1 < t_2 \leq \tau$. If the discount rate is denoted by d , then the problem may be stated mathematically as follows:

$$\begin{aligned} \text{DCP:} \quad & \text{minimize} \quad \int_0^b g(x) d(x, y_1) dx + \int_{I_1} h(x) d(x, y_1) dx \\ & \quad + \int_{I_2} h(x) d(x, y_2) dx \\ & \text{subject to } I_1 \cup I_2 = [0, b] \end{aligned}$$

where

$$g(\cdot) = \sum_{t=1}^{t_2-1} \left(\frac{1}{1+d}\right)^{t-1} f^t(\cdot),$$

$$h(\cdot) = \sum_{t=t_2}^{\tau} \left(\frac{1}{1+d}\right)^{t-1} f^t(\cdot),$$

and where the decision variables are

y_i = location of the i th facility, $i = 1, 2$

I_i = subinterval of $[0, b]$ served by facility i after $t = t_2$, $i = 1, 2$.

It should be clear that there are two cases which must be addressed, namely, (i) $y_1 < y_2$ and (ii) $y_2 < y_1$. Here, we will consider the case when $y_1 < y_2$ and note that case (ii) can be solved as a mirror image of case (i). The following theorem establishes a necessary optimality condition for problem DCP when $y_1 < y_2$.

Theorem 4.8. Let $y^* = (y_1^*, y_2^*)$ be an optimal solution to Problem DCP in the case when $y_1 < y_2$ and let $z^* = (y_1^* + y_2^*)/2$. Then it is necessary that $I_1 = [0, z^*]$, $I_2 = [z^*, b]$ and that

$$\int_0^{y_1^*} (g(x) + h(x)) dx = \int_{y_1^*}^b g(x) dx + \int_{y_1^*}^{z^*} h(x) dx \quad (4.14)$$

and

$$\int_{z^*}^{y_2^*} h(x) dx = \int_{y_2^*}^b h(x) dx. \quad (4.15)$$

Proof. Clearly, I_1 and I_2 are as defined in the theorem. Now rewrite

the total cost (TC) as follows for any $y_1 < y_2$ with $z = (y_1 + y_2)/2$:

$$\begin{aligned}
TC = & \int_0^{y_1} g(x)(y_1-x)dx + \int_{y_1}^b g(x)(x-y_1)dx \\
& + \int_0^{y_1} h(x)(y_1-x)dx + \int_{y_1}^z h(x)(x-y_1)dx \\
& + \int_z^{y_2} h(x)(y_2-x)dx + \int_{y_2}^b h(x)(x-y_2)dx
\end{aligned}$$

Taking the right and left partial derivatives with respect to y_1 and y_2 , it can be seen that TC is differentiable and that (4.14) and (4.15) precisely state the necessary first order optimality condition, namely, $\partial TC / \partial y_1 = 0$ and $\partial TC / \partial y_2 = 0$, respectively. This completes the proof.

As was the case in Section 4.2, Problem DCP is, in general, nonconvex and thus to find a global optimum, we need to determine all solutions (y_1, y_2) of (4.14) and (4.15). We will do this by enforcing condition (4.14) at all times and finding solutions which also satisfy (4.15). Toward this end, let us rewrite (4.14) (with * suppressed) by adding $\int_0^{y_1} g(x)dx$ to both sides to get

$$\int_0^{y_1} (2g(x)+h(x))dx = \int_0^b g(x)dx + \int_{y_1}^z h(x)dx.$$

Now letting $q(x) = 2g(x)+h(x)$ and $K = \int_0^b g(x)dx$ we get

$$\int_0^{y_1} q(x)dx = K + \int_{y_1}^z h(x)dx \quad (4.16)$$

Observe that for (4.16) to hold, y_1 must lie in some subinterval of $[0, b]$. The following result establishes these bounds on y_1 .

Lemma 4.9. Let $0 \leq y_1 < y_2 \leq b$ satisfy (4.16). Then $y_{1\min} \leq y_1 \leq y_{1\max}$ where

$$\int_0^{y_{1\min}} q(x) dx = K \quad (4.17)$$

$$\int_0^{y_{1\max}} (g(x)+h(x)) dx = \frac{1}{2} \int_0^b (g(x)+h(x)) dx \quad (4.18)$$

Proof. Equation (4.17) follows clearly from (4.16) and the fact that

$\int_{y_1}^{z_1} h(x) dx \geq 0$. Next consider $y_{1\max}$. Adding $\int_0^{y_1} h(x) dx$ to both sides

of (4.16) we get

$$2 \int_0^{y_1} (g(x)+h(x)) dx = K + \int_0^z h(x) dx.$$

Since z must be no greater than b , we get

$$2 \int_0^{y_1} (g(x)+h(x)) dx \leq \int_0^b (g(x)+h(x)) dx$$

and clearly if $y_{1\max}$ is chosen as defined by (4.18) then $y_1 \leq y_{1\max}$ and the proof is complete.

Now, consider the following strategy. Suppose that we start with $y_1 = y_{1\min}$ and increase it to the value $y_{1\max}$. Then, for each value of y_1 in this range, the quantity z is readily available from (4.16), and noting that $y_2 = 2z - y_1$, the variable y_2 may be accordingly considered as an implicit, continuous function of y_1 . In particular, taking right hand derivatives in (4.16), we obtain

$$q^+(y_1) = \frac{1}{2} h^+(z) \left[1 + \frac{d^+ y_2}{dy_1} \right] - h^+(y_1)$$

where $q^+(\cdot)$ and $h^+(\cdot)$ respectively denote right hand limits of $q(\cdot)$ and $h(\cdot)$, and d^+y_2/dy_1 denotes the right-hand derivative of y_2 with respect to y_1 . Hence, we obtain

$$\frac{d^+y_2}{dy_1} = \frac{2q^+(y_1) + 2h^+(y_1) - h^+(z)}{h^+(z)} . \quad (4.19)$$

Note from the proof of Lemma 4.9 that for $y_{1\min} \leq y_1 \leq y_{1\max}$, (4.16) gives $y_1 \leq z \leq b$ and so (4.19) is well defined.

In a similar manner, $\partial TC / \partial y_2$ may be considered as a function $D(y_1)$ of y_1 . Hence,

$$D(y_1) = \int_z^{y_2} h(x) dx - \int_{y_2}^b h(x) dx . \quad (4.20)$$

Observe from (4.14), (4.15) that we are interested in finding those values of $y_1 \in [y_{1\min}, y_{1\max}]$ for which $D(y_1) = 0$. Furthermore, taking right hand derivatives in (4.20), it follows that

$$G(y_1) = \frac{d^+D(y_1)}{dy_1} = [2h^+(y_2) - \frac{1}{2} h^+(z)] \frac{d^+y_2}{dy_1} - \frac{1}{2} h^+(z) . \quad (4.21)$$

An important point to note regarding (4.21) is that whereas z increases with y_1 since $(1 + d^+y_2/dy_1) > 0$ from (4.19), y_2 itself may change in either direction as y_1 increases. Hence in (4.21), $h^+(y_2)$ must be regarded as the right hand limit of $h(\cdot)$ if y_2 increases with y_1 and as the left hand limit of $h(\cdot)$ otherwise.

Now, in the spirit of the approach in Section 4.2, the strategy to be adopted should be evident. Theorem 4.10 below lays the groundwork of the approach, analogous to Theorem 4.4 in Section 4.2. Namely, as y_1 increases, so long as y_1 and z , obtained through (4.16), remain within certain bounds, then d^+y_2/dy_1 from (4.19) remains a constant.

Furthermore, in order to have (4.20) and (4.21) simply defined when $y_2 > b$, let $h(x) = 0$ for $x > b$. Then so long as the increase in y_1 also keeps y_2 within certain bounds, $G(y_1)$ is also a constant. Therefore,

$D(y_1)$ is linear within this bounded range $[\bar{y}_1, \tilde{y}_1)$ for y_1 , where $\tilde{y}_1$ is defined in Theorem 4.10 below. Consequently, the piecewise linear form of $D(y_1)$ is readily traced and within each subinterval of the type $[\bar{y}_1, \tilde{y}_1) \subseteq [y_{1\min}, y_{1\max})$, zeroes of $D(y_1)$ may be found, if they exist.

Note that since

$$\frac{d^+TC}{dy_1} = \frac{\partial TC}{\partial y_1} + \frac{\partial TC}{\partial y_2} \frac{d^+y_2}{dy_1} = D(y_1) \frac{d^+y_2}{dy_1},$$

whenever a point $\hat{y}_1$ is found such that $D(\hat{y}_1) = 0$, whence the

corresponding $\hat{y}_2 < b$ necessarily, the objective value at the point need not be computed when

$$G(\hat{y}_1) \left[\frac{d^+y_2}{dy_1} \right]_{y_1=\hat{y}_1} < 0.$$

This follows because as y_1 just increases beyond $\hat{y}_1$,

$D(y_1) = 0 + (y_1 - \hat{y}_1)G(\hat{y}_1)$ and d^+y_2/dy_1 remains a constant (by Theorem 4.10 below), hence d^+TC/dy_1 becomes negative and such a point could not be optimal. The limiting bounds on y_1 are now derived and the algorithm is stated formally below.

Theorem 4.10. Given $y_1 = \bar{y}_1 \in [y_{1\min}, y_{1\max})$, compute $z = \bar{z}$ from (4.16) and let $\bar{y}_2 = 2\bar{z} - \bar{y}_1$. Accordingly, compute $\bar{\delta} = d^+y_2/dy_1$ at $\bar{y}_1$ from (4.19) and compute $\bar{\Delta} = G(\bar{y}_1)$ from (4.21). Let $\alpha_q > \bar{y}_1$, $\alpha_h > \bar{y}_1$, and $\beta_h > \bar{z}$ be respectively the largest values for which $q(x) = q^+(\bar{y}_1)$ for $\bar{y}_1 < x < \alpha_q$, $h(x) = h^+(y_1)$ for $\bar{y}_1 < x < \alpha_h$ and $h(x) = h^+(\bar{z})$ for $\bar{z} < x < \beta_h$.

Furthermore, compute

$$\frac{\gamma_h}{|\bar{\delta}|} = \begin{cases} \text{largest value such that } h(x) = h^+(\bar{y}_2) \text{ for} \\ \quad \bar{y}_2 < x < \bar{y}_2 + \gamma_h \text{ if } \bar{y}_2 < b \text{ and } \bar{\delta} > 0 \\ \text{largest value such that } h(x) = h^+(\bar{y}_2) \text{ for} \\ \quad \bar{y}_2 - \gamma_h < x < \bar{y}_2 \text{ if } \bar{\delta} < 0 \\ \infty \text{ if } \bar{\delta} = 0 \text{ or if } \bar{y}_2 > b \text{ and } \bar{\delta} > 0, \end{cases} \quad (4.22)$$

and

$$\tilde{y}_1 = \min\{\alpha_q, \alpha_h, \bar{y}_1 + \frac{2(\beta_h - \bar{z})}{(1+\bar{\delta})}, \bar{y}_1 + \frac{\delta_h}{|\bar{\delta}|}\} . \quad (4.23)$$

Then $\tilde{y}_1 > \bar{y}_1$ and moreover, $d^+y_2/dy_1 = \bar{\delta}$ and $G(y_1) = \bar{\Delta}$ for $\bar{y}_1 \leq y_1 < \tilde{y}_1$.

Proof. Clearly $\bar{y}_1 < \tilde{y}_1$. Now, for $y_1 > \bar{y}_1$, $q^+(y_1)$ is a constant if $y_1 < \alpha_q$, $h^+(y_1)$ is a constant if $y_1 < \alpha_h$, and $h^+(z)$ is a constant if $z < \beta_h$. Hence, from (4.19), d^+y_2/dy_1 is then a constant. But in this case, $dz/dy_1 = (1/2)(1+\bar{\delta})$ or $z = \bar{z} + (y_1 - \bar{y}_1)(1/2)(1+\bar{\delta})$. Thus $z < \beta_h$ whenever $y_1 < \bar{y}_1 + 2(\beta_h - \bar{z})/(1+\bar{\delta})$. Noting (4.22), we get that $d^+y_2/dy_1 = \bar{\delta}$ when $\bar{y}_1 \leq y_1 < \tilde{y}_1$, and moreover, $y_2 = \bar{y}_2 + (y_1 - \bar{y}_1)\bar{\delta}$. But when $y_1 < \bar{y}_1 + \delta_h/|\bar{\delta}|$ as in (4.23), then $|y_2 - \bar{y}_2| < \delta_h$, or from (4.22), $h^+(y_2)$ remains a constant. Noting (4.21), this in turn means that $G(y_1)$ remains a constant and the proof is complete.

We now state an algorithm for solving Problem DCP in the case when $y_1 < y_2$. The other case is a mirror image of this one and is therefore omitted.

Algorithm DCP

Initialization. Determine $y_{1\min}$ and $y_{1\max}$ as in Lemma 4.9 and let

$$\bar{y}_1 = y_{1\min}.$$

Step 1. Compute $\bar{z}$ and $\bar{y}_2$ from (4.16) and determine $\tilde{y}_1$ as in (23). If

$\bar{y}_2 \leq b$, compute a zero $\hat{y}_1$ of the linear function

$$D(y_1) = D(\bar{y}_1) + (y_1 - \bar{y}_1)G(\bar{y}_1) \text{ in the interval } \bar{y}_1 \leq y_1 < \tilde{y}_1, \text{ if it exists,}$$

and also compute $\hat{\lambda} = G(\hat{y}_1)[d^+y_2/dy_1]$ at $y_1 = \hat{y}_1$. If a zero

$y_1 \in [\bar{y}_1, \tilde{y}_1]$ does not exist or if $\hat{\lambda} < 0$, go to Step 3.

Step 2. Compute $TC(\hat{y}_1, \hat{y}_2)$ and update the incumbent if necessary.

Step 3. If $\tilde{y}_1 > y_{1\max}$, stop with the incumbent as an optimal solution.

Otherwise, set $\bar{y} = \tilde{y}$ and return to Step 1.

4.5 An Example

Let interval $[0, b] = [0, 20]$ with demand density defined by

$$f^t(x) = \begin{cases} 35 + t & \text{if } 0 \leq x < 2 \\ 5 + t & \text{if } 2 \leq x < 12 \\ 25 + t & \text{if } 12 \leq x < 16 \\ t & \text{if } 16 \leq x \leq 20 \end{cases}, \quad t = 1, \dots, 5$$

and consider the problem of locating two facilities to serve $[0, b]$.

Further, suppose that the first facility is to be located in period $t_1 = 1$ and the second facility is to be located in period $t_2 = 3$. We will also assume that the demand in period $t = 5$ represents the long-range stationary demand.

Table 4.1(a) shows the optimal locations achieved by the strategies of Sections 4.2, 4.3, and 4.4. Here y_i represents the facility located in period t_i , $i = 1, 2$ and it was assumed that $1/(1+d) = 0.9$. A comparison of the single-period costs associated with the three different solutions appears in Table 4.1(b).

Note that as expected, the myopic strategy is preferred for the early periods while the long-range strategy is most efficient in the latter periods. The total discounted present worth cost (for periods

Table 4.1. Results of ExampleTable 4.1(a). Optimal Locations

Strategy	y_1	y_2
Myopic	10.0	1.3889
Long-range	13.6098	1.7073
Discounted	12.6302	1.5859

Table 4.1(b). Single-period Costs

Period	Myopic	Long-range	Discounted
1	1300.0	1397.613	1349.451
2	1400.0	1543.044	1456.369
3	750.556	498.586	521.699
4	820.058	555.879	580.607
5	889.560	613.171	639.516

$t = 1, \dots, 5$) is 4349.413 for the myopic strategy, 3997.745 for the long-range, and 3925.608 for the discounted case.

If the demand pattern remains stationary for all subsequent time periods after $t = 5$, then it is easily seen that for time periods after and including $t = 9$, the long-range strategy becomes preferable to the discounted strategy.

4.6 A Myopic Strategy for the Multiperiod 3-facility Problem on a Tree

In this section, we consider a myopic strategy when the underlying graph is an undirected tree $T = T(N, A)$ with discrete demands h_v^t associated with the nodes $v \in N$ in periods $t = 1, \dots, \tau$ and with uniform demand densities w_ℓ^t associated with the links $\ell \in A$ in periods $t = 1, \dots, \tau$.

As in Section 4.2, we wish to determine that facility location which minimizes present period costs in the presence of q existing facilities. In this regard, we will consider only those cases with $q = 0, 1$, or 2 .

The case $q = 0$ is of course the standard 1-median location problem on a tree as discussed in Section 3.3.

In the case when $q = 1$, we want to optimally locate a new facility on T in the presence of one existing facility. Making an observation similar to that in Section 3.3, we note that the new facility must lie on the path connecting some end node and the existing facility location. Thus as before, an optimal solution can be found by solving a subproblem on each such path. Of course, in this case paths are of the form $P(y_1, e_1)$, where y_1 denotes the existing facility location and $e_1 \in E$,

the set of end nodes of T . Also path subproblems are of the type P_r or P_ℓ in Section 4.2.

To reduce the number of such path subproblems, note that we may apply Reduction Algorithm 1 to each subtree which results when T is disconnected at y_1 . (Here y_1 plays the role of $\bar{y}$ in the algorithm.)

Before formally presenting an algorithm for the case $q = 1$ on a tree, let us make an additional observation. Consider some subtree T_j (with end node y_1) obtained by disconnecting T at y_1 . Clearly y_1 has only one incident link, say $\bar{\ell}$. Denote the length of $\bar{\ell}$ by $L_{\bar{\ell}}$ and let η represent that point on $\bar{\ell}$ such that $d(y_1, \eta) = L_{\bar{\ell}}/3$. Let H_η be the total demand weight of that subtree of T_j not containing y_1 when T_j is disconnected at η . Let $\hat{y}$ represent the sought optimal myopic solution on T_j , and note that $\hat{y}$ may serve at most H_η . Then, in the spirit of (4.12) and (4.13), a bound on the total decrease in cost realized by locating a new facility on T_j is $H_\eta \cdot d(y_1, \hat{y})$. However by Theorem 3.7 and Reduction Algorithm 1, we can eliminate portions of T_j from consideration as possible location sites for $\hat{y}$. Let T_j^R represent T_j after reduction and denote by $E(T_j^R)$ the set of end nodes of T_j^R . Then $d(y_1, \hat{y}) \leq \max_{e \in E(T_j^R)} d(y_1, e)$ and a bound B_j for ΔTC_j is given by

$$\Delta TC_j \leq B_j = H_n \cdot \max_{e \in E(T_j^R)} d(y_1, e) \quad (4.24)$$

An algorithm can now be stated for solving the myopic case on a tree T with $q = 1$.

Algorithm MT1

Step 1. Using the existing facility location, decompose T into $n \geq 2$ subtrees, perform a reduction on each subtree using Reduction Algorithm 1 and compute the bound B_j for each subtree, $j = 1, \dots, n$ as in (4.24). Let $S = \{B_1, \dots, B_n\}$.

Step 2. Let $B_k = \max_{B_j \in S} \{B_j\}$. Compute the optimal location $\hat{y}$ on T_k along with ΔTC_k . This is done by solving path subproblems of the type P_r or P_ℓ . Update the incumbent and the best savings ΔTC^* obtained thus far. Replace S by $S - \{B_k\} \cup \{B_j : B_j \leq \Delta TC^*\}$.

Step 3. If $S = \emptyset$; stop with the incumbent as an optimal solution. Otherwise, return to Step 2.

Now consider the case with $q = 2$ existing facilities, y_1 and y_2 , on T . If we decompose T by disconnecting at y_1 and y_2 , then since there is a unique path $P(y_1, y_2)$, there will be one subtree, say T_0 , with both existing facilities as end nodes, and all other subtrees being as described in the $q = 1$ case. Therefore, to solve the $q = 2$ case, we only need to discuss the additional problem of locating a new facility on a tree with two existing facilities located at distinct end nodes. Consider the following reduction theorem.

Theorem 4.11. Let y_1 and y_2 denote existing facility locations at two distinct end nodes of a tree $T_0(N_0, A_0)$. Let G represent the union of all subtrees rooted on $P(y_1, y_2)$. Consider any point $s \in G$ such that s is not an end node of T_0 . Let T_s be that subtree of T_0 contained in G which results when T_0 is disconnected at s . Treat s as a node of T_0 and hence an end node of T_s . Find a node $e(s)$ in T_s which is furthest away from the node s . Let M_1 denote the midpoint of the path $P(y_1, e(s))$ in T_0 and let M_2 denote the midpoint of the path $P(y_2, e(s))$ in T_0 . Denote by T_M that subtree of T_0 containing $e(s)$ which is obtained by disconnecting T_0 at M_1 and M_2 . If $M_1 \in P(M_2, e(s))$, treat M_1 as an end node of T_M . If $M_2 \in P(M_1, e(s))$, treat M_2 as an end node of T_M . Otherwise, treat both M_1 and M_2 as end nodes of T_M . By neglecting $T_0 - T_M$, optimally locate a single facility on T_M , and let y_M denote this location. Treat y_M as a node of T_M . Then an equivalent problem of locating a single new facility on T_0 results by performing the following reductions, designated $R_2(s, T_s)$ on T_0 :

- (a) If $y_M \notin T_s$, then collapse T_s into s .
- (b) If $y_M \in T_s$, then for each node $v \in P(y_M, M_1) \cap P(y_M, M_2)$, including y_M , if a subtree T_v of T_s is rooted at v on $P(y_M, M_1) \cap P(y_M, M_2)$ then collapse T_v into v .

Proof. The proof parallels that of Theorem 3.7 and is therefore omitted.

One implementation of the reduction strategy of Theorem 4.11 is as follows.

Reduction Algorithm 2

Initialization. Let s be the existing node of $P(y_1, y_2)$ which is closest to y_1 and has degree ≥ 3 . Determine a subtree T_s of T_0 , not containing y_1 and y_2 , which is obtained by disconnecting T_0 at s .

Step 1. Perform the reduction $R_2(s, T_s)$ of Theorem 4.11. Designate a point ξ as being either the point s or the midpoint M_2 of the path from y_2 to $e(s)$ defined in Theorem 4.11, whichever is further from y_2 .

Step 3. Find an original node k of T_0 closest to ξ on the path $P(y_2, \xi)$, which may be ξ itself such that when T_0 is disconnected at k , a subtree T_k of T_0 results which does not contain any reduced portion of T_0 . If no such node k exists, stop. Otherwise, set $r = k$, $T_r = T_k$, and return to Step 1.

Once an equivalent problem has been determined by Reduction Algorithm 2, it should be clear that if the new facility is to be located on T_0 , then it must be located on either $P(y_1, y_2)$ or on some remaining subtree rooted on $P(y_1, y_2)$. The optimal solution on $P(y_1, y_2)$ can be found by solving a problem of type $P_{\ell r}$ in Section 4.2, after collapsing all rooted subtrees into their respective nodes on $P(y_1, y_2)$.

On the other hand, let v be a node on $P(y_1, y_2)$ with a subtree T_v rooted at v . Then the optimal solution on T_v can be found by solving a path subproblem for each end node e of T_v as depicted in Figure 4.1. Of course, it is assumed here that the remainder of the tree is collapsed as described earlier. We will denote such a subproblem as P_e and the subtree in Figure 4.1 as T_e .

In Figure 4.1, b , c , and d represent actual measured distances from the end node e , which is assumed to be located at the origin. Without loss of generality, let us assume that $c \leq d$.

Problem P_e consists of optimally locating a new facility on the interval $[0, b]$ in the presence of existing facilities y_1 and y_2 . As were Problems $P_{\ell r}$, P_ℓ , and P_r , this problem is in general nonconvex.

Let y represent a facility location on $[0, b]$ as measured from the origin. Observe that if $0 \leq y \leq b - (c - b)$, where it is assumed that $b - (c - b) > 0$, then facility y serves only a portion of interval $[0, b]$ while y_1 and y_2 serve the remainder. However, if $b - (c - b) < y \leq b$, then y serves all of $[0, b]$ along with portions of $[b, c]$ and $[b, d]$. Of course, if $b - (c - b) \leq 0$, then y serves only as in the latter case. Let us denote the piecewise continuous demand density on $[0, b]$ as $f(\cdot)$, on $[b, c]$ as $f_1(\cdot)$ and on $[b, d]$ as $f_2(\cdot)$. Then the following theorem gives a necessary optimality condition for Problem P_e .

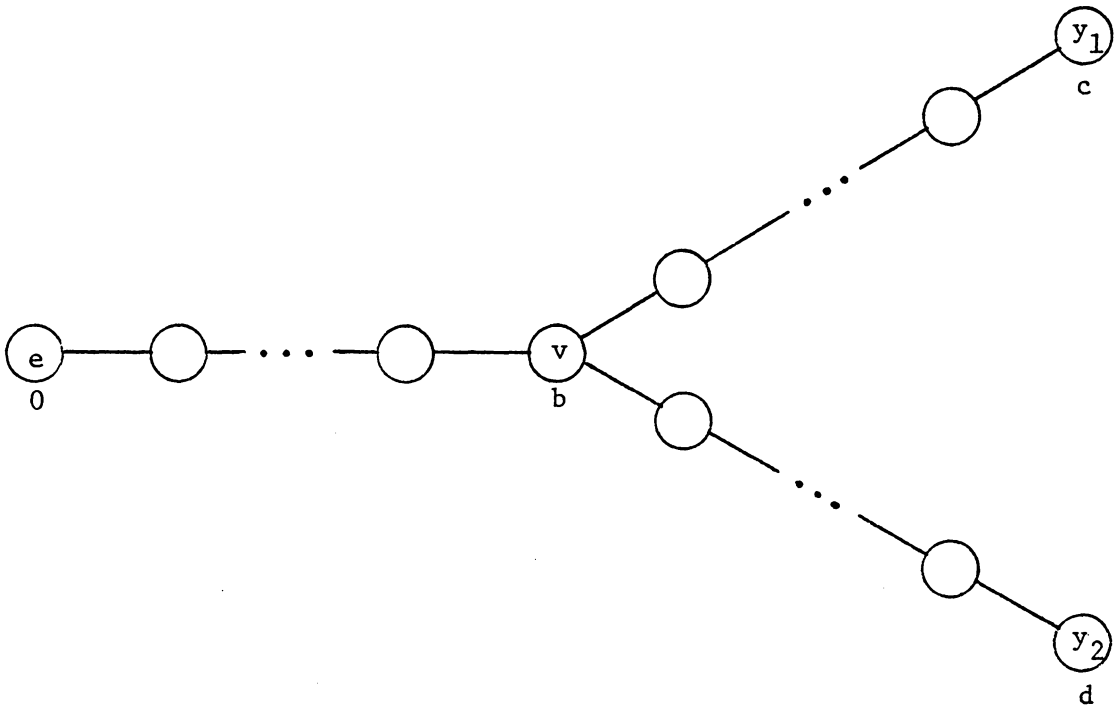


Figure 4.1. Subtree T_e

Theorem 4.12. Let y^* be an optimal facility location on $[0, b]$, and let $\gamma = \max\{0, b - (c-b)\}$, $z_c^* = (y^* + c)/2$, and $z_d^* = (y^* + d)/2$. Then,

(i) if $0 < y^* < \gamma$, it is necessary that y^* serves the interval

$[0, z_c^*] \subseteq [0, b]$, and furthermore, that

$$\int_0^{y^*} f(x)dx = \int_y^{z_c^*} f(x)dx \quad (4.25)$$

(ii) if $\gamma < y^* < b$, it is necessary that y^* serves

$[0, b] \cup [b, z_c^*] \cup [b, z_d^*]$, and furthermore, that

$$\int_0^{y^*} f(x)dx = \int_y^b f(x)dx + \int_b^{z_c^*} f_1(x)dx + \int_b^{z_d^*} f_2(x)dx \quad (4.26)$$

Proof. It should be clear that the region served by facility y^* is as defined by the theorem. Writing the total cost function $TC(y)$ for any $y \in [0, b]$ we get

(i) if $0 < y < \gamma$, then

$$\begin{aligned} TC(y) &= \int_0^y f(x)(y-x)dx + \int_y^{z_c^*} f(x)(x-y)dx \\ &+ \int_{z_c^*}^b f(x)(c-x)dx + \int_b^c f_1(x)(c-x)dx \\ &+ \int_b^{\frac{d-(c-d)-2b}{2}} f_2(x)(x-b+(c-b))dx \\ &+ \int_{\frac{d-(c-d)-2b}{2}}^d f_2(x)(d-x)dx \end{aligned}$$

(ii) if $\gamma \leq y \leq b$, then

$$\begin{aligned} TC(y) = & \int_0^y f(x)(y-x)dx + \int_y^b f(x)(x-y)dx \\ & + \int_b^{z_c} f_1(x)(x-y)dx + \int_b^{z_d} f_2(x)(x-y)dx \\ & + \int_{z_c}^c f_1(x)(c-x)dx + \int_{z_d}^d f_2(x)(d-x)dx. \end{aligned}$$

Taking the right and left hand derivatives with respect to y , it is readily verified that $TC(\cdot)$ is differentiable and that

(i) if $0 \leq y \leq \gamma$, then

$$\frac{dTC}{dy} = \int_0^y f(x)dx - \int_y^{z_c} f(x)dx \quad (4.27)$$

(ii) if $\gamma \leq y \leq b$, then

$$\frac{dTC}{dy} = \int_0^y f(x)dx - \int_y^b f(x)dx - \int_b^{z_c} f_1(x)dx - \int_b^{z_d} f_2(x)dx \quad (4.28)$$

Since $\frac{dTC}{dy} = 0$ is a necessary optimality condition, conditions (4.25) and (4.26) follow immediately from (4.27) and (4.28) respectively and the proof is complete.

Note again that at optimality, conditions (4.25) and (4.26) assert that the new facility serves an equal demand weight on either side.

Observe that in case (i), the necessary condition (4.25) is precisely that of Problem P_r (See (4.5).). Also in case (ii), note that

$$\frac{d^+}{dy} \left(\frac{dTC}{dy} \right) = 2f^+(y) - \frac{1}{2} (f_1^+(z_c) + f_2^+(z_d))$$

which is of the same form as (4.7). Thus Problem P_e admits to the same solution method as P_{lr} and P_r .

Hence, the myopic case when $q = 2$ can be solved as in Algorithm MT1 with appropriate modifications for handling the subtree T_0 . In this context, as far as a bound B_0 for the decrease in total cost ΔTC_0 on subtree T_0 is concerned, it may be readily verified that this bound may be obtained as follows. Ignoring the existing facilities at y_1, y_2 , find the optimal 1-median y_0 on T_0 and let θ represent the closest point to y_0 on $P(y_1, y_2)$. Denote by M_1 and M_2 the midpoints of $P(\theta, y_1)$ and $P(\theta, y_2)$ respectively. Let ℓ_1, ℓ_2 denote the links incident to y_1, y_2 respectively with lengths L_1 and L_2 . Define η_1 as that point on ℓ_1 such that $d(y_1, \eta_1) = L_1/3$ and similarly define η_2 such that $d(y_2, \eta_2) = L_2/3$. Let H_1 be the demand of that subtree consisting of $P(\eta_1, M_2)$ along with all subtrees rooted on $P(\eta_1, M_2)$ not containing y_1 or y_2 . H_2 is defined similarly. Then in the spirit of (4.10), $B_0 = d_0 \cdot \max\{H_1, H_2\}$, where

$$d_0 = \max_{y \in T_0^R} [\min\{d(y, y_1), d(y, y_2)\}],$$

with T_0^R being the tree which results from T_0 upon the application of Reduction Algorithm 2.

4.7 A Discounted Present Worth Strategy for the Multiperiod 2-facility Problem on a Tree

In this section, we consider a discounted strategy for the 2-facility problem on an undirected tree T as described in Section 4.5. We assume, as in Section 4.4, that facility locations occur in periods

$t_1 = 1$ and t_2 where $t_1 = 1 < t_2 < \tau$. For each $v \in N$, define

$$d_v^1 = \sum_{t=1}^{\tau} \alpha^{t-1} h_v^t, \quad d_v^2 = \sum_{t=t_2}^{\tau} \alpha^{t-1} h_v^t.$$

Also for each link $\ell \in A$, define

$$u_\ell^1 = \sum_{t=1}^{\tau} \alpha^{t-1} w_\ell^t, \quad u_\ell^2 = \sum_{t=t_2}^{\tau} \alpha^{t-1} w_\ell^t.$$

Again, we assume $u_\ell^1 > 0$, $u_\ell^2 > 0$ for all $\ell \in A$. Now since both

facilities must lie on the path connecting some two end nodes, we may again find the global optimum by solving problems of the type

(i) $y_1 < y_2$ and (ii) $y_2 < y_1$ described in Section 4.4 along each such

path. Then at most $|E|(|E|-1)$ subproblems need be solved. However, as before, we may reduce the number of path subproblems by performing

suitable reductions on T . Toward this end, let y^1 and y^2 denote the optimal 1-medians on T with respect to the demands defined by

$\{d_v^1, u_\ell^1\}$ and $\{d_v^2, u_\ell^2\}$ respectively. Treat y^1 and y^2 as (end) nodes.

Furthermore, let T^1 be the subtree of T composed of y^1 along with all subtrees rooted at y^1 except that subtree containing y^2 . Similarly,

define T^2 . Consider the following result.

Theorem 4.13. Let y_1^* and y_2^* denote optimal solutions to the discounted

2-facility problem on tree T . Then (a) if $y_1^* \notin T^1$, then y_1^* cannot serve

all of T^1 and (b) if $y_2^* \notin T^2$, then y_2^* cannot serve all of T^2 .

Proof. Consider (a). Since the total demand weight of T^1 with respect to $\{d_v^1, u_\ell^1\}$ is at least half the total demand, then if y_1^* serves all of T^1 , necessary condition (4.14) cannot be satisfied for a subproblem associated with any path $P(e_i, e_j)$ containing y_1^* and y_2^* , where $e_i, e_j \in E$. Similarly case (b) follows from (4.15) and the proof is complete.

Let V represent the set of nodes on $P(y_1^1, y_2^2)$ and denote by $E(T_k)$ the set of end nodes of subtree $T_k \subseteq T$ which are also end nodes of T . The following results, which are useful for eliminating paths from consideration, follow directly from Theorem 4.13 and are presented as corollaries.

Corollary 4.14. If y_1^* and $y_2^* \in T_0$, then $P(y_1^*, y_2^*)$ and $P(y_1^1, y_2^2)$ have a common link.

Corollary 4.15. Let T_u and T_v be two subtrees rooted at u, v respectively, where $u, v \in V - \{y_1, y_2\}$. And suppose $d(y_1^1, u) < d(y_1^1, v)$.

Let $e_i \in E(T_u)$ and $e_j \in E(T_v)$. Then y_1^* and y_2^* cannot be located on $P(e_i, e_j)$ such that $d(e_i, y_1^*) < d(e_i, y_2^*)$.

Note that Corollary 4.15 specifies that not both cases (i) $y_1 < y_2$ and (ii) $y_2 < y_1$ of Section 4.4 need to be solved for these paths. The following corollaries give similar results.

Corollary 4.16. Let T_v be a subtree rooted at $v \in V - \{y_1, y_2\}$, and suppose that $e_i \in E(T^1)$ and $e_j \in E(T_v)$ such that $d(y_1^1, v) > d(v, e_j)$.

Then y_1^* and y_2^* cannot be located on $P(e_i, e_j)$ such that

$$d(e_i, y_1^*) < d(e_i, y_2^*).$$

Corollary 4.17. Let T_v be a subtree rooted at $v \in V - \{y_1, y_2\}$, and

suppose that $e_i \in E(T_v)$ and $e_j \in E(T^2)$ such that $d(y^2, v) > d(v, e_i)$.

Then y_1^* and y_2^* cannot be located on $P(e_i, e_j)$ such that

$$d(e_i, y_1^*) < d(e_i, y_2^*).$$

Not let S_1 denote the set of subtrees of T^1 rooted at y_1^1 and let

S_2 denote the set of subtrees of T^2 rooted at y_2^2 . The following

theorems further eliminate subproblems.

Theorem 4.18. Let T_v be a subtree rooted at $v \in V$. Then both y_1^* and y_2^* cannot lie on T_v .

Proof. Suppose $y_1^*, y_2^* \in T_v \in S_1$. If $d(y^1, y_1^*) < d(y^1, y_2^*)$, then y_1^*

serves $T - T_v$ which contains at least half the total demand with

respect to $\{d_v^1, u_\ell^1\}$. And thus necessary condition (4.14) cannot be

satisfied for a subproblem associated with any path $P(e_i, e_j) \supseteq P(y_1^*, y_2^*)$

where $e_i, e_j \in E(T)$. If $d(y^1, y_1^*) > d(y^1, y_2^*)$, then Theorem 4.13(b) is

contradicted. Similarly $y_1^*, y_2^* \in T_j \in S_2$. Finally, if $y_1^*, y_2^* \in T_v \subseteq T_0$,

then Corollary 4.14 is contradicted. This completes the proof.

Theorem 4.19. Suppose $y_1^* \in T_k \in S_2$. Let $\bar{y}$ represent the optimal 1-median on $T - T_k$ with respect to demands $\{d_v^2, u_\ell^2\}$. Let $\bar{y} \in T_m$ where $T_m \in S_2$ or $T_m \subseteq T_0 \cup T^1$. Then for all $T_j \in S_2 - \{T_m\}$, $y_2^* \in T_j$.

Proof. First suppose $T_m \subseteq T_0 \cup T^1$. Clearly $y_2^* \notin T_k$ by Theorem 4.18.

Suppose $y_2^* \in T_j \in S_2 - \{T_k\}$. Note that $d(y_1^*, y^2) > d(y_2^*, y^2)$, else y_1^* would serve T^1 , which contradicts Theorem 4.13. Therefore y_2^* serves at least $T - \{T_k\}$ and by Lemma 3.6, y_2^* must lie on $P(\bar{y}, y^2) \subseteq T_0 \cup T^1$. This is a contradiction.

Next suppose $T_m \in S_2$. Again, $y_2^* \notin T_k$. Hence, suppose

$y_2^* \in T_j \in S_2 - \{T_m, T_k\}$. Then as before, y_2^* must serve at least $T - \{T_k\}$. Thus $y_2^* \in P(\bar{y}, y^2) \subseteq T_m$ and the proof is complete.

The proof of the following theorem parallels that of Theorem 4.19 and the theorem is therefore stated without proof.

Theorem 4.20. Suppose $y_2^* \in T_k \in S_1$. Let $\bar{y}$ represent the optimal

1-median on $T - T_k$ with respect to demands $\{d_v^1, u_\ell^1\}$. Let $\bar{y} \in T_m$ where $T_m \in S_1$ or $T_m \subseteq T_0 \cup T^2$. Then for all $T_j \in S_1 - \{T_m\}$, $y_1^* \notin T_j$.

To summarize, whenever a pair of end nodes $e_i, e_j \in E(T)$ are considered for solving subproblems, the following conditions may be checked. If $e_i, e_j \in T_0$ and if $P(e_i, e_j)$ and $P(y^1, y^2)$ do not have a

common link, the pair (e_i, e_j) may be disregarded by Corollary 4.14.

Otherwise in this case, Corollary 4.15 may be checked to ascertain if one of the two cases on $P(e_i, e_j)$ may be eliminated. Corollaries 4.16 and 4.17 provide similar eliminations of one of the pair of subproblems associated with $P(e_i, e_j)$, where $e_i \in E(T_0)$ and $e_j \in E(T^1)$ or $E(T^2)$.

Furthermore, Theorem 4.18 asserts that pairs of end nodes in the same subtree rooted at nodes on $P(y^1, y^2)$ need not be considered. Moreover, Theorem 4.19 and 4.20 eliminate one (or possibly both) of the pair of subproblems associated with paths connecting end nodes in either T^1 or T^2 , both of which need not be end nodes of the same subtree in S_1 or S_2 respectively, as in Theorem 4.18. In addition to these results, some further reductions along the following lines are possible. Suppose that we wish to solve case (i) subproblems along paths from $E(T_u)$ to $E(T_v)$, where T_u and T_v are subtrees rooted at $u, v \in V$ respectively. Since both y_1^* and y_2^* cannot belong to T_v , we can validly perform reductions (with

Reduction Algorithm 1) on T_v assuming demands defined by $\{d_v^2, u_l^2\}$,

pretending that y_1^* is at v , since y_2^* must balance demand weights with

respect to these demands. In a likewise manner, we may use Reduction

Algorithm 1 on T_u for case (i) subproblems, pretending $y_2^* = u$, and using

demands $\{d_v^1, u_l^1\}$. The reverse holds for case (ii) subproblems. Finally,

observe that the collapsing of a tree in order to solve path subproblems

yields certain fixed costs. These fixed costs along with some suitable bounds on the path subproblem may be used to eliminate additional subproblems by comparing lower bounds with known incumbent solutions.

CHAPTER V

EUCLIDEAN DISTANCE LOCATION-ALLOCATION PROBLEMS WITH UNIFORM DEMANDS OVER CONVEX POLYGONS

5.1 Introduction

Since their origination with the work of Cooper [1963, 1964], location-allocation problems have received considerable attention and are still an area of active research. Chapter II outlines and discusses several variations which have been considered in the literature.

The minisum location-allocation problem we consider involves a demand region which is a convex polygon having a continuous demand distribution, and which may also contain discrete demand points. A given number of facilities have to be located, and the subregion which each facility serves has to be determined, so as to minimize total transportation costs using a Euclidean distance measure.

Leamer [1968] has addressed this very same problem in the context of a demand region which is either a square, an equilateral triangle or a circle. A heuristic is employed in which facilities are perturbed one at a time until no further improvement in objective value results. No claim of either convergence or optimality is made. Rather, the principal observation is that as the number of facilities increase, the areas served by each facility tends to be hexagonal, except for distorting effects at the boundary of the region. Later in Section 5.6, we show that our algorithm generates better solutions for some of Leamer's test problems.

Single facility, Euclidean distance (pure) location problems in which the demand can occur either at discrete points or be uniformly

distributed over lines and rectangular regions, have been considered by several authors. Love [1972] solves such a problem using a nonlinear optimization technique, and Bennett and Mirakhor [1974] solve an approximation of this problem in which the area destinations are collapsed to points which are their respective centers of gravity. A more recent contribution which employs ℓ_p distances and permits more general area shapes than rectangles is addressed by Drezner and Wesolowsky [1980]. A two-stage iterative procedure based on the Weiszfeld [1937] technique is developed. However, no proof of optimality or convergence is provided. Although it appears that the method would converge, we do give an example in Section 5.3.4, in which a single facility has to be located to serve a triangle, to show that the algorithm may converge to a nonoptimal solution. Of course, the strength of the Drezner-Wesolowsky algorithm lies in that it provides an effective heuristic for very general single facility problems of this type.

Analogous problems using the rectilinear distance norm have also been considered. Wesolowsky and Love [1971] develop a gradient search algorithm to solve a multifacility version of the foregoing problem. A noteworthy contribution to the rectilinear distance case is due to Maruchek and Aly [1981]. They consider a location-allocation problem in which customer demands are characterized by bivariate uniform probability density functions over rectangular regions. A branch and bound algorithm is developed to minimize an expected weighted distance objective function. For other formulations and discussions on

continuous destination or demand problems, the reader is referred to Francis [1967a, 1967b].

This chapter is organized as follows. In the next section, we formulate the problem considered herein, and establish some basic properties of this problem. Thereafter, we consider a single facility version of the problem. An algorithm is described which is shown to converge to the global optimal solution to the problem. An efficient method for implementing the algorithm using a shifting-origin polar coordinate system is presented. We also show via an example that the only existing competing alternative algorithm may converge to a non-optimal solution. This analysis is then extended to the nonconvex multifacility problem. A convergent iterative heuristic is described along with techniques for efficiently implementing and accelerating the procedure. In order to enhance the likelihood of finding an optimal solution using this heuristic, a method for finding good starting solutions is described. Finally, we provide computational experience and comparisons.

5.2 Problem Formulation and Fundamental Characteristics

Consider a situation in which the demand for a given service or product is uniformly distributed over a convex polygon P with demand density w . Hence, the cost of serving an incremental area dA of P from a point at a Euclidean distance D from it is given by $wDdA$. In addition, suppose there exist m discrete demand points in P , represented according to some x - y coordinate system as $Q_k \equiv (a_k, b_k)$, $k=1, \dots, m$, with respective demand weights w_k , $k=1, \dots, m$. The interpretation of the

demand weights is that the associated cost of serving Q_k from a point at a Euclidean distance D from it is given by $w_k D$. The problem is to determine the locations $Z_i \equiv (x_i, y_i)$, $i=1, \dots, n$ of n uncapacitated facilities, and the corresponding allocations R_i , $i=1, \dots, n$ of subregions of P to each facility, and allocations S_i , $i=1, \dots, n$ of subsets of the discrete demand point set $\{1, \dots, m\}$ to each facility, so as to minimize total costs. Note that since the facilities are uncapacitated, each demand point will be allocated to the closest facility, using the Euclidean distance norm. Denoting the Euclidean distance between two points P_1 and P_2 as $d(P_1, P_2)$, the above location-allocation problem may be mathematically stated as

$$\begin{aligned} \text{LAP}(n, m): \quad & \text{minimize} \quad \sum_{i=1}^n \left[w \int_{R_i} \int d(Z_i, Z) dA + \sum_{k \in S_i} w_k d(Z_i, Q_k) \right] \\ & \text{subject to} \quad \bigcup_{i=1}^n R_i = P \end{aligned} \tag{5.1}$$

$$\bigcup_{i=1}^n S_i = \{1, \dots, m\} \tag{5.2}$$

where $Z \equiv (x, y)$ and $dA \equiv dx dy$.

From the viewpoint of ease in presentation, as well as to demonstrate a convenient method to prove convergence of algorithms based on the Weiszfeld [1937] iterative technique, we will for the most part address the problem in the absence of the discrete demand points. Our development of course extends readily to accommodate discrete demand points in P , and remarks to this order will be made at relevant points in the discussion. For the sake of reference, the area demand location allocation problem is stated below as the special case $\text{LAP}(n, 0)$

$$\underline{\text{ADLAP}}(n): \text{ minimize } \left\{ \sum_{i=1}^n \int_{R_i} \int d(Z_i, Z) dA : \text{ constraint (5.1)} \right\}$$

with $w \equiv 1$ without loss of generality.

Note that when $n = 1$, Problem LAP(1,m) becomes a pure location problem in that all demand points must be allocated to this single facility. Moreover, as shown by Love [1972], this problem is a convex programming problem. However, when $n \geq 2$, the problem is generally nonconvex and may have local optimal solutions different from global optimal solutions. In such a case, one can conceptualize a heuristic based on Cooper's [1964] approach of iterating between the location and allocation aspects of the problem. In other words, given a location of facilities, one may determine corresponding allocations R_i, S_i , $i=1, \dots, n$, using the fact that each demand point is served by the closest facility, and then one may relocate each facility i to best serve its allocated demand region $R_i \cup S_i$, $i=1, \dots, n$. This would continue until a fixed point of the iterative algorithm is obtained, if at all. Later in Section 5.4 we describe a convergent implementation of this approach. Toward the end of developing such a method, let us first of all characterize the demand allocations corresponding to a given set of facility locations.

Theorem 5.1. Let $\{Z_1, \dots, Z_n\} \subseteq P$ denote a set of locations for the facilities $1, \dots, n$, and assume that $\bar{Z}_i \neq \bar{Z}_j$ for $i \neq j$. Then

Problem LAP(n,m), with $Z_i = \bar{Z}_i$ fixed for $i=1, \dots, n$, determines the associated optimal area allocations $R_1, \dots, R_n$ as convex polygons with

non-empty interiors, and determines a set of optimal discrete point allocations $S_1, \dots, S_n$ satisfying $S_i \subseteq R_i$ for $i=1, \dots, n$.

Proof. Consider a facility i , and for each facility $j \neq i$, let

$c_{ij}Z = d_{ij}$ be the perpendicular bisector of the line segment joining $\bar{Z}_i$

and $\bar{Z}_j$, with $c_{ij}\bar{Z}_i < d_{ij}$ and $c_{ij}\bar{Z}_j > d_{ij}$. Define the sets

$$\tau_{ij} = \{Z: c_{ij}Z \leq d_{ij}\}, \quad j=1, \dots, n, \quad j \neq i \quad (5.3)$$

and note that $Z \in \tau_{ij}$ if and only if $d(\bar{Z}_i, Z) \leq d(\bar{Z}_j, Z)$, with the latter

inequality being strict if and only if $Z \in \text{int}(\tau_{ij})$. (Here, $\text{int}(\tau)$

denotes the interior of a set τ .) Consequently, given a set of facility locations, since Problem LAP(n,m) allocates each demand point to the closest facility, we must have

$$R_i = P \cap \left[\bigcap_{j \neq i} \tau_{ij} \right]. \quad (5.4)$$

Note that R_i is a convex polygon and moreover, $\text{int}(R_i) \neq \phi$ because

otherwise, since $\bar{Z}_i \in R_i$, it must be true that $c_{ij}\bar{Z}_i = d_{ij}$ for at least one $j \neq i$, a contradiction. (Furthermore, observe that by defining R_i

as in (5.4) above for each $i=1, \dots, n$, it follows that

$\text{int}(R_i) \cap \text{int}(R_j) = \phi$ for $i \neq j$, and $\bigcup_{i=1}^n R_i = P$, since it is impossible

for a point Z to satisfy $c_{ij}Z < d_{ij}$ and $c_{ij}Z > d_{ij}$, and since any $Z \in P$

must satisfy $d(\bar{Z}_i, Z) \leq d(\bar{Z}_j, Z)$ $j=1, \dots, n$ for some $i \in \{1, \dots, n\}$, which

would mean that $Z \in R_i$ from (5.3), (5.4).) Finally, since each discrete

demand point must get allocated to a closest facility (with ties broken

arbitrarily), any partition $S_1, \dots, S_n$ of $\{1, \dots, m\}$ is optimal in $LAP(n, m)$ if and only if $S_i \subseteq R_i$, $i=1, \dots, n$. This completes the proof.

Notice that in stating Theorem 5.1, we chose a candidate set of facility locations satisfying two properties. One that $\bar{Z}_i \in P$, $i=1, \dots, n$ and second that $\bar{Z}_i \neq \bar{Z}_j$ for $i \neq j$. The following two results justify this choice, and also provide some additional fundamental characterizations for the problem.

Theorem 5.2. Let $\{Z_1^*, \dots, Z_n^*\}$ denote an optimal set of facility

locations for Problem $LAP(n, m)$. Then $Z_i^* \in P$, $i=1, \dots, n$ and $Z_i^* \neq Z_j^*$,

$i \neq j$. Furthermore, if $m = 0$, then $Z_i^* \in \text{int}(P)$, $i=1, \dots, n$.

Proof. Let $\{Z_1^*, \dots, Z_n^*\}$ solve $LAP(n, m)$. Then since each facility i

must necessarily be serving some region of P at optimality, it must be

true that $Z_i^* \in P$ for otherwise, by the convexity of P , the objective

function can be strictly improved by moving Z_i to the (unique) point in

P closest to Z_i^* , and thereby making every point in P strictly closer to

this new location than to Z_i^* . Further, if $Z_i^* = Z_j^*$ for any $i \neq j$, then

since the facilities are uncapacitated, facilities i and j behave as

only a single facility and hence the objective function can be strictly

improved by moving one of the facilities to any point in P not occupied

by another facility.

Finally, consider the case $m = 0$. By Theorem 5.1, each Z_1^* is an optimal solution to a problem of the type LAP(1,0) (or ADLAP(1)) with the associated region R_1 playing the role of the convex polygon P .

Hence, to complete the proof, it is sufficient to show that Problem ADLAP(1): minimize $TC(Z_1) \equiv \int_P d(Z_1, Z) dA$ has an interior optimum.

Toward this end, consider any point $\bar{Z}_1$ on the boundary of P . Since P is a convex polygon there exists a supporting hyperplane $g(\bar{Z}_1 - Z) = 0$ of P at $\bar{Z}_1$ such that $g(\bar{Z}_1 - Z) \leq 0$ for all $Z \in P$ with the inequality being strict if $Z \in \text{int}(P)$, and such that $\bar{Z}_1 + \lambda g \in P$ for all $0 \leq \lambda \leq \delta$, for some $\delta > 0$. But, denoting the gradient of $TC(\cdot)$ by $\nabla TC(\cdot)$,

$$g \nabla TC(\bar{Z}_1) = \int_P \frac{g(\bar{Z}_1 - Z)}{d(\bar{Z}_1, Z)} dA < 0 \quad (5.5)$$

and so g is a feasible, improving direction at Z_1 . Hence Z_1 could not be optimal and the proof is complete.

Corollary 5.3. Let Z_1^* solve LAP(1,m). Then, Z_1^* belongs to the boundary of P only if $Z_1^* = Q_k$ for some $k \in \{1, \dots, m\}$.

Proof. Consider any point $\bar{Z}_1$ on the boundary of P such that $\bar{Z}_1 \neq Q_k$ for any $k \in \{1, \dots, m\}$. Denoting the total cost function of LAP(1,m) by

$$TC_m(Z_1) \equiv w \int_P d(Z_1, Z) dA + \sum_{k=1}^m w_k d(Z_1, Q_k),$$

and defining g as in the proof of Theorem 5.2 above, we get

$$g\text{VTCm}(\bar{Z}_1) = \int_P \frac{g(\bar{Z}_1 - Z)}{d(\bar{Z}_1, Z)} dA + \sum_{k=1}^m \frac{w_k g(\bar{Z}_1 - Q_k)}{d(\bar{Z}_1, Q_k)} < 0$$

and hence, $Z_1 = \bar{Z}_1$ cannot be an optimal solution. This completes the proof.

Theorem 5.4. Let $\{\bar{Z}_1, \dots, \bar{Z}_n\} \subseteq P$ be a set of facility locations such that $\bar{Z}_i \neq \bar{Z}_j$ for $i \neq j$. Let $\{\hat{Z}_1, \dots, \hat{Z}_n\}$ be obtained by first finding the optimal demand allocations $R_i, S_i, i=1, \dots, n$ corresponding to $\{\bar{Z}_1, \dots, \bar{Z}_n\}$ in $\text{LAP}(n, m)$, and then relocating each Z_i to optimally serve its associated demand allocation $R_i \cup S_i$, for $i=1, \dots, n$. Then,

$$\{\hat{Z}_1, \dots, \hat{Z}_n\} \subseteq P, \text{ with } \hat{Z}_i \neq \hat{Z}_j \text{ for } i \neq j.$$

Proof. From Theorem 5.1, notice that each Z_i is determined as an optimal solution to a problem of the type $\text{LAP}(1, |S_i|)$ with demand region R_i and with discrete demand points in R_i designated by S_i . Hence,

$\hat{Z}_i \in R_i, i=1, \dots, n$ and so $\{\hat{Z}_1, \dots, \hat{Z}_n\} \subseteq P$. Further, if $\hat{Z}_i = \hat{Z}_j$ for any $i \neq j$, then both $\hat{Z}_i$ and $\hat{Z}_j$ belong to the boundaries of their respective regions R_i and R_j . Hence by Corollary 5.3, $\hat{Z}_i = Q_{k1}$ for some

$k1 \in S_i$ and $\hat{Z}_j = Q_{k2}$ for some $k2 \in S_j$. But this means that $k1 = k2$

which contradicts that $S_i \cap S_j = \emptyset$ and this completes the proof.

The above results indicate that in an iterative implementation of Cooper's [1964] heuristic, one can commence with some distinct facility locations in P and determine the demand allocations as in Theorem 5.1. Each such demand allocation defines a single facility problem of the type $LAP(1, |S_i|)$, $i=1, \dots, n$. On solving these problems, a distinct set of facility locations within P are reobtained, and hence the procedure may be repeated until a fixed point results. In the following section, we develop a method to optimally locate a single facility in a convex polygon with a uniform demand distribution and with discrete demand points. Thereafter in Section 5.4, we present a convergent implementation of the above heuristic scheme.

5.3 A Convergent Algorithm for Optimally Locating a Single Facility

In this section we first develop an algorithm for Problem ADLAP(1). This facilitates the presentation as well as illustrates a convenient tool for proving convergence. Of course, both the implementation strategy as well as the convergence analysis readily extend to Problem LAP(1,m). Also, before proceeding we remark here that the algorithm due to Drezner and Wesolowsky [1980], although designed to accommodate more general demand patterns, is not applicable here. As demonstrated in Section 5.3.4, it may converge to a nonoptimal solution.

5.3.1 The Basic Algorithm and Its Convergence

For the sake of convenience, let us restate ADLAP(1)

$$\text{ADLAP}(1): \text{ minimize } TC(Z_1) = \int_P \int d(Z_1, Z) dA.$$

Love [1972] has shown that the objective function $TC(\cdot)$ is convex; in fact, it is readily shown to be strictly convex. Hence, since $TC(\cdot)$ is

bounded from below and is differentiable, its unique minimizer is given by the solution Z_1 to the equation

$$\nabla TC(Z_1) \equiv \int_P \int \frac{(Z_1 - Z) dA}{d(Z_1, Z)} = 0. \quad (5.6)$$

Note that a solution to (5.6) is not readily available and an iterative numerical procedure must be employed. Following the Weiszfeld [1937] technique, (5.6) can be rewritten as

$$Z_1 = \int_P \int \frac{Z dA}{d(Z_1, Z)} / \int_P \int \frac{1}{d(Z_1, Z)} dA. \quad (5.7)$$

Now, as usual, a strategy for solving (5.7) iteratively is as follows, where the superscript index indicates the iteration count. Letting $k = 1$ and selecting any $Z_1^1 \in P$, determine

$$Z_1^{k+1} = T(Z_1^k), \text{ where} \quad (5.8)$$

$$T(Z_1^k) \equiv \int_P \int \frac{Z dA}{d(Z_1^k, Z)} / \int_P \int \frac{1}{d(Z_1^k, Z)} dA.$$

Increment k by one and reapply (5.8) until for some iteration k ,

$Z_1^{k+1} = Z_1^k$ is obtained. (Realistically, one would terminate if

$d(Z_1^k, Z_1^{k+1}) < \epsilon$ for some tolerance $\epsilon > 0$.) The fact that such a fixed

point of the algorithmic map $T(\cdot)$ defined in (5.8) is an optimal

solution to ADLAP(1) is readily seen by rewriting this map using (5.6)

as

$$Z_1^{k+1} \equiv T(Z_1^k) = Z_1^k - \nabla TC(Z_1^k) / \int_P \int \frac{1}{d(Z_1^k, Z)} dA. \quad (5.9)$$

Hence, Z_1^k is a fixed point of $T(\cdot)$ if and only if $\nabla TC(Z_1^k) = 0$, i.e., if and only if it solves Problem ADLAP(1). Of course, the first issue is whether the above iterative procedure would converge to a fixed point of $T(\cdot)$. Kuhn [1973] and Ostresh [1978] prove the convergence of the Weiszfeld technique in the context of discrete demand points. Although a limiting case argument of the method of Ostresh [1978] in particular, can be given here, it is more instructive to construct a direct convergence proof based on Zangwill's [1969] Theorems. Toward this end, we first need to show that every iteration provides a strict decrease in objective function value, in case an optimal solution has not yet been obtained. Observe from (5.9) that Z^{k+1} is obtained from Z^k by taking a prescribed step length along the negative gradient direction. The following result asserts that this step guarantees a decrease in objective value.

Lemma 5.5. If $\nabla TC(Z_1^k) \neq 0$, then $TC(Z_1^{k+1}) < TC(Z_1^k)$.

Proof. Let us first rewrite (5.9) as

$$-\nabla TC(Z_1^k) = (Z_1^{k+1} - Z_1^k) \int_P \frac{1}{d(Z_1^k, Z)} dA. \quad (5.10)$$

Adding $2\nabla TC(Z_1^k)$ to both sides of (5.10) and using the fact from (5.9)

that $(Z_1^{k+1} - Z_1^k) \nabla TC(Z_1^k) < 0$, we get

$$(Z_1^{k+1} - Z_1^k) \left[(Z_1^{k+1} - Z_1^k) \int_P \frac{1}{d(Z_1^k, Z)} dA + 2\nabla TC(Z_1^k) \right] < 0. \quad (5.11)$$

Expanding and rearranging (5.11) and adding $\int_P d(Z, Z) dA / d(Z_1^k, Z)$ to both

sides, yields

$$\int_P \int \frac{d^2(z_1^{k+1}, Z) dA}{d(z_1^k, Z)} < \int_P \int \frac{d^2(z_1^k, Z)}{d(z_1^k, Z)} \equiv TC(z_1^k)$$

or

$$\int_P \int \frac{[d(z_1^{k+1}, Z) - d(z_1^k, Z) + d(z_1^k, Z)]^2}{d(z_1^k, Z)} dA < TC(z_1^k)$$

or upon further simplification,

$$2TC(z_1^{k+1}) + \int_P \int \frac{[d(z_1^{k+1}, Z) - d(z_1^k, Z)]^2 dA}{d(z_1^k, Z)} < 2TC(z_1^k)$$

and this completes the proof.

Next, one must show that the sequence $\{z_1^k\}$ generated by the algorithmic map $T(\cdot)$ is contained within a compact set. In fact here, as shown below, $\{z_1^k\} \subseteq P$.

Lemma 5.6. The sequence $\{z_1^k\}$ generated by $T(\cdot)$ lies in P .

Proof. Let P be defined by a set of inequalities $A_j Z \leq B_j$, $j=1, \dots, J$.

Then from (5.8), we have,

$$A_j z_1^{k+1} = \int_P \int \frac{A_j Z dA}{d(z_1^k, Z)} / \int_P \int \frac{1}{d(z_1^k, Z)} dA \leq B_j \int_P \int \frac{dA}{d(z_1^k, Z)} / \int_P \int \frac{dA}{d(z_1^k, Z)} = B_j$$

for each $j=1, \dots, J$. Hence, each iterate generated by $T(\cdot)$ lies in P and the proof is complete.

Theorem 5.7. Let $T(\cdot)$ be the algorithmic map defined by (5.8) and let Ω be the solution set $\{Z: T(Z) = Z\}$, namely, $\Omega = \{Z: \nabla TC(Z) = 0\}$. Given

z_1^1 , $k = 1$, generate a sequence $\{z_1^k\}$ iteratively as follows: If $z_1^k \in \Omega$, then stop; otherwise, let $z_1^{k+1} = T(z_1^k)$, increment k by 1, and repeat.

Then either the algorithm converges in a finite number of steps to a point in Ω , or else an infinite sequence $\{z_1^k\}$ is generated such that

$$\{z_1^k\} \rightarrow z^* \in \Omega.$$

Proof. Since the algorithmic map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a continuous function, Zangwill's convergence theorem (see Bazaraa and Shetty [1979], Theorem 7.2.3, p. 233) asserts that if any infinite sequence is generated, then every convergent subsequence of $\{z_1^k\}$ has a limit point in Ω , and $\{TC(z_1^k)\}$ approaches the optimal objective value. In fact, since Ω is a singleton in our case, the corollary to the theorem in Bazaraa and Shetty [1979] asserts that the entire sequence $\{z_1^k\}$ converges. This completes the proof.

5.3.2 Implementation of the Algorithmic Map $T(\cdot)$

We will now describe an efficient implementation of the algorithmic map $T(\cdot)$ defined in (5.8). Given $z_1^k \equiv (x_1^k, y_1^k)$, we begin by shifting the origin to z_1^k and then applying the algorithmic map $T(\cdot)$ to obtain, with respect to the new coordinate system,

$$\hat{x}_1^{k+1} = \int_P \int \frac{x dx dv}{\sqrt{x^2 + v^2}} / \int_P \int \frac{dx dv}{\sqrt{x^2 + v^2}} \equiv x_1^{k+1} - x_1^k$$

$$\hat{y}_1^{k+1} = \int_P \int \frac{v dx dv}{\sqrt{x^2 + v^2}} / \int_P \int \frac{dx dv}{\sqrt{x^2 + v^2}} \equiv y_1^{k+1} - y_1^k.$$

Switching over to polar coordinates by writing $Z \equiv (x, y) \equiv (r \cos \theta, r \sin \theta)$, the angle θ being measured counterclockwise with respect to the positive x-axis, say, and the distance r representing the distance of Z from the origin, we get

$$\hat{x}_1^{k+1} = \int_P \int r \cos \theta dr d\theta / \int_P \int dr d\theta \equiv x_1^{k+1} - x_1^k \quad (5.12)$$

$$\hat{y}_1^{k+1} = \int_P \int r \sin \theta dr d\theta / \int_P \int dr d\theta \equiv y_1^{k+1} - y_1^k \quad (5.13)$$

where, of course, P is now represented with respect to the new polar coordinate system. Let us now derive closed form expressions for (5.12) and (5.13). Note that Lemma 5.6 asserts that the present origin (Z_1^k) is a point in P . Hence, suppose that we construct lines connecting this origin to the vertices of P . This decomposes P into t triangles $T_1, \dots, T_t$, each triangle having a vertex at the origin (see Figure 5.1). Let the lengths a_i, b_i, c_i and the angles $\alpha_i, \beta_i, \phi_i$ characterize triangle $T_i, i=1, \dots, t$, as shown in Figure 5.1 where the angle ϕ_i is measured counterclockwise with respect to the positive x-axis, say. Then, if $\ell_i(\theta), \phi_i \leq \theta \leq \phi_i + \alpha_i$ denotes the distance from the origin to a point on the opposite side (of length a_i) of the triangle T_i as a function of θ , Equations (5.12) and (5.13) become

$$\hat{x}_1^{k+1} = \sum_{i=1}^t \int_{\theta=\phi_i}^{\phi_i+\alpha_i} \frac{\ell_i(\theta)^2}{2} \cos \theta d\theta / \sum_{i=1}^t \int_{\theta=\phi_i}^{\phi_i+\alpha_i} \ell_i(\theta) d\theta \equiv x_1^{k+1} - x_1^k \quad (5.14)$$

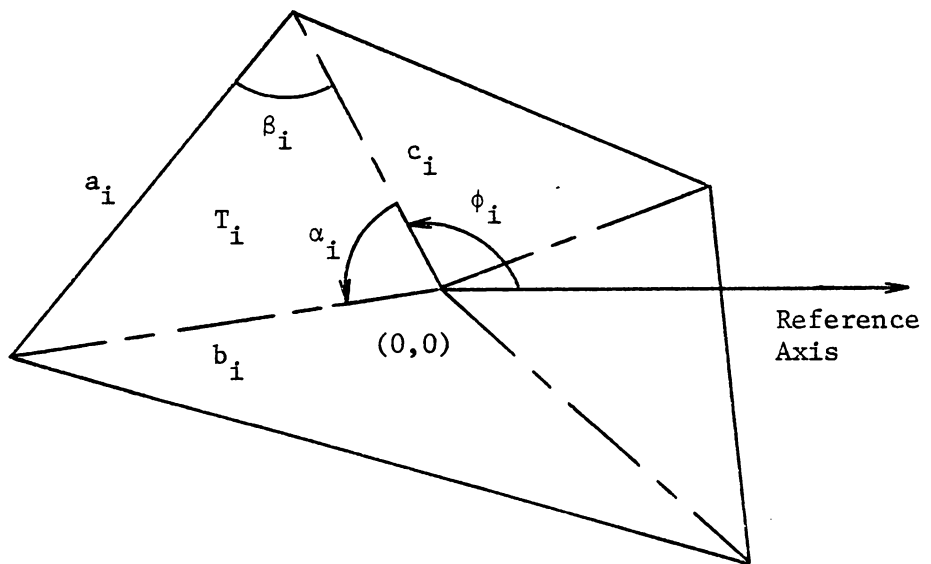
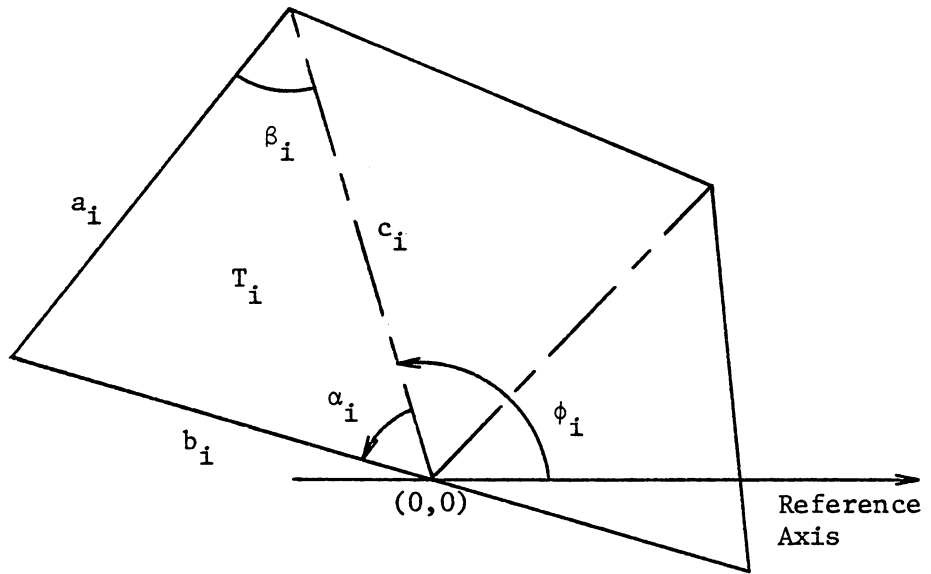


Figure 5.1. Alternative Decompositions of P

$$\hat{y}_1^{k+1} = \frac{\sum_{i=1}^t \int_{\theta=\phi_i}^{\phi_i+\alpha_i} \frac{\ell_i(\theta)^2}{2} \sin\theta d\theta}{\sum_{i=1}^t \int_{\theta=\phi_i}^{\phi_i+\alpha_i} \ell_i(\theta) d\theta} \equiv y_1^{k+1} - y_1^k. \quad (5.15)$$

Noting that the distance function $\ell_i(\theta)$ is given by

$$\ell_i(\theta) = \frac{b_i c_i \sin \alpha_i}{[b_i \sin(\phi_i + \alpha_i) - c_i \sin \phi_i] \cos \theta + [c_i \cos \phi_i - b_i \cos(\phi_i + \alpha_i)] \sin \theta},$$

$$\phi_i \leq \theta \leq \phi_i + \alpha_i$$

and using standard integral tables, Gradshteyn and Ryzhik [1980], we derive

$$x_1^{k+1} = \left\{ \sum_{i=1}^t U_i [N_i (c_i - b_i) + M_i V_i] / \sum_{i=1}^t V_i \right\} + x_1^k \quad (5.16)$$

$$y_1^{k+1} = \left\{ \sum_{i=1}^t U_i [M_i (c_i - b_i) + N_i V_i] / \sum_{i=1}^t V_i \right\} + y_1^k \quad (5.17)$$

where,

$$M_i = b_i \sin(\phi_i + \alpha_i) - c_i \sin \phi_i \quad (5.18)$$

$$N_i = c_i \cos \phi_i - b_i \cos(\phi_i + \alpha_i) \quad (5.19)$$

$$U_i^2 = b_i c_i \sin \alpha_i / a_i \quad (5.20)$$

$$V_i = \frac{b_i c_i \sin \alpha_i}{a_i} \ln \left\{ \frac{\tan[(\alpha_i + \beta_i)/2]}{\tan[\beta_i/2]} \right\}. \quad (5.21)$$

Hence to summarize, one begins with some starting facility location

$Z_1^k = (x_1^k, y_1^k) \in P$, with $k = 1$. Shifting the origin to Z_1^k and

decomposing P into triangles $T_1, \dots, T_t$ one uses the formulae (5.16) and

(5.17) to compute $Z_1^{k+1} \equiv (x_1^{k+1}, y_1^{k+1})$. If $d(Z_1^k, Z_1^{k+1}) < \varepsilon$ for some

tolerance $\varepsilon > 0$, then one terminates with Z_1^{k+1} as an optimal solution

(within a certain tolerance). Otherwise, one increments k by one and

repeats. Theorem 5.7 above asserts that for any specified $\epsilon > 0$, such a scheme would converge finitely. (Computationally, we used $\epsilon = 10^{-4}$.)

5.3.3 Extension to Include Discrete Demand Points

The foregoing analysis is readily extended to include discrete demand points as well. In lieu of (5.8), one now defines an algorithmic map $\tilde{T}(\cdot)$ as

$$z_1^{k+1} \equiv \tilde{T}(z_1^k) = \frac{\{w \int_P \frac{z dA}{d(z_1^k, Z)} + \sum_{j=1}^m \frac{w_j Q_j}{d(z_1^k, Q_j)}\}}{\{w \int_P \frac{dA}{d(z_1^k, Z)} + \sum_{j=1}^m \frac{w_j}{d(z_1^k, Q_j)}\}} \quad (5.22)$$

Note that this map is undefined if z_1^k coincides with some Q_j , $j \in \{1, \dots, m\}$. Hence, as in Kuhn [1973], it is possible that the algorithm may stop prematurely with z_1^k coinciding with a discrete demand point, and in this case, z_1^k would have to be perturbed and the iterative procedure restarted. However, as shown by Kuhn [1973], an infinite sequence tending toward a nonoptimal discrete demand point location cannot occur. Hence, Zangwill's convergence theorem again applies with the algorithm either stopping finitely, or else generating a sequence tending toward the optimal solution to the problem. Note that the implementation of the foregoing section applies here also, with the summations in the numerator and denominator of (5.22) being incorporated respectively into the numerators and denominators of (5.16) and (5.17).

Alternatively, in order to avoid the interruption when $z_1^k = Q_j$ for

some $j \in \{1, \dots, m\}$ one can redefine the algorithmic map $\tilde{T}(\cdot)$ as in Ostresh [1978] in this case so that $z_1^k = Q_j$ is a fixed point of the map in case a zero sub-gradient of the objective function exists at this point, and the algorithmic map takes an appropriate step along a negative subgradient direction otherwise.

5.3.4 Illustrative (Counter) Example

Consider the example depicted in Figure 5.2 in which the polygon P is a triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$ in the $x - y$ plane, and having a (unit) uniform demand distribution. The optimal solution in this case is the point $(0.3244, 0.3244)$. If one uses the algorithm of Drezner and Wesolowsky [1980], then starting with any nonoptimal point on the line $x = y$, Equation (8) of Drezner and Wesolowsky [1980] gives the indeterminate line as " $y_1 = x_1$ ". Hence, replacing the triangle with the equivalent single point on $y = x$ closest to the center of the rectangle bounding P , we obtain the solution $(1/2, 1/2)$. Repeating this argument, it is easily seen that $(1/2, 1/2)$ is a fixed point. Using our algorithm from, say, the point $z_1^1 = (0, 0)$, we get $t = 1$ (single triangle) with $\phi_1 = 0$, $\alpha_1 = \pi/2$, $\beta_1 = \pi/4$, $a_1 = \sqrt{2}$, $b_1 = c_1 = 1$, which gives from (5.20) - (5.23) that $M_1 = 1$, $N_1 = 1$, $U_1 = 1/2$, and $V_1 = (1/\sqrt{2}) (\ln[\tan(3\pi/8) \div \tan(\pi/8)])$. This gives from (5.16), (5.17) $z_1^2 = (1/2, 1/2)$. Now, $t = 2$ and we get two triangles T_1 and T_2 as shown in Figure 5.2. Here, shifting the origin to $(1/2, 1/2)$, we get $\phi_1 = 3\pi/4$, $\alpha_1 = \pi/2$, $\beta_1 = \pi/4$, $a_1 = 1$, $b_1 = c_1 = 1/\sqrt{2}$, $\phi_2 = 5\pi/4$, $\alpha_2 = \pi/2$,

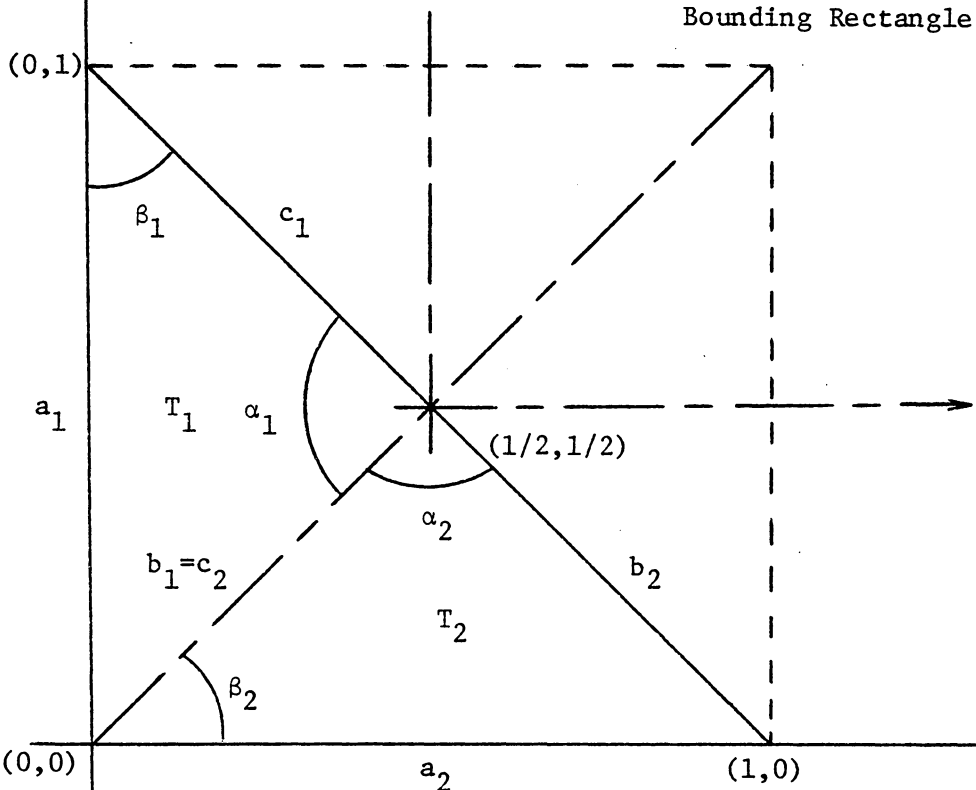


Figure 5.2. Illustrative Example

$\beta_2 = \pi/4$, $a_2 = 1$, $b_2 = c_2 = 1/\sqrt{2}$. Consequently, $M_1 = -1$, $M_2 = 0$, $N_1 = 0$, $N_2 = -1$, $U_1 = 1/2$, $U_2 = 1/2$, $V_1 = V_2 = (1/2) (\ln[\tan(3\pi/8) \div \tan(\pi/8)])$.

Consequently, (5.16), (5.17) give $Z_1^3 = (1/4, 1/4)$. One would now continue and obtain a sequence converging to (0.3244, 0.3244).

5.4 Convergent Iterative Heuristic for the Multifacility Problem

In this section, we begin by addressing the problem ADLAP(n), and then later extend it for the problem LAP(n,m). Note that for either problem, Theorem 5.2 asserts that an optimal solution must have a distinct set of facility locations. Further, given any set of distinct facility locations in P, Theorem 5.1 provides the demand allocations which, when fixed, result in n separable single facility problems of the type ADLAP(1) or LAP(1, |S_i|), i=1,...,n, as the case may be. Moreover, when the facilities are relocated with respect to these demand allocations, Theorem 5.4 asserts that a distinct set of new facility locations results. Below, we first discuss convergence issues, and then provide an efficient implementation scheme, and finally remark on extending this method to include discrete demand points.

5.4.1 The Basic Algorithm and Its Convergence

The iterative algorithm that is being proposed generates, as seen above, a sequence of locations $\{Z^k\}$, where $Z^k \equiv (Z_1^k, \dots, Z_n^k)$ is the kth iterate satisfying $Z_i^k \neq Z_j^k$ for $i \neq j$. Since $\{Z^k\} \subseteq P$, a compact set, it has a convergent subsequence. Let us first of all show that no such convergent subsequence can have a limit point $\bar{Z}$ in which some facility locations coincide.

Lemma 5.8. Let $\bar{Z}$ be a limit point of any convergent subsequence of the sequence $\{Z^k\}$ generated by the proposed algorithm. Then, $\bar{Z}_i \neq \bar{Z}_j$ for all $i \neq j$.

Proof. On the contrary, suppose that for some fixed $\Delta_1 > 0$, and some index set $J \subseteq \{1, \dots, n\}$ with $|J| \geq 2$, given any arbitrarily small $\epsilon > 0$, there exists an iterate Z^k such that $|Z_i^k - Z_j^k| < \epsilon$ for all $i, j \in J$ and $|Z_i^k - Z_\ell^k| > \Delta_1$ for all $i \in J, \ell \notin J$. Observe that the total demand area allocated to the facilities $i \in J$ is bounded away from zero and each facility $i \in J$ gets arbitrarily close to the boundary of the region it serves as $\epsilon \rightarrow 0$. Hence, by virtue of Theorem 5.2 (and the analogue of (5.5) for near-boundary points), whenever the facilities in J get sufficiently close to each other, at least one facility in the cluster J which serves a region of area bounded away from zero must move away from the cluster at the next iteration. In other words, for k large enough, and for some $\Delta_2 > 0$ which depends on the limiting configuration of the assumed subsequence, $|Z_i^{k+1} - Z_j^{k+1}| > \Delta_2$ for some $i, j \in J$. Since the only way in which such a cluster J could now form is if the regions allocated to $i \in J$ are adjacent and their areas approach zero, such a subsequence cannot exist, and this completes the proof.

Now, in order to use Zangwill's Convergence Theorems as in Section 5.3.1, let us describe the algorithmic map that is being employed. Unlike the map of Section 5.3.1, the present map is a composition of two maps, namely, one which yields the allocation given a location, and the

other which gives the new location corresponding to this allocation.

First, consider the allocation map. Given $Z^k = (Z_1^k, \dots, Z_n^k)$, let $F(Z^k)$ yield an optimal allocation $R^{k+1} \equiv (R_1^{k+1}, \dots, R_n^{k+1})$ via Problem ADLAP(n) with $Z_i = Z_i^k$ fixed. Note that since $Z_i^k \neq Z_j^k$ for $i \neq j$ by Theorem 5.4 (even in a limiting sense by Lemma 5.8), F is a point to point map for the iterates of our algorithm, and so we can write the functional relationship $R^{k+1} = F(Z^k)$. Next, given $R^{k+1} = (R_1^k, \dots, R_n^k)$, let $G(R^{k+1})$ yield optimal facility locations $Z^{k+1} \equiv (Z_1^{k+1}, \dots, Z_n^{k+1})$ via Problem ADLAP(n), for fixed area demand allocations R^{k+1} . Again, $\text{int}(R_i^k) \neq \emptyset$ for all iterates of our algorithm and so by Theorems 5.1 and 5.4 (even in a limiting sense by Lemma 5.8), G is a point to point map or a function, and we may write $Z^{k+1} = G(R^{k+1})$. The algorithmic map A is the composition $A = GF$. We emphasize that A is a point to point map as far as the iterates of our algorithm and their subsequence limits are concerned. We are interested in the fixed points of A .

Before proceeding, one feature of the map G should be pointed out. Namely, this map is comprised of single facility problems which are infinitely convergent as proven in Section 5.3.1. Here, however, G is treated as an oracle, i.e., given the region allocations, the corresponding optimal locations are assumed available via G . Realistically, the single facility problems would be terminated with some tolerance, and likewise, the overall algorithm would also be terminated using a perhaps less stringent tolerance.

We now prove the main result of this section. It may be noted that the proof of Zangwill's theorem (see Bazaraa and Shetty [1979], Zangwill [1969]) requires that the map A be closed at all accumulation points of the sequence generated. Usually, since this characterization is too specialized, one requires A to be closed at all points on the complement of the solution set. However, we do have through Lemma 5.8 a good enough characterization of such accumulation points, and we exploit this fact below.

Theorem 5.9. Consider the algorithmic map $A = GF$ defined above, and let the solution set Ω be the set of fixed points of A with distinct facility coordinate locations, namely, $\Omega = \{Z \equiv (Z_1, \dots, Z_n) : Z = A(Z), \text{ and } Z_i \neq Z_j \text{ for all } i \neq j\}$. Given $Z^1 = (Z_1^1, \dots, Z_n^1)$ with $Z_i^1 \neq Z_j^1$ for all $i \neq j$, generate a sequence iteratively as follows, starting with $k = 1$. If $Z^k \in \Omega$, stop, otherwise, let $Z^{k+1} = A(Z^k)$, increment k by one, and repeat. Then, either the algorithm stops in a finite number of steps, or else an infinite sequence $\{Z^k\}$ is generated such that $\{Z^k\} \rightarrow Z^* \in \Omega$.

Proof. Assuming that an infinite sequence $\{Z^k\}$ is generated, since $\{Z^k\} \subseteq P$ by Theorems 5.2, 5.4, it has a convergent subsequence. Let this be indexed by K and suppose that $\{Z^k\}_K \rightarrow \bar{Z}$. Consider the sequence $\{Z^{k+1}\}_K$, where $Z^{k+1} = A(Z^k)$, $k \in K$. Since $\{Z^{k+1}\} \subseteq P$, it has a convergent subsequence indexed by some set $K' \subseteq K$, say. Hence, $\{Z^{k+1}\}_{K'} \rightarrow \hat{Z}$, say, and of course, $\{Z^k\}_K \rightarrow \bar{Z}$. The proof of Zangwill's Theorem (see Bazaraa and Shetty [1979], Theorem 7.2.3, p. 233) applies

here if we can first of all show that $\hat{Z} = A(\bar{Z})$. Toward this end, consider the sequence $\{R^{k+1}\}_K$, where $R^{k+1} = F(Z^k)$, $k \in K'$. Note that R^{k+1} is determined as in Theorem 5.1 via a set of inequalities which are perpendicular bisectors of lines joining pairs of points in Z^k . By the continuity of the functional form of these bisectors, and since $\bar{Z}_i \neq \bar{Z}_j$ for $i \neq j$ by Lemma 5.8, it follows that $\{R^{k+1}\}_K \rightarrow \bar{R}$ such that $\bar{R} = F(\bar{Z})$. Now, for each $k \in K'$, $Z^{k+1} = G(R^{k+1})$ is a unique minimizer of a strictly convex function (determined by R^{k+1}). Since $\{R^{k+1}\}_K \rightarrow \bar{R}$ with $\text{int}(\bar{R}_i) \neq \emptyset$ for all i by Lemma 5.8 and Theorem 5.1, and since $\{Z^{k+1}\}_K \equiv \{G(R^{k+1})\}_K \rightarrow \hat{Z}$, it must be true that $\hat{Z}$ is an optimal location with respect to the area allocations $\bar{R}$, i.e., $\hat{Z} = G(\bar{R})$. Hence, $\bar{R} = F(\bar{Z})$ and $\hat{Z} = G(\bar{R})$, and so, $\hat{Z} = GF(\bar{Z}) \equiv A(\bar{Z})$.

Next, define $C(Z^k)$ to be the objective function value of ADLAP(n) with $Z_i = Z_i^k$ fixed. Clearly, $C(\cdot)$ is a continuous function. Hence, Zangwill's theorem applies if we can show that $Z^k \notin \Omega$ implies that $C(Z^{k+1}) < C(Z^k)$, for each iterate Z^k generated by the algorithm. Given Z^k , $C(Z^k)$ is evaluated by determining $R^{k+1} = F(Z^k)$ via ADLAP(n) with $Z_i = Z_i^k$ fixed. However, $Z^{k+1} \neq Z^k$ gives the unique optimal set of locations with respect to R^{k+1} , i.e., $Z^{k+1} = G(R^{k+1})$, and $C(Z^{k+1})$ is

the value of the best allocation with respect to $Z_i = Z_i^{k+1}$ fixed.

Hence, $C(Z^{k+1}) < C(Z^k)$. Therefore, Zangwill's Theorem concludes that every convergent subsequence of $\{Z^k\}$ converges to a point in Ω .

Finally, let us show that the entire sequence converges to a point in Ω . Toward this end, consider any convergent subsequence $\{Z^k\}_K \rightarrow \bar{Z}$ and for the sequence $\{Z^{k+1}\}_K$, consider any corresponding convergent subsequence $\{Z^{k+1}\}_{K'} \rightarrow \hat{Z}$, say. Now, as shown above, $\hat{Z} = A(\bar{Z})$. But $\bar{Z} = A(\bar{Z})$, and so, $\hat{Z} = \bar{Z}$. Since the choice of a convergent subsequence of $\{Z^{k+1}\}_K$ was arbitrary, we have $\{Z^{k+1}\}_K \rightarrow \bar{Z}$. Continuing in this fashion, we get $\{Z^{k+2}\}_K \rightarrow \bar{Z}$, $\{Z^{k+3}\}_K \rightarrow \bar{Z}, \dots$. Consequently, it is impossible to have another subsequence $\{Z^k\}_W$ with a limit point distinct from $\bar{Z}$, since such a subsequence would have to be distinct from all the foregoing sequences for k large enough. This completes the proof.

5.4.2 Remarks for the Case When Discrete Demand Points are Included

Here again, Lemma 5.8 holds since by the Corollary to Theorem 5.2 and by Theorem 5.4, the algorithm generates iterates having distinct facility coordinates, and the only facilities which may be located on boundary points of allocated subregions must coincide with some allocated discrete demand point. However, caution must be exercised in allocating discrete demand points to facilities in case of a tie. Namely, a discrete demand point once allocated to a facility should be

reallocated only when the closest facility to this demand point turns out to be at a strictly smaller distance than that from the facility presently serving it. Consequently, the map F is now defined as $(R^{k+1}, S^{k+1}) = F(Z^k, S^k)$, i.e., given facility locations Z^k and the previous discrete demand point allocation S^k , a new demand allocation is determined, using the above mentioned caution. Then, the map $G(R^{k+1}, S^{k+1})$ determines (Z^{k+1}, S^{k+1}) as the new set of facility locations Z_i^{k+1} which optimally serve $R_i^{k+1} \cup S_i^{k+1}$ for $i=1, \dots, n$. The map $A \equiv GF$ therefore determines $(Z^{k+1}, S^{k+1}) = A(Z^k, S^k)$. As in Theorem 5.9, one may now show that A is a closed point to point map at limit points of convergent subsequences of the sequence $\{Z^k, S^k\}$ generated by the algorithm, and that the continuous function $C(\cdot)$ satisfies $C(Z^{k+1}) < C(Z^k)$ in case $(Z^k, S^k) \notin \Omega \equiv \{(Z, S) : (Z, S) = A(Z, S), Z_i \neq Z_j \text{ and } S_i \cap S_j = \emptyset \text{ for all } i \neq j\}$. Hence, one may show that $\{Z^k, S^k\} \rightarrow \{Z^*, S^*\} \in \Omega$.

5.4.3 Efficient Implementation of the Algorithm

The present algorithm sequentially solves single facility problems, and allocates demand regions. For the former process, the expedients of Section 5.3.3 may be used and the latter operation is done in closed form as the proof of Theorem 5.1 describes. However, it turns out empirically that at some time in the overall algorithm, the iterates begin to converge along an approximately linear trajectory roughly according to a geometric series. This fact is exploited below to

accelerate the rate of convergence, by attempting to recognize when this begins to happen, and then by predicting the limit point of the sequence. Formally, for $k \geq 4$, the acceleration step may be described as follows. Let $D^k = (Z^k - Z^{k-1})$ be the step taken from Z^{k-1} to Z^k , and denote its i^{th} component $D_i^k = (d_i^k, e_i^k)$, $i=1, \dots, n$. Define a ratio vector $\gamma_i^k \equiv (d_i^k/d_{i-1}^k, e_i^k/e_{i-1}^k) \equiv (r_i^k, s_i^k)$, say, for $i=1, \dots, n$. Now, using the last three steps taken by the algorithm, if

$$\frac{1}{2n} \sum_{i=1}^n (|r_i^k - r_i^{k-1}| + |s_i^k - s_i^{k-1}|) < \delta \quad (5.23)$$

where δ is sufficiently small, then one may assert that the algorithm is converging along an approximately linear trajectory. An acceleration step may now be taken by computing new iterates according to the sum of an appropriate geometric series:

$$\tilde{x}_i = x_i^{k-1} + d_i^k \{(1 - (r_i^k)^p)/(1 - r_i^k)\}, \quad i=1, \dots, n \quad (5.24)$$

$$\tilde{y}_i = y_i^{k-1} + e_i^k \{(1 - (s_i^k)^p)/(1 - s_i^k)\}, \quad i=1, \dots, n \quad (5.25)$$

where p is the number of terms used in the series. Empirically, we

chose $\delta = 0.02$ and $p = 30$ if the ratio r_i^k or s_i^k was less than unity,

and $p = 15$ otherwise. If $\tilde{Z}_i \equiv (\tilde{x}_i, \tilde{y}_i)$, $i=1, \dots, n$ results in a

reduction in total cost, then we set $Z^{k+1} = \tilde{Z}$ and continue. Otherwise,

we use $Z^{k+1} = A(Z^k)$ as before. Computationally, we used the criterion

$||Z^{k+1} - Z^k|| < \sqrt{n} (10^{-4})$ to recognize a fixed point of $A(\cdot)$, where Z^k and

Z^{k+1} are treated as vectors in R^{2n} .

We conclude this section by making an observation of some interest. The analogue of ADLAP(n) on a line (instead of over a polygon) with a uniform demand density is a convex program (although not if there is more than one segment having a different uniform demand density on the line). Hence the added dimensionality in which the demand density is related to the area of the region rather than to the length of a line, appears to lend nonconvexity to the problem. In the following section, we provide a method for determining a good starting solution in order to enhance the likelihood of converging to a global optimum.

5.5 Obtaining a Starting Solution

The procedure described in this section has been derived through empirical observations over a variety of examples. For small values of n we try to make the facilities satisfy certain symmetry properties, and for larger values of n , we attempt to distribute the facilities to serve equal demand areas. The method proceeds as follows.

Step 1. If $n \geq 5$ or $n = 4$ and the polygon is a triangle, go to Step 6.

Step 2. Find the optimal single facility location Z^* , starting with the center of gravity of P . (If $n = 1$, terminate). If $n = 3$, go to Step 4. Otherwise, go to Step 3.

Step 3. If $n = 2$, construct a line through Z^* perpendicular to the longest edge of P . If $n = 4$, construct in addition a line through Z^* perpendicular to the first line. Go to Step 5.

Step 4. Construct a 120° tripod centered at Z^* with one leg parallel to the longest edge of P , in the direction which gives the longer projection onto this side.

- Step 5. For the 2, 3 or 4 subregions which P has been divided into, locate a single facility optimally in each subregion, and terminate.
- Step 6. Compute $\Lambda = (\text{area of } P)/n$. Pick the vertex v of P with the smallest subtended angle. Set $k = 0$, $P_0 = P$.
- Step 7. Shift the origin to v and rotate the x - y axes such that the positive x -axis is the bisector of the angle subtended at v in P_k . Obtain the coordinates of the vertices of P_k with respect to the new coordinate system (done simply through an appropriate subtraction and matrix premultiplication). Order these vertices in increasing x -coordinate values. Next, find a scalar $\theta = \bar{\theta}$ such that with $Z(\theta) \equiv P_k \cap \{Z = (x,y): x \leq \theta\}$, the area of $Z(\bar{\theta})$ is Λ . (This is done easily by accumulating areas, first of a triangle, and then of trapezoids as θ increases from zero, with a new trapezoid forming each time an ordered vertex of P_k is crossed.)
- Step 8. Locate an optimal single facility to serve $Z(\bar{\theta})$. (Represent this location in the original coordinate system.) Let $P_{k+1} = P_k \cap \{Z = (x,y): x \geq \bar{\theta}\}$, and increment k by one.
- Step 9. If $k = n$, terminate. If $k = n - 1$, denote P_{k+1} as " $Z(\bar{\theta})$ " and go to Step 8. Otherwise pick a vertex v of P_{k+1} which has the smallest subtended angle, and for which $x > \bar{\theta}$. In case of ties, pick the one with the largest x -coordinate. Return to Step 7.

5.6. Computational Experience

The single and multifacility algorithms described respectively in Sections 5.3 and 5.4 were coded in Fortran IV and implemented on an IBM 3081 series D24 computer, using the Fortran G1 compiler with the double precision option, under normal load batch operation. This section reports on the computational experience.

The (exact) single facility algorithm converged to a fixed point (within a tolerance of 0.0001 on the distance between consecutive iterates) extremely fast. We tried problems involving convex polygons having up to 15 sides, and the execution time was no more than 0.01 cpu seconds for any of the problems attempted.

Next, we tested the multifacility algorithm of Section 5.4 on two types of polygons, namely, a square with side of length 10 (as attempted by Leamer [1968]), and on a randomly generated heptagon. (Here, the seven vertices were generated with random polar coordinates, taking care that each vertex is at least at a distance of 10 units from the origin, and that the convex hull of the seven vertices generated is a heptagon.) Table 5.1 reports the experience with locating 3, 8, 13 or 15 facilities on each of these polygons. The algorithm was terminated when

$$||z^{k+1} - z^k|| < \sqrt{n} 10^{-4}.$$

The initial cost column in Table 5.1 gives the objective value of the starting solution generated by the method of Section 5.5. Table 5.1 shows that the value of the starting solution turns out to be within 3% to 4% of the final objective value obtained. Moreover, the effort for generating a starting solution was no more than 0.07 cpu seconds for any of the problems attempted. We also tried to generate starting solutions

randomly. At an average, the algorithm of Section 5.5 results in a 40.1% savings in overall effort.

Table 5.1 also shows the effect of using the acceleration technique described in Section 5.4.3. Except for the case with 8 facilities being located on a heptagon, there was generally a significant reduction in overall cpu effort. This saving ranges from 13% to 50% with an average savings of 20.6% (including the 8 facility on a heptagon case).

Finally, we attempted a comparison with the solutions obtained by Leamer [1968] for the case of a square polygon. Although Leamer presents his results only pictorially, we measured the facility coordinate locations in his solution using an enlargement of the figure in Leamer [1968]. The objective values of these solutions turned out to be 252.893, 137.759, 108.312 and 100.363 respectively for the case of 3, 8, 13 and 15 facilities. When used as starting solutions in our algorithm, the objective values improved to 235.617, 136.538, 107.351 and 98.584 respectively. We note that the facility configurations for the latter two solutions were visually quite different from those in the figures drawn by Leamer [1968].

Table 5.1. Computational Experience With the Multifacility Algorithm

Region	No. of Facil- ities (n)	Initial Cost	Final Cost	No. of Itera- tions	No. of Acceler- ation Steps	Execution Time CPU (secs)
Square	3	235.765	235.617	7	Not Used	0.15
			235.617	7	1	0.13
	8	138.101	136.538	37	Not Used	2.35
			136.484	21	2	1.38
	13	111.109	107.396	76	Not Used	9.42
			107.396	62	1	8.10
	15	102.626	100.130	60	Not Used	9.03
			100.130	50	1	7.78
7 Sided Polygon	3	520.941	515.500	46	Not Used	1.77
			515.500	23	2	0.89
	8	320.222	308.528	70	Not Used	6.36
			308.528	70	0	9.33
	13	248.058	240.471	57	Not Used	8.68
			240.471	35	1	5.84
	15	231.917	223.933	324	Not Used	55.65
			223.933	148	7	29.95

CHAPTER VI

CONCLUSIONS AND APPLICATIONS

6.1 Conclusions

A solution procedure which exploits the special structure of the necessary conditions was developed for the p-median problem on a chain graph with probabilistic link demands. This procedure enables one to find a global optimum to this nonconvex problem by identifying a discrete set of candidate points satisfying the necessary conditions. This is done by solving a series of trivial linear programming problems.

This methodology, along with various localization and reduction techniques, was then used to develop exact solution procedures for similar problems on trees and isolated cycle graphs.

Exact algorithms were then developed for dynamic, multiperiod extensions of the above problems with special attention given to the formulation of the objective function. Methods were developed to effectively handle myopic, long-range, and discounted present worth strategies.

Finally, a globally convergent algorithm was developed for the single facility problem involving uniform demands over convex polygons. Here, an efficient implementation of the Weiszfeld technique was employed which used a shifting-origin polar coordinate system. This analysis formed the basis for a convergent heuristic for the nonconvex multifacility case.

6.2 Extensions

Further research can be directed at investigating capacitated versions of the foregoing problems as well as versions which involve more general probability density functions.

Also, with regard to the network problems, considerations similar to those in Chapters III and IV can be addressed using a minimax rather than a minisum criterion.

Variations of the area problems which can be considered include using the rectilinear distance norm, minimizing the maximum total cost or the maximum distance associated with any facility, and minimizing the number of facilities subject to separation constraints.

6.3 Applications

Consider a situation in which emergency call boxes need to be located on a highway network. Since accidents may occur anywhere on the network, the incidence of accidents along the links may be suitably approximated by uniform probability density functions derived from transportation studies. The optimal locations to minimize the sum of expected travel distances may then be found by using one of the solution procedures developed herein.

The locations of emergency service facilities may be sought in a similar fashion. In this case, however, it may be more desirable to minimize expected travel times rather than distances.

The location of maintenance facilities for a highway network is another possible application.

Finally, in the instance when the region to be served is either large or permits roughly straight line travel paths, the Euclidean

distance measure may be a suitable approximation for distance traveled. Hence, a model such as that in Chapter V may be applicable when the demand is approximately uniform. A possible use would be the location of emergency medical facilities in military applications.

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