Examining Connections among Instruction, Conceptual Metaphors, and Beliefs of Instructors and Students

Rachel Lynn Rupnow

Dissertation submitted to the faculty of the Virginia Polytechnic Institute and State University in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
In
Mathematics

Estrella M. S. Johnson. Chair
Anderson H. Norton
Megan J. Wawro
Jesse L. Wilkins

May 30, 2019
Blacksburg, Virginia

Keywords: Conceptual Metaphor, Isomorphism, Homomorphism, Beliefs, Motivation

Examining Connections among Instruction, Conceptual Metaphors, and Beliefs of Instructors and Students

Rachel Lynn Rupnow

ABSTRACT

In this study, I will examine the beliefs and conceptual understanding of instructors and students from two abstract algebra classes. This research takes the form of a case study in which I answer four research questions, each addressing a relationship between instruction and beliefs or conceptual understanding. Specifically, these research questions are:

1. What beliefs do the instructors have about math, teaching, and learning and what relationship exists between these beliefs and instructional practice?
2. What is the relationship between instructional practice and students' beliefs about math, teaching, and learning?
3. What conceptual metaphors do the professors use to describe isomorphisms and homomorphisms and what relationship exists between these metaphors and the mathematical content in instruction?
4. What is the relationship between the mathematical content in instruction and conceptual metaphors the students use to describe isomorphisms and homomorphisms?

In terms of beliefs, the instructors articulated considered positions on the nature of math, math learning, and math teaching. These beliefs were clearly reflected in their overall approaches to teaching. However, their instruction shifted in practice over the course of the semester. Students' beliefs seemed to shift slightly as a result of the ways their instructors taught. However, their core beliefs about math seemed unchanged and some lessons students took away were similar in the two classes.

In terms of conceptual understanding, the instructors provided many conceptual metaphors that related to how they understood isomorphism. They struggled more to provide an image for homomorphism, which requires thinking about a more complicated mathematical object. Their understandings of isomorphism and homomorphism were largely reflected in their instruction with some notable differences. Students took away similar understandings of isomorphism to the instructors, but did not all take away the same level of structural understanding of homomorphism.

In short, relationships between instructors' beliefs and instruction and between instructors' conceptual understanding and instruction were evident. However, certain elements were not made as clear as they perhaps intended. Relationships between instruction and students' beliefs and between instruction and students' conceptual understanding were also evident. However, relationships between instruction and beliefs were subtler than between instruction and conceptual understanding.

Examining Connections among Instruction, Conceptual Metaphors, and Beliefs of Instructors and Students

Rachel Lynn Rupnow

GENERAL AUDIENCE ABSTRACT

In this study, I will examine the beliefs and conceptual understanding of instructors and students from two abstract algebra classes. I address four relationships: between instructors' beliefs and instruction, between instruction and students' beliefs, between instructors' conceptual understanding and instruction, and between instruction and students' conceptual understanding.

Relationships between instructors' beliefs and instruction and between instructors' conceptual understanding and instruction were evident. However, certain elements were not made as clear as they perhaps intended. Relationships between instruction and students' beliefs and between instruction and students' conceptual understanding were also evident. However, relationships between instruction and beliefs were subtler than between instruction and conceptual understanding.

ACKNOWLEDGEMENTS

I have been blessed with a wonderful community of faculty and students in my time at Virginia Tech. This listing of supportive people is by no means exhaustive. However, I would specifically like to thank:

My advisor, Estrella Johnson, whose calmness, feedback, professional advice, and attention to life outside graduate school made this work possible;
My committee members, Megan, Andy, and Jay, whose constructive feedback pushed me to improve and gave me tools to do so;
My fellow math education graduate students, Sarah, Kevin, Tiffany, Brooke, Karen, Kaitlyn, Ahsan, Brigitte, Rachel, Vlad, and Marcie, whose feedback on presentations and interactions in classes and at conferences have been enlightening and rewarding;
My husband, Vahid, whose emotional support and academic advice have helped me keep going;
My professors at Virginia Tech, including Katy, Dr. Linnell, Dr. Childs, Dr. Jones, Dr. Biscotte, and Dr. Doolittle, whose instruction shaped my view of mathematics, education, and the world broadly;
My fellow math graduate students, whose willingness to play intramurals and other games, to discuss in the office, and to take road trips made graduate school bearable;
Other mentors in the math department and the math education community who have given me academic models and encouragement; and
My participants, whose willingness to be part of this research made this work possible.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

Chapter 1 – Introduction

As math educators, we want to understand how teachers' backgrounds and instruction influence students' learning so that we can improve instruction. I specifically care about how teachers' beliefs and content knowledge affect their instruction and how that instruction affects students' beliefs, content knowledge, and achievement. In this study, I examine the conceptual understanding and beliefs about math, teaching, and learning that students took away in two classes. To better understand why they may have taken such beliefs and content knowledge away, I examine teachers' beliefs and content knowledge as well as what happened in instruction in order to see how these aspects might be related.

Many characteristics of teachers influence their instruction, and these characteristics have been reflected in many conceptual models relating teachers' beliefs and instruction. Fennema, Carpenter, and Peterson (1989) created a model for curriculum development that related teachers' knowledge and beliefs to their decisions, which in turn ultimately influenced students' learning. Anderson, White, and Sullivan (2005) created a schematic model, or a framework, relating teachers' objective and subjective knowledge, practices, and the social context of teaching to show factors impacting teachers' beliefs and practices. Many other frameworks and models have also highlighted the importance of teachers' beliefs and knowledge when considering their instruction and eventual impact on students' learning (e.g. Ernest, 1991; Raymond, 1997). Because of this importance to instruction, two aspects of teachers' backgrounds I will focus on in this study are beliefs and content knowledge. This will provide opportunities to look at the interplay of beliefs and instruction as well as content knowledge and instruction.

In addition to examining relationships between teachers' beliefs and instruction and teachers' content knowledge and instruction, I will examine these relationships for students as well. This is because I am interested in how teachers and students connect through instruction in addition to my interest in the specific components of beliefs and content knowledge. My long-term interests are summarized in Figure 1.1 below.

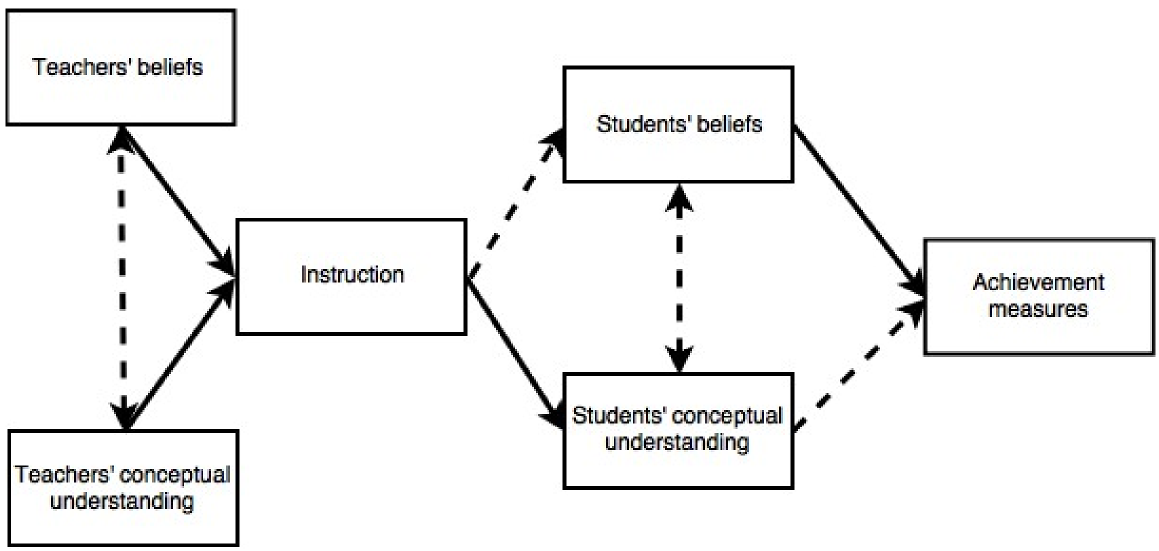

Figure 1.1. Broad framework of influence of teachers on students through instruction.

Abstract Algebra Context

The context in which I examine beliefs, content knowledge, and instruction is Abstract Algebra. Special attention is paid to the instruction of isomorphism and homomorphism, but instruction throughout a 3000-level Abstract Algebra course is examined. The two classes chosen were taught using different instructional approaches. One teacher used the Inquiry Oriented Abstract Algebra (IOAA) instructional materials, based on students reinventing the course content. The other teacher used both lecture and "lab" sessions in which students worked on problems together.

Abstract algebra is important because it is a class required for graduation for 89% of math departments training secondary pre-service teachers, and either required or generally taken in 95% of programs (Blair, Kirkman, & Maxwell, 2013, p. 54). For math majors in general, Abstract Algebra is required by more than 85% of math departments across types of institutions (Blair, et al., 2013, p. 53). Not only is Abstract Algebra important, but it might also be considered a "best case scenario" for implementing and studying a variety of types of instruction. Many research-based reform materials, including the Inquiry Oriented Abstract Algebra (IOAA) curriculum (Larsen, Johnson, & Bartlo, 2013), have been developed for Abstract Algebra, providing more materials for active instruction in this course than for many other upper-level math courses (Johnson, Keller, & Fukawa-Connelly, 2017). Additionally, Abstract Algebra is often a terminal course, allowing more flexibility with the course material.

Previous research has characterized what a lecturer teaching Abstract Algebra could look like, but instruction using inquiry-oriented materials or blended lecture and lab sessions has not been examined in detail. I seek to give descriptions of what these types of instruction in Abstract Algebra could look like to fill this gap in the literature.

Abstract Algebra provides special advantages for studying university teachers' and students' beliefs and conceptual knowledge as well. Because Abstract Algebra is often a terminal course, instructors can use their flexibility with the course material to allow students to look back at the courses they have taken and examine their views of mathematics as a whole. This could lead to students reflecting on their beliefs about mathematics. Specific content knowledge that is addressed in Abstract Algebra, such as isomorphism and homomorphism, has not been studied extensively either, making a study in Abstract Algebra well-suited to examining content knowledge of teachers and students.

Beliefs

Teachers' beliefs about mathematics, teaching, and learning have been shown to influence their instructional practices at the K-12 level (e.g., Ernest, 1991). These instructional practices include, for example, choosing to teach via lecture, through problem solving sessions, or by having students reinvent the mathematics. Wilkins (2008) created a model relating elementary teachers' content knowledge, attitudes towards mathematics, beliefs about the effectiveness of inquiry-based instruction, and the use of inquiry-based instruction. He found that teachers' beliefs about the effectiveness of inquiry-based instruction had the strongest effect on teachers' practice.

In addition to teachers' beliefs influencing instruction, teachers' beliefs may also influence students' beliefs via instruction. Gresalfi (2009) noted differences in two eighth grade algebra classes that both used group work, but used group work differently. In one class, students were left to work in groups as they sought fit, and if any problems arose for students, the teacher would step in and explain the content, which I interpret as indicating a belief in the teacher as the mathematical authority. In the other class, the teacher explicitly directed students in groups to explain their mathematics to their fellow students until everyone was satisfied that they understood, which I interpret as giving some of the mathematical authority to the students. By the end of the year, students seemed to reflect the teachers' beliefs. Students in the first class would give an explanation, but if other group members did not understand, they did not feel the need to elaborate. In the second class, students would demand full explanations from their group members and would receive them.

One reason researchers want to be able to influence students' beliefs is because some student beliefs are correlated with better outcomes than others (Hofer, 1999; Muis, 2004; Schommer, 1998; Schutz, Pintrich, and Young, 1993). One way to consider beliefs is through the dichotomy of availing beliefs, which are beliefs that are correlated with positive outcomes, and non-availing beliefs, those beliefs associated with negative learning outcomes or having no effect (Muis, 2004). Some examples of availing beliefs are the belief that the structure of mathematics is complex, the belief that there are many ways to solve problems, and a belief in incremental learning over time (Schommer, 1998). These beliefs are associated with positive outcomes such as a mastery goal orientation (Schutz, et al., 1993) and higher intrinsic motivation, self-efficacy, self-regulation, and course grades (Hofer, 1999). Because of the potential for positive outcomes, researchers want to know how to foster availing beliefs, including motivation to learn and a growth mindset.

One important area of research is motivation. It is well established that motivation is linked to higher achievement. Uguroglu and Walberg (1979) synthesized work from 40 studies using analysis of variance and regression techniques to estimate 11.4 percent of the variance in achievement measures could be attributed to motivational factors. In math and science, Singh, Granville, and Dika (2002) showed positive effects of motivation on achievement as well.

This study seeks to explore how teachers' and students' beliefs may be connected through instruction and the forms these connections may take at the university level. This includes seeing if students' beliefs change during the semester, especially to be more similar (or more different from) their instructor's beliefs. For observed relationships, the manner by which beliefs were affected is examined.

To summarize, numerous studies at the K-12 level have shown teachers' beliefs influence their instructional practices. Instructional practices may influence students' beliefs about mathematics, teaching, and learning. Students' beliefs in turn affect their achievement. In this study, possible influences on beliefs and a characterization of Abstract Algebra students' beliefs are explored. This study considers relationships among students' beliefs about math, teaching, and learning during the semester in both qualitative and quantitative ways.

Content Knowledge

Teachers need to coordinate many different types of knowledge as they teach. Shulman (1986) noted that teachers need to know the subject they are expected to teach (content knowledge), have knowledge for teaching (pedagogical content knowledge), and be familiar with other classes students take and possible materials that can be used for teaching (curricular knowledge). Building on this foundation, Ball, Thames, and Phelps (2008) considered six types of content knowledge: common content knowledge, horizon content knowledge, specialized content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. The first three branches address subject matter knowledge and the last three address pedagogical content knowledge. A way I will look at teachers' content knowledge is through teachers' use of conceptual metaphors. A "conceptual metaphor is a cognitive mechanism for allowing us to reason about one kind of thing as if it were another" (Lakoff & Núñez, 2000, p. 6). These conceptual metaphors were considered likely to be a major influence on the examples they chose and the informal discussions of formal mathematical ideas they had during class.

Although an instructor could have many different reasons for using a specific metaphor in a specific context, some metaphors may be more helpful for students than others. A way to look at the usefulness of metaphors is to see if there are any relationships between certain metaphors and success (as measured by content inventories, grades, or ability to solve problems). Another possibility is that having access to many metaphors may be more helpful for students than just having a few metaphors available to them. These hypotheses arise from Rupnow (2017), in which students who used more metaphors tended to be more successful. Based on the limited number of participants, I hoped to see whether these findings generalized or whether other patterns could be detected with students who had been instructed on both isomorphisms and homomorphisms.

In a classroom setting, there are interactions between different individuals, providing opportunities for metaphors to spread in different ways. These interactions could include the teacher's metaphors influencing the metaphors students use or some vocal students influencing the metaphors other students (or even the teacher) uses. Seeing how metaphors appear to be transmitted around the classroom will also be considered in this study.

In summary, teachers' content knowledge influences their use of language. The conceptual metaphors teachers use in instruction likely influence students' conceptual metaphors. Students may also influence each other's use of metaphors. Different types of metaphors or numbers of metaphors may be linked to students' success on achievement measures. Which metaphors were used for isomorphism and homomorphism, what contexts they were used in, how metaphors appeared to spread, and which metaphors might be linked to better outcomes will be explored in this study.

Tying everything together, teachers' beliefs and content knowledge have previously been shown to affect their instruction at the K-12 level. This instruction has also been shown to influence students' beliefs and content knowledge, which in turn affect their achievement. For this study, I examine four relationships in a university level setting: between teachers' beliefs and instruction, teachers' content knowledge and instruction, instruction and students' beliefs, and instruction and students' content knowledge. In the next chapter, previous work on beliefs, content knowledge, conceptual metaphors, college instruction, and the teaching and learning of Abstract Algebra will be explored.

Chapter 2 – Literature Review

In this chapter, I will open by providing my epistemological perspective and how I see conceptual understanding and beliefs interacting. Next, I will examine prior work on beliefs and belief systems in general, on teachers' beliefs, and on students' beliefs. Then I will examine previous literature characterizing college instruction. This will include the role of instruction as mediator between teachers' and students' beliefs and characterizations of classes using different curricular materials. Finally, I will examine the literature on the teaching and learning of a particular Abstract Algebra topic, isomorphism, as well as the ways in which instruction could influence the conceptual metaphors used when discussing isomorphism.

Epistemological Perspective

Metaphors undergird how we see the world (Lakoff & Johnson, 1980). As explored in their book, Lakoff and Johnson (1980) put forward the thesis that people's conceptual systems are metaphorical and that the metaphorical language that individuals use can be examined as evidence of what their conceptual system looks like. Conceptual metaphors are a way to link one relatively developed domain of knowledge (source domain) to another less developed domain (target domain) in order to develop one's thinking about the target domain. Specifically, "Conceptual metaphor is a cognitive mechanism for allowing us to reason about one kind of thing as if it were another" (Lakoff & Núñez, 2000, p. 6). Elsewhere, Lakoff and Núñez provide more details as they describe conceptual metaphors as "cross-domain conceptual mappings" that "*project* the structure of a source domain onto a target domain" (Lakoff & Núñez, 1997, p. 32).

The source domain in a conceptual metaphor provides much of the structure for better understanding the target domain, though some interaction between the domains is necessary to be able to utilize the metaphor effectively. As Sfard (1997) notes, in order to extract the useful part of an analogy between a mathematical concept (target domain) and a source domain, some knowledge of the target is necessary: "Indeed, how can one say that something is similar to something else when this "something else" has not yet been born?" (p. 354). She recognized that this leads to a problem akin to the learning paradox, but suggested a potential way out through the two domains informing one another:

> …creation of new concepts should be viewed as a zig-zag movement between relatively familiar source and the emergent target. More often than not, this is a process of mutual adaptation which, with its every swing, does not only strengthen the target but also alters and adds new dimensions to the source (Sfard, 1997, p. 355).

Because of their connection to structuring thought, conceptual metaphors are a natural lens for analyzing abstract concepts. Because we do not experience abstract concepts like isomorphism and homomorphism directly, it is useful to link to domains we can concretely experience (called grounding metaphors in Lakoff and Núñez, 1997) or to other abstract domains (called linking metaphors in Lakoff and Núñez, 1997) that we have robustly constructed.

Furthermore, metaphors are already embedded in our language and thinking. For example, consider Lakoff and Johnson's (1980) example of the metaphor "argument as war". When talking about aspects of arguments, they note that people make statements like "Your claims are *indefensible*" or "He *attacked every weak point* in my argument" (p. 4). These statements reflect a conceptual structure of how people think about arguments, not just artistic language choices. Although argument (verbal discussion) and war (literal armed conflict) are not the same thing, our thought processes for arguments can be understood and structured in terms of our fundamental understanding of war, which is grounded in sensorimotor experience and, thus, a more primitive concept. Later they go on to explain that war or physical conflict is a concept better understood by people than verbal argument, based on experience in fighting for the resources they desire. Thus war is an example of a gestalt, or "a whole that we human beings find more basic than the parts" (p. 70).

Based on experiences in the world and in their culture, people form gestalts that influence the way that they understand other concepts. Furthermore, the metaphors linking wars and arguments could be the standard way of using language (as chosen above), could utilize unused parts of the standard metaphors (Arguments between Bob and Sue reduce to a siege in which each waits for the other to make a mistake.), or could be novel metaphors that present a new way of thinking about a concept (Arguments in this family are CDs that are saved to be replayed when specific topics arise.). The non-novel embedded "dead" or "invisible" metaphors are often overlooked as metaphors because they have become the standard way of discussing certain concepts within a culture. Although speakers may not choose to use novel metaphors in their conversations, they cannot avoid metaphors embedded in the structure of concepts so metaphors can be used to study conceptual understanding.

In a similar manner, metaphors are threaded throughout conceptual understanding of mathematics, and, therefore, mathematical language. Consider Sfard's (1997) example of the idea of rational number. She combines the metaphors "fraction as partitioning," "fraction as piece," and "fraction as number" to construct the concept of rational number. The first two metaphors relate to concrete actions that can be taken to create specific images of fractions; the third (in conjunction with the first two) links fractions to the broader discourse of what numbers are. Using the previous vocabulary, “fraction as partitioning” and “fraction as piece” are examples of grounding metaphors, as they are based in concrete actions that aid in meaning-making. “Fraction as number” is an example of a linking metaphor that places fractions in the broader dialogue about numbers. While “fraction as number” may not appear to be a metaphor at first glance, this is because this metaphor has “died” or become invisible after being incorporated (with the other two metaphors) into the standard way of viewing fractions. Sfard (1997) explains:

> We only acquired a well-developed sense of understanding these [rational] numbers later, when we managed to integrate the different metaphors. At that time the metaphor *fraction as a number* died: expressions like “2/5 is a number” lost their metaphorical flavor and turned into statements of facts.

Because metaphors are embedded in the conceptual structure of mathematics, even the most dedicated formalists use metaphors to structure a definition or describe an abstract mathematical concept. This does not mean that all mathematicians or students use the exact same metaphors, however, because they may not all have the exact same conceptual structure for a given concept. Previous studies have also used conceptual metaphors as a lens for examining conceptual understanding in mathematics. For example, Núñez (2005) used conceptual metaphors while examining infinity. More recently, Zandieh, Ellis, and Rasmussen (2016) noted five different metaphorical clusters used by students in the context of linear transformations and functions in high school and a linear algebra course.

Furthermore, because conceptual understanding can be understood through metaphors both inside and outside the context of mathematics, one can look at an individual’s ways of reasoning as a sensible system through metaphors. When instructing, professors make decisions about the conceptual knowledge to invoke based on their beliefs about what is good for their students, though they may or may not intentionally invoke specific metaphors. Through instruction, students learn different ways to structure their understanding of new concepts. Thus, this perspective is well-suited to my intentions in this study.

Beliefs

Beliefs are important. Peoples' beliefs impact the ways they perceive and interpret situations and the ways they behave (Pajares, 1992). Studies of classroom interactions thus need to take the beliefs of both teachers and students into account in order to interpret events. For this reason, it is not surprising that numerous studies have been done on both students' and teachers' beliefs, including four summary chapters meant to give an overview of literature on teachers' affect and beliefs in math education (McLeod, 1992; Thompson, 1992; Richardson, 1996; Philipp, 2007).

Many early studies did not give a definition of belief; instead they assumed everyone knew what was intended by the term (Thompson, 1992). However, this led to unclear characterizations of beliefs that varied how much they overlapped with knowledge, values, and affect. In order to address this problem, researchers in education in general tried to characterize beliefs and belief systems. Rokeach (1968) considered a belief to be a disposition to action in his general work on beliefs. Beswick (2012) chose to essentially equate knowledge and beliefs, concluding that they could not be disentangled in her constructivist paradigm when studying mathematics. Because of the many definitions given in different contexts, Pajares (1992) elected to give a list of sixteen characteristics most education researchers would agree apply to beliefs. His list included items addressing beliefs' resistance to contradictions, the role of beliefs in interpreting knowledge, and the influence of beliefs on perception and behavior, including in the domain of teaching.

In his synthesis of the literature, Philipp (2007) provided the following definition of beliefs:

> Psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes. (p. 259)

This definition incorporates Rokeach's (1968) and Pajares' (1992) characteristics, among others, while rejecting the idea of fully equating knowledge and beliefs as Beswick (2012) did. In this study, I will adopt Philipp's (2007) definition of beliefs.

Many general frameworks have been developed to describe how multiple beliefs are coordinated into a belief system. One early description of belief systems was given by Green (1971), where he noted three characteristics. First, beliefs together form a quasi-logical structure: some beliefs are derived from others. Second, he pictured a spatial or psychological dimension to beliefs in which related beliefs "cluster" together. Third, he recognized some beliefs are held more tightly than others, and he called this characteristic psychological strength. He also recognized beliefs could be based on different foundations and that differing foundations could impact how they could be changed. Beliefs based on evidence (evidential beliefs) could be influenced by logical argument whereas beliefs held axiomatically, perhaps held as a result of indoctrination, would not be susceptible to change through logical arguments.

Another common belief system framework was developed by Nespor. Nespor (1987) considered groupings of beliefs in his belief system framework that included six features: existential presumption; alternativity; affective and evaluative loading; episodic storage; non-consensuality; and unboundedness. Existential presumption refers to a belief in existence or non-existence of something. Alternativity requires being able to envision alternatives. Affective and evaluative loading refers to the impact of affect and to the impact of being evaluated. Episodic storage refers to remembering according to specific memory episodes. Non-consensuality notes that beliefs could be held by one person and not by another, but that would be considered acceptable to both parties. Unboundedness stresses that there are no clear logical rules that link beliefs to events. His first four features focused on separating belief from knowledge while the last two points addressed how belief systems are structured.

Philipp (2007) synthesized previous belief system characterizations in the following way, largely drawing on Green (1971):

> A metaphor for describing the manner in which one's beliefs are organized in a cluster, generally around a particular idea or object. Beliefs systems are associated with three aspects: (a) Beliefs within a beliefs system may be primary or derivative; (b) beliefs within a beliefs system may be central or peripheral; (c) beliefs are never held in isolation and might be thought of as existing in clusters (p. 259).

Though his three main points mirror Green, Nespor's six features do not conflict with this definition. Furthermore, his two structural features of non-consensuality and unboundedness mirror the foundational points Green made that were not highlighted in Philipp's definition. Like the definition of beliefs, I adopt Philipp's general description of how belief systems are structured. However, I also note that beliefs may come from different sources and that the source of a belief will influence its centrality and its likelihood of change. In addition, drawing on Leatham (2006), though beliefs may seem inconsistent to an outside observer, beliefs are not generally inconsistent to the person that holds them; rather they form "sensible systems." In these sensible systems, ideas that seem inconsistent to an outside observer may be held in different clusters and not interact for the person that holds the belief or ideas that seem inconsistent to an observer have not been fully explained, but logical connections between the ideas do exist.

Teachers' beliefs

In this section, I will examine frameworks that have addressed teachers' beliefs and belief systems, as well as results on ways to influence teachers' beliefs. Next, I will examine ways in which teachers' beliefs about teaching or their students influence their instructional practices. Then I will examine results on teachers' beliefs from two perspectives: researchers looking for inconsistencies in teachers' words and actions and researchers seeking to harmonize teachers' words and actions. Finally, varied perspectives on examining beliefs, including scaling up studies through quantitative methods and examining different populations, will be considered. Because of the limited literature on university instructors' beliefs, especially for non-tenure track instructors, examining university teachers' beliefs will address a gap in the literature.

Some frameworks have been developed to specifically address teachers' beliefs and beliefs systems. Anderson, White, & Sullivan (2005) analyzed six sets of researchers' frameworks of teaching practices and teachers' beliefs (Romberg, 1984; Guskey, 2002; Fennema, Carpenter, & Peterson, 1989; Flexer, Cumbo, Borko, Mayfield, & Marion, 1994; Ernest, 1991; and Raymond, 1997) before providing and testing their own. Common characteristics of these frameworks were examining the impact of beliefs on practice as well as the impact of outside factors on beliefs. Although the purposes of the frameworks varied, from modeling changes in teachers' beliefs (Guskey, 2002) to modeling the relationships between espoused and enacted beliefs (Ernest, 1991) to modeling relationships between beliefs and practice (Raymond, 1997), these diverse frameworks all focused on activities and constructs that may influence teachers' beliefs. Guskey (2002) created a framework in which professional development leads to changes in teachers' practices and students' outcomes, which leads to changes in teachers' beliefs and attitudes. Fennema and colleagues (1989) conceptualized knowledge and beliefs as mutually interacting in curriculum development. Raymond (1997) modeled mathematics beliefs as being influenced by prior school experiences, early family experiences, teacher education programs, and immediate classroom situations and mutually influencing teaching practices. Anderson and colleagues (2005) noted the influence of the social context of teaching on beliefs. The social context of teaching includes constraints and opportunities based on the students' development, knowledge, and understanding.

Ernest (1991) conceptualized teachers' espoused beliefs of learning and teaching mathematics as being influenced by their overall epistemological and ethical perspective and their view of the nature of mathematics. In his 1989 paper, he noted three main views of the nature of mathematics: the problem-solving view, the Platonist view, and the instrumentalist view. The problem-solving view pictures math as a continually changing and growing field that can be created and revised. The Platonist view pictures math as an unchanging, unified entity to be discovered. The instrumentalist view pictures math as a useful but disconnected collection of facts and tools. These views of mathematics can influence teachers' views of learning or students' mathematics. For example, a teacher with an instrumentalist view could focus on one solution path as the only valid solution path for a problem based on a belief that a specific tool or algorithm should be used to address the problem.

Although the previously stated studies developed formal frameworks addressing influences on teachers' beliefs, a number of studies have theorized influences on beliefs without developing formal frameworks. In the conclusion of their study, Barkatsas and Malone (2005) noted that the "broad social and cultural climate of the classroom may impact the teachers' enacted beliefs about mathematics, and mathematics learning, teaching, and assessment" (p. 88). Gilmore, Maher, Feldon, and Timmerman (2014) found that having teaching mentors that directly observed teaching and were involved in helping graduate students improve instruction had a positive effect on developing student-centered teaching orientations in STEM Graduate Teaching Assistants (GTAs). Phillipou & Christou (1998) found that learning about the history of math positively impacted pre-service teachers' attitudes and beliefs about the usefulness of math and satisfaction from math.

Beliefs have also been found to influence teachers' instructional practices in many ways. Beliefs directly impact teachers' plans, which indirectly impacts classroom actions and student performance according to Romberg (1984). Ernest (1991) noted teachers' espoused models of teaching and learning impacted their enacted beliefs of teaching and learning as mediated by social context. Wilkins (2008) noted that teachers' beliefs had the strongest direct effect on instructional practice while also mediating knowledge and attitudes in his quantitative model. Beliefs directly impact teachers' knowledge and decisions, which indirectly influence classroom instruction, students' behavior, cognition, and learning according to Fennema and colleagues (1989). One example of how teachers' beliefs can influence instructional practices is given by Russ and Luna (2013). Russ and Luna (2013) noted differences between what a biology teacher noticed in discussion classes and during labs due to her different views of the purposes of the two types of classes.

Teachers' beliefs about their students can also affect their instruction. Cross (2009) noted the potential impact of having students who were successful, as measured by standardized tests, on teachers' beliefs about the quality of their teaching or the need to change their practices. Turner, Warzon, and Christensen (2011) noted teachers may have pre-conceived notions about students of low ability or low socio-economic status that influence their beliefs about students' ability to learn in certain ways.

One of the major distinctions between different studies incorporating teachers' beliefs is how researchers address perceived inconsistencies in teachers' statements and actions. In many early studies, the purpose was to show the differences between what a teacher claimed about their practice and what they did. Cohen (1990) examined the case of Mrs. Oublier, who claimed to have revolutionized her teaching to reformist methods. However, Cohen only saw superficial changes to her teaching rather than changes in the mathematical authority and mathematical argumentation. Cooney (1985) followed a novice teacher, Fred, from his pre-service teacher program to his first year of independent teaching and noted a disconnect between Fred's stated belief of math being about problem solving and his classroom practice of teaching heuristics.

Many other researchers have noted other seeming inconsistencies between practice and stated beliefs as well (e.g., Thompson, 1984; Barkatsas & Malone, 2005; Schoenfeld, 1999; Cross, 2009). However, other researchers have emphasized that teachers' belief systems do make sense to them in some way and simply stating that a teachers' beliefs are inconsistent likely means analysis has been oversimplified. For example, Leatham (2006) outlined the theoretical framework of "sensible systems," based on Green's (1971) framework. Because teachers' beliefs could be clustered in ways that do not cause beliefs that seem inconsistent to a researcher to come into contact with each other, a teacher could consider their behavior perfectly consistent. Furthermore, certain beliefs could be held as ideal while others are considered more appropriate for specific situations. Generally, he suggested that if a researcher concluded that a teacher's beliefs were inconsistent, the researcher did not have all of the information. Using similar reasoning, Skott (2001) noted that teachers should be observed for a long period of time before coming to conclusions on the relationship between practice and beliefs.

Later studies have also emphasized the importance of examining both teachers' beliefs and practices before drawing conclusions. Schoenfeld (2003), Speer (2005), and Speer (2008) noted the importance of studying both beliefs and practices before making claims about how they were connected to each other. Speer (2005) and Speer (2008) both emphasized the need to ascertain the meanings of words used were the same for researchers and teachers because teachers and researchers could have very different interpretations of what "problem solving" or "group work" mean. Speer (2005) further noted no belief can be fully "professed" by a teacher because a researcher attributes beliefs to teachers based on their statements.

Although asking teachers about their beliefs and observing their practice is helpful for studying individual teachers and for avoiding attributing inconsistencies to teachers, quantitative methods have been used to examine teachers' beliefs on a wider scale. Wilkins (2008) used questions from the Third International Mathematics and Science Study (TIMSS, 1998a,b), the Longitudinal Study of American Youth (LSAY; Miller, Kimmel, Hoffer, & Nelson, 2000), the Second International Mathematics Study (SIMS, 1995), and self-developed items that examined content knowledge, attitudes towards mathematics teaching, beliefs about the importance of specific strategies for effective math instruction, and the frequency of occurrence of specific instructional practices. Based on giving this survey to 481 early elementary (K-2) and upper elementary (grades 3-5) in-service teachers, he found that there was a consistent relationship between beliefs and practice for most teachers. Specifically, teachers' attitudes towards math and math teaching had a positive effect on the use of inquiry based instructional materials whereas content knowledge had a negative effect on the use of inquiry-based materials. He theorized that this may be due to teachers with more content knowledge recalling their own success in math classes and figuring their students could learn that way too. Beliefs had the strongest direct effect on instructional practice and also mediated knowledge and attitudes. He therefore concluded that pre-service and in-teacher development programs should help teachers examine their own beliefs.

In addition to the variety of approaches to measuring beliefs, a variety of teacher populations have been examined. Numerous studies have been conducted on pre-service teachers' beliefs, often with an aim of changing teachers' beliefs (e.g., Jao, 2017; Phillipou & Christou, 1998; Van Zoest, Jones, & Thornton, 1994). Others have addressed in-service teachers, mostly at the K-12 level (e.g., Aguirre & Speer, 1999; Barkatsas & Malone, 2005; Beswick, 2005; Beswick, 2012; Cohen, 1990; Cooney, 1985; Cross, 2009; Erickson, 1993; Skott, 2001; Thompson, 1984; Turner, Warzon, & Christensen, 2011; Wilkins, 2008).

Only a few studies have examined teachers' beliefs at the university level. Gilmore and colleagues (2014) examined the beliefs of GTAs from Science, Technology, Engineering, and Mathematics that were traditional university GTAs, co-taught in a middle school, or were full time teachers in K-12 education or local colleges. Speer (2005, 2008) examined the beliefs of math doctoral GTAs who were teaching calculus. Weber (2004) examined the beliefs underpinning a real analysis professor's teaching presentation choices. He noted that even though the real analysis professor taught in a Definition-Theorem-Proof (DTP) format, his style of teaching varied based on the material and he had specific reasons for teaching different material in different ways; his teaching was based on a coherent belief system. While studies have been conducted on teaching at the university level, only Weber (2004) appears to address professors' beliefs and there does not appear to be any literature on non-tenure track instructors' beliefs about teaching or mathematics.

Students' Beliefs

A number of studies have examined students' epistemological beliefs, their beliefs about the nature, basis, and breadth of knowledge and how we justify those beliefs (Honderich, 1995). Muis (2004) reviewed 33 articles on students' epistemological beliefs about mathematics and found five themes: beliefs about mathematics, development of beliefs, the effects of beliefs on behavior, domain differences, and changing beliefs. For the purposes of this study, the first three themes are the most relevant, and these three themes provide the structure for the first portion of this section.

Epistemological beliefs about mathematics. Epistemological beliefs about mathematics include what Muis (2004) called "nonavailing" beliefs about mathematics: beliefs that are not correlated with more learning gains (as opposed to "availing" beliefs that are correlated with higher learning gains). Some nonavailing beliefs students have about mathematics include believing mathematical knowledge as a field does not change, mathematical knowledge can be passively received from authorities, and that people have or do not have the ability to do mathematics. This idea of who can and cannot do mathematics is related to self-theories of intelligence. Self-theories of intelligence include students' views of the malleability of their intelligence: as a fixed entity or as a malleable quality that can incrementally grow over time (Yeager & Dweck, 2012). Multiple studies have examined students' views of mathematical intelligence, and have noted that students with a fixed mindset often have more performance oriented goals that can feed into stereotype threat for African Americans (Aronson, Fried, & Good, 2002) and women (Dweck, 2007; Dweck & Master, 2009), leading to lower performance, thus making it a nonavailing belief.

Epistemological beliefs about mathematics also include students' views on the nature of mathematics. A number of studies have asked students and faculty members to describe their views of mathematics directly or through metaphors. Markovits and Forgasz (2017) asked fourth and sixth grade students in Israel to say what animal they think math is like and to explain their answer, using these responses to develop themes such as relating math to wisdom, associating math with characteristics of an animal, or providing an animal representing a negative connotation for math. Although directly asking young students about their attitudes towards math might have been difficult, using metaphors allowed researchers to examine relatively young students' attitudes towards math. Schinck, Neale, Pugalee, and Cifarelli (2008) asked ninth and tenth grade American students in a private school to give a metaphor for math and to describe how the metaphor represents math. Students' answers were coded and classified according to the themes perseverance, structure, journey, tool, and student role. Again, although directly asking students' attitudes toward math might have been difficult, this allowed students to express their thoughts in a creative manner. Szydlik (2013) asked general education college students in a problem-based course what "mathematics is". When students' answers were categorized according to themes, themes changed from highly computational answers focused on numbers and equations to more nuanced answers like a way to understand the world or being about logic or proof. Similarly, Ward, Campbell, Goodloe, Miller, Kleja, Kombe, and Torres (2010) asked first year college students in a math appreciation course how they would respond to someone who asked what math is. They noted themes related to math helping understand the world and math being about language, systems, and rules. The studies involving college students' statements and descriptions of math gave students an opportunity to critically consider their beliefs about math. In addition, by asking students about their beliefs about math, researchers have an opportunity to consider if those beliefs are availing or not, and if interventions should be designed to change students' beliefs.

Students' beliefs about the nature of mathematics can affect their beliefs about what math classes should look like, including the forms of classroom norms and sociomathematical norms. Boaler (1999) noted students in "traditional" classes scored more poorly on standardized exams than students in a class focused on contextualizing math. She theorized this score discrepancy happened because students in the "traditional" class saw math in real-life and classroom math as being disconnected. Gresalfi (2009) noted the difference in students' approaches and explanations to classmates in need of help based on the established norms for group work in each classroom. In one class, helping fellow classmates was expected and the burden of understanding was placed on the person explaining a solution method rather than on the person listening to an explanation. Thus, students explaining to others were forced to organize their statements in a flexible way meant to be understood by others. Similarly, Cobb, Gresalfi, and Hodge (2009) noted students' different types of agency that were activated in two different classes. In an Algebra I class, students were given only disciplinary agency, in which they could explain solutions with given solution methods, whereas in a statistics class, students were also given conceptual agency, in which they could choose methods and develop meanings between concepts. Students who were in both classes experienced very different sociomathematical norms in the two classes. Although the researchers had hoped that students would develop a need to fully understand their work in both classes, this did not happen. Thus, the classroom norms established through instruction can influence students' beliefs. How students' beliefs are affected, however, is not well understood.

Development of beliefs. As noted above, a number of students adhere to nonavailing beliefs of mathematics. Unfortunately, these nonavailing beliefs are often at least partially a result of instruction. Boaler (2000) observed that students felt the math classroom was monotonous and lacking meaning; a lack of meaning was not surprising, given most students felt memorization was key to math. Schoenfeld (1988) noted dangers could arise even from "well-taught" classes in which students participated and performed well as measured by standardized tests. Although many outside observers would consider such a geometry class to be well-taught, students may have generated a number of nonavailing beliefs from the class. These nonavailing beliefs included proof processes not being connected to discovery, students thinking it should be possible to solve any problem in five minutes, a belief that only geniuses could truly understand math and others should just accept procedures "from above", and that success is measured by following the teacher's directions instead of by learning.

Although many nonavailing beliefs can be developed through instruction, availing beliefs can also be developed through instruction. De Corte, Op't Eynde, & Verschaffel (2002) cited studies in which students' beliefs about mathematical problem solving were improved by teaching through nontraditional, challenging problems and having students discuss the problems. Gresalfi (2009) noted that two successful students in two classes had similar beliefs about participation at the beginning of the year, but that the ways in which the students interacted with their classmates changed by the end of the year. In the class in which students were responsible for working together and insuring all students in one's group understood content, the strong student would regularly explain her thoughts and solutions in a manner that could be understood by others. In the class in which groups were not given explicit instructions on the purpose of groups, the strong student would explain her solution path and if others did not understand, that was their problem. By structuring group work, the teacher was able to assist her student in developing mathematical communication and reasoning skills rather than only focusing on answers.

Some authors have also noted more availing beliefs about math arise as students become more educated (Muis, 2004). Muis, Trevors, Duffy, Ranellucci, and Foy (2016) examined the beliefs of (mostly psychology) students from secondary school (15-16 years old) to graduate school (23-30 years old) on math, psychology, and general knowledge. Students regularly had absolutist beliefs about math, meaning they expressed views of math knowledge as certain, relied on experts for the source of math knowledge, believed in an objective mathematical reality and were certain of their beliefs. However, they were less absolutist about general knowledge and even less absolutist about psychology. They noted that students at increasing educational levels held less absolutist beliefs, especially the graduate students. The psychology graduate students noted that even though their beliefs reflected certainty in mathematics, it was possible that mathematicians would hold less certain views. The authors posited that this was because the graduate students were linking knowledge of the uncertainty of their own domain of psychology to what high level mathematics might be like. In his study, Szydlik (2013) examined the views of junior/senior math majors and math faculty as well as students in his problem-based class. He noted that junior senior math majors had higher scoring (more availing) beliefs on a beliefs questionnaire and faculty had far higher (more availing) beliefs. Additionally, Schommer (1990) observed that age was correlated with believing the ability to learn is acquired and taking more college-level math classes was correlated with believing knowledge is tentative.

Effects of beliefs on behavior. The precise connections between students' beliefs about math education, the self, and the social context of mathematics is not yet clear (De Corte, et al., 2002). However, some empirical studies have demonstrated correlations between availing beliefs and learning and between availing beliefs and achievement (Muis, 2004). For example, Schommer (1998) suggested students who believed in simple, certain knowledge were more likely to look for a single path to an answer; believers in quick learning may not spend as much time studying because such “extra” time would be a waste; and students who believe in fixed intelligence are more likely to give up when faced with challenges rather than persisting. Phan (2008) partially tested this with both a cross-lag model and a causal-mediating model using structural equation modeling (SEM). In his study with first year arts and second year math students in a Pacific university, he showed beliefs and learning predicted performance and reflective thinking and that beliefs predicted approaches to learning. However, learning approaches did not predict epistemological beliefs. Because of the direct impact of students’ beliefs on behavior, students’ beliefs can directly affect student outcomes.

In addition to the direct impact of students’ beliefs on outcomes, students’ beliefs can influence their motivation to learn or to work hard in a class, thereby influencing student outcomes as mediated by motivation. Because high motivation is correlated with higher achievement across multiple educational studies (Uguroglu and Walberg, 1979) and specifically in math and science (Singh, Granville, and Dika, 2002), different motivational theories may inform interpretations of students’ behavior. To operationalize the motivation construct, I adopt Jones’ (2015) definition of motivation: “Motivation is the extent to which one intends to engage in an activity” (p. 3).

Rendered in this way, similarities between beliefs and motivation are evident. Although motivation is often considered separately from beliefs, both share the trait of being “dispositions toward action” that Philipp (2007) included in his definition of beliefs (p. 259). For the purpose of seeing how ideas about classes relate to experiences in the classroom, I consider motivation to be part of what I am considering when examining students' beliefs.

To structure this discussion about motivational theories, I draw on Jones' (2009) work with the MUSIC Model of Academic Motivation. In this model, he puts forward five components of motivation: eMpowerment, Usefulness, Success, Interest, and Caring. Empowerment refers to how much control students feel they have over their learning. Usefulness refers to students' perceptions of how much practical value the content will have for them in the future. Success refers to students' perceptions of their ability to do well if they learn the content and put forth the effort to learn the content. Interest addresses both short and long-term stimulation provided by an activity. Caring centers on students' perceptions of how much the instructor attends to and is interested in students meeting course objectives and their well-being. In his framework, Jones synthesized previous motivation research to create these five key components of student motivation.

One study in math education related to aspects of empowerment is Kiemar, Groschner, Pehmer, & Seidel (2015), in which teachers engaged in professional development designed to improve motivation, which was measured using the lens of self-determination theory. (One aspect of self-determination theory (Deci & Ryan, 2000) is students' need for autonomy, which is related to empowerment.) In this study, specifically students' feelings of autonomy rose through the intervention, as did their intrinsic motivation.

The usefulness of mathematics is another topic in math education literature. In their review, De Corte and colleagues (2002) noted that task value beliefs, such as beliefs about the importance of learning material, address why students engage in learning. In their study with children's animal metaphors, Markovits and Forgasz (2017) highlighted the theme that girls were more sensitive to the usefulness of math than boys were.

Numerous studies have addressed motivation issues related to success. Control beliefs, related to having the control to do well in a course by studying appropriately, and self-efficacy beliefs, related to one's confidence that a specific task like adding three digit numbers can be done, can be linked to how much a student wants to engage in the course or addition problems (De Corte, et al., 2002). Goal orientation beliefs, or beliefs focused on the purpose for activity, ranging from avoiding looking incompetent to seeking to develop competence have focused different studies (De Corte, et al., 2002; Lazarides & Watt, 2015). Lazarides and Watt (2015) used a multilevel SEM to relate gender, achievement, perceived teacher beliefs, goal structures, and mathematics career intentions. They found that mastery goal orientations were positive predictors for math-based career plans at the classroom level, but performance contexts were positive predictors at the individual level. This may be because students who excel in competitive contexts intend to pursue math-based careers, whereas other students who do not like competition are turned away by such classroom atmospheres. The performance goal orientation had a weak but direct affect on students' math career intentions at the individual level. The mastery goal orientation had an indirect effect through task values at the individual level and a strong direct effect at the classroom level. Self-efficacy, or one's confidence in being able to accomplish a specific task, has also been studied to examine students' motivation. Bonne & Johnston (2016) used self-efficacy theory to examine motivations of students age 7-9, and Hekimoglu & Kittrell (2010) used self-efficacy theory to examine motivations of community college students after watching a documentary on Andrew Wiles' proof of Fermat's last theorem.

Self-theories of intelligence or students' beliefs about knowledge being fixed or being able to grow, can also be linked to students' success. Numerous studies have examined students' mindsets in this way and have noted connections to achievement or other constructs like self-efficacy (e.g. Aronson, Fried, & Good, 2002; Bonne & Johnston, 2016; Dweck, 2007; Dweck & Master, 2009; Yeager & Dweck, 2012). The main theory examining success that will be used in this study will be examinations of mindsets.

The fourth component of the MUSIC model, interest, has been modeled in general by Hidi and Renninger (2006). In their model of interest development, they note that interest can be triggered and maintained through interesting classroom activities like group work or puzzles. Domain identification, or how much a student makes an activity or subject part of their identity can also play a role in students' persistence and interest in a subject or class. Cribbs, Hazari, and Sonnert (2015) specifically examined college math students' interest, recognition, and competence and their relation to math identity. Using SEM, they concluded that competence/performance has an indirect effect on math identity, mediated through recognition and interest. Interest and recognition had direct effects on math identity. Performance/competence had a stronger effect on recognition. This means the more students believe they can do math, the more likely they are to think that others like their parents and teachers view them as a math person. Also, the more students think they can do math, the more likely they are to be interested in math. Students who were more interested in math were also more likely to identify with math.

The final component of the MUSIC model, Caring, has also been studied in education, though not as much in the context of math education. Katz (2016) used the "need for relatedness" component of self-determination theory to examine students' perceptions of support in the classroom. Even though more interactions happened between boys and the teacher than girls and the teacher, the girls felt more supported in the class. She suggested this might be because women often perceive the intentions of others more positively than men do (according to the Social Relations Model).

Because students' beliefs can influence achievement outcomes directly or as mediated by motivation, I have examined different ways students' beliefs and motivation can affect outcomes. In addition, students' beliefs can be influenced by teachers' beliefs as relayed through instruction. Because instruction can serve as a medium for influencing students' beliefs, and, thereby, achievement, it is important to characterize the forms instruction can take.

Characterizing College Instruction

As stated above, students' beliefs about the nature of mathematics can affect their beliefs about what math classes should look like, including the forms of classroom norms and sociomathematical norms. Discussions about the influence of teachers' beliefs on students' beliefs, especially as mediated by instruction, have been noted widely in the K-12 literature (e.g., Boaler, 1999; Boaler, 2000; De Corte, Op't Eynde, & Verschaffel, 2002; Gresalfi, 2009; Cobb, Gresalfi, and Hodge, 2009; Schoenfeld, 1988).

On the other hand, historically, teaching practices at the college level were an "unexamined practice" (Speer, Smith, & Horvath, 2010). When Speer and colleagues (2010) conducted their review, only five studies met their criteria of being empirical research that also examined teaching practice. They defined teaching practice as "what teachers do and think daily, in class and out, as they perform their teaching work" (p. 99) which includes aspects such as planning, in-class decisions, and assessment. Teaching practices were contrasted with instructional activities like lecturing, group work, and lab work.

Teaching practices. A few studies have examined teaching practices in college math "lecture" classes and have found that "lecturers" engage in a variety of teaching practices. Weber (2004) examined the practice of "Dr. T" in a real analysis course. Although Dr. T taught in a definition-theorem-proof (DTP) format, his teaching style varied among logico-structural, procedural, and semantic styles based on the topic. Fukawa-Connelly (2012) examined Dr. Tripp's instruction in Abstract Algebra. Although Dr. Tripp self-identified as a lecturer, she would ask questions while proceeding through proofs. These questions would often focus on funneling students to correct answers, but they did give students an idea of how to think about approaching proofs.

Other studies have examined teaching practices when implementing non-lecture forms of instruction. Fukawa-Connelly (2016) observed an instructor teaching an inquiry-based Abstract Algebra course that focused on defining and proving. In this course, the instructor was more successful in giving students responsibility in proving than defining. Fukawa-Connelly theorized this might be because the professor feared students would choose non-standard definitions and suggested that using a research-based curriculum might allow teachers to feel more confident in letting students take the lead in generating definitions.

A specific type of inquiry-based learning is inquiry-oriented (IO) instruction, and some studies have specifically examined instruction when the professor has used IO materials. In a differential equations class taught by a researcher in differential equations, Speer and Wagner (2009) noted the professor struggled to use students' contributions productively during class.

They concluded there was a need for developing teachers' techniques for analytic scaffolding, which is scaffolding mathematical ideas for students. Other studies have examined the use of IO materials in Abstract Algebra. Johnson (2013) examined how a teacher's pedagogical content knowledge can influence a teacher's ability to interpret and use students' contributions, which thereby affords or constrains future opportunities for students to engage in mathematical activity and influences the discourse in the classroom. Johnson and Larsen (2012) considered Dr. Bond's responses to students' questions and misconceptions through the lens of knowledge of content and students, noting the need to know likely student conceptions/misconceptions and, especially, the consequences of those conceptions/misconceptions as well as the need to situate this knowledge of students (p. 128).

Abstract algebra instruction. Especially in recent years, studies have characterized approaches to instruction taken by Abstract Algebra teachers as a whole. Fukawa-Connelly, Johnson, and Keller (2016) and Johnson, Keller, and Fukawa-Connelly (2017) used surveys of Abstract Algebra teachers to examine instructional practices. The 2016 study noted that 85% of instructors self-reported teaching in a lecture format. The 2017 study built on the same data set to try to characterize pedagogical practices in Abstract Algebra instruction, to note affordances and constraints on lecturers' use of non-lecture practices, and to examine commonalities and differences between lecturers and non-lecturers. In terms of pedagogical practices, the main difference between lecturers and non-lecturers seemed to be the amount of time students had to work individually or collectively during class. Both types of instructors reported presenting theorems and proofs and using examples and counterexamples.

Teachers' beliefs and instruction. The different forms of instruction at the college level proceed from a number of different teacher beliefs about math and instruction. As stated above, only a few studies have directly examined teachers' beliefs at the university level. As Weber (2004) examined the beliefs underpinning Dr. T's teaching presentation choices, he noted that Dr. T's style of teaching varied based on the material and he had specific reasons for teaching different material in different ways. Johnson, Caughman, Fredericks, and Gibson (2013) also examined teachers' priorities for instruction while using IO materials, specifically noting concerns of content coverage, goals for student learning, and student opportunities to discover mathematics.

Some work has also been done to see how teachers' beliefs influence their instructional practices at the college level. In their survey of Abstract Algebra instructors, Johnson et al. (2017) examined lecturers' perceptions of constraints on adopting non-lecture practices. The main reason given was a lack of time to change their practice. However, many constraints, like the ability to utilize professional development and department coverage pressure, seemed to be more internalized constraints than external constraints. Some reasons given for lecturing were that they did not have time to teach in other ways or lacked materials/resources to do so. Many noted they would never switch from lecture because they needed to get through a certain amount of content, thought it would go poorly, or did not think it was appropriate for their students. Resources that instructors considered to be very influential to teaching (in order of being most reported) were their experience as a teacher, experience as a student, and talking to colleagues.

In order to determine the influence of specific beliefs on instructors, quantitative models based on factor analysis and logistic regression models were developed to see whether lecturers and non-lecturers could be distinguished. The models were better at predicting lecturers than non-lecturers. Based on these models, they concluded that people who think lecture is best for instruction would lecture. However, there is a factor missing in terms of why non-lecturers choose not to lecture. In a further follow-up study, Johnson, Keller, Peterson, and Fukawa-Connelly (2017) sent the same survey to institutions where the terminal math degree was a bachelor's. Instead of dividing professors into the two categories of lecturers and non-lecturers, they instead divided professors into three categories according to instructional practices: alternative, mixed, and lecture. To divide instructors into these three groups, lecturers were defined as those believing lecture was the best way to teach. They discovered there were differences in how professors believed students learn and in students' capabilities. Interest in research also differed, with lecturers indicating the most interest in Abstract Algebra research and alternative teachers indicating the most interest in education research. In terms of situational factors impacting professors, differences were found in instructional type in different institutions. Of the alternative teachers, 50% were in bachelor's institutions whereas roughly 18% were in PhD granting institutions. Conversely, of the traditional teachers, roughly 21% were in bachelor's institutions whereas roughly 59% were in PhD granting institutions. (There were 79 instructors from PhD granting institutions and 96 instructors from bachelor's only institutions.) Departmental circumstances did not appear to be significant, however. Despite being able to state some trends in the effect of teachers' beliefs on their instructional practices, the many unanswered questions in the models show we still have much to learn about the influence of teachers' beliefs on their instruction.

Inquiry-oriented instruction. Although lecture is the dominant instructional practice used for teaching Abstract Algebra, the IOAA curriculum provides a well-researched alternative to lecture. IO instruction is based on four guiding principles from Kuster, Johnson, Keene, & Andrews-Larson (2017): *generating* student ways of reasoning, *building* on student contributions, developing a *shared understanding*, and *connecting* to standard mathematical language and notation. Generating student ways of reasoning includes engaging students in mathematical tasks so their thinking is shared and explored with the class. Building on student contributions involves taking students' ideas and using them to direct class discussion, potentially in unforeseen ways. Developing a shared understanding describes helping individual students understand one another's thinking, reasoning, and notation so that a common experience can be "taken-as-shared" in the classroom (Stephan & Rasmussen, 2002). Connecting to standard mathematical language and notation involves transitioning students from the idiosyncratic mathematical notation and terms used in class to standard descriptions and notation, such as "groups" or phase planes.

Teaching and Learning of Isomorphism

Teaching isomorphism. In addition to teaching using different instructional practices, IO and lecture based classes may present material differently. The IO Abstract Algebra (IOAA) materials were developed in three main stages: developing local instructional theories, generalizing to classrooms, and generalizing to new instructors. In order to develop local instructional theories (LITs), teaching experiments were conducted to allow analysis of students' approaches to the tasks presented to them. The three main units of the IOAA curriculum focus on groups (Larsen, 2013), isomorphism (Larsen, 2013), and quotient groups (Larsen & Lockwood, 2013). Here, special attention will be paid to the learning of isomorphism.

As summarized in Larsen (2013) and Larsen and Lockwood (2013), students develop a conception of groups in the first unit, of isomorphism in the second unit, and of quotient groups in the third unit of the IOAA curriculum. Students' previous work with the symmetries of an equilateral triangle are used to remind students that when they chose how to represent the symmetries, they could have represented the symmetries differently. This leads to the idea that there should be ways to link the different representations, and this linking is done through activities in which students explicitly test different mappings. This leads to examining alternative maps that would and would not be consistent ways to model the symmetries of an equilateral triangle, which leads to the development of the concept of homomorphism and to students finding an expression for the homomorphism property. Finally, recognition that this mapping that has been created is a bijective mapping is used to formalize a definition for isomorphism.

Although the IOAA curriculum presents one way to develop conceptions of isomorphism and homomorphism, not all curriculum materials introduce these materials in the same way. A common textbook, Gallian (2013), situates an introduction to isomorphisms after providing information on groups, subgroups, finite groups, and permutation groups. To start his sixth chapter, which is titled "Isomorphisms", Gallian provides a language example in which an American and a German are asked to count a collection of objects, so the American counts in English and the German counts in German (p. 127). This language example in which the two people are describing the same concept, just with slightly different labels, is used to motivate the concept of isomorphism. After this introduction, the definition for group isomorphism is presented along with a picture showing specific elements mapped from one set to another (p. 128). Later in this chapter, properties of isomorphisms are stated, both for isomorphisms acting on elements and groups, leading to the statement of a definition for automorphism. In chapters 7-9, Cosets and Lagrange's Theorem, External Direct Products, and Normal Subgroups and Factor Groups are examined. The definition for group homomorphisms is not stated until chapter 10. The only development of homomorphism in chapter 10 before stating the definition for group homomorphism is the etymology of the word homomorphism and that there is an "intimate connection between the factor groups of a group and homomorphisms of a group" (p. 208).

Student learning. Experts have identified isomorphism and homomorphism as two of the most central topics to Abstract Algebra (Melhuish, 2015). Although some research has been done on how students approach isomorphism, including designing an inquiry-oriented curriculum that addresses isomorphism (Larsen et al., 2013), research explicitly on students' understanding of homomorphism has been scarce.

Because both instructors taught isomorphism before homomorphism, I define them in that order as well. A group isomorphism is defined as follows:

> Two groups $(G, \cdot)$ and $(H, \circ)$ are isomorphic if there exists a one-to-one and onto map $\phi: G \to H$ such that the group operation is preserved; that is, $\phi(a \cdot b) = \phi(a) \circ \phi(b)$ for all a, b in G. If G is isomorphic to H, we write $G \cong H$. The map ϕ is called an isomorphism (Judson, 2017, p. 113).

Thus, a group isomorphism is a bijective function that shows two groups are essentially the same. This framing of sameness has been called "naïve isomorphism" (Leron et al., 1995). A more general, but related, relationship between groups can be found in group homomorphism:

“A homomorphism between groups (G, $\cdot$) and (H, $\circ$) is a map ϕ: $G \rightarrow H$ such that $\phi(g_1 \cdot g_2) = \phi(g_1) \circ \phi(g_2)$ for all $g_1, g_2 \in G$” (Judson, 2017, p. 132). Thus, a homomorphism is a map that preserves the structure of the original group in the domain of the second group. It does not require the groups to have the same cardinality; group G may be larger or smaller than group H. There is always at least one homomorphism between groups; namely, the trivial homomorphism, in which every element of G is mapped to the identity in H.

Relating the two mappings, isomorphisms are a specific type of homomorphism in which the cardinalities of both groups are the same and all elements of the first group are mapped to distinct elements of the second group. For example, V, the Klein four-group, is isomorphic to $Z_2 \times Z_2$. However, $Z_2 \times Z_2$ is not isomorphic to Z_4 because the homomorphism property is not satisfied by any bijective map. While homomorphisms are mappings that preserve the structure of groups, or share sameness of some aspect of the group, isomorphisms are mappings that verify the two groups are essentially the same. Some people have also used a theorem known by many names to link isomorphism and homomorphism via quotient groups. One name for this theorem is the Fundamental Homomorphism Theorem (FHT), which states: “If ψ: $G \rightarrow H$ is a group homomorphism with $K = \ker(\psi)$, then K is normal in G. Let $\phi : G \rightarrow G/K$ be the canonical [natural] homomorphism [$\phi : G \rightarrow G/H$ such that $\phi(g) = gH$]. Then there exists a unique isomorphism η: $G/K \rightarrow \psi(G)$ such that $\psi = \eta\phi$” (Judson, 2017, p. 135).

Previous studies have examined isomorphism in different ways. Early studies mostly provided students with two Cayley tables or stated two groups and asked if they were isomorphic or how they could tell they were isomorphic. Dubinsky, Dautermann, Leron, and Zazkis (1994) indicated that when students considered isomorphisms between groups, they considered the cardinality of each group, but not whether the homomorphism property was satisfied. Leron, Hazzan, and Zazkis (1995) also noted students' tendency to check the cardinality of a group, but their students continued to test multiple other properties by finding the identity element (in Cayley tables), the orders of individual elements (the smallest positive integer m such that a^m is the identity for each element a), the number of elements of given orders in each group, whether a group is generated by a single element, and whether it is commutative. Despite the many factors to check, students would still struggle if more than one way to construct an isomorphism existed, demonstrating a "craving for canonical procedures" (p. 168).

Other studies have considered isomorphism in the context of proof. In related studies, Weber and Alcock (2004) and Weber (2002) asked undergraduate and doctoral students to prove a number of theorems related to isomorphism and to prove or disprove specific groups were isomorphic. While both doctoral and undergraduate students were able to prove the initial simple propositions, the doctoral students continued to be successful in proving the remaining five propositions, while collectively only two proofs (out of a possible 20) of the remaining propositions given by undergraduates were successful. Much like in Dubinsky et al. (1994), these difficulties largely related to undergraduates' tendency to form arbitrary mappings once they had ascertained that a bijection between groups could be formed. They would not apply other properties of the groups when trying to find or disprove the existence of an isomorphism.

Recent studies on isomorphism have shifted the focus to developing LITs that could be transformed into an IO curriculum that included topics such as isomorphism. In the process of examining how students used their existing ways of reasoning to engage with mathematically rich tasks, other student views of isomorphism have come to light. In 2009, Larsen recorded a teaching experiment in which participants were expected to generate a definition of isomorphism. In that study, participant Jessica noted that the definition of isomorphism should include bijection because "…it has to go both ways" (p. 133). Her statement brought to light another approach to isomorphism: symmetry in mapping. Later, Larsen et al. (2013) noted that the homomorphism property was more challenging for students to unpack than the bijection property. Additionally, Larsen (2013) noted, "students' use of the homomorphism property is usually largely or completely implicit" (p. 722). Thus, a number of tasks in the IOAA curriculum engage students in forming an explicit homomorphism in order to help students formulate the homomorphism property and, later, write a complete definition for isomorphism. These studies highlight that students attend to the properties of the two groups, but that generating mappings between the groups was either ignored or very difficult for students.

In these isomorphism studies, some research has been conducted on homomorphism in the process of researching isomorphism. However, a few studies have examined homomorphism more closely in the context of proof. Nardi (2000) noted students' struggles in proving the First Isomorphism Theorem for Groups (equivalent to the FHT) stemmed from three major sources: an inability to recall definitions or a lack of understanding of definitions, poor conceptions of mapping (such as thinking a homomorphism was an element of a group), and not realizing specifically what each part of the proof was proving. Weber (2001) observed that despite undergraduates' ability to recall relevant theorems, they struggled to move past syntactic, "definition unpacking" techniques when trying to prove theorems related to isomorphism and homomorphism, such as proving a group was abelian given the map was a surjective (onto) homomorphism. He also noted doctoral students tended to use conceptual knowledge to formulate proofs more strategically and experienced more success in proving.

More recently, Hausberger (2017) examined the role of the homomorphism concept in tying together ideas across abstract algebra. In this paper, he discussed the common, but vague, description of homomorphism as a “structure-preserving function” and noted that which structure is being referenced is not necessarily clear to students because the word structure is used in multiple contexts. Furthermore, he discussed the rationale of the homomorphism concept as having three main aspects: a way to link two isomorphic objects (one isomorphic to a quotient of the other) based on the FHT; a general procedure that applies across structures (e.g. groups, rings); and the sets of interest are kernels of homomorphisms.

Conceptual metaphors. A theoretical lens for analyzing mappings and isomorphism is the conceptual metaphor construct (e.g. Lakoff & Núñez, 2000). Conceptual metaphors have been used to examine students’ reasoning about many topics including linear transformations (Zandieh, et al., 2016). Zandieh and colleagues examined the properties and metaphorical expressions students used within five metaphorical clusters: Input/Output, Traveling, Morphing, Mapping, and Machine. Rupnow (2017) previously examined students’ metaphorical expressions in the context of isomorphism and utilized the same five categories as Zandieh, et al. (2016) as well as a sixth category of Matching. These six clusters of metaphorical expressions were considered when naming the metaphors used by instructors and students in this study.

Conclusion

Although many factors, such as students’ previous encounters with the concept of isomorphism and the instructors’ previous experiences could influence their conceptual metaphors to describe isomorphisms, it is possible that the curricular materials could also influence the ways instructors and students talk about isomorphism. One of the purposes of this study is to examine which metaphors are commonly used by instructors and students and what contexts they used those metaphors in as well as how different conceptual metaphors or variety of metaphors might provide affordances or constraints on students' achievement.

In addition to instruction potentially mediating the use of metaphors, instruction can influence students' beliefs. Boaler and Greeno (2000) described different ways of knowing that developed in classes taught didactically and discursively. Students in didactic classes were taught in a manner that encouraged "received knowing" or a manner of knowing in which an individual views their knowledge as depending on an authority outside themselves (p. 174). Conversely, students in discursive classes were taught in a manner that encouraged "connected knowing" or knowing in which the individual views knowledge as being constructed by themselves through interaction with others (p. 174).

Gresalfi (2009) examined students from two different 8th grade algebra classes that both engaged in group work. However one teacher specifically taught students that math is a joint enterprise in which students should fully explain their work until others are satisfied with a solution, whereas the other teacher just told students to work in groups without explaining further and would take the role of explaining to students that were confused herself. At the beginning of the year, two successful students (one in each class) had similar group participation tendencies, but at the end of the year, the girl in the joint enterprise class would give thorough explanations meant to help the students who did not understand. On the other hand, the girl where the teacher explained would explain her solution path to her group in a way that made sense to her and if other students did not understand, that was not her problem. Based on other evidence as well, Gresalfi concluded that dispositions are not fully determined by individuals' traits or by classroom practices (though they are influenced by both), that dispositions can change over time, and that some classroom practices are better suited for developing good mathematical dispositions than others (such as the importance of creating sound arguments that everyone understands).

Cilli-Turner (2017) noted differences in undergraduates' beliefs about the functions of proof between students taught using the Moore method, a form of inquiry-based learning (IBL), and more "traditional" lectures. Based on correlations from a survey on beliefs about proofs, she concluded that students in the IBL classes held more availing beliefs about the functions of proof, especially in communication, intellectual challenge, and providing autonomy functions. She theorized this may have been because students regularly presented proofs in class and had opportunities to see the conclusions they were able to come to without the instructor.

Although instruction can influence students' conceptions, a professor's intended message in instruction may not always be the message students hear. Lew, Fukawa-Connelly, Mejia-Ramos, and Weber (2016) presented a case study on an expert instructor's real analysis lecture and what main ideas six above-average students (in three pairs) took away from the lecture versus what mathematicians felt the main points were. Despite a good lecture being presented (according to various measures), students did not take away most of the main points that the lecturer intended or that other mathematicians gleaned. One possible interpretation for this is that many of the main ideas of class were presented verbally whereas students attended more to what was written on the board. Additionally, colloquial terms like "small" were understood differently by the students than they were intended by the professor. The researchers believed this case study contributes to the limited understanding we have of college mathematics teaching, including a need for more theoretical work on the processes by which students interpret what their professors say, connect to their own understanding, and refine their own mental models (p. 167). This dissertation will attempt to build on this foundational work.

In summary, this study examines the role of instruction in mediating teachers' and students' beliefs and content knowledge. Two classes taught using different instructional methods, inquiry-oriented instruction and lecture/lab sessions, are considered to see the influence of teachers' beliefs and content knowledge on their instruction. Conceptual metaphors are examined as a lens for content knowledge. Finally, the influence of instruction on students' beliefs and conceptual metaphors is examined as well.

Chapter 3 – Methods

In this chapter, the overall approach, participants and setting, data collection methods, outlined interview protocols, and analytical framework are addressed. The goal of this chapter is to outline a plan for addressing the following four research questions.

1. What beliefs do the instructors have about math, teaching, and learning and what relationship exists between these beliefs and instructional practice?
2. What is the relationship between instructional practice and students' beliefs about math, teaching, and learning?
3. What conceptual metaphors do the professors use to describe isomorphisms and homomorphisms and what relationship exists between these metaphors and the mathematical content in instruction?
4. What is the relationship between the mathematical content in instruction and conceptual metaphors the students use to describe isomorphisms and homomorphisms?

Case Study Approach

In this project, I am interested in studying how teachers' beliefs and conceptual metaphors influence instruction and how instruction influences students' beliefs and mathematical conceptions. In order to address this overall aim, I will utilize a case study approach using two Abstract Algebra classes as the two cases. According to Yin (2009), "A case study is an empirical inquiry that investigates a contemporary phenomenon in depth and within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident" (p. 18). Because instruction is difficult to study outside its real-life context, a case study approach is appropriate.

Although both cases were 3000-level Abstract Algebra classes at the same university and were taught in the same room with roughly the same number of students, the cases do offer contrasts. Most notably, one instructor taught using the Inquiry Oriented Abstract Algebra (IOAA) instructional materials and sought to guide students to reinvent the mathematics of interest by advancing their mathematical activity through horizontal and vertical mathematization (e.g., Gravemeijer, 1999; Rasmussen, Zandieh, King, & Teppo, 2005). In the other section, the instructor used both lecture and lab days for instruction. Many other contrasts exist and are discussed below.

In each case I look at the teacher's beliefs and metaphors and each set of students' beliefs and uses of metaphors. Each case will provide some evidence for a theory of how beliefs, instruction, metaphors, and outcomes are connected. Having two cases can provide some evidence for ruling out alternative explanations for how these aspects are connected. I will be using a partially embedded design because the student could be considered an extra unit of analysis for impacts of instruction. Although any theories created from two cases should be considered tentative, Yin (2009) notes the merits of examining even two cases:

> Analytic conclusions independently arising from two cases, as with two experiments, will be more powerful than those coming from a single case (or single experiment) alone. Alternatively you may have deliberately selected your two cases because they offered contrasting situations, and you were not seeking a direct replication. In this design, if the subsequent findings support the hypothesized contrast, the results represent a strong start toward theoretical replication—again vastly strengthening your findings compared to those from a single case alone (p. 61).

Thus, having two cases with similarities and differences will afford chances for repetition and contrast, which may provide more evidence in support of theoretical explanations.

Participants and Setting

This project includes two cases: two 3000-level Abstract Algebra classes at a large university in the Eastern United States. Embedded within each case are an instructor and roughly 20 students. The class itself was taught with groups before rings in both classes. Four students from each class were selected for interviews. Students in these classes were mostly math majors or double majors that had previously taken an introduction to proofs course. All students in both classes were solicited for class-wide surveys. By ascertaining many students' beliefs and previous knowledge, a selection of students with availing and non-availing beliefs and relatively stronger and weaker math backgrounds were selected to provide multiple perspectives from each class. Descriptions of participants are provided below.

Instructor A was a tenure-track math education researcher who was teaching Abstract Algebra for the sixth time. Her class met twice each week in 75-minute classes. She used the IOAA curricular materials in her instruction. She had students draw a card to see where they would sit each day, so students changed seats and tablemates nearly every class period.

Student 8a was a senior math major with a concentration in mathematics education. She had previously taken linear algebra, real analysis, multivariable calculus, vector calculus, and a Math for Secondary Teachers course. She had previously taken Instructor A's Math for Secondary Teachers course and intentionally chose Instructor A's section of Abstract Algebra.

Student 10a was a sophomore double major in math and computer science. He had previously taken vector calculus and operational methods and was concurrently in linear algebra. He chose Instructor A's section because it fit well in his schedule.

Student 11a was a sophomore math major. He had previously taken linear algebra, differential equations, and multivariable calculus. He landed in Instructor A's section because he forgot to do course request and that was the section with an opening.

Student 15a was a junior math major with an Applied Computational Mathematics concentration. He had previously taken linear algebra, real analysis, multivariable calculus, vector calculus, introduction to numerical analysis, and calculus of several variables. He was in Instructor A's section because a friend was in this section and he wanted to take it with his friend; the time also fit well in his schedule. His first interview happened in Week 13 because he was a replacement for a student who dropped out of the study.

Instructor B was a full-time instructor who had done his PhD work in logic and was teaching Abstract Algebra for the second time. His class met three times per week in 50-minute classes. He used a lecture/lab format in class where he would lecture for two days each week and give students a sheet of problems to work through in a lab one day each week. Students were allowed to choose their own seats every day and usually sat in the same seats every class period, so they usually had the same tablemates.

Student 3b was a senior aerospace engineering and math double major. He had previously taken linear algebra, real analysis, multivariable calculus, vector calculus, Fourier Series partial differential equations, and numerical analysis. He was in Instructor B's section because this was one of the sections that fit in his schedule, and he had heard from other students that Instructor B was a good teacher.

Student 11b was a sophomore double major in math and classical studies. He had previously taken linear algebra and had started abstract algebra but dropped it in a previous semester. He was in Instructor B's section because it fit well in his schedule and Instructor B had been his advisor before so he knew him.

Student 18b was a sophomore math and physics double major. He had previously taken linear algebra, multivariable calculus, and differential equations and was concurrently in vector calculus and advanced calculus (real analysis). He was in Instructor B's section because the time fit well in his schedule, and Instructor B did not seem to have any terrible reviews online.

Student 19b was a sophomore math major with a concentration in mathematics education. She had previously taken multivariable calculus and vector calculus and was concurrently in linear algebra and advanced calculus (real analysis). She was in Instructor B's section because it fit well with her schedule.

Data Collection

In order to examine instructors' beliefs, metaphors, and intentions in instruction, data were collected through four 45-60 minute face-to-face semi-structured interviews (Fylan, 2005) with each instructor. Instructors received the planned interview questions a day or two before each interview and received $10 compensation for their participation at each interview session. These interviews were dispersed through the semester and had different foci. Collection times for data that was analyzed for this study are shown in Table 3.1. Instructors' statements regarding their beliefs about mathematics, teaching, and learning were compared with class recordings.

In addition to stating their beliefs about teaching, learning, and mathematics, instructors were asked about how they think about isomorphisms and homomorphisms and their ways of describing isomorphisms and homomorphisms to others, if different. This follow-up question was asked in case instructors found that certain metaphors helped them, but not their students or vice versa.

Table 3.1. Data Collection Times and Topics.

	Instructor interviews	Class recordings	Student interviews	Class-wide data collection
Weeks 3-5	Beliefs; Growth mindset inventory; Instructional practices	Recorded a week of class periods	N/A	(consent forms) Previous grade information; Math background; Questions on math, teaching, & learning beliefs
Weeks 5-8	Isomorphism definition/ metaphors; Student motivation perceptions	Recorded all classes on isomorphism/ homomorphism	Beliefs	MUSIC inventory; General questions on motivation in this course
Weeks 9-13	Beliefs	Recorded all classes on quotient groups/ homomorphism	Isomorphism/ homomorphism solving	Isomorphism/ homomorphism ideas; Growth mindset inventory
Weeks 13-15	Student motivation perceptions	N/A	Growth mindset/ motivation follow-up	Questions on role of student and teacher in a "typical" class and this one

Another form of data collection was classroom video data. Whole class discussion data were collected using the video cameras and whole class microphones installed in the classroom.

Additionally, students' conversations in groups were recorded with audio recorders placed at students' tables. Students' written work on boards were recorded using recording technology in the room or by having a picture taken at the end of class. These class recordings were used to determine the amount of class time spent on different types of activities and to classify the nature of instruction using the Inquiry Oriented Instructional Measure (IOIM) and the Toolkit for Assessing Mathematics Instruction—Observation Protocol (TAMI-OP). The IOIM focuses on the extent to which a class adheres to IO instruction, which provided a measure of the extent to which the instructor teaching with the IOAA materials adhered to the principles of IO instruction as outlined in the IOIM, as well as giving a way to compare the lecture/lab class' active learning characteristics to the IO class. The TAMI-OP is a tool designed to allow the types of activities, such as group work or lecture, engaged in by the students and instructor to be recorded separately in 2-minute segments. This also aided in characterizing how class time was used and what activities were common in each class. The class periods recorded during different portions of the semester are stated in Table 3.1.

A third form of data collection was whole class surveys. In order to select students for interviews, students who were willing to participate were asked to respond to four questions about their beliefs on math, teaching, and learning; to define or describe "isomorphism" to the best of their ability; and to provide information on their math background early in the semester. Later surveys addressed students' motivation in the class, the ways they would describe and define isomorphisms and homomorphisms after instruction, and the roles of teachers and students in class. The timing of these surveys is stated in Table 3.1. These surveys were administered through Qualtrics.

The final form of data collection were interviews with eight students. Based on students' responses to the first set of whole class surveys and class observations, four students were selected from each class for 45-60 minute face-to-face semi-structured interviews (Fylan, 2005) dispersed through the semester as shown in Table 3.1. Students were compensated $10 per interview session. Some interviews focused on students' beliefs about mathematics, teaching, and learning. Others were mostly clinical interviews (Hunting, 1997) about isomorphisms and homomorphisms. The interviews were used to examine the impact of instruction on students' beliefs and outcomes as measured by metaphors and content assessments.

Data Collection Instruments

In order to conduct semi-structured interviews with instructors and students, interview protocols were developed. To provide a sense of the types of questions in the protocols, see Table 3.2 and Table 3.3. Questions used in analysis from the interviews with instructors and students are in Appendix A and Appendix B, respectively.

Table 3.2. Sample Questions to Instructors.

Beliefs	What is mathematics?
	In what ways does your teaching style reflect your beliefs about mathematics, teaching, and learning? (Do you perceive any points of tension? If so, in what ways and why?)
	What do you hope an A student will get out of your class? C student? Why?
Metaphors	How would you define an isomorphism between groups?
	How would you describe an "isomorphism" to a ten-year-old child?
	How are the way(s) you personally think about isomorphism the same or different from the ways you describe them to students?

Table 3.3. Sample Questions to Students.

Beliefs	What is mathematics?
	How would you characterize a "typical" math class? How would you characterize this class?
	How would you characterize a "good" Abstract Algebra student? Why?
Metaphors	How would you define an isomorphism between groups?
	How would you describe an isomorphism to others?
	For each of the following pairs of groups, is it possible to form an isomorphism between them? Why or why not?

Some of the class-wide data collection instruments were previously developed and validated by other researchers. The MUSIC Model of Academic Motivation (Jones, 2009) was used to assess students' motivation with respect to their feelings of eMpowerment, Usefulness, Success, Interest, and Caring. The MUSIC Inventory, which was administered mid-semester, has 26 items where responses are given on a 6-point Likert scale (Jones, 2017). The MUSIC Inventory has been validated with college students (Jones & Skaggs, 2016). The Implicit Theories of Intelligence inventory (De Castella & Byrne, 2015) is the result of a slight modification of Dweck, Chiu, and Hong's (1995) original inventory. Both inventories exhibit high reliability as measured by Cronbach's alpha, but De Castella and Byrne's scale, which shifts items from being phrased in a general way (you) to a more personal way (I) had slightly higher reliability scores. For the Entity Self-Beliefs Subscale, $\alpha = .90$ for De Castella and Byrne and for the Incremental Self-Beliefs Subscale $\alpha = .92$. The inventory items are scored on a 7-point Likert scale.

Other questions were related to students' previous math classes, the roles of teachers and students in math classes, and descriptions of isomorphisms and homomorphisms. These questions were usually asked in conjunction with previously developed inventories following the timeline in Table 3.1.

Analytical Framework

The overarching structure tying together the four research questions is a framework of the influence of teachers' beliefs and conceptual metaphors on instruction, which in turn influences students' beliefs and metaphor use. This framework is shown in Figure 3.1. The linchpin of this framework is instruction. My hypothesis was that teachers' beliefs and conceptual understanding would influence their instruction in some ways and that Abstract Algebra instruction would influence students' conceptual understanding of mathematics. I also expected the sum total of students' instructional experiences to influence their beliefs about teaching, learning, and mathematics. However, one Abstract Algebra course would not necessarily have a measurable influence on their beliefs about math, teaching, or learning.

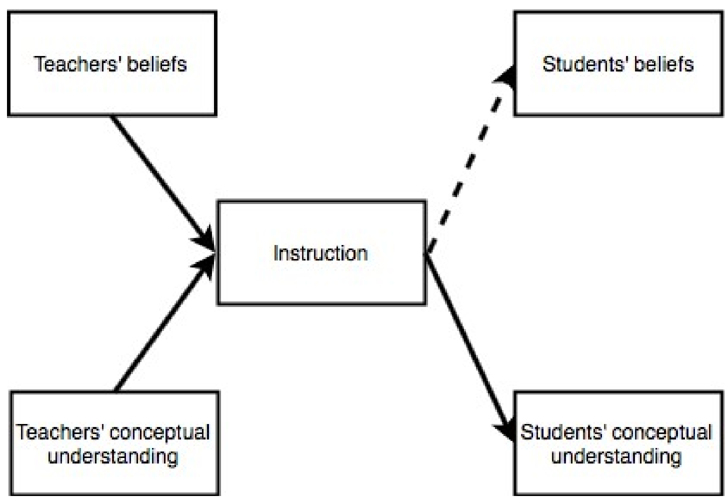

Figure 3.1. Research question framework.

Beliefs. Schoenfeld (2003), Speer (2005), and Speer (2008) stated methodological considerations that should be accounted for when studying beliefs. All three papers emphasized the importance of studying both beliefs and practices before making claims about how they were connected to each other. Schoenfeld (2003) also noted the need to triangulate meanings and interpretations, look at appropriate grain sizes for relationships and mechanisms, not focus exclusively on the factors being looked for, and look for theoretical explanations based on data instead of just correlations.

The teachers' beliefs about mathematics, teaching, and learning are all examined through interview responses and examinations of classroom practice. Interviews were coded using thematic analysis (Braun & Clarke, 2006). In keeping with Leatham (2006), teachers' beliefs were considered sensible systems. If a teacher's statement appeared to contradict their actions in class, clarifying questions were asked based on the interpretation of video clips and contextual considerations and analysis looked to harmonize statements as much as possible.

The teachers' beliefs about mathematics that were studied most closely related to what mathematics is. This includes the purpose of mathematics and its uses, the structure of mathematics as a field of study, mathematics' defining or most salient characteristics, and what it means for an individual to "do" mathematics. To analyze these aspects of beliefs, I started from Ernest's (1989) three common views of the nature of mathematics: problem-solving, Platonist, and instrumental. These beliefs were of interest because they can influence instructional decisions. For instance, a teacher who thinks about mathematics as a discrete set of tools (instrumental view) may drill specific concepts or techniques in class whereas a teacher who views mathematics as an interconnected system that is growing and changing (problem-solving view) may focus more on having students discover new knowledge for themselves.

Teachers' beliefs about learning mathematics focused on how they learn mathematics and how they think their students learn mathematics. Particular foci for beliefs about learning mathematics included existing and preferred student practices (e.g. expected time spent on the class, manner of studying for class, and expected group size for addressing problems in and out of class), recalling their own practices as a student in Abstract Algebra and other math classes, perceptions of students' aptitude for learning mathematics, and the role of students' attitudes towards math and learning in the learning process. Special emphasis was placed on instructors' view of intelligence as fixed or possible to grow incrementally and on their perceptions of students' motivation. This information was used to paint a picture of what instructors viewed as important for learning mathematics and were compared and contrasted with their beliefs about mathematics and teaching in order to see which concepts appear to be linked in their systems of beliefs.

I suspected that teachers' perceptions of students' motivation and aptitude to learn could influence the amount of effort they put into instruction. Instructors that believe students do not care might be more inclined to teach as they had been taught or in a way requiring limited interaction between the teacher and students because it is easier and they do not want to waste their time on something students will not appreciate. Alternatively, instructors could feel that motivating students or changing students' attitudes about math is very important and therefore focus extra time and attention on the course in general or on being engaging specifically.

Teachers' beliefs about teaching mathematics focused on how they facilitate students' learning. Specific points of emphasis for this type of beliefs were the types of exercises and proofs addressed in class, assigned in homework, or intended to be read outside class and the reasons for those choices; how class activities are chosen to structure class and why class is structured that way; how participation is structured in class and why it is structured in that way; and what content and skills are most important for the course and what makes them important.

This data was used to paint a picture of each instructor's classroom, including their perception of the roles of the teacher and the students in the class.

Data on the teachers' beliefs about mathematics, teaching, and learning were compared and contrasted with instructional practice to see what relationships, if any, connect them. I hypothesized that teachers' beliefs about mathematics and learning were more abstract than their beliefs about teaching because teachers make specific decisions about how they want to structure class periods and courses, but they do not necessarily think about the structure of mathematics regularly. Thus, I expected a strong relationship between teachers' beliefs about teaching and their instructional practices and that what happened in instruction likely had a strong influence on teachers' beliefs about instruction. I expected a more indirect relationship between beliefs about math or learning and instructional practice, though I suspected beliefs about math and learning could influence each other. I also suspected that beliefs about math and learning have a stronger influence on beliefs about teaching than the reverse because the abstract ideas held about math influence their choices in how they set-up their course and interact with students.

In order to examine the influence of instruction on students' beliefs, students were asked questions about their beliefs and motivation throughout the course. Select students were also asked about their responses to the growth mindset inventory near the end of the semester. Students' motivation in each class was assessed using the MUSIC inventory and class averages on each component were compared. The MUSIC inventory is designed to provide snapshot information about students' motivation within a single class, not about their views of math broadly. Thus, this inventory was intended to aid in contrasting the two class cases in my study and give a window into students' perceptions of instruction in this course. I hypothesized that teachers' beliefs, as reflected by their instruction, would influence the extent to which students were motivated. Analysis of students' beliefs followed a similar pattern to teachers' beliefs.

The students' beliefs about mathematics that were studied most closely were related to what mathematics is. This includes the purpose of mathematics and its uses, the structure of mathematics as a field of study, mathematics' defining or most salient characteristics, and what it means for an individual to "do" mathematics. To analyze these aspects of beliefs, I started from Ernest's (1989) three common views of the nature of mathematics: problem-solving, Platonist, and instrumental. These beliefs were of interest because they could help students organize their knowledge within the course. Students adhering to an instrumental view might be more narrowly focused on how to do specific problems whereas students with a problem-solving view might be more interested in how their previous knowledge fits together and place more emphasis on conceptual understanding.

Students' beliefs about learning mathematics focused on how they learn mathematics though some attention was be paid to how they think other students learn mathematics. Particular foci for beliefs about learning mathematics included their study practices and their perceptions of others' practices in this class and other classes (e.g. time spent on the class, manner of studying for class, note taking style, and group size for addressing problems in and out of class), perceptions of their aptitude for learning Abstract Algebra, and their attitudes towards math and learning. Emphasis was also placed on students' view of intelligence as fixed or possible to grow incrementally and on their motivation. This information was used to paint a picture of what students viewed as important for learning mathematics and were compared and contrasted with their beliefs about mathematics and teaching in order to see which concepts appeared to be linked in their systems of beliefs.

I suspected that students' own perceived competence and motivation and their perceptions of their teachers' motivation could influence the amount of effort they put into learning. Students that felt confident of their own abilities could put forward less effort because they feel it is unnecessary for them as a smart person (fixed mindset) or could push themselves to develop their understanding further (growth mindset) depending on their beliefs about the nature of learning. Students could enjoy Abstract Algebra and therefore choose to put forward extensive effort or choose to put forward minimal effort in order to make sure their other classes go well. I expected students that perceived their teacher's motivation to be low would also have low motivation; however a student may or may not have high motivation to learn Abstract Algebra if their teacher has high motivation.

Students' beliefs about teaching mathematics focused on how they felt their learning is best facilitated. Specific points of emphasis for this type of beliefs were the types of exercises and proofs they felt should be addressed in class, assigned in homework, or intended to be read outside class and the reasons for those opinions; how class activities should structure a class and why; how participation should be structured in class and why it should be structured that way; and what content and skills are most important for the course and what makes them important. This data was used to paint a picture of how each student felt a math class should be taught, including their perception of the roles of the teacher and the students in the class.

Data on the students' beliefs about mathematics, teaching, and learning were compared and contrasted with instructional practice to see what relationships, if any, connected them. I hypothesized that students' beliefs about mathematics were not as developed as their beliefs about teaching and learning because they are not typically, if ever, prompted to think about the structure of mathematics. Students do, however, experience many different classes and teaching styles so they were likely to have considered which courses to take and which professors to take courses from based on their teaching and learning preferences. Because students had taken a number of courses previously, the impact of this course on their beliefs about teaching and learning were not expected to be strong so I suspected there may not be a clear relationship between instruction and students' beliefs. I expected stronger relationships between students' different types of beliefs than between instructional practices and beliefs. While I acknowledged the possibility that students' beliefs would influence the instructional practices the teachers choose to use, this was not the main focus in this study.

Conceptual metaphors. In order to analyze instructors' and students' conceptual metaphors related to isomorphism and homomorphism, relevant interviews and student answers on surveys were analyzed using thematic analysis (Braun & Clarke, 2006). Specifically, the second interview conducted with students and the beginning of the second interview with instructors comprise the main data sources for this analysis. The instructors were asked questions about what isomorphism and homomorphism are and how they think about concepts versus how they teach them. For more details on questions, see questions 1-8 from interview two in Appendix A. The students were asked similar questions about what isomorphism and homomorphism were at the beginning of their interview, but were also asked to find isomorphisms and homomorphisms between specific groups (assumed to be under addition). For more details on questions, see the whole second student interview protocol in Appendix B.

For coding, I initially used simultaneous, structural, and in vivo coding in keeping with Saldaña (2016). In this initial stage, I sought to familiarize myself with the data from two students' interviews. This included marking passages relevant for understanding isomorphism and homomorphism, gathering passages that seemed similar into themes, and considering what conceptual domains seemed to describe each set of gathered passages for targets isomorphism and homomorphism. The naming of themes was influenced by previous work in Rupnow (2017), Zandieh et al. (2016), and Hausberger (2017).

Next, I checked these themes against the two instructors' interviews and revised themes accordingly. This included both splitting and collapsing some of the themes that had emerged from the first two student interviews. Then, I coded the other six students' interviews and selected transcriptions from class. Some minor splitting and collapsing occurred during this coding process as well. After this, I grouped themes at a higher level into clusters related to mappings, sameness, and the formal definition. Most themes belonged to only one cluster, but a few used components of both mappings and sameness.

The relationship between teachers' conceptual metaphors and students' metaphors were examined through the classes of metaphors teachers used in interviews and especially during instruction. The types of metaphors used by students about isomorphism before and, especially, after instruction were examined to see what similarities and differences existed between teachers' and students' metaphor usage. Students' metaphors were compared to their instructor's metaphors to see if their origin could be ascertained. Students' metaphor use while problem solving in interviews was examined for patterns in the number of classes or specific classes of metaphors in addressing problems.

Summary

Instructors and students from two Abstract Algebra classes were observed and interviewed to answer four research questions relating instructors' conceptual metaphors and beliefs to instruction and relating instruction to students' beliefs and content knowledge through conceptual metaphors and achievement measures. Teachers were directly asked about their beliefs in interviews and were compared to their actions to see how they aligned. A seeming lack of consistency was a cue to ask questions or consider other factors that could be influencing their practice. Students' beliefs were measured through inventories and interviews to look for relationships among constructs and to give a measure to compare students in the two classes. Teachers were also directly asked about their metaphors in instruction and were observed in class to see what types of metaphors they frequently used. Students' metaphor use before instruction, during instruction, and after instruction were examined to see what, if any, influence the instructor's metaphors had on students' metaphors.

Chapter 4—Instruction and Beliefs

In order to answer research questions one and two, which focus on relationships between instructors' beliefs and instruction and between instruction and students' beliefs, I need to examine students' and instructors' beliefs within each class. Therefore, I organize this chapter by case, meaning I will characterize Class A and Class B separately. Then I will see what can be learned about the relationships between instructors' beliefs and instruction and between instruction and students' beliefs by comparing across cases.

Beliefs are made evident through what is said about math, teaching, learning and what is done in the classroom. Thus, within each class I will first examine statements teachers made about their beliefs in interviews in abstract (math and learning) and concrete (teaching and motivation) settings. Then I will characterize time spent on different activities in the classroom as well as describing the interactions of teachers and students to give a description of instruction. Next, I will examine how teachers' beliefs and instruction align. Finally, I will characterize students' beliefs about math, teaching, and learning and examine how instruction aligns with these beliefs.

Class A

In this section, I analyze Instructor A's beliefs as stated in her interview and instruction, and alignment between these contexts. Then, I will examine Class A students' beliefs followed by alignment between instruction and students' beliefs.

Instructor beliefs. Instructor A provided a detailed picture of her views on math, learning, and teaching. She provided a number of images, including unsolicited metaphors and drawings, to give a detailed picture of her views.

Nature of math. When discussing her view of the nature of math, she focused on math as something we do and as a creative endeavor, much like Freudenthal's (1973) description of math as a "human activity". Though Freudenthal seemed to be her main inspiration, if we consider Ernest's (1989) three views of math, she also fits well into one of his categorizations: a problem-solving view of math. For example, she blended these ideas together in the following excerpt:

> Mathematics is a human activity so I believe that it's a way that people make sense and operationalize their world using symbols and numbers and notations and the rules that we have developed in order to manipulate those symbols and ideas and notations.

She also viewed math as including many tools, such as "concepts and logical proving techniques," though she did not personally make use of math as a tool to solve problems. In fact, she almost espoused an anti-instrumentalist view that did not look for math to be useful. For example:

> With my experiences with mathematics, any time I was presented a scenario in which I was supposed to use it to do something, it felt very artificial and forced, and I know that's not the case. I know there's researchers here that very authentically use mathematics to solve problems and that was just never the math I did. So I don't typically think of mathematics as a useful thing.

She also provided different metaphors to describe her view of mathematics. She gave an unsolicited analogy to music that tied together many of her views of mathematics as including different tools and notations, while also having a creative aspect:

> Like we have, like the violin, and we have the clarinet, but we also have sheets and we have notes. So yeah there's the notes on the board or on paper that in one way accurately represent the music that we create or hear, and there's rules about what those notes mean

> and how we use them, and then there's also like the creative aspects of playing an instrument and that different instruments create different sounds and feelings and what not.

When asked what animal math was like, she said a dog as she again highlighted views of math that connect to acquiring specific skills or tools for doing math, but later become better understood and more amenable to flexible use within a branch of mathematics:

> I think math is like a puppy or a dog pet. Like when you first get them it's all about learning like the tricks and how to kind of get math to do what you want it to do, get your puppy to do what you wanted it to do: sit, stand, whatever. But then over time, you develop an intuition and a relationship where it's more authentic engagement with the animal based on an understanding of its needs and your needs and then fostering that relationship.

She later expanded on this metaphor as she noted different branches of math might be viewed as different dogs and that different relationships would be developed with the different dogs or branches of math.

In summary, Instructor A's main view of math was that it was a human activity, meaning that math is created by people to make sense of the world and address their needs. This idea is based on Freudenthal, but also aligns with Ernest's problem-solving view of math. This also aligns with the initial dog metaphor she gave, which focused on developing a relationship with math so that you can understand what is needed to accomplish a goal and then improving on that understanding. She also noted differences between branches of math in the extension of her metaphor.

She also utilized a number of metaphors as she described the nature of math. These included the dog metaphor as well as a music metaphor. The music metaphor again considered the underlying structure of a discipline, in this case music instead of math, but in a discipline people typically view as creative. In so doing, she again focused on the creative nature of math.

Nature of learning math. The most salient aspects of learning math that Instructor A talked about related to how one learns math and what it means to learn math. Much of the focus of her statements was on learning math as being about doing math, which included behaving more like a mathematician and developing definitions and theorems, and being about acquiring content knowledge and tools.

In terms of doing math, she considered doing math central to how she learned math and related this to making sense of the work done in a math class:

> …playing with it. Figuring out its bounds and how it works, what it includes and what it doesn't include, and the same with theorems. Like, it's defining a very specific relationship, so if I can understand the bounds of that relationship that usually provides insight into why it might be true….And then of course, I think there is an aspect of practicing and proving and using the things that they have learned in new and novel settings to continue learning more about the situation….I think some people are just better at doing those activities hidden, if that makes sense. Like I got, like I said, I was very good in my lecture courses cuz they would give me a definition and I would do all those activities, I just didn't do it in class, and I didn't do it because the teacher asked me to, right? Like that was just part of my own personal meaning making, so I feel like yeah, if you're like authentically learning mathematics, you are doing all this meaning making.

In addition to doing math broadly and specifically developing definitions and theorems, she viewed behaving more like a mathematician as a measure of learning, and these activities were not mutually exclusive. For example, she noted "…the ways in which my students are active is that they are defining, they're conjecturing, they're proving, they're doing things like a mathematician would so they're engaging in discipline practices." Another way she viewed students as becoming more like mathematicians was through the development of intuition: "Another one that I try to foster is intuition, and I feel like that's part of becoming fluent in a mathematical discipline, is having some intuition about how things will play out." She also noted other activities that related to math learning, such as doing homework, studying, and being an active note-taker "filling in ideas and conjectures," which also related to doing mathematics.

In terms of characteristics of students who learned math, she viewed anyone as being able to learn math or at least improve from where they were. She indicated she had a growth mindset through her responses to the Likert-scale questions. When asked to explain her responses to these growth mindset questions, she made an analogy to running:

> I'm gonna run at whatever speed I run at. If I keep running I'm gonna get faster. I'm never gonna be Usain Bolt but I will get faster, and I feel like intelligence is similar. Like we might have some predispositions or preferences but with practice and effort I believe we can improve our intelligence.

Thus, ability may influence one's skill with a discipline, but does not preclude improvement. Instead of ability being the most important characteristic to learn math, she viewed dispositions like curiosity as important.

> When we're in class together, I'd much rather have an eager curious student than a "high ability" [air quotes] student. Those things usually go together, right, like the more

> previous math or success with math I've had, I'm more likely to want to engage or be curious or know what questions to ask or have ideas about how to proceed. So it's not that ability isn't important, it's just not enough I guess. And my hopes, my hope is that everyone who walks into my class every day can access the material, regardless of "ability." [air quotes]

She also believed curiosity connected to work ethic: "I feel like [work ethic] also speaks to their curiosity and their willingness to engage."

She did not view other non-prerequisite courses as essential to learning algebra, though they could be generally useful.

> I think [other courses] are useful in that the more I…take kind of raises the level of mathematical sophistication, but I wouldn't necessarily think that they're learning particular topics, or concepts, or even proving techniques that they would be able to bring directly into algebra.

This relates to her view of math as many different dogs that each require developing their own relationship.

In summary, Instructor A focused on doing math as essential to her learning of math and to how she believed others learned math. She defined doing math as engaging in discipline practices like defining and proving or generally behaving more like a mathematician. Acting more like a mathematician also included developing intuition to be able to approach problems more effectively. She indicated she had a growth mindset as she noted that dispositions like curiosity were more important for learning than ability or other non-prerequisite coursework. She also utilized another unsolicited metaphor about running to explain her view of learning with a growth mindset.

Connecting beliefs about the nature of math and the nature of learning math, Instructor A consistently noted the importance of doing math in both abstract math belief contexts. In the nature of math, she focused on math being made to address needs in some way. This aligned with the learning belief that doing creative mathematical work like defining and proving was important. To an extent, a growth mindset also aligns with this belief that doing math is central to learning math. Rather than correctness or speed being the measure of learning, making progress (like a mathematician working on a proof) was held as the standard.

She noted differences between branches of math in the nature of math and nature of learning math contexts as well. In the math context, she noted different relationships needed to be developed with different branches of math in the dog metaphor. In the learning math context, she did not place an emphasis on taking non-prerequisite classes to be able to learn abstract algebra.

Nature of teaching math. Instructor A explored many topics as she discussed the nature of teaching math. These topics included exploring the role of different types of instruction within a class period, classroom norms, and classroom community.

Different types of instruction. Instructor A believed in using different types of instruction in the class period to assist students' learning. Specifically, she saw individual work time, discussion in pairs or small groups, whole class discussion, and lecture as all having their place in the classroom. She viewed individual work time, discussion in small groups, and whole class discussion as ways of letting students do math in class through the tasks she provided, which, as noted above, she viewed as important for learning math.

In my class…there is lots and lots of time devoted to students thinking and working and

> engaging in mathematics. Sometimes the students do that individually and I call it private time and I don't let them talk to each other. Sometimes I ask them to share with a buddy or with their table. I usually think of that more akin to the…pair in the think-pair-share idea. And I see that separate from group work….Group work for me...is when the students are given a large problem and they need to work collaboratively to solve that problem. I usually don't give large problems in my class….What I have is more like they privately think of a question, try to gain some traction, think about how they would start it, building intuition, and then talk to a buddy for consensus-building almost….And then the share part is where the class comes together to share these different ideas…and the class comes together on a consensus.

She saw lecture as a way to tie together the ideas students were meant to have gotten from the math they did: "So that's where I see lectures coming in as a way to like wrap up, provide closure, let them know what the mathematical point of their activity was." Her definition of lecture follows:

> …what I'm summarizing, when I'm tying together, when I'm rehashing things that might have happened in the previous class, so like when I'm talking, but the point isn't to provide a new task or provide a new learning situation for the students.

Although she thought she only spent 15-20% of the lesson on lecture, she felt it was a "high on importance" component of any lesson.

Based on how she spent the time in class and the sequencing of activities her students engaged in, she considered her instruction to be "alternative" to traditional instruction:

> For two reasons. One is the students are engaged, I believe, in authentic mathematical activity, and the other one is because the concepts are developed kind of from the bottom

> up, from the students' understanding to the formal instead of from the formal through examples or something….So by their activities in class and because of the development of the content.

Classroom norms. Instructor A noted some specific classroom norms she sought to establish in her classroom, including expecting students to participate actively in class, transferring the mathematical authority to students, and the expectations of rigor for the class.

She frequently noted the importance she placed on students participating in class. For example, she said:

> I believe I set high expectations for class participation. I believe every student understands that they are expected to engage with the ideas in class and engage with each other's ideas. I also tried to set up the norm early, I even put it in the syllabus, that there's like, this didactic contract that if they stopped asking questions I can assume they understand what's going on. I…try to let them know that I'm there for their learning. They're not here for my content.

She went on to explain this participation was expected to extend to asking questions in class: "I mean, you know, that's a hard one to stop the class and say "I don't understand" but I do expect my students to do that." She also stated her expectation that all students would share their mathematics publicly: "…keep pushing that envelope to make sure that all students are being heard from….Students usually don't get a pass from talking to each other or from sharing in front of class."

The establishment of students' participation was essential to another classroom norm: giving students the mathematical authority in the class. Students needed to do math and make those ideas public for their ideas to push the mathematical agenda: "…in my class I try to make it that the…idea[s] start with the students and then get more and more formal."

She also established the norm that students would write their mathematics formally as befit an abstract algebra class.

> So this is where I enforce the mathematical rigor that is consistent with the 3000-level math course. Like, we're developing intuitive understandings to get to formal definitions. Once we have the formal definitions, they need to be writing formal mathematics, so that's what I look for in the homework and the exams, these are they communicating formal mathematics. Are they communicating, yeah, the formal version of their intuitive understandings.

Classroom community. Instructor A noted the importance of affective and interpersonal factors that impacted her actions in the classroom. This included ideas related to students' level of confusion or frustration at any given point, attention to equity in the classroom, and the role of the interpersonal relationships developed during the semester.

Instructor A noted that students might feel frustrated by the tasks or feeling lost in the class, but she tried to attend to this and mitigate frustration by having students reflect on what their struggle has accomplished. For example, she noted:

> …like every day they're doing something new and confusing but if they take a second and look back at how far they've come in the semester, they're like, "Oh yeah like I do know what a group is. Like every day is confusing but I know what a group is."

She also attended to perceptions students might have about her in terms of the effort investment she was making in her teaching:

> I feel like students…they went from a very like structured, somebody was telling me, I knew where I stood, I knew what I was supposed to be responsible for. And now we're… it's less clear. I feel like students might interpret that as like me taking less responsibility for their learning, when they might feel like a lecture is taking more responsibility. So right like if I was in a class and it felt like everyone around me was getting it and I was struggling, it might feel like the teacher didn't care about my struggle. And I feel like with my class it hits them harder, like maybe they expect the lecturers to not care or don't notice that lecturers don't care, but when they feel like the odd one out in a discussion class, I feel like it hurts a lot harder.

However, she also noted she typically received positive end of semester evaluations from students, so she did not feel pressured to change her instruction radically based on student evaluations.

Instructor A also noted the importance she placed on creating an equitable and inclusive environment for students, though she knew she also had areas to improve in this regard. On the positive side, she noted an end-of-semester evaluation she had received from a student: "I had like this one student writing [a] review that was like, 'This was the first time I've ever been in a math classroom that actually felt inclusive, that I was included, that it was for me.'" On the other hand, she also noted areas related to gender that she wanted to continue thinking about and focusing on in her teaching:

> One thing that has changed over the years is a greater awareness and attention to diversity issues in the classroom. So for instance, I think I have like 20 students in class towards the end of the semester, probably 16 are showing up on a regular basis. When I think about those 16 and then I think about who I'm calling on, or who I'm going to for

answers, there are times when I'm disappointed in myself about who I'm going to, and I feel like that's a new thing that I'm attending to….At the end of class, I've always been in the habit of thinking back about who I heard from and who I didn't hear from. So that's not new that I'm like trying to pay attention to that, but that it has this like gendered overlay on it is new.

She also noted the importance of creating a classroom community and rapport with students throughout the semester. This could manifest itself through specifically reaching out to students who were struggling or members of "easily marginalized groups." She observed rapport seemed to help with motivation for students to keep working: "I think there was just interpersonal rapport that has built up that they will work for me harder, or more, or longer, or more eager than they would for a stranger."

In summary, Instructor A made a concerted effort to vary the size of groups engaged in building consensus about math, from working individually to discussing in pairs, to discussing at the full class level. This related to her desire for students to retain mathematical authority, as their mathematics would drive progress in the lesson. Also, though she did not feel she lectured much, she felt it was important for summing up what students were meant to get from their work on tasks. Putting together these components of instruction, she felt she taught in an alternative way as considered on the scale from Johnson et al. (2017). These beliefs directly align with her previously stated beliefs about the nature of math and of learning math. In particular, in class she wanted students to do math and discuss it in larger and larger groups. This aligns with her beliefs that math is a human activity and that we learn math by doing math. The acts of proving and defining are also some of the activities intended to be addressed in tasks.

Instructor A considered potential affective effects of her alternative instruction. These included making students feel included in the class, rapport with students leading to better work in class, managing students' frustration at tasks, and expecting the same level of rigor as would be expected in any abstract algebra class despite a different class format. These aspects connect more directly to motivation.

Motivation. Instructor A was specifically asked about how she defined motivation and how different aspects from the MUSIC model might affect or be related to her students' motivation. She also seemed to use participation as a measure for how motivated students were.

In terms of defining motivation, she was asked this twice and her responses were slightly different, though related. When asked in interview 2, she said, "I would define motivation as the will [Rosie the Riveter gesture] or the will to work towards a goal, and I think the goal can change, and that different tasks or activities align with different goals." In the fourth interview, she said,

> I guess motivation for me is ... an eagerness or a desire to succeed? So like, I know you can succeed in lots of different ways. Right? So there is an eagerness and a desire to succeed in the grade. I feel like there is an eagerness and a desire to meet my expectations in class.

Though the second definition had more of a success flavor, both definitions focused on accomplishing some sort of goal. This may also relate to why she linked motivation and participation: "If they're participating, I feel like that is evidence of motivation. So if I need them to participate, I need them to be motivated to participate."

In terms of the MUSIC model's components, she was given the same inventory students received as part of the second class-wide survey and asked to fill it out as she thought a student would fill it out. In general, she underestimated the scores they gave her, though she had the same order as students (from giving the lowest score on usefulness to highest on caring). Her scores and averages from the class are reported in Table 4.1. (Recall the scale is from 1 to 6, with a 1 indicating strong disagreement and a 6 indicating strong agreement.) The averages were based on responses from 19 students.

Table 4.1. MUSIC Score Comparison for Instructor A and Class.

	Instructor A	Class A	Standard Deviation	Standard Deviations away
eMpowerment	2.4	4.67	.81	-2.79
Usefulness	1.8	4.55	.90	-3.06
Success	4	4.74	.86	-.86
Interest	4.5	4.98	.70	-.69
Caring	5	5.68	.42	-1.62

For empowerment, she noted she gave students the appearance of control as their contributions would push the class forward, though she did have a set of intended tasks for them to work through:

> I try to give them the appearance that they control the class. Most of the contributions they make I am anticipating and I have planned tasks to address them, but I try to make them feel like they're leading the class, to the extent possible.

Thus, in the follow-up interview in which she was asked to reflect on differences between her expected scores and students' responses, she was not surprised that students felt empowered in retrospect.

For usefulness, she did not see much usefulness for abstract algebra content besides process skills: "I think problem-solving is important. So I think the process skills are important. The content I don't know how it's relevant to anyone." Thus, she was surprised that students said they found the class useful: "Why do they think it's useful? That's super surprising," and later in the interview, "but the really surprising one is useful. Like that's shocking."

For success, she considered a number of measures of success that students might use. These included the speed with which work could be completed as well as the grades obtained:

> I think they believe that if they understand something it should be fast and easy. And like that understanding is what they would characterize as success…. so they will think they are successful if they are able to easily and correctly complete homeworks and exams and if they are able to easily and correctly complete homework and exams, then they will get an A…

Because of this expected connection between students' perceptions of their ability to succeed and their grades, she had the policy that students could redo their homework to improve their grade as long as they had handed it in completed and on time:

> And that's part of the reason that I let them redo their homework, cuz the first time through it's hard. If I can give them a second time through, I think it's more likely to be successful. It's also my sneaky little way of getting them to keep thinking about ideas.

This policy also allowed her to link students' perceptions of their ability to succeed to continued engagement with the material:

> There's a motivation to increase their grade by re-looking at those homework problems with my notes, and trying to redo them. So I try to motivate both their grades and their like continued study of that material with that homework policy.

This policy may also have contributed to greater alignment of scores on this component, as there was broad agreement that students could be successful in the class.

For interest, before seeing student scores she thought she might stimulate interest, especially at the task level: "I don't know if they're motivated, like, to learn abstract algebra more generally, but I feel like on the task level sometimes I can really get them hooked." She was glad that there was alignment between her assessment and students' assessment.

For caring, she showed she cared by reaching out to students in a way that focused on inclusivity:

> If you don't reach out, you lose students and the students you lose are the ones who are like suffering, from students in college or first-generation students or easily marginalized groups, right? So I feel like caring is my way of trying to keep those groups safe and keep them in school. So I try to care a lot.

Her students agreed that she cared, as they averaged more than "agree" as shown in Table 4.1.

In summary, she focused on motivation as a will to accomplish a goal and perceived students' motivation in terms of their participation in class. In terms of the MUSIC components that could foster motivation, she provided specific ways she looked to foster motivation in each context. She gave students the appearance of control through their work on the tasks (M), helped students develop their proof writing and problem-solving skills (U), provided homework redos to allow students to have more chances to work on assignments and be successful (S), worked through tasks in class that were stimulating (I), and worked to make students feel they belonged in class (C). However, she gave herself lower marks on these components than her students gave her, especially for usefulness. Though she noted ways that students would improve process skills and that this could be useful to students, she generally did not find abstract algebra useful.

Connecting across beliefs about teaching math and motivation, Instructor A gave consistent answers to questions on the nature of teaching math and how she fostered components of motivation in her class across different interviews. How she said she promoted empowerment related to ceding the mathematical authority to students, which she did through having students participate in tasks that guided the mathematical work in class. This was a fundamental part of the structure of her class. She promoted usefulness through helping students develop problem solving skills that would enable them to act more like mathematicians. However, she did not find this particularly useful, perhaps because she did not expect most of her students would in fact be mathematicians. How she helped students be successful, offering homework redos, related to helping students' grades improve, which she believed students would see as helping them be successful. However, redos also encouraged students to review the math required to do the problem, meaning they were engaged in doing more math, which was how she viewed people as learning math. She felt interest was stimulated by the tasks done in class, and she had previously noted that she wanted students to be actively involved in doing math. She felt she demonstrated caring through reaching out to students, and she had noted her desire to establish rapport with students and to build a community that supported learning at other times.

How teaching reflects beliefs. Instructor A reflected on how her teaching mirrors her beliefs about the nature of math and learning math. She also mentioned a number of places where her research interests influenced her beliefs about teaching. A number of times, she noted how interconnected her beliefs were. For example:

> I feel like [my teaching style] directly reflects [my beliefs about math teaching and learning]. Yeah, I believe math is a human activity. I believe students' engagement in activity is learning, and I teach by engaging them in activity in the hopes that that changes and is evident on the learning. I feel like they're very intertwined.

She also noted her choice to have students' activity be central to lessons displayed her belief that math was based on sense-making: "So like the belief that the student understanding can motivate, be tied to, underpin the formal definition, I think there's a belief about mathematics."

In addition to the influence of her beliefs about math and learning on her beliefs about teaching, she observed different places where her research or research in her field influenced how she approached the classroom. For example, differences in outcomes for students of different genders had been observed in classes using inquiry-oriented curricular materials, and as noted above, she attended more to who she was calling on in class as a result. She also noted how she had read about influences on women going to grad school and how that influenced her approach to talking with women in her classes:

> Every once and a while a paper comes through…there is one about women going to grad school and how if they have the same grades as men that's not enough. They need somebody reaching out and like telling them, so like that, I think that really spoke to me like, oh, I need to reach out and tell them.

This also reflects some beliefs about inclusivity in the math classroom, as this belief leads her to make a concerted effort to foster a sense of belonging in grad school in math for women in her class.

She also mentioned readings she had done in graduate school had influenced her beliefs about teaching and learning that were reflected in other statements about her beliefs:

> Some very influential Math Ed research when I was a grad student. So one of them is this, Freudenthal's idea of realistic mathematics education, that math is a human activity that we engage in. I think the other one was Lave and Wenger's participation metaphor for learning that, though, that really spoke to me. That it's not about the things I have in my head; it's about the things I can do that signify learning.

This also relates to her measure of students' learning: acting more like a mathematician. Specifically, behaving more like a mathematician involves doing mathematics like defining and proving, and these activities are observable.

Characterizing instruction. In this section, I will characterize instruction in three units in terms of general instructional approaches, patterns of engagement with students, and common activity within classes. This characterization of instruction is intended to give details on the Instruction component used in research questions one and two. These characterizations draw on analysis aided by the TAMI-OP, IOIM, and statements in interviews.

The TAMI-OP provided ways to organize the number of 2-minute segments spent on different activities like lecturing, as well as the number of questions asked within each 2-minute segment. Terms, such as "lecturing" will be used in tables to describe how often events happened in class. Lecturing referred to the instructor standing near the document camera or a whiteboard and addressing the class as a whole. This also included times when the instructor facilitated a classroom discussion, but all questions and responses were directed through them, as opposed to students freely responding to other students' statements in a whole class setting. Students freely responding to each other in a whole class setting was considered "whole class discussion." Neither lecturing nor whole class discussion included managing group discussions or one-on-one conversations with students. However, students working individually or in groups in such a setting was considered "students working." Student presentations were distinct from this category, as they involved a student going to a whiteboard or the document camera to share their work publicly and then explaining that work to the class. Questions to the class refer to a non-rhetorical question being asked by the instructor to the whole class or to a specific student while in a lecturing or whole class discussion context. Student answers count the number of responses such questions received; some questions received more than one student answer, and some received no answer. These contexts are not distinguished. Questions to the instructor were only counted in whole class discussions or lecture contexts.

When presenting data, all rates are rounded to the nearest hundredth (or whole percent) and counts are given to the nearest whole. Counts of time blocks refer to the number of 2-minute blocks (e.g. 9/31 segments lecturing means 9 of the 31 2-minute segments had at least a portion of the time spent on lecturing). Statistics on the numbers of questions asked and answered and the number of 2-minute intervals in which different activities occurred are found in a number of tables below. Questions asked and answered are taken only from 2-minute segments in which whole class discussion, student presentations to the class, or lecturing occurred. This excludes questions that were asked and answered only in small group settings, meaning they were not audible to the whole class.

The IOIM provided a way to characterize how inquiry-oriented a class was. This was measured on a five-point scale (low to high) across seven different practices. These scores were intended to provide an overarching sense of what happened in each unit, as the IOIM is intended to be used to score a full unit of material.

Recall Class A met twice each week in 75-minute class periods. I provide analysis from 3 times in the semester: Week 3, in the middle of a unit on groups; Weeks 7-8, through the whole unit on group isomorphism; and Weeks 9-12, through the whole unit on quotient groups and the Fundamental Homomorphism Theorem (FHT).

In Week 3, the IOIM shows medium-high and high scores for all components from two days of the group unit. Specifically, she received high scores on facilitating student engagement (practice 1) and supporting formalizing of ideas and notation (practice 7); and medium high scores on eliciting student reasoning and contributions (practice 2), inquiring into student thinking (practice 3), being responsive to student contributions (practice 4), engaging students in one another's thinking (practice 5), and guiding and managing the development of the mathematical agenda (practice 6). (These scores are later summarized in Table 4.8.) These scores indicate that across all practices the class aligned at a medium-high to high level with the qualities of inquiry-oriented instruction. Thus, the class exemplified the four components underlying the seven practices: generating student contributions, building on those student contributions, creating a shared understanding of the mathematics throughout the class, and formalizing students' mathematics so that notation for the ideas discussed would be understandable to a mathematician outside the class.

The results from the IOIM make the summary statistics from this week unsurprising. Table 4.2 demonstrates that while some instructional time was spent on having the instructor lecture, more time was spent facilitating whole class discussion, in which students freely responded to each other's ideas. This aligns with the first three components of generating student ideas (so there was something to discuss), building on student ideas, and creating a shared understanding (through the ideas being discussed at the whole class level). Additionally, students had extensive time to work individually or in groups (generating student contributions) and students were given opportunities to present their work publicly to the class (building on student contributions and creating a shared understanding).

Table 4.2. Partial Group Unit Time Averages for Instructor A.

Day	Segments lecturing	Segments students working	Segments student presenting	Segments whole class discussion
Day 1	9/31	17/31	3/31	19/31
Day 2	5/38	22/38	9/38	10/38
Total	14/69	39/69	12/69	29/69
Average	20%	57%	17%	42%

Furthermore, Table 4.3 shows questions were being asked and answered regularly, on average more than one question every two minutes being asked by the instructor. Similarly, a student would answer an instructor's question at least once every two minutes, on average. Students asked the instructor a question roughly once every six minutes on average. This again highlights the interactive exchange of ideas necessary for creating a shared understanding in class.

Table 4.3. Partial Group Unit Question Averages for Instructor A.

Day	Questions to Class	Student Answers	Questions to Instructor
Day 1	62	48	14
Day 2	44	39	10
Total	106	87	24
Average	.77/minute	.63/minute	.17/minute

In Weeks 7 and 8, the IOIM shows medium and medium-high scores for all components from the isomorphism unit. Specifically, she received medium-high scores on facilitating student engagement (practice 1), eliciting student reasoning and contributions (practice 2), being responsive to student contributions (practice 4), engaging students in one another's thinking (practice 5), guiding and managing the development of the mathematical agenda (practice 6), and supporting formalizing of ideas and notation (practice 7); and a medium score on inquiring into student thinking (practice 3). In comparison to the first unit, scores on practices 1, 3, and 7 decreased by one. Practice 1 relates to generating student contributions, practice 3 relates to generating and building on student contributions, and practice 7 relates to creating a shared understanding and connecting to standard mathematical language and notation. Thus, these three practices span all four underlying components of inquiry-oriented instruction.

These slightly lower IOIM scores are not surprising, given a shift in how time was spent in class. From Table 4.4, we see a larger amount of time spent lecturing and a much smaller amount of time spent on whole class discussion than in the previous unit. There was also less time spent on student presentations, but the amount of time students were working was essentially the same.

Table 4.4. Isomorphism Unit Time Averages for Instructor A.

Day	Segments lecturing	Segments students working	Segments student presenting	Segments whole class discussion
Day 1	21/37	20/37	2/37	13/37
Day 2	25/38	25/38	1/38	0/38
Day 3	20/37	19/37	7/37	0/37
Day 4	25/38	22/38	0/38	1/38
Total	91/150	86/150	10/150	14/150
Average	61%	57%	7%	9%

In terms of questions being asked and answered, questions were asked and answered slightly less often, but still more than one every two minutes. Students asked questions slightly more frequently than in the last unit, once every four minutes on average, as shown in Table 4.5.

Table 4.5. Isomorphism Unit Question Averages for Instructor A.

Day	Questions to Class	Student Answers	Questions to Instructor
Day 1	47	49	11
Day 2	51	43	23
Day 3	38	23	19
Day 4	65	53	22
Total	201	168	75
Average	.67/minute	.56/minute	.25/minute

In weeks 9-12, the IOIM shows similar but slightly lower scores on the quotient group and FHT unit than the isomorphism unit. Specifically, she received medium-high scores on facilitating student engagement (practice 1) and guiding and managing the development of the mathematical agenda (practice 6); and medium scores on eliciting student reasoning and contributions (practice 2), inquiring into student thinking (practice 3), being responsive to student contributions (practice 4), engaging students in one another's thinking (practice 5), and supporting formalizing of ideas and notation (practice 7). Relative to unit 2, scores on practices

2, 5, 6, and 7 decreased by 1. Practice 2 relates to generating student contributions and building on those contributions, practice 5 relates to developing a shared understanding, practice 6 relates to building on student contributions, and practice 7 relates to developing a shared understanding and connecting to standard mathematical language and notation. Thus, the practices with decreased scores relate to all four underlying components of inquiry-oriented instruction.

The lower IOIM scores are again reflected in the time spent on tasks in class. The amount of time in whole class discussion or having students present decreased again while the amount of time lecturing increased slightly. The amount of time students were working was similar to the previous units, but slightly lower, as shown in Table 4.6.

Table 4.6. Quotient Group/Homomorphism Unit Time Averages.

Day	Segments lecturing	Segments students working	Segments student presenting	Segments whole class discussion
Day 1	23/38	24/38	0/38	0/38
Day 2	24/38	26/38	0/38	1/38
Day 3	30/38	15/38	3/38	0/38
Day 4	25/36	19/36	2/38	0/38
Day 5	24/37	18/37	0/37	1/38
Day 6	27/38	18/38	0/38	0/38
Total	153/225	120/225	5/225	2/225
Average	68%	53%	2%	1%

In terms of questions and answers, the instructor asked questions slightly less often, though still more than once every two minutes, and received fewer replies, slightly less than one answer every two minutes. Questions to the instructor were similar to the previous unit, roughly one every four minutes.

Table 4.7. Quotient Group/Homomorphism Unit Question Averages.

Day	Questions to Class	Student Answers	Questions to Instructor
Day 1	67	47	16
Day 2	34	27	24
Day 3	31	20	21
Day 4	51	31	11
Day 5	51	43	15

Day 6	31	22	26
Total	265	190	113
Average	.59/minute	.41/minute	.24/minute

IOIM scores across all three units are summarized in Table 4.8. Looking across the IOIM table, notice the scores hold steady or decrease across the three units for all seven practices. However, no score is ever below a medium (3), indicating some alignment with inquiry-oriented instruction. Furthermore, we can compare her scores to the average of 13 other Abstract Algebra instructors who used the IOAA materials. These instructors were previously scored with the IOIM on the quotient group unit as reported in Kuster, Johnson, Rupnow, and Wilhelm (2019). Comparing Instructor A's quotient group unit scores to these instructors, we see that her scores were at or above the medians for each practice with only 2 exceptions: the practice 6 and 7 scores on the quotient group unit. The score of 3 on practice 6 matched the lowest score from the other instructors, and the score of 3 for practice 7 matched the first quartile division. The 3's on practices 2 and 5 matched the median. The 4 on practice one and the 3 on practice 3 matched the third quartile cutoff, and the 4 on practice 4 exceeded the third quartile cutoff. Thus, Instructor A's scores were average or above average for 5 of the 7 practices, again indicating alignment with how others have implemented the IOAA materials.

Table 4.8. IOIM Scores for Instructor A.

	P1	P2	P3	P4	P5	P6	P7
Groups	5	4	4	4	4	4	5
Isomorphism	4	4	3	4	4	4	4
Quotient Groups	4	3	3	4	3	3	3

Summary time usage statistics from the three units are presented in Table 4.9 below. Notice the time lecturing increases across the three units, while the time students were presenting and in whole class discussion decreases across the three units. The time students were working individually or in groups held fairly steady across the three units. This shift in how time was spent may be indicative of the shift in difficulty of the material, leading the instructor to feel more scaffolding and direct presentation of material was appropriate. Alternatively, this shift could be the result of wanting to make sure all desired content was addressed in class.

Table 4.9. Average Time Usage across Units for Instructor A.

	Segments lecturing	Segments students working	Segments student presenting	Segments whole class discussion
Groups	20%	57%	17%	42%
Isomorphism	61%	57%	7%	9%
Quotient Groups	68%	53%	2%	1%

Summary question statistics from the three units are presented in Table 4.10 below. The frequency of questions from the instructor to the class and of student answers in whole class settings decreased while the questions to the instructor increased slightly and then held steady. The decreased questions to the class and student answers speak to decreased interactivity in the class, much like the previous table. The shift in questions to the instructor may be a product of more lecturing, meaning students did not have time to work out answers to their questions in small group settings, or may be indicative of the increased difficulty of the material.

Table 4.10. Average Questions across Units for Instructor A.

	Questions to Class	Student Answers	Questions to Instructor
Groups	.77/minute	.63/minute	.17/minute
Isomorphism	.67/minute	.56/minute	.25/minute
Quotient Groups	.59/minute	.41/minute	.24/minute

Putting together the information from the TAMI-OPs and the IOIM scores, we see a class where there is a high level of interaction between students and between students and the teacher. Students discussed their work with each other in small groups and in whole class discussions, and students' ideas were used to move the class forward. Throughout the three units, students were given extensive time to work individually or in small groups on tasks. These conclusions are based on the medium to medium-high scores across all inquiry-oriented instruction practices, students working in over half of all segments, and the high frequency of questions being asked and answered.

Nevertheless, we can see an increase in lecturing and a decrease in the interactivity of the class at the whole class level across the semester. This is shown through the decreased scores in the IOIM as the instructor gave less room for students' mathematics to guide instruction and kept more of the mathematical authority. This is especially apparent in the practice 7 scores, which had the largest drop across the semester. At the beginning of the semester, students' work was guiding the notation and the development of the concept of a group. However, as the semester progressed, students were given less time to work with new concepts before they were given a definition or theorem. This was especially true of the FHT, where the instructor gave strong guidance through a set of small tasks to make sure the theorem was introduced on that day. These decreases in students' mathematical authority and interactivity could be a result of the greater complexity of the concepts being studied, perceived time constraints from the instructor, or students pushing back on the freedom they had received and wanting more guidance through the tasks.

Alignment of Instructor A's beliefs and instruction. Looking only at Case A, I address my first research question, related to the relationship between instructors' beliefs and their instructional practice. To a large extent, Instructor A's stated beliefs and instruction clearly aligned in terms of the activities she had students engage with. However, especially in the quotient group and FHT unit, she fostered less-interactive classes.

Instructor A's stated beliefs about the nature of mathematics related to math as a human activity and something that we do. This related to her belief that we learn math by doing math, specifically defining concepts and generating theorems. This also linked to her idea that a measure for teaching is how much more students are behaving like mathematicians than before and that students should be given opportunities to do meaningful mathematical activity in class by participating in working and discussing their mathematics.

In class, these linked beliefs were evident in the choice of and enactment of the curricular materials used. She used the IOAA curriculum that used a series of tasks intended to help students generate definitions like the definition for a group or isomorphism. In class, she gave students time to work on the tasks themselves and then to discuss them with peers. Specifically, students were working individually or in small groups for more than 50% of the two-minute intervals across all three units. Students were also given some opportunities to present their work publicly at the board. Furthermore, the medium to medium-high scores on the IOIM indicate that the curriculum was at least somewhat used as intended, allowing students to generate knowledge, build upon this knowledge, create a shared understanding with their peers, and then formalize their new knowledge, which are the underlying components of inquiry-oriented instruction (Kuster et al., 2017). Unlike the IBL instructor in Fukawa-Connelly (2016), students did not seem to be have more freedom in proving than defining. At the beginning of the semester, students had a lot of freedom in both choosing how to prove theorems and in deciding notation and definitions. However, as the semester went on, students' freedom in both of these areas decreased. Because teachers are sensible, in keeping with Leatham (2006), this change over the course of the semester indicates unarticulated beliefs, perhaps about the need to cover content or the amount of scaffolding needed for more complex content.

Although her scores indicate some alignment with her stated desire to give students time to do math and make connections for themselves, her IOIM scores and time given to students to present their work publicly declined over the course of the semester and the amount of time she lectured increased. Furthermore, interactivity decreased as the frequency with which she asked questions and received questions and answers decreased across the semester. In keeping with Leatham (2006), this shift could be emblematic of the increasing difficulty of the material or the fact that it was later in the semester so students were just tired.

In terms of inclusion, her belief that she showed her students that she cared seems to be borne out by the scores students gave on the MUSIC inventory. She had an average of 5.68 for caring, and no student gave her an average caring score lower than 4.5 (between somewhat agree and agree).

Student beliefs and affect. In this section, I characterize Class A students' beliefs in order to address relationships between instruction and students' beliefs. This will allow me to begin to address research question two. As noted in the research question framework, this research question was more exploratory in nature and was less certain to have observable relationships.

To describe students' beliefs, I will first examine results from the first class-wide survey, accessible in Weeks 3-4, and expansions on responses to the survey in the first student interviews (Week 5, 6, 7, or 13 depending on the student). Questions analyzed from this survey and interview addressed what students said math is and how they learn math. Questions from the second class-wide survey, accessible Weeks 5-6, addressed students' motivation. Questions from the third class-wide survey, accessible Weeks 11-12, addressed students' view of the nature of intelligence. I address changes during the semester through questions about MUSIC components and a question on perceived changes in beliefs from the final interview with the four interview participants. These interviews took place in Week 14 or 15, depending on the student.

Nature of math. Students in Class A provided a variety of words to describe what math is on a survey given in Weeks 3-4, as shown in the word cloud in Figure 4.1. While "numbers" was the most common word included (9 instances), "study," "world," and "real" also are prominent, indicating some expectation of math being an entity as well as being useful for practical applications.

Figure 4.1. Word cloud on what math is from Class A.

In Table 4.11, the four interview participants' responses to what math is and the nature of math are spelled out more fully as a window into their views on math at the beginning of the semester. Notice all of these students talked about math as the "study" of something, whether numbers, logic, relationships, applications, or properties. These responses provide some context for other students' responses in the word cloud. The four interview participants' animal metaphors for what math is like were largely affective (10a, 11a, and 15a) and/or focused on problem solving (8a, 10a and 11a).

Table 4.11. Class A Students' Views of the Nature of Math.

Student	Math animal	What math is
8a	Math is like a fox because they are clever!	Math is the study of numbers and logic to make sense of the world around us.

10a	German Shepard because if you don't play with it right it's vicious	The study of numbers and their relations with each other
11a	Math is like a mule. It can, after much training, learn how to do tasks with a heavy amount of human input. However, it can never truly change how it behaves in nature.	Mathematics is the study of converting real world information into numbers and manipulating those numbers to discover more information about them.
15a	Math is like a cat because everyone either loves cats, or hates them. Everyone loves math, or never wants to touch it in their life.	Mathematics is the study of properties such as operations, spaces, and proving them.

Comparing responses to the two questions, Student 11a focused on applications of math in both of his responses. In terms of Ernest's (1989) views of math, he seems to align most with the instrumentalist view, as he focuses most on usefulness and does not seem interested in how ideas relate to each other. Student 8a elaborated on her animal metaphor in her interview, saying you need to be clever in how you approach problems:

> Cuz it's not always obvious. Sometimes, it's sort of tricky how you have to solve a problem, and I don't know if foxes are really clever, but that's always the impression they say. So yeah, it's not always easy, or in your face of how it's going to work out, but there's sort of strategies you have to use with that.

This aligns with the sense-making part of her description of what math is. She also appears to have an instrumentalist view, because her view of sense-making seems to align with modeling real world events, which still focuses on usefulness, and she does not seem to focus on connections between different parts of math. In the interview, Student 10a explained his response about the German Shepherd was about how you need to follow mathematical rules well, especially in the number theory context he discussed, or else you will not reach a correct solution. This also relates to his understanding of math as relating numbers to each other, and, like the other students, seems to indicate an instrumentalist view of math. Student 15a's responses are less obviously related. However, his responses complement each other as they provide windows to general people's perceptions of math as well as what he understands math to be about. His definition of math seems to outline a structure and connections, potentially indicating a Platonist view of math. However, as Ernest himself pointed out, people can have more than one view of math, and it is possible all of the students identify with other views of math as well.

From the students' responses, we see that three of them focused on math as being about problem-solving in some way, whether the approach to math (Students 8a and 10a) or applications of math (Student 11a), and these three students seemed to possess an instrumentalist view of math. All four of them also thought about how different people perceive math, with Student 8a discussing the type of approach needed from a person that does math, Student 10a addressing potential negative consequences of approaching math badly, Student 11a focused on the effort required to make math work for you, and Student 15a focused on the general public's perceptions. In these statements, Students 8a, 10a, and 11a expressed instrumental views of math, whereas Student 15a expressed more of a Platonist view.

Nature of learning and teaching math. Students were also asked about their views of how they learn math and characteristics of a good teacher on the first survey to the class. Selected responses that relate to changes they noted later in the semester are collected here. Student 8a related math to working through problems, saying, "I learn [math] from examples of other's work and time to work out confusion and get it right by myself." Student 10a mentioned "realizing we're taking more classes outside of theirs" was an important attribute for a teacher and that he learned math by "showing up to class." This seemed to indicate he found teachers who cared about their general workload to be a good characteristic. He also seemed to think coming to class would lead to learning. Student 11a said a characteristic of a good teacher was that they “explain in a way where it is the student who solves it before the teacher tells.” This suggests that the student appreciated having time to make some connections for himself before being told how to do problems. Student 15a said, “I learn math by writing scratch work on paper, looking at notes, and doing examples.” This aligns with a view of working through problems and making connections in a way that could be done individually.

In the last interview with each of these students, they were asked how their beliefs about math, teaching, or learning had changed or been strengthened by the class. Student 8a, a pre-service teacher, said her belief that a classroom could be non-traditional had been confirmed and that she appreciated having a chance to see guiding and questioning students in this way, which helped make the suggestions on guiding students less abstract. Student 10a realized that some teachers do actually care about their students and he was beginning to believe Instructor A might be more like the norm rather than an outlier in her caring. Additionally, he noted a change in his approach to how a class should be taught. During the first week of class, he had been skeptical of a non-lecture class. However, he liked that the guided instruction meant his interactions with math were “like salsa dancing, you feel like you’re dancing with mathematics.” Student 11a strengthened the belief that you should build your math knowledge yourself. In practice he stated that he would have appreciated a little more direct guidance in class, however. Student 15a changed his view on how a math class could be conducted, allowing that interactivity could be incorporated effectively and that there was more than one way for math to be taught.

From these early and late semester snapshots, we see can see that three of the four students in Class A focused on working through notes and problems individually or listening to lectures when they thought of how they learned or math should be taught. However, at the end of the semester, students seemed open to other ways of thinking about math. Students were more open to learning math by reasoning through problems themselves under guidance and by discussing math with others. Interestingly, Student 11a, who originally mentioned that he liked having a chance to solve problems before the teacher explained, would have actually appreciated more guidance on the tasks during the semester.

Motivation. Students also responded to a survey in Weeks 5 and 6 that assessed components of motivation under the MUSIC model. Each interviewed student's average for each component and the class average are presented in Table 4.12. Overall, the students being interviewed gave slightly higher scores than the class as a whole. In the final interview (Week 14 or 15), students provided some interpretations for their scores.

Table 4.12. MUSIC Score Comparison for Class A Students.

	Student 8a	Student 10a	Student 11a	Student 15a	Class A
eMpowerment	5.2	5	6	3.8	4.67
Usefulness	5.2	4.8	6	4.4	4.55
Success	5.25	3.75	5.75	5	4.74
Interest	6	4.83	5.83	5	4.98
Caring	6	5.67	5.5	5.67	5.68

A prominent way empowerment manifested itself was feeling control around their learning on the tasks. Student 8a noted that being able to focus the conversation on the part of the problem they wanted to focus on or the direction of the class gave her a sense of control. Student 15a mentioned his (relatively) low score for empowerment reflected his lack of control over which units of material would be covered, though he would likely give a higher score if he filled out the survey again, based on the influence on the flow of the class.

Usefulness, the component Instructor A did not intend to instill in her students, was perceived through students' increasing confidence in writing proofs. Student 11a said that as a math major, "it's extremely important to know how to write proofs properly and that this is what most of this class is, just sort of like perfecting the proof in a way." Student 10a also mentioned he was considering going on to get his Master's and that he would probably need at least graduate level algebra, so he would apply knowledge from this class there.

Success seemed to be measured according to the grades students had or thought they would get at the end of the semester. Student 10a mentioned that he had heard "horror stories" about abstract algebra "being worse than watching *Inception* for the first time." However, he gave himself an 8.5 to 9 out of 10 on his success in the class. He also thought the reason he had originally given a low score for this component had been because he was going through a rough patch, but that he felt more comfortable now. Other students like Student 8a said they would probably adjust their success score down slightly then because they had struggled more as the semester progressed.

Interest was piqued by the tasks and instructional method in class for Student 8a and Student 15a. Student 10a and Student 11a seemed to have self-generated curiosity and enjoyment of learning about new math facts. Student 10a seemed more focused on isomorphism and function, as he saw applications for them in computer science. Student 11a seemed to like that the integers and even integers were isomorphic even though one group seemed larger than the other.

All four students indicated that they felt their professor cared about their learning. They mentioned the ability to ask questions of the professor and their peers allowed them to feel connected to others in the class. Student 10a also appreciated that Instructor A would "crack a joke every now and then" and that she was the type of person he would stop and talk to if he saw her outside class.

In summary, these students and the class as a whole largely at least “somewhat agreed” that Instructor A fostered their empowerment, usefulness, success, interest and caring in the course. Reasons for this included feeling some control over the task sequence, perceiving usefulness in practice writing proofs, feeling they could get decent grades in the class, being interested in the tasks, and feeling the instructor cared about them and that they had opportunities to connect with their peers in class.

Summary of students’ beliefs. While students’ views of what math is were similar to each other, focused on being a study relating numbers or ideas, responses to what animal math is like varied a bit more. Even there, however, we can observe the four interviewed students still focused on affective issues and/or problem-solving in their view of the nature of math. Three students seemed to have instrumental views of math and one seemed to have a Platonist view at the beginning of the course. Students did not directly mention any changes in these beliefs about the nature of math in the end of semester interview.

However, the course seemed to influence their view of the nature of teaching and learning math. In particular, students expressed more openness to different ways of learning math, including being guided through approaching the problem rather than being told how to do it in a lecture. Additionally, students seemed open specifically to interactive classrooms where they had opportunities to discuss with other students.

Furthermore, the students seemed reasonably motivated in the course, as measured by the MUSIC components of motivation. While different students gave higher scores to different components for different reasons, all students generally agreed that their empowerment, sense of usefulness, ability to succeed, interest in material, and perceptions of care from the instructor were fostered in the course. The interviewed students cited specific aspects of instruction, such as control over the pace of the class, interest in the tasks being discussed, and a sense that the instructor genuinely cared about their well-being.

Alignment between instruction and students' beliefs. Looking only at Case A, I address my second research question, as I examine relationships between instruction and students' beliefs in Class A. There are some clear connections between what happened in class and what they said changed about their beliefs. However, many of their previous beliefs, especially about the nature of math, seemed unchanged.

One of Instructor A's main beliefs about how we learn math and that she reflected in her instruction was that we learn math by doing math. In class, students were given time to work on problems before being told procedures for addressing the problems. By the end of the semester, the students seemed to recognize this was a potential way to learn math. For Student 8a and Student 11a, the class strengthened this belief, whereas for Student 10a and Student 15a this seemed to be a new approach. Student 10a especially seemed to latch onto this idea, as he noted now he was having opportunities to "danc[e] with mathematics."

Similarly, Instructor A promoted an interactive classroom, where students would build consensus around ideas with growing groups of people, as a way for students to develop their mathematical thinking. Students seemed on-board with this style of instruction by the end of the semester as well. While Student 15a did not directly address his comfort with the guided instruction, he noted that interactivity seemed more acceptable to him in a math class than before.

Despite some alignment on doing math and discussing math in class, Instructor A's idea related to math as a human activity was the idea that math is a creative endeavor. While this can be perceived in instruction, where tasks were open-ended and different correct solutions were highlighted in class, this was not as clear of a student belief. Student 8a had originally mentioned "being clever" was important in solving problems, but focused more on having strategies to deal with such situations than seeing them as an opportunity for creativity. Additionally, Student 11a said he would have appreciated more guidance on activities, not less, which does not align with a desire to have more room for mathematical creativity.

In terms of motivation components, Instructor A did not feel Abstract Algebra was useful beyond helping students develop their problem-solving abilities, so she was surprised that students said they found it to be useful. However, this was not far out of alignment, because the students cited proof-writing practice and problem-solving as the most useful parts of the course. Furthermore, it had the lowest average component score from the class as a whole.

Instructor A also claimed she cared about students a lot and would reach out to them, especially vulnerable students, to make sure they were ok. Students agreed with this assessment, as they gave caring the highest component score on average. Student 10a specifically noted that Instructor A was the type of person he would stop and talk to outside of class, indicating he felt he could interact comfortably with his professor.

Empowerment and interest were stimulated in the ways noted by Instructor A, namely through control over pacing in the tasks and interest being stimulated by the task sequence for some students. (Students 10a and 11a seemed interested in the underlying math besides just the novel presentation of material.) Instructor A also accurately predicted that students would use their grades as a measure of success, though some students seemed less confident in their ability to succeed late in the semester.

In summary, to a large extent, students' beliefs about teaching and learning math seemed to be altered or strengthened according to what happened in class. Students were more comfortable with being guided through learning for themselves and in interacting with their peers while they learned math. Students' motivation was also largely stimulated in the ways the instructor expected from her instruction. However, root beliefs about the nature of mathematics did not shift noticeably for students. Though Instructor A also believed creativity is important to math, this belief was not clearly represented in students' statements at the end of the semester. This may be because students felt their ways of learning math were still compatible with their beliefs about math without requiring creativity. Alternatively, the nature of one's view of math might be too abstract of a belief to be clearly communicated by an instructor.

Class B

In this section, I analyze Instructor B's beliefs as stated in his interview and instruction, and alignment between these contexts. Then, I will examine Class B students' beliefs followed by alignment between instruction and students' beliefs.

Instructor beliefs. Instructor B provided concrete language and examples to illustrate his beliefs about math, learning, and teaching. In so doing, his background in logic and proof was evident.

Nature of math. Instructor B focused on math as both an object of study and something we do, especially in the context of proof, as well as a tool for addressing new problems. Furthermore, he noted that he had views of the nature of mathematics, but that others were free to hold other views.

Of Ernest's (1989) three views of the nature of mathematics, Instructor B seemed most aligned with the Platonist view as he noted mathematicians' search for theorems: "So I think mathematics is the search for theorems which…I would take to mean things that both can be proven…and then also the actual pursuit of proof…" In this statement he also made clear his view of math as both an entity to be studied and something we do, though he spent more time emphasizing what is done. Furthermore, his original description of what math is follows: "…to answer kind of circularly, anything that is pursued with the methods of mathematical inquiry, which I would take to mean, you know, things that you can…prove rigorously."

He also noted that math did not need applications, but proof and logical thinking were useful tools that could be viewed as a purpose of math. For example,

> I don't think mathematics necessarily has to have a purpose. I think, you know, it can be pursued for its own sake, but, you know, I think that some of the benefits of mathematics are training…one's mind to think rigorously to, you know, take both a rigorous and creative approach to the problems. You know, certainly there's all sorts of applications for mathematics which I think would also fall into the realm of the purpose of mathematics.

He also observed that though math as a field did not change, its presentation over different levels shifts from more computational in nature to more logical and proof-based in nature:

> This idea of just sort of looking at abstract structures [in abstract algebra] and reasoning about them, trying to make general statements, trying to find other conditions that you can put on them to…let you make, in turn, other statements, that…whole idea attracted me a lot, so it was, you know, sort of a big rupture from…just the calculus and differential equations that I had…seen up until that point….I think it's more the… undergraduate presentation….The differential equations class I took there, as it is here, is largely a course for engineers….It was just we want to solve this equation and maybe apply it to this sort of problem, so you know, certainly it was more a matter of level than a matter of the actual subject matter.

While he noted this shift in presentation, he did not feel it was appropriate for him to specifically emphasize his own views about math; rather students could define for themselves what math is based on their own experiences:

> I definitely think it's, it's good not to impose one's views about math. I think probably by taking different courses throughout our curriculum they'll be exposed to different views about math, so I don't know that it's necessarily my, my role to, you know, sort of give a philosophical overview about what math is in this particular course, although it's, you know, if there were a course to do such a thing, this probably would be a good course to do that, on the other hand….But yeah, I don't know, I wasn't really thinking of talking about different views of what math is in this course, but certainly if, we'd also, I also say that [I] wouldn't want to discourage, you know, students from talking about that or having their own independent views about that if that was what they chose to do.

When asked what animal math was like, he highlighted ideas related to proof and logic with ideas of doing math, much as he had noted when asked direct questions, and highlighted that this idea lacked mathematicians' agency in searching for theorems, when asked about limitations for the metaphor. He also mentioned that he asked others about metaphors, and chose the one he liked best rather than devising his own.

> Ok, in the interest of full disclosure I'm gonna admit I cheated on this question and talked to some other people, but the best answer I heard was an ant because it's, it is sort of a very simple thing that's not capable of very much on its own, but that they can combine in very interesting and complex ways. So I think that, you know, sort of fits with this idea of…at the basic level you have…your individual axioms and inference rules but you can sort of combine these things in ways to generate much more complexity than is apparent

> at the most base level of sort of definition, logic….I mean it obviously, it's gonna have limitations as, as any analogy does. I guess…maybe there's no conscious entities trying to push the ants in any particular direction to produce new more interesting things whereas in mathematics…the I guess the human beings are missing from the ants analogy, right, the actual mathematicians….So in that analogy I guess…the ants they're sort of your axioms, your inference rules, and the complex patterns they're producing are the, are the theorems, right but…mathematicians are producing theorems and they're not actually ants, right, so, so there's something missing there, right?

In summary, Instructor B's main view of math centered on the search for proofs of theorems based on the logical structure of mathematics. This view of math aligns with Ernest's Platonist view of math, as well as the animal metaphor for math he most identified with, which focused on the structure of mathematics and how axioms and inference rules could be combined in complex, interesting ways. His view of math centered on math both as a field of study (the logical structure being made) and as the mathematical activity needed to create that field. He also noted math did not have to have applications, but greater understanding of proof was one possible application. Additionally, math presentation in school shifts from computational to logical as studies advance even though the nature of math itself is not changing. Finally, he noted he had his own beliefs about the nature of math, and he felt students should be free to have their own views; he does not seek to proselytize students to share his beliefs.

Nature of learning math. The most salient aspects of learning math that Instructor B talked about related to how one learns math and what it means to learn math, much like Instructor A, though he focused more on ways of learning math. Furthermore, he noted characteristics of students that can aid in learning math.

Some of the broad ways he noted that people can learn math included actively doing math both in and out of class. For the in-class context he stated:

> I'm a firm believer in learning by doing is best, so…every class I try to give the students something to do even if it's just…here I'm gonna put this…particular example on the board for two minutes, let you guys work on it…but just take a pause…, so I think…they have to be active at some level.

He also noted having a good work ethic was a necessary component of learning: "I think they have to be willing to do the work, both inside and outside of class, and they're probably not going to do very well if they're not putting in the work."

He also provided more specific ideas of how people learn, including coming to lectures, working through the details of proofs, and working through examples, while also noting how much each of these methods helped depended on the person. He based these ideas on how he learned. For example:

> You know for most of my life I've mostly been more of a book learner, and so I would, I would kind of go to class just to sort of see what was important. And then go, you know, go read the book later and just really learn it and ingrain it….I think as I've grown older I've become way more capable to sort of assimilate as I listen.

Later he provided more context for what doing math meant for him as he noted how he would do math in lectures as a student:

> …but I'm the kind of person who…in a lecture, I'll try to verify all, all the details and I can easily get hung up on one detail and just figure it out and then look up ten minutes later and, you know, the class has moved on without me. So ultimately I also kind of, you know, I have to be, I have to be writing and sort of not just writing notes that someone

> else is giving me but just, sort of, coming up with my own examples or coming up with my own proofs and just really synthesizing for it, for it to stick. That's how I learn mathematics.

He was also very cognizant that how he learned could differ from how others learn, just as people have different preferences and ways of thinking in other contexts.

> I think minds, people work…differently in, you know, any aspect of what you're talking about. You know, different people all like different foods….Different people have different frames for interpreting politics for example right, so I think, you know, I think the same applies to learning, and so there's, you know, just different, different procedures, different processes are gonna work for, for, for different people when it comes to learning, just like it, it comes to any other aspect of, of human behavior….So I guess that's…just sort of a baseline that I, I'm coming from.

In terms of what he believed it meant to learn math, and especially what it meant to learn abstract (modern) algebra, he noted specific knowledge and tools students should acquire. He also mentioned the ability to apply knowledge to new contexts as a sign of learning. For instance, he combined these ideas in the following extended excerpt:

> So I think if you kind of go back to the fundamental principles of algebra that I mentioned before and then I want [an A] student to kind of understand those, be able to describe how, how they manifest in different situations, and also be able to deal with the content as it's been presented so to, you know, have a good sense of the major undergraduate theorems of group theory. [They] could handle all the examples of groups that we went over, same with rings, fields, finite fields as we get to that stuff….So [a] C student, I mean I guess I would still hope that they, you know, they have a sense of what

> modern algebra is, you know. Think maybe that would come in a little lower level right, so you know, they may not have as high a level understanding of you know what, what algebraic structure is…how those things are preserved with different…mappings and all that stuff I talked about before, but at least be able to describe you know, say something like, "Well in modern algebra oh we're interested in looking at things like, you know, structures like groups and rings and these things are, are defined by various axioms and we can come up with, with examples of such structures, right, satisfying the examples…" We're putting, you know, different numbers of operations on those structures, so I think hopefully a C student would be able to at least internalize that much, and, you know, probably would have, however, a more difficult time actually applying that, right. So you know, whereas you know an A student would be able to…take an example of a group with a homomorphism, you know, find all the factor groups, if I mod out by the kernel of that homomorphism come up with the Cayley table, you know do all that seamlessly no problem. You know a C student…they have a have a notion of a group…they're able to kind of deal with the elementary problems. They may struggle a bit more with, with an example and the order of difficulty with what I just described….

He also noted that students were being given an opportunity to apply their previous knowledge from their Proofs course, thereby showing learning of material by applying it to a new context.

> And I know some of them have recently taken Proofs, right so, so kind of applying what they learn in Proofs in a more content-specific environment, so hopefully then they'll be able to, you know, think abstractly, reason abstractly about algebraic structures or maybe even not just, just algebra. I think, you know, probably help them in other areas of mathematics as well.

In terms of characteristics that are useful for learning mathematics, he provided similar characteristics to Instructor A, such as curiosity, motivation, a willingness to explore ideas, and willingness to invest the time rather than innate ability. For example, he summarized his beliefs in the following statement:

> I think at this level, I think pretty much anyone that has the willingness to ... the motivation, the willingness, the curiosity, what have you, I think if they're willing to put in the work, they can be successful. I think we talked about this a little bit last time. I certainly think there are different levels of ability and aptitude, but I think any student who's in that classroom right now, any student that I've had in this course throughout this year, has the ability to be successful if they're willing to put the work in.

In summary, Instructor B believed there were a number of ways to learn math that varied depending on the person. Some ways of learning included coming to lectures, working through the details of proofs and examples, and looking through the textbook. Doing this work outside class showed students' work ethic. He personally felt he learned by synthesizing work with examples and creating examples and proofs of concepts. Because he was focused on the details of proofs as a student, he found lectures to be unhelpful except for knowing what material he was expected to learn. However, he recognized that different people learn differently and that his ability to learn from lecture has increased over the years.

Connecting beliefs about the nature of math and the nature of teaching math, in both settings he emphasized the role of doing mathematics, where doing math is defined as working on given examples or creating examples or proofs for a problem. He viewed the nature of math as logically structured according to axioms and rules of inference and focused much of his energy in learning math on the logical details of proofs as a student. This is also reflected in his emphasis on creating examples to make sure he fully understood the definitions and theorems he was working with.

He also displayed openness to many perspectives on the nature of math and of learning math. While he acknowledged he had specific beliefs about the nature of math related to the search for theorems, he allowed students might have different views of the nature math. Though he thought exposure to more math might affect students' view of the nature of math, he did not feel it was his place to tell students what they should believe about math. Similarly, though he knew he learned in certain ways, he was open to the possibility that students could think differently and therefore learn differently.

Nature of teaching math. Instructor B explored many topics as he discussed the nature of teaching math. These topics included exploring the role of different types of instruction within a class period, classroom norms, and clear communication between the instructor and students.

Different types of instruction. Instructor B chose to use a combination of lecture and lab periods in his instruction, while also including some active components on lecture days. He explained the roles of each of the types of class periods as well as the role of discussion at other times.

For the lab days, he said he got the idea from watching another instructor teach an abstract algebra class and wanted to expand the interaction that students were engaged in, especially for addressing examples.

> So I came up with the idea last semester, the first time I taught the class, of you know just try this out and see how it works, and maybe it doesn't work at all, but I found I really found that the students, the students were talking with you know with each other, they were engaging a lot.

Thus, the lab days provided an opening for students to experience longer examples, whereas lecture days would focus more on exploring the definitions and proofs in the class with a few smaller examples worked in.

> So what I was calling lecture, right, I'm writing stuff on the board and talking pretty much the whole class. You know, like I said, punctuated with just little examples probably that I ask them to work out for a few minutes. So with, with, with the labs, I know I give them a handout with, with some questions and some things that they're supposed to discuss, and so they're written, hand written right up at the end of class and then I'm just, I'm circulating, you know, talking to people individually, talking to groups…if there's a good group conversation table, maybe trying to not insert myself too much, right, but just sort of observing. But if there's a particular table, nobody seems to be talking, maybe trying to prompt them to start a little bit of discussion. Yeah, so I guess for me it's, this, I'm doing very different things, you know, they are doing very different things because they're talking to each other. Right, and not talking to each other so much when I'm lecturing so to speak I guess.

He also made it clear that he valued lecture as a way to make sure he taught all of the material he wanted to teach, and he was satisfied with this balance being struck with two lecture periods and one lab period each week.

> Although I mentioned that the course is sort of non-content specific, there, you know, there are things I want to cover…I'm kind of balancing things to the extent where I want to lecture enough to sort of get to all the material I want them to see, but also put in enough group work where I'm getting them to, to kind of interact and, you know, come up with their own ideas…

In terms of characterizing his instruction, he felt he spent roughly 60% of the class time lecturing, based on one class period of student work on labs and a few moments of working on examples during lecture days. Based on this breakdown of the time spent lecturing and having students do activities, he considered his class to be mixed when given the options traditional, alternative, and mixed, and said he felt that was a good characterization of his teaching.

Classroom norms. Instructor B identified homework-focused norms and participation as classroom norms he established. Whereas the homework-focused norms were non-negotiable, forms of participation were open to more variation.

When asked about the expectations he set for students' written work on homework and tests, he emphasized the importance of academic honesty, in that whatever students wrote on the page should be reflective of their own work, even if they discussed the problem with other students before writing solutions. He also noted that once he graded an assignment, he would not permit grade bargaining:

> Grades are not negotiable. So once I grade an assignment, you know the only reason anything's ever changed is, I mean, I made a mistake adding points, or I didn't see something written on the paper that was clearly correct. I never go and change the amount of partial credit ever.

He also hoped to establish the idea that students could come to conclusions about their mathematics independently, rather than relying on him to affirm their work.

> I think it's you know, to get, big thing to get the students to interact with each other or bounce ideas off each other, you know, I think it kind of gives them confidence when they're answering each other's questions. It gives them a sense…that they can come up with ideas about mathematics, that their and their ideas are valuable, that they have the

> ability to refine their, their peers' ideas and…come to a conclusion and without, without necessarily, you know, the instructor having to tell them this is, this is the way things have to be done.

He also believed that participation was a norm that was established in class. In lecture, he had a minimum expectation that students would pay attention, and he would enforce this through general statements to the class.

> You know and I know people, you know people phase out, the lights phase out every once in a while, as do people [lights off and on in room]. If, you know, if somebody's texting in class, I'm happy to kind of say it general, make a general statement like "Please stop texting," but, you know, I'm also not gonna, you know, call on someone who's obviously not paying attention just to embarrass them.

He also wanted students to engage in lecture by asking questions and doing the short problems he would include during lectures. If there was minimal participation, he would also call on students by name. He expected similar engagement with tasks on the lab days, as shown here, where he connected his view of participating with examples in the lab and lecture days respectively.

> Minimally would be to just be, you know, be engaged with the task….Ideally, you know, meaningfully interacting with other students and I guess also finishing the assignment….Yeah, so again if it's…an example and I asked them to work, it'd be the same thing basically, just you know be engaged trying to work something out. You know ideally get to an answer but, you know, that I don't mind so much as long as they're actually thinking about whatever problem I give them. And you know like so if I ask questions to the class, you know, answering every once in a while. Like I said, a lot of

times, it's the same few students but I you know, I try to try to call on students by name every once in a while, so I guess minimally it'd just…be engaged to the level where you can answer a question. Or you know don't, not necessarily answer correctly but have something to say about the question that I posed to the class.

Clear communication. Instructor B valued clear bi-directional communication with students. For his side of communication, he said, "I think definitely we have to…be clear about what our expectations are. We have to communicate the ideas that are important and let the students know what it is that we expect them to know." On the flip side, he also encouraged students to provide informal feedback during the semester as well as formal student evaluations at the end of the semester so that he could adjust his instruction if needed. For example, he noted his previous semester's algebra students had found the labs useful, encouraging him to keep using them: "I put more labs in but that was really just because students last semester were telling me that they found that helpful and so I thought I'd do a few more of them." He also noted some formal feedback he had received at the end of the previous semester that had influenced his teaching:

> In one class…I was doing projects, there I think there were a lot of useful comments on how to how to improve the projects, kind of reorganize the groups. I had the group size was too large. If you give students more choice in what, in what topics they could they could do for the projects. So just like I look for, you know, very specific things about what I'm doing that I, could be improved.

In summary, Instructor B noted the role of different types of instruction, class norms, and clear communication in his beliefs about teaching. He used labs to give students opportunities to do longer examples and discuss their math with other students. He used lecture to make sure that he could get to all of the material he wanted his students to see. He established grading norms that were non-negotiable, specifically students should be academically honest and grades were not open to discussion. While he wanted students to participate by paying attention and being able to engage with questions in lecture and by engaging with tasks and discussing with other students in labs, students were not expected to be able to answer every question. He also recognized that sometimes minds wander and he would not look to embarrass students. For clear communication, he felt it was his job to be clear about what he expected students to know. He hoped students would clearly tell him how he could improve, both through formal and informal feedback.

His beliefs about teaching math have clear connections to his beliefs about the nature of math and learning math. He emphasized logic and proof in his view of the nature of math and in how he learned math. In his beliefs about teaching, he noted his use of lecture days where he would speak for most of the class to be sure to get to all of the material he wanted. In his presentation of material, much of what he did would be definition and proof-focused. This also related to his expectation for himself of making the important content known to students. This connects to his main reason for attending lecture as a student: knowing what material was important. His expectation of participation from students related to his beliefs about doing math. Because he learned math by working through examples and creating examples, he wanted to give students opportunities to do so on the labs and in short periods on lecture days. Finally, his choice of using different types of instruction that allow students to learn in different ways connects to his beliefs that people view the nature of math differently and, especially, that people learn differently. Thus, he provided opportunities for students to engage with the material and learn in many ways so that hopefully at least one of the ways would connect with each of his students.

Motivation. Instructor B was specifically asked about how he defined motivation and how different aspects from the MUSIC model might affect or be related to his students' motivation. Like Instructor A, he seemed to use participation as a measure of how motivated students were.

Instructor B defined motivation slightly differently when asked at the second and fourth interviews of the semester. In the second interview, he focused more on aspects related to initiative and interest:

> I would define it as taking, well yeah them taking initiative to, sort of baseline do their work, do their assignments, come to class. Signs of, of, sort of higher-level motivation, you know, maybe coming to office hours, asking more questions during class, sort of, just coming in to talk about something that's, you know, maybe a little beyond what we've seen or tangentially related to what we've seen in the class. Right, just kind of showing signs of interest, other, other things I've seen from students that I would consider signs of motivation. So last semester I had a student who would just sort of, ask sort of open questions, or at least open questions to him on his homework. He would solve the problem and kind of say what about this? And, so I guess, you know, sort of thinking a little bit beyond exactly what's going on the coursework, right, would be a sign of motivation to me.

In the fourth interview, he focused more on curiosity and success components of motivation:

> I'd say that they exhibit some sort of desire to learn and I guess curiosity could spur motivation. Sort of ... I guess students could be motivated by other things. Grades, for

example, are always a big motivator. But I think you have to recognize that that's the case.

While they have different emphases, both definitions include an impulse to act in a way that leads to acquiring new information.

In terms of the MUSIC model components, he was also asked to fill out the MUSIC survey as he thought a student would fill it out. He was fairly close to his students on all but the empowerment component, which he vastly underestimated. His scores and the average from his class are shown in Table 4.13. The class average comes from 14 students' responses.

Table 4.13. MUSIC Score Comparison for Instructor B and Class.

	Instructor B	Class B	Standard Deviation	Standard Deviations away
eMpowerment	2	4.90	.87	-3.41
Usefulness	4.8	4.49	.97	.31
Success	4.75	4.91	.83	-.19
Interest	4.17	4.76	.75	-.77
Caring	5	5.33	.63	-.53

For empowerment, he initially thought he allowed his students freedom through the labs and through other opportunities to discuss with him:

> So I think, sort of doing participative activities in class, I'll just try to get them thinking about an idea for the first time, you know, as opposed to me presenting all these, the ideas, trying to trying to give them opportunities to, to work together. Trying to give them an opportunity to voice questions or concerns or comments if they have them. Encouraging them to come to office hours to, to talk about things and just giving them space to talk in office hours or in class and trying to…I'm not always great at this but trying to let them fully articulate a thought…before I jump in and try to interpret what they're saying for example.

In interview four, when I pointed out that he gave himself a much lower score than the students had, he was initially confused.

> I mean, I guess overall, maybe I was erring on the side of being conservative, so maybe that's setting myself up for failure….Yeah, the empowerment ... I have no idea. And I don't know which specific questions were the empower…Were they kind of interspersed, the questions?

When I gave examples of the empowerment related questions (e.g. "I have the opportunity to decide for myself how to meet the course goals."), he understood the score better, and came back to some of the empowerment ideas he had originally shared in the second interview:

> So, how to meet the course goals, maybe I interpreted that a little more rigidly. So I guess maybe they were thinking they can study in their own way, read in their own way, etc. whereas I was maybe thinking, well completing the course goals means you have to do all the assignments, and so I'm writing assignments and at the very least, I guess, maybe they feel more empowered because they can work with who they want to work with on the assignments, they can come to office hours or not. During the labs, they could, they don't, but they could get up and change their groups if they wanted to or submit an individual assignment, work with a group, talk with me, not talk with me, whatever. So, I guess maybe from that they were seeing things that were empowering and allowing them to approach the work in a different way, whereas I was seeing, "Well, I'm writing all the questions and I'm saying the lab is going to be during a certain day and that the exam's going to be on a certain day and they're going to have a certain amount of time for that." So, I guess it could just be a difference in perspective there.

Usefulness was the one component he slightly overestimated, though not by much. He initially thought of ways that they were applying previous Proof course knowledge as well as places that the reasoning explored in the class could be applied elsewhere:

> The type of, type of reasoning they're seeing here, at this sort of rigor of the method I think could, could apply in other areas as well. But, you know, I sort of view it as…it's a pure math class where they're sort of learning to sort of refine and focus their mathematical reasoning, so certainly something that would sort of be down to their, to their benefit in other mathematics classes going forward.

Later in the semester, when asked to reflect on students' scores, he reflected on more specific usefulness aspects and interest aspects:

> I try to sell the subject matter a little bit I guess, so when we get to a result that's maybe striking or surprising or that I find inherently interesting, try to point that out, and try to also...Here and there, it's a pretty abstract subject matter, but say we're talking about getting the finite field, say, point out, "Hey, these kind of things are used all over the place in cryptography, coding theory, for example." So I guess try to stimulate intrinsic interest in the material and also point out what the material could be used for further down the line.

For success, he thought students felt they could be successful based on their behavior as of the second interview:

> There are students who, who clearly are, you know, doing well, good things, have, success is being indicated in their performance on the assignment so far so, I think that since they have been successful up until now they probably believe they can be successful and, you know, the other ones seem to be, for the most part, taking, taking

> things in their hands and doing something about it which indicates to me that they believe they could be successful as well.

Later in that interview, he indicated that success to students likely revolved around grades, though he hoped "assimilating the concepts" was also important to them. In the final interview, he centered more on grades as a way that students measure success: "Their metric of success could be getting a good grade in the class for whatever value of good grade is appropriate to them."

For interest, he especially focused on students who showed interest that went "above and beyond" as well as his efforts to project interest:

> When I can, try to get them to, you know, to discover things on their own. I know one of the, I think one of the labs, I think it was lab four where I had this question about a cyclic group of prime order. And so the students had to figure out what the subgroups were, that one of the students expressed appreciation that "Hey, like you know we're really sort of figuring…this theorem out on our own here in this question." And so, he seemed, seemed very motivated by that. So I think, you know, the essence, sort of, you know combination of you know, trying to get them to discover things on their own, I think stimulates their interest. Trying to trying to project interest and sort of, sort of try to show them what, what may lie beyond the course also….

His statements later in the semester reflected the same ideas again.

For caring, he believed it was important to show that he cared through positive interactions in class and office hours:

> So if they show up to office hours and I just am trying to, you know, gradually push them out the door then it's, you know, probably not sending a…signal up that they're, that

> they're welcome…that we encourage their, their participation, you know, that we also value our interaction with them, right. So I'd say it's, it's highly important to do so.

He also noted other ways that he spent time on the class that he felt showed that he cared:

> So I think it's being prepared, kind of knowing, kind of having the notes in order before each class, kind of having a plan of what we're, what we're going to go through to sort of do activities in class, being available for, for office hours, trying to make sure that students get their questions addressed or anything during office hours, making sure they feel comfortable showing up and talking about things there.

He did not add anything new on how he demonstrated caring later in the semester.

In summary, Instructor B defined motivation in a way related to students' curiosity or initiative along with their interest in the material or perceived ability to succeed with the material, though both of his definitions related to students seeking out new information. For each of the components of motivation, he provided ways that his teaching could foster students' motivation. For empowerment, he initially did not know what students found empowering about his instruction. However, he thought the fact that they could choose how much to discuss or whether or not they wanted to come to office hours might give students a sense of empowerment. For usefulness, he perceived more usefulness than his class, though only slightly. He felt that the course gave students an opportunity to refine their mathematical reasoning and that he alerted students to places they could apply this knowledge, such as cryptography. For success, he assumed students' measure of success was their grades, and it seemed that students were either getting decent grades or were coming to office hours, indicating they felt they could improve if they continued working with the material. He hoped understanding the concepts was important to them too. For interest, he liked to project interest in the material by pointing out applications in other settings. However, he also noted some students were intrinsically interested by the material as shown by the questions they asked that went beyond the expectations for the course. For caring, he felt it was important to be welcoming in office hours and to be prepared to teach, thereby showing that he cared enough about them to spend time crafting a good lesson.

Some parallels can be drawn between his beliefs about teaching math and his ways of fostering motivation. He felt empowerment could be stimulated by students' freedom to choose how much to participate in class and how much to engage with material outside of class, such as in office hours. This relates to his belief that he should use multiple types of instruction, which also relates to his belief that people learn in different ways, because this empowerment comes from students' ability to choose how they want to learn. Second, his view of usefulness related to students' development of their reasoning skills. Because his view of math centered on logical structures, it is natural that he would consider development of logical thinking to be a useful skill. His way of demonstrating caring through attending to students in office hours could also be viewed as supporting students in learning in different ways, as students could ask more questions or work through part of a problem and then ask questions as suited their needs. This could also contribute to his view of students' sense that they could be successful in the course.

How teaching reflects beliefs. Instructor B identified two main ways that his beliefs about the nature of math, learning math, or teaching math were reflected in his instruction. He particularly noted his use of different types of instruction to reach different types of learners:

> I think it's just trying to mix things up. It reflects my belief that people learn in different ways, and so, you know, try not to use the same style throughout and also do different things….All my undergraduate mathematics classes were, you know, what I've been referring to as lecture. Every single one of them I never, never had any sort of group work

> in those classes, and…like I said…I wasn't great at following what was going on in the lectures at that point in time. The group work is the kind of thing that would have would helped me, so that's you know just kind of putting in that different element for maybe people who do learn in a different way. Trying to be helpful to different types of learners.

As connected above, his beliefs about creating different types of learning opportunities for his students sprang from his own experiences as a learner. In this case, the lack of alignment between his experiences and what would have helped him appeared to be formative. This relates to Johnson et al. (2017), which noted the second most reported influence on teachers' instruction was their experience as a student.

The other way Instructor B noted the influence of his beliefs on his teaching was in students working on mathematics. He highlighted his specific desire to have students be active in the classroom:

> I try to refine things, so I kind of, you know, if I find one way of explaining things, something, this is not working, then I'll, you know, try to come back and explain it a different way.…There was something we did in class this semester where it was just clear that absolutely no one had any idea of what I was talking about….So I just gave them a bunch of examples to, to work on…like I guess then that would be kind of an example where, you know, I want them to be active, I want them to be thinking about things.

Here we see Instructor B's solution to students not connecting with his lecturing was to have them do specific examples so they could make more connections for themselves. This connects to his belief that students learn by doing math.

Finally, although Instructor B did not mention this connection himself, based on his other statements about his math, learning, and teaching beliefs, his beliefs about the structure of math seem to color his view of what content is important to share with students. Specifically, because he viewed math as a logical structure and studied logic in his previous work, it is reasonable to expect that this makes presenting proofs and working through the details of proofs important in his instruction, especially on the lecture days.

Characterizing instruction. Like in Class A, in this section, I will characterize instruction in terms of general instructional approaches, patterns of engagement with students, and common activity within classes. This characterization of instruction is intended to give details on the Instruction component of research questions one and two. These characterizations draw on analysis aided by the TAMI-OP, IOIM, and statements in interviews.

Recall Class B met three times each week in 50-minute class periods. I provide analysis from three times in the semester: Weeks 3-4, in the middle of a unit on groups; Week 7, through the whole unit on group isomorphism; and Weeks 8-10, through the whole unit on quotient groups and the Fundamental Homomorphism Theorem (FHT).

On lecture days, the instructor generally presented material as he was writing notes on the board. Questions to the class and questions from the class were occasionally interspersed between these presentations. There was a slight decrease across the semester in how much students talked during the class period, as will be shown from both the TAMI-OP and IOIM measures.

Although the IOIM is designed to give a single score to describe an instructor's overall performance during a unit, I chose to separate the scores on lecture and lab days because there were distinct differences in behavior that could be more clearly shown by providing separate scores for the two types of days. In two consecutive lessons from Weeks 3 and 4, the lectures received a medium score on guiding and managing the development of the mathematical agenda (practice 6); medium-low scores on facilitating student engagement (practice 1), eliciting student reasoning and contributions (practice 2), being responsive to student contributions (practice 4), and engaging students in one another's thinking (practice 5); and low scores on inquiring into student thinking (practice 3) and supporting formalizing of ideas and notation (practice 7). In Week 4 of the semester, the lab day received medium-high scores on facilitating student engagement (practice 1) and eliciting student reasoning and contributions (practice 2); a medium score on being responsive to student contributions (practice 4) and engaging students in one another's thinking (practice 5); and a medium-low score on actively inquiring into student thinking (practice 3), guiding and managing the development of the mathematical agenda (practice 6), and supporting formalizing of ideas and notation (practice 7). (These scores are later summarized in Table 4.20.)

These IOIM scores indicate that the lecture days were not very much aligned with any of the components of inquiry-oriented instruction, which is not particularly surprising. All practices except the practice related to guiding and managing the development of the mathematical agenda received medium-low or low scores. The higher score for managing the mathematical agenda stemmed from having an orderly progression to the lesson, which would be expected from a lecture. The scores on the lab day vary more, but show more alignment with inquiry-oriented instruction. Practices 1 and 2, which both relate to generating student thinking, received medium-high scores, indicating students were working on approaches for the tasks at hand. Practices 4 and 5, which both relate to creating a shared understanding in the class, received medium scores. Practices 3 and 6, which relate to building on student thinking, as well as practice 7, which relates to connecting to standard mathematical notation, all received medium-low scores. While some of the components overlap across practices, this gives an initial sense that the lab days most aligned with inquiry-oriented instruction with regard to generating student ideas.

The results from the IOIM are also reflected in the distinct difference between how time was spent on lecture and lab days, as shown in Table 4.14. From this, we see lecture days were dominated by the instructor presenting at the board (lecturing) and included less time for students to work individually or in groups, whereas on the lab day the allocation of time is flipped. This aligns with a lack of generating student thinking, building on it, sharing it, or connecting students' work to standard mathematical notation on the lecture day, but increases in generating student thinking on the lab day. Overall, this led to the instructor lecturing roughly two-thirds of class time and students working roughly two-fifths of the time. No time was spent on students presenting solutions at the board or discussing ideas across tables in whole class discussion.

Table 4.14. Partial Group Unit Time Averages for Instructor B.

Day	Segments lecturing	Segments students working	Segments student presenting	Segments whole class discussion
Day 1 (lecture)	23/25	3/25	0/25	0/25
Day 2 (lecture)	26/26	6/26	0/26	0/25
Day 3 (lab)	3/25	23/25	0/25	0/25
Total	52/76	32/76	0/76	0/76
Average	68%	42%	0%	0%

In Table 4.15, we see the questions being asked and answered in the first unit. Overall, the instructor directed questions to the class roughly once every three minutes, though the average was closer to once every two minutes on lecture days. Students answered a question publicly roughly once every four minutes, though the average was closer to once every three minutes on lecture days. Students asked questions infrequently (once every ten minutes on lecture days, less overall). This provides a possible explanation for the relatively lower scores for building on student thinking in the lab day, as students' ideas were not made public for building, but students did have opportunities to create (limited) shared understandings at their tables.

Table 4.15. Partial Group Unit Question Averages for Instructor B.

Day	Questions to Class	Student Answers	Questions to Instructor
Day 1 (lecture)	16	8	4
Day 2 (lecture)	29	25	6
Day 3 (lab)	4	2	0
Total	49	35	10
Average	.32/minute	.23/minute	.07/minute

Similar scores and characterizations occurred in Week 7 during the isomorphism unit. On the IOIM, isomorphism and homomorphism lessons in Week 7 when evaluated on lecture days received similar scores to in Weeks 3-4: a medium score on guiding and managing the development of the mathematical agenda (practice 6); medium-low scores on facilitating student engagement (practice 1), eliciting student reasoning and contributions (practice 2), inquiring into student thinking (practice 3), being responsive to student contributions (practice 4), and supporting formalizing of ideas and notation (practice 7); and a low score on engaging students in one another's thinking (practice 5). Thus, relative to Weeks 3-4, practices 3 and 7 went up slightly and practice 5 went down slightly. Thus, there was slightly more probing of students' given thinking (practice 3) and more use of students' mathematics before presenting formal mathematics (practice 7), but less encouragement to have students think about each other's thinking (practice 5).

The lab day received lower scores on three practices of the IOIM relative to Week 4. Specifically, it received medium scores on facilitating student engagement (practice 1) and being responsive to student contributions (practice 4); and medium-low scores on all other practices. This indicates that students were engaged in similar practices, but that discussions between students were not generating as many student ideas and aspects of sharing their ideas were also reduced relative to the previous lab.

Much like the IOIM, the statistics on time were similar to the first unit, as shown in Table 4.16. On average, slightly more than two-thirds of class time included lecturing and roughly two-fifths involved students working in small groups, with most of the time working in small groups happening on the lab day. No time was spent on students presenting solutions at the board or discussing ideas across tables in whole class discussion.

Table 4.16. Isomorphism Unit Time Averages for Instructor B.

Day	Segments lecturing	Segments students working	Segments student presenting	Segments whole class discussion
Day 1 (lab)	6/26	26/26	0/26	0/26
Day 2 (lecture)	27/27	0/27	0/27	0/27
Day 3 (lecture)	20/24	5/24	0/24	0/24
Total	53/77	31/77	0/51	0/51
Average	69%	40%	0%	0%

From Table 4.17, we see the instructor asked students questions roughly once every four minutes (closer to once every three minutes on lecture days) and students answered questions once every five minutes (a bit less than once every three minutes on lecture days). Questions were asked publicly to the instructor at similar rates across all three classes, a bit less than once every ten minutes. Thus, the interactivity in the class decreased slightly relative to the first unit.

Table 4.17. Isomorphism Unit Question Averages for Instructor B.

Day	Questions to Class	Student Answers	Questions to Instructor
Day 1 (lab)	1	0	4
Day 2 (lecture)	22	18	3
Day 3 (lecture)	15	12	5
Total	38	30	12
Average	.25/minute	.19/minute	.08/minute

In Weeks 8-10, during the quotient group and FHT unit, there was much more time spent on instructor presentations to the class than in previous weeks. For the IOIM, when evaluated on lecture days, the unit in Weeks 8-10 received the same or slightly lower scores than in Week 7: a medium score on guiding and managing the development of the mathematical agenda (practice 6); medium-low scores on facilitating student engagement (practice 1), eliciting student reasoning and contributions (practice 2), and inquiring into student thinking (practice 3); and low scores on being responsive to student contributions (practice 4), engaging students in one another's thinking (practice 5), and supporting formalizing of ideas and notation (practice 7). Relative to unit 2, the score for practices 4 and 7 decreased, and the other scores remained the same. Practices 4 and 7 relate to creating a shared understanding of the mathematics (and practice 7 is the only practice related to connecting to standard mathematical language and notation).

The partial lab days received scores similar to lecture days in these weeks: a medium score on guiding and managing the development of the mathematical agenda (practice 6); medium-low scores on facilitating student engagement (practice 1), eliciting student reasoning and contributions (practice 2), inquiring into student thinking (practice 3), and being responsive to student contributions (practice 4); and low scores on engaging students in one another's thinking (practice 5) and supporting formalizing of ideas and notation (practice 7). The only difference between the scores for lectures and labs in this time frame was on practice 4, where the score was slightly higher for the labs.

Students on lab days in these weeks engaged in less discussion with each other at all but one table, which depressed the IOIM scores relative to previous weeks. Specifically, practices 1, 4, 5, and 7 decreased, and practice 6 increased. Thus, some scores across generating student thinking, building on that thinking, and developing a shared understanding decreased relative to the previous unit. One score related to building, the practice focused on developing the mathematical agenda in an orderly way, increased.

Looking at this longer unit, much more time was spent lecturing on average, roughly four-fifths of class time, as shown in Table 4.18. Students also spent less time working than in previous units, closer to one-fifth of the time. Students still did not present solutions publicly or engage in whole class discussion. Notice unlike the previous units where labs received one full day each time, the labs in this unit received only partial days or spread over two days. Specifically, day 1 started as a lecture and then finished with the lab. On day 6, the lab did not start until partway into class and students did not finish. On day 7, there was a lecture on what students were supposed to have taken away from the lab so far and then students had time to finish the lab at the end of class.

Table 4.18. Quotient Group Unit Time Averages for Instructor B.

Day	Segments lecturing	Segments students working	Segments student presenting	Segments whole class discussion
Day 1 (lecture/lab)	10/26	16/26	0/26	0/26
Day 2 (lecture)	25/25	0/25	0/25	0/25
Day 3 (lecture)	25/25	0/25	0/25	0/25
Day 4 (lecture)	25/25	0/25	0/25	0/25
Day 5 (lecture)	18/27	11/27	0/27	0/27
Day 6 (lecture/lab)	18/26	13/26	0/26	0/26
Day 7 (lecture/lab)	18/26	8/26	0/26	0/26
Day 8 (lecture)	25/25	0/25	0/25	0/25
Day 9 (lecture)	8/8	0/8	0/8	0/8
Total	172/213	38/213	0/213	0/213
Average	81%	18%	0%	0%

In terms of questions, the instructor asked questions roughly once every two to three minutes while students gave answers roughly once every four minutes, as shown in Table 4.19. Questions asked of the instructor happened slightly less than once every ten minutes, much like previous units. Notice, the frequency of instructor questions almost doubled, but the frequency of student answers only increased by .04/minute. This relates to the fact that the instructor asked questions twice (phrased differently) when he did not receive a response from the class, whereas he had less need to do that in previous units.

Table 4.19. Quotient Group Unit Question Averages for Instructor B.

Day	Questions to Class	Student Answers	Questions to Instructor
Day 1 (lecture/lab)	11	5	2
Day 2 (lecture)	21	13	6
Day 3 (lecture)	32	14	3
Day 4 (lecture)	30	14	0
Day 5 (lecture)	18	13	4
Day 6 (lecture/lab)	16	8	5
Day 7 (lecture/lab)	21	9	6
Day 8 (lecture)	26	16	4
Day 9 (lecture)	11	6	0
Total	186	98	30
Average	.44/minute	.23/minute	.07/minute

IOIM scores across the three units are given in Table 4.20. For the labs, scores held steady or decreased across the three units except for practice six, management of the mathematical agenda. This increase can be attributed to greater organization and tying together of ideas from the lab in the third unit, whereas students were left hanging a bit without being told what they were supposed to have learned in a whole class setting in the previous units. For the lectures, scores largely held steady or decreased, with the exception of practice 7 in unit 2. In that case, the lab preceded the unit, allowing some informal notation and ideas to come from the students before the definition of isomorphism was fully introduced and explained. Comparing to Kuster et al. (2019), we see Instructor B's scores from the quotient group unit were all lower than the average scores from that study. In fact, the scores are at or below the lowest recorded score for all practices. However, Instructor B was not seeking to enact inquiry-oriented instruction, so it is not surprising that his scores indicate a less inquiry-oriented classroom.

Table 4.20. IOIM Scores for Instructor B in Lectures/Labs.

	P1	P2	P3	P4	P5	P6	P7
Groups	2/4	2/4	1/2	2/3	2/3	3/2	1/2
Isomorphism	2/3	2/2	2/2	2/3	1/2	3/2	2/2
Quotient Groups	2/2	2/2	2/2	1/2	1/1	3/3	1/1

Summary statistics for time usage are given in Table 4.21. In looking across the three units, we see an increase in the amount of time lecturing, especially between units two and three, and a decrease in the amount of time students were working individually or in groups, especially between units two and three. Students did not present at the board or engage in cross-group discussion in whole class settings in any unit.

Table 4.21. Average Time Usage across Units for Instructor B.

	Segments lecturing	Segments students working	Segments student presenting	Segments whole class discussion
Groups	68%	42%	0%	0%
Isomorphism	69%	40%	0%	0%
Quotient Groups	81%	18%	0%	0%

Summary statistics for questions across units are given in Table 4.22. In terms of the average frequency of questions, we see a slight decrease in the frequency of questions being asked and answered from unit 1 to 2 and an increase, especially from the instructor, for units two to three. This can be partially explained by fewer lab days in the third unit, since the three partial lab days were only partially devoted to the labs, and more questions tended to be asked in lecture days. Based on notes in the TAMI-OP, however, the instructor also had to ask questions twice a number of times because no one responded to his initial question, leading a number of his questions to be asked multiple times. The frequency of questions to the instructor in whole class discussion held fairly steady across the three units.

Table 4.22. Average Questions across Units for Instructor B.

	Questions to Class	Student Answers	Questions to Instructor
Groups	.32/minute	.23/minute	.07/minute
Isomorphism	.25/minute	.19/minute	.08/minute
Quotient Groups	.44/minute	.23/minute	.07/minute

Putting together the information from the IOIM and the TAMI-Ops paints a picture of a class strongly guided by the instructor's mathematical knowledge but with some opportunities for students to begin building intuition on topics before being shown how to approach topics. Thus, there were limited opportunities for students to create their own mathematical discoveries, but there were opportunities for students to begin exploring. The ultimate mathematical authority rested with the instructor, and he was in charge of moving the class forward at the pace he wanted to set.

Across the units, we see a decrease in the richness of students' mathematics. Students were given less time to work individually or in small groups, especially in unit 3, and the amount of time the instructor spent lecturing increased, especially in unit 3. We also see decreases in the richness of the math being generated across the three units, which is likely linked to the decreases in other IOIM scores.

Alignment of Instructor B's beliefs and instruction. Looking only at Case B, I address my first research question, related to the relationship between instructors' beliefs and their instructional practice. To a large extent, Instructor B's stated beliefs and instruction clearly aligned in terms of the activities he had students engage with. However, especially in the quotient group unit, he gave less time for students to do mathematics.

Instructor B's stated beliefs about the nature of mathematics focused on the structure of mathematics and the search for theorems. These theorems could potentially be used to address problems or could just be interesting. In terms of learning math, he focused on writing proofs and creating or working through examples. When he spoke about teaching, he emphasized students should be given opportunities to see a variety of examples. His instruction reflected this belief in math as the search for theorems through his emphasis on proof in lecture, which he addressed by lecturing twice as much as he provided labs. The lectures included a bit of time for students to work on problems, largely individually, but most of the time on lecture days was devoted to presenting proofs of theorems and thinking through implications of the work the instructor did at the board. Thus, the logical structure of math could be emphasized through proofs done by the instructor.

In his interviews, Instructor B emphasized that students should have opportunities to do math and to discuss with one another in addition to the lecture periods, so that students could experience different types of instruction. He did give students opportunities to do this during labs, and he stated that the labs were intended to be interactive. However, most groups did not interact with each other much on the labs, but would wait until the instructor came to their table to ask questions of him. Though the different types of instructional days, opportunities to work on problems for extended periods, and opportunities to interact aligned with Instructor B's stated desire to use many types of instruction to reach many types of learners, in practice, most groups experienced largely lecture and individual work time in class instead of varied amounts of discussion. This was still more instructional variety than might be expected in a "typical" lecture class, even though the discussion was minimal for some tables. Instructor B noted that his previous semester's section had been more interactive, so it is possible this was more due to the students' preferences than the instructor's intention. Here we seem to have a tension between Instructor B's belief that students should be interactive and students should be free to make choices about how they want to learn. It seems Instructor B placed more value on the latter belief.

Instructor B also believed math is often learned to be applied in other contexts, whether a content application or an increased ability to reason logically. This applicability was evident in the connections he made to other courses, like cryptography, that he noted in class. He also considered the development of logical thinking skills to be a valuable tool, and he did give students opportunities to develop their proof writing skills in assignments and through suggesting next steps for proofs he did at the board.

Student beliefs and affect. In this section, I characterize Class B students' beliefs in order to address relationships between instruction and students' beliefs. This will allow me to begin to address research question two. As noted in the research question framework, this research question was more exploratory in nature and was less certain to have observable relationships.

To describe students' beliefs, I will first examine results from the first class-wide survey, accessible in Weeks 3-4, and expansions on responses to the survey in the first student interviews (Week 6 or 7 depending on the student). Questions analyzed from this survey and interview addressed what students said math is and how they learn math. Questions from the second class-wide survey, accessible Weeks 5-6, addressed students' motivation. Questions from the third class-wide survey, accessible Weeks 10-11, addressed students' view of the nature of intelligence. I address changes during the semester through questions about MUSIC components and a question on perceived changes in beliefs from the final interview with the four interview participants. These interviews took place in Week 14.

Nature of math. Students in Class B provided a variety of words to describe what math is, as shown in the word cloud in Figure 4.2. While "can" was the most common word included (7 instances), "world," "study," and "abstract" also are prominent, indicating some expectation of math being both an entity and an activity as well as being both practical and theoretical.

Figure 4.2. Word cloud on what math is from Class B.

In Table 4.23, the four interview participants' responses to what math is and the nature of math are spelled out more fully as a window into their views on math at the beginning of the semester. Notice all four students referred to math in terms of connections to the real world, with Students 3b, 11b, and 19b focused more on applications of math. Specifically, Student 3b focused on applications or just enjoying math, Student 11b focused on finding "a quantifiable solution to life's problems," and Student 19b focused on modeling with math ("explaining phenomen[a]") as well as using math as a way to communicate information ("language"). Student 18b was more structurally focused, but gave a possible origin of mathematical knowledge outside pure logic ("empirical observations"). These responses provide some context for the other students' responses in the word cloud. The four students' animal metaphors reflected affective considerations (Students 11b and 19b), attention to solving problems (Students 3b, 11b, and 19b), and a focus on the nature of reality (Student 18b).

Table 4.23. Class B Students' Views of the Nature of Math.

Student	Math animal	What math is
3b	A horse. It can be used for many things, and while most use it (and view it) as a means of doing something else, if you take the time to learn about the horse it can be something in its self. I mean this because most people, especially engineers, view math as a mechanism for doing engineering; however, there are facets of it that are not seen by the average person using math as a mechanism, that are quite interesting.	Its whatever you make it. You can use it as a mechanism for doing other things, or you can use it as a proof that you CAN do other things, or you can simply enjoy the knowledge.
11b	I think math is like a spider. Both are seen everywhere and help us throughout our lives without many people realizing it. Both are also huge fears that people have.	Mathematics is the study that uses formulas, properties, numbers, and more to find a quantifiable solution to life's problems.
18b	I do not think it is like any animal, since math is in some sense metaphysical. It reminds me of humans the most out of any animal.	A set of ideas generated by (logically deduced from) axioms which are often but not always based on empirical observations/intuition.
19b	I think math is like a cat. It can seem to hate you at times, but then completely change. Math is a giant mystery and we have to solve, just like a cat's true feelings.	Mathematics is the abstract concept of problem solving. It is the bare boned, direct explanation for many phenomenons in the world. It is the language we use to explain the world we live in.

Comparing responses to the two questions, observe Students 3b and 11b had a clear focus on applications of math in both contexts, though Student 11b added an affective component with his animal. Student 19b also added an affective component with her animal response, but highlighted problem-solving in both contexts. Her problem-solving emphasis was slightly different from Students 3b and 11b, which she elaborated more in her interview.

> I've always seen math as a big puzzle that's like…the more time I put into it, like, I'll understand it better and I can actually figure it out…I mean Social Studies, you know, your teacher doesn't agree with your opinion, you're kind of done for. But I always liked math because of the simplicity in it. Now that I am in higher level math classes, I'm

> realizing that it's not always the case….It is still the puzzle mindset that I, you know, given a problem…you can look at it through a bunch of different angles and fit all the pieces together.

She went on to explain that cats are temperamental if you leave them alone for a weekend. Relating cats and math, she said,

> I guess with a cat…they can be very unpredictable, like math, like, you never know what kind of problem you're gonna get. Especially like last fall I started Proofs, like since then a lot of my classes are proof-based classes so, you know, some of them…you get right away and you're like, "Great, I understand this." Then get to the next one, and you're like, "What the heck? I don't know."

Across the two contexts, we see an emphasis on solving puzzle-type problems in math, but that these puzzles could vary a bit in ease of solving. Student 19b focused on structural aspects of math in both responses as he focused on axioms and ideas in his description of math. He explained his response of "humans" for the math animal by saying math is a "product of human consciousness" and that math is "very analytical and precise…it was invented to serve humans."

In responding to these questions, Students 3b, 11b, and 19b seemed most aligned with Ernest's (1989) instrumental view of math. This is because they focused on accomplishing specific tasks with math and seemed less interested in its overarching structure. Student 18b seemed most aligned with the problem-solving view of math, as he was focused on how math fit together and, especially in his interview, focused on people as makers of mathematics instead of discoverers of math. However, as Ernest himself pointed out, people can have more than one view of math and it is possible the students could identify with other views of math.

From the students' responses, we see that the four interviewed students all connected math to the world in some way. This connection was made through applications of math (Students 3b and 11b), the approach to math being about problem solving (Student 19b), or experimental results as a basis for making new math (Student 18b). Three students focused on people's views of math, whether affective reactions (Students 11b and 19b) or what general people know about math (Student 3b). Student 18b was much more structural in his view of math than the other students. In these statements, Students 3b, 11b, and 19b expressed instrumental views of math, whereas Student 18b expressed a problem-solving view of math.

Nature of teaching and learning math. Students were also asked about their views of how they learn math and characteristics of a good teacher on the first survey to the class. Selected responses that relate to changes they noted later in the semester are collected here. Student 3b noted that he was "better at math than most other things," that he usually studied by reviewing homework, quizzes, and examples, and that this generally sufficed. Student 11b said he learned math by receiving organized notes:

> By being given fundamental information and example problems/solutions in class, then taking it all home to reference while working on new problems. Works best when given typed note outlines with proofs/blanks to fill in. That way I can more easily distinguish between definitions/formulas and problem solving.

Student 18b said he learned math in individual contexts: "I practice and do math, and if I struggle with an idea I allow it to sit in the back of my mind while I do other things and think about it time to time until I understand it." Student 19b focused on how she studied when explaining how she learned math:

> I learn math through practice. When a teacher explains a concept in class, I often have trouble understanding it until after we have gone through some examples. When I am on my own, after classes, I can reflect on the material and using the examples, I can begin to understand the original concept.

In the last interview with each of these students, they were asked how their beliefs about math, teaching, or learning had changed or been strengthened by the class. Student 3b did not really feel his beliefs had changed, but that this class had been a different experience for him, as the content was outside his focus area of aerospace engineering. Thus, in this course he had been forced to apply the idea that anyone can learn math if they try hard (use of growth mindset), unlike previous classes where content had come more naturally to him. The class also opened his eyes to how broad math is, because he was being exposed to so many new ideas and this was only an introduction to abstract algebra.

Student 11b felt his belief that teaching styles matter had been strengthened. He specifically mentioned that he would not have felt comfortable in Class A based on the clips of the class that he had seen, though such a class might be good for students seeing the material for the first time. He did appreciate the labs that had happened in his own class though and thought if he had had labs in the class the first time he took it, he might have done better: "[It's] easier to know how a machine works if you build it yourself, which is sort of how the labs work if they're before the lecture."

Student 18b mostly commented on changes to his perceptions of math majors. He was surprised by the fact that others in the class learned differently from him. Specifically, he was surprised by how much they relied on the class notes instead of looking for other materials to expand their understanding, such as other textbooks. He also engaged in more group studying and work than he had done before, and he felt group study sessions were reasonably well-focused.

As a preservice teacher, Student 19b said she generally paid attention to how teachers approach instruction and appreciated the lab days because they provided a day off from new material, which allowed time to apply new ideas or processes. However, she also would have appreciated more structure on the lab days because if you did not understand what was going on, you just struggled unproductively. This was a contrast to her previous experience with lab-like activities in middle and high school, where she generally had not struggled and did not see any downsides to doing activities. On the whole, she enjoyed having the mix of different types of instruction in class so that she was not bored by repetitious lectures and so that teamwork skills could be built through group work, but she did take away downsides of labs that she had not observed before.

From these early and late semester snapshots, we see can see that these students in Class B entered the class with application-focused views of the nature of math that focused on doing math (e.g. used “to find a quantifiable solution to life’s problems”). They also left room for the abstractness of math (e.g. math as “ideas…logically deduced from axioms”). At the end of the semester, students’ views of math as a field did not seem to have changed, but students seemed to be more reflective about how they learned math. Some students were more open to learning math in new ways (Students 11b and 18b) while others were more attuned to the variety of affective math experiences a student could have (Students 3b and 19b).

Motivation. Students also responded to a survey in Weeks 5 and 6 that assessed components of motivation under the MUSIC model. Each interviewed student’s average for each component and the class average are presented in Table 4.24. Overall, the students being interviewed gave slightly higher scores than the class as a whole. In the final interview (Week 14), students provided some interpretations for these scores.

Table 4.24. MUSIC Score Comparison for Class B Students.

	Student 3b	Student 11b	Student 18b	Student 19b	Class B
eMpowerment	6	5.2	5	4	4.90
Usefulness	4	5	5.4	4	4.49
Success	6	6	5	4.5	4.91
Interest	5.33	5.33	5.33	4.67	4.76
Caring	6	6	4.67	5	5.33

A prominent way empowerment manifested itself was through how students approached the course content and how much time they spent on the class. Student 3b felt he generally had control over the course and if he did not understand the material, he could get help in office hours. Student 11b focused on being able to do homework when he wanted to, to come to class when he wanted to, and that he felt he had control over the grade he got in the class, though he felt this was similar to his view on other classes. Student 19b felt she had control over how much she participated as well as how much work she wanted to put into the course. Student 18b also focused on control over his understanding, though he took this further than others by actively seeking out other resources:

> I definitely had control over to the extent that I would understand the material. If the lectures didn't give anything, then maybe I'd be in a slightly different position. But, I think that even disregarding any class policies or quality of lectures, which are all actually fine, I mean that I had the resources and the books and stuff on my own….at the end of the day, it was really just on me to learn the material….In general I try to use as many resources as possible considering books present things in a different order or maybe they just present it in a different light…. It's interesting to see how these professionals

> think the subject should be laid out, and that way I get all of the different viewpoints so that I can sort of construct my own nature of where all these pieces fit together.

This was a practice he had thought more math majors did, as noted above.

All four students seemed to view the general problem-solving and reasoning aspects as the most useful elements of the class, though the importance they placed on these skills varied. Students 3b and 19b did not see any direct applications of the content for their future careers. Student 3b said:

> I'm gonna say yes and no. Yes in the sense that it was interesting in the context of "Hey I like learning maybe I'll take other classes…." But in like the context of like "Hey do you think you're gonna use this at your job?" Probably not. I'm a vibrational… analyst I guess but I don't know that I'll be using group theory in vibrations data.

Student 19b mentioned she did not see any connections between the class and her intended career of teaching sixth through twelfth graders. The most use she took away was growth in her problem-solving skills. On the other hand, Student 18b saw potential connections to future research if he decided to go to graduate school in math instead of physics. Student 11b also thought the course would be useful considering he was planning to take the 4000-level algebra class and that more practice with proof was useful.

Like the interviewed students from Class A, three of the four students interviewed in Class B seemed to view grades as their central measure of success. Students 3b, 11b, and 19b referenced their feelings about their grades to address how successful they had been in meeting their goals for the class. However, Student 3b added that he had learned "a ton of stuff in this class." Student 18b focused more on his approach to the course in considering his success and goals for the course: “I don’t know too much about what I would have changed in how I approached the course, so I’m happy with it.”

Three of the students viewed the material they were going through at the time of the interview, ring theory, to be the most interesting. Students 11b and 19b specifically said ring polynomials were interesting. For Student 11b, this was because the foundational principles behind algebra concepts from seventh grade were being explained. For Student 19b, she appreciated the connection to greatest common denominators and that this was a semi-familiar procedure but with a “twist” or “spin” so she was still being challenged by it. Student 18b was interested in general applications of ring theory as well as being more intrigued by the structure itself, as there were now two operations to work with. Student 3b was more interested in the applications to cryptography that his instructor noted in class, in part because his instructor seemed interested in them.

Office hours and labs had the most prominent places in students’ feelings of connecting to other students and the professor. Students 3b and 19b both mentioned that they felt their professor cared about them based on their experiences with him in office hours. Student 11b had not felt the need to use office hours as much this semester, so he had not felt the need to develop as strong of a connection to his professor that semester. He also felt most connected to his classmates during the labs and less connected during the lectures, though he never felt isolated in the class. Student 3b felt both most connected to and most isolated from classmates during the lab days. When he was able to “bounce ideas off” others and work at the same pace, he felt connected, whereas if he was a step behind as others bounced ideas off each other, it was an isolating experience. He also noted falling a step behind while in lecture could be isolating for similar reasons. Student 19b noted similar ideas of feeling connected to others on lab days and when studying outside class, but feeling isolated when she felt others understood and she did not. She also noted she was typically the person at her table who was willing to ask the professor questions as he came around, whereas the others would try to figure the problems out themselves. Student 18b also noted feeling connected to other students by interacting with them. He reflected on a different way of disconnecting with other students, though he did not consider this isolating:

> Some of the people that I'm friends with in the course probably have slightly less interest in math than me, based on the discussions I've had. And some of them aren't so interested in learning the materials for the material's sake. It's more of like a, "I have to take this class, I have to get a good grade, I like math in general but I'm not that interested in it." So, there's kind of like a disconnect there with some of the people that I'm interested with. But I don't know if I'd describe it as isolated. But, there's one disconnect.

In summary, these students and the class as a whole largely at least "somewhat agreed" that Instructor A fostered their empowerment, usefulness, success, interest and caring in the course. Reasons for this included feeling some control over how much work they put into the course (e.g. coming to office hours or seeking out other resources if needed); perceiving usefulness in problem-solving and potential applications to future course work (for those considering graduate school); feeling they could get decent grades in the class; being interested in the course material on rings or material their instructor found interesting like cryptography; and feeling the chance to connect with their peers in class or with their instructor in office hours.

Summary of students' beliefs. The interviewed students all connected their view of math to the world in some way, whether by focusing on applications or as a source of inspiration for math. In their description of what animal math is like, students gave some similar answers, but added more details as they noted affective reactions people have to math, more specific ways applications can be seen, or more abstract views of the nature of math. Students did not directly mention any changes in these beliefs about the nature of math in the end of semester interview.

However, students' views of how they learned math seemed to undergo reflection. Some students expressed more openness to learning in non-lecture settings, while others considered how struggling aimlessly during class might affect students' views of mathematics. Nevertheless, all interviewed students seemed open to learning math in a variety of ways.

Furthermore, the students seemed reasonably motivated in the course, as measured by the MUSIC components of motivation. While different students gave higher scores to different components for different reasons, all students generally agreed that their empowerment, sense of usefulness, ability to succeed, interest in material, and perceptions of care from the instructor were fostered in the course. The interviewed students cited some specific aspects of instruction as stimulating motivation, such as the instructor's references to other content the algebra they were learning connected to, opportunities to connect with the professor in office hours, and opportunities to connect with other students in labs.

However, some of the reasons they cited for feeling supported could have referred to many classes. They could choose how much effort to put into any class (M), and many math classes support improving your ability to reason or preparation for graduate school (U). Even the specific instructional component of labs, which were cited as helping students feel cared for, were also cited as making the students feel isolated, depending on if they were working at the same pace as their classmates.

Alignment between instruction and students' beliefs. Looking only at Case B, I address my second research question, focused on relationships between instruction and students' beliefs in Class B. There are some clear connections between what happened in class and what they said changed about their beliefs. However, many of their previous beliefs, especially about the nature of math, seemed unchanged.

Alignment between instruction and students' beliefs was clearest around types of instruction. Instructor B used different types of instruction in his classes, varying between lectures roughly twice each week and labs roughly once each week. Students' beliefs seemed most affected by this class set-up, which was novel for them, at least in a college math setting. After experiencing this different class set-up, students seemed to place a higher value on having variety in instruction and were more open to non-lecture class periods. However, the main shift in students' beliefs seemed to be in valuing variety in instruction, not specifically valuing an alternative to lecture. Furthermore, the approved variety in instruction seemed focused on the format of the tasks more than the discussion level. Student 11b noted that having a lab before new material gave insight into what they were about to learn, which he found helpful. Similarly, Student 19b found a day to process old material before seeing more material to be helpful, which was what she highlighted as most beneficial about the labs.

Related to the variety of instruction types, Instructor B noted that students learn in different ways and he wanted to support different types of learning. This belief was expressed through allowing students to discuss however much or little they wanted to at their tables, even though he did encourage them to talk with their tablemates. In instruction, this presented itself as different tables interacting at different levels. This seems to be a place where students' beliefs influenced instruction at the table level, as students at different tables experienced different interactivity levels. Specifically, Student 18b was at the most interactive table, and he would animatedly discuss problems with other students. Students 3b and 19b were at the second-most interactive table and would discuss with peers at times. Student 11b was at one of the less interactive tables and often worked independently, though he was willing to discuss with others at his table.

On the other hand, students' alignment with the pace of their table seemed to relate to their feelings of connection with peers. Student 18b was at the most active table and noted he felt more connected with peers when interacting with them; if he had any problem with connecting to others, it was in an interest gap, not a mathematical discussion problem. Students 3b and 19b were at the second-most interactive table and both noted a connection with peers when they worked at the same pace as others, but isolation when they were behind. Students at this table seemed to check in periodically, but would not fully explain their thinking if they were ahead of other students. Furthermore, Student 19b said she was the only person at her table who was willing to ask questions of the instructor when he came around, which may have contributed to feeling different from her peers and thereby less connected. Student 11b was at one of the less interactive tables, but never noted feelings of isolation, just reduced connection during lecture.

Finally, students and Instructor B shared a view of what was useful for life from instruction in the class, though they disagreed on how useful it was. Recall, students shared some beliefs about the nature of math with their instructor, such as a focus on connections to the real world (all four students), a focus on problem-solving (Students 3b, 11b, and 19b), and a focus on the structure of math (Student 18b). Furthermore, Instructor B thought helping students "refine and focus their mathematical reasoning" was the most useful aspect of the class. This refining happened through the "type of reasoning they're seeing here," which seems to indicate proofs done in lecture as well as problems students would do. However, students' expected future work influenced how useful they actually found this instruction, with students considering higher level math courses finding it valuable (Students 11b and 18b) and students with engineering and middle/high-school teaching interests finding it less so (Students 3b and 19b).

In summary, students' beliefs about teaching and learning math seemed to be altered or strengthened with regard to their openness to variety in instruction. Students appreciated having different ways of learning in keeping with the experience they had in the class. Students also affected instruction to some extent through the ways they chose to interact at their table. They were given the freedom to choose how interactive they wanted to be and different tables made different choices. However, these choices had ramifications for students' perceptions of connection. Students aligned with the level of interaction at their table expressed feelings of satisfaction with labs, whereas students working at a different pace from their peers and who were not having others' thoughts explained to them felt disconnected instead. Finally, students' perceptions of usefulness aligned with what the instructor intended, though the level of importance assigned to what was learned varied based on students' future career interests.

Chapter Discussion

This section focuses on addressing research questions one and two by conducting a cross-case analysis. First, I will compare and contrast instructors' stated beliefs. Next, I will compare and contrast what happened in their instruction. Then, I will use this comparison to inform research question one, examining relationships between instructor beliefs and instruction. Finally, I will compare and contrast students' beliefs and compare to instruction in order to examine relationships between instruction and students' beliefs, which is research question two.

Comparison of instructors' beliefs. Instructors A and B shared some general beliefs about math, learning, and teaching, but also have different emphases and ways of acting on those beliefs that influence their instruction. Similarities include viewing math as both an object of study and something that we do, that doing math and participating in class are important for students' learning, and that anyone can improve their knowledge of math. They also had similar views of how their students would view their facilitation of empowerment, usefulness, success, interest, and caring in the course. Differences include their main views of the nature of math, what doing math entails, the importance of math applications, the purpose of discussion, and the ways they promote inclusivity in their classes.

Instructors A and B shared broad beliefs about the nature of math, learning, and teaching. Both spoke of math as a field of study and something we do, though both instructors placed more emphasis on what is done in math than the fact that math is a field of study. Both instructors also emphasized the role of participation and students being active in their classrooms. This included an expectation of attention from students and working on problems when they were given. Both instructors displayed a growth mindset when they filled out a survey and when they responded to questions related to the changeability of intelligence. Specifically, both noted that working on problems would improve one's understanding of the material and seemed to believe that if a student was not doing well, it was because they were not doing the work, not because they were fundamentally incapable of it. Finally, they had similar expectations for how students would score the MUSIC components of motivation with the exception of usefulness; Instructor A did not expect students to find the course useful, whereas Instructor B did.

The instructors also noted a similar source of influence on their teaching that aligned with the Johnson et al. (2017) study: experiences as a student. Instructor A's experiences in graduate school, including reading Freudenthal's (1973) work on realistic mathematics education, influenced how she felt math should be experienced and taught. Instructor B noted his experiences in lecture classes were not very beneficial, as he would get stuck thinking about one detail in a proof as the class moved on to other material. Thus, he believed giving opportunities for students to work together in class could be beneficial for students like him. This may connect to a finding by Wilkins (2008), where he noted that content knowledge had a negative effect on the use of inquiry-based materials. Here, we see an instructor who did not feel he learned well from the setup of the class choosing to use a mixed method of instruction, partly in reaction to his experience.

For differences, Instructor A held a "problem-solving" view of math as defined by Ernest (1989): she focused on math as a human activity in which mathematical knowledge is created. This aligned with her definition of doing math as behaving more like a mathematician, by which she meant engaging in defining terms and generating theorems. Thus, studying math becomes a tool for thinking more rigorously, and may also be applicable to other contexts. She also gave limited space for students to hold other views of what math is, such as that math is just about following procedures. She emphasized that students were expected to engage in the tasks and participate in the way that she defined for the class. This would allow students to build consensus at various levels, from self to small group to whole class to math recognizable to mathematicians. Though she had a firm view of what math, math learning, and math teaching should look like, she focused on building relationships with students and being inclusive of students in the classroom. She would reflect on who was speaking in the classroom and how she could support students by showing that she cared about their welfare.

Instructor B held a Platonist view of math, in which mathematicians search for or discover theorems. This aligned with his view of doing math as proving theorems and working on examples and applications of those theorems. This also linked to his greater emphasis on the applicability and usefulness of math. While he acknowledged he had his own views of math, he felt it was important for students to come to their own conclusions about what math is and not impose his own views. This related to his idea of providing different ways for students to participate and engage with material through both lecture and lab components. Students were encouraged to discuss with each other so that they could benefit from each other's knowledge, but discussion could be mainly directed to the instructor or students could work individually for long stretches of class. Allowing students to interact with him and with each other as they felt comfortable became a way to try to include everyone and address their needs.

From this analysis, we see the expressed beliefs and teaching characterizations of Instructors A and B differ in a number of respects. This lends credence to the idea that the two classrooms do present distinct cases in this case study. In addition to espousing slightly different views of what "doing mathematics" means and having different views of how to approach instruction, they characterized their instruction differently. Instructor A expected she lectured roughly 15-20% of the class and characterized her instruction as alternative, whereas Instructor B estimated 60% of the time was spent lecturing in class and characterized his instruction as mixed.

Comparison of instruction. While there are some similarities in the two instructors' teaching, differences are much more salient. Looking first at the comparison of IOIM scores in Table 4.25, scores for Instructor A match or exceed those of Instructor B in each unit. Given that Instructor A was trying to teach in an inquiry-oriented way and Instructor B was not, this is not particularly surprising. However, it does indicate differences in the interactivity in the classes and the extent to which students' ideas were involved in the development of the mathematics.

Table 4.25. IOIM Scores for Both Instructors.

	P1 A	P1 B	P2 A	P2 B	P3 A	P3 B	P4 A	P4 B	P5 A	P5 B	P6 A	P6 B	P7 A	P7 B
Groups	5	2/4	4	2/4	4	1/2	4	2/3	4	2/3	4	3/2	5	1/2
Isomorphism	4	2/3	4	2/2	3	2/2	4	2/3	4	1/2	4	3/2	4	2/2
Quotient Groups	4	2/2	3	2/2	3	2/2	4	1/2	3	1/1	3	3/3	3	1/1

In examining time usage from Table 4.26, both instructors increased their time lecturing across the semester, and both instructors lectured similar amounts of time during the isomorphism unit. However, Instructor A spent less time lecturing in each unit than Instructor B, and the sharp increase in lecture time was between units 1 and 2 for Instructor A, whereas the sharp increase was between units 2 and 3 for Instructor B. Both instructors held the amount of time students worked fairly steady during the first two units, but Instructor B allowed less time for students to work in unit 3, partly due to time constraints, as he chose to finish lecture material before moving to the labs. A difference between the instructors throughout the semester was in whether or not students engaged in whole class discussion or presented their work. Instructor B never had his students do this. Instructor A did in all units, though the amount of time spent on this activity fell sharply, especially between units 1 and 2.

Table 4.26. Average Time Usage across Units for Instructors A/B.

	Segments lecturing	Segments students working	Segments student presenting	Segments whole class discussion
Groups	20%/68%	57%/42%	17%/0%	42%/0%
Isomorphism	61%/69%	57%/40%	7%/0%	9%/0%
Quotient Groups	68%/81%	53%/18%	2%/0%	1%/0%

For question frequencies, we turn to Table 4.27. Instructor A asked questions more frequently, received answers more frequently, and received questions more frequently than Instructor B across all units. For the first two units, these frequencies were 2-3 times greater. In the last unit, Instructor A's frequency of asking questions decreased while Instructor B's increased, but the ratio of student answer rates was still nearly two-to-one.

Table 4.27. Average Questions across Units for Instructors A/B.

	Questions to Class	Student Answers	Questions to Instructor
Groups	.77/minute	.63/minute	.17/minute
	.32/minute	.23/minute	.07/minute
Isomorphism	.67/minute	.56/minute	.25/minute
	.25/minute	.19/minute	.08/minute
Quotient Groups	.59/minute	.41/minute	.24/minute
	.44/minute	.23/minute	.07/minute

These tables show distinct differences in the instructors' choices of how to spend their time in class. Instructor A gave students more time to work individually or in groups as well as to present their thoughts publicly than Instructor B. She also had much higher rates of asking and receiving answers to questions in whole class contexts. Both instructors spent large amounts of time lecturing in the last two units as well. Thus, we can see that instructors not only said they had different approaches to instruction; qualitative differences are also apparent in the amount of time spent on different activities in class.

The instructors seemed to intend to enact the interactive classrooms they talked about in the interviews. However, as the semester wore on, other factors seem to have gotten in the way. The instructors, though given very broad guidance on required content, felt the need to cover certain material in class. (The course catalog simply says the course is an introductory course in groups, rings, and fields.) Instructor B had even noted his content freedom in the course, and yet seemed to feel it was important to cover all of the content he intended to cover at the beginning of the semester.

When instructors were behind their internal schedule, they pressed to finish by reducing the time given to students to work. In Instructor A's class, this meant the Fundamental Homomorphism Theorem (FHT) only received a partial day of instruction while students also reviewed for an upcoming exam. Furthermore, this instruction was largely guided by the instructor instead of being student-discovered as much of the other content in class was taught. In Instructor B's class, this took the form of giving half days instead of full days to work on the labs. This bears out one of the conclusions of Johnson et al. (2017), that noted that instructors internalized constraints such as how much content needed to be covered in the class.

Comparison of relationships between beliefs and instruction. In terms of research question one, both instructors' beliefs influenced their instruction. These differences were evident in how they structured their courses and in their day to day instruction. Their different beliefs also manifested in different ways.

Differences in structure were evident in the different curricula and formats chosen for their classes. Instructor A chose to use the IOAA curriculum because she viewed math as a human activity and wanted students to generate definitions and proofs as a mathematician would. This aligned with a curriculum based on Freudenthal's (1973) work in realistic mathematics education. Instructor B formatted his class using two days of lecture and one day of lab, which aligned with his view of math as an interconnected structure. Because Instructor B viewed math as discovering axioms and theorems, much as might be expected from a logician, he lectured twice each week, giving opportunities to present many theorems and proofs to his students. However, he also budgeted time for students to explore concepts and make some connections for themselves. Here we see a similarity between the instructors' beliefs and instruction: both put structures in place that allowed students to work together and do mathematics, which they both considered important.

Different beliefs were apparent in day to day instructional choices as well. Instructor A wanted to help students be able to act more like mathematicians. Thus, she was less interested in applications for students' mathematical work outside improving the ability to reason, especially in mathematics. On the other hand, Instructor B made a point to note applications and connections to cryptography and number theory as they arose in class. Additionally, he seemed to view improving the ability to think logically as an application of math already. This aligned with his view that math was a search for theorems and applications of those theorems.

We see a similar change in instruction over the semester that could be a result of similar beliefs. Specifically, as instructors might have felt content coverage pressure near the end of the semester and the material became more complicated, instruction shifted in both classes. The instructors did not state a desire to do this, so it is possible they did not even notice they were shifting how much time they spent on different activities or how often they asked students questions. Nevertheless, this raises questions for further research on the influence of other instructional pressures.

Comparison of students' beliefs and connections to instruction. In this section, I examine students' beliefs and compare them to instruction. This is meant to further address research question two through a cross-case analysis. Because results in the two classes were similar, I address the comparison of classes and connections to instruction together in this section. I compare the two classes using the MUSIC score comparisons, growth mindset scores, and statements on students' beliefs about math, teaching, and learning.

Motivation. In many ways the students in the two classes were very similar in their responses about how motivating their course was, as of Weeks 5-6 when survey 2 was open. Based on the scores for the five MUSIC components, shown in Table 4.28, students averaged at least some level of agreement on all components. Notice many of the scores are highly similar, and no differences are statistically significant.

Table 4.28. MUSIC Score Comparison for All Students.

	Class A	Class B	Whitney-Mann U score	z-score	p-value
eMpowerment	4.67	4.90	110	-.820	.412
Usefulness	4.55	4.49	130	.091	.928
Success	4.74	4.91	117	-.565	.575
Interest	4.98	4.76	112.5	.729	.465
Caring	5.68	5.33	88.5	1.60	.110

The only ordering difference between the classes is that interest received the second highest score in Class A and the second lowest in Class B. This might be attributable to differences in instruction, as the interviewed students in Class A mentioned the tasks they did were interesting, whereas the main sources of interest mentioned Class B student interviews related to intrinsic motivation or places the material might be applied.

The fact that the students in the two classes gave similar motivation scores despite different instruction reveals more factors than just what happens in the classroom are important to students' motivation. For example, recall Instructor A felt she fostered students' empowerment through giving students some control over the pace of tasks and discussion in class, and some students also mentioned this in their interviews (Students 8a and 15a). On the other hand, interviewed students in Class B noted their freedom to consult outside resources (Student 18b) or to come to class when they wanted to (Student 11b), which seemed less instruction dependent. However, students in Class A were encouraged not to Google terms related to what they were doing in class so that they could reinvent definitions independently. Also, part of students' grades in Class A was having solutions to the tasks from class written in their notes, meaning students' attendance was more important to this class than some others. Thus, even though Instructor A seemed to foster empowerment through instruction in the ways she intended, students may have perceived less empowerment overall through norms for the class outside instruction.

Growth mindset. As we examine responses to the growth mindset inventory, we see students held similar beliefs about the changeability of intelligence (growth mindset). The growth mindset inventory was asked on survey 3 (Weeks 11-12 for Class A and Weeks 10-11 for Class B) for students and during interview 1 (Week 4) for instructors. All items were asked on a seven-point scale and scores were reformatted so that 1 indicated a fixed mindset and 7 indicated a growth mindset, with 4 neutral. (Questions that were written so a high score indicated a fixed mindset were given the other score equidistant from neutral center 4 when computing averages.)

From Table 4.29 we see all interviewed people had growth mindsets of varying degrees. Comparing to Table 4.30, we see the interviewed students expressed a higher degree of alignment with a growth mindset than students on the whole, but most students in both classes indicated a growth mindset. In fact, no students who responded to the survey in Class A (n = 15) had an average score less than 4, indicating all students were closer to a growth mindset than a fixed mindset. In Class B (n=14), two students had average scores under 4, but other students gave higher growth mindset scores, leading to a higher overall average. Comparing the students' and instructors' scores, both instructors had growth mindsets and on the whole students had growth mindsets in both classes. This inventory was given late in the semester, so it is possible students' beliefs about the changeability of intelligence shifted to align more with their instructors over the semester. However, more research is necessary to confirm such a conclusion.

Table 4.29. Growth Mindset Scores from Interviewees.

Class A		Class B	
Student 8a	7	Student 3b	7
Student 10a	5	Student 11b	-
Student 11a	6.75	Student 18b	4.88
Student 15a	4.5	Student 19b	6.88
Student interview average	5.81	Student interview average	6.25
Instructor A	6	Instructor B	7

Table 4.30. Growth Mindset Score Comparison for All Students.

	Class A	Class B	Whitney-Mann U score	z-score	p-value
Growth mindset	5.45	5.57	90	.633	.529

Nature of math, learning, and teaching. To compare students' beliefs about the nature of math, I consider the two word clouds in Figure 4.3, created from students' responses to the first survey (Weeks 3-4). Words were taken from students' responses to the question "What is mathematics?" Class A had 12 responses and Class B had 15 responses.

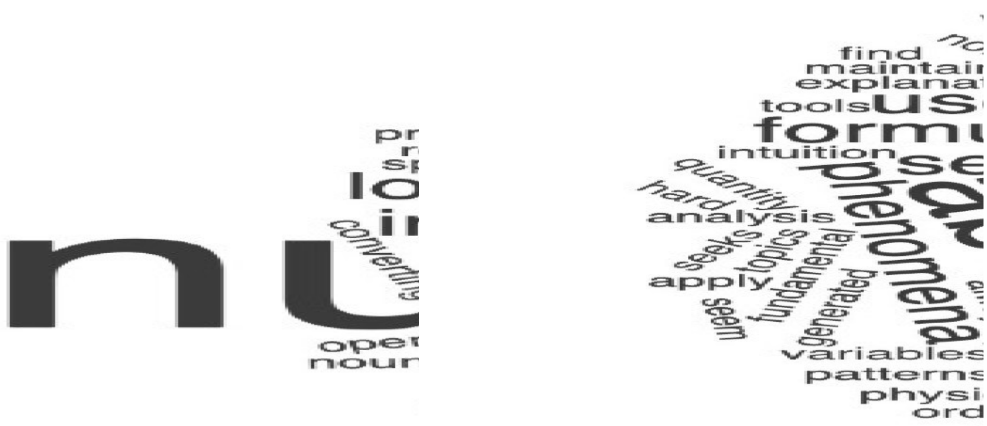

Figure 4.3. Word clouds from Class A (left) and Class B (right).

Based on Figure 4.3, we see bigger emphases were placed on “numbers,” “study,” and “real” in Class A, bigger emphases were placed on “can,” “abstract,” and “axioms” in Class B, and both classes placed a fair amount of emphasis on “world,” “explain,” and “logic.” Thus, in initial analysis, students from Class A were more focused on concrete applications and contexts of math, while acknowledging the role of logic, whereas students in Class B were more focused on abstract and structural aspects of math, while acknowledging the role of applications. Alignment in Class B with Instructor B’s view of math is easy to see, as the logician emphasized structure and application in his view of math and aligned his class accordingly (as noted above). However, alignment in Class A is less clear, as Instructor A focused on math as a creative human activity. While this can include uses for numbers, she de-emphasized the usefulness of math outside improving reasoning.

A few explanations for the difference in alignment present themselves. Because this question was asked on the first survey, it is possible that students reflected the beliefs they entered class with and had not had time to change their views of math based on instruction yet. Furthermore, the students in Class A reported lower scores on previous math classes than in Class B. Specifically, 4 students listed a score in the C range as the grade on their most recent math class, 5 listed B range, 2 listed A range, and 1 listed C and B range scores (from two classes). In Class B, no one listed a grade in the C range, 6 students listed a score in the B range, 7 students listed a score in the A range, and two students listed B and A range scores. It is possible that these more complex views of mathematics are in fact availing beliefs, which are linked to higher student achievement (e.g. Muis, 2004; Phan, 2008). Previously, Szydlik (2013) noted variance in the views of what math is provided by general education students, math majors, and math faculty with increasingly nuanced answers provided by math majors and math faculty. However, firm conclusions on whether or not these more nuanced answers are availing beliefs have not been reached. This finding suggests further research in this area is warranted. Furthermore, the students that expressed simpler views of math could have shifted their beliefs by the end of the course or after taking more classes. More study on when students' beliefs about the nature of math shift and whether this view of the nature of math leads students to drop a math major seems warranted.

I do not have any measures that permit easy comparison of the classes as a whole with regard to students' beliefs about teaching and learning. Thus, I examine the interviewed students' responses here. Recall, the interviewed students from Class A expressed more openness to learning math through guidance while reasoning through problems themselves and through discussing math with others. Students in Class B seemed to be more reflective about how they learned math with some students expressing openness to learning math in new ways (Students 11b and 18b) while others were more attuned to the variety of affective math experiences a student could have (Students 3b and 19b).

Notice students in both classes expressed openness to learning math in new ways. However, the ways students expressed their increased openness seemed more similar to the instruction they had received than what happened in the other class, indicating some influence from instruction on their beliefs. Students from Class A mentioned being more open to non-lecture formats and having opportunities to discover new paths for themselves. Students in Class B noted openness to multiple types of instruction within a course rather than just lecture or just group work. Though we see openness to new types of instruction in both contexts, what students were open to is more similar to the type of instruction they received than what happened in the other course.

Thus, to return to research question two, there does appear to be a subtle relationship between instruction and students' beliefs about what is possible in a college math class. Specifically, students seemed more open to expanding their methods of learning math according the experiences they had in the class. However, broad beliefs about the changeability of intelligence, valuing non-lecture formats, and students' motivation seemed similar in the two classes, despite different choices being made in instruction.

Chapter 5—Metaphors

In order to address research questions three and four, which focus on relationships between instructors' interview metaphors and instructional metaphors and between instructional metaphors and students' metaphors, I need to examine metaphors used by instructors and students within each class. Thus, I begin by defining the conceptual metaphors used across all contexts. Then, I organize this chapter by case, meaning I will characterize Class A and Class B separately. Finally, I will compare and contrast the two classes to further address the relationships between instructors in interview and class settings (research question three) and between metaphors used in class and students' metaphors (research question four).

Conceptual Metaphors

In this section, I will define the conceptual metaphors I found for isomorphism and homomorphism. I will also provide the frequency with which they were used. These definitions and frequencies will be used to address relationships between the metaphors used in class and the metaphors used by the instructors (research question three) and the students (research question four).

Defining conceptual metaphors. Looking across all interviews and class periods, 3 general metaphors and 12 specific metaphors were evident for both isomorphism and homomorphism. (Two of the specific metaphors were different for isomorphism and homomorphism.) In my coding scheme, phrases were not allowed to be double-coded, though a paragraph of text could receive multiple non-overlapping codes. Consecutive statements with the same code would be coded together as one code as long as they were not interrupted. Below I order the metaphors according to their general categorization in Figure 5.1. I also define and provide an example of what each metaphor looks like in context.

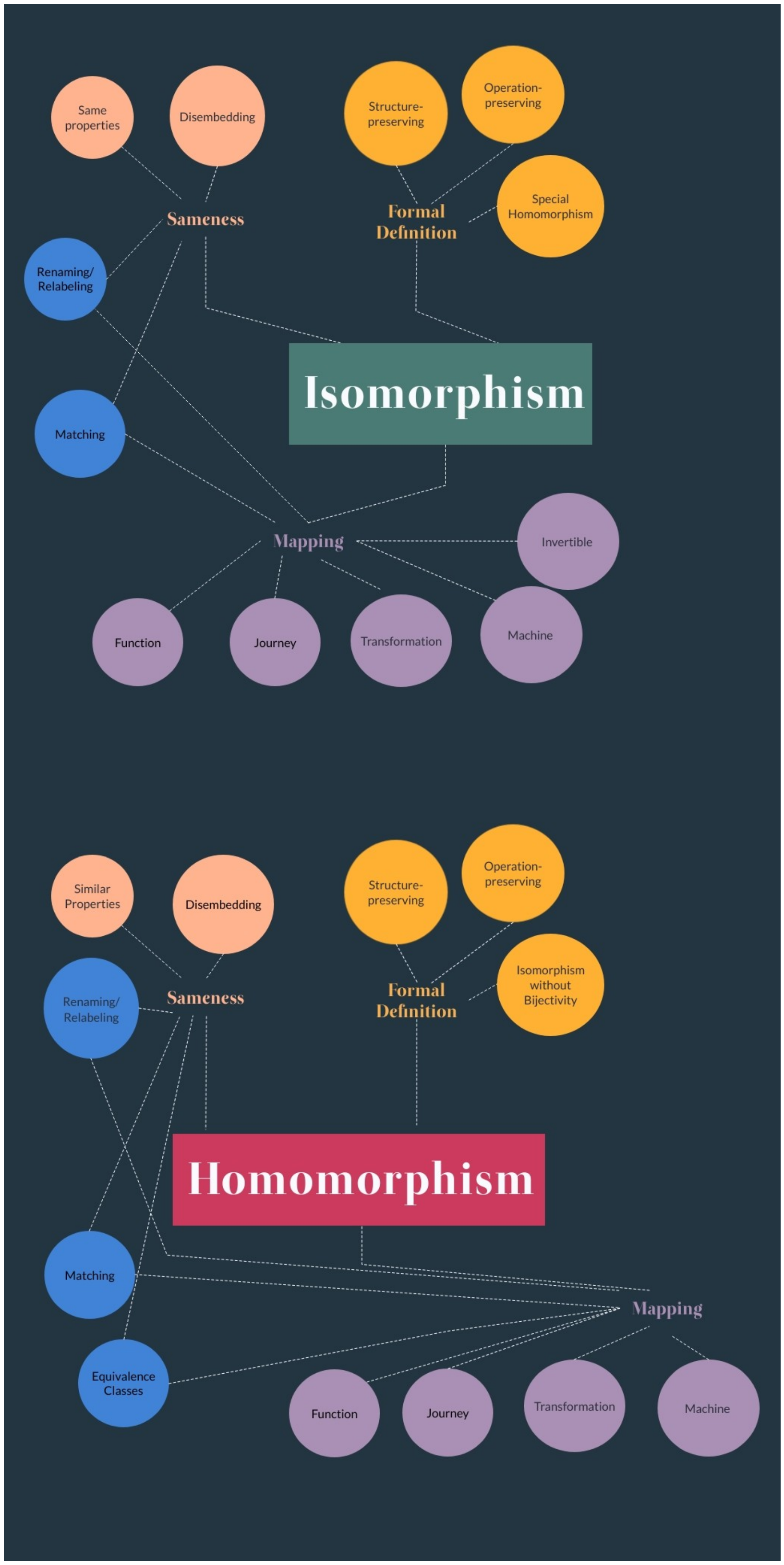

Figure 5.1. Isomorphism and Homomorphism metaphors organized thematically.

Codes related to sameness. In this section, I will discuss metaphors related to groups being the same or similar if an isomorphism or homomorphism could be formed between them. These are the peach-colored codes from Figure 5.1. These codes focus on sameness in terms of the characteristics of the groups being discussed. I will defend including codes in this cluster that do not contain "same" in their title.

An isomorphism/A homomorphism is a mapping between groups that are the same/similar. This code focused on generic references to groups being the same or similar, whether at the whole group level or as general statements about relationships between elements. An example of general sameness from Student 3b is: "So going back to the two different groups of just things…you would have to have the same, like, relationship equivalencies between the one and the other group." Note that in this example, there is no explicit reference to what is the same between the groups such as the property of two groups having the same number of elements, which would be considered same properties (below). Rather, "relationship equivalencies" are the same in this example, which is a vague term that connotes sameness in some capacity, but does not clearly show how the groups are the same.

An isomorphism/A homomorphism is a mapping between groups with the same/similar properties. This code was more specific than the previous reference to "being the same." Instead, use of properties that are the same for all isomorphic groups including size, being abelian or not abelian, orders of elements, etc. or properties of homomorphisms, such as mapping to a range that is the same size or smaller than the domain triggered this code. For instance, Student 18b observed: "…they're two different sizes so there can't exist an isomorphism between them."

An isomorphism/A homomorphism is a disembedding. This code used structure-focused language to describe extraction or highlighting of properties from the domain in the codomain.

This could more specifically focus on revealing a structure that is shared between groups. Student 18b noted: “A homomorphism is just a function that preserves the structure, or some structure, not necessarily…all of the structures; it might just preserve one structure. Like the integers map to Z mod 2 or something, that could preserve the structure of like the evens and the odds, but it destroys a lot of the other properties of the integers.” Notice the student highlighted a specific aspect of a group that would be shared by the other group and, in effect, pulled it out for inspection so what was the same could be more clearly shown. This code is included in the “sameness” cluster because it highlights a property within the groups of interest that is shared by the domain and range of the isomorphism or homomorphism.

Codes related to mapping. In this section, I will discuss metaphors related to finding mappings between groups. These are the purple codes from Figure 5.1. These codes focus on connecting the groups at the element or whole group level. Unlike the sameness codes that focused on identifying similarities between the groups, the mapping codes connect groups via formulas or general relations without requiring the groups to possess specific characteristics. I will defend including codes in this cluster that do not contain “mapping” in their title.

An isomorphism/A homomorphism is a type of mapping. This code was used whenever there was a generic reference to an isomorphism or homomorphism as a function or mapping without further details about how an individual was thinking about the mapping or explicitly utilizing properties of functions. Alternatively, the use of a set of drawings/lines or formulas to show a mapping was an indication of this metaphor. Ordinarily, this code was used when students gave a general idea of having a rule for mapping or assigning outputs. For example, Student 11b noted, “I can multiply by different multiples of 2 and get homomorphisms.” This demonstrates that the student had an idea of how to create new mappings, but did not provide details on how he thought about the mapping outside mathematical computations.

An isomorphism/A homomorphism is a function. This code was applied when participants utilized a specific function property to draw conclusions about isomorphisms or homomorphisms. This included all elements of the domain being mapped (everywhere defined) and no elements of the domain being mapped multiple places (well-defined) even if there was not a direct statement that this was a property of functions. For example, Student 11b observed, "…you can't have something mapping to nothing. Like everything has to map to something." This is a metaphor because the student was using the everywhere defined property of functions to structure his reasoning about homomorphism. "An isomorphism/A homomorphism is a function" is a linking metaphor much like Sfard's (1997) example of "fraction as number." While mathematicians might be inclined to view this code more as a statement of fact than a metaphor, an individual's understanding of function, the source domain, would provide structure for reasoning about isomorphism or homomorphism in much the way understanding a fraction as a number would provide a structure for reasoning. Because this code highlighted properties of maps rather than properties of the groups, this was considered a mapping-focused metaphor.

An isomorphism is an invertible mapping. This way of thinking was used to refer to isomorphism as a function with an inverse or more vaguely as a possibility of reversing a mapping, such as the colloquial idea of hopping back and forth. For example, Student 3b used the hopping back and forth image: "Each of the elements from the first set that gets mapped to the second set, you had, there's a reverse, goes back..." An example of the function with an inverse metaphor comes from Student 8a: "If you can go from Z to 2Z, then this would be sort of undoing that, so inverse of the isomorphism, sort of like an inverse function." Because either scenario focuses on how the isomorphism is mapping elements, this belongs to the mapping cluster.

An isomorphism/A homomorphism is a journey. This metaphor gives a sense of traveling from a specific starting point to a specific ending point, which can include a direction or path to travel. For instance, Instructor A stated: “I have my function that goes over to my range, and now this set is sent to a single element over here.” In this example, we have a set that starts in the first group, travels (“goes over” and “sent”), and ends in an element in the other group. Because the journey metaphor is more focused on how elements in the two groups are connected than structural properties of the groups being connected, this was considered to be a mapping metaphor.

*An isomorphism/A homomorphism is a machine***.** This metaphor forms connections to how a machine works (e.g. takes inputs and produces output) or to a machine’s programming (e.g. following a rule). For example, Student 10a observed, “You can just apply the function of both and it should spit out the same thing, so it’s kind of like the distributive property of functions.” This student focused on using the machine (“apply the function”) and the output that results (“spit out the same thing”). While the student’s statement also noted ideas related to sameness, the main point of his statement seemed to be the mechanism by which the function machine changed the inputs to the outputs. This code also focused on how maps work rather than the structure of the groups, leading to its placement in the mapping cluster.

An isomorphism/A homomorphism is a transformation. This code referred to unspecified methods or states of changing, such as Student 18b’s assertion: “I guess I would describe it as a transformation of one object to another object that only keeps a few of the properties of the original object." This metaphor focused on methods of changing rather than similarities between groups, indicating membership in the mapping cluster.

Codes spanning sameness and mapping. Some codes highlighted the sameness of groups by finding a mapping to connect the groups. These include the blue codes from Figure 5.1. These codes use sameness to find mappings or use mappings to demonstrate sameness, meaning they do not fit neatly in the sameness or mapping categories alone.

An isomorphism/A homomorphism is a matching. This code refers to connecting or corresponding specific elements in two groups or lining up elements in order to create such a correspondence. This is exemplified by Instructor B's statement: "I guess I would say [isomorphism]'s a correspondence that…matches like things with like things." Here we see an emphasis on aligning similar objects with each other. This code is included in the "sameness" cluster because it focuses on alignment between elements in the two groups; if such a correspondence can be found for all relevant elements, then the groups share a structure. This code is also included in the "mapping" cluster because the correspondence itself is a mapping.

An isomorphism/A homomorphism is a renaming/relabeling. This code refers to the possibility of giving new names/labels to elements to show equivalence between groups. For example, Instructor B indicated: "If you just took these elements and attached these other labels instead of the labels you originally had and you get the same exact structure [then you have an isomorphism]." Whereas the matching metaphor aligns elements and seems to assign some importance to the names of elements in alignment, the renaming/relabeling metaphor abstracts further to recognize the names/labels being given to elements are somewhat arbitrary. This code highlights sameness because the goal of the relabeling is to show the sameness of structure. However, the relabeling itself is a mapping, indicating membership in the mapping cluster.

A homomorphism is a mapping defined by equivalence classes. This metaphor involves leveraging knowledge of the structure of groups to find similar elements in the domain that can be mapped to the same place in the range. This includes describing homomorphism in terms of which sets in the domain are collapsed to single representatives. This could be a mechanism for finding sets to map via the Fundamental Homomorphism Theorem (FHT), though just mentioning the FHT is not sufficient to be coded in this way. This code was often used to determine which specific elements should be mapped to specified elements, such as here by Student 11b: "So my idea for the homomorphism is just taking it from Z mod 6 to Z mod 3 sort of directly so I mean like, theta of 4 mod 6 would equal 1 mod 6 because they'd both be 1 mod 3…" Thus, he chose to map both 1 and 4 from Z_6 to 1 in Z_3 because 1 and 4 would both be 1 mod 3, indicating they could be treated in a similar way. This metaphor highlights sameness because the shared structure of the groups is being leveraged to find equivalence classes. However, the homomorphism is also the mechanism for translating between groups, making it a mapping.

Codes related to the formal definition. Metaphors in this cluster focus on symbolic manipulation based on text of the formal definition. These are the yellow-orange codes from Figure 5.1. Because the formal definition is a way of reasoning about isomorphisms and homomorphisms, this is considered to be a conceptual metaphor. These codes focus on direct application of the formal definition and/or phrases used as a euphemism for the formal definition that are not unpacked to reveal another meaning besides the formal definition. I will defend inclusion of metaphors that are not literally the formal definition.

An isomorphism is a bijective mapping that satisfies $\Theta(a+b) = \Theta(a) * \Theta(b)$./A homomorphism is a mapping that satisfies $\Theta(a+b) = \Theta(a) * \Theta(b)$ [string of symbols]. This code indicates use of the string of symbols to talk about isomorphism or homomorphism or use of clarifying notation within the string of symbols. Additionally, use of words related to bijective, onto, or 1-1 (or a mapping lacking those properties) to talk about isomorphism (with or without the string of symbols) was placed in this code. An instance of this code comes from Student 10a: "A homomorphism between groups is simply one where the combination of two elements of the function of the combination of the two elements is the same as the function of one element combined with the function of the second element." He simply stated the string of symbols in word form. While the formal definition might not seem like a metaphor in the literary sense, it fits the definition of conceptual metaphor by providing a string of symbols (and bijectivity for isomorphism) that can frame reasoning. Specifically, individuals doing abstract algebra have had extensive exposure to algebraic notation and have ways of interpreting and reasoning about that notation which can be used for reasoning about the specific concepts of isomorphism and homomorphism.

An isomorphism/A homomorphism is operation-preserving. This code highlighted the use of the term "operation-preserving" or "respect the group operation" (or a tense variation) without interpretation. Alternatively, the use of a specific operation while talking about preserving (e.g. preserving addition) or preserving the homomorphism property is included in this code. This could also be highlighted as an "order of operating" focus (e.g. operating in the domain and then mapping versus mapping and then operating in the range). Consider Student 15a's statement: "I would describe [homomorphism] as another function that…preserves the operation of two groups with respect to their individual groups." Because "operation-preservation" is a short-hand way to refer to the string of symbols, this code is considered a variation of the formal definition.

*An isomorphism/A homomorphism is structure-preserving***.** This metaphor used the term structure-preserving or a slight variation without interpretation including to describe the homomorphism property. For instance, Student 18b noted: "…that, that's the structure right there that's being preserved: things still will be nice and well-defined and play nicely I guess." Because the structure being referred to in this code was generally the homomorphism property itself, this was considered a variation of the formal definition much as Hausberger (2017) described "structure-preserving".

An isomorphism is a special homomorphism./A homomorphism is an isomorphism without bijectivity. This metaphor was used by participants to draw on aspects of isomorphism to talk about homomorphism or vice versa. For example, Student 10a said, "I mean isomorphism is a fancy case of homomorphism…." This comparison to the other mapping is a metaphor because reasoning is structured around the individual's understanding of the other concept and what aspects can be shared between them. This metaphor highlighted what was shared or not shared between the formal definitions of isomorphism and homomorphism, leading to its inclusion in the formal definition cluster.

Novel metaphors. At times, participants intentionally drew an analogy between their thinking and an alternate context, such as relating isomorphism to naming stuffed animals or to a mirror. These contexts were not coded with the metaphor codes above, but will be analyzed separately below with attention to how they connect to the other metaphors.

Frequency of conceptual metaphors. The metaphors defined above were used to varying degrees across the interviews and in class. I will discuss individuals' frequency of use of these metaphors and the contexts in which they were commonly used below. Here, in Table 5.1, the number of times each metaphor was used in the isomorphism and homomorphism context across all interviews is listed. Rows with a bold heading give the sum of the frequencies of codes within the cluster. Times during instruction are excluded from this tally because sometimes the instructor would say the same thing to different groups separately. Also, instructional time was selectively transcribed.

Table 5.1. Frequencies of Codes Across All Interviews.

Theme	Isomorphism Frequency	Homomorphism Frequency
Sameness	112	45
Generic Sameness	39	10
Same/Similar Properties	70	28
Disembedding	3	7
Mapping	74	177
Generic Mapping	40	106
Journey	14	46
Transformation	2	4
Invertible	11	0
Function	2	14
Machine	5	7
Sameness with Mapping	39	68
Renaming/Relabeling	15	5
Matching	24	22
Equivalence Classes	0	41
Formal Definition	68	137
Literal Formal Definition	44	67
Operation-preserving	12	25
Structure-preserving	4	1
Compare to other mapping	8	44
Total	293	427

Code discussion. Some of the metaphors found in this project were inspired by previous literature. Specifically, the Zandieh et al. (2016) work on transformations as functions highlighted input/output, traveling, morphing, mapping, and machine metaphors. The metaphors as defined here combine input/output and machine into what is called “machine” here. Traveling is renamed as “journey,” morphing is renamed as transformation, and mapping is much as it was. In previous work, I noted a matching metaphor (Rupnow, 2017) which is used here as well. Hausberger (2017) also highlights structure-preservation as central to the ideas of isomorphism and homomorphism, and is incorporated here.

Class A

In this section, I will introduce the metaphors used by Instructor A in her interview. Then I will describe interviews used in class and the settings in which they were invoked. Next, I will compare metaphors in these contexts, to answer research question three in the context of Case A. Subsequently, I address the four interviewed students’ metaphors from their problem-solving interview. Finally, I examine similarities and differences between metaphors in class and students’ metaphors, to address research question four in the context of Case A.

Instructor’s metaphors. In this section, I use data from Instructor A’s second interview. This took place in Week 7, just after starting the unit on isomorphism in class. Instructor A was not asked to solve specific problems during interviews. Rather, she was asked about how she would describe and define isomorphism and homomorphism in different ways, as well as if she thought about these topics differently than how she presented them in class. I examine isomorphism and homomorphism metaphors separately.

Isomorphism. When discussing isomorphism, Instructor A often said this meant the groups were the same. Her initial description of isomorphism was as follows: “When I think about groups, if they’re isomorphic, it means that they are the same group just notated with different names or notated with a different operation, but that the groups are essentially the same.” She went on to link this general idea of sameness to the specific idea of finding a renaming function to demonstrate this sameness: “I would verify that two things are the same by finding a renaming function, an isomorphism between the two groups. So I kind of feel like that is the test…” She later emphasized the homomorphism property through her discussion of operation-preservation: “…the operation also needs to be the same, so that’s the operation-preservation part of it. So the operation in one group and the operation in another has to be preserved under the renaming function.” While she also noted the formal definition of isomorphism at times, the recurrent emphasis of her descriptions of isomorphism related back to sameness as shown through the renaming function that would preserve how the elements within the groups related to each other under the given group operation. Frequencies of metaphors are listed in Table 5.2.

Table 5.2. Code Frequencies for Isomorphism in Interview for Instructor A.

Code	Frequency
Generic Sameness	10
Renaming/Relabeling	7
Operation-preserving	3
Literal Formal Definition	2
Generic Mapping	1
Total	23

When discussing whether she thought about isomorphism in the same way as she described it to students, she felt her ideas were very similar. Specifically, her goal was to help them see the formal definition as a renaming function in order to see the sameness of the groups:

> I want them to get to the formal definition, but even then I want them to understand the formal definition as like a renaming function. I feel like that was not at all obvious to me as a student. And so it was really hard to unpack…why this…seemingly arbitrary function would prove that two things were the same.

In addition to standard descriptions given to describe isomorphism, Instructor A also related isomorphism to naming stuffed animals when asked how she would explain isomorphism to a child:

> So I might say, like, their collection of stuffed animals, each of their stuffed animals has a name, but it would be the same bear even if I called it a different name….It's the same bear if I call them Fred or Sam.

Comparing this intentional novel metaphor to her standard language for isomorphism, we see that she chose to highlight the sameness aspect once again, which is in keeping with her most common choice of metaphor above. Specifically, she highlighted the arbitrariness of the name of the bear in keeping with the renaming metaphor.

Homomorphism. When discussing homomorphisms, Instructor A frequently spoke in terms of equivalence classes, specifically with a goal of thinking about what elements were behaving in a similar way. She often linked general thoughts of mapping and sameness through this image. A table of code frequencies for homomorphism is below in Table 5.3.

Table 5.3. Code Frequencies for Homomorphism in Interview for Instructor A.

Code	Frequency
Equivalence classes	16
Generic Sameness	5
Generic Mapping	4
Operation-preserving	3
Similar properties	2
Literal Formal Definition	2
Journey	1
Matching	1
Renaming/Relabeling	1
Total	35

When she initially described homomorphism, she focused on equivalence classes as a way to identify which elements from the domain group were the same in some way:

> That one's harder….I think equivalence classes, like the idea that I could pick one representative for a set, and the homomorphism kind of gives me a way to think about what's equivalent. So either like, apply the same name to a group of things that are equivalent or collapsing a set into a single element, those are the two ways that I think of homomorphism.

When asked to repeat what she said, she rephrased:

> So I think I said like a homomorphism is a rule for how I would determine that the same set is the same. So 2 is the same as 4 is the same as 6 under this rule, or it's a way of taking that set, {2,4,6}, and collapsing it down into a single representative like 2. So it's kind of the rule that tells me what's the same, and I can either think of it as like, populating a set where they're all the same or collapsing a set down into a representative. But either way, the rule tells me what is the same.

Thus, for homomorphism, much like isomorphism, she viewed sameness as central to understanding. However, in this case the sameness focused within groups, to see what elements acted in the same way instead of globally identifying the two groups as being the same in some way. When directly asked about how she viewed sameness with isomorphisms and homomorphisms she clarified:

> No, I don't mean the same type of sameness. So with isomorphism I mean this collection and the operation is the same as this collection and operation and the only thing different about them is the names that I chose for the elements in operation. What I mean same in homomorphism, I mean…I'm saying what things in my domain are the same under that

mapping. So it's kind of like I take my domain, call everything the same, and then that collection of same things maps to a single element in the range. And then there's an isomorphism between those, between my collapsed domain into these representatives and those representatives to the range.

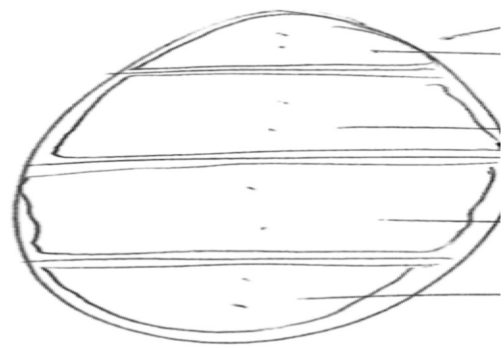

Figure 5.2. Easter egg diagram.

She went on to draw a picture of an "Easter egg" in which the bands of the domain "egg" were connected to a single element in the range (see Figure 5.2 below). She verbally noted that there might be some other elements of the range that "were not hit" but that the matching set and element were then defined to be the same.

When asked to define a homomorphism, Instructor A's answer changed to "a mapping that preserves operation," though she also noted she did not think the formal definition clearly "capture[d] this idea of collapsing or sameness or reduction. I feel like it's all kind of lost in operation preservation, so I don't feel like that's a great, meaningful definition for what that function does." Thus, she seemed to utilize "operation preservation" language to talk about homomorphism because she felt it was a standard way to talk about homomorphism rather than from a sense that it was a useful way to think about homomorphism.

When asked to describe a homomorphism to a ten-year-old, she struggled to find an adequate picture. She finally decided the collapsing or class of same name idea could be captured with stuffed animals again, though as a novel metaphor this was not officially coded with the equivalence class metaphor:

> So going back to stuffed animals, like I could sort things into bears and dogs or whatever, but the idea that it partitions the set, or that these are equal sizes, like there it quickly falls apart with that collection of stuffed animals. But I could get at…I can group these things, that they function the same.

She also thought about the possibility of relating evens and odds to children in terms of sorting the integers and how addition of evens and odds works. However, she was not sure what other accessible example for children could be related to this idea. Considering these intentional analogies, we again see that she focused on creating equivalence classes, much like a lot of the time she spent discussing homomorphisms in general.

When asked if she thought about homomorphism in the same way she tried to relate it to students, she felt she relied more on the formal definition than with isomorphism. Although she often shared the Easter egg picture, she believed she focused more class time on the formal definition. She also noted that though the homomorphism property was presented when discussing isomorphism, she only discussed homomorphisms as their own entity after discussing quotient groups, so she did not think the way that she talked about homomorphisms really changed over time during the semester.

Summary. Instructor A used sameness and equivalence ideas when describing both isomorphism and homomorphism. However, the sameness she focused on was global in the isomorphism context and narrow in the homomorphism context. In addition to general sameness language, she seized on specific images she felt illustrated that sameness for isomorphism and homomorphism. For isomorphism, the renaming function was a way of showing the general sameness of the two groups. For homomorphism, gathering elements into equivalence classes according to which elements were similar in order to match a class of elements to another element used sameness in two different ways.

Metaphors in instruction. In this section, I will record the classes of metaphors used by the instructor (and, if made public, by students). This is intended to characterize metaphors in Instruction, the linchpin of my research questions, in order to eventually examine relationships with the instructor interview (research question three) and students' interviews (research question four). Data for this section come from recordings from Instructor A's course in Weeks 7-12. Moments when a metaphor was made publicly were recorded.

Because I was interested in metaphors being communicated in class on a large scale, only discussion that seemed to be audible for the whole class was included, not discussions between the instructor and students at tables, even if it was audible on the recording, unless the same idea was discussed with all groups. Also, because students could refer to notes that they took in class at later dates, and the scale of use over multiple class periods could be very different from a targeted interview, I am not recording the number of times a metaphor was used in a class. For metaphors used in class, contexts in which they were used and their relative importance are considered.

In the four class periods within Weeks 7-12 focused mostly or entirely on developing ideas related to isomorphism and homomorphism, many metaphors were utilized in some context. However, shifts in the importance of metaphors from the sameness cluster, mapping cluster, and sameness/mapping cluster are evident between isomorphism and homomorphism.

Isomorphism. I begin with looking at the ways metaphors from the different clusters were used in class in an isomorphism context. The presence or absence of a metaphor for isomorphism is summarized in Table 5.4.

Table 5.4. Codes Used for Isomorphism in Instructor A's Class.

Metaphors	Present
Generic Sameness	X
Same properties	X
Disembedding	
Generic Mapping	X
Invertible	
Function	X
Journey	X
Machine	X
Transformation	
Literal Formal Definition	X
Operation-preserving	X
Structure-preserving	X
Special Homomorphism	X
Renaming/Relabeling	X
Matching	X

Sameness was generally used in a general approach context, such as when defining isomorphism and when thinking about how to approach whether or not groups were isomorphic (e.g. "Always start these with like, are they the same?"). Same properties language was especially used when verifying groups were not isomorphic (e.g. different orders or cyclic versus non-cyclic groups). Disembedding was not used.

Mapping language was used in generic contexts, such as creating a formulaic or discrete mapping representation for an isomorphism. Extended time was spent talking about the well-defined and everywhere defined properties of functions. Journey language often referred to where elements or whole groups were "going to" or being "sent to", as well as which elements of the range were being "hit." Machine language was often used to refer to what was being input in functions, especially when focusing on understanding what well-defined and everywhere defined meant. For example, almost an entire class period was spent discussing differences between well-defined, everywhere defined, one-to-one, and onto with set diagrams in order to clarify what was needed for a function before discussing these terms in the context of isomorphism. The invertible and transformation metaphors were not used.

In terms of the sameness/mapping metaphors, renaming language was only used when first presenting the formal definition as a colloquial way to understand the idea. Matching was used many times before the formal definition was given to reason about whether groups in Cayley tables were the "same" in some way, but was rarely used after the formal definition was given.

The formal definition was used mostly when discussing why isomorphisms should require being one-to-one and onto as well as the form the homomorphism property should take, but was not used much outside of proofs. Operation-preserving language was used a few times to summarize what happened in the homomorphism property. Structure-preserving language was used twice by the same student to note a "same structure" being shared; the instructor noted "structural differences" would indicate groups were not isomorphic. Otherwise this metaphor was not observed. Isomorphisms were referred to as special homomorphisms twice after the definition of homomorphism (as its own entity) was given.

In summary, sameness was used as a lens for approaching isomorphism, both in defining the concept and in thinking through how to approach verifying if groups were (or especially were not) isomorphic. Mapping language was ubiquitous, but was only central to discussions about the nature of isomorphisms being functions. Sameness/mapping language was used when initially defining for renaming and throughout tasks for matching, but was used less after the formal definition was given. The formal definition was largely used in proof contexts.

Homomorphism. For homomorphism, a greater variety of mapping-related metaphors were used whereas there was less variety in the sameness metaphors than were used for isomorphism. I now examine the metaphors used for homomorphism according to cluster. Presence and absence of homomorphism metaphors are summarized in Table 5.5.

Table 5.5. Codes Used for Homomorphism in Instructor A's Class.

Metaphors	Present
Generic Sameness	
Similar Properties	X
Disembedding	
Generic Mapping	X
Function	X
Journey	X
Machine	X
Transformation	
Literal Formal Definition	X
Operation-preserving	X
Structure-preserving	
Isomorphism without Bijectivity	X
Renaming/Relabeling	
Matching	
Equivalence Classes	X

Similar properties was largely used indirectly as properties of homomorphisms were being derived. Generic sameness and disembedding metaphors were not used.

Mapping was used in the context of general relations between groups. The well-defined and everywhere defined properties of functions were revisited again as back to back questions would consider first if a mapping was a function and then if it satisfied the homomorphism property. Journey language was used frequently, especially when saying where the kernel was "sent to" and what image was being "hit." The input-output language of machines was used a few times in generic contexts. Transformation metaphors were not used.

Equivalence classes were largely highlighted on the last day of the unit, the class period before an exam, in the context of the Fundamental Homomorphism Theorem (FHT). While discussing this theorem, pictures were drawn on the board illustrating bands that would be mapped to specific places according to the maps in the FHT. Renaming and matching metaphors were not used.

The formal definition was used regularly, especially when verifying whether a mapping was a homomorphism. Operation-preserving language was used once, but was pre-typed specifically for notes. Isomorphism without bijectivity was used a number of times, often to emphasize that one could no longer assume the mapping was one-to-one and onto. Structure-preserving metaphors were not used.

Summary. The formal definition was the dominant metaphor used in class for content purposes. Mapping metaphors were also used prominently, though more when expressing ideas common to any function than properties specific to homomorphism. The equivalence class metaphors were only introduced during a review day on the day before an exam, though they featured prominently on that day when the FHT was also discussed.

Overall, Instructor A used sameness metaphors to communicate her main ideas about isomorphism. She also used matching metaphors in the tasks before presenting the formal definition, and generic mapping language was used throughout the unit. Emphasis was placed on addressing essential qualities of functions, though this knowledge was not specific to isomorphism. For homomorphism, the formal definition was more prominent, though mapping metaphors were used frequently in passing. Sameness and sameness/mapping metaphors were also used, but many of these references happened the day before an exam.

Comparing interview and classroom metaphors. In this section, I will tie together results from the previous two sections in order to address research question three for Case A, which focuses on the relationship between the instructor's conceptual understanding and how she presented material in instruction. Many parallels between metaphors in the interview and instruction are clear, especially for isomorphism. However, many differences between types of metaphors in the contexts are apparent.

All metaphors that were used in the interview to discuss isomorphism were also used at least once in class, as shown in Table 5.6. However, many other metaphors were also used in class. Some, such as function and journey language were even used regularly. However, the variety of topics addressed during class likely explains the inclusion of other types of metaphors. In the interview, Instructor A was only asked about describing isomorphisms and thinking about how she would teach them, not how she would think through solving problems or proving relevant theorems.

Table 5.6. Codes Used for Isomorphism in Instructor A Contexts.

Metaphors	Presence in Class	Frequency in Interview
Generic Sameness	X	10
Same Properties	X	
Disembedding		
Generic Mapping	X	1
Invertible		
Function	X	
Journey	X	
Machine	X	
Transformation		
Literal Formal Definition	X	2
Operation-preserving	X	3
Structure-preserving	X	
Special Homomorphism	X	
Renaming/Relabeling	X	7
Matching	X	

Addressing the research question for Instructor A, there was fairly clear alignment at the conceptual level in addressing isomorphism. The instructor focused on sameness and the sameness/mapping metaphor of renaming in the interview as she described the core of what an isomorphism is. In class, she again used sameness and sameness/mapping metaphors to build the idea of what an isomorphism is, though the sameness/mapping metaphor that came across more clearly in class through the tasks was matching, not renaming.

The greater frequency of mapping metaphors in class than in the interviews has a few potential explanations. One reason is that in the interview, she was asked to define and describe isomorphism, not to find isomorphisms between groups or prove theorems. Thus, there was more time spent on the big picture of what isomorphism is about instead of working with specific isomorphisms (functions) that fit required criteria like was done in different examples and proofs in class. An additional explanation is that the instructor likely wanted to make sure the students understood the definition of function and could distinguish between the requirements for that definition (well-defined and everywhere defined) and the definitions of one-to-one and onto that are also relevant to the definition of isomorphism. This was especially salient in the class period where the class used set diagrams to demonstrate what each definition meant. In the interviews, I think she assumed I knew what one-to-one, onto, and function meant so that discussion was unnecessary. Thus, the main differences between mapping metaphors in the interview and in class can be explained by differences in the tasks being accomplished in the two settings as well as the audiences; it seems less rooted in differences in Instructor A’s thinking.

The formal definition was present in both the interview and the class. However, it was not a focal point in either context. While time was spent in class developing the informal ideas around sameness into the formal definition, the way students were encouraged to think about isomorphism was still rooted in sameness. In the interview, the definition was also mentioned in passing, but more time was spent thinking about what that meant, largely in terms of sameness.

For homomorphism, there was more variation between the language used in the interview and in class, as shown in Table 5.7. Sameness was used to discuss homomorphism a number of times in the interview, but was not used in class. Renaming and matching were also used (sparingly) in the interview but were not observed in class. However, the times renaming and matching were used in the interview were in the context of forming an isomorphism between equivalence classes in order to address a homomorphism as structured by the Fundamental Homomorphism Theorem (FHT). Additionally, equivalence classes were used the most of any metaphor in the interview, but were not used much until the final day of the unit. This may be because the instructor viewed the FHT as important for her understanding of homomorphism, but because of its complexity could not discuss it until the end of the unit, after quotient groups had been introduced. Thus, there was not much class time left in the unit when the topic was introduced, leading to limited opportunities to use equivalence class metaphors.

Table 5.7. Codes Used for Homomorphism in Instructor A Contexts.

Metaphors	Presence in Class	Frequency in Interview
Generic Sameness		5
Similar Properties	X	2
Disembedding		
Generic Mapping	X	4
Function	X	
Journey	X	1
Machine	X	
Transformation		
Literal Formal Definition	X	2
Operation-preserving	X	3
Structure-preserving		
Isomorphism without Bijectivity	X	
Renaming/Relabeling		1
Matching		1
Equivalence Classes	X	16

In terms of the research question in the homomorphism context, the sameness understanding of homomorphism that was clearly articulated in the interview was not nearly as salient in instruction. Equivalence classes as a way of understanding homomorphism were brought up on the last day of the unit rather than being integrated throughout the unit. Thus, the instructor's sameness approach to homomorphism was obscured in the class. Instead, the formal definition and uses of mapping language that were present but not emphasized in the interview became the focus of most of the time spent on homomorphism in the class. As noted above, this disconnect could stem from the number of concepts that need to be coordinated to think about equivalence classes, including an understanding of quotient groups. The instructor may have felt that culminating with this equivalence class-focused idea of homomorphism would show its importance without needing to repeat it on other occasions.

Student metaphors. First, I will examine each student's descriptions and definitions of isomorphism and homomorphism as well as their approaches to each problem separately. For each student, I will highlight the most common metaphors used in the interviews and compare with the initial descriptions given by the students. Then I will compare across students in Class A. The material from this section will allow me examine relationships between the metaphors in instruction and the metaphors students used in Class A, which relates to research question four.

Data for this section are taken from the second interview with my four interviewed students from Class A. These interviews were conducted in Week 12 or 13, depending on the student. All students were interviewed after taking their class' exam on group isomorphism and homomorphism content. This interview involved asking students to define and describe isomorphism and homomorphism. Students were also asked to find isomorphisms and homomorphisms between groups. I used thematic analysis to define and examine the metaphors used in the students' interviews.

The conceptual picture given to students for isomorphism and, to an extent, homomorphism, focused on sameness. Mapping metaphors were ubiquitous in discussions for both contexts in class, while proofs were the main place the formal definition was used. This class characterization is meant to give context for comparing and contrasting with the metaphors students used and the ways they used them.

Student 8a. When discussing definitions and descriptions of isomorphism, Student 8a used same, mapping, matching, renaming, and the formal definition. When initially describing isomorphism she said, "I guess the isomorphism itself is almost like a mapping or a function [gesturing] between two things that act similarly. They may not be the same but you can sort of match it up that way." Later she added, "Here's two groups and they may not be exactly the same, but the parts act the same way. I'm trying to think of two specific groups. But I guess when we were doing all the charts with different letters, you know, the symmetries may be different, like whatever thing you're talking about, but you can sort of rename the things as something else and it'll still act the same way." When pressed for a more formal definition, she noted an isomorphism is "a function that maps one group to another and has to be onto and one-to-one." When switching to describing an isomorphism to a 10-year old, she used more matching language: "I would say, kind of like I was saying before, that there's a way to match up two different sets of things. It could be numbers or just other objects, a way to kind of overlap [gestures] that they're the same I guess."

When discussing definitions and descriptions of homomorphism, she used isomorphism without bijectivity, mapping, the formal definition, and operation-preserving metaphors. She initially described a homomorphism by contrasting with isomorphism, but immediately jumped to how she would explain a homomorphism to a child:

> So all I know is it's also a mapping but it doesn't have to be one-to-one and onto. But as far as, I'm thinking ahead, if I had to explain it to a child, I don't know how I would explain that differently because I guess the difference is the one-to-one and onto part which I don't think the average person knows, so I don't know how I would, in just like normal language, make a difference between the two.

Later she expanded on how she might distinguish a homomorphism from an isomorphism as she used operation-preservation language:

> Things have been shaky lately, the difference between isomorphism and homomorphism. But I guess it's, that's almost like saying they should be the same no matter what order you do it in. Like whether you do the operation then do the mapping or if you map and then do the operation I guess [gesturing] is what needs to stay the same.

In the problem-solving portion of the interview, Student 8a addressed 9 pairs of groups. She was successful in all 6 of the isomorphism problems in which she used same property language. She also observed that all isomorphisms are homomorphisms.

When addressing Z_5 to 5Z, she used matching, same properties, and sameness metaphors to reach a correct conclusion that the groups (sets) of different sizes could not be matched evenly. For homomorphisms, she used isomorphism without bijectivity, mapping, journey, and function metaphors to reach a mostly correct conclusion. She started by seeing if it was possible to make categories in order to map, realized it might be easier to go the other direction, but then noticed there were too many elements in 5Z to be able to map to all elements. At that point she recalled homomorphisms did not have to "do everything" (be onto) so flipping the axes of the function diagram she created should be fine. When she was asked to verify her mapping, she started checking elements with the definition individually and saw a problem. Based on this, she figured there probably is not a homomorphism because you would run out of elements.

For examining Z_5 to Z/5Z, she struggled to gain traction because she could not figure out what Z/5Z looked like. Thus, she used no language to discuss isomorphism and just mentioned mapping for homomorphism without reaching much of a conclusion.

For examining Z_5 to Z_6, she used the same properties metaphor to reach a correct conclusion, specifically the groups have different numbers of elements so there could not be an isomorphism. For homomorphism, she used mapping and matching metaphors to test the "identity" homomorphism (e.g. 0 to 0, 1 to 1, 2 to 2, etc.) and to try mapping everything to 0 and 1. She concluded neither of those two specific maps worked and from this concluded no homomorphisms were likely.

To examine Z_3 to Z_6, she used the same properties metaphor in the form of a different number of elements to again conclude there was no isomorphism. For homomorphism, she used mapping metaphors as she created the identity map (0-0, 1-1, 2-2) which she discovered did not work. Then she figured she did not know any way to make it work unless maybe she would find a workable map at random.

To examine Z_6 to Z_3, she said there was no isomorphism again for the same reason as the previous groups. For homomorphism, mapping, journey, function, formal definition, and similar properties metaphors were used. Initially, she thought she could not repeat where elements go, but then realized this is a requirement for being one-to-one, not for functions broadly. Then she created a mapping based on modular arithmetic, but decided it did not work. Next, she realized she can map everything to 0, and it will satisfy the formal definition. When asked for another map, she came up with the natural mapping; at first her computations led her astray as she lost track of which mod she was working in, but then the three combinations she checked worked. She figured the mapping worked at this point, but did not know any way to check other than looking at all of the combinations. When asked if she was alright with a homomorphism existing one way but not the other, she said it seemed reasonable that one way worked but not the other. She said nothing about the trivial homomorphism.

To address Z to 2Z, she used same properties, the formal definition, mapping, matching, and journey metaphors as she reached a standard, correct mapping of the form $f(x) = 2x$. She specifically noted that both groups were infinite so sizes of the groups were the same. When prompted to look for other mappings, she thought she could scramble her mapping, such as mapping 0 to 2 and reordering a few other elements. However, when she tested this map with specific computations related to the formal definition, she saw this caused problems and concluded her original map was the only isomorphism. For homomorphism, she used isomorphism without bijectivity, journey, formal definition, matching, similar properties, and mapping metaphors as she reached some correct conclusions. This included realizing an isomorphism is a homomorphism. She also mapped to subgroups of 2Z such as multiples of 4 (4Z). She also noted that the trivial homomorphism works.

To look at the possibility of isomorphisms from 2Z to Z, she used mapping, invertible, and journey metaphors as she reached some correct conclusions. Using formulaic notation, she realized that $y=1/2\ x$ should work as a mapping. At first, she thought the identity mapping was also an isomorphism but then realized this was incorrect. She also believed there should be isomorphisms both ways (Z to 2Z and 2Z to Z) because it's like an inverse function going back to where things came from. For homomorphism, she used journey and mapping metaphors to conclude that the identity mapping would work here. She also thought multiples of 3 or any multiples (positive integers) should work, though only multiples that are subgroups of 2Z (multiples of 2) would work.

As she addressed the Cayley tables, she used same properties to realize that one group had all self-inverses and the other did not, meaning they could not be isomorphic groups. For homomorphism, she used journey, mapping, and formal definition metaphors as she tried to gain traction on the prompt. However, she struggled to disambiguate symbols having the same name (e.g. a in group + and a in group * potentially being different elements). Most of her maps were also onto even though she clearly stated there was no isomorphism.

To address Z_8 to D_8 (dihedral group of 8 elements based on Class A notation), she used same properties to recognize Z_8 has two self-inverses whereas D_8 has five self-inverses, indicating they cannot be isomorphic. For homomorphism, she used mapping language as she described the problem, but never attempted constructing a map. She noted she did not know where to start other than just guessing some maps and that homomorphism was not her thing.

After completing these problems, we reviewed her previous work. I wanted to see if she would point out the trivial homomorphism as a mapping between any other groups. However, she did not add or change any maps in this review. After I finished recording and we discussed "correct answers," she mentioned the trivial homomorphism was always a homomorphism between groups.

In summary, Student 8a seemed to have a robust understanding of isomorphism and used a variety of metaphors to discuss isomorphism. For homomorphism, she used much more generic mapping and journey language and leaned on the formal definition more heavily. Frequencies of isomorphism and homomorphism metaphors are in Tables 5.8 and 5.9, respectively. The differences in her comfort with isomorphism and homomorphism were also evident from her initial descriptions of concepts and her ability (or limited ability) to condense her understanding to something understandable to a child. She used definitions but also matching-focused language to describe isomorphism, whereas she directly stated that she struggled to distinguish homomorphism from isomorphism and focused on describing the formal definition. This pattern held in her problem-solving, where she was generally successful in addressing isomorphism (as long as she understood what the group looked like), but frequently struggled to extend past the formal definition when looking for homomorphisms.

Table 5.8. Code Frequencies for Isomorphism for Student 8a.

Metaphor	Frequency
Same Properties	7
Generic Mapping	7
Matching	5
Same	5
Literal Formal Definition	4
Invertible	2
Journey	2
Renaming/Relabeling	1
Total	33

Table 5.9. Code Frequencies for Homomorphism for Student 8a.

Metaphor	Frequency
Generic Mapping	18
Journey	8
Literal Formal Definition	8
Isomorphism without Bijectivity	7
Function	3
Similar Properties	2
Matching	2
Operation-preserving	1
Total	49

In terms of connecting to instruction, she used sameness metaphors that had been used a number of times for isomorphism in class (same, same properties, and matching). Her language for homomorphism also connected to what was used frequently in class, mapping and formal definition metaphors, even though these were not the clusters Instructor A found central in her interview description of homomorphism.

Student 10a. When discussing definitions and describing isomorphism he highlighted one-to-one and onto properties and related isomorphism to homomorphism: “An isomorphism between two groups is a homomorphism that is also one-to-one and onto, or a homomorphism that is bijective.” When I asked what a homomorphism was, he used journey and formal definition language:

> A homomorphism is a mapping from a group to another group [gestured] where the property the function evaluated at a plus b is the same as, is equal to, the function evaluated at a plus the function evaluated at b, or times the function evaluated at b, depending on what the operation is for group A and group B.

When asked for a description apart from the formal definition, he used a journey metaphor again: “I just think combining these two elements here should give you, should make it, like, that you can separate it in the other place. That’s one way I think about it.” He later elaborated on how he pictured function evaluation as he used machine and formal definition language:

> When you do the function evaluate out the combination, then you can still separate the combination into two different parts. Kind of like a reverse distributive property, maybe….Like the distributive property’s just if you have a constant on the outside and then you have like a plus b times, or a plus b in parentheses times a constant, then you have ca plus cb. That, well, it’s the same thing, but like kind of, it’s very messy because that’s not really how functions work, but that’s kind of how I see it. Like, you have a plus

b inside. You can just apply the function of both and it should spit out the same thing, so it's kind of like the distributive property of functions.

When I asked whether this description referred more to isomorphism or homomorphism, he again related isomorphism to homomorphism: "I mean, isomorphism is a fancy case of homomorphism, so both."

When I asked how he would describe isomorphism to a ten-year-old, he created a novel analogy focused on different colored marbles. The analogy focused on being one-to-one and onto as he looked to match marbles, but seemed to neglect the homomorphism property:

> You have blue marbles and red marbles, and say you put your blue marbles in a circle and you number them, like, 1 through 10, and then you put your red marbles in a circle, and you number them also 1 through 10, but instead you just kind of rotate it so you don't have 1's matching up with the 1's, and you just kind of like, you say, well this 1 matches up with this 3, so the 1 in the red matches up with the 3 in the blue, and then you keep going and then you figure out if you have 1+3, that'll get you to marble 4. Well, marble 4 matches to marble 6 or whatever, so something like that.

He did later note that you could rotate the marbles further or not at all so that marble 1 was mapped to marble 1.

When asked to describe a homomorphism to a ten-year-old, he initially gave an analogy he later concluded was better suited to isomorphism. This analogy related to courses he had taken when he started in the mechanical engineering major before switching to math:

> The idea of kind of like how work is independent of path, so it's not something a ten-year-old would understand, but I would say when you're going up a mountain there could be multiple trails, right? Like you could have trail A that takes you on the left side and

> then like steep ascent or trail B that takes you on the right side and then…shallower descent or ascent, and like, regardless, so they're, they're the same, you get to the same place.

He later brought over his thermos as a prop to further explain this example.

> I guess for isomorphism I was gonna say the hiking example where, like, it doesn't matter if you're hiking up a mountain and then you can decide to go this path. At some point, like a time, whatever, 60 minutes, you could be here or you could be here. It doesn't really matter, but you're at the same elevation give or take. It depends on the trail but like, the idea is, like, regardless of how you go, it's the same ending spot, so what you're doing is actually the same operation, this just looks different. So that's kind of like what an isomorphism is: you're doing the same thing, it just looks different.

He returned to the marble analogy for a homomorphism example that seemed to focus just on mapping all of the marbles (being a function), not on the homomorphism property:

> I guess you kind of do the same thing, but instead of having the same number of marbles you can have less, and you just feel like now you're allowed to overlap, so if you want to map this marble to that one, or if you want to like say this marble is associated with that you, then you can. So I guess just reuse the marble example.

In the problem-solving portion of the interview, he used the formal definition, same properties, same, special homomorphism, matching, journey, and mapping metaphors. He used the formal definition, similar properties, matching, equivalence classes, function, machine, journey, mapping, and isomorphism without bijectivity when discussing homomorphism. He addressed 8 problems in total. This included correct conclusions about isomorphism in the four examples he applied same properties to, and recognizing that isomorphisms are homomorphisms.

He also noted the trivial homomorphism on the first two examples and for Cayley tables, but not explicitly elsewhere.

To address mapping from Z_5 to 5Z, for isomorphism he used the same properties metaphor to note that the groups were of different sizes (finite and infinite) so they could not be isomorphic. For homomorphism, he used mapping, the formal definition, and function metaphors to reach a mostly correct conclusion. He found the trivial homomorphism, and through computations realized that other maps he proposed were not homomorphisms. He then assumed no other homomorphisms would exist, though he was not certain.

For Z_5 to Z/5Z, he used same, mapping, matching, formal definition, and same properties metaphors to conclude there would be an isomorphism between them. He noticed that the cosets of Z/5Z and the elements of Z_5 would match and act the same way so he was satisfied there would be an isomorphism between them. However, he did not realize rearrangements of the isomorphism were possible and specifically said the isomorphism was unique. For homomorphism, he used isomorphism without bijectivity, mapping, and matching metaphors to reach largely correct conclusions. He knew the isomorphism he found was a homomorphism and the trivial homomorphism should be present. He also realized no other options were possible for subgroups (besides the identity or the whole group).

For Z_5 to Z_6, he used the same properties metaphor in the form of different size groups to conclude there were no isomorphisms. For homomorphism, he used journey, formal definition, mapping, and matching metaphors as he reached some reasonable conclusions. He started with a focus on orders instead of sizes of subgroups, then shifted to focusing on generators though this did not lead to a usable mapping. When I proposed the identity (i.e. f(x) = x) mapping, he recognized immediately that the homomorphism property would not be satisfied.

For Z_3 to Z_6, he just said there would be no isomorphism, and he was not pressed to explain further (because he had already noted different size groups could not be isomorphic multiple times previously). For homomorphism, he used matching and mapping metaphors but did not find any homomorphisms. Like the previous scenario, he focused on generators and the mapping he used did not work. He realized this mapping did not work, but he did not seek any other mappings after that and moved on.

For Z_6 to Z_3, he never explicitly addressed isomorphism, though this seemed to be an implicit no to the existence of isomorphisms. For homomorphism, he used mapping and journey metaphors as he created two homomorphisms: (1) the identity mapping of 0 to 0, 1 to 1, and 2 to 2 and (2) mapping 0 to 0, 1 to 2, and 2 to 1. When asked if he thought it was acceptable for there to be homomorphisms from Z_6 to Z_3 but not Z_3 to Z_6, he did not consider this to be a problem.

For Z to 2Z, he used the formal definition to verify the identity homomorphism was one-to-one and onto and that the homomorphism property was satisfied. He did not state any other isomorphisms, but he was not prompted to do so. For homomorphism, he used isomorphism without bijectivity, mapping, and similar properties as he found multiple homomorphisms. This included noting that the isomorphism was a homomorphism. He also stated that any mapping of the form f(a) = 2ka would work as a homomorphism but not an isomorphism (because the map would not be onto but the subsets would be fine), though no specific mappings were verified or pressed for.

For 2Z to Z, mapping and formal definition metaphors were used as he found the standard f(x) = ½ x mapping. He demonstrated this mapping was one-to-one and onto and satisfied the homomorphism property. He was not pressed to find other isomorphisms. When asked if it was to be expected that there were isomorphisms in both directions, he said he had not thought about this previously, and that this "does not keep [him] up at night." For homomorphism, he used mapping and similar properties metaphors as he found other mappings by changing the ½ constant to other values. He was not pressed to verify these mappings were homomorphisms.

He spent the remainder of the interview thinking about the two groups presented as Cayley tables. For isomorphism, he used the formal definition and the same properties to correctly conclude the groups were not isomorphic because they had elements of different orders. For homomorphism, he used mapping, similar properties, journey, and matching language as he tried to reconcile his order argument for isomorphism with the mapping he thought might be a homomorphism. Specifically, his proposed homomorphism mapped the identity to the identity and all other elements to one other element (of order 2), based on the assumption that elements had to be mapped to elements of the same order. However, he also noted the trivial homomorphism should exist and did not find this to contradict his order argument.

In summary, Student 10a largely utilized same properties and the formal definition for isomorphism. He mostly used mapping and formal definition-related language when discussing homomorphism (see Tables 5.10 and 5.11). When initially asked to describe isomorphism and homomorphism, his descriptions focused on the framing of the formal definition, whereas the description for a ten-year-old focused more on matching. Also of interest, his novel metaphor focused on matching marbles violated the homomorphism property, and when he used matching metaphors to address Z_3 to Z_6 and the Cayley tables, he struggled to make progress. In general, he used a lot of formal definition- and mapping-focused language, though he occasionally made use of sameness, especially in the context of leveraging specific properties. He was largely successful in these endeavors, though his thinking about orders of elements seemed to get in his way at times.

Table 5.10. Code Frequencies for Isomorphism for Student 10a.

Metaphor	Frequency
Same Properties	7
Literal Formal Definition	6
Special Homomorphism	2
Generic Mapping	2
Matching	1
Journey	1
Generic Sameness	1
Total	20

Table 5.11. Code Frequencies for Homomorphism for Student 10a.

Metaphor	Frequency
Generic Mapping	19
Literal Formal Definition	9
Isomorphism without Bijectivity	5
Journey	5
Matching	5
Similar Properties	3
Machine	2
Function	1
Total	49

In terms of connecting to instruction, he used the same properties reasoning that had been modeled a number of times for isomorphism in class, though he relied more on the formal definition than had been emphasized in class. His language for homomorphism also connected to what was used frequently in class, mapping and formal definition metaphors, even though these were not the clusters Instructor A found central in her interview description of homomorphism. Even his most common sameness-related metaphor was matching, which Instructor A had not even used in class for homomorphism.

Student 11a. When initially describing isomorphism, he highlighted mapping and sameness: "Isomorphism is sort of when each of the elements in one group can be mapped to the other, to another element in the other group such that they share the same behaviors." When prompted for a formal definition, he provided a standard definition: "An isomorphism is basically a function that maps one group to another group such that the function is one-to-one and onto and such that the function of the combination of two values in the first group is equal to the function of the first value combined with the function of the second value." Of note, rather than referring to an isomorphism as a mapping, he said this mapping was a function. When asked what he would say to a ten-year-old, he just focused on sameness: "If two groups of numbers or anything are the same."

When asked to describe a homomorphism, he began with the formal definition: "A homomorphism between groups is simply one where the combination of two elements or the function of the combination of two elements is the same as the function of one element combined with the function of the second element." When asked to give a more intuitive description, he compared to isomorphism: "A homomorphism is an isomorphism that doesn't have to be one-to-one and onto." To explain to a ten-year-old, he struggled to create an accessible analogy and noted he had struggled when trying to create one for the survey: "I was trying to think about that, because when you, when you explain that it's two groups don't have to be the same, then it gets really confusing on what is a homomorphism and what isn't a homomorphism." Later he had a partial response that used transforming language: "Maybe it might be sort of just, a function that changes one value to another…value."

In the problem-solving section, he addressed 8 problems. For isomorphism, he used formal definition, same properties, same, invertible, matching and mapping metaphors. For homomorphism, he used formal definition, renaming, transformation, equivalence classes, function, machine, mapping, and isomorphism without bijectivity metaphors. All five problems when he used same properties for isomorphism had correct conclusions. He also noted that isomorphisms are homomorphisms. He noted the trivial homomorphism a number of times as well, including saying it's been mentioned "500 times so we need to stop" so he probably just got sick of mentioning it on the other questions.

For Z_5 to 5Z, he used same properties, formal definition, mapping, and matching metaphors as he concluded they could not be isomorphic groups because they were of different sizes. For homomorphism, he used mapping and the formal definition to recognize the trivial homomorphism should be present but did not believe any others would be because the computations implied a closure problem.

For Z_5 to Z/5Z, he used matching and mapping language while addressing isomorphism. He struggled to manipulate the homomorphism property equation appropriately and concluded that the problem that afflicted the previous problem also affected this scenario. For homomorphism, he used mapping, isomorphism without bijectivity, and formal definition metaphors. He had the same problem with homomorphism as with isomorphism in this scenario. He did not mention the trivial homomorphism with this scenario.

For Z_5 to Z_6, he used the different sizes of groups (same properties) to conclude that the groups were not isomorphic. For homomorphism, he used equivalence classes, mapping, and formal definition language as he reached correct conclusions. He mentioned that homomorphisms should only be possible between a group of order 5 and one of order 1 or 5 (subgroups) just like one of 6 elements could go to a group of order 3 or 1 (6 not mentioned), though it was unclear whether this knowledge was focused on his knowledge of Lagrange's theorem or something else. He also used the homomorphism property to show his proposed map of f(x)=x would not be a homomorphism because there would be closure problems.

For Z_3 to Z_6, he used same properties (different sizes of groups) to conclude no isomorphism was possible. For homomorphism, he used mapping, formal definition, equivalence classes, and function language as he reasoned. He initially said this felt "like it's almost the same exact thing as the Z_5 to Z_6. He initially tried f(x)=x, but quickly realized this did not work. He then elaborated on what he had said about possible sizes of groups before by relating to divisibility, saying Z_3 could only be mapped to something of Z_1 or Z_3, but Z_6 could map to Z_1, Z_3, or Z_6. Specifically, a group could not map to a group of twice as many elements because you would need "to have one element map to two different ones;" this seemed to imply he believed homomorphisms must be onto, though this is not a required property of homomorphisms. He did not mention the trivial homomorphism this time.

For Z_6 to Z_3, he again used same properties (size of groups) to conclude that they were not isomorphic. For homomorphism, he used the formal definition and mapping language as he found a map he incorrectly concluded did not work based on misapplying modular arithmetic. He specifically noted, "I know this is a homomorphism. I just can't figure out what the function is for it to work." He did not mention the trivial homomorphism here.

For Z to 2Z, he used the formal definition and mapping language as he demonstrated that the function f(x) = 2x was an isomorphism. He formally proved that this function was one-to-one and onto and that it satisfied the homomorphism property. He noted that f(x) = -2x should also be an isomorphism. For homomorphism, he used isomorphism without bijectivity and mapping metaphors as he noted that the previously stated isomorphisms would be homomorphisms. He also included the trivial homomorphism and mapping to other multiples of 2 (f(x) = 4x, f(x) =

6x, etc.). Notice here he noted that this was not an onto mapping, but was satisfied that this would be a homomorphism.

For 2Z to Z, he used invertible, formal definition, and mapping metaphors. He started by inverting the previous mapping to f(x) = x/2:

> It's sort of the same thing the other way around. It's almost like the, well it is the inverse of the one we just did before….Like the inverse isomorphism where it goes from 2Z to Z instead of from Z to 2Z, so this brings it back.

He also specifically noted that isomorphism should be expected to be reflexive: "I mean if it's one-to-one and onto and if the f(a+b) is equal to the f(a) plus the f(b), then it should be sort of assumed that you could do it backwards." He also noted f(x) = -x/2 should be an isomorphism. For homomorphism, he used mapping language as he concluded you could do the same thing for homomorphisms (isomorphisms as homomorphisms implied) though you could not just keep dividing by larger values as that could lead to non-integers. He stated the trivial homomorphism, f(x) = x, and f(x) = -x should be homomorphisms as could larger multiples of integers (e.g. f(x) = 2x) though no justification was given.

For the Cayley tables, he used matching and same properties metaphors as he concluded that the number of self-inverses was different in the two groups so they could not be isomorphic groups. For homomorphism, he used mapping, formal definition, renaming, and machine metaphors as he found different homomorphisms. The first homomorphism he found mapped the group isomorphic to Z_2 x Z_2 to the two-element subgroup of the group isomorphic to Z_4. He also realized he could choose the identity with any of the order 2 elements to map to the identity in Z_4 and then the other two order 2 elements from Z_2 x Z_2 would map to the element of order 2 in Z_4.

He was not pushed to verify these mappings were homomorphisms past his initial justification that seemed to (at least informally) pull on how cosets work: operating with the identity and an order 2 element in Z_2 x Z_2 would always produce themselves and the other two elements would produce themselves. He also noted the trivial homomorphism would work here again.

In summary, he often used the formal definition and mapping as he reasoned about both isomorphism and homomorphism (see Tables 5.12 and 5.13). He also used lots of sameness related language to talk about isomorphism. These were also borne out when first describing isomorphism and homomorphism outside the problem-solving context. For homomorphism, he mostly related to isomorphism or used mapping-related language to reason about proposed mappings. The occasions in which he used equivalence class related reasoning met with mixed success. Many of his struggles related to problems with using modular arithmetic.

Table 5.12. Code Frequencies for Isomorphism for Student 11a.

Metaphor	**Frequency**
Literal Formal Definition	7
Generic Mapping	7
Same Properties	6
Matching	3
Invertible	3
Generic Sameness	2
Total	28

Table 5.13. Code Frequencies for Homomorphism for Student 11a.

Metaphor	**Frequency**
Generic Mapping	25
Literal Formal Definition	10
Isomorphism without Bijectivity	4
Equivalence Classes	4
Function	2
Machine	1
Renaming	1
Transformation	1
Total	48

In terms of connecting to instruction, he used same properties, which had been used a number of times for isomorphism in class, though he relied more on the formal definition than had been emphasized in class. His language for homomorphism also connected to what was used frequently in class, mapping and formal definition metaphors, even though these were not the clusters Instructor A found central in her interview description of homomorphism. Unlike the previous two students, he did utilize equivalence classes when solving. The maps he created when using equivalence class language were homomorphisms, and he could have seen this if he had computed correctly with modular arithmetic.

Student 15a. When defining isomorphism, he gave a standard definition focused on functions and then elaborated:

> An isomorphism between two groups is a bijective function that has the homomorphism property. It's a bijective function that preserves the property of two groups from one group to another [gestures] and is invertible and onto and has the homomorphic property.

When asked to give information about the homomorphic property, he described the computations involved using journey and machine metaphors:

> So if a function of group 1, if an isomorphism is a function from group 1 to group 2, a homomorphism is that function from group 1 to group 2 that if you operate two elements in group 1 [gestures], plug that into the homomorphism, then that, the output of those two elements operated on each other will be preserved in, in the output.

When asked to describe isomorphism, his language still focused on the formal definition via operation preservation: "I would say it's a function that preserves, preserves the property of the two groups. I just described it as a function that preserves the property of two groups." When asked how he would describe isomorphism to a ten-year-old, he highlighted sameness: “I would describe it as a function, or I would describe it as relating two groups that are the same.”

When asked to define a homomorphism, he gave a similar response to what he had said about the homomorphic property, but with more of an operation-preserving emphasis instead of a journey emphasis:

> Homomorphism loops, like it’s a…function between two groups that if you operate, the operating in the operation in the domain is preserved in the, or the operation of that you plug it in such that you plug it into the input, same as the output. So when you operate in the input and in the domain, in the co-domain it’ll operate the same. So with respect to their two groups, so the operation will be preserved.

When asked to describe a homomorphism, his language again reflected a machine metaphor:

> Kind of that same way. If you plug something into the, if you operate something in the domain and that homomorphism, or spit out an output such that the inputs and the outputs when operated on each other will be the same.

When asked to describe a homomorphism to a ten-year-old, he returned to the operation-preservation language he had used earlier: “I would describe it as another function that preserves the, preserves the operation of two groups with respect to their individual groups.”

In the problem-solving portion of the interview, he addressed eight scenarios. In six of the isomorphism scenarios, he used the same properties metaphor, and he successfully determined the groups were or were not isomorphic in all of those cases. He also realized that isomorphisms are homomorphisms and noted the existence of the trivial homomorphism on the last example. In general, he used formal definition, operation-preserving, same properties, same, special homomorphism, invertible, matching, function, machine, journey, and mapping metaphors to address isomorphism. He used formal definition, operation-preserving, similar properties, matching, function, machine, journey, mapping, and isomorphism without bijectivity metaphors for homomorphism.

For Z_5 to 5Z, he used same properties to note differing numbers of generators in the two groups, indicating different behavior in the two groups and that there could not be an isomorphism between them. (Though he believed there was only 1 generator, there are two in 5Z. Nevertheless, his broader argument held.) For homomorphism, he used formal definition and similar properties as he made an argument focused on the orders of elements. Specifically, he (incorrectly) believed that the orders of elements needed to be preserved under homomorphism. He used this reasoning to correctly conclude that the identity mapping was not a homomorphism. However, he missed the trivial homomorphism as he over-generalized to saying that no homomorphisms existed between the groups.

For Z_5 to Z/5Z, he used machine, same, operation-preserving, matching, formal definition, and mapping language to create a workable isomorphism and to verify that the natural map was one-to-one, onto, and satisfied the homomorphism property. However, he went on to say you could move the mapping around into any order and as long as it stayed one-to-one and onto it would still be a homomorphism. I then explicitly asked if mapping 0 to 3, 1 to 2, etc. could work and he said yes without checking. When I asked if any other isomorphisms exist, he said Z_5 was finite so there would be a finite number of mappings. While talking about the isomorphisms, his last step in verifying his work was to check the homomorphism property, and he said this meant the groups were homomorphic. When I asked later asked if his work showed he had a homomorphism as well, he said yes, and I did not pursue this prompt further. Thus, he indirectly made use of the idea that an isomorphism is a special homomorphism, but did not explicitly make that statement or speak exclusively about homomorphism.

For Z_5 to Z_6, he used same properties in the form of different size groups to conclude the groups could not be isomorphic. For homomorphism, he used similar properties, matching, and isomorphism without bijectivity as he concluded that the groups could not be homomorphic. Again, he claimed the groups could not be homomorphic because the orders did not match (non-identity elements in Z_5 were of order 5 whereas non-identity elements in Z_6 had orders 2, 3, and 6). He also made the general statement that all orders matching was needed for homomorphism, but did not guarantee homomorphism.

For Z_3 to Z_6, he used same properties, journey, and the formal definition as he concluded same size groups were needed to satisfy one-to-one and onto simultaneously. His argument would have been better served by juxtaposing function and onto, but he still reached the conclusion that the groups needed the same cardinalities for isomorphism. For homomorphism, he used journey, similar properties, matching, operation-preserving, and mapping metaphors as he found the doubling mapping (0 to 0, 1 to 2, 2 to 4). He said orders were preserved under that map. He also noted that swapping what maps to 4 and 2 produced another homomorphism. Initially he claimed you could change what mapped to the identity, but then changed to say that identities needed to be mapped to each other so that same orders would map to same orders.

For Z_6 to Z_3, he used same properties, formal definition, and function metaphors as he concluded that the groups were still not isomorphic for the same reasons as the previous example. For homomorphism, he used matching, isomorphism without bijectivity, and journey metaphors as he concluded there could not be a homomorphism between the groups because there were no elements of order 2 in Z_3. When asked if it was surprising that he found a homomorphism in one direction but not the other with these groups, he leveraged his knowledge of one-to-one functions to make sense of the result:

> No, not surprising now because I kind of guessed it would come up, but it makes sense because for an isomorphism, an isomorphism is a bijective function and, which is one-to-one, and one-to-one functions are invertible, so a homomorphism, it can, it can be isomorphic, but since these sizes are of different sets they can't be one-to-one and onto and so there may not be a function that's invertible such that you can find a homomorphism this way but not that other way.

Note his reasoning about whether homomorphisms should be expected to go both ways is accurate.

For Z to 2Z, he used matching, formal definition, machine, same properties, mapping, function, and operation-preserving metaphors as he addressed isomorphism. He created the mapping f(n) = 2n and showed the mapping was one-to-one and onto and satisfied the homomorphism property. Then he said there should be an infinite number of ways to map so you could scale down to 2Z by adding constants k because the map was a linear function, though this was not accurate. He even verified (for himself) that adding a constant still satisfies the homomorphism property. He also mentioned again that the same number of elements of each order must be present, though this is true for isomorphism. He also made a claim about generators, but did not seem to see how it related to ideas of sameness:

> 15a: There's one generator so for this, for Z, it's the integer 1 that generates all of Z. For this one, it's the integer 2 that generates all of the even integers. So, therefore, every other

element is, what there wouldn't be a generator, but the homomorphism property would still be preserved because each element, each set, has the same order amount in them.

I: Ok, so you mentioned 1 as a generator and 2 as a generator in respective groups, so are you saying that those have to map to each other or just like…

15a: They don't have to map to each other, but they have to exist such that they have to have the same, the same amount of elements of a certain order, the same amount of generators in one have to appear in another for the homomorphism property to exist.

For homomorphism, he used isomorphism without bijectivity, mapping, and formal definition metaphors as he noted that the isomorphisms he found should also be homomorphisms.

For 2Z to Z, he used invertible, machine, matching, formal definition, mapping, and special homomorphism metaphors to address isomorphism. He saw the standard mapping of dividing by 2 and showed this map was one-to-one and onto and satisfied the homomorphism property. He also scaled by 1/2, 2/2, 3/2 and said these will produce isomorphisms. This was an overgeneralization for isomorphisms, though true for homomorphisms. For homomorphism, he additionally used a journey metaphor as he noted that the isomorphisms were homomorphisms because order was preserved. He did not provide any additional homomorphisms.

For the Cayley tables, he used same properties to conclude the groups were not isomorphic, as one group had all self-inverses and the other group did not. For homomorphism, he used operation-preserving, mapping, formal definition, journey, function, machine, and isomorphism without bijectivity metaphors. He again claimed that orders should be preserved so it should be possible to find a homomorphism. However, his initial maps were onto mappings in which order 2 elements were mapped to order 2 elements and the identity was mapped to the identity. He eventually stated that everything could be mapped to the identity (trivial homomorphism), but did not seem to think this contradicted his previous statements about order-preservation.

In summary, Student 15a used the formal definition and same properties frequently with isomorphism (see Tables 5.14 and 5.15), often successfully. He also used operation-preserving language extensively in both contexts, and he seemed to have invented the property that satisfying the homomorphism property requires orders to match. This idea of preservation was threaded through his initial descriptions and definitions as well, so this is not surprising.

Table 5.14. Code Frequencies for Isomorphism for Student 15a.

Metaphors	**Frequency**
Literal Formal Definition	11
Same Properties	9
Operation-preserving	5
Matching	5
Machine	4
Generic Sameness	3
Journey	3
Generic Mapping	3
Function	2
Invertible	1
Special Homomorphism	1
Total	47

Table 5.15. Code Frequencies for Homomorphism for Student 15a.

Metaphors	**Frequency**
Operation-preserving	8
Matching	8
Generic Mapping	5
Similar Properties	5
Journey	5
Machine	4
Isomorphism without Bijectivity	4
Literal Formal Definition	3
Function	2
Total	44

In terms of connecting to instruction, he used same properties, which had been used a number of times for isomorphism in class, though he relied more on the formal definition than had been emphasized in class. His language for homomorphism relied more on operation-preservation and matching than had been used in class.

Summary of students in Class A. Students in Class A frequently used same properties and the formal definition to reason about isomorphism, though mapping, matching and general sameness were also used reasonably often, as shown in Table 5.16. Students also used most of these metaphors in similar ways. Same properties was largely used to invoke a property that was not shared by the groups (such as number of elements) to demonstrate the groups were not isomorphic, whereas generic sameness was used more often in the describing/defining part of the interview. The formal definition was generally used to verify a mapping that had been created. Mapping and journey metaphors were used to describe what happened to specific elements in mappings. Matching was often used in connection to a Cayley table, whether the problem that was explicitly given in Cayley table form (Students 8a and 11a) or by making a table to reason about for themselves (Students 10a, 11a, and 15a). Invertibility was used in the context of the 2Z to Z problem after they had done the Z to 2Z problem (Students 8a, 11a, and 15a). In terms of differences, Student 15a was the only one to use machine or operation-preservation language. Only Students 10a and 15a reasoned from homomorphism as the fundamental mapping to conclude something about isomorphism.

Table 5.16. Code Frequencies across Class A Students for Isomorphism.

Metaphors	**Frequency**
Same Properties	29
Literal Formal Definition	27
Generic Mapping	19
Matching	14
Generic Sameness	11
Journey	6

Invertible	6
Operation-preserving	5
Machine	4
Special Homomorphism	3
Function	2
Renaming	1
Structure-preserving	0
Transformation	0
Disembedding	0
Total	127

For homomorphism, mapping was by far the most commonly used metaphor, though the formal definition, isomorphism without bijectivity, journey, matching, and similar properties were also used reasonably often, as shown in Table 5.17. Of note, general sameness language was not used by any of the students to talk about homomorphism, despite the word root homo- meaning "same."

Table 5.17. Code Frequencies across Class A Students for Homomorphism.

Metaphors	**Frequency**
Generic Mapping	67
Literal Formal Definition	30
Isomorphism without Bijectivity	20
Journey	18
Matching	15
Similar Properties	10
Operation-preserving	9
Function	8
Machine	7
Equivalence Classes	4
Renaming	1
Transformation	1
Disembedding	0
Generic Sameness	0
Structure-preserving	0
Total	190

Comparing students to each other, all four students used mapping and other mapping cluster language throughout their problem solving, largely while describing how specific elements were being treated under mappings. The formal definition was used to verify the mappings students generated. In some cases, especially for Student 8a, testing with the formal definition seemed to be the first thing to check. Other students would use divisibility rules to see what mappings might be possible (Student 11a), or use orders of elements as a property or to assist in creating a matching to, not always correctly, assert whether or not homomorphisms were possible (Students 10a and 15a).

Comparing isomorphism and homomorphism, same properties was the go-to way to think about whether or not groups were isomorphic, and this strategy was generally successful. The exception was for the 2Z to Z problem, where 3 of the 4 students used invertibility language instead, which built off previous work from Z to 2Z. In general, students were fairly successful in dealing with the isomorphism problems they were given. The only problem arose with the question that involved quotient groups, and the problem was related to understanding the groups more than understanding what isomorphism is.

For homomorphism, there was more variety in how students chose to approach problems. Similar properties, as invoked in “same order” or matching contexts (Students 10a and 15a) was not a real property so it led to some incorrect conclusions. Plunging blindly into checking maps with the formal definition after trying the identity mapping (Student 8a) did not lead to good results either. Students seemed to have the most success when they made use of real properties like divisibility (Student 11a) or when the most obvious mapping (identity map) was in fact a homomorphism.

Relating instructors and instruction to students in Class A. In this section, I seek to answer research question four in the context of Class A. I will relate Instructor A's metaphors in her interview and especially in instruction to metaphors used by students in Class A. This will allow me to address relationships between conceptual understanding (as understood through metaphors) in instruction and students' metaphors.

Many isomorphism metaphors used by the students are clearly related to what happened in instruction. (Code appearances and frequencies across Class A contexts are given in Table 5.18.) The exception, invertible, is a property of one-to-one functions that students see in high school and linear algebra. It is possible students were able to make this connection for themselves based on prior knowledge or that this information was given in office hours or comments on homework.

Table 5.18. Metaphors Used for Isomorphism in Class A.

Metaphors	Presence in Class	Instructor Interview	Student Frequency
Generic Sameness	X	10	11
Same Properties	X		29
Disembedding			
Mapping	X	1	19
Invertible			6
Function	X		2
Journey	X		6
Machine	X		4
Transformation			
Literal Formal Definition	X	2	27
Operation-preserving	X	3	5
Structure-preserving	X		
Special Homomorphism	X		3
Renaming/Relabeling	X	7	1
Matching	X		14

Given the questions being asked in the two interview contexts, the appearance and frequencies of most isomorphism metaphors were not surprising. Same properties, mapping, formal definition, operation-preserving, and special homomorphism usage are as might be expected given the types of questions asked. Specifically, because Instructor A was asked about her definitions and descriptions of isomorphism, most of her interview language focused on sameness rather than applying sameness via same properties in a specific isomorphism. On the other hand, students mentioned generic sameness at the beginning of the interview and not after that. Similarly, the instructor was not asked to solve problems, so there was no need for her to discuss same properties, but students were asked to solve problems so they used this language frequently. This same reasoning could address disparities for mapping and the formal definition.

Renaming and matching form more interesting contrasts. Although the instructor used renaming language a number of times in her interview and raised the idea in class, only one student used this metaphor once. Because all of the times the code was used were in the context of the definition itself, it is possible the students did not have enough repetition to incorporate this language into their regular thinking and speaking about isomorphism. Interestingly, Instructor A never used matching language to describe isomorphism in her interview, yet this type of thinking was woven throughout the build-up to the formal definition of isomorphism. It is possible that because the students' initial introduction to isomorphism was through matching, they felt comfortable returning to this approach when problem-solving, especially with the Cayley table problem.

Alternatively, it is possible that Instructor A viewed her renaming language and the matching-focused tasks in class as developing the same idea of renaming. Both metaphors link ideas of sameness and mapping. However, renaming might be viewed as the more abstract version of matching, in which the names of elements are arbitrary, whereas in matching the names of elements can still seem to carry important information. Thus, while the instructor utilized renaming language and, thus, likely recognized the arbitrariness of the names of elements while using them, students might not have viewed the names of elements in the same way.

For homomorphism, some metaphors followed expected patterns and others did not. As might be expected, students used similar properties, mapping, journey, formal definition, and operation-preserving more times than Instructor A, who also used these metaphors in class, as shown in Table 5.19. Given that there are four students being counted and that the problems the students were asked included a number of computational questions, these are not surprising results. Additionally, the use of function, machine, and isomorphism without bijectivity language by the instructor in class only and their use by students is reasonable given the types of questions being asked, as explained above.

Table 5.19. Codes Used for Homomorphism in Class A.

Metaphors	Presence in Class	Instructor Interview	Student Frequency
Generic Sameness		5	
Similar Properties	X	2	10
Disembedding			
Generic Mapping	X	4	67
Function	X		8
Journey	X	1	18
Machine	X		7
Transformation			1
Literal Formal Definition	X	2	30
Operation-preserving	X	3	9
Structure-preserving			
Isomorphism without Bijectivity	X		20
Renaming/Relabeling		1	1
Matching		1	15
Equivalence Classes	X	16	4

Unexpected results include a student using transformation language to talk about homomorphism, though this language was not used by the instructor in her interview or in class.

Perhaps this can be attributed to the student's prior knowledge of functions and language that is common when discussing functions broadly. Students also used renaming and matching language like the instructor did in her interview, though this language was not used in class. This could be due to isomorphism and homomorphism ideas transferring for the students or some information that was not captured from small group work or office hours. Notice, though the instructor spoke of sameness in the homomorphism context in her interview, this was not communicated in class. The students also did not use this language, even though the prefix homo- does mean "same." The instructor used equivalence classes language more often than the students did together. This may be because the students did not fully digest the material in the short time between being introduced to that way of thinking and their exam.

Class B

In this section, I will introduce the metaphors used by Instructor B in his second interview. Then I will describe interviews used in class and the settings in which they were invoked. Next, I will compare metaphors in these contexts, to answer research question three in the context of Case B. Subsequently, I address the four interviewed students' metaphors from their problem-solving interview. Finally, I examine similarities and differences between metaphors in class and students' metaphors, to address research question four in the context of Case B.

Instructor's metaphors. In this section, I use data from Instructor B's second interview. This took place in Week 7, just after starting the unit on isomorphism in class. Instructor B was not asked to solve specific problems during interviews. Rather, he was asked about how he would describe and define isomorphism and homomorphism in different ways, as well as if he thought about these topics differently than how he presented them in class. I examine isomorphism and homomorphism metaphors separately.

Isomorphism. Though it was not initially obvious, Instructor B favored isomorphism language focused on general sameness as well as relabeling. When first asked about the words or phrases that came to mind to describe isomorphism, he provided three images: "structure-preserving map, equivalence of structures, relabeling of elements." When defining isomorphism, Instructor B focused on structure-preservation without further elaboration: "I'd define [isomorphism] as a mapping between two algebraic structures that preserves the structure." He incorporated matching metaphors as he answered his next question of how to describe isomorphism to a ten-year-old: "It's a correspondence that matches like things with like things." When pressed for his preferred way of thinking about isomorphism, he wove together relabeling, sameness, formal definition, and operation-preserving images:

> I really prefer to think of it as a relabeling so that, you know, you're really from an algebraic point of view, there's really no difference between these structures, and so…if you just took these elements and attached these other labels instead of the labels you originally had and you get the same exact structure. So that's the idea I try to get across more than…that you have a…bijective function that…preserves such and such operation. So I think it's really the relabeling is the most natural way to think of it.

Based on this, he seemed to view relabeling as a specific way to show structures like groups were the same. Furthermore, he seemed to view the sameness and relabeling ideas as more valuable for students than the formal definition alone. He incorporated disembedding as well as general mapping and sameness language as he expanded on what relabeling meant to him.

> The isomorphism itself can just sort of disappear to the background and you can…really just identify…these structures, right? And you could start talking about *the* cyclic group with n elements, right? And you don't need to know…what cyclic group, you don't need to, to know that there could possibly be two different cyclic groups hanging around, and, you know, a function mapping elements to another. You can just say, "Well, if…I had a different instance of a cyclic group with n elements, if I wanted to, I could just change those labels to these labels, and so really it's the same underlying structure."

From this statement, we see that his concept of isomorphism was about a shared structure, allowing us to see the groups were *isomorphic*, as opposed to a focus on the function that connected them (*isomorphism*). Overall, we see the breakdown of code frequencies for isomorphism in Table 5.20.

Table 5.20. Code Frequencies for Isomorphism in Interview for Instructor B.

Code	Frequency
Renaming/Relabeling	6
Generic Sameness	6
Generic Mapping	3
Structure-preserving	2
Disembedding	1
Matching	1
Operation-preserving	1
Literal Formal Definition	1
Total	21

Homomorphism. When discussing homomorphisms, Instructor B initially only mentioned operation-preserving maps. When asked what word came to mind he initially said, "Operation-preserving map. I guess that's all I have." When roughly defining, he drew on similar language: "A map from one structure to another structure of the same type that preserves whatever operations around the structure." (In this context it appears his use of the word structure is meant to stand in for a group, ring, or other algebraic structure.) When asked to explain what was being preserved in homomorphism, he contrasted his view of homomorphism with his view of isomorphism as a relabeling:

> Since you…lose the bijectiveness, you sort of lose this, this other way of thinking about it as just…being able to take an element here and then just attach the label that you were using over here instead of the, the original label.

However, after contrasting with isomorphism, he created an image focused on collapsing by mapping that fit the equivalence relation type:

> I guess you could sort of view it as threads condensing into a single…element in the codomain and…then those would become equivalence classes modulo the kernel of, of the map, etc. etc. But then I think we're sort of into…a few more subtleties that aren't really present in the case of isomorphism.

In this description, we see multiple elements from the domain being mapped to a single element in the codomain. Each set of elements being mapped to the same place forms an equivalence class and these sets all have the same size as the kernel, which is the equivalence class mapped to the identity. When asked how he would explain homomorphism to a ten-year-old, he could not come up with anything specific, besides saying he would try to look for an analogy that would make sense to a ten-year-old.

He noted his way of discussing homomorphism with his students shifted over the course of the semester. Initially he would focus on homomorphisms as isomorphisms without bijectivity. However, later in the semester when the first isomorphism theorem (another name for the FHT) would be discussed, more details related to equivalence classes emerge:

> When you look at the seven elements that get mapped to a particular element, then what we really have is this, this equivalence class modulo the kernel, and then we can…if we

mod out by the kernel, then we can take any one of those things as a…representative. So I think…by the point where we actually get to proving… the first isomorphism theorem, then I'm sort of describing to them what I'm thinking about when I think about a homomorphism. But initially it's just sort of, "Well, what if we take an isomorphism and then just drop this…condition of bijectivity?" Kind of don't really initially see how the… structure within the…domain group is reflected in the…co-domain whereas with isomorphism we…see that right away. Right, we just see that it's…equivalence of structures.

From this, we observe Instructor B made it a point to introduce the term homomorphism soon after introducing isomorphism, but because students did not have an understanding of quotient groups to draw upon, he would initially motivate homomorphism just by showing it was distinct from isomorphism. However, this initial distinction that he shared with students was not his structural view of homomorphism. Once students learned about quotient groups and could understand the FHT, the structure he saw in homomorphism would become accessible and a point of his instruction. Specifically, the structure similarity could be revealed through equivalence relations. This was also why he viewed homomorphism as more complex than isomorphism: structural similarities could be seen easily in isomorphism, whereas more information needed to be coordinated to see the structural similarity for homomorphism.

Table 5.21. Code Frequencies for Homomorphism in Interview for Instructor B.

Code	Frequency
Isomorphism without Bijectivity	8
Equivalence Classes	2
Operation-preserving	2
Disembedding	1
Generic Mapping	1
Total	14

Overall for homomorphism, we see Instructor B compared homomorphism to isomorphism a lot when describing and defining homomorphism, as shown in Table 5.21. However, he also leveraged equivalence classes and disembedding when he sought to give more structure to his images of homomorphism. These language patterns also reflect the way he intended to teach the material: start with comparing and contrasting homomorphism with isomorphism, then highlight how a homomorphism reflects similarity of structure through quotient groups.

Summary. Instructor B had a clear focus of isomorphism as revealing algebraic structures to be the same. In fact, he viewed the main point of isomorphism as revealing when algebraic structures were essentially the same rather than the fact that there was a way to move between such structures via a mapping (isomorphism). He felt isomorphic structures were clearly revealed through the idea of relabeling. For homomorphism, he did not feel that shared structure was obvious; rather the initial ways he spoke of homomorphism related to the formal definition, including ways to compare back to isomorphism. However, as the interview progressed, he elaborated on the hidden structure he saw in homomorphism through equivalence classes. This progression mirrored his intended method of teaching homomorphism in contrast to isomorphism and later emphasizing the structure that could be observed if one used quotient groups.

Metaphors in instruction. In this section, I will record the classes of metaphors used by the instructor (and, if made public, by students). This is intended to characterize metaphors in Instruction, the linchpin of my research questions, in order to examine relationships with the instructor interview (research question three) and students' interviews (research question four).

Data for this section come from recordings from Instructor B's course in Weeks 7-10. Moments when a metaphor was used publicly were recorded.

Because I was interested in metaphors being communicated in class on a large scale, only discussion that seemed to be audible for the whole class was included, not discussions between the instructor and students at tables, even if it was audible on the recording, unless the same idea was discussed with all groups. Also, because students could refer to notes that they took in class at later dates, and the scale of use over multiple class periods could be very different from a targeted interview, I am not recording the number of times a metaphor was used in a class. For metaphors used in class, contexts in which they were used and their relative importance are considered.

Instructor B used a variety of conceptual metaphors across the six class periods focused mostly on isomorphism and homomorphism in Weeks 7-10. However, when describing the concepts, he focused more on sameness in the isomorphism context, whereas he used more combined sameness/mapping metaphors and the formal definition to address homomorphism.

Isomorphism. I begin with looking at the ways metaphors from the different clusters were used in class in an isomorphism context. The presence or absence of a metaphor for isomorphism is summarized in Table 5.22.

Table 5.22. Codes Used for Isomorphism in Instructor B's Class.

Metaphors	Present
Generic Sameness	X
Same Properties	X
Disembedding	
Generic Mapping	X
Invertible	
Function	X
Journey	X
Machine	
Transformation	
Literal Formal Definition	X

Operation-preserving	X
Structure-preserving	X
Special Homomorphism	X
Renaming/Relabeling	X
Matching	X

Sameness was used to refer to groups being “essentially the same” on a number of occasions, especially when the definition was initially given. Same properties were used especially during the lab day when students were being prompted to look for properties that should hold in both groups if they were isomorphic. A list of properties that should hold for isomorphic groups was presented and proved in another class period and used in subsequent proofs and examples. Disembedding language was not used.

Mapping was used when looking for formulaic representations of isomorphisms or when seeing what to “map to” specific elements. Being well-defined, a property of functions, was referenced once in passing when considering a specific mapping, but was not a major focus in class for isomorphism. Journey metaphors that referenced going from one group to another or elements being sent were used in multiple periods. Invertible, machine, and transformation metaphors were not used.

Relabeling was used numerous times to refer to the isomorphism (mapping) being used to show groups were isomorphic. This language was especially used on the lab day (as a prompt or part of a question at every table) and to introduce the lab. Matching was used a few times as students were prompted to look at which element got “matched up” with each element when creating a discrete mapping.

The formal definition was used to verify maps were one-to-one, onto, and had the homomorphism property. After the definition of homomorphism was given, one example of homomorphism that was given was “any isomorphism,” indicating isomorphisms are special homomorphisms. Operation-preservation was used a bit more broadly to help students think about their goals in the labs or to summarize what they were trying to accomplish in an exercise as a class (e.g. we need to "show the bijection respects the group operation" before verifying the homomorphism property). Structure-preserving language was noted once as a "relabeling that preserves the structure."

In summary, sameness metaphors were largely used when speaking in general terms or to show groups were not isomorphic. Mapping metaphors were used often, but were not a content focus. Sameness/mapping metaphors, especially relabeling, were used often on the initial lab day, but less later. Formal definition metaphors were used most in proof contexts, though operation-preservation was used on other occasions as well and seemed to be a proxy for the homomorphism property.

Homomorphism. I begin with looking at the ways metaphors from the different clusters were used in class in a homomorphism context. The presence or absence of a metaphor for homomorphism is summarized in Table 5.23.

Table 5.23. Codes Used for Homomorphism in Instructor B's Class.

Metaphors	Present
Generic Sameness	
Similar Properties	X
Disembedding	X
Generic Mapping	X
Function	X
Journey	X
Machine	
Transformation	
Literal Formal Definition	X
Operation-preserving	X
Structure-preserving	X
Isomorphism without Bijectivity	X
Renaming/Relabeling	
Matching	
Equivalence Classes	X

Similar properties were used indirectly as different properties of homomorphisms were derived and the Fundamental Homomorphism Theorem (FHT) was used to find all possible homomorphisms. Disembedding was used in the context of the FHT as a way to think about how groups were related:

> If you've got a surjective homomorphism, then the range H essentially is already living inside of G somehow. All the information about H is already here, and in fact we can recover H purely in terms of G by taking the factor group of G mod the kernel.

Another way of saying this is if we have an onto homomorphism, then the structure of the range exists in the domain somehow and can be extracted through quotient groups. This metaphor was also used in practice as a template for finding all possible homomorphisms between groups. Generic sameness was not used.

Mapping was used on numerous occasions to refer to the homomorphism or, for example, to note what the kernel would be mapped to. Being well-defined, a property of functions, was a quality of specific homomorphisms that was checked regularly. Journey metaphors that referenced going from one group to another or elements being sent were used in multiple periods. Machine and transformation metaphors were not used.

Equivalence classes were modeled through specific examples that used congruence as a potential homomorphism mapping, though the instructor never explained why these precise mappings were chosen to show homomorphism. For example, in class, the instructor showed a homomorphism existed from Z_6 to Z_3 by using congruence classes as the mapping. However, the rationale for choosing this function was not stated. After the FHT was given, equivalence classes were given as the "big picture" takeaway: "the range of the homomorphism has to be a factor group of the original group." Relabeling and matching metaphors were not used.

The formal definition was used numerous times during proofs and to verify examples. Operation-preserving language was used a few times as a stand-in for checking the formal definition (e.g. needing to show determinants respect the group operation, and then using the string of symbols from the definition to check this). Structure-preserving language was used to preview what would be observed through the FHT (e.g. "it's related to the structure of H somehow"). Isomorphism without bijectivity was used when first introducing homomorphism and in the short review of material at the beginning of the subsequent class period.

Summary. Instructor B used mapping metaphors the most frequently in class, though many usages were in passing and could have been used with any function, not just a homomorphism. He also used formal definition language often, especially in the context of proofs. Some sameness metaphors were used, though generic sameness was not. Though used less frequently, the sameness/mapping metaphor of equivalence classes was worked into a number of examples and structured his second teaching of homomorphism (after quotient groups had been taught).

Overall, Instructor B used sameness and sameness/mapping metaphors to make his main concept points for isomorphism and homomorphism even though he did not directly invoke sameness when talking about homomorphism. Mapping language was ubiquitous in both contexts, though it seemed focused on showing that isomorphisms and homomorphisms were examples of functions or was used in the same way one would talk about any function. The formal definition was used often in proof contexts, though operation-preservation language was also used at the beginning of tasks or examples to highlight the goal of the task.

Comparing interview and classroom metaphors. For Instructor B, all of the isomorphism metaphors used in the interview were also used in class except for disembedding, as shown in Table 5.24. However, the example he gave in the interview that used disembedding was also utilized in class, but with the disembedding context removed. Specifically, in his interview he observed you could talk about "the cyclic group of n elements and you don't need to know…what cyclic group…it's the same underlying structures." In class, he often used the conclusion that there is one cyclic group of each order without the disembedding context. In terms of the other direction, he did not use same properties, function, journey, or special homomorphism in his interview, though he used them in class. This is likely because he was not working with any specific groups or examples in the interview and most places he used those metaphors in class were in the context of working through examples, not giving general interpretations of the meaning of isomorphism.

Table 5.24. Codes Used for Isomorphism in Instructor B Contexts.

Metaphors	Presence in Class	Frequency in Interview
Generic Sameness	X	6
Same Properties	X	
Disembedding		1
Mapping	X	3
Invertible		
Function	X	
Journey	X	
Machine		
Transformation		
Literal Formal Definition	X	1
Operation-preserving	X	1
Structure-preserving	X	2
Special Homomorphism	X	
Renaming/Relabeling	X	6
Matching	X	1
Equivalence Classes		

In terms of the research question, for isomorphism there was fairly clear alignment at the conceptual level. The instructor focused on sameness and the sameness/mapping metaphor of relabeling in the interview as he described the core of what an isomorphism is. In class, he again used sameness and sameness/mapping metaphors to build the idea of what an isomorphism is, especially on the lab day as he talked about relabeling as a way to approach the problem at each of the tables.

The greater frequency and variety of mapping metaphors in class than in the interviews is likely because of the difference in the types of tasks involved. In the interview, he was asked to define and describe isomorphism, not to find isomorphisms between groups or prove theorems. Thus, there was more time spent on the big picture of what isomorphism is about instead of working with specific isomorphisms (functions) that fit required criteria like was done in different examples and proofs in class. Much like Instructor A, the difference between metaphors in the interview and in class can be explained by differences in the tasks being accomplished in the two settings more easily than differences in Instructor B's thinking.

The formal definition was present in both the interview and the class. However, it was not a focal point in either context. While time was spent in class developing the informal ideas around sameness into the formal definition, the way students were encouraged to think about isomorphism was still rooted in sameness. In the interview, the definition was mentioned in passing, but more time was spent thinking about what that meant, largely in terms of sameness. Even the structure-preservation language that was used in the interview seemed to be related to his views of sameness which were elaborated upon in the interviews and in class.

For homomorphism, he did not use any metaphor in the interview only (see Table 5.25). He did use similar properties, function, journey, formal definition, and structure-preserving in class but not his interview. In class, he used structure-preserving to foreshadow the FHT, but in the interview he assumed I knew about the FHT and immediately talked about the FHT, which may be the reason structure-preserving was not used in the interview. For the other metaphors, they were largely used in a problem-solving setting, which may be why they were not used in the interview, where I did not ask him to solve particular problems or to construct homomorphisms between groups.

Table 5.25. Codes Used for Homomorphism in Instructor B Contexts.

Metaphors	Presence in Class	Frequency in Interview
Generic Sameness		
Similar Properties	X	
Disembedding	X	1
Mapping	X	1
Function	X	
Journey	X	
Machine		
Transformation		
Literal Formal Definition	X	
Operation-preserving	X	2
Structure-preserving	X	
Isomorphism without Bijectivity	X	8
Renaming/Relabeling		
Matching		
Equivalence Classes	X	2

In terms of the research question, Instructor B's responses in the interview were expanded upon in class. In particular, he focused on the formal definition, especially at the beginning of the interview, as he struggled to come up with a picture for homomorphism. Thus, he initially compared to isomorphism or used operation-preserving language. However, he eventually articulated an image of homomorphism from disembedding and equivalence classes that was developed more in class when discussing how to find the structure of the range that was "already living inside" the domain that could be found through quotient groups. This mirrored his approach to instruction in which students used the formal definition and worked with examples to compare and contrast with isomorphism. Then after quotient groups, the FHT was presented, allowing the more complicated and more structured view of homomorphism to emerge. He also used more mapping language in the classroom setting, but like the isomorphism context, this seems to be a product of different tasks in class than in the interview.

Student metaphors. First, I will examine each student's descriptions and definitions of isomorphism and homomorphism as well as their approaches to each problem separately. For each student, I will highlight the most common metaphors used in the interviews and compare with the initial descriptions given by the students. Then I will compare across students in Class B. The material from this section will allow me examine relationships between the metaphors in instruction and the metaphors students used in Class B, which relates to research question four.

Data for this section are taken from the second interview with my four interviewed students from Class B. These interviews were conducted in Week 12 or 13, depending on the student. All students were interviewed after taking their class' exam on isomorphism and homomorphism content. This interview involved asking students to define and describe isomorphism and homomorphism. Students were also asked to find isomorphisms and homomorphisms between groups. I used thematic analysis to define and examine the metaphors used in the students' interviews.

Recall in Class B, Instructor B used sameness and sameness/mapping metaphors to make his main concept points for isomorphism and homomorphism even though he did not directly invoke sameness when talking about homomorphism. Mapping language was ubiquitous in both contexts, though it seemed focused on showing that isomorphisms and homomorphisms were examples of functions or was used in the same way one would talk about any function. The formal definition was used often in proof contexts, though operation-preservation language was also used at the beginning of tasks or examples to highlight the goal of the task. This class characterization is meant to give context for comparing and contrasting with the metaphors students used and the ways they used them.

Student 3b. When describing isomorphism initially, he used same, operation-preserving, formal definition, invertible, and same properties as he linked together ideas related to relationships among elements:

> So an isomorphism between groups is a relationship between those two groups in which the group operation from the first, well, ok. In a more broad sense, it is linking the two groups and saying that they're the same but a little bit different, that they're, all the elements in them are, have the same relationships between each other so that you can draw a relationship between the overall two groups, and then a little bit more formal of it is that the group operation has to be preserved when you go through this isomorphism so if you have two elements in the first set and they have a certain relationship, the two equivalent elements in the other group also have that same relationship. And then the other part of it is that it has to be a bijective function, so you can't have more elements in one set or the other one. And each of the elements from the first set that gets mapped to the second set you had, there's a reverse goes back so you don't have unequal number of elements between the two.

When asked what he meant by operation-preserving, he stated the homomorphism property:

> Okay, so when you go, like I guess like the formal side of it is if you have an isomorphism you're gonna call theta, if you have…and let's just say the groups are additively written, if you have like elements in the first group, you know, a and b, a plus b has to be like a plus b and then, well, theta of a plus b and to be equal to theta of a plus theta b so that you're taking this sum of these two elements and applying an isomorphism

> to them is the same as applying the isomorphism to just one or just the other and then adding them after the fact, so that that addition is preserved across the, well, across the isomorphism….Like the respective group operation is what I mean, so when I said like a plus b over here and then theta a plus theta b, I'm kind of assuming that they're both additive groups, but it's just whatever the group operation is on the two.

When asked how he would explain isomorphism to a ten-year-old, he came back to the idea of same relationships:

> I would say that if given two groups of stuff, if you can treat the two groups of stuff the same in how they might be related to each other. It's kind of a tough question to put in the frame of a ten-year-old, so I guess if you have two groups of objects, whatever they may be, you know, however those two groups, the objects within each group interact with each other, what they are doing, there's an equivalent. Like the other group has the same relationships between all of the elements within the groups.

When asked to describe a homomorphism between groups, he made a last note about same properties in isomorphism and isomorphism being a special type of homomorphism before highlighting the sameness still present in homomorphism:

> So I guess to expand a little bit on the isomorphism one, that is that you would have to have the same number of things in each group so that the distinction could be made in the homomorphism group, but that doesn't necessarily have to be the case, that if you have a lot of elements in a…set, more elements in the second group than the first group, you can say that, well, this element is the same as, you know, these multiple elements because they all interact, they all have the same relationships as this one element in the first group.

When I asked if this meant one element was being mapped to multiple elements in the other group, he said this was what he meant (though this is not acceptable for functions). When asked to define homomorphism, he condensed to operation-preservation:

> So the homomorphism, it preserves that same group operation. There's the word I'm looking for, the same group operation, preserving properties of it, of the…isomorphism, that is still continuing into the homomorphism. However, you no longer have to have the bijectivity between the two groups.

When asked how he would describe homomorphism to a ten-year-old, he came back to sameness and relating to isomorphism:

> So going back to the two different groups of just things, you would have to have the same, like, relationship equivalencies between the one and the other group [gestures]. However, they no longer have to have the same number of elements or of things in each group.

In the problem-solving portion of the interview, he addressed nine scenarios. He used the formal definition, operation-preserving, same properties, same, special homomorphism, invertible, machine, journey, and mapping metaphors while reasoning about isomorphism. In 7 of the 8 problems in which he used same properties to talk about isomorphism, he came to correct conclusions. He used the formal definition, operation-preserving, similar properties, same, matching, equivalence classes, journey, mapping, and isomorphism without bijectivity while reasoning about homomorphism. Over time, it became apparent that he had an incorrect definition of homomorphism in mind, as he believed homomorphisms had to be onto. This meant groups that were not isomorphic often received the same conclusion that there were no homomorphisms. He also believed isomorphisms and homomorphisms were mutually exclusive in this part of the interview, so when he found isomorphisms in infinite groups, he thought there were no homomorphisms. However, he was later given the correct definition of homomorphism as a separate exercise at the end of the session. His conclusions on the second pass on these problems will be stated within the appropriate problem scenarios. He never mentioned the trivial homomorphism.

For Z_5 to 5Z, he used same properties to conclude that the groups were of different sizes (5 elements versus infinite) so they could not be isomorphic. For homomorphism, he initially used operation-preserving, matching, and journey metaphors as he concluded that because you cannot continue adding the same element to return to the identity in 5Z but you can in Z_5, there could not be a homomorphism. However, it probably would be possible to construct a homomorphism in the other direction, because then you could create an onto mapping. In the second pass on this problem with the correct definition, he used mapping metaphors as he created the standard mapping (0 to 0, 1 to 5, etc.) and observed this still did not work. He did not consider the trivial homomorphism.

For Z_5 to Z/5Z, he used same properties, journey, special homomorphism, and formal definition metaphors while addressing isomorphism. He initially believed it should be possible to construct an isomorphism because there were two 5-element groups present. However, he could not construct a mapping that satisfied him, partially because he struggled to understand the group operation in Z/5Z and was trying to create an isomorphism that still took "infinite sums" of sets into account. Of note, his struggle with this problem seemed to stem from uncertainty in picturing the groups in a useful way rather than his use of properties. For homomorphism, he used equivalence classes, similar properties, isomorphism without bijectivity, journey, operation-preserving, and formal definition metaphors. Much of the discussion for homomorphism overlapped the discussion for isomorphism. He noted that he wanted to use the Fundamental Homomorphism Theorem because there was a quotient group present, but because Z and 5Z are infinite, he could not divide indices and he abandoned this line of reasoning.

For Z_5 to Z_6, he used same properties to conclude that the groups were not isomorphic because they had different numbers of elements and a different maximum order of element. For homomorphism, he used similar properties and operation-preserving as he concluded a homomorphism was impossible. Specifically, he believed because the groups had different maximum orders, an “operation-preserving element” was not possible.

For Z_3 to Z_6, he used same properties (different numbers of elements) to conclude that no isomorphism could be formed. For homomorphism the first time, he used similar properties to say that because only one group had an element of order 6, there could not be a homomorphism. The second time he addressed this problem, with the correct definition, he used mapping, equivalence classes, operation-preserving, and similar properties while finding different mappings. He started by trying the identity mapping (0 to 0, 1 to 1, and 2 to 2) which he realized was not a homomorphism. Then he tried the standard modular mapping (0 to 0, 1 to 2, 2 to 4) which he concluded was a homomorphism. The fact that the range was a subgroup of Z_6 seemed to be a striking concept to him. However, he used this to know that his chosen range had closure and concluded that this was the only homomorphism.

For Z_6 to Z_3, he used same properties and correctly concluded the groups were not isomorphic for the same reason as the previous example. For homomorphism, he used mapping, operation-preserving, journey, equivalence classes, and the formal definition when finding the standard modular map, which is a homomorphism. He then suggested alternate maps could be made by adding 3’s and 6’s because they would mod out, though these result in essentially the same map. He also suggested elements could be rearranged (e.g. map 2 and 5 to the identity), though when pressed to verify this was a homomorphism, he correctly concluded it was not a homomorphism.

For Z to 2Z, he used same properties and mapping metaphors as he realized multiplying by 2 produced an isomorphism. He did not believe any other isomorphisms existed. For homomorphism, he used mapping metaphors and at this point seemed to believe that isomorphism and homomorphism were mutually exclusive so he did not count his isomorphism as a homomorphism. He created the map that multiplies elements by 4 (essentially $f(x) = 4x$), but concluded this could not be a homomorphism because it was not an onto mapping.

For 2Z to Z, he used mapping and same metaphors to make an appropriate isomorphism by dividing by two, but also concluded this was the only isomorphism. When I came back to isomorphism to ask if it should be expected to have isomorphisms in both directions, he said this should be expected because if "equivalence is in this direction, it's also gonna be in the reverse direction." He also went on a tangent about the possibility of losing identities through multiplication in multiplicative groups. For homomorphism, he just said there were none, with the implication that this scenario was like the previous one.

For the Cayley tables, he used same properties to correctly conclude the groups were not isomorphic. Specifically, he noticed that one group's elements always reached the identity when operated with themselves, but the other group had different answers result. He then went on a tangent about how maybe the 4-element group presented was actually a two-element group, but then realized not all the same elements were returned so it was not. For homomorphism, he directly stated there were no homomorphisms based on what he saw for isomorphism.

For Z_8 to D_4 (the dihedral group of 8 elements according to the notation in class), he noticed only one element in Z_8 yields the identity when squared, but he found two elements (without even checking all elements) in D_4 with order 2, leading him to conclude the groups were not isomorphic. For homomorphism, he used similar properties to reach an incorrect conclusion that because they were same size groups they had the same issue as isomorphism had.

In summary, for isomorphism, he leaned on the same properties metaphor most, though he also used other metaphors. For homomorphism, he used operation-preservation, equivalence classes, mapping, and similar properties most (see Tables 5.26 and 5.27). This student's work with homomorphism is difficult to analyze because he was working with an incorrect definition of homomorphism for much of the interview. Nevertheless, we see that his focus on operation-preservation with isomorphism and especially homomorphism was reflected in his later work on specific problems.

Table 5.26. Code Frequencies for Isomorphism for Student 3b.

Metaphors	**Frequency**
Same Properties	11
Literal Formal Definition	4
Generic Sameness	4
Operation-preserving	2
Special Homomorphism	2
Journey	2
Generic Mapping	2
Machine	1
Invertible	1
Total	29

Table 5.27. Code Frequencies for Homomorphism for Student 3b.

Metaphors	**Frequency**
Operation-preserving	7
Equivalence Classes	6
Generic Mapping	6
Similar Properties	5

Journey	5
Isomorphism without Bijectivity	3
Literal Formal Definition	3
Generic Sameness	2
Matching	1
Total	38

In terms of connecting to instruction, he used sameness metaphors that had been used a number of times for isomorphism in class (same and same properties). For both isomorphism and homomorphism, he used a number of mapping cluster metaphors. This largely mirrored what was done in class, though he also used other metaphors, such as machine and invertible. This may have come from his general knowledge of functions. His metaphors for homomorphism focused on operation-preserving and, when given the real definition of homomorphism, equivalence classes. The heavy use of equivalence classes mirrors what was done near the end of the unit in class, though his reliance on operation-preservation from the formal definition was more salient than what was done in class.

Student 11b. When defining an isomorphism between groups, Student 11b initially used same and transformation language to call it "a transition between two identical things that one can sort of identify." When asked for a more formal definition, he used transformation and matching language: "An operation through which you would transform an element of one group to the corresponding element in an identical group." When asked to describe isomorphism to a ten-year-old, he used a novel metaphor that highlighted sameness and transformation: "Imagine you had an identical twin, like exact same behaviors, looks, everything, but it's still a different person. Now imagine there's one way you transform into them and back: that would be an isomorphism."

When asked to describe a homomorphism, he used transformation and similar properties language: "A transformation or a[n] operation through which you would transform an element from one group to a subset of the group retaining similar properties but not exactly in the same group. It doesn't have to be the same group." When asked to expand on the caveat about not being the same group, he highlighted the connection between isomorphism and homomorphism: "Because, like, isomorphisms are homomorphisms, so it can be the exact same group, but it doesn't have to be. I just left it out which was incorrect." When asked for a more formal definition, he highlighted transformation and similar properties metaphors: "An operation through which you would transform an element in one group to a group with similar characteristics that is of lesser or equal size." When asked how he would describe a homomorphism to a ten-year-old, he provided a novel metaphor related to condensation:

> Look at your dad and then look at yourself. Imagine what your dad, like what part of your dad went to you sort of as a homomorphism. So like he took a part of himself and sort of condensed it to create you, ignoring, like, the combination of the mother's genes.

When asked if he had any last thoughts on isomorphism or homomorphism, he gave another novel metaphor linking both concepts together while distinguishing according to size:

> They're almost like mirrors. Like isomorphism is…closer to the mirror, but sometimes you have mirrors that make you look smaller, like at the corners of hallways and hospitals. Like, you get the same thing back. Sometimes it's a little bit smaller, that's like a homomorphism. But you look in, just like a regular mirror straight on, it's pretty much the exact same thing back but it's not you. It's just an image of you that retains all the characteristics.

In the problem-solving section, he addressed nine scenarios. Across these scenarios, he used that fact that isomorphisms are homomorphisms and reached appropriate conclusions in all six isomorphism contexts where he used same properties. Across isomorphism, he used the formal definition, same properties, same, invertible, matching, morphing, and mapping metaphors. When addressing homomorphism, he used formal definition, similar properties, matching, transformation, equivalence classes, function, journey, mapping, and isomorphism without bijectivity. On the second-to-last problem, he recalled that the trivial homomorphism always exists and went back and showed what it would look like on some previous examples.

For Z_5 to 5Z, he used same properties as he noted one group was finite and the other was infinite so there could not be an isomorphism between them. Unlike some other students, he had not directly stated the homomorphism property earlier in the interview. Thus, when he asked if homomorphisms have to be onto, I stated the definition of homomorphism. From this, he said "I can't think of a homomorphism. I can think against it so probably no." He wrote on the paper the mapping $\theta(x)=5x$, and showed an example demonstrating it was not a homomorphism. Thus, his only metaphor in this section is mapping, as he reasoned by creating a specific mapping as an example.

For Z_5 to Z/5Z, he used matching and invertible metaphors as he concluded an isomorphism would be constructible. He figured that the cosets and bar elements would be the same thing and that you could also map the reverse direction with a similar map (from Z/5Z to Z_5). For homomorphism, he used isomorphism without bijectivity to immediately state that the isomorphism he had found was also a homomorphism. He was not pressed to provide alternate isomorphisms or homomorphisms.

For Z_5 to Z_6, he used same properties as he noted different group sizes meant there would not be an isomorphism. For homomorphism, he used the formal definition as he recognized that multiplication by two would lead to the groups “looping” at different times so, for example, 2+3 mod 5 would be 0, whereas 2(2) + 2(3) = 10 would be 4 mod 6 which would not be 0.

For Z_3 to Z_6, he used same properties again as he noted these were different size groups. For homomorphism, he used mapping and formal definition metaphors as he derived the doubling mapping (0 to 0, 1 to 2, and 2 to 4). He only checked one pair of elements, but figured it would continue to work if he checked all pairs; he recognized he could not be certain based on checking only one pair of elements. He was not aware of any other homomorphisms at the time.

For Z_6 to Z_3, he did not specifically say anything about isomorphism; rather, he jumped straight into homomorphism. (Because he had already noted different size groups ruled out isomorphisms a number of times, I assume he thought this again, but did not feel the need to say it again.) For homomorphism, he used equivalence classes, journey, formal definition, similar properties, function, mapping, and isomorphism without bijectivity. He immediately formed a map based on what the numbers 0 through 5 would be mod 3. He then verified his conclusion in general and called this a “more proper proof.” When I asked if any other homomorphisms were possible, he tried mapping 0 and 1 to 0, 2 and 3 to 1, and 4 and 5 to 2, though he did not feel confident in it. Using the formal definition, he then showed it was not a homomorphism. When I asked if he was comfortable with getting homomorphisms in both directions, he said that was fine and noted that compensation needs to happen somehow for homomorphisms: “When it comes to like, different group sizes, you have to compensate somehow, and since this one’s smaller you have to leave some out, and cuz this one’s bigger, you have to double up.” He then started talking about properties of functions and mapping: “Since you can’t have something mapping to nothing. Like everything has to map to something and you can't have two of them, two different ones mapping to the same thing when it goes from smaller to bigger." Note the first part of this statement refers to functions being everywhere defined, though his second statement is not generally true for homomorphisms. He then went on a tangent thinking about whether you could have an onto function that went from smaller a sized group to larger group.

For Z to 2Z, he used mapping, invertible, same properties, and formal definition to form the standard multiply by 2 mapping. He then verified the homomorphism property and verbally noted that both groups were "infinite so I can't say that they have different group sizes even though one is bigger, but also not bigger" and thus the mapping was technically onto and one-to-one. For homomorphism, he used isomorphism without bijectivity, mapping, and journey metaphors as he quickly concluded that his isomorphism was a homomorphism. He also said multiplying by other multiples of two would produce homomorphisms, because they were subsets.

For 2Z to Z, he had already noted the mapping in the other direction (multiply by 1/2) when given the prompt for Z to 2Z. Follow-up questions on homomorphism were not asked.

For the Cayley tables, he used same properties as he noted all elements were self-inverses in one group but not the other so there could not be an isomorphism. For homomorphism, he used similar properties, matching, isomorphism without bijectivity, and mapping metaphors. Initially, he believed there would not be any homomorphisms, but he was still trying to create an onto mapping. Then, he rewrote the second table so the identity was the first element listed to see if he could match elements, but he noticed the pattern of the 2x2 square of the first two elements still did not match. When I asked if he was looking for an isomorphism or homomorphism at this point, he realized that he did not need his mapping to be onto and that the trivial homomorphism always exists. He then showed me what the trivial homomorphism would look like in this context. He did not believe any other homomorphisms would exist for these groups.

For Z_8 to D_4, he used same properties to note that Z_8 was abelian, whereas only the rotations were "abelian" in D_4. For homomorphism, he used function reasoning to rule out a number of maps that just mapped the even components to the rotations, because that would "leave out the reflections," indicating he realized his homomorphism needed to be everywhere defined. He concluded the trivial homomorphism worked, but could not figure out what to do with odd elements in any other scenario.

In summary, Student 11b spoke much more about homomorphism than isomorphism, as shown in Tables 5.28 and 5.29. For isomorphism, he largely used same properties, though matching, invertibility, and mapping also appeared. Notice the two appearances of transformation in the interview happened describing isomorphism in general, not when problem-solving. For homomorphism, the formal definition, mapping, isomorphism without bijectivity, function, and similar properties were the most common. The focus on relating isomorphism and homomorphism was evident in his mirror example at the beginning. He also focused on sizes of groups for both isomorphism and homomorphism (same/similar properties) at the beginning.

Table 5.28. Code Frequencies for Isomorphism for Student 11b.

Metaphors	**Frequency**
Same Properties	6
Matching	3
Invertible	2
Transformation	2
Generic Mapping	2
Generic Sameness	1
Literal Formal Definition	1
Total	17

Table 5.29. Code Frequencies for Homomorphism for Student 11b.

Metaphors	**Frequency**
Literal Formal Definition	9
Generic Mapping	9
Isomorphism without Bijectivity	6
Function	5
Similar Properties	4
Journey	3
Matching	3
Transformation	2
Equivalence Classes	1
Total	42

In terms of connecting to instruction, he used sameness-related metaphors that had been used a number of times for isomorphism in class (same properties and matching). His metaphors for homomorphism focused on the formal definition and mapping cluster language, though he did reflect specifically on his knowledge of functions as he was deciding if specific maps he made could be homomorphisms. Furthermore, his use of mapping metaphors extended outside what was used in class, such as with transformation metaphors. However, this might be explained by the fact that transformations are a common way to refer to functions, especially after taking linear algebra. Also, he had started and dropped abstract algebra with a different instructor in a previous semester, and that instructor could have used transformation metaphors. Although he only spelled out his thinking using equivalence classes once, when he explained where his homomorphism from Z_6 to Z_3 came from, he utilized this idea seemingly effortlessly as he thought about how numbers work, and may have been thinking about this idea at other times, even if he went directly to verifying his idea with the formal definition.

Student 18b. When asked to define an isomorphism, he both defined and described as he used the formal definition, structure-preserving, and relabeling language together: "I guess an isomorphism would be a function which is bijective and it's structure-preserving in a specific, I mean it, basically you can just relabel the Cayley table, but that's formalized as f(ab) = f(a)*f(b)." When asked how he would explain an isomorphism to a ten-year-old, he used mapping, disembedding, and same metaphors:

> I wouldn't describe an isomorphism, but I would describe what it means to be isomorphic because I think it gives a, just, an intuitive understanding of what mathematicians are actually saying. They're just saying that they recognize structures in these two things that are essentially, on some other level, makes them the same. So two things are isomorphic if they are essentially the same in some way.

I then asked how he saw isomorphic as different from isomorphism, and he used mapping and formal definition language: "Well, two things are isomorphic, or two groups are isomorphic if there's a group isomorphism between them. Once you get down to the function's bijective and stuff with a ten-year-old, I probably wouldn't describe that." When I asked for more clarification he used same, mapping, and formal definition metaphors:

> Well I feel like if you say two things are isomorphic, there's definitely a specific trait that you're talking about that makes them essentially the same thing. So the isomorphism would just be a way of point that out without any doubt about it. And then the way that takes place in mathematics and group theory is through the, these more rigorous definitions.

When I asked how he would define a homomorphism between groups, he used disembedding to focus on the structure-preservation choice he saw:

> A homomorphism is just a function that preserves the structure, or some structure, not necessarily all of the structures; it might just preserve one structure. Like the integers map to Z mod 2 or something: that could preserve the structure of, like, the evens and

odds, but it destroys a lot of the other properties of the integers….If you construct a homomorphism, you can almost pick which structure to preserve, at least if you're dealing with easy enough groups that you can see.

When asked what he meant by "preserving the structure," he turned to the formal definition, journey, and structure-preserving metaphors:

> It would be that definition: that f(a*b) = f(a)*f(b), but…it's intuitive for me to go back and think about the Cayley tables. Because they're just saying that where the product of these two things gets mapped to gets mapped to where the product of where these two others things gets mapped to [gestures]. So that's the structure right there that's being preserved: things still will be nice and well-defined and play nicely I guess.

When asked how he would explain a homomorphism to a ten-year-old, he used a transformation metaphor: "I guess I would describe it as a transformation of one object to another object that only keeps a few of the properties of the original object."

In the problem-solving portion, he addressed nine scenarios. For isomorphism, he used the formal definition, structure-preserving, disembedding, same properties, same, invertible, relabeling, function, journey, and mapping metaphors. In the six problems in which he used same properties, he reached appropriate conclusions about the existence of an isomorphism each time. He also noted isomorphisms are homomorphisms. For homomorphism, he used relabeling, formal definition, structure-preserving, disembedding, similar properties, transformation, equivalence classes, journey, mapping, and isomorphism without bijectivity metaphors. He noted the trivial homomorphism from the second example onwards.

For Z_5 to 5Z, he used same properties as he noted the groups were of different sizes so there could not be an isomorphism between them. For homomorphism he used an unusual approach as he used journey and disembedding metaphors as he tried to find an intermediate group he could map to:

> Well, I was also…trying to see if I could see, like, isomorphisms between them and any other group. Not with Z mod 5, but maybe 5Z, because it's isomorphic to the integers. So I was trying to think about any isomorphism that I know of. And then of course you can take the…integers to Z mod 5 pretty easily I think. And then the kernel would be 5Z of that one. But I don't know of anything that way.

He did not mention the existence of any homomorphisms on this problem.

For Z_5 to Z/5Z, he used mapping, journey, and same properties metaphors as he created an isomorphism in two parts: a homomorphism from Z to Z_5 using congruence to make equivalence classes and then the observation that the kernel of that mapping would be multiples of 5 (5Z) so the two groups should be isomorphic. (The equivalence classes metaphor was used to talk about creating a homomorphism in the service of creating an isomorphism, but because he used it to create a homomorphism, I classified this under homomorphism.) He then specified which elements would map to each other. He did not believe any other isomorphisms would exist. For homomorphism, he used equivalence classes, isomorphism without bijectivity, and mapping metaphors. He immediately stated two maps: that the isomorphism was a homomorphism and that you could map everything to the identity. He could not think of any other homomorphisms.

For Z_5 to Z_6, he used same properties to note you could not form an isomorphism with different size groups. For homomorphism, he used mapping, journey, and similar properties as he observed that 5 was prime so Z_5 should be cyclic, meaning it would have no nontrivial subgroups. Also, for any homomorphism, the kernel is a normal subgroup of the domain group and a subgroup of the co-domain, and there was not any alignment between the groups besides the trivial homomorphism.

For Z_3 to Z_6, he just said there was no isomorphism and was not pressed for further details because he had already noted the need for having the same size groups a few times. For homomorphism, he used equivalence classes, formal definition, and journey metaphors as he noted that the 6-element group had a 3-element subgroup (by divisibility, which is true of this group) and it should be possible to map to these elements. He struggled somewhat in determining what the 3-element group should be, but seemed satisfied with what he found. When asked how he could verify his work, he noted the formal definition would need to be checked. He also noted the existence of the trivial homomorphism. At the end of the interview, we returned to this problem, and I suggested swapping where 1 and 2 were mapped. At this point, he used disembedding, formal definition, similar properties, and mapping as he concluded that this suggested mapping seemed to work. He did not find this surprising because generators should act similarly, though you could not change where the identity was mapped (and he verified this in general).

For Z_6 to Z_3, he said no to isomorphism and was not pressed for the same reasons as before. For homomorphism, he used equivalence classes, journey, formal definition, and disembedding. He recognized the trivial homomorphism should be present immediately. He realized there should be another map because 3 divides 6, and the map he created doubled up in the standard way based on congruence classes. When I prompted him to verify this mapping worked, he struggled. He concluded the map had to work by the Fundamental Homomorphism Theorem and the fact that the kernel would be Z_2. When I asked for further justification on the specific mapping he gave, he handwaved, saying operating in mod 3 should still yield the same congruence classes.

For Z to 2Z, he used same properties, mapping, journey, disembedding, and formal definition metaphors when addressing isomorphism. He concluded that they must be isomorphic because both groups were infinite and cyclic. When asked for a specific isomorphism, he gave the standard mapping ("every integer k will get sent to 2k in [2Z]"). He also noted you could probably map to -2k, though he felt his first map was "the most obvious one, the one that most reveals, probably like, why it's true…I probably wouldn't use another one." When asked to verify his conclusions, he demonstrated his map followed the homomorphism property and stated that a proof that the map was bijective "should be pretty straightforward." He also noted "I've seen the proof that any two cyclic groups of an infinite size are isomorphic, and then I can just find a generator in each one, and then feel reasonably confident in saying they're isomorphic." To address homomorphisms, he used isomorphism without bijectivity, journey, and mapping metaphors. He noted the isomorphism was a homomorphism immediately. After thinking a bit, he noted mapping to multiples of 4 should also work.

For 2Z to Z, he used invertibility as he noted "isomorphism is symmetric….You can just take the inverse of that function" and he wrote $\phi_1 = \frac{x}{2}$. For homomorphism, he used journey, disembedding, mapping, function, similar properties, and formal definition metaphors. He initially just reversed his homomorphism mappings as well, but then realized his map was not defined on the whole domain. He then switched to a "function like 2x again," which would work. I then suggested mapping to multiples of 2 in Z, which he identified as the identity mapping, and said that should work. He was not pressed to verify this would work.

For the Cayley tables, he used same properties to compare what happened to elements squared in each group, and concluded they could not be isomorphic. For homomorphism, he used equivalence classes, mapping, and journey metaphors as he made different observations. He believed (correctly) there were only two groups of order 4 so one of them had to be Z_4. However, he used this information to map only parts of the domain in his initial map. He was not confident about this map. He did note the trivial homomorphism in which he would map all elements to the identity.

For Z_8 to D_4, he used same properties to note one group was abelian and the other was not so they could not be isomorphic. For homomorphism, he used equivalence classes, relabeling formal definition, mapping, and similar properties as he created multiple homomorphisms. He noted you could map to the rotation subgroup by taking elements mod 4. He noted he was rotating clockwise as he spoke about the rotation subgroup. When I asked if it mattered which direction he rotated, he said no. When I asked if that was the only homomorphism, he noted you could view rotation 90 degrees one way as rotation the other way 270 degrees. When I asked if this changed the mapping in some way, he thought that would just change the labelings, and it might still work. He was still more confident in his first map, though the reason for his confidence seemed to be that he knew the rotations were isomorphic to Z_4 because it was a 4-element cyclic group, which would apply equally well in the other case. He also noted you could map to rotation by 360 degrees and 180 degrees according to what they were congruent to mod 2. He did not specifically note the trivial homomorphism.

In summary, Student 18b used a variety of metaphors throughout the interview, including a strong emphasis on same properties and the formal definition for isomorphism and the use of journey, equivalence classes, mapping, and the formal definition for homomorphism. However, his emphasis on structure that was evident through both his structure-preserving and disembedding language at the beginning of the interview also came through during the interview, especially as he created the structures by thinking about equivalence classes. Frequencies of metaphors used are given in Tables 5.30 and 5.31.

Table 5.30. Code Frequencies for Isomorphism for Student 18b.

Metaphors	**Frequency**
Same Properties	8
Literal Formal Definition	6
Generic Mapping	6
Generic Sameness	2
Journey	2
Disembedding	2
Invertible	1
Relabeling	1
Structure-preserving	1
Total	29

Table 5.31. Code Frequencies for Homomorphism for Student 18b.

Metaphors	**Frequency**
Journey	13
Equivalence Classes	12
Generic Mapping	10
Literal Formal Definition	9
Disembedding	6
Similar Properties	5
Relabeling	3
Isomorphism without Bijectivity	2
Function	1
Transformation	1
Structure-preserving	1
Total	63

In terms of connecting to instruction, he used sameness metaphors that had been used a number of times for isomorphism in class (same and same properties). He also picked up on the idea of disembedding, which none of the other students used, though it had been part of Instructor B's conceptual frame for isomorphism and homomorphism. He utilized many mapping cluster metaphors as he was explaining his approaches to problems, though they were not generally the point of what he was saying, much like in class. The heavy use of equivalence classes mirrors what was done near the end of the unit in class. He made use of the formal definition as well, though he was not inclined to use it unless forced to verify his mapping was a homomorphism; this was how the formal definition was used in class as well.

Student 19b. When asked to define an isomorphism between groups, she used special homomorphism and same metaphors:

> The definition they give you in class is that it's a group homomorphism that's also a bijection, so that's basically, I kind of think of it as like a relation between two groups where they're not, you know, identical groups; they have different elements in them, but they have similar characteristics and kind of the same structure.

When asked to describe an isomorphism, she again highlighted sameness: "I guess I would definitely talk about how it's basically you take two groups and they're related to each other [gestures]. They're not…identical, but they've got, you know, this is more like similar characteristics to it." When asked how she would describe isomorphism to a ten-year-old, she used a novel metaphor:

> I'm trying to think of, like, a real-life example, because I'd want to give them an example….I mean, they always get, like, vocab words in their classes, so if they had, they usually have to, like, categorize them by, like, the different sounds that the words make, like at and bat and hat, like, all make the same sounds, but they're not the same word. So, like, all those go in the same category, so that could be kind of like an isomorphism.

When I asked her to interpret her analogy, she highlighted both sameness and same properties.

> They have similar, like, the words they would have like a similar structure of similar… it's not always, you know, same amount of letters, but like, a lot of times it is. So same number of elements, I guess you could say, in them. And similar sounds, so the elements themselves have similar characteristics, like how you say them.

When asked to describe a homomorphism between groups, she used operation-preserving, same, isomorphism without bijectivity, and matching language:

> So I mean, I know like, it's more just like it respects the group operation [air quotes gesture] is what they tell us in class. That one I have, like, a harder time, you know. Like I understand the definition and how to show that something's homomorphism, but I have a harder time, you know, putting that into words just because it's…I mean yes, the groups like have similar…It's not functions. It's like similar operations to them, but they're not always…it's not always the bijection, so it's not always the same…exact structure, so you can have some elements that are kind of outliers in each group that don't really match with the other group that you're talking about.

When I asked her to clarify if she meant "similar" in the same way for isomorphism and homomorphism, she said it was the same way, and then clarified.

> I guess the isomorphism would be just a more…accurate similarity, so there's, like, more common traits between the two groups, whereas a homomorphism they can, I mean they still have some similarities, but they're not always identical. They can…vary a little bit more.

When asked for a more formal definition of homomorphism, she used operation-preserving and formal definition language:

> That's something where it would respect your group operation. So you'd have, like, if you have two elements in your domain, like x and y, then the function evaluating, like, x operated on y, is the same as the function of x operated on the function of y.

When asked for clarification on "operate," this was meant as the respective groups' operations. When asked to describe a homomorphism to a ten-year-old, she highlighted a desire to be operation-preserving before using a novel metaphor focused on order of operations:

> I guess, so if you want something that's they're similar, but it's more like when you're operating on them it's the same thing. I guess you could talk about, like, baking….So if you have, like, a recipe where you're making a cake and you mix all of your liquid ingredients and then all of your dry ingredients separately and then you combine them, then you still get cake batter, but if you do the ingredients in a different order, then you still end up with cake batter. Might not be exactly the same, but kind of defeats the purpose in the analogy so, something like that where you don't always have to mix things in the same order and you still get the same result at the end.

For the problem-solving portion, she addressed eight scenarios. She used the formal definition, structure-preserving, same properties, same, special homomorphism, invertible, matching, journey, and mapping metaphors while addressing isomorphism. All seven problems in which she used same properties also had correct conclusions about the presence or absence of isomorphisms. She noted that isomorphisms are homomorphisms. She happened across the trivial homomorphism in one problem, but did not generalize its existence to any other problem. For homomorphism, she used the formal definition, operation-preserving, similar properties, same, matching, journey, mapping, and isomorphism without bijectivity metaphors.

For Z_5 to 5Z, she used same properties, matching, journey, and mapping as she noted you need the same order of groups to be one-to-one, or else, in this case, the same element would need to go multiple places. (Technically, she was assuming her map needed to be onto also and then got a contradiction on one-to-one for this map, or she was using the fact that functions should be well-defined. Her ultimate conclusion that elements can only map to one element was correct.) For homomorphism, she used matching, formal definition, journey, mapping, and isomorphism without bijectivity. It took some time for her to determine how to operate within the groups. Once she did, she tried the standard mapping (0 to 0, 1 to 5, etc.) and tested a few cases with the homomorphism property that happened to work. When I prompted her to test a different pair of elements, she realized this pair did not work and so the map did not work. She seemed uncertain whether any map could work or not.

For Z_5 to Z/5Z, she used same properties, formal definition, mapping, operation-preserving, and special homomorphism while addressing isomorphism. She found the standard mapping and verified this mapping was one-to-one and onto and satisfied the homomorphism property. For homomorphism, she used isomorphism without bijectivity and the formal definition as she noted that isomorphisms are homomorphisms. She did not believe there would be any other homomorphisms.

For Z_5 to Z_6, she used same properties to note the groups had different numbers of elements so they could not be isomorphic, though she technically said this violated being one-to-one, similar to her argument above. For homomorphism, she used formal definition and journey metaphors. She began by thinking that the identity mapping would produce a homomorphism because “they use the same operation.” However, when prompted to look at specific examples, she realized this map did not work and did not expect any other homomorphism to exist.

For Z_3 to Z_6, she used the formal definition to again state that an isomorphism would have to map elements more than one place so it could not be one-to-one. For homomorphism, she used similar properties, mapping, and formal definition metaphors. Her initial intuition led her to believe that divisibility should allow for a homomorphism. First, she tried the identity homomorphism (0 to 0, 1 to 1, 2 to 2), which did not work. Then she used $f(x) = 2x$ in mod 6 which worked. She also thought multiplying by any even number should work because values would reduce to the same thing, but this was not probed further. (While multiplication by other even numbers would also work, such multiplication would not always produce an equivalent map.)

For Z_6 to Z_3, she used same properties to note there was still a size disparity so there would not be an isomorphism. For homomorphism, she used journey, formal definition, mapping, and isomorphism without bijectivity. She created the map according to values in mod 3 immediately, which was a working homomorphism. In the process of testing multiplication by different values (like the previous problem) she realized that multiplication by 3 led all elements to be mapped to 0, which is the trivial homomorphism. She recognized this map worked, though she did not generalize this map to any other example. When asked if it was alright that certain elements in the co-domain were mapped to multiple times and others were not mapped to at all, she noted that homomorphisms do not require bijectivity so this was fine.

For Z to 2Z, she used same properties, mapping, journey, and special homomorphism as she addressed homomorphism. Though she noted she wanted to say 2Z has half as many elements as Z, she knew they were both infinite groups so the $f(x) = 2x$ map should work. She noted each element would only map one place, which she believed satisfied one-to-one. She decided all elements in the co-domain would have an element go to them so the map was onto.

She was not pressed to verify the homomorphism property. For homomorphism, she used mapping and isomorphism without bijectivity. She immediately recognized that her isomorphism would be a homomorphism. She also believed that multiplying by any even integer (e.g. f(x) = 4x) would result in a homomorphism.

For 2Z to Z, she used mapping, journey, same properties, invertible, and special homomorphism to address isomorphism. She immediately noted you could divide by 2 to create a map that would still go to 0, 1, 2, etc. She also noted that they had the "same order" [with air quotes] meaning they would be one-to-one. She called this the "inverse" mapping of what she had just done. When asked what she meant by inverse mapping, she used the language of function composition:

> So if you had, like if the other one was f and then the function f in this case was g, then if you take something from 2Z and you map it to Z, and then you, like if you compose the 2 functions together, then you're just gonna be mapping something from 2Z to itself. It like, I don't want to say it reduces the function back down to where you're just like mapping each element [gesturing] directly to itself, but if you had f of Z going to 2Z and g going from 2Z to Z, then f compose with g would just mean that you start with taking, so this is like f of g, so then like g of x would take an x from 2Z and send it to the integers and then f would take it from the integers back to 2Z, and it would be the same, so you'd like get back whatever you input.

For homomorphism, she used mapping as she noted that the isomorphism she found was also a homomorphism. She also recognized she could not divide by larger numbers because that would lead to non-integers. She did not consider multiplying by larger integers or note the trivial homomorphism.

For the Cayley tables, she used matching, same properties, and structure-preserving while addressing isomorphism. She initially realized each element operated with itself gave the same result in one group and different results in the other group, leading her to say there would not be an isomorphism. While working on the homomorphism prompt, she believed she found an isomorphism because she created a one-to-one and onto mapping. Then she realized that the orders of elements were different and that one group was cyclic and the other was not so the map could not be an isomorphism. For homomorphism, she used similar properties, formal definition, same, and operation-preserving metaphors. Her initial belief was that the identities needed to have the same symbol to have a homomorphism, but she reframed this to say the identities simply had to match to each other in the mapping. She also noted that the homomorphism needed to respect the group operation, but did not seem to know how to use this information. At the end, she said she felt like there could be a homomorphism, but was not sure how to get there.

In summary, she used more metaphors in discussing isomorphism than homomorphism. She frequently made use of same properties, and often made use of mapping and matching language for isomorphism, as shown in Table 5.32. This aligns with her descriptions and analogies at the beginning that focused on sameness, especially same properties. For homomorphism, she used the formal definition, mapping, and journey metaphors most often, as shown in Table 5.33. In her descriptions, she spent some time describing how she saw the formal definition. Although she did not use the operation-preservation language as often while solving problems as she had at the beginning of the interview, she did fall back on that idea in the context of the Cayley tables.

Table 5.32. Code Frequencies for Isomorphism for Student 19b.

Metaphors	**Frequency**
Same Properties	16
Generic Mapping	7

Matching	6
Generic Sameness	5
Journey	4
Literal Formal Definition	3
Special Homomorphism	3
Operation-preserving	1
Structure-preserving	1
Invertible	1
Total	47

Table 5.33. Code Frequencies for Homomorphism for Student 19b.

Metaphors	**Frequency**
Literal Formal Definition	14
Generic Mapping	9
Journey	6
Isomorphism without Bijectivity	5
Operation-preserving	4
Generic Sameness	3
Matching	2
Similar Properties	2
Total	45

In terms of connecting to instruction, she used sameness-related metaphors that had been used a number of times for isomorphism in class (same, same properties, and matching). Her metaphors for homomorphism focused heavily on the formal definition cluster (formal definition, isomorphism without bijectivity, and operation-preserving), and were used throughout problem-solving rather than being relegated to the proof context as they were used in class. She also used mapping cluster language, much as it had been used in class.

Summary of students in Class B. In looking across the students in Class B, more metaphors were employed to talk about homomorphism than isomorphism. For isomorphism, the most common metaphor was same properties, distantly followed by mapping, the formal definition, and same as shown in Table 5.34.

Table 5.34. Code Frequencies across Students in Class B for Isomorphism.

Metaphors	**Frequency**
Same Properties	41
Generic Mapping	17
Literal Formal Definition	14
Generic Sameness	12
Matching	9
Journey	8
Special Homomorphism	5
Invertible	5
Operation-preserving	3
Transformation	2
Disembedding	2
Structure-preserving	2
Machine	1
Relabeling	1
Function	0
Total	122

Students generally used the metaphors in similar ways when they used the same metaphors. Much like Class A, same properties was largely used to invoke a property that was not shared by the groups (such as number of elements) to demonstrate the groups were not isomorphic, whereas generic sameness was used more often in the describing/defining part of the interview. The formal definition was generally used to verify a mapping that had been created. Mapping and journey metaphors were used to describe what happened to specific elements in mappings. Matching was only used by Students 11b and 19b, and most of those instances came when Student 19b was working with the Cayley table problem. Invertibility was used by Students 11b, 18b, and 19b in the context of the 2Z to Z problem after they had done the Z to 2Z problem, whereas it was used when generally describing isomorphism by Student 3b.

For homomorphism, the most common metaphors were the formal definition, mapping, journey, equivalence classes, similar properties, isomorphism without bijectivity, and operation-preserving, as shown in Table 5.35. Of note, all but three of the metaphors identified in this study were used by at least one student in each context (function and equivalence classes for isomorphism and machine for homomorphism). The nature of the questions asked may have prevented students from needing to lean on their knowledge of functions to talk about isomorphism.

Table 5.35. Code Frequencies across Students in Class B for Homomorphism.

Metaphors	**Frequency**
Literal Formal Definition	35
Generic Mapping	34
Journey	27
Equivalence Classes	19
Similar Properties	16
Isomorphism without Bijectivity	16
Operation-preserving	11
Matching	6
Disembedding	6
Function	6
Generic Sameness	5
Transformation	3
Relabeling	3
Structure-preserving	1
Machine	0
Total	188

Comparing students to each other, all four students used mapping and other mapping cluster language throughout their problem solving, largely to describe how specific elements were being treated under mappings. The formal definition was largely used to verify the mappings students generated, though Student 11b tended to state a mapping and figure it would probably work in general, and Student 18b would use the FHT to assert a mapping must exist and not want to give a specific map or verify it. Students 3b, 11b, and 18b all used equivalence classes to reason through mappings. Student 11b only mentioned this aloud once, and Student 3b only used it after receiving a modified homomorphism definition. Student 18b, however, used it regularly throughout his interview.

Comparing isomorphism and homomorphism, same properties was the go-to way to think about whether or not groups were isomorphic, and this strategy was generally successful, much like in Class A. The exception was for the 2Z to Z problem, where 3 of the 4 students used invertibility instead, which built off previous work from Z to 2Z. In general, students were fairly successful in dealing with the isomorphism problems they were given. The only problem arose with the question that involved quotient groups, and the problem was related to understanding how to operate on quotient groups more than understanding what isomorphism is.

For homomorphism, there was more variety in how students chose to approach problems. Student 11b tended to sit in silence for a while and then just state a map. Student 18b would start from equivalence classes to decide if such a map could exist. Student 19b would generally start from the identity homomorphism and see where that led her. If it did not work, then other considerations like similar properties of the groups would be considered.

Relating instructors and instruction to students in Class B. In this section, I seek to answer research question four in the context of Class B. I will relate Instructor B's metaphors in his interview and especially in instruction to metaphors used by students in Class B. This will allow me to address relationships between conceptual understanding (as understood through metaphors) in instruction and students' metaphors.

Table 5.36. Codes Used for Isomorphism in Class B.

Metaphors	Presence in Class	Frequency in Interview	Student Frequency
Generic Sameness	X	6	12
Same Properties	X		41
Disembedding		1	2
Generic Mapping	X	3	17
Invertible			5
Function	X		
Journey	X		8
Machine			1
Transformation			2

Literal Formal Definition	X	1	14
Operation-preserving	X	1	3
Structure-preserving	X	2	2
Special Homomorphism	X		5
Renaming/Relabeling	X	6	1
Matching	X	1	9

In Class B, many metaphors were used across all three isomorphism contexts, as shown in Table 5.36. These metaphors include sameness, mapping, formal definition, operation-preserving, structure-preserving, relabeling, and matching. Many others were used in problem-solving contexts in class and by the students, including same properties, journey, and special homomorphism. The unusual cases are function (used only by Instructor B in class), disembedding (used only in interviews), and invertible, machine, and transformation (used only in student interviews). However, the function metaphor may explain most of the apparent discrepancies. The instructor used properties of functions and acted under the assumption that isomorphisms were functions throughout the class periods. Machine and transformation metaphors have been used by other students to describe functions (e.g. Zandieh et al., 2016) and some students have displayed the ability to extend their knowledge of function to abstract algebra contexts (e.g. Melhuish, Lew, Hicks, & Kandasamy, 2019). Thus, these metaphors may be artifacts of students' connection of isomorphism with other functions. Although not all functions are invertible, we expect one-to-one functions to be invertible from high school algebra and the one-to-one nature of isomorphisms was emphasized in class, which may also explain the invertible metaphor.

Disembedding was also used in the homomorphism context so disembedding may have transferred from the homomorphism setting or could be a product of discussions in office hours. Additionally, the instructor used relabeling more than the students did, both using the language at each table in class and more times in his interview. This may be attributed to students in this class associating relabeling with Cayley tables, which was the context in which they were used most frequently.

Table 5.37. Codes Used for Homomorphism in Class B.

Metaphors	Presence in Class	Frequency in Interview	Student Frequency
Generic Sameness			5
Similar Properties	X		16
Disembedding	X	1	6
Generic Mapping	X	1	34
Function	X		6
Journey	X		27
Machine			
Transformation			3
Literal Formal Definition	X		35
Operation-preserving	X	2	11
Structure-preserving	X		1
Isomorphism without Bijectivity	X	8	16
Renaming/Relabeling			3
Matching			6
Equivalence Classes	X	2	19

For homomorphism in Class B, metaphors largely align with what might be expected. Disembedding, operation-preserving, isomorphism without bijectivity, and equivalence classes were used across all contexts, as shown in Table 5.37. (It should be noted that although disembedding was used across all contexts, only one student used this metaphor.) Similar properties, function, journey, formal definition, structure-preserving language were used in class and by students, likely because of the similarity of the tasks in those contexts. Same, transformation, relabeling, and matching language were only used by students. Like for isomorphism, transformation and matching can likely be explained by students' understanding of function. Same and relabeling are likely transferred from the isomorphism context. Unlike in Class A, students in Class B used the equivalence classes metaphor fairly frequently. This may be because Instructor B modeled its use through the Fundamental Homomorphism Theorem (FHT) with specific examples across multiple class periods, whereas in Class A, students only interacted with the FHT in proof contexts the day before an exam.

Chapter Discussion

This section focuses on addressing research questions three and four by conducting a cross-case analysis. First, I will compare and contrast instructors' metaphors and what metaphors were used in their instruction. Then, I will use this comparison to inform research question three, examining relationships between conceptual understanding and metaphors in instruction. Next, I will compare and contrast students' metaphors. Finally, I will compare instructional metaphors and students' metaphors in order to examine relationships between instruction and students' metaphors, which is research question four.

Comparison of instructors' interview metaphors. In this section, I characterize instructors' metaphors in their interviews. This will connect to my answer to research question three, which focuses on the relationship between instructors' metaphors in interviews and in class. Recall, instructors were not asked to solve specific problems during interviews. Rather, they were asked about how they would describe and define isomorphism and homomorphism in different ways, as well as if they thought about these topics differently than how they presented them in class.

As we examine the two instructors' metaphors, parallels are evident in the isomorphism context as we examine Table 5.38. Both instructors used many sameness-related metaphors to talk about isomorphism, whether broadly discussing how groups were the "same" or more specifically using renaming or relabeling language. They also utilized language from the formal definition, whether adhering closely to stated definitions or using preservation language. However, both instructors' use of mapping-type language was minimal for isomorphism.

Table 5.38. Code Frequencies for Isomorphism in Instructor Interviews.

Code	Instructor A	Instructor B
Generic Sameness	10	6
Disembedding	0	1
Generic Mapping	1	3
Matching	0	1
Renaming/relabeling	7	6
Literal Formal Definition	2	1
Operation-preserving	3	1
Structure-preserving	0	2
Total	23	21

The majority of both instructors' metaphors related to sameness or combined sameness and mapping for isomorphism. This language mostly focused on general sameness and the specific mapping/sameness image of renaming or relabeling. Both also shared a greater emphasis on the groups being isomorphic than on the isomorphism mapping or function that connected the two groups. Instructor A mentioned the isomorphism was the way to verify groups were isomorphic, but placed more emphasis on that group structure. Similarly, Instructor B focused on how we can think of types of groups (e.g. "*the* cyclic group of order n") and allow the isomorphism mapping to fade into the background.

Both used formal definition language in a secondary capacity, mainly when asked to define an isomorphism. Both also made limited use of exclusively mapping metaphors. This seems to be because they both had a concrete image of sameness and they felt the structures themselves were more important than the mappings connecting them.

For homomorphism, both instructors again used formal definition language and there was limited focus on mapping language in this context. Both instructors also used equivalence class language to provide imagery, though Instructor A expanded more on this metaphor. However, there was less sameness language used outside the context of equivalence classes for Instructor B, whereas Instructor A utilized similar properties and general sameness language again.

Table 5.39. Code Frequencies for Homomorphism in Instructor Interviews.

Code	Instructor A	Instructor B
Generic Sameness	5	0
Similar Properties	2	0
Disembedding	0	1
Generic Mapping	4	1
Journey	1	0
Matching	1	0
Renaming/relabeling	1	0
Equivalence classes	16	2
Literal Formal Definition	2	0
Operation-preserving	3	2
Isomorphism without Bijectivity	0	8
Total	35	14

In examining Table 5.39, we see that Instructor A utilized language from all four clusters, but focused on sameness related clusters. She especially leaned on equivalence class metaphors as she described finding similar elements in the domain (sameness part of the metaphor) and then mapping them to corresponding elements in the codomain. However, she focused more of her descriptions on the sameness side of the equivalence classes. She used the formal definition, and said that she thought she leaned on the formal definition more than other pictures when teaching.

In contrast, Instructor B used formal definition related language through most of his interview. However, at the end he utilized equivalence class and disembedding language as he drew on the structure of the FHT. This is the same progression he intended to use in his class, as he noted the similarity of structures was not as obvious in the context of homomorphism, whereas it was very salient for isomorphism.

Comparing isomorphism and homomorphism, both instructors used sameness language in both contexts. However, sameness was the central picture for both instructors in isomorphism, whereas it was only part of a more complicated picture for homomorphism. For Instructor A, sameness was global in isomorphism as it described the whole groups involved; for homomorphism, it was localized to elements within equivalence classes and, to an extent, that connected to an element in the other group. Instructor B had a similar view of isomorphism, but he did not directly use sameness language to describe homomorphism. Instead, he seemed to view any similarities between groups linked by a homomorphism as hidden similarities, despite a similar structure being shared.

Comparison of instructors' metaphors in class. The instructors used many of the same isomorphism metaphors in class. They both skipped using disembedding, invertible, transformation, and equivalence classes (see Table 5.40), and both used same, same properties, mapping, function, journey, formal definition, operation-preserving, structure-preserving, special homomorphism, renaming/relabeling, and matching. The only difference is that Instructor A also incorporated machine language on a number of occasions, largely in the context of using set diagrams to illustrate function properties like well-defined and everywhere-defined. For example, when the class was discussing what everywhere-defined meant in the context of domain set X and codomain set Y, Instructor A said the operative question was "Can I put all my x's into [relation] g and get something that lives in Y?"

The contexts in which the different clusters of isomorphism metaphors were used were similar as well. Both instructors largely used sameness metaphors to address the big picture ideas of isomorphism and its meaning. Mapping language was used throughout the unit, though it was not usually the point of the discussion. The exception to this was Instructor A's discussion of set diagrams and characteristics of functions. The formal definition was invoked in proof contexts. The main, though subtle, difference was that the instructors used different sameness/mapping metaphors in class. Instructor A incorporated matching metaphors in the tasks before presenting the formal definition, whereas Instructor B highlighted relabeling in the lab day.

Table 5.40. Codes Used for Isomorphism in Instructors' Classes.

Metaphors	Instructor A	Instructor B
Generic Sameness	X	X
Same Properties	X	X
Disembedding		
Generic Mapping	X	X
Invertible		
Function	X	X
Journey	X	X
Machine	X	
Transformation		
Literal Formal Definition	X	X
Operation-preserving	X	X
Structure-preserving	X	X
Special Homomorphism	X	X
Renaming/Relabeling	X	X
Matching	X	X

For homomorphism, the instructors again used similar metaphors. Neither used sameness, transformation, renaming/relabeling, or matching to refer to homomorphism as shown in Table 5.41. Both used similar properties, mapping, function, journey, formal definition, operation-preserving, isomorphism without bijectivity, and equivalence classes. Just Instructor B used disembedding and structure-preserving language, and just Instructor A used machine language. Instructor A used input-output machine language a few times when referring to specific elements in groups that were of interest. Instructor B used disembedding and structure-preserving language to foreshadow and to explain the FHT.

Table 5.41. Codes Used for Homomorphism in Instructors' Classes.

Metaphors	Instructor A	Instructor B
Generic Sameness		
Similar Properties	X	X
Disembedding		X
Generic Mapping	X	X

Function	X	X
Journey	X	X
Machine	X	
Transformation		
Literal Formal Definition	X	X
Operation-preserving	X	X
Structure-preserving		X
Isomorphism without Bijectivity	X	X
Renaming/Relabeling		
Matching		
Equivalence classes	X	X

The instructors used many of the same metaphor clusters for the same purposes in the homomorphism context as well. Mapping metaphors were embedded in the language used to talk about homomorphism, though the instructors did not emphasize these metaphors as the way to think about the concept of homomorphism. Formal definition language was mostly used in proof contexts. Sameness and sameness/mapping language were used to communicate the big ideas for homomorphism, especially in the context of the FHT. However, Instructor B provided this structure in some of his examples when initially looking at examples of homomorphisms as well as when he returned to homomorphism after quotient groups to present the FHT on a regular class day, whereas Instructor A provided this structure on a review day, the day before an exam. Additionally, the choices of sameness related metaphors differed. Whereas Instructor A focused on equivalence classes alone in a collapsing image, Instructor B used both equivalence classes and disembedding to provide a hidden-structure-revealed sense to homomorphism.

Comparing instructors' interview and class metaphors. Both instructors related isomorphism to sameness very clearly in both interview and classroom contexts. They both utilized the renaming/relabeling metaphor that linked sameness and mapping contexts in their interview. Instructor A used this metaphor in class to an extent, but emphasized matching, another sameness/mapping metaphor, more often in class. Instructor B used relabeling when talking to students while working on the lab much as he had used it in his interview. They both used some mapping language and the formal definition in their interviews, but used them much more in class. Mapping language in class was threaded throughout discussions whereas formal definition language was used in proof contexts.

For homomorphism, both instructors articulated sophisticated views of homomorphism centered on the Fundamental Homomorphism Theorem through sameness and sameness/mapping metaphors. However, their uses of sameness-related metaphors were not identical. Instructor A largely used equivalence class metaphors in her interview and articulated a rich image of collapsing a class of elements to a single element and then mapping that class of elements to a single element in the domain. However, this image was only shared with students on the last day of the unit. Instructor B used both equivalence class and disembedding metaphors to reveal a shared structure between groups in an image shared occasionally during the unit, but mostly near the end of the unit. Both instructors started teaching with the formal definition and worked up to their more complicated image of the concept, likely because students did not have the quotient group machinery to understand their more complicated views of homomorphism. Both instructors also used mapping and formal definition language more in class than they had used them in interviews (especially for mapping). Mapping undergirded much of their discussion whereas the formal definition was used in proof-based contexts.

Comparing students' metaphors. The total number of metaphors used by students in class A and B were similar to each other for isomorphism (127 to 122) and for homomorphism (190 to 188). The popularity of specific metaphors within each class varied some. For isomorphism, students in Class B used same properties more than in Class A, whereas students in Class A used the formal definition, matching, and machine more, as shown in Table 5.42.

Students in the two classes used mapping, sameness, journey, invertible, operation-preserving, special homomorphism roughly the same amount. Function, transformation, disembedding, structure-preserving, and renaming/relabeling were not used often in either class.

Table 5.42. Code Frequencies across All Students for Isomorphism.

Metaphors	**Frequency Total**	**Frequency A**	**Frequency B**
Same Properties	70	29	41
Literal Formal Definition	41	27	14
Generic Mapping	36	19	17
Generic Sameness	23	11	12
Matching	23	14	9
Journey	14	6	8
Invertible	11	6	5
Operation-preserving	8	5	3
Special Homomorphism	8	3	5
Machine	5	4	1
Function	2	2	0
Transformation	2	0	2
Disembedding	2	0	2
Structure-preserving	2	0	2
Renaming/Relabeling	2	1	1
Total	249	127	122

In general, students were successful in determining if an isomorphism was present, with successful conclusions in 65/68 scenarios across students. The only scenario students struggled with was looking at if there was an isomorphism between Z_5 and Z/5Z, and most problems stemmed from being unsure how to understand or map to and from a quotient group. "Same properties" was the most common approach overall, as it was used in 48/68 posed problems for isomorphism to note size or order disparities or groups being abelian or not. Students came to correct conclusions in all but one of the scenarios in which it was used (Student 3b for Z_5 and Z/5Z) and the problem for him was related to finding the map; based on same properties he believed there should be an isomorphism. Unlike the other problems, where same properties was generally used, invertibility or the formal definition were the most common ways to approach mapping from 2Z to Z after mapping from Z to 2Z.

Placing isomorphism metaphors in their clusters, students in both classes used sameness metaphors most followed by mapping, formal definition, and same/mapping metaphors, as shown in Table 5.43. However, this ordering does not tell the whole story. As we consider the proportions of use within each class in Figure 5.3, we see 53% of metaphors related to sameness (same and same/mapping) in Class B where only 43% did in Class A. This difference mostly came from the relative use of the formal definition.

Table 5.43. Cluster Frequencies across All students for Isomorphism.

Metaphors	**Frequency Total**	**Frequency A**	**Frequency B**
Sameness	95	40	55
Mapping	70	37	33
Formal Definition	59	35	24
Same/Mapping	25	15	10
Total	249	127	122

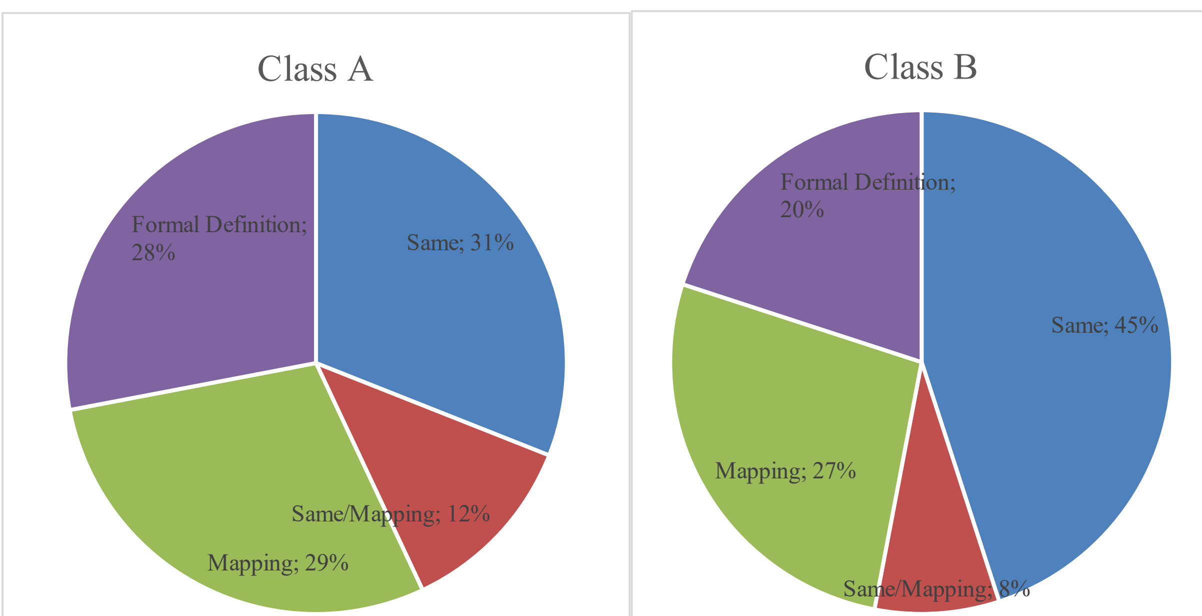

Figure 5.3: Student isomorphism cluster comparisons.

For homomorphism, students in Class A used mapping, matching, and machine more than in Class B, whereas students in Class B used journey, disembedding, same, and, especially, equivalence classes more than in Class A (see Table 5.44). Both classes used the formal definition, isomorphism without bijectivity, similar properties, operation-preserving, and function metaphors roughly the same amount. Neither class used renaming/relabeling, transformation, or structure-preserving metaphors often.

Table 5.44. Code Frequencies across All Students for Homomorphism.

Metaphors	**Frequency Total**	**Frequency A**	**Frequency B**
Generic Mapping	101	67	34
Literal Formal Definition	65	30	35
Journey	45	18	27
Isomorphism without Bijectivity	36	20	16
Similar Properties	26	10	16
Equivalence Classes	23	4	19
Matching	21	15	6
Operation-preserving	20	9	11
Function	14	8	6
Machine	7	7	0
Disembedding	6	0	6
Generic Sameness	5	0	5
Renaming/Relabeling	4	1	3
Transformation	4	1	3
Structure-preserving	1	0	1
Total	378	190	188

For homomorphism, all but Student 3b used the “isomorphism is a special homomorphism” idea to address at least one homomorphism scenario. Because Student 3b used an incorrect definition of homomorphism for most of the interview, it is not surprising he did not make this connection. The use of “similar properties” led to mixed results when addressing homomorphism, unlike the “same properties” metaphor for isomorphism. When the properties invoked for homomorphism were valid, there was not a problem; however, students invented properties when dealing with homomorphism, such as requiring the orders of elements or groups to match (e.g. Student 15a) or needing the same name for the identity (e.g. Student 19b). Seven of the eight students noted the existence of the trivial homomorphism at some point, three across multiple problems and fairly consistently, four more sporadically. Interestingly, four of the students also used the "equivalence classes" metaphor as they leveraged numerical patterns to find homomorphisms. Of these four students, three were the consistent trivial homomorphism students (Students 11a, 11b, and 18b) and the fourth was Student 3b after being given the correct definition of homomorphism. This may suggest that the equivalence class metaphor gave students a structured approach for thinking about how to find homomorphisms, which allowed them to regularly note the existence of the trivial homomorphism. However, this potential relationship requires further research.

Placing homomorphism metaphors in their clusters, both classes used mapping metaphors most, followed by formal definition, same/mapping, and same, as shown in Table 5.45. Relative to isomorphism, these clusters followed the same order, except instead of same metaphors being used the most often, they were used the least often for homomorphism. Once again, this ordering does not tell the whole story. As we consider the proportions of use within each class in Figure 5.4, formal definition metaphors were used roughly a third of the time in both classes. However, in Class B, mapping and sameness-related metaphors were also used roughly a third of the time, whereas in Class A, mapping metaphors were used over half of the time and sameness-related metaphors were used roughly a sixth of the time.

Table 5.45. Cluster Frequencies across All Students for Homomorphism.

Metaphors	**Frequency Total**	**Frequency A**	**Frequency B**
Mapping	171	101	70
Formal Definition	122	59	63
Same/Mapping	48	20	28
Same	37	10	27
Total	378	190	188

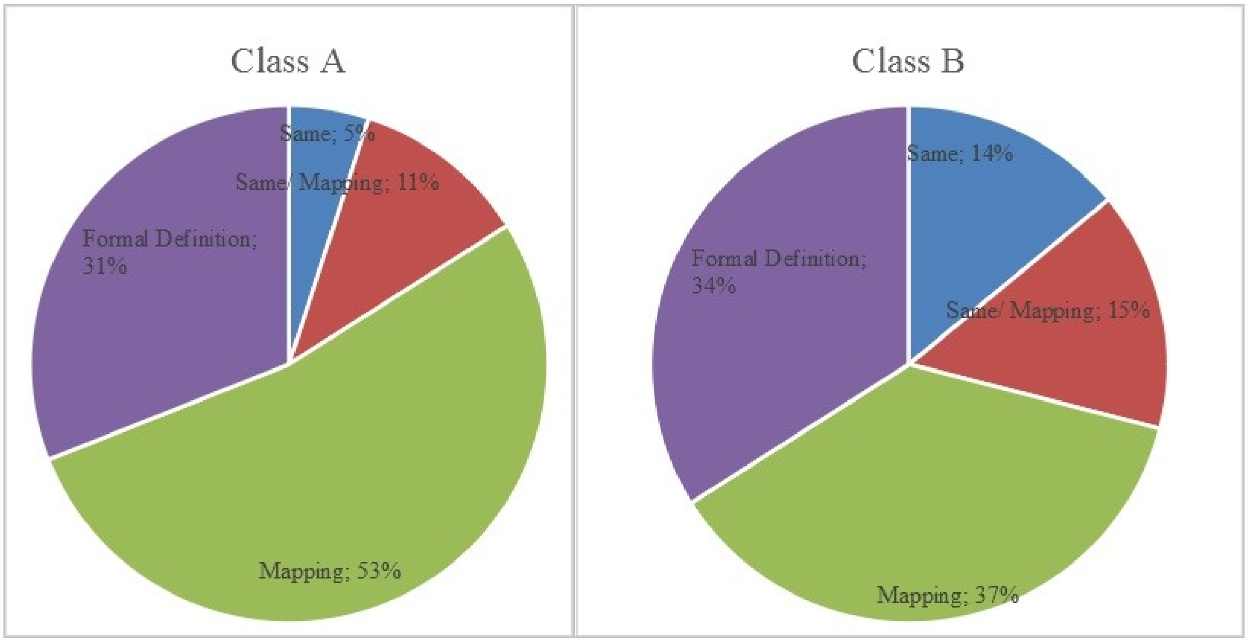

Figure 5.4: Student homomorphism cluster comparisons.

Comparing to the literature, when looking for isomorphisms, students did consider cardinality first, much as Dubinsky et al. (1994) noted. However, they also attended to the orders of elements and the number of elements of those orders, which element was the identity, and whether or not groups were abelian when cardinalities could not answer the question for them, such as when examining the Cayley tables or mapping Z_8 to the symmetries of a square. This aligns with the work of Leron et al. (1995).

For homomorphism, limited work has been done with students finding homomorphisms. However, some students did behave like those from Weber and Alcock (2004) and Weber (2002), in which students tried finding homomorphisms seemingly arbitrarily. This would include students like Student 8a. However, other students like Students 11a, 11b, and 18b, used more methodical approaches to finding homomorphisms, seemingly structured by their knowledge of equivalence classes. This knowledge of equivalence classes links to a way to interpret “structure-preservation” in the homomorphism context, as mentioned by Hausberger (2017).

Relating instructors and instruction to students. Much like comparing the instructor interview and instruction contexts, relationships between what happened in instruction and students' interviews is more clearly aligned for isomorphism than homomorphism. Isomorphism showed conceptual alignment around sameness. For homomorphism, there was much more variety in the language and approaches used by students.

For isomorphism, there was clear alignment in the idea of sameness and the use of same properties between instruction and interview contexts. The conceptual structure of isomorphism suggested in class was centered on sameness. Students exhibited this knowledge through their general descriptions of isomorphism as well as their use of same properties when solving problems.

However, there were two main ways in which the students were more similar to each other than to their instruction: invertibility and renaming/relabeling. Students across both classes commonly used invertibility to address finding a mapping from 2Z to Z even though this metaphor was not used in either class. This may be because invertibility is a property of one-to-one functions that students see in high school and linear algebra. It is possible students were able to make this connection for themselves based on prior knowledge. Students in the two classes also used the renaming/relabeling metaphor at similar frequencies even though there was more emphasis placed on it in Class B than in Class A. This may indicate the conceptual difficulty in renaming/relabeling that allows element names/labels to be arbitrary. Neither instructor emphasized automorphism, which focuses on the number of isomorphisms from a group to itself, which can be viewed as the number of relabelings possible for a group. It is possible that this more advanced concept, treated more in higher level abstract algebra, is integral to the instructors' understanding of isomorphism because they know it is useful for automorphism, but not viewed as essential to students who have not focused on automorphism yet.

For homomorphism, there was even more variation between instruction and students' metaphors. Students transferred more metaphors from other contexts. Transformation, renaming/relabeling, matching, and generic sameness language for homomorphism, which was not used in students' classes, appeared in their interviews. Transformation metaphors appear to have transferred from students' knowledge of functions. Renaming/relabeling, matching, and generic sameness language may have transferred from students' understanding of isomorphism.

However, there was some alignment with regard to the equivalence classes metaphor. Instructor A only used this metaphor in class on the last day of the unit and only one student used it in their interview. Instructor B used disembedding and equivalence class metaphors to foreshadow the FHT and gave students time to process these images, and three of his students used this metaphor. Furthermore, the method of verifying a homomorphism, the formal definition, was used in class and by students regularly in both classes. Students also utilized lots of mapping cluster language throughout their interviews. Mapping cluster language was threaded throughout discussions in class as well.

Chapter 6—Conclusions

In this study, I sought to examine four research questions focused on relationships among beliefs, instruction, and metaphors. To examine these relationships, I looked at the individual components separately for instructors and students. I coordinated the component characterizations to address my research questions and found both alignment and differences between components. While there were many clear connections between the instructors' responses in interviews, there were differences as well. Similarly, students appeared to be influenced by what happened in instruction, though more in their content understanding than in their beliefs about math, learning, and teaching. I end this chapter with the main takeaways and directions for future research.

Research Question Framework

I sought to examine four research questions focused on relationships between instructors' beliefs and instruction, instruction and students' beliefs, instructors' metaphors and instruction, and instruction and students' metaphors. This was highlighted through my research question framework, reproduced below in Figure 6.1. Components for research questions one and two, the upper three boxes, were addressed in chapter four. Components for research questions three and four were addressed in chapter five, the lower three boxes. Instruction is addressed twice to make connections to beliefs and metaphors clearer. Recall, my research questions were:

1. What beliefs do the instructors have about math, teaching, and learning and what relationship exists between these beliefs and instructional practice?
2. What is the relationship between instructional practice and students' beliefs about math, teaching, and learning?

3. What conceptual metaphors do the professors use to describe isomorphisms and homomorphisms and what relationship exists between these metaphors and the mathematical content in instruction?
4. What is the relationship between the mathematical content in instruction and conceptual metaphors the students use to describe isomorphisms and homomorphisms?

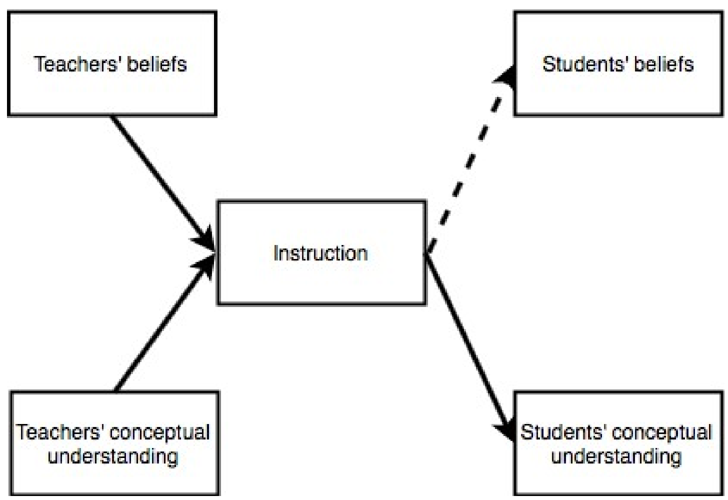

Figure 6.1. Research question framework.

Belief Components

There are some clear connections between what instructors said about their beliefs and how they chose to teach, especially in terms of the overall structures they adopted for their classes. Connections between what happened in instruction and students' beliefs were not as strongly connected, but were still noticeable.

Instructor beliefs. Instructors A and B shared some general beliefs about math, learning, and teaching. Both spoke of math as a field of study and something we do, both emphasized the role of participation and students being active in their classrooms, and both possessed a growth mindset. They also expected their students to perceive components of motivation similarly.

However, the instructors did not possess the exact same beliefs. Instructor A held a "problem-solving" view of math as defined by Ernest (1989). This aligned with her definition of doing math as behaving more like a mathematician. She gave limited space for students to hold other views of what math is, such as that math is about following procedures. She emphasized that students were expected to engage in the tasks and participate in the way that she defined for the class in order for them to build ever-widening consensus on their mathematics. Though she had a firm belief of what math, math learning, and math teaching should look like, she focused on building relationships with students and being inclusive of students in the classroom in order to show she cared and that they belonged in math.

In contrast, Instructor B held a Platonist view of math. This aligned with his view of doing math as proving theorems and applying math to real world contexts, which linked to his belief in the usefulness of math. While he knew he had his own beliefs about math, he felt it was important for students to come to their own conclusions about what math is and not impose his own views. This related to his idea of providing different ways for students to participate and engage with material through both lecture and lab components. Students were encouraged to participate through discussion, but they were not forced to do so. Allowing students to interact with him and with each other as they felt comfortable became a way to also try to include everyone and address their needs.

Instruction. There were some similarities in the two instructors' teaching throughout the semester. Both instructors increased their amount of lecture time across the semester, and both instructors lectured for similar amounts of time during the isomorphism unit. The sharper increases in lecture time occurred between different units for the instructors. Both instructors held the amount of time students worked fairly steady during the first two units, but Instructor B allowed less time for students to work in unit 3, partly due to time constraints.

Differences between instructional practices were more salient. A difference between the instructors throughout the semester was Instructor B never had students explain their work publicly at the board or engage in whole class discussions with students at other tables. Instructor A did, though the time devoted to both of these types of activities decreased across the semester. Differences in questioning patterns were also apparent. Instructor A asked questions more frequently, received answers more frequently, and received questions more frequently than Instructor B across all units.

Student beliefs. Students in Class A entered the course focused on math as problem-solving and attentive to how people reacted to math. These beliefs about the nature of math did not undergo obvious changes. However, the course seemed to influence students' view of teaching and learning math. In particular, students expressed more openness to different ways of learning math, including being guided through approaching problems rather than being told how to do them in a lecture. Furthermore, the students seemed reasonably motivated in the course, as measured by the MUSIC components of motivation, and had growth mindsets.

In Class B, students' views of the nature of math did not seem to change much during the course either. However, they did seem to reflect on how they learned math. All interviewed students seemed open to learning math in a variety of ways. Furthermore, the students seemed reasonably motivated in the course, as measured by the MUSIC components of motivation, and possessed growth mindsets.

Research Question One

Instructors' beliefs influenced their instruction in many ways. The instructors' respective beliefs influenced the large-scale approaches to instruction used in their classes. Furthermore, both instructors put structures in place that allowed students to work together and do mathematics, which they both considered important. However, as other class circumstances shifted, instruction shifted as well. The instructors did not state a desire to do this, so it is possible they did not even notice they were changing the amount of time spent lecturing during the semester.

Differences in the instructors' backgrounds and beliefs were also noticeable in their instruction. Instructor A viewed math as a human activity and focused on having students generate definitions and proofs as a mathematician would. She was less interested in applications for the math she taught outside improving reasoning skills. Instructor B viewed math as discovering axioms and theorems and noted connections to other mathematical disciplines. He also spent at least two class periods each week in a largely lecture format in which he would present theorems and proofs of theorems to the class.

Thus, to summarize the answer to this research question, there were relationships between what instructors said in their interviews and what they did in class, especially with regard to their large-scale instructional choices. This included their curriculum and class formatting choices. However, the relationship between beliefs in interviews and day-to-day instructional choices, like how much time to spend lecturing or allowing students to work, became less aligned as the semester wore on.

Research Question Two

There appears to be a subtle relationship between instruction and students' beliefs about what is possible in a college math class. Specifically, students seemed more open to new methods of instruction at the end of the course. In Class A, this took the form of openness to guided instruction, whereas in Class B this took the form of wanting to change-up what happened in class each day. However, students' beliefs about the nature of math did not noticeably change despite different choices being made in instruction by the instructors. This suggests some beliefs about math, learning, and teaching may be largely set by the time a student takes an upper-level math class for math majors. Alternatively, the measures I used may not have been able to perceive the changes that occurred. Future research examining students' beliefs more frequently could be considered to address this question.

More intriguingly, broad beliefs about the changeability of intelligence and students' motivation seemed similar in the two classes despite different methods of instruction. This is especially intriguing as the instructors noted different ways of fostering components of motivation, but received scores that were not significantly different from each other. For example, Students in Class A perceived empowerment through some control over the pace of the tasks in class, whereas Students in Class B perceived empowerment through outside class options, like access to outside resources or being able to decide how often they wanted to attend class.

Summarizing the answer to this research question, instruction seemed to have some effect on students' beliefs about possibilities for college math instruction. Namely, students seemed more open to non-lecture methods of instruction in line with what they experienced in their class, whether guided instruction or a variety of instructional types. However, students in the two classes were similar on many measures despite experiencing different types of instruction, indicating some limitations to the influence of instruction on students' beliefs, and raising questions about how similar math majors' beliefs and motivations are by the time they reach their upper-level math classes.

Metaphors

Clear connections were observed between instructors' metaphors in interviews and the metaphors they chose to use in class. Most metaphors used in interviews were also used in class, and some metaphors were added in class when a greater variety of activities happened. Students did not use many metaphors that had not been used in class. Most metaphors of this type related to functions, which they had previously been exposed to in high school and other math classes or were moved from the isomorphism to homomorphism context or vice versa.

Instructor metaphors. Instructor A used sameness and equivalence ideas when describing both isomorphism and homomorphism. However, the sameness she focused on was global in the isomorphism context and narrow in the homomorphism context. Instructor B had a clear focus of isomorphism as revealing mathematical objects to have the same structure. He viewed the main point of isomorphism as revealing when structures were essentially the same rather than the fact that there was a way to move between such structures via a mapping. For homomorphism, he did not feel that the shared structure was obvious; however, he noted the hidden structure he saw in homomorphism through equivalence classes. He progressed from discussing the formal definition to this structural image in both his interview and instruction.

The majority of both instructors' metaphors related to sameness or combined sameness and mapping for isomorphism. This language mostly focused on general sameness and the specific mapping/sameness image of renaming or relabeling. Both also shared a greater emphasis on the groups being isomorphic than on the isomorphism function that connected the two groups. Both used formal definition and mapping language in a secondary capacity.

For homomorphism, both instructors used formal definition language and had a limited focus on mapping language in their interviews. Both instructors also used equivalence class language to provide imagery, though Instructor A expanded more on this metaphor in her interview. She also used the formal definition, and said that she thought she leaned on the formal definition more than other pictures when teaching. In contrast, Instructor B largely used formal definition related language through most of his interview. However, at the end he utilized equivalence class and disembedding language as he drew on the structure of the FHT. This is the same progression he ultimately used in his class.

Both instructors used sameness language to describe both isomorphism and homomorphism. However, sameness was the central picture for both instructors in isomorphism, whereas it was only part of a more complicated picture for homomorphism. For Instructor A, sameness was global in isomorphism but localized to elements within equivalence classes in homomorphism. Instructor B had a similar view of isomorphism, but he did not directly use sameness language to describe homomorphism. Instead, he seemed to view homomorphisms as revealing hidden similarities through disembedding and equivalence classes.

Metaphors in instruction. For Instructor A, sameness was used as a lens for approaching defining and verifying the existence of an isomorphism. Mapping language was used constantly, but was only central to discussions about isomorphisms being functions. Sameness/mapping language was used when initially defining for renaming and throughout tasks for matching, but was used less after the formal definition was given. The formal definition was largely used in proof contexts.

For homomorphism, the formal definition was the dominant metaphor used in class. Mapping metaphors were also used prominently, though more to express ideas common to any function than properties specific to homomorphism. The equivalence class metaphors were only introduced during a review day on the day before an exam, though they featured prominently on that day when the Fundamental Homomorphism Theorem (FHT) was also discussed.

For Instructor B, sameness metaphors were largely used when speaking in general terms or to show groups were not isomorphic. Mapping metaphors were used often, but were not a content focus. Sameness/mapping metaphors, especially relabeling, were used often on the initial lab day, but less later. Formal definition metaphors were used most in proof contexts, though operation-preservation was used as a proxy for the homomorphism property at other times.

Instructor B used mapping metaphors the most frequently for homomorphism, though many usages were in passing and could have been used with any function, not just a homomorphism. He also used formal definition language often, especially in the context of proofs. Some sameness metaphors were used, though sameness itself was not. The sameness/mapping metaphor of equivalence classes was worked into a number of examples and structured his second teaching of homomorphism (after quotient groups had been taught).

Both instructors largely used sameness metaphors to address the big picture ideas of isomorphism and its meaning. The main, though subtle, difference was that the instructors used different sameness/mapping metaphors in class. Instructor A incorporated matching metaphors in the tasks before presenting the formal definition, whereas Instructor B highlighted relabeling in the lab day. The instructors used similar approaches to the homomorphism context as well. Sameness and sameness/mapping language were used to communicate the big ideas for homomorphism, especially in the context of the FHT. However, Instructor B provided this structure in examples throughout the unit as well as when teaching the FHT. Instructor A provided this structure on a review day, the day before an exam. Additionally, the choices of sameness related metaphors differed. Whereas Instructor A focused on equivalence classes alone in a collapsing image, Instructor B used both equivalence classes and disembedding to provide a hidden-structure-revealed sense to homomorphism.

Student metaphors. In general, students were successful in determining if an isomorphism was present. The only scenario students struggled with involved a quotient group, and most problems stemmed from being unsure how to understand the quotient group. “Same properties” was the most common approach overall, and students came to correct conclusions in all but one of the scenarios in which it was used. Unlike the other problems, where same properties was generally used, invertibility or the formal definition were the most common ways to approach mapping from 2Z to Z after mapping from Z to 2Z.

Students in both classes used sameness metaphors most followed by mapping, formal definition, and same/mapping metaphors. However, more metaphors related to sameness (same and same/mapping) were used in Class B than Class A. This difference mostly came from the relative use of the formal definition.

For homomorphism, all but the student with the incorrect definition of homomorphism used the “isomorphism is a special homomorphism” idea to address at least one homomorphism scenario. When using true properties, students were often successful. However, students also invented properties when dealing with homomorphism, such as requiring the orders of elements to match or needing the same name for the identity. Seven of the eight students noted the existence of the trivial homomorphism at some point, three across multiple problems and fairly consistently, four more sporadically. There appears to be a tentative link between consistent use of the trivial homomorphism and equivalence classes. This may suggest that the equivalence class metaphor gave students a structured approach for thinking about how to find homomorphisms, which allowed them to regularly note the existence of the trivial homomorphism. However, this potential relationship requires further investigation.

Students in both classes used mapping cluster metaphors most, followed by formal definition, same/mapping, and sameness. Relative to isomorphism, these clusters followed the same order, except instead of sameness metaphors being used the most often, they were used the least often for homomorphism. This ordering does not tell the whole story. Formal definition metaphors were used roughly a third of the time in both classes. However, in Class B mapping and sameness-related metaphors were also used roughly a third of the time, whereas in Class A, mapping metaphors were used over half of the time and sameness-related metaphors were used roughly a sixth of the time.

Research Question Three

For Instructor A, there was fairly clear alignment at the conceptual level in addressing isomorphism. The instructor focused on sameness and the sameness/mapping metaphor of renaming in the interview as she described the core of what an isomorphism is. In class, she again used sameness and sameness/mapping metaphors to build the idea of what an isomorphism is, though matching, not renaming was the common metaphor used. Instructor B also had fairly clear alignment at the conceptual level. He focused on sameness and the sameness/mapping metaphor of relabeling in the interview as he described the core of what an isomorphism is. In class, he again used sameness and sameness/mapping metaphors to build the idea of what an isomorphism is.

However, differences in the frequency of mapping cluster and formal definition metaphors were obvious between the interview and instructional contexts. These differences seem attributable to the differences in the tasks being accomplished and questions being posed in the two contexts. In the interview, instructors were asked about their main conceptual views of isomorphism and were not asked to solve problems. In class, they provided conceptual views, but also solved problems and did proofs, leading to contexts to use more metaphors.

For homomorphism, the sameness understanding that Instructor A clearly articulated in her interview was not nearly as clear in instruction. Equivalence classes as a way of understanding homomorphism were brought up on the last day of the unit. Thus, her sameness approach to homomorphism was obscured in the class. This mismatch could connect to the number of concepts that need to be coordinated to think about equivalence classes, including an understanding of quotient groups. Instructor B's discussions in class were clearer than his statements in the interview. He articulated an image of homomorphism from disembedding and equivalence classes that he fleshed out further in class when discussing how to find the structure of the range that was "already living inside" the domain that could be found through quotient groups.

Summarizing conclusions, for isomorphism there was reasonably clear alignment between what instructors said in interviews and instruction, with most differences attributable to the greater variety of tasks being addressed in instruction. For homomorphism, the relationship between contexts was less clear. Instructor A focused on sameness, especially through equivalence classes, in her interview, but this view of homomorphism only appeared in instruction on the last day of the unit. Instead, she focused on the formal definition as a way to reason about homomorphism for most of the unit. Instructor B struggled to articulate a view of homomorphism outside the formal definition for most of his interview, though he eventually provided limited disembedding and equivalence class views. In class, he articulated these views more clearly and threaded them through his instruction, though he did start from the formal definition as well. Thus, homomorphism metaphors were only limitedly aligned between the two contexts and seemed to be more context dependent.

Research Question Four

Much like comparing the instructor interview and instruction contexts, relationships between what happened in instruction and students' interviews is more clearly aligned for isomorphism than homomorphism. Isomorphism showed conceptual alignment around sameness. For homomorphism, there was more variety in the language and approaches used by students.

For isomorphism, there was clear alignment in the idea of sameness and the use of same properties between instruction and interview contexts. The conceptual structure of isomorphism suggested in class was centered on sameness. Students exhibited this knowledge through their general descriptions of isomorphism as well as their use of same properties when solving problems.

However, there were two main ways in which the students were more similar to each other than to their instruction: invertibility and renaming/relabeling. Students across both classes commonly used invertibility to address finding a mapping from 2Z to Z even though this metaphor was not used in either class. Students in the two classes also used the renaming/relabeling metaphor at similar frequencies even though there was more emphasis placed on it in Class B than in Class A. Neither instructor emphasized automorphism in their course. It is possible that this more advanced concept, treated more in higher level abstract algebra, is integral to the instructors' understanding of isomorphism because they know it is useful for automorphism, but not viewed as essential to students who have not focused on automorphism yet.

For homomorphism, there was even more variation between instruction and students' metaphors, though some overall approaches were shared. The formal definition, which was used to verify homomorphisms, was used in class and by students from both courses regularly. Students also utilized mapping cluster language throughout their interviews, much as had happened in class discussions. Usage of the equivalence classes metaphor also followed the pattern that might be expected given class instruction. Instructor A only used this metaphor in class on the last day of the unit and only one student used it in their interview. Instructor B used disembedding and equivalence class metaphors to foreshadow the FHT and gave students time to process these images, and three of his students used this metaphor.

However, there was more metaphor transference for homomorphism than isomorphism. Transformation, renaming/relabeling, matching, and generic sameness language for homomorphism, which was not used in students' classes, appeared in their interviews. Transformation metaphors appear to have transferred from students' knowledge of functions. Renaming/relabeling, matching, and generic sameness language may have transferred from students' understanding of isomorphism.

Summarizing the response to the research question, most metaphors for isomorphism and homomorphism used by students were also used by the instructor at some point, possibly in the context of the other mapping. Thus, some relationship between what happened in class and students' metaphors exists. However, the relative frequencies and importance of metaphors in class did not always transfer to students' metaphor use. Furthermore, students transferred some metaphors between isomorphism and homomorphism contexts, and also seemed to use outside knowledge of functions to reason about isomorphism and homomorphism, indicating instruction was not the only influence on their understanding of isomorphism and homomorphism.

Conclusions and Future Work

This study aimed to push the field forward by giving more details on what happens in an abstract algebra classroom, how beliefs connect to instruction at the college level, and what understandings of isomorphism and homomorphism are brought to and taken from the classroom.

This work extends the literature on characterizing college instruction in three main ways. First, these instructors characterized their instruction differently than “Dr. Tripp,” (Fukawa-Connelly, 2012), who characterized herself as a lecturer. This allowed characterizations of other types of abstract algebra instruction to be brought to the literature. Breakdowns of time spent on different activities and measures of interactivity can give other researchers and interested instructors something to compare to, especially because research had not previously focused on specifying how much time instructors spent on different activities in upper-level college math courses. Second, this research suggests that taking a one-unit snapshot of an instructor’s teaching can be misleading. As content and time demands change, instructors may vary how much they allow students to explore mathematics before explaining what students should take away. Third, examining the instruction of non-tenure track/tenured instructors can potentially bring new perspectives to instruction and who the audience of our research is.

Although the instructors employed different methods of instruction, students took away similar beliefs about the nature of math, teaching, and learning to what they brought with them. This suggests that students’ beliefs about math may be somewhat set by the time they are taking upper-level math courses in college. However, students’ views about math teaching were slightly shifted based on classes that were different from normal. Although students’ beliefs were shifted in slightly different ways, this may suggest that helping instructors change their teaching to include a little more active learning components could already provide benefits to students, in terms of changing their view of what teaching and learning math can look like. Thus, we need not encourage instructors to try methods well outside their comfort zones when benefits could come from more limited changes.

Although outside my research questions, the disconnected experience of Student 18b raises questions about the role of students' beliefs on each other. Recall Student 18b had noted feeling disconnected from his peers when his interest in the material was deeper than his theirs and when he would seek out other resources and they would be content with the notes they were given. As a mathematician and math educator, I am enthused by Student 18b's interest and want to encourage him to stay in the math major and continue his graduate work in math. However, the literature suggests high-performing students like Student 18b can be pushed out of majors if not led to feel a sense of belonging (e.g. Danielak, Gupta, & Elby, 2014 in an engineering context). Further research on what factors can compensate for poor experiences with peers or in classes should be done to help retain these high-powered students.

In addition to implications for instruction and beliefs, this study also suggests that there are connections between the language used in class and the ways students structure their understanding of concepts. Future research should address which metaphors are the most helpful for fostering an understanding of isomorphism and homomorphism in students. In future, I intend to compare students' individual scores on achievement measures to their usage of metaphors to look for any correlations between specific metaphor use and achievement as another way of considering if any class(es) of metaphor were more powerful for students than others.

Instructors were largely consistent in their language in interviews and in class. Some new metaphors were introduced in class, but the greater variety of tasks in class largely explains this discrepancy. The most unusual difference was that Instructor A used matching language for isomorphism in class, but not in her interview. This is likely due to the nature of tasks in class and the complexity of the relabeling metaphor that she favored in her interview. While she could leverage her advanced mathematical knowledge to connect her image to automorphisms (self-isomorphisms) students were not thinking about this alternative as they were just being introduced to isomorphism. For homomorphism, her emphasis on sameness that was not taken up by students was also interesting, especially because students like Student 11a struggled to disambiguate isomorphism and homomorphism relative to sameness. Moving forward, explicitly addressing students' and instructors' views of the meaning of sameness could prove enlightening.

Instructor B was largely consistent in his use of language in the interview and in class. However, it is interesting that he could not find an analogy for a ten-year-old to describe homomorphism that he was satisfied with, though he used disembedding elsewhere in his interview and more clearly in class. Perhaps he considered the ideas around disembedding still too complex for a ten-year-old to understand. Alternatively, the disembedding metaphor he invoked might be situated in his classroom teaching knowledge, and not something that he naturally describes when thinking about homomorphism (Greeno, Moore, & Smith, 1993).

For the students, there was variety in the metaphors individual students used in different contexts and across students. However, general ideas of sameness related to isomorphism were communicated by all students, and same properties was a standard approach to isomorphism prompts. Homomorphism problems induced more variety in the approaches used. Using equivalence classes seemed to be beneficial for the students that used it. Additionally, the students who consistently noted the trivial homomorphism were also the students that used this equivalence class metaphor. This may be because students were connecting the FHT's permissible sizes of ranges to possible mappings. This may also relate to students' "craving for canonical procedures" expressed in isomorphism (Leron et al., 1995). Having the structure of the FHT may have allowed students to feel more comfortable in approaching the search for mappings. However, further research on this topic is needed.

In terms of connecting student work to the instructors' statements, most metaphors used by students were also used by the instructors sometime in class. The few metaphors not used by instructors were connected to functions, were used for the other mapping, or were extensions of sameness or the formal definition. The most interesting of these disconnects, however, is invertibility for isomorphism. Invertibility is central to the definition of isomorphism given in category theory. For example:

> Definition 1.1.9: An isomorphism in a category is a morphism $f: X \to Y$ for which there exists a morphism $g: Y \to X$ so that $gf = 1_X$ and $fg = 1_Y$. The objects X and Y are isomorphic whenever there exists an isomorphism between X and Y, in which case one writes $X \cong Y$ (Riehl, 2016, p. 7).

Yet neither instructor focused on this idea related to isomorphism. Some students did leverage the idea, especially in the context of going to and from the same groups, but it was not a universally understood or expected idea. Further research is needed on whether other instructors with algebra or category theory focused research programs use similar language or different language to instructors who do not do research in these areas.

Another direction for future research relates to the sameness/mapping metaphors. Renaming/relabeling and matching are similar metaphors. They both integrate elements of sameness and matching in the context of isomorphism. However, the matching metaphor retains some emphasis on the names of elements. For example, the tasks in Class A encouraged students to see if the dihedral group of six elements (symmetries of an equilateral triangle) were equivalent to a given six element group. In practice, this led to students trying to align elements from the groups to each other. However, students were satisfied when they found some alignment that worked, and did not make the connection to the fact that some elements of the same order, if named differently, could have been aligned with the group of interest instead. Renaming/relabeling pushes past the names of elements and allows for the arbitrariness of names and starts laying the groundwork for thinking about group automorphisms. While the instructors seemed to have this understanding, students did not seem attuned to these differences. Future research should examine how students begin understanding automorphism.

The FHT served as a structure for both instructors' view of homomorphism through equivalence classes. However, they structured their view of homomorphism differently. Instructor A mostly focused on collapsing equivalence classes to a single representative element and then mapping to a corresponding element. Instructor B focused on collapsing and mapping simultaneously as threads condensed to a single element. He also utilized disembedding as he noted the structure that was hidden in the domain could be made visible through a homomorphism. While these instructors focused on the FHT to structure their thinking of homomorphism, it remains to be seen whether other instructors do so as well or have other ways to do so. Future research should examine whether other instructors, including algebraists think about homomorphism in the same or different ways.

The FHT structure behind equivalence classes also appears to have helped students think through more possibilities for homomorphisms. Students with the equivalence class metaphor noted the trivial homomorphism more regularly than other students. Further research should examine how this knowledge of equivalence classes could be leveraged to aid students' understanding of homomorphisms.

References

Aguirre, J., & Speer, N. M. (1999). Examining the relationship between beliefs and goals in teacher practice. *Journal of Mathematical Behavior, 18*(3), 327–356.

Anderson, J., White, P., & Sullivan, P. (2005). Using a schematic model to represent influences on, and relationships between, teachers' problem-solving beliefs and practices. *Mathematics Education Research Journal, 17*(2), 9–38.

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M., Weller, K. (2014). Genetic decomposition. In *APOS theory: A framework for research and curriculum development in mathematics education*, 27–55. New York, NY: Springer.

Aronson, J., Fried, C. B., & Good, C. (2002). Reducing the effects of stereotype threat on African American college students by shaping theories of intelligence. *Journal of Experimental Social Psychology, 38*(2), 113–125. doi: 10.1006/jesp.2001.1491

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. *Research in Collegiate Mathematics Education II, CBMS Issues in Mathematics Education, 6,* 1–32.

Asiala, M., Dubinsky, E., Mathews, D. M., Morics, S., Oktac, A. (1997). Development of students' understanding of cosets, normality, and quotient groups. *Journal of Mathematical Behavior, 16*(3), 241–309.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407. doi: 10.1177/0022487108324554

Barkatsas, A., & Malone, J. (2005). A typology of mathematics teachers' beliefs about teaching and learning mathematics and instructional practices. *Mathematics Education Research Journal, 17*(2), 69–90.

Beswick, K. (2005). The beliefs/practice connection in broadly defined contexts. *Mathematics Education Research Journal, 17*(2), 39–68.

Beswick, K. (2012). Teachers' beliefs about school mathematics and mathematicians' mathematics and their relationship to practice. *Educational Studies in Mathematics, 79*(1), 127–147.

Blair, R., Kirkman, E. E., & Maxwell, J. W. (2013). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States: Fall 2010 CBMS Survey.* Washington DC: American Mathematical Society.

Boaler, J. (1999). Participation, knowledge, and beliefs: A community perspective on mathematics learning. *Educational Studies in Mathematics, 40*(3), 259–281.

Boaler, J. (2000). Mathematics from another world: Traditional Communities and the alienation of learners. *Journal of Mathematical Behavior, 18*(4), 379–397.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple Perspectives on Mathematics Teaching and Learning*, 171–200. Westport, CT: Ablex Publishing.

Bonne, L., & Johnston, M. (2016). Students' beliefs about themselves as mathematics learners. *Thinking Skills and Creativity, 20*, 17–28.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101.

Calderhead, J. (1981). Stimulated Recall: a Method for Research on Teaching. *Br. J. Educ. Psychol*, *51*, 211–217. https://doi.org/10.1111/j.2044-8279.1981.tb02474.x

Cilli-Turner, E. (2017). Impacts of inquiry pedagogy on undergraduate conceptions of the function of proof. *Journal of Mathematical Behavior, 48*, 14–21. doi: 10.1016/j.jmathb.2017.07.001

Cobb, P., Gresalfi, M., & Hodge, L., L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education, 40*(1), 40–68.

Cohen, D. K. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis, 12*(3), 311–329.

Cooney, T. (1985). A beginning teacher's view of problem solving. *Journal for Research in Mathematics Education, 16*(5), 324–336.

Cribbs, J. D., Hazari, Z., Sonnert, g., & Sadler, P. M. (2015). Establishing an explanatory model for mathematics identity. *Child Development, 86*(4), 1048–1062.

Cross, D. I. (2009). Alignment, cohesion, and change: Examining mathematics teachers' belief structures and their influence on instructional practices. *Journal of Mathematics Teacher Education, 12*, 325–346.

Danielak, B. A., Gupta, A., & Elby, A. (2014). Marginalized identities of sense-makers: Reframing engineering student retention. *Journal of Engineering Education, 103*(1), 8-44. doi:10.1002/jee.20035

De Castella, K., & Byrne, D. (2015). My intelligence may be more malleable than yours: the revised implicit theories of intelligence (self-theory) scale is a better predictor of achievement, motivation, and student disengagement. *European Journal of Psychology of Education*, *30*(3), 245–267. https://doi.org/10.1007/s10212-015-0244-y

De Corte, E., Op't Eynde, P., & Verschaffel, L. (2002). Knowing what to believe: The relevance of students' mathematical beliefs for mathematics education. In B. K. Hofer & P. Ro. Pintrich (Eds.), *Personal Epistemology: The Psychology of Beliefs about Knowledge and Knowing* (pp. 297 – 320). Mahwah, NJ: Lawrence Erlbaum Associates.

Deci, E. L., & Ryan, R. M. (2000). The "what" and "why" of goal pursuits: Human needs and the self-determination of behavior. *Psychological Inquiry, 11*(4), 227–268.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27*(3), 267–305.

Dubinsky, E., Dauterman, J., Leron, U., & Zazkis, R. (1997). A reaction to Burn's "What are the fundamental concepts of group theory?" *Educational Studies in Mathematics, 34*(3), 249–253.

Dummit, D. S., & Foote, R. M. (2004). *Abstract Algebra* (3rd ed.). Hoboken, NJ: Wiley.

Dweck, C. S. (2007). Is math a gift? Beliefs that put females at risk. In S. J. Ceci & W. M. Williams (Eds.), *Why aren't more women in science?: Top researchers debate the evidence* (pp. 47–55). doi: 10.1037/11546-004

Dweck, C. S., Chiu, C., & Hong, Y. (1995). Implicit Theories and Their Role in Judgments and Reactions: A Word From Two Perspectives. *Psychological Inquiry*, *6*, 267–285. https://doi.org/10.1207/s15327965pli0604_1

Dweck, C. S., & Master, A. (2009). Self-theories and motivation: Students' beliefs about intelligence. In K. R. Wentzel & A. Wigfield (eds.), *Handbook of Motivation at School*, 123–140. New York, NY: Routledge.

Ernest, P. (1989). The knowledge, beliefs, and attitudes of the mathematics teacher: A model. *Journal of Education for Teaching, 15*(1), 13–33. doi: 10.1080/0260747890150102

Ernest, P. (1991). *The Philosophy of Mathematics Education.* London: The Falmer Press.

Fennema, E., Carpenter, T. P., & Peterson, P. (1989). Teachers' decision making and cognitively guided instruction: A new paradigm for curriculum development. In N. F. Ellerton & M. A. Clements (Eds.), *School Mathematics: The challenge to change* (pp. 174–187). Geelong: Deakin University Press.

Flexer, R. J., Cumbo, K., Borko, H., Mayfield, V., & Marion, S. F. (1994, April). *How "messing about" with performance assessment in mathematics affects what happens in classrooms.* Paper presented at the annual meeting of the American Educational Research Association and the National Council on Measurement in Education, New Orleans.

Fukawa-Connelly, T. P. (2012). A case study of one instructor's lecture-based teaching of proof in Abstract Algebra: Making sense of her pedagogical moves. *Educational Studies in Mathematics, 81*(3), 325–345.

Fukawa-Connelly, T. (2016). Responsibility for proving and defining in Abstract Algebra. *International Journal of Mathematical Education in Science and Technology, 47*(5), 733–749. doi: 10.1080/0020739X.2015.1114159

Fukawa-Connelly, T., Johnson, E., & Keller, R. (2016). Can math education research improve the teaching of Abstract Algebra? *Notices of the AMS, 63*(3), 276–281. doi: 10.1090/noti1339

Fylan, F. (2005). Semi structured interviewing. In *A handbook of research methods for clinical and health psychology* (pp. 65–78).

Gallian, J. A. (2013). *Contemporary Abstract Algebra* (8th ed.). Stamford, CT: Cengage Learning.

Gilmore, J., Maher, M. A., Feldon, D. F., & Timmerman, B. (2014). Exploration of factors related to the development of science, technology, engineering, and mathematics graduate teaching assistants' teaching orientations. *Studies in Higher Education, 39*(10), 1910–1928.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning 1*(2), 155–177.

Green, T. F. (1971). Teaching and the formation of beliefs. In *The Activities of Teaching*, 41–63. New York, NY: McGraw-Hill.

Greeno, J. G., Moore, J. L., & Smith, D. R. (1993). Transfer of situated learning. In D. K. Detterman & R. J. Sternberg (Eds.), *Transfer on trial: Intelligence, cognition, and instruction* (pp. 99-167). Westport, CT, US: Ablex Publishing.

Gresalfi, M. S. (2009). Taking up opportunities to learn: Constructing dispositions in mathematics classrooms. *The Journal of the Learning Sciences, 18*(3), 327–369.

Guskey, T. R. (2002). Professional development and teacher change. *Teachers and Teaching: Theory and Practice, 8*(3/4), 381–391.

Hausberger, T. (2017). The (homo)morphism concept: Didactic transposition, meta-discourse, and thematisation. *International Journal of Research in Undergraduate Mathematics Education, 3*(3), 417–443. doi: 10.1007/s40753-017-0052-7

Hekimoglu, S., & Kittrell, E. (2010). Challenging students' beliefs about mathematics: The use of documentary to alter perceptions of efficacy. *PRIMUS, 20*(4), 299–331.

Hidi, S., & Renninger, K. A. (2006). The four-phase model of interest development. *Educational Psychologist, 41*(2), 111–127.

Hofer, B. K. (1999). Instructional Context in the College Mathematics Classroom: Epistemological Beliefs and Student Motivation. *Journal of Staff, Program & Organization Development*, *16*(2), 73-82.

Honderich, T. (ed.). (1995). *The Oxford Companion to Philosophy.* New York: Oxford University Press.

Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *The Journal of Mathematical Behavior*, *16*(2), 145–165. https://doi.org/10.1016/S0732-3123(97)90023-7

Jao, L. (2017). Shifting pre-service teachers' beliefs about mathematics teaching: The contextual situation of a mathematics methods course. *International Journal of Science and Mathematics Education, 15*(5), 895–914.

Johnson, E. (2013). Teachers' mathematical activity in inquiry-oriented instruction. *The Journal of Mathematical Behavior, 32*(4), 761–775.

Johnson, E., Caughman, J., Fredericks, J., & Gibson, L. (2013). Implementing inquiry-oriented curriculum: From the mathematicians' perspective. *The Journal of Mathematical Behavior, 32*(4), 743–760.

Johnson, E., Keller, R., & Fukawa-Connelly, T. (2017). Results from a survey of Abstract Algebra instructors across the United States: Understanding the choice to (not) lecture. *International Journal of Research in Undergraduate Mathematics Education*. doi: 10.1007/s40753-017-0058-1

Johnson, E., Keller, R., Peterson, V., & Fukawa-Connelly, T. (2017). Individual and situational factors related to undergraduate mathematics instruction. Manuscript.

Johnson, E., Andrews-Larson, C., Keene, K., Melhuish, K., Keller, R., & Fortune, N. (2019). Inquiry does not guarantee equity. In Weinberg, A., Moore-Russo, D., Soto, H., and Wawro, M. (Eds.), *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education.* (pp. 282–291), Oklahoma City, OK.

Johnson, E. M. S., & Larsen, S. P. (2012). Teacher listening: The role of knowledge of content and students. *The Journal of Mathematical Behavior, 31*, 117–129. doi: 10.1016/j.jmathb.2011.07.003

Jones, B. D. (2009). Motivating students to engage in learning: The MUSIC Model of Academic Motivation. *International Journal of Learning in Higher Education, 21*, 272–285.

Jones, B. D. (2015). *Motivating students by design: Practical strategies for professors.* Charleston, SC: CreateSpace.

Jones, B. D. (2017). User guide for assessing the components of the MUSIC model of academic motivation. Retrieved from http://www.themusicmodel.com

Jones, B. D., & Skaggs, G. (2016). Measuring students' motivation: Validity evidence for the MUSIC Model of Academic Motivation Inventory, *International Journal for the Scholarship of Teaching and Learning, 10*(1), Article 7. doi: 10.20429/ijsotl.2016.100107

Katz, I. (2016). In the eye of the beholder: Motivational effects of gender differences in perceptions of teachers. *The Journal of Experimental Education, 0*(0), 1–18. doi: 10.1080/00220973.2015.1101533

Kiemar, K., Groschner, A., Pehmer, A-K, & Seidel, T. (2015). Effects of a classroom discourse intervention on teachers' practice and students' motivation to learn mathematics and science. *Learning and Instruction, 35*, 94 – 103. doi: 10.1016/j.learninstruc.2014.10.003

Kuster, G., Johnson, E., Keene, K., & Andrews-Larson, C. (2017). Inquiry-oriented instruction: A conceptualization of the instructional principles, *PRIMUS, 28*(1), 13-30. doi: 10.1080/10511970.2017.1338807

Kuster, G., Johnson, E., Rupnow, R., & Wilhelm, A. G. (2019). The inquiry-oriented instructional measure. *International Journal of Research in Undergraduate Mathematics Education. doi: 10.1007/s40753-019-00089-2*

Lakoff, G. (1987). *Women, fire, and dangerous things: What categories reveal about the mind.* Chicago: The University of Chicago Press.

Lakoff, G. & Johnson, M. (1980). *Metaphors we live by.* Chicago: The University of Chicago Press.

Lakoff, G., & Núñez, R. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In Lyn D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 21–89). Mahwah, NJ: Erlbaum.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York: Basic Books.

Larsen, S. (2009). Reinventing the concepts of group and isomorphism: The case of Jessica and Sandra. *The Journal of Mathematical Behavior, 28*(2-3), 119–137.

Larsen, S. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*, *32*(4), 712–725.

Larsen, S., Johnson, E., & Bartlo, J. (2013). Designing and scaling up an innovation in Abstract Algebra. *The Journal of Mathematical Behavior, 32*(4), 693–711.

Larsen, S., & Lockwood, E. (2013). A local instructional theory for the guided reinvention of the quotient group concept. *The Journal of Mathematical Behavior, 32*(4), 726–742.

Lazarides, R. & Watt, H. M. G. (2015). Girls' and boys' perceived mathematics teacher beliefs, classroom learning environments and mathematical career intentions. *Contemporary Educational Psychology, 41*, 51–61.

Leatham, K. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education, 9*(1), 91–102. doi: 10.1007/s10857-006-9006-8

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics, 29*(2), 153–174.

Lew, K., Fukawa-Connelly, T. P., Mejia-Ramos, J. P., & Weber, K. (2016). Lectures in advanced mathematics: Why students might not understand what the mathematics professor is trying to convey. *Journal for Research in Mathematics Education, 47*(2), 162–198.

Liu, P., & Liu, S. (2011). A cross-subject investigation of college students' epistemological beliefs of physics and mathematics. *The Asia-Pacific Education Researcher, 20*(2), 336–351.

Markovits, Z., & Forgasz, H. (2017). "Mathematics is like a lion": Elementary students' beliefs about mathematics. *Educational Studies in Mathematics, 96*, 49–64. doi: 10.1007/s10649-017-9759-2

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (ed.), *Handbook of Research on Mathematics Teaching and Learning*, 575–596. New York: Macmillan Publishing Company.

Melhuish, K. M. (2015). *The design and validation of a group theory concept inventory: A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics Education* (doctoral dissertation). Retrieved from ProQuest Dissertations Publishing. (ProQuest Number: 3722708)

Melhuish, K. (2015). Determining what to assess: A methodology for concept domain analysis as applied to group theory. In T. Fukawa-Connelly, N. Infante, K. Keene, and M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate*

Mathematics Education (pp. 736–744), Pittsburgh, PA: SIGMAA on RUME.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2019). Abstract algebra students' function-related understanding and activity. In Weinberg, A., Moore-Russo, D., Soto, H., and Wawro, M. (Eds.), *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education.* (pp. 419–427), Oklahoma City, OK.

Miller, J. D., Kimmel, L., Hoffer, T. B., & Nelson, C. (2000). *Longitudinal study of American youth: User's manual.* Chicago, IL: International Center for the Advancement of Scientific Literacy, Northwestern University.

Muis, K. R. (2004). Personal epistemology and mathematics: A critical review and synthesis of research. *Review of Educational Research, 74*(3), 317–377.

Muis, K., Bendixen, L. D., & Haerle, F. C. (2006). Domain-generality and domain-specificity in personal epistemology research: Philosophical and empirical reflections in the development of a theoretical framework. *Educational Psychology Review, 18*(1), 3–54.

Muis, K. R., Trevors, G., Duffy, M., Ranellucci, J., & Foy, M. J. (2016). Testing the TIDE: Examining the nature of students' epistemic beliefs using a multiple methods approach. *The Journal of Experimental Education, 84*(2), 264–288. doi: 10.1080/00220973.2015.1048843

Nardi, E. (2000). Mathematics undergraduates' responses to semantic abbreviations, 'geometric images' and multi-level abstractions in group theory. *Educational Studies in Mathematics, 43*(2), 169–189.

Nespor, J. (1987). The role of beliefs in the practice of teaching. *Journal of Curriculum Studies, 19*(4), 317–328.

Nuñez, R. E. (2005). Creating mathematical infinities: Metaphor, blending, and the beauty of transfinite cardinals. *Journal of Pragmatics, 37*(10), 1717-1741.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*(3), 307–332.

Phan, H. P. (2008). Predicting change in epistemological beliefs, reflective thinking and learning styles: A longitudinal study. *British Journal of Educational Psychology, 78*, 75–93.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. *Second Handbook of Research on Mathematics Teaching and Learning, 1*, 257–315.

Phillipou, G. N. & Christou, C. (1998). The effects of a preparatory mathematics program in changing prospective teachers' attitudes towards mathematics. *Educational Studies in Mathematics, 35*(2), 189–206.

Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning, 7*(1), 51–73.

Raymond, A. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal for Research in Mathematics Education, 28*(5), 550–576.

Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula (Ed.), *Handbook of Research on Teacher Education,* 102–119. New York: Macmillan Publishing Company.

Riehl, E. (2016). *Category Theory in Context.* Mineola, New York: Dover Publications, Inc.

Rokeach, M. (1968). *Beliefs, Attitudes, and Values: A Theory of Organization and Change.* San Francisco: Jossey-Bass.

Romberg, T. A. (1984). Classroom tasks, instructional episodes, and performance in mathematics. In B. Southwell, R. Eyland, M. Cooper, J. Conroy, & K. Collis (Eds.), *Proceedings of the 8th International Conference for the Psychology of Mathematics Education* (pp. 116–126). Sydney: PME.

Rupnow, R. (2017). Students' conceptions of mappings in abstract algebra. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education Conference on Research in Undergraduate Mathematics Education,* (pp. 259–273), San Diego, CA.

Russ, R. S., & Luna, M. J. (2013). Inferring teacher epistemological framing from local patterns in teacher noticing. *Journal of Research in Science Teaching, 50*(3), 284–314.

Saldaña, J. (2016). *The coding manual for qualitative researchers.* Thousand Oaks, CA: SAGE Publications, Inc.

Schinck, A. G., Neale, Jr., H. W., Pugalee, D. K., Cifarelli, V. V. (2008). Using metaphors to unpack student beliefs about mathematics. *School Science and Mathematics, 108*(7), 326–333. doi: 10.111/j.1949-8594.2008.tb17845.x

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of 'well-taught' mathematics courses. *Educational Psychologist, 23*(2), 145–166. doi: 10.1207/s15326985ep2302_5

Schoenfeld, A. H. (1999). Models of the teaching process. *Journal of Mathematical Behavior, 18*(3), 243–261.

Schoenfeld, A. H. (2003). How can we examine the connections between teachers' world views and their educational practices? *Issues in Education, 8*(2), 217–227.

Schommer, M. (1990). Effects of beliefs about the nature of knowledge on comprehension. *Journal of Educational Psychology, 82*(3), 498–504.

Schommer, M. (1998). The role of adults' beliefs about knowledge in school, work, and everyday life. In M. C. Smith & T. Pourchot (Eds.), *Adult learning and development: Perspectives from educational psychology* (pp. 127–143). Mahwah, NJ: Erlbaum.

Schunk, D. H., Meece, J. L., & Pintrich, P. R. (2014). *Motivation in education: Theory, research, and applications.* Columbus: Pearson.

Schutz, P. A., Pintrich, P. R., & Young, A. J. (1993, April). Epistemological beliefs, motivation, and student learning. In *annual meeting of the American Educational Research Association, Atlanta, GA*.

Second International Mathematics Study (1995). *Technical report IV: Instrument book.* Urbana, IL: University of Illinois.

Sfard, A. (1997). Commentary: On metaphorical roots of conceptual growth. In Lyn D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 339–371). Mahwah, NJ: Erlbaum.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Skott, J. (2001). The emerging practices of a novice teacher: The roles of his school mathematics images. *Journal of Mathematics Teacher Education, 4*, 3–28.

Singh, K., Granville, M., & Dika, S. (2002). Mathematics and science achievement: Effects of motivation, interest, and academic engagement. *The Journal of Educational Research*, *95*(6), 323-332.

Speer, N. M. (2005). Issues of methods and theory in the study of mathematics teachers' professed and attributed beliefs. *Educational Studies in Mathematics, 58*(3), 361–391.

Speer, N. M. (2008). Connecting beliefs and practices: A fine-grained analysis of a college mathematics teacher's collections of beliefs and their relationship to his instructional practices. *Cognition and Instruction, 26*(2), 218–267.

Speer, N. M., Smith, J. P. III, & Horvath, A. (2010). Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior, 29*, 99–114. doi: 10.1016/j.jmathb.2010.02.001

Speer, N. M., & Wagner, J. F. (2009). Knowledge needed by a teacher to provide analytic scaffolding during undergraduate mathematics classroom discussions. *Journal for Research in Mathematics Education, 40*(5), 530–562.

Szydlik, S. D. (2013). Beliefs of liberal arts mathematics students regarding the nature of mathematics. *Teaching Mathematics and Its Applications, 32*, 95–111. doi: 10.1093/teamat/hrt002

Tall, D. & Vinner, S. (1981). Concept image and concept definition with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151–169.

Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics, 15*, 105–127.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (ed.), *Handbook of Research on Mathematics Teaching and Learning*, 127–146. New York: Macmillan Publishing Company.

TIMSS (1998a). *TIMSS mathematics items for the middle school years: Released set for population 2 (seventh and eighth grades).* Chestnut Hill, MA: Boston College.

TIMSS (1998b). *TIMSS mathematics items for the final year of secondary school: Released item set for population 3.* Chestnut Hill, MA: Boston College.

Turner, J., Warzon, K. B., & Christensen, A. (2011). Motivating mathematics learning: Changes in teachers' practices and beliefs during a nine-month collaboration. *American Educational Research Journal, 48*(3), 718–762.

Uguroglu, M. E., & Walberg, H. J. (1979). Motivation and achievement: A quantitative synthesis. *American educational research journal*, *16*(4), 375-389.

Van Zoest, L. R., Jones, G. A., & Thornton, C. A. (1994). Beliefs about mathematics teaching held by pre-service teachers involved in a first grade mentorship program. *Mathematics Education Research Journal, 6*(1), 37–55.

Vidic, A. D. (2015). First-year students' beliefs about context problems in mathematics in university science programmes. *International Journal of Science and Mathematics Education, 13*, 1161–1187.

Ward, B. B., Campbell, S. R., Goodloe, M. R., Miller, A. J., Kleja, K. M., Kombe, E. M., & Torres, R. E. (2010). Assessing a mathematical inquiry course: Do students gain an appreciation for mathematics? *PRIMUS, 20*(3), 183–203.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101–119.

Weber, K. (2002). The role of instrumental and relational understanding in proofs about group isomorphisms. In *Proceedings from the 2nd International Conference for the Teaching of Mathematics*, Hersonisoss, Crete: Wiley & Sons.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior, 23*(2), 115–133.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*(2-3), 209–234.

Wilkins, J. L. M. (2008). The relationship among elementary teachers' content knowledge, attitudes, beliefs, and practices. *Journal of Mathematics Teacher Education, 11*(2), 139–164.

Yeager, D. S., & Dweck, C. S. (2012). Mindsets that promote resilience: When students believe that personal characteristics can be developed. *Educational Psychologist, 47*(4), 302–314. doi: 10.1080/00461520.2012.722805

Yin, R. K. (2009). *Case Study Research* (4th ed.). Washington DC: SAGE.

Zandieh, M., Ellis, J., & Rasmussen, C. (2016). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics,* 1–18. doi: 10.1007/s10649-016-9737-0

Appendix A

Relevant Interview Protocol for First Interview with Instructors

1. What is mathematics? (Follow-up with purpose(s) and use(s) of math.)
2. What animal do you think math is like? (Explain.) (Markovits & Forgasz, p. 56)
3. What limitations does this animal analogy have?
4. How do you learn mathematics?
5. How do students learn mathematics?
6. How would you characterize your teaching style?
7. What led you to adopt this style of teaching?
8. What is the role of lecture in your instruction?
9. What is the role of "group work" in your instruction?
10. How does the arrangement of the room help or hinder your instruction?
11. In what ways does your teaching style reflect your beliefs about mathematics, teaching, and learning? (Do you perceive any points of tension? If so, in what ways and why?)
12. What expectations (both ideally and minimally) do you set for students' in-class participation?
13. What expectations do you set for students' written work on homework and tests?
14. To what extent can students negotiate with you on class expectations?
15. What do you hope an A student will get out of your class? C student? Why?

[Give growth mindset inventory]

Relevant Interview Protocol for Second Interview with Instructors

1. What words or phrases come to mind for you when you hear the word "isomorphism"?
2. How would you define an "isomorphism"?
3. How would you describe an "isomorphism" to a ten-year-old child?
4. How are the way(s) you personally think about isomorphism the same or different from the ways you describe them to students?
5. What words or phrases come to mind for you when you hear the word "homomorphism"?
6. How would you define a "homomorphism"?
7. How would you describe a "homomorphism" to a ten-year-old child?
8. Do you think about homomorphism in the same way(s) as you describe them to students? (If so, could you describe homomorphisms in a different way if needed? If not, why do you describe homomorphisms in this way instead of the way you think about them?)
9. To what extent do you think students are motivated in your class? (Why do you think that? I didn't define motivation for you. How do you define that word, at least in a classroom/teaching context?) (Overall opinion)
10. What, if anything, do you try to do to give your students control over their learning? (How successful do you think you are?) (M)
11. In what ways do you think this class will be useful for your students? Why? (U)
12. What aspects do you think students consider useful while in the class? Why? (U)

13. To what extent do you think instructors deserve credit or blame for students' success or failure? (Why? How do you define success or failure in teaching?) (S)
14. Do you think your students think they can be successful in this class? Why or why not? (How do you think your students define success?) (S)
15. What, if anything, do you try to do in your class to stimulate students' interest in the course? In mathematical content? (I)
16. So far in the semester, how interested in the class do you think your students are? Why? (I)
17. To what extent do you think it is important for university instructors to demonstrate that they care about their students? Why? (C)
18. For you personally, what aspects of your teaching (both in and out of the classroom) do you think might demonstrate to your students that you care about your students? Why? What aspects might demonstrate a lack of caring about your students? Why? (C)

Relevant Interview Protocol for Third Interview with Instructors

1. In our first interview, you filled out an 8-item questionnaire on the changeability of intelligence and I wanted to follow up on your responses. You consistently chose options indicating that intelligence is changeable. Could you talk more about how you view the nature of intelligence?
2. What led you to think about intelligence in that way for yourself?
3. Do you hold similar beliefs about your students' ability to change their intelligence? Why or why not?
4. Returning to a different aspect of the first interview, how, if at all, do the SPOT evaluations influence your instruction?
5. Linking aspects of the first two interviews, in what ways, if any, do you think your beliefs about math, teaching, and learning manifest in your explanations while teaching specific content?

Relevant Interview Protocol for Fourth Interview with Instructors

1. What characteristics of students do you view as most important for students to be able to learn in your course (e.g., students' abilities, students' prior coursework, students' motivation, students' participation, students' work ethic, etc.)? Why those aspects?
2. Do you feel that students' motivation in this course has remained the same throughout, increased, or decreased during the semester? Why?
3. When I asked you about the answers you thought students would give to this survey [MUSIC inventory] earlier this semester, these were the answers you gave. Do you believe these answers were pretty close to the class average, well above, or well below? Why?
4. What role, if any, does student motivation play in your approach to instruction?
5. To what extent do you view motivation to learn as important for student learning?
6. What, if any, relationship do you see between students' motivation and the course evaluations they give?
7. What, if any, changes in students' beliefs about math, teaching, and learning have you observed during the semester?
8. Throughout these interviews we've talked about your beliefs about math, teaching, and learning and about metaphors for how you think about specific content areas (isomorphism and homomorphism). In what ways, if any, do you see these aspects of your thinking relating to each other?

Appendix B

Relevant Interview Protocol for First Interview with Students

1. What is mathematics?
2. How do people learn mathematics? [Compare/contrast follow-up based on previous written response on how they learn math]
3. How would you characterize a “typical” math class?
4. How would you characterize this math class?
5. How do you participate in this class?
6. What expectations does your instructor have for students’ in-class participation?
7. What expectations does your instructor have for written work on homework and tests?
8. To what extent do you feel you have control over your learning and success in this class?
9. How would you characterize a “good” math student? This may be the same or different: how would you characterize a “good” Abstract Algebra student? Why? [Pursue similarities and differences]

Interview Protocol for Second Interview with Students

1. How would you define an isomorphism (homomorphism) between groups?
2. How would you describe an isomorphism (homomorphism) to others?
3. For each of the following pairs of groups, is it possible to form an isomorphism between them? Why or why not? Is it possible to form a homomorphism between them? Why or why not?
   a. $\mathbb{Z}_5 \to 5\mathbb{Z}$
   b. $\mathbb{Z}_5 \to \mathbb{Z}/5\mathbb{Z}$
   c. $\mathbb{Z}_5 \to \mathbb{Z}_6$
   d. $\mathbb{Z}_3 \to \mathbb{Z}_6$
   e. $\mathbb{Z}_6 \to \mathbb{Z}_3$
   f. $\mathbb{Z} \to 2\mathbb{Z}$
   g. $2\mathbb{Z} \to \mathbb{Z}$

*	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

+	a	b	c	d
a	b	a	d	c
b	a	b	c	d
c	d	c	a	b
d	c	d	b	a

h.

i. $Z_8 \rightarrow D_4$

Relevant Interview Protocol for fourth interview with students

1. What is necessary to be successful in math?
2. Which of those qualities are also important for success in music? In psychology? In life?
3. On a scale of 1 to 10 (low to high), how motivated to learn have you been in this class? Why?
4. What aspects of this class have motivated you? Hindered your motivation?
5. Which aspects of this class have you felt you had control over?
6. How useful do you think this class will be to you in future?
7. What were your goals for this course? How successful have you been in meeting those goals?
8. What aspects of this class have piqued your interest? Why?
9. When, if ever, have you felt connected to your teacher or classmates in this course? Why did you feel that connection?
10. When, if ever, have you felt isolated from your teacher or classmates in this course? Why did you feel that isolation?
11. In what ways, if any, have your beliefs about math, teaching, and learning changed or been strengthened by this class?
12. During these interviews, we have talked about your beliefs about math, teaching, and learning as well as spending some time on content knowledge of isomorphism and homomorphism. In what ways, if any, do you see these aspects of your thinking relating to each other?