Forced Capillary-Gravity Water Waves in a 2D Rectangular Basin

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(ABSTRACT)

This dissertation concerns capillary-gravity surface waves in a two-dimensional rectangular basin that is partially filled with water. To generate the surface waves, a harmonic forcing is applied to the vertical side walls of the basin. The dissertation consists of four parts which work with different assumptions on the frequencies of the forcing.

The first part discusses the linearized model with Hocking's edge condition and gives an eigenvalue equation and an asymptotic expansion for the eigenvalues. Then, for the nonlinear problem, it is assumed that the frequency of the forcing is close to an eigenfrequency and the solution has an asymptotic expansion using a two time-scales approach. Under an edge condition, the first- and second-order approximations of the solution and a solvability condition from the third-order equations yield an ordinary differential equation for the amplitude of the solution.

In part two, it is assumed that the frequency of the forcing applied to the boundary is close to the sum of two eigenfrequencies. In this case, the solvability conditions give a system of two differential equations for the complex valued amplitudes of the two eigenmodes. The system can be reduced to one real-valued differential equation. Its solutions yield the solutions of the original system and their properties. A condition for the existence of homoclinic orbits connecting the trivial equilibrium is obtained. These results are confirmed by numerical experiments.

The third part is based on the results in the second part. Here, one of the eigenfrequencies is chosen to be much larger than the other one, and different orders of the amplitudes of the eigenmodes are assumed. The orders of the coefficients of the system found in the second part are obtained, and the resulting special case is discussed in detail. In particular, numerical examples of orbits that can be associated with homoclinic orbits connecting non-trivial equilibria are given. The behavior of solutions close to those orbits is demonstrated.

In the fourth part, an additional frequency for the forcing terms given in parts two and three is introduced. In each situation, the modified systems are presented and discussed.

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Chapter 1

Introduction

1.1 Previous work on capillary-gravity water waves in a basin

The theory of water waves has attracted scientists in fluid mechanics and applied mathematics for at least one and a half centuries and has been a source of intriguing - and often difficult - mathematical problems. Apart from being important in various branches of engineering and applied sciences, many water-wave phenomena happen in every-day experience. Waves generated by ships in rivers and waves generated by wind or earthquakes in oceans are probably the most familiar examples. The mathematical theory of water waves consists of the equations of fluid mechanics, the concepts of wave propagation, and the critically important role of boundary conditions. The results obtained from theory may give some explanation of a natural phenomenon or provide a description that can be tested whenever an expanse of water is at hand: a river or pond, the ocean, or simply the household bath or sink. However, obtaining a thorough understanding of the relevant physical mechanisms presents fluid dynamicists and applied mathematicians with a great challenge.

The problem considered here concerns linear and nonlinear surface waves on water under the action of a harmonic forcing together with the influence of gravity and surface tension in a two-dimensional rectangular basin. The frequency of the forcing is assumed to be close to a resonance frequency or some combination of the resonance frequencies. By “water”, we mean an incompressible, inviscid fluid with constant density with an irrotational fluid motion. It is assumed that the fluid is bounded by a free surface, two side walls oscillating under a harmonic forcing, and a horizontal rigid bottom. The free surface marks the boundary between the fluid and the “air” which we assume has negligible density and viscosity. We are interested in linear and nonlinear waves, called resonance waves, on the free surface and their mathematical description when the frequency of the forcing is close to one of the natural frequencies of the basin or some combinations of the natural frequencies.

The first research describing resonance waves caused by a forcing whose frequency was twice a resonance frequency was done by Michael Faraday [13]. He observed that when a water-filled basin is placed on a plane that oscillates vertically with approximately twice a resonance frequency, waves are generated on the surface. By introducing a coordinate system moving with the basin, we can interpret this problem as that of a stationary basin with oscillating gravity. In the literature, these waves are named "Faraday waves" after their first observer. By the 1990s, Faraday waves had been discussed by [4], [11], [14], [19], [27], [28], [35] and others. More recent publications include [25], [32], [34], [44].

While the phenomenon of Faraday waves may be one of the most famous, water wave phenomena in a basin had certainly been observed long before: Wang [42], describes an entertainment device which works with water waves and had been popular in ancient China. The device, known as "fish wash" or - in a variation - "dragon wash", was a water-filled basin with two handles attached on opposite sides of the rim. Rubbing the handles would cause horizontal oscillations of the side walls of the basin. At low frequencies, a standing wave would be observed on the surface. With increasing frequency, the free surface would break and resonance would be observed. Fishes or dragons were engraved on the bottom of the basins in a way that made it seem like the water was sputtering out of their mouths, thus the names "fish wash" and "dragon wash". The mechanical properties of this device were studied by Wang [42].

Wang [43] also showed that in a circular container a low-frequency axisymmetric wave could arise as a result of high-frequency forcing applied to the side walls. For his experiments, he used two circular, water-filled, steel alloy containers of different radii and attached vibrators to opposite sides of the containers. They can be seen as thin elastic shells with experimentally determined discrete resonance frequencies. Standing waves of a high resonance frequency corresponding to an asymmetric mode were observed when the forcing frequency was close to that high resonance frequency. If the forcing frequency was increased and close to the sum of that high frequency and the lowest frequency of axisymmetric modes waves of that low frequency were observed as well. Self-excited vibration of shell-liquid coupled systems were discussed by Liu et al. [23]. It was shown by Sun et al. [40] that for an asymmetric mode by itself the center of the surface is very calm, whereas the center of the region is where the maximum amplitude for an axisymmetric mode is observed.

These observations motivated more theoretical studies. An exact solution of the linearized problem of harmonic forcing applied to the side walls of a circular basin was found by Shen et al. [37], using Green's functions. The weakly nonlinear problem was discussed in [40] under the assumption that the forcing was close to a resonance frequency or twice the fundamental frequency. A multi-scale asymptotic expansion approach was used to find the equations governing surface waves in each of these cases. It was shown that the amplitude of such surface waves is bounded, except for when twice the fundamental frequency also happens to be a resonance frequency. This case is called internal resonance and is not included in [40].

In more general settings, waves generated by motion of the side boundaries of a basin were studied by Havelock [17] based on linear equations and the assumption of no surface tension. Surface tension was then accounted for by Evans [12] who also stated the importance of an "edge condition" for the amplitude of the surface wave at the intersection of the surface with the side boundary of the basin. Evans suggested that the normal derivative of the surface displacement at the boundary should be proportional to the applied forcing terms. This condition is known as "Evans' edge condition". Later, Hocking carried out some experiments which led to "Hocking's edge condition" [20]. Hocking's condition suggests that the normal derivative at the boundary is proportional to the time-derivative of surface displacement. In [33], Miles makes a strong statement in favor of Hocking's edge condition, because it resembles experimental observations better than Evans' edge condition.

To study this problem mathematically Shen and Yeh [38] used Green's functions and found an exact solution for the linear problem with Hocking's edge condition. They ran into problems when the forcing frequency was equal to a resonance frequency, but solved them in [39]. In [7] and [8], a nonlinear theory for capillary-gravity waves under Evans' and Hocking's edge conditions was developed. It was shown that a low-frequency axisymmetric eigenmode can be generated by applying a high-frequency asymmetric harmonic forcing to the side wall of a circular basin, which can be used to explain a recent experimental observation. Here, we note that the numerical calculations for this case with a circular basin involve the integration of Bessel functions which is sometimes hard to do accurately because Bessel functions for high eigenmodes have very strong oscillations. To avoid this difficulty we consider the case of a two-dimensional rectangular basin so that the functions for eigenmodes involve only trigonometric functions and can be integrated easily. Despite this simplification, the problem still captures the features of a three-dimensional circular basin.

Some research that deals with rectangular containers was done by Huntley [21] and Mahony and Smith [24]. In [24], the deep water problem for a rectangular organ pipe is considered. The flow field was assumed to be two-dimensional. It was shown that surface water waves may be excited by high-frequency acoustic fields. The work of [24] was confirmed by experiment [21]. More recently, Yoshimatsu and Funakoshi [45] investigated the resonance waves in square containers caused by horizontal oscillations. It was shown that the kind of waves observed depends on the angle of the direction of oscillation with a container wall. The resulting bifurcations were also discussed.

As we indicated above, we restrict our problem to two dimensions by considering a cross section of a rectangular cylinder. The resulting eigenvalue problem for the linearized equations is presented by Miles in [31]. The objective of this dissertation is to develop a weakly nonlinear theory by an asymptotic approach for excited capillary-gravity waves in a water-filled two-dimensional rectangular basin under some edge condition at the contact line. We assume that the forcing frequency is near a resonance frequency or some combination of the resonance frequencies. By using a two time-scale asymptotic expansion of the solution and solvability conditions for the equations of the third-order approximations in the expansion, the amplitude equations of the excited surface waves at the resonance frequencies are derived. Then, the solutions of the amplitude equations are studied in great detail and the solutions of these equations are also calculated numerically. Many interesting behaviors of the solutions are found. In particular, we find that in some cases the amplitude of the solution for the amplitude equations suddenly becomes very large when the forcing passes a critical value. Moreover, in general, for the frequency of the forcing near the sum of two eigenfrequencies the solutions exhibit chaotic behavior in the phase planes and the trajectories of solutions are dense in an annulus region. A more detailed overview of this dissertation is given as follows.

- In the remaining part of Chapter 1, we show the setting of the problem and the governing equations by following [31], but restrict ourselves to the case of a two-dimensional rectangular basin. All of the other chapters are based on those exact governing equations.

- In Chapter 2, we first consider the linearized equations with Hocking's edge condition. We assume that the velocity potential as well as the surface displacement can be written as a product of $e^{i\omega t}$ and a function of the spatial variables. An equation for the eigenvalues ω is derived, see [31]. Because this equation contains an infinite sum of expressions of ω, we expect an infinite number of solutions. It is impossible to express the solutions explicitly. We approximate the first ten eigenvalues for a specific set of parameters. Then, for the deep-water problem, we find an asymptotic expansion for large eigenvalues. This allows us to find an asymptotic expansion for the norm of the surface displacement function.

  Using a simplified edge condition, we consider the nonlinear governing equations with a forcing of third order whose frequency is near an eigenfrequency. Under the assumption that the velocity potential and the surface displacement are both small and of the same order, we use a two time-scale approach and assume an asymptotic expansion for both velocity potential and surface displacement. After presenting the equations for the first- to third-order approximations, we calculate the first- and second-order approximations. The first- and second-order approximations both contain a complex-valued function $p(\tau)$ of the slower time. A differential equation for $p(\tau)$ is obtained by making use of a solvability condition from the third-order equations. In addition, the simplified edge condition is replaced with some special cases of Hocking's edge condition. The modifications to the differential equation induced by this replacement are discussed.

- Chapter 3 deals with a forcing of second order whose frequency is close to the sum of two eigenfrequencies ω_m and ω_n. First, we state the modifications that have to be made to the equations in Section 2.3. The first-order approximation is assumed to be a sum of terms with frequencies ω_m and ω_n containing complex-valued functions $p_m(\tau)$ and $p_n(\tau)$ dependent on the slower time scale. For the second-order approximation, three cases depending on the form of the forcing terms are distinguished. Then solvability conditions from the third-order equations are used to obtain two coupled differential equations for $p_m(\tau)$ and $p_n(\tau)$. In [22], Knobloch et al. have shown that those differential equations can be reduced to one differential equation for $|p_m|^2$. A similar approach yields a differential equation for $\rho = \frac{\beta_1}{2}|p_m|^2 + \frac{\beta_2}{2}|p_n|^2$. All solutions of this differential equation are obtained. Most of the solutions are periodic, the other ones can be interpreted as homoclinic orbits. In the case of a periodic solution, it is shown that the radii of p_m and p_n have the same period as ρ and that the angles of p_m and p_n are sums of linear functions and functions that have the same period as ρ. The behavior of the radii and angles for other cases is also discussed. Next, we find the equilibria of the system for the case that the forcing frequency is exactly equal to the sum of two eigenfrequencies. For other cases, some periodic orbits of the system are found. There are no heteroclinic orbits, but in some cases there are homoclinic orbits. A condition for homoclinic orbits connecting the trivial equilibrium is established. The discussion of other homoclinic orbits is delayed to Chapter 4. A numerical experiment is used to confirm the predictions we have made.

- Up to now, we have assumed that the frequencies ω_m and ω_n both are assumed to be $O(1)$. In experiments, one of the frequencies, say ω_n, is typically much larger than the other one. In Chapter 4, we do not assume that the amplitudes of both frequencies have the same order. The orders of the frequencies, the order of the forcing and the order of the slower time scale τ are written as powers of ϵ. First, we assume absence of surface tension and a simplified edge condition. Then, we introduce a change of variables so that it is easier to compare the orders of the terms that have to be estimated later. After writing down the new equations for the first- to third-order approximations, we solve the equations for forcing of a special form. The solvability conditions from the third-order equations yield differential equations for the amplitude functions. These are similar to those in Chapter 3, but we have much more information about the orders of the coefficients. Then the exponents introduced at the beginning of this chapter are chosen in an appropriate way to obtain the model equations that are discussed in the second part of this chapter.

  We discuss what happens in absence of forcing and how the exponents need to be chosen in order to have the high-frequency amplitude smaller than the low-frequency amplitude. Then we state how large surface tension can be before inducing any changes to the coefficients of our model equations. The simplified edge condition is replaced by a variation of Hocking's edge condition. If there is no forcing, the differential equations can be solved exactly. From this, the behavior of the solutions of the system for small forcing is predicted. After finding the equilibria and periodic orbits for this system, we try to obtain orbits that converge to circular orbits. These orbits correspond to homoclinic orbits connecting non-trivial equilibria of a transformation of the system. Using a numerical example, it is determined when these orbits are likely to occur. We also see the behavior of the solutions for initial conditions and parameters close to those orbits.

- In Chapter 5, an additional forcing of frequency ω_n is introduced. For the system in

Chapter 3, the equations are modified in an appropriate way and the discussion of equilibria and periodic orbits is repeated. A numerical example is used to show how these calculations work.

For the system in Chapter 4, the equations are also modified in a suitable way. With the restriction of having forcing of frequency ω_n only, the solution for small parameters ϵ is approximated by perturbing the solution for $\epsilon = 0$. In [40], a critical value for the forcing is found. In numerical examples, the changes in the solution are shown, when the forcing parameter crosses the critical value.

1.2 Basic formulation

In setting up and describing the problem, we follow Miles [31], but restrict ourselves to the special case of a two-dimensional rectangular basin. Waves are considered in a region $[b_1, a+b_2] \times [-h, \Theta(x,t)]$, where $\Theta(x,t)$ is the displacement of the free boundary in z-direction. The bottom boundary is fixed, the top boundary is free and for the side boundaries we allow some forcing terms that can move the boundary.

Let $u(x,z,t)$ and $v(x,z,t)$ be the time-dependent velocities in x- and z-direction, respectively. As there is no liquid created or destroyed, the mass flux through any fixed closed line S enclosing a region has to vanish.

Let $f(x,z,t) = (u(x,z,t), v(x,z,t))$. Assuming a constant density of the fluid ρ, we then have

$$\int_S \rho f_n dS = 0\,, \tag{1.1}$$

where n is a vector normal to the line.

Using Gauss's divergence theorem on this, we obtain

$$\iint_R div(\rho f) dr = 0 \tag{1.2}$$

for any region $R \subset [b_1, a+b_2] \times [-h, \Theta]$. We assume incompressibility of the fluid. As the integral of the divergence over any region R has to vanish, the divergence of f has to vanish identically:

$$div(f(x,z,t)) = u_x(x,z,t) + v_z(x,z,t) = 0\,. \tag{1.3}$$

This is also known as the law of conversation of mass.

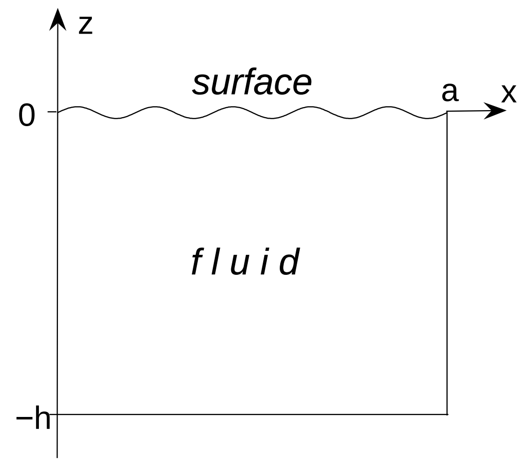

Figure 1.1: Setting

Furthermore, the flow of the liquid is assumed to be irrotational, i.e. the circulation for all closed curves is zero. This means

$$u_z(x,z,t) = v_x(x,z,t)\,. \tag{1.4}$$

Thus we can find a function $\phi(x,z,t)$, called the velocity potential, such that

$$f(x,z,t) = grad\,(\phi(x,z,t)) = (\phi_x(x,z,t), \phi_z(x,z,t))\,. \tag{1.5}$$

Using (1.3), we know that $\phi(x,z,t)$ has to satisfy the Laplace equation

$$\triangle\phi(x,z,t) = 0 \qquad \text{for}\,(x,z) \in [b_1, a+b_2] \times [-h, \Theta], t > 0\,, \tag{1.6}$$

where $\triangle\phi = \phi_{xx} + \phi_{zz}$.

Now, consider the boundaries. Assuming that the bottom of the basin is fixed and the sides of the basin are displaced with $b_1(z,t)$ and $b_2(z,t)$, respectively, the fluid at the boundary should be moving at the same speed as the boundary itself. For the left boundary

$$\begin{aligned} \frac{\partial}{\partial t}(x - b_1) &= 0 \quad \text{at } x = b_1 \\ \Rightarrow \quad \phi_x - b_{1,t} - b_{1,z}\phi_z &= 0 \quad \text{at } x = b_1 \,. \end{aligned} \tag{1.7}$$

Similarly, for the right boundary

$$\begin{aligned} \frac{\partial}{\partial t}\left(x - (a + b_2)\right) &= 0 \quad \text{at } x = a + b_2 \\ \Rightarrow \quad \phi_x - b_{2,t} - b_{2,z}\phi_z &= 0 \quad \text{at } x = a + b_2 \,. \end{aligned} \tag{1.8}$$

The bottom boundary is rigid, thus

$$\begin{aligned} \frac{\partial z}{\partial t} &= 0 \quad \text{at } z = -h \\ \Rightarrow \quad \phi_z &= 0 \quad \text{at } z = -h \,. \end{aligned} \tag{1.9}$$

Now consider the top free surface boundary. The particles on the boundary have to move at the same speed as the boundary itself.

$$\begin{aligned} \frac{\partial}{\partial t}(\Theta - z) &= 0 \quad \text{at } z = \Theta \\ \Rightarrow \quad \Theta_t + \Theta_x\phi_x - \phi_z &= 0 \quad \text{at } z = \Theta \,. \end{aligned} \tag{1.10}$$

Bernoulli's law yields

$$g\Theta + \phi_t(x,\Theta) + \frac{1}{2}\left(\phi_x(x,\Theta)^2 + \phi_z(x,\Theta)^2\right) + \frac{p(x)}{\rho} = 0 \,,$$

where $p(x)$ is the pressure given by

$$p(x) = s\left(\frac{\Theta_x}{\sqrt{1 + \Theta_x^2}}\right)_x \,, \qquad s = \text{const.}$$

and g is the gravity constant.

Introducing $T = \frac{s}{\rho}$ as the ratio of surface tension to density, the two previous equations together become

$$\phi_t + g\Theta + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_z^2 - T\frac{\Theta_{xx}}{(1 + \Theta_x^2)^{3/2}} = 0 \,.$$

Using the Taylor expansion of the denominator - only the first term is needed - yields

$$\phi_t + g\Theta + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_z^2 - T\Theta_{xx}\left(1 - \frac{3}{2}\Theta_x^2\right) = 0 \qquad \text{at} \quad z = \Theta \,. \tag{1.11}$$

Hocking's edge condition can be stated as

$$\begin{aligned} c\Theta_x &= \Theta_t \quad \text{for } x = a + b_2\,, \\ c\Theta_x &= -\Theta_t \quad \text{for } x = b_1 \end{aligned} \tag{1.12}$$

for some experimentally determined constant c. This condition means that the movement of the free surface at the side boundary is proportional to its normal derivative. In other words, the faster the surface at the side boundary is moving, the steeper it is.

The equations (1.6) to (1.12) together make up the governing equations on which the upcoming chapters are based.

Chapter 2

Linearized equations and simplified edge conditions

2.1 Introduction

In this chapter, the governing equations are linearized. Following [31], an equation for the eigenvalues is found. This equation has infinitely many solutions. Approximations for the ten smallest solutions for a specific set of parameters are presented, and an asymptotic expansion of the eigenvalues for the deep-water problem is derived.

Based on a simplified edge condition, the nonlinear equations are discussed for forcing of third order with a frequency that is close to an eigenfrequency of the system. It is assumed that both the velocity potential and the surface displacement are of first order and have asymptotic expansions. Using a two time-scale approach, the first- and second-order approximations which contain a slowly varying function $p(\tau)$ are found. Then the third-order equations yield a solvability condition that is used to obtain a complex-valued differential equation for $p(\tau)$. It is shown how the coefficients of this differential equation change if the simplified edge conditions are replaced with special cases of Hocking's edge condition.

2.2 Eigenvalue equations

2.2.1 Linearized equations and derivation of eigenvalue equations

We linearize equations (1.6) to (1.12) and assume

$$\phi(x,z,t) = e^{i\omega t}\Phi(x,z) \qquad \text{and} \qquad \Theta(x,t) = e^{i\omega t}\zeta(x)\,.$$

Thus $\phi_t(x,z,t) = i\omega\phi(x,z,t)$ and $\zeta_t(x,t) = i\omega\zeta(x,t)$. Cancelling the factor $e^{i\omega t}$ in each equation and assuming that b_1 and b_2 contain higher-order terms only, the linearized equations for $\Phi(x,z)$ and $\zeta(x)$ are

$$\begin{aligned}
\Delta\Phi(x,z) &= 0 \quad \text{for} \quad (x,z) \subset [0,a]\times[-h,0]\,, && (2.1)\\
\Phi_x(x,z) &= 0 \quad \text{for} \quad x=0, x=a\,, && (2.2)\\
\Phi_z(x,z) &= 0 \quad \text{for} \quad z=-h\,, && (2.3)\\
\Phi_z(x,z) &= i\omega\zeta(x) \quad \text{for} \quad z=0\,, && (2.4)\\
i\omega\Phi(x,z) + g\zeta(x) &= T\zeta_{xx}(x) \quad \text{for} \quad z=0\,, && (2.5)\\
c\zeta_x(x) &= i\omega\zeta(x) \quad \text{for} \quad x=a\,, && (2.6)\\
c\zeta_x(x) &= -i\omega\zeta(x) \quad \text{for} \quad x=0\,. && (2.7)
\end{aligned}$$

Let $k_n = \frac{n\pi}{a}$ and $\xi_n(x) = \cos(k_n x)$. The Fourier transforms of Φ and ζ are then given by

$$\begin{aligned}
\Phi_n(z) &= \int_0^a \Phi_n(x,z)\xi_n(x)dx\,, && (2.8)\\
\zeta_n &= \int_0^a \zeta(x)\xi_n(x)dx\,. && (2.9)
\end{aligned}$$

The inverse Fourier transforms are

$$\Phi(x,z) = \sum_{n=1}^{\infty} \frac{\Phi_n(z)\xi_n(x)}{N_n}\,, \qquad (2.10)$$

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{\zeta_n\xi_n(x)}{N_n}\,. \qquad (2.11)$$

Here, $N_n = \int_0^a \xi_n^2(x)dx$ is a normalization factor. With $\xi_n(x) = \cos(k_n x)$, we have $N_n = \frac{1}{2}a$. For the Fourier transformation of the equations, we have to use integration by parts. Taking into account that the boundary terms vanish and that we have $\xi_{n,xx}(x) = -k_n^2\xi_n(x)$, we get

$$\int_0^a \xi_n(x)\Phi_{xx}(x,z)dx = \int_0^a -k_n^2\xi_n(x)\Phi(x,z)dx = -k_n^2\Phi_n(z)\,.$$

Furthermore, the derivatives of $\Phi_n(z)$ are given by

$$\begin{aligned}
\Phi_{n,z}(z) &= \int_0^a \Phi_z(x,z)\xi_n(x)dx \quad \text{and} && (2.12)\\
\Phi_{n,zz}(z) &= \int_0^a \Phi_{zz}(x,z)\xi_n(x)dx\,. && (2.13)
\end{aligned}$$

Multiplying (2.1) with $\xi_n(x)$ and integrating with respect to x (from 0 to a), we obtain

$$\begin{aligned}\int_0^a \Delta\Phi(x,z)\xi_n(x)dx &= \int_0^a \Phi_{zz}\xi_n(x)dx + \int_0^a \Phi_{xx}\xi_n(x)dx \\ &= \Phi_{n,zz}(z) - k_n^2\Phi_n(z) = 0 \quad \text{for } z \in [-h,0]\,. \end{aligned} \tag{2.14}$$

Applying the same steps, (2.3) can be written as

$$\int_0^a \Phi_z(x,z)\xi_n(x)dx = \Phi_{n,z}(z) = 0 \qquad \text{for } z = -h \tag{2.15}$$

and (2.4) becomes

$$\int_0^a \Phi_z(x,z)\xi_n(x)dx = i\omega \int_0^a \zeta(x)\xi_n(x)dx \qquad \text{for } z = 0\,,$$

$$\text{thus} \qquad \Phi_{n,z}(0) = i\omega\zeta_n\,. \tag{2.16}$$

For (2.5), we need to use integration by parts and the conditions (2.6) and (2.7). Then

$$\int_0^a \zeta_{xx}(x)\xi_n(x)dx = \int_0^a -k_n^2\zeta(x)\xi_n(x)dx - \left[\frac{i\omega}{c}\zeta(x)\xi_n(x)\right]_0^a \tag{2.17}$$

(where the corner brackets denote the *sum* of the evaluation at the upper and lower value) and

$$i\omega\int_0^a \Phi_z(x,0)\xi_n(x)dx + g\int_0^a \zeta(x)\xi_n(x)dx = T\int_0^a \zeta_{xx}\xi_n(x)dx \tag{2.18}$$

together become

$$i\omega\Phi_n(0) + (g + T\,k_n^2)\zeta_n = -\left[\frac{i\omega}{c}\zeta(x)\xi_n(x)\right]_0^a\,. \tag{2.19}$$

The general solution of (2.14) can be written as

$$\Phi_n(z) = a_n\cosh(k_n(z+h)) + b_n\sinh(k_n(z+h))\,. \tag{2.20}$$

Consider the condition (2.15) to see that $b_n = 0$. Furthermore, a_n is now determined by (2.16) :

$$\Phi_{n,z}(0) = a_n k_n \sinh(k_n h) = i\omega\zeta_n \Rightarrow a_n = \frac{i\omega\zeta_n}{k_n}\,.$$

Thus, we finally have

$$\begin{aligned} \Phi_n(z) &= \frac{i\omega\zeta_n}{k_n \sinh(k_n h)} \cosh(k_n(z+h))\,, && (2.21)\\ \text{especially} \qquad \Phi_n(0) &= \frac{i\omega\zeta_n}{k_n \tanh(k_n h)}\,. && (2.22) \end{aligned}$$

Plugging (2.22) into (2.19), we get

$$-\frac{\omega^2}{k_n \tanh(k_n h)}\zeta_n + (g + T\,k_n^2)\zeta_n = -\frac{i\omega T}{c}\left[\zeta(x)\xi_n(x)\right]_0^a\,,$$

$$\zeta_n \frac{-\omega^2 + (g + T\,k_n^2)k_n \tanh(k_n h)}{k_n \tanh(k_n h)} = -\frac{i\omega T}{c}\left[\zeta(x)\xi_n(x)\right]_0^a\,.$$

Introducing

$$\omega_n^2 = (g + T\,k_n^2)k_n \tanh(k_n h)\,, \tag{2.23}$$

this becomes

$$\zeta_n = \frac{k_n \tanh(k_n h)}{\omega^2 - \omega_n^2}\frac{i\omega T}{c}\left[\zeta(x)\xi_n(x)\right]_0^a\,. \tag{2.24}$$

To simplify notation, let

$$\Omega_n = \frac{k_n \tanh(k_n h)}{\omega^2 - \omega_n^2}\,. \tag{2.25}$$

Introduce $l = \sqrt{\frac{T}{g}}$ and $\gamma = \frac{c}{\omega l}$. Then we have $\frac{\omega T}{c} = \frac{g\,l}{\gamma}$. Using (2.24) and (2.25) with the inverse Fourier transformation(2.11), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \tfrac{2}{a}\Omega_n i \tfrac{g\,l}{\gamma}\left[\zeta(\overline{x})\cos(k_n\overline{x})\right]_0^a \cos(k_n x) &= \zeta(x)\,,\\ \tfrac{2g\,l}{a} \sum_{n=1}^{\infty} \Omega_n \left[(-1)^n\zeta(a) + \zeta(0)\right]\cos(k_n x) &= -i\gamma\zeta(x)\,. \end{aligned} \tag{2.26}$$

It is important to keep in mind that γ is dependent on ω.

Plugging in $x = 0$ and $x = a$, (2.26) yields two linear equations in $\zeta(0)$ and $\zeta(a)$:

$$\begin{aligned} \zeta(a) \left[\tfrac{2g\,l}{a}\sum_{n=1}^{\infty}(-1)^n\Omega_n\right] + \zeta(0)\left[\tfrac{2g\,l}{a}\sum_{n=1}^{\infty}\Omega_n + i\gamma\right] &= 0\,,\\ \zeta(a) \left[\tfrac{2g\,l}{a}\sum_{n=1}^{\infty}\Omega_n + i\gamma\right] + \zeta(0)\left[\tfrac{2g\,l}{a}\sum_{n=1}^{\infty}(-1)^n\Omega_n\right] &= 0\,. \end{aligned} \tag{2.27}$$

We want nontrivial solutions for $\zeta(0)$ and $\zeta(a)$ to be possible, so the determinant of the coefficient matrix has to vanish. For convenience, let $\sigma = \sum_{n=1}^{\infty} \Omega_{2n}$ and $\tau = \sum_{n=1}^{\infty} \Omega_{2n-1}$. We get

$$\left[\frac{2g\,l}{a}(\sigma-\tau)\right]^2 - \left[\frac{2g\,l}{a}(\sigma+\tau)+i\gamma\right]^2 = 0$$

$$\begin{array}{lccccccc} \Rightarrow & \frac{2g\,l}{a}(\sigma-\tau) & + & (\frac{2g\,l}{a}(\sigma+\tau)+i\gamma) & = & \frac{4g\,l}{a}\sigma+i\gamma & = & 0 \\ \text{or} & \frac{2g\,l}{a}(\sigma-\tau) & - & (\frac{2g\,l}{a}(\sigma+\tau)+i\gamma) & = & -\frac{4g\,l}{a}\tau-i\gamma & = & 0\,. \end{array} \tag{2.28}$$

Recalling the definitions of σ, τ and Ω_n, see (2.25), we obtain the equations for the eigenvalues from (2.28) :

$$\frac{4g\,l}{a}\sum_{n=1}^{\infty}\frac{k_{2n}\tanh(k_{2n}h)}{\omega^2-\omega_{2n}^2} = -i\gamma \tag{2.29}$$

$$\textbf{or} \qquad \frac{4g\,l}{a}\sum_{n=1}^{\infty}\frac{k_{2n-1}\tanh(k_{2n-1}h)}{\omega^2-\omega_{2n-1}^2} = -i\gamma\,. \tag{2.30}$$

It should be emphasized that an eigenvalue only needs to satisfy **one** of these equations.

2.2.2 Discussion of the equations for the eigenvalues

Note that the equations (2.29,2.30) contain infinitely many terms. The consequences are:

- We expect the equations to have infinitely many solutions.
- We cannot expect to find all solutions in an explicit form.
- It is possible to find numerical approximations to some solutions.
- We can obtain an asymptotic expansion for very large solutions.

If ω is supposed to be small, we can obtain solutions by assuming that

$$\tanh(k_n h) \approx 1 \qquad \text{and} \qquad \frac{1}{\omega^2-\omega_n^2} = \frac{1}{-\omega_n^2} \qquad \text{for} \quad n > N\,.$$

We use *Mathematica* to obtain approximate values for the first ten solutions of the equations for the following set of parameters:

$$N = 50; c = 1; g = 9.81; a = 1; h = 1; l = 1; T = 1\,.$$

$$\begin{array}{lllllll} 13.7212 & + & 0.3177 & i\,, & 25.507 & + & 0.3235 & i\,, \\ 40.6831 & + & 0.3457 & i\,, & 57.528 & + & 0.3472 & i\,, \\ 76.729 & + & 0.3569 & i\,, & 97.414 & + & 0.3581 & i\,, \\ 119.982 & + & 0.3645 & i\,, & 143.88 & + & 0.3660 & i\,, \\ 169.384 & + & 0.3709 & i\,, & 196.03 & + & 0.3726 & i\,. \end{array} \tag{2.31}$$

Here, the values in the left column are solutions corresponding to (2.29), whereas in the second column there are solutions of (2.30).

Having found some small solutions, we now want to find an asymptotic expansion for solutions with $|\omega| \to \infty$. For this purpose, we restrict the discussion to the deep-water problem, i.e. $h = \infty$ or, in order to maintain relevance for applications, very large. Thus we have

$$\tanh(k_n h) = 1 \qquad \text{for all } n\,. \tag{2.32}$$

Set

$$\omega^2 = g\,k_1 m(1 + \kappa m^2) \qquad \text{where } \kappa = k_1 l\,. \tag{2.33}$$

Plugging this into (2.29, 2.30) and taking into account (2.32), we get

$$\kappa \sum_n \frac{n}{m(1+\kappa^2 m^2) - n(1+\kappa^2 n^2)} + \frac{1}{4} i\pi\gamma = 0$$

or, after rewriting the denominator,

$$\kappa \sum_n \frac{n}{(m-n)[(1+\kappa^2(m^2+mn+n^2)]} + \frac{1}{4} i\pi\gamma = 0\,.$$

Here, in both equations as well as in the following equations, the sum is taken either over all even or all odd integers.

After partial fraction expansion, this becomes

$$\sum_n \left(\frac{\kappa m}{1+3\kappa^2 m^2} \frac{1}{m-n} - \frac{\kappa m}{\rho} Im \left[\frac{\frac{1}{2}\kappa - i\rho}{\frac{3}{2}\kappa + i\rho} \frac{1}{(n+\frac{1}{2}m)\kappa - i\rho} \right] \right) + \frac{1}{4} i\pi\gamma = 0\,, \tag{2.34}$$

where $\rho = \sqrt{1 + \frac{3}{4}\kappa^2 m^2}$.

For further work on the infinite sums, the logarithmic derivative of the gamma function, $\psi(z)$, is useful. Some of its properties are stated in Section 5.3 of [1]:

$$
\begin{aligned}
\psi(n+1+z) &= \psi(z) + \sum_{j=0}^{n} \frac{1}{z+j}\,, && (2.35)\\
\psi(1-z) &= \psi(z) + \pi\cot(\pi z)\,, && (2.36)\\
\psi(z) &\approx \ln(z) - \frac{1}{2z} + O\left(\frac{1}{z^2}\right) \qquad \text{for } z \to \infty\,. && (2.37)
\end{aligned}
$$

Using these formulas, we can say

$$
\begin{aligned}
\textstyle\sum_n^N \frac{1}{m-n} &= -\tfrac{1}{2}\psi(\tfrac{N}{2}+1-\tfrac{m}{2}) + \tfrac{1}{2}\psi(1+\tfrac{m}{2}-\tfrac{\nu}{2}) + \tfrac{1}{2}\pi\cot(\tfrac{\pi}{2}(m-\nu))\,,\\
\textstyle\sum_n^N \frac{1}{n+\frac{m}{2}-i\frac{\rho}{\kappa}} &= \tfrac{1}{2}\psi(\tfrac{N}{2}+1+\tfrac{m}{4}-\tfrac{\rho}{2\kappa}i) - \tfrac{1}{2}\psi(\tfrac{m}{4}-\tfrac{\nu}{2}-\tfrac{\rho}{2\kappa}i)\,,
\end{aligned}
\qquad (2.38)
$$

where $\nu = 0$ and N even if n is summed over the even integers,
$\nu = 1$ and N odd if n is summed over the odd integers.

Applying (2.38) to (2.34) and taking the limit $N \to \infty$, the evaluations of ψ at terms depending on N cancel and we obtain

$$
\begin{aligned}
&\tfrac{\kappa m}{1+3\kappa^2 m^2}\Bigg(\tfrac{1}{2}\psi(1+\tfrac{m}{2}-\tfrac{\nu}{2}+\tfrac{\pi}{2}\cot(\tfrac{\pi}{2}(m-\nu))\\
&\qquad +\tfrac{1}{2\rho}\mathrm{Im}\left[\tfrac{1+\frac{3}{2}\kappa^2m^2-\frac{1}{2}i\kappa m\rho}{1+\kappa^2m^2}\psi(\tfrac{m}{4}-\tfrac{\rho}{2\kappa}i+\tfrac{\nu}{2}\right]\Bigg) = -\tfrac{1}{4}i\pi\gamma\,.
\end{aligned}
\qquad (2.39)
$$

For the asymptotic expansion, we only consider the leading terms of this equation. First, we consider the leading order terms for parts of the equation, making use of the Taylor expansions:

$$
\begin{aligned}
\gamma &= \tfrac{c}{\omega l} = \tfrac{c}{l\sqrt{m(1+\kappa^2m^2)}} \approx \tfrac{c}{\kappa l}m^{-\frac{3}{2}}\,,\\
\rho &= \sqrt{1+\tfrac{3}{4}\kappa^2m^2} \approx \tfrac{\sqrt{3}}{2}\kappa m\,,\\
\tfrac{\kappa m}{1+3\kappa^2m^2} &\approx \tfrac{1}{3\kappa}\tfrac{1}{m}\,,\\
\tfrac{1}{2}\psi(1+\tfrac{m}{2}-\tfrac{\nu}{2}) &\approx \tfrac{1}{2}\ln(m)+\tfrac{1}{2}\ln(\tfrac{1}{2})\,,\\
\tfrac{\pi}{2}\cot(\tfrac{\pi}{2}(m-\nu)) &\approx \tfrac{\pi}{2}\left[\tfrac{i-e^{b\pi}\cos((a+\nu)\pi)i+e^{b\pi}\sin((a+\nu)\pi)}{1+e^{2b\pi}-2e^{b\pi}\cos((a+\nu)\pi)}\right]\,,\\
\tfrac{1}{2\rho} &\approx \tfrac{\sqrt{3}}{3\kappa}\,,\\
\tfrac{1+\frac{3}{2}\kappa^2m^2-i\rho\kappa m}{1+3\kappa^2m^2} &\approx \tfrac{1}{2}-\tfrac{1}{2\sqrt{3}}i\,,\\
\psi_{\frac{m}{4}}\ \tfrac{\rho}{2\kappa}i+\tfrac{\nu}{2} &\approx \ln(m)+\ln(\tfrac{1}{4}-\tfrac{\sqrt{3}}{4}i)\,.
\end{aligned}
\qquad (2.40)
$$

In the fifth of these equations, the substitution $m = a + bi$ has been used.

Substituting all of these into (2.39), we get

$$\begin{aligned}\frac{1}{3\kappa m}\frac{1}{2}\ln(\tfrac{m}{2})+\frac{\pi(i-ie^{b\pi}\cos((a+\nu)\pi)+e^{b\pi}\sin((a+\nu)\pi))}{1+e^{2b\pi}-2e^{b\pi}\cos((a+\nu)\pi)}&\\ +\frac{1}{\sqrt{3}\kappa m}\mathrm{Im}[(\tfrac{1}{2}-\tfrac{i}{2\sqrt{3}})(\ln(\tfrac{1}{4}-\tfrac{\sqrt{3}}{4}i)+\ln(m))] &= -\frac{c\pi i}{4l\kappa}m^{-\frac{3}{2}}\,.\end{aligned}\tag{2.41}$$

We expand this equation and set $m = a + bi$ to get

$$\begin{aligned}\frac{m^{\frac{1}{2}}}{\sqrt{3}\kappa}\Bigg(\frac{\pi e^{b\pi}\sin((a+\nu)\pi)}{2\sqrt{3}(1+e^{2b\pi}-2e^{b\pi}\cos((a+\nu)\pi))}+\tfrac{1}{2}\arctan(-\sqrt{3})+\tfrac{1}{2}\arctan(\tfrac{b}{a})&\\ +i\left[\frac{1-e^{b\pi}\cos((a+\nu)\pi)}{2\sqrt{3}(1+e^{2b\pi}-2e^{b\pi}\cos((a+\nu)\pi))}+\frac{1}{2\sqrt{3}}\arctan(\tfrac{b}{a})\right]\Bigg) &= -\frac{c\pi i}{4l\kappa}\,.\end{aligned}\tag{2.42}$$

As we make m large, it is likely that the evaluations of the trigonometric functions converge to a certain value, thus we assume

$$\begin{aligned} a &\equiv 2n-\nu+A(n)\,,\\ b &\equiv B(n)\,.\end{aligned}\tag{2.43}$$

In (2.42), the terms in big parentheses need to go to zero for $|m| \to \infty$ to make up for the factor $m^{\frac{1}{2}}$. However, they must not converge too fast, as the right side of the equation does not vanish. The assumption $B(n) = o(n)$ is thus justified. This means $\arctan(\frac{b}{a}) \to 0$.

Using (2.43), we set the imaginary and the real part of the big parentheses in (2.42) to zero:

$$\begin{aligned}\frac{\pi e^{b\pi}\sin(A\pi)}{2\sqrt{3}(1+e^{2b\pi}-2e^{b\pi}\cos(A\pi))} &= -\tfrac{1}{2}\arctan(-\sqrt{3}) = \tfrac{\pi}{6}\,,\\ \frac{1-e^{b\pi}\cos(A\pi)}{2\sqrt{3}(1+e^{2b\pi}-2e^{b\pi}\cos(A\pi))} &= 0\,.\end{aligned}\tag{2.44}$$

From the second of these equations, we conclude $\cos(A\pi) = e^{-B\pi}$ and plug this into the first of these equations. To solve for B, set $x = e^{B\pi}$ and solve for x first:

$$\frac{x\sqrt{1-x^{-2}}}{1+x^2-2} = \frac{1}{\sqrt{3}} \Rightarrow \frac{\sqrt{x^2-1}}{x^2-1} = \frac{1}{\sqrt{3}} \Rightarrow x = +2\,.$$

This means $B = \frac{\ln(2)}{\pi} = 0.2206356$ and thus $A = \frac{1}{\pi}\arccos(\frac{1}{x}) = \frac{1}{3}$. The asymptotic expansion for m is then

$$m = a + ib = 2n - \nu + A + iB = 2n - \nu + \frac{1}{3} + \frac{\ln(2)}{\pi}i\,.\tag{2.45}$$

Now we go back to (2.33) and plug in the expansion for m to finally get the solutions for (2.29, 2.30). Looking again at leading order terms only, we get

$$\omega = 2\sqrt{2}l\sqrt{\frac{g\pi^3}{a^3}}n^{\frac{3}{2}} + 3\ln(2)l\sqrt{\frac{g\pi}{2a^3}}n^{\frac{1}{2}}i \tag{2.46}$$

as the desired asymptotic expansion for eigenvalues.

2.2.3 Asymptotic expansion of $\|\zeta\|$

Solving (2.27) for $\zeta(a)$ and using the definition of σ and τ introduced there, we get

$$\begin{aligned} \zeta(a) &= \left(\tfrac{2g\,l}{a}\textstyle\sum_{n=1}^{\infty}\Omega_n + i\gamma\right)\left(\tfrac{2g\,l}{a}\textstyle\sum_{n=1}^{\infty}(-1)^n\Omega_n\right)^{-1}\zeta(0) \\ &= \left(\tfrac{2g\,l}{a}(\sigma+\tau) + i\gamma\right)\left(\tfrac{2g\,l}{a}(\sigma-\tau)\right)^{-1}\zeta(0)\,. \end{aligned} \tag{2.47}$$

Using the first equation of (2.28), this would simplify to $\zeta(a) = \zeta(0)$, whereas using the other equation would yield $\zeta(a) = -\zeta(0)$.

Squaring (2.46) we get the leading order terms for ω_m^2 :

$$\omega_m^2 = 8l^2\frac{g\pi^3}{a^3}m^3 + 12i\sqrt{2}\ln(2)\frac{g\pi^2}{a^3}l^2m^2\,. \tag{2.48}$$

Then using those leading order expressions together with (2.24), we get

$$\begin{aligned} \zeta_n &= -2\left[\tfrac{n\pi}{a}iT\left(2\sqrt{2}l\sqrt{\tfrac{g\pi^3}{a^3}}m^{3/2} + 3\ln(2)l\tfrac{g\pi^3}{2a^3}m^{1/2}i\right)\right] \\ &\quad / \left[c\left(8l^2\tfrac{g\pi^3}{a^3}m^3 + 12i\sqrt{2}\ln(2)\tfrac{g\pi^2}{a^3}l^2m^2 - T\tfrac{\pi^3}{a^3}\right)\right] \end{aligned} \tag{2.49}$$

for n even or odd, depending on which equation of (2.28) had been used, the other coefficients vanish. Recalling $g\,l^2 = T$, this can be simplified to

$$\zeta_n = \beta n\frac{i(4m^{3/2} + 3\ln(2)m^{1/2})}{8\pi m^3 + 12i\sqrt{2}\ln(2)m^2 - \pi n^3} \tag{2.50}$$

for some β independent of m and n.

Now use the Inverse Fourier Formula (2.11) to get a formula for $\zeta(x)$. We can now calculate the norm of ζ :

$$\begin{aligned} \|\zeta\|^2 &= \textstyle\int_0^a\left(\sum_n \tfrac{2}{a}\zeta_n\cos(k_nx)\right)^2dx = \tfrac{2}{a}\sum_n|\zeta_n^2| \\ &= \textstyle\sum_n \tfrac{2}{a}\beta^2n^2\tfrac{16m^3+9\ln(2)^2m^2}{\pi^2(8m^3-n^3)^2+288\ln(2)^2m^4} \end{aligned} \tag{2.51}$$

where n is summed over even or odd integers only. To find the sum, we split it up:

- for $n < 2m - m^p - 1$, we have terms of order $O(m^2 \frac{m^3}{m^6}) = O(\frac{1}{m})$, so the first sum is $O(1)$.
- for $2m + m^p + 1 < n < 3m$, we have terms of order $O(m^2 \frac{m^3}{m^{6p}})$, so the sum is $O(\sqrt{m})$, if $p > \frac{5.5}{6}$.
- for $n > 3m$, we have terms of order $O(\sum_{n=3m}^{\infty} \frac{n^2 m^3}{n^6}) \leq O(1)$.

For the remaining terms (which will turn out to be dominating), we write

$$
\begin{aligned}
& \textstyle\sum_n \frac{2}{a}\beta^2 n^2 \frac{16m^3 + 9\ln(2)^2 m^2}{\pi^2(8m^3 - n^3)^2 + 288\ln(2)^2 m^4} \\
\approx\ & \textstyle\frac{2}{a}\beta^2 \sum_{n=2m-m^p}^{2m+m^p} n^2 \frac{16m^3}{\pi^2(n-2m)^2(n^2+2mn+4m^2)2+288\ln(2)^2 m^4} \\
\approx\ & \textstyle\frac{2}{a}\beta^2 \sum_{n=2m-m^p}^{2m+m^p} n^2 \frac{16m^{-1}}{\pi^2(n-2m)^2(4+4+4)^2+288\ln(2)^2} \\
\approx\ & \textstyle\frac{2}{a}\beta^2 \sum_{n=-m^p}^{m^p} \frac{64m}{144\pi^2 n^2 + 288\ln(2)^2}\,.
\end{aligned}
\tag{2.52}
$$

Now we take $m \to \infty$. For n summed over even integers only, the limit is $\frac{128\beta^2}{a} m \times 0.00779662$, whereas in the case of summing over odd integers only and $m \to \infty$, the limit is $\frac{128\beta^2}{a} m \times 0.00160927$.

So, these are the dominating terms, and $\|\zeta\|$ is the square root of the above result, in each case:

- n summed over even integers: $\|\zeta\| = 0.99796736 \frac{\beta}{\sqrt{a}} \sqrt{m}\,,$
- n summed over odd integers: $\|\zeta\| = 0.20598656 \frac{\beta}{\sqrt{a}} \sqrt{m}\,.$

2.3 Nonlinear theory for simplified edge conditions

2.3.1 Equations for first- to third-order approximations

To simplify higher-order approximations, simplified edge conditions are used. The modified model consists of (1.6) to (1.11) and the edge conditions

$$
\Theta_x(x) = 0 \qquad \text{for} \quad x = b_1, x = a + b_2\,,
\tag{2.53}
$$

where b_1 and b_2 are the time-dependent movements of the side boundaries.

A second slower time scale $\tau = \epsilon^2 t$ is introduced, where ϵ is meant to be small. The functions b_1, b_2 are assumed to be of third order, i.e. $b_i(z,t,\epsilon) = \epsilon^3 B_i(z,t)$ for $i = 1,2$. The solution is assumed to be of the form

$$\begin{aligned} \phi(x,z,t,\tau) &= \epsilon\phi_1(x,z,t,\tau) + \epsilon^2\phi_2(x,z,t,\tau) + \epsilon^3\phi_3(x,z,t,\tau)\,, \\ \Theta(x,t,\tau) &= \epsilon\Theta_1(x,t,\tau) + \epsilon^2\Theta_2(x,t,\tau) + \epsilon^3\Theta_3(x,t,\tau)\,. \end{aligned} \tag{2.54}$$

Expanding (1.6) to (1.11) and (2.53), we obtain the following equations for the functions $\phi_i(x,z)$ and $\Theta_i(x)$ where i = 1, 2, 3:

For $(x,z) \in [b_1, a+b_2] \times [-h, \Theta(x)]$,

$$\triangle\phi = 0$$

$$\begin{aligned} \Rightarrow\quad \Delta\phi_1 &= 0\,, \\ \Delta\phi_2 &= 0\,, \\ \Delta\phi_3 &= 0\,. \end{aligned} \tag{2.55}$$

At $x = b_1$,

$$\phi_x - b_{1,t} - b_{1,z}\phi_z = 0$$

$$\begin{aligned} \Rightarrow\quad \phi_{1,x} &= 0\,, \\ \phi_{2,x} &= 0\,, \\ \phi_{3,x} &= B_{1,t}\,, \end{aligned} \tag{2.56}$$

with all evaluations at $(0,z,t,\tau)$.

At $x = a + b_2$,

$$\phi_x - b_{2,t} - b_{2,z}\phi_z = 0$$

$$\begin{aligned} \Rightarrow\quad \phi_{1,x} &= 0\,, \\ \phi_{2,x} &= 0\,, \\ \phi_{3,x} &= B_{2,t}\,, \end{aligned} \tag{2.57}$$

with all evaluations at (a,z,t,τ).

At $z = -h$,

$$\phi_z = 0$$

$$\begin{aligned} \Rightarrow\quad \phi_{1,z} &= 0\,, \\ \phi_{2,z} &= 0\,, \\ \phi_{3,z} &= 0\,, \end{aligned} \tag{2.58}$$

with all evaluations at $(x, -h, t, \tau)$.

At $z = \Theta(x)$,

$$\Theta_t + \Theta_x \phi_x - \phi_z = 0\,.$$

To have evaluations at 0 rather than at $\Theta(z)$, Taylor expansions are used:

$$\Theta_t + \Big(\phi_x + \Theta\phi_{xz} + O(\epsilon^3)\Big)\,\Theta_x - \Big(\phi_z + \Theta\phi_{zz} + \frac{1}{2}\Theta^2\phi_{zzz}\Big) = 0\,.$$

Collecting terms of the same order yields

$$\begin{aligned}
\Theta_{1,t} - \phi_{1,z} &= 0\,,\\
\Theta_{2,t} - \phi_{2,z} &= -\phi_{1,x}\Theta_{1,x} + \phi_{1,zz}\Theta_1\,,\\
\Theta_{3,t} - \phi_{3,z} &= -\left(\phi_{2,x}\Theta_{1,x} + \phi_{1,xz}\Theta_{1,x}\Theta_1 + \phi_{1,x}\Theta_{2,x}\right)\\
&\quad + \Big(\phi_{1,zz}\Theta_2 + \phi_{2,zz}\Theta_1 + \tfrac{1}{2}\phi_{1,zzz}\Theta_1^2\Big) - \Theta_{1,\tau}\,,
\end{aligned} \tag{2.59}$$

with all evaluations at $(x, 0, t, \tau)$.

Also at $z = \Theta(x)$,

$$\phi_t + g\Theta + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_z^2 - T\Theta_{xx}\left(1 - \frac{3}{2}\Theta_x^2\right) = 0\,.$$

In addition, we have

$$\begin{aligned}
\phi_x^2(x, \Theta, t, \tau) &= (\phi_x + \Theta\phi_{xz} + O(\epsilon^3))^2 = \phi_x^2 + 2\Theta\phi_x\phi_{xz} + O(\epsilon^4)\,,\\
\phi_x^2(x, 0, t, \tau) &= \epsilon^2\phi_{1,x}^2 + 2\epsilon^3\phi_{1,x}\phi_{2,x} + O(\epsilon^4)\,,\\
\phi_t(x, \Theta, t, \tau) &= \phi_t + \Theta\phi_{tz} + \tfrac{1}{2}\Theta^2\phi_{tzz} + O(\epsilon^4)
\end{aligned}$$

and similar expressions for $\phi_z^2(x, \Theta, t, \tau)$. The evaluations are at $z = 0$ unless explicitly stated differently. Thus we have

$$\begin{aligned}
\phi_{1,t} + g\Theta_1 - T\Theta_{1,xx} &= 0\,,\\
\phi_{2,t} + g\Theta_2 - T\Theta_{2,xx} &= -\Theta_1\phi_{1,tz} - \tfrac{1}{2}\phi_{1,x}^2 - \tfrac{1}{2}\phi_{1,z}^2\,,\\
\phi_{3,t} + g\Theta_3 - T\Theta_{3,xx} &= -(\phi_{1,x}\phi_{2,x} + \Theta_1\phi_{1,x}\phi_{1,xz}) - (\phi_{1,z}\phi_{2,z} + \Theta_1\phi_{1,z}\phi_{1,zz})\\
&\quad - \Big(\Theta_1\phi_{2,tz} + \Theta_2\phi_{1,tz} + \tfrac{1}{2}\Theta_1^2\phi_{1,tzz}\Big) - \tfrac{3}{2}T\Theta_{1,xx}\Theta_{1,x}^2 - \phi_{1,\tau}\,.
\end{aligned} \tag{2.60}$$

Finally, (2.53) gives us for $x = 0$ and $x = a$

$$\begin{aligned}
\Theta_{1,x} &= 0\,,\\
\Theta_{2,x} &= 0\,,\\
\Theta_{3,x} &= 0\,.
\end{aligned} \tag{2.61}$$

2.3.2 First-order approximation

Now we assume

$$\begin{aligned} \phi_1(x,z,t,\tau) &= e^{i\omega t+i\lambda\tau}\Phi_1(x,z,\tau)\,, \\ \Theta_1(x,t,\tau) &= e^{i\omega t+i\lambda\tau}\zeta_1(x,\tau)\,. \end{aligned} \tag{2.62}$$

Proceeding the same way as with the linearized equations with Hocking's edge condition, equations (2.14) to (2.16) can be established. Since we now have the simplified edge conditions, we can replace (2.17) with

$$\int_0^a \zeta_{xx}(x)\xi_n(x)dx = \int_0^a -k_n^2\zeta(x)\xi_n(x)dx\,. \tag{2.63}$$

As (2.18) is unchanged, we now get

$$i\omega\Phi_{1,n}(0) + \left(g + T\,k_n^2\right)\zeta_{1,n} = 0\,. \tag{2.64}$$

Equations (2.20) to (2.22) remain unchanged. However, we can now solve for $\zeta_{1,n}$ by plugging (2.22) into (2.64) :

$$\begin{aligned} -\tfrac{\omega^2}{k_n\tanh(k_n h)}\zeta_{1,n} + (g+T\,k_n^2)\zeta_{1,n} &= 0\,, \\ \zeta_{1,n}\tfrac{-\omega^2+(g+T\,k_n^2)k_n\tanh(k_n h)}{k_n\tanh(k_n h)} &= 0\,. \end{aligned} \tag{2.65}$$

To avoid the trivial case of all ζ_n being zero, we need the second factor to vanish, i.e. we need

$$\begin{aligned} \omega^2 &= (g+T\,k_n^2)k_n\tanh(k_n h) && \text{for some } n\,, \\ \text{so} \qquad \omega &= \pm\sqrt{(g+T\,k_n^2)k_n\tanh(k_n h)} && \text{for some } n\,. \end{aligned} \tag{2.66}$$

Thus $\zeta_{1,m} = 0$ for $m \neq n$, $\zeta_{1,m} = const.$ for $m = n$. Going back to (2.10) and (2.11) and using (2.62), the solutions are

$$\begin{aligned} \Theta_1(x,t,\tau) &= e^{i\omega_m t+i\lambda\tau}\tfrac{2}{a}p(\tau)\cos(k_m x)\,, \\ \phi_1(x,z,t,\tau) &= e^{i\omega_m t+i\lambda\tau}\tfrac{2}{a}\tfrac{i\omega_m p(\tau)}{k_m\sinh(k_m h)}\cosh(k_m(z+h))\cos(k_m x)\,, \end{aligned} \tag{2.67}$$

where $p(\tau)$ is a complex-valued function to be determined.

2.3.3 Second-order approximation

The solutions should be real-valued, thus we let $\phi_1^R = \phi_1 + \overline{\phi_1}$ and $\Theta_1^R = \Theta_1 + \overline{\Theta_1}$ instead of ϕ_1, Θ_1. From now on, "$+c.c.$" stands for adding the complex conjugate of the previous expression.

The right side of the second-order equation in (2.59) then is

$$\begin{aligned}
& -\phi^R_{1,x}(x,0,t,\tau)\Theta^R_{1,x}(x,t,\tau) + \phi^R_{1,zz}(x,0,t,\tau)\Theta^R_1(x,t,\tau) = \\
= \; & -\left(\phi_{1,x}(x,0,t,\tau) + \overline{\phi_{1,x}(x,0,t,\tau)}\right)\left(\Theta_{1,x}(x,t,\tau) + \overline{\Theta_{1,x}(x,t,\tau)}\right) \\
& + \left(\phi_{1,zz}(x,0,t,\tau) + \overline{\phi_{1,zz}(x,0,t,\tau)}\right)\left(\Theta_1(x,t,\tau) + \overline{\Theta_1(x,t,\tau)}\right) \\
= \; & -\phi_{1,x}\Theta_{1,x} - \phi_{1,x}\overline{\Theta_{1,x}} + \phi_{1,zz}\Theta_1 + \phi_{1,zz}\overline{\Theta_1} + c.c. \\
= \; & \tfrac{4ik_m\omega_m}{a^2}\coth(k_m h)\cos(2k_m x)\left(e^{2i(\omega_m t + \lambda\tau)}p^2(\tau) + p(\tau)\overline{p(\tau)}\right) + c.c. \\
= \; & \tfrac{4ik_m\omega_m}{a^2}\coth(k_m h)\cos(2k_m x)e^{2i(\omega_m t + \lambda\tau)}p^2(\tau) + c.c.
\end{aligned} \tag{2.68}$$

The right side of the second-order equation in (2.60) becomes

$$\begin{aligned}
& -\Theta^R_1\phi^R_{1,tz} - \tfrac{1}{2}\left(\phi^R_{1,x}\right)^2 - \tfrac{1}{2}\left(\phi^R_{1,z}\right)^2 \\
= \; & -\left(\Theta_1 + \overline{\Theta_1}\right)\left(\phi_{1,tz} + \overline{\phi_{1,tz}}\right) - \tfrac{1}{2}\left(\phi_{1,x} + \overline{\phi_{1,x}}\right)^2 - \tfrac{1}{2}\left(\phi_{1,z} + \overline{\phi_{1,z}}\right)^2 \\
= \; & \left(-\Theta_1\phi_{1,tz} - \tfrac{1}{2}\phi^2_{1,x} - \tfrac{1}{2}\phi^2_{1,z}\right) + \left(-\Theta_1\overline{\phi_{1,tz}} - \tfrac{1}{2}\phi_{1,x}\overline{\phi_{1,x}} - \tfrac{1}{2}\phi_{1,z}\overline{\phi_{1,z}}\right) + c.c. \\
= \; & \tfrac{\omega^2\operatorname{csch}^2(k_m h)p^2(\tau)}{a^2}e^{2i(\omega_m t + \lambda\tau)} \\
& \times\Big(-1 - 2\cos(2k_m x) + 2\cosh(2k_m h) + \cos(2k_m x)\cosh(2k_m h)\Big) \\
& + \tfrac{\omega^2\operatorname{csch}^2(k_m h)}{a^2}\left(-p(\tau)\overline{p(\tau)} + p(\tau)\overline{p(\tau)}\cos(2k_m x)\cosh(2k_m h)\right) + c.c.
\end{aligned} \tag{2.69}$$

This leads us to the assumption that the second-order approximation has the following form:

$$\begin{aligned}
\Theta_2(x,t,\tau) &= c_1\cos(2k_m x) + c_{1,2\omega}e^{2i(\omega_m t + \lambda\tau)}\cos(2k_m x) \\
&\quad + c_2 + c_{2,2\omega}e^{2i(\omega_m t + \lambda\tau)} + c.c.\,, \\
\phi_2(x,z,t,\tau) &= \left(d_1\cos(2k_m x) + d_{1,2\omega}e^{2i(\omega_m t + \lambda\tau)}\cos(2k_m x)\right)\cosh(2k_m(z+h)) \\
&\quad + d_{2,2\omega}e^{2i(\omega_m t + \lambda\tau)} + c.c.
\end{aligned} \tag{2.70}$$

Because of (2.61),

$$\int_0^a \Theta_{xx}\cos(2k_m x)dx = -4k_m^2\int_0^a \Theta\cos(2k_m x)dx\,.$$

Both sides of (2.59) and (2.60) are now multiplied by $\cos(2k_m x)$ and integrated from 0 to a. Using the previous relation and defining $\alpha_m = g + 4T\,k_m^2$, the following equations for the coefficients are obtained:

$$\begin{aligned}
-2k_m \sinh(2k_m h) d_1 &= 0\,, \\
2i\omega_m c_{1,2\omega} - 2k_m \sinh(2k_m h) d_{1,2\omega} &= \tfrac{4ik_m\omega_m p^2(\tau)}{a^2} \coth(k_m h)\,, \\
2i\omega_m c_{2,2\omega} &= 0\,, \\
\alpha_m c_1 &= \tfrac{\omega_m^2}{a^2} p(\tau)\overline{p(\tau)} \cosh(2k_m h) \operatorname{csch}^2(k_m h)\,, \\
g\, c_2 &= -\tfrac{\omega_m^2}{a^2} p(\tau)\overline{p(\tau)} \operatorname{csch}^2(k_m h)\,, \\
\alpha_m c_{1,2\omega} + 2i\omega_m \cosh(2k_m h) d_{1,2\omega} &= \tfrac{\omega_m^2 \operatorname{csch}^2(k_m h) p^2(\tau)}{a^2}(-2 + \cosh(2k_m h))\,, \\
g\, c_{2,2\omega} + 2i\omega_m d_{2,2\omega} &= \tfrac{\omega_m^2 \operatorname{csch}^2(k_m h) p^2(\tau)}{a^2}(2\cosh(2k_m h) - 1)\,.
\end{aligned} \tag{2.71}$$

Solving these equations yields

$$\begin{aligned}
c_1 &= \tfrac{\omega_m^2 p(\tau)\overline{p(\tau)} \cosh(2k_m h) \operatorname{csch}^2(k_m h)}{a^2\alpha_m}\,, \\
c_2 &= -\tfrac{\omega_m^2 p(\tau)\overline{p(\tau)} \operatorname{csch}^2(k_m h)}{a^2 g}\,, \\
c_{1,2\omega} &= \tfrac{k\omega_m^2 p^2(\tau)(5\cosh(k_m h)+\cosh(3k_m h)) \operatorname{csch}(k_m h)}{a^2(2\omega_m^2\cosh(2k_m h) - k_m\alpha_m \sinh(2k_m h))}\,, \\
c_{2,2\omega} &= 0\,, \\
d_1 &= 0\,, \\
d_{1,2\omega} &= \tfrac{i\omega_m p^2(\tau) \operatorname{csch}^2(k_m h)}{a^2(2\omega_m^2\cosh(2k_m h) - k_m\alpha_m \sinh(2k_m h))} \\
&\quad \Big(2\omega_m^2 - \omega_m^2\cosh(2k_m h) + k_m\alpha_m \sinh(2k_m h)\Big)\,, \\
d_{2,2\omega} &= \tfrac{-i\omega_m p^2(\tau)(2\cosh(2k_m h)-1)\operatorname{csch}^2(k_m h)}{2a^2}\,.
\end{aligned} \tag{2.72}$$

2.3.4 Solvability condition from third-order equations

The third-order approximation ϕ_3 needs to satisfy (2.55). Multiplying by $\cos(k_m x)\cosh(k_m(z+h))$ and integrating over the whole domain, we get

$$\int_{-h}^{0}\left(\int_0^a (\phi_{3,xx} + \phi_{3,zz})\cos(k_m x)\cosh(k_m(z+h))dx\right)dz = 0\,.$$

After doing integration by parts twice, we have

$$\begin{aligned}
&\textstyle\int_{-h}^{0}\left(\phi_{3,x}\cos(k_m x)|_0^a - k_m^2\int_0^a \phi_3\cos(k_m x)dx\right)\cosh(k_m(z+h))dz \\
&\quad + \textstyle\int_0^a\Big(\phi_{3,z}\cosh(k_m(z+h))|_{-h}^0 - k_m\phi_3\sinh(k_m(z+h))|_{-h}^0 \\
&\quad\quad + k_m^2\textstyle\int_{-h}^0 \phi_3\cosh(k_m(z+h))dz\Big)\cos(k_m x)dx \quad = \quad 0\,.
\end{aligned} \tag{2.73}$$

Making use of (2.56) to (2.61), we obtain after rearranging

$$\begin{aligned}
&\textstyle\int_{-h}^{0}((-1)^m B_{2,t}-B_{1,t})\cosh(k_m(z+h))dz\\
&\quad+\textstyle\int_0^a\Big(i\omega_m\cosh(k_mh)+\frac{k_m}{i\omega_m}(g+T\,k_m^2)\sinh(k_mh)\Big)\,\Theta_3\cos(k_mx)dx\\
&\quad-\textstyle\int_0^a\Big([\text{r.s. (2.59)}]\cosh(k_mh)+\frac{k_m}{i\omega_m}[\text{r.s. (2.60)}]\sinh(k_mh)\Big)\cos(k_mx)dx \quad = \quad 0\,.
\end{aligned} \tag{2.74}$$

Note that the expression in parentheses in the second line vanishes by (2.66). Also, taking the complex conjugate has to be applied to the factor $\frac{1}{i\omega_m}$ in the last line as well. For simplification, we only consider the terms that are relevant for discussing resonance, which are the terms with frequency ω_m. The terms in the second-order approximation are split up into terms that are constant with respect to time and terms that have frequency $2\omega_m$:

$$\begin{aligned}
\Theta_2(x,t,\tau) &= \Theta_{2con}(x,t,\tau)+\Theta_{2osc}(x,t,\tau)\,,\\
\phi_2(x,z,t,\tau) &= \phi_{2con}(x,z,t,\tau)+\phi_{2osc}(x,z,t,\tau)\,.
\end{aligned} \tag{2.75}$$

The right side of the third-order equation in (2.59) is then

$$\begin{aligned}
&-(\phi_{2,x}\Theta_{1,x}+\phi_{1,xz}\Theta_{1,x}\Theta_1+\phi_{1,x}\Theta_{2,x})+\Big(\phi_{1,zz}\Theta_2+\phi_{2,zz}\Theta_1+\tfrac{1}{2}\phi_{1,zzz}\Theta_1^2\Big)-\Theta_{1,\tau}\\
=\;&-\Big(\Theta_{1,x}\phi_{2con,x}+\overline{\Theta_{1,x}}\phi_{2osc,x}+\overline{\phi_{1,xz}}\Theta_{1,x}\Theta_1+\phi_{1,xz}\overline{\Theta_{1,x}}\Theta_1+\phi_{1,xz}\Theta_{1,x}\overline{\Theta_1}\\
&\quad+\phi_{1,x}\Theta_{2con,x}+\overline{\phi_{1,x}}\Theta_{2osc,x}\Big)\\
&+\Big(\phi_{1,zz}\Theta_{2con}+\overline{\phi_{1,zz}}\Theta_{2osc}+\phi_{2con,zz}\Theta_1+\overline{\Theta_1}\phi_{2osc,zz}+\phi_{1,zzz}\Theta_1\overline{\Theta_1}+\tfrac{1}{2}\overline{\phi_{1,zzz}}\Theta_1^2\Big)\\
&\quad-\Theta_{1,\tau}+c.c.\,,
\end{aligned} \tag{2.76}$$

where only terms of frequency ω_m have been considered. Similarly, the right side of the third-order equation in (2.60) is now

$$\begin{aligned}
&-(\phi_{1,x}\phi_{2,x}+\Theta_1\phi_{1,x}\phi_{1,xz})-(\phi_{1,z}\phi_{2,z}+\Theta_1\phi_{1,z}\phi_{1,zz})\\
&\quad-\Big(\Theta_1\phi_{2,tz}+\Theta_2\phi_{1,tz}+\tfrac{1}{2}\Theta_1^2\phi_{1,tzz}\Big)-\tfrac{3}{2}T\Theta_{1,xx}\Theta_{1,x}^2-\phi_{1,\tau}\\
=\;&-\Big(\overline{\phi_{1,x}}\phi_{2osc,x}+\phi_{1,x}\phi_{2con,x}+\overline{\Theta_1}\phi_{1,x}\phi_{1,xz}+\Theta_1\overline{\phi_{1,x}}\phi_{1,xz}+\Theta_1\phi_{1,x}\overline{\phi_{1,xz}}\Big)\\
&-\Big(\overline{\phi_{1,z}}\phi_{2osc,z}+\phi_{1,z}\phi_{2con,z}+\overline{\Theta_1}\phi_{1,z}\phi_{1,zz}+\Theta_1\overline{\phi_{1,z}}\phi_{1,zz}+\Theta_1\phi_{1,z}\overline{\phi_{1,zz}}\Big)\\
&-\Big(\overline{\Theta_1}\phi_{2osc,tz}+\Theta_1\phi_{2con,tz}+\Theta_{2osc}\overline{\phi_{1,tz}}+\Theta_{2con}\phi_{1,tz}+\Theta_1\overline{\Theta_1}\phi_{1,tzz}+\tfrac{1}{2}\Theta_1^2\overline{\phi_{1,tzz}}\Big)\\
&-\Big(\tfrac{3}{2}T\overline{\Theta_{1,xx}}\Theta_{1,x}^2+3T\Theta_{1,xx}\Theta_{1,x}\overline{\Theta_{1,x}}\Big)-\phi_{1,\tau}+c.c.
\end{aligned} \tag{2.77}$$

Working out the equations and assuming that the forcing terms are of the form

$$B_i(z,t,\tau)=(i\omega_m)^{-1}e^{i(\omega_mt+\lambda\tau)}f_i(z)\,, \tag{2.78}$$

equation (2.74) becomes a differential equation for $p(\tau)$:

$$c_1 p' + c_2 |p|^2 p + c_3 p + c_4 = 0\,. \tag{2.79}$$

The constants in this equation are given by

$$\begin{aligned}
c_1 &= 2\cosh(k_m h)\,,\\
c_2 &= -i\,m\pi\ \operatorname{csch}^3(k_m h)\\
&\quad \times\Big(-72 g k_m^4 T \alpha_m \omega_m^2 + 12 g k_m^2 \alpha_m^2 \omega_m^2\\
&\qquad +2\omega_m^2 (32\alpha_m \omega_m^4 + 63 g k_m^4 T \alpha_m + 4 g k_m \alpha_m^2 - 3 g k_m^2 \alpha_m^2 + 12 g \omega_m^4)\cosh(2k_m h)\\
&\qquad -12 g k_m^2 \alpha_m (6k_m^2 T + \alpha_m)\omega_m^2 \cosh(4k_m h)\\
&\qquad +(18 g k_m^4 T \alpha_m \omega_m^2 - 8 g k_m \alpha_m^2 \omega_m^2 + 6 g k_m^2 \alpha_m^2 \omega_m^2 + 8 g \omega_m^6)\cosh(6k_m h)\\
&\qquad +(-45 g k_m^5 T \alpha_m^2 + 8 g \alpha_m \omega_m^4 - 76 g k_m \alpha_m \omega_m^4 - 32 k_m \alpha_m^2 \omega_m^4)\sinh(2k_m h)\\
&\qquad +(36 g k_m^5 T \alpha_m^2 - 32 g \alpha_m \omega_m^4 + 24 g k_m \alpha_m \omega_m^4)\sinh(4k_m h)\\
&\qquad +(-9 g k_m^5 T \alpha_m^2 + 8 g \alpha_m \omega_m^4 - 12 g \alpha_m k_m \omega_m^4)\sinh(6k_m h)\Big)\\
&\quad \Big/\Big(32 a^3 g \alpha_m \omega_m (-2\omega_m^2 \cosh(2k_m h) + k_m \alpha_m \sinh(2k_m h))\Big)\,,\\
c_3 &= 2i\lambda\cosh(k_m h)\,,\\
c_4 &= \textstyle\int_{-h}^{0}((-1)^m f_2(z) - f_1(z))\cosh(k_m(z+h))dz\,.
\end{aligned} \tag{2.80}$$

2.4 Modifications for some special cases of Hocking's edge condition

Here we discuss the modified governing equations consisting of (1.6) to (1.11) together with special cases of Hocking's edge condition. We use (1.12), but with additional assumptions.

2.4.1 Assumption: $\epsilon = \frac{\omega}{c}$

First we assume that $\epsilon = \frac{\omega}{c}$. We get equations (2.55) to (2.60) as in Section 2.3 , but (2.61) now splits up into

$$\begin{aligned}
\Theta_{1,x} &= 0\,,\\
\Theta_{2,x} &= i\Theta_1(x)\,,\\
\Theta_{3,x} &= 2i\Theta_2(x)\,,
\end{aligned} \tag{2.81}$$

evaluated at $x = a$, and

$$
\begin{aligned}
\Theta_{1,x} &= 0\,, \\
\Theta_{2,x} &= -i\Theta_1(x)\,, \\
\Theta_{3,x} &= -2i\Theta_2(x)\,,
\end{aligned}
\tag{2.82}
$$

evaluated at $x = 0$.

These changes also affect the work on (2.60). Assume now that the solution has the form

$$
\begin{aligned}
\Theta_2(x,t,\tau) &= c_1\cos(2k_m x) + c_{1,2\omega}e^{2i(\omega_m t+\lambda\tau)}\cos(2k_m x) + c_2 \\
&\quad + c_{2,2\omega}e^{2i(\omega_m t+\lambda\tau)} + c_\omega e^{i(\omega_m t+\lambda\tau)} + c.c.\,, \\
\phi_2(x,z,t,\tau) &= \Big(d_1\cos(2k_m x) + d_{1,2\omega}e^{2i(\omega_m t+\lambda\tau)}\cos(2k_m x)\Big)\cosh(2k_m h) \\
&\quad + d_2 + d_{2,2\omega}e^{2i(\omega_m t+\lambda\tau)} + d_\omega e^{i(\omega_m t+\lambda\tau)} + c.c.
\end{aligned}
\tag{2.83}
$$

To find the solution of (2.60), this equation is multiplied by $\cos(2k_m x)$ and integrated from 0 to a. Its left side now is

$$
\begin{aligned}
&\textstyle\int_0^a(\phi_{2,t} + g\Theta_2 - T\Theta_{2,xx})\cos(2k_m x)dx = \\
=\;& i\omega_m a\, e^{2i(\omega_m t+\lambda\tau)}\cosh(k_m h)d_{1,2\omega} + (g+4Tk_m^2)\tfrac{a}{2}c_1 \\
&\quad + (g+4Tk_m^2)\tfrac{a}{2}e^{2i(\omega_m t+\lambda\tau)}c_{1,2\omega} - \tfrac{2iT}{a}e^{i(\omega_m t+\lambda\tau)}p(\tau)((-1)^m+1)\,,
\end{aligned}
\tag{2.84}
$$

where we have used

$$
\begin{aligned}
&\textstyle\int_0^a\Theta_{2,xx}\cos(2k_m x)dx \\
=\;& \Theta_{2,x}\cos(2k_m x)|_0^a - \textstyle\int_0^a\Theta_{2,x}(-2k_m\sin(2k_m x))dx \\
=\;& i\Theta_1(a) - (-i\Theta_1(0)) - 4k_m^2\textstyle\int_0^a\Theta_2\cos(2k_m x)dx \\
=\;& \tfrac{2i}{a}e^{i(\omega_m t+\lambda\tau)}p(\tau)((-1)^m+1) - 4k_m^2\textstyle\int_0^a\Theta_2\cos(2k_m x)dx\,.
\end{aligned}
\tag{2.85}
$$

As this setting is very similar to the one in Section 2.3 , the coefficients appearing in (2.72) turn out to be the same. From (2.59) and (2.60), we get for the remaining coefficients:

$$
\begin{aligned}
i\omega_m c_\omega &= 0\,, \\
gc_\omega + i\omega_m d_\omega &= \tfrac{2iT((-1)^m+1)p(\tau)}{a}\,.
\end{aligned}
\tag{2.86}
$$

Solving this yields

$$
\begin{aligned}
c_\omega &= 0\,, \\
d_\omega &= \tfrac{2(1+(-1)^m)T\,p(\tau)}{a\omega_m}\,.
\end{aligned}
\tag{2.87}
$$

Note that the only change compared to the second-order approximation in Section 2.3.2 is the additional part with frequency ω_m. However, those terms do not influence the terms in the third-order equations that have frequency ω_m, the frequency which is critical when the possibility of resonance is considered. Thus the system of equations and the solutions for $p(\tau)$ are the same in this case as in Section 2.3.4.

2.4.2 Assumption: $\epsilon^2 = \frac{\omega}{c}$

This time we assume that $\epsilon^2 = \frac{\omega}{c}$. We have equations (2.55) to (2.60) from Section 2.3 . Instead of (2.61), we should use

$$\begin{aligned} \Theta_{1,x} &= 0\,, \\ \Theta_{2,x} &= 0\,, \\ \Theta_{3,x} &= i\Theta_1(x)\,, \end{aligned} \tag{2.88}$$

evaluated at $x = a$, and

$$\begin{aligned} \Theta_{1,x} &= 0\,, \\ \Theta_{2,x} &= 0\,, \\ \Theta_{3,x} &= -i\Theta_1(x)\,, \end{aligned} \tag{2.89}$$

evaluated at $x = 0$.

Thus, the first-order approximation and the second-order approximations remain unchanged, when compared to the ones found in Section 2.3 . For the third-order solvability condition, nothing changes but the left side of (2.60). It is now

$$\begin{aligned} & \textstyle\int_0^a (\phi_{3,t} + g\Theta_3 - T\Theta_{3,xx}) \cos(k_m x) dx = \\ = \; & \textstyle\int_0^a (\phi_{3,t} + (g + T\,k_m^2)\Theta_3) \cos(k_m x) dx - \frac{4iT}{a} e^{i(\omega_m t + \lambda\tau)} p(\tau)\,, \end{aligned} \tag{2.90}$$

where we have used

$$\begin{aligned} & \textstyle\int_0^a \Theta_{3,xx} \cos(k_m x) dx \\ = \; & \textstyle\Theta_{3,x} \cos(k_m x)|_0^a - \int_0^a \Theta_{3,x}(-k_m \sin(k_m x)) dx \\ = \; & \textstyle i(-1)^m \Theta_1(a) - (-i\Theta_1(0)) - k_m^2 \int_0^a \Theta_3 \cos(k_m x) dx \\ = \; & \textstyle\frac{4i}{a} e^{i(\omega_m t + \lambda\tau)} p(\tau) - k_m^2 \int_0^a \Theta_3 \cos(k_m x) dx\,. \end{aligned} \tag{2.91}$$

This yields a differential equation of the form (2.79) with c_1, c_2 and c_4 given by (2.80), and

$$c_3 = 2i\lambda \cosh(k_m h) + \frac{4k_m T}{a\omega_m} \sinh(k_m h)\,. \tag{2.92}$$

Chapter 3

Forcing frequencies that are sums of eigenfrequencies

3.1 Introduction

In this chapter, the case of simplified edge conditions is studied for a forcing that is of second order with frequency close to $\omega_m + \omega_n$ where ω_m and ω_n are eigenfrequencies. This forcing has an effect on the eigenmodes of frequency ω_m and ω_n. As before, the first- and second-order approximations are found. These now depend on two slowly varying functions $p_m(\tau)$ and $p_n(\tau)$. The third-order equations yield two solvability conditions which in turn yield complex-valued differential equations for $p_m(\tau)$ and $p_n(\tau)$. In [7] , [8] and [22], this system of differential equations has been discussed. Knobloch et al. [22] were the first to show that the system can be reduced to one real-valued differential equation for $\rho = |p_m|^2$. To avoid restrictions that need to be satisfied using $\rho = |p_m|^2$, a real-valued differential equation for $\rho = \beta_1|p_m|^2 + \beta_2|p_n|^2$ is used. All of the solutions to this differential equation are presented. For each case, the behavior of the radii and angles of p_m and p_n is analyzed. We also find the equilibria and some periodic orbits of the original system. A condition for existence of homoclinic orbits connecting the trivial equilibrium is given and the points that are on those homoclinic orbits are described. These results are confirmed by a numerical example.

3.2 First- to third-order approximation and solvability conditions

The boundary forcing terms are assumed to be of the form

$$b_k(z,t,\tau) \;=\; \epsilon^2 B_k(z,t,\tau) = \epsilon^2 \left(i(\omega_m+\omega_n)\right)^{-1} e^{i(\omega_m+\omega_n)t+i\lambda\tau} f_k(z) + c.c. \quad (k=1,2)\,. \tag{3.1}$$

To have second-order terms of frequency $\omega_m + \omega_n$, the first-order solution has to be a linear combination of the first-order solutions for frequencies ω_m and ω_n.

Equations (2.56, 2.57) have to be replaced. To get conditions at $x = 0$ and $x = a$ rather than at the moving side boundaries, Taylor series are used. The new equations are

$$\begin{aligned} \phi_{1,x} &= 0\,, \\ \phi_{2,x} &= B_{1,t}\,, \\ \phi_{3,x} &= B_{1,z}\phi_{1,z} - B_1\phi_{1,xx}\,, \end{aligned} \tag{3.2}$$

with all evaluations at $(0,z,t,\tau)$, and

$$\begin{aligned} \phi_{1,x} &= 0\,, \\ \phi_{2,x} &= B_{2,t}\,, \\ \phi_{3,x} &= B_{2,z}\phi_{1,z} - B_2\phi_{1,xx}\,, \end{aligned} \tag{3.3}$$

with all evaluations at (a,z,t,τ).

3.2.1 First-order approximation

As the first-order equations are linear in Θ_1, ϕ_1 and their derivatives, the first-order approximation is the sum of the approximations in (2.67) :

$$\begin{aligned} \Theta_1(x,t,\tau) &= e^{i\omega_m t}\tfrac{2}{a}p_m(\tau)\cos(k_m x) + e^{i\omega_n t}\tfrac{2}{a}p_n(\tau)\cos(k_n x) + c.c.\,, \\ \phi_1(x,z,t,\tau) &= e^{i\omega_m t}\tfrac{2}{a}\tfrac{i\omega_m p_m(\tau)}{k_m\sinh(k_m h)}\cosh(k_m(z+h))\cos(k_m x) \\ &\quad + e^{i\omega_n t}\tfrac{2}{a}\tfrac{i\omega_n p_n(\tau)}{k_n\sinh(k_n h)}\cosh(k_n(z+h))\cos(k_n x) + c.c. \end{aligned} \tag{3.4}$$

3.2.2 Second-order approximation

The procedure here depends on the f_i in (3.1).

a) f_i can be written as sums of $\cos(j\frac{\pi}{h}z)$

To get back from the second-order equations (3.2, 3.3) to the homogeneous second-order equations (2.56, 2.57), a function Ψ_2 such that $\psi_2 = \phi_2 - \Psi_2$ satisfies (2.56, 2.57) is considered. The equations for Ψ_2 are

$$\begin{array}{rcll} \Delta\Psi_2 & = & 0 & \text{on } [0,a]\times[-h,0]\,, \\ \Psi_{2,x} & = & B_{1,t} & \text{for } x=0\,, \\ \Psi_{2,x} & = & B_{2,t} & \text{for } x=a\,, \\ \Psi_{2,z} & = & 0 & \text{for } z=-h\,, \\ \Psi_{2,z} & = & 0 & \text{for } z=0\,. \end{array} \tag{3.5}$$

Doing separation of variables, we assume

$$\Psi_2(x,z,t,\tau) = e^{i(\omega_m+\omega_n)t+i\lambda\tau}\textstyle\sum_{j=0}^{\infty} X_j(x)Z_j(z) + c.c. \tag{3.6}$$

Plugging this into the equations and assuming $l_j \neq 0$ yields

$$\begin{array}{lrcl} \text{(i)} & X_j''(x) & = & l_j^2 X_j(x)\,, \\ \text{(ii)} & Z_j''(z) & = & -l_j^2 Z_j(z)\,, \\ \text{(iii)} & e^{i(\omega_m+\omega_n)t+i\lambda\tau}\sum_{j=0}^{\infty} X_j'(0)Z_j(z) & = & B_{1,t}\,, \\ \text{(iv)} & e^{i(\omega_m+\omega_n)t+i\lambda\tau}\sum_{j=0}^{\infty} X_j'(a)Z_j(z) & = & B_{2,t}\,, \\ \text{(v)} & Z_j'(-h) & = & 0\,, \\ \text{(vi)} & Z_j'(0) & = & 0\,. \end{array} \tag{3.7}$$

In order to be able to handle (iii) and (iv) and considering the remaining equations, it is assumed that

$$\begin{array}{rcl} X_j(x) & = & v_j\cosh(l_j x) + w_j\cosh(l_j(x-a))\,, \\ Z_j(z) & = & \cos(l_j z)\,, \end{array} \tag{3.8}$$

where $l_j = j\frac{\pi}{h}$. Now plug (3.6) into (iii) and (iv) of (3.7) to get

$$\begin{array}{rcl} \sum_{j=0}^{\infty} w_j\, l_j \sinh(-l_j a)\cos(l_j z) & = & f_1(z)\,, \\ \sum_{j=0}^{\infty} v_j\, l_j \sinh(l_j a)\cos(l_j z) & = & f_2(z)\,. \end{array} \tag{3.9}$$

Multiplying both sides by $\cos(l_j z)$ and integrating from $-h$ to 0 yields

$$\begin{array}{rcl} w_j & = & -\frac{2}{l_j \sinh(l_j a)h}\int_{-h}^{0} f_1(z)\cos(l_j z)dz\,, \\ v_j & = & \frac{2}{l_j \sinh(l_j a)h}\int_{-h}^{0} f_2(z)\cos(l_j z)dz\,. \end{array} \tag{3.10}$$

For ϕ_2 and Θ_2 to satisfy (2.55, 2.58 to 2.61, 3.2, 3.3), ψ_2 and Θ_2 need to satisfy

$$
\begin{aligned}
\Delta\psi_2 &= 0 && \text{on } [0,a]\times[-h,0]\,,\\
\psi_{2,x} &= 0 && \text{for } x=0,a\,,\\
\psi_{2,z} &= 0 && \text{for } z=-h\,,\\
\Theta_{2,t}-\psi_{2,z} &= -\phi_{1,x}\Theta_{1,x}+\phi_{1,zz}\Theta_1 && \text{for } z=0\,,\\
\psi_{2,t}+g\Theta_2-T\Theta_{2,xx} &= -\Theta_1\phi_{1,tz}-\tfrac{1}{2}\phi_{1,x}^2-\tfrac{1}{2}\phi_{1,z}^2-\Psi_{2,t} && \text{for } z=0\,,\\
\Theta_{2,x} &= 0 && \text{for } x=0,a\,.
\end{aligned}
\tag{3.11}
$$

The right side of the fourth equation in (3.11) is now

$$
\begin{aligned}
&\tfrac{4i\pi}{a^3}\overline{p_n(\tau)}p_n(\tau)\omega_n n\cos(2k_nx)\coth(k_nh)\\
&+\tfrac{4i\pi}{a^3}\overline{p_m(\tau)}p_m(\tau)\omega_m m\cos(2k_mx)\coth(k_mh)\\
&+e^{i(\omega_m-\omega_n)t}\tfrac{4i\pi}{a^3}\coth(k_mh)\\
&\qquad\times\left(m\cos(k_nx)\cos(k_mx)-n\sin(k_nx)\sin(k_mx)\right)\overline{p_n(\tau)}p_m(\tau)\omega_m\\
&+e^{2i\omega_m t}\tfrac{4i\pi}{a^3}m\cos(2k_mx)\coth(k_mh)p_m^2(\tau)\omega_m\\
&+e^{2i\omega_n t}\tfrac{4i\pi}{a^3}n\cos(2k_nx)\coth(k_nh)p_n^2(\tau)\omega_n\\
&+e^{i(\omega_m+\omega_n)t}\tfrac{4i\pi}{a^3}p_n(\tau)p_m(\tau)\\
&\quad\times\Big[\coth(k_nh)\omega_n\left(n\cos(k_nx)\cos(k_mx)-m\sin(k_nx)\sin(k_mx)\right)\\
&\qquad\coth(k_mh)\omega_m\left(m\cos(k_nx)\cos(k_mx)-n\sin(k_nx)\sin(k_mx)\right)\Big]\\
&+e^{i(\omega_n-\omega_m)t}\tfrac{4i\pi}{a^3}\coth(k_nh)\\
&\qquad\times\left(n\cos(k_nx)\cos(k_mx)-m\sin(k_nx)\sin(k_mx)\right)\overline{p_m(\tau)}p_n(\tau)\omega_n+c.c.\\
=\;&e^{i(\omega_m-\omega_n)t}\tfrac{2i\pi}{a^3}p_m(\tau)\overline{p_n(\tau)}\\
&\quad\times\Big[\cos(k_{m-n}x)(\omega_m\coth(k_mh)+\omega_n\coth(k_nh))(m-n)\\
&\qquad+\cos(k_{m+n}x)(\omega_m\coth(k_mh)-\omega_n\coth(k_nh))(m+n)\Big]\\
&+e^{2i\omega_m t}\tfrac{4i\pi}{a^3}m\cos(2k_mx)\coth(k_mh)p_m^2(\tau)\omega_m\\
&+e^{2i\omega_n t}\tfrac{4i\pi}{a^3}n\cos(2k_nx)\coth(k_nh)p_n^2(\tau)\omega_n\\
&+e^{i(\omega_m+\omega_n)t}\tfrac{2i\pi}{a^3}p_n(\tau)p_m(\tau)\\
&\quad\times\Big[\cos(k_{m-n}x)(\omega_m\coth(k_mh)-\omega_n\coth(k_nh))(m-n)\\
&\qquad+\cos(k_{m+n}x)(\omega_m\coth(k_mh)+\omega_n\coth(k_nh))(m+n)\Big]+c.c.
\end{aligned}
\tag{3.12}
$$

Here, the constant terms with respect to t are cancelled out. For the terms with frequency $i(\omega_m-\omega_n)t$, we added the terms of that frequency and the complex conjugate of the terms with frequency $i(\omega_n-\omega_m)t$. Using trigonometric formulas, we obtained

$$
\begin{aligned}
& \tfrac{4i\pi}{a^3}\coth(k_m h)\left(m\cos(k_n x)\cos(k_m x) - n\sin(k_n x)\sin(k_m x)\right)\overline{p_n(\tau)}p_m(\tau)\omega_m \\
& -\tfrac{4i\pi}{a^3}\coth(k_n h)\left(n\cos(k_n x)\cos(k_m x) - m\sin(k_n x)\sin(k_m x)\right)p_m(\tau)\overline{p_n(\tau)}\omega_n \\
= \; & \tfrac{2i\pi}{a^3}p_m(\tau)\overline{p_n(\tau)} \\
& \quad \times\Big[\coth(k_m h)\omega_m\left(m(\cos(k_{m-n}x) + \cos(k_{m+n}x)) - n(\cos(k_{m-n}x) - \cos(k_{m+n}x))\right) \\
& \qquad - \coth(k_n h)\omega_n\left(n(\cos(k_{m-n}x) + \cos(k_{m+n}x)) - m(\cos(k_{m-n}x) - \cos(k_{m+n}x))\right)\Big] \\
= \; & \tfrac{2i\pi}{a^3}p_m(\tau)\overline{p_n(\tau)} \\
& \quad \times\Big[\cos(k_{m-n}x)(\omega_m\coth(k_m h) + \omega_n\coth(k_n h))(m-n) \\
& \qquad + \cos(k_{m+n}x)(\omega_m\coth(k_m h) - \omega_n\coth(k_n h))(m+n)\Big].
\end{aligned}
\tag{3.13}
$$

The right side of the fifth equation in (3.11) is

$$
\begin{aligned}
& \tfrac{2}{a^2}\overline{p_m(\tau)}p_m(\tau)\omega_m^2\left(\cos^2(k_m x) - \coth^2(k_m h)\sin^2(k_m x)\right) \\
& +\tfrac{2}{a^2}\overline{p_n(\tau)}p_n(\tau)\omega_n^2\left(\cos^2(k_n x) - \coth^2(k_n h)\sin^2(k_n x)\right) \\
& +e^{i(\omega_n-\omega_m)t}\tfrac{2}{a^2}\overline{p_m(\tau)}p_n(\tau)\omega_m \\
& \quad \times\Big(2\cos(k_m x)\cos(k_n x)\omega_m - \cos(k_m x)\cos(k_n x)\omega_n \\
& \qquad - \coth(k_m h)\coth(k_n h)\sin(k_m x)\sin(k_n x)\omega_n\Big) \\
& +e^{2i\omega_m t}\tfrac{2}{a^2}p_m^2(\tau)\omega_m^2\left(3\cos^2(k_m x) + \coth^2(k_m h)\sin^2(k_m x)\right) \\
& +e^{2i\omega_n t}\tfrac{2}{a^2}p_n^2(\tau)\omega_n^2\left(3\cos^2(k_n x) + \coth^2(k_n h)\sin^2(k_n x)\right) \\
& +e^{i(\omega_m-\omega_n)t}\tfrac{2}{a^2}p_m(\tau)\overline{p_n(\tau)}\omega_n \\
& \quad \times\Big(2\cos(k_m x)\cos(k_n x)\omega_n - \cos(k_m x)\cos(k_n x)\omega_m \\
& \qquad - \coth(k_m h)\coth(k_n h)\sin(k_m x)\sin(k_n x)\omega_m\Big) \\
& +e^{i(\omega_m+\omega_n)t}\tfrac{4}{a^2}p_m(\tau)p_n(\tau) \\
& \quad \times\Big((\omega_m^2+\omega_n^2)\cos(k_m x)\cos(k_n x) + \cos(k_m x)\cos(k_n x)\omega_m\omega_n \\
& \qquad + \coth(k_m h)\coth(k_n h)\sin(k_m x)\sin(k_n x)\omega_m\omega_n\Big) \\
& -e^{i(\omega_m+\omega_n)t+i\lambda\tau}i(\omega_m+\omega_n) \\
& \qquad\qquad \times\textstyle\sum_{j=0}^{\infty}\left(v_j\cosh(l_j x) + w_j\cosh(l_j(x-a))\right)\cos(l_j 0) + c.c.
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{1}{a^2}\left(|p_m(\tau)|^2\omega_m^2(1-\coth^2(k_m h)) + |p_n(\tau)|^2\omega_n^2(1-\coth^2(k_n h))\right)\\
&\quad + \cos(2k_m x)\tfrac{1}{a^2}|p_m(\tau)|^2\omega_m^2\left(1+\coth^2(k_m h)\right)\\
&\quad + \cos(2k_n x)\tfrac{1}{a^2}|p_n(\tau)|^2\omega_n^2\left(1+\coth^2(k_n h)\right)\\
&\quad + e^{2i\omega_m t}\tfrac{1}{a^2}p_m^2(\tau)\omega_m^2\left(3+\coth^2(k_m h)\right)\\
&\quad + e^{2i\omega_m t}\cos(2k_m x)\tfrac{1}{a^2}p_m^2(\tau)\omega_m^2\left(3-\coth^2(k_m h)\right)\\
&\quad + e^{2i\omega_n t}\tfrac{1}{a^2}p_n^2(\tau)\omega_n^2\left(3+\coth^2(k_n h)\right)\\
&\quad + e^{2i\omega_n t}\cos(2k_n x)\tfrac{1}{a^2}p_n^2(\tau)\omega_n^2\left(3-\coth^2(k_n h)\right)\\
&\quad + e^{i(\omega_m-\omega_n)t}\cos(k_{m-n}x)\tfrac{1}{a^2}p_m(\tau)\overline{p_n(\tau)}\\
&\qquad \times\left(2\omega_m^2+2\omega_n^2-2\omega_m\omega_n-\coth(k_m h)\coth(k_n h)(\omega_m^2+\omega_n^2)\right)\\
&\quad + e^{i(\omega_m-\omega_n)t}\cos(k_{m+n}x)\tfrac{1}{a^2}p_m(\tau)\overline{p_n(\tau)}\\
&\qquad \times\left(2\omega_m^2+2\omega_n^2-2\omega_m\omega_n+\coth(k_m h)\coth(k_n h)(\omega_m^2+\omega_n^2)\right)\\
&\quad + e^{i(\omega_m+\omega_n)t}\cos(k_{m-n}x)\tfrac{2}{a^2}p_m(\tau)p_n(\tau)\\
&\qquad \times\left(\omega_m^2+\omega_n^2+\omega_m\omega_n+\coth(k_m h)\coth(k_n h)\omega_m\omega_n\right)\\
&\quad + e^{i(\omega_m+\omega_n)t}\cos(k_{m+n}x)\tfrac{2}{a^2}p_m(\tau)p_n(\tau)\\
&\qquad \times\left(\omega_m^2+\omega_n^2+\omega_m\omega_n-\coth(k_m h)\coth(k_n h)\omega_m\omega_n\right)\\
&\quad - e^{i(\omega_m+\omega_n)t+i\lambda\tau}i(\omega_m+\omega_n)\\
&\qquad\qquad \times\textstyle\sum_{j=0}^{\infty}\left(v_j\cosh(l_j x)+w_j\cosh(l_j(x-a))\right)+c.c.
\end{aligned}
\tag{3.14}
$$

This leads to the assumption that the second-order approximation is of the form

$$
\begin{aligned}
\Theta_2(x,t,\tau) &= c_1 + c_2 e^{2i\omega_m t} + c_3 e^{2i\omega_n t}\\
&\quad +(c_{1,2m}+c_{2,2m}e^{2i\omega_m t})\cos(2k_m x)\\
&\quad +(c_{1,2n}+c_{2,2n}e^{2i\omega_n t})\cos(2k_n x)\\
&\quad +c_{-,-}e^{i(\omega_m-\omega_n)t}\cos(k_{m-n}x)+c_{+,-}e^{i(\omega_m+\omega_n)t}\cos(k_{m-n}x)\\
&\quad +c_{-,+}e^{i(\omega_m-\omega_n)t}\cos(k_{m+n}x)+c_{+,+}e^{i(\omega_m+\omega_n)t}\cos(k_{m+n}x)+c.c.\,,\\
\phi_2(x,z,t,\tau) &= d_2 e^{2i\omega_m t}+d_3 e^{2i\omega_n t}\\
&\quad +(d_{1,2m}+d_{2,2m}e^{2i\omega_m t})\cos(2k_m x)\cosh(2k_m(z+h))\\
&\quad +(d_{1,2n}+d_{2,2n}e^{2i\omega_n t})\cos(2k_n x)\cosh(2k_n(z+h))\\
&\quad +d_{-,-}e^{i(\omega_m-\omega_n)t}\cos(k_{m-n}x)\cosh(k_{m-n}(z+h))\\
&\quad +d_{+,-}e^{i(\omega_m+\omega_n)t}\cos(k_{m-n}x)\cosh(k_{m-n}(z+h))\\
&\quad +d_{-,+}e^{i(\omega_m-\omega_n)t}\cos(k_{m+n}x)\cosh(k_{m+n}(z+h))\\
&\quad +d_{+,+}e^{i(\omega_m+\omega_n)t}\cos(k_{m+n}x)\cosh(k_{m+n}(z+h))\\
&\quad +\textstyle\sum_{\sigma=0}^{\infty}u_\sigma e^{i(\omega_m+\omega_n)t+i\lambda\tau}\cos(k_\sigma x)+\Psi_2+c.c.
\end{aligned}
\tag{3.15}
$$

The equations for the coefficients are:

$$
\begin{aligned}
gc_1 &= \gamma_1 \,, \\
2i\omega_m c_2 &= 0 \,, \\
2i\omega_m d_2 + gc_2 &= \tfrac{1}{a^2} p_m(\tau)^2 \omega_m^2 (3 + \coth^2(k_m h)) \,, \\
2i\omega_n c_3 &= 0 \,, \\
2i\omega_n d_3 + gc_3 &= \tfrac{1}{a^2} p_n(\tau)^2 \omega_n^2 (3 + \coth^2(k_n h)) \,, \\
-2k_m \sinh(2k_m h) d_{1,2m} &= 0 \,, \\
\alpha_{2m} c_{1,2m} &= \tfrac{1}{a^2} |p_m(\tau)|^2 \omega_m^2 (1 + \coth^2(k_m h)) \,, \\
2i\omega_m c_{2,2m} - 2k_m \sinh(2k_m h) d_{2,2m} &= \tfrac{4i\pi}{a^3} m \coth(k_m h) p_m^2(\tau) \omega_m \,, \\
\alpha_{2m} c_{2,2m} + 2i\omega_m \cosh(2k_m h) d_{2,2m} &= \tfrac{1}{a^2} p_m^2(\tau) \omega_m^2 (3 - \coth^2(k_m h)) \,, \\
-2k_n \sinh(2k_n h) d_{1,2n} &= 0 \,, \\
\alpha_{2n} c_{1,2n} &= \tfrac{1}{a^2} |p_n(\tau)|^2 \omega_n^2 (1 + \coth^2(k_n h)) \,, \\
2i\omega_n c_{2,2n} - 2k_n \sinh(2k_n h) d_{2,2n} &= \tfrac{4i\pi}{a^3} n \coth(k_n h) p_n^2(\tau) \omega_n \,, \\
\alpha_{2n} c_{2,2n} + 2i\omega_n \cosh(2k_n h) d_{2,2n} &= \tfrac{1}{a^2} p_n^2(\tau) \omega_n^2 (3 - \coth^2(k_n h)) \,, \\
i(\omega_m - \omega_n) c_{-,-} - k_{m-n} \sinh(k_{m-n} h) d_{-,-} &= \gamma_2 \,, \\
\alpha_{m-n} c_{-,-} + i(\omega_m - \omega_n) \cosh(k_{m-n} h) d_{-,-} &= \gamma_3 \,, \\
i(\omega_m - \omega_n) c_{-,+} - k_{m+n} \sinh(k_{m+n} h) d_{-,+} &= \gamma_4 \,, \\
\alpha_{m+n} c_{-,+} + i(\omega_m - \omega_n) \cosh(k_{m+n} h) d_{-,+} &= \gamma_5 \,, \\
i(\omega_m + \omega_n) c_{+,-} - k_{m-n} \sinh(k_{m-n} h) d_{+,-} &= \gamma_6 \,, \\
\alpha_{m-n} c_{+,-} + i(\omega_m + \omega_n) \cosh(k_{m-n} h) d_{+,-} &= \gamma_7 \,, \\
i(\omega_m + \omega_n) c_{+,+} - k_{m+n} \sinh(k_{m+n} h) d_{+,+} &= \gamma_8 \,, \\
\alpha_{m+n} c_{+,+} + i(\omega_m + \omega_n) \cosh(k_{m+n} h) d_{+,+} &= \gamma_9 \,, \\
\textstyle\sum_{\sigma=0}^{\infty} u_\sigma \cos(k_\sigma x) &= \gamma_{10} \,.
\end{aligned}
\tag{3.16}
$$

The constants γ_i are given by

$$
\begin{aligned}
\gamma_1 &= \tfrac{1}{a^2} |p_m(\tau)|^2 \omega_m^2 (1 - \coth^2(k_m h)) + \tfrac{1}{a^2} |p_n(\tau)|^2 \omega_n^2 (1 - \coth^2(k_n h)) \,, \\
\gamma_2 &= \tfrac{2i\pi}{a^3} p_m(\tau) \overline{p_n(\tau)} \left(\omega_m \coth(k_m h) + \omega_n \coth(k_n h) \right) (m - n) \,, \\
\gamma_3 &= \tfrac{1}{a^2} p_m(\tau) \overline{p_n(\tau)} \left(2\omega_m^2 + 2\omega_n^2 - 2\omega_m \omega_n - \coth(k_m h) \coth(k_n h) (\omega_m^2 + \omega_n^2) \right) \,, \\
\gamma_4 &= \tfrac{2i\pi}{a^3} p_m(\tau) \overline{p_n(\tau)} \left(\omega_m \coth(k_m h) - \omega_n \coth(k_n h) \right) (m + n) \,, \\
\gamma_5 &= \tfrac{1}{a^2} p_m(\tau) \overline{p_n(\tau)} \left(2\omega_m^2 + 2\omega_n^2 - 2\omega_m \omega_n + \coth(k_m h) \coth(k_n h) (\omega_m^2 + \omega_n^2) \right) \,, \\
\gamma_6 &= \tfrac{2i\pi}{a^3} p_n(\tau) p_m(\tau) \left(\omega_m \coth(k_m h) - \omega_n \coth(k_n h) \right) (m - n) \,, \\
\gamma_7 &= \tfrac{2}{a^2} p_m(\tau) p_n(\tau) \left(\omega_m^2 + \omega_n^2 + \omega_m \omega_n + \coth(k_m h) \coth(k_n h) \omega_m \omega_n \right) \,, \\
\gamma_8 &= \tfrac{2i\pi}{a^3} p_n(\tau) p_m(\tau) \left(\omega_m \coth(k_m h) + \omega_n \coth(k_n h) \right) (m + n) \,, \\
\gamma_9 &= \tfrac{2}{a^2} p_m(\tau) p_n(\tau) \left(\omega_m^2 + \omega_n^2 + \omega_m \omega_n - \coth(k_m h) \coth(k_n h) \omega_m \omega_n \right) \,, \\
\gamma_{10} &= -i(\omega_m + \omega_n) \textstyle\sum_{j=0}^{\infty} \left(v_j \cosh(l_j x) + w_j \cosh(l_j (x - a)) \right) \,.
\end{aligned}
\tag{3.17}
$$

Solving those equations yields

$$
\begin{aligned}
c_1 &= \tfrac{1}{a^2 g}|p_m(\tau)|^2\omega_m^2(1-\coth^2(k_m h)) + \tfrac{1}{a^2 g}|p_n(\tau)|^2\omega_n^2(1-\coth^2(k_n h))\,,\\
c_2 &= 0\,,\\
d_2 &= \tfrac{1}{2i\omega_m}\tfrac{1}{a^2}|p_m(\tau)|^2\omega_m^2(3+\coth^2(k_m h))\,,\\
c_3 &= 0\,,\\
d_3 &= \tfrac{1}{2i\omega_n}\tfrac{1}{a^2}|p_n(\tau)|^2\omega_n^2(3+\coth^2(k_n h))\,,\\
d_{1,2m} &= 0\,,\\
c_{1,2m} &= \tfrac{1}{a^2(g+(2k_m)^2T)}p_m(\tau)\overline{p_m(\tau)}\omega_m^2(1+\coth^2(k_m h))\,,\\
c_{2,2m} &= \tfrac{2m\pi\coth(k_m h)p_m^2(\tau)(-2+\cosh(2k_m h)-2\omega_m)\omega_m}{m\pi(a^2g+4m^2\pi^2T)\sinh(2k_m h)-2a^3\omega_m^2}\,,\\
d_{2,2m} &= \tfrac{i\,\mathrm{csch}^2(k_m h)p_m^2(\tau)\omega_m\left(m\pi(a^2g+4m^2\pi^2T)\sinh(2k_m h)-a^3\omega_m(-2+\cosh(2k_m h))\right)}{-a^2m\pi(a^2g+4m^2\pi^2T)\sinh(2k_m h)+2a^5\omega_m^2}\,,\\
d_{1,2n} &= 0\,,\\
c_{1,2n} &= \tfrac{1}{a^2(g+(2k_n)^2T)}p_n(\tau)\overline{p_n(\tau)}\omega_n^2(1+\coth^2(k_n h))\,,\\
c_{2,2n} &= \tfrac{2n\pi\coth(k_n h)p_n^2(\tau)(-2+\cosh(2k_n h)-2\omega_n)\omega_n}{n\pi(a^2g+4n^2\pi^2T)\sinh(2k_n h)-2a^3\omega_n^2}\,,\\
d_{2,2n} &= \tfrac{i\,\mathrm{csch}^2(k_n h)p_n^2(\tau)\omega_n\left(n\pi(a^2g+4n^2\pi^2T)\sinh(2k_n h)-a^3\omega_n(-2+\cosh(2k_n h))\right)}{-a^2n\pi(a^2g+4n^2\pi^2T)\sinh(2k_n h)+2a^5\omega_n^2}\,,\\
c_{-,-} &= -\tfrac{i}{(\omega_m-\omega_n)}\left(\gamma_2-\tfrac{\sinh(k_{m-n}h)k_{m-n}[(g+Tk_{m-n}^2)\gamma_2-i\gamma_3(\omega_m-\omega_n)]}{\sinh(k_{m-n}h)k_{m-n}(g+Tk_{m-n}^2)-\cosh(k_{m-n}h)(\omega_m-\omega_n)}\right)\,,\\
d_{-,-} &= \tfrac{(g+Tk_{m-n}^2)\gamma_2-i\gamma_3(\omega_m-\omega_n)}{\sinh(k_{m-n}h)k_{m-n}(g+Tk_{m-n}^2)-\cosh(k_{m-n}h)(\omega_m-\omega_n)}\,,\\
c_{-,+} &= -\tfrac{i}{(\omega_m-\omega_n)}\left(\gamma_4-\tfrac{\sinh(k_{m+n}h)k_{m+n}[(g+Tk_{m+n}^2)\gamma_4-i\gamma_5(\omega_m-\omega_n)]}{\sinh(k_{m+n}h)k_{m+n}(g+Tk_{m+n}^2)-\cosh(k_{m+n}h)(\omega_m-\omega_n)}\right)\,,\\
d_{-,+} &= \tfrac{(g+Tk_{m+n}^2)\gamma_4-i\gamma_5(\omega_m-\omega_n)}{\sinh(k_{m+n}h)k_{m+n}(g+Tk_{m+n}^2)-\cosh(k_{m+n}h)(\omega_m-\omega_n)}\,,\\
c_{+,-} &= -\tfrac{i}{(\omega_m+\omega_n)}\left(\gamma_6-\tfrac{\sinh(k_{m-n}h)k_{m-n}[(g+Tk_{m-n}^2)\gamma_6-i\gamma_7(\omega_m+\omega_n)]}{\sinh(k_{m-n}h)k_{m-n}(g+Tk_{m-n}^2)-\cosh(k_{m-n}h)(\omega_m+\omega_n)}\right)\,,\\
d_{+,-} &= \tfrac{(g+Tk_{m-n}^2)\gamma_6-i\gamma_7(\omega_m+\omega_n)}{\sinh(k_{m-n}h)k_{m-n}(g+Tk_{m-n}^2)-\cosh(k_{m-n}h)(\omega_m+\omega_n)}\,,\\
c_{+,+} &= -\tfrac{i}{(\omega_m+\omega_n)}\left(\gamma_8-\tfrac{\sinh(k_{m+n}h)k_{m+n}[(g+Tk_{m+n}^2)\gamma_8-i\gamma_9(\omega_m+\omega_n)]}{\sinh(k_{m+n}h)k_{m+n}(g+Tk_{m+n}^2)-\cosh(k_{m-n}h)(\omega_m+\omega_n)}\right)\,,\\
d_{+,+} &= \tfrac{(g+Tk_{m+n}^2)\gamma_8-i\gamma_9(\omega_m+\omega_n)}{\sinh(k_{m+n}h)k_{m+n}(g+Tk_{m+n}^2)-\cosh(k_{m-n}h)(\omega_m+\omega_n)}\,,\\
u_\sigma &= -\tfrac{2i}{a(\omega_m+\omega_n)}\int_0^a\left(\textstyle\sum_{j=0}^\infty v_j\cosh(l_jx)+w_j\cosh(l_j(x-a))\right)\cos(k_\sigma x)dx\,.
\end{aligned}
\tag{3.18}
$$

b) f_1 and f_2 are constants

Because according to Neumann's condition the integral of the normal derivatives over the boundary vanishes, (3.5) has to be modified. We do so by dropping one condition and consider

$$
\begin{aligned}
\Delta\Psi_2 &= 0\,,\\
\Psi_{2,x} &= B_{1,t} \quad \text{for } x=0\,,\\
\Psi_{2,x} &= B_{2,t} \quad \text{for } x=a\,,\\
\Psi_{2,z} &= 0 \qquad \text{for } z=-h\,.
\end{aligned}
\tag{3.19}
$$

Assuming $f_1 = a_1$ and $f_2 = a_2$, we can choose

$$\Psi_2 = e^{i(\omega_m+\omega_n)t+i\lambda\tau}\left(\tfrac{a_2-a_1}{2a}(x^2-z^2)+a_1x-\tfrac{h(a_2-a_1)}{a}z\right)+c.c. \qquad (3.20)$$

Now the condition in the equations for ψ_2 that was dropped to satisfy Neumann's condition [compare (3.11)] has to be incorporated

$$\begin{aligned}
\Delta\psi_2 &= 0\,, & \\
\psi_{2,x} &= 0 & \text{for } x=0,a\,,\\
\psi_{2,z} &= 0 & \text{for } z=-h\,,\\
\Theta_{2,t}-\psi_{2,z} &= -\phi_{1,x}\Theta_{1,x}+\phi_{1,zz}\Theta_1+\Psi_{2,z} & \text{for } z=0\,,\\
\psi_{2,t}+g\Theta_2-T\Theta_{2,xx} &= -\Theta_1\phi_{1,tz}-\tfrac{1}{2}\phi_{1,x}^2-\tfrac{1}{2}\phi_{1,z}^2-\Psi_{2,t} & \text{for } z=0\,,\\
\Theta_{2,x} &= 0 & \text{for } x=0,a\,.
\end{aligned} \qquad (3.21)$$

Note that

$$\begin{aligned}
\Psi_{2,z}|_{z=0} &= e^{i(\omega_m+\omega_n)t+i\lambda\tau}\left(-\tfrac{h(a_2-a_1)}{a}\right)+c.c.\,,\\
\Psi_{2,t}|_{z=0} &= e^{i(\omega_m+\omega_n)t+i\lambda\tau}i(\omega_m+\omega_n)\left(\tfrac{a_2-a_1}{2a}x^2+a_1x\right)+c.c.
\end{aligned}$$

While $\Psi_{2,z}|_{z=0}$ is already in the form needed, $\Psi_{2,t}|_{z=0}$ has to be rewritten using a cosine expansion

$$\frac{a_2-a_1}{2a}x^2+a_1x=\frac{1}{6}a(2a_1+a_2)-\sum_{j=1}^{\infty}\frac{2a(a_1-(-1)^ja_2)}{j^2\pi^2}\cos(k_jx)\,.$$

Now it is assumed that ψ_2 and Θ_2 are of the form

$$\begin{aligned}
\Theta_2(x,t,\tau) &= c_1+c_2e^{2i\omega_mt}+c_3e^{2i\omega_nt}\\
&\quad+(c_{1,2m}+c_{2,2m}e^{2i\omega_mt})\cos(2k_mx)\\
&\quad+(c_{1,2n}+c_{2,2n}e^{2i\omega_nt})\cos(2k_nx)\\
&\quad+c_{-,-}e^{i(\omega_m-\omega_n)t}\cos(k_{m-n}x)+c_{+,-}e^{i(\omega_m+\omega_n)t}\cos(k_{m-n}x)\\
&\quad+c_{-,+}e^{i(\omega_m-\omega_n)t}\cos(k_{m+n}x)+c_{+,+}e^{i(\omega_m+\omega_n)t}\cos(k_{m+n}x)\\
&\quad+e^{i(\omega_m+\omega_n)+i\lambda\tau}\textstyle\sum_{j=0}^{\infty}\beta_{1,j}\cos(k_j\,x)+c.c.\,,\\
\phi_2(x,z,t,\tau) &= d_2e^{2i\omega_mt}+d_3e^{2i\omega_nt}\\
&\quad+(d_{1,2m}+d_{2,2m}e^{2i\omega_mt})\cos(2k_mx)\cosh(2k_m(z+h))\\
&\quad+(d_{1,2n}+d_{2,2n}e^{2i\omega_nt})\cos(2k_nx)\cosh(2k_n(z+h))\\
&\quad+d_{-,-}e^{i(\omega_m-\omega_n)t}\cos(k_{m-n}x)\cosh(k_{m-n}(z+h))\\
&\quad+d_{+,-}e^{i(\omega_m+\omega_n)t}\cos(k_{m-n}x)\cosh(k_{m-n}(z+h))\\
&\quad+d_{-,+}e^{i(\omega_m-\omega_n)t}\cos(k_{m+n}x)\cosh(k_{m+n}(z+h))\\
&\quad+d_{+,+}e^{i(\omega_m+\omega_n)t}\cos(k_{m+n}x)\cosh(k_{m+n}(z+h))\\
&\quad+e^{i(\omega_m+\omega_n)t+i\lambda\tau}\textstyle\sum_{j=0}^{\infty}\beta_{2,j}\cos(k_j\,x)\cosh(k_j\,(z+h))+\Psi_2+c.c.
\end{aligned} \qquad (3.22)$$

with the constants given by (3.18). In addition to that, for $j = 0$

$$\begin{array}{rcrcl} i(\omega_m+\omega_n)\beta_{1,0} & & & = & -\frac{h(a_2-a_1)}{a}\,, \\ g\beta_{1,0} & + & i(\omega_m+\omega_n)\beta_{2,0} & = & -i(\omega_m+\omega_n)\frac{1}{6}a(2a_1+a_2)\,, \end{array}$$

and for $j \neq 0$

$$\begin{array}{rcrcl} i(\omega_m+\omega_n)\beta_{1,j} & - & k_j\sinh(k_j\,h)\beta_{2,j} & = & 0\,, \\ (g+Tk_j^2)\beta_{1,j} & + & i(\omega_m+\omega_n)\cosh(k_j\,h)\beta_{2,j} & = & i(\omega_m+\omega_n)\frac{2a(a_1-(-1)^j a_2)}{j^2\pi^2}\,. \end{array}$$

Solving this yields

$$\begin{array}{rcll} \beta_{1,0} & = & i\frac{h(a_2-a_1)}{(\omega_m+\omega_n)a}\,, & \\ \beta_{2,0} & = & -\frac{1}{6}a(2a_1+a_2)+\frac{(a_1-a_2)g\,h}{(\omega_m+\omega_n)^2 a}\,, & \\ \beta_{1,j} & = & i\frac{2a(a_1-(-1)^j a_2)k_j(\omega_m+\omega_n)\sinh(k_j h)}{-j^2\pi^2[\cosh(k_j\,h)(\omega_m+\omega_n)^2-k_j(g+k_j^2T)\sinh(k_j h)]} & \text{for } j\neq 0\,, \\ \beta_{2,j} & = & \frac{2a(a_1-(-1)^j a_2)(\omega_m+\omega_n)^2}{j^2\pi^2[\cosh(k_j\,h)(\omega_m+\omega_n)^2-k_j(g+k_j^2T)\sinh(k_j h)]} & \text{for } j\neq 0\,. \end{array} \tag{3.23}$$

c) Other expressions for f_1 and f_2

If the expressions for f_1 and f_2 have forms different from a) and b), the previously found solutions can be combined. To do so, the cosine expansion of f_1 and f_2 is needed. Assuming that a_1 and a_2 are the constant terms in the cosine expansions of f_1 and f_2, the solution can be written as

$$\begin{array}{rcl} \Theta_2(x,t,\tau) & = & c_1+c_2e^{2i\omega_m t}+c_3e^{2i\omega_n t} \\ & & +(c_{1,2m}+c_{2,2m}e^{2i\omega_m t})\cos(2k_m x) \\ & & +(c_{1,2n}+c_{2,2n}e^{2i\omega_n t})\cos(2k_n x) \\ & & +c_{-,-}e^{i(\omega_m-\omega_n)t}\cos(k_{m-n}x)+c_{+,-}e^{i(\omega_m+\omega_n)t}\cos(k_{m-n}x) \\ & & +c_{-,+}e^{i(\omega_m-\omega_n)t}\cos(k_{m+n}x)+c_{+,+}e^{i(\omega_m+\omega_n)t}\cos(k_{m+n}x) \\ & & +e^{i(\omega_m+\omega_n)+i\lambda\tau}\sum_{j=0}^{\infty}\beta_{1,j}\cos(k_j x)+c.c.\,, \\ \phi_2(x,z,t,\tau) & = & d_2e^{2i\omega_m t}+d_3e^{2i\omega_n t} \\ & & +(d_{1,2m}+d_{2,2m}e^{2i\omega_m t})\cos(2k_m x)\cosh(2k_m(z+h)) \\ & & +(d_{1,2n}+d_{2,2n}e^{2i\omega_n t})\cos(2k_n x)\cosh(2k_n(z+h)) \\ & & +d_{-,-}e^{i(\omega_m-\omega_n)t}\cos(k_{m-n}x)\cosh(k_{m-n}(z+h)) \\ & & +d_{+,-}e^{i(\omega_m+\omega_n)t}\cos(k_{m-n}x)\cosh(k_{m-n}(z+h)) \\ & & +d_{-,+}e^{i(\omega_m-\omega_n)t}\cos(k_{m+n}x)\cosh(k_{m+n}(z+h)) \\ & & +d_{+,+}e^{i(\omega_m+\omega_n)t}\cos(k_{m+n}x)\cosh(k_{m+n}(z+h)) \\ & & +e^{i(\omega_m+\omega_n)t+i\lambda\tau}\sum_{j=0}^{\infty}(u_j+\beta_{2,j})\cos(k_j\,x)\cosh(k_j\,(z+h))+\Psi_2+c.c. \end{array} \tag{3.24}$$

with the constants given by (3.18) and (3.23) and Ψ_2 being the sum of the definitions in (3.6, 3.20).

3.2.3 Third-order approximation and solvability condition

The third-order approximation ϕ_3 needs to satisfy (2.55). To find a condition for the terms of frequency ω_m, we multiply by $\cos(k_m x)\cosh(k_m(z+h))$ and integrate over the domain:

$$\int_{-h}^{0}\left(\int_{0}^{a}(\phi_{3,xx}+\phi_{3,zz})\cos(k_m x)\cosh(k_m(z+h))dx\right)dz = 0\,.$$

As in Section 2.3.4, we get (2.73). Using (2.55 ,2.58 to 2.61, 3.2, 3.3), this becomes

$$\begin{aligned}&\textstyle\int_{-h}^{0}((-1)^m[B_{2,z}\phi_{1,z}-B_2\phi_{1,xx}]|_{x=a}-[B_{1,z}\phi_{1,z}-B_1\phi_{1,xx}]|_{x=0})\cosh(k_m(z+h))dz\\ &\quad+\textstyle\int_0^a\left(i\omega_m\cosh(k_m h)+\frac{k_m}{i\omega_m}(g+T\,k_m^2)\sinh(k_m h)\right)\Theta_3\cos(k_m x)dx\\ &\quad-\textstyle\int_0^a\left([\text{r.s. (2.59)}]\cosh(k_m h)+\frac{k_m}{i\omega_m}[\text{r.s. (2.60)}]\sinh(k_m h)\right)\cos(k_m x)dx \quad =0\,.\end{aligned} \tag{3.25}$$

The middle line of this equation again vanishes by the definition of ω_m. In order to find the terms of the third-order equations in (2.59) and (2.60) that have frequency ω_m, use the notation

$$\begin{array}{ll}\Theta_{1,m} & \text{terms of } \Theta_1 \text{ that have frequency } \omega_m\,,\\ \Theta_{1,n} & \text{terms of } \Theta_1 \text{ that have frequency } \omega_n\,,\\ \Theta_{2,con} & \text{terms of } \Theta_2 \text{ that are constant with respect to } t\,,\\ \Theta_{2,2m} & \text{terms of } \Theta_2 \text{ that have frequency } 2\omega_m\,,\\ \Theta_{2,2n} & \text{terms of } \Theta_2 \text{ that have frequency } 2\omega_n\,,\\ \Theta_{2,m-n} & \text{terms of } \Theta_2 \text{ that have frequency } \omega_m-\omega_n\,,\\ \Theta_{2,m+n} & \text{terms of } \Theta_2 \text{ that have frequency } \omega_m+\omega_n\,,\\ \Theta_{2,n-m} & \text{terms of } \Theta_2 \text{ that have frequency } \omega_n-\omega_m\,.\end{array} \tag{3.26}$$

An equivalent notation is introduced for ϕ_1 and ϕ_2. Then the terms on the right side of (2.59) are

$$
\begin{aligned}
&-\Big(\phi_{2,con,x}\Theta_{1,m,x} + \phi_{2,2m,x}\overline{\Theta_{1,m,x}} + \phi_{2,m-n,x}\Theta_{1,n,x} + \phi_{2,m+n,x}\overline{\Theta_{1,n,x}}\\
&\quad +\phi_{1,m,xz}\Theta_{1,m,x}\overline{\Theta_{1,m}} + \phi_{1,m,xz}\overline{\Theta_{1,m,x}}\Theta_{1,m} + \phi_{1,m,xz}\Theta_{1,n,x}\overline{\Theta_{1,n}}\\
&\quad +\phi_{1,m,xz}\overline{\Theta_{1,n,x}}\Theta_{1,n} + \overline{\phi_{1,m,xz}}\Theta_{1,m,x}\Theta_{1,m} + \phi_{1,n,xz}\Theta_{1,m,x}\overline{\Theta_{1,n}}\\
&\quad +\phi_{1,n,xz}\overline{\Theta_{1,n,x}}\Theta_{1,m} + \overline{\phi_{1,n,xz}}\Theta_{1,n,x}\Theta_{1,m} + \overline{\phi_{1,n,xz}}\Theta_{1,m,x}\Theta_{1,n}\\
&\quad +\Theta_{2,con,x}\phi_{1,m,x} + \Theta_{2,2m,x}\overline{\phi_{1,m,x}} + \Theta_{2,m-n,x}\phi_{1,n,x} + \Theta_{2,m+n,x}\overline{\phi_{1,n,x}}\Big)\\
&-\Big(\Theta_{2,con}\phi_{1,m,zz} + \Theta_{2,2m}\overline{\phi_{1,m,zz}} + \Theta_{2,m-n}\phi_{1,n,zz} + \Theta_{2,m+n}\overline{\phi_{1,n,zz}}\\
&\quad +\phi_{2,con,zz}\Theta_{1,m} + \phi_{2,2m,zz}\overline{\Theta_{1,m}} + \phi_{2,m-n,zz}\Theta_{1,n} + \phi_{2,m+n,zz}\overline{\Theta_{1,n}}\\
&\quad +\phi_{1,m,zzz}\Theta_m\overline{\Theta_m} + \phi_{1,m,zzz}\Theta_n\overline{\Theta_n} + \tfrac{1}{2}\overline{\phi_{1,m,zzz}}\Theta_m^2 + \phi_{1,n,zzz}\Theta_m\overline{\Theta_n} + \overline{\phi_{1,n,zzz}}\Theta_m\Theta_n\Big)\,.
\end{aligned}
\tag{3.27}
$$

To check this, it should be noted that the product of a second-order term and a first-order term splits up into four terms, while the product of three first-order terms splits up into nine terms. In the later case, if two of the first-order terms are equal (i.e. a product of two first-order terms, one being squared), this reduces to five terms.

Similarly, the terms on the right side of (2.60) are

$$
\begin{aligned}
&-\Big(\phi_{2,con,x}\phi_{1,m,x} + \phi_{2,2m,x}\overline{\phi_{1,m,x}} + \phi_{2,m-n,x}\phi_{1,n,x} + \phi_{2,m+n,x}\overline{\phi_{1,n,x}}\\
&\quad +\phi_{1,m,xz}\phi_{1,m,x}\overline{\Theta_{1,m}} + \phi_{1,m,xz}\overline{\phi_{1,m,x}}\Theta_{1,m} + \phi_{1,m,xz}\phi_{1,n,x}\overline{\Theta_{1,n}}\\
&\quad +\phi_{1,m,xz}\overline{\phi_{1,n,x}}\Theta_{1,n} + \overline{\phi_{1,m,xz}}\phi_{1,m,x}\Theta_{1,m} + \phi_{1,n,xz}\phi_{1,m,x}\overline{\Theta_{1,n}}\\
&\quad +\phi_{1,n,xz}\overline{\phi_{1,n,x}}\Theta_{1,m} + \overline{\phi_{1,n,xz}}\phi_{1,m,x}\Theta_{1,n} + \overline{\phi_{1,n,xz}}\phi_{1,n,x}\Theta_{1,m}\Big)\\
&-\Big(\phi_{2,con,z}\phi_{1,m,z} + \phi_{2,2m,z}\overline{\phi_{1,m,z}} + \phi_{2,m-n,z}\phi_{1,n,z} + \phi_{2,m+n,z}\overline{\phi_{1,n,z}}\\
&\quad +\phi_{1,m,zz}\phi_{1,m,z}\overline{\Theta_{1,m}} + \phi_{1,m,zz}\overline{\phi_{1,m,z}}\Theta_{1,m} + \phi_{1,m,zz}\phi_{1,n,z}\overline{\Theta_{1,n}}\\
&\quad +\phi_{1,m,zz}\overline{\phi_{1,n,z}}\Theta_{1,n} + \overline{\phi_{1,m,zz}}\phi_{1,m,z}\Theta_{1,m} + \phi_{1,n,zz}\phi_{1,m,z}\overline{\Theta_{1,n}}\\
&\quad +\phi_{1,n,zz}\overline{\phi_{1,n,z}}\Theta_{1,m} + \overline{\phi_{1,n,zz}}\phi_{1,m,z}\Theta_{1,n} + \overline{\phi_{1,n,zz}}\phi_{1,n,z}\Theta_{1,m}\Big)\\
&-\Big(\phi_{2,con,tz}\Theta_{1,m} + \phi_{2,2m,tz}\overline{\Theta_{1,m}} + \phi_{2,m-n,tz}\Theta_{1,n} + \phi_{2,m+n,tz}\overline{\Theta_{1,n}}\\
&\quad +\Theta_{2,con}\phi_{1,m,tz} + \Theta_{2,2m}\overline{\phi_{1,m,tz}} + \Theta_{2,m-n}\phi_{1,n,tz} + \Theta_{2,m+n}\overline{\phi_{1,n,tz}}\\
&\quad +\phi_{1,m,tzz}\Theta_m\overline{\Theta_m} + \phi_{1,m,tzz}\Theta_n\overline{\Theta_n} + \tfrac{1}{2}\overline{\phi_{1,m,tzz}}\Theta_m^2 + \phi_{1,n,tzz}\Theta_m\overline{\Theta_n} + \overline{\phi_{1,n,tzz}}\Theta_m\Theta_n\Big)\\
&-T\Big(3\Theta_{1,m,xx}\Theta_m\overline{\Theta_m} + 3\Theta_{1,m,xx}\Theta_n\overline{\Theta_n} + \tfrac{3}{2}\overline{\Theta_{1,m,xx}}\Theta_m^2 + 3\Theta_{1,n,xx}\Theta_m\overline{\Theta_n} + 3\overline{\Theta_{1,n,xx}}\Theta_m\Theta_n\Big)\,.
\end{aligned}
\tag{3.28}
$$

Plugging these terms into equation (3.25), we get an equation of the form

$$
r_1 p_m' + r_2|p_m|^2 p_m + r_3|p_n|^2 p_m + r_4 e^{i\lambda\tau}\overline{p_n} = 0\,. \tag{3.29}
$$

The coefficients of this differential equation are given by

$$
\begin{aligned}
r_1 &= 2\cosh(k_m h)\,, \\
r_2 &= \tfrac{ik_m}{8a^4\omega_m}\operatorname{csch}^3(k_m h) \\
&\quad \times\Big[108g\,m^2\pi^2 T\sinh^4(k_m h) + 2ia^3 d_{2,2m} g\,m\pi\sinh^3(2k_m h)\omega_m \\
&\quad\quad + a^2\omega_m^2\Big(8 + a^2 g c_{1,2m} + 2a^2 c_{2,2m} g - 2a^2(c_{1,2m} + c_{2,2m}) g\cosh(2k_m h) \\
&\quad\quad + a^2 c_{1,2m} g\cosh(4k_m h)\Big)\Big]\,, \\
r_3 &= \tfrac{k_m}{2a^5 g\omega_m}\,, \\
&\Big[2ia^2\operatorname{csch}(k_m h)\operatorname{csch}^2(k_n h)\omega_m^2 - a^3 g\coth(k_n h)\omega_m \\
&\quad \times\Big([(d_{+,-} + d_{-,-})(m-n) + (d_{+,+} + d_{-,+})(m+n)]\,\pi\sinh(k_n h) \\
&\quad\quad + ia(c_{-,-} - c_{-,+} - c_{+,-} + c_{+,+})\cosh(k_m h)\omega_n\Big) \\
&+ g\sinh(k_m h)\Big(12i(m^2 + 2n^2)\pi^2 T - a^3\pi\coth(k_n h)\omega_n \\
&\quad\quad \times[(d_{-,-} - d_{+,-})(m-n)\cosh(k_{m-n}h) - (d_{-,+} - d_{+,+})(m+n)\cosh(k_{m+n}h)] \\
&\quad + i\omega_n^2\,[a^4(c_{-,-} + c_{-,+} + c_{+,-} + c_{+,+}) - 8a\;n\pi\coth(k_n h)]\Big)\Big]\,, \\
r_4 &= -\tfrac{2\omega_n}{a(\omega_m+\omega_n)}\operatorname{csch}(k_n h) \\
&\quad \textstyle\int_{-h}^{0}\Big(((-1)^m f_2'(z) - f_1'(z))\sinh(k_n(z+h)) \\
&\quad\quad + ((-1)^m f_2(z) - f_1(z))k_n\cosh(k_n(z+h))\Big)\cosh(k_m(z+h))dz \\
&+ \tfrac{e^{-i\omega_m t - i\lambda\tau}}{\overline{p_n}}\textstyle\int_0^a\Big[\Big(\phi_{2,x}^*\overline{\Theta_{1,n,x}} + \overline{\phi_{1,n,z}}\Theta_{2,x}^* - \overline{\phi_{1,n,zz}}\Theta_2^* - \phi_{2,zz}^*\overline{\Theta_{1,n}}\Big)\cosh(k_m h) \\
&\quad + \Big(\overline{\Theta_{1,n}}\phi_{2,tz}^* + \overline{\phi_{1,n,x}}\phi_{2,x}^* + \overline{\phi_{1,n,z}}\phi_{2,z}^* + \Theta_2^*\overline{\phi_{1,n,tz}}\Big)\tfrac{k_m}{i\omega_m}\sinh(k_m h)\Big]\cos(k_m x)dx\,.
\end{aligned}
\tag{3.30}
$$

Here, ϕ_2^* and Θ_2^* are the parts of ϕ_2 and Θ_2 that have the factor $e^{i(\omega_m+\omega_n)t+i\lambda\tau}$.

To get a condition for the terms of frequency ω_n, equation (2.55) is multiplied by $\cos(k_n x)\cosh(k_n(z+h))$ and integrated over the whole domain. Using the same steps as to obtain (3.25), we get

$$
\begin{aligned}
&\textstyle\int_{-h}^{0}\left((-1)^n[B_{2,z}\phi_{1,z} - B_2\phi_{1,xx}]|_{x=a} - [B_{1,z}\phi_{1,z} - B_1\phi_{1,xx}]|_{x=0}\right)\cosh(k_n(z+h))dz \\
&\quad + \textstyle\int_0^a\Big(i\omega_n\cosh(k_n h) + \tfrac{k_n}{i\omega_n}(g + T\,k_n^2)\sinh(k_n h)\Big)\Theta_3\cos(k_n x)dx \\
&\quad - \textstyle\int_0^a\Big([\text{r.s. (2.59)}]\cosh(k_n h) + \tfrac{k_n}{i\omega_n}[\text{r.s. (2.60)}]\sinh(k_n h)\Big)\cos(k_n x)dx \quad = \quad 0\,.
\end{aligned}
\tag{3.31}
$$

When working out the terms of frequency ω_n on the right side of the third-order equations in (2.59) and (2.60), we see that we get the same expressions as in (3.27, 3.28) with just the roles of m and n being switched. Below is a table of what gets switched to what, in detail:

$$
\begin{aligned}
\Theta_{1,m} &\leftrightarrow \Theta_{1,n}\,, \\
\Theta_{2,2m} &\leftrightarrow \Theta_{2,2n}\,, \\
\Theta_{2,m-n} &\leftrightarrow \Theta_{2,n-m}\,.
\end{aligned}
\tag{3.32}
$$

As they are symmetric in m and n or independent of them, $\Theta_{2,m+n}$ and $\Theta_{2,con}$ are not changed. Naturally, the same patterns are applied to ϕ_1, ϕ_2 and the respective different frequencies. We recall that

$$\Theta_{2,m-n} = e^{i(\omega_m-\omega_n)t}\left(c_{-,-}p_m\overline{p_n}\cos(k_{m-n}x) + c_{-,+}p_m\overline{p_n}\cos(k_{m+n}x)\right) .$$

Since $\Theta_{2,n-m}$ is the complex conjugate of $\Theta_{2,m-n}$ and the coefficients $c_{-,\pm}$ are real-valued, we have

$$\Theta_{2,n-m} = e^{i(\omega_n-\omega_m)t}\left(c_{-,-}\overline{p_m}p_n\cos(k_{m-n}x) + c_{-,+}\overline{p_m}p_n\cos(k_{m+n}x)\right) . \tag{3.33}$$

Similarly,

$$\begin{aligned}\phi_{2,m-n} = e^{i(\omega_m-\omega_n)t}\Big(& d_{-,-}p_m\overline{p_n}\cos(k_{m-n}x)\cosh(k_{m-n}h) \\ & +d_{-,+}p_m\overline{p_n}\cos(k_{m+n}x)\cosh(k_{m+n}h)\Big) .\end{aligned} \tag{3.34}$$

With the same reasoning as above and because the coefficients $d_{-,\pm}$ are purely imaginary, we get

$$\begin{aligned}\phi_{2,n-m} = -e^{i(\omega_n-\omega_m)t}\Big(& d_{-,-}\overline{p_m}p_n\cos(k_{m-n}x)\cosh(k_{m-n}h) \\ & +d_{-,+}\overline{p_m}p_n\cos(k_{m+n}x)\cosh(k_{m+n}h)\Big) .\end{aligned} \tag{3.35}$$

This means that by plugging everything into (3.31), an equation of the type

$$s_1p_n' + s_2|p_n|^2p_n + s_3|p_m|^2p_n + s_4e^{i\lambda\tau}\overline{p_m} = 0 \tag{3.36}$$

is obtained where the coefficients can be found by switching m and n. When doing so, we also have to take the negative of coefficients of the type $d_{-,\pm}$. The resulting coefficients are

$$
\begin{aligned}
s_1 &= 2\cosh(k_n h)\,,\\
s_2 &= \tfrac{ik_n}{8a^4\omega_n}\operatorname{csch}^3(k_n h)\\
&\quad\times\Big[108g\,n^2\pi^2 T\sinh^4(k_n h) + 2ia^3 d_{2,2n} g\,n\pi\sinh^3(2k_n h)\omega_n\\
&\qquad a^2\omega_n^2\Big(8 + a^2 g c_{1,2n} + 2a^2 c_{2,2n} g - 2a^2(c_{1,2n} + c_{2,2n})g\cosh(2k_n h)\\
&\qquad +a^2 c_{1,2n} g\cosh(4k_n h)\Big)\Big]\,,\\
s_3 &= \tfrac{k_n}{2a^5 g\omega_n}\\
&\quad\times\Big[2ia^2\operatorname{csch}(k_n h)\operatorname{csch}^2(k_m h)\omega_n^2 - a^3 g\coth(k_m h)\omega_n\\
&\qquad\Big([(d_{+,-} - d_{-,-})(n-m) + (d_{+,+} + d_{-,+})(m+n)]\,\pi\sinh(k_m h)\\
&\qquad\quad +ia(c_{-,-} - c_{-,+} - c_{+,-} + c_{+,+})\cosh(k_n h)\omega_m\Big)\\
&\quad +g\sinh(k_n h)\Big(12i(n^2+2m^2)\pi^2 T - a^3\pi\coth(k_m h)\omega_m\\
&\qquad\times[(-d_{-,-} - d_{+,-})(n-m)\cosh(k_{m-n}h) - (-d_{-,+} - d_{+,+})(m+n)\cosh(k_{m+n}h)]\\
&\qquad\quad +i\omega_n^2\,[a^4(c_{-,-} + c_{-,+} + c_{+,-} + c_{+,+}) - 8a\;m\pi\coth(k_m h)]\Big)\Big]\,,\\
s_4 &= -\tfrac{2\omega_m}{a(\omega_m+\omega_n)}\operatorname{csch}(k_m h)\\
&\qquad\textstyle\int_{-h}^{0}\Big(((-1)^n f_2'(z) - f_1'(z))\sinh(k_m(z+h))\\
&\qquad\quad +((-1)^n f_2(z) - f_1(z))k_m\cosh(k_m(z+h))\Big)\cosh(k_n(z+h))dz\\
&\quad +\tfrac{e^{-i\omega_n t - i\lambda\tau}}{\overline{p_m}}\textstyle\int_0^a\Big[\Big(\phi^*_{2,x}\overline{\Theta_{1,m,x}} + \overline{\phi_{1,m,z}}\Theta^*_{2,x} - \overline{\phi_{1,m,zz}}\Theta^*_2 - \phi^*_{2,zz}\overline{\Theta_{1,m}}\Big)\cosh(k_n h)\\
&\qquad +\Big(\overline{\Theta_{1,m}}\phi^*_{2,tz} + \overline{\phi_{1,m,x}}\phi^*_{2,x} + \overline{\phi_{1,m,z}}\phi^*_{2,z} + \Theta^*_2\overline{\phi_{1,m,tz}}\Big)\tfrac{k_n}{i\omega_n}\sinh(k_n h)\Big]\cos(k_n x)dx\,.
\end{aligned}
\tag{3.37}
$$

Recall that ϕ_2^* and Θ_2^* are the parts of ϕ_2 and Θ_2 that contain the factor $e^{i(\omega_m+\omega_n)t+i\lambda\tau}$.

3.3 Discussion of the amplitude equations

3.3.1 Reduction to one ODE

We notice from (3.30, 3.37) that r_1, r_4 and s_1, s_4 are real-valued. Since the coefficients $c_{\pm,\pm}$ are real and the $d_{\pm,\pm}$ are purely imaginary, every part of the sums for r_2, r_3 and s_2, s_3 is purely imaginary, thus those coefficients have to be purely imaginary. Solving for the derivatives in each equation, we can write (3.30, 3.37) in the form

$$
\begin{aligned}
p_m' &= c_{11} i p_m |p_m|^2 + c_{12} i p_m |p_n|^2 + c_{13} e^{i\lambda\tau}\overline{p_n}\,,\\
p_n' &= c_{21} i p_n |p_n|^2 + c_{22} i p_n |p_m|^2 + c_{23} e^{i\lambda\tau}\overline{p_m}\,.
\end{aligned}
\tag{3.38}
$$

The c_{ij} are all real-valued and given by

$$\begin{array}{rclrcl} c_{11} & = & -\frac{r_2}{r_1 i}\,, & c_{21} & = & -\frac{s_2}{s_1 i}\,, \\ c_{12} & = & -\frac{r_3}{r_1 i}\,, & c_{22} & = & -\frac{s_3}{s_1 i}\,, \\ c_{13} & = & -\frac{r_4}{r_1}\,, & c_{23} & = & -\frac{s_4}{s_1}\,. \end{array} \tag{3.39}$$

Next, we introduce new variables $P_m(\tau), P_n(\tau)$ through

$$\begin{array}{rcl} p_m & = & P_m e^{i\frac{\lambda}{2}\tau}\,, \\ p_n & = & P_n e^{i\frac{\lambda}{2}\tau}\,. \end{array} \tag{3.40}$$

Substitute this into (3.38) to obtain

$$\begin{array}{rcl} (i\frac{\lambda}{2}P_m + P_m')e^{i\frac{\lambda}{2}\tau} & = & c_{11}iP_m e^{i\frac{\lambda}{2}\tau}|P_m|^2 + c_{12}iP_n e^{i\frac{\lambda}{2}\tau}|P_m|^2 + c_{13}e^{i\lambda\tau}\overline{P_n}e^{-i\frac{\lambda}{2}\tau}\,, \\ (i\frac{\lambda}{2}P_n + P_n')e^{i\frac{\lambda}{2}\tau} & = & c_{21}iP_n e^{i\frac{\lambda}{2}\tau}|P_n|^2 + c_{22}iP_m e^{i\frac{\lambda}{2}\tau}|P_n|^2 + c_{23}e^{i\lambda\tau}\overline{P_m}e^{-i\frac{\lambda}{2}\tau}\,. \end{array}$$

Cancelling $e^{i\frac{\lambda}{2}\tau}$ and solving for $P_m(\tau)$ and $P_n(\tau)$, this yields autonomous differential equations for P_m and P_n :

$$\begin{array}{rcl} P_m' & = & -i\frac{\lambda}{2}P_m + c_{11}iP_m|P_m|^2 + c_{12}iP_m|P_n|^2 + c_{13}\overline{P_n}\,, \\ P_n' & = & -i\frac{\lambda}{2}P_n + c_{21}iP_n|P_n|^2 + c_{22}iP_n|P_m|^2 + c_{23}\overline{P_m}\,. \end{array} \tag{3.41}$$

Polar coordinates for $P_m(\tau), P_n(\tau)$ are introduced:

$$\begin{array}{rcl} P_m & = & r_m e^{i\alpha_m}\,, \\ P_n & = & r_n e^{i\alpha_n}\,. \end{array} \tag{3.42}$$

Here, r_m, r_n and α_m, α_n are functions of τ. The equations in the polar coordinates are

$$\begin{array}{rcl} r_m' e^{i\alpha_m} + i\alpha_m' r_m e^{i\alpha_m} & = & -i\frac{\lambda}{2}r_m e^{i\alpha_m} + c_{11}ir_m^3 e^{i\alpha_m} + c_{12}ir_m r_n^2 e^{i\alpha_m} + c_{13}r_n e^{-i\alpha_n}\,, \\ r_n' e^{i\alpha_n} + i\alpha_n' r_n e^{i\alpha_n} & = & -i\frac{\lambda}{2}r_n e^{i\alpha_n} + c_{21}ir_n^3 e^{i\alpha_n} + c_{22}ir_n r_m^2 e^{i\alpha_n} + c_{23}r_m e^{-i\alpha_m}\,. \end{array} \tag{3.43}$$

The first equation is divided by $e^{i\alpha_m}$ and the second equation by $e^{i\alpha_n}$. Considering the real parts only and using Re $[e^{-i(\alpha_m+\alpha_n)}] = \cos(\alpha_m + \alpha_n)$, we get

$$\begin{array}{rcl} r_m' & = & c_{13}r_n\cos(\alpha)\,, \\ r_n' & = & c_{23}r_m\cos(\alpha)\,, \end{array} \tag{3.44}$$

where $\alpha = \alpha_m + \alpha_n$. To obtain equations for the angles, divide the first equation in (3.43) by $e^{i\alpha_m}r_m$ and the second equation by $e^{i\alpha_n}r_n$ and compare the imaginary parts

$$\begin{array}{rcl} \alpha_m' &=& -\frac{\lambda}{2} + c_{11} r_m^2 + c_{12} r_n^2 - c_{13} \frac{r_n}{r_m} \sin(\alpha)\,, \\ \alpha_n' &=& -\frac{\lambda}{2} + c_{21} r_n^2 + c_{22} r_m^2 - c_{23} \frac{r_m}{r_n} \sin(\alpha)\,. \end{array} \tag{3.45}$$

Summing up those two equations gives an ODE for α :

$$\alpha' = -\lambda + (c_{11} + c_{22}) r_m^2 + (c_{12} + c_{21}) r_n^2 - \frac{1}{r_m r_n} (c_{13} r_n^2 + c_{23} r_m^2) \sin(\alpha)\,. \tag{3.46}$$

In [22], it was shown that the system consisting of (3.44) and (3.46) has two invariant functionals

$$\begin{array}{rcl} J &=& \frac{1}{2c_{23}} (c_{23} r_m^2 - c_{13} r_n^2)\,, \\ E &=& \frac{\lambda}{2c_{13}^2 + 2c_{23}^2} (c_{13} r_m^2 + c_{23} r_n^2) + r_m r_n \sin(\alpha) - \frac{c_{11}+c_{22}}{4c_{13}} r_m^4 - \frac{c_{12}+c_{21}}{4c_{23}} r_n^4\,. \end{array} \tag{3.47}$$

This is true, because

$$\begin{array}{rcl} \frac{dJ}{d\tau} &=& \frac{1}{2c_{23}} (2c_{23} r_m r_m' - 2c_{13} r_n r_n') \\ &=& \frac{1}{2c_{23}} \left(2c_{13} c_{23} r_m r_n \cos(\alpha) - 2c_{13} c_{23} r_m r_n \cos(\alpha)\right) = 0 \end{array}$$

and

$$\begin{array}{rcl} \frac{dE}{d\tau} &=& \frac{\lambda}{2c_{13}^2 + 2c_{23}^2} (2c_{13} r_m r_m' + 2c_{23} r_n r_n') + r_m' r_n \sin(\alpha) + r_m r_n' \sin(\alpha) \\ && \quad + r_m r_n \cos(\alpha) \alpha' - \frac{c_{11}+c_{22}}{4c_{13}} 4 r_m^3 r_m' - \frac{c_{12}+c_{21}}{4c_{23}} 4 r_n^3 r_n' \\ &=& \frac{\lambda}{2c_{13}^2 + 2c_{23}^2} \left(2c_{13}^2 r_m r_n \cos(\alpha) + 2c_{23}^2 r_m r_n \cos(\alpha)\right) + c_{13} r_n^2 \sin(\alpha)\cos(\alpha) \\ && \quad + c_{23} r_m^2 \sin(\alpha)\cos(\alpha) - \lambda r_m r_n \cos(\alpha) - (c_{23} r_m^2 + c_{13} r_n^2) \sin(\alpha)\cos(\alpha) \\ && \quad + (c_{11} + c_{22}) r_m^3 r_n \cos(\alpha) + (c_{12} + c_{21}) r_n^3 r_m \cos(\alpha) \\ && \quad - \frac{c_{11}+c_{22}}{4c_{13}} \cdot 4c_{13} r_m^3 r_n \cos(\alpha) - \frac{c_{12}+c_{21}}{4c_{23}} \cdot 4c_{23} r_n^3 r_m \cos(\alpha) \\ &=& 0\,. \end{array}$$

Now define $\rho = \frac{\beta_1}{2} r_m^2 + \frac{\beta_2}{2} r_n^2$ and the functional $Q(\rho)$ by

$$Q(\rho) \;=\; (\beta_1 c_{13} + \beta_2 c_{23})^2 \left[r_m^2 r_n^2 - \left(E + \frac{\lambda c_{23}^2}{c_{13}(c_{13}^2 + c_{23}^2)} J - \frac{\lambda}{2c_{13}} r_m^2 + \frac{c_{11}+c_{22}}{4c_{13}} r_m^4 + \frac{c_{12}+c_{21}}{4c_{23}} r_n^4 \right)^2 \right] \tag{3.48}$$

with substitutions

$$\begin{array}{rcl} r_m^2 &\to& \frac{2c_{13}}{c_{13}\beta_1 + c_{23}\beta_2}\, \rho + \frac{2c_{23}\beta_2}{c_{13}\beta_1 + c_{23}\beta_2}\, J\,, \\ r_n^2 &\to& \frac{2c_{23}}{c_{13}\beta_1 + c_{23}\beta_2}\, \rho - \frac{2c_{23}\beta_1}{c_{13}\beta_1 + c_{23}\beta_2}\, J\,. \end{array} \tag{3.49}$$

Note that, since J and E are invariants given by initial conditions, Q is a polynomial of ρ with coefficients depending on c_{ij}, λ, J and E only. By substituting in the terms for E and J according to (3.47) and simplifying, we see that

$$\left(\frac{d\rho}{d\tau}\right)^2 = (c_{13}\beta_1 + c_{23}\beta_2)^2 r_m^2 r_n^2 \cos^2(\alpha) = Q(\rho)\,. \tag{3.50}$$

Thus the system (3.44, 3.46) has been reduced to one differential equation for ρ

$$\left(\frac{d\rho}{d\tau}\right)^2 = Q(\rho) = d_0 + d_1\rho + d_2\rho^2 + d_3\rho^3 + d_4\rho^4\,. \tag{3.51}$$

With $C_1 = c_{11} + c_{22}$ and $C_2 = c_{12} + c_{21}$, the coefficients turn out to be

$$\begin{aligned}
d_0 &= -E^2(c_{13}\beta_1 + c_{23}\beta_2)^2 - \tfrac{c_{23}(c_{13}\beta_1^2 + c_{23}\beta_2^2)^2}{c_{13}^2(c_{13}\beta_1 + c_{23}\beta_2)^2} J^4 + \tfrac{2c_{23}^2(c_{13}\beta_2 - c_{23}\beta_1)(C_2 c_{13}\beta_1^2 + C_1 c_{23}\beta_2^2)}{c_{13}(c_{13}^2 + c_{23}^2)(c_{13}\beta_1 + c_{23}\beta_2)} \lambda J^3 \\
&\quad + J^2\left(-\tfrac{2c_{23}(c_{13}C_2\beta_1^2 + C_1 c_{23}\beta_2^2)}{c_{13}} E + c_{23}^2\left[-4\beta_1\beta_2 - \tfrac{(c_{23}\beta_1 - c_{13}\beta_2)^2\lambda^2}{c_{13}(c_{13}^2 + c_{23}^2)(c_{13}\beta_1 + c_{23}\beta_2)}\right]\right)\,, \\
d_1 &= 2E(c_{13}\beta_1 + c_{23}\beta_2) + \tfrac{4c_{23}^2(C_2\beta_1 - C_1\beta_2)(C_2 c_{13}\beta_1^2 + C_1 c_{23}\beta_2^2)}{c_{13}(c_{13}\beta_1 + c_{23}\beta_2)^2} J^3 \\
&\quad + \tfrac{2c_{23}\left(C_2 c_{13}(c_{13}^2 + 3c_{23}^2)\beta_1^2 - 2c_{13}c_{23}(C_2 c_{13} + C_1 c_{23})\beta_1\beta_2 + C_1 c_{23}(3c_{13}^2 + c_{23}^2)\beta_2^2\right)}{c_{13}(c_{13}^2 + c_{23}^2)(c_{13}\beta_1 + c_{23}\beta_2)^2} J^2\lambda \\
&\quad + J\left(4c_{23}E(C_2\beta_1 - C_1\beta_2) + 4c_{23}(c_{23}\beta_2 - c_{13}\beta_1) + \tfrac{2c_{23}(c_{23}\beta_1 - c_{13}\beta_2)\lambda^2}{c_{13}^2 + c_{23}^2}\right)\,, \\
d_2 &= -\tfrac{2c_{23}\left(C_2 c_{13}\beta_1^2(C_1 c_{13} + 3C_2 c_{23}) - 4C_1 C_2 c_{13}c_{23}\beta_1\beta_2 + C_1 c_{23}(3C_1 c_{13} + C_2 c_{23})\right)}{c_{13}(c_{13}\beta_1 + c_{23}\beta_2)^2} J^2 \\
&\quad + \tfrac{2c_{23}\left(-\beta_1(2C_2 c_{13}^2 + C_1 c_{13}c_{23} + 3C_2 c_{23}^2) + \beta_2(3C_1 c_{13}^2 + C_2 c_{13}c_{23} + 2C_1 c_{23}^2)\right)}{(c_{13}^2 + c_{23}^2)(c_{13}\beta_1 + c_{23}\beta_2)} J\lambda \\
&\quad - 2E(C_1 c_{13} + C_2 c_{23}) + 4c_{13}c_{23} - \lambda^2\,, \\
d_3 &= -\tfrac{4c_{23}(C_1 c_{13} + C_2 c_{23})(C_1\beta_2 - C_2\beta_1)}{(c_{13}\beta_1 + c_{23}\beta_2)^2} J + \tfrac{2(C_1 c_{13} + C_2 c_{23})}{c_{13}\beta_1 + c_{23}\beta_2}\lambda\,, \\
d_4 &= -\tfrac{(C_1 c_{13} + C_2 c_{23})^2}{(c_{13}\beta_1 + c_{23}\beta_2)^2}\,.
\end{aligned} \tag{3.52}$$

The existence of solutions of (3.51) depends on the roots ρ_i (i=1,2,3,4) of the polynomial $Q(\rho) = d_4\Pi(\rho_i - \rho)$. The following discussion is based on the assumption that $d_4 \neq 0$.

a) $Q(\rho)$ has real roots only

Before discussing particular cases, some properties of the Jacobian elliptic function sn should be recalled. For this purpose, let $u = \int_0^\phi (1 - k^2\sin^2 t)^{-\frac{1}{2}} dt$. Then define $\mathrm{sn}(u,k) = \sin(\phi)$. $\mathrm{sn}(u,k)$ has similar properties as the sin function: It is an odd function with range [-1,1] (if restricted to real arguments), and it has period $4K$ where $K = \int_0^{\frac{\pi}{2}} (1 - k^2\sin^2 t)^{-\frac{1}{2}} dt$. Note that we have $\mathrm{sn}(u + 2K)$ = -sn(u), so $\mathrm{sn}^2(u + 2K) = (-\mathrm{sn}(u))^2 = \mathrm{sn}^2 u$. This means sn^2 is periodic with period $2K$.

- $\rho_1 < \rho_2 < \rho_3 < \rho_4$: Then we have

$$\frac{d\rho}{d\tau} = \pm\sqrt{-d_4}\Big((\rho_4-\rho)(\rho_3-\rho)(\rho-\rho_2)(\rho_1-\rho)\Big)^{\frac{1}{2}}.$$

Because $Q(\rho) > 0$ and $d_4 < 0$, we know that either

$$\rho_1 \leq \rho \leq \rho_2 \text{ or } \rho_3 \leq \rho \leq \rho_4\,.$$

In the first case, integrating and using formula (252.00) in [6] yields

$$\int_{\rho_1}^{\rho}\Big((\rho_4-t)(\rho_3-t)(t-\rho_2)(\rho_1-t)\Big)^{-\frac{1}{2}}dt = g_1\,\mathrm{sn}^{-1}(\sin(\phi_1),k_1) = \pm\sqrt{-d_4}(\tau-\tau_0)\,,$$

where

$$\begin{aligned} g_1 &= 2\left((\rho_4-\rho_2)(\rho_3-\rho_1)\right)^{-\frac{1}{2}}\,,\\ \sin(\phi_1) &= \left(\tfrac{(\rho_4-\rho_2)(\rho-\rho_1)}{(\rho_2-\rho_1)(\rho_4-\rho)}\right)^{\frac{1}{2}}\,,\\ k_1 &= \left(\tfrac{(\rho_4-\rho_3)(\rho_2-\rho_1)}{(\rho_4-\rho_2)(\rho_3-\rho_1)}\right)^{\frac{1}{2}}\,. \end{aligned}$$

Solving for ρ, we get

$$\begin{aligned} \tfrac{(\rho_4-\rho_2)(\rho-\rho_1)}{(\rho_2-\rho_1)(\rho_4-\rho)} &= \sin^2(\phi) = \mathrm{sn}^2\left(\sqrt{-d_4}\,g_1^{-1}(\tau-\tau_0),k_1\right)\,,\\ (\rho_4-\rho_2)\rho-(\rho_4-\rho_2)\rho_1 &= \mathrm{sn}^2\left(\sqrt{-d_4}\,g_1^{-1}(\tau-\tau_0),k_1\right)(\rho_2-\rho_1)\rho_4\\ &\quad -\mathrm{sn}^2\left(\sqrt{-d_4}\,g_1^{-1}(\tau-\tau_0),k_1\right)(\rho_2-\rho_1)\rho\,. \end{aligned}$$

Finally, we obtain

$$\rho = \frac{(\rho_4-\rho_2)\rho_1+\mathrm{sn}^2\left(\sqrt{-d_4}\,g_1^{-1}(\tau-\tau_0),k_1\right)(\rho_2-\rho_1)\rho_4}{\rho_4-\rho_2+\mathrm{sn}^2\left(\sqrt{-d_4}\,g_1^{-1}(\tau-\tau_0),k_1\right)(\rho_2-\rho_1)}\,. \tag{3.53}$$

Since $\mathrm{sn}(u,k)$ has range [-1,1], $\mathrm{sn}^2(u,k)$ takes values [0,1]. Thus ρ takes values between ρ_1 (also the initial value here) and $\frac{(\rho_4-\rho_2)\rho_1+(\rho_2-\rho_1)\rho_4}{\rho_4-\rho_2+(\rho_2-\rho_1)} = \rho_2$. As $\mathrm{sn}(C\tau+D,k_1)$ has period $\frac{4K_1}{C}$, where

$$K_1 = \int_0^{\frac{\pi}{2}}(1-k_1^2\sin^2 t)^{-\frac{1}{2}}dt\,,$$

we conclude that $\mathrm{sn}^2(C\tau+D,k_1)$ has period $\frac{2K_1}{C}$. Thus ρ here has period $2K_1g_1(-d_4)^{-\frac{1}{2}}$.

Now consider the case $\rho_3 \leq \rho \leq \rho_4$. Integrating and using (256.00) from [6], we get

$$\int_{\rho_3}^{\rho}\Big((t-\rho_4)(\rho_3-t)(\rho_2-t)(\rho_1-t)\Big)^{-\frac{1}{2}}dt = g_2\,\mathrm{sn}^{-1}(\sin(\phi_2),k_2) = \pm\sqrt{-d_4}(\tau-\tau_0)$$

with

$$\begin{aligned} g_2 &= 2\left((\rho_4-\rho_2)(\rho_3-\rho_1)\right)^{-\frac{1}{2}}, \\ \sin(\phi_2) &= \left(\tfrac{(\rho_4-\rho_2)(\rho-\rho_3)}{(\rho_4-\rho_3)(\rho-\rho_2)}\right)^{\frac{1}{2}}, \\ k_2 &= \left(\tfrac{(\rho_4-\rho_3)(\rho_2-\rho_1)}{(\rho_4-\rho_2)(\rho_3-\rho_1)}\right)^{\frac{1}{2}}. \end{aligned}$$

Solving for ρ yields

$$\begin{aligned} \tfrac{(\rho_4-\rho_2)(\rho-\rho_3)}{(\rho_4-\rho_3)(\rho-\rho_2)} &= \sin^2(\phi) = \mathrm{sn}^2\left(\sqrt{-d_4}\, g_2^{-1}(\tau-\tau_0), k_2\right), \\ (\rho_4-\rho_2)\rho-(\rho_4-\rho_2)\rho_3 &= \mathrm{sn}^2\left(\sqrt{-d_4}\, g_2^{-1}(\tau-\tau_0), k_2\right)(\rho_4-\rho_3)\rho \\ &\quad -\mathrm{sn}^2\left(\sqrt{-d_4}\, g_2^{-1}(\tau-\tau_0), k_2\right)(\rho_4-\rho_3)\rho_2 . \end{aligned}$$

Then

$$\rho = \frac{(\rho_4-\rho_2)\rho_3-\mathrm{sn}^2\left(\sqrt{-d_4}\, g_2^{-1}(\tau-\tau_0), k_2\right)(\rho_4-\rho_3)\rho_2}{\rho_4-\rho_2-\mathrm{sn}^2\left(\sqrt{-d_4}\, g_2^{-1}(\tau-\tau_0), k_2\right)(\rho_4-\rho_3)} . \tag{3.54}$$

With the same reasoning as in the previous case, we conclude that ρ this time oscillates between ρ_3 and ρ_4 with period $2K_2 g_2(-d_4)^{-\frac{1}{2}}$, where

$$K_2 = \int_0^{\frac{\pi}{2}} (1-k_2^2\sin^2 t)^{-\frac{1}{2}} dt .$$

- $\rho_1 = \rho_2 < \rho_3 < \rho_4$

  As we need $Q(\rho) > 0$ and $d_4 < 0$ is known, $\rho_3 \le \rho \le \rho_4$. This is a special case of the previous one. Setting $\rho_1 = \rho_2$ implies $k_2 = 0$. Since $\mathrm{sn}(u,0) = \sin(u)$, the solution in (3.54) becomes

$$\rho = \frac{(\rho_4-\rho_2)\rho_3-\sin^2\left(\sqrt{-d_4}\, g_2^{-1}(\tau-\tau_0)\right)(\rho_4-\rho_3)\rho_2}{\rho_4-\rho_2-\sin^2\left(\sqrt{-d_4}\, g_2^{-1}(\tau-\tau_0)\right)(\rho_4-\rho_3)} . \tag{3.55}$$

  The range of ρ again is $[\rho_3, \rho_4]$. Since the sin function is periodic with range [-1,1], $\sin^2(C\tau + D)$ has range [0,1] and period $\frac{\pi}{C}$. In this case, the squared sine function has period $\pi g_2(-d_4)^{-\frac{1}{2}}$, and so does ρ.

- $\rho_1 < \rho_2 = \rho_3 < \rho_4$ Comparing with the case of four distinct roots, we would have $k_1 = 1$ which would then imply $K_1 = \infty$, thus this needs to be treated separately.

  The condition $Q(\rho) > 0$ here implies that $\rho_1 < \rho < \rho_2$ or $\rho_2 < \rho < \rho_4$. In the first case, we have

$$\begin{aligned} &\textstyle\int_{\rho_1}^{\rho} \left((\rho_4-t)(\rho_2-t)^2(t-\rho_1)\right)^{-\frac{1}{2}} dt \\ &\quad = \tfrac{2}{\sqrt{(\rho_4-\rho_2)(\rho_2-\rho_1)}} \operatorname{arctanh}\sqrt{\tfrac{(\rho-\rho_1)(\rho_4-\rho_2)}{(\rho_4-\rho)(\rho_2-\rho_1)}} = \pm\sqrt{-d_4}(\tau-\tau_0) . \end{aligned}$$

Solving for ρ yields

$$\begin{aligned}
\sqrt{\tfrac{(\rho-\rho_1)(\rho_4-\rho_2)}{(\rho_4-\rho)(\rho_2-\rho_1)}} &= \tanh\left(\pm\tfrac{1}{2}\sqrt{-d_4}(\tau-\tau_0)\sqrt{(\rho_4-\rho_2)(\rho_2-\rho_1)}\right)\,,\\
(\rho-\rho_1)(\rho_4-\rho_2) &= (\rho_4-\rho)(\rho_2-\rho_1)\\
&\quad\times\tanh^2\left(\pm\tfrac{1}{2}\sqrt{-d_4}(\tau-\tau_0)\sqrt{(\rho_4-\rho_2)(\rho_2-\rho_1)}\right)\,.
\end{aligned}$$

The solution in this case is

$$\rho = \frac{\rho_1(\rho_4-\rho_2)+(\rho_2-\rho_1)\rho_4\tanh^2\left(\frac{1}{2}\sqrt{-d_4}(\tau-\tau_0)\sqrt{(\rho_4-\rho_2)(\rho_2-\rho_1)}\right)}{\rho_4-\rho_2+(\rho_2-\rho_1)\tanh^2\left(\frac{1}{2}\sqrt{-d_4}(\tau-\tau_0)\sqrt{(\rho_4-\rho_2)(\rho_2-\rho_1)}\right)}\,. \tag{3.56}$$

Because $0 \leq \tanh^2(.) < 1$, we see that ρ takes values between ρ_1 and ρ_2. With

$$C(\tau) = \frac{1}{2}\sqrt{-d_4}(\tau-\tau_0)\sqrt{(\rho_4-\rho_2)(\rho_2-\rho_1)}\,,$$

we have

$$\rho' = \frac{(\rho_4-\rho_1)(\rho_2-\rho_1)(\rho_4-\rho_2)\,\mathrm{sech}^2(C(\tau))\tanh(C(\tau))}{\left[\rho_4-\rho_2+(\rho_2-\rho_1)\tanh^2(C(\tau))\right]^2}C'(\tau)\,.$$

This means that ρ is decreasing for $\tau < \tau_0$, increasing for $\tau > \tau_0$ with $\lim_{\tau\to\pm\infty}\rho = \rho_2$ and $\rho(\tau_0) = \rho_1$.

Now $\rho_2 < \rho < \rho_4$ is assumed.

$$\begin{aligned}
&\textstyle\int_{\rho_4}^{\rho}\left((\rho_4-t)(t-\rho_2)^2(t-\rho_1)\right)^{-\frac{1}{2}}dt =\\
&= -\tfrac{2}{\sqrt{(\rho_4-\rho_2)(\rho_2-\rho_1)}}\mathrm{arctanh}\sqrt{\tfrac{(\rho_4-\rho)(\rho_2-\rho_1)}{(\rho-\rho_1)(\rho_4-\rho_2)}} = \pm\sqrt{-d_4}(\tau-\tau_0)\,.
\end{aligned}$$

This has to be solved for ρ

$$\begin{aligned}
\sqrt{\tfrac{(\rho_4-\rho)(\rho_2-\rho_1)}{(\rho-\rho_1)(\rho_4-\rho_2)}} &= \tanh\left(\mp\tfrac{1}{2}\sqrt{-d_4}(\tau-\tau_0)\sqrt{(\rho_4-\rho_2)(\rho_2-\rho_1)}\right)\,,\\
(\rho_4-\rho)(\rho_2-\rho_1) &= (\rho-\rho_1)(\rho_4-\rho_2)\\
&\quad\times\tanh^2\left(\mp\tfrac{1}{2}\sqrt{-d_4}(\tau-\tau_0)\sqrt{(\rho_4-\rho_2)(\rho_2-\rho_1)}\right)\,.
\end{aligned}$$

Solving for ρ, we get

$$\rho = \frac{\rho_4(\rho_2-\rho_1)+(\rho_4-\rho_2)\rho_1\tanh^2\left(\frac{1}{2}\sqrt{-d_4}(\tau-\tau_0)\sqrt{(\rho_4-\rho_2)(\rho_2-\rho_1)}\right)}{\rho_2-\rho_1+(\rho_4-\rho_2)\tanh^2\left(\frac{1}{2}\sqrt{-d_4}(\tau-\tau_0)\sqrt{(\rho_4-\rho_2)(\rho_2-\rho_1)}\right)}\,. \tag{3.57}$$

With

$$C(\tau) = \frac{1}{2}\sqrt{-d_4}(\tau-\tau_0)\sqrt{(\rho_4-\rho_2)(\rho_2-\rho_1)}\,,$$

the derivative of ρ is

$$\rho' = -\frac{(\rho_4-\rho_1)(\rho_2-\rho_1)(\rho_4-\rho_2)\,\mathrm{sech}^2(C(\tau))\tanh(C(\tau))}{[\rho_2-\rho_1+(\rho_4-\rho_2)\tanh^2(C(\tau))]^2}C'(\tau)\,.$$

Hence, ρ is increasing for $\tau < \tau_0$, decreasing for $\tau > \tau_0$ with $\lim_{\tau\to\pm\infty}\rho = \rho_2$ and $\rho(\tau_0) = \rho_4$.

- $\rho_1 < \rho_2 < \rho_3 = \rho_4$

  Here we need $\rho_1 \le \rho \le \rho_2$. Equation (3.53) of the case $\rho_1 < \rho_2 < \rho_3 < \rho_4$ is used with $\rho_3 = \rho_4$. This implies $k_1 = 0$. The solution from (3.53) for this special case becomes

$$\rho = \frac{(\rho_4-\rho_2)\rho_1 + \sin^2\left(\sqrt{-d_4}\,g_1^{-1}(\tau-\tau_0)\right)(\rho_2-\rho_1)\rho_4}{\rho_4-\rho_2+\sin^2\left(\sqrt{-d_4}\,g_1^{-1}(\tau-\tau_0)\right)(\rho_2-\rho_1)}\,. \tag{3.58}$$

  This solution oscillates between ρ_1 and ρ_2 with period $\pi g_1(-d_4)^{-\frac{1}{2}}$.

- $\rho_1 < \rho_2 = \rho_3 = \rho_4$

  Now we have

$$\frac{d\rho}{d\tau} = \pm\sqrt{-d_4}\left((\rho_4-\rho)^3(\rho-\rho_1)\right)^{\frac{1}{2}}\,.$$

  $Q(\rho) > 0$ in this case implies $\rho_1 \le \rho \le \rho_4$. Integrating yields

$$\int_{\rho_1}^{\rho}\left((\rho_4-t)^3(t-\rho_1)\right)^{-\frac{1}{2}}dt = \frac{2}{\rho_4-\rho_1}\left(\frac{\rho-\rho_1}{\rho_4-\rho}\right)^{\frac{1}{2}} = \sqrt{-d_4}(\tau-\tau_0)\,.$$

  Solving for ρ, we get

$$\begin{aligned}\rho-\rho_1 &= \left(|d_4|(\tau-\tau_0)^2\cdot\tfrac{1}{4}(\rho_4-\rho_1)^2\right)(\rho_4-\rho)\,,\\ \rho\left[1+\tfrac{1}{4}|d_4|(\rho_4-\rho_1)^2(\tau-\tau_0)^2\right] &= \tfrac{1}{4}|d_4|(\rho_4-\rho_1)^2\rho_4(\tau-\tau_0)^2+\rho_1\,.\end{aligned}$$

  The solution now is

$$\rho = \frac{\rho_1+\frac{1}{4}|d_4|(\rho_4-\rho_1)^2\rho_4(\tau-\tau_0)^2}{1+\frac{1}{4}|d_4|(\rho_4-\rho_1)^2(\tau-\tau_0)^2}\,. \tag{3.59}$$

Using quotient rule, the numerator of ρ' is

$$\begin{aligned} & \tfrac{1}{2}|d_4|(\rho_4-\rho_1)^2\rho_4(\tau-\tau_0)\left(1+\tfrac{1}{4}|d_4|(\rho_4-\rho_1)^2(\tau-\tau_0)^2)\right) \\ & \quad -\tfrac{1}{2}|d_4|(\rho_4-\rho_1)^2(\tau-\tau_0)\left(\rho_1+\tfrac{1}{4}|d_4|(\rho_4-\rho_1)^2\rho_4(\tau-\tau_0)^2)\right) \\ = & \ \tfrac{1}{2}|d_4|(\rho_4-\rho_1)^3(\tau-\tau_0)\,. \end{aligned}$$

Thus

$$\rho' = \frac{\frac{1}{2}|d_4|(\rho_4-\rho_1)^3(\tau-\tau_0)}{\left(1+\frac{1}{4}|d_4|(\rho_4-\rho_1)^2(\tau-\tau_0)^2\right)^2}\,.$$

This means the solution in this case is decreasing for $\tau < \tau_0$ and increasing for $\tau > \tau_0$, and $\lim_{\tau\to\pm\infty}\rho = \rho_4$ with $\rho(\tau_0) = \rho_1$.

- $\rho_1 = \rho_2 = \rho_3 < \rho_4$

  Here we have

$$\frac{d\rho}{d\tau} = \pm\sqrt{-d_4}\left((\rho-\rho_1)^3(\rho_4-\rho)\right)^{\frac{1}{2}}\,.$$

  $Q(\rho) > 0$ in this case implies $\rho_1 \le \rho \le \rho_4$.

$$\int_{\rho_4}^{\rho}\left((t-\rho_1)^3(\rho_4-t)\right)^{-\frac{1}{2}}dt = -\frac{2}{\rho_4-\rho_1}\left(\frac{\rho_4-\rho}{\rho-\rho_1}\right)^{\frac{1}{2}} = \sqrt{-d_4}(\tau-\tau_0)\,.$$

  Solving for ρ, we get

$$\begin{aligned} \rho_4-\rho &= \left(|d_4|(\tau-\tau_0)^2\cdot\tfrac{1}{4}(\rho_4-\rho_1)^2\right)(\rho-\rho_1)\,, \\ \rho\left[1+\tfrac{1}{4}|d_4|(\rho_4-\rho_1)^2(\tau-\tau_0)^2\right] &= \tfrac{1}{4}|d_4|(\rho_4-\rho_1)^2\rho_1(\tau-\tau_0)^2+\rho_4\,. \end{aligned}$$

  The solution now is

$$\rho = \frac{\rho_4+\frac{1}{4}|d_4|(\rho_4-\rho_1)^2\rho_1(\tau-\tau_0)^2}{1+\frac{1}{4}|d_4|(\rho_4-\rho_1)^2(\tau-\tau_0)^2}\,. \tag{3.60}$$

  As the solution is identical to the one in the previous case with roles of ρ_1 and ρ_4 switched, ρ is increasing for $\tau < \tau_0$, decreasing for $\tau > \tau_0$. Besides, we have $\lim_{\tau\to\pm\infty}\rho = \rho_1$ and $\rho(\tau_0) = \rho_4$.

- $\rho_1 = \rho_2 \le \rho_3 = \rho_4$

  Here we have $Q(\rho) = d_4(\rho_1-\rho)^2(\rho_3-\rho)^2$. Since we need $Q(\rho) > 0$, but have $d_4 < 0$, the only solutions are the constant solutions $\rho = \rho_1$ or $\rho = \rho_3$. In the case of $\rho_1 = \rho_2 = \rho_3 = \rho_4$, these solutions coincide.

b) $Q(\rho)$ has at least one complex root

For this case, we need the Jacobian elliptic function cn. With $u = \int_0^\phi (1 - k^2 \sin^2 t)^{-\frac{1}{2}} dt$, define $\mathrm{cn}(u,k) = \cos(\phi)$. $\mathrm{cn}(u,k)$ has similar properties as the cosine function: It is an even function whose range (for real arguments) is [-1,1], and it is periodic with period $4K$ where $K = \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2 t)^{-\frac{1}{2}} dt$. Because $\mathrm{cn}(u + 2K) = -\mathrm{cn}(u)$, we have $\mathrm{cn}^2(u + 2K) = (-\mathrm{cn}(u))^2 = \mathrm{cn}^2 u$. We conclude that cn^2 is periodic with period $2K$.

Here we have to distinguish three cases: two pairs of complex roots, one pair of complex roots and two real roots that are equal or not.

- Two pairs of complex roots

  We write $\rho_k = \xi_{k,r} + i\,\xi_{k,i}$ (k=1,2) and ρ_3, ρ_4 to be the complex conjugates of ρ_1, ρ_2 respectively. This means

  $$Q(\rho) = d_4 \left((\rho - \rho_{1,r})^2 + \rho_{1,i}^2 \right) \left((\rho - \rho_{2,r})^2 + \rho_{2,i}^2 \right) < 0\,.$$

  Since we would need $Q(\rho) \geq 0$, no solutions exist in this case.

- Two real roots $\rho_1 < \rho_2$ and a pair of conjugate roots ρ_3, ρ_4

  Setting $\rho_3 = \xi_r + i\,\xi_i$, we get

  $$\left(\frac{d\rho}{d\tau}\right)^2 = d_4(\rho - \rho_1)(\rho - \rho_2)((\rho - \xi_r)^2 + \xi_i^2)\,.$$

  We notice that we need $\rho_1 \leq \rho \leq \rho_2$ once again. Integrating this and using formula (259.00) from [6] yields

  $$\int_{\rho_1}^{\rho} \left((t - \rho_1)(\rho_2 - t)((t - \xi_r)^2 + \xi_i^2) \right)^{-\frac{1}{2}} dt = g_3 \mathrm{cn}^{-1}(\cos(\phi_3), k_3) = \sqrt{-d_4}(\tau - \tau_0)\,,$$

  where

  $$\begin{aligned} A^2 &= (\rho_2 - \xi_r)^2 + \xi_i^2\,, \\ B^2 &= (\rho_1 - \xi_r)^2 + \xi_i^2\,, \\ g_3 &= (AB)^{-\frac{1}{2}}\,, \\ \cos(\phi_3) &= \tfrac{(\rho_2 - \rho)B - (\rho - \rho_1)A}{(\rho_2 - \rho)B + (\rho - \rho_1)A}\,, \\ k_3^2 &= \tfrac{(\rho_2 - \rho_1)^2 - (A - B)^2}{4AB}\,. \end{aligned}$$

  Solving for ρ, we obtain

  $$\begin{aligned} \tfrac{(\rho_2 - \rho)B - (\rho - \rho_1)A}{(\rho_2 - \rho)B + (\rho - \rho_1)A} &= \cos(\phi_3) = \mathrm{cn}\left(\sqrt{-d_4}\, g_3^{-1}(\tau - \tau_0), k_3\right)\,, \\ \rho(-A - B) + (\rho_1 A + \rho_2 B) &= \mathrm{cn}\left(\sqrt{-d_4}\, g_3^{-1}(\tau - \tau_0), k_3\right)(A - B)\rho \\ &\quad + \mathrm{cn}\left(\sqrt{-d_4}\, g_3^{-1}(\tau - \tau_0), k_3\right)(\rho_2 B - \rho_1 A)\,. \end{aligned}$$

The solution then is

$$\rho = \frac{\rho_1 A + \rho_2 B + (\rho_1 A - \rho_2 B)\,\mathrm{cn}\left(\sqrt{-d_4} g_3^{-1}(\tau - \tau_0), k_3\right)}{A + B + (A - B)\,\mathrm{cn}\left(\sqrt{-d_4} g_3^{-1}(\tau - \tau_0), k_3\right)} . \tag{3.61}$$

As the range of $\mathrm{cn}(u,k)$ is [-1,1], ρ takes values between ρ_1 and ρ_2. Because the function $\mathrm{cn}(C\tau + D, k_3)$ has period $\frac{4K_3}{C}$ with

$$K_3 = \int_0^{\frac{\pi}{2}} (1 - k_3^2 \sin^2 t)^{-\frac{1}{2}} dt \, ,$$

we conclude that ρ here has period $4K_3 g_3 (-d_4)^{-\frac{1}{2}}$.

- Two real roots $\rho_1 = \rho_2$ and a pair of conjugate roots ρ_3, ρ_4

  This is a special case of the previous case. Using the same notation, we now have

  $$\left(\frac{d\rho}{d\tau}\right)^2 = d_4(\rho - \rho_1)^2((\rho - \xi_r)^2 + \xi_i^2) \, .$$

  We notice that the right side is nonnegative only for $\rho = \rho_1$, thus there is only the constant solution $\rho = \rho_1$ in this case.

3.3.2 Solutions where r_m or r_n are zero

In the previous section, we implicitly assume that neither r_m or r_n are zero. A brief investigation is added for the case that either r_m or r_n vanish.

First, suppose $r_m(0) = 0$, meaning $P_m(0) = 0$, and $P_n(0) = P_{n,0}$. If we assume $|P_m|$ and c_{21} negligible, (3.41) simplifies to

$$\begin{aligned} P_m' &= c_{13}\overline{P_n} \, , \\ P_n' &= -i\tfrac{\lambda}{2} P_n \, . \end{aligned}$$

Solving the second equation yields

$$P_n(\tau) = e^{-i\frac{\lambda}{2}\tau} P_{n,0} \, .$$

Integrating the first equation

$$\begin{aligned} P_m(\tau) &= \textstyle\int c_{13} e^{i\frac{\lambda}{2}\tau}\overline{P_{n,0}} \\ &= \tfrac{2}{i\lambda} c_{13} e^{i\frac{\lambda}{2}\tau}\overline{P_{n,0}} - \tfrac{2}{i\lambda} c_{13}\overline{P_{n,0}} \\ &= \tfrac{2}{i\lambda}(e^{i\frac{\lambda}{2}\tau} - 1) c_{13}\overline{P_{n,0}} \\ &\approx \tfrac{2}{i\lambda}(i\tfrac{\lambda}{2}\tau) c_{13}\overline{P_{n,0}} \\ &= \tau c_{13}\overline{P_{n,0}} \, . \end{aligned}$$

Assume that the angle of $P_n(0)$ is $\alpha_{n,0}$. Then for $\tau > 0$ we have

$$\alpha = \alpha_m + \alpha_n = -\alpha_{n,0} + (-\frac{\lambda}{2}\tau + \alpha_{n,0}) = -\frac{\lambda}{2}\tau \,.$$

Thus, using $\sin(\alpha) \approx \alpha$ for small α, we have

$$\frac{\sin(\alpha)}{r_m} = \frac{-\frac{\lambda}{2}\tau}{|c_{13}||P_{n,0}|\tau} = -\frac{\lambda}{2|c_{13}||P_{n,0}|} \,,$$

which means that α' is bounded for r_m small, and the singularity $r_m = 0$ can be removed. For $\tau < 0$, we can obtain the same result using $\sin(\theta + \pi) \approx -\theta$ for small θ.

A similar result can be established if we assume $P_n(0) = 0$ and $P_m(0) = P_{m,0}$. For $|P_n|$ and c_{12} small, (3.41) then simplifies to

$$\begin{aligned} P'_m &= -i\tfrac{\lambda}{2}P_m \,, \\ P'_n &= c_{23}\overline{P_m} \,. \end{aligned}$$

The approximation to the solution for this system is

$$\begin{aligned} P_m(\tau) &= e^{-i\frac{\lambda}{2}\tau}P_{m,0} \,, \\ P_n(\tau) &= \tau c_{23}\overline{P_{m,0}} \,. \end{aligned}$$

With the same reasoning as before we conclude that in this case

$$\frac{\sin(\alpha)}{r_n} = \frac{-\frac{\lambda}{2}\tau}{|c_{23}||P_{m,0}|\tau} = -\frac{\lambda}{2|c_{23}||P_{m,0}|} \,,$$

and it does not cause a problem when we consider r_n small, or even $r_n = 0$.

3.3.3 Solutions of the original system

In order to see what the solutions of the original system are, we consider (3.44, 3.45).

a) ρ is periodic

We assume that ρ is periodic with period Ω. From (3.49), we can conclude that r_m^2 and r_n^2 also have period Ω. Furthermore, if ρ was found to be bound by ρ_{min} and ρ_{max}, bounds for

r_m^2 and r_n^2 can be obtained by plugging in the upper and lower bounds for ρ. To find out more about α_m and α_n, we use (3.47) to express $\sin(\alpha)$ in terms of r_m and r_n.

$$\sin(\alpha) = \frac{1}{r_m r_n}\left(E - \frac{\lambda}{2c_{13}^2 + 2c_{23}^2}(c_{13}r_m^2 + c_{23}r_n^2) + \frac{c_{11}+c_{22}}{4c_{13}}r_m^4 + \frac{c_{12}+c_{21}}{4c_{23}}r_n^4\right). \tag{3.62}$$

Putting this into (3.45) yields

$$\begin{aligned} \alpha_m' &= -\tfrac{\lambda}{2} + c_{11}r_m^2 + c_{12}r_n^2 \\ &\quad -\tfrac{c_{13}}{r_m^2}\left(E - \tfrac{\lambda}{2c_{13}^2+2c_{23}^2}(c_{13}r_m^2 + c_{23}r_n^2) + \tfrac{c_{11}+c_{22}}{4c_{13}}r_m^4 + \tfrac{c_{12}+c_{21}}{4c_{23}}r_n^4\right), \\ \alpha_n' &= -\tfrac{\lambda}{2} + c_{21}r_n^2 + c_{22}r_m^2 \\ &\quad -\tfrac{c_{23}}{r_n^2}\left(E - \tfrac{\lambda}{2c_{13}^2+2c_{23}^2}(c_{13}r_m^2 + c_{23}r_n^2) + \tfrac{c_{11}+c_{22}}{4c_{13}}r_m^4 + \tfrac{c_{12}+c_{21}}{4c_{23}}r_n^4\right). \end{aligned} \tag{3.63}$$

Since r_m^2 and r_n^2 are periodic with the same period and the right sides are dependent on τ only in the way it appears implicitly in r_m^2 and r_n^2, (3.63) can be considered as

$$\begin{aligned} \alpha_m' &= F_1[r_m^2(\tau), r_n^2(\tau)], \\ \alpha_n' &= F_2[r_m^2(\tau), r_n^2(\tau)]. \end{aligned}$$

Here, F_1 and F_2 have period Ω if thought of as functions of τ. Integrating shows that

$$\begin{aligned} \alpha_m(\tau+\Omega) - \alpha_m(\tau) &= \textstyle\int_\tau^{\tau+\Omega} F_1(t)dt \\ &= \textstyle\int_\tau^{\Omega} F_1(t)dt + \int_\Omega^{\Omega+\tau} F_1(t)dt \\ &= \textstyle\int_\tau^{\Omega} F_1(t)dt + \int_\Omega^{\Omega+\tau} F_1(t-\Omega)dt \\ &= \textstyle\int_\tau^{\Omega} F_1(t)dt + \int_0^{\tau} F_1(t)dt \quad = \int_0^{\Omega} F_1(t)dt = B_1, \end{aligned}$$

where B_1 is a constant independent of τ and similarly

$$\alpha_n(\tau+\Omega) - \alpha_n(\tau) = \int_0^{\Omega} F_2(t)dt = B_2.$$

To see what this means for α_m and α_n, define

$$\beta_m(\tau) = \alpha_m(\tau) - \frac{B_1}{\Omega}\tau.$$

Then

$$\begin{aligned} \beta_m(\tau+\Omega) - \beta_m(\tau) &= [\alpha(\tau+\Omega) - \tfrac{B_1}{\Omega}(\tau+\Omega)] - [\alpha(\tau) - \tfrac{B_2}{\Omega}\tau] \\ &= B_1 + \alpha_m(\tau) - \tfrac{B_1}{\Omega}(\tau+\Omega) - \alpha_m(\tau) + \tfrac{B_1}{\Omega}\tau \quad = 0. \end{aligned}$$

This means that $\beta_m(\tau)$ has period Ω. The same way it can be seen that if $\beta_n(\tau)$ is defined by

$$\beta_n(\tau) = \alpha_n(\tau) - \frac{B_2}{\Omega}\tau\,,$$

it also turns out to be a function with period Ω. The result is that α_m and α_n can be written as

$$\begin{array}{rcl} \alpha_m(\tau) & = & \frac{B_1}{\Omega}\tau + \beta_m(\tau)\,, \\ \alpha_n(\tau) & = & \frac{B_2}{\Omega}\tau + \beta_n(\tau)\,, \end{array}$$

where β_m and β_n are functions with period Ω.

Recall that this discussion was about the system (P_m,P_n). According to (3.40), the radii and angles of (p_m, p_n) are the same, except for that we need to add $\frac{\lambda}{2}\tau$ to the angles. Note that this only changes the constants of the linear terms in the previous equations.

The behavior of the radii and angles implies that the behavior of these solutions in the phase plane is chaotic and that the trajectories are dense in an annulus.

b) Other cases

If ρ is not periodic, ρ is monotone on $(-\infty, \tau_0)$ and (τ_0, ∞) and the limits $\lim_{\tau\to\pm\infty}\rho$ and exist and are known. From (3.49) we see that r_m^2 and r_n^2 are monotone functions on the same intervals and that those limits also exist for them as well.

Equations (3.62,3.63) can still be used and tell us that the limits to $\pm\infty$ of α_m' and α_n' exist as well. Even more, we know that $\cos(\alpha_m + \alpha_n) \to 0$ as $\tau \to \pm\infty$. As r_m is monotone for $\tau < \tau_0$ and $\tau > \tau_0$, $\cos(\alpha_m + \alpha_n)$ cannot change its sign when we take the limit, hence its derivative also goes to zero as $\tau \to \pm\infty$. This means that $-\sin(\alpha_m + \alpha_n)(\alpha_m' + \alpha_n') \to 0$. Since the first factor cannot go to zero, the second factor needs to do so, thus

$$\lim_{\tau\to\infty} \alpha_n' = -\lim_{\tau\to\infty} \alpha_m'\,.$$

If $\mathcal{A}_m$ and $\mathcal{A}_n$ are the angles of p_m and p_n, this relationship means

$$\begin{array}{l} \lim_{\tau\to\infty} \mathcal{A}_n' = \lim_{\tau\to\infty} \alpha_n' + \frac{\lambda}{2} = -\lim_{\tau\to\infty} \alpha_m' + \frac{\lambda}{2} \\ \quad = -\left(\lim_{\tau\to\infty} \alpha_m' + \frac{\lambda}{2}\right) + \lambda = -\lim_{\tau\to\infty} \mathcal{A}_m' + \lambda\,. \end{array}$$

The same relationships can be established for the limit $\tau \to -\infty$.

A special case of this is $\rho = const$. Then r_m, r_n, α_m' and α_n' are also constants. This means that the angles α_m and α_n and thus also the angles $\mathcal{A}_m$ and $\mathcal{A}_n$ are linear.

3.3.4 Analysis of the dynamical system

We want to discuss (3.38) in greater detail.

Equilibria and periodic orbits

In the case of $\lambda \neq 0$ we should have $\overline{p_n} = \overline{p_m} = 0$ to account for the changing τ in order to get an equilibrium. This means $p_m = p_n = 0$ is the only equilibrium for $\lambda \neq 0$.

Let us focus on the case $\lambda = 0$. Then we have $p_m = P_m$ and $p_n = P_n$ in (3.40), of course. If p_m and p_n should be at an equilibrium, so should be their radii. To avoid the trivial equilibrium, from (3.44) we see that we need $\cos(\alpha) = \cos(\alpha_m + \alpha_n) = 0$ or $\alpha_n = \pm\frac{\pi}{2} - \alpha_m$. In other words, we can assume that

$$p_n = iK\overline{p_m} \quad \text{for some } K \in \Re\,. \tag{3.64}$$

Then to have an equilibrium in (3.38) we need

$$\begin{aligned} c_{11} i p_m |p_m|^2 + c_{12} i p_m K^2 |p_m|^2 - c_{13} i K p_m &= 0\,, \\ -c_{21} K^3 \overline{p_m} |p_m|^2 - c_{22} K \overline{p_m} |p_m|^2 + c_{23} \overline{p_m} &= 0\,. \end{aligned}$$

We drop the trivial solution and let $r_m = |p_m|$. Then this simplifies to

$$\begin{aligned} (c_{11} + c_{12}K^2) r_m^2 - c_{13} K &= 0\,, \\ (c_{21}K^3 + c_{22}K) r_m^2 - c_{23} &= 0\,. \end{aligned} \tag{3.65}$$

Solving the first equation for r_m^2, we obtain

$$r_m^2 = \frac{c_{13}K}{(c_{11} + c_{12}K^2)}\,. \tag{3.66}$$

Plugging this into the second equation of (3.65), we get

$$(c_{21}K^3 + c_{22}K)\frac{c_{13}K}{(c_{11} + c_{12}K^2)} - c_{23} = 0$$

or

$$c_{21}c_{13}u^2 + (c_{22}c_{13} - c_{12}c_{23})u - c_{11}c_{23} = 0\,, \tag{3.67}$$

where $u = K^2$.

Since K is real, we are only interested in positive solutions of this quadratic equation. There may be zero, one or two such solutions. Going back to (3.66), this means that for each positive solution u of (3.67), one solution for r_m^2 can be obtained by picking the right sign in $K = \pm\sqrt{u}$.

Note that any p_m with the right absolute value now satisfies the equation. Thus equilibria are described by

$$\begin{aligned} p_m &= e^{i\alpha_m} r_m \, , \\ p_n &= e^{-i\alpha_m} i \, K \, r_m \end{aligned} \tag{3.68}$$

for pairs $(u = K^2, r_m)$ that solve (3.67), (3.66) and a free choice of α_m.

To get more information for the case $\lambda \neq 0$, equilibria of the system for P_m and P_n which are introduced in (3.40) are studied. With the same reasoning as for the case $\lambda = 0$, it is assumed that $\alpha_m + \alpha_n = \pm\frac{\pi}{2}$ and thus

$$P_n = iK \, \overline{P_m} \quad \text{for some } K \in \Re \, . \tag{3.69}$$

Hence, to have an equilibrium in (3.41), we need

$$\begin{aligned} -i\tfrac{\lambda}{2} P_m + c_{11} i P_m |P_m|^2 + c_{12} i P_m K^2 |P_m|^2 - c_{13} i K P_m &= 0 \, , \\ \tfrac{\lambda}{2} K \overline{P_m} - c_{21} K^3 \overline{P_m} |P_m|^2 - c_{22} K \overline{P_m} |P_m|^2 + c_{23} \overline{P_m} &= 0 \, . \end{aligned}$$

Cancelling out iP_m in the first equation and $\overline{P_m}$ in the second one yields

$$\begin{aligned} (c_{11} + c_{12} K^2) r_m^2 - c_{13} K - \tfrac{\lambda}{2} &= 0 \, , \\ (c_{21} K^3 + c_{22} K) r_m^2 - c_{23} - \tfrac{\lambda}{2} K &= 0 \end{aligned} \tag{3.70}$$

with $r_m = |P_m|$. Solving the first equation for r_m^2 yields

$$r_m^2 = \frac{c_{13} K + \frac{\lambda}{2}}{(c_{11} + c_{12} K^2)} \, . \tag{3.71}$$

Plugging this into the second equation, we get

$$(c_{21} K^3 + c_{22} K) \frac{c_{13} K + \frac{\lambda}{2}}{(c_{11} + c_{12} K^2)} - c_{23} - \frac{\lambda}{2} K = 0 \, .$$

Finding the common denominator and setting the numerator equal to zero yields

$$c_{21}c_{13}K^4 + \frac{1}{2}(c_{21} - c_{12})\lambda K^3 + (c_{22}c_{13} - c_{12}c_{23})K^2 + \frac{1}{2}(c_{22} - c_{11})\lambda K - c_{11}c_{23} = 0\,. \quad (3.72)$$

For each of the real solutions K_i of this equation, go back to (3.71) to check whether the right side is positive for that solution. If yes, we can solve for r_m - if no, discard that solution. We conclude that equilibrium solutions for P_m and P_n are given by

$$\begin{aligned} P_m &= e^{i\alpha_m} r_m\,, \\ P_n &= e^{-i\alpha_m} iK r_m \end{aligned} \quad (3.73)$$

for pairs of (K, r_m) that satisfy (3.71) and (3.72) and a free choice of α_m. Corresponding periodic solutions of the original system are then given by

$$\begin{aligned} p_m &= P_m e^{i\frac{\lambda}{2}\tau}\,, \\ p_n &= P_n e^{i\frac{\lambda}{2}\tau}\,. \end{aligned} \quad (3.74)$$

Stability of the equilibria and periodic orbits

First consider the trivial equilibrium. The Jacobian matrix is

$$J = \begin{pmatrix} 0 & \frac{\lambda}{2} & c_{13} & 0 \\ -\frac{\lambda}{2} & 0 & 0 & -c_{13} \\ c_{23} & 0 & 0 & \frac{\lambda}{2} \\ 0 & -c_{23} & -\frac{\lambda}{2} & 0 \end{pmatrix} \quad (3.75)$$

with the definitions of c_{ij} from (3.39). Its eigenvalues are $\pm\sqrt{c_{13}c_{23} - \frac{1}{4}\lambda^2}$ which means that the trivial equilibrium can be stable only if $c_{13}c_{23} - \frac{1}{4}\lambda^2 < 0$.

For other equilibria and periodic orbits, respectively, we consider the system for P_m and P_n, see equation (3.41). For $\lambda = 0$, the system is identical to (3.38), while for $\lambda \neq 0$, periodic orbits of (3.38) correspond to equilibria (3.41). A periodic orbit of (3.38) is stable, if and only if the corresponding equilibrium of (3.41) is determined to be stable.

If non-trivial equilibria exist, the characteristic polynomial of the Jacobian is given by

$$\chi(\mu) = \mu^4 + C_1\mu^2 + C_2 \quad (3.76)$$

with

$$
\begin{aligned}
C_1 &= -2c_{13}c_{23} + 2(c_{13}c_{22} + c_{12}c_{23})Kr_m^2 \\
&\quad + \left(3c_{11}^2 + c_{22}^2 + 4(c_{11}c_{12} + c_{21}c_{22})K^2 + (c_{12}^2 + 3c_{21}^2)K^4\right) r_m^4 \\
&\quad -[2c_{11} + c_{22} + (c_{12} + 2c_{21})K^2]r_m^2\lambda + \tfrac{1}{2}\lambda^2 \,, \\
C_2 &= \Big(c_{13}c_{23} - [(c_{11} + c_{12}K^2)r_m^2 - \tfrac{1}{2}\lambda][(c_{22} + c_{21}K^2)r_m^2 - \tfrac{1}{2}\lambda]\Big) \\
&\quad \times\Big(c_{13}c_{23} - 2(c_{13}c_{22} + c_{12}c_{23})Kr_m^2 \\
&\qquad -3[c_{11}c_{22} + (3c_{11}c_{21} - c_{12}c_{22})K^2 + c_{12}c_{21}K^4]r_m^4 \\
&\qquad +\tfrac{1}{2}[3c_{11} + c_{22} + (c_{12} + 3c_{21})K^2]r_m^2\lambda - \tfrac{1}{4}\lambda^2\Big) \,.
\end{aligned} \tag{3.77}
$$

In this formula, C_2 deserves closer attention. We can rewrite the condition for equilibria of the system (P_m, P_n), equation (3.70), as

$$
\begin{aligned}
(c_{11} + c_{12}K^2)r_m^2 - \tfrac{1}{2}\lambda &= c_{13}K \,, \\
(c_{21}K^3 + c_{22}K)r_m^2 - \tfrac{1}{2}\lambda K &= c_{23} \,.
\end{aligned}
$$

Multiply the left and right sides of each equation and cancel out the factor K to see

$$
\left[(c_{11} + c_{12}K^2)r_m^2 - \frac{1}{2}\lambda\right]\left[(c_{21}K^2 + c_{22})r_m^2 - \frac{1}{2}\lambda\right] = c_{13}c_{23} \,.
$$

This implies $C_2 = 0$. In the context of (3.76), this means that at non-trivial equilibria, there is a double eigenvalue 0. If $C_1 > 0$, the remaining eigenvalues are purely imaginary. Otherwise, there is one positive and one negative eigenvalue.

3.3.5 Homoclinic and heteroclinic orbits

The discussion in Section 3.3.1 shows that there are no heteroclinic orbits. To find homoclinic orbits for non-trivial equilibria, we would need to find initial conditions that yield double roots of the polynomial $Q(\rho)$ introduced in (3.48). There is no obvious way to do this analytically.

Thus, we focus on homoclinic orbits for the trivial equilibrium. To get such a homoclinic orbit, it can be seen from 3.3.1 that we need roots $\rho_1 \leq \rho_2 = \rho_3 = 0 < \rho_4$.

On orbits that go through the trivial equilibrium, the invariants J, E introduced in (3.47) vanish. For the polynomial $Q(\rho)$ and its coefficients given in (3.51) and (3.52), this means $d_0 = d_1 = 0$. The polynomial $Q(\rho)$ now takes the form

$$
Q(\rho) = \rho^2\left(d_2 + d_3\rho + d_4\rho^2\right) \,,
$$

which already gives us the double root 0. Now we need to make sure that the remaining roots have opposite signs. They are given by

$$\rho_{1,4} = \frac{1}{2d_4}\left(-d_3 \pm \sqrt{d_3^2 - 4d_2d_4}\right).$$

Substituting $c_{23} = K^2 c_{13}$ and using $\beta_1 = 2$, $\beta_2 = 0$, we get

$$\rho_{1,4} = \frac{2(\lambda \pm 2c_{13}K)}{c_{11} + c_{22} + (c_{12} + c_{21})K^2}. \tag{3.78}$$

A condition for having roots ρ_1, ρ_4 of different signs then is

$$\lambda^2 \leq 4c_{13}^2 K^2 = 4c_{13}c_{23}. \tag{3.79}$$

It should be noted that this means that the eigenvalues of the Jacobian at the trivial equilibrium are real.

The conclusion: there are homoclinic orbits connecting the trivial equilibrium if and only if $\lambda^2 \leq 4c_{13}c_{23}$. In fact, since $Q(\rho)$ can be expressed in terms of E and J only, any point with $E = J = 0$ then lies on such a homoclinic orbit. However, the angle α_m can be chosen freely. Thus we expect an infinite number of homoclinic orbits.

3.3.6 Numerical Experiment

For parameters $m = 1; n = 15; a = 10; h = 1; g = 9.81; T = 0.1; f_1 = 10, f_2 = -10$, we get

$$\begin{aligned} r_1 &= 2.0995, & s_1 &= 111.3, \\ r_2 &= -0.0474i, & s_2 &= -101.2i, \\ r_3 &= 2.4763i, & s_3 &= 828.3i, \\ r_4 &= -4.2620, & s_4 &= -240.4, \end{aligned}$$

in (3.29, 3.36) and thus

$$\begin{aligned} c_{11} &= 0.22585, & c_{21} &= 0.90941, \\ c_{12} &= -1.17944, & c_{22} &= -7.44031, \\ c_{13} &= 2.03001, & c_{23} &= 2.19039 \end{aligned}$$

in (3.39). As an example, we consider initial conditions $r_m = 1$; $r_n = 0.01$; $\alpha = -\frac{\pi}{2}$ and $\lambda = 1$. With $\beta_1 = 2$ and $\beta_2 = 0$, we get

$$Q(\rho) = -19.545 - 11.519\rho + 50.467\rho^2 - 5.3191\rho^3 - 14.084\rho^4.$$

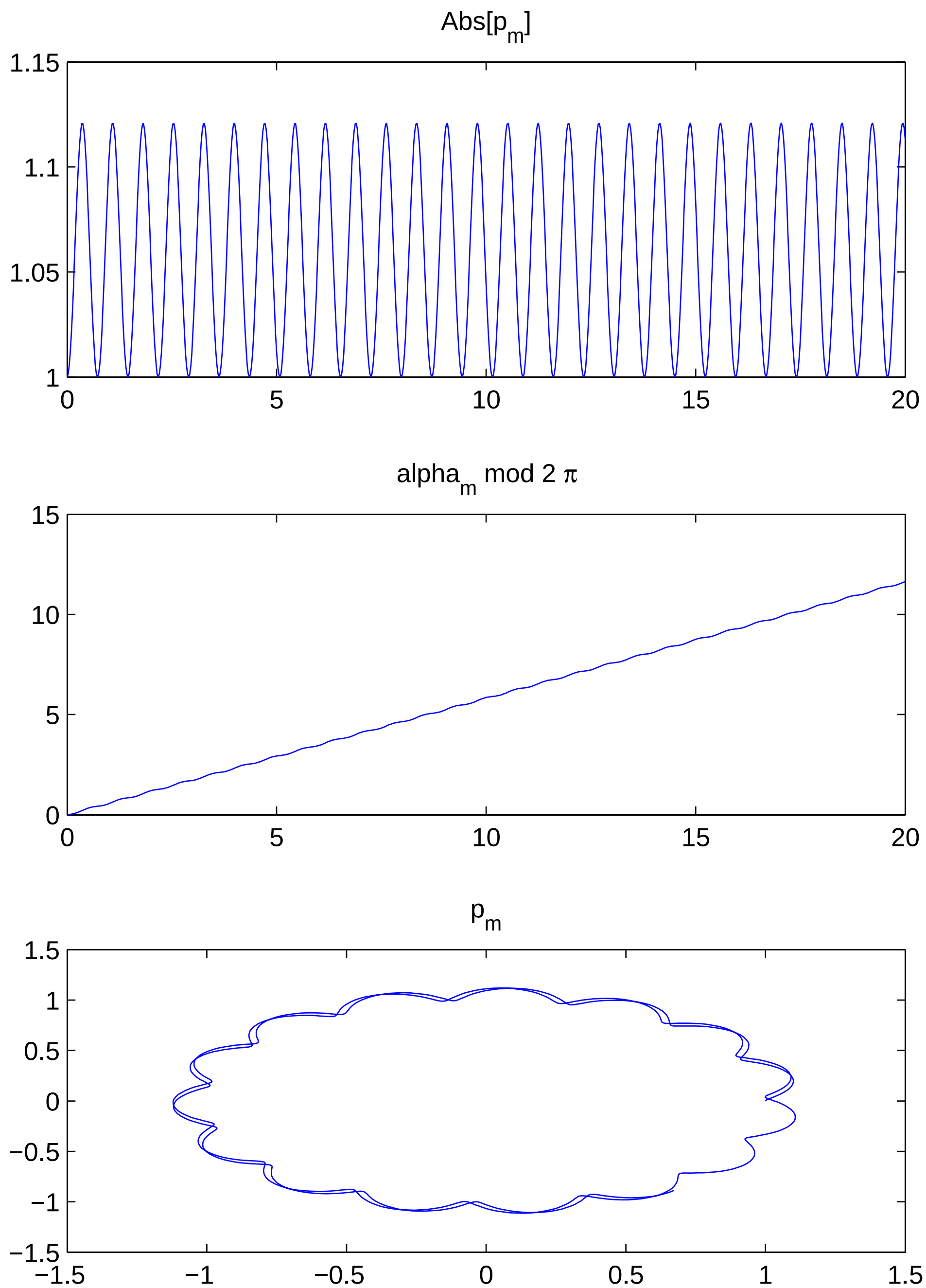

Figure 3.1: Radius, angle and phase portrait of p_m

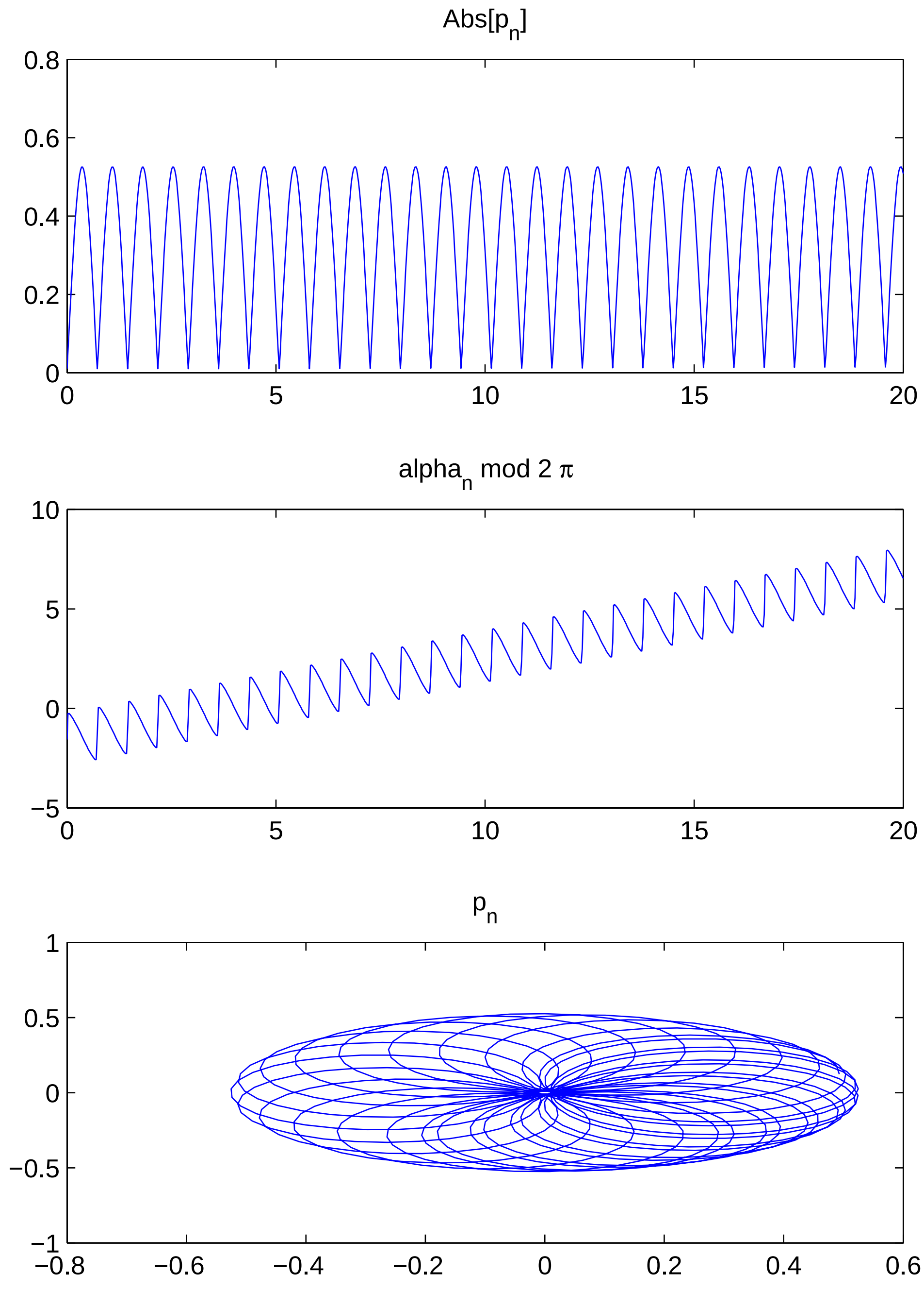

Figure 3.2: Radius, angle and phase portrait of p_n

The roots of this quartic polynomial are $-2.111, -0.5232, 1, 1.2565$. According to the previous section, the solution should show oscillations of $|p_m|$ between 1 and 1.12 and $|p_n|$ between 0.01 and 0.53 with period 0.725 .

Figures 3.1 and 3.2 show the radii, angles and phase portraits of p_m and p_n. These solutions have been calculated using the ODE23S command of MATLAB with absolute and relative tolerance of 10^{-8}. It can be seen that the radii oscillate between the bounds we have given and oscillate with the predicted frequency. The angle functions turn out to be sums of linear functions and functions of period 0.725, as expected.

The eigenvalues at the trivial equilibrium in this case are ± 2.11 for $\lambda = 0$, ± 2.05 for $\lambda = 1$. Now focus on non-trivial equilibria for $\lambda = 0$. Equation (3.67) in this case has solutions $u_1 = -0.03928, u_2 = 6.82137$. Being interested in positive solutions only, we focus on the second solution. Choosing $K = -\sqrt{u_2}$ yields $r_m = 0.82343$. The equilibrium solutions are

$$\begin{aligned} p_m &= 0.82343\, e^{i\alpha_m}\,, \\ p_n &= -2.15061\, i\, e^{-i\alpha_m}\,, \end{aligned} \tag{3.80}$$

where the angle α_m can be chosen freely. At each of those equilibria, the Jacobian has eigenvalues 0 (double) and $\pm 8.5957i$. Stability thus depends on whether the Jacobian has two linear independent eigenvectors for the double eigenvalue 0.

For $\lambda \neq 0$, periodic orbits can be found. Using $\lambda = 1$, the solutions of (3.72) for the set of parameters given in the beginning of this section are

$$K_1 = -2.7619, \quad K_2 = 2.4985, \quad K_{3,4} = -0.151185 \pm 0.126397\, i\,.$$

Only K_1 gives us a real solution for r_m in (3.71), and we obtain $r_m = 0.763038$. Using equations (3.73, 3.74), the periodic orbits for this case are thus given by

$$\begin{aligned} p_m &= 0.763038 e^{(\alpha_m + \frac{\tau}{2})i}\,, \\ p_n &= -2.10742 i\, e^{(-\alpha_m + \frac{\tau}{2})i} \end{aligned} \tag{3.81}$$

with free choice of α_m.

The eigenvalues of the corresponding equilibrium of the system (P_m, P_n) are 0 (double) and $\pm 8.4955i$. As before, stability depends on the existence of two linearly independent eigenvectors for the double eigenvalue 0.

Since (3.79) is satisfied, there should be homoclinic orbits. For the given set of parameters, the initial values $r_m = 10^{-6}$, $r_n = 1.03875 \cdot 10^{-6}$, $\alpha = -0.33033437$ satisfy $J = E = 0$, thus such points should be on a homoclinic orbit. Figures 3.3 and 3.4 show the phase portraits of homoclinic orbits through these points with choices of $\alpha_m = 0$ (blue), $\alpha_m = \frac{\pi}{3}$ (black), $\alpha_m = \frac{2\pi}{3}$ (red), $\alpha_m = \pi$ (green), $\alpha_m = \frac{4\pi}{3}$ (cyan) and $\alpha_m = \frac{5\pi}{3}$ (magenta).

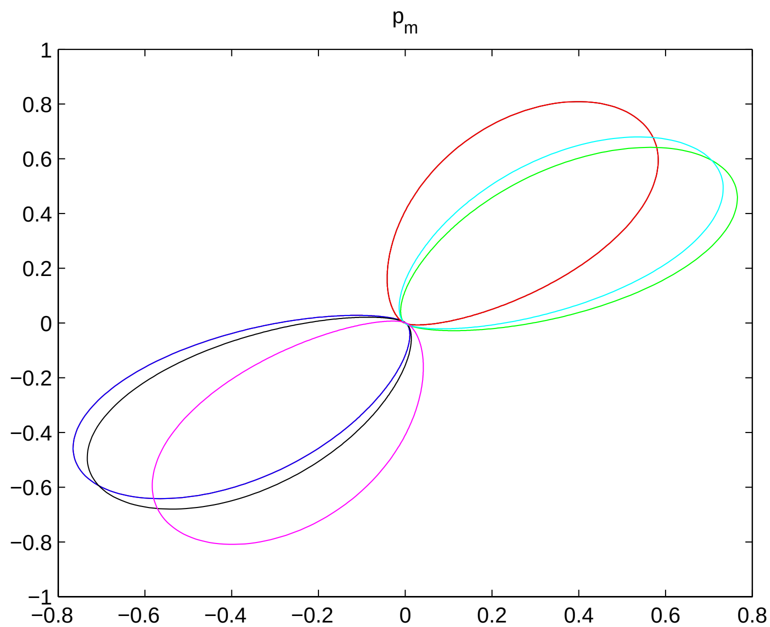

Figure 3.3: Phase portrait of homoclinic orbits, showing p_m

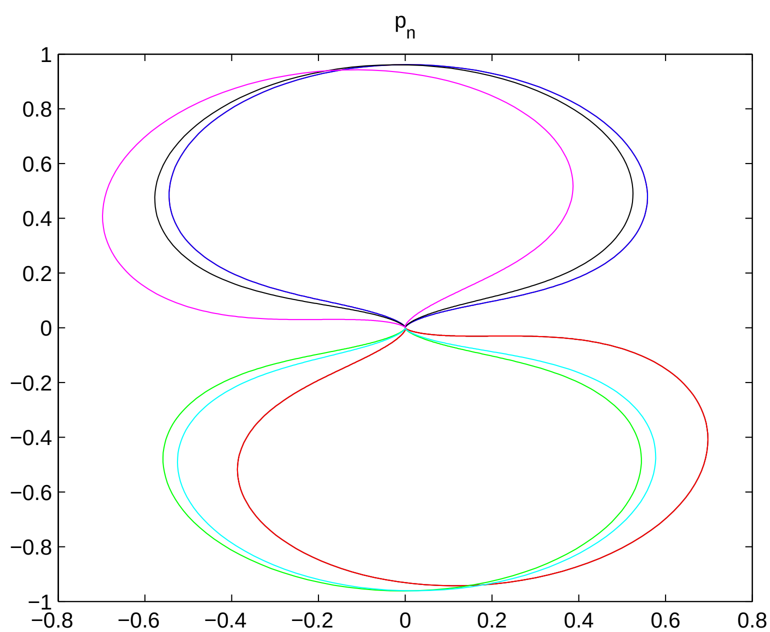

Figure 3.4: Phase portrait of homoclinic orbits, showing p_n

Chapter 4

Forcing frequencies that are sums of eigenfrequencies with different orders

4.1 Introduction

We again consider forcing frequencies close to $\omega_m + \omega_n$ where ω_m and ω_n are eigenfrequencies. In experiments, one eigenfrequency is usually significantly larger than the other one, say $\omega_n \gg \omega_m$. This changes the orders of the coefficients in the previous system (3.38). The assumptions and governing equations are modified appropriately. A change of variables is used to make it easier to compare the orders of different terms that come into play later.

Under the above assumptions, the process of finding the first- and second-order approximations and solvability conditions from the third-order equations is repeated under the assumption of negligible surface tension and simplified edge conditions. The result is a system of the form (3.38), but with coefficients that have different orders. We find some suitable choices for the constants α_i that represent the different orders of the various factors involved in the equations. Surface tension and a variation of Hocking's edge condition are included, and it is shown what order surface tension can be so that it does not become a relevant factor in the equations.

For the chosen set of α_i, the analysis of the system in terms of equilibria and homoclinic orbits is repeated, and the reduced differential equation is discussed in this setting. A way to obtain orbits that converge to circular orbits and can be associated with homoclinic orbits connecting non-trivial equilibria is given. We illustrate this by a numerical example and show how the solutions behave when approaching those orbits.

4.2 Derivation of equations for absence of surface tension and simplified edge conditions

4.2.1 Formulation

As a generalization, we now want the boundary terms to be of the form

$$b_k(z,t,\tau) \quad = \quad \epsilon^{\alpha_4} B_k(z,t,\tau) = \epsilon^{\alpha_4} e^{i(\omega_m+\omega_n)t+i\lambda\tau} \tfrac{1}{i} f_k(z) + c.c. \tag{4.1}$$

where we assume f_k to be real-valued and now, more general than before,

$$\tau = \epsilon^{\alpha_5} t \tag{4.2}$$

is the slower time scale. Also, if the frequency ω_n is considerably larger than the frequency ω_m, it is not justified anymore to assume that they are both of order 1. Since ω_n contains k_n, it is assumed that $n = \epsilon^{-\alpha_1} m$ $(\alpha_1 > 0)$ which leads to

$$k_n = \frac{n\pi}{a} = \epsilon^{-\alpha_1} \frac{m\pi}{a} = \epsilon^{-\alpha_1} k_m \,. \tag{4.3}$$

In order to be able to determine which terms are the dominating ones, it is suitable to introduce a change of variables:

$$\begin{aligned} x^* &:= k_n x \,, \\ z^* &:= k_n z \,, \\ t^* &:= \omega_n t \,. \end{aligned} \tag{4.4}$$

We need to start with the original equations (1.6 - 1.11) and apply this change of variables to them. Note that

$$\phi_x = \frac{\partial \phi}{\delta x} = \frac{\partial \phi}{\partial x^*} \frac{\partial x^*}{\partial x} = \phi_{x^*} k_n \,.$$

The same way we have $\phi_z = \phi_{z^*} k_n$ and $\phi_t = \phi_{t^*} \omega_n$. This, of course, works for θ as well and can be continued for higher-order derivatives. The original equations then become

$$\begin{aligned}
k_n^2\phi_{x^*x^*} + k_n^2\phi_{z^*z^*} &= 0\,, \\
k_n\phi_{x^*} - \omega_n b_{1,t^*} - k_n^2 b_{1,z^*}\phi_{z^*} &= 0 \text{ at } x^* = k_n b_1\,, \\
k_n\phi_{x^*} - \omega_n b_{2,t^*} - k_n^2 b_{2,z^*}\phi_{z^*} &= 0 \text{ at } x^* = k_n b_2 + k_n a\,, \\
k_n\phi_{z^*} &= 0 \text{ at } z^* = -k_n h\,, \\
\omega_n\theta_{t^*} + k_n^2\theta_{x^*}\phi_{x^*} - k_n\phi_{z^*} &= 0 \text{ at } z^* = k_n\theta\,, \\
\tfrac{\omega_n}{k_n^2}\phi_{t^*} + \tfrac{g}{k_n^2}\theta + \tfrac{1}{2}\phi_{x^*}^2 + \tfrac{1}{2}\phi_{z^*}^2 - T\theta_{x^*x^*}(1 - \tfrac{3}{2}k_n^2\theta_{x^*}^2) &= 0 \text{ at } z^* = k_n\theta\,, \\
k_n\theta_{x^*} &= 0 \text{ at } x^* = 0, k_n a\,.
\end{aligned} \tag{4.5}$$

We now set up θ and ϕ as

$$\begin{aligned}
\theta &= \theta_1 + \theta_2 + \theta_R\,, \\
\phi &= \phi_1 + \phi_2 + \phi_R\,.
\end{aligned}$$

The intention of doing this is to have low-order terms θ_1 and ϕ_1 which have frequencies ω_m and ω_n. In θ_2 and ϕ_2, no terms of frequency ω_m or ω_n should be contained. Terms of those frequencies are expected to show up in θ_R and ϕ_R and will be used to find solvability conditions.

In the case $T = 0$, we have

$$\omega_n = \sqrt{gk_n \tanh(k_n h)} \approx \sqrt{gk_n}\,.$$

With the substitutions $b_i^* = k_n b_i$ and $\theta^* = k_n\theta$ and using Taylor expansions, the equations in (4.5) become

$$\begin{aligned}
\phi_{x^*x^*} + \phi_{z^*z^*} &= 0\,, \\
(\phi_{x^*} + \phi_{x^*x^*}b_1^* + \tfrac{1}{2}\phi_{x^*x^*x^*}(b_1^*)^2 + HO) & \\
-\tfrac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}b_{1,t^*}^* - b_{1,z^*}^*(\phi_{z^*} + \phi_{z^*x^*}b_1^* + HO) &= 0 \text{ at } x^* = 0\,, \\
(\phi_{x^*} + \phi_{x^*x^*}b_2^* + \tfrac{1}{2}\phi_{x^*x^*x^*}(b_2^*)^2 + HO) & \\
-\tfrac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}b_{2,t^*}^* - b_{2,z^*}^*(\phi_{z^*} + \phi_{z^*x^*}b_2^* + HO) &= 0 \text{ at } x^* = k_n a\,, \\
\phi_{z^*} &= 0 \text{ at } z^* = -k_n h\,, \\
\tfrac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\theta_{t^*}^* + \theta_{x^*}^*(\phi_{x^*} + \phi_{x^*z^*}\theta^* + HO) & \\
-(\phi_{z^*} + \phi_{z^*z^*}\theta^* + \tfrac{1}{2}\phi_{z^*z^*z^*}(\theta^*)^2 + HO) &= 0 \text{ at } z^* = 0\,, \\
\tfrac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}(\phi_{t^*} + \phi_{t^*z^*}\theta^* + \tfrac{1}{2}\phi_{t^*z^*z^*}(\theta^*)^2 + HO) + \tfrac{g}{k_n^3}\theta^* & \\
+\tfrac{1}{2}(\phi_{x^*} + \phi_{x^*z^*}\theta^* + HO)^2 + \tfrac{1}{2}(\phi_{z^*} + \phi_{z^*z^*}\theta^* + HO)^2 &= 0 \text{ at } z^* = 0\,, \\
\theta_{x^*}^* &= 0 \text{ at } x^* = 0, k_n a\,.
\end{aligned} \tag{4.6}$$

Here, "HO" stands for higher-order terms. In order to have converging Taylor series, we require $O(b_1^*) < 1$ and $O(\theta^*) < 1$. The first condition means $\alpha_4 > \alpha_1$, the second condition is taken care of when we find the first-order approximation. The equations for θ_1^* and ϕ_1 then are

$$\begin{aligned}
\phi_{1,x^*x^*} + \phi_{1,z^*z^*} &= 0\,, \\
\phi_{1,x^*} &= 0 \text{ at } x^* = 0\,, \\
\phi_{1,x^*} &= 0 \text{ at } x^* = k_n a\,, \\
\phi_{1,z^*} &= 0 \text{ at } z^* = -k_n h\,, \\
\frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\theta^*_{1,t^*} - \phi_{1,z^*} &= 0 \text{ at } z^* = 0\,, \\
\frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\phi_{1,t^*} + \frac{g}{k_n^3}\theta^*_1 &= 0 \text{ at } z^* = 0\,, \\
\theta^*_{1,x^*} &= 0 \text{ at } x^* = 0, k_n a\,.
\end{aligned} \tag{4.7}$$

The equations for ϕ_2 and θ_2^* are

$$\begin{aligned}
\phi_{2,x^*x^*} + \phi_{2,z^*z^*} &= 0\,, && \\
\phi_{2,x^*} &= \frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}} b^*_{1,t^*} && \text{at } x^* = 0\,, \\
\phi_{2,x^*} &= \frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}} b^*_{2,t^*} && \text{at } x^* = k_n a\,, \\
\phi_{2,z^*} &= 0 && \text{at } z^* = -k_n h\,, \\
\frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\theta^*_{2,t^*} - \phi_{2,z^*} &= -\theta^*_{1,x^*}\phi^*_{1,x^*} + \phi^*_{1,z^*z^*}\theta^*_1 && \text{at } z^* = 0\,, \\
\frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\phi_{2,t^*} + \frac{g}{k_n^3}\theta^*_2 &= -\frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\phi_{1,t^*z^*}\theta^*_1 - \frac{1}{2}\phi^2_{1,x^*} - \frac{1}{2}\phi^2_{1,z^*} && \text{at } z^* = 0\,, \\
\theta^*_{2,x^*} &= 0 && \text{at } x^* = 0, k_n a\,.
\end{aligned} \tag{4.8}$$

The equations for the remaining terms ϕ_R and θ_R^* finally are

$$
\begin{aligned}
\phi_{R,x^*x^*} + \phi_{R,z^*z^*} &= 0\,, \\
\phi_{R,x^*} &= -\phi_{1,x^*x^*} b_1^* - \tfrac{1}{2}\phi_{1,x^*x^*x^*}(b_1^*)^2 \\
&\quad + b_{1,z^*}^* \phi_{1,z^*} + b_{1,z^*}^* \phi_{1,z^*x^*} b_1^*\,, \\
\phi_{R,x^*} &= -\phi_{1,x^*x^*} b_2^* - \tfrac{1}{2}\phi_{1,x^*x^*x^*}(b_2^*)^2 \\
&\quad + b_{2,z^*}^* \phi_{1,z^*} + b_{2,z^*}^* \phi_{1,z^*x^*} b_2^*\,, \\
\phi_{R,z^*} &= 0\,, \\
\frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\theta_{R,t^*}^* - \phi_{R,z^*} &= -\theta_{1,x^*}^* \phi_{2,x^*} - \theta_{2,x^*}^* \phi_{1,x^*} - \theta_{1,x^*}^* \phi_{1,x^*z^*} \theta_1^* \\
&\quad + \phi_{1,z^*z^*}\theta_2^* + \phi_{2,z^*z^*}\theta_1^* + \tfrac{1}{2}\phi_{1,z^*z^*z^*}(\theta_1^*)^2 - \frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\epsilon^{\alpha_5}\theta_{1,\tau}^*\,, \\
\frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\phi_{R,t^*} + \frac{g}{k_n^3}\theta_R^* &= -\frac{g^{\frac{1}{2}}}{2k_n^{\frac{3}{2}}}\left(\phi_{1,t^*z^*}\theta_2^* - \phi_{2,t^*z^*}\theta_1^* - \phi_{x^*z^*z^*}(\theta^*)^2\right) \\
&\quad - \phi_{1,x^*}\phi_{2,x^*} - \phi_{1,x^*}\phi_{1,x^*z^*}\theta_1^* \\
&\quad - \phi_{1,z^*}\phi_{2,z^*} - \phi_{1,z^*}\phi_{1,z^*z^*}\theta_1^* - \frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\epsilon^{\alpha_5}\phi_{1,\tau}^*\,, \\
\theta_{R,x^*}^* &= 0\,.
\end{aligned}
\tag{4.9}
$$

Terms that do not have frequency ω_m or ω_n are not of interest for the solvability conditions and thus are not listed.

4.2.2 First- and second-order approximation

For the first-order approximation, there are no major changes compared to the situation in Chapter 3. The first-order approximation can be written as

$$
\begin{aligned}
\Theta_1^* &= \epsilon^{\alpha_2}\Theta_{1,m}^* + \epsilon^{\alpha_3}\Theta_{1,n}^* \\
\text{and} \qquad \phi_1 &= \epsilon^{\alpha_2}\phi_{1,m} + \epsilon^{\alpha_3}\phi_{1,n}\,, \\
\text{where} \quad \Theta_{1,m}^* &= e^{i\frac{\omega_m}{\omega_n}t^*}\epsilon^{-\alpha_1}\frac{2k_m}{a}p_m(\tau)\cos(\epsilon^{\alpha_1}x^*)\,, \\
\phi_{1,m} &= e^{i\frac{\omega_m}{\omega_n}t^*}\frac{2}{a}\frac{i\omega_m p_m(\tau)}{k_m\sinh(\epsilon^{\alpha_1}h^*)}\cosh(\epsilon^{\alpha_1}(z^*+h^*))\cos(\epsilon^{\alpha_1}x^*)\,, \\
\Theta_{1,n}^* &= e^{it^*}\epsilon^{-\alpha_1}\frac{2k_m}{a}p_n(\tau)\cos(x^*)\,, \\
\phi_{1,n} &= e^{it^*}\epsilon^{\frac{1}{2}\alpha_1}\frac{2}{k_m a}\frac{i(gk_m)^{\frac{1}{2}}p_n(\tau)}{\sinh(h^*)}\cosh(z^*+h^*)\cos(x^*)\,.
\end{aligned}
\tag{4.10}
$$

When recalling the condition $O(\theta^*) < 1$ which we needed to have converging Taylor series, we see that this means $min(\alpha_2, \alpha_3) > \alpha_1$. To obtain a second order approximation for forcing that is constant with respect to z, we proceed in a similar way as in Section 3.2.2, part b). Set $\phi_2 = \psi_2 + \Psi_2$ and find a solution to

$$
\begin{aligned}
\Psi_{2,x^*x^*} + \Psi_{2,z^*z^*} &= 0\,, \\
\Psi_{2,x^*} &= e^{i(1+\frac{\omega_m}{\omega_n})t^*+i\lambda\tau}\epsilon^{\alpha_4+\frac{1}{2}\alpha_1}\frac{g^{\frac{1}{2}}}{k_m^{\frac{1}{2}}}f_1(z) \quad \text{at } x=0\,, \\
\Psi_{2,x^*} &= e^{i(1+\frac{\omega_m}{\omega_n})t^*+i\lambda\tau}\epsilon^{\alpha_4+\frac{1}{2}\alpha_1}\frac{g^{\frac{1}{2}}}{k_m^{\frac{1}{2}}}f_2(z) \quad \text{at } x=k_n a\,, \\
\Psi_{2,z^*} &= 0 \quad \text{at } z=-k_n h\,.
\end{aligned} \tag{4.11}
$$

With $f_1 = a_1$ and $f_2 = a_2$, Ψ_2 can be chosen to be

$$
\begin{aligned}
\Psi_2 = e^{i(1+\frac{\omega_m}{\omega_n})t^*+i\lambda\tau} \\
\times\epsilon^{\alpha_4+\frac{1}{2}\alpha_1}\frac{g^{\frac{1}{2}}}{k_m^{\frac{1}{2}}}\left(\epsilon^{\alpha_1}\tfrac{a_2-a_1}{2k_m a}\left((x^*)^2-(z^*)^2\right)+a_1x^*-\tfrac{h(a_2-a_1)}{a}z^*\right)+c.c.
\end{aligned} \tag{4.12}
$$

Hence, the equations for ψ_2 and θ_2^* are

$$
\begin{aligned}
\psi_{2,x^*x^*}+\psi_{2,z^*z^*} &= 0\,, \\
\psi_{2,x^*} &= 0 \quad \text{at } x^*=0\,, \\
\psi_{2,x^*} &= 0 \quad \text{at } x^*=k_n a\,, \\
\psi_{2,z^*} &= 0 \quad \text{at } z^*=-k_n h\,, \\
\frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\theta^*_{2,t^*}-\psi_{2,z^*} &= -\theta^*_{1,x^*}\phi^*_{1,x^*}+\phi^*_{1,z^*z^*}\theta^*_1+\Psi_{2,z^*} \quad \text{at } z^*=0\,, \\
\frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\psi_{2,t^*}+\frac{g}{k_n^3}\theta^*_2 &= -\frac{g^{\frac{1}{2}}}{k_n^{\frac{3}{2}}}\phi_{1,t^*z^*}\theta^*_1-\tfrac{1}{2}\phi^2_{1,x^*}-\tfrac{1}{2}\phi^2_{1,z^*}-\Psi_{2,t^*} \quad \text{at } z^*=0\,, \\
\theta^*_{2,x^*} &= 0 \quad \text{at } x^*=0,k_n a\,.
\end{aligned} \tag{4.13}
$$

The derivatives of Ψ_2 are

$$
\begin{aligned}
\Psi_{2,z^*}|_{z^*=0} &= e^{i(1+\frac{\omega_m}{\omega_n})t^*+i\lambda\tau}\epsilon^{\alpha_4+\frac{1}{2}\alpha_1}\frac{g^{\frac{1}{2}}}{k_m^{\frac{1}{2}}}\left(-\tfrac{h(a_2-a_1)}{a}\right)+c.c.\,, \\
\Psi_{2,t^*}|_{z^*=0} &\approx e^{i(1+\frac{\omega_m}{\omega_n})t^*+i\lambda\tau}\epsilon^{\alpha_4+\frac{1}{2}\alpha_1}\frac{ig^{\frac{1}{2}}}{k_m^{\frac{1}{2}}}\left(\epsilon^{\alpha_1}\tfrac{a_2-a_1}{2k_m a}(x^*)^2+a_1x^*\right)+c.c.
\end{aligned}
$$

The second of these expressions is rewritten using cosine expansions

$$
\epsilon^{\alpha_1}\frac{a_2-a_1}{2k_m a}(x^*)^2+a_1x^* \approx \frac{1}{3}\epsilon^{-\alpha_1}(2a_1+a_2)k_m-\epsilon^{-\alpha_1}\sum_{j=1}^{\infty}\frac{2a(a_1-(-1)^j a_2)k_m}{j^2\pi^2}\cos(\epsilon^{\alpha_1}\frac{j}{m}X)\,.
$$

The right sides of the second-order equations in (4.13) now are r.h.s. of 5^{th} equation

$$
\begin{aligned}
&= e^{2i\frac{\omega_m}{\omega_n}t^*}\epsilon^{2\alpha_2+\alpha_1}\tfrac{4i}{a^2}\cos(2\epsilon^{\alpha_1}x^*)\coth(\epsilon^{\alpha_1}h^*)\omega_m p_m^2\\
&\quad +e^{2it^*}\epsilon^{2\alpha_3-\frac{1}{2}\alpha_1}\tfrac{4i}{a^2}\cos(2x^*)\coth(h^*)(gk_m)^{\frac{1}{2}}p_n^2\\
&\quad +e^{i(1+\frac{\omega_m}{\omega_n})t^*}\epsilon^{\alpha_2+\alpha_3-\frac{1}{2}\alpha_1}\tfrac{4i}{a^2}\cos(\epsilon^{\alpha_1}x^*)\cos(x^*)\coth(h^*)(gk_m)^{\frac{1}{2}}p_m p_n\\
&\quad -e^{i(\frac{\omega_m}{\omega_n}-1)t^*}\epsilon^{\alpha_2+\alpha_3-\frac{1}{2}\alpha_1}\tfrac{4i}{a^2}\cos(\epsilon^{\alpha_1}x^*)\cos(x^*)\coth(h^*)(gk_m)^{\frac{1}{2}}p_m\overline{p_n}\,,
\end{aligned}
$$

r.h.s. of 6^{th} equation

$$
\begin{aligned}
&= \epsilon^{2\alpha_2+2\alpha_1}\tfrac{2\omega_m^2}{a^2k_m^2}\Big(\cos^2(\epsilon^{\alpha_1}x^*)-\coth(\epsilon^{\alpha_1}h^*)\sin^2(\epsilon^{\alpha_1}x^*)\Big)\overline{p_m}p_m\\
&\quad +\epsilon^{2\alpha_3+\alpha_1}\tfrac{2g}{a^2k_m}\Big(\cos^2(x^*)-\coth^2(h^*)\sin^2(h^*)\Big)\overline{p_n}p_n\\
&\quad +e^{2i\frac{\omega_m}{\omega_n}t^*}\epsilon^{2\alpha_2+2\alpha_1}\tfrac{2\omega_m^2}{a^2k_m^2}\Big(3\cos^2(\epsilon^{\alpha_1}x^*)+\coth^2(\epsilon^{\alpha_1}h^*)\sin^2(\epsilon^{\alpha_1}x^*)\Big)p_m^2\\
&\quad +e^{2it^*}\epsilon^{2\alpha_3+\alpha_1}\tfrac{2g}{a^2k_m}\Big(3\cos^2(x^*)+\coth^2(h^*)\sin^2(x^*)\Big)p_n^2\\
&\quad +e^{i(\frac{\omega_m}{\omega_n}+1)t^*}\epsilon^{\alpha_2+\alpha_3+\frac{1}{2}\alpha_1}\tfrac{4\omega_m^2}{a^2(gk_m)^{\frac{1}{2}}}\cos(\epsilon^{\alpha_1}x^*)\cos(x^*)p_m p_n\\
&\quad +e^{i(\frac{\omega_m}{\omega_n}-1)t^*}\epsilon^{\alpha_2+\alpha_3+\frac{1}{2}\alpha_1}\tfrac{4\omega_m^2}{a^2(gk_m)^{\frac{1}{2}}}\cos(\epsilon^{\alpha_1}x^*)\cos(x^*)p_m\overline{p_n}\,.
\end{aligned}
$$

In a similar fashion as in (3.22), the form of those right sides leads to the assumption that

$$
\begin{aligned}
\Theta_2(x,t,\tau) &= c_1+c_2e^{2i\frac{\omega_m}{\omega_n}t^*}+c_3e^{2it^*}\\
&\quad +(c_{1,2m}+c_{2,2m}e^{2i\frac{\omega_m}{\omega_n}t^*})\cos(2\epsilon^{\alpha_1}x^*)\\
&\quad +(c_{1,2n}+c_{2,2n}e^{2it^*})\cos(2x^*)\\
&\quad +c_{-,-}e^{i(\frac{\omega_m}{\omega_n}-1)t^*}\cos((1-\epsilon^{\alpha_1})x^*)\\
&\quad +c_{+,-}e^{i(\frac{\omega_m}{\omega_n}+1)t^*}\cos((1-\epsilon^{\alpha_1})x^*)\\
&\quad +c_{-,+}e^{i(\frac{\omega_m}{\omega_n}-1)t^*}\cos((1+\epsilon^{\alpha_1})x^*)\\
&\quad +c_{+,+}e^{i(\frac{\omega_m}{\omega_n}+1)t^*}\cos((1+\epsilon^{\alpha_1})x^*)\\
&\quad +e^{i(\frac{\omega_m}{\omega_n}+1)+i\lambda\tau}\textstyle\sum_{j=0}^{\infty}\beta_{1,j}\cos(\frac{k_j}{k_n}x^*)+c.c.\,,\\
\phi_2(x,z,t,\tau) &= d_2e^{2i\frac{\omega_m}{\omega_n}t^*}+d_3e^{2it^*}\\
&\quad +(d_{1,2m}+d_{2,2m}e^{2i\frac{\omega_m}{\omega_n}t^*})\cos(2\epsilon^{\alpha_1}x^*)\cosh(2\epsilon^{\alpha_1}(z^*+h^*))\\
&\quad +(d_{1,2n}+d_{2,2n}e^{2it^*})\cos(2x^*)\cosh(2(z^*+h^*))\\
&\quad +d_{-,-}e^{i(\frac{\omega_m}{\omega_n}-1)t^*}\cos((1-\epsilon^{\alpha_1})x^*)\cosh((1-\epsilon^{\alpha_1})(z^*+h^*))\\
&\quad +d_{+,-}e^{i(\frac{\omega_m}{\omega_n}+1)t^*}\cos((1-\epsilon^{\alpha_1})x^*)\cosh((1-\epsilon^{\alpha_1})(z^*+h^*))\\
&\quad +d_{-,+}e^{i(\frac{\omega_m}{\omega_n}-1)t^*}\cos((1+\epsilon^{\alpha_1})x^*)\cosh((1+\epsilon^{\alpha_1})(z^*+h^*))\\
&\quad +d_{+,+}e^{i(\frac{\omega_m}{\omega_n}+1)t^*}\cos((1+\epsilon^{\alpha_1})x^*)\cosh((1+\epsilon^{\alpha_1})(z^*+h^*))\\
&\quad +e^{i(\frac{\omega_m}{\omega_n}+1)t^*+i\lambda\tau}\textstyle\sum_{j=0}^{\infty}\beta_{2,j}\cos(\frac{k_j}{k_n}x^*)\cosh(\frac{k_j}{k_n}(z^*+h^*))+\Psi_2+c.c.
\end{aligned}
\tag{4.14}
$$

The equations for those coefficients are

$$\begin{aligned}
g\epsilon^{3\alpha_1}k_m^{-3}c_{1a} &= \gamma_{1a}\,,\\
g\epsilon^{3\alpha_1}k_m^{-3}c_{1b} &= \gamma_{1b}\,,\\
2i\epsilon^{2\alpha_1}\tfrac{\omega_m}{k_m^2}c_2 &= 0\,,\\
2i\epsilon^{2\alpha_1}\tfrac{\omega_m}{k_m^2}d_2+\epsilon^{3\alpha_1}\tfrac{g}{k_m^3}c_2 &= \epsilon^{2\alpha_2+2\alpha_1}\tfrac{\omega_m^2}{a^2k_m^2}p_m(\tau)^2(3+\coth^2(\epsilon^{\alpha_1}h^*))\,,\\
2i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}c_3 &= 0\,,\\
2i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}d_3+\epsilon^{3\alpha_1}\tfrac{g}{k_m^3}c_3 &= \epsilon^{2\alpha_3+\alpha_1}\tfrac{4g}{a^2k_m}p_n(\tau)^2\,,\\
-2\epsilon^{\alpha_1}\sinh(2\epsilon^{\alpha_1}h^*)d_{1,2m} &= 0\,,\\
\epsilon^{3\alpha_1}\tfrac{g}{k_m^3}c_{1,2m} &= \epsilon^{2\alpha_2+2\alpha_1}\tfrac{\omega_m^2}{a^2k_m^2}|p_m(\tau)|^2(1+\coth^2(\epsilon^{\alpha_1}h^*))\,,\\
2i\epsilon^{2\alpha_1}\tfrac{\omega_m}{k_m^2}c_{2,2m}-2\epsilon^{\alpha_1}\sinh(2\epsilon^{\alpha_1}h)d_{2,2m} &= \epsilon^{2\alpha_2+\alpha_1}\tfrac{4i\omega_m}{a^2}\coth(\epsilon^{\alpha_1}h^*)\,,\\
2i\epsilon^{2\alpha_1}\tfrac{\omega_m}{k_m^2}\cosh(\epsilon^{\alpha_1}h^*)d_{2,2m}+\epsilon^{3\alpha_1}\tfrac{g}{k_m^3}c_{2,2m} &= \epsilon^{2\alpha_2+2\alpha_1}\tfrac{\omega_m^2}{a^2k_m^2}(3-\coth^2(\epsilon^{\alpha_1}h^*))p_m^2(\tau)\,,\\
-2\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\sinh(2h^*)d_{1,2n} &= 0\,,\\
\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\cosh(2h^*)d_{1,2n}+\epsilon^{3\alpha_1}\tfrac{g}{k_m^3}c_{1,2n} &= \epsilon^{2\alpha_3+\alpha_1}\tfrac{2g}{a^2k_m}|p_n(\tau)|^2\,,\\
2i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}c_{2,2n}-2\epsilon^{\alpha_1}\sinh(2h^*)d_{2,2n} &= \epsilon^{2\alpha_3-\frac{1}{2}\alpha_1}\tfrac{4i}{a^2}(gk_m)^{\frac{1}{2}}p_n^2(\tau)\,,\\
2i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\cosh(2h^*)d_{2,2n}+\epsilon^{3\alpha_1}\tfrac{g}{k_m^3}c_{2,2n} &= \epsilon^{2\alpha_3+\alpha_1}\tfrac{2g}{a^2k_m}p_n^2(\tau)\,,\\
-i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}c_{-,-}-\sinh(h^*)d_{-,-} &= \gamma_2\,,\\
-i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\cosh(h^*)d_{-,-}+\epsilon^{3\alpha_1}\tfrac{g}{k_m^3}c_{-,-} &= \gamma_3\,,\\
-i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}c_{-,+}-\sinh(h^*)d_{-,+} &= \gamma_4\,,\\
-i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\cosh(h^*)d_{-,+}+\epsilon^{3\alpha_1}\tfrac{g}{k_m^3}c_{-,+} &= \gamma_5\,,\\
i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}c_{+,-}-\sinh(h^*)d_{+,-} &= \gamma_6\,,\\
i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\cosh(h^*)d_{+,-}+\epsilon^{3\alpha_1}\tfrac{g}{k_m^3}c_{+,-} &= \gamma_7\,,\\
i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}c_{+,+}-\sinh(h^*)d_{+,+} &= \gamma_8\,,\\
i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\cosh(h^*)d_{+,+}+\epsilon^{3\alpha_1}\tfrac{g}{k_m^3}c_{+,+} &= \gamma_9\,,\\
i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\beta_{1,0} &= \epsilon^{\alpha_4+\frac{1}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{1}{2}}}\left(-\tfrac{h(a_2-a_1)}{a}\right)\,,\\
i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\beta_{2,0}+\epsilon^{3\alpha_1}\tfrac{g}{k_m^3}\beta_{1,0} &= \epsilon^{\alpha_4-\frac{1}{2}\alpha_1}\tfrac{i\,g^{\frac{1}{2}}}{k_m^{\frac{1}{2}}}\tfrac{(2a_1+a_2)k_m}{3}\,,\\
i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\beta_{1,j}-\epsilon^{\alpha_1}\tfrac{k_j}{k_m}\beta_{2,j} &= 0\,,\\
i\epsilon^{\frac{3}{2}\alpha_1}\tfrac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\beta_{2,j}+\epsilon^{3\alpha_1}\tfrac{g}{k_m^3}\beta_{1,j} &= -\epsilon^{\alpha_4-\frac{1}{2}\alpha_1}\tfrac{i\,g^{\frac{1}{2}}}{k_m^{\frac{1}{2}}}\tfrac{2a(a_1-(-1)^ja_2)k_m}{j^2\pi^2}\,.
\end{aligned}\tag{4.15}$$

The constants are given by

$$
\begin{aligned}
\gamma_{1a} &= \epsilon^{2\alpha_1+2\alpha_2}\frac{(1-\coth^2(\epsilon^{\alpha_1}h^*))\omega_m^2}{a^2k_m^2}p_m\overline{p_m}\,, \\
\gamma_{1b} &\approx 0\,, \\
\gamma_2 = \gamma_4 &= -\epsilon^{\alpha_2+\alpha_3-\frac{1}{2}\alpha_1}\frac{2i(gk_m)^{\frac{1}{2}}}{a^2}p_m\overline{p_n}\,, \\
\gamma_3 = \gamma_5 &= \epsilon^{\alpha_2+\alpha_3+\frac{1}{2}\alpha_1}\frac{2\omega_m^2}{a^2(gk_m)^{\frac{1}{2}}}p_m\overline{p_n}\,, \\
\gamma_6 = \gamma_8 &= \epsilon^{\alpha_2+\alpha_3-\frac{1}{2}\alpha_1}\frac{2i(gk_m)^{\frac{1}{2}}}{a^2}p_mp_n\,, \\
\gamma_7 = \gamma_9 &= \epsilon^{\alpha_2+\alpha_3+\frac{1}{2}\alpha_1}\frac{2\omega_m^2}{a^2(gk_m)^{\frac{1}{2}}}p_mp_n\,.
\end{aligned}
\tag{4.16}
$$

Using $\coth(h^*) \approx 1$ yields

$$
\begin{aligned}
c_{1a} &= -\epsilon^{2\alpha_2-\alpha_1}\frac{\omega_m^2k_m}{a^2g}|p_m|^2\,, \\
c_{1b} &= 0\,, \\
c_2 &= 0\,, \\
d_2 &= -\epsilon^{2\alpha_2}\frac{i\omega_m}{2a^2}(3+\coth^2(\epsilon^{\alpha_1}h^*))p_m^2\,, \\
c_3 &= 0\,, \\
d_3 &= -\epsilon^{2\alpha_3-\frac{1}{2}\alpha_1}\frac{2ig^{\frac{1}{2}}k_m^{\frac{1}{2}}}{a^2}p_n^2\,, \\
c_{1,2m} &= \epsilon^{2\alpha_2-\alpha_1}\frac{\omega_m^2k_m}{a^2g}(1+\coth^2(\epsilon^{\alpha_1}h^*))|p_m|^2\,, \\
d_{1,2m} &= 0\,, \\
c_{2,2m} &= \epsilon^{2\alpha_2-\alpha_1}\frac{k_m^2}{a^2}\frac{2+\cosh(2\epsilon^{\alpha_1}h^*)}{\sinh^2(\epsilon^{\alpha_1}h^*)}\coth(\epsilon^{\alpha_1}h^*)p_m^2\,, \\
d_{2,2m} &= \epsilon^{2\alpha_2}\frac{3i\omega_m^{\frac{1}{2}}}{2a^2}\operatorname{csch}(\epsilon^{\alpha_1}h^*)p_m^2\,, \\
c_{1,2n} &= \epsilon^{2\alpha_3-2\alpha_1}\frac{2k_m^2}{a^2}|p_n|^2\,, \\
d_{1,2n} &= 0\,, \\
c_{2,2n} &= \epsilon^{2\alpha_3-2\alpha_1}\frac{2k_m^2}{a^2}p_n^2\,, \\
d_{2,2n} &= 0\,, \\
c_{-,\pm} &= \epsilon^{\alpha_2+\alpha_3-3\alpha_1}\frac{\omega_mk_m^3}{a^2g}p_m\overline{p_n}\,, \\
d_{-,\pm} &= -\epsilon^{\alpha_2+\alpha_3-\frac{3}{2}\alpha_1}\frac{\omega_mk_m^{\frac{3}{2}}}{i\,a^2g^{\frac{1}{2}}}\operatorname{sech}((\epsilon^{\alpha_1}\pm 1)h^*)p_m\overline{p_n}\,, \\
c_{+,\pm} &= -\epsilon^{\alpha_2+\alpha_3-3\alpha_1}\frac{\omega_mk_m^3}{a^2g}p_mp_n\,, \\
d_{+,\pm} &= -\epsilon^{\alpha_2+\alpha_3-\frac{3}{2}\alpha_1}\frac{\omega_mk_m^{\frac{3}{2}}}{i\,a^2g^{\frac{1}{2}}}\operatorname{sech}((\epsilon^{\alpha_1}\pm 1)h^*)p_mp_n\,, \\
\beta_{1,0} &= \epsilon^{\alpha_4-\alpha_1}\frac{i(a_2-a_1)k_mh}{a}\,, \\
\beta_{2,0} &= \epsilon^{\alpha_4-2\alpha_1}\frac{1}{3}(2a_1+a_2)k_m^2\,, \\
\beta_{1,j} &= \epsilon^{\alpha_4-\frac{5}{2}\alpha_1}\frac{2i\,a(a_1-(-1)^ja_2)k_m^{\frac{7}{2}}}{g^{\frac{1}{2}}j\pi(ak_m-\epsilon^{\alpha_1}j\pi)}\,, \\
\beta_{2,j} &= \epsilon^{\alpha_4-2\alpha_1}\frac{2a^2(a_1-(-1)^ja_2)k_m^3}{j^2\pi^2(\epsilon^{\alpha_1}j\pi-ak_m)}\,.
\end{aligned}
\tag{4.17}
$$

4.2.3 Solvability conditions from third-order equations

The solvability condition for frequency ω_m is now

$$\begin{aligned}
&\int_{-h^*}^{0}\Big(\phi_{R,x^*}\cos(\epsilon^{\alpha_1}x^*)|_0^{a^*} - \epsilon^{2\alpha_1}\int_0^{a^*}\phi_R\cos(\epsilon^{\alpha_1}x^*)dx^*\Big)\cosh(\epsilon^{\alpha_1}(z^*+h^*))dz^* \\
&\quad + \int_0^{a^*}\Big(\phi_{R,z^*}\cosh(\epsilon^{\alpha_1}(z^*+h^*))|_{-h^*}^{0} - \epsilon^{\alpha_1}\phi_R\sinh(\epsilon^{\alpha_1}(z^*+h^*))|_{-h^*}^{0} \\
&\quad\quad + \epsilon^{2\alpha_1}\int_{-h^*}^{0}\phi_R\cosh(\epsilon^{\alpha_1}(z^*+h^*))dz^*\Big)\cos(\epsilon^{\alpha_1}x^*)dx^* \qquad = 0\,.
\end{aligned}$$

Making use of (4.9), we obtain after rearranging

$$\begin{aligned}
&\int_{-h}^{0}\Big((-1)^m\left[b^*_{1,z^*}\phi_{1,z^*} + b_{1,z^*}\phi_{1,z^*x^*}b_1^* - \phi_{1,x^*x^*}b_1^* - \tfrac{1}{2}\phi_{x^*x^*x^*}(b_1^*)^2\right]_{x^*=a^*} \\
&\quad\quad - \left[b^*_{2,z^*}\phi_{1,z^*} + b_{2,z^*}\phi_{1,z^*x^*}b_2^* - \phi_{1,x^*x^*}b_2^* - \tfrac{1}{2}\phi_{x^*x^*x^*}(b_2^*)^2\right]_{x^*=0}\Big)\cosh(\epsilon^{\alpha_1}(z^*+h^*))dz^* \\
&+ \int_0^{a^*}\Big(\epsilon^{2\alpha_1}i\tfrac{\omega_m}{k_m^2}\cosh(\epsilon^{\alpha_1}h^*) - \epsilon^{2\alpha_1}i\tfrac{g}{k_m\omega_m}\sinh(\epsilon^{\alpha_1}h^*)\Big)\Theta_R\cos(\epsilon^{\alpha_1}x^*)dx^* \\
&- \int_0^{a^*}\Big([\text{r.s. }(4.9_5)]\cosh(\epsilon^{\alpha_1}h^*) + \epsilon^{-\alpha_1}\tfrac{k_m^2}{i\omega_m}[\text{r.s. }(4.9_6)]\sinh(\epsilon^{\alpha_1}h^*)\Big)\cos(\epsilon^{\alpha_1}x^*)dx^* \quad = 0\,.
\end{aligned}$$

Recalling the definition of ω_m , the middle line vanishes. Besides, only terms of approximate frequency ω_m should be considered.

We again get a differential equation of the form

$$R_1 p_m' + R_2|p_m|^2 p_m + R_3|p_n|^2 p_m + R_4 e^{i\lambda\tau}\overline{p_n} = 0 \tag{4.18}$$

with

$$\begin{aligned}
R_1 &= \epsilon^{\alpha_2+\alpha_5-\frac{1}{2}\alpha_1}2\cosh(k_m h)g^{\frac{1}{2}}k_m^{\frac{1}{2}}\,,\\
R_2 &= \epsilon^{3\alpha_2}i\,C\,,\\
R_3 &= \epsilon^{\alpha_2+2\alpha_3-\frac{5}{2}\alpha_1}(-i)\frac{k_m^{\frac{7}{2}}\omega_m}{g^{\frac{1}{2}}a^2}\sinh(k_m h)\,,\\
R_4 &= \epsilon^{\alpha_3+\alpha_4-\frac{1}{2}\alpha_1}\left[-\frac{(gk_m)^{\frac{1}{2}}}{a^2}\cosh(k_m h)\left((2a_1-a_2)a + (-1)^{m+n}(a\,a_1 + 2(a_1-a_2)m\pi)\right)\right].
\end{aligned} \tag{4.19}$$

The constant C is given by

$$\begin{aligned}
C = -\frac{k_m\,\mathrm{csch}(k_m h)\omega_m}{64a^3 g}\Big(&gm\pi[25\sinh(2k_m h) + 10\sinh(4k_m h) + \sinh(6k_m h)] \\
&+3a\omega_m^2\left[8 + 16g^{\frac{3}{2}} + (32g^{\frac{3}{2}} - 7)\cosh(2k_m h) + 16g^{\frac{3}{2}}\cosh(4k_m h) - \cosh(6k_m h)\right]\Big)\,.
\end{aligned}$$

We note that, for reasonably small h, we have $C < 0$. However, as h increases and other parameters remain unchanged, the terms with hyperbolic functions in $6k_m h$ become dominant. With the simplification $\tanh(k_m h) = 1$ which is valid for large h/a, we have

$$3a\omega_m^2 = 3gm\pi > gm\pi\,.$$

Thus we can see that, as h increases, C eventually becomes positive.

The solvability condition for frequency ω_n is

$$\begin{aligned}&\textstyle\int_{-h^*}^0 \left(\phi_{R,x^*}\cos(x^*)|_0^{a^*} - \int_0^{a^*}\phi_R\cos(x^*)dx^*\right)\cosh((z^*+h^*))dz^*\\ &\quad + \textstyle\int_0^{a^*}\left(\phi_{R,z^*}\cosh((z^*+h^*))|_{-h^*}^0 - \phi_R\sinh((z^*+h^*))|_{-h^*}^0\right.\\ &\quad \left.+ \textstyle\int_{-h^*}^0\phi_R\cosh((z^*+h^*))dz^*\right)\cos(x^*)dx^* \qquad = 0\,.\end{aligned}$$

Using (4.9) and rearranging yields

$$\begin{aligned}&\textstyle\int_{-h}^0\Big((-1)^n\left[b^*_{1,z^*}\phi_{1,z^*} + b_{1,z^*}\phi_{1,z^*x^*}b_1^* - \phi_{1,x^*x^*}b_1^* - \frac{1}{2}\phi_{x^*x^*x^*}(b_1^*)^2\right]_{x^*=a^*}\\ &\quad - \left[b^*_{2,z^*}\phi_{1,z^*} + b_{2,z^*}\phi_{1,z^*x^*}b_2^* - \phi_{1,x^*x^*}b_2^* - \frac{1}{2}\phi_{x^*x^*x^*}(b_2^*)^2\right]_{x^*=0}\Big)\cosh(z^*+h^*)dz^*\\ &+ \textstyle\int_0^{a^*}\left(i\epsilon^{\frac{3}{2}\alpha_1}\frac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\cosh(h^*) - i\epsilon^{\frac{3}{2}\alpha_1}\frac{g^{\frac{1}{2}}}{k_m^{\frac{3}{2}}}\sinh(h^*)\right)\Theta_R\cos(x^*)dx^*\\ &- \textstyle\int_0^{a^*}\left([\text{r.s. }(4.9_5)]\cosh(h^*) + \epsilon^{-\frac{3}{2}\alpha_1}\frac{k_m^{\frac{3}{2}}}{ig^{\frac{1}{2}}}[\text{r.s. }(4.9_6)]\sinh(h^*)\right)\cos(x^*)dx^* \quad = 0\,.\end{aligned}$$

Here, only terms of approximate frequency ω_n are considered. Using $\sinh(h^*) = \cosh(h^*)$, the middle line vanishes. The resulting differential equation is

$$S_1 p_n' + S_2|p_n|^2 p_n + S_3|p_m|^2 p_n + S_4 e^{i\lambda\tau}\overline{p_m} = 0 \tag{4.20}$$

with

$$\begin{aligned} S_1 &= \epsilon^{\alpha_3+\alpha_5-\frac{1}{2}\alpha_1}2\cosh(h^*)g^{\frac{1}{2}}k_m^{\frac{1}{2}}\,,\\ S_2 &= \epsilon^{3\alpha_3-\frac{5}{2}\alpha_1}i\cosh(h^*)\frac{k_m^{\frac{5}{2}}}{g^{\frac{1}{2}}a^2}\,,\\ S_3 &= \epsilon^{2\alpha_2+\alpha_3-3\alpha_1}i\cosh(h^*)\frac{k_m^3}{a^2 g}\,,\\ S_4 &= \epsilon^{\alpha_2+\alpha_4+\alpha_1}\cosh(h^*)\frac{\coth(k_m h)\omega_m}{a^2}\\ &\qquad \times[([(-1)^{m+n}-2]a_1+a_2)a + 2(-1)^{m+n}(a_2-a_1)m\pi]\,. \end{aligned} \tag{4.21}$$

4.2.4 Suitable choices for α_i and model equations

In experiments, the low-frequency wave usually has a larger amplitude than the high-frequency wave, thus we assume $\alpha_3 > \alpha_2$. The condition mentioned after equation (4.6) means that we should have $\alpha_1 < \alpha_2$ (which implies $\alpha_1 < \alpha_3$) and $\alpha_1 < \alpha_4$. The goal in choosing α_i is to keep as many terms in (4.19) and (4.21) as possible, and the terms containing the derivatives should not be higher-order compared to other terms.

Comparing R_1 and S_1 with the other R_i's and S_i's and solving for α_5 yields

$$\begin{array}{rcll} \text{(i)} & \alpha_5 & \leq & 2\alpha_2 + \frac{1}{2}\alpha_1 \,, \\ \text{(ii)} & \alpha_5 & \leq & 2\alpha_3 - 2\alpha_1 \,, \\ \text{(iii)} & \alpha_5 & \leq & \alpha_3 - \alpha_2 + \alpha_4 \,, \\ \text{(iv)} & \alpha_5 & \leq & 2\alpha_3 - 2\alpha_1 \,, \\ \text{(v)} & \alpha_5 & \leq & 2\alpha_2 - \frac{5}{2}\alpha_1 \,, \\ \text{(vi)} & \alpha_5 & \leq & \alpha_2 - \alpha_3 + \alpha_4 + \frac{3}{2}\alpha_1 \,. \end{array} \tag{4.22}$$

As many of those as possible should be equalities, but satisfying (v) and having $\alpha_1 > 0$ does not allow (i) to become an equality. Also, we want at least one of R_2, R_3 and R_4 to have the same order as R_1.

R_2 cannot have the same order as R_1, because we cannot have equality in (i). So, assume that R_3 has the same order as R_1, i.e. equality in (ii). After substituting α_5 with $2\alpha_3 - 2\alpha_1$, inequality (v) becomes

$$2\alpha_3 - 2\alpha_2 \leq -\frac{1}{2}\alpha_1 \,.$$

Since the left side of this is positive, but the right side is negative, this is not possible. Thus, assume now that R_4 has the same order as R_1, meaning equality in (iii). Now replace α_5 with $\alpha_3 - \alpha_2 + \alpha_4$ in inequalities (ii),(v) and (vi). Solving the first two of those for α_4 and the last one for α_1, we get

$$\begin{array}{rcll} \text{(ii)'} & \alpha_4 & \leq & \alpha_2 + \alpha_3 - 2\alpha_1 \,, \\ \text{(v)'} & \alpha_4 & \leq & 3\alpha_2 - \alpha_3 - \frac{5}{2}\alpha_1 \,, \\ \text{(vi)'} & \alpha_1 & \geq & \frac{4}{3}(\alpha_3 - \alpha_2) \,. \end{array}$$

It has been shown that there cannot be equality in (ii)', so we can try to get equality in (v)' and (vi)'. Assuming equality in (v)', plugging into (ii)' and solving for α_1 yields

$$\text{(ii)"} \quad \frac{1}{2}\alpha_1 \geq 2(\alpha_2 - \alpha_3) \,,$$

which is always true under our assumptions. Remains to check whether $\alpha_1 < \alpha_4$ is satisfied. Plugging the expression of (v)' for α_4 into that and solving for α_1, we get

$$\alpha_1 < \frac{2}{7}(3\alpha_2 - \alpha_3) \,.$$

Assume equality also in (vi). Then

$$\frac{4}{3}(\alpha_3 - \alpha_2) < \frac{2}{7}(3\alpha_2 - \alpha_3)\,.$$

This simplifies to

$$\alpha_3 < \frac{23}{17}\alpha_2\,.$$

This condition has to be satisfied independent of the choices of α_1 and α_4. The condition $\alpha_1 < \alpha_2$ implies

$$\frac{4}{3}(\alpha_3 - \alpha_2) < \alpha_2$$

or $\alpha_3 < \frac{7}{4}\alpha_2$ which is already granted by the previous condition.

Hence, with $\alpha_3 < \frac{23}{17}\alpha_2$, set

$$\begin{array}{rcll} \alpha_1 &=& \frac{4}{3}\alpha_3 - \frac{4}{3}\alpha_2\,, & \\ \alpha_4 &=& 3\alpha_2 - \alpha_3 - \frac{5}{2}\alpha_1 & = \frac{19}{3}\alpha_2 - \frac{13}{3}\alpha_3\,, \\ \alpha_5 &=& \alpha_3 - \alpha_2 + \alpha_4 & = \frac{16}{3}\alpha_2 - \frac{10}{3}\alpha_3\,. \end{array}$$

Equations (4.18) and (4.20) become

$$\begin{array}{rcl} p_m' &=& C_{13}e^{i\lambda\tau}\overline{p_n} + i\epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)}C_{12}|p_n|^2p_m + i\epsilon^{4(\alpha_3-\alpha_2)}C_{11}|p_m|^2p_m\,, \\ p_n' &=& C_{23}e^{i\lambda\tau}\overline{p_m} + iC_{22}|p_m|^2p_n + i\epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)}C_{21}|p_n|^2p_n\,. \end{array} \tag{4.23}$$

The C_{ij} are constants independent of ϵ and given by

$$\begin{array}{rcl} C_{11} &=& -\frac{C}{2\cosh(k_mh)(gk_m)^{\frac{1}{2}}}\,, \\ C_{12} &=& \frac{k_m^3\omega_m}{2a^2g}\tanh(k_mh)\,, \\ C_{13} &=& \frac{1}{a^2}\left((2a_1-a_2)a + (-1)^{m+n}(a\,a_1 + 2(a_1-a_2)m\pi)\right)\,, \\ C_{21} &=& -\frac{k_m^2}{2a^2g}\,, \\ C_{22} &=& -\frac{k_m^{\frac{5}{2}}}{2a^2g^{\frac{3}{2}}}\,, \\ C_{23} &=& -\frac{1}{2(gk_m)^{\frac{1}{2}}}\frac{\coth(k_mh)\omega_m}{a^2}\left[([(-1)^{m+n}-2]a_1 + a_2)a + 2(-1)^{m+n}(a_2-a_1)m\pi\right]\,. \end{array}$$

In some cases, e.g. $a_1 = a_2$ and $m+n$ even, it may seem that $C_{13} = 0$ or $C_{23} = 0$. In these cases, lower-order terms that have been neglected become dominant.

For example, take $\alpha_2 = 1$ and $\alpha_3 = \frac{4}{3}$. Then we have $\alpha_1 = \frac{4}{9}$, $\alpha_4 = \frac{5}{9}$ and $\alpha_5 = \frac{8}{9}$. Notice that this means that the amplitude of the forcing is larger than the low-frequency amplitude.

4.3 Discussion of the model equations

4.3.1 Special cases

There are some questions that should be answered.

- What happens if there is no forcing?

  This means that $R_4 = S_4 = 0$. We only have inequalities (i),(ii),(iv),(v) left in (4.22). With the same reasoning as before, we cannot have equality in (i) or (ii). The consequence is that, by having no forcing, we are losing the leading order term in the right side of the equation for p'_m in (4.23).

- Can we have the amplitude of the forcing smaller than the amplitude of the low-frequency wave?

  In other words

  $$\alpha_2 < \alpha_4 = \frac{19}{3}\alpha_2 - \frac{13}{3}\alpha_3 \, .$$

  Solving for α_3 yields

  $$\alpha_3 < \frac{16}{13}\alpha_2 \, ,$$

  which is slightly stricter than the previously obtained condition $\alpha_3 < \frac{23}{17}\alpha_2$. For example, choose $\alpha_2 = 1$ and $\alpha_3 = \frac{6}{5}$. This means that $\alpha_1 = \frac{4}{15}$, $\alpha_4 = \frac{17}{15}$ and $\alpha_5 = \frac{4}{3}$.

4.3.2 Small surface tension and Hocking's edge condition

We want to see how small T (ratio of surface tension to density) needs to be in order not to have any effect on the differential equations.

Suppose that $T = \epsilon^{\alpha_7}$. As an edge condition, we now assume

$$\begin{aligned} \theta_{x^*} &= \epsilon^{\alpha_8}\theta_{t^*} \quad \text{at } x^* = k_n a \, , \\ \theta_{x^*} &= -\epsilon^{\alpha_8}\theta_{t^*} \quad \text{at } x^* = 0 \, . \end{aligned}$$

Consider the part $T\theta_{x^*x^*}$. Multiplying by $\cos(\epsilon^{\alpha_1}x^*)$ and integrating yields

$$\begin{aligned}
& \textstyle\int_0^{k_n a} \theta_{x^*x^*} \cos(\epsilon^{\alpha_1}x^*)dx^* \\
= \; & \theta_{x^*} \cos(\epsilon^{\alpha_1}x^*)|_0^{k_n a} - \textstyle\int_0^{k_n a} \theta_{x^*}(-\epsilon^{\alpha_1}\sin(\epsilon^{\alpha_1}x^*))dx^* \\
= \; & \epsilon^{\alpha_8}[(-1)^m \theta_{t^*}|_{x^*=k_n a} + \theta_{t^*}|_{x^*=0}] - \epsilon^{2\alpha_1} \textstyle\int_0^{k_n a} \theta \cos(\epsilon^{\alpha_1}x^*)dx^* \,.
\end{aligned}$$

The terms resulting from $T\theta_{x^*x^*}$ should be small compared to $\frac{g}{k_n^2}\theta$, hence

$$\begin{aligned}
& |\textstyle\int_0^{k_n a} \frac{g}{k_m^2}\epsilon^{2\alpha_1}\theta\cos(\epsilon^{\alpha_1}x^*)dx^*| \\
\gg \; & |\epsilon^{\alpha_7+\alpha_8}[(-1)^m\theta_{t^*}|_{x^*=k_n a} + \theta_{t^*}|_{x^*=0}]| + |\epsilon^{\alpha_7+2\alpha_1} \textstyle\int_0^{k_n a} \theta\cos(\epsilon^{\alpha_1}x^*)dx^*| \,.
\end{aligned}$$

Thus α_7 and α_8 should satisfy $\alpha_7 \gg 0$ and $\alpha_8 \gg 2\alpha_1 - \alpha_7$.

The same procedure for multiplying by $\cos(x^*)$:

$$\begin{aligned}
& \textstyle\int_0^{k_n a} \theta_{x^*x^*} \cos(x^*)dx^* \\
= \; & \epsilon^{\alpha_8}[(-1)^n \theta_{t^*}|_{x^*=k_n a} + \theta_{t^*}|_{x^*=0}] - \textstyle\int_0^{k_n a} \theta \cos(x^*)dx^* \,.
\end{aligned}$$

In this case we should have

$$\begin{aligned}
& |\textstyle\int_0^{k_n a} \frac{g}{k_m^2}\epsilon^{2\alpha_1}\theta\cos(x^*)dx^*| \\
\gg \; & |\epsilon^{\alpha_7+\alpha_8}[(-1)^n\theta_{t^*}|_{x^*=k_n a} + \theta_{t^*}|_{x^*=0}]| + |\epsilon^{\alpha_7} \textstyle\int_0^{k_n a} \theta\cos(x^*)dx^*| \,.
\end{aligned}$$

Hence, we need $\alpha_7 \gg 2\alpha_1$, the requirement for α_8 is the same.

Additional restrictions result from making sure that the $\frac{3}{2}Tk_n^2\theta_{x^*x^*}\theta_{x^*}^2$ term does not contain any dominating terms. The conditions to be satisfied are

$$\begin{aligned}
\alpha_7 + 3\alpha_2 - \alpha_1 &> 3\alpha_2 \,, \\
\alpha_7 + \alpha_2 + 2\alpha_3 - 4\alpha_1 &> \alpha_2 + 2\alpha_3 - \tfrac{5}{2}\alpha_1 \,, \\
\alpha_7 + 3\alpha_3 - 5\alpha_1 &> 3\alpha_3 - \tfrac{5}{2}\alpha_1 \,, \\
\alpha_7 + 2\alpha_2 + \alpha_3 - 3\alpha_1 &> 2\alpha_2 + \alpha_3 - 3\alpha_1 \,.
\end{aligned}$$

Solving each of those for α_7, the most restrictive inequality is the third one, thus

$$\alpha_7 \gg \frac{5}{2}\alpha_1 \qquad \text{and} \qquad \alpha_8 \gg 2\alpha_1 - \alpha_7 \,. \tag{4.24}$$

are the conditions on surface tension and the terms from Hocking's edge condition needed to ensure that neither effect contributes to the dominating terms of the equations.

4.3.3 Approximations to solutions of the new system

We want to discuss a system of the form (4.23). The steps of Section 3.3 can be repeated with setting

$$\begin{aligned} c_{11} &= \epsilon^{4(\alpha_3-\alpha_2)}C_{11}\,, \\ c_{12} &= \epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)}C_{12}\,, \\ c_{13} &= \delta\,, \\ c_{21} &= \epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)}C_{21}\,, \\ c_{22} &= C_{22}\,, \\ c_{23} &= \kappa\,\delta\,. \end{aligned} \tag{4.25}$$

Now we investigate the roots of the polynomial $Q(\rho)$ defined in (3.50-3.52). In order to obtain simpler expressions for the roots, simplifications are made.

All terms with δ or ϵ are neglected

In this case, $Q(\rho)$ has the roots

$$\rho_{1,2} = \frac{1}{2}(\beta_1 r_m^2 + \beta_2 r_n^2)\,, \qquad \rho_{3,4} = \frac{C_{22}(r_n^2\beta_2 - r_m^2(\beta_1 + 2\kappa\beta_2)) + 2(\beta_1+\kappa\beta_2)\lambda}{2C_{22}}\,.$$

These are two double real roots which, in the case of $r_m^2 = \frac{\lambda}{C_{22}}$ coincide.

All terms with δ neglected

We can still determine the roots exactly:

$$\begin{aligned} \rho_{1,2} &= \tfrac{1}{2}(\beta_1 r_m^2 + \beta_2 r_n^2)\,, \\ \rho_{3,4} &= \Big[C_{22}(r_n^2\beta_2 - r_m^2(\beta_1+2\kappa\beta_2)) + 2(\beta_1+\kappa\beta_2)\lambda \\ &\quad -\epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)}\Big(-(C_{12}+C_{21})(\kappa r_m^2 - 2r_n^2)\beta_1 + C_{11}r_m^2\beta_1\epsilon^{\frac{4}{3}(\alpha_3-\alpha_2)} \\ &\qquad +[(C_{12}+C_{21})\kappa r_n^2 + C_{11}(2\kappa r_m^2 - r_n^2)\epsilon^{\frac{4}{3}(\alpha_3-\alpha_2)}]\beta_2\Big)\Big] \\ &\quad \Big/\Big[2\Big(C_{22} + (C_{12}+C_{21})\kappa\epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)} + C_{11}\epsilon^{4(\alpha_3-\alpha_2)}\Big)\Big]\,. \end{aligned}$$

So, we still have double roots. They coincide if

$$r_m^2 = \frac{\lambda - \epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)}(C_{12}+C_{21})r_n^2}{C_{22} + \epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)}(C_{12}+C_{21})\kappa + \epsilon^{4(\alpha_3-\alpha_2)}C_{11}}\,.$$

Approximation of roots neglecting ϵ only

Calculating the roots directly fails. The expressions for the roots become very complicated, and some derivatives do not exist, which is why Taylor expansions are not too helpful.

We use the previously obtained roots as a first approximation and perturb each of those roots by $R\,\delta$, plug in the modified candidates for roots and choose R in a way that the leading order error vanishes. To do so, powers of δ are collected and the terms with δ^2 are set equal to zero. The approximations of the roots are

$$\begin{aligned} \rho_{1,2} &= \tfrac{1}{2}(\beta_1 r_m^2 + \beta_2 r_n^2) + \tfrac{2 r_m r_n (\beta_1+\kappa\beta_2)(\sin(\alpha)\pm 1)}{\lambda - C_{22} r_m^2}\delta\,, \\ \rho_{3,4} &= \tfrac{C_{22}(r_n^2\beta_2 - r_m^2(\beta_1+2\kappa\beta_2))+2(\beta_1+\kappa\beta_2)\lambda}{2C_{22}} \\ &\quad +(\beta_1+\kappa\beta_2)\tfrac{2C_{22} r_m r_n \sin(\alpha)\pm 2\sqrt{(C_{22}r_m^2-2\lambda)(2C_{22}\kappa r_m^2 - C_{22} r_n^2 - 2\kappa\lambda)}}{C_{22}^2 r_m^2 - C_{22}\lambda}\delta\,. \end{aligned} \tag{4.26}$$

These roots are approximations in the way that $Q(\rho_i) = O(\delta^3)$ for all of the roots.

In the case $r_m^2 = \frac{\lambda}{C_{22}}$, the denominators are zero, so that case has to be considered separately. The original roots are perturbed by $S_1\,\delta^{\frac{1}{2}} + S_2\,\delta$ and plugged in. Powers of δ are collected and the terms with δ^2 and $\delta^{\frac{5}{2}}$ set equal to zero and solved for S_1 and S_2. Now the approximations are

$$\begin{aligned} \rho_{1,2,3,4} &= \tfrac{1}{2}(\beta_1 r_m^2 + \beta_2 r_n^2) \pm (\beta_1+\kappa\beta_2)\sqrt{\sqrt{\tfrac{\lambda}{C_{22}^3}}\, r_n (1+\sin(\alpha))}\,\delta^{\frac{1}{2}} \\ &\quad \pm \sqrt{\tfrac{(\beta_1+\kappa\beta_2)(C_{22} r_n^2 + \lambda\kappa)}{2 r_n \sqrt{C_{22}^3 \lambda}}}\,\delta\,. \end{aligned} \tag{4.27}$$

This is to be understood in the way that there are four distinct roots by using all possible combinations of choices of '+' and '-'. Again, these roots are approximations in the way that $Q(\rho_i) = O(\delta^3)$ for all of the roots.

4.3.4 Analysis of the dynamical system

Equilibria of the new system and their stability

We still have the trivial equilibrium $r_m = r_n = 0$. Other equilibria need to satisfy (3.72). Using once again (4.25), this means we need to solve

$$\begin{aligned} C_{22}L^2\delta + \tfrac{1}{2}C_{22}L\lambda + \epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)}\left(C_{21}\delta L^4 + \tfrac{1}{2}(C_{21}-C_{12})\lambda L^3 - C_{12}\delta\kappa L^2\right) \\ +\epsilon^{4(\alpha_3-\alpha_2)}\left(-C_{11}\kappa\delta - \tfrac{1}{2}\lambda C_{11} L\right) = 0\,. \end{aligned}$$

We can approximate this equation by dropping the part with $\epsilon^{4(\alpha_3-\alpha_2)}$. The solutions that we obtain that way are two complex conjugate solutions (which are dropped) and two real solutions.

$$\begin{aligned} L_1 &= 0\,, \\ L_2 &\approx -\tfrac{\lambda}{2\delta} + \epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)}\tfrac{C_{12}\lambda^3}{8C_{22}\delta^3}\,. \end{aligned}$$

In L_2, the first two terms of the Taylor expansion are given. It is observed that the small solution L_1 is approximated better by the solution of

$$\frac{1}{2}C_{22}\lambda L + \epsilon^{4(\alpha_3-\alpha_2)}\left(-C_{11}\delta\kappa - \frac{1}{2}C_{11}\lambda L\right) = 0\,.$$

Using a Taylor series, the solution of this is given by

$$L_1 \approx \frac{2C_{11}\delta\kappa}{C_{22}\lambda}\epsilon^{4(\alpha_3-\alpha_2)}\,.$$

However, when changing the equation, the effect on the roots may be large compared to the portion neglected. So, we go back to the original equation. Starting out with

$$\begin{aligned} L_1 &= \tfrac{2C_{11}\delta\kappa}{C_{22}\lambda}\epsilon^{4(\alpha_3-\alpha_2)}\,, \\ L_2 &= -\tfrac{\lambda}{2\delta}\,, \end{aligned}$$

we plug in $L_1+\epsilon^{8(\alpha_3-\alpha_2)}B$ and $L_2+\epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)}D_1+\epsilon^{4(\alpha_3-\alpha_2)}D_2+\epsilon^{\frac{16}{3}(\alpha_3-\alpha_2)}D_3$. After collecting expressions with same powers of ϵ, we set the terms with leading order powers equal to zero and solve for B and D_i.

The roots obtained that way are

$$\begin{aligned} L_1 &= \tfrac{2C_{11}\delta\kappa}{C_{22}\lambda}\epsilon^{4(\alpha_3-\alpha_2)} - \tfrac{2C_{11}^2\delta\kappa(4\delta^2\kappa-\lambda^2)}{C_{22}^2\lambda^3}\epsilon^{8(\alpha_3-\alpha_2)}\,, \\ L_2 &= -\tfrac{\lambda}{2\delta} + \tfrac{C_{12}\lambda(-4\delta^2\kappa+\lambda^2)}{8C_{22}\delta_3}\epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)} + \tfrac{-4C_{11}\delta^2\kappa+C_{11}\lambda^2}{2C_{22}\delta\lambda}\epsilon^{4(\alpha_3-\alpha_2)} \\ &\quad -\tfrac{C_{12}\lambda(-4\delta^2\kappa+\lambda^2)(-4C_{12}\delta^2\kappa+2C_{12}\lambda^2+C_{21}\lambda^2)}{32C_{22}^2\delta^5}\epsilon^{\frac{16}{3}(\alpha_3-\alpha_2)}\,. \end{aligned} \tag{4.28}$$

Using (3.71), we obtain the equilibria

$$\begin{aligned} \rho_1 &\approx \tfrac{\lambda}{2C_{11}}\epsilon^{-4(\alpha_3-\alpha_2)} + \tfrac{2\delta^2\kappa}{C_{22}\lambda}\,, \\ \rho_2 &\approx \tfrac{-4\delta^2\kappa+\lambda^2}{2C_{22}\lambda} + \tfrac{(4\delta^2\kappa-\lambda^2)(4C_{12}\delta^2\kappa+C_{21}\lambda^2)}{8C_{22}\delta^2\lambda}\epsilon^{\frac{8}{3}(\alpha_3-\alpha_2)}\,. \end{aligned} \tag{4.29}$$

The leading order terms need to be positive in order to actually have an equilibrium. We plug in the pairs (r_1, L_1) and (r_2, L_2) to see that, for the above approximations to equilibria, the errors are $O\left(\epsilon^{\frac{14}{3}(\alpha_3-\alpha_2)}\right)$ and $O\left(\epsilon^{4(\alpha_3-\alpha_2)}\right)$, respectively.

The eigenvalues of the trivial equilibrium are $\pm\sqrt{\kappa\delta^2 - \frac{\lambda^2}{4}}$. Depending on the sign of the expression under the root, the eigenvalues are either purely imaginary or real.

For the other equilibria, consider (3.76) and (3.77). For L_1 and ρ_1, we get

$$C_1 \approx \epsilon^{-8(\alpha_3-\alpha_2)}\frac{C_{22}^2\lambda^2}{4C_{11}^2}\,.$$

For L_2 and ρ_2, we have

$$C_1 \approx \frac{4\delta^4\kappa^2}{\lambda^2} - \frac{\lambda^2}{4}\,.$$

Homoclinic and heteroclinic orbits

There are still no heteroclinic orbits, of course, and the homoclinic orbits connecting the trivial equilibrium have been found in Section 3.3.5 .

It is now of interest to find out whether there are homoclinic orbits connecting the other equilibria. Recalling Section 3.3.1, the system (r_m, r_n, α) has homoclinic orbits if we have four real roots $\rho_1 \le \rho_2 = \rho_3 \le \rho_4$ with at least two distinct roots.

Considering the roots obtained in (4.26), this could particularly be happening if, for $\delta = 0$, r_m^2 and $\frac{-C_{22}r_m^2+2\lambda}{C_{22}}$ are close to each other and two roots are moving towards each other, as δ is modified.

We choose the parameters $m = 1; a = 1; h = 1.0527; g = 9.81; a_1 = 1; a_2 = -1; n$ even; $\alpha_3 - \alpha_2 = \frac{1}{5}$ and $\kappa = 1$, $\lambda = -0.1$. For $\epsilon = 0.01$, $r_m = \frac{1}{2}\sqrt{\frac{\lambda}{c_{22}}}$, $r_n = 1$ and $\sin(\alpha) = 0.5$, figure 4.1 then shows a plot of the roots of $Q(\rho)$ in terms of δ.

We can see that for some value of δ between 0.12 and 0.13 there is a double root, thus a homoclinic orbit of the system (r_m, r_n, α). Figure 4.1 also displays another interesting effect: r_m^2 oscillates between the real roots closest to the initial value of r_m^2, which is indicated by the dashed line. The double arrows in the figure are meant to demonstrate that r_m^2 reaches values up to 0.2 for $\delta = 0.1$, but values up to 2 for $\delta = 0.15$. In particular, there is a big jump in the oscillation of r_m^2 at the value of δ that gives the homoclinic orbit. In the following figures we use parameters as above with $\epsilon = 0.002, 0.01, 0.05$ and plot the oscillation amplitude of r_m^2 in dependance of δ and the initial value of r_m. At the jumps there are combinations of parameters for which we get orbits converging to circular orbits for $\tau \to \pm\infty$. In order to better see the curves along which the jumps occur, we have also included the corresponding plots of the curve in the phase plane.

It should be noted that the oscillation has been evaluated only for discrete values of δ and r_m. Thus it is a subjective decision what is a relatively steep incline and what is a jump by defining an appropriate threshold value. We have set the threshold value to 0.1 for $\epsilon = 0.002$ and $\epsilon = 0.01$, and 0.05 for $\epsilon = 0.05$.

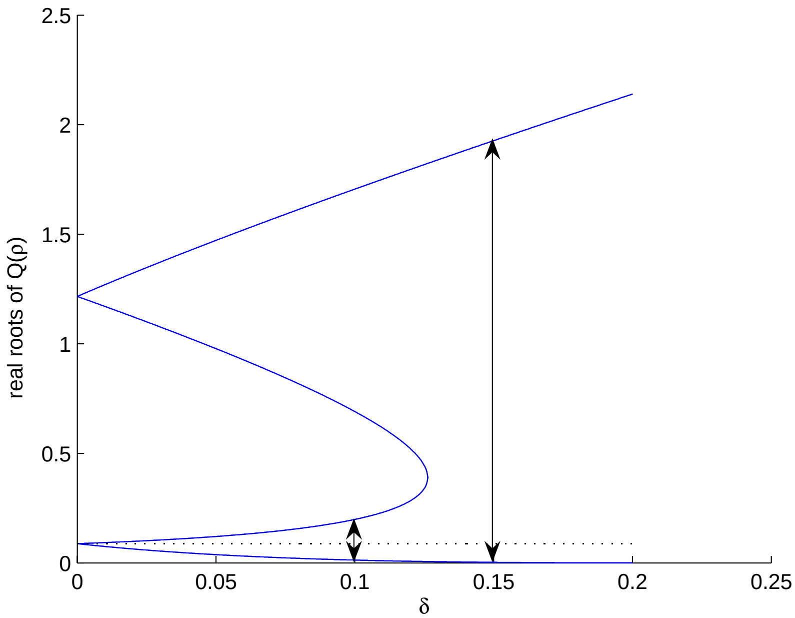

Figure 4.1: Roots of $Q(\rho)$ in terms of δ

It would be interesting to see what these special orbits look like. While it is not possible to numerically find the exact value of δ for which $Q(\rho)$ has a double real root, we can get arbitrarily close. Figures 4.8, 4.9 show the radii, angles and phase portraits for $\delta = 0.126225$ and figures 4.10, 4.11 show the same for $\delta = 0.1262255$. The critical value of δ for which $Q(\rho)$ would have the real double root is between those values of δ.

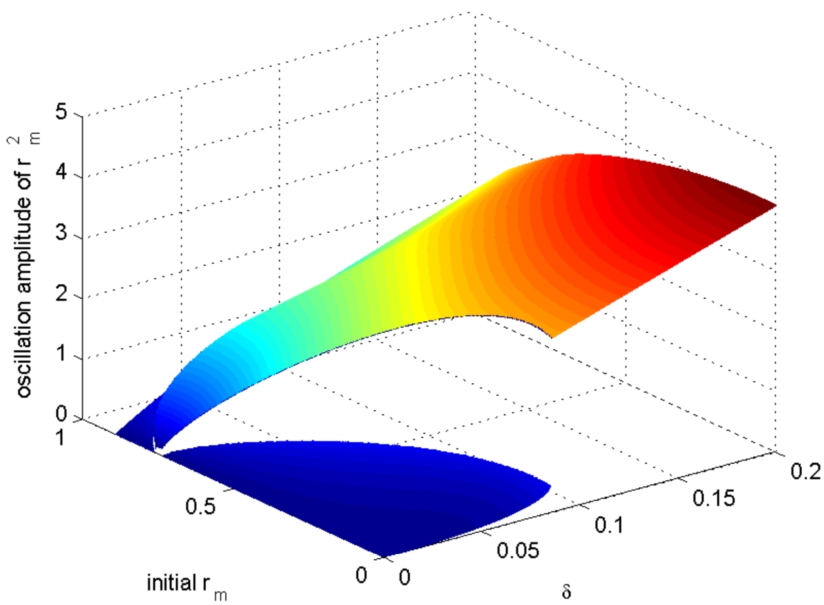

Figure 4.2: Homoclinic orbits and jumps in oscillation amplitude for $\epsilon = 0.002$

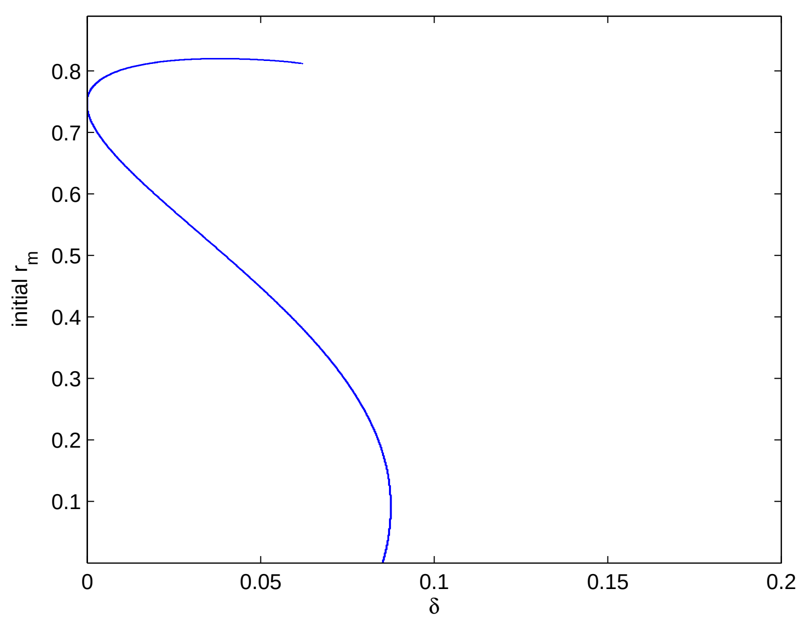

Figure 4.3: Curve along which jumps occur for $\epsilon = 0.002$

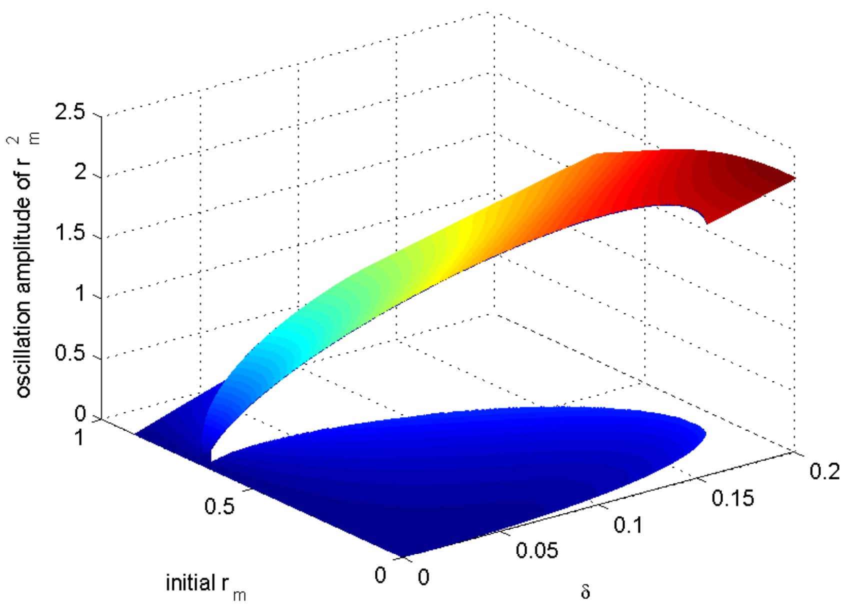

Figure 4.4: Homoclinic orbits and jumps in oscillation amplitude for $\epsilon = 0.01$

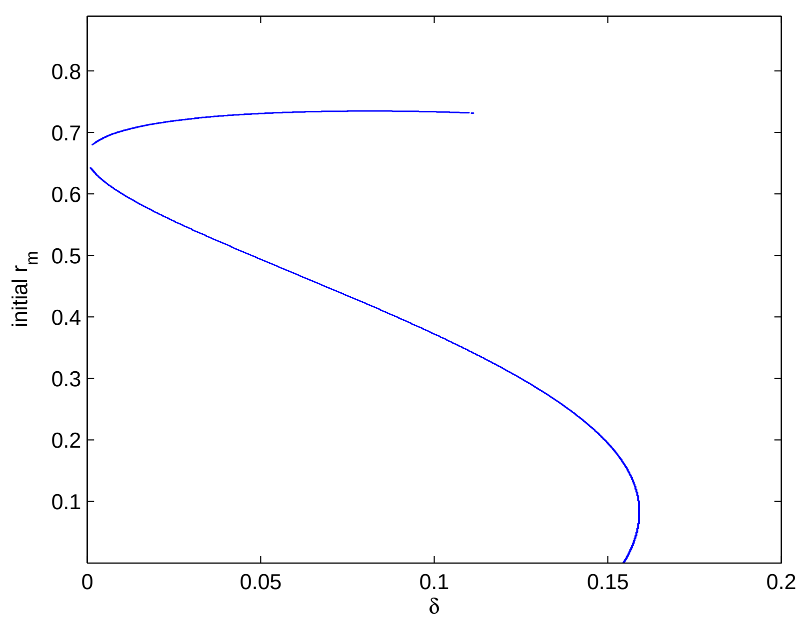

Figure 4.5: Curve along which jumps occur for $\epsilon = 0.01$

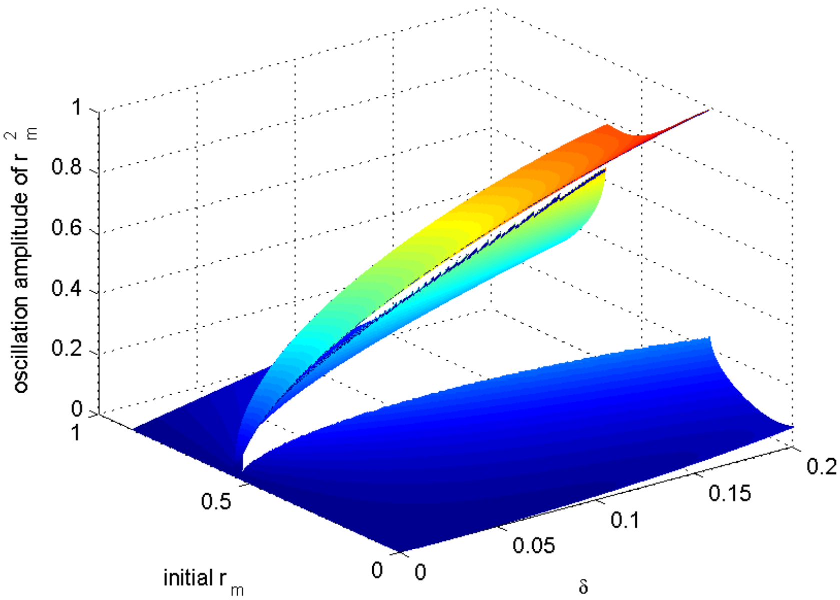

Figure 4.6: Homoclinic orbits and jumps in oscillation amplitude for $\epsilon = 0.05$

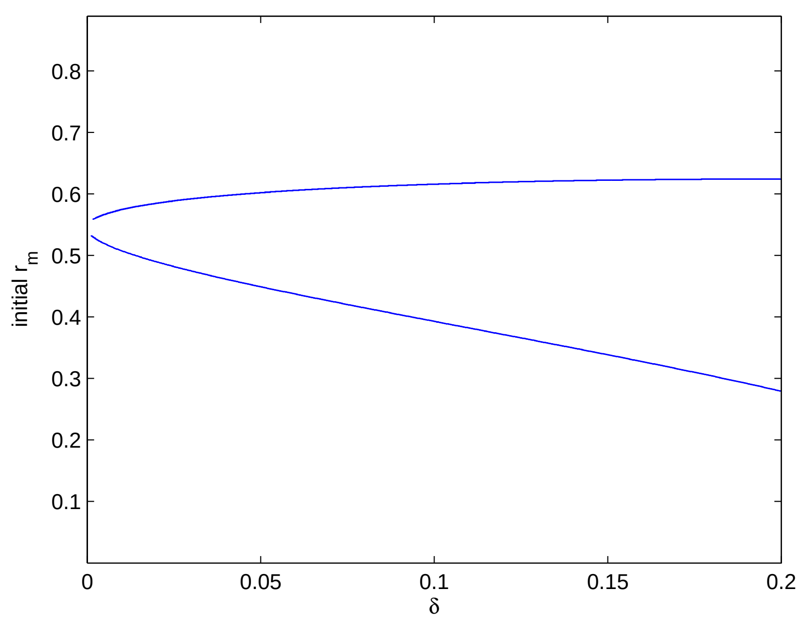

Figure 4.7: Curve along which jumps occur for $\epsilon = 0.05$

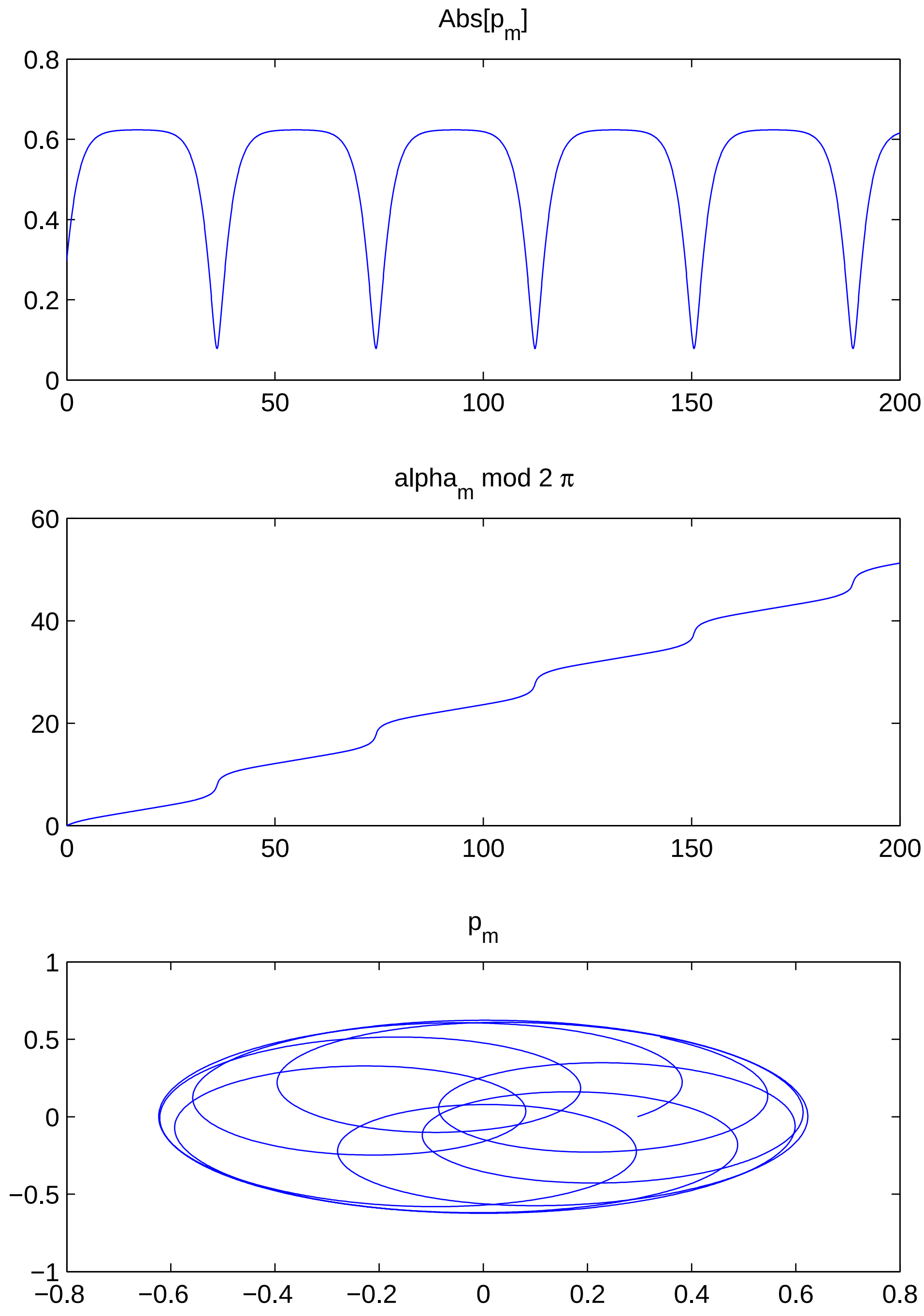

Figure 4.8: Orbit of p_m for $\delta = 0.126225$

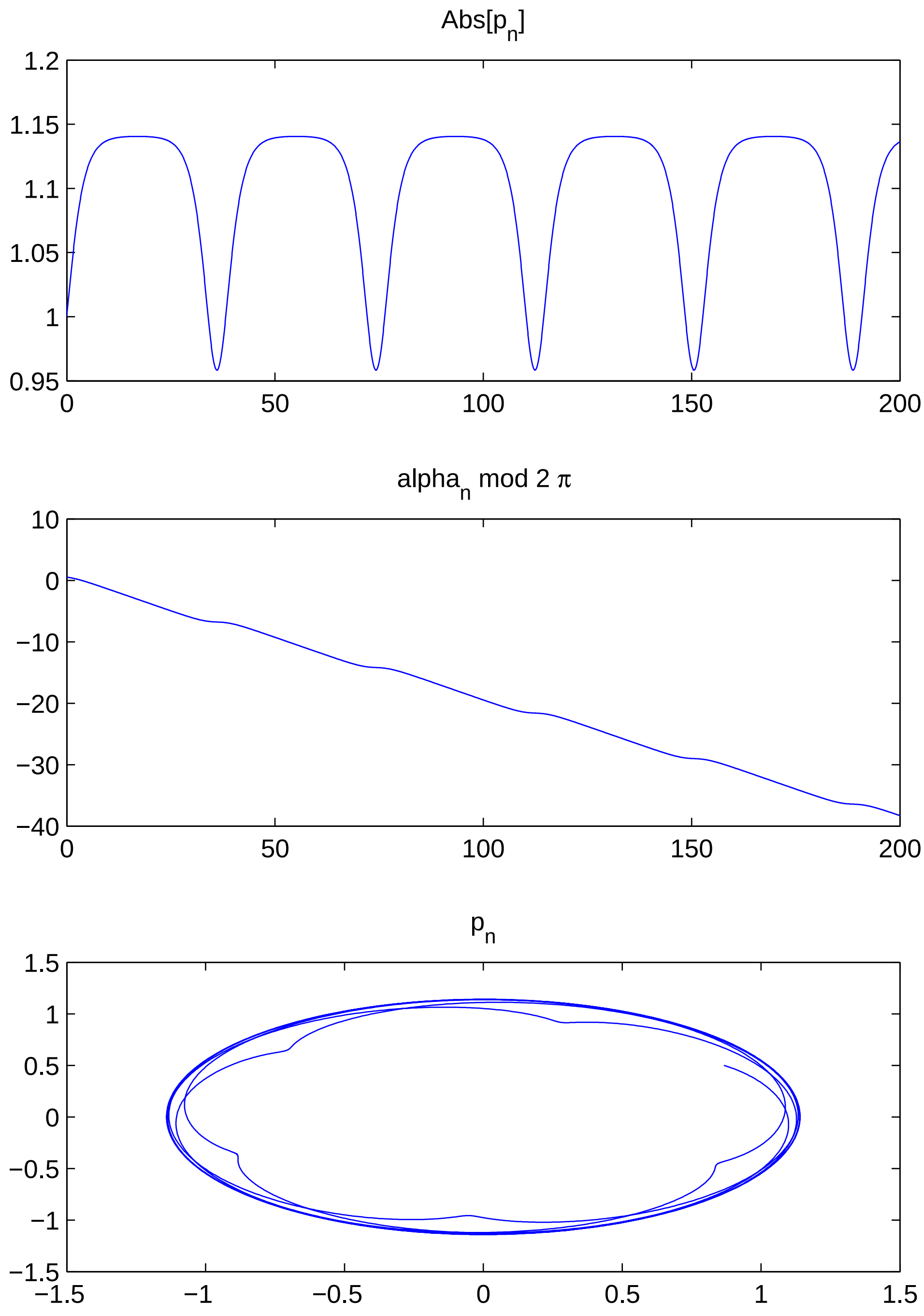

Figure 4.9: Orbit of p_n for $\delta = 0.126225$

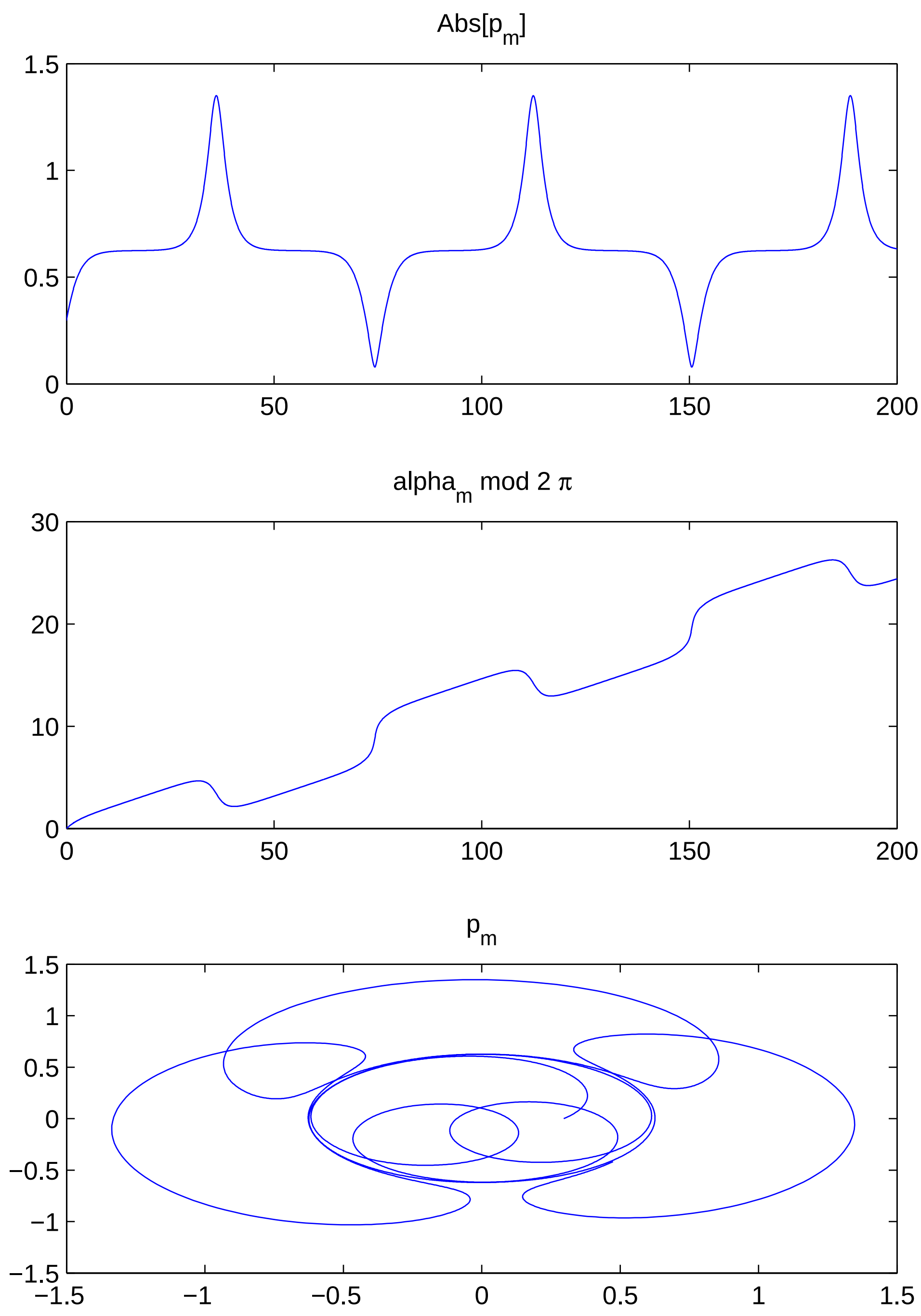

Figure 4.10: Orbit of p_m for $\delta = 0.1262255$

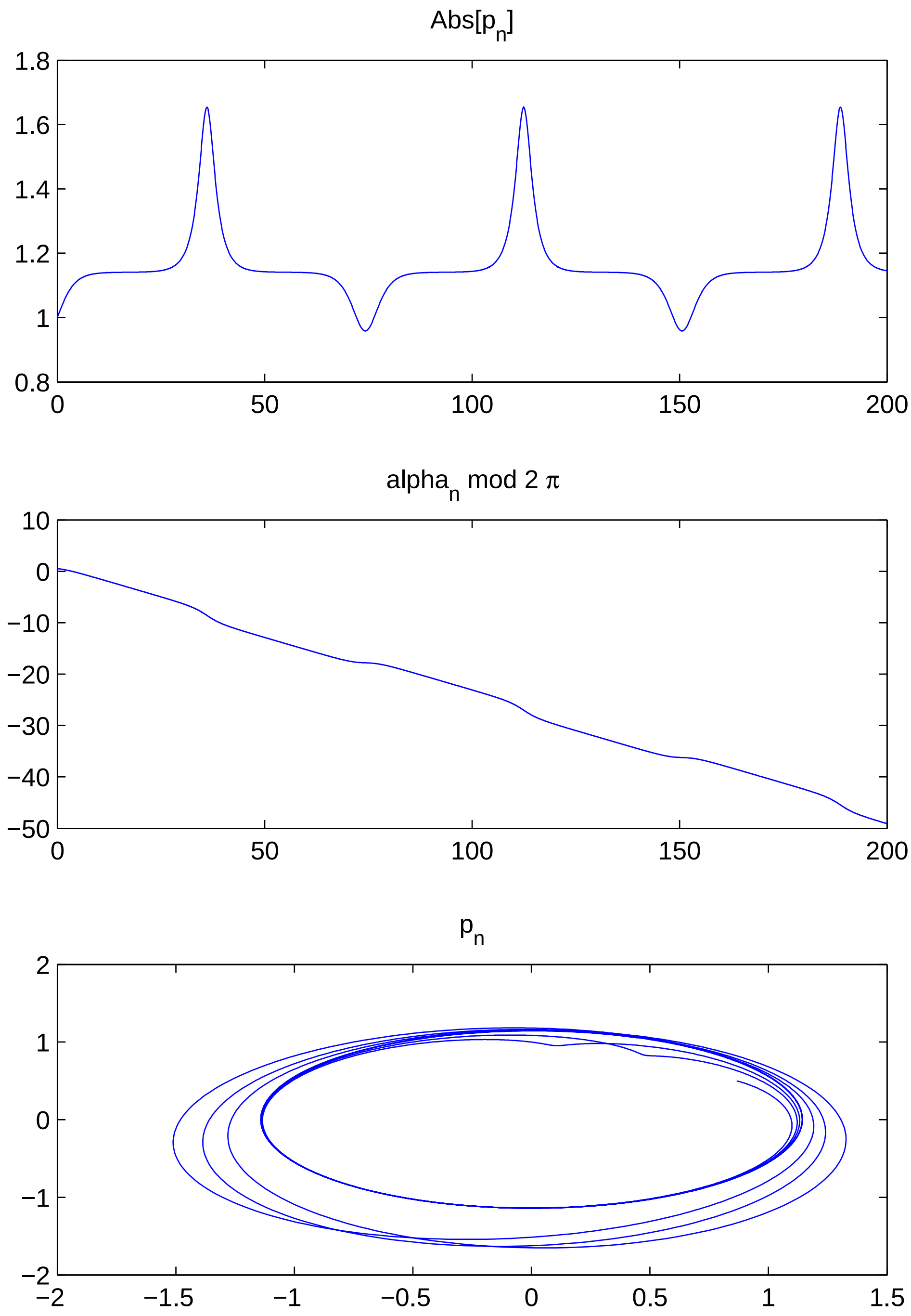

Figure 4.11: Orbit of p_n for $\delta = 0.1262255$

Chapter 5

Additional forcing with frequency close to the larger eigenfrequency

5.1 Introduction

In addition to the forcing terms introduced in Chapters 3 and 4, a forcing of third order with frequency close to the larger eigenfrequency ω_n is applied.

For the system in Chapter 3, we show how the governing equations are modified by this extra forcing, and the equilibria and periodic orbits are found. The special case of applying only third-order forcing of frequency close to ω_n is considered. A numerical example for the calculation of equilibria and periodic orbits is given.

For the system in Chapter 4, the modified equations are presented as well. We focus on the case of having forcing of frequency ω_n only and find approximations to the solutions for ϵ small. In [40], critical values of the forcing were determined. A numerical example is presented that shows how the amplitudes of the solutions change rapidly when the forcing parameter surpasses the critical value.

5.2 Additional forcing for the system in Chapter 3

5.2.1 Modifications needed for the equations

For now, we work in a similar setting as in Chapter 3. However, we consider boundary terms of the form

$$b_k(z,t) = \epsilon^2 \left(i(\omega_m + \omega_n)\right)^{-1} e^{i(\omega_m+\omega_n)t+i\lambda\tau} f_k(z) + \epsilon^3 (i\omega_n)^{-1} e^{i\omega_n t+i\lambda^*\tau} g_k(z)\,. \tag{5.1}$$

Since the only change is in the third-order terms of frequency ω_n, neither the first- and second-order approximation nor the third-order solvability condition for frequency ω_m are affected. This also means that the differential equation for p_m remains unchanged. The third-order solvability condition for frequency ω_n is updated and results in a new differential equation for p_n of the form

$$s_1 p_n' + s_2 |p_n|^2 p_n + s_3 |p_m|^2 p_n + s_4 e^{i\lambda\tau} \overline{p_m} + s_5 e^{i\lambda^*\tau} = 0 \,, \tag{5.2}$$

where s_1 through s_4 are given by (3.37) and

$$s_5 = \int_{-h}^{0} ((-1)^n g_2(z) - g_1(z)) \cosh(k_n(z+h)) dz \,. \tag{5.3}$$

When solving the differential equations for p_m and p_n, equation (3.38) becomes

$$\begin{aligned} p_m' &= c_{11} i p_m |p_m|^2 + c_{12} i p_m |p_n|^2 + c_{13} e^{i\lambda\tau} \overline{p_n} \,, \\ p_n' &= c_{21} i p_n |p_n|^2 + c_{22} i p_n |p_m|^2 + c_{23} e^{i\lambda\tau} \overline{p_m} + c_{24} e^{i\lambda^*\tau} \,. \end{aligned} \tag{5.4}$$

The coefficients are given by (3.39) and

$$c_{24} = -\frac{s_5}{s_1} \,.$$

To make the system autonomous, we use the transformation

$$\begin{aligned} p_m &= e^{i(\lambda-\lambda^*)\tau} P_m \,, \\ p_n &= e^{i\lambda^*\tau} P_n \,. \end{aligned} \tag{5.5}$$

The equations in P_m, P_n then are

$$\begin{aligned} \left(P_m' + i(\lambda - \lambda^*) P_m\right) e^{i(\lambda-\lambda^*)\tau} &= \left(c_{11} i P_m |P_m|^2 + c_{12} i P_m |P_n|^2 + c_{13} \overline{P_n}\right) e^{i(\lambda-\lambda^*)\tau} \,, \\ \left(P_n' + i\lambda^* P_n\right) e^{i\lambda^*\tau} &= \left(c_{21} i P_n |P_n|^2 + c_{22} i P_n |P_m|^2 + c_{23} \overline{P_m} + c_{24}\right) e^{i\lambda^*\tau} \,. \end{aligned}$$

Cancelling the exponential terms and solving for P_m', P_n' yields

$$\begin{aligned} P_m' &= -i(\lambda - \lambda^*) P_m + c_{11} i P_m |P_m|^2 + c_{12} i P_m |P_n|^2 + c_{13} \overline{P_n} \,, \\ P_n' &= -i\lambda^* P_n + c_{21} i P_n |P_n|^2 + c_{22} i P_n |P_m|^2 + c_{23} \overline{P_m} + c_{24} \,. \end{aligned} \tag{5.6}$$

Changing to polar coordinates as in (3.42) gives

$$\begin{aligned} r_m' e^{i\alpha_m} + i\alpha_m' r_m e^{i\alpha_m} &= -i(\lambda-\lambda^*) r_m e^{i\alpha_m} + c_{11} i r_m^3 e^{i\alpha_m} + c_{12} i r_m r_n^2 e^{i\alpha_m} + c_{13} r_n e^{-i\alpha_n}\,, \\ r_n' e^{i\alpha_n} + i\alpha_n' r_n e^{i\alpha_n} &= -i\lambda^* r_n e^{i\alpha_n} + c_{21} i r_n^3 e^{i\alpha_n} + c_{22} i r_n r_m^2 e^{i\alpha_n} + c_{23} r_m e^{-i\alpha_m} + c_{24}\,. \end{aligned} \tag{5.7}$$

Assuming $r_m r_n \neq 0$, the resulting system of differential equations for r_m, r_n, α_m and α_n then is

$$\begin{aligned} r_m' &= c_{13} r_n \cos(\alpha_m+\alpha_n)\,, \\ r_n' &= c_{23} r_m \cos(\alpha_m+\alpha_n) + c_{24}\cos(\alpha_n)\,, \\ \alpha_m' &= -(\lambda-\lambda^*) + c_{11} r_m^2 + c_{12} r_n^2 - c_{13}\tfrac{r_n}{r_m}\sin(\alpha_m+\alpha_n)\,, \\ \alpha_n' &= -\lambda^* + c_{21} r_n^2 + c_{22} r_m^2 - c_{23}\tfrac{r_m}{r_n}\sin(\alpha_m+\alpha_n) - c_{24}\tfrac{1}{r_n}\sin(\alpha_n)\,. \end{aligned} \tag{5.8}$$

The functionals J and E are not invariants anymore, thus a reduction to one ODE as in Section 3.3.1 is not possible.

5.2.2 Equilibria and certain periodic solutions

Going back to (5.4), it is clear that we can only have equilibria for $\lambda = \lambda^* = 0$. For $\lambda \neq 0$ and/or $\lambda^* \neq 0$, we get periodic solutions instead. Since the final solution for surface displacement is periodic anyway, this just means an adjustment of the period.

The first equation of (5.8) implies that either $\cos(\alpha_m + \alpha_n) = 0$ or $r_n = 0$. For the later case, recall (5.4). While the first equation requires $p_m = 0$, the second one implies $\overline{p_m} = -\frac{c_{24}}{c_{23}} = p_m$, since the fraction is real. Assuming $c_{24} \neq 0$ this is a contradiction. Thus we can focus on the case $\cos(\alpha_m + \alpha_n) = 0$.

The second equation in (5.8) then yields $\cos(\alpha_n) = 0$. Together with $\cos(\alpha_m + \alpha_n) = 0$, this implies $\sin(\alpha_m) = 0$. Thus it is assumed that

$$\begin{aligned} P_m &= R_m\,, \\ P_n &= i\,M R_m\,. \end{aligned}$$

Here, R_m and M are real-valued, but not necessarily positive . Plugging this into (5.6) yields

$$\begin{aligned} -i(\lambda-\lambda^*)R_m + c_{11} i R_m^3 + c_{12} i R_m^3 M^2 - c_{13} i M R_m &= 0\,, \\ \lambda^* M R_m - c_{21} M^3 R_m^3 - c_{22} M R_m^3 + c_{23} R_m + c_{24} &= 0\,. \end{aligned}$$

Cancelling $i\,R_m$ in the first equation and collecting terms, we obtain

$$\begin{aligned} R_m^2(c_{11} + c_{12} M^2) - (c_{13} M + (\lambda-\lambda^*)) &= 0\,, \\ R_m^3(c_{21} M^3 + c_{22} M) - R_m(c_{23} + \lambda^* M) - c_{24} &= 0\,. \end{aligned} \tag{5.9}$$

Solving the first equation for R_m^2 gives

$$R_m^2 = \frac{c_{13}M + (\lambda - \lambda^*)}{c_{11} + c_{12}M^2}\,. \tag{5.10}$$

Take c_{24} to the right side in the second equation and square to get

$$R_m^6(c_{21}M^3 + c_{22}M)^2 - 2R_m^4(c_{21}M^3 + c_{22}M)(c_{23} + \lambda^* M) + R_m^2(c_{23} + \lambda^* M)^2 = c_{24}^2\,.$$

Substituting in the expression for R_m^2, we get an equation in M only. Finding common denominator and collecting terms yields

$$a_9 M^9 + a_8 M^8 + ... + a_1 M + a_0 = 0\,. \tag{5.11}$$

The coefficients a_k are given by

$$\begin{aligned}
a_0 &= -c_{11}^2(c_{11}c_{24}^2 + c_{23}^2(\lambda^* - \lambda))\,,\\
a_1 &= c_{11}c_{23}\left(-2c_{22}(\lambda-\lambda^*)^2 + c_{11}(c_{13}c_{23} + 2(\lambda-\lambda^*)\lambda^*)\right)\,,\\
a_2 &= c_{22}^2(\lambda-\lambda^*)^3 - 2c_{11}(\lambda-\lambda^*)\left(2c_{13}c_{22}c_{23} - c_{12}c_{23}^2 + c_{22}(\lambda-\lambda^*)\lambda^*\right)\\
&\qquad + c_{11}^2\left(-3c_{12}c_{24}^2 + \lambda^*(2c_{13}c_{23} + (\lambda-\lambda^*)\lambda^*)\right)\,,\\
a_3 &= c_{22}(3c_{13}c_{22} - 2c_{12}c_{23})(\lambda-\lambda^*)^2 + c_{11}^2 c_{13}(\lambda^*)^2\\
&\qquad - 2c_{11}\left(c_{13}^2c_{22}c_{23} + c_{23}(\lambda-\lambda^*)(c_{21}(\lambda-\lambda^*) - 2c_{12}\lambda^*)\right)\,,\\
a_4 &= (\lambda-\lambda^*)\left(3c_{13}^2c_{22}^2 - 4c_{12}c_{13}c_{22}c_{23} + c_{12}^2c_{23}^2 + 2c_{12}c_{22}\lambda^*(-\lambda+\lambda^*)\right)\\
&\qquad - c_{11}\Big(3c_{12}^2c_{24}^2 + 2(2c_{13}c_{21}c_{23}(\lambda-\lambda^*) + c_{13}^2c_{22}\lambda^* + c_{21}(\lambda-\lambda^*)^2\lambda^*)\\
&\qquad + 2c_{12}\lambda^*(-2c_{13}c_{23} + \lambda^*(-\lambda+\lambda^*))\Big)\,,\\
a_5 &= c_{13}^3c_{22}^2 - 2c_{13}^2(c_{11}c_{21} + c_{12}c_{22})c_{23} + 2c_{12}c_{23}(\lambda-\lambda^*)(-c_{21}\lambda + (c_{12}+c_{21})\lambda^*)\\
&\qquad + c_{13}(c_{12}^2c_{23}^2 + 2c_{21}(\lambda-\lambda^*)(3c_{22}\lambda - 2c_{11}\lambda^* - 3c_{22}\lambda^*)\\
&\qquad + 2c_{12}\lambda^*(-2c_{22}\lambda + c_{11}\lambda^* + 2c_{22}\lambda^*))\,,\\
a_6 &= -c_{12}^3c_{23}^2 + c_{12}^2\lambda^*(2c_{13}c_{23} + (\lambda-\lambda^*)\lambda^*) - 2c_{12}(2c_{13}c_{21}c_{23}(\lambda-\lambda^*) + c_{13}^2c_{22}\lambda^*\\
&\qquad + c_{21}(\lambda-\lambda^*)^2\lambda^*) + c_{21}(c_{21}(\lambda-\lambda^*)^3 + c_{13}^2(6c_{22}\lambda - 2(c_{11}+3c_{22})\lambda^*))\,,\\
a_7 &= c_{13}\left(2c_{13}^2c_{21}c_{22} - 2c_{12}c_{13}c_{21}c_{23} + (c_{21}\lambda - (c_{12}+c_{21})\lambda^*)(3c_{21}\lambda - (c_{12}+3c_{21})\lambda^*)\right)\,,\\
a_8 &= c_{13}^2c_{21}(3c_{21}(\lambda-\lambda^*) - 2c_{12}\lambda^*)\,,\\
a_9 &= c_{13}^3c_{21}^2\,.
\end{aligned}$$

Each real solution M_i of (5.11) yields a R_m from (5.10). Since the second equation in (5.9) was squared, the sign was lost. Thus, the sign of R_m needs to be chosen in a way that this second equation is satisfied.

The case with forcing of frequency ω_n only, i.e. $c_{13} = c_{23} = 0$, should be considered separately. Set $\lambda = \lambda^*$ in (5.5) to simplify the transformation. Equation (5.8) in this case becomes

$$
\begin{aligned}
r'_m &= 0\,, \\
r'_n &= c_{24}\cos(\alpha_n)\,, \\
\alpha'_m &= c_{11}r_m^2 + c_{12}r_n^2\,, \\
\alpha'_n &= -\lambda^* + c_{21}r_n^2 + c_{22}r_m^2 - c_{24}\tfrac{1}{r_n}\sin(\alpha_n)\,.
\end{aligned} \tag{5.12}
$$

The second equation implies $\cos(\alpha_n) = 0$. The equation for α'_m shows that there can be equilibria only if c_{11} and c_{12} have different signs. In that case,

$$r_m^2 = -\frac{c_{12}}{c_{11}}r_n^2$$

can be plugged into the equation for α'_n :

$$\left(c_{21} - \frac{c_{12}}{c_{11}}c_{22}\right) r_n^3 - \lambda^* r_n - c_{24}\sin(\alpha_n) = 0\,. \tag{5.13}$$

We can choose $\sin(\alpha_n) = 1$ and in turn allow negative r_n. Then this boils down to a cubic equation in r_n. Note that α_m can be chosen freely.

5.2.3 Numerical Experiment

For the numerical experiment, the previous set of parameters is used again.

$$m = 1; n = 15; a = 10; h = 1; g = 9.81; T = 0.1; f_1 = 10, f_2 = -10\,.$$

In addition, choose $g_1 = g_2 = 1$. Then $c_{24} = -\frac{s_5}{s_1} = 0.212172$ following (5.3). First, the equilibria in the case $\lambda = \lambda^* = 0$ are found. The polynomial equation in (5.11) yields

$$
\begin{aligned}
&-0.0005186 + 0.4968M + 0.008125M^2 + 25.1475M^3 - 0.04243M^4 \\
&\quad +314.522M^5 + 0.07386M^6 - 93.8444M^7 + 6.91855M^9 = 0\,.
\end{aligned}
$$

The solutions of this equation are

$$
\begin{aligned}
M_1 &= -2.64226\,, \\
M_2 &= -2.58136\,, \\
M_3 &= 0.0010383\,, \\
M_{4,5} &= -0.00693 \pm 0.19153\,i\,, \\
M_{6,7} &= 0.00648 \pm 0.20488\,i\,, \\
M_{8,9} &= 2.61174 \pm 0.03046\,i\,.
\end{aligned}
$$

Choosing the correct signs yields

$$\begin{aligned}(R_m)_1 &= -0.818394\,,\\ (R_m)_2 &= 0.828551\,,\\ (R_m)_3 &= -0.096862\,.\end{aligned}$$

The three equilibrium solutions in this case thus are

$$\begin{aligned}(p_m)_1 &= -0.818394 & (p_m)_2 &= 0.828551 & (p_m)_3 &= -0.096862\,,\\ (p_n)_1 &= 2.16241\,i & (p_n)_2 &= -2.13878\,i & (p_n)_3 &= -0.0001011\,i\,.\end{aligned} \tag{5.14}$$

The eigenvalues of the equilibria are

- $\pm 8.63322i$ and $\pm 0.802148i$ for the first equilibrium, thus it is stable,
- $\pm 8.55971i$ and ± 0.79166 for the second equilibrium, thus it is unstable,
- $2.10846 \pm 0.037i$ and $-2.10846 \pm 0.037i$ for the third equilibrium, thus it is unstable.

For the case $\lambda \neq 0$ and/or $\lambda^* \neq 0$, the case $\lambda = 1$, $\lambda^* = 2$ is considered as an example . In this case, (5.11) becomes

$$\begin{aligned}&-0.24525 + 7.41137M - 75.257M^2 + 300.73M^3 - 533.405M^4\\ &\quad +366.142M^5 + 22.901M^6 - 94.9312M^7 + 7.45605M^8 + 6.91855M^9 = 0\,.\end{aligned}$$

The roots are

$$\begin{aligned}M_1 &= -3.27146\,,\\ M_2 &= -3.21595\,,\\ M_3 &= 0.49248\,,\\ M_{4,5} &= 0.08294 \pm 0.0047805\,i\,,\\ M_{6,7} &= 0.54305 \pm 0.0067989\,i\,,\\ M_{8,9} &= 1.83263 \pm 0.0439191\,i\,.\end{aligned}$$

Choose the correct sign to get

$$\begin{aligned}(R_m)_1 &= -0.785089\,,\\ (R_m)_2 &= 0.792981\,,\\ (R_m)_3 &= -0.066487\,.\end{aligned}$$

Thus the periodic solutions in this case are given by

$$\begin{array}{rcrcl} (p_m)_1 & = & -0.785089 & & e^{-i\tau}\,, \\ (p_n)_1 & = & 2.56839 & i & e^{2i\tau} \\ \text{or} & & & & \\ (p_m)_2 & = & 0.792981 & & e^{-i\tau}\,, \\ (p_n)_2 & = & -2.55019 & i & e^{2i\tau} \\ \text{or} & & & & \\ (p_m)_3 & = & -0.06649 & & e^{-i\tau}\,, \\ (p_n)_3 & = & -0.03274 & i & e^{2i\tau}\,. \end{array} \tag{5.15}$$

To find out about the stability of these periodic solutions, we consider the stability of the corresponding equilibria of the system for (P_m, P_n).

- The eigenvalues of the equilibrium corresponding to the first periodic orbit are $\pm 10.6949i$ and $\pm 4.82357i$, thus the first periodic orbit is stable.
- The eigenvalues of the equilibrium corresponding to the second periodic orbit are $\pm 10.6016i$ and $\pm 4.57503i$, thus the second periodic orbit is stable.
- The eigenvalues of the equilibrium corresponding to the third periodic orbit are $2.11772 \pm 0.01443i$ and $-2.11772 \pm 0.01443i$, thus the third periodic orbit is unstable.

5.3 Additional forcing for the system of Chapter 4

5.3.1 Modifications of the system

In a similar way as in the last section, an additional high-order forcing of frequency ω_n can be introduced. The modified forcing is assumed to be of the form

$$\begin{aligned} b_j(z,t,\tau) &= \epsilon^{\alpha_4} e^{i(\omega_m+\omega_n)t+i\lambda\tau} \tfrac{1}{i} f_j(z) + \epsilon^{\alpha_6} e^{i\omega_n t+i\lambda^*\tau} F_j(z) + c.c. \\ &=: b_{j,\omega_m+\omega_n} + b_{j,\omega_n} \,. \end{aligned} \tag{5.16}$$

Here, $b_{j,\omega_m+\omega_n}$ and b_{j,ω_n} are the components of frequencies $\omega_n+\omega_n$ and ω_n , respectively.

There are no changes to (4.6). Considering equations (4.7-4.9), b_j should be replaced with $b_{j,\omega_m+\omega_n}$. In addition to that, (4.9) needs to be updated with the new component of the forcing. The equations that change are now

$$\begin{aligned} \phi_{R,x^*} &= -\phi_{1,x^*x^*} b^*_{1,\omega_m+\omega_n} - \tfrac{1}{2}\phi_{1,x^*x^*x^*}(b^*_{1,\omega_m+\omega_n})^2 \\ &\quad + b^*_{1,\omega_m+\omega_n,z^*}\phi_{1,z^*} + b^*_{1,\omega_m+\omega_n,z^*}\phi_{1,z^*x^*} b^*_{1,\omega_m+\omega_n} + b^*_{1,\omega_n} \quad \text{at } x^*=0\,, \\ \phi_{R,x^*} &= -\phi_{1,x^*x^*} b^*_{2,\omega_m+\omega_n} - \tfrac{1}{2}\phi_{1,x^*x^*x^*}(b^*_{2,\omega_m+\omega_n})^2 \\ &\quad + b^*_{2,\omega_m+\omega_n,z^*}\phi_{1,z^*} + b^*_{2,\omega_m+\omega_n,z^*}\phi_{1,z^*x^*} b^*_{2,\omega_m+\omega_n} + b^*_{2,\omega_n} \quad \text{at } x^*=k_n a\,. \end{aligned} \tag{5.17}$$

The components of b^*_j are again given by $b^*_{j,\cdot} = k_n b_{j,\cdot}$ (j=1,2). If F_1 and F_2 are chosen to be constants A_1 and A_2, respectively, the updated differential equation for p_n is

$$S_1 p_n' + S_2|p_n|^2 p_n + S_3|p_m|^2 p_n + S_4 e^{i\lambda_1\tau}\overline{p_m} + S_5 e^{i\lambda_2\tau} = 0\,, \tag{5.18}$$

where S_1 through S_4 are given by (4.21) and

$$S_5 = ((-1)^n A_2 - A_1)\epsilon^{\alpha_6-\alpha_1} k_m \cosh(h^*)\,.$$

For S_5 to have the same order as S_1, it is required that

$$\alpha_6 = \alpha_3 + \alpha_5 + \frac{1}{2}\alpha_1 = \frac{14}{3}\alpha_2 - \frac{5}{3}\alpha_3\,.$$

For the number examples in 4.2.4 and 4.3.1, this would yield $\alpha_6 = \frac{22}{9}$ and $\alpha_6 = \frac{8}{3}$, respectively.

5.3.2 Approximation of solutions for forcing of frequency ω_n only and small ϵ

For simplification, we assume no forcing of frequency $\omega_m + \omega_n$ and use

$$\begin{aligned} p_m' &= i\epsilon_2^2 C_{12}|p_n|^2 p_m + i\epsilon_2^3 C_{11}|p_m|^2 p_m\,, \\ p_n' &= iC_{22}|p_m|^2 p_n + i\epsilon_2^2 C_{21}|p_n|^2 p_n + C_{24}e^{i\lambda^*\tau} \end{aligned} \tag{5.19}$$

with $C_{24} = O(1)$ and $\epsilon_2 = \epsilon^{\frac{4}{3}(\alpha_3-\alpha_2)}$ for simpler notation.

We set $\epsilon = 0$ and obtain the simpler equations

$$\begin{aligned} p_m' &= 0\,, \\ p_n' &= iC_{22}|p_m|^2 p_n + C_{24}e^{i\lambda^*\tau}\,. \end{aligned} \tag{5.20}$$

This can be solved explicitly:

$$\begin{aligned} p_m &= p_m(0)\,, \\ p_n &= \tfrac{C_{24}}{i(C_{22}|p_m(0)|^2-\lambda^*)}(e^{iC_{22}|p_m(0)|^2\tau} - e^{i\lambda^*\tau}) + p_n(0)e^{iC_{22}|p_m(0)|^2\tau}\,. \end{aligned} \tag{5.21}$$

In a first attempt to better approximate the system for $\epsilon \neq 0$, consider the system

$$\begin{aligned} p_m' &= i\,\epsilon_2^2\,C_{12}|p_n|^2 p_m\,, \\ p_n' &= i\,C_{22}|p_m|^2 p_n + C_{24}e^{i\lambda^*\tau}\,. \end{aligned} \tag{5.22}$$

Once $|p_n|$ is known, the first equation can be integrated to get

$$p_m = p_m(0)e^{i\epsilon_2^2 C_{12}\int_0^\tau |p_n(t)|^2 dt}\,. \tag{5.23}$$

Since the radius of p_m is still constant, p_n is still given by (5.21). In turn, we now just need to find $|p_n|^2$ for the solution in (5.21) and plug into (5.23) to get a solution for p_m.

Now consider the first equation of (5.19). An approximate solution is set up as

$$p_m = |p_m(0)|e^{i(\alpha_m(0)+\epsilon_2^2 C_{12}\int_0^\tau |p_n(t)|^2 dt+\mathcal{A}_m(\tau)}\,. \tag{5.24}$$

with the function $\mathcal{A}_m(\tau)$ to be determined.

Plugging this setup into the first equation yields

$$i|p_m(0)|\mathcal{A}_m'(\tau) = iC_{11}\epsilon_2^3|p_m(0)|^3\,.$$

Hence the solution which satisfies the initial condition $\mathcal{A}'_m(\tau) = 0$ is

$$\mathcal{A}_m(\tau) = C_{11}\epsilon_2^3|p_m(0)|^2\tau\,.$$

With this, p_m turns out to be

$$p_m = p_m(0)e^{i\left(\epsilon_2^2 C_{12}\int_0^\tau |p_n(t)|^2 dt + \epsilon_2^3 C_{11}|p_m(0)|^2\tau\right)}\,. \tag{5.25}$$

On the second equation, transformation (5.5) is used. The differential equation turns into

$$P'_n = i(C_{22}|p_m(0)|^2 - \lambda^*)P_n + i\epsilon_2^2 C_{21}|P_n|^2 P_n + C_{24}\,. \tag{5.26}$$

The improved approximation is set up as a perturbed solution of the form

$$P_n = -\frac{iC_{24}}{D_1} + e^{iD_1\tau + \epsilon_2^2\mathcal{A}_n(\tau)}D_2(1 + \epsilon_2^2\mathcal{R}_n(\tau))\,, \tag{5.27}$$

where

$$\begin{aligned} D_1 &= C_{22}|p_m(0)|^2 - \lambda^*\,, \\ D_2 &= \tfrac{iC_{24}}{D_1} + P_n(0)\,. \end{aligned}$$

To find $\mathcal{A}_n(\tau)$ and $\mathcal{R}_n(\tau)$, P_n is plugged into the differential equation, taking into account $|p_m(\tau)| = |p_m(0)|$. In order to find $|P_n|^2$, $P_n(\tau)$ is split up into its real and imaginary parts.

$$\begin{aligned} |P_n|^2 &= \Big((1 + \epsilon_2^2\mathcal{R}_n^2)\Big[Re[D_2]\cos(D_1\tau + \epsilon_2^2\mathcal{A}_n) - Im[D_2]\sin(D_1\tau + \epsilon_2^2\mathcal{A}_n)\Big]\Big)^2 \\ &\quad + \Big(\tfrac{C_{24}}{D_1} + (1 + \epsilon_2^2\mathcal{R}_n^2)\Big[Re[D_2]\sin(D_1\tau + \epsilon_2^2\mathcal{A}_n) + Im[D_2]\cos(D_1\tau + \epsilon_2^2\mathcal{A}_n)\Big]\Big)^2\,. \end{aligned}$$

Now, if P_n and $|P_n|^2$ are plugged into (5.26), the error terms are

$$\epsilon_2^2\left(C_{21}e^{-iD_1\tau}\Big[C_{24} - iD_1 e^{iD_1\tau}D_2\Big]^2\Big[C_{24}e^{iD_1\tau} + iD_1\overline{D_2}\Big] + iD_1^3 e^{iD_1\tau}D_2(\mathcal{A}'_n - i\mathcal{R}'_n)\right) + O(\epsilon_2^4)\,.$$

Collect the terms with ϵ_2^2 and set the real and imaginary parts of that equal to zero. The result is two differential equations in $\mathcal{A}_n(\tau)$ and $\mathcal{R}_n(\tau)$. The integration constants are chosen to satisfy the initial conditions $\mathcal{A}_n(0) = \mathcal{R}_n(0) = 0$. The solution then is

$$
\begin{aligned}
\mathcal{A}_n(\tau) = \ & \tfrac{C_{21}}{2D_1^4|D_2|^2}\Big(2C_{24}Re[D_2](C_{24}^2 + 2C_{24}D_1Im[D_2] + 3D_1^2|D_2|^2) \\
& \quad +2D_1^2|D_2|^2(2C_{24}^2 + D_1^2|D_2|^2)\tau \\
& \quad -2C_{24}[Re[D_2]\cos(D_1\tau) - Im[D_2]\sin(D_1\tau)] \\
& \quad [C_{24}^2 + 3D_1^2|D_2|^2 + C_{24}D_1(Im[D_2]\cos(D_1\tau) + Re[D_2]\sin(D_1\tau)]\Big) \\
\mathcal{R}_n(\tau) = \ & \tfrac{2C_{21}C_{24}}{D_1^4|D_2|^2}\sin(\tfrac{D_1}{2}\tau)\Big(Im[D_2]\sin(\tfrac{D_1}{2}\tau) - Re[D_2]\cos(\tfrac{D_1}{2}\tau)\Big) \\
& \Big(C_{24}^2 + C_{24}D_1Im[D_2] + D_1^2|D_2|^2 + C_{24}D_1(Im[D_2]\cos(\tfrac{D_1}{2}\tau) + Re[D_2]\sin(\tfrac{D_1}{2}\tau)\Big).
\end{aligned}
\tag{5.28}
$$

In [40], a slight variation of the equation for p_n is discussed. We replace C_{24} with $iC_{24}e^{-i\frac{\pi}{2}}$ in our original equation (5.26) and modify transformation (5.5) to $p_n = P_n e^{i(\frac{\pi}{2} - \lambda^*\tau)}$ to turn this into

$$
P_n' = i(C_{22}|p_m(0)|^2 - \lambda^*)P_n + i\epsilon_2^2 C_{21}|P_n|^2 P_n + iC_{24}\,.
$$

Then we just need to multiply by i to obtain the kind of differential equation discussed in Section 3.1 of [40], with parameters

$$
\begin{aligned}
\lambda_2 &= C_{22}|p_m(0)|^2 - \lambda^*\,, \\
c &= -\epsilon_2^2 C_{21}\,, \\
\beta &= -C_{24}\,.
\end{aligned}
$$

Setting $P_n = u + iv$, equilibria are given by

$$
v = 0, \qquad -\lambda_2 u + cu^3 = -\beta\,.
$$

Now, if $\beta < 0$, replace each of the parameters with its negative. This justifies the assumption $\beta > 0$. In [40] it is stated that for $\lambda_2 c < 0$, there is one equilibrium solution and periodic solutions only. Otherwise, there are three (real) solutions if $\beta < \beta^* = \frac{2}{3}\sqrt{\frac{\lambda_2^3}{3c}}$, one solution for $\beta > \beta^*$. Thus,β^* is a critical value of the forcing parameter β.

In our setting, it should be noted that, because c is small, β^* is large, unless λ_2 is small, which would mean $|p_m(0)^2| \approx \frac{\lambda^*}{C_{22}}$.

5.3.3 Numerical examples

For the following numerical example, the set of parameters is

$$
m = 1; a = 1; h = 1.0527; g = 9.81; n \text{ even}, \lambda^* = -1, \alpha_3 - \alpha_2 = \frac{1}{5}\,.
$$

The resulting system parameters are

$$\begin{aligned} C_{11} &= -62.2289\,, & C_{12} &= 8.7380\,, \\ C_{21} &= -0.5030\,, & C_{22} &= -0.2847\,. \end{aligned}$$

With the choice of $r_m = 1$, this means that

$$\lambda_2 = 0.7153\,, \qquad c = 0.0431\,.$$

The critical value is then $\beta^* = 1.1211$. Figures 5.1 and 5.2 show the radii, angles and orbits of p_m and p_n for $\beta = 0.9\beta^*$, while figures 5.3 and 5.4 show the same for $\beta = 1.1\beta^*$.

We observe strong differences especially in the amplitudes of p_n, which in turn causes differences in the angles of p_m. To understand the behavior at the critical value of the forcing parameter, the phase portraits of the equation for P_n are plotted. Figure 5.5 shows the phase portrait for $\beta = 0.9\beta^*$. Equilibria occur at around (-4.5,0), (1.7,0) and (2.9,0), with a homoclinic orbit connecting the third equilibrium which encloses a region around the second of these equilibria. In figure 5.6, only the equilibrium at (-4.5,0) is still there, it has just moved a bit to the left. The two other equilibria have disappeared, which causes major changes of the solution especially for initial conditions that used to be inside the homoclinic orbit seen in figure 5.5.

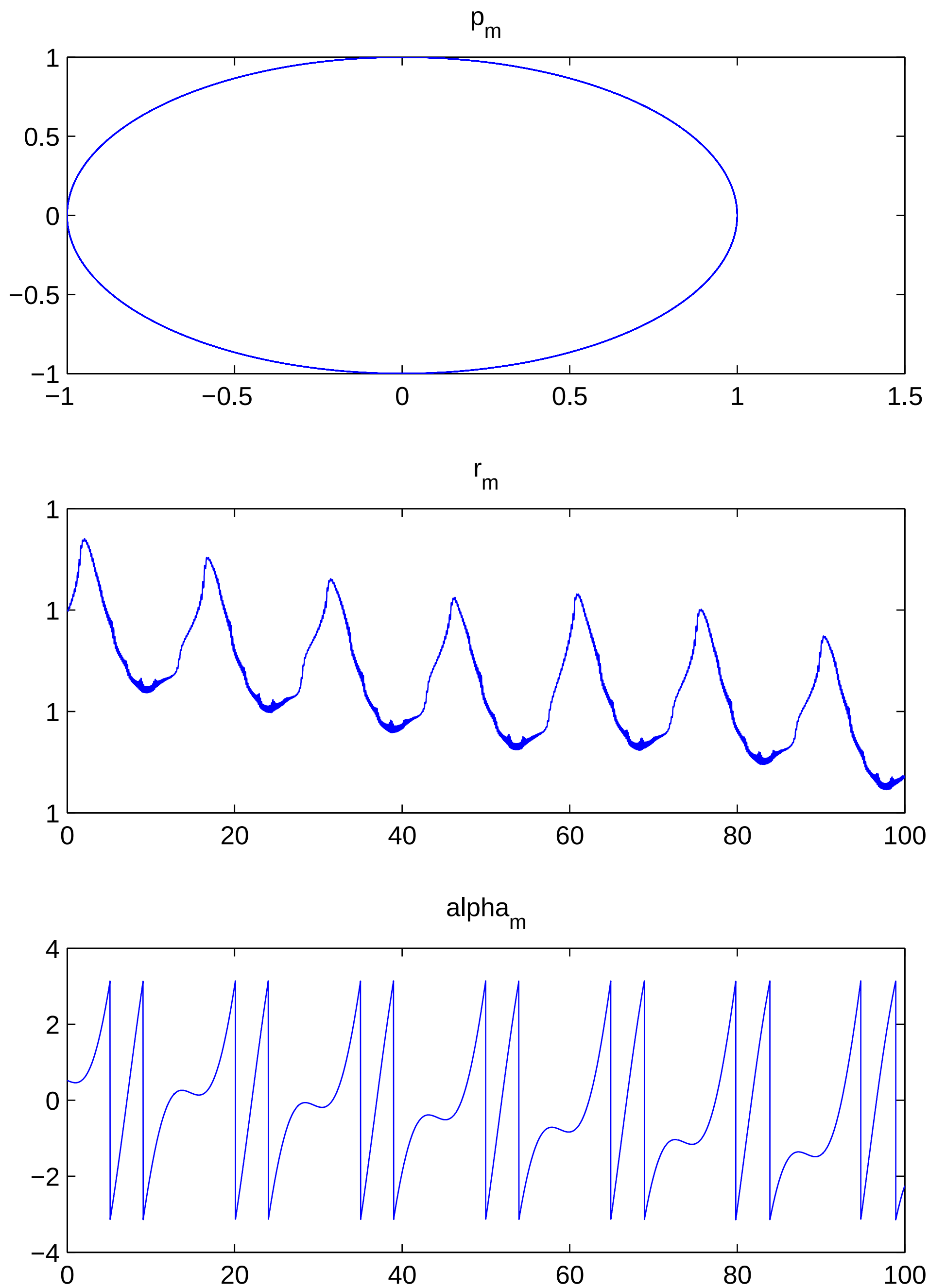

Figure 5.1: Radius, angle and orbit of p_m for $\beta = 0.9\beta^*$

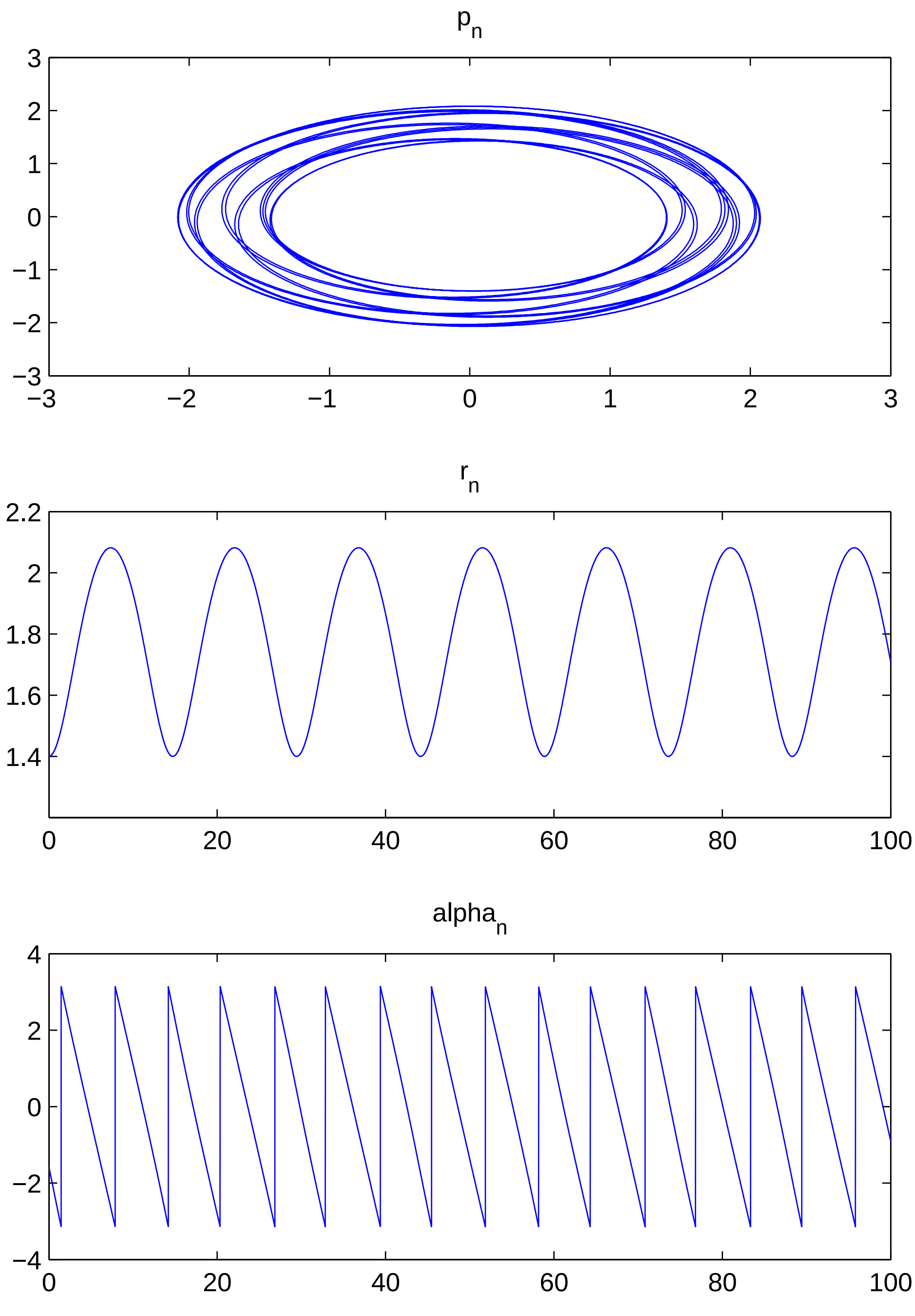

Figure 5.2: Radius, angle and orbit of p_n for $\beta = 0.9\beta^*$

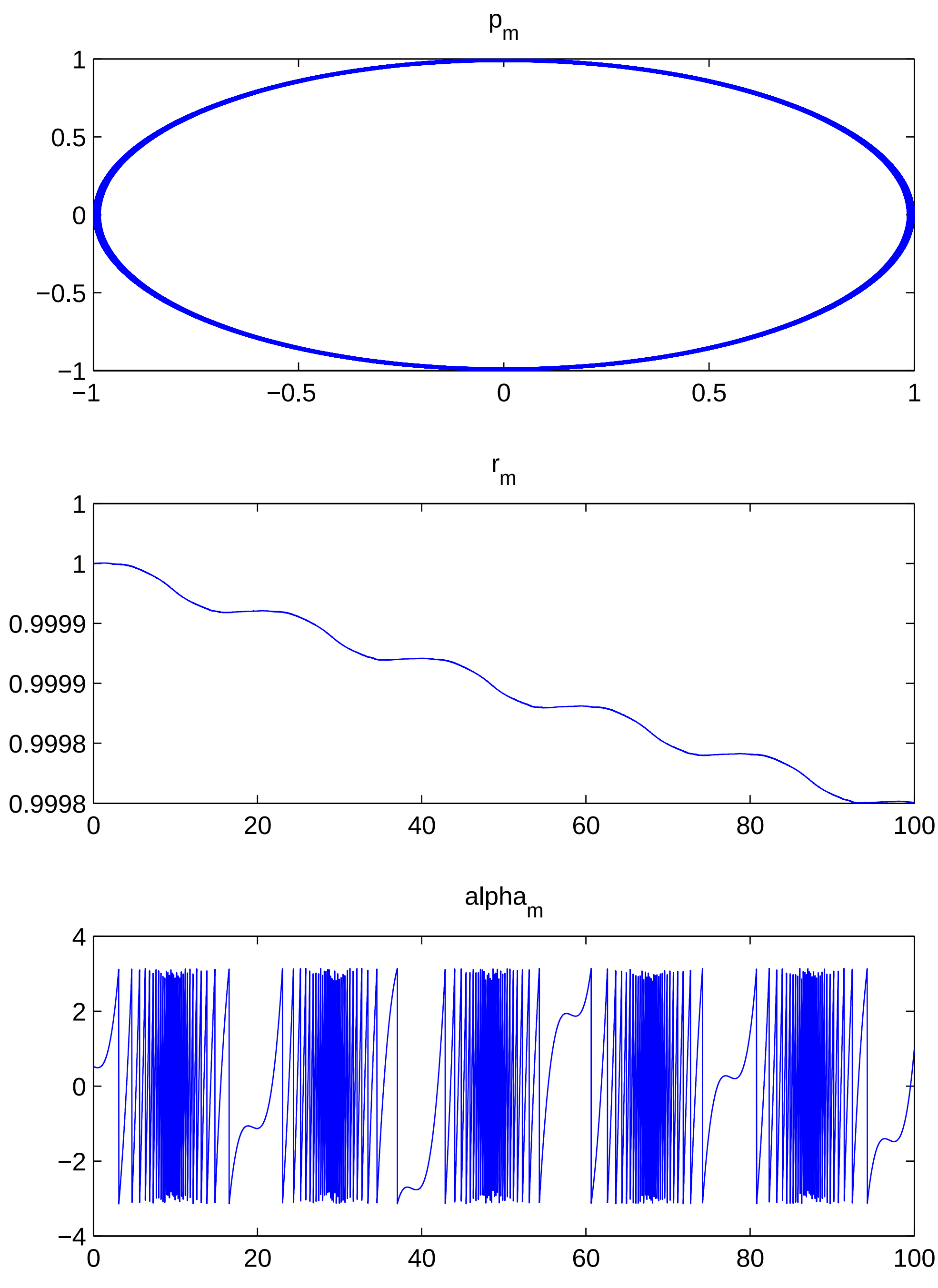

Figure 5.3: Radius, angle and orbit of p_m for $\beta = 1.1\beta^*$

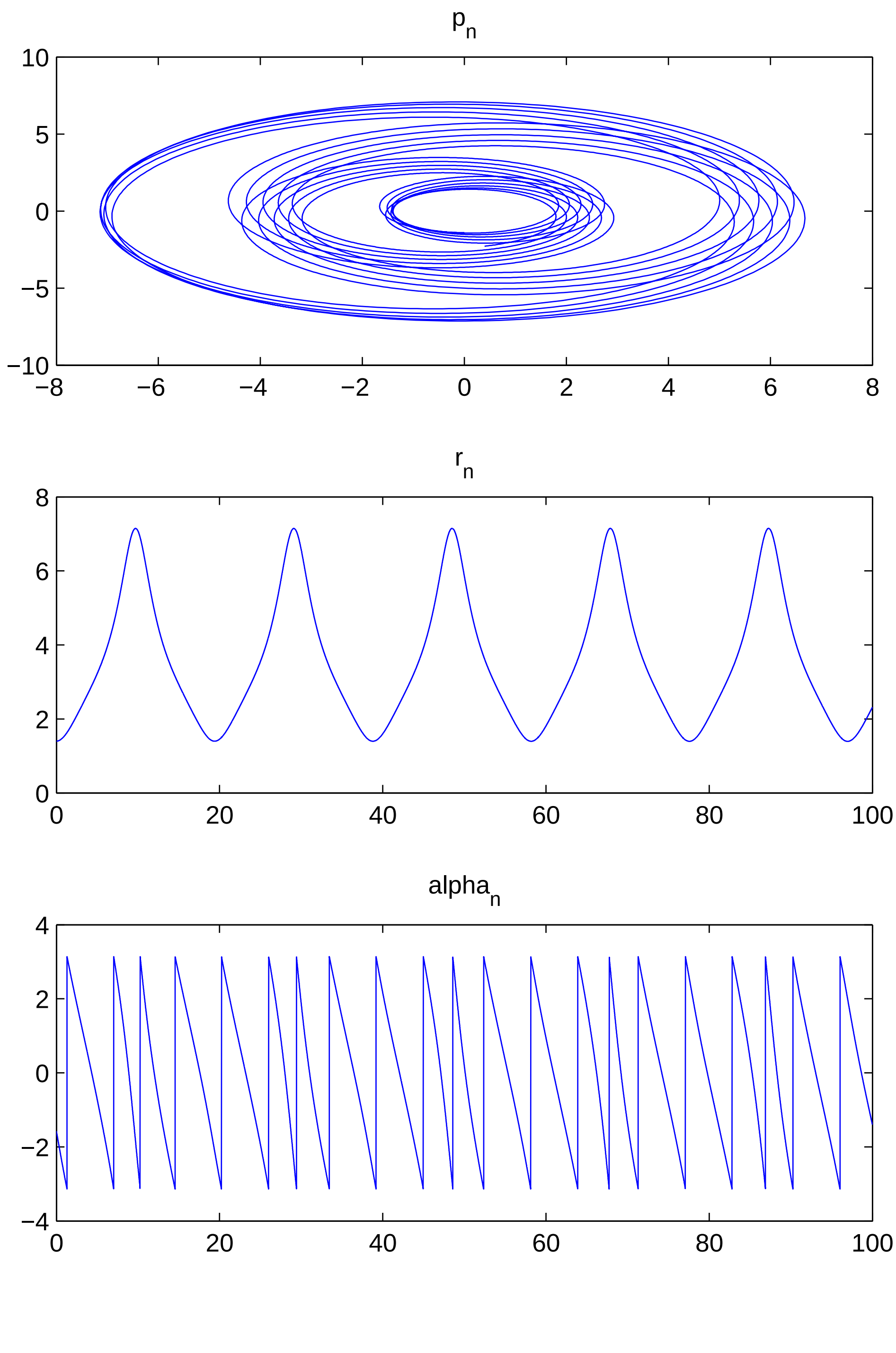

Figure 5.4: Radius, angle and orbit of p_n for $\beta = 1.1\beta^*$

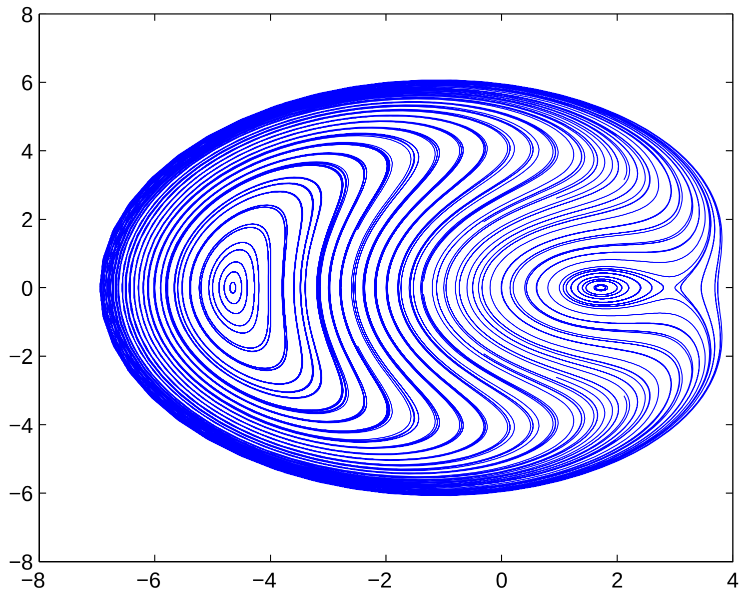

Figure 5.5: Phase portrait for $\beta = 0.9\beta^*$

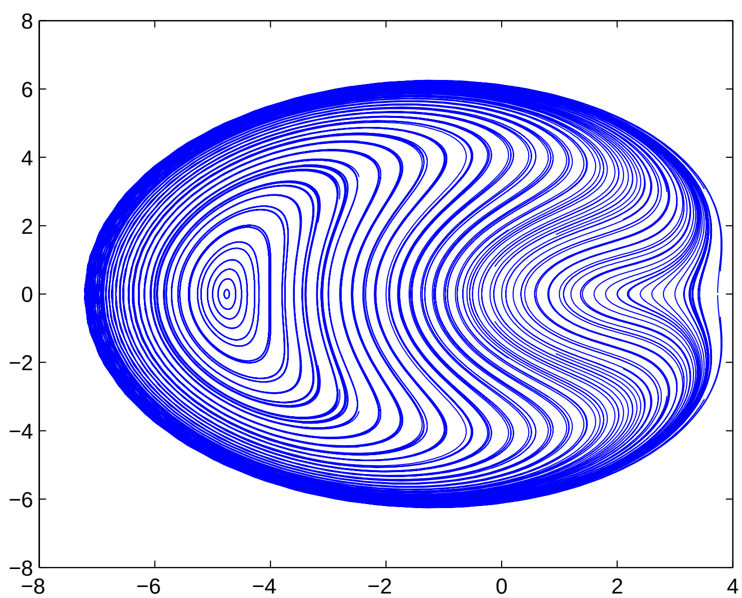

Figure 5.6: Phase portrait for $\beta = 1.1\beta^*$

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