The $Ph_t/Ph_t/s/c$ Queueing Model and Approximation

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(ABSTRACT)

Time-dependent queueing models are important since most of real-life problems are time-dependent. We develop a numerical approximation algorithm for the mean, variance and higher-order moments of the number of entities in the system at time t for the $Ph_t/Ph_t/s/c$ queueing model. This model can be thought as a reparameterization to the $G_t/GI_t/s$. Our approach is to partition the state space into known and identifiable structures, such as the $M_t/M_t/s/c$ or $M_t/M_t/1$ queueing models. We then use the Polya-Eggenberger distribution to approximate certain unknown probabilities via a two-moment matching algorithm. We describe the necessary steps to validate the approximation and measure the accuracy of the model.

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Chapter 1

Introduction

We study the $Ph_t/Ph_t/s/c$ queueing model and build computational approximation algorithms. The $Ph_t/Ph_t/s/c$ model is the time-dependent generalization of the $Ph/Ph/s/c$, which itself is a generalization of the familiar $M/M/s/c$ basic queueing model. Here the Ph notation refers to the "phase-type" distributions/processes, and Ph_t refers to their time-dependent generalizations. Phase-type random variables represent the time-till-absorption for finite-state Markov processes having exactly one absorbing state [9].

The phase-type distribution family has the attractive property of being a "dense" family of distributions. Denseness is a distributional property defined as the ability of the family of distributions to approximate any distribution shape arbitrarily closely [5]. Thus use of the Ph_t process in the $Ph_t/Ph_t/s/c$ queueing model can be thought of as an attractive computational time-dependent generalization of the $G/GI/s/c$ model. These models are useful in a wide variety of application settings.

Evaluating any specific instance of the $Ph_t/Ph_t/s/c$ model can, of course, be accomplished via numerical integration of the Kolmogorov forward equations. The problem of dimensionality prevents this approach from being practical for all but the smallest of problem instances.

As will be shown, the number of Kolmogorov forward equations for the $Ph_t/Ph_t/s/c$ model is greater than $c\, m_A\, m_B + m_A$, where m_A and m_B represent the number of phases in the arrival and service time dependent-distributions, respectively.

In our approximation we decompose the state space into $2\, m_A\, m_B$ subspaces and then use a two-moment matching algorithm to approximate the random process behavior in each subspace of the system. Our approach is a type of *closure* approximation that has been used successfully in a number of different types of Ph-based queueing models [3] and [4].

In the next Chapter we describe the basic closure-type queueing approximation and introduce the necessary notation. In Chapter 3 we describe the development of the necessary Partial Moment Differential Equations and the approximations. We present the actual algorithm in Chapter 4. Then we present actual results and validation of the model in Chapter 5. Finally the Virtual Waiting time and the conclusion are presented in Chapters 6 and 7 respectively.

Chapter 2

Background

First we review some basic results for some elementary time-dependent Markovian queueing models. We look at the basic models, and discuss practical methods for evaluating their time-dependent behavior.

2.1 The $M_t/M_t/\infty$ queueing model

The $M_t/M_t/\infty$ queueing model clearly has an infinite number of states. In order to evaluate time-dependent performance measures for this model we would need to evaluate an infinite set of differential difference equations called the Kolmogorov forward equations (KFEs). Those equations are:

$$\begin{aligned} d\mathrm{P}\left(N(t)=i\right)/dt &= -\left(\lambda_t + i\mu_t\right)\mathrm{P}\left(N(t)=i\right) \\ &\quad +\lambda_t \mathrm{P}\left(N(t)=i-1\right)\delta_{[i\,>\,0]} \\ &\quad +\mu_t\left(i+1\right)\mathrm{P}\left(N(t)=i+1\right) \end{aligned} \tag{2.1}$$

for $i = 0, \cdots$

where

$$\delta_{[i>0]} = \begin{cases} 0, & i \leq 0 \\ 1, & i > 0 \end{cases}$$

Thus for even this simple model we are seemingly faced with an infinite amount of computational work.

If we are primarily interested in time-dependent moments of the number of entities in the system (such as mean and variance) we can develop an alternative approach that is more computationally feasible.

Consider the time-dependent expected number of entities in the system:

$$\mathrm{E}\left[N(t)\right] \equiv \sum_{i=0}^{\infty} i\mathrm{P}\left(N(t)=i\right)$$

and thus

$$\begin{aligned} d\mathrm{E}\left[N(t)\right]/dt &\equiv d\sum_{i=0}^{\infty} i\mathrm{P}\left(N(t)=i\right)/dt \\ &= \sum_{i=0}^{\infty} i\, d\mathrm{P}\left(N(t)=i\right)/dt \end{aligned}$$

or in a tidier form

$$\mathrm{E}'\left[N(t)\right] \equiv \sum_{i=0}^{\infty} iP_i'\left(t\right).$$

Where: $P_i(t) \equiv \mathrm{P}(N(t) = i)$ and $\mathrm{E}'[N(t)] \equiv d\mathrm{E}[N(t)]/dt$

After some algebraic manipulation this simplifies to

$$d\mathrm{E}[N(t)]/dt = \lambda_t - \mu_t \mathrm{E}[N(t)] \tag{2.2}$$

Notice that this single differential equation can be evaluated numerically and thus, given $\mathrm{E}[N(0)]$, we have the entire time-dependent path of $\mathrm{E}[N(t)]$. So we have the time-trajectory of the desired performance measure without having done an "infinite" amount of work; we evaluated just *one* differential equation. If we had observed terms on the right-hand-side (RHS) of this moment-differential equation (MDE) that included state probabilities, such as $P_i(t)$, then we would have needed to also numerically integrate the KFE for this state probability. Then of course we would have needed to numerically integrate the KFEs for states $N(t) = i + 1$ and $N(t) = i - 1$, etc. and in fact we would have needed to numerically integrate all of the KFEs. The fact that (2.2) has no probabilities on the RHS is called a "closure" property. Since (2.2) is closed we can avoid an infinite amount of work and evaluate the time-dependent performance measure in just one differential equation. This MDE-closure approach is one of the important constructs we will make use of in developing approximations for the more complex $Ph_t/Ph_t/s/c$ model.

We should note that for the simple $M/M/\infty$ model it is well known that the equilibrium state distribution is a Poisson distribution having mean λ/μ, and the $M_t/M_t/\infty$ has a time-dependent Poisson distribution. We include this brief discussion of this model because it is the simplest case and we build upon insights gained from evaluating this model. For instance we can interpret the positive term in the RHS of the MDE as the input rate to the system, and the negative term in the RHS as the system departure rate (the service rate per server times the expected number of busy servers). This interpretation of the positive and negative flux in this MDE will have natural extensions in the more complex cases.

2.2 The $M_t/M_t/1$ model

The MDE for the $M_t/M_t/1$ queueing model can be developed in a manner analogous to the development of the MDE for the $M_t/M_t/\infty$ model [4]. The resulting MDE is:

$$\mathrm{E}'\left[N\left(t\right)\right] = \lambda_t - \mu_t\left[1 - P_0\left(t\right)\right] \tag{2.3}$$

This MDE is *not* closed. For this MDE to be evaluated numerically we would need to have a value for $P_0\left(t\right)$ available for all values of t. And of course in order to have values of $P_0\left(t\right)$ available we would need to numerically evaluate all of the KFEs.

Taaffe and Ong [4] developed an approximation for the $M_t/M_t/1/c$. In their approximation algorithm, for any value of t, they approximate $P_0\left(t\right)$ with $\mathrm{P}\left(X = 0\right)$, where X is a Polya-Eggenberger (PE) integer-valued random variable having range $(0, \cdots, c)$ and its first two moments matched to the values of $\mathrm{E}\left[N(t)\right]$, and $\mathrm{E}\left[N^2(t)\right]$, respectively.

2.2.1 Polya-Eggenberger

The Polya-Eggenberger (PE) distribution is a distribution based on urn model with stochastic replacement[2]. In one urn there is w white balls and b black balls. Then in the first trial a ball is drawn, then for the second trial this ball is replaced with s more balls of the same color. This procedure is repeated n times. In general s does not need to be integer; i.e., it can be a real number. The probability mass function is

$$\mathrm{P}\left(X=i\right)=\binom{n}{i}\frac{b\left(b+s\right)\cdots\left(b+\left(i-1\right)s\right)w\left(w+s\right)\cdots\left(w+\left(n-i-1\right)s\right)}{\left(b+w\right)\left(b+w+s\right)\cdots\left(b+w+\left(n-1\right)s\right)},$$

where $\mathrm{P}\left(X=i\right)$ is the probability of drawing i black balls after n trials.

Then

$$\mathrm{P}\left(X=i\right)=\binom{n}{i}\frac{\left(\prod_{j=0}^{i-1}\left(\theta+j\gamma\right)\right)\left(\prod_{j=0}^{n-i-1}\left(\left(1-\theta\right)+j\gamma\right)\right)}{\left(\prod_{j=0}^{n-1}\left(1+j\gamma\right)\right)},$$

where $\theta=\frac{b}{b+w}$ and $\gamma=\frac{s}{b+w}$.

Then it can be proven that the first and second moments for the PE are

$$\mathrm{E}\left[X\right]=n\,\theta,$$

and

$$\mathrm{E}\left[X^2\right]=\frac{n\,\theta\left(n\,\left(\theta+\gamma\right)+\left(1-\theta\right)\right)}{\left(1+\gamma\right)},$$

respectively.

One advantage of the PE distribution is that some limiting cases are known distributions.

1. If $b, w \to \infty$ with $b(b+w)^{-1} \to \bar{p}$ where $0 < \bar{p} < 1$, and n is fixed, then

$$\mathrm{P}\left(X = i\right) \to \binom{n}{i} \bar{p}^{i} \left(1 - \bar{p}\right)^{n-i}$$

   $i = 0, 1, \cdots, n.$

   So the number of black draws out of n is Binomial with parameter $\bar{p}$.

2. If $b, w \to \infty$ with $b(b+w)^{-1} \to 0$ and $n \to \infty$ with $nb(b+w)^{-1} \to \bar{\lambda}$ where $0 < \bar{\lambda} < \infty$ and $(ns)^{-1}(b+w) \to \bar{\gamma}(0 < \bar{\gamma} < \infty)$, then

$$\mathrm{P}\left(X = i\right) \to \binom{\bar{\lambda}\bar{\gamma} + i - 1}{i} \left(\frac{\bar{\gamma}}{1 + \bar{\gamma}}\right)^{\bar{\lambda}\bar{\gamma}} \left(\frac{1}{1 + \bar{\gamma}}\right)^{i}$$

   $i = 0, 1, \cdots$

   So the number of black draws out of n is Negative Binomial with parameters $\bar{\lambda}\bar{\gamma}$ and $(1 + \bar{\gamma})^{-1}$.

3. There are three ways that PE distributions can converge to the Poisson distribution with parameter $\bar{\lambda}$.

$$\mathrm{P}\left(X = i\right) \to \frac{\bar{\lambda}^{j}}{j!} e^{-\bar{\lambda}}$$

   $j = 0, 1, 2, \cdots$:

   (a) If $n \to \infty$ with $n^{-2}s^{-1}(b+w) \to \bar{\theta}$ where $0 < \bar{\theta} < \infty$ and $nb(b+w)^{-1} \to \bar{\lambda}$ where $0 < \lambda < \infty$.

   (b) If $n \to \infty$ with $n^{-2}s^{-1}(b+w) \to 0, n^{-3}s^{-2}(b+w)^{2} \to \infty$ and $nb(b+w)^{-1} \to \bar{\lambda}$ where $0 < \lambda < \infty$.

(c) If $n \to \infty$ with $n^{-2}s^{-1}(b+w) \to \infty, n^{-3}s^{-1}(b+w) \to 0$ and $nb(b+w)^{-1} \to \bar{\lambda}$ where $0 < \lambda < \infty$.

4. Other special cases for the PE distribution are when $b = w = s$, it becomes a discrete rectangular distribution and if $s = -1$ it becomes a hypergeometric distribution.

2.2.2 Approximation Algorithm

The moment-matching approximation approach is called a *closure* approximation. The MDE together with this closure approximation is referred to as a pseudo-closed MDE.

The steps of the algorithm can be summarized as follows:

1. Set initial conditions $\mathrm{E}\left[N\left(0\right)\right]$ and $\mathrm{E}\left[N^2\left(0\right)\right]$ to $N\left(0\right)$ and $N^2\left(0\right)$, respectively.

2. At time t, match $\mathrm{E}\left[N\left(t\right)\right]$ and $\mathrm{E}\left[N^2\left(t\right)\right]$ to the first and second moments of a PE having support on $(0, \cdots, c)$, respectively.

3. Approximate the probability $\mathrm{P}\left[N\left(t\right)=0\right]$ with $\mathrm{P}\left(X=0\right)$ where X has a PE distribution with range $(0, \cdots, c)$ and first two moments $\mathrm{E}\left[N\left(t\right)\right]$, $\mathrm{E}\left[N^2\left(t\right)\right]$.

4. Numerically integrate the first and second MDEs using the approximated probability from time t to $t + \Delta t$.

5. Set $t = t + \Delta t$ and go to Step 2.

2.3 The $M/M/s/c$ queueing model and approximation

We next consider $M/M/s/c$ queueing model. Here we see that the steady-state distribution of the number of entities in the system has two functional forms [7].

$$P_i = \left\{ \begin{array}{ll} \frac{(\lambda/\mu)^i}{i!} P_0 & , \quad 0 \leq i < s \\ \\ \frac{s^s \lambda}{(s\mu)^i s!} P_0 & , \quad s \leq i \leq c \end{array} \right\} \tag{2.4}$$

where

$$P_0 = \left\{ \begin{array}{ll} \left(\sum_{i=0}^{s-1} \frac{(\lambda/\mu)^i}{i!} + \frac{(\lambda/\mu)^s}{s!} \frac{1-\left(\frac{\lambda}{c\mu}\right)^{c-s+1}}{1-\left(\frac{\lambda}{c\mu}\right)} \right)^{-1} & , \quad \frac{\lambda}{c\mu} \neq 1 \\ \\ \left(\sum_{i=0}^{s-1} \frac{(\lambda/\mu)^i}{i!} + \frac{(\lambda/\mu)^s}{s!} (c-s+1) \right)^{-1} & , \quad \frac{\lambda}{c\mu} = 1 \end{array} \right\} \tag{2.5}$$

This structure in the steady-state distribution suggests a natural partitioning of the state space into two disjoint subspaces. The state space, Ω can be written as

$$\Omega \equiv \{0, \cdots, s-1\} \bigcup \{s, \cdots, c\}$$

Observe that the first part of (2.4), the steady-state distribution for the first subspace, can be viewed as a truncated Poisson distribution; i.e., the steady-state distribution for the $M/M/s/s$ model having service rate μ and arrival rate λ. And observe that the second part of (2.4), the steady-state distribution for the second subspace, can be viewed as a shifted, truncated-geometric distribution (shift the support of the distribution to $(0, \cdots, c-s)$ from $(s, \cdots, c)$, and thus truncate at $c-s$); i.e., the steady-state distribution for the $M/M/1/c-s$

model where the service rate is $s\mu$ and the arrival rate is λ. The structure of the steady-state distribution (2.4) and (2.5) suggests an approach to approximating the time-dependent generalization of this model.

2.4 The $M_t/M_t/s/c$ queueing model and approximation

Rothkopf and Oren [8], Clark [1], and others have approximated this model using subspace-partitioning and moment-matching approach. We generalize the state-space partitioning, moment-matching, and closure approach to approximate the time-dependent behavior of the $Ph_t/Ph_t/s/c$.

2.5 The $Ph_t/Ph_t/1$ queueing model and approximation

Consider a queuing model having phase-type distributions for both the arrival and service times.

2.5.1 Phase-type Arrival Process

The phase-type nonstationary process is represented by its underlying Markov chain. Let $\{A(t);t \geq 0\}$ be the random process representing the arrival phase of the next arrival to the system at time t, where $A(t) \in \{1, \cdots, m_A\}$. The instantaneous absorbing state need not be explicitly represented. Let

$$\mathcal{A}(t) = \left\{ \begin{array}{cc} \mathcal{A}_1(t) & \mathcal{A}_2(t) \\ \boldsymbol{\alpha}(t)^T & 0 \end{array} \right\}$$

where

$$\mathcal{A}_1(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1m_A}(t) \\ \vdots & \ddots & \vdots \\ a_{m_A 1}(t) & \cdots & a_{m_A m_A}(t) \end{pmatrix}$$

is the matrix of the one-step transition matrix for the transient-to-transient state transitions, and

$$\mathcal{A}_2(t) = \begin{pmatrix} a_{1,m_A+1}(t) \\ \vdots \\ a_{m_A,m_A+1}(t) \end{pmatrix}$$

represents the vector transition probabilities from transient states to the absorbing state. The vector $\boldsymbol{\alpha}(t) = [\alpha_1(t), \cdots, \alpha_{m_A}(t)]^T$ represents the initial arrival-phase probabilities for the next entity. Let $\boldsymbol{\lambda}(t) = [\lambda_1(t), \cdots, \lambda_{m_A}(t)]^T$ be the vector of real-valued integrable rate functions for transient states of the arrival process. Then, the vector for the entire arrival process is $[\boldsymbol{\lambda}(t), \infty]$.

2.5.2 Phase-type Service Process

Similarly, let $\{B(t); t \geq 0\}$ be the random process representing the service phase of the next service to the system at time t, where $B(t) \in \{1, \cdots, m_B\}$. The instantaneous absorbing state need not be explicitly represented. Let

$$\mathcal{B}(t) = \left\{ \begin{array}{cc} \mathcal{B}_1(t) & \mathcal{B}_2(t) \\ \boldsymbol{\beta}(t)^T & 0 \end{array} \right\}$$

where

$$\mathcal{B}_1(t) = \begin{pmatrix} b_{11}(t) & \cdots & b_{1m_B}(t) \\ \vdots & \ddots & \vdots \\ b_{m_B 1}(t) & \cdots & b_{m_B m_B}(t) \end{pmatrix}$$

is the matrix of the one-step transition matrix for the transient-to-transient state transitions, and

$$\mathcal{B}_2(t) = \begin{pmatrix} b_{1,m_B+1}(t) \\ \vdots \\ b_{m_B,m_B+1}(t) \end{pmatrix}$$

represents the vector transition probabilities from transient states to the absorbing state. The vector $\boldsymbol{\beta}(t) = [\beta_1(t), \cdots, \beta_{m_B}(t)]^T$ represents the initial service-phase probabilities for the next entity. Let $\boldsymbol{\mu}(t) = [\mu_1(t), \cdots, \mu_{m_B}(t)]^T$ be the vector of real-valued integrable rate functions for transient states of the service process. Then, the vector for the entire service process is $[\boldsymbol{\mu}(t), \infty]$.

2.5.3 Approximation Algorithm

Taaffe and Ong [4] considered this model and developed an approximation for it. Their approach was to again partition the state space in $(m_A m_B + 1)$ subspaces (where m_A is the number of phases in the arrival process and m_B is the number of phases in the service process); one subspace for each combination of arrival and service phases. This partitioning scheme resulted in the following set of MDEs:

$$\begin{aligned} d\mathrm{E}_{i,j}^{(p)}(t)/dt &= -\left[\lambda_i(t)+\mu_j(t)\right]\mathrm{E}_{i,j}^{(p)}(t) \\ &+\sum_{k=1}^{m_A} a_{k,m_A+1}(t)\,\alpha_i(t)\,\lambda_k(t)\,\beta_j(t)\,P_{0,k,0}(t) \\ &+\sum_{k=1}^{m_A} a_{k,m_A+1}(t)\,\alpha_i(t)\,\lambda_k(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}_{k,j}^{(q)}(t) \\ &-\sum_{k=1}^{m_A} a_{k,m_A+1}(t)\,\alpha_i(t)\,\lambda_k(t)\,P_{c,i,j}(t)\sum_{q=0}^{p-1}\binom{p}{q}c^q \\ &+\sum_{k=1}^{m_A} a_{ki}(t)\,\lambda_k(t)\,\mathrm{E}_{k,j}^{(p)}(t)+\sum_{k=1}^{m_B} b_{kj}(t)\,\mu_k(t)\,\mathrm{E}_{i,k}^{(p)}(t) \\ &+\sum_{k=1}^{m_B} b_{k,m_B+1}(t)\,\beta_j(t)\,\mu_k(t)\sum_{q=0}^{p}\binom{p}{q}(-1)^{p-q}\,\mathrm{E}_{i,k}^{(q)}(t) \\ &-\sum_{k=1}^{m_B} b_{k,m_B+1}(t)\,\beta_j(t)\,\mu_k(t)\,P_{1,i,k}(t) \end{aligned} \tag{2.6}$$

where

$$\mathrm{E}_{i,j}^{(p)}(t) \equiv \sum_{n=0}^{c} n^p \mathrm{P}\left(N(t)=n, A(t)=i, B(t)=j\right),$$

$$P_{n,i,j}(t) \equiv \mathrm{P}\left(N(t)=n, A(t)=i, B(t)=j\right)$$

and $N(t)$ is the number of entities in the system at time t, $A(t)$ is the phase of the arrival process at time t, and $B(t)$ is the phase of service of the entity being served at time t or 0 if there is no entity in service.

Now using this set of MDEs, the approximation algorithm follows:

1. Initialize $\mathrm{E}\left[N(0)\right]$ and $\mathrm{E}^2\left[N(0)\right]$ to $N(0)$ and $N^2(0)$, respectively.

2. At time t, match $\mathrm{E}_{i,j}^{(1)}(t)/\mathrm{E}_{i,j}^{(0)}(t)$ and $\mathrm{E}_{i,j}^{(2)}(t)/\mathrm{E}_{i,j}^{(0)}(t)$ to the first two moments of $PE(i,j;t)$ for every pair (i,j) where $PE(i,j;t)$ is a Polya-Eggenberger distribution, $i=1,\cdots,m_A$ and $j=1,\cdots,m_B$.

3. Approximate the $P_{1,i,j}(t)$ and $P_{c,i,j}(t)$ by $X = 1$ in the $PE(i, j; t)$ distribution times $\mathrm{E}^{(0)}_{i,j}(t)$ and the probability of $X = c$ in the same distribution times $\mathrm{E}^{(0)}_{i,j}(t)$, respectively.

4. Simultaneously numerically integrate the MDEs for $p = 0, 1, 2$ and the KFEs corresponding to states $(0, i, 0)$ (the empty states) for $i = 1, \cdots, m_A$ from time t to time $t + \Delta t$.

5. Sum the partial moments to obtain the actual first two moments of the number of entities in the system.

6. Set $t = t + \Delta t$ and go to Step (2).

Notice that in this approximation algorithm the individual MDEs are *partial* MDEs (PMDEs); i.e., the summation does not cover the full support of the distribution. The PMDEs for this queueing model are not closed since there are unknown terms on the RHS. If the unknown terms for one particular subspace are found in another subspace and the union of the PMDEs (with the union of the few KFEs representing an empty system) are closed, then we say that the PMDEs are quasi-closed. Thus the combination of the PMDEs, a few KFEs (for the states representing an empty system), and the approximations for $P_{1,i,j}(t)$ and $P_{c,i,j}(t)$ form a pseudo-closed set.

Observe that we only have, at most, one entity in service at any point of time. We next consider the model that is the subject of this proposal, the $Ph_t/Ph_t/s/c$ model, and the associated proposed approximation.

Chapter 3

The $Ph_t/Ph_t/s/c$ Queueing Model and Approximation

The $Ph_t/Ph_t/s/c$ model is described in detail in this section and the proposed approximation algorithm is laid out. The approximation is a combination of the decomposition methods, pseudo-closure, and quasi-closure approaches described above for the simpler models.

First we look at the KFEs for the model:

3.1 Kolmogorov forward equations

The state space, Ω for this model can be partitioned into two disjoint subspaces as follows:

Let $\Omega \equiv \Omega_1 \bigcup \Omega_2$, where

$$\begin{aligned}
\Omega_1 &= \left\{ (\mathbf{N}(t) = \mathbf{n}, A(t) = \ell, Q(t) = 0) \mid 0 \le \textstyle\sum_{i=1}^{m_B} N_i(t) < s, \ell = 1, 2, \cdots, m_A \right\}, \text{and} \\
\Omega_2 &= \left\{ (\mathbf{N}(t) = \mathbf{n}, A(t) = \ell, Q(t) = q) \mid \textstyle\sum_{i=1}^{m_B} N_i(t) = s, \ell = 1, 2, \cdots, m_A, q = 0, \cdots, c - s \right\}
\end{aligned}$$

where $\mathbf{N}(t) \equiv (N_1(t), \cdots, N_{m_B}(t))$. Let

$$P_{n_1 \cdots n_{m_B}; \ell, 0}(t) \equiv \mathrm{P}\left(N_1(t) = n_1, \cdots, N_{m_B}(t) = n_{m_B}, A(t) = \ell, Q(t) = 0\right)$$

for $N(t) \le s - 1$, where $N(t) \equiv \sum_{j=1}^{m_B} N_j(t) + Q(t)$, and

$$P_{n_1 \cdots n_{m_B}; \ell, q}(t) \equiv \mathrm{P}\left(N_1(t) = n_1, \cdots, N_{m_B}(t) = n_{m_B}, A(t) = \ell, Q(t) = q\right)$$

for $N(t) = s$, $N_j(t)$ is the number of entities in phase j of their service at time t, $A(t)$ is the phase of the arrival process at time t, and $Q(t)$ is defined as the number of entities waiting in the queue. Notice that observing that $Q(t) = 0$ is not sufficient to determine which of the two subspaces the process is in.

The resulting KFEs for subspace Ω_1 are:

$$\begin{aligned}
P_{n_1 \cdots n_{m_B}; \ell, 0}(t)' = & -\lambda_\ell(t)(1 - a_{\ell\ell}) P_{n_1 \cdots n_{m_B}; \ell, 0}(t) - \sum_{i=1}^{m_B} n_i \mu_i(t) [1 - b_{ii}(t)] P_{n_1 \cdots n_{m_B}; \ell, 0}(t) \\
& + \sum_{\substack{i=1 \\ i \ne \ell}}^{m_A} a_{i\ell}(t) \lambda_i(t) P_{n_1 \cdots n_{m_B}; i, 0}(t) \\
& + \sum_{i=1}^{m_B} \delta_{[n_i > 0]} \sum_{\substack{j=1 \\ j \ne i}}^{m_B} b_{ji}(t) [n_j + 1] \mu_j(t) P_{n_1 \cdots n_i - 1 \cdots n_j + 1 \cdots n_{m_B}; \ell, 0}(t) \\
& + \sum_{i=1}^{m_A} a_{i, m_A + 1}(t) \alpha_\ell(t) \lambda_i(t) \left\{ \sum_{j=1}^{m_B} \delta_{[n_j > 0]} \beta_j(t) P_{n_1 \cdots n_j - 1 \cdots n_{m_B}; i, 0}(t) \right\} \\
& + \sum_{i=1}^{m_B} b_{i, m_B + 1}(t) [n_i + 1] \mu_i(t) P_{n_1 \cdots n_i + 1 \cdots n_{m_B}; \ell, 0}(t)
\end{aligned} \qquad (3.1)$$

for $0 \leq n_i \leq s-1$; $i = 1, \cdots, m_B$; $0 \leq \sum_{j=1}^{m_B} n_j \leq s-1$; $\ell = 1, \cdots, m_A$ and $t \geq 0$.

where

n_i represents the number of entities in the specific phase i of the service

ℓ represents the phase in which the arrival process is in .

The final subscript represents the number in the queue, which for partition 1 is 0.

The KFEs for Ω_1 can be simplified to:

$$\begin{aligned}
P_{n_1 \cdots n_{m_B};\ell,0}(t)' \quad = \quad & -\lambda_\ell(t) P_{n_1 \cdots n_{m_B};\ell,0}(t) - \sum_{i=1}^{m_B} n_i \mu_i(t) [1 - b_{ii}(t)] P_{n_1 \cdots n_{m_B};\ell,0}(t) \\
& + \sum_{i=1}^{m_A} a_{i\ell}(t) \lambda_i(t) P_{n_1 \cdots n_{m_B};i,0}(t) \\
& + \sum_{i=1}^{m_B} \delta_{[n_i>0]} \sum_{\substack{j=1 \\ j \neq i}}^{m_B} b_{ji}(t) [n_j + 1] \mu_j(t) P_{n_1 \cdots n_i - 1 \cdots n_j + 1 \cdots n_{m_B};\ell,0}(t) \\
& + \sum_{i=1}^{m_A} a_{i,m_A+1}(t) \alpha_\ell(t) \lambda_i(t) \left\{ \sum_{j=1}^{m_B} \delta_{[n_j>0]} \beta_j(t) P_{n_1 \cdots n_j - 1 \cdots n_{m_B};i,0}(t) \right\} \\
& + \sum_{i=1}^{m_B} b_{i,m_B+1}(t) [n_i + 1] \mu_i(t) P_{n_1 \cdots n_i + 1 \cdots n_{m_B};\ell,0}(t)
\end{aligned} \tag{3.2}$$

for $0 \leq n_i \leq s-1$; $i = 1, \cdots, m_B$; $0 \leq \sum_{j=1}^{m_B} n_j \leq s-1$; $\ell = 1, \cdots, m_A$ and $t \geq 0$.

The KFEs for subspace Ω_2 are:

$$
\begin{aligned}
P_{n_1 \cdots n_{m_B};\ell,q}(t)' \quad = \quad & -\lambda_\ell(t) \sum_{\substack{i=1 \\ i \neq \ell}}^{m_A} a_{\ell i}(t) P_{n_1 \cdots n_{m_B};\ell,q}(t) - \delta_{[q<c-s]} \lambda_\ell(t) a_{\ell,m_A+1} P_{n_1 \cdots n_{m_B};\ell,q}(t) \\
& - \left(1 - \delta_{[q<c-s]}\right) \lambda_\ell(t) a_{\ell,m_A+1}(t) \left(1 - \alpha_\ell(t)\right) P_{n_1 \ldots n_{m_B};\ell,q}(t) \\
& - \sum_{i=1}^{m_B} n_i \mu_i(t) [1 - b_{ii}(t)] P_{n_1 \cdots n_{m_B};\ell,q}(t) \\
& + \sum_{\substack{i=1 \\ i \neq \ell}}^{m_A} a_{i\ell}(t) \lambda_i(t) P_{n_1 \cdots n_{m_B};i,q}(t) \\
& + \sum_{i=1}^{m_B} \delta_{[n_i>0]} \sum_{\substack{j=1 \\ j \neq i}}^{m_B} b_{ji}(t) \delta_{[n_j<s]} [n_j + 1] \mu_j(t) P_{n_1 \cdots n_i - 1 \cdots n_j + 1 \cdots n_{m_B};\ell,q}(t) \\
& + \sum_{i=1}^{m_A} a_{i,m_A+1}(t) \alpha_\ell(t) \lambda_i(t) \Bigg\{ \left(1 - \delta_{[q>0]}\right) \sum_{j=1}^{m_B} \delta_{[n_j>0]} \beta_j(t) \\
& \quad P_{n_1 \cdots n_j - 1 \cdots n_{m_B};i,q}(t) + \delta_{[q>0]} P_{n_1 \cdots n_{m_B};i,q-1}(t) \Bigg\} \\
& + \left(1 - \delta_{[q<c-s]}\right) \sum_{\substack{i=1 \\ i \neq \ell}}^{m_A} a_{i,m_A+1}(t) \alpha_\ell(t) \lambda_i(t) P_{n_1 \ldots n_{m_B};i,q}(t) \\
& + \delta_{[q<c-s]} \sum_{i=1}^{m_B} \sum_{\substack{j=1 \\ j \neq i}}^{m_B} \delta_{[n_j>0]} b_{i,m_B+1}(t) \beta_j(t) \delta_{[n_i<s]} [n_i + 1] \mu_i(t) \\
& \quad P_{n_1 \cdots n_i + 1 \cdots n_j - 1 \cdots n_{m_B};\ell,q+1}(t) \\
& + \delta_{[q<c-s]} \sum_{i=1}^{m_B} b_{i,m_B+1}(t) \beta_i(t) n_i \mu_i(t) P_{n_1 \cdots n_{m_B};\ell,q+1}(t)
\end{aligned}
\tag{3.3}
$$

for $0 \leq n_i \leq s$; $i = 1, \cdots, m_B$; $0 \leq \sum_{j=1}^{m_B} n_j = s$; $\ell = 1, \cdots, m_A$; $q = 0, \cdots, c - s$ and $t \geq 0$.

where

$$
\delta_{[i<j]} \equiv \begin{cases} 0, & i \geq j \\ 1, & i < j \end{cases}
$$

In the same manner the KFEs can be simplified to:

$$
\begin{aligned}
P_{n_1\cdots n_{m_B};\ell,q}(t)' \;=\;& -\lambda_\ell(t)\,P_{n_1\cdots n_{m_B};\ell,q}(t) \\
& -\sum_{i=1}^{m_B} n_i\mu_i(t)\,[1-b_{ii}(t)]P_{n_1\cdots n_{m_B};\ell,q}(t) \\
& +\sum_{i=1}^{m_A} a_{i\ell}(t)\,\lambda_i(t)\,P_{n_1\cdots n_{m_B};i,q}(t) \\
& +\sum_{i=1}^{m_B}\delta_{[n_i>0]}\sum_{\substack{j=1\\ j\neq i}}^{m_B} b_{ji}(t)\,\delta_{[n_j<s]}\,[n_j+1]\,\mu_j(t)\,P_{n_1\cdots n_i-1\cdots n_j+1\cdots n_{m_B};\ell,q}(t) \\
& +\sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\left\{\left(1-\delta_{[q>0]}\right)\sum_{j=1}^{m_B}\delta_{[n_j>0]}\beta_j(t)\right. \\
& \left. P_{n_1\cdots n_j-1\cdots n_{m_B};i,q}(t)+\delta_{[q>0]}P_{n_1\cdots n_{m_B};i,q-1}(t)\right\} \\
& +\left(1-\delta_{[q<c-s]}\right)\sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\,P_{n_1\ldots n_{m_B};i,q}(t) \\
& +\delta_{[q<c-s]}\sum_{i=1}^{m_B}\sum_{\substack{j=1\\ j\neq i}}^{m_B}\delta_{[n_j>0]}b_{i,m_B+1}(t)\,\beta_j(t)\,\delta_{[n_i<s]}\,[n_i+1]\,\mu_i(t) \\
& P_{n_1\cdots n_i+1\cdots n_j-1\cdots n_{m_B};\ell,q+1}(t) \\
& +\delta_{[q<c-s]}\sum_{i=1}^{m_B} b_{i,m_B+1}(t)\,\beta_i(t)\,n_i\mu_i(t)\,P_{n_1\cdots n_{m_B};\ell,q+1}(t)
\end{aligned}
\tag{3.4}
$$

for $0\leq n_i\leq s$; $i=1,\cdots,m_B$; $0\leq\sum_{j=1}^{m_B} n_j = s$; $\ell=1,\cdots,m_A$; $q=0,\cdots,c-s$ and $t\geq 0$.

The numbers of KFEs for example systems are illustrated in Table 3.1.

Table 3.1: Number of KFEs as a function of s and c.

Number of Service Phases (m_B)	Number of KFEs
1	$(c+1)\, m_A$
2	$\left(\frac{s^2}{2}+\frac{3s}{2}+1+(c-s)\,(s+1)\right) m_A$
3	$\left(\frac{s^3}{6}+s^2+\frac{11s}{6}+1+(c-s)\left(\frac{s^2}{2}+\frac{3s}{2}+1\right)\right) m_A$
4	$\left(\frac{s^4}{24}+\frac{5s^3}{12}+\frac{35s^2}{24}+\frac{25s}{12}+1+(c-s)\left(\frac{s^3}{6}+s^2+\frac{11s}{6}+1\right)\right) m_A$
5	$\left(\frac{s^5}{120}+\frac{s^4}{8}+\frac{17s^3}{24}+\frac{15s^2}{8}+\frac{137s}{60}+1+(c-s)\left(\frac{s^4}{24}+\frac{5s^3}{12}+\frac{35s^2}{24}+\frac{25s}{12}\right)\right) m_A$

Table 3.2: Number of KFEs.

m_B	s	Number of KFEs
2	3	$(4c-2)m_A$
2	5	$(6c-9)m_A$
2	10	$(11c-44)m_A$
3	3	$(10c-10)m_A$
3	5	$(21c-49)m_A$
3	10	$(66c-374)m_A$
5	3	$(35c-49)m_A$
5	5	$(126c-378)m_A$
5	10	$(1001c-7007)m_A$

3.2 The Moment Differential Equations for the $Ph_t/Ph_t/s/c$ queueing model

For each of the subspaces a PMDE can be developed in a manner analogous (but more tedious) to the development of the MDE for the simple $M_t/M_t/\infty$ model.

3.2.1 The First PMDES

Define:

$$\mathrm{E}'\left[N_i\left(t\right), \ell, 1\right] \equiv \mathrm{E}'\left[N_i\left(t\right), A\left(t\right) = \ell, I\left(t\right) = 1\right]$$

$$\mathrm{E}'\left[N_i\left(t\right), A\left(t\right) = \ell, I\left(t\right) = 1\right] \equiv \sum_{n_1=0}^{s-1} \sum_{n_2=0}^{s-1-n_1} \cdots \sum_{n_{m_B}=0}^{s-1-n_1-\cdots-n_{m_B-1}} n_i P'_{n_1 \cdots n_{m_B};\ell,0}\left(t\right)$$

Theorem 1 *For Ω_1 the first PMDE*

$$\begin{aligned}
\mathrm{E}'\left[N_i(t), \ell, 1\right] &= -\lambda_\ell(t)\,\mathrm{E}\left[N_i(t), \ell, 1\right] \\
&+ \sum_{j=1}^{m_A} a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t), j, 1\right] \\
&- \mu_i(t)\,\mathrm{E}\left[N_i(t), \ell, 1\right] \\
&+ \sum_{j=1}^{m_B} b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_j(t), \ell, 1\right] \\
&+ \sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t), j, 1\right] \\
&+ \sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\,\mathrm{P}\left(I(t)=1, A(t)=j\right) \\
&- \sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t), N(t)=s-1, A(t)=j\right] \\
&- \sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\,\mathrm{P}\left(N(t)=s-1, A(t)=j\right) \\
&- b_{i,m_B+1}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t), N(t)=s, A(t)=\ell\right] \\
&+ \sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t), N(t)=s, A(t)=\ell\right]
\end{aligned}$$

and, for Ω_2 the first PMDE

$$\begin{aligned}
\mathrm{E}'\left[N(t), \ell, 2\right] &= -\lambda_\ell(t)\,\mathrm{E}\left[N(t), \ell, 2\right] \\
&+ \sum_{i=1}^{m_A} a_{i\ell}(t)\,\lambda_i(t)\,\mathrm{E}\left[N(t), i, 2\right] \\
&+ \sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\,sP\left(N(t)=s-1, A(t)=i\right) \\
&+ \sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\,\mathrm{E}\left[N(t), i, 2\right] \\
&+ \sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\,P\left(I(t)=2, A(t)=i\right) \\
&- \sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\,P\left(N(t)=c, A(t)=i\right) \\
&- \sum_{i=1}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t), I(t)=2, A(t)=\ell\right] \\
&- \sum_{i=1}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\,(s-1)\,\mathrm{E}\left[N_i(t), N(t)=s, A(t)=\ell\right]
\end{aligned}$$

for $i = 1, \cdots, m_B$; $\ell = 1, \cdots, m_A$ and $t \geq 0$.

where

$$I(t) = \begin{pmatrix} 1, & 0 \leq \sum_{i=1}^{m_B} N_i(t) \leq s-1 \\ 2, & \sum_{i=1}^{m_B} N_i(t) = s \end{pmatrix}$$

and $\mathrm{P}(I(t) = 1, A(t) = i) \equiv \sum_{n_1=0}^{s-1} \sum_{n_2=0}^{s-1-n_1} \cdots \sum_{n_{m_B}=0}^{s-1-n_1-\cdots-n_{m_B-1}} P_{n_1 \cdots n_{m_B};\ell,0}(t)$.

Proof: See Appendix 1.

Clearly, the first PMDEs for the two subspaces are not closed. We develop the zero PMDEs for the two subspaces.

3.2.2 The Zero PMDES

Theorem 2 *For Ω_1 the zero PMDE*

$$\begin{aligned} P'(I(t) = 1, A(t) = \ell) &= -\lambda_\ell(t) P(I(t) = 1, A(t) = \ell) \\ &+ \sum_{i=1}^{m_A} a_{i\ell}(t) \lambda_i(t) P(I(t) = 1, A(t) = i) \\ &+ \sum_{i=1}^{m_A} a_{i,m_A+1}(t) \alpha_\ell(t) \lambda_i(t) P(I(t) = 1, A(t) = i) \\ &- \sum_{i=1}^{m_A} a_{i,m_A+1}(t) \alpha_\ell(t) \lambda_i(t) P(N(t) = s-1, A(t) = i) \\ &+ \sum_{i=1}^{m_B} b_{i,m_B+1}(t) \mu_i(t) \mathrm{E}[N_i(t), N(t) = s, A(t) = \ell] \end{aligned}$$

and, for Ω_2 the zero PMDE

$$\begin{aligned}
P'\left(I\left(t\right)=2,A\left(t\right)=\ell\right) \;=\; & -\lambda_{\ell}\left(t\right)P\left(I\left(t\right)=2,A\left(t\right)=\ell\right)\\
&+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)P\left(I\left(t\right)=2,A\left(t\right)=i\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)P\left(N\left(t\right)=s-1,A\left(t\right)=i\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)P\left(I\left(t\right)=2,A\left(t\right)=i\right)\\
&-\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\mathrm{E}\left[N_i\left(t\right),N\left(t\right)=s,A\left(t\right)=\ell\right]
\end{aligned}$$

Proof: see Appendix 2.

Again, these zero PMDEs are not closed. We develop the necessary approximations for the probabilities and *moments* that are necessary in order to have pseudo-closure.

3.2.3 The *pth* PMDES

In order to match the first and second moments of the system to the PE distributions. We develop the *pth* moment PMDEs for both subspaces.

Theorem 3 *For Ω_1 the pth PMDE*

$$\begin{aligned}
\mathrm{E}'\left[N_i^p(t),\ell,1\right] &= -\lambda_\ell(t)\,\mathrm{E}\left[N_i^p(t),\ell,1\right] \\
&+\sum_{j=1}^{m_A} a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i^p(t),j,1\right] \\
&+\sum_{\substack{j=1\\ j\neq i}}^{m_B} b_{ji}(t)\,\mu_j(t)\sum_{h=1}^{p}\binom{p}{h}\mathrm{E}\left[N_i^{p-h}(t)\,N_j(t),\ell,1\right] \\
&+\left(1-b_{ii}(t)\right)\mu_i(t)\sum_{h=0}^{p-1}\binom{p}{h+1}(-1)^{h+1}\,\mathrm{E}\left[N_i^{p-h}(t),\ell,1\right] \\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i^p(t),j,1\right] \\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\sum_{h=1}^{p}\binom{p}{h}\mathrm{E}\left[N_i^{p-h}(t),j,1\right] \\
&-\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\sum_{h=1}^{p}\binom{p}{h}\mathrm{E}\left[N_i^{p-h}(t),N(t)=s-1,A(t)=j\right] \\
&-\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i^p(t),N(t)=s-1,A(t)=j\right] \\
&+b_{i,m_B+1}(t)\,\mu_i(t)\sum_{h=0}^{p-1}\binom{p}{h+1}(-1)^{h+1}\,\mathrm{E}\left[N_i^{p-h}(t),N(t)=s,A(t)=\ell\right] \\
&+\sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\mathrm{E}\left[N_i^p(t)\,N_j(t),N(t)=s,A(t)=\ell\right]
\end{aligned}$$

and, for Ω_2 the pth PMDE

$$\begin{aligned}
\mathrm{E}'\left[N^p(t),\ell,2\right] &= -\lambda_\ell(t)\,\mathrm{E}\left[N^p(t),\ell,2\right] \\
&+\sum_{i=1}^{m_A} a_{i\ell}(t)\,\lambda_i(t)\,\mathrm{E}\left[N^p(t),i,2\right] \\
&+\sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\,s^pP\left(N(t)=s-1,A(t)=i\right) \\
&+\sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\sum_{h=0}^{p}\binom{p}{h}\mathrm{E}\left[N^{p-h}(t),i,2\right] \\
&-\sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\sum_{h=1}^{p}\binom{p}{h}c^{p-h}P\left(N(t)=c,A(t)=i\right) \\
&+\sum_{i=1}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\sum_{h=1}^{p}\binom{p}{h}(-1)^h\,\mathrm{E}\left[N_i(t)\,N^{p-h}(t),I(t)=2,A(t)=\ell\right] \\
&-\sum_{i=1}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\,(s-1)^p\,\mathrm{E}\left[N_i(t),N(t)=s,A(t)=\ell\right]
\end{aligned}$$

Proof: see Appendix 3.

The first cross product PMDEs are necessary to evaluate the Variance.

3.2.4 The cross-product PMDES

Theorem 4 *For Ω_1 the first cross product PMDE*

$$
\begin{aligned}
&\mathrm{E}'\left[N_i(t)\,N_j(t),\ell,1\right] = -\lambda_\ell(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right]\\
&-\mu_i(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right]\\
&-\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right]\\
&+\sum_{k=1}^{m_A} a_{k\ell}(t)\,\lambda_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),k,1\right]\\
&-b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_j(t),\ell,1\right]\\
&-b_{ij}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t),\ell,1\right]\\
&+\sum_{k=1}^{m_B} b_{ki}(t)\,\mu_k(t)\,\mathrm{E}\left[N_j(t)\,N_k(t),\ell,1\right]\\
&+\sum_{k=1}^{m_B} b_{kj}(t)\,\mu_k(t)\,\mathrm{E}\left[N_i(t)\,N_k(t),\ell,1\right]\\
&+\sum_{k=1}^{m_A} a_{k,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),k,1\right]\\
&+\sum_{k=1}^{m_A} a_{k,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\left(\beta_i(t)\,\mathrm{E}\left[N_j(t),k,1\right]+\beta_j(t)\,\mathrm{E}\left[N_i(t),k,1\right]\right)\\
&-\sum_{k=1}^{m_A} a_{k,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),N(t)=s-1,A(t)=k\right]\\
&-\sum_{k=1}^{m_A} a_{k,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_i(t)\,\mathrm{E}\left[N_j(t),N(t)=s-1,A(t)=k\right]\\
&-\sum_{k=1}^{m_A} a_{k,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_j(t)\,\mathrm{E}\left[N_i(t),N(t)=s-1,A(t)=k\right]\\
&-\left(b_{i,m_B+1}(t)\,\mu_i(t)+b_{j,m_B+1}(t)\,\mu_j(t)\right)E\left[N_i(t)\,N_j(t),N(t)=s,A(t)=\ell\right]\\
&+\sum_{k=1}^{m_B} b_{k,m_B+1}(t)\,\mu_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t)\,N_k(t),N(t)=s,A(t)=\ell\right]
\end{aligned}
$$

and, for Ω_2 the first cross product PMDE

$$E'\left[N_i(t)\,N(t),\ell,2\right] = -\lambda_\ell(t)\,\mathrm{E}\left[N_i(t)\,N(t),\ell,2\right]$$

$$-\mu_i(t)\,\mathrm{E}\left[N_i(t)\,N(t),\ell,2\right]$$

$$+\sum_{j=1}^{m_A} a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t)\,N(t),j,2\right]$$

$$+\sum_{j=1}^{m_B} b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_j(t)\,N(t),\ell,2\right]$$

$$+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,s\mathrm{E}\left[N_i(t),N(t)=s-1,A(t)=j\right]$$

$$+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\,sP\left(N(t)=s-1,A(t)=j\right)$$

$$+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t)\,N(t),j,2\right]$$

$$+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t),I(t)=2,A(t)=j\right]$$

$$-\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t),N(t)=c,A(t)=j\right]$$

$$+b_{i,m_B+1}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t),I(t)=2,A(t)=\ell\right]$$

$$+\sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\,\mathrm{E}\left[N_j(t)\,N(t),\ell,2\right]$$

$$-\sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),I(t)=2,A(t)=\ell\right]$$

$$-\sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\,\mathrm{E}\left[N_j(t),I(t)=2,A(t)=\ell\right]$$

$$-\sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\,(s-1)\,\mathrm{E}\left[N_j(t),N(t)=s,A(t)=\ell\right]$$

$$-\sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,(s-1)\,\mathrm{E}\left[N_i(t)\,N_j(t),N(t)=s,A(t)=\ell\right]$$

$$+b_{i,m_B+1}(t)\,\mu_i(t)\,(s-1)\,\mathrm{E}\left[N_i(t),N(t)=s,A(t)=\ell\right]$$

Proof: see Appendix 4.

3.3 The Approximations

The PMDEs have expressions on the RHS which need to be approximated, and by doing so we have a pseudo-closure type approximation.

The probabilities and conditional moments that are necessary to approximate are:

$$P\left(N\left(t\right)=s-1,A\left(t\right)=\ell\right)$$

$$P\left(N\left(t\right)=s,A\left(t\right)=\ell\right)$$

$$P\left(N\left(t\right)=c,A\left(t\right)=\ell\right)$$

$$\mathrm{E}\left[N_i^p\left(t\right),N\left(t\right)=k,A\left(t\right)=\ell\right]$$

$$\mathrm{E}\left[N_i^p\left(t\right)N_j,N\left(t\right)=k,A\left(t\right)=\ell\right]$$

$$\mathrm{E}\left[N_i^p\left(t\right),I\left(t\right)=2,A\left(t\right)=\ell\right]$$

$$\mathrm{E}\left[N_i\left(t\right)N_j,I\left(t\right)=2,A\left(t\right)=\ell\right]$$

for $i=1,\cdots,m_B$, $j=1,\cdots,m_B$, $\ell=1,\cdots,m_A$, $p=1,2,3$ and $k=s-1,s,c$.

3.3.1 The Probabilities

We approximate the probabilities as follows:

Match $\mathrm{E}\left[N\left(t\right),A\left(t\right)=\ell,I\left(t\right)=2\right]/\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=\ell\right)$ and $\mathrm{E}\left[N^2\left(t\right),A\left(t\right)=\ell,I\left(t\right)=2\right]/\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=\ell\right)$ to the first and second moment of a shifted $PE\left(n,\ell,2\right)\equiv\mathrm{P}\left[N\left(t\right)=n,A\left(t\right)=\ell,I\left(t\right)=2\right]$, respectively.

Approximate $\mathrm{P}\left[N\left(t\right)=s,A\left(t\right)=\ell,I\left(t\right)=2\right]$ and $\mathrm{P}\left[N\left(t\right)=c,A\left(t\right)=\ell,I\left(t\right)=2\right]$ with $PE\left(s,\ell,2\right)$ and $PE\left(c,\ell,2\right)$, respectively.

Match $\mathrm{E}\left[N\left(t\right),A\left(t\right)=\ell,I\left(t\right)=1\right]/\mathrm{P}\left(I\left(t\right)=1,A\left(t\right)=\ell\right)$ and $\mathrm{E}\left[N^2\left(t\right),A\left(t\right)=\ell,I\left(t\right)=1\right]/\mathrm{P}\left(I\left(t\right)=1,A\left(t\right)=\ell\right)$ to the first and second moment of a $PE\left(n,\ell,1\right)\equiv\mathrm{P}\left[N\left(t\right)=n,A\left(t\right)=\ell,I\left(t\right)=1\right]$, respectively.

Approximate $\mathrm{P}\left[N\left(t\right)=s,A\left(t\right)=\ell,I\left(t\right)=2\right]$ with $PE\left(s-1,\ell,1\right)$.

3.3.2 The Conditional Moments

Now we consider the conditional Moments necessary to approximate.

Since these probabilities are not the desired probabilities; i.e. the probability $\mathrm{P}\left[N\left(t\right)=s,A\left(t\right)=\ell,I\left(t\right)=2\right]$ represents the probability that the total number of entities in the system is s, but it does not specify in which phase of service each of these entities are in at time t, we use the *multinomial distribution* (MN) in order to approximate and specify to which phase of service the entities are in at time t. It is not necessary to find each probability since what we need are the partial moments of the MN. The parameters for approximating the MN are:

1. $p_i\equiv$ the probability that a particular entity is in service phase i given that the entity is in some phase of service,

where

$$p_i\left(t\right) = \frac{\mathrm{E}\left[N_i\left(t\right), I\left(t\right) = 1\right]}{\mathrm{E}\left[N\left(t\right), I\left(t\right) = 1\right]}$$

and $i = 0, \cdots, m_B$.

2. The number of trials for the MN are the specific number of entities being served in the system, $\{s-1, s\}$.

If we consider that the system is only in subspace Ω_2, then the parameters for the MN are $p_i\left(t\right) = \beta_i\left(t\right)$.

For example:

$$\mathrm{E}\left[N_i\left(t\right), N\left(t\right) = s, A\left(t\right) = \ell\right] \approx \mathrm{E}\left[N_i\left(t\right) | N\left(t\right) = s, A\left(t\right) = \ell\right] PE\left(s, \ell, 2\right) \mathrm{P}\left(I\left(t\right) = 2, A\left(t\right) = \ell\right)$$

where

$$\mathrm{E}\left[N_i\left(t\right) | N\left(t\right) = s, A\left(t\right) = \ell\right] = sp_i\left(t\right).$$

Chapter 4

The Algorithm

The algorithm is as follows:

1. Initialize $\mathrm{E}\left[N\left(0\right)\right]$ and $\mathrm{E}^2\left[N\left(0\right)\right]$ to $N\left(0\right)$ and $N^2\left(0\right)$, respectively. Introduce the necessary conditions whether the system is in the subspace Ω_1 or Ω_2 for the initialization of each of the different PMDEs.

2. At time t, match $\mathrm{E}\left[N\left(t\right), I\left(t\right) = 1, A\left(t\right) = \ell\right] / \mathrm{P}\left(I\left(t\right) = 1, A\left(t\right) = \ell\right)$ and $\mathrm{E}\left[N^2\left(t\right), I\left(t\right) = 1, A\left(t\right) = \ell\right] / \mathrm{P}\left(I\left(t\right) = 1, A\left(t\right) = \ell\right)$ to the first two moments of $PE(\ell, 1; t)$ for every ℓ, $\ell = 1, \cdots, m_A$, where $PE(\ell, 1; t)$ is a Polya-Eggenberger distribution with support $0, \cdots, s-1$.

3. At time t, match $\mathrm{E}\left[N\left(t\right), I\left(t\right) = 2, A\left(t\right) = \ell\right] / \mathrm{P}\left(I\left(t\right) = 2, A\left(t\right) = \ell\right) - s$ and $E\left[N^2\left(t\right), I\left(t\right) = 2, A\left(t\right) = \ell\right] / \mathrm{P}\left(I\left(t\right) = 2, A\left(t\right) = \ell\right)$

$- 2\mathrm{E}\left[N\left(t\right), I\left(t\right) = 2, A\left(t\right) = \ell\right] / \mathrm{P}\left(I\left(t\right) = 2, A\left(t\right) = \ell\right) + s^2$ to the first two moments of $PE(\ell, 2; t)$ for every ℓ, $\ell = 1, \cdots, m_A$, where $PE(\ell, 2; t)$ is a Polya-Eggenberger distribution with support $0, \cdots, c - s$.

4. Compute the parameters $p_i(t)$ from $\mathrm{E}\left[N_i\left(t\right), I\left(t\right) = 1\right] / \mathrm{E}\left[N\left(t\right), I\left(t\right) = 1\right]$ or the vector $\beta(t)$ for $i = 1, \cdots, m_B$.

5. Approximate the $\mathrm{P}\left(N\left(t\right) = s - 1, A\left(t\right) = \ell\right)$, $\mathrm{P}\left(N\left(t\right) = s, A\left(t\right) = \ell\right)$ and $\mathrm{P}\left(N\left(t\right) = c, A\left(t\right) = \ell\right)$ with $PE\left(s - 1, \ell, 1\right) P\left(I\left(t\right) = 1, A\left(t\right) = \ell\right)$, $PE\left(s, \ell, 2\right) \mathrm{P}\left(I\left(t\right) = 2, A\left(t\right) = \ell\right)$ and $PE\left(c, \ell, 2\right) \mathrm{P}\left(I\left(t\right) = 2, A\left(t\right) = \ell\right)$, respectively. $\ell = 1, \cdots, m_A$

6. Compute the *moments* $\mathrm{E}\left[N_i^p\left(t\right) | N\left(t\right) = k, A\left(t\right) = \ell\right]$ with the multinomial moments with parameter $p_i(t)$ for $i = 1, \cdots, m_B$, and $\{s - 1, s\}$ number of trials.

7. Simultaneously numerically integrate the PMDEs $m_A\left(4 + 2m_B + m_B^2\right)$ using the approximations computed in Steps (5) and (6).

8. Sum the partial moments to obtain the actual first two moments of the number of entities in the system.

9. Set $t = t + \Delta t$ and go to Step (2).

The actual code was inplemented in Matlab, see Appendix 7.

Chapter 5

The Results

As shown before the total number of KFEs grows fast with s and m_B, for this reason we compare our approximation with the actual results (numerically integrate all the KFEs) for $s = 3$ and $m_B = 2, 3$. For other cases we use simulation in order to validate our approximation.

Table 5 shows the different cases we implemented in order to validate our approximation. The specific arrival and service processes used for each of the cases are presented in Appendix 5. For Case One the specific arrival and service process are

$$\mathcal{A}(t) = \left\{ \begin{array}{cccc} 0.1 & 0.1 & 0.3 & 0.5 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 \end{array} \right\}$$

$$\boldsymbol{\lambda}(t) = \left[3 + 0.5\sin\left(t/3\pi\right), 3 + 0.5\sin\left(t/3\pi\right), 2 + 0.5\sin\left(t/3\pi\right)\right]$$

and

$$\mathcal{B}(t) = \left\{\begin{array}{ccc} 0.1 & 0.6 & 0.3 \\ 0.3 & 0.1 & 0.6 \\ 0.5 & 0.5 & 0 \end{array}\right\}$$

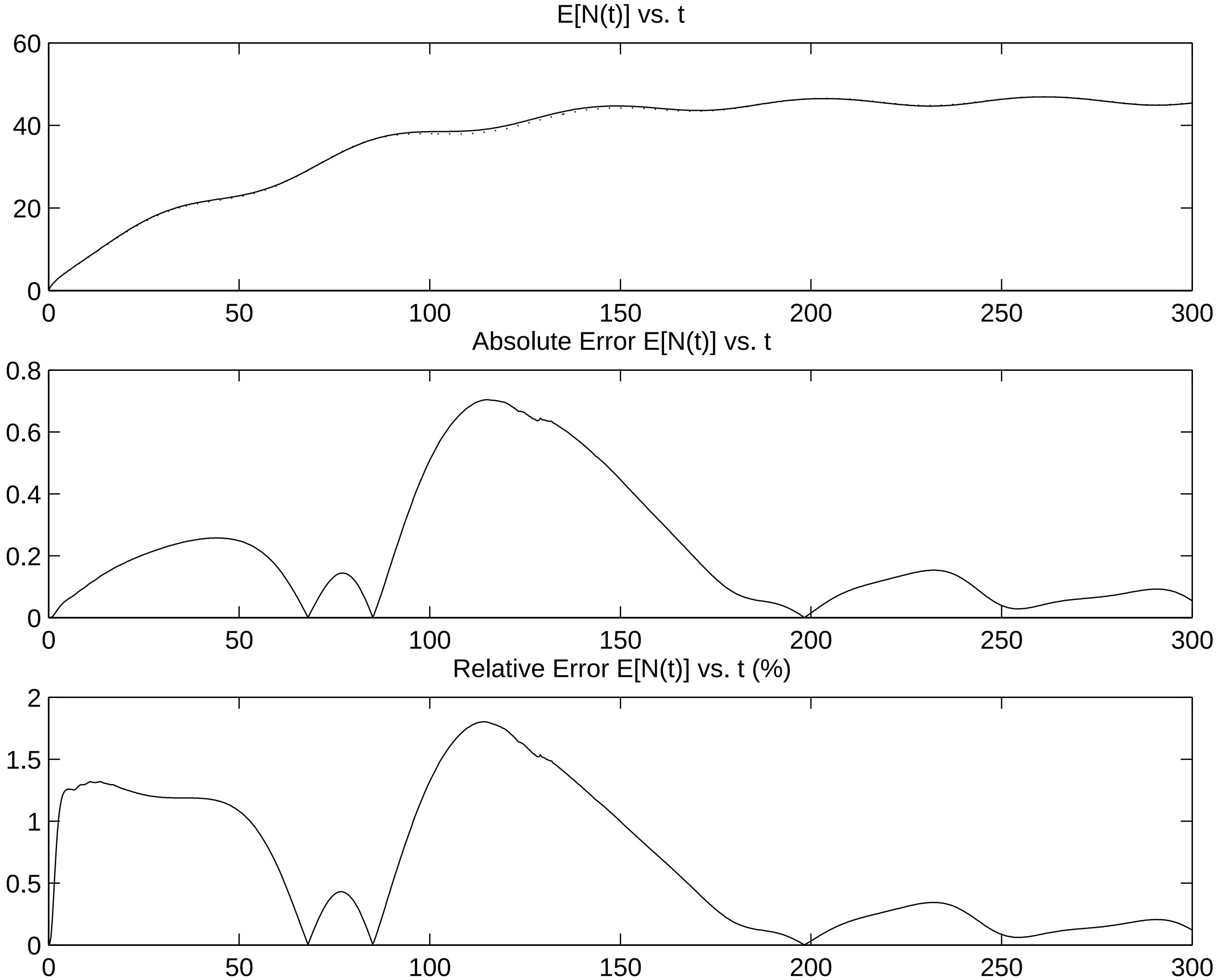
$$\boldsymbol{\mu}(t) = [1, 1]$$

respectively.

Table 5.1: Example cases implemented.

No	m_A	m_B	s	c	ρ
1	3	2	3	5	$<< 1$
2	3	2	3	6	$<< 1$
3	3	2	3	30	$<< 1$
4	3	2	3	30	~ 1
5	3	2	3	50	~ 1
6	3	3	3	5	$<< 1$
7	3	3	3	6	$<< 1$
8	3	3	3	30	$<< 1$
9	3	3	3	30	~ 1
10	3	3	3	50	~ 1
11	3	3	5	30	both
12	3	3	5	50	both
13	3	3	10	30	both
14	3	3	10	50	both
15	3	5	5	30	both
16	3	5	5	50	both
17	3	5	10	30	both
18	3	5	10	50	both

Figure 5.1: $\mathrm{E}[N(t)]$ vs. t and errors, Case 5.



5.1 The $\mathrm{E}[N(t)]$

Table 5.1 presents the results and errors for the expectation of number of entities in the system across the time interval $(0, T]$, where $T = 300, 1000, 12$, compared with the method of validation for each of the cases presented in Table 5. In Figures 5.1 and 5.2 the $\mathrm{E}[N(t)]$ for the Cases 5 and 7 are shown. Notice that the doted line is the approximation.

Table 5.2: Maximum Absolute and Relative Errors for the $\mathrm{E}[N(t)]$.

No	Max. Abs. Error	Rel. Error (%)	Max. Rel. Error (%)	Abs. Error	Method
1	0.0094	0.4377	0.4379	0.0093	KFEs
2	0.0252	1.1350	1.2602	0.0242	KFEs
3	0.0948	2.4436	2.7319	0.0780	KFEs
4	0.3098	1.3084	1.3486	0.2992	KFEs
5	0.7038	1.8013	1.8021	0.7034	KFEs
6	0.0150	0.8425	0.8447	0.0150	KFEs
7	0.0321	1.5892	1.8324	0.0269	KFEs
8	0.1644	2.9459	3.2578	0.0681	KFEs
9	0.6203	3.0882	3.0925	0.6198	KFEs
10	0.9079	2.3883	2.5103	0.8801	KFEs
11	0.5181	2.3069	2.3069	0.5181	Simulation
12	0.8453	2.8544	2.2519	0.3463	Simulation
13	0.6110	3.9688	3.9688	0.6110	Simulation
14	0.5185	2.5424	2.5424	0.5185	Simulation
15	0.5642	4.3134	4.3134	0.5642	Simulation
16	1.3105	8.2956	8.2956	1.3105	Simulation
17	0.8419	5.5949	5.5949	0.8419	Simulation
18	1.0688	5.5198	5.5198	1.0688	Simulation

Figure 5.2: $\mathrm{E}[N(t)]$ vs. t and errors, Case 7.

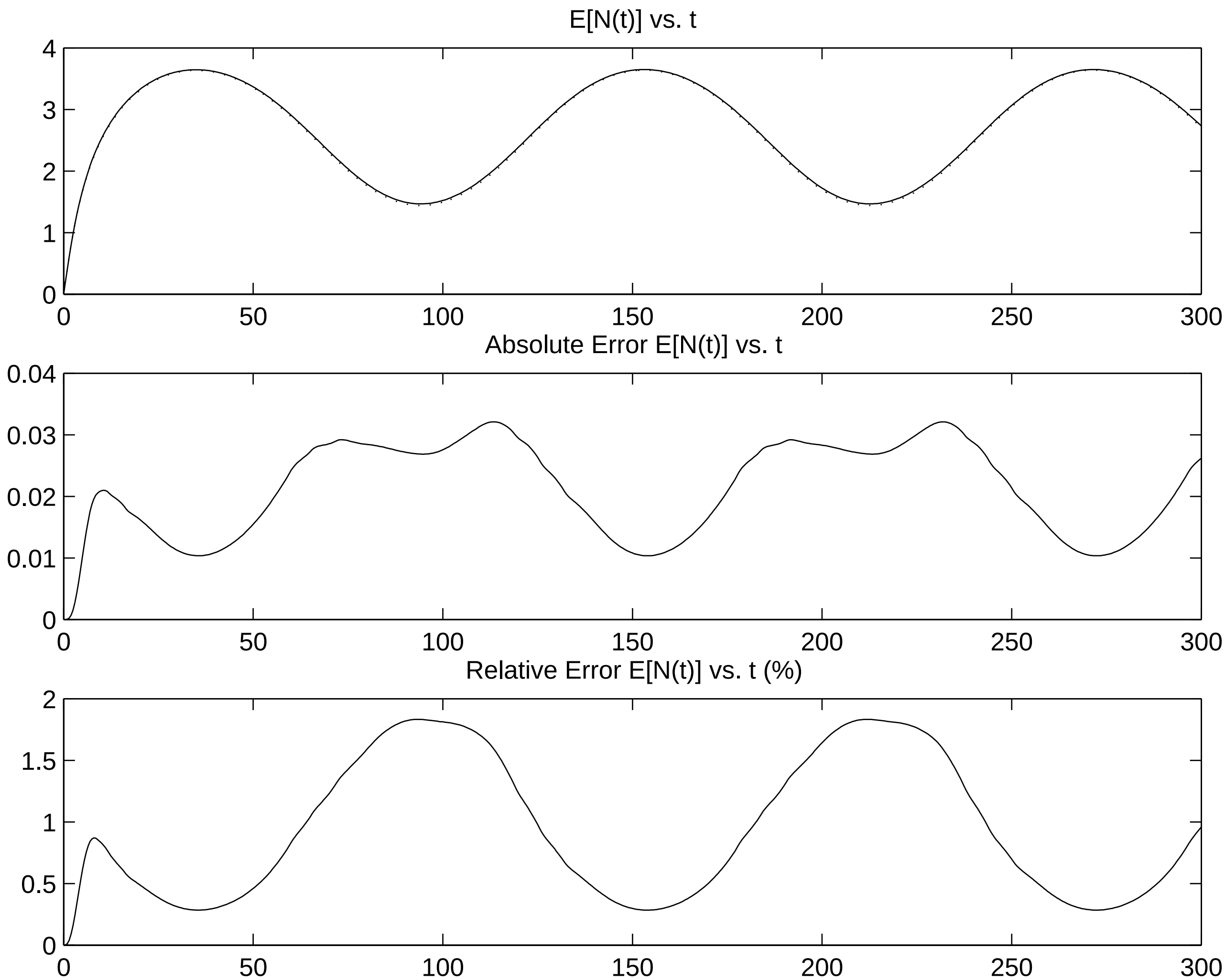
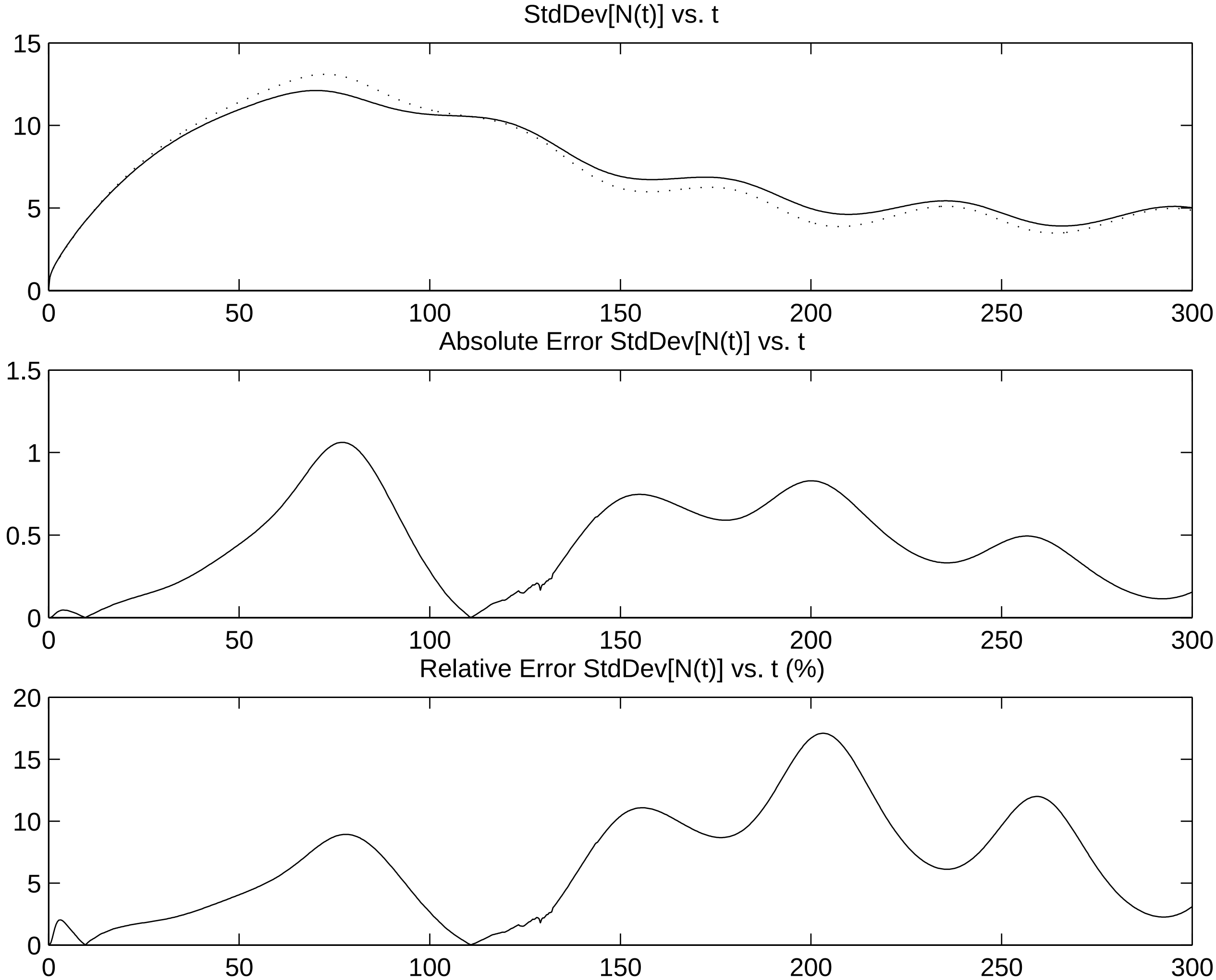


Figure 5.3: SD $[N(t)]$ vs. t and errors, Case 5.



5.2 The SD $[N(t)]$

Table 5.2 presents the results and errors for the standard deviation of number of entities in the system across the time interval $(0, T]$, where $T = 300, 1000, 12$,compared with the method of validation for each of the cases presented in Table 5. In Figures 5.3 and 5.4 the SD $[N(t)]$ for the Cases 5 and 7 are shown. Notice that the doted line is the approximation.

Table 5.3: Maximum Absolute and Relative Errors for the SD $[N(t)]$.

No	Max. Abs. Error	Rel. Error (%)	Max. Rel. Error (%)	Abs. Error	Method
1	0.0107	0.7121	0.7123	0.0107	KFEs
2	0.0185	1.2167	1.2167	0.0185	KFEs
3	0.1030	5.3211	5.7352	0.1000	KFEs
4	0.5631	11.3610	11.5672	0.5521	KFEs
5	1.0616	8.9045	17.0929	0.8144	KFEs
6	0.1503	11.9284	12.2357	0.1461	KFEs
7	0.1324	10.4121	11.2520	0.1274	KFEs
8	0.1202	10.1775	10.6400	0.1112	KFEs
9	0.4391	5.8179	11.2234	0.1186	KFEs
10	1.1553	9.3867	11.0070	0.1334	KFEs
11	0.4766	9.4154	9.4154	0.4766	Simulation
12	0.8359	13.1448	13.1448	0.8359	Simulation
13	0.2547	3.6421	3.6421	0.2547	Simulation
14	1.2046	10.9788	10.9788	1.2046	Simulation
15	0.3336	4.3367	4.3367	0.3336	Simulation
16	0.6528	10.3768	10.3768	0.6528	Simulation
17	0.8932	13.5039	13.5039	0.8932	Simulation
18	0.7127	6.4990	6.4990	0.7127	Simulation

Figure 5.4: SD $[N(t)]$ vs. t and errors, Case 7.

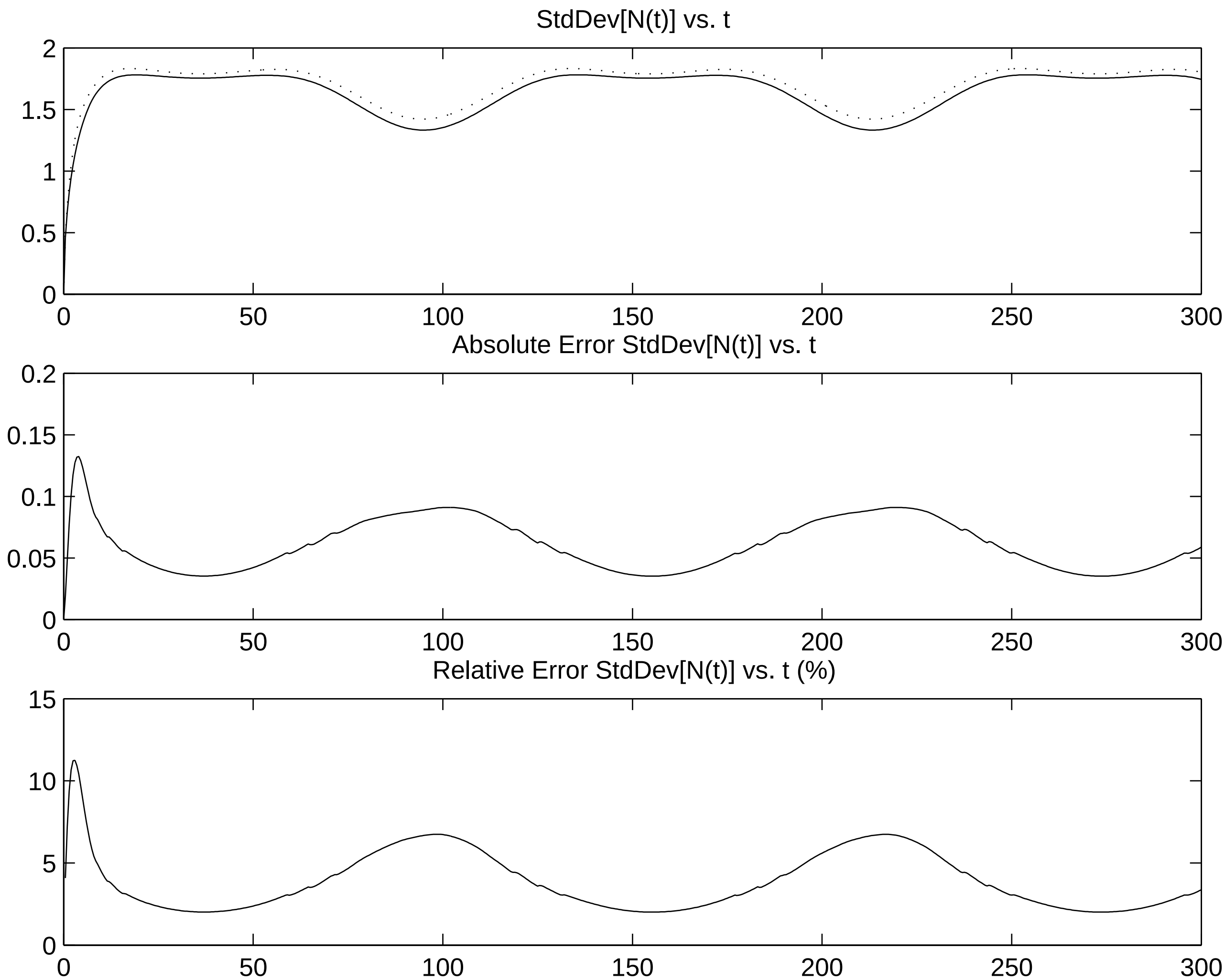
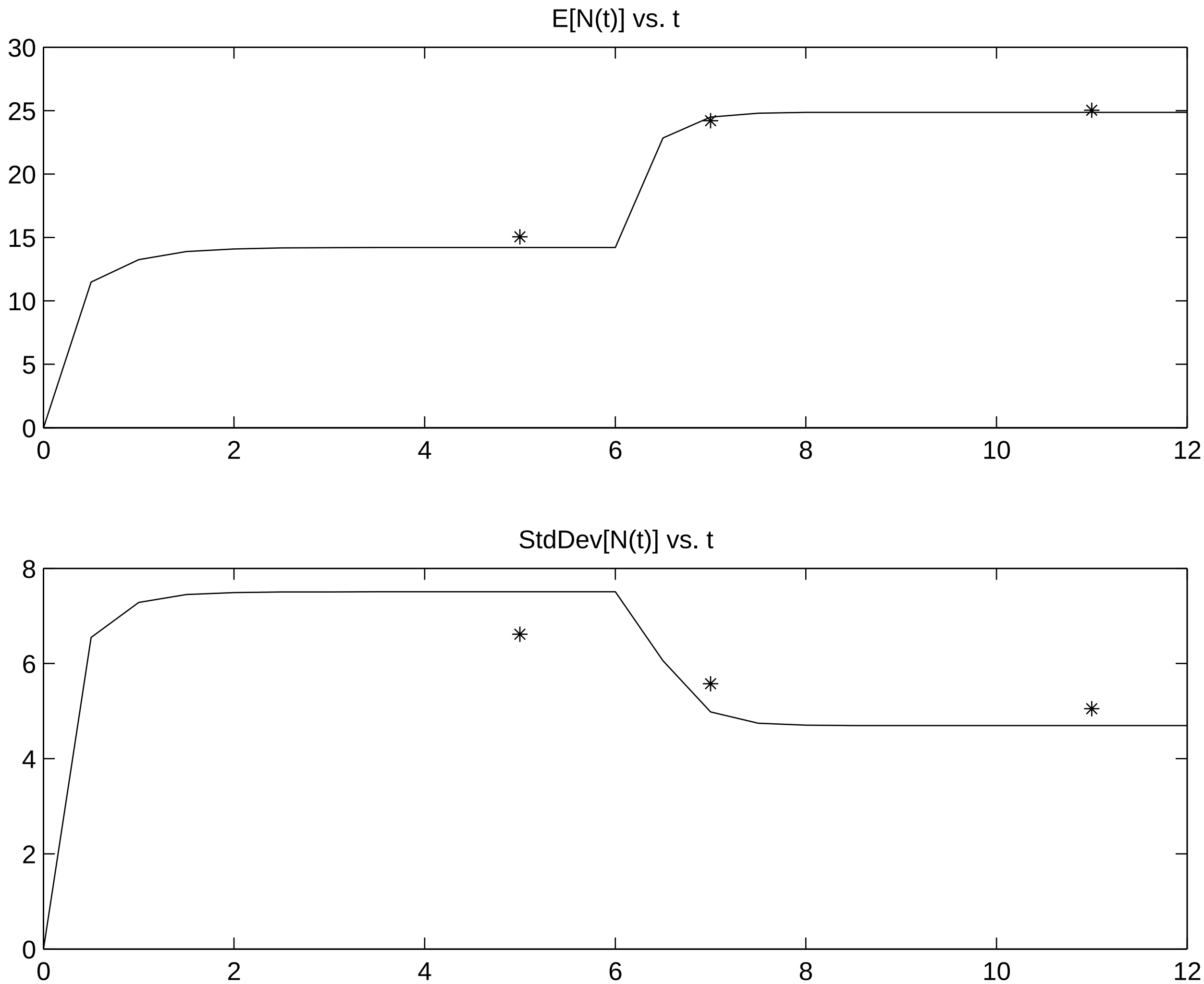


Figure 5.5: $\mathrm{E}\left[N(t)\right]$ vs. t and $\mathrm{SD}\left[N(t)\right]$ vs. t, Case 17.



Finally, in Figure 5.5 the $\mathrm{E}\left[N(t)\right]$ and $\mathrm{SD}\left[N(t)\right]$ for the Case 17 are shown. The three points are the points compare to the simulation. Four hundred replications were executed for each of the simulations. This produce a half-width (95%) less or equal to 1.3 for the mean, and in the worst Case (half-width equal to 1.3) the expected number of entities in the system is approximately 29. The half-width (95%) of the standard deviation for the worst Case is equal to 0.72 and the actual value of the standard deviation for the coresponding Case is

13.16.

Observe that in the Cases (5 and 7) shown above the solid line (Actual) and the doted line (Approximation) is indistinguishable and always quite close. In Case 17 the approximation (solid line) is close to the simulation (three points) as well. We highlight the Cases above because those are the worst cases that we found from all of the cases studied and the approximation continue giving a good performance.

See Appendix 6 for the figures of the other cases.

Chapter 6

The Virtual Waiting Time

Finally develop the differential equations and approximations for the moments and distribution function of waiting time for entities who entered the system at time t. Consider a subsidiary stochastic process $\mathbf{N}_t = \{N_t(t+\tau) : \tau \geq 0\}$ where $N_t(t+\tau) \equiv N(t)$ and no arrivals are allowed into the system after time t. Let $W_t \equiv \inf\{\tau : \tau \geq 0, N_t(t+\tau) \leq s-1\}$ and we want to compute the cdf of W_t as well as its moments.

Therefore

$$1 - F_{W_t}(\tau) \equiv \mathrm{P}(W_t > \tau)$$

$$1 - F_{W_t}(\tau) = \mathrm{P}(N_t(t+\tau) > s-1)$$

and we know that

$$\mathrm{P}(N_t(t+\tau) > s-1) = \mathrm{P}(I_t(t+\tau) = 2)$$

where

$$\mathrm{P}\left(I_t\left(t+\tau\right)=2\right)=\sum_{\ell=1}^{m_A}\mathrm{P}\left(I_t\left(t+\tau\right)=2,A_t\left(t+\tau\right)=\ell\right)$$

and

$$\mathrm{P}\left(I_t\left(t+\tau\right)=2,A_t\left(t+\tau\right)=\ell\right)=\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=\ell\right)$$

$\ell=1,\cdots,m_A$ and no arrivals are allowed after time t, e.g. $\lambda_\ell\left(t+\tau\right)=0$ for $\tau\geq 0$. Initialize the system with the actual number of entities in the system at time t, $N(t)$.

All the PMDEs from the actual queueing system still hold and they simplify since $\lambda_\ell\left(t+\tau\right)=0$ for $\tau\geq 0$ and $\ell=1,\cdots,m_A$. The *pth* PMDEs are simplify to:

For subspace Ω_1

$$\begin{aligned}
\mathrm{E}'\left[N_i^p\left(t\right),\ell,1\right] \;=\;& +\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{ji}\left(t\right)\mu_j\left(t\right)\sum_{h=1}^{p}\binom{p}{h}\mathrm{E}\left[N_i^{p-h}\left(t\right)N_j\left(t\right),\ell,1\right]\\
&+\left(1-b_{ii}\left(t\right)\right)\mu_i\left(t\right)\sum_{h=0}^{p-1}\binom{p}{h+1}\left(-1\right)^{h+1}\mathrm{E}\left[N_i^{p-h}\left(t\right),\ell,1\right]\\
&+b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\sum_{h=0}^{p-1}\binom{p}{h+1}\left(-1\right)^{h+1}\mathrm{E}\left[N_i^{p-h}\left(t\right),N\left(t\right)=s,A\left(t\right)=\ell\right]\\
&+\sum_{j=1}^{m_B} b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\mathrm{E}\left[N_i^p\left(t\right)N_j\left(t\right),N\left(t\right)=s,A\left(t\right)=\ell\right]
\end{aligned}$$

and, for Ω_2

$$\begin{aligned}
\mathrm{E}'\left[N^p\left(t\right),\ell,2\right] \;=\;& +\sum_{i=1}^{m_B} b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\sum_{h=1}^{p}\binom{p}{h}\left(-1\right)^h\mathrm{E}\left[N_i\left(t\right)N^{p-h}\left(t\right),I\left(t\right)=2,A\left(t\right)=\ell\right]\\
&-\sum_{i=1}^{m_B} b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\left(s-1\right)^p\mathrm{E}\left[N_i\left(t\right),N\left(t\right)=s,A\left(t\right)=\ell\right].
\end{aligned}$$

The Cross-Product PMDEs are simplify to:

For subspace Ω_1

$$
\begin{aligned}
&\mathrm{E}'\left[N_i(t)\,N_j(t),\ell,1\right] = -\mu_i(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right] \\
&-\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right] \\
&-b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_j(t),\ell,1\right] \\
&-b_{ij}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t),\ell,1\right] \\
&+\sum_{k=1}^{m_B} b_{ki}(t)\,\mu_k(t)\,\mathrm{E}\left[N_j(t)\,N_k(t),\ell,1\right] \\
&+\sum_{k=1}^{m_B} b_{kj}(t)\,\mu_k(t)\,\mathrm{E}\left[N_i(t)\,N_k(t),\ell,1\right] \\
&-\left(b_{i,m_B+1}(t)\,\mu_i(t)+b_{j,m_B+1}(t)\,\mu_j(t)\right)E\left[N_i(t)\,N_j(t),N(t)=s,A(t)=\ell\right] \\
&+\sum_{k=1}^{m_B} b_{k,m_B+1}(t)\,\mu_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t)\,N_k(t),N(t)=s,A(t)=\ell\right]
\end{aligned}
$$

and, for Ω_2

$$
\begin{aligned}
&E'\left[N_i(t)\,N(t),\ell,2\right] = -\mu_i(t)\,\mathrm{E}\left[N_i(t)\,N(t),\ell,2\right] \\
&+\sum_{j=1}^{m_B} b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_j(t)\,N(t),\ell,2\right] \\
&+b_{i,m_B+1}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t),I(t)=2,A(t)=\ell\right] \\
&+\sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\,\mathrm{E}\left[N_j(t)\,N(t),\ell,2\right] \\
&-\sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),I(t)=2,A(t)=\ell\right] \\
&-\sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\,\mathrm{E}\left[N_j(t),I(t)=2,A(t)=\ell\right] \\
&-\sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\,(s-1)\,\mathrm{E}\left[N_j(t),N(t)=s,A(t)=\ell\right] \\
&-\sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,(s-1)\,\mathrm{E}\left[N_i(t)\,N_j(t),N(t)=s,A(t)=\ell\right] \\
&+b_{i,m_B+1}(t)\,\mu_i(t)\,(s-1)\,\mathrm{E}\left[N_i(t),N(t)=s,A(t)=\ell\right].
\end{aligned}
$$

It is necessary to numerically integrate all the PMDEs in order to evaluate the first and second moments, then evaluate the zeroth PMDEs using the approximation.

Finally, in order to evaluate the *pth* moment of the virtual waiting time at time t

$$\mathrm{E}\left[W_t^p\right] = \int_{0^+}^{\infty} p\tau^{p-1}\left(1 - F_{W_t}(\tau)\right) d\tau$$

we need to evaluate the PMDEs of the regular system $N(t)$, $(m_A(4 + 2m_B + m_B^2))$, up until time t, then evaluate the simplified PMDEs for the new subsidiary system, $N_t(t+\tau)$, until $\mathrm{P}\left(I_t(t+\tau) = 2\right) \leq \epsilon$, and finally we evaluate the differential equation:

$$\frac{d}{d\tau}\mathrm{E}\left[W_t^p(\tau)\right] = p\tau^{p-1}\left(1 - F_{W_t}(\tau)\right) = p\tau^{p-1}P\left(I_t(t+\tau) = 2\right).$$

Chapter 7

Conclusion

We developed an efficient numerical algorithm and approximation to analyze the time-dependent behavior of the $Ph_t/Ph_t/s/c$ queueing system. The algorithm is efficient since the total number of PMDEs to numerically integrate is small in comparison with the classic procedure of numerically integrating all of the KFEs. In addition the number of PMDEs does not depend on the number of servers (s), nor does it depend on the capacity of the system (c). The test cases demonstrate that even if the absolute value of the error for the standard deviation is *big*, more than 5%, the corresponding absolute value of the error is *small*, less than 5 entities at time t.

The approximation gives accurate results for the test cases we considered and thus we conclude that it is sufficiently accurate for many practical applications. The time needed to produce the results is reasonable, ~ 5 minutes for the cases we examined, and is certainly more efficient that a simulation experiment.

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Appendix 1

Derivation of the First Moment Subspace Ω_1

$$\mathrm{E}'\left[N_i(t), \ell, 1\right] \equiv \mathrm{E}'\left[N_i(t), A(t)=\ell, I(t)=1\right]$$

$$\mathrm{E}'\left[N_i(t), A(t)=\ell, I(t)=1\right] \equiv \sum_{n_1=0}^{s-1} \sum_{n_2=0}^{s-1-n_1} \cdots \sum_{n_{m_B}=0}^{s-1-n_1-\cdots-n_{m_B-1}} n_i P'_{n_1\cdots n_{m_B};\ell,0}(t)$$

Using the KFEs

$$\begin{aligned}
&\mathrm{E}'\left[N_i(t), \ell, 1\right] = \\
&-\lambda_\ell(t)\,\mathrm{E}\left[N_i(t), \ell, 1\right] \\
&-\left(1-b_{ii}(t)\right)\mu_i(t)\,\mathrm{E}\left[N_i^2(t), \ell, 1\right] \\
&-\sum_{\substack{j=1\\ j\neq i}}^{m_B} \left(1-b_{jj}(t)\right)\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t), \ell, 1\right] \\
&+\sum_{j=1}^{m_A} a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t), j, 1\right] \\
&+\sum_{\substack{j=1\\ j\neq i}}^{m_B} b_{ji}(t)\,\mu_j(t)\left\{\sum_{n_i=1}^{s-1}\sum_{n_j=0}^{s-1-n_i}\sum_{n_1=0}^{s-1-n_i-n_j}\cdots\sum_{n_{i-1}=0}^{s-1-n_1\cdots-n_{i-2}-n_i-n_j}\sum_{n_{i+1}=0}^{s-1-n_1\cdots-n_i-n_j}\cdots\right. \\
&\left.\sum_{n_{j-1}=0}^{s-1-n_1\cdots-n_{j-2}-n_j}\sum_{n_{j+1}=0}^{s-1-n_1\cdots-n_j}\cdots\sum_{n_{m_B}=0}^{s-1-n_1-\cdots-n_{m_B-1}} n_i\left(n_j+1\right)P_{n_1\cdots n_i-1\cdots n_j+1\cdots n_{m_B};\ell,0}(t)\right\}
\end{aligned}$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{ij}(t)\,\mu_i(t)\left\{\sum_{n_j=1}^{s-1}\sum_{n_i=0}^{s-1-n_j}\sum_{n_1=0}^{s-1-n_i-n_j}\cdots\sum_{n_{j-1}=0}^{s-1-n_1\cdots-n_{j-2}-n_i-n_j}\sum_{n_{j+1}=0}^{s-1-n_1\cdots-n_i-n_j}\cdots\right.$$

$$\left.\sum_{n_{i-1}=0}^{s-1-n_1\cdots-n_{i-2}-n_i}\sum_{n_{i+1}=0}^{s-1-n_1\cdots-n_i}\cdots\sum_{n_{m_B}=0}^{s-1-n_1-\cdots-n_{m_B-1}} n_i\,(n_i+1)\,P_{n_1\cdots n_j-1\cdots n_i+1\cdots n_{m_B};\ell,0}(t)\right\}$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{kj}(t)\,\mu_k(t)\left\{\sum_{n_j=1}^{s-1}\sum_{n_k=0}^{s-1-n_j}\sum_{n_1=0}^{s-1-n_j-n_k}\cdots\sum_{n_{j-1}=0}^{s-1-n_1\cdots-n_{j-2}-n_k-n_j}\sum_{n_{j+1}=0}^{s-1-n_1\cdots-n_k-n_j}\cdots\right.$$

$$\left.\sum_{n_{k-1}=0}^{s-1-n_1\cdots-n_{k-2}-n_k}\sum_{n_{k+1}=0}^{s-1-n_1\cdots-n_k}\cdots\sum_{n_{m_B}=0}^{s-1-n_1-\cdots-n_{m_B-1}} n_i\,(n_k+1)\,P_{n_1\cdots n_j-1\cdots n_k+1\cdots n_{m_B};\ell,0}(t)\right\}$$

$$+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\left\{\sum_{n_i=1}^{s-1}\sum_{n_1=0}^{s-1-n_i}\cdots\sum_{n_{i-1}=0}^{s-1-n_1\cdots-n_{i-2}-n_i}\sum_{n_{i+1}=0}^{s-1-n_1\cdots-n_i}\cdots\right.$$

$$\left.\sum_{n_{m_B}=0}^{s-1-n_1-\cdots-n_{m_B-1}} n_i P_{n_1\cdots n_i-1\cdots n_{m_B};j,0}(t)\right\}$$

$$+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{\substack{k=1\\k\neq i}}^{m_B}\beta_k(t)\left\{\sum_{n_k=1}^{s-1}\sum_{n_1=0}^{s-1-n_k}\cdots\sum_{n_{k-1}=0}^{s-1-n_1\cdots-n_{k-2}-n_k}\sum_{n_{k+1}=0}^{s-1-n_1\cdots-n_k}\cdots\right.$$

$$\left.\sum_{n_{m_B}=0}^{s-1-n_1-\cdots-n_{m_B-1}} n_i P_{n_1\cdots n_k-1\cdots n_{m_B};j,0}(t)\right\}$$

$$+b_{im_B+1}(t)\,\mu_i(t)\sum_{n_1=0}^{s-1}\cdots\sum_{n_{m_B}=0}^{s-1-n_1-\cdots-n_{m_B-1}} n_i\,(n_i+1)\,P_{n_1\cdots n_i+1\cdots n_{m_B};\ell,0}(t)$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\sum_{n_1=0}^{s-1}\cdots\sum_{n_{m_B}=0}^{s-1-n_1-\cdots-n_{m_B-1}} n_i\,(n_j+1)\,P_{n_1\cdots n_j+1\cdots n_{m_B};\ell,0}(t)$$

Then we manipulate the limits for some terms

$$\begin{aligned}
\mathrm{E}'[N_i(t),\ell,1] &= -\lambda_\ell(t)\,\mathrm{E}[N_i(t),\ell,1]\\
&-(1-b_{ii}(t))\,\mu_i(t)\,\mathrm{E}[N_i^2(t),\ell,1]\\
&-\sum_{\substack{j=1\\j\neq i}}^{m_B}(1-b_{jj}(t))\,\mu_j(t)\,\mathrm{E}[N_i(t)\,N_j(t),\ell,1]\\
&+\sum_{j=1}^{m_A} a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}[N_i(t),j,1]\\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{ji}(t)\,\mu_j(t)\,\mathrm{E}[N_i(t)\,N_j(t),\ell,1]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{j=1\\ j\neq i}}^{m_B} b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_j(t),\ell,1\right]\\
&+\sum_{\substack{j=1\\ j\neq i}}^{m_B} b_{ij}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i^2(t),\ell,1\right]\\
&-\sum_{\substack{j=1\\ j\neq i}}^{m_B} b_{ij}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t),\ell,1\right]\\
&+\sum_{\substack{j=1\\ j\neq i}}^{m_B}\sum_{\substack{k=1\\ k\neq i,j}}^{m_B} b_{kj}(t)\,\mu_k(t)\,\mathrm{E}\left[N_i(t)\,N_k(t),\ell,1\right]\\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\,\mathrm{E}\left[N_i(t),j,1\right]\\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\,\mathrm{P}\left(I(t)=1,A(t)=j\right)\\
&-\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\sum_{n_1=0}^{s-1}\cdots\sum_{n_{m_B-1}=0}^{s-1-n_1-\cdots-n_{m_B-2}}(n_i+1)\,P_{n_1\cdots n_{m_B}=s-1-n_1-\cdots-n_{m_B-1};j,0}(t)\\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{\substack{k=1\\ k\neq i}}^{m_B}\beta_k(t)\,\mathrm{E}\left[N_i(t),j,1\right]\\
&-\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{\substack{k=1\\ k\neq i}}^{m_B}\beta_k(t)\sum_{n_1=0}^{s-1-n_k}\cdots\sum_{n_{m_B-1}=0}^{s-1-n_1-\cdots-n_{m_B-2}} n_i P_{n_1\cdots n_{m_B}=s-1-n_1-\cdots-n_{m_B-1};j,0}(t)\\
&+b_{im_B+1}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i^2(t),\ell,1\right]\\
&-b_{i,m_B+1}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t),\ell,1\right]\\
&+b_{i,m_B+1}(t)\,\mu_i(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\cdots-n_{m_B-2}} n_i(n_i-1)\,P_{n_1\cdots n_{m_B}=s-n_1-\cdots-n_{m_B-1};\ell,0}(t)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right]\\
&+\sum_{\substack{j=1\\ j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\cdots-n_{m_B-2}} n_i n_j P_{n_1\cdots n_{m_B}=s-n_1-\cdots-n_{m_B-1};\ell,0}(t)
\end{aligned}
$$

Canceling some terms

$$
\begin{aligned}
&\mathrm{E}'\left[N_i\left(t\right),\ell,1\right]=-\lambda_\ell\left(t\right)\mathrm{E}\left[N_i\left(t\right),\ell,1\right]\\
&+\sum_{j=1}^{m_A}a_{j\ell}\left(t\right)\lambda_j\left(t\right)\mathrm{E}\left[N_i\left(t\right),j,1\right]\\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}\left(t\right)\mu_j\left(t\right)\mathrm{E}\left[N_j\left(t\right),\ell,1\right]\\
&-\left(1-b_{ii}\left(t\right)\right)\mu_i\left(t\right)\mathrm{E}\left[N_i\left(t\right),\ell,1\right]\\
&+\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\mathrm{E}\left[N_i\left(t\right),j,1\right]\\
&+\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\beta_i\left(t\right)\mathrm{P}\left(I\left(t\right)=1,A\left(t\right)=j\right)\\
&-\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\sum_{n_1=0}^{s-1}\cdots\sum_{n_{m_B-1}=0}^{s-1-n_1-\cdots-n_{m_B-2}}n_iP_{n_1\cdots n_{m_B}=s-1-n_1-\cdots-n_{m_B-1};j,0}\left(t\right)\\
&-\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s-1-n_k}\cdots\sum_{n_{m_B-1}=0}^{s-1-n_1-\cdots-n_{m_B-2}}P_{n_1\cdots n_{m_B}=s-1-n_1-\cdots-n_{m_B-1};j,0}\left(t\right)\\
&+b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\cdots-n_{m_B-2}}n_i\left(n_i-1\right)P_{n_1\cdots n_{m_B}=s-n_1-\cdots-n_{m_B-1};\ell,0}\left(t\right)\\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\cdots-n_{m_B-2}}n_in_jP_{n_1\cdots n_{m_B}=s-n_1-\cdots-n_{m_B-1};\ell,0}\left(t\right)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\mathrm{E}'\left[N_i(t),\ell,1\right] = & -\lambda_\ell(t)\,\mathrm{E}\left[N_i(t),\ell,1\right] \\
& +\sum_{j=1}^{m_A} a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t),j,1\right] \\
& +\sum_{\substack{j=1\\ j\neq i}}^{m_B} b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_j(t),\ell,1\right] \\
& -\left(1-b_{ii}(t)\right)\mu_i(t)\,\mathrm{E}\left[N_i(t),\ell,1\right] \\
& +\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t),j,1\right] \\
& +\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\,\mathrm{P}\left(I(t)=1,A(t)=j\right) \\
& -\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t),N(t)=s-1,A(t)=j\right] \\
& -\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\,\mathrm{P}\left(N(t)=s-1,A(t)=j\right) \\
& +b_{i,m_B+1}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t)\left(N_i(t)-1\right),N(t)=s,A(t)=\ell\right] \\
& +\sum_{\substack{j=1\\ j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),N(t)=s,A(t)=\ell\right]
\end{aligned}
$$

where

$$
\mathrm{E}\left[N_i(t),N(t)=s-1,A(t)=j\right]\equiv\sum_{n_1=0}^{s-1}\cdots\sum_{n_{m_B-1}=0}^{s-1-n_1-\cdots-n_{m_B-2}} n_i P_{n_1\cdots n_{m_B}=s-1-n_1-\cdots-n_{m_B-1};j,0}(t),
$$

and

$$
\mathrm{E}\left[N_i(t)\,N_j(t),N(t)=s,A(t)=\ell\right]\equiv\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\cdots-n_{m_B-2}} n_i n_j P_{n_1\cdots n_{m_B}=s-n_1-\cdots-n_{m_B-1};\ell,0}(t).
$$

Finally,

$$
\begin{aligned}
\mathrm{E}'\left[N_i\left(t\right),\ell,1\right] = & -\lambda_\ell\left(t\right)\mathrm{E}\left[N_i\left(t\right),\ell,1\right] \\
& +\sum_{j=1}^{m_A} a_{j\ell}\left(t\right)\lambda_j\left(t\right)\mathrm{E}\left[N_i\left(t\right),j,1\right] \\
& -\mu_i\left(t\right)\mathrm{E}\left[N_i\left(t\right),\ell,1\right] \\
& +\sum_{j=1}^{m_B} b_{ji}\left(t\right)\mu_j\left(t\right)\mathrm{E}\left[N_j\left(t\right),\ell,1\right] \\
& +\sum_{j=1}^{m_A} a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\mathrm{E}\left[N_i\left(t\right),j,1\right] \\
& +\sum_{j=1}^{m_A} a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\beta_i\left(t\right)\mathrm{P}\left(I\left(t\right)=1,A\left(t\right)=j\right) \\
& -\sum_{j=1}^{m_A} a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\mathrm{E}\left[N_i\left(t\right),N\left(t\right)=s-1,A\left(t\right)=j\right] \\
& -\sum_{j=1}^{m_A} a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\beta_i\left(t\right)\mathrm{P}\left(N\left(t\right)=s-1,A\left(t\right)=j\right) \\
& -b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\mathrm{E}\left[N_i\left(t\right),N\left(t\right)=s,A\left(t\right)=\ell\right] \\
& +\sum_{j=1}^{m_B} b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\mathrm{E}\left[N_i\left(t\right)N_j\left(t\right),N\left(t\right)=s,A\left(t\right)=\ell\right]
\end{aligned}
$$

Derivation of the First Moment Subspace Ω_2

$$
\mathrm{E}'\left[N\left(t\right),\ell,2\right] \equiv \mathrm{E}'\left[N\left(t\right),A\left(t\right)=\ell,I\left(t\right)=2\right]
$$

$$
\mathrm{E}'\left[N\left(t\right),A\left(t\right)=\ell,I\left(t\right)=2\right] \equiv \sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}\left(s+q\right)P'_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}\left(t\right)
$$

Using the KFEs

$$\mathrm{E}'\left[N\left(t\right),\ell,2\right]=-\lambda_{\ell}\left(t\right)\mathrm{E}\left[N\left(t\right),\ell,2\right]$$
$$-\sum_{i=1}^{m_B}\left(1-b_{ii}\left(t\right)\right)\mu_i\left(t\right)\sum_{n_1=0}^{s}...\sum_{n_{m_B-1}=0}^{s-n_1-...-n_{m_B-2}}\sum_{q=0}^{c-s}n_i\left(s+q\right)P_{n_1...n_{m_B}=s-n_1-...-n_{m_B-1};\ell,q}\left(t\right)$$
$$+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{E}\left[N\left(t\right),i,2\right]$$
$$+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}\left(t\right)\mu_j\left(t\right)\Bigg\{\sum_{n_j=0}^{s-1}\sum_{n_i=1}^{s-n_j}\sum_{n_1=0}^{s-n_i-n_j}...\sum_{n_{j-1}=0}^{s-n_1...-n_{j-2}-n_j-n_i}\sum_{n_{j+1}=0}^{s-n_1...-n_j-n_i}...\sum_{n_{i-1}=0}^{s-n_1...-n_{i-2}-n_i}$$
$$\sum_{n_{i+1}=0}^{s-n_1...-n_i}...\sum_{n_{m_B-1}=0}^{s-n_1-...-n_{m_B-2}}\sum_{q=0}^{c-s}\left(n_j+1\right)\left(s+q\right)P_{n_1...n_i-1...n_j+1...n_{m_B}=s-n_1-...-n_{m_B-1};\ell,q}\left(t\right)\Bigg\}$$
$$+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\Bigg\{\sum_{j=1}^{m_B}\beta_j\left(t\right)\sum_{n_j=1}^{s}\sum_{n_1=0}^{s-n_j}...\sum_{n_{j-1}=0}^{s-n_1...-n_{j-2}-n_j}\sum_{n_{j+1}=0}^{s-n_1...-n_j}...$$
$$\sum_{n_{m_B-1}=0}^{s-n_1-...-n_{m_B-2}}sP_{n_1...n_j-1...n_{m_B}=s-n_1-...-n_{m_B-1};i,0}\left(t\right)\Bigg\}$$
$$+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s}...\sum_{n_{m_B-1}=0}^{s-n_1-...-n_{m_B-2}}\sum_{q=1}^{c-s}\left(s+q\right)P_{n_1...n_{m_B}=s-n_1-...-n_{m_B-1};i,q-1}\left(t\right)$$
$$+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s}...\sum_{n_{m_B-1}=0}^{s-n_1-...-n_{m_B-2}}cP_{n_1...n_{m_B}=s-n_1-...-n_{m_B-1};i,c-s}\left(t\right)$$
$$+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_j\left(t\right)\Bigg\{\sum_{n_i=0}^{s-1}\sum_{n_j=1}^{s-n_i}\sum_{n_1=0}^{s-n_i-n_j}...\sum_{n_{i-1}=0}^{s-n_1...-n_{i-2}-n_j-n_i}\sum_{n_{i+1}=0}^{s-n_1...-n_j-n_i}...\sum_{n_{j-1}=0}^{s-n_1...-n_{j-2}-n_j}$$
$$\sum_{n_{j+1}=0}^{s-n_1...-n_j}...\sum_{n_{m_B-1}=0}^{s-n_1-...-n_{m_B-2}}\sum_{q=0}^{c-s-1}\left(n_i+1\right)\left(s+q\right)P_{n_1...n_i+1...n_j-1...n_{m_B}=s-n_1-...-n_{m_B-1};\ell,q+1}\left(t\right)\Bigg\}$$
$$+\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s}...\sum_{n_{m_B-1}=0}^{s-n_1-...-n_{m_B-2}}\sum_{q=0}^{c-s-1}n_i\left(s+q\right)P_{n_1...n_{m_B}=s-n_1-...-n_{m_B-1};\ell,q+1}\left(t\right)$$

Then we manipulate the limits for some terms

$$
\begin{aligned}
&\mathrm{E}'\left[N\left(t\right),\ell,2\right]=-\lambda_{\ell}\left(t\right)\mathrm{E}\left[N\left(t\right),\ell,2\right]\\
&-\sum_{i=1}^{m_B}\left(1-b_{ii}\left(t\right)\right)\mu_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_i\left(s+q\right)P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{E}\left[N\left(t\right),i,2\right]\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\ j\neq i}}^{m_B}b_{ji}\left(t\right)\mu_j\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_j\left(s+q\right)P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}sP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,0}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{E}\left[N\left(t\right),i,2\right]\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=i\right)\\
&-\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\left(c+1\right)P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,c-s}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}cP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,c-s}\left(t\right)\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\ j\neq i}}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_j\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_i\left(s+q\right)P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}\left(t\right)\\
&-\sum_{i=1}^{m_B}\sum_{\substack{j=1\\ j\neq i}}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_j\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}\left(t\right)\\
&-\sum_{i=1}^{m_B}\sum_{\substack{j=1\\ j\neq i}}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_j\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\left(s-1\right)n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}\left(t\right)\\
&+\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_i\left(s+q\right)P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}\left(t\right)\\
&-\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}\left(t\right)\\
&-\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\left(s-1\right)n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}\left(t\right)
\end{aligned}
$$

Canceling some terms

$$\begin{aligned}
&\mathrm{E}'\left[N\left(t\right),\ell,2\right]=-\lambda_{\ell}\left(t\right)\mathrm{E}\left[N\left(t\right),\ell,2\right]\\
&+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{E}\left[N\left(t\right),i,2\right]\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}sP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,0}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{E}\left[N\left(t\right),i,2\right]\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=i\right)\\
&-\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,c-s}\left(t\right)\\
&-\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}\left(t\right)\\
&-\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\left(s-1\right)n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}\left(t\right)
\end{aligned}$$

Finally

$$\begin{aligned}
\mathrm{E}'\left[N\left(t\right),\ell,2\right] &= -\lambda_{\ell}\left(t\right)\mathrm{E}\left[N\left(t\right),\ell,2\right]\\
&+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{E}\left[N\left(t\right),i,2\right]\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)s\mathrm{P}\left(N\left(t\right)=s,A\left(t\right)=i\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{E}\left[N\left(t\right),i,2\right]\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=i\right)\\
&-\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(N\left(t\right)=c,A\left(t\right)=i\right)\\
&-\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\mathrm{E}\left[N_i\left(t\right),I\left(t\right)=2,A\left(t\right)=\ell\right]\\
&-\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\left(s-1\right)\mathrm{E}\left[N_i\left(t\right),N\left(t\right)=s,A\left(t\right)=\ell\right]
\end{aligned}$$

Appendix 2

Derivation of the Zero Moment Subspace Ω_1

$$\mathrm{P}'\left(I\left(t\right)=1,A\left(t\right)=\ell\right)\equiv\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}}P'_{n_1\dots n_{m_B};\ell,0}\left(t\right)$$

Using the KFEs

$$\mathrm{P}'\left(I\left(t\right)=1,A\left(t\right)=\ell\right)=-\lambda_\ell\left(t\right)\mathrm{P}\left(I\left(t\right)=1,A\left(t\right)=\ell\right)$$
$$-\sum_{i=1}^{m_B}\left(1-b_{ii}\left(t\right)\right)\mu_i\left(t\right)\mathrm{E}\left[N_i\left(t\right),\ell,1\right]$$
$$+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}'\left(I\left(t\right)=1,A\left(t\right)=i\right)$$
$$+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}\left(t\right)\mu_j\left(t\right)\left\{\sum_{n_j=1}^{s-1}\sum_{n_k=0}^{s-1-n_j}\sum_{n_1=0}^{s-1-n_j-n_k}\dots\sum_{n_{j-1}=0}^{s-1-n_1\dots-n_{j-2}-n_j-n_i}\sum_{n_{j+1}=0}^{s-1-n_1\dots-n_j-n_i}\dots\right.$$
$$\left.\sum_{n_{i-1}=0}^{s-1-n_1\dots-n_{i-2}-n_i}\sum_{n_{i+1}=0}^{s-1-n_1\dots n_i}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}}\left(n_j+1\right)P_{n_1\dots n_i-1\dots n_k+1\dots n_{m_B};\ell,0}\left(t\right)\right\}$$
$$+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_i\left(t\right)\sum_{j=1}^{m_B}\beta_j\left(t\right)\left\{\sum_{n_j=1}^{s-1}\sum_{n_1=0}^{s-1-n_j}\dots\sum_{n_{j-1}=0}^{s-1-n_1\dots-n_{j-2}-n_j}\sum_{n_{j+1}=0}^{s-1-n_1\dots-n_j}\dots\right.$$
$$\left.\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}}P_{n_1\dots n_j-1\dots n_{m_B};i,0}\left(t\right)\right\}$$
$$+\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}}\left(n_i+1\right)P_{n_1\dots n_i+1\dots n_{m_B};\ell,0}\left(t\right)$$

Then we manipulate the limits for some terms

$$\mathrm{P}'\left(I\left(t\right)=1,A\left(t\right)=\ell\right)=-\lambda_{\ell}\left(t\right)\mathrm{P}\left(I\left(t\right)=1,A\left(t\right)=\ell\right)$$
$$-\sum_{i=1}^{m_B}\left(1-b_{ii}\left(t\right)\right)\mu_i\left(t\right)\mathrm{E}\left[N_i\left(t\right),\ell,1\right]$$
$$+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}'\left(I\left(t\right)=1,A\left(t\right)=i\right)$$
$$+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\ j\neq i}}^{m_B}b_{ji}\left(t\right)\mu_j\left(t\right)\mathrm{E}\left[N_j\left(t\right),\ell,1\right]$$
$$+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=1,A\left(t\right)=i\right)$$
$$-\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}P_{n_1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};i,0}\left(t\right)$$
$$+\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\mathrm{E}\left[N_i\left(t\right),\ell,1\right]$$
$$+\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}\left(t\right)$$

Canceling some terms

$$\mathrm{P}'\left(I\left(t\right)=1,A\left(t\right)=\ell\right)=-\lambda_{\ell}\left(t\right)\mathrm{P}\left(I\left(t\right)=1,A\left(t\right)=\ell\right)$$
$$+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}'\left(I\left(t\right)=1,A\left(t\right)=i\right)$$
$$+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=1,A\left(t\right)=i\right)$$
$$-\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}P_{n_1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};i,0}\left(t\right)$$
$$+\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}\left(t\right)$$

Finally

$$\begin{aligned}
\mathrm{P}'\left(I\left(t\right)=1,A\left(t\right)=\ell\right) \;=\; & -\lambda_{\ell}\left(t\right)\mathrm{P}\left(I\left(t\right)=1,A\left(t\right)=\ell\right) \\
& +\sum_{i=1}^{m_A} a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}'\left(I\left(t\right)=1,A\left(t\right)=i\right) \\
& +\sum_{i=1}^{m_A} a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=1,A\left(t\right)=i\right) \\
& -\sum_{i=1}^{m_A} a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(N\left(t\right)=s-1,A\left(t\right)=i\right) \\
& +\sum_{i=1}^{m_B} b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\mathrm{E}\left[N_i\left(t\right),N\left(t\right)=s,A\left(t\right)=\ell\right]
\end{aligned}$$

Derivation of the Zero Moment Subspace Ω_2

$$\mathrm{P}'\left(I\left(t\right)=2,A\left(t\right)=\ell\right)\equiv\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}P'_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}\left(t\right)$$

Using the KFEs

$$
\begin{aligned}
&\mathrm{P}'\left(I\left(t\right)=2,A\left(t\right)=\ell\right)=-\lambda_{\ell}\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=\ell\right)\\
&-\sum_{i=1}^{m_B}\left(1-b_{ii}\left(t\right)\right)\mu_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=i\right)\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}\left(t\right)\mu_j\left(t\right)\Bigg\{\sum_{n_j=0}^{s-1}\sum_{n_i=1}^{s-n_j}\sum_{n_1=0}^{s-n_i-n_j}\ldots\sum_{n_{j-1}=0}^{s-n_1\ldots-n_{j-2}-n_j-n_i}\sum_{n_{j+1}=0}^{s-n_1\ldots-n_j-n_i}\ldots\sum_{n_{i-1}=0}^{s-n_1\ldots-n_{i-2}-n_i}\\
&\sum_{n_{i+1}=0}^{s-n_1\ldots-n_i}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}\left(n_j+1\right)P_{n_1\ldots n_i-1\ldots n_j+1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}\left(t\right)\Bigg\}\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\Bigg\{\sum_{j=1}^{m_B}\beta_j\left(t\right)\sum_{n_j=1}^{s}\sum_{n_1=0}^{s-n_j}\ldots\sum_{n_{j-1}=0}^{s-n_1\ldots-n_{j-2}-n_j}\sum_{n_{j+1}=0}^{s-n_1\ldots-n_j}\ldots\\
&\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}P_{n_1\ldots n_j-1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,0}\left(t\right)\Bigg\}\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=1}^{c-s}P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,q-1}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,c-s}\left(t\right)\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_j\left(t\right)\Bigg\{\sum_{n_i=0}^{s-1}\sum_{n_j=1}^{s-n_i}\sum_{n_1=0}^{s-n_i-n_j}\ldots\sum_{n_{i-1}=0}^{s-n_1\ldots-n_{i-2}-n_i-n_j}\sum_{n_{i+1}=0}^{s-n_1\ldots-n_i-n_j}\ldots\sum_{n_{j-1}=0}^{s-n_1\ldots-n_{j-2}-n_j}\\
&\sum_{n_{j+1}=0}^{s-n_1\ldots-n_j}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s-1}\left(n_i+1\right)P_{n_1\ldots n_i+1\ldots n_j-1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q+1}\left(t\right)\Bigg\}\\
&+\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s-1}n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q+1}\left(t\right)
\end{aligned}
$$

Then we manipulate the limits for some terms

$$
\begin{aligned}
&\mathrm{P}'\left(I\left(t\right)=2,A\left(t\right)=\ell\right)=-\lambda_{\ell}\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=\ell\right)\\
&-\sum_{i=1}^{m_B}\left(1-b_{ii}\left(t\right)\right)\mu_i\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_iP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=i\right)\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}\left(t\right)\mu_j\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_jP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B-1}=0}^{s-1-n_1-\dots-n_{m_B-2}}P_{n_1\dots n_{m_B}=s-1-n_1-\dots-n_{m_B-1};i,0}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=i\right)\\
&-\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}P_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};i,c-s}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}P_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};i,c-s}\left(t\right)\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_j\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_iP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}\left(t\right)\\
&-\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_j\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}n_iP_{n_1\dots n_i+1\dots n_j-1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,0}\left(t\right)\\
&+\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_iP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}\left(t\right)\\
&-\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}n_iP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,0}\left(t\right)
\end{aligned}
$$

Canceling some terms

$$\begin{aligned}
&\mathrm{P}'\left(I\left(t\right)=2,A\left(t\right)=\ell\right)=-\lambda_{\ell}\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=\ell\right)\\
&+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=i\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B-1}=0}^{s-1-n_1-\dots-n_{m_B-2}}P_{n_1\dots n_{m_B}=s-1-n_1-\dots-n_{m_B-1};i,0}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=i\right)\\
&-\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}n_iP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,0}\left(t\right)
\end{aligned}$$

Finally

$$\begin{aligned}
\mathrm{P}'\left(I\left(t\right)=2,A\left(t\right)=\ell\right) &= -\lambda_{\ell}\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=\ell\right)\\
&+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=i\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(N\left(t\right)=s-1,A\left(t\right)=i\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{P}\left(I\left(t\right)=2,A\left(t\right)=i\right)\\
&-\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\mathrm{E}\left[N_i\left(t\right),N\left(t\right)=s,A\left(t\right)=\ell\right]
\end{aligned}$$

Appendix 3

Derivation of the *pth* Moment Subspace Ω_1

$$\mathrm{E}'\left[N_i^p\left(t\right),\ell,1\right] \equiv \mathrm{E}'\left[N_i^p\left(t\right),A\left(t\right)=\ell,I\left(t\right)=1\right]$$

$$\mathrm{E}'\left[N_i^p\left(t\right),A\left(t\right)=\ell,I\left(t\right)=1\right] \equiv \sum_{n_1=0}^{s-1} \ldots \sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_i^p P'_{n_1\ldots n_{m_B};\ell,0}\left(t\right)$$

Using the KFEs

$$
\begin{aligned}
&\mathrm{E}'\left[N_i^p(t),\ell,1\right] = -\lambda_\ell(t)\,\mathrm{E}\left[N_i^p(t),\ell,1\right] \\
&-\left(1-b_{ii}(t)\right)\mu_i(t)\,\mathrm{E}\left[N_i^{p+1}(t),\ell,1\right] \\
&-\sum_{\substack{j=1\\j\neq i}}^{m_B}\left(1-b_{jj}(t)\right)\mu_j(t)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_i^p n_j P_{n_1\ldots n_{m_B};\ell,0}(t) \\
&+\sum_{j=1}^{m_A} a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i^p(t),j,1\right] \\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{ji}(t)\,\mu_j(t)\Bigg\{\sum_{n_i=1}^{s-1}\sum_{n_1=0}^{s-1-n_i}\ldots\sum_{n_{i-1}=0}^{s-1-n_1\ldots-n_{i-2}-n_i}\sum_{n_{i+1}=0}^{s-1-n_1\ldots-n_i}\ldots \\
&\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_i^p\left(n_j+1\right)P_{n_1\ldots n_i-1\ldots n_j+1\ldots n_{m_B};\ell,0}(t)\Bigg\} \\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{ij}(t)\,\mu_i(t)\Bigg\{\sum_{n_j=1}^{s-1}\sum_{n_1=0}^{s-1-n_j}\ldots\sum_{n_{j-1}=0}^{s-1-n_1\ldots-n_{j-2}-n_j}\sum_{n_{j+1}=0}^{s-1-n_1\ldots-n_j}\ldots \\
&\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_i^p\left(n_i+1\right)P_{n_1\ldots n_j-1\ldots n_i+1\ldots n_{m_B};\ell,0}(t)\Bigg\} \\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B}\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{kj}(t)\,\mu_k(t)\Bigg\{\sum_{n_j=1}^{s-1}\sum_{n_k=0}^{s-1-n_j}\sum_{n_1=0}^{s-1-n_j-n_k}\ldots\sum_{n_{j-1}=0}^{s-1-n_1\ldots-n_{j-2}-n_j-n_k}\sum_{n_{j+1}=0}^{s-1-n_1\ldots-n_j-n_k}\ldots \\
&\sum_{n_{k-1}=0}^{s-1-n_1\ldots-n_{k-2}-n_k}\sum_{n_{k+1}=0}^{s-1-n_1\ldots-n_k}\ldots\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_i^p\left(n_k+1\right)P_{n_1\ldots n_j-1\ldots n_k+1\ldots n_{m_B};\ell,0}(t)\Bigg\} \\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\Bigg\{\sum_{n_i=1}^{s-1}\sum_{n_1=0}^{s-1-n_i}\ldots\sum_{n_{i-1}=0}^{s-1-n_1\ldots-n_{i-2}-n_i}\sum_{n_{i+1}=0}^{s-1-n_1\ldots-n_i}\ldots \\
&\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_i^p P_{n_1\ldots n_i-1\ldots n_{m_B};j,0}(t)\Bigg\} \\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{\substack{k=1\\k\neq i}}^{m_B}\beta_k(t)\Bigg\{\sum_{n_k=1}^{s-1}\sum_{n_1=0}^{s-1-n_k}\ldots\sum_{n_{k-1}=0}^{s-1-n_1\ldots-n_{k-2}-n_k}\sum_{n_{k+1}=0}^{s-1-n_1\ldots-n_k}\ldots \\
&\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_i^p P_{n_1\ldots n_k-1\ldots n_{m_B};j,0}(t)\Bigg\} \\
&+b_{i,m_B+1}(t)\,\mu_i(t)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_i^p\left(n_i+1\right)P_{n_1\ldots n_i+1\ldots n_{m_B};\ell,0}(t) \\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_i^p\left(n_j+1\right)P_{n_1\ldots n_j+1\ldots n_{m_B};\ell,0}(t)
\end{aligned}
$$

Then we manipulate the limits for some terms

$$\mathrm{E}'\left[N_i^p\left(t\right),\ell,1\right]=-\lambda_\ell\left(t\right)\mathrm{E}\left[N_i^p\left(t\right),\ell,1\right]$$

$$-\left(1-b_{ii}\left(t\right)\right)\mu_i\left(t\right)\mathrm{E}\left[N_i^{p+1}\left(t\right),\ell,1\right]$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m_B}\left(1-b_{jj}\left(t\right)\right)\mu_j\left(t\right)\sum_{n_1=0}^{s-1}...\sum_{n_{m_B}=0}^{s-1-n_1-...-n_{m_B-1}}n_i^p n_j P_{n_1...n_{m_B};\ell,0}\left(t\right)$$

$$+\sum_{j=1}^{m_A}a_{j\ell}\left(t\right)\lambda_j\left(t\right)\mathrm{E}\left[N_i^p\left(t\right),j,1\right]$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}\left(t\right)\mu_j\left(t\right)\sum_{n_1=0}^{s-1}...\sum_{n_{m_B}=0}^{s-1-n_1-...-n_{m_B-1}}\left(n_i+1\right)^p n_j P_{n_1...n_{m_B};\ell,0}\left(t\right)$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ij}\left(t\right)\mu_i\left(t\right)\sum_{n_1=0}^{s-1}...\sum_{n_{m_B}=0}^{s-1-n_1-...-n_{m_B-1}}\left(n_i-1\right)^p n_i P_{n_1...n_{m_B};\ell,0}\left(t\right)$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}\sum_{\substack{k=1\\k\neq i,j}}^{m_B}b_{kj}\left(t\right)\mu_k\left(t\right)\sum_{n_1=0}^{s-1}...\sum_{n_{m_B}=0}^{s-1-n_1-...-n_{m_B-1}}n_i^p n_k P_{n_1...n_{m_B};\ell,0}\left(t\right)$$

$$+\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s-1}...\sum_{n_{m_B}=0}^{s-1-n_1-...-n_{m_B-1}}\left(n_i+1\right)^p P_{n_1...n_{m_B};j,0}\left(t\right)$$

$$-\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s-1}...\sum_{n_{m_B-1}=0}^{s-1-n_1-...-n_{m_B-2}}\left(n_i+1\right)^p P_{n_1...n_i-1...n_{m_B}=s-1-n_1-...-n_{m_B-1};j,0}\left(t\right)$$

$$+\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\sum_{\substack{k=1\\k\neq i}}^{m_B}\beta_k\left(t\right)\mathrm{E}\left[N_i^p\left(t\right),j,1\right]$$

$$-\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\sum_{\substack{k=1\\k\neq i}}^{m_B}\beta_k\left(t\right)\sum_{n_1=0}^{s-1}...\sum_{n_{m_B-1}=0}^{s-1-n_1-...-n_{m_B-2}}n_i^p P_{n_1...n_{m_B}=s-1-n_1-...-n_{m_B-1};j,0}\left(t\right)$$

$$+b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\sum_{n_1=0}^{s-1}...\sum_{n_{m_B}=0}^{s-1-n_1-...-n_{m_B-1}}\left(n_i-1\right)^p n_i P_{n_1...n_{m_B};\ell,0}\left(t\right)$$

$$+b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\sum_{n_1=0}^{s}...\sum_{n_{m_B-1}=0}^{s-n_1-...-n_{m_B-2}}\left(n_i-1\right)^p n_i P_{n_1...n_{m_B}=s-n_1-...-n_{m_B-1};\ell,0}\left(t\right)$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\sum_{n_1=0}^{s-1}...\sum_{n_{m_B}=0}^{s-1-n_1-...-n_{m_B-1}}n_i^p n_j P_{n_1...n_{m_B};\ell,0}\left(t\right)$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\sum_{n_1=0}^{s}...\sum_{n_{m_B-1}=0}^{s-n_1-...-n_{m_B-2}}n_i^p n_j P_{n_1...n_{m_B}=s-n_1-...-n_{m_B-1};\ell,0}\left(t\right)$$

Expanding some terms

$$\mathrm{E}'\left[N_i^p(t), \ell, 1\right] = -\lambda_\ell(t)\,\mathrm{E}\left[N_i^p(t), \ell, 1\right]$$

$$-\left(1-b_{ii}(t)\right)\mu_i(t)\,\mathrm{E}\left[N_i^{p+1}(t), \ell, 1\right]$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m_B}\left(1-b_{jj}(t)\right)\mu_j(t)\,\mathrm{E}\left[N_i^p(t)\,N_j(t), \ell, 1\right]$$

$$+\sum_{j=1}^{m_A} a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i^p(t), j, 1\right]$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_i^p(t)\,N_j(t), \ell, 1\right]$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{ji}(t)\,\mu_j(t)\sum_{h=1}^{p}\binom{p}{h}\mathrm{E}\left[N_i^{p-h}(t)\,N_j(t), \ell, 1\right]$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{ij}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i^{p+1}(t), \ell, 1\right]$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{ij}(t)\,\mu_i(t)\sum_{h=0}^{p-1}\binom{p}{h+1}(-1)^{h+1}\,\mathrm{E}\left[N_i^{p-h}(t), \ell, 1\right]$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{kj}(t)\,\mu_k(t)\,\mathrm{E}\left[N_i^p(t)\,N_k(t), \ell, 1\right]$$

$$+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\,\mathrm{E}\left[N_i^p(t), j, 1\right]$$

$$+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\sum_{h=1}^{p}\binom{p}{h}\mathrm{E}\left[N_i^{p-h}(t), j, 1\right]$$

$$-\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}(n_i+1)^p\,P_{n_1\ldots n_i-1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};j,0}(t)$$

$$+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{\substack{k=1\\k\neq i}}^{m_B}\beta_k(t)\,\mathrm{E}\left[N_i^p(t), j, 1\right]$$

$$-\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{\substack{k=1\\k\neq i}}^{m_B}\beta_k(t)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}} n_i^p P_{n_1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};j,0}(t)$$

$$+b_{i,m_B+1}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i^{p+1}(t),\ell,1\right]$$
$$+b_{i,m_B+1}(t)\,\mu_i(t)\sum_{h=0}^{p-1}\binom{p}{h+1}(-1)^{h+1}\,\mathrm{E}\left[N_i^{p-h}(t),\ell,1\right]$$
$$+b_{i,m_B+1}(t)\,\mu_i(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}(n_i-1)^p\,n_iP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,0}(t)$$
$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{j,m_B+1}(t)\,\mu_j(t)\,\mathrm{E}\left[N_i^p(t)\,N_j(t),\ell,1\right]$$
$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{j,m_B+1}(t)\,\mu_j(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}n_i^pn_jP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,0}(t)$$

Canceling some terms

$$\mathrm{E}'\left[N_i^p(t),\ell,1\right]=-\lambda_\ell(t)\,\mathrm{E}\left[N_i^p(t),\ell,1\right]$$
$$+\sum_{j=1}^{m_A}a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i^p(t),j,1\right]$$
$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}(t)\,\mu_j(t)\sum_{h=1}^{p}\binom{p}{h}\mathrm{E}\left[N_i^{p-h}(t)\,N_j(t),\ell,1\right]$$
$$+(1-b_{ii})\,\mu_i(t)\sum_{h=0}^{p-1}\binom{p}{h+1}(-1)^{h+1}\,\mathrm{E}\left[N_i^{p-h}(t),\ell,1\right]$$
$$+\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i^p(t),j,1\right]$$
$$+\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\sum_{h=1}^{p}\binom{p}{h}\mathrm{E}\left[N_i^{p-h}(t),j,1\right]$$
$$-\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B-1}=0}^{s-1-n_1-\dots-n_{m_B-2}}(n_i+1)^p\,P_{n_1\dots n_i-1\dots n_{m_B}=s-1-n_1-\dots-n_{m_B-1};j,0}(t)$$
$$-\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{\substack{k=1\\k\neq i}}^{m_B}\beta_k(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B-1}=0}^{s-1-n_1-\dots-n_{m_B-2}}n_i^pP_{n_1\dots n_{m_B}=s-1-n_1-\dots-n_{m_B-1};j,0}(t)$$
$$+b_{i,m_B+1}(t)\,\mu_i(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}(n_i-1)^p\,n_iP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,0}(t)$$
$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{j,m_B+1}(t)\,\mu_j(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}n_i^pn_jP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,0}(t)$$

Finally

$$\begin{aligned}
\mathrm{E}'\left[N_i^p(t),\ell,1\right] &= -\lambda_\ell(t)\,\mathrm{E}\left[N_i^p(t),\ell,1\right] \\
&+\sum_{j=1}^{m_A} a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i^p(t),j,1\right] \\
&+\sum_{\substack{j=1\\ j\neq i}}^{m_B} b_{ji}(t)\,\mu_j(t)\sum_{h=1}^{p}\binom{p}{h}\mathrm{E}\left[N_i^{p-h}(t)\,N_j(t),\ell,1\right] \\
&+\left(1-b_{ii}(t)\right)\mu_i(t)\sum_{h=0}^{p-1}\binom{p}{h+1}(-1)^{h+1}\,\mathrm{E}\left[N_i^{p-h}(t),\ell,1\right] \\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i^p(t),j,1\right] \\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\sum_{h=1}^{p}\binom{p}{h}\mathrm{E}\left[N_i^{p-h}(t),j,1\right] \\
&-\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\sum_{h=1}^{p}\binom{p}{h}\mathrm{E}\left[N_i^{p-h}(t),N(t)=s-1,A(t)=j\right] \\
&-\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i^p(t),N(t)=s-1,A(t)=j\right] \\
&+b_{i,m_B+1}(t)\,\mu_i(t)\sum_{h=0}^{p-1}\binom{p}{h+1}(-1)^{h+1}\,\mathrm{E}\ \left[N_i^{p-h}(t),N(t)=s,A(t)=\ell\right] \\
&+\sum_{j=1}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\mathrm{E}\left[N_i^p(t)\,N_j(t),N(t)=s,A(t)=\ell\right]
\end{aligned}$$

Derivation of the *pth* Moment Subspace Ω_2

$$\mathrm{E}'\left[N^p(t),\ell,2\right]\equiv\mathrm{E}'\left[N^p(t),A(t)=\ell,I(t)=2\right]$$

$$\mathrm{E}'\left[N^p(t),A(t)=\ell,I(t)=2\right]\equiv\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\ \sum_{q=0}^{c-s}(s+q)^p\,P'_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)$$

Using the KFEs

$$
\begin{aligned}
&\mathrm{E}'\left[N^{p}\left(t\right),\ell,2\right]=-\lambda_{\ell}\left(t\right)\mathrm{E}\left[N^{p}\left(t\right),\ell,2\right]\\
&-\sum_{i=1}^{m_B}\left(1-b_{ii}\left(t\right)\right)\mu_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_i\left(s+q\right)^{p}P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_i\left(t\right)\mathrm{E}\left[N^{p}\left(t\right),i,2\right]\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}\left(t\right)\mu_j\left(t\right)\left\{\sum_{n_j=0}^{s-1}\sum_{n_i=1}^{s-n_j}\sum_{n_1=0}^{s-n_i-n_j}\ldots\sum_{n_{j-1}=0}^{s-n_1\ldots-n_{j-2}-n_j-n_i}\sum_{n_{j+1}=0}^{s-n_1\ldots-n_j-n_i}\ldots\sum_{n_{i-1}=0}^{s-n_1\ldots-n_{i-2}-n_i}\right.\\
&\left.\sum_{n_{i+1}=0}^{s-n_1\ldots-n_i}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}\left(n_j+1\right)\left(s+q\right)^{p}P_{n_1\ldots n_i-1\ldots n_j+1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}\left(t\right)\right\}\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\left\{\sum_{j=1}^{m_B}\beta_j\left(t\right)\sum_{n_j=1}^{s}\sum_{n_1=0}^{s-n_j}\ldots\sum_{n_{j-1}=0}^{s-n_1\ldots-n_{j-2}-n_j}\right.\\
&\left.\sum_{n_{j+1}=0}^{s-n_1\ldots-n_j}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}s^{p}P_{n_1\ldots n_j-1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,0}\left(t\right)\right\}\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=1}^{c-s}\left(s+q\right)^{p}P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,q-1}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}c^{p}P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,c-s}\left(t\right)\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_j\left(t\right)\left\{\sum_{n_i=0}^{s-1}\sum_{n_j=1}^{s-n_i}\sum_{n_1=0}^{s-n_i-n_j}\ldots\sum_{n_{i-1}=0}^{s-n_1\ldots-n_{i-2}-n_i-n_j}\sum_{n_{i+1}=0}^{s-n_1\ldots-n_i-n_j}\ldots\sum_{n_{j-1}=0}^{s-n_1\ldots-n_{j-2}-n_j}\right.\\
&\left.\sum_{n_{j+1}=0}^{s-n_1\ldots-n_j}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s-1}\left(n_i+1\right)\left(s+q\right)^{p}P_{n_1\ldots n_i+1\ldots n_j-1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q+1}\left(t\right)\right\}\\
&+\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s-1}n_i\left(s+q\right)^{p}P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q+1}\left(t\right)
\end{aligned}
$$

Then we manipulate the limits for some terms

$$
\begin{aligned}
&\mathrm{E}'\left[N^p(t),\ell,2\right]=-\lambda_\ell(t)\,\mathrm{E}\left[N^p(t),\ell,2\right]\\
&-\sum_{i=1}^{m_B}(1-b_{ii}(t))\,\mu_i(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_i(s+q)^p\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)\\
&+\sum_{i=1}^{m_A}a_{i\ell}(t)\,\lambda_i(t)\,\mathrm{E}\left[N^p(t),i,2\right]\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}(t)\,\mu_j(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_j(s+q)^p\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\sum_{j=1}^{m_B}\beta_j(t)\,s^p\sum_{n_1=0}^{s-1}\cdots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}P_{n_1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};i,0}(t)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}(s+q+1)^p\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,q}(t)\\
&-\sum_{i=1}^{m_A}a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\,(c+1)^p\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,c-s}(t)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}c^pP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,c-s}(t)\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{i,m_B+1}(t)\,\mu_i(t)\,\beta_j(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_i(s+q-1)^p\,P_{n_1..n_j-1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)\\
&-\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{i,m_B+1}(t)\,\mu_i(t)\,\beta_j(t)\,(s-1)^p\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)\\
&+\sum_{i=1}^{m_B}b_{i,m_B+1}(t)\,\mu_i(t)\,\beta_i(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_i(s+q-1)^p\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)\\
&-\sum_{i=1}^{m_B}b_{i,m_B+1}(t)\,\mu_i(t)\,\beta_i(t)\,(s-1)^p\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)
\end{aligned}
$$

Expanding some terms

$$
\begin{aligned}
&\mathrm{E}'\left[N^{p}\left(t\right),\ell,2\right]=-\lambda_{\ell}\left(t\right)\mathrm{E}\left[N^{p}\left(t\right),\ell,2\right]\\
&-\sum_{i=1}^{m_B}\left(1-b_{ii}\left(t\right)\right)\mu_{i}\left(t\right)\mathrm{E}\left[N_{i}\left(t\right)N^{p}\left(t\right),I\left(t\right)=2,A\left(t\right)=\ell\right]\\
&+\sum_{i=1}^{m_A}a_{i\ell}\left(t\right)\lambda_{i}\left(t\right)\mathrm{E}\left[N^{p}\left(t\right),i,2\right]\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}\left(t\right)\mu_{j}\left(t\right)\mathrm{E}\left[N_{j}\left(t\right)N^{p}\left(t\right),I\left(t\right)=2,A\left(t\right)=\ell\right]\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_{i}\left(t\right)\sum_{j=1}^{m_B}\beta_{j}\left(t\right)s^{p}\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}P_{n_1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};i,0}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_{i}\left(t\right)\sum_{h=0}^{p}\binom{p}{h}\mathrm{E}\left[N^{p-h}\left(t\right),i,2\right]\\
&-\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_{i}\left(t\right)\sum_{h=0}^{p}\binom{p}{h}c^{p-h}\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,c-s}\left(t\right)\\
&+\sum_{i=1}^{m_A}a_{i,m_A+1}\left(t\right)\alpha_{\ell}\left(t\right)\lambda_{i}\left(t\right)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}c^{p}P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,c-s}\left(t\right)\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{i,m_B+1}\left(t\right)\mu_{i}\left(t\right)\beta_{j}\left(t\right)\mathrm{E}\left[N_{i}\left(t\right)N^{p}\left(t\right),I\left(t\right)=2,A\left(t\right)=\ell\right]\\
&+\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{i,m_B+1}\left(t\right)\mu_{i}\left(t\right)\beta_{j}\left(t\right)\sum_{h=1}^{p}\binom{p}{h}\left(-1\right)^{h}\mathrm{E}\left[N_{i}\left(t\right)N^{p-h}\left(t\right),I\left(t\right)=2,A\left(t\right)=\ell\right]\\
&-\sum_{i=1}^{m_B}\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{i,m_B+1}\left(t\right)\mu_{i}\left(t\right)\beta_{j}\left(t\right)\left(s-1\right)^{p}\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}n_{i}P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}\left(t\right)\\
&+\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_{i}\left(t\right)\beta_{i}\left(t\right)\mathrm{E}\left[N_{i}\left(t\right)N^{p}\left(t\right),I\left(t\right)=2,A\left(t\right)=\ell\right]\\
&+\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_{i}\left(t\right)\beta_{i}\left(t\right)\sum_{h=1}^{p}\binom{p}{h}\left(-1\right)^{h}\mathrm{E}\left[N_{i}\left(t\right)N^{p-h}\left(t\right),I\left(t\right)=2,A\left(t\right)=\ell\right]\\
&-\sum_{i=1}^{m_B}b_{i,m_B+1}\left(t\right)\mu_{i}\left(t\right)\beta_{i}\left(t\right)\left(s-1\right)^{p}\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}n_{i}P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}\left(t\right)
\end{aligned}
$$

Canceling some terms

$$
\begin{aligned}
&\mathrm{E}'\left[N^p(t),\ell,2\right] = -\lambda_\ell(t)\,\mathrm{E}\left[N^p(t),\ell,2\right] \\
&+\sum_{i=1}^{m_A} a_{i\ell}(t)\,\lambda_i(t)\,\mathrm{E}\left[N^p(t),i,2\right] \\
&+\sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\,s^p\sum_{n_1=0}^{s-1}\cdots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}} P_{n_1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};i,0}(t) \\
&+\sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\sum_{h=0}^{p}\binom{p}{p}\mathrm{E}\left[N^{p-h}(t),i,2\right] \\
&-\sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\sum_{h=1}^{p}\binom{p}{h}c^{p-h}\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}} P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};i,c-s}(t) \\
&+\sum_{i=1}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\sum_{h=1}^{p}\binom{p}{h}(-1)^h\,\mathrm{E}\left[N_i(t)\,N^{p-h}(t),I(t)=2,A(t)=\ell\right] \\
&-\sum_{i=1}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\,(s-1)^p\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}} n_i P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)
\end{aligned}
$$

Finally

$$
\begin{aligned}
\mathrm{E}'\left[N^p(t),\ell,2\right] &= -\lambda_\ell(t)\,\mathrm{E}\left[N^p(t),\ell,2\right] \\
&+\sum_{i=1}^{m_A} a_{i\ell}(t)\,\lambda_i(t)\,\mathrm{E}\left[N^p(t),i,2\right] \\
&+\sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\,s^p\mathrm{P}\left(N(t)=s-1,A(t)=i\right) \\
&+\sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\sum_{h=0}^{p}\binom{p}{h}\mathrm{E}\left[N^{p-h}(t),i,2\right] \\
&-\sum_{i=1}^{m_A} a_{i,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_i(t)\sum_{h=1}^{p}\binom{p}{h}c^{p-h}\mathrm{P}\left(N(t)=c,A(t)=i\right) \\
&+\sum_{i=1}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\sum_{h=1}^{p}\binom{p}{h}(-1)^h\,\mathrm{E}\left[N_i(t)\,N^{p-h}(t),I(t)=2,A(t)=\ell\right] \\
&-\sum_{i=1}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\,(s-1)^p\,\mathrm{E}\left[N_i(t),N(t)=s,A(t)=\ell\right]
\end{aligned}
$$

Appendix 4

Derivation of the Cross-product Moments Subspace Ω_1

$$\mathrm{E}'\left[N_i\left(t\right)N_j\left(t\right),\ell,1\right]\equiv\mathrm{E}'\left[N_i\left(t\right)N_j\left(t\right),A\left(t\right)=\ell,I\left(t\right)=1\right]$$

$$\mathrm{E}'\left[N_i\left(t\right)N_j\left(t\right),A\left(t\right)=\ell,I\left(t\right)=1\right]\equiv\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}}n_in_jP'_{n_1\ldots n_{m_B};\ell,0}\left(t\right)$$

Using the KFEs

$$\begin{aligned}
&\mathrm{E}'\left[N_i(t)\,N_j(t),\ell,1\right] = -\lambda_\ell(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right] \\
&-\left(1-b_{ii}(t)\right)\mu_i(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i^2 n_j P_{n_1\dots n_{m_B};\ell,0}(t) \\
&-\left(1-b_{jj}(t)\right)\mu_j(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j^2 P_{n_1\dots n_{m_B};\ell,0}(t) \\
&-\sum_{\substack{k=1\\k\neq i,j}}^{m_B}\left(1-b_{kk}(t)\right)\mu_k(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j n_k P_{n_1\dots n_{m_B};\ell,0}(t) \\
&+\sum_{k=1}^{m_A} a_{k\ell}(t)\,\lambda_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),k,1\right] \\
&+b_{ji}(t)\,\mu_j(t)\left\{\sum_{n_i=1}^{s-1}\sum_{n_1=0}^{s-1-n_i}\dots\sum_{n_{i-1}=0}^{s-1-n_1\dots-n_{i-2}-n_i}\sum_{n_{i+1}=0}^{s-1-n_1\dots-n_i}\dots\right. \\
&\left.\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j\left(n_j+1\right)P_{n_1\dots n_i-1\dots n_j+1\dots n_{m_B};\ell,0}(t)\right\} \\
&+b_{ij}(t)\,\mu_i(t)\left\{\sum_{n_j=1}^{s-1}\sum_{n_1=0}^{s-1-n_j}\dots\sum_{n_{j-1}=0}^{s-1-n_1\dots-n_{j-2}-n_j}\sum_{n_{j+1}=0}^{s-1-n_1\dots-n_j}\dots\right. \\
&\left.\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j\left(n_i+1\right)P_{n_1\dots n_j-1\dots n_i+1\dots n_{m_B};\ell,0}(t)\right\} \\
&+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{ki}(t)\,\mu_k(t)\left\{\sum_{n_i=1}^{s-1}\sum_{n_1=0}^{s-1-n_i}\dots\sum_{n_{i-1}=0}^{s-1-n_1\dots-n_{i-2}-n_i}\sum_{n_{i+1}=0}^{s-1-n_1\dots-n_i}\dots\right. \\
&\left.\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j\left(n_k+1\right)P_{n_1\dots n_i-1\dots n_k+1\dots n_{m_B};\ell,0}(t)\right\} \\
&+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{ik}(t)\,\mu_i(t)\left\{\sum_{n_k=1}^{s-1}\sum_{n_1=0}^{s-1-n_k}\dots\sum_{n_{k-1}=0}^{s-1-n_1\dots-n_{k-2}-n_k}\sum_{n_{k+1}=0}^{s-1-n_1\dots-n_k}\dots\right. \\
&\left.\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j\left(n_i+1\right)P_{n_1\dots n_k-1\dots n_i+1\dots n_{m_B};\ell,0}(t)\right\} \\
&+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{kj}(t)\,\mu_k(t)\left\{\sum_{n_j=1}^{s-1}\sum_{n_1=0}^{s-1-n_j}\dots\sum_{n_{j-1}=0}^{s-1-n_1\dots-n_{j-2}-n_j}\sum_{n_{j+1}=0}^{s-1-n_1\dots-n_j}\dots\right. \\
&\left.\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j\left(n_k+1\right)P_{n_1\dots n_j-1\dots n_k+1\dots n_{m_B};\ell,0}(t)\right\} \\
&+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{jk}(t)\,\mu_j(t)\left\{\sum_{n_k=1}^{s-1}\sum_{n_1=0}^{s-1-n_k}\dots\sum_{n_{k-1}=0}^{s-1-n_1\dots-n_{k-2}-n_k}\sum_{n_{k+1}=0}^{s-1-n_1\dots-n_k}\dots\right. \\
&\left.\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j\left(n_j+1\right)P_{n_1\dots n_k-1\dots n_j+1\dots n_{m_B};\ell,0}(t)\right\}
\end{aligned}$$

$$+\sum_{\substack{k=1\\k\neq i,j}}^{m_B}\sum_{\substack{l=1\\l\neq i,j,k}}^{m_B} b_{lk}\left(t\right)\mu_l\left(t\right)\left\{\sum_{n_k=1}^{s-1}\sum_{n_1=0}^{s-1-n_k}\ldots\sum_{n_{k-1}=0}^{s-1-n_1\ldots-n_{k-2}-n_k}\sum_{n_{k+1}=0}^{s-1-n_1\ldots-n_k}\ldots\right.$$
$$\left.\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_in_j\left(n_l+1\right)P_{n_1\ldots n_k-1\ldots n_l+1\ldots n_{m_B};\ell,0}\left(t\right)\right\}$$
$$+\sum_{k=1}^{m_A} a_{km_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_k\left(t\right)\beta_i\left(t\right)\left\{\sum_{n_i=1}^{s-1}\sum_{n_1=0}^{s-1-n_i}\ldots\sum_{n_{i-1}=0}^{s-1-n_1\ldots-n_{i-2}-n_i}\sum_{n_{i+1}=0}^{s-1-n_1\ldots-n_i}\ldots\right.$$
$$\left.\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_in_jP_{n_1\ldots n_i-1\ldots n_{m_B};j,0}\left(t\right)\right\}$$
$$+\sum_{k=1}^{m_A} a_{km_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_k\left(t\right)\beta_j\left(t\right)\left\{\sum_{n_j=1}^{s-1}\sum_{n_1=0}^{s-1-n_j}\ldots\sum_{n_{j-1}=0}^{s-1-n_1\ldots-n_{j-2}-n_j}\sum_{n_{j+1}=0}^{s-1-n_1\ldots-n_j}\ldots\right.$$
$$\left.\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_in_jP_{n_1\ldots n_j-1\ldots n_{m_B};j,0}\left(t\right)\right\}$$
$$+\sum_{k=1}^{m_A} a_{km_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_k\left(t\right)\sum_{\substack{l=1\\l\neq i,j}}^{m_B}\beta_l\left(t\right)\left\{\sum_{n_l=1}^{s-1}\sum_{n_1=0}^{s-1-n_l}\ldots\sum_{n_{l-1}=0}^{s-1-n_1\ldots-n_{l-2}-n_l}\sum_{n_{l+1}=0}^{s-1-n_1\ldots-n_l}\ldots\right.$$
$$\left.\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_in_jP_{n_1\ldots n_l-1\ldots n_{m_B};j,0}\left(t\right)\right\}$$
$$+b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_in_j\left(n_i+1\right)P_{n_1\ldots n_i+1\ldots n_{m_B};\ell,0}\left(t\right)$$
$$+b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_in_j\left(n_j+1\right)P_{n_1\ldots n_j+1\ldots n_{m_B};\ell,0}\left(t\right)$$
$$+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{km_B+1}\left(t\right)\mu_k\left(t\right)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_in_j\left(n_k+1\right)P_{n_1\ldots n_k+1\ldots n_{m_B};\ell,0}\left(t\right)$$

Then manipulating the limits of some terms and expanding

$$
\begin{aligned}
&\mathrm{E}'\left[N_i(t)\,N_j(t),\ell,1\right] = -\lambda_\ell(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right] \\
&-(1-b_{ii}(t))\,\mu_i(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i^2 n_j P_{n_1\dots n_{m_B};\ell,0}(t) \\
&-(1-b_{jj}(t))\,\mu_j(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j^2 P_{n_1\dots n_{m_B};\ell,0}(t) \\
&-\sum_{\substack{k=1\\k\neq i,j}}^{m_B}(1-b_{kk}(t))\,\mu_k(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j n_k P_{n_1\dots n_{m_B};\ell,0}(t) \\
&+\sum_{k=1}^{m_A} a_{k\ell}(t)\,\lambda_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),k,1\right] \\
&+b_{ji}(t)\,\mu_j(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j^2 P_{n_1\dots n_{m_B};\ell,0}(t) \\
&+b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_j^2(t),\ell,1\right] \\
&-b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right] \\
&-b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_j(t),\ell,1\right] \\
&+b_{ij}(t)\,\mu_i(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i^2 n_j P_{n_1\dots n_{m_B};\ell,0}(t) \\
&+b_{ij}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i^2(t),\ell,1\right] \\
&-b_{ij}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right] \\
&-b_{ij}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t),\ell,1\right] \\
&+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{ki}(t)\,\mu_k(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j n_k P_{n_1\dots n_{m_B};\ell,0}(t) \\
&+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{ki}(t)\,\mu_k(t)\,\mathrm{E}\left[N_j(t)\,N_k(t),\ell,1\right] \\
&+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{ik}(t)\,\mu_i(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i^2 n_j P_{n_1\dots n_{m_B};\ell,0}(t) \\
&-\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{ik}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right]
\end{aligned}
$$

$$+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{kj}(t)\,\mu_k(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j n_k P_{n_1\dots n_{m_B};\ell,0}(t)$$
$$+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{kj}(t)\,\mu_k(t)\,\mathrm{E}\left[N_i(t)\,N_k(t),\ell,1\right]$$
$$+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{jk}(t)\,\mu_j(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j^2 P_{n_1\dots n_{m_B};\ell,0}(t)$$
$$-\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{jk}(t)\,\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right]$$
$$+\sum_{\substack{k=1\\k\neq i,j}}^{m_B}\sum_{\substack{l=1\\l\neq i,j,k}}^{m_B} b_{lk}(t)\,\mu_l(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i n_j n_k P_{n_1\dots n_{m_B};\ell,0}(t)$$
$$+\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_i(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),k,1\right]$$
$$+\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_i(t)\,\mathrm{E}\left[N_j(t),k,1\right]$$
$$-\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_i(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B-1}=0}^{s-1-n_1-\dots-n_{m_B-2}} n_i n_j P_{n_1\dots n_{m_B}=s-1-n_1-\dots-n_{m_B-1};k,0}(t)$$
$$-\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_i(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B-1}=0}^{s-1-n_1-\dots-n_{m_B-2}} n_j P_{n_1\dots n_{m_B}=s-1-n_1-\dots-n_{m_B-1};k,0}(t)$$
$$+\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),k,1\right]$$
$$+\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_j(t)\,\mathrm{E}\left[N_i(t),k,1\right]$$
$$-\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_j(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B-1}=0}^{s-1-n_1-\dots-n_{m_B-2}} n_i n_j P_{n_1\dots n_{m_B}=s-1-n_1-\dots-n_{m_B-1};k,0}(t)$$
$$-\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_j(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B-1}=0}^{s-1-n_1-\dots-n_{m_B-2}} n_i P_{n_1\dots n_{m_B}=s-1-n_1-\dots-n_{m_B-1};k,0}(t)$$
$$+\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\sum_{\substack{l=1\\l\neq i,j}}^{m_B}\beta_l(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),k,1\right]$$
$$-\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\sum_{\substack{l=1\\l\neq i,j}}^{m_B}\beta_l(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B-1}=0}^{s-1-n_1-\dots-n_{m_B-2}} n_i n_j P_{n_1\dots n_{m_B}=s-1-n_1-\dots-n_{m_B-1};k,0}(t)$$
$$+b_{i,m_B+1}(t)\,\mu_i(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B}=0}^{s-1-n_1-\dots-n_{m_B-1}} n_i^2 n_j P_{n_1\dots n_{m_B};\ell,0}(t)$$

$$-b_{i,m_B+1}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t)\,N_j(t)\,,\ell,1\right]$$
$$+b_{i,m_B+1}(t)\,\mu_i(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}} n_i\left(n_i-1\right)n_j P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)$$
$$+b_{j,m_B+1}(t)\,\mu_j(t)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_i n_j^2 P_{n_1\ldots n_{m_B};\ell,0}(t)$$
$$-b_{j,m_B+1}(t)\,\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t)\,,\ell,1\right]$$
$$+b_{j,m_B+1}(t)\,\mu_j(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}} n_i n_j\left(n_j-1\right)P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)$$
$$+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{km_B+1}(t)\,\mu_k(t)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_i n_j n_k P_{n_1\ldots n_{m_B};\ell,0}(t)$$
$$+\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{km_B+1}(t)\,\mu_k(t)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B}=0}^{s-1-n_1-\ldots-n_{m_B-1}} n_i n_j n_k P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)$$

Canceling some terms

$$
\begin{aligned}
&\mathrm{E}'\left[N_i(t)\,N_j(t),\ell,1\right]=-\lambda_\ell(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right]\\
&-\left(1-b_{ii}(t)\right)\mu_i(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right]\\
&-\left(1-b_{jj}(t)\right)\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),\ell,1\right]\\
&+\sum_{k=1}^{m_A}a_{k\ell}(t)\,\lambda_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),k,1\right]\\
&+b_{ji}(t)\,\mu_j(t)\left(\mathrm{E}\left[N_j^2(t),\ell,1\right]-\mathrm{E}\left[N_j(t),\ell,1\right]\right)\\
&+b_{ij}(t)\,\mu_i(t)\left(\mathrm{E}\left[N_i^2(t),\ell,1\right]-\mathrm{E}\left[N_i(t),\ell,1\right]\right)\\
&+\sum_{\substack{k=1\\k\neq i,j}}^{m_B}b_{ki}(t)\,\mu_k(t)\,\mathrm{E}\left[N_j(t)\,N_k(t),\ell,1\right]\\
&+\sum_{\substack{k=1\\k\neq i,j}}^{m_B}b_{kj}(t)\,\mu_k(t)\,\mathrm{E}\left[N_i(t)\,N_k(t),\ell,1\right]\\
&+\sum_{k=1}^{m_A}a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t),k,1\right]\\
&+\sum_{k=1}^{m_A}a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\left(\beta_i(t)\,\mathrm{E}\left[N_j(t),k,1\right]+\beta_j(t)\,\mathrm{E}\left[N_i(t),k,1\right]\right)\\
&-\sum_{k=1}^{m_A}a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}n_in_jP_{n_1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};k,0}(t)\\
&-\sum_{k=1}^{m_A}a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_i(t)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}n_jP_{n_1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};k,0}(t)\\
&-\sum_{k=1}^{m_A}a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_j(t)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}n_iP_{n_1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};k,0}(t)\\
&+b_{i,m_B+1}(t)\,\mu_i(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}n_i\left(n_i-1\right)n_jP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)\\
&+b_{j,m_B+1}(t)\,\mu_j(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}n_in_j\left(n_j-1\right)P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)\\
&+\sum_{\substack{k=1\\k\neq i,j}}^{m_B}b_{km_B+1}(t)\,\mu_k(t)\sum_{n_1=0}^{s-1}\ldots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}n_in_jn_kP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)
\end{aligned}
$$

Finally

$$\mathrm{E}'\left[N_i(t)\,N_j(t)\,,\ell,1\right] = -\lambda_\ell(t)\,\mathrm{E}\left[N_i(t)\,N_j(t)\,,\ell,1\right]$$

$$-\mu_i(t)\,\mathrm{E}\left[N_i(t)\,N_j(t)\,,\ell,1\right]$$

$$-\mu_j(t)\,\mathrm{E}\left[N_i(t)\,N_j(t)\,,\ell,1\right]$$

$$+\sum_{k=1}^{m_A} a_{k\ell}(t)\,\lambda_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t)\,,k,1\right]$$

$$-b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_j(t)\,,\ell,1\right]$$

$$-b_{ij}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t)\,,\ell,1\right]$$

$$+\sum_{k=1}^{m_B} b_{ki}(t)\,\mu_k(t)\,\mathrm{E}\left[N_j(t)\,N_k(t)\,,\ell,1\right]$$

$$+\sum_{k=1}^{m_B} b_{kj}(t)\,\mu_k(t)\,\mathrm{E}\left[N_i(t)\,N_k(t)\,,\ell,1\right]$$

$$+\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t)\,,k,1\right]$$

$$+\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\left(\beta_i(t)\,\mathrm{E}\left[N_j(t)\,,k,1\right]+\beta_j(t)\,\mathrm{E}\left[N_i(t)\,,k,1\right]\right)$$

$$-\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t)\,,N(t)=s-1,A(t)=k\right]$$

$$-\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_i(t)\,\mathrm{E}\left[N_j(t)\,,N(t)=s-1,A(t)=k\right]$$

$$-\sum_{k=1}^{m_A} a_{km_A+1}(t)\,\alpha_\ell(t)\,\lambda_k(t)\,\beta_j(t)\,\mathrm{E}\left[N_i(t)\,,N(t)=s-1,A(t)=k\right]$$

$$-\left(b_{i,m_B+1}(t)\,\mu_i(t)+b_{j,m_B+1}(t)\,\mu_j(t)\right)\mathrm{E}\left[N_i(t)\,N_j(t)\,,N(t)=s,A(t)=\ell\right]$$

$$+\sum_{k=1}^{m_B} b_{km_B+1}(t)\,\mu_k(t)\,\mathrm{E}\left[N_i(t)\,N_j(t)\,N_k(t)\,,N(t)=s,A(t)=\ell\right]$$

Derivation of the Cross-product Moments Subspace Ω_2

$$\mathrm{E}'\left[N_i(t)\,N(t)\,,\ell,2\right] \equiv \mathrm{E}'\left[N_i(t)\,N(t)\,,A(t)=\ell,I(t)=2\right]$$

$$\mathrm{E}'\left[N_i(t)\,N(t)\,,A(t)=\ell,I(t)=2\right] \equiv \sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i\,(s+q)\,P'_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)$$

Using the KFEs

$$
\begin{aligned}
&\mathrm{E}'\left[N_i(t)\,N(t),\ell,2\right] = -\lambda_\ell(t)\,\mathrm{E}\left[N_i(t)\,N(t),\ell,2\right] \\
&-\left(1-b_{ii}(t)\right)\mu_i(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i^2\,(s+q)\,P_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}(t) \\
&-\sum_{\substack{j=1\\j\neq i}}^{m_B}\left(1-b_{jj}(t)\right)\mu_j(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i n_j\,(s+q)\,P_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}(t) \\
&+\sum_{j=1}^{m_A} a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t)\,N(t),j,2\right] \\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{ji}(t)\,\mu_j(t)\Bigg\{\sum_{n_j=0}^{s-1}\sum_{n_i=1}^{s-n_j}\sum_{n_1=0}^{s-n_i-n_j}\dots\sum_{n_{j-1}=0}^{s-n_1\dots-n_{j-2}-n_j-n_i}\sum_{n_{j+1}=0}^{s-n_1\dots-n_j-n_i}\dots\sum_{n_{i-1}=0}^{s-n_1\dots-n_{i-2}-n_i}\sum_{n_{i+1}=0}^{s-n_1\dots-n_i}\dots \\
&\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i\,(n_j+1)\,(s+q)\,P_{n_1\dots n_i-1\dots n_j+1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}(t)\Bigg\} \\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{ij}(t)\,\mu_i(t)\Bigg\{\sum_{n_i=0}^{s-1}\sum_{n_j=1}^{s-n_i}\sum_{n_1=0}^{s-n_i-n_j}\dots\sum_{n_{i-1}=0}^{s-n_1\dots-n_{i-2}-n_j-n_i}\sum_{n_{i+1}=0}^{s-n_1\dots-n_j-n_i}\dots\sum_{n_{j-1}=0}^{s-n_1\dots-n_{j-2}-n_j}\sum_{n_{j+1}=0}^{s-n_1\dots-n_j}\dots \\
&\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i\,(n_i+1)\,(s+q)\,P_{n_1\dots n_j-1\dots n_i+1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}(t)\Bigg\} \\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B}\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{kj}(t)\,\mu_k(t)\Bigg\{\sum_{n_k=0}^{s-1}\sum_{n_j=1}^{s-n_k}\sum_{n_1=0}^{s-n_j-n_k}\dots\sum_{n_{k-1}=0}^{s-n_1\dots-n_{k-2}-n_j-n_k}\sum_{n_{k+1}=0}^{s-n_1\dots-n_j-n_k}\dots\sum_{n_{j-1}=0}^{s-n_1\dots-n_{j-2}-n_j} \\
&\sum_{n_{j+1}=0}^{s-n_1\dots-n_j}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i\,(n_k+1)\,(s+q)\,P_{n_1\dots n_j-1\dots n_k+1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}(t)\Bigg\} \\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\Bigg\{\sum_{n_i=1}^{s}\sum_{n_1=0}^{s-n_i}\dots\sum_{n_{i-1}=0}^{s-n_1\dots-n_{i-2}-n_i}\sum_{n_{i+1}=0}^{s-n_1\dots-n_i}\dots \\
&\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}} n_i s P_{n_1\dots n_i-1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};j,0}(t)\Bigg\} \\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{\substack{k=1\\k\neq i}}^{m_B}\beta_k(t)\Bigg\{\sum_{n_k=1}^{s}\sum_{n_1=0}^{s-n_k}\dots\sum_{n_{i-1}=0}^{s-n_1\dots-n_{i-2}-n_i}\sum_{n_{i+1}=0}^{s-n_1\dots-n_i}\dots \\
&\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}} n_i s P_{n_1\dots n_k-1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};j,0}(t)\Bigg\} \\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=1}^{c-s} n_i\,(s+q)\,P_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};j,q-1}(t) \\
&+\sum_{j=1}^{m_A} a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}} n_i c P_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};j,c-s}(t)
\end{aligned}
$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\,\beta_j(t)\Bigg\{\sum_{n_i=0}^{s-1}\sum_{n_j=1}^{s-n_i}\sum_{n_1=0}^{s-n_i-n_j}\ldots\sum_{n_{i-1}=0}^{s-n_1\ldots-n_{i-2}-n_j-n_i}\sum_{n_{i+1}=0}^{s-n_1\ldots-n_j-n_i}\ldots\sum_{n_{j-1}=0}^{s-n_1\ldots-n_{j-2}-n_j}$$

$$\sum_{n_{j+1}=0}^{s-n_1\ldots-n_j}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s-1} n_i\,(n_i+1)\,(s+q)\,P_{n_1\ldots n_i+1\ldots n_j-1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q+1}(t)\Bigg\}$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\Bigg\{\sum_{n_j=0}^{s-1}\sum_{n_i=1}^{s-n_j}\sum_{n_1=0}^{s-n_i-n_j}\ldots\sum_{n_{j-1}=0}^{s-n_1\ldots-n_{j-2}-n_j-n_i}\sum_{n_{j+1}=0}^{s-n_1\ldots-n_j-n_i}\ldots\sum_{n_{i-1}=0}^{s-n_1\ldots-n_{i-2}-n_i}$$

$$\sum_{n_{i+1}=0}^{s-n_1\ldots-n_i}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s-1} n_i\,(n_j+1)\,(s+q)\,P_{n_1\ldots n_j+1\ldots n_i-1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q+1}(t)\Bigg\}$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_k(t)\Bigg\{\sum_{n_j=0}^{s-1}\sum_{n_k=1}^{s-n_j}\sum_{n_1=0}^{s-n_j-n_k}\ldots\sum_{n_{j-1}=0}^{s-n_1\ldots-n_{j-2}-n_j-n_k}\sum_{n_{j+1}=0}^{s-n_1\ldots-n_j-n_k}\ldots\sum_{n_{k-1}=0}^{s-n_1\ldots-n_{k-2}-n_k}$$

$$\sum_{n_{k+1}=0}^{s-n_1\ldots-n_k}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s-1} n_i\,(n_j+1)\,(s+q)\,P_{n_1\ldots n_j+1\ldots n_k-1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q+1}(t)\Bigg\}$$

$$+b_{i,m_B+1}(t)\,\mu_i(t)\,\beta_i(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s-1} n_i^2\,(s+q)\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q+1}(t)$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_j(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s-1} n_i n_j\,(s+q)\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q+1}(t)$$

Then we manipulate the limits for some terms

$$
\begin{aligned}
&\mathrm{E}'\left[N_i(t)\,N(t),\ell,2\right]=-\lambda_\ell(t)\,\mathrm{E}\left[N_i(t)\,N(t),\ell,2\right]\\
&-(1-b_{ii}(t))\,\mu_i(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_i^2\,(s+q)\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)\\
&-\sum_{\substack{j=1\\j\neq i}}^{m_B}(1-b_{jj}(t))\,\mu_j(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_in_j\,(s+q)\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)\\
&+\sum_{j=1}^{m_A}a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t)\,N(t),j,2\right]\\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}(t)\,\mu_j(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_in_j\,(s+q)\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)\\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_j(t)\,N(t),\ell,2\right]\\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ij}(t)\,\mu_i(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_i^2\,(s+q)\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)\\
&-\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ij}(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t)\,N(t),\ell,2\right]\\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B}\sum_{\substack{k=1\\k\neq i,j}}^{m_B}b_{kj}(t)\,\mu_k(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_in_k\,(s+q)\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)\\
&+\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\sum_{n_1=0}^{s-1}\cdots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}sn_iP_{n_1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};j,0}(t)\\
&+\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\sum_{n_1=0}^{s-1}\cdots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}sP_{n_1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};j,0}(t)\\
&+\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{\substack{k=1\\k\neq i}}^{m_B}\beta_k(t)\sum_{n_1=0}^{s-1}\cdots\sum_{n_{m_B-1}=0}^{s-1-n_1-\ldots-n_{m_B-2}}sn_iP_{n_1\ldots n_{m_B}=s-1-n_1-\ldots-n_{m_B-1};j,0}(t)\\
&+\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t)\,N(t),j,2\right]\\
&+\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s}n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};j,q}(t)\\
&-\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}(c+1)\,n_iP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};j,c-s}(t)\\
&+\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{n_1=0}^{s}\cdots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}n_icP_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};j,c-s}(t)
\end{aligned}
$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\,\beta_j(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i^2\,(s+q)\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\,\beta_j(t)\,\mathrm{E}\left[N_i(t)\,N(t)\,,\ell,2\right]$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\,\beta_j(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i^2 P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\,\beta_j(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{i,m_B+1}(t)\,\mu_i(t)\,\beta_j(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}(s-1)\,n_i\,(n_i-1)\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i n_j\,(s+q)\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\,\mathrm{E}\left[N_j(t)\,N(t)\,,\ell,2\right]$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i n_j P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s} n_j P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}(s-1)\,(n_i+1)\,n_j P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_k(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i n_j\,(s+q)\,P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m_B}\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_k(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}\sum_{q=0}^{c-s} n_i n_j P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,q}(t)$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m_B}\sum_{\substack{k=1\\k\neq i,j}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_k(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}(s-1)\,n_i n_j P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)$$

$$
\begin{aligned}
&+b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_i^2\left(s+q\right)P_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}\left(t\right)\\
&-b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_i^2P_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}\left(t\right)\\
&-b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\beta_i\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\left(s-1\right)n_i^2P_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,0}\left(t\right)\\
&+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\beta_j\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_in_j\left(s+q\right)P_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}\left(t\right)\\
&-\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\beta_j\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_in_jP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}\left(t\right)\\
&-\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\beta_j\left(t\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\left(s-1\right)n_in_jP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,0}\left(t\right)
\end{aligned}
$$

Canceling some terms

$$\mathrm{E}'\left[N_i(t)\,N(t),\ell,2\right]=-\lambda_\ell(t)\,\mathrm{E}\left[N_i(t)\,N(t),\ell,2\right]$$

$$-\left(1-b_{ii}(t)\right)\mu_i(t)\,\mathrm{E}\left[N_i(t)\,N(t),\ell,2\right]$$

$$+b_{i,m_B+1}(t)\,\beta_i(t)\,\mu_i(t)\,\mathrm{E}\left[N_i(t)\,N(t),\ell,2\right]$$

$$+\sum_{j=1}^{m_A}a_{j\ell}(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t)\,N(t),j,2\right]$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{ji}(t)\,\mu_j(t)\,\mathrm{E}\left[N_j(t)\,N(t),\ell,2\right]$$

$$+\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B-1}=0}^{s-1-n_1-\dots-n_{m_B-2}}sn_iP_{n_1\dots n_{m_B}=s-1-n_1-\dots-n_{m_B-1};j,0}(t)$$

$$+\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\beta_i(t)\sum_{n_1=0}^{s-1}\dots\sum_{n_{m_B-1}=0}^{s-1-n_1-\dots-n_{m_B-2}}sP_{n_1\dots n_{m_B}=s-1-n_1-\dots-n_{m_B-1};j,0}(t)$$

$$+\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\,\mathrm{E}\left[N_i(t)\,N(t),j,2\right]$$

$$+\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_iP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};j,q}(t)$$

$$-\sum_{j=1}^{m_A}a_{j,m_A+1}(t)\,\alpha_\ell(t)\,\lambda_j(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}n_iP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};j,c-s}(t)$$

$$-b_{i,m_B+1}(t)\,\mu_i(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_i^2P_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}(t)$$

$$+b_{i,m_B+1}(t)\,\mu_i(t)\left(1-\beta_i(t)\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_iP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}(t)$$

$$-b_{i,m_B+1}(t)\,\mu_i(t)\left(1-\beta_i(t)\right)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}(s-1)\,n_i\left(n_i-1\right)P_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,0}(t)$$

$$+\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\,\mathrm{E}\left[N_j(t)\,N(t),\ell,2\right]$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{j,m_B+1}(t)\,\mu_j(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_in_jP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}(t)$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m_B}b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\sum_{n_1=0}^{s}\dots\sum_{n_{m_B-1}=0}^{s-n_1-\dots-n_{m_B-2}}\sum_{q=0}^{c-s}n_jP_{n_1\dots n_{m_B}=s-n_1-\dots-n_{m_B-1};\ell,q}(t)$$

$$
\begin{aligned}
&-\sum_{\substack{j=1\\ j\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_i(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}(s-1)(n_i+1)\,n_j P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)\\
&-\sum_{\substack{j=1\\ j\neq i}}^{m_B}\sum_{\substack{k=1\\ k\neq i}}^{m_B} b_{j,m_B+1}(t)\,\mu_j(t)\,\beta_k(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}(s-1)\,n_i n_j P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)\\
&-b_{i,m_B+1}(t)\,\mu_i(t)\,\beta_i(t)\sum_{n_1=0}^{s}\ldots\sum_{n_{m_B-1}=0}^{s-n_1-\ldots-n_{m_B-2}}(s-1)\,n_i^2 P_{n_1\ldots n_{m_B}=s-n_1-\ldots-n_{m_B-1};\ell,0}(t)
\end{aligned}
$$

Finally

$$\mathrm{E}'\left[N_i\left(t\right)N\left(t\right),\ell,2\right]=-\lambda_\ell\left(t\right)\mathrm{E}\left[N_i\left(t\right)N\left(t\right),\ell,2\right]$$

$$-\mu_i\left(t\right)\mathrm{E}\left[N_i\left(t\right)N\left(t\right),\ell,2\right]$$

$$+\sum_{j=1}^{m_A}a_{j\ell}\left(t\right)\lambda_j\left(t\right)\mathrm{E}\left[N_i\left(t\right)N\left(t\right),j,2\right]$$

$$+\sum_{j=1}^{m_B}b_{ji}\left(t\right)\mu_j\left(t\right)\mathrm{E}\left[N_j\left(t\right)N\left(t\right),\ell,2\right]$$

$$+\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)s\mathrm{E}\left[N_i\left(t\right),N\left(t\right)=s-1,A\left(t\right)=j\right]$$

$$+\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\beta_i\left(t\right)s\mathrm{P}\left(N\left(t\right)=s-1,A\left(t\right)=j\right)$$

$$+\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\mathrm{E}\left[N_i\left(t\right)N\left(t\right),j,2\right]$$

$$+\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\mathrm{E}\left[N_i\left(t\right),I\left(t\right)=2,A\left(t\right)=j\right]$$

$$-\sum_{j=1}^{m_A}a_{j,m_A+1}\left(t\right)\alpha_\ell\left(t\right)\lambda_j\left(t\right)\mathrm{E}\left[N_i\left(t\right),N\left(t\right)=c,A\left(t\right)=j\right]$$

$$+b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\mathrm{E}\left[N_i\left(t\right),I\left(t\right)=2,A\left(t\right)=\ell\right]$$

$$+\sum_{j=1}^{m_B}b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\beta_i\left(t\right)\mathrm{E}\left[N_j\left(t\right)N\left(t\right),\ell,2\right]$$

$$-\sum_{j=1}^{m_B}b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\mathrm{E}\left[N_i\left(t\right)N_j\left(t\right),I\left(t\right)=2,A\left(t\right)=\ell\right]$$

$$-\sum_{j=1}^{m_B}b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\beta_i\left(t\right)\mathrm{E}\left[N_j\left(t\right),I\left(t\right)=2,A\left(t\right)=\ell\right]$$

$$-\sum_{j=1}^{m_B}b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\beta_i\left(t\right)\left(s-1\right)\mathrm{E}\left[N_j\left(t\right),N\left(t\right)=s,A\left(t\right)=\ell\right]$$

$$-\sum_{j=1}^{m_B}b_{j,m_B+1}\left(t\right)\mu_j\left(t\right)\left(s-1\right)\mathrm{E}\left[N_i\left(t\right)N_j\left(t\right),N\left(t\right)=s,A\left(t\right)=\ell\right]$$

$$+b_{i,m_B+1}\left(t\right)\mu_i\left(t\right)\left(s-1\right)\mathrm{E}\left[N_i\left(t\right),N\left(t\right)=s,A\left(t\right)=\ell\right]$$

Appendix 5

In the next sections are presented the input arrival and service processes for each of the cases.

Case 2

The arrival process

$$\mathcal{A}(t) = \left\{ \begin{array}{cccc} 0.1 & 0.1 & 0.3 & 0.5 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 \end{array} \right\}$$

$$\boldsymbol{\lambda}(t) = \left[2.7 + 0.5 \sin\left(t/3\pi\right), 2.7 + 0.5 \sin\left(t/3\pi\right), 1.7 + 0.5 \sin\left(t/3\pi\right)\right]$$

The service process

$$\mathcal{B}(t) = \begin{Bmatrix} 0.1 & 0.6 & 0.3 \\ 0.3 & 0.1 & 0.6 \\ 0.5 & 0.5 & 0 \end{Bmatrix}$$

$$\boldsymbol{\mu}(t) = [1, 1]$$

Case 3

The arrival process

$$\mathcal{A}(t) = \begin{Bmatrix} 0.1 & 0.1 & 0.3 & 0.5 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 \end{Bmatrix}$$

$$\boldsymbol{\lambda}(t) = [2.4 + 0.5\sin(t/3\pi), 2.4 + 0.5\sin(t/3\pi), 1.4 + 0.5\sin(t/3\pi)]$$

The service process

$$\mathcal{B}(t) = \begin{Bmatrix} 0.1 & 0.6 & 0.3 \\ 0.3 & 0.1 & 0.6 \\ 0.5 & 0.5 & 0 \end{Bmatrix}$$

$$\boldsymbol{\mu}(t) = [1, 1]$$

Cases 4 and 5

The arrival process

$$\mathcal{A}(t) = \left\{ \begin{array}{cccc} 0.1 & 0.1 & 0.3 & 0.5 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 \end{array} \right\}$$

$$\boldsymbol{\lambda}(t) = [4 + 0.5\sin(t/3\pi), 4 + 0.5\sin(t/3\pi), 3.5 + 0.5\sin(t/3\pi)]$$

The service process

$$\mathcal{B}(t) = \left\{ \begin{array}{ccc} 0.1 & 0.6 & 0.3 \\ 0.3 & 0.1 & 0.6 \\ 0.5 & 0.5 & 0 \end{array} \right\}$$

$$\boldsymbol{\mu}(t) = [1, 1]$$

Case 6

The arrival process

$$\mathcal{A}(t) = \left\{ \begin{array}{cccc} 0.3 & 0.1 & 0.3 & 0.3 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 \end{array} \right\}$$

$$\boldsymbol{\lambda}(t) = \left[1.4 + 0.5\sin\left(t/6\pi\right), 2.2 + 0.8\sin\left(t/6\pi\right), 1.6 + 0.6\sin\left(t/6\pi\right)\right]$$

The service process

$$\mathcal{B}(t) = \left\{\begin{array}{cccc} 0.2 & 0.2 & 0.2 & 0.4 \\ 0.2 & 0.1 & 0.2 & 0.5 \\ 0.2 & 0.3 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0.5 & 0 \end{array}\right\}$$

$$\boldsymbol{\mu}(t) = [0.5, 1, 0.6]$$

Case 7

The arrival process

$$\mathcal{A}(t) = \left\{\begin{array}{cccc} 0.3 & 0.1 & 0.3 & 0.3 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 \end{array}\right\}$$

$$\boldsymbol{\lambda}(t) = \left[1.2 + 0.5\sin\left(t/6\pi\right), 2 + 0.8\sin\left(t/6\pi\right), 1.4 + 0.6\sin\left(t/6\pi\right)\right]$$

The service process

$$\mathcal{B}(t) = \begin{Bmatrix} 0.2 & 0.2 & 0.2 & 0.4 \\ 0.2 & 0.1 & 0.2 & 0.5 \\ 0.2 & 0.3 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0.5 & 0 \end{Bmatrix}$$

$$\boldsymbol{\mu}(t) = [0.5, 1, 0.6]$$

Case 8

The arrival process

$$\mathcal{A}(t) = \begin{Bmatrix} 0.3 & 0.1 & 0.3 & 0.3 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 \end{Bmatrix}$$

$$\boldsymbol{\lambda}(t) = [1 + 0.5\sin(t/6\pi), 1.8 + 0.8\sin(t/6\pi), 1.2 + 0.6\sin(t/6\pi)]$$

The service process

$$\mathcal{B}(t) = \begin{Bmatrix} 0.2 & 0.2 & 0.2 & 0.4 \\ 0.2 & 0.1 & 0.2 & 0.5 \\ 0.2 & 0.3 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0.5 & 0 \end{Bmatrix}$$

$$\boldsymbol{\mu}(t) = [0.5, 1, 0.6]$$

Cases 9 and 10

The arrival process

$$\mathcal{A}(t) = \begin{Bmatrix} 0.3 & 0.1 & 0.3 & 0.3 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 \end{Bmatrix}$$

$$\boldsymbol{\lambda}(t) = \left[1.8 + 0.5\sin\left(t/6\pi\right), 3.5 + 0.8\sin\left(t/6\pi\right), 2.1 + 0.6\sin\left(t/6\pi\right)\right]$$

The service process

$$\mathcal{B}(t) = \begin{Bmatrix} 0.2 & 0.2 & 0.2 & 0.4 \\ 0.2 & 0.1 & 0.2 & 0.5 \\ 0.2 & 0.3 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0.5 & 0 \end{Bmatrix}$$

$$\boldsymbol{\mu}(t) = [0.5, 1, 0.6]$$

Cases 11 and 12

The arrival process

$$\mathcal{A}(t) = \left\{ \begin{array}{cccc} 0.1 & 0.1 & 0.3 & 0.5 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 \end{array} \right\}$$

$$\boldsymbol{\lambda}(t) = \left\{ \begin{array}{cc} [160, 120, 160]\,, & t < 6 \\ [200, 150, 200]\,, & 6 \leq t \leq 12 \end{array} \right\}$$

The service process

$$\mathcal{B}(t) = \left\{ \begin{array}{cccc} 0.2 & 0.2 & 0.2 & 0.4 \\ 0.2 & 0.1 & 0.2 & 0.5 \\ 0.2 & 0.3 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0.5 & 0 \end{array} \right\}$$

$$\boldsymbol{\mu}(t) = [40, 30, 40]$$

Cases 13 and 14

The arrival process

$$\mathcal{A}(t) = \left\{ \begin{array}{cccc} 0.1 & 0.1 & 0.3 & 0.5 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 \end{array} \right\}$$

$$\boldsymbol{\lambda}(t) = \left\{ \begin{array}{lr} [320, 240, 320]\,, & t < 6 \\ \\ [400, 300, 400]\,, & 6 \leq t \leq 12 \end{array} \right\}$$

The service process

$$\mathcal{B}(t) = \left\{ \begin{array}{cccc} 0.2 & 0.2 & 0.2 & 0.4 \\ \\ 0.2 & 0.1 & 0.2 & 0.5 \\ \\ 0.2 & 0.3 & 0.2 & 0.3 \\ \\ 0.4 & 0.1 & 0.5 & 0 \end{array} \right\}$$

$$\boldsymbol{\mu}(t) = [40, 30, 40]$$

Cases 15 and 16

The arrival process

$$\mathcal{A}(t) = \left\{ \begin{array}{cccc} 0.1 & 0.1 & 0.3 & 0.5 \\ \\ 0.3 & 0.1 & 0.1 & 0.5 \\ \\ 0.3 & 0.2 & 0.1 & 0.4 \\ \\ 0.3 & 0.3 & 0.4 & 0 \end{array} \right\}$$

$$\boldsymbol{\lambda}(t) = \left\{ \begin{array}{lr} [160, 120, 165]\,, & t < 6 \\ \\ [200, 150, 210]\,, & 6 \leq t \leq 12 \end{array} \right\}$$

The service process

$$\mathcal{B}(t) = \left\{ \begin{array}{cccccc} 0.05 & 0.15 & 0.2 & 0.2 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.05 & 0.15 & 0.1 & 0.4 \\ 0.08 & 0.12 & 0.05 & 0.3 & 0.2 & 0.25 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.5 \\ 0.09 & 0.13 & 0.08 & 0.3 & 0.1 & 0.3 \\ 0.3 & 0.1 & 0.2 & 0.1 & 0.3 & 0 \end{array} \right\}$$

$$\boldsymbol{\mu}(t) = [40, 30, 40, 50, 45]$$

Cases 17 and 18

The arrival process

$$\mathcal{A}(t) = \left\{ \begin{array}{cccc} 0.1 & 0.1 & 0.3 & 0.5 \\ 0.3 & 0.1 & 0.1 & 0.5 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.3 & 0.4 & 0 \end{array} \right\}$$

$$\boldsymbol{\lambda}(t) = \left\{ \begin{array}{cc} [320, 240, 330], & t < 6 \\ [400, 300, 420], & 6 \leq t \leq 12 \end{array} \right\}$$

The service process

$$\mathcal{B}(t) = \left\{ \begin{array}{cccccc} 0.05 & 0.15 & 0.2 & 0.2 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.05 & 0.15 & 0.1 & 0.4 \\ 0.08 & 0.12 & 0.05 & 0.3 & 0.2 & 0.25 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.5 \\ 0.09 & 0.13 & 0.08 & 0.3 & 0.1 & 0.3 \\ 0.3 & 0.1 & 0.2 & 0.1 & 0.3 & 0 \end{array} \right\}$$

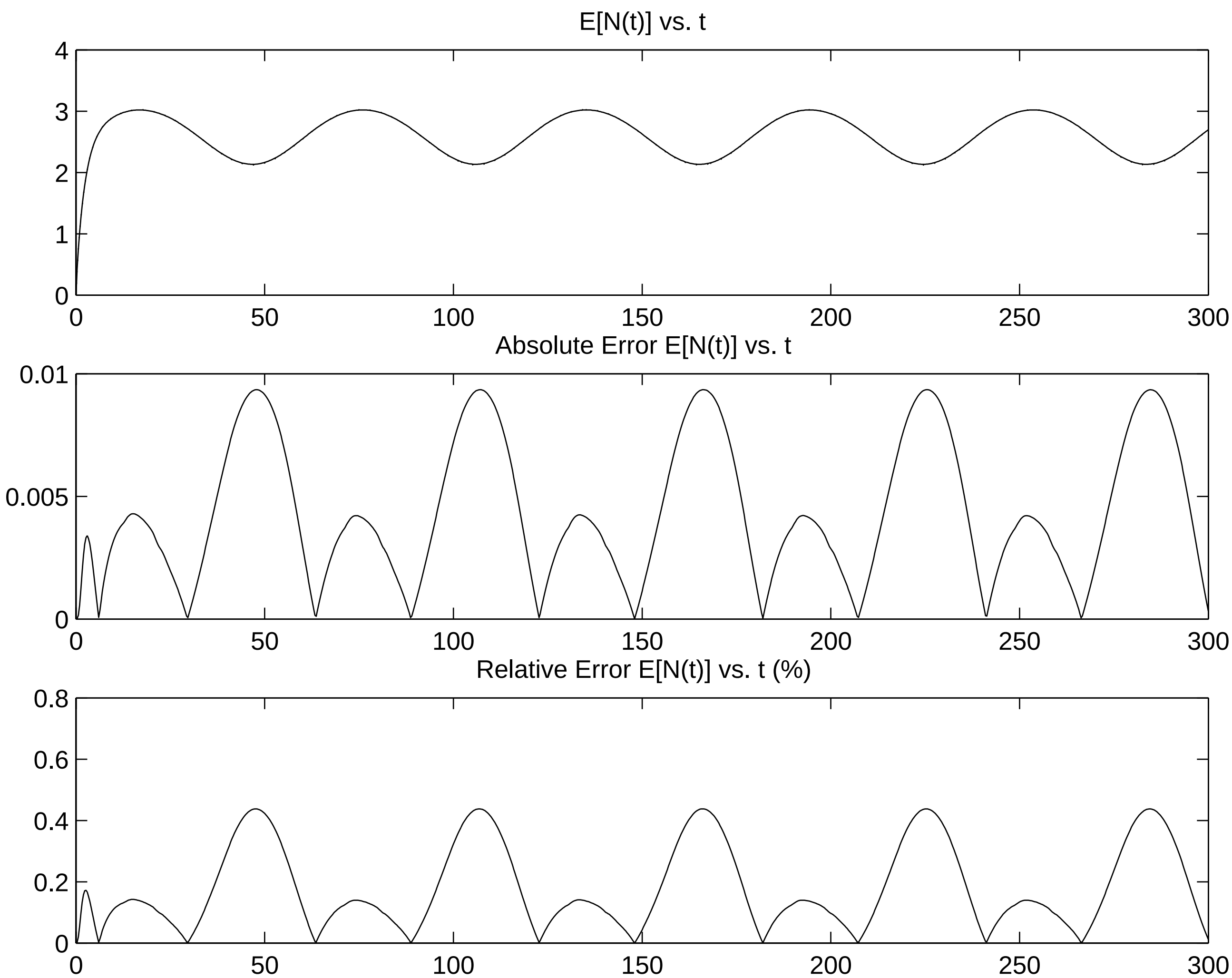
$$\boldsymbol{\mu}(t) = [40, 30, 40, 50, 45]$$

Appendix 6

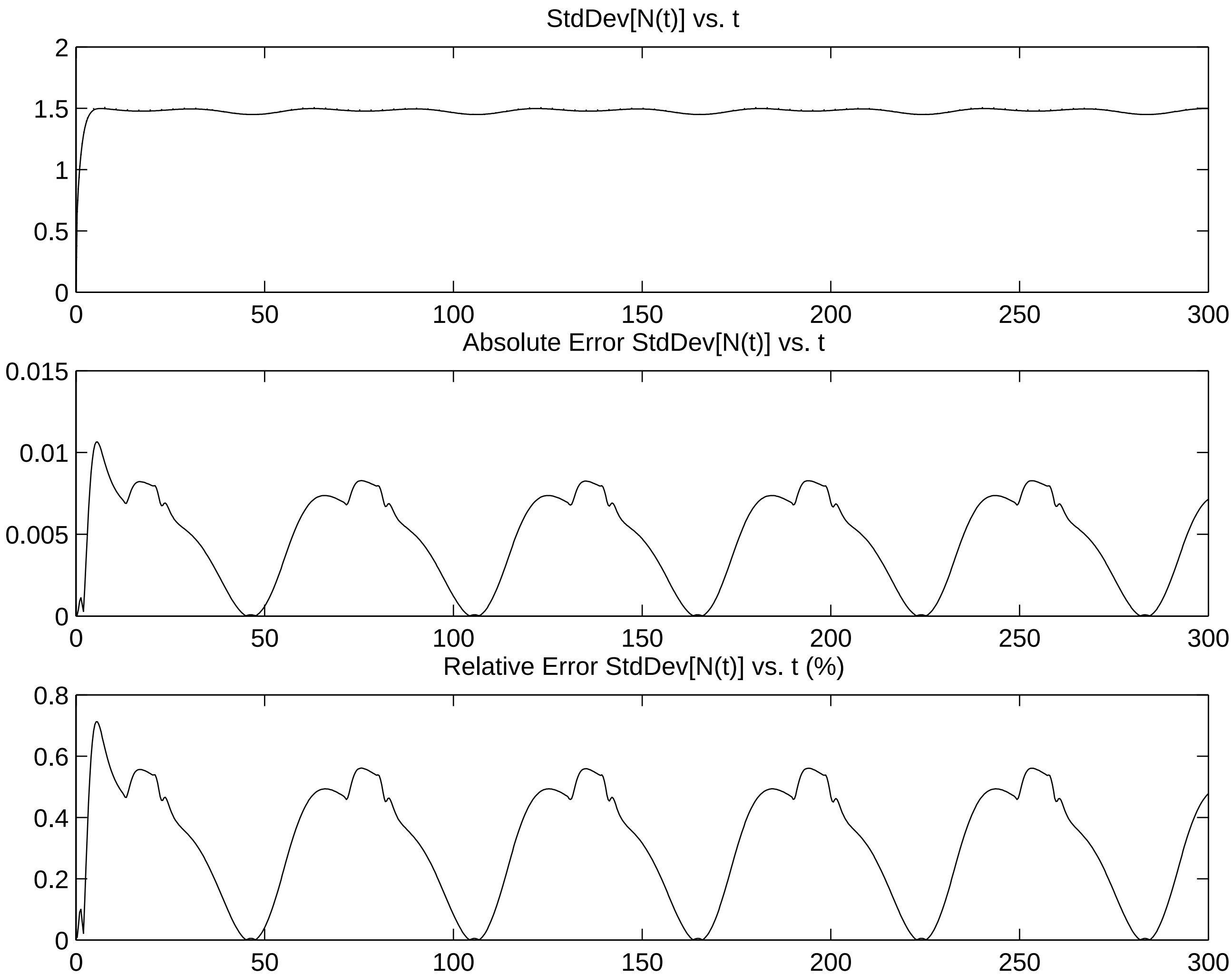
Notice that in Cases 1 trough 10 the solid line represents the actual values and the doted line represents the approximation values. In addition, Cases 11 trough 18 the solid line represents the values of the approximation and the three points are the specific points extracted from the simulation.

Case 1

$\mathrm{E}[N(t)]$ vs. t and errors.

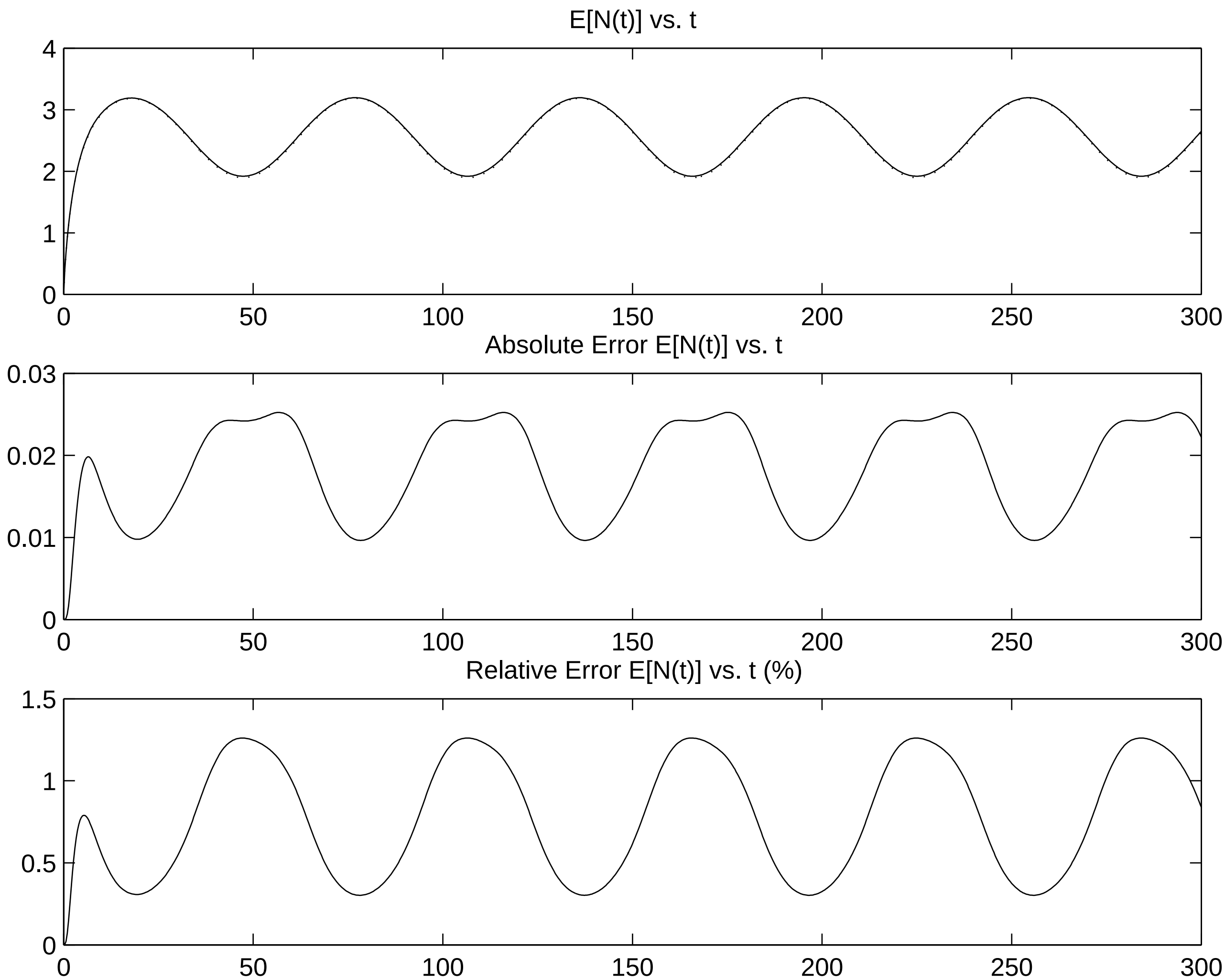


$\mathrm{SD}\left[N(t)\right]$ vs. t and errors.

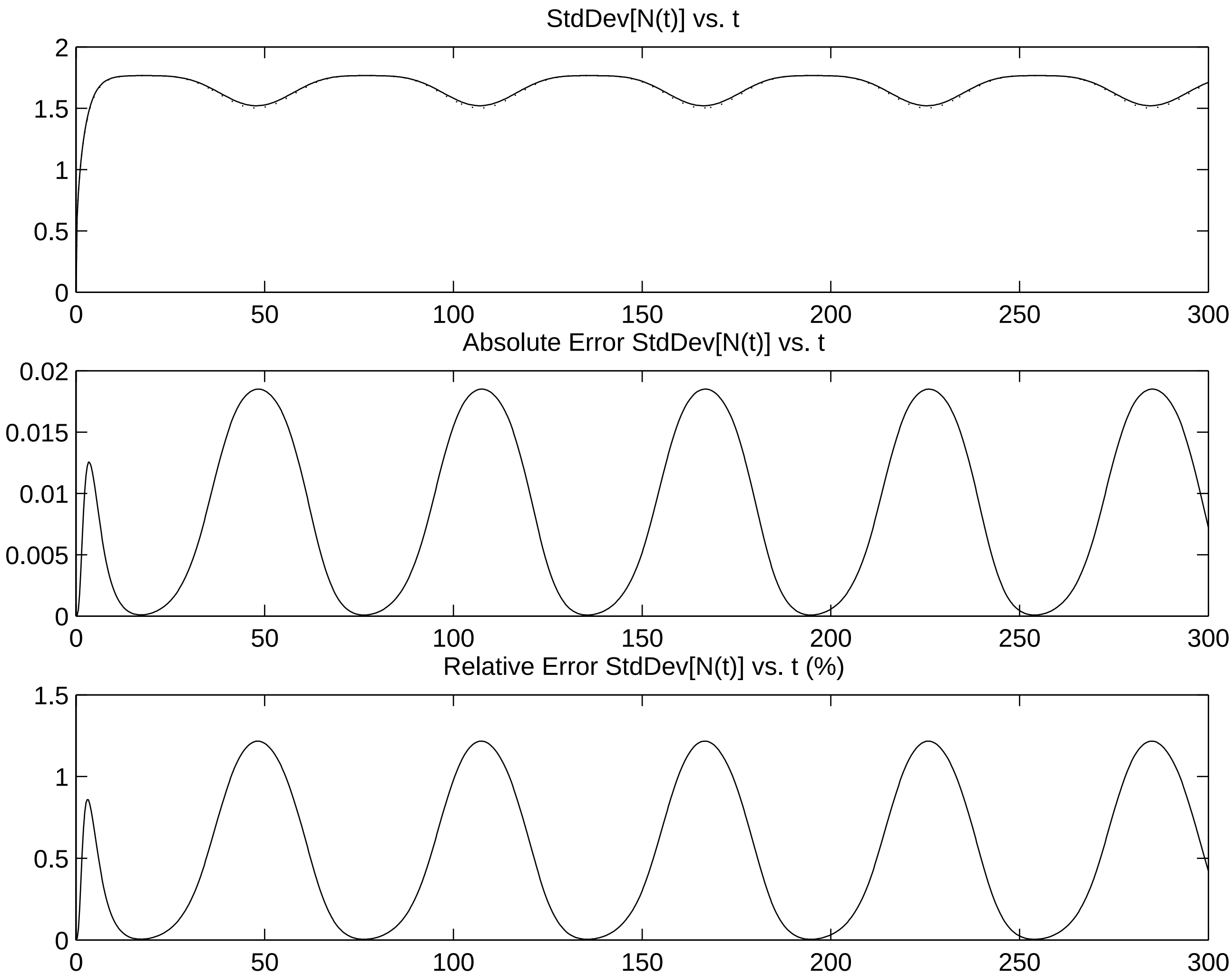


Case 2

$\mathrm{E}\left[N(t)\right]$ vs. t and errors.

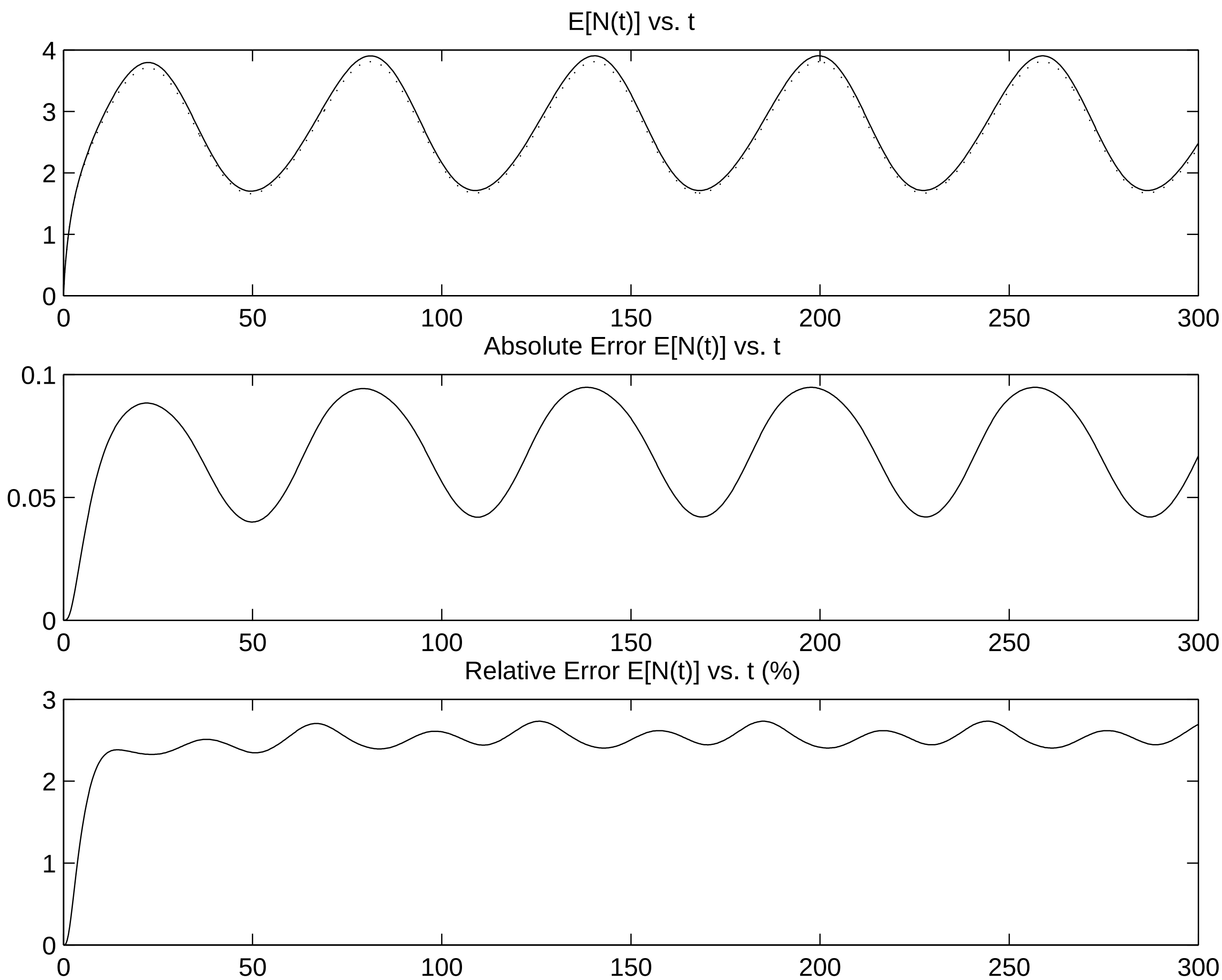


SD $[N(t)]$ **vs.** t **and errors.**

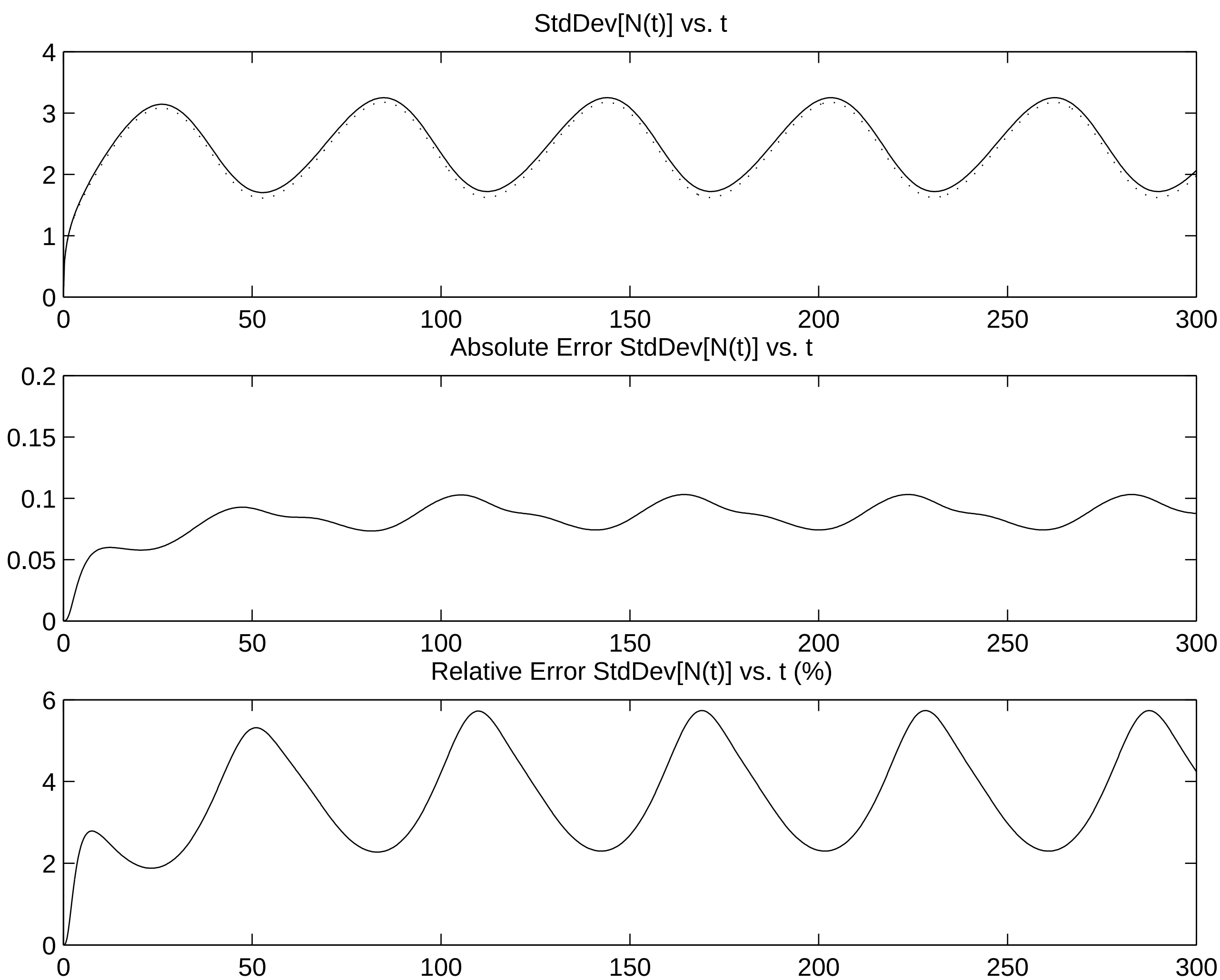


Case 3

$\mathrm{E}[N(t)]$ vs. t and errors.

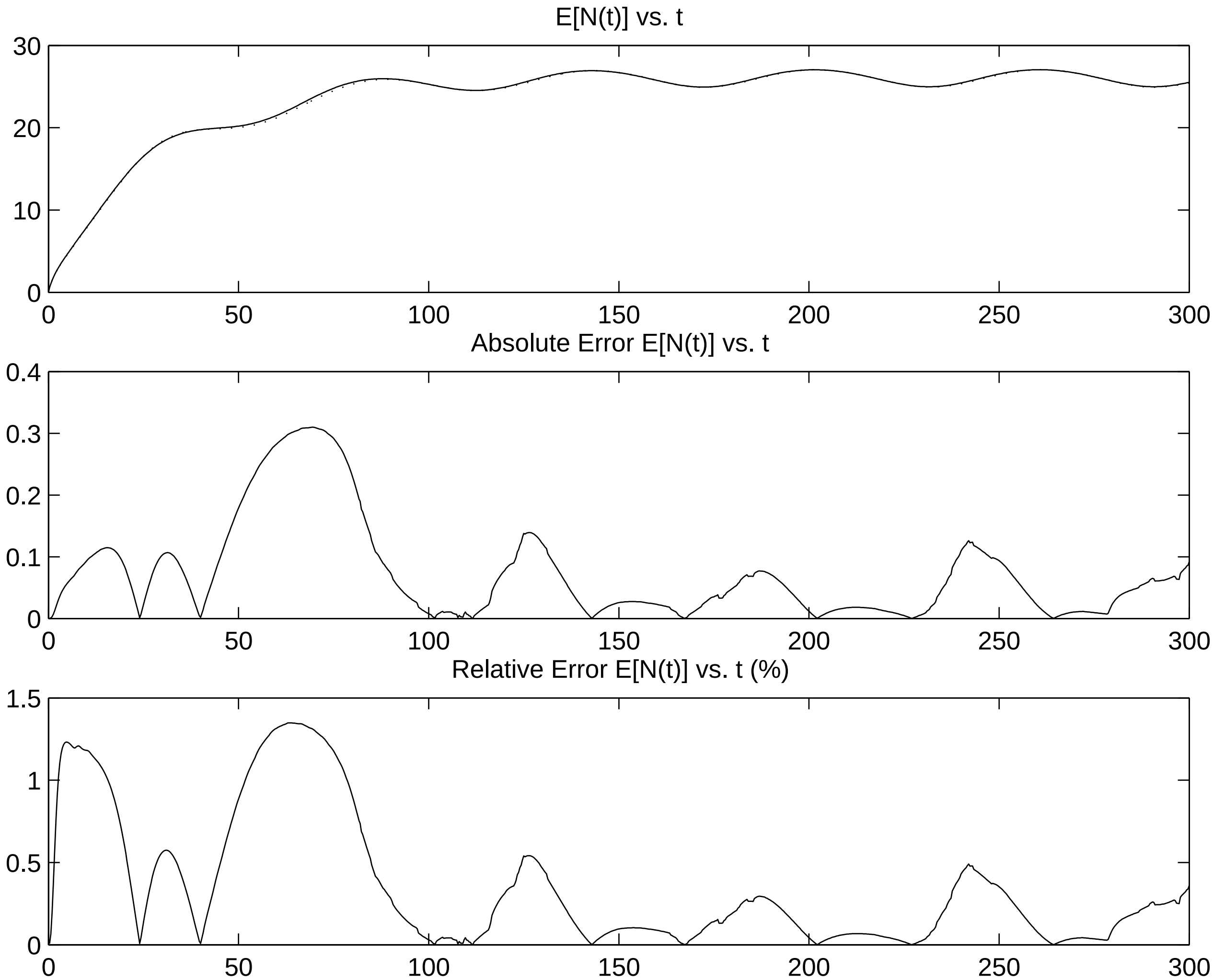


SD $[N(t)]$ **vs.** t **and errors.**

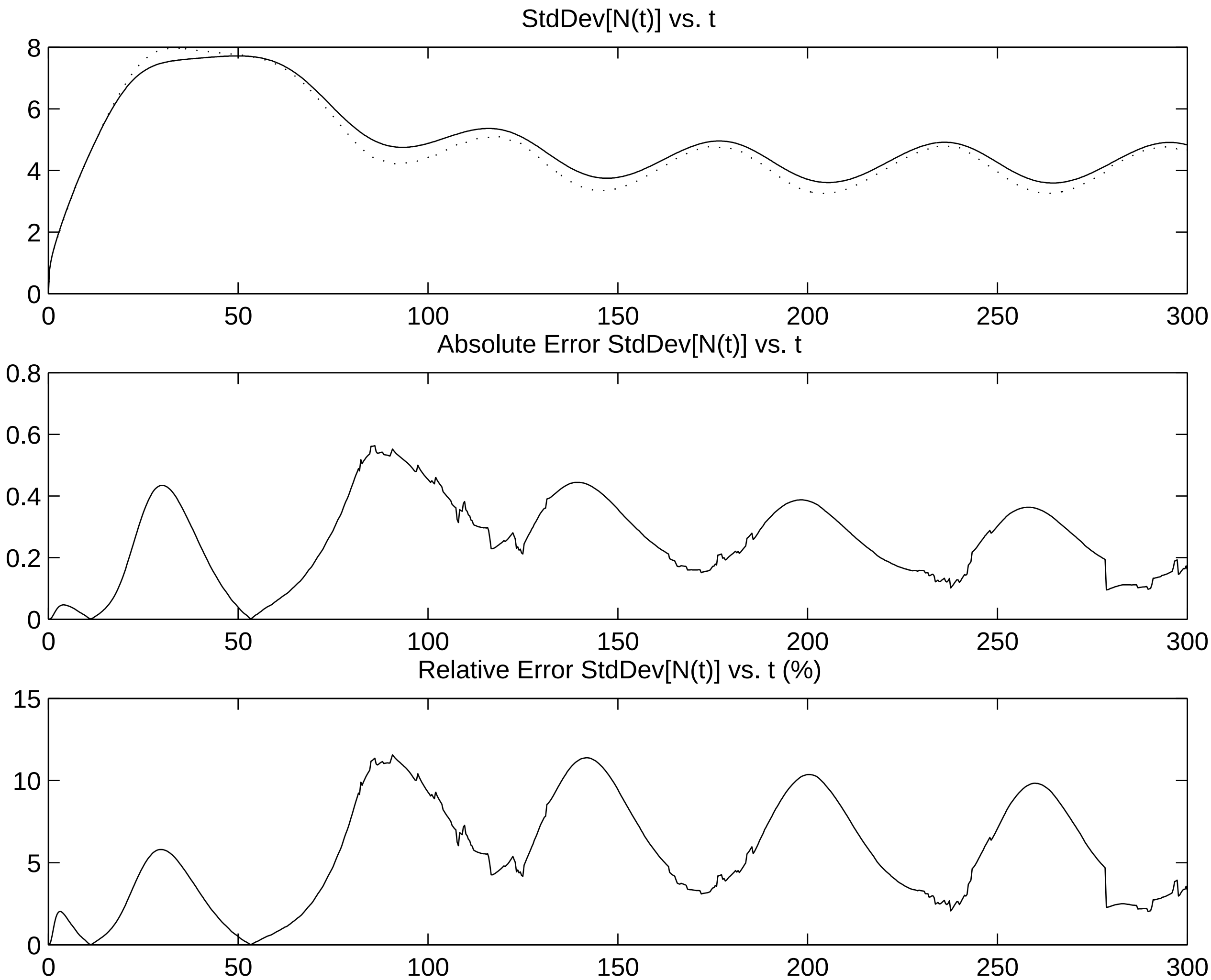


Case 4

$\mathrm{E}[N(t)]$ vs. t and errors.

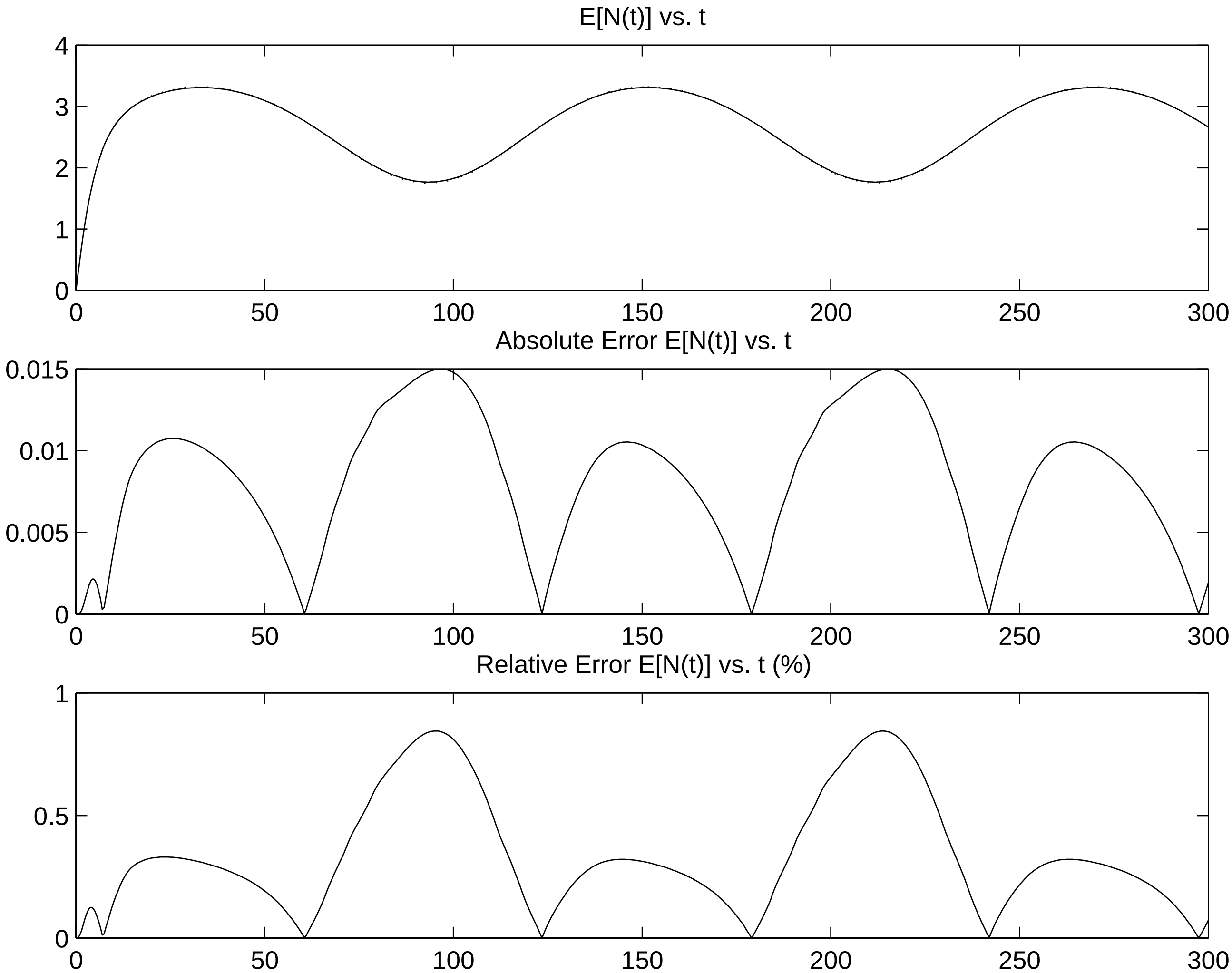


$\mathrm{SD}\,[N(t)]$ vs. t and errors.

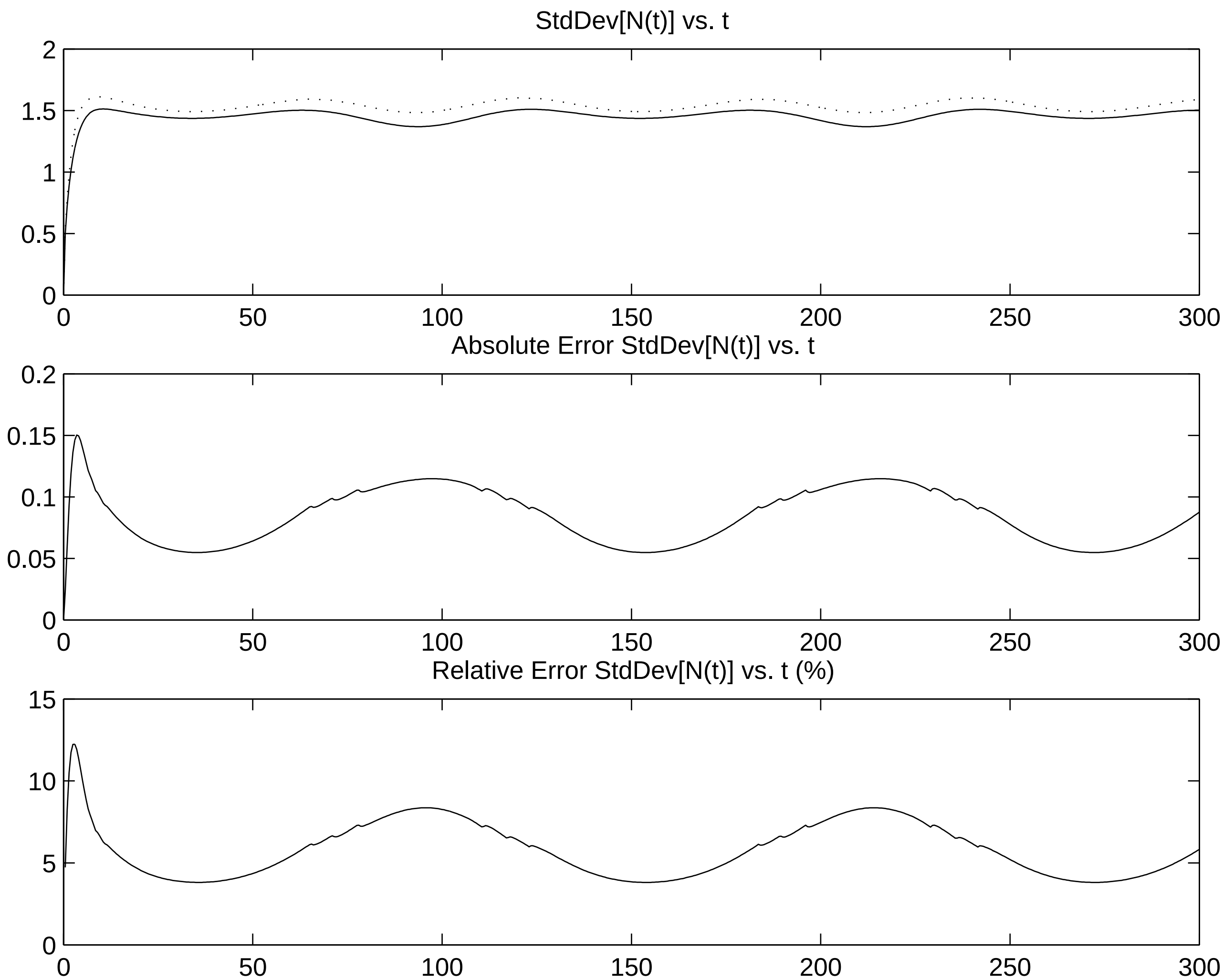


Case 6

$\mathrm{E}[N(t)]$ vs. t and errors.

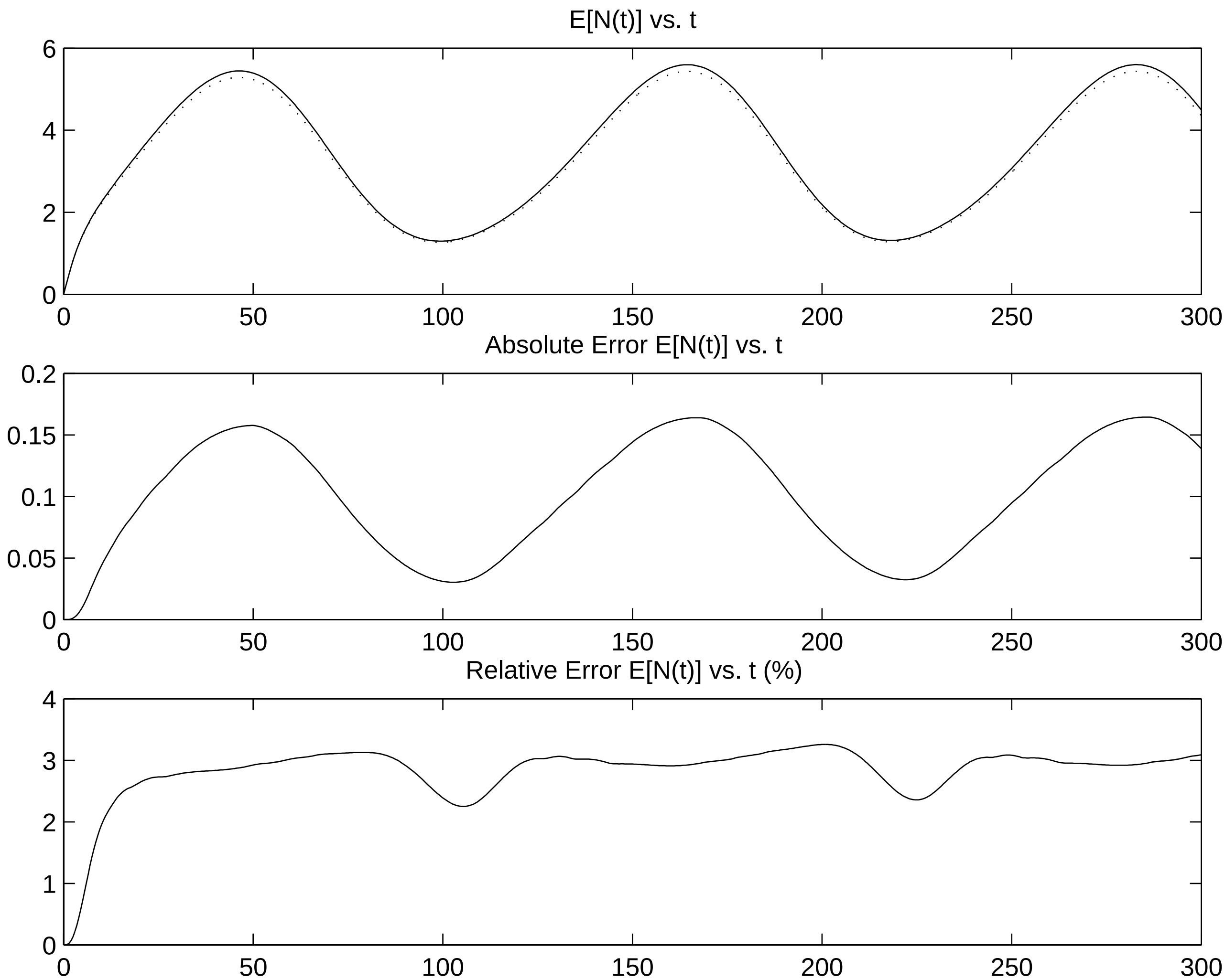


SD $[N(t)]$ vs. t and errors.

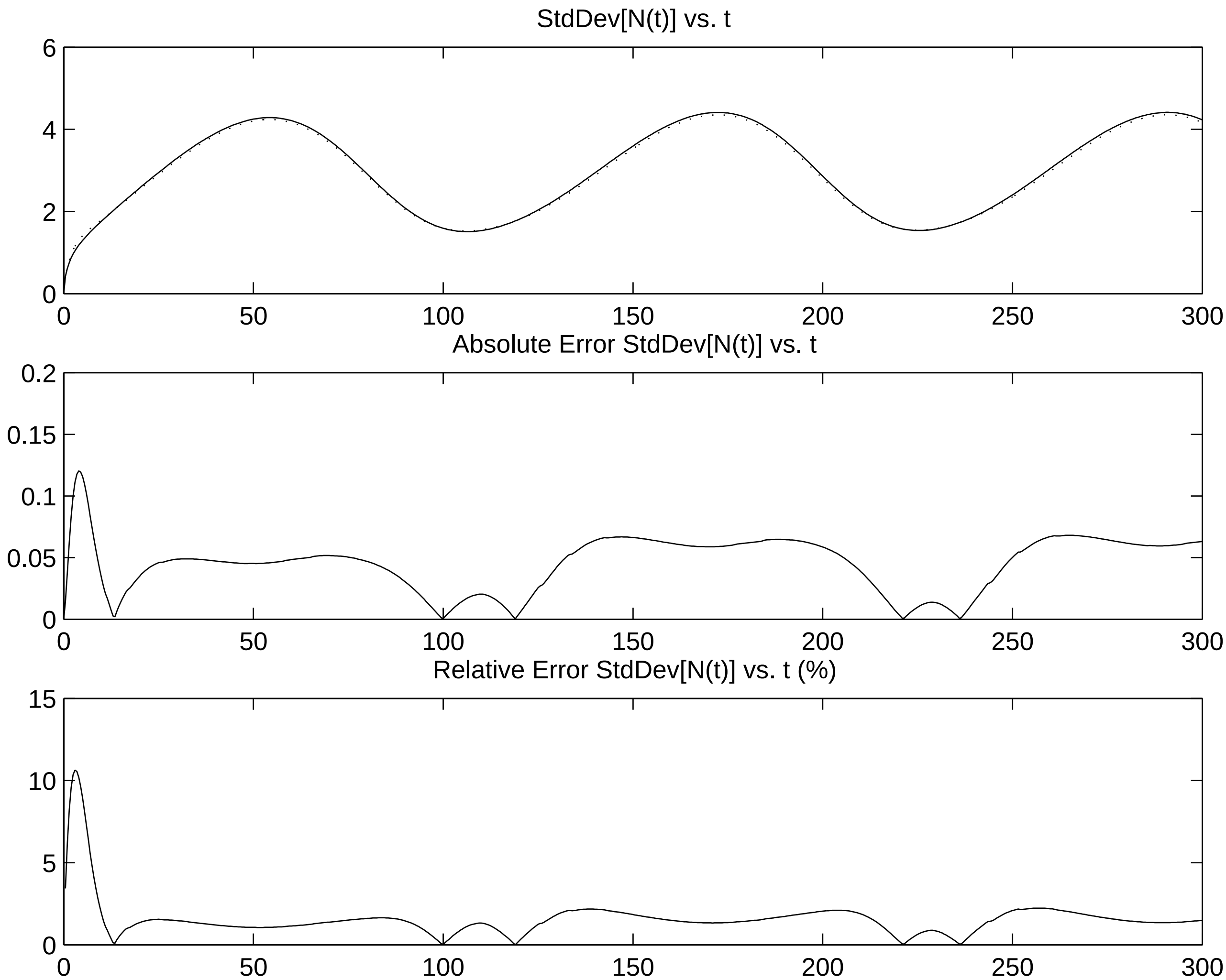


Case 8

$\mathrm{E}\left[N(t)\right]$ vs. t and errors.

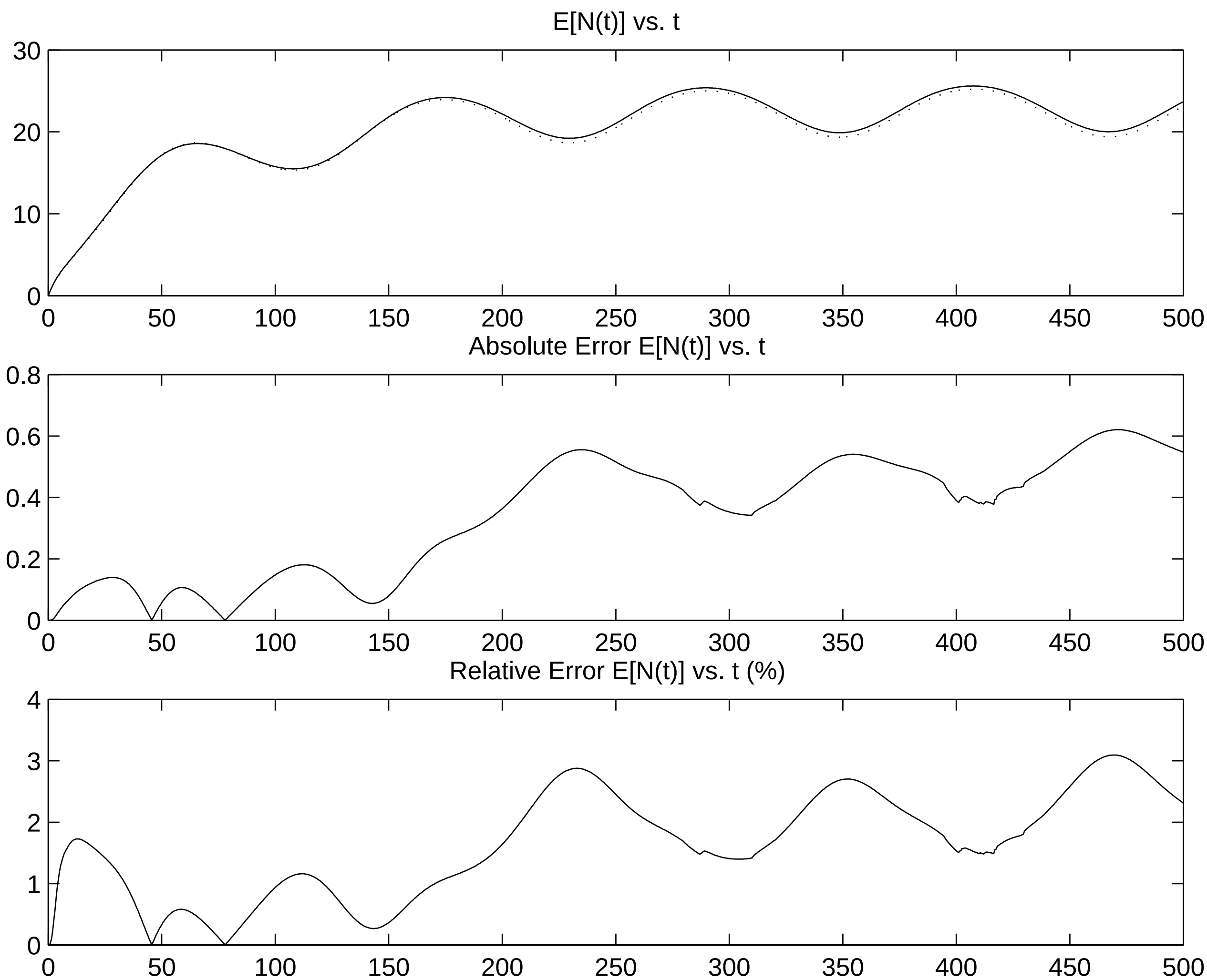


$\mathrm{SD}\left[N(t)\right]$ vs. t and errors.

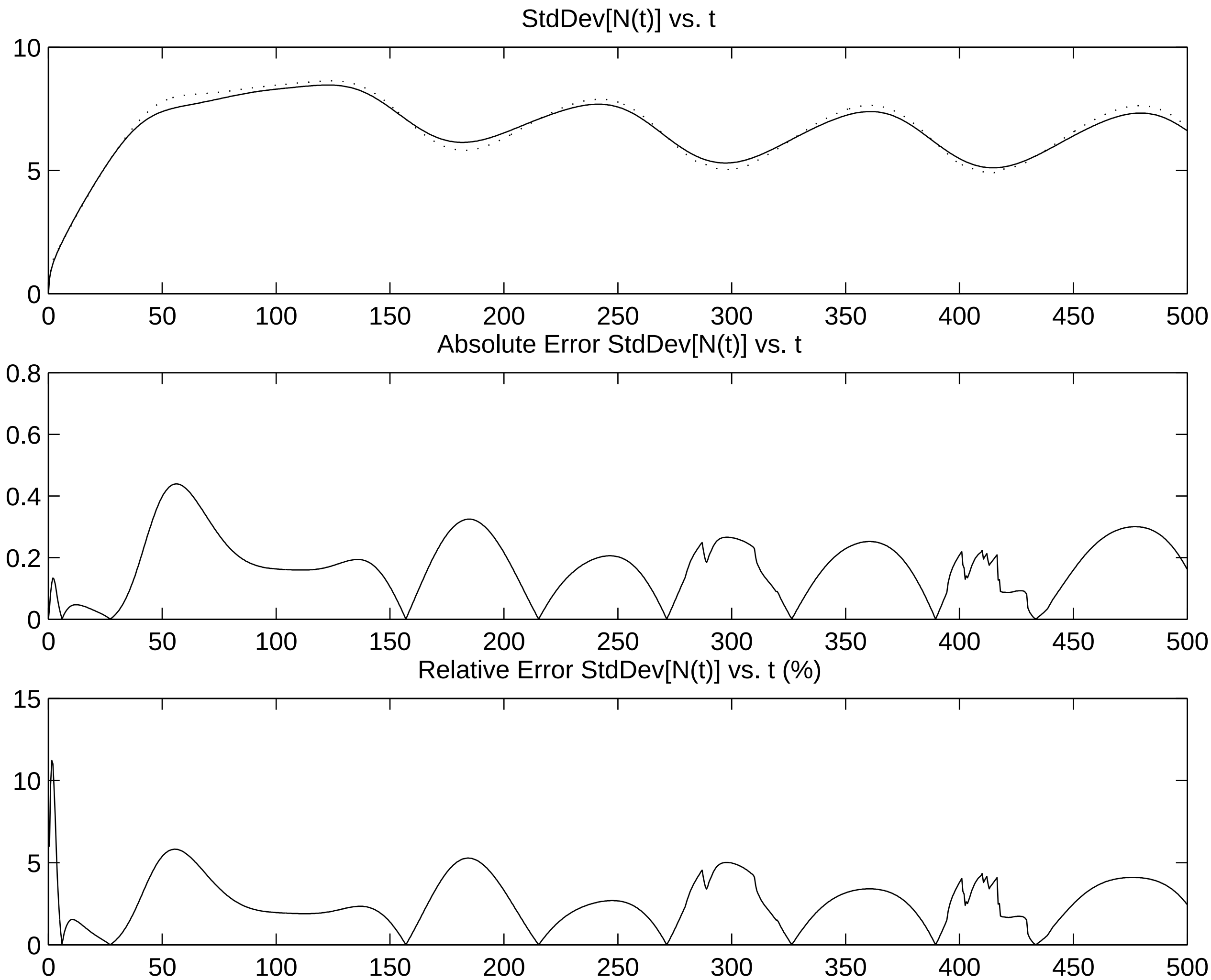


Case 9

E $[N(t)]$ vs. t and errors.

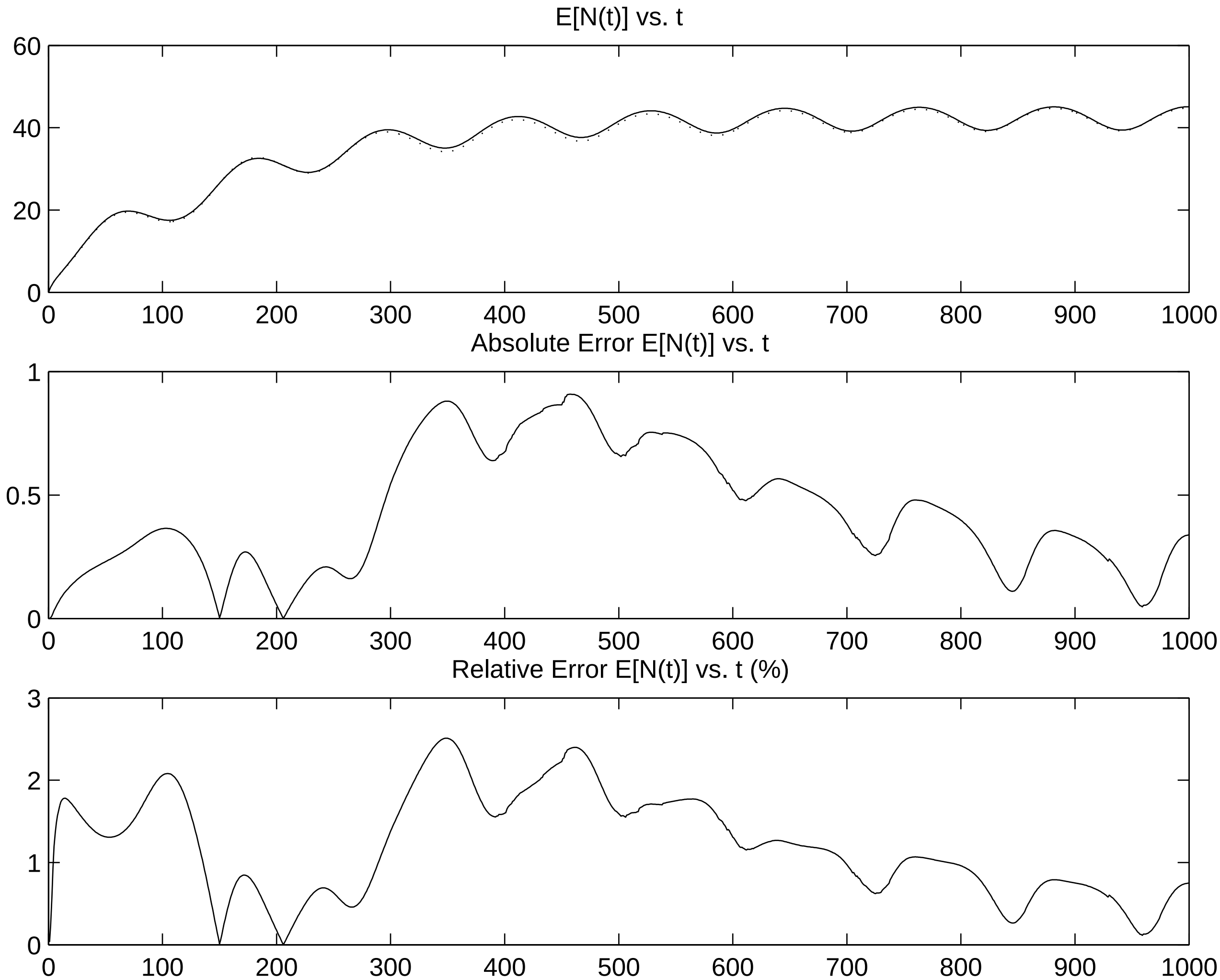


SD $[N(t)]$ vs. t and errors.

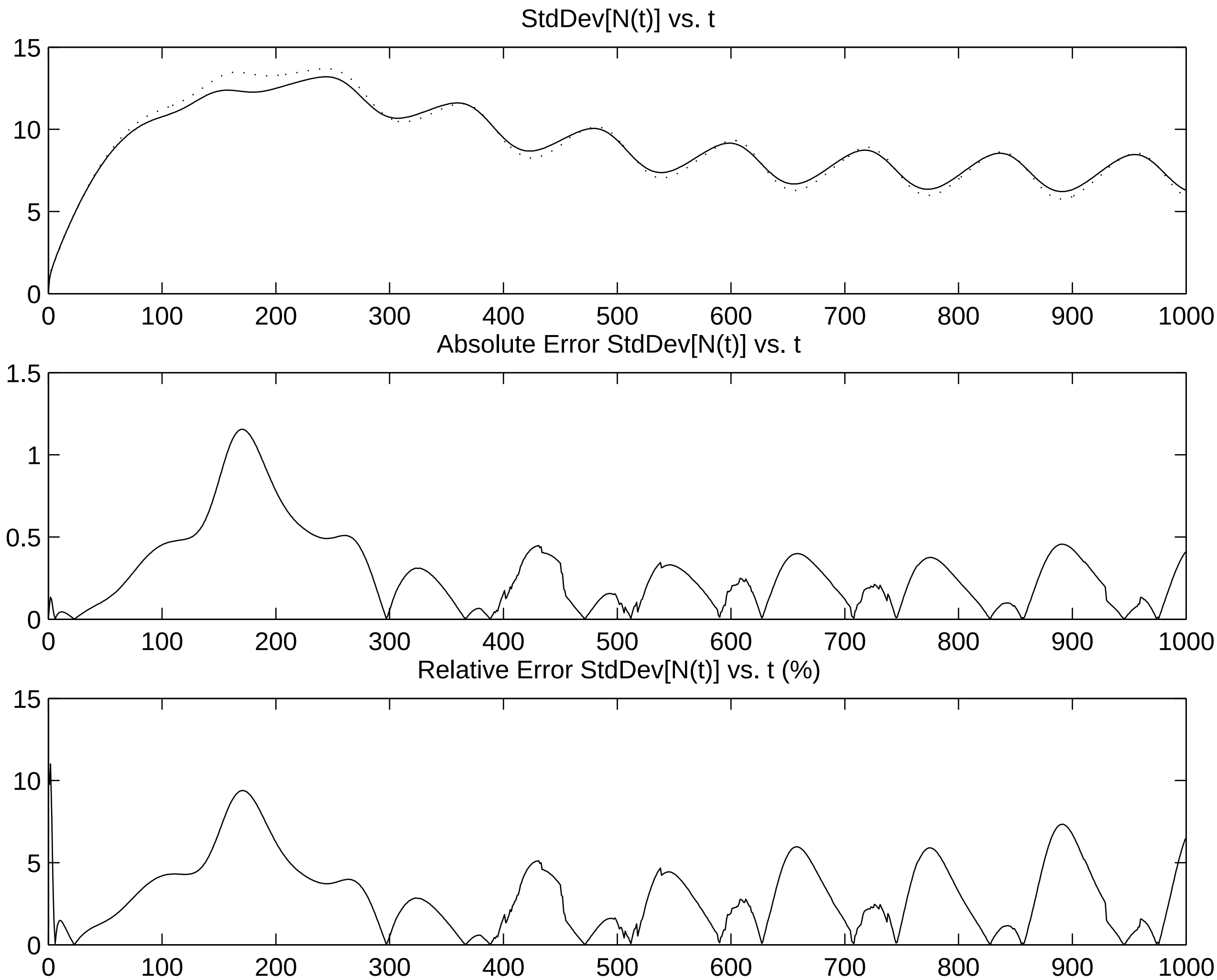


Case 10

$\mathrm{E}\left[N(t)\right]$ vs. t and errors.

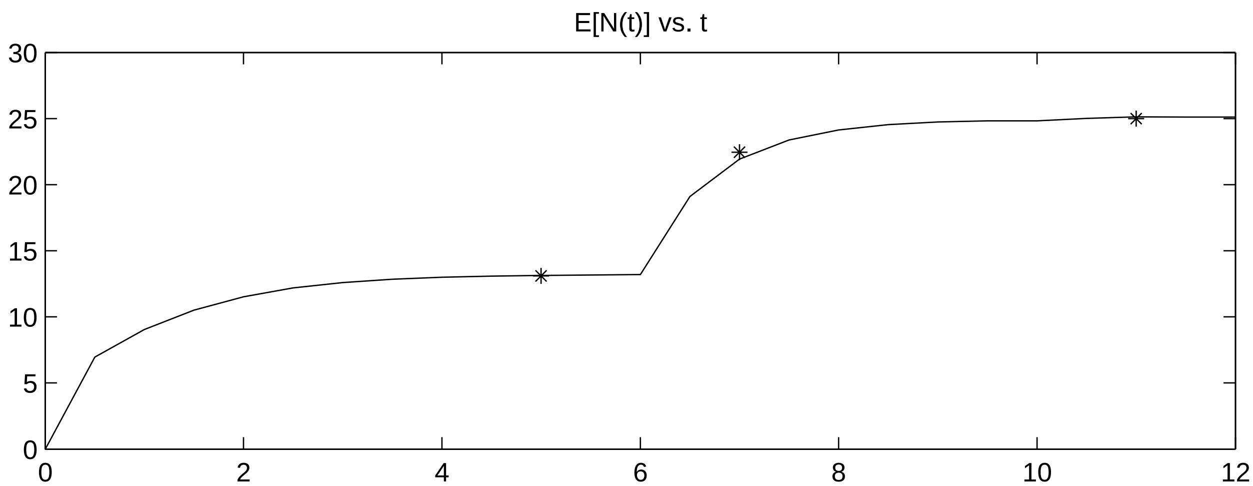


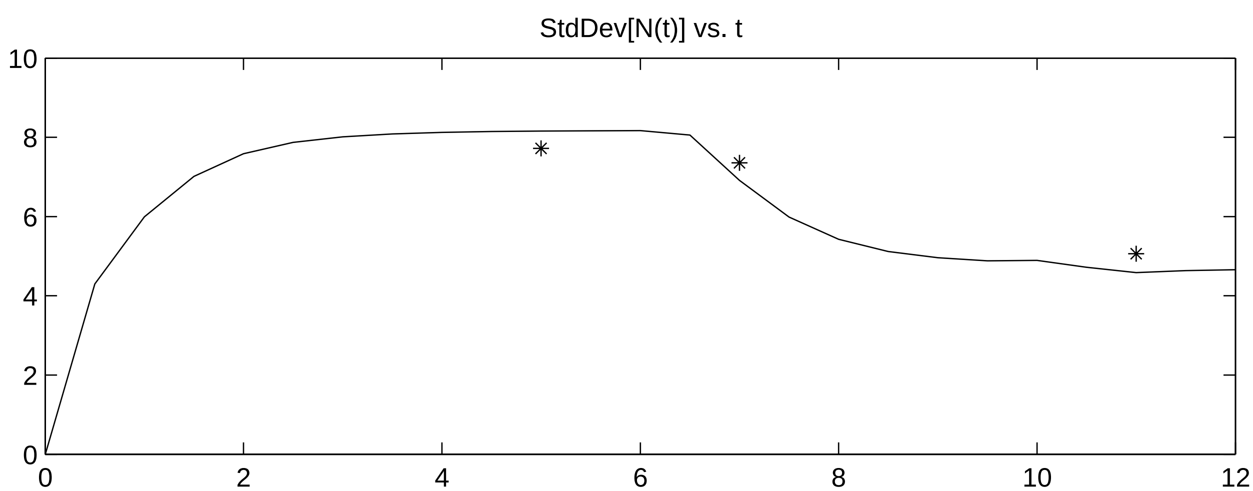
$\mathrm{SD}\,[N(t)]$ **vs.** t **and errors.**



Case 11

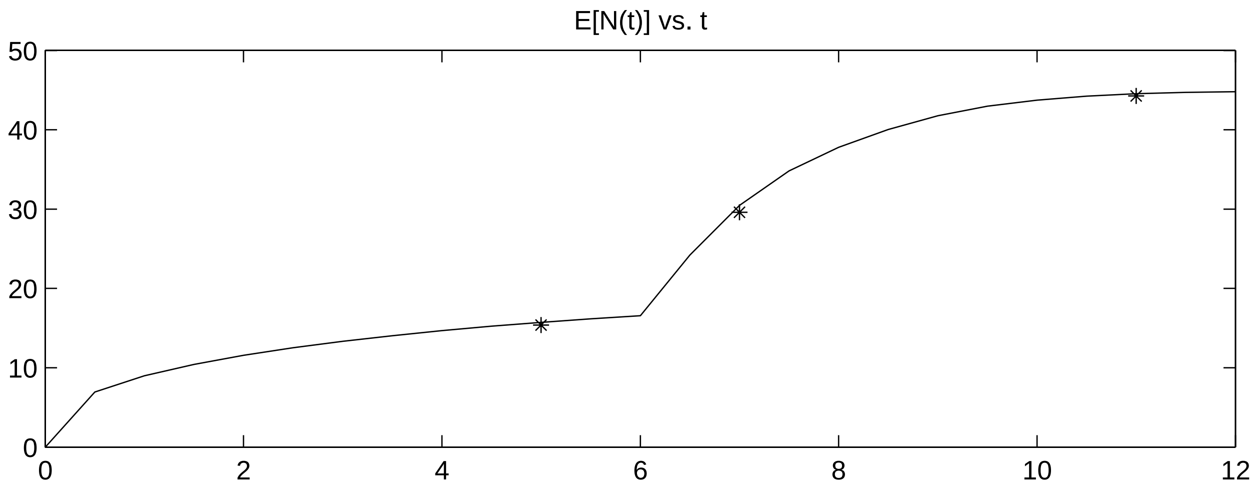
$\mathrm{E}[N(t)]$ vs. t and $\mathrm{SD}[N(t)]$ vs. t.

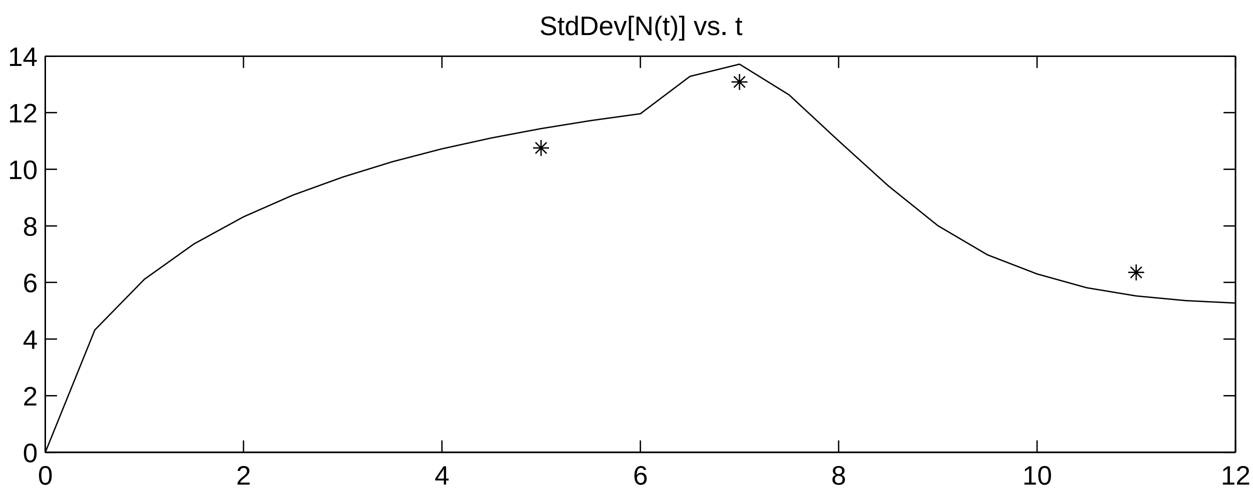




Case 12

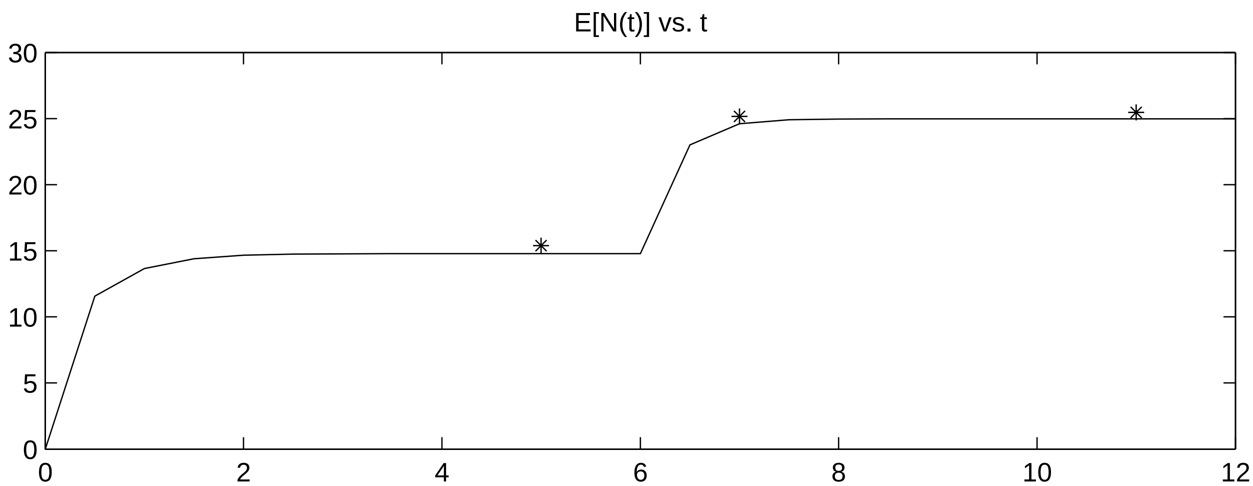
$\mathrm{E}\left[N(t)\right]$ **vs.** t **and** $\mathrm{SD}\left[N(t)\right]$ **vs.** t**.**

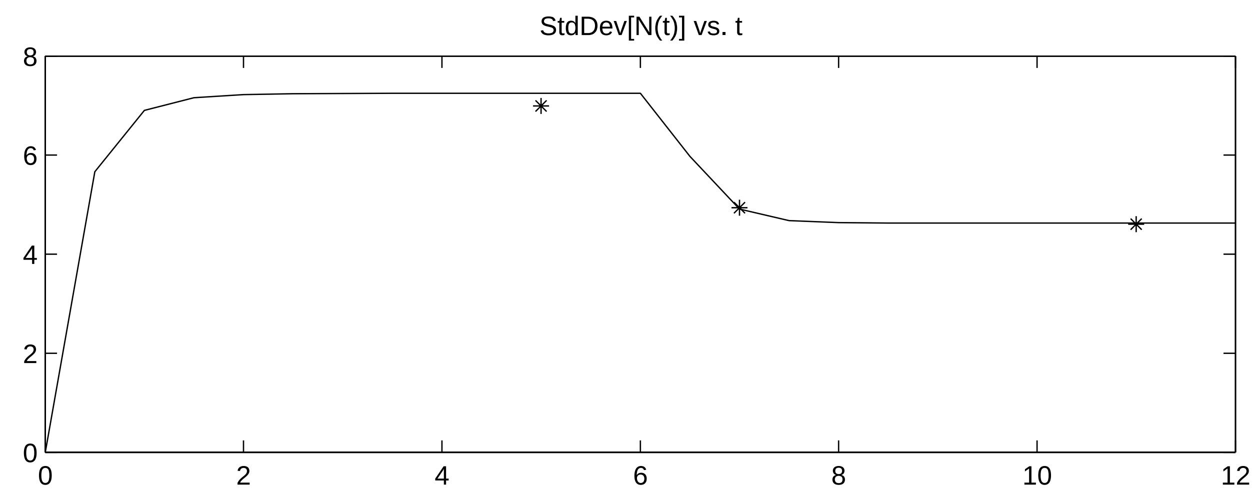




Case 13

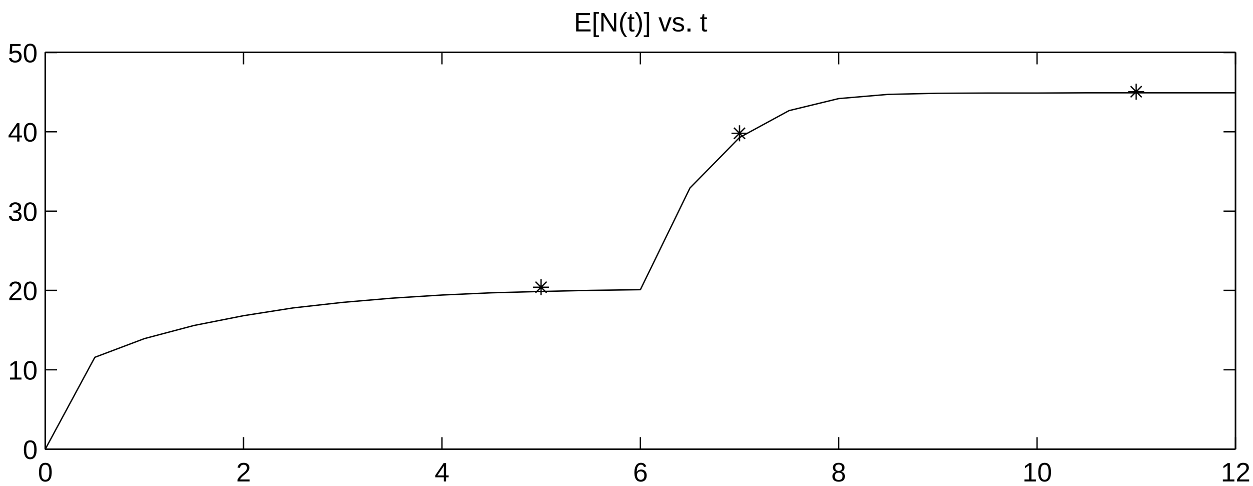
$\mathrm{E}[N(t)]$ vs. t and $\mathrm{SD}[N(t)]$ vs. t.

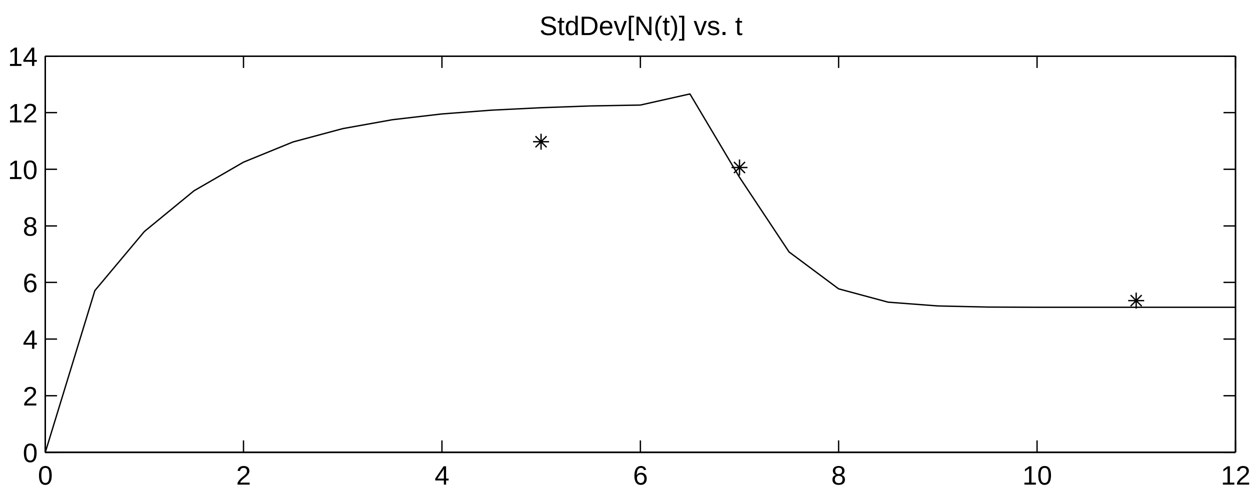




Case 14

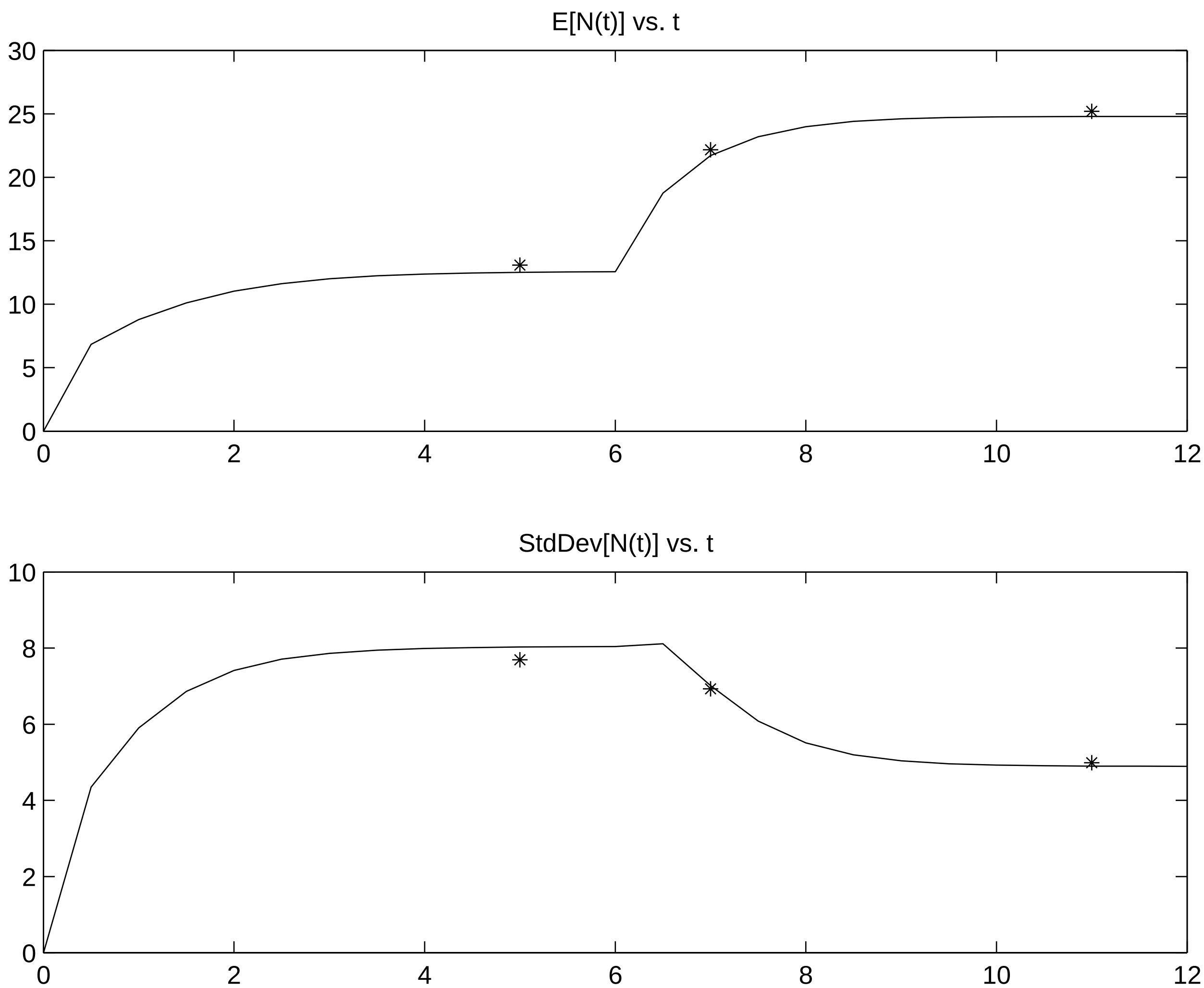
$\mathrm{E}\left[N(t)\right]$ vs. t and $\mathrm{SD}\left[N(t)\right]$ vs. t.





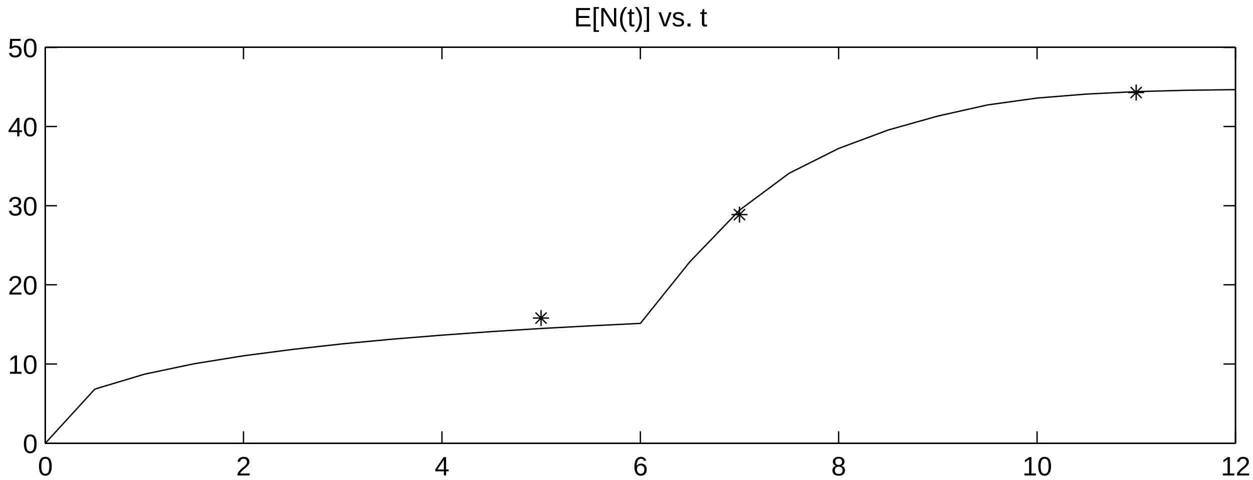
Case 15

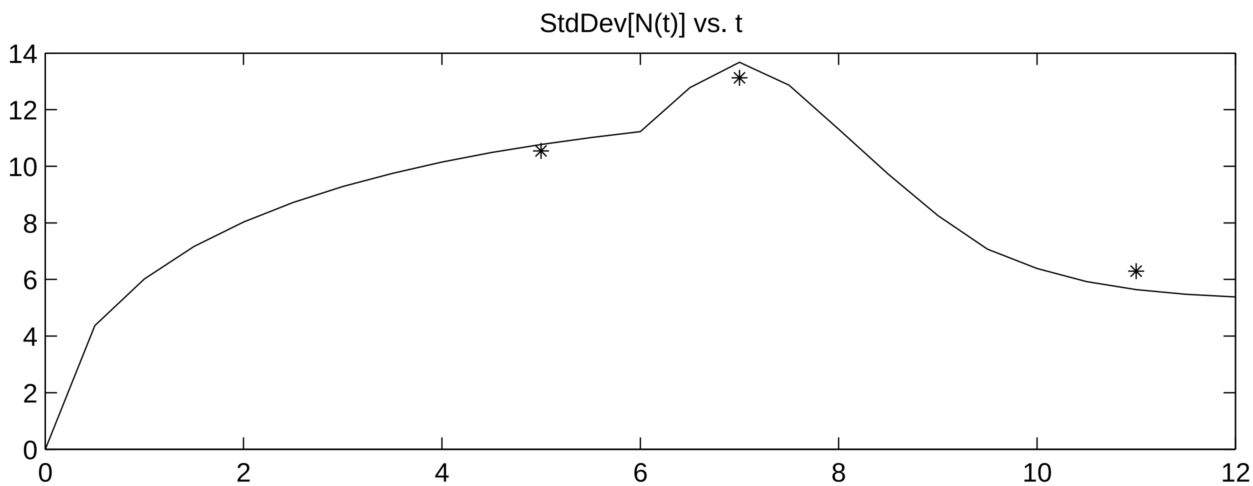
$\mathrm{E}[N(t)]$ vs. t and $\mathrm{SD}[N(t)]$ vs. t.



Case 16

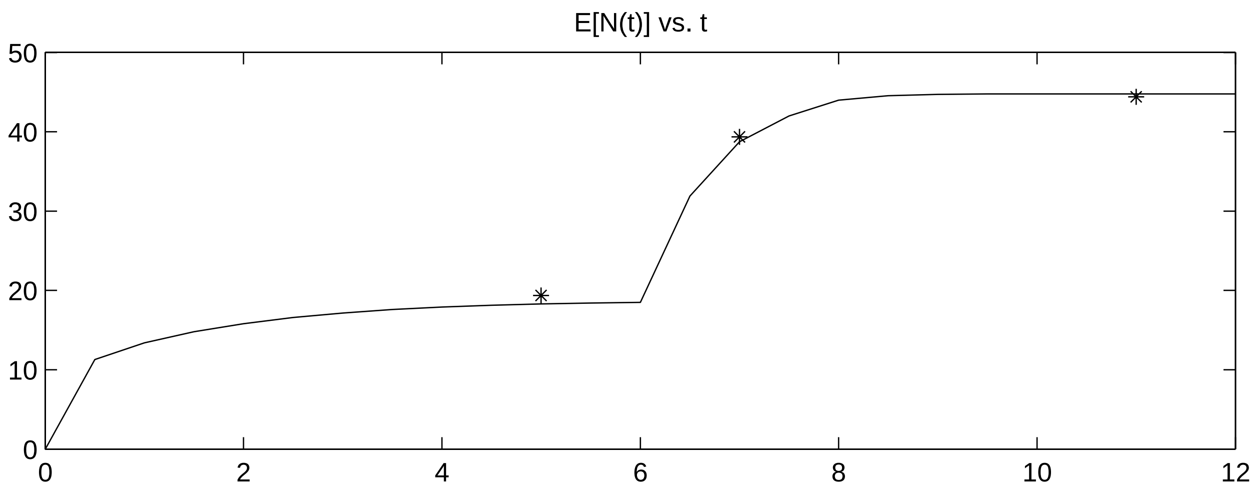
$\mathrm{E}[N(t)]$ vs. t and $\mathrm{SD}[N(t)]$ vs. t.

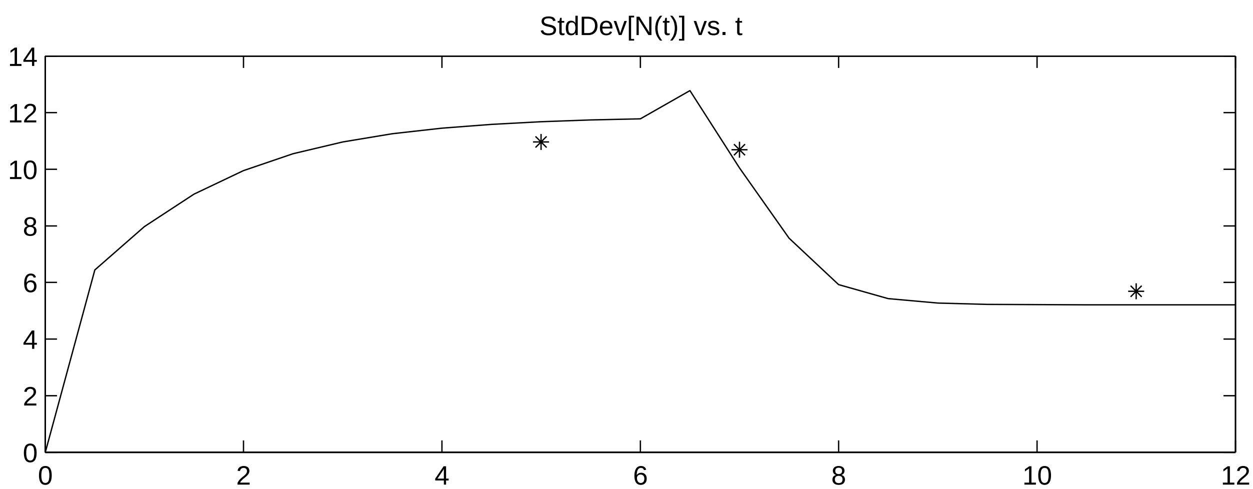




Case 18

$\mathrm{E}[N(t)]$ **vs.** t **and** $\mathrm{SD}[N(t)]$ **vs.** t**.**





Appendix 7

The PE approximation subrutine

```
function PEAprox = f(M1, M2, c, h)

eita = 10000;

epsilon = 1/eita;

if M1 < epsilon

p = 0;

else

p = M1/c;

end

q = 1 - p;

x = -min(p,q)/(c-1);

d = M2-c*M1;

if or(M1 < epsilon, M1 > c-epsilon)
```

```
alpha = 0;
elseif M2-M1*M1 < epsilon
alpha = -1/c + epsilon;
elseif abs(d) < epsilon
alpha = eita;
elseif (M1*(M1+(1-p))-M2)/d < x
alpha = x + epsilon;
else
alpha = (M1*(M1+(1-p))-M2)/d;
end
if h == 0
if q < epsilon
prob = 0;
else
prob = 1;
for i = 0:c-1,
prob = prob*(q+i*alpha)/(1+i*alpha);
end
end
else
if p < epsilon
```

```
prob = 0;
else
prob = 1;
for i = 0:c-1,
prob = prob*(p+i*alpha)/(1+i*alpha);
end
end
end
PEAprox = prob;
```

The introduction of the PMDEs and the approximations

```
function dy = PhtPhtsc(t,y)
s = 3;
c = 5;
m_a = 3;
m_b = 3;
epsilon = 10^(-4);
Num_PMDEs = 4*m_a+m_a*m_b*(2+m_b);
alpha = [0.3;0.3;0.4];
beta = [0.4;0.1;0.5];
```

```
a = [0.3,0.1,0.3,0.3; 0.3,0.1,0.1,0.5; 0.3,0.2,0.1,0.4];

b = [0.2,0.2,0.2,0.4; 0.2,0.1,0.2,0.5; 0.2,0.3,0.2,0.3];

lambda = [1.4+0.5*sin(t/(6*pi));2.2+0.8*sin(t/(6*pi));1.6+0.6*sin(t/(6*pi))];

mu = [0.5;1;0.6];

dy = zeros(Num_PMDEs,1);

h = m_a*2;

for i = 1:m_a,

if y(i) < epsilon

for j = 1:m_b,

cM1L(i,j) = 0;

cM2L(i,j) = 0;

h = h + 1;

end

else

for j = 1:m_b,

cM1L(i,j) = y(h+1)/y(i);

cM2L(i,j) = y(h+1+m_a*(1+m_b))/y(i);

h = h + 1;

end

end

end
```

```
for i = 1:m_a,

if y(m_a+i) < epsilon

cM1U(i) = 0;

cM2U(i) = 0;

else

cM1U(i) = y(m_a*(2+m_b)+i)/y(m_a+i);

cM2U(i) = y(m_a*(3+2*m_b)+i)/y(m_a+i);

end

end

for i = 1:m_a,

cM1LA(i) = 0;

cM2LA(i) = 0;

for j = 1:m_b,

cM1LA(i) = cM1LA(i) + cM1L(i,j);

cM2LA(i) = cM2LA(i) + cM2L(i,j);

end

end

for i = 1:m_a,

cp_sm1(i) = PEAprox(cM1LA(i),cM2LA(i),s-1,1);

p_sm1(i) = cp_sm1(i)*y(i);

cp_s(i) = PEAprox(cM1U(i)-s,cM2U(i)-2*s*cM1U(i)+s2̂,c-s,0);
```

```
p_s(i) = cp_s(i)*y(m_a+i);

cp_c(i) = PEAprox(cM1U(i)-s,cM2U(i)-2*s*cM1U(i)+s2̂,c-s,1);

p_c(i) = cp_c(i)*y(m_a+i);

end

for i = 1:m_b,

M1LB(i) = 0;

for j = 1:m_a,

M1LB(i) = M1LB(i) + y(2*m_a+i+(j-1)*m_b);

end

end

M1Lt = 0;

for i = 1:m_b,

M1Lt = M1Lt + M1LB(i);

end

for i = 1:m_b,

if M1Lt ¡= epsilon

p(i) = beta(i);

else

p(i) = M1LB(i)/M1Lt;

end

end
```

```
for i = 1:m_b,

aM1ism1(i) = (s-1)*p(i);

aM1is(i) = s*p(i);

aM2ism1(i) = (s-1)*(s-2)*p(i)2̂ + (s-1)*p(i);

aM2is(i) = s*(s-1)*p(i)2̂ + s*p(i);

aM3is(i) = s*(s-1)*((s-2)*p(i)3̂ + 3*p(i)2̂) + s*p(i);

for j = 1:m_b,

if i ≃ j

aM1ijsm1(i,j) = (s-1)*(s-2)*p(i)*p(j);

aM1ijs(i,j) = s*(s-1)*p(i)*p(j);

aM2ijs(i,j) = s*(s-1)*p(i)*p(j)*((s-2)*p(i)+1);

for k = 1:m_b,

if k ≃ j

aM1ijks(i,j,k) = s*(s-1)*(s-2)*p(i)*p(j)*p(k);

end

end

end

end

end

for i = 1:m_a,

dy(i) = - lambda(i)*y(i);
```

```
for j = 1:m_a,
dy(i) = dy(i) + a(j,i)*lambda(j)*y(j) + a(j,m_a+1)*alpha(i)*lambda(j)*(y(j)-p_sm1(j));
end
for j = 1:m_b,
dy(i) = dy(i) + b(j,m_b+1)*mu(j)*p_s(i)*aM1is(j);
end
end
for i = 1:m_a,
dy(m_a+i) = -lambda(i)*y(m_a+i);
for j = 1:m_a,
dy(m_a+i) = dy(m_a+i) + a(j,i)*lambda(j)*y(m_a+j) + a(j,m_a+1)*alpha(i)*lambda(j)*
(p_sm1(j)+y(m_a+j));
end
for j = 1:m_b,
dy(m_a+i) = dy(m_a+i) - b(j,m_b+1)*mu(j)*p_s(i)*aM1is(j);
end
end
h = 2*m_a;
l = 2*m_a;
for i = 1:m_a,
for j = 1:m_b,
```

```
dy(h+1) = -lambda(i)*y(h+1) - mu(j)*y(h+1) - b(j,m_b+1)*mu(j)*p_s(i)*aM1is(j);
for k = 1:m_a,
dy(h+1) = dy(h+1) + a(k,i)*lambda(k)*y(l+j+(k-1)*m_b) + a(k,m_a+1)*alpha(i)*
lambda(k)*(y(l+j+(k-1)*m_b)+beta(j)*y(k)-p_sm1(k)*aM1ism1(j)-beta(j)*p_sm1(k));
end
for k = 1:m_b,
if k == j
dy(h+1) = dy(h+1) + b(k,j)*mu(k)*y(h+1-j+k)+b(j,m_b+1)*mu(j)*p_s(i)*aM2is(j);
else
dy(h+1) = dy(h+1) + b(k,j)*mu(k)*y(h+1-j+k)+b(k,m_b+1)*mu(k)*p_s(i)*aM1ijs(j,k);
end
end
h = h + 1;
end
end
h = m_a*(2+m_b);
for i = 1:m_a,
dy(h+i) = -lambda(i)*y(h+i);
for j = 1:m_a,
dy(h+i) = dy(h+i) + a(j,i)*lambda(j)*y(h+j) + a(j,m_a+1)*alpha(i)*lambda(j)*
(s*p_sm1(j)+y(h+j)+y(m_a+j)-p_c(j));
```

```
end

for j = 1:m_b,

dy(h+i) = dy(h+i) - b(j,m_b+1)*mu(j)*(y(m_a+i)*aM1is(j)+(s-1)*p_s(i)*aM1is(j));

end

end

h = m_a*(3+m_b);

l = m_a*(3+m_b);

for i = 1:m_a,

for j = 1:m_b,

dy(h+1) = -lambda(i)*y(h+1) - 2*mu(j)*y(h+1)+(1-2*b(j,j))*mu(j)*y(h+1-m_a*(1+m_b))

+b(j,m_b+1)*mu(j)*p_s(i)*(aM1is(j)-2*aM2is(j));

for k = 1:m_b,

if k == j

dy(h+1) = dy(h+1) + b(k,j)*mu(k)*y(l-m_a*(1+m_b)+k+(i-1)*m_b)

+b(j,m_b+1)*mu(j)*p_s(i)*aM3is(j);

else

dy(h+1) = dy(h+1) + b(k,j)*mu(k)*y(l-m_a*(1+m_b)+k+(i-1)*m_b)

+b(k,m_b+1)*mu(k)*p_s(i)*aM2ijs(j,k);

end

end

for k = 1:m_a,
```

```
dy(h+1) = dy(h+1) + a(k,i)*lambda(k)*y(l+j+(k-1)*m_b)+a(k,m_a+1)*alpha(i)*
lambda(k)* (y(l+j+(k-1)*m_b)+2*beta(j)*y(l-m_a*(1+m_b)+j+(k-1)*m_b)+beta(j)*y(k)-
p_sm1(k)*aM2ism1(j) -2*beta(j)*p_sm1(k)* aM1ism1(j)-beta(j)*p_sm1(k));
end
d = 0;
for k = 1:m_b,
if j == k
dy(h+1) = dy(h+1) + 2*b(j,j)*mu(j)*y(h+1);
else
d = d + 1;
if d ¡ m_b
dy(h+1) = dy(h+1) + 2*b(k,j)*mu(k)*y(m_a*(4+2*m_b)+(j-1)*(m_b-1)+(i-1)*m_b*(m_b-
1)+d);
end
end
end
h = h + 1;
end
end
h = m_a*(3+2*m_b);
for i = 1:m_a,
```

```
dy(h+i) = -lambda(i)*y(h+i);

for j = 1:m_a,

dy(h+i) = dy(h+i) + a(j,i)*lambda(j)*y(h+j) + a(j,m_a+1)*alpha(i)*lambda(j)*

(s^2*p_sm1(j)+y(h+j)+2*y(h-m_a*(1+m_b)+j)+y(m_a+j)-2*c*p_c(j)-p_c(j));

end

for j = 1:m_b,

dy(h+i) = dy(h+i) + b(j,m_b+1)*mu(j)*(y(m_a+i)*aM1is(j)

-2*y(h+m_a*(1+m_b^2-m_b)+j+(i-1)*m_b)-(s-1)^2*p_s(i)*aM1is(j));

end

end

h = m_a*(4+2*m_b);

l = m_a*(4+2*m_b);

for i = 1:m_a,

g = 0;

for j = 1:m_b,

for k = 1:m_b,

if j ~= k

g = g + 1;

dy(h+1) = -lambda(i)*y(h+1)-(mu(j)+mu(k))*y(h+1)+b(k,j)*mu(k)*(y(l-m_a*(1+m_b)+k+(i-

1)*m_b)-y(l-2*m_a*(1+m_b)+k+(i-1)*m_b))+b(j,k)*mu(j)*(y(l-m_a*(1+m_b)+j+(i-1)*m_b)-

y(l-2*m_a*(1+m_b)+j+(i-1)*m_b))-(b(j,m_b+1)*mu(j)+b(k,m_b+1)*mu(k))*p_s(i)*aM1ijs(j,k);
```

```
for f = 1:m_a,

dy(h+1) = dy(h+1) + a(f,i)*lambda(f)*y(l+g+(f-1)*m_b*(m_b-1))

+a(f,m_a+1)*alpha(i)*lambda(f)*(y(l+g+(f-1)*m_b*(m_b-1))

+beta(j)*y(l-2*m_a*(1+m_b)+k+(f-1)*m_b)+beta(k)*y(l-2*m_a*(1+m_b)+j+(f-1)*m_b)

-p_sm1(f)*aM1ijsm1(j,k)-p_sm1(f)*(beta(j)*aM1ism1(k)+beta(k)*aM1ism1(j)));

end

e = 1;

for d = 1:m_b,

if and(e < m_b, d~=k)

dy(h+1) = dy(h+1) + b(d,j)*mu(d)*y(l+e+(k-1)*(m_b-1)+(i-1)*m_b*(m_b-1));

e = e + 1;

end

end

e = 1;

for d = 1:m_b,

if and(e < m_b, d~=j)

dy(h+1) = dy(h+1) + b(d,k)*mu(d)*y(l+d+(j-1)*(m_b-1)+(i-1)*m_b*(m_b-1));

e = e + 1;

end

end

for d = 1:m_b,
```

```
if and(d~=j,d~=k)

dy(h+1) = dy(h+1) + b(d,m_b+1)*mu(d)*p_s(i)*aM1ijks(j,k,d);

elseif d == j

dy(h+1) = dy(h+1) + b(d,m_b+1)*mu(d)*p_s(i)*aM2ijs(j,k);

elseif d == k

dy(h+1) = dy(h+1) + b(d,m_b+1)*mu(d)*p_s(i)*aM2ijs(k,j);

end

end

h = h + 1;

end

end

end

end

h = m_a*(4+m_b+m_b^2);

l = m_a*(4+m_b+m_b^2);

for i = 1:m_a,

for j = 1:m_b,

dy(h+1) = -lambda(i)*y(h+1)- mu(j)*y(h+1) + b(j,m_b+1)*mu(j)*aM1is(j)*(y(m_a+i)+(s-

1)*p_s(i));

for k = 1:m_a,

dy(h+1) = dy(h+1) + a(k,i)*lambda(k)*y(l+j+(k-1)*m_b)+a(k,m_a+1)*
```

```
alpha(i)*lambda(k)*(s*p_sm1(k)*aM1ism1(j) + beta(j)*s*p_sm1(k)+y(l+j+(k-1)*m_b)
+y(m_a+k)*aM1is(j)-p_c(k)*aM1is(j));
end
for k = 1:m_b,
dy(h+1) = dy(h+1) + b(k,j)*mu(k)*y(l+k+(i-1)*m_b)+b(k,m_b+1)*mu(k)*beta(j)*
(y(l+k+(i-1)*m_b)-y(m_a+i)*aM1is(k)-(s-1)*p_s(i)*aM1is(k));
end
for k = 1:m_b,
if k == j
dy(h+1) = dy(h+1) - b(j,m_b+1)*mu(j)*aM2is(j)*(y(m_a+i) + (s-1)*p_s(i));
else
dy(h+1) = dy(h+1) - b(k,m_b+1)*mu(k)*aM1ijs(j,k)*(y(m_a+i) + (s-1)*p_s(i));
end
end
h = h + 1;
end
end
```

The Initial Conditions

```
function ics = f(Init_number,m_a,m_b,s)

y = zeros(4*m_a+m_a*m_b*(2+m_b),1);

if Init_number < s

y(1) = 1;

y(2*m_a+1) = Init_number;

y(m_a*(3+m_b)+1) = Init_number^2; y(m_a*(4+2*m_b)+1) = Init_number^2; else

y(m_a+1) = 1;

y(m_a*(2+m_b)+1) = Init_number;

y(m_a*(3+2*m_b)+1) = Init_number^2;

y(4*m_a+m_a*m_b*(2+m_b)+1) = Init_number*s;

end

ics = y;
```

The Actual commands for the numerical integration and plots

```
clear;

s = 3;
```

```
c = 5;

m_a = 3;

m_b = 3;

relativetolerance = 1e-1;

t0 = 0;

t1 = 300;

td = 2;

ies = 0;

options = odeset('RelTol',relativetolerance);

for i =t0:td*(t1-t0), tspan(i+1)=i/td; end

[T,Y] = ode45(@PhtPhtsc,tspan,ics(ies,m_a,m_b,s),options);

moment1 = Y(:,2*m_a+1);for i=2*m_a+2:3*m_a+m_a*m_b, moment1=moment1+Y(:,i); end

moment2 = Y(:,3*m_a+m_a*m_b+1);for j=3*m_a+m_a*m_b+2:4*m_a+2*m_a*m_b,

moment2=moment2+Y(:,j); end

variL = Y(:,3*m_a+m_a*m_b+1);

for i = 3*m_a+m_a*m_b+2:3*m_a+2*m_a*m_b,

variL = variL + Y(:,i);

end

for i = 4*m_a+2*m_a*m_b+1:4*m_a+m_a*m_b+m_a*m_b^2,

variL = variL + Y(:,i);

end
```

```
h = 2*m_a;
for i = 1:m_a,
for j = 1:m_b,
for k = 1:m_b,
variL = variL - Y(:,h+(i-1)*m_b + j).*Y(:,h+(i-1)*m_b + k);
end
end
end
variU = Y(:,3*m_a+2*m_a*m_b+1);
newfm = Y(:,2*m_a+m_a*m_b+1);
for i = 3*m_a+2*m_a*m_b+2:4*m_a+2*m_a*m_b,
variU = variU + Y(:,i);
end
for i = 2*m_a+m_a*m_b+2:3*m_a+m_a*m_b,
newfm = newfm + Y(:,i);
end
newfm1 = newfm.^2;
variU = variU - newfm1;
expL = Y(:,2*m_a+1);
for i = 2*m_a+2:2*m_a+m_a*m_b,
expL = expL + Y(:,i);
```

```
end

crossp = expL .* newfm;

vari = variL + variU - 2*crossp;

stddevi = vari.^(1/2);

subplot(2,1,1), plot(T,moment1,'k-');title('E[N(t)] vs. t')

subplot(2,1,2), plot(T,stddevi,'k-');title('StdDev[N(t)] vs. t')

print -dbitmap plot

save T.dat T -ascii

save Moment1.dat moment1 -ascii

save vari.dat vari -ascii

save Y.dat Y -ascii
```

Vita

Javier E. Rueda was born in Quito-Ecuador on May 10, 1979. He attended elementary school at Pensionado Borja N-3 in Quito-Ecuador. Early in high school his interests were in physics and mathematics at C.O.T.A.C. Quito-Ecuador. He continued his education at Universidad San Francisco de Quito, Quito-Ecuador in Industrial Engineering. He decided to broaden his horizons and travel as an exchange student to Virginia Tech in Blacksburg, Virginia, his senior year. There he decided to continue towards a Master's degree since he was introduced to research as an undergraduate research assistant. With the continuing intent of further broadening his horizons he chose to persue the Dual Master's degree at Virginia Tech in Blacksburg, VA, with Ecole des Mines des Nantes in Nantes-France. This thesis is the final step in completing the Dual Master's program.