Distance Sets and Gap Lemmas

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(ABSTRACT)

Many problems in geometric measure theory are centered around finding conditions and structures on a set to guarantee that its distance set must be large. Two notions of structure that are of importance in this work are Hausdorff dimension and thickness. Recent progress has been made on generalizing the notion of thickness so part of this work also generalizes previous results using this new upgraded version of thickness. We also show why a famous conjecture about distance sets does not hold on the real line and thus, why this conjecture needs to happen in higher dimensions. Furthermore, we give explicit distance set and thickness calculations for a special class of self-similar sets.

Distance Sets and Gap Lemmas

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(GENERAL AUDIENCE ABSTRACT)

Part of the study of geometric measure theory is centered around creating interesting structures to place on a set and determining what sort of threshold on that structure allows you to guarantee that some interesting geometric property exists for that set. An example of this is determining when you can guarantee that a set contains many unique distances between elements in that set. This work presents various types of structures that help to investigate the problem of when you can guarantee that a set has the previously mentioned geometric property.

Dedication

To my mom; I'm not sure how I ended up doing what I'm doing, but I love it and have you to thank for giving me the freedom and support to do this.

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First I want to thank my advisor Dr. Eyvi Palsson. Thank you for your patience while I painfully stumbled through these past two years of graduate school and also for your guidance in not only mathematical matters but general academia matters as well. I also want to thank Sean Sovine, Francisco Romero Acosta, and Jose Gaitan for all the interesting mathematical conversations we've had. Also, thank you to Aidan Murphy for the many conversations we've had about anything related to graduate school as your advice as helped to keep me sane. Lastly, thank you to my family who have been there for me at all times, the good and the bad, and did everything they could to support me.

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Chapter 1

Introduction

In 1946, Erdős [5] considered the problem of determining the minimal number of distinct distances between N points in the plane. In that paper, Erdoős conjectured that this number is $\gtrsim \frac{N}{\sqrt{\log N}}$ as $N \to \infty$. This problem has now been coined the Erdős distinct distances problem. The problem proved to be difficult as it was not considered resolved until 2010 when Guth and Katz [9] proved that this number is $\gtrsim \frac{N}{\log N}$ as $N \to \infty$. A continuous version of this problem was formulated in 1985 by Falconer, who stated the problem in terms of Hausdorff dimension and Lebesgue measure where Hausdorff dimension will be defined later.

Definition 1.1 (Distance set)**.** Let $E \in \mathbb{R}^d$. The distance set of E, denoted $\Delta(E)$, is defined as $\Delta(E) := \{|x - y| : x, y \in E\}$.

In [6], Falconer proved that when $E \in \mathbb{R}^d$ is a compact set for $d \geq 2$, if $\dim_H(E) > \frac{d+1}{2}$ then $\mathcal{L}(\Delta(E)) > 0$ where $\mathcal{L}$ is Lebesgue measure. Also in [6], Falconer conjectured that we can lower this threshold to $\frac{d}{2}$. Specifically, his conjecture is that for $E \in \mathbb{R}^d$ for $d \geq 2$, if $\dim_H(E) > \frac{d}{2}$ then $\mathcal{L}(\Delta(E)) > 0$, now known as Falconer's distance conjecture. Similar to the Erdős distinct distances problem, Falconer's distance conjecture has also proven to be difficult and has not been resolved. Various improvements on the dimensional threshold have been found though. Currently, the best known threshold when $d \geq 3$ is $\frac{d^2}{2d-1}$ which was first achieved by Du, Guth, Ou, Wang, Wilson and Zhang in [3] when $d = 3$ and generalized to higher dimensions by Du and Zhang in [2].

Interestingly, this threshold can be improved for even dimensions. When restricting d to even integers and for $d \geq 2$, the current threshold is $\frac{d}{2} + \frac{1}{4}$. For $d = 2$, this threshold was obtained by Guth, Iosevich, Ou and Wang in [10] and generalized to higher even dimensions by Du, Iosevich, Ou, Wang and Zhang in [4].

An interesting and similar result was proven by Mattila and Sjölin in [12]. They proved that for $E \in \mathbb{R}^d$ compact with $d \geq 2$, if $\dim_H(E) > \frac{d+1}{2}$ then not only do we have $\mathcal{L}(\Delta(E)) > 0$ but we also have that $\Delta(E)$ has non-empty interior. This result is now known as the Mattila-Sjölin theorem. Proving results of this type have since become popular. An example of this type of result which is of high importance to this thesis is that McDonald and Taylor in [14] prove that if two Cantor sets, K_1 and K_2, satisfy $\tau(K_1)\tau(K_2) > 1$, where τ represents thickness which will be defined in Chapter 2, then not only can we say that $\Delta(K_1 \times K_2)$ has non-empty interior, but $\Delta_T(K_1 \times K_2)$ contains non-empty interior where Δ_T denotes a tree structure, which will be defined in Chapter 3, placed on the distance set. One part of this thesis is generalizing this result of McDonald and Taylor to compact sets in $\mathbb{R}$ and the main result of this thesis is showing that $\Delta_{x^0}(C^1 \times C^2)$ contains an interval where we now allow $C^1, C^2 \in \mathbb{R}^d$ and where $\Delta_x(\cdot)$ represents the pinned distance set.

Along these same lines, there are a number of papers that establish Mattila-Sjölin type results for sets that have product structure. For example, as a Corollary of the Mattila-Sjölin theorem, for a set of the form $A \times A \times \cdots \times A = A^d \subset \mathbb{R}^d$, we have that if $\dim_H(A) > (d+1)/2d$ then $\Delta(A^d)$ has non-empty interior. Furthermore, the authors in [1] improve upon this threshold where for $d \geq 10$, they establish a better threshold of $\frac{d+1}{2d} - \frac{23d-228}{114d(d-4)}$ and when $d \geq 5$ they establish a threshold of $\frac{d+1}{2d} - \frac{d-4}{2d(3d-4)}$. They also note that when $d \geq 27$ the former threshold is better than the latter. Note that in the previous paragraph discussing the main results of this thesis, we do not assume our sets have "complete" product structure as just discussed, rather we only assume structure on sets of the form $C^1 \times C^2$ for $C^1, C^2 \in \mathbb{R}^d$.

The notion of thickness was created by Newhouse [15] only for Cantor sets, however a recent paper by Falconer and Yavicoli [8] generalizes Newhouse's definition and result to compact sets and thus, the machinery in [8] is what we use to generalize the result of McDonald and Taylor.

The rest of this thesis is organized as follows: In Chapter 2 we give background material on the necessary terminology. Chapter 3 provides the main result of this thesis which generalizes the result McDonald and Taylor previously mentioned to compact sets and some higher dimensional compact sets. In Chapter 4 we provide an example a compact set in $\mathbb{R}^1$ which has Hausdorff dimension equal to one but whose distance set has zero Lebesgue measure, thus illustrating why Falconer's conjecture needs to be stated for dimensions greater than or equal to two. Lastly, in Chapter 5 we show a way to calculate the distance set of Cantor-type sets and how large their distance sets are, and also calculating the thickness of these Cantor-type sets.

Chapter 2

Background Material

We first start off by defining Hausdorff dimension and measure.

Definition 2.1. Let $s \geq 0$ and $\delta \in (0, \infty]$. Given a set $E \in \mathbb{R}^d$, a **δ-cover** of E is any countable family of sets $\{U_i\}_{i \in \mathbb{N}}$ such that $E \subseteq \bigcup_i U_i$ and $\operatorname{diam}(U_i) \leq \delta$ for all i. Then the **s-dimensional Hausdorff δ-measure** is

$$H^s_\delta := \inf \left\{ \sum_{i \in \mathbb{N}} \operatorname{diam}(U_i)^s : \{U_i\}_{i \in \mathbb{N}} \text{ is a } \delta\text{-cover of } E \right\}.$$

Note that as $\delta \to 0^+$, the condition of begin a δ-cover becomes more restrictive which means that there are fewer options for covers of E and since this definition is an infimum, we get that if $\delta_1 \leq \delta_2$ then $H^s_{\delta_2}(E) \leq H^s_{\delta_1}(E)$. As a result of this fact, we can state the definition of Hausdorff measure.

Definition 2.2. The **s-dimensional Hausdorff measure** of a set E is defined to be

$$H^s(E) := \lim_{\delta \to 0^+} H^s_\delta(E) = \sup_{\delta > 0} H^s_\delta(E) \in [0, \infty].$$

The definition of Hausdorff dimension relies on the following theorem:

Theorem 2.3. *For $0 \leq s < t$ and a set E,*

(i) if $H^s(E) < \infty$ then $H^t(E) = 0$,

(ii) if $H^t(E) > 0$ *then* $H^s(E) = \infty$.

Proof. This proof is the same as found in Chapter 4 of [13]. For (i), let $\{U\}_{i\in\mathbb{N}}$ be a δ-cover of E with $\sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^s \leq H^s_\delta(E)+1$. Note that we can do this by properties of infimums. Then

$$H^t_\delta(E) \leq \sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^t \leq \delta^{t-s} \sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^s \leq \delta^{t-s}\left(H^s_\delta(E)+1\right)$$

and thus, this shows (i) as $\delta \to 0^+$.

Lastly, note that (ii) is the contrapositive of (i). We present the theorem this way because this emphasizes how Hausdorff dimension is defined. □

By Theorem 2.3 we can now define Hausdorff dimension.

Definition 2.4. The **Hausdorff dimension** of a set $E \in \mathbb{R}^d$ is denoted as $\dim_H(E)$ and is defined as

$$\dim_H(E) := \sup\{s : H^s(E) > 0\} = \sup\{s : H^s(E) = \infty\} = \inf\{t : H^t(E) < \infty\} = \inf\{t : H^t(E) = 0\}.$$

A key theorem that is used in Chapter 4 is called the mass distribution principle. Before giving stating and giving a proof of this principle, we first need to define what a mass distribution is.

Definition 2.5. A measure μ defined on a bounded subset of $\mathbb{R}^d$ for which

$$0 < \mu(\mathbb{R}^d) < \infty$$

is called a **mass distribution**.

Now we can state the mass distribution principle

Theorem 2.6. *(Mass distribution principle) Let μ be a mass distribution on a set E and suppose that for some number $s \geq 0$ there exists $C > 0$ and $\varepsilon > 0$ such that*

$$\mu(U) \leq C \operatorname{diam}(U)^s$$

for all sets U with $\operatorname{diam}(U) \leq \varepsilon$. Then $\mathcal{H}^s(E) \geq \mu(E)/C$. In particular, this means $s \leq \dim_H(E)$.

Proof. The proof of this is the same as found in Chapter 3 of [7]. Let $\{U_i\}_{i\in\mathbb{N}}$ be any cover of E. Then by assumption

$$0 < \mu(E) \leq \mu\left(\bigcup_i U_i\right) \leq \sum_{i\in\mathbb{N}} \mu(U_i) \leq C \sum_{i\in\mathbb{N}} \operatorname{diam}(U)^s.$$

Then after taking infima, for δ small enough we have $H^s_\delta(E) \geq \mu(E)/C$ and since this is true for all δ small enough, we may conclude that $H^s(E) \geq \mu(E)/C$. Lastly, because $0 < \mu(E)$ we may conclude that $s \leq \dim_H(E)$. □

Finding lower bounds on Hausdorff dimension is difficult because, in general, you need to take infimums over arbitrary coverings. The mass distribution principle allows us to slightly relax these conditions. In particular, for a wide array of sets, as in the case that is encountered in Chapter 4, it is fairly easy to construct probability measures supported on these sets which immediately gives us access to the mass distribution principle.

Chapter 5 is concerned with calculating a special class of self-similar sets, so now we give definitions related to these concepts.

Definition 2.7. A mapping $S : \mathbb{R}^d \to \mathbb{R}^d$ is called a **similitude or similarity function** if

there exists r, $0 < r < 1$ such that

$$|S(x) - S(y)| = r|x - y|$$

for all $x, y \in \mathbb{R}^d$. Furthermore, r is called the **contraction ratio** of S.

An interesting note to make is that in [11], Hutchinson proved that every similarity is of the form

$$S(x) = rg(x) + z$$

where $z \in \mathbb{R}^d$ and $g \in O(n)$ where $O(n)$ denotes the orthogonal group of dimension n.

Definition 2.8. Suppose $\mathcal{I} = \{S_1, \cdots, S_N\}$, $N \geq 2$, is a finite sequence of similarities with contraction rations $r_1, \cdots, r_N$. We call $\mathcal{I}$ an **iterated function system** and if $r_1 = \cdots = r_N$ then $\mathcal{I}$ is called a **homogeneous** iterated function system. Then the unique non-empty compact K such that

$$K = \bigcup_{i=1}^{N} S_i(K)$$

is called **self-similar**

Note that in this definition, K is said to be unique. The proof of this can also be found in [11].

An example of a self-similar set is the ternary Cantor set, C, is generated by the similarities $S_1(x) = \frac{x}{3}$ and $S_2(x) = \frac{2}{3} + \frac{x}{3}$ where $x \in [0, 1]$. Then note that

$$S_1[0, 1] \cup S_2[0, 1]$$

corresponds to the first iteration of C,

$$(S_1 \circ S_1[0,1]) \cup (S_1 \circ S_2[0,1]) \cup (S_2 \circ S_1[0,1]) \cup (S_2 \circ S_2[0,1])$$

corresponds to the second iteration of C and so on. In particular, C is generated by the set of similarities

$$\{S_{i_1} \circ \cdots S_{i_n} : i_1, \cdots, i_n \in \{1,2\}\}.$$

Note that at the nth stage of the construction of C, the interval $[0,1]$ is being shrunk to an interval of length $1/3^n$ by some composition $S_{i_1} \circ \cdots S_{i_n}$ and each interval at the nth state of construction produces two children intervals because we start out with two similarities. For many cases, self-similar sets on the real line have an easy intuition for how they are structured. For example, suppose we have the similarity functions $S_1(x) = r(x)$ and $S_2(x) = \alpha + rx$ such that $r < \alpha$ for $x \in [0,1]$. Then r tells you how much you are shrinking the interval $[0,1]$ at each stage in the construction and because $r < \alpha$, the images $S_1[0,1]$ and $S_2[0,1]$ do not overlap. Furthermore, α tells you where the left endpoint of the set $S_2[0,1]$ will be. So in this case, the set generated by S_1 and S_2 will "look" much like the Cantor set.

Another theme of this thesis is the notion of thickness which is what we will now discuss. The general definition of a Cantor set is defined as a set which is compact, perfect, and totally disconnected. The dynamical systems community is often interested in knowing when two Cantor sets on the real line intersect. In [15], Newhouse created the notion of thickness, denoted τ, specifically for Cantor sets. We can naturally think about the "gaps" of a Cantor set and Newhouse showed that if two Cantor sets K_1 and K_2 are such that neither of them is contained in a gap of the other and $\tau(K_1)\tau(K_2) >$ '1, then $K_1 \cap K_2 \neq \emptyset$. Of course, one will then think if we can extend this definition to more general sets and to higher dimensional

sets. In [8], Falconer and Yavicoli did just that. We do not present the original definition of thickness as is defined in [15] because it is quite clunky, but rather we present the definition as found in [8] which not only applies to general compact sets, but also to a compact set in any dimension.

Definition 2.9. Given a compact set $C \subset \mathbb{R}^d$, we define $\{G_n\}_{n=1}^{\infty}$ to be the, at most, countably many open bounded path-connected components of C^C (the complement of C) and E to be the unbounded open path-connected component of C^C (except when $d = 1$ when E consists of two unbounded intervals). We call E the **unbounded gap** of C and $\{G_n\}_{n=1}^{\infty}$ the **unbounded gaps** of C. Furthermore, we assume that the sequence of bounded gaps $\{G_n\}_{n=1}^{\infty}$ is ordered by non-increasing diameter.

We will write dist for the usual distance between points of non-empty subsets of $\mathbb{R}^d$ and diam for the diameter of a non-empty subset of $\mathbb{R}^d$.

Definition 2.10 (Thickness in $\mathbb{R}^d$)**.** The **thickness** of C is

$$\tau(C) := \inf_{n\in\mathbb{N}} \frac{\operatorname{dist}\left(G_n, \bigcup_{1\le i\le n-1} G_i \cup E\right)}{\operatorname{diam}(G_n)},$$

provided that E is not the only path-connected component of C^C. When the only complementary path-connected component is E, we define

$$\tau(C) = \begin{cases} +\infty & \text{if } C^\circ \neq \emptyset \\ 0 & \text{if } C^\circ = \emptyset \end{cases}$$

We say C is **thick** if $\tau(C) > 0$.

Falconer and Yavicoli in [8] prove the following result:

Theorem 2.11 (Gap lemma in $\mathbb{R}^d$)**.** *Let $C_1, C_2 \in \mathbb{R}^d$ be compact such that neither of them is contained in a gap of the other and $\tau(C_1)\tau(C_2) > 1$. Then $C_1 \cap C_2 \neq \emptyset$.*

To end this chapter, we present another way writing τ.

Lemma 2.12. *Let $C \in \mathbb{R}^d$ be compact with bounded gaps $\{G_n\}_{n=1}^{\infty}$. Set $\Lambda_n := \{i \neq n : \operatorname{diam}(G_n) \leq \operatorname{diam}(G_i)\}$. Then we can also write τ as*

$$\tau(C) = \inf_{n\in\mathbb{N}} \frac{\operatorname{dist}\left(G_n, \bigcup_{i\in\Lambda_n} G_i \cup E\right)}{\operatorname{diam}(G_n)}.$$

Proof. To see this, assume that we cannot. Set

$$\widehat{\tau}(C) := \inf_{n\in\mathbb{N}} \frac{\operatorname{dist}\left(G_n, \bigcup_{i\in\Lambda_n} G_i \cup E\right)}{\operatorname{diam}(G_n)}.$$

First note that since $\operatorname{diam}(G_n) \geq \operatorname{diam}(G_{n+1})$ we have $\widehat{\tau}(C) \leq \tau(C)$ as we could only be increasing the number gaps in our analysis using Λ_n (as would be the case if we had $\operatorname{diam}(G_i) = \operatorname{diam}(G_n)$ with $i > n$). So therefore we may assume $\widehat{\tau}(C) < \tau(C)$. Thus, there exists $\eta_1 > 0$ such that $\widehat{\tau}(C) + \eta_1 = \tau(C)$. Since $\widehat{\tau}$ is an infimum, by properties of infimums we can find $N \in \mathbb{N}$ such that

$$\frac{\operatorname{dist}\left(G_N, \bigcup_{i\in\Lambda_N} G_i \cup E\right)}{\operatorname{diam}(G_N)} < \widehat{\tau}(C) + \eta_1 = \tau(C).$$

Also note that this means

$$\frac{\operatorname{dist}\left(G_N, \bigcup_{i\in\Lambda_N} G_i \cup E\right)}{\operatorname{diam}(G_N)} < \tau(C) \leq \frac{\operatorname{dist}\left(G_N, \bigcup_{1\leq i\leq N-1} G_i \cup E\right)}{\operatorname{diam}(G_N)}$$

and therefore, the gap with the smallest distance is not achieved by E since E is present in

both the LHS and RHS of the above inequality. Now set $\eta_2 > 0$ such that

$$\frac{\operatorname{dist}\left(G_N, \bigcup_{i\in\Lambda_N} G_i \cup E\right)}{\operatorname{diam}(G_N)} + \eta_2 = \tau(C).$$

Because dist is an infimum property, because the minimal distance to G_N is not achieved by G_i with $1 \leq i \leq N-1$, and because we are always assuming $\operatorname{diam}(G_n) \geq \operatorname{diam}(G_{n+1})$, we can find $M \in \Lambda_N$ such that $M > N$, $\operatorname{diam}(G_M) = \operatorname{diam}(G_N)$, and

$$\frac{\operatorname{dist}(G_N, G_M)}{\operatorname{diam}(G_M)} < \frac{\operatorname{dist}\left(G_N, \bigcup_{i\in\Lambda_N} G_i \cup E\right)}{\operatorname{diam}(G_N)} + \eta_2 = \tau(C).$$

But since $M > N$ and $\operatorname{diam}(G_M) = \operatorname{diam}(G_N)$ this implies

$$\frac{\operatorname{dist}(G_N, G_M)}{\operatorname{diam}(G_N)} = \frac{\operatorname{dist}(G_M, G_N)}{\operatorname{diam}(G_M)} \geq \frac{\operatorname{dist}\left(G_M, \bigcup_{1\leq i\leq M-1} G_i \cup E\right)}{\operatorname{diam}(G_M)}$$

which means

$$\frac{\operatorname{dist}\left(G_M, \bigcup_{1\leq i\leq M-1} G_i \cup E\right)}{\operatorname{diam}(G_M)} < \tau(C).$$

However this is a contradiction since τ is an infimum over all such elements and thus, we can write

$$\tau(C) = \inf_n \frac{\operatorname{dist}\left(G_n, \bigcup_{1\leq i\leq n-1} G_i \cup E\right)}{\operatorname{diam}(G_n)} = \inf_n \frac{\operatorname{dist}\left(G_n, \bigcup_{i\in\Lambda_n} G_i \cup E\right)}{\operatorname{diam}(G_n)}.$$

$\square$

Chapter 3

Intersections of Compact Sets

The main part of this chapter is generalizing the result of McDonald and Taylor, and the last part of this chapter gives a small discussion about the intersection of three compact sets.

3.1 Products of Thick Compact Sets

The strategy we present in this chapter heavily follows that of McDonald and Taylor in [14]. The result they prove is for Cantor sets, and here we show the same claim for compact sets. Along the way, we will point out differences in their paper from this one. First we begin with some definitions which come from [14]. The language and notation of graphs is convenient for this discussion, so this is where we start.

Definition 3.1 (Graphs)**.** A (finite) **graph** is a pair $G = (V, E)$ where V is a (finite) set and E is a set of 2-element subsets of V. If $\{i, j\} \in E$ we say i and j are **adjacent** and write $i \sim j$.

Definition 3.2 (Chain and tree graphs)**.** The k-**chain** is the graph on the vertex set $\{1, \cdots, k+1\}$ with $i \sim j$ if and only if $|i - j| = 1$. A **tree** is a connected, acyclic graph; equivalently, a tree is a graph in which any two vertices are connected by exactly one path. If T is a tree, the **leaves** of T are the vertices which are adjacent to exactly one other vertex of T.

The following is fairly obvious proposition.

Proposition 3.3 (Tree structure)**.** *If T is a tree with $k+1$ vertices, then T has k edges. Moreover, given such a tree T there is a sequence of trees $T_1, \cdots, T_k, T_{k+1}$ such that $T_1 = T$, T_{k+1} consists of only one vertex, and each T_{i+1} is obtained from T_i by removing one leaf and its corresponding edge.*

This description of a tree comes in handy for the proof strategy of Theorem 3.7.

Definition 3.4 (G distance sets)**.** Let G be a graph on the vertex set $\{1, \cdots, k+1\}$ with m edges, and let $\sim$ denote the adjacency relation on G. Define the G **distance set** of B to be

$$\Delta_G(B) := \left\{ \left(|x^i - x^j|\right)_{i\sim j} : x^1, \cdots, x^{k+1} \in B, x^i \neq x^j \right\},$$

where $(a_{i,j})_{i\sim j}$ denotes a vector in $\mathbb{R}^m$ with coordinates indexed by the edges of G.

The usual definition of the distance set is a 1-chain except with the element 0 removed, i.e. if we make G a 1-chain then $\Delta(B) = \Delta_G(B) \cup \{0\}$. The element 0 is removed from G distance sets because the proof of Theorem 3.7 uses a non-degeneracy condition.

We now define the main theorem of this chapter which will proven later.

Theorem 3.5. *Let $C_1, C_2 \in \mathbb{R}$ be compacts sets such that $\tau(C_1)\tau(C_2) > 1$. Then for any finite tree T, the set $\Delta_T(C_1 \times C_2)$ has non-empty interior.*

Before moving further, we will be assuming that we are working compact sets $C_1, C_2 \in \mathbb{R}$ such that $\tau(C_1)\tau(C_2) > 1$ and where at least one of C_1 or C_2 has at least one bounded gap. If both C_1 and C_2 have no bounded gaps, then both are non-degenerate intervals and thus, $C_1 \times C_2$ is a filled in rectangle which would mean Theorem 3.5 is trivial. The next proposition ensures also that we will have plenty of points to choose from in a thick compact set.

Proposition 3.6. *Let $C \in \mathbb{R}$ be a thick compact set, i.e. $\tau(C) > 0$. Then C is uncountable*

Proof. Assume C is countable. Then the elements in C are isolated points. So now let $x \in C$. Then since x is isolated, x is the right endpoint of a gap G_i and the left endpoint of a gap G_j both of which will have positive diameter since x is isolated. Without loss of generality, we assume G_i and G_j are ordered so that $j > i$. Then

$$\frac{\operatorname{dist}\left(G_i, \bigcup_{1 \le i \le n-1} G_j \cup E\right)}{\operatorname{diam}(G_i)} = 0$$

and this equality holds even if either $G_i = E$ or $G_j = E$ which is a contradiction since τ is an infimum. □

McDonald and Taylor present a way to convert the problem of finding non-empty interior of a set to proving what they call a pin wiggling lemma

Let $\phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\phi(x, y) = |x - y|$. Then given a point x and a set E, we use the notation

$$\phi(x, E) := \{\phi(x, y) : y \in E\}.$$

Given a tree T on vertices $\{1, \cdots, k+1\}$ and distinct set of points $x^1, \cdots, x^{k+1} \in E$, define

$$\Phi(x^1, \cdots, x^{k+1}) = \left(\phi(x^1, x^j)\right)_{i \sim j}$$

where $\sim$ denotes the adjacency relation of the graph T. Thus, the sets $\Delta_T(E)$ are the images of E^{k+1} under for Φ for the function $\phi(x, y) = |x - y|$. Then a pig wiggling lemma is a lemma which finds conditions on a set E to guarantee that the set

$$\bigcap_{x \in S} \phi(x, E)$$

has non-empty interior for some neighborhood of pins S. The next theorem gives conditions needed for which the sets $\Phi(E^{k+1})$ will have non-empty interior. This theorem comes from [14] and the proof strategy does not change.

Theorem 3.7 (Tree building mechanism)**.** *Fix a tree T on vertices $\{1, \cdots, k+1\}$ and take the map $\Phi : (\mathbb{R}^2)^{k+1} \to \mathbb{R}^k$ as previously defined. Let $C_1, C_2 \in \mathbb{R}$ be compact sets satisfying $\tau(C_1)\tau(C_2) > 1$ and let $x^1, \cdots, x^{k+1} \in C_1 \times C_2$ be distinct points. Suppose that for any compact set $\widetilde{C_j} \subset C_j$, there exists open neighborhoods S_i of x^i such that the set*

$$\bigcap_{x \in S_i} \phi\left(x, \widetilde{C_1} \times \widetilde{C_2}\right)$$

has non-empty interior. Then $\Phi\left((C_1 \times C_2)^{k+1}\right)$ has non-empty interior. Moreover, $\Phi(x^1, \cdots, x^{k+1})$ is in the closure of $\Phi\left((C_1 \times C_2)^{k+1}\right)^\circ$.

Proof. Since we do not allow 0 to be in the distance set, we can let $2\varepsilon > 0$ denote the minimal distance:

$$\varepsilon = \frac{1}{2} \min\left\{|x^i - x^j| : i \neq j \in \{1, \cdots, k+1\}\right\} > 0,$$

and for each $i = 1, \cdots, k+1$ define the ε-box about x^i by

$$\begin{aligned} B(x^i, \varepsilon) &= x^i + [-\varepsilon, \varepsilon]^2 \\ &= [x_1^i - \varepsilon, x_1^i + \varepsilon] \times [x_2^i - \varepsilon, x_2^i + \varepsilon] \\ &= B_1(x^i, \varepsilon) \times B_2(x^i, \varepsilon), \end{aligned}$$

where $B_1(x^i, \varepsilon), B_2(x^i, \varepsilon)$ are the closed ε-intervals about the coordinates of x^i. Next choose any leaf of T and without loss of generality we may assume we have labeled the vertices so

that $k+1$ is our leaf. Now let n denote the unique vertex which satisfies $n \sim k+1$. Let $\widetilde{C_j} = C_j \cap B_j(x^{k+1}, \varepsilon)$. Then $\widetilde{C_j}$ is compact, so by assumption there exists a neighborhood S_n of x^n so that the set

$$\bigcap_{x \in S_n} \phi\left(x, \widetilde{C_1} \times \widetilde{C_2}\right) \tag{3.1}$$

has non-empty interior. Further, because we are taking the intersection of all $x \in S_n$ and because the above set has non-empty interior, we may assume $S_n \subset B(x^n, \varepsilon)$ which guarantees that the points in S_n and the points in $\widetilde{C_1} \times \widetilde{C_2} \subset B(x^{k+1}, \varepsilon)$ are distinct. Again, because the set (3.1) has non-empty interior, we can choose $\varepsilon_2 \in (0, \varepsilon]$ so that $B(x^n, \varepsilon_2) \subset S_n$, and hence (3.1) still holds with $B(x^n, \varepsilon_2)$ in place of S_n. So for simplicity, we replace each of the ε-boxes about $x^1, \cdots, x^{k+1}$ by potentially smaller boxes $B(x^i, \varepsilon_2)$ for each $i \in \{1, \cdots, k+1\}$.

To conclude let $E_i = B(x^i, \varepsilon_2) \cap (C_1 \times C_2)$, let T_2 be the tree obtained from T by removing the vertex $k+1$ and its corresponding edge, and let Φ_2 be the function as in the statement of the theorem corresponding to the tree T_2. We have shown that there exists a non-empty open interval I_1 so that

$$\Phi(E_1 \times \cdots \times E_{k+1}) \supset \Phi_2(E_1 \times \cdots \times E_k) \times I_1.$$

Running this argument successively on each of the trees $T_1, \cdots, T_k$ as in Proposition 1, we conclude that $\Phi\left((C_1 \times C_2)^{k+1}\right)$ contains a set of the form $I_1 \times \cdots \times I_k$ for non-empty open intervals $I_1, \cdots, I_k$. By construction, $\Phi(x^1, \cdots, x^{k+1})$ is in the closure of $I_1 \times \cdots \times I_k$. $\square$

Now we create another notion of thickness that gives us some wiggle room to work with in the later calculations. The following definition is adapted from Definition 3.2 in [14].

Definition 3.8. Let $C \in \mathbb{R}^d$ be a compact set with bounded gaps $\{G_n\}_{n=1}^{\infty}$. Let $u_n \in \partial G_n$

and define $H_{\varepsilon,n} := \{i \neq n : (1-\varepsilon)\operatorname{diam}(G_n) < \operatorname{diam}(G_i)\}$. Then the ε-thickness of C at u_n is defined to be

$$\tau_\varepsilon(C, u_n) := \frac{\operatorname{dist}\left(u_n, \bigcup_{i \in H_{\varepsilon,n}} G_i \cup E\right)}{\operatorname{diam}(G_n)}$$

and the **ε-thickness** of C is

$$\tau_\varepsilon(C) = \inf_{n \in \mathbb{N}} \inf_{u_n} \tau_\varepsilon(C, u_n)$$

the infimum being taken over all boundary points of all gaps.

This definition leads to two properties, which we quickly prove.

Proposition 3.9. *Let $C \in \mathbb{R}^d$ be compact. Then*

(i) If $\varepsilon_1 < \varepsilon_2$ then $\tau_{\varepsilon_2}(C) \leq \tau_{\varepsilon_1}(C)$

(ii) $\tau_\varepsilon(C) \to \tau(C)$ as $\varepsilon \to 0$.

Proof. For the first claim, if $(1-\varepsilon_1)\operatorname{diam}(G_n) < \operatorname{diam}(G_i)$ then

$$(1-\varepsilon_2)\operatorname{diam}(G_n) < (1-\varepsilon_1)\operatorname{diam}(G_n) < \operatorname{diam}(G_i).$$

So if $i \in H_{\varepsilon_1,n}$ then $i \in H_{\varepsilon_2,n}$. Therefore, we include possibly more gaps into consideration with indices in the set $H_{\varepsilon_2,n}$ and thus, the distance to G_n can only shrink. So $\tau_{\varepsilon_2}(C) \leq \tau_{\varepsilon_1}(C)$.

Now for the second claim, we can rewrite $\tau(C)$ as is done in Lemma 2.12. Then note, in terms of lim sup and lim inf sets, we have $\lim_{m\to\infty} H_{1/m,n} = \Lambda_n$ and thus, as $\varepsilon \to 0$ we get $H_{\varepsilon,n} \to \Lambda_n$ which shows $\tau_\varepsilon(C) \to \tau(C)$. □

Now we state a key lemma which tells us how thickness is affected under continuously differentiable mappings. This is Lemma 3.4 in [14] where the only difference is that their

definition of ε-thickness of stated differently than what is done here and this is for compact sets.

Lemma 3.10. *Let $C \subset \mathbb{R}$ be compact, let u be a right endpoint of some gap of C, and let g be a function which is continuously differentiable on a neighborhood of u which satisfies $g'(u) \neq 0$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\tau\left(g\left(C \cap [u, u+\delta]\right)\right) > \tau_\varepsilon(C)(1-\varepsilon).$$

Proof. For the purposes of this proof, let $|\cdot| = \operatorname{diam}(\cdot)$. Fix $\varepsilon > 0$. Since g' is continuous and because $g'(u) \neq 0$, we get that $g'(x) \neq 0$ in an entire neighborhood of u. So by continuity we can find $\delta > 0$ such that for all $x_1, x_2 \in [u, u+\delta]$we have

$$\left|\frac{|g'(x_1)|}{|g'(x_2)|} - 1\right| < \varepsilon.$$

Also make δ, possibly, smaller so that g is monotone on the interval $[u, u+\delta]$. So for any subinterval $I \subset [u, u+\delta]$ with right and left endpoints x_r and x_l respectively, we can write $|g(I)| = |g(x_r) - g(x_l)|$. Then by the mean value theorem, there exists $x_I \in I$ such that $|g(I)| = |I| \cdot |g'(x_I)|$. Let v be the right endpoint of some gap G in $C \cap [u, u+\delta]$. Note that if there are no gaps in $C \cap [u, u+\delta]$ then $C \cap [u, u+\delta]$ is an interval and therefore, $g(C \cap [u, u+\delta])$ is an interval which implies $\tau(g(C \cap [u, u+\delta])) = +\infty$ and the lemma is trivially true. Now let $G_0 = (a_0, b_0)$ be the closest gap such that $(1-\varepsilon)\operatorname{diam}(G) < \operatorname{diam}(G_0)$. Then there is a line segment, which we will call $B_\varepsilon(v)$, such that $B_\varepsilon(v) = [v, a_0]$.

We will first show that $|g(B_\varepsilon(v))| \leq |B_{\varepsilon^2}(g(v))|$. To see this, note that any gap in $g(C \cap [u, u+\delta])$ is the image of a gap in $C \cap [u, u+\delta]$. Therefore, it suffices to prove that any gap $H \subset B_\varepsilon(v)$ satisfies $|g(H)| \leq (1-\varepsilon^2)|g(G)|$. Then since $H \subset B_\varepsilon(v)$ we have $|H| \leq (1-\varepsilon)|G|$.

Thus

$$\begin{aligned}
|g(H)| &= |H| \cdot |g'(x_H)| \\
&\leq (1-\varepsilon)|G| \cdot \frac{|g'(x_H)|}{|g'(x_G)|} \cdot |g'(x_G)| \\
&< (1-\varepsilon)|G| \cdot (1+\varepsilon) \cdot |g'(x_G)| \\
&= (1-\varepsilon^2)|g(G)|.
\end{aligned}$$

Now we get

$$\begin{aligned}
\tau_{\varepsilon^2}\left(g(C \cap [u, u+\delta]), g(v)\right) &= \frac{|B_{\varepsilon^2}(g(v))|}{|g(G)|} \\
&\geq \frac{|g(B_\varepsilon(v))|}{|g(G)|} \\
&= \frac{|B_\varepsilon(v)|}{|G|} \cdot \frac{|g'(x_{B_\varepsilon(v)})|}{|g'(x_G)|} \\
&\geq \frac{|B_\varepsilon(v)|}{|G|} \cdot (1-\varepsilon).
\end{aligned}$$

Then note that $|B_\varepsilon(v)|$ represents the distance from v (which is a boundary point of G) to the nearest gap and therefore $|B_\varepsilon(v)| = \operatorname{dist}(G, \bigcup_{i \in H_\varepsilon} G_i \cup E)$ where $H_\varepsilon = \{i : G_i \neq G, (1-\varepsilon)|G| < |G_i|\}$. Thus,

$$\tau_{\varepsilon^2}\left(g(C \cap [u, u+\delta]), g(v)\right) \geq \tau_\varepsilon(C \cap [u, u+\delta], v) \cdot (1-\varepsilon).$$

Taking the infimum over v, we get

$$\tau(g\left(C \cap [u, u+\delta]\right)) > \tau_{\varepsilon^2}(g\left(C \cap [u, u+\delta]\right)) \geq \tau_\varepsilon(C) \cdot (1-\varepsilon)$$

which shows the claim. $\square$

Now we only need one more ingredient to prove Theorem 3.5. However, checking that a compact set is not contained in a gap of another compact set is tricky (which is a hypothesis of Theorem 3.5), so the next lemma and theorem gives a sufficient criterion and easier to work with criterion. Let $\mathrm{conv}(\cdot)$ denote the convex hull of a set.

Lemma 3.11. *Let $C \in \mathbb{R}^d$ be a compact set such that $\tau(C) > 0$. Let $U = \mathrm{conv}(C)^\circ$. Then U is non-empty.*

Proof. Assume for contradiction that $U = \emptyset$. Then note that $\mathrm{conv}(C)$ has no bounded gaps. Then since $U = \emptyset$, by definition of thickness, this implies $\tau(\mathrm{conv}(C)) = 0$. But note that $\tau(C) \leq \tau(\mathrm{conv}(C))$ since $\mathrm{conv}(C)$ has simply removed all of the bounded gaps of C which then means

$$0 < \tau(C) \leq \tau(\mathrm{conv}(C)) = 0$$

a contradiction. □

Theorem 3.12. *Let C_1 and C_2 be compact sets in $\mathbb{R}^d$ with $\tau(C_1), \tau(C_2) > 0$ and such that their convex hulls are linked. That is, by setting $U_1 = \mathrm{conv}(C_1)^\circ$ and $U_2 = \mathrm{conv}(C_2)^\circ$, we have that $U \cap V \neq \emptyset$, $(\partial U) \setminus V \neq \emptyset$, and $(\partial V) \setminus U \neq \emptyset$. Then neither C_1 or C_2 is contained in a gap of the other.*

Proof. Set $U_1 = \mathrm{conv}(C_1)^\circ$ and $U_2 = \mathrm{conv}(C_2)^\circ$. By Lemma 3.11 both U_1 and U_2 are non-empty. Assume for contradiction that C_1 is contained in a gap of C_2, say G^2. So $C_1 \subseteq G^2$. Note that $C_1 \subset G^2$ since G^2 is open and C_1 is closed. Then $\mathrm{conv}(C_1) \subseteq \mathrm{conv}(G^2)$. Note that since G^2 is open, $\mathrm{conv}(G^2)$ is open. Thus,

$$U_3 := \mathrm{conv}(G^2)^\circ = \mathrm{conv}(G^2).$$

Note that $\partial U_1 \subseteq \partial \operatorname{conv}(C_1)$. Then if $\partial \operatorname{conv}(C_1) \subset U_3$ we're done since this would imply

$$\emptyset \neq (\partial U_1) \backslash U_2 \subseteq (\partial \operatorname{conv}(C_1)) \backslash U_2 \subseteq (\partial \operatorname{conv}(C_1)) \backslash U_3 = \emptyset$$

which yields a contradiction. So now assume $\partial \operatorname{conv}(C_1) \supseteq U_3$. To see that this implies these sets are equal, take $x \in \partial \operatorname{conv}(C_1)$. Then since C_1 is closed, $\operatorname{conv}(C_1)$ is also closed and thus we have $\partial \operatorname{conv}(C_1) \subseteq \operatorname{conv}(C_1)$ and therefore, $x \in \operatorname{conv}(C_1)$. But

$$x \in \operatorname{conv}(C_1) \subseteq \operatorname{conv}(G^2) = \operatorname{conv}(G^2)^\circ = U_3$$

and therefore, $U_3 = \partial \operatorname{conv}(C_1)$ which is a contradiction since U_3 is open and $\partial \operatorname{conv}(C_1)$ is closed. This gives the desired conclusion. □

Now we present the last lemma needed to prove Theorem 3.5. This is Lemma 3.5 in [14].

Lemma 3.13. *Let C_1, C_2 be compact sets such that C_1 has at least one bounded gap and $\tau(C_1)\tau(C_2) > 1$. For any $x^0 \in \mathbb{R}^2$, there exists an open set S about x^0 such that*

$$\bigcap_{x \in S} \Delta_x(C_1 \times C_2)$$

has non-empty interior.

Proof. For $(x, t) \in \mathbb{R}^2 \times (0, \infty)$, define

$$g_{x,t}(z) := x_2 + \sqrt{t^2 - (z - x_1)^2},$$

and note that if $C_2 \cap g_{x,t}(C_1) \neq \emptyset$ then $t \in \Delta_x(C_1 \times C_2)$. Let u_j be a right endpoint of a bounded gap of C_j, and without loss of generality assume $u_j > x_j^0$ where $x^0 = (x_1^0, x_2^0)$. Let

$t_0 = |x^0 - (u_1, u_2)|$. Let $\widetilde{C}_j = C_j \cap [u_j, u_j + \delta_j]$ where δ_1, δ_2 are chosen so that $\tau(\widetilde{C}_2)\tau(g(\widetilde{C}_1)) > 1$, which is possible by Lemma 3.10, and so that $\widetilde{C}_1$ is in the domain of $g_{x,t}$ whenever (x,t) is sufficiently close to (x^0, t_0). Furthermore, we can also assume $u_j + \delta_j \in C_j$ by making δ_j possibly smaller. Then by Theorem 2.11 and Theorem 3.12, we will have $\widetilde{C}_2 \cap g_{x,t}(\widetilde{C}_1) \neq \emptyset$ whenever the parameters (x,t) are such that $\widetilde{C}_2$ and $g_{x,t}(\widetilde{C}_1)$ are linked, that is, when their convex hulls are linked. Note that $g_{x,t}$ is a decreasing function. Then because we want to know when $\widetilde{C}_2$ and $g_{x,t}(\widetilde{C}_1)$ are linked, consider the set

$$U = \left\{(x,t) \in \mathbb{R}^2 \times \mathbb{R} : g_{x,t}(u_1 + \delta_1) < u_2 < g_{x,t}(u_1) < u_2 + \delta_2\right\}.$$

Then by construction, if $(x,t) \in U$ then we have $\widetilde{C}_2$ and $g_{x,t}(\widetilde{C}_1)$ are linked (this is where the assumption that $u_j + \delta_j \in C_j$ comes in) and hence, $t \in \Delta_x(C_1 \times C_2)$. We claim that U is an open set containing a point of the form (x^0, t) for some t. This would complete the proof because we can then take open neighborhoods S, T of x^0, t respectively such that

$$T \subset \bigcap_{x \in S} \Delta_x(C_1 \times C_2).$$

To show this claim, note that for a fixed z, the quantity $g_{x,t}(z)$ is a continuous function of (x,t) and thus, U is open. To find a point of the form $(x^0, t) \in U$, by construction we have $g_{x^0,t_0}(u_1) = u_2$ and because the quantity $g_{x,t}(z)$ is strictly increasing in t, for any $t > t_0$ we will have $g_{x^0,t_0}(u_1) > u_2$. On the other hand, by continuity in t we will also have $g_{x^0,t}(u_1 + \delta_1) < u_2$ and $g_{x^0,t}(u_1) < u_2 + \delta_2$ whenever t is sufficiently close to t_0. Therefore, we can find t such that $(x_0, t) \in U$ which completes the proof. $\square$

Theorem 3.5 follows from Theorem 3.7 and Lemma 3.13.

3.2 Extension to $\mathbf{R}^d$

Here we extend the result of McDonald and Taylor to $\mathbb{R}^d$ for pinned distance sets rather than trees.

Theorem 3.14. *Let $C^1, C^2 \subset \mathbb{R}^d$ be compacts sets satisfying $\tau(C^1)\tau(C^2) > 1$. Then for any $x^0 \in C^1 \times C^2$ there exists an open neighborhood T such that $T \subset \Delta_{x^0}(C^1 \times C^2)$, i.e. $\Delta_{x^0}(C^1 \times C^2)$ has non-empty interior.*

We now present a d-dimensional analogue of the $g_{x,t}$ function. Fix $(x,t) \in \mathbb{R}^{2d} \times [0,\infty)$ and for $i \in \{1, \cdots, d\}$ define $\widetilde{g}^i : \mathbb{R}^d \to \mathbb{R}$ by

$$\begin{aligned}
\widetilde{g}^i(y_1, \cdots, y_d) &:= x_{i+d} + \sqrt{\begin{aligned}&\frac{1}{d}t^2 - 0(x_1 - y_1)^2 - \cdots - 0(x_{i-1} - y_{i-1})^2 \\ &\qquad - (x_i - y_i)^2 - 0(x_{i+1} - y_{i+1})^2 - \cdots - 0(x_d - y_d)^2\end{aligned}} \\
&= x_{i+d} + \sqrt{\frac{1}{d}t^2 - (x_i - y_i)^2} \\
&=: g^i(y_i).
\end{aligned}$$

We do this because if we set $y_{i+d} = g^i(y_i)$ then $\frac{1}{d}t^2 = (y_{i+d} - x_{i+d})^2 + (y_i - x_i)^2$ for all i, which then means that by summing each of these terms we will have

$$\sum_{i=1}^{2d} (y_i - x_i)^2 = t^2$$

or in other words, the distance between $(y_1, \cdots, y_{2d})$ and $(x_1, \cdots, x_{2d})$ is t. Finally defined $g_{x,t} : \mathbb{R}^d \to \mathbb{R}^d$ by

$$g_{x,t}(y_1, \cdots, y_d) = (\widetilde{g}^1(y_1, \cdots, y_d), \cdots, \widetilde{g}^d(y_1, \cdots, y_d)) = (g^1(y_1), \cdots, g^d(y_d)).$$

Now we list an interesting property of $g_{x,t}$. From here on out, $g'_{x,t}$ will denote the Jacobian matrix of $g_{x,t}$. Then for $z = (z_1, \cdots, z_d) \in \mathbb{R}^d$ we have

$$g'_{x,t}(z) = \begin{bmatrix} \frac{dg^1}{dz_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{dg^2}{dz_2} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & \frac{dg^d}{dz_d} \end{bmatrix} = \begin{bmatrix} \frac{x_1-z_1}{\sqrt{\frac{1}{d}t^2-(z_1-x_1)^2}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{x_2-z_2}{\sqrt{\frac{1}{d}t^2-(z_2-x_2)^2}} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & \frac{x_d-z_d}{\sqrt{\frac{1}{d}t^2-(z_d-x_d)^2}} \end{bmatrix}.$$

Thus, $g'_{x,t}(z)$ is a diagonal matrix. Furthermore, the norm that we will be working with for $g'_{x,t}(z)$ is the operator norm which is the square root of the largest eigenvalue of $g'_{x,t}(z)^T g'_{x,t}(z)$ where T denotes the transpose. Then because $g'_{x,t}(z)$ is diagonal, we have $g'_{x,t}(z)^T = g'_{x,t}(z)$ and therefore

$$g'_{x,t}(z)^T g'_{x,t}(z) = \begin{bmatrix} \frac{(x_1-z_1)^2}{\frac{1}{d}t^2-(z_1-x_1)^2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{(x_2-z_2)^2}{\frac{1}{d}t^2-(z_2-x_2)^2} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & \frac{(x_d-z_d)^2}{\frac{1}{d}t^2-(z_d-x_d)^2} \end{bmatrix}$$

and thus,

$$\|g'_{x,t}(z)\|_{\text{op}} = \max\left\{ \frac{|x_1 - z_1|}{\sqrt{\frac{1}{d}t^2 - (z_1 - x_1)^2}}, \cdots, \frac{|x_d - z_d|}{\sqrt{\frac{1}{d}t^2 - (z_d - x_d)^2}} \right\} = \max\left\{ \left|\frac{dg^1}{dz_1}\right|, \cdots, \left|\frac{dg^d}{dz_d}\right| \right\}. \tag{3.2}$$

This estimate will come in later. The function $g_{x,t}$ also satisfies another useful property.

Lemma 3.15 (Mean Value Theorem for $g_{x,t}$)**.** *Let $a, b \in \mathbb{R}^d$ be given such that $a_i < b_i$ for all $i \in \{1, \cdots, d\}$ and $g_{x,t}$ is defined on $[a_1, b_1] \times \cdots \times [a_d, b_d]$. Then there exists $z_i \in (a_i, b_i)$*

such that

$$(b-a)g'_{x,t}(z) = g_{x,t}(b) - g_{x,t}(a)$$

where $z = (z_1, \cdots, z_d)$.

Proof. Since each g^i is a mapping from $\mathbb{R}$ to $\mathbb{R}$ and will be differentiable in $[a_i, b_i]$, by the one-dimensional mean value theorem, there exists $z_i \in (a_i, b_i)$ such that $(b_i - a_i)\frac{dg^i}{dz_i} = g^i(b_i) - g^i(a_i)$. So

$$\begin{aligned}
(b-a)g'_{x,t}(z) &= \begin{bmatrix} b_1 - a_1 \\ \vdots \\ b_d - a_d \end{bmatrix} \begin{bmatrix} \frac{dg^1}{dz_1} & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & \frac{dg^d}{dz_d} \end{bmatrix} \\
&= \begin{bmatrix} (b_1 - a_1)\frac{dg^1}{dz_1} \\ \vdots \\ (b_d - a_d)\frac{dg^2}{dz_d} \end{bmatrix} \\
&= \begin{bmatrix} g^1(b_1) - g^1(a_1) \\ \vdots \\ g^d(b_d) - g^d(a_d) \end{bmatrix} \\
&= \begin{bmatrix} g^1(b_1) \\ \vdots \\ g^d(b_d) \end{bmatrix} - \begin{bmatrix} g^1(a_1) \\ \vdots \\ g^d(a_d) \end{bmatrix} \\
&= g_{x,t}(b) - g_{x,t}(a).
\end{aligned}$$

□

Since we are now working in $\mathbb{R}^d$ we now need to deal with more complicated sets and in particular, more complicated boundaries. But the next series of lemmas, which is presented

without proof, allows us nicely characterize how the boundary and diameter of a set is affected by $g_{x,t}$.

Lemma 3.16. *Suppose f is continuous and H is open where $f(H) = H'$ are open sets with H having compact closure. Then $f(\partial H) = \partial H'$.*

In our language, this will mean $g_{x,t}(\partial H) = \partial g_{x,t}(H)$ where H is a gap of C.

Lemma 3.17. *If H is open, then* $\operatorname{diam}(H) = \operatorname{diam}(\partial H)$.

Combining these two Lemmas gives us the next lemma.

Lemma 3.18. *Since $g_{x,t}$ is continuous we have*

$$\operatorname{diam}(g(H)) = \operatorname{diam}(\partial g(H)) = \operatorname{diam}(g(\partial H)).$$

Throughout the proof of Theorem 3.14, it will be convenient for our pinned point to be the origin and for us to be able work with boundary points of gaps of C^1 and C^2 to exist on the diagonal. Fortunately, we can do this because thickness is preserved under similarities. The next Lemma characterizes this shift.

Lemma 3.19. *Let $C^1, C^2 \subset \mathbb{R}^d$ with $\tau(C^1)\tau(C^2) > 1$. Let $x^0 \in C^1 \times C^2$ be given and let $u^1 \in C^1$, $u^2 \in C^2$ be such that they are boundary points of some gap (could also be the unbounded gaps) of C^1 and C^2 respectively. Then there exists similarities $S_1 : \mathbb{R}^d \to \mathbb{R}^d$ and $S_2 : \mathbb{R}^d \to \mathbb{R}^d$ with the following properties:*

1. *$x^0 \mapsto \vec{0}$ under S_1 and S_2*
2. *$u^1 \mapsto w^1 = (w_1^1, \cdots, w_d^1)$ under S_1 such that $w_i^1 = w_1^1$ for all i and $u^2 \mapsto w^2 = (w_1^2, \cdots, w_d^2)$ under S_2 such that $w_i^2 = w_1^2$ for all i*

3. $\tau(S_1(C^1)) = \tau(C^1)$ *and* $\tau(S_2(C^2)) = \tau(C^2)$

4. $\Delta_{\vec{0}}(S_1(C^1) \times S_2(C^2)) = \Delta_{x^0}(C^1 \times C^2)$.

Proof. Let $x^{0,1} := (x_1^0, \cdots, x_d^0)$ and $x^{0,2} := (x_{d+1}^0, \cdots, x_{2d}^0)$. Find $z^1, z^2 \in \mathbb{R}^d$ such that $x^{0,1} + z^1 = \vec{0}$ and $x^{0,2} + z^2 = \vec{0}$. Let $SO(d)$ denote the group of d-dimensional rotations. Then there exists $A_1, A_2 \in SO(d)$ such that $A_1(u^1 + z^1) = w^1 = (w_1^1, \cdots, w_d^1)$ such that $w_i^1 = w_1^1$ for all $i \in \{1, \cdots, d\}$ and $A_2(u^2 + z^2) = w^2(w_1^2, \cdots w_d^2)$ such that $w_i^2 = w_1^2$ for all i. Furthermore, elements in $SO(d)$ are linear. So for any $x \in \mathbb{R}^d$ we have that the function $S_i : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$S_i(x) = A_i(x + z^i) = A_i x + A_i z^i$$

and thus, S_i is a similarity. Therefore, since thickness is preserved under similarities we get $\tau(S_i(C^i)) = \tau(C^i)$. Furthermore, both translations by z^i and rotation by A_i are isometries which means that S_i is an isometry. Thus, for $t \in \Delta_{\vec{0}}(S_1(C^1) \times S_2(C^2))$ we can find $y^1 \in C^1$ and $y^2 \in C^2$ such that

$$\begin{aligned} t^2 &= \left[\mathrm{dist}(S_1(x^{0,1}), S_1(y^1))\right]^2 + \left[\mathrm{dist}(S_2(x^{0,2}), S_2(y^2))\right]^2 \\ &= \left[\mathrm{dist}(x^{0,1}, y^1)\right]^2 + \left[\mathrm{dist}(x^{0,2}, y^2)\right]^2 \\ &= \sum_{i=1}^{d} (x_i^0 - y_i^1)^2 + \sum_{j=d+1}^{2d} (x_j^0 - y_j^2)^2 \end{aligned}$$

and therefore, $t \in \Delta_{x^0}(C^1 \times C^2)$. Furthermore, since equality held throughout, we can say $\Delta_{x^0}(C^1 \times C^2) = \Delta_{\vec{0}}(S_1(C^1) \times S_2(C^2))$. □

Therefore, throughout this document we will assume we are in the situation where $x^0 = \vec{0}$, $u^1 = (u_1^1, u_1^1, \cdots, u_1^1)$, and $u^2 = (u_1^2, u_1^2, \cdots, u_1^2)$.

We now show that even in the higher dimensional case, we still have that thickness is nearly

preserved under $g_{x,t}$. Furthermore, in the next lemma we will take $x = (x_1, \cdots, x_{2d}) \in \mathbb{R}^{2d}$ such that $x_1 = x_i$ for all $i \in \{1, \cdots, d\}$. But this is not a concern for us by the previous lemma as we can always rotate and translate our sets so that we are in this case.

Lemma 3.20 (Thickness is nearly preserved)**.** *Let $C \in \mathbb{R}^d$ compact and let u be a boundary point of some bounded gap of C. Find $x \in \mathbb{R}^{2d}$ such that $x_1 = x_i$ for all $i \in \{1, \cdots, d\}$ and for which u is in the domain of $g_{x,t}$ and such that $\frac{dg^i}{du_i} \neq 0$ for all $i \in \{1, \cdots, d\}$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\tau\left(g_{x,t}(C \cap ([u_1, u_1 + \delta] \times \cdots \times [u_d, u_d + \delta]))\right) > \tau_{2\varepsilon - \varepsilon^2}(C)(1 - \varepsilon).$$

Proof. First note that since $g_{x,t}$ is continuous, the image of any compact set is also compact. So it is valid for us to talk about the thickness of the image of a compact set under $g_{x,t}$. Let $\varepsilon > 0$ be given. Note that since $x_1 = x_i$ for $i \in \{1, \cdots, d\}$, this means

$$\frac{dg^i}{dz_i} = \frac{x_1 - z_i}{\sqrt{\frac{1}{2}t^2 - (x_1 - z_i)^2}}.$$

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(z) = \frac{x_1 - z}{\sqrt{\frac{1}{2}t^2 - (x_1 - z)^2}}$. Then f is continuous and for $z \in \mathbb{R}^d$ we have $f(z_i) = dg^i/dz_i$ for $i \in \{1, \cdots, d\}$. Furthermore, since each dg^i/du_i is nonzero, f will also be nonzero in a neighborhood of u. With this and because f is continuous, there exists $\widetilde{\delta} > 0$ such that for any $z \in [u_1, u_1 + \delta] \times \cdots \times [u_d, u_d + \delta]$ where $\delta = \widetilde{\delta}/\sqrt{d}$, we have

$$\left| \frac{|dg^i/dz_i|}{|dg^j/dz_j|} - 1 \right| = \left| \frac{|f(z_i)|}{|f(z_j)|} - 1 \right| < \varepsilon \tag{3.3}$$

for any $i, j \in \{1, \cdots, d\}$ since this means $|z_i - z_j| \leq \sqrt{d} \cdot \delta = \widetilde{\delta}$ which invokes continuity of f. Also note that any gap in $A_1 := g_{x,t}(C \cap ([u_1, u_1 + \delta] \times \cdots \times [u_d, u_d + \delta]))$ is the image of a gap in $A_2 := C \cap ([u_1, u_1 + \delta] \times \cdots \times [u_d, u_d + \delta])$. In particular, all gaps in A_1 will be of the

form $g_{x,t}(G_n)$ where G_n is a gap in A_2. After possibly reordering the original gaps, we then produce a new list of gaps of A_1, say $\{g_{x,t}(G_n)\}_{n=1}^{\infty}$, with $\operatorname{diam}(g_{x,t}(G_n)) \geq \operatorname{diam}(g_{x,t}(G_{n+1}))$. By Lemma 3.18, $\operatorname{diam}(g_{x,t}(G_n)) = \operatorname{diam}(\partial g_{x,t}(G_n)) = \operatorname{diam}(g_{x,t}(\partial G_n))$. Thus, for ε-thickness we can take $u_n \in \partial G_n$, for valid G_n, and have

$$\tau_\varepsilon(A_1, g_{x,t}(u_n)) = \frac{\operatorname{dist}(g_{x,t}(u_n), \bigcup_{i \in H_{\varepsilon,n}} g_{x,t}(G_i) \cup E)}{\operatorname{diam}(g_{x,t}(G_n))}$$

where E now denotes the unbounded gap of A_1 and $H_{\varepsilon,n} = \{i \neq n : (1-\varepsilon)\operatorname{diam}(g_{x,t}(G_n)) < \operatorname{diam}(g_{x,t}(G_i))\}$. Now let $v_n \in \partial G_n$ such that $g_{x,t}(G_n)$ is a gap in A_1. Note that since we are assuming that we have thick compact sets, then the thickness of neither is equal to zero. So we always get nonzero values in the numerator for the expression of $\tau_\varepsilon(A_1, g(v_n))$. So we can find some element, say $v_j \in \partial G_j$ where G_j could also be the unbounded gap of A_1 such that $(1-\varepsilon)\operatorname{diam}(g_{x,t}(G_n)) < \operatorname{diam}(g_{x,t}(G_j))$ and $\operatorname{dist}(g_{x,t}(v_n), \bigcup_{i \in H_{\varepsilon,n}} g_{x,t}(G_i) \cup E) = \|g_{x,t}(v_n) - g_{x,t}(v_j)\|$. Set $A_3 := [u_1, u_1+\delta] \times \cdots \times [u_d, u_d+\delta]$. By Lemma 3.15, there exists $z \in A_3$ such that $\|g_{x,t}(v_n) - g_{x,t}(v_j)\| = \|g'_{x,t}(z)(v_n - v_j)\|$. Furthermore, $\operatorname{diam}(g_{x,t}(G_n)) = \|g_{x,t}(a_n) - g_{x,t}(b_n)\|$ for some $a_n, b_n \in \partial G_n$. Again by Lemma 3.15 we have that there exists some $w \in A_3$ such that $\|g_{x,t}(a_n) - g_{x,t}(b_n)\| = \|g'_{x,t}(w)(a_n - b_n)\|$. This gives

$$\begin{aligned}
\tau_\varepsilon(A_1, g_{x,t}(v_n)) &= \frac{\|g'_{x,t}(z)(v_n - v_j)\|}{\operatorname{diam}(g_{x,t}(G_n))} \\
&= \frac{\|g'_{x,t}(z)(v_n - v_j)\|}{\|g'_{x,t}(w)(a_n - b_n)\|} \\
&\geq \frac{\|g'_{x,t}(z)(v_n - v_j)\|}{\|g'_{x,t}(w)\|_{op}\|a_n - b_n\|}.
\end{aligned}$$

To get a further lower estimate on this quantity, assume WLOG that $\left|\frac{dg^2}{dz_2}\right| = \min\left\{\left|\frac{dg^1}{dz_1}\right|, \cdots, \left|\frac{dg^d}{dz_d}\right|\right\}$.

Then

$$\begin{aligned}
\|g'_{x,t}(z)(v_n - v_j)\| &= \left\| \begin{bmatrix} \frac{d\tilde{g}^1}{dz_1} & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & \frac{dg^d}{dz_d} \end{bmatrix} \begin{bmatrix} v_{n,1} - v_{j,1} \\ \vdots \\ v_{n,d} - v_{j,d} \end{bmatrix} \right\| \\
&= \sqrt{\left(\frac{dg^1}{dz_1}\right)^2 (v_{n,1} - v_{j,1})^2 + \cdots + \left(\frac{dg^d}{dz_d}\right)^2 (v_{n,d} - v_{j,d})^2} \\
&\geq \sqrt{\left(\frac{dg^2}{dz_2}\right)^2 (v_{n,1} - v_{j,1})^2 + \cdots + \left(\frac{dg^2}{dz_2}\right)^2 (v_{n,d} - v_{j,d})^2} \\
&= \left|\frac{dg^2}{dz_2}\right| \|v_n - v_j\|.
\end{aligned}$$

Then

$$\begin{aligned}
\tau_\varepsilon(A_1, g_{x,t}(v_n)) &\geq \frac{\left|\frac{dg^2}{dz_2}\right| \|v_n - v_j\|}{\|g'_{x,t}(w)\|_{op} \|a_n - b_n\|} \\
&\geq \frac{\left|\frac{dg^2}{dz_2}\right| \|v_n - v_j\|}{\|g'_{x,t}(w)\|_{op} \operatorname{diam}(G_n)}.
\end{aligned}$$

Also WLOG we can assume $\|g'_{x,t}(w)\|_{op} = \left|\frac{dg^1}{dw_1}\right|$. By equation 3.3 we have

$$\frac{\left|\frac{dg^2}{dz_2}\right|}{\left|\frac{dg^1}{dw_1}\right|} > 1 - \varepsilon$$

which yields

$$\begin{aligned}
\tau_\varepsilon(A_1, g_{x,t}(v_n)) &\geq \frac{\left|\frac{dg^2}{dz_2}\right| \|v_n - v_j\|}{\left|\frac{dg^1}{dw_1}\right| \operatorname{diam}(G_n)} \\
&> (1 - \varepsilon) \frac{\|v_n - v_j\|}{\operatorname{diam}(G_n)}.
\end{aligned}$$

Note that v_n and v_j were chosen such that $\|g_{x,t}(v_n)-g_{x,t}(v_j)\| = \operatorname{dist}\left(g_{x,t}(v_n), \bigcup_{i\in H_{\varepsilon,n}} g_{x,t}(G_i) \cup g_{x,t}(E)\right)$ and recall that since $j \in H_{\varepsilon,n}$ we have $(1-\varepsilon)\operatorname{diam}(g_{x,t}(G_n)) < \operatorname{diam}(g_{x,t}(G_j))$. Note that j being in $H_{\varepsilon,n}$ refers to the ordering that was placed on $\{g_{x,t}(G_n)\}_{n=1}^{\infty}$, not necessarily the ordering that we placed on $\{G_n\}_{n=1}^{\infty}$. We will now try to find $\eta > 0$ such that $j \in H_{\eta,n}$ where this now refers to the ordering that we placed on $\{G_n\}_{n=1}^{\infty}$ which will provide a lower estimate on $\|v_n - v_j\|$. Let $\alpha^1, \beta^1 \in \partial G_n$ such that $\operatorname{diam}(G_n) = \|\alpha^1 - \beta^1\|$ and $\alpha^2, \beta^2 \in \partial G_j$ such that $\operatorname{diam}(g_{x,t}(G_j)) = \|g_{x,t}(\alpha^2) - g_{x,t}(\beta^2)\|$. Then

$$(1-\varepsilon)\|g_{x,t}(\alpha^1) - g_{x,t}(\beta^1)\| \leq (1-\varepsilon)\|g_{x,t}(a_n) - g_{x,t}(b_n)\| < \|g_{x,t}(\alpha^2) - g_{x,t}(\beta^2)\|.$$

As before, we can find $\rho^1, \rho^2 \in A_3$ such that $\|g_{x,t}(\alpha^1) - g_{x,t}(\beta^1)\| = \|g'_{x,t}(\rho^1)(\alpha^1 - \beta^1)\|$ and $\|g_{x,t}(\alpha^2) - g_{x,t}(\beta^2)\| = \|g'_{x,t}(\rho^2)(\alpha^2 - \beta^2)\|$. By a similar calculations as above, we can assume WLOG that

$$\|g'_{x,t}(\rho^1)(\alpha^1 - \beta^1)\| \geq \left|\frac{dg^1}{d\rho_1^1}\right| \|\alpha^1 - \beta^1\|$$

and

$$\|g'_{x,t}(\rho^2)(\alpha^2 - \beta^2)\| \leq \|g'_{x,t}(\rho^2)\|_{op}\|\alpha^2 - \beta^2\| = \left|\frac{dg^2}{d\rho_2^2}\right| \|\alpha^2 - \beta^2\|.$$

Along with equation 3.3 this implies

$$\begin{aligned}
\operatorname{diam}(G_j) &\geq \|\alpha^2 - \beta^2\| \\
&> (1-\varepsilon)\frac{\left|\frac{dg^1}{d\rho_1^1}\right|}{\left|\frac{dg^2}{d\rho_2^2}\right|}\|\alpha^1 - \beta^1\| \\
&> (1-\varepsilon)(1-\varepsilon)\|\alpha^1 - \beta^1\| \\
&= (1-(2\varepsilon - \varepsilon^2))\operatorname{diam}(G_n).
\end{aligned}$$

Therefore, $j \in H_{2\varepsilon-\varepsilon^2,n}$ where this refers to the ordering placed on $\{G_n\}_{n=1}^{\infty}$. So

$$\begin{aligned}\tau_\varepsilon(A_1, g_{x,t}(v_n)) &\geq (1-\varepsilon)\frac{\|v_n - v_j\|}{\operatorname{diam}(G_n)} \\ &\geq (1-\varepsilon)\frac{\operatorname{dist}\left(v_n, \bigcup_{i\in H_{2\varepsilon-\varepsilon^2,n}} G_i \cup E\right)}{\operatorname{diam}(G_n)} \\ &= (1-\varepsilon)\tau_{2\varepsilon-\varepsilon^2}(C, v_n).\end{aligned}$$

Taking infimums produces

$$\tau(A_1) \geq \tau_\varepsilon(A_1) > \tau_{2\varepsilon-\varepsilon^2}(C)(1-\varepsilon)$$

which finishes the proof. □

In the proof of Theorem 3.14 it will be critical that when we work with boundary points u^1 of some gap of C^1 and u^2 of some gap of C^2 that we can find $\delta^1, \delta^2 \in \mathbb{R}^d$ such that $u^1 + \delta^1 \in C^1$ and $u^2 + \delta^2 \in C^2$ as this allows us to characterize the convex hull of shrinked versions of C^1 and C^2. The next lemma allows us to do this.

Lemma 3.21. *Let $C \in \mathbb{R}^d$ compact such that $\tau(C) > 0$. Let $u \in C$ be a boundary point of some gap of C. Then we can find $\delta = (\delta_1, \cdots, \delta_d)$ such that $u + \delta \in C$. Furthermore, for any $\eta > 0$ we can, possibly, shrink δ so that $|\delta| < |\eta|$ and $|\delta_1| = |\delta_i|$ for all $i \in \{1, \cdots, d\}$.*

Proof. Assume for contradiction that we cannot find δ such that $u + \delta \in C$. Let G_n denote the gap for which $u \in \partial G_n$. It is clear that any gap (bounded or unbounded) will have an element on the boundary, say $w \in \partial G_n$ such that for any small $\delta > 0$ we will have $w + (\delta, \cdots, \delta) \notin G_n$, i.e. w sits at the "top-right" of G_n. Clearly, it is possible that u does not satisfy this requirement however it is clear that for any small $\delta > 0$ we will have $u + (\pm\delta, \cdots, \pm\delta) \notin G_n$. So we assume WLOG that u sits at the "top-right"of G_n for

simplicity of notation. By our contradiction assumption, since no such $\delta \in \mathbb{R}^d$ exists such that $u + \delta \in C$, we must have that there exists a gap, say G_i such that $\text{dist}(u, G_i) = 0$. To see this, assume this was not the case. Then there would be a "minimal" G_i, i.e. there exists G_i such that $0 < \text{dist}(u, G_i) \leq \text{dist}(u, H)$ where H is any gap of C. But then by setting $c = \text{dist}(u, G_i)$ we simply choose $\delta > 0$ such that $\sqrt{d}\delta < c$ which means

$$0 < \sqrt{(u_1 - (u_1 + \delta))^2 + \cdots + (u_d - (u_d + \delta))^2} = \sqrt{d}\delta < c$$

and this implies implies $u + (\delta, \cdots, \delta) \in C$ since G_i was the minimal gap. However this contradicts our assumption that there is no $\delta \in \mathbb{R}^d$ such that $u + \delta \in C$. Thus, we can assume there exists G_i such that $\text{dist}(u, G_i) = 0$ and therefore, $\text{dist}(G_n, G_i) = 0$. WLOG we can assume $\text{diam}(G_i) \leq \text{diam}(G_n)$. But this means

$$\frac{\text{dist}(G_i, \bigcup_{j \in \Lambda_i} G_j \cup E)}{\text{diam}(G_i)} \leq \frac{\text{dist}(G_i, G_n)}{\text{diam}(G_i)} = 0$$

which implies $\tau(C) = 0$, a contradiction. Note that Λ_i is being defined as in Lemma 2.12. Thus, we can find $\delta = (\delta_1, \cdots \delta_d)$ such that $u + \delta \in C$ and $|\delta_1| = |\delta_i|$ for $i \in \{1, \cdots, d\}$. Furthermore, it is clear that we could have initially chosen δ_1 such that $|\delta| = |(\delta_1, \cdots, \delta_d)| < |\eta|$. □

We now present the proof of Theorem 3.14.

Proof of Theorem 3.14. Recall that we may assume without loss of generality that $x^0 = \vec{0}$. Furthermore, we have $t \in \Delta_{x_0}(C^1 \times C^2)$ provided $C^2 \cap g_{x_0,t}(C^1) \neq \emptyset$. Let u^1 and u^2 be a boundary point of some gap (not necessarily bounded gaps) of C^1 and C^2 respectively. Also without loss of generality we may assume (u^1, u^2) lives to the upper right of x^0. By Lemma 3.21 we find $\delta^j \in \mathbb{R}^d$ such that $u^j, u^j + \delta^j \in C^j$ for which $\delta^j_1 = \delta^j_i$ for all i. So

set $\widetilde{C^j} := C^j \cap ([u_1^i, u_1^i + \delta_1^i] \times \cdots \times [u_d^i, u_d^i + \delta_1^i])$. In particular we can also choose δ^i small enough so that $\tau(\widetilde{C^2})\tau(g_{x,t}(\widetilde{C^1})) > 1$ by Lemma 3.21 and Lemma 3.20. Recall that by Lemma 3.19 we may also assume without loss of generality that $u_i^1 = u_1^1$ and $u_i^2 = u_1^2$ for all $i \in \{1, \cdots d\}$. To use the Gap Lemma we find t such that $\widetilde{C^2}$ and $g_{x_0,t}(\widetilde{C^1})$ are linked. The parameter t is important for this proof so let g_t^i denote g^i. Note that each g_t^i is decreasing in its argument. So consider the set

$$U := \{t \in \mathbb{R} : g_t^1(u_1^1 + \delta_1^1) < u_1^2 < g_t^1(u_1^1) < u_1^2 + \delta_1^2, \cdots, g_t^d(u_d^1 + \delta_d^1) < u_d^2 < g_t^d(u_d^1) < u_d^2 + \delta_d^2\}.$$

Then if $t \in U$, we have that $\widetilde{C^2}$ and $g_{x_0,t}(\widetilde{C^1})$ are linked because their convex hulls will be rectangles in $\mathbb{R}^d$ and for $t \in U$, this guarantees that these rectangles intersect each other and exist outside of each other. Note this means we are using Lemma 3.21.

We will show that U is a non-empty open set. To see that U is non-empty, set $\frac{1}{2}(t_1)^2 = (x_1^0 - u_1^1)^2 + (x_{d+1}^0 - u_1^2)^2$. By construction, we have $g_{t_1}^1(u_1^1) = u_1^2$ and because each g_t^i is increasing in t, for any $t > t_1$ we will have $g_t^1(u_1^1) > u_1^2$. Furthermore, we also have that g_t^1 is continuous in t for a fixed input and therefore, when t is sufficiently close to t_1 we will get $g_t^1(u_1^1 + \delta_1^1) < u_1^2$ and $g_t^1(u_1^1) < u_1^2 + \delta_1^2$.

Now set $\frac{1}{2}(t_i)^2 = (x_i^0 - u_i^1)^2 + (x_{i+d}^0 - u_i^2)^2$. But by assumption, since $x^0 = \vec{0}$, $u_i^1 = u_1^1$, and $u_i^2 = u_2^2$ we get that the same t which worked for g_t^1 will work for g_t^2 as well. Therefore U is non-empty. Furthermore, note that for a fixed input, both g_t^1 and g_t^2 are continuous in t and therefore, U is open. Thus, there is an open neighborhood T such that the linked condition on $\widetilde{C^2}$ and $g_{x^0,t}(\widetilde{C^1})$ is satisfied and therefore, by the Gap Lemma, we get that $\widetilde{C^2} \cap g_{x^0,t}(\widetilde{C^1})$ is non-empty for every $t \in T$, and this shows the claim. □

We end this section with a note that wraps up everything. In Lemma 3.20 we required the boundary point we were looking at to be a boundary point of a bounded gap. However,

Theorem 3.14 still holds when C^1 does not contain any bounded gaps. To see this, assume C^1 has no bounded gaps. Then because we are assuming $\tau(C^1)\tau(C^2) > 1$ this implies $\tau(C^1) > 0$ and because C^1 has no bounded gaps, this implies $\tau(C^1) = +\infty$. Also because C^1 has no bounded gaps, we have that C^1 has non-empty interior. So there exists a ball $B_1 \subset C^1$ and therefore we can find a closed box R_1 such that $R_1 \subset B_1$. Then recall that g^1 and g^2 only take in one coordinate as their arguments. So because of this and because each g^i is decreasing and continuous, $g_{x,t}(R_1)$ is a filled-in rectangle and therefore, $g_{x,t}(R_1) = +\infty$. Therefore in the Lemma 3.20, instead of looking at the box formed by a boundary point of some bounded gap of C^1 we can observe the quantity $\tau(g_{x,t}(R_1))$. Then we do the same process as in the proof of Theorem 3.14.

3.3 Intersection of Three Compact Sets

In this section, we discuss issues that arise when trying to come up with a gap lemma for the intersection of three compact sets. So given three compact sets C_1, C_2, C_3 what properties of thickness is needed so that we can say $C_1 \cap C_2 \cap C_3 \neq \emptyset$? One would hope to be able to use the strategy as in [8], however issues arise. The crux of the issue seems to be that when doing an analysis on three sets, one may be able to obtain control over two of the sets, however the other set can get lost in the analysis and may not be able to be recovered. So a guess as to the conditions needed for non-empty intersection could be that none of the compact sets are contained in a gap of any of the others while imposing the condition

$$\tau(C_1)\tau(C_2)\tau(C_3) > 1. \tag{3.4}$$

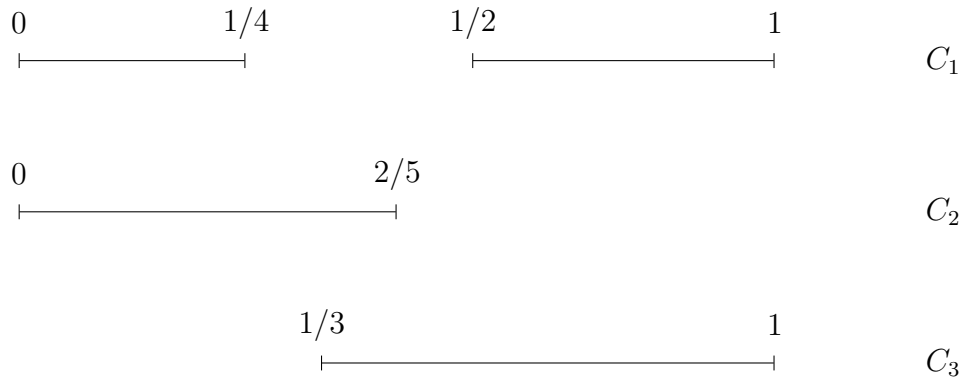

Figure 3.1: Compact sets which satisfy proposed hypotheses but have empty intersection.

If this doesn't work, we could maybe even relax this condition to

$$\tau(C_i)\tau(C_j) > 1 \tag{3.5}$$

for all $i \neq j$. Note that condition is more relaxed since, by taking products, this implies $\tau(C_1)^2\tau(C_2)^2\tau(C_3)^2 > 1$ and taking the square root results in (3.4). However, this isn't true and coming up with examples is unfortunately very easy. An example which satisfies the above hypotheses but has empty intersection is given by Figure 3.1.

In Figure 3.1, none of the compact sets are contained in a gap of the other compact sets, C_1 has thickness equal to one and C_2, C_3 have infinite thickness because they are non-degenerate intervals, but $C_1 \cap C_2 \cap C_3 = \emptyset$. Thus, it seems like the ideas used to guarantee that the intersection of two compact sets being non-empty do not hold well when thinking about the intersection of three compact sets.

Chapter 4

Why $d \geq 2$ is Necessary in Falconer's Conjecture

We give an example of a compact set in $\mathbb{R}$ with Hausdorff dimension one and the Lebesgue measure of the distance being zero. Take

$$A := \{x \in [0,1] : 10^k\text{th digit is 0 for all } k\}.$$

Then $A := \bigcap_{k\geq 1} A_k$ where

$$A_k = \{x \in [0,1] : 10^k\text{th digit is 0}\}.$$

A useful property to have is nestedness, so we then define $G_0 = A_0$ and take $G_i = A_i \cap G_{i-1}$ so that $G_0 \supset G_1 \supset \cdots$ and $A = \bigcap_{k\geq 1} G_k$. To understand the structure of each G_k, we first observe G_1. An example of an element in G_1 will be $0.0i_2i_3\cdots i_9\overline{0}$ where i_k represents the kth decimal place of x. Then observe that any element of the form $0.0i_2i_3\cdots i_90i_{11}i_{12}\cdots$ where i_k for $k \geq 11$ can represent any integer in $\{0, 1, \cdots, 9\}$. Thus, we get that the interval

$$[0.0i_2i_3\cdots i_9\overline{0},\ 0.0i_2i_3\cdots i_91)$$

is in G_1 since the first bad number after $0.0i_2i_3\cdots i_9\overline{0}$ will be $0.0i_2i_3\cdots i_91$. Thus, G_1 will be a union of intervals. Furthermore, since each i_k with $k \in \{2, \cdots, 9\}$ has 10 choices for what they may be, and because i_1 and i_{10} are fixed, we get that there will be 10^8 of these intervals since the left endpoint of the intervals will always be of the form $0.0i_2i_3\cdots i_9\overline{0}$. Now if we let $x = 0.0i_2\cdots i_9\overline{0}$ and $y \in [0, 10^{-10})$, then the interval $[x, x+y]$ will be in G_1. So this shows that the length of each interval in G_1 is 10^{-10}. Then also note that these intervals will be disjoint and the distance between each succeeding interval will be $9(10^{-10})$. But because we want each G_1 to be compact, we add in all elements of the form $0.0i_2i_3\cdots i_91$ so that each interval will be of the form $[0.0i_2i_3\cdots i_9\overline{0},\ 0.0i_2i_3\cdots i_91]$. Then call this new set $\widetilde{G_1}$. For $\widetilde{G_2}$ we now get intervals of the form

$$[0.0i_2\cdots i_90i_{11}\cdots i_{99}\overline{0}, 0.0i_2\cdots i_90\cdots i_{11}\cdots i_{99}1]$$

and since there are 3 fixed digits (and therefore 97 free digits), we get there will be 10^{97} intervals. Generally, for $\widetilde{G_k}$, it will consist of $10^{10^k-(k+1)}$ disjoint intervals each of length 10^{-10^k}. We then set $\widetilde{A} = \bigcap_{k\geq 1} \widetilde{G_k}$.

Before moving further, we make one quick remark. Each interval in $\widetilde{G_{k-1}}$ will produce $10^{9\cdot 10^{k-1}-1}$ intervals and the spacing between these children intervals (so we are now in the kth level) will be $9\cdot 10^{-10^k}$, then after these $10^{9\cdot 10^k-1}$ there will be a gap of $9\cdot 10^{-10^{k-1}}$ between the next intervals and this pattern continues so that the spacing between all intervals in $\widetilde{G_k}$ (other than $\widetilde{G_1}$) will have large sequences of evenly spaced intervals, then jumps of larger gaps.

Because each $\widetilde{G}_k$ is nested, this shows that $\widetilde{A}$ has similar behaviors to a Cantor-type set. However, we cannot make $\widetilde{A}$ into a self-similar set because the definition of a self-similar set requires you to have a finite set of similarities. Because each $\widetilde{G}_k$ is not producing a fixed

number intervals, we cannot hope for $\widetilde{A}$ to be generated by a finite number of similarities.

Theorem 4.1. $\dim_H(\widetilde{A}) = 1$

Proof. We show this using the mass distribution principle. To use the mass distribution principle, we need μ to be a mass distribution on $\widetilde{A}$, i.e. a measure which takes on a finite and positive measure on $\widetilde{A}$. Note that if $I_{i,k}$ is one of the intervals in $\widetilde{G_k}$, then $|I_{i,k}| = 10^{-10^k}$ where $|\cdot|$ denotes diameter. Since there are $10^{10^k-(k+1)}$ of these intervals, we set $\mu(I_{i,k}) = 10^{-10^k+(k+1)}$ which gives $\mu(\widetilde{G_k}) = 1$ for all k. By Proposition 1.7 in [**?**], μ extends to a mass distribution. Let $\varepsilon > 0$ be given and define $\varepsilon(k) = \frac{k+1}{10^k}$. More precise requirements on k will be given throughout these calculations, but for now, we require k large enough so that $\varepsilon(k-1) \leq \varepsilon$. Now choose U such that $|U| < 10^{-10^{k-1}}$. We now break up the proof into cases; the first being as follows. Suppose that $10^{-10^k-1} \leq |U| \leq 10^{-10^k}$. Then since the distance between each interval is $9(10^{-10^k})$ we have that U intersects at most one interval in $\widetilde{G_k}$ and therefore,

$$\begin{aligned}
\mu(U) &\leq 10^{-10^k+(k+1)} \\
&= 10^{-10^k(1-(k+1)/10^k)} \\
&= 10^{-10^k(1-\varepsilon(k))} \\
&\leq (10|U|)^{1-\varepsilon(k)} \\
&\leq (10|U|)^{1-\varepsilon} \\
&\leq 10|U|^{1-\varepsilon}.
\end{aligned}$$

Similarly, if $10^{-10^{k-1}-1} \leq |U| \leq 10^{-10^{k-1}}$ then $\mu(U) \leq 10|U|^{1-\varepsilon}$. Now we show that if $|U|$ has a length in between 10^{-10^k} and $10^{-10^{k-1}-1}$ then it will also satisfy bounds that do not

depend on k. Let m_k denote the midpoint between 10^{-10^k-1} and $10^{-10^{k-1}-1}$. That is,

$$m_k = \frac{10^{-10^{k-1}-1} + 10^{-10^k-1}}{2} = \frac{10^{-10^{k-1}} + 10^{-10^k}}{20}.$$

Set $S_k = 10^{-10^k} - 10^{-10^k-1}$. Then we break up the interval $[10^{-10^k-1}, m_k]$ into a decomposition where the length of each interval in this decomposition is $\frac{9}{5}10^{-10^k}$. The number $\frac{9}{5}10^{-10^k}$ isn't particularly special here, we just choose this number because if U is a set with

$$10^{-10^k-1} + n \cdot S_k \leq |U| \leq 10^{-10^k} + n \cdot S_k$$

where n is a natural number, then the length of $[10^{-10^k-1} + n \cdot S_k, 10^{-10^k} + n \cdot S_k]$ is $\frac{9}{5}10^{-10^k}$. So based on this, we could break up the interval $[10^{-10^k}, 10^{-10^{k-1}-1}]$ into pieces of kind

$$[10^{-10^k-1} + n \cdot S_k, 10^{-10^k} + n \cdot S_k]$$

where $n \in \mathbb{N}$. In other words, S_k represents a shift in the interval and n represents how many shifts away we are from the "base" interval $[10^{-10^k-1}, 10^{-10^k}]$. Now instead of breaking up the interval $[10^{-10^k}, 10^{-10^{k-1}-1}]$ we will break up the interval $[10^{-10^k}, m_k]$ into intervals of length $\frac{9}{5}10^{-10^k}$ so that the last interval in this decomposition will be of the form

$$[10^{-10^k-1} + i \cdot S_k, m_k]$$

where i is some positive real number, not necessarily an integer, and such that

$$\ell\left([10^{-10^k-1} + i \cdot S_k, m_k]\right) = \frac{9}{5}10^{-10^k}$$

where ℓ represents length. Suppose we have U such that $|U| \in [10^{-10^k-1} + i \cdot S_k, m_k]$.

Now ideally, what we would like to say is that if $|U_1| \leq |U_2|$ then $\mu(U_1) \leq \mu(U_2)$ as this would immediately imply that $\mu(V) \leq \mu(U)$ for any V with $|V| \in [10^{-10^k}, m_k]$ and thus, an estimate on $\mu(U)$ would provide a worst case scenario for sets whose diameter is in $[10^{-10^k}, m_k]$. However, this is not necessarily true but an estimate on $\mu(U)$ will still provide a worst case scenario for the following reason: the method we perform takes a set, say V, and provides an estimate that looks like

$$\mu(V) \leq \left(\text{largest possible number of intervals V intersects in } \widetilde{G_k}\right) \cdot 10^{-10^k+(k+1)}.$$

So the estimate acts as if V itself is an interval, no matter what the structure of V actually is. Thus, providing an estimate on $\mu(U)$ provides a worst case scenario for sets with diameter in $[10^{-10^k}, m_k]$. Now we find i where is the number in $[10^{-10^k-1} + i \cdot S_k, m_k]$.

As before, we want

$$\ell\left([10^{-10^k-1} + i \cdot S_k, m_k]\right) = \frac{9}{5}10^{-10^k}.$$

So expanding the definitions of S_k and m_k we have that this means

$$\frac{1}{20}10^{-10^{k-1}} + \frac{1}{20}10^{-10^k} - 10^{-10^k-1} - i\left(10^{-10^k} - 10^{-10^k-1}\right) = \frac{9}{5}10^{-10^k}$$

and after some algebra, this yields

$$i = -\frac{37 \cdot 10^{-10^k} - 10^{-10^{k-1}}}{18 \cdot 10^{-10^k}}.$$

Thus, we are supposing

$$10^{-10^k-1} - \frac{37 \cdot 10^{-10^k} - 10^{-10^{k-1}}}{18 \cdot 10^{-10^k}}\left(10^{-10^k} - 10^{-10^k-1}\right) \leq |U| \leq \frac{10^{-10^{k-1}} + 10^{-10^k}}{20}.$$

Before going further, we simplify the lower bound on $|U|$.

$$\begin{aligned}
10^{-10^k-1} - \left(\frac{37 \cdot 10^{-10^k}}{18 \cdot 10^{-10^k}} - \frac{10^{-10^{k-1}}}{18 \cdot 10^{-10^k}}\right)\left(10^{-10^k} - 10^{-10^k-1}\right) &= \frac{1}{10} \cdot 10^{-10^k} \\
&\quad - \left(\frac{37}{18} - \frac{1}{18}10^{\frac{9}{10}10^k}\right)\left(10^{-10^k} - \frac{1}{10}10^{-10^k}\right) \\
&= \frac{1}{10} \cdot 10^{-10^k} - \frac{37}{18} \cdot 10^{-10^k} + \frac{37}{180} \cdot 10^{-10^k} \\
&\quad + \frac{1}{18} \cdot 10^{-\frac{1}{10}10^k} + \frac{1}{180} \cdot 10^{-\frac{1}{10}10^k} \\
&= -\frac{315}{180} \cdot 10^{-10^k} + \frac{11}{180} \cdot 10^{-10^{k-1}} \\
&= \frac{11}{180} \cdot 10^{-10^{k-1}} - \frac{7}{4} \cdot 10^{-10^k}.
\end{aligned}$$

So we have

$$\frac{11}{180}10^{-10^{k-1}} - \frac{7}{5}10^{-10^k} \leq |U| \leq \frac{10^{-10^{k-1}} + 10^{-10^k}}{20}.$$

Next, we estimate how many intervals U may possibly intersect in $\widetilde{G_k}$. Recall that each interval in $\widetilde{G_k}$ is of length 10^{-10^k}. Also recall that the spacing between intervals can be $9 \cdot 10^{-10^n}$ for any $0 \leq n \leq k$. But for the sake of simplicity, we provide an estimate acting as if the spacing between each interval is $9 \cdot 10^{-10^k}$. Then note that by doing this, we will actually be providing an overestimate on how many intervals U may intersect since $9 \cdot 10^{-10^k} \leq 9 \cdot 10^{-10^n}$ for $0 \leq n \leq k$. So now let $\ell = 10^{-10^k}$ and $d = 9 \cdot 10^{-10^k}$ and observe three intervals:

$$\overbrace{[\ \cdot\ ,\ \cdot\ ]}^{\ell}\underbrace{\qquad\qquad}_{d}\overbrace{[\ \cdot\ ,\ \cdot\ ]}^{\ell}\underbrace{\qquad\qquad}_{d}\overbrace{[\ \cdot\ ,\ \cdot\ ]}^{\ell}.$$

So if V is a set with

$$|V| = 18 \cdot 10^{-10^k} + 10^{-10^k} = 19 \cdot 10^{-10k}$$

then it may intersect three intervals. For four intervals,

$$\overbrace{[\ \cdot\ ,\ \cdot\ ]}^{\ell}\underbrace{\qquad\qquad}_{d}\overbrace{[\ \cdot\ ,\ \cdot\ ]}^{\ell}\underbrace{\qquad\qquad}_{d}\overbrace{[\ \cdot\ ,\ \cdot\ ]}^{\ell}\underbrace{\qquad\qquad}_{d}\overbrace{[\ \cdot\ ,\ \cdot\ ]}^{\ell}$$

so that if

$$|V| = 27 \cdot 10^{-10^k} + 2 \cdot 10^{-10^k} = 29 \cdot 10^{-10^k}$$

then V may intersect 4 intervals. For five intervals,

$$\overbrace{[\ \cdot\ ,\ \cdot\ ]}^{\ell}\underbrace{\qquad\qquad}_{d}\overbrace{[\ \cdot\ ,\ \cdot\ ]}^{\ell}\underbrace{\qquad\qquad}_{d}\overbrace{[\ \cdot\ ,\ \cdot\ ]}^{\ell}\underbrace{\qquad\qquad}_{d}\overbrace{[\ \cdot\ ,\ \cdot\ ]}^{\ell}\underbrace{\qquad\qquad}_{d}\overbrace{[\ \cdot\ ,\ \cdot\ ]}^{\ell}$$

so that if

$$|V| = 36 \cdot 10^{-10^k} + 3 \cdot 10^{-10^k} = 39 \cdot 10^{-10^k}$$

then V may intersect 5 intervals. In general, if a set V has $|V| = (10 \cdot n + 9) \cdot 10^{-10^k}$, then it may intersect $n + 2$ intervals in $\widetilde{G_k}$. Since $|U| \leq (10^{-10^{k-1}} + 10^{-10^k})/20$, we can estimate n by

$$\frac{10^{-10^{k-1}} + 10^{-10^k}}{20} = (10 \cdot n + 9) \cdot 10^{-10^k} = n \cdot 10^{-10^k+1} + 9 \cdot 10^{-10^k}$$

implying

$$\begin{aligned} n \cdot 10^{-10^k+1} &= \frac{10^{-10^{k-1}} + 10^{-10^k}}{20} + \frac{9 \cdot 20 \cdot 10^{-10^k}}{20} \\ &= \frac{10^{-10^{k-1}} + 10^{-10^k} - 180 \cdot 10^{-10^k}}{20} \\ &= \frac{10^{-10^{k-1}} - 179 \cdot 10^{-10^k}}{20} \end{aligned}$$

implying

$$n = \frac{10^{-10^{k-1}} - 179 \cdot 10^{-10^k}}{200 \cdot 10^{-10^k}}.$$

Now clearly it is possible that n is not an integer, however recall that the spacing between intervals in $\widetilde{G_k}$ ranges over values $9 \cdot 10^{-10^n}$ for $0 \leq n \leq k$ and therefore, for large enough k, this value for n will already provide an overestimation in terms of however intervals U could intersect. Then by previous statements, we get that U intersects at most $n+2$ intervals in $\widetilde{G_k}$. We now provide the estimate on $\mu(U)$. Recall that $\mu(I_k) = 10^{-10^k+(k+1)}$. So since U intersects at most $n+2$ intervals in $\widetilde{G_k}$ we get

$$\begin{aligned}
\mu(U) &\leq \left(\frac{10^{-10^{k-1}} - 179 \cdot 10^{-10^k}}{200 \cdot 10^{-10^k}} + 2\right) 10^{-10^k+(k+1)} \\
&= \left(\frac{10^{9 \cdot 10^{k-1}} - 179}{200} + 2\right) 10^{-10^k+(k+1)} \\
&= \frac{1}{200} \cdot 10^{\frac{9}{10}10^k - 10^k + (k+1)} - \frac{179}{200} \cdot 10^{-10^k+(k+1)} + 2 \cdot 10^{-10^k+(k+1)} \\
&= \frac{1}{200} \cdot 10^{-10^k+(k+1)} + \frac{221}{200} \cdot 10^{-10^k+(k+1)} \\
&\leq \frac{1}{200} \cdot 10^{-10^k+(k+1)} + \frac{221}{200} \cdot 10^{-10^{k-1}+(k+1)} \\
&= \frac{111}{100} \cdot 10^{-10^{k-1}+(k+1)} \\
&= \frac{111}{100} \cdot 10^{-10^{k-1}\left(1 - \frac{k+1}{10^{k-1}}\right)} \\
&= \frac{111}{100} \cdot 10^{-10^{k-1}(1-10\varepsilon(k))}.
\end{aligned}$$

Recall that the lower bound on $|U|$ is

$$\frac{11}{180} \cdot 10^{-10^{k-1}} - \frac{7}{4} \cdot 10^{-10^k} = 10^{-10^{k-1}} \left(\frac{11}{180} - \frac{7}{4} \cdot 10^{-9 \cdot 10^{k-1}}\right).$$

So make k possibly larger so that

$$-\frac{7}{4} \cdot 10^{-9 \cdot 10^{k-1}} > -\frac{1}{180}$$

and therefore,

$$\frac{11}{180} - \frac{7}{4} \cdot 10^{-9 \cdot 10^{k-1}} > \frac{11}{180} - \frac{1}{180} = \frac{1}{18}.$$

So,

$$10^{-10^{k-1}} \leq \frac{|U|}{\frac{11}{180} - \frac{7}{4} \cdot 10^{-9 \cdot 10^{k-1}}} < \frac{|U|}{\frac{1}{18}} = 18|U|.$$

Again, by choosing k possibly larger so that $10\varepsilon(k) \leq \varepsilon$ and $18|U| < 1$ we obtain

$$\begin{aligned} \mu(U) &\leq \frac{111}{100} \left(18|U|\right)^{1-10\varepsilon(k)} \\ &\leq \frac{111 \cdot 18}{100} |U|^{1-10\varepsilon(k)} \\ &\leq \frac{999}{50} \cdot |U|^{1-\varepsilon}. \end{aligned}$$

Now assume $m_k \leq |U| \leq 10^{10^{k-1}-1}$. Then U intersects at most interval in $\widetilde{G_{k-1}}$ and therefore,

$$\begin{aligned} \mu(U) &\leq 10^{-10^{k-1}+((k-1)+1)} \\ &= 10^{-10^{k-1}+k} \\ &= 10^{-10^{k-1}(1-\varepsilon(k-1))}. \end{aligned}$$

Recall that $m_k = \left(10^{-10^{k-1}} + 10^{-10^k}\right)/20$. So

$$\frac{1}{20} \cdot 10^{-10^{k-1}} \leq m_k \leq |U|$$

and therefore,

$$10^{-10^{k-1}} \leq 20|U|.$$

Thus

$$\mu(U) \leq (20|U|)^{1-\varepsilon(k-1)} \leq 20|U|^{1-\varepsilon}.$$

Thus, we have shown that for any U with $|U| \in [10^{-10^k-1}, 10^{-10^{k-1}}]$ that we get an upper bound $\mu(U)$ with a constant that does not depend on k. So in general, if we took any set W with $|W| \leq 10^{-10^{k-1}}$ we can find a positive integer n with $n \geq k-1$ such that $|W| \in [10^{-10^n-1}, 10^{-10^{n-1}}]$. Then we would perform the same process as above to get a bound on $\mu(W)$ that does not depend on n. So choose our constant to be 20 and by the mass distribution principle, we have $\dim \widetilde{A} \geq 1-\varepsilon$ and since ε was arbitrary we may conclude that $\dim \widetilde{A} = 1$. □

Theorem 4.2. $\mathcal{L}(\Delta(\widetilde{A})) = 0$.

Proof. To do this, we first show

$$\Delta(\widetilde{A}) = \Delta\left(\bigcap_{k\geq 1} \widetilde{G_k}\right) \subseteq \bigcap_{k\geq 1} \Delta(\widetilde{G_k}).$$

So take $x \in \Delta(\widetilde{A})$. Then there exists $y, z \in \bigcap_{k\geq 1} \widetilde{G_k}$ such that $x = y - z$. Then because $y, z \in \bigcap_{k\geq 1} \widetilde{G_k}$ we get $y - z \in \Delta(\widetilde{G_k})$ for all k. Thus, $x \in \bigcap_{k\geq 1} \Delta(\widetilde{G_k})$. One note to make here is that this proof did not rely on the properties of $\widetilde{G_k}$, it only utilizes properties of distance sets. We can now work on $\Delta(\widetilde{G_k})$ instead of $\Delta(\widetilde{A})$. First off, recall that $\widetilde{G_k}$ consists of $10^{10^k-(k+1)}$ disjoint intervals each of length 10^{-10^k}. So,

$$\mathcal{L}(\widetilde{G_k}) = 10^{10^k-(k+1)} \cdot 10^{-10^k} = 10^{-(k+1)}.$$

We first do an upper estimate on $\Delta(\widetilde{G_1})$. So take any $x, y \in \widetilde{G_1}$ and assume for simplicity that $x \geq y$. Then only three scenarios can happen for $x - y$; we have either $x - y = 0.0i_2 \cdots i_9 0 i_{11} \cdots$, $x - y = 0.0i_2 \cdots i_9 9 i_{11} \cdots$, or $x - y = 0.0i_2 \cdots i_9 1$. The first scenario happens when either we are subtracting right endpoints of intervals in $\widetilde{G_1}$ (that is, points of the form $0.0i_2 \cdots i_9 1$) or when the 11th digit of x is larger than the 11th digit of y. The

second scenario happens if x is not a right endpoint but y is or when the 11th digits of y is larger than the 11th digit of x. The third scenario only happens when x is a right endpoint and y is a left endpoint. Then notice that if scenarios one or three happen, then $x - y \in \widetilde{G_1}$. So now we just have to consider when scenario two happens. Since we know it's possible for $x - y = 0.0i_2 \cdots i_9 9 i_{11} \cdots$ but we don't have an understanding of what the i_n's look like for $n \geq 11$, we simply include all of these numbers and put them into an interval, say $[0.0i_2 \cdots i_9 9, 0.0i_2 \cdots (i_9 + 1)0)$. We get this right endpoint because this would be the first element that doesn't have a 9 in the 10th decimal place. Now obviously it is possible for $i_9 = 9$ and in that case $(1_9 + 1)$ wouldn't be valid notation, however if this does occur then we simply find the first digit i_n, $n \in \{1, \cdots, 8\}$ which is not equal to 9 then add 1 to it. Either way, the length of the interval $[0.0i_2 \cdots i_9 9, 0.0i_2 \cdots (i_9 + 1)0)$ is 10^{-10} even if $i_9 = 9$. Then since there are 10 choices for the digits i_n, $n \in \{2, \cdots 9\}$ and because there are 8 of these digits, we get there exists 10^8 intervals of the form $[0.0i_2 \cdots i_9 9 i_{11} \cdots, 0.0i_2 \cdots (i_9 + 1)0)$. Let F_1 denote the collection of these intervals. Then because if scenario one or three happens, $x - y$ will be an element of $\widetilde{G_1}$ and if scenario two happens, $x - y$ will be an element of F_1 we get

$$\begin{aligned}
\mathcal{L}(\Delta(\widetilde{G_1})) &\leq \mathcal{L}(\widetilde{G_1}) + \mathcal{L}(F_1) \\
&= 10^8 \cdot 10^{-10} + 10^8 \cdot 10^{-10} \\
&= 2 \cdot 10^{-2}.
\end{aligned}$$

We can do something similar with $\Delta(\widetilde{G_2})$. That is, if we take $x, y \in \widetilde{G_2}$ with $x \geq y$ then $x - y$ will give five scenarios:

$$(i) \quad x - y = 0.0i_2 \cdots i_9 0 i_{11} \cdots i_{99} 0 i_{101} \cdots$$

$$(ii) \quad x - y = 0.0i_2 \cdots i_9 9 i_{11} \cdots i_{99} 0 i_{101} \cdots$$

$$(iii) \quad x - y = 0.0i_2 \cdots i_9 0 i_{11} \cdots i_{99} 9 i_{101} \cdots$$

$$(iv) \quad x - y = 0.0i_2 \cdots i_9 9 i_{11} \cdots i_{99} 9 i_{101} \cdots$$

$$(v) \quad x - y = 0.0i_2 \cdots i_9 0 i_{11} \cdots i_{99} 1$$

where these scenarios similarly as they did in the case with $\Delta(\widetilde{G_1})$. That is, the relation between the 99th and 101th digits of x and y will give these scenarios. If scenarios (i) or (v) happen, then $x - y \in \widetilde{G_2}$. Then we form intervals of the form

$$[0.0i_2 \cdots i_9 9 i_{11} \cdots i_{99} 0, 0.0i_2 \cdots i_9 9 i_{11} \cdots i_{99} 1),$$

$$[0.0i_2 \cdots i_9 0 i_{11} \cdots i_{99} 9, 0.0i_2 \cdots i_9 0 i_{11} \cdots (i_{99} + 1)), \text{ and}$$

$$[0.0i_2 \cdots i_9 9 i_{11} \cdots i_{99} 9, 0.0i_2 \cdots i_9 9 i_{11} \cdots (i_{99} + 1))$$

where we still give the same caveats on $(i_{99} + 1)$ as we did on $(i_9 + 1)$ as in the $\Delta(\widetilde{G_1})$ case. Then each of these intervals are of length 10^{-100}. Furthermore, because there are 10 choices for digits i_n where $n \in \{2, \cdots 9, 11, \cdots 99\}$ we get that there will be 10^{97} intervals of the form $[0.0i_2 \cdots i_9 9 i_{11} \cdots i_{99} 0, 0.0i_2 \cdots i_9 9 i_{11} \cdots i_{99} 1)$. Similarly we have 10^{97} intervals of the form $[0.0i_2 \cdots i_9 0 i_{11} \cdots i_{99} 9, 0.0i_2 \cdots i_9 0 i_{11} \cdots (i_{99} + 1))$ and 10^{97} intervals of the form $[0.0i_2 \cdots i_9 9 i_{11} \cdots i_{99} 9, 0.0i_2 \cdots i_9 9 i_{11} \cdots (i_{99} + 1))$. Then let F_2 denote the set of all of these intervals. Then,

$$\begin{aligned}
\mathcal{L}(\Delta(\widetilde{G_2})) &\leq \mathcal{L}(\widetilde{G_2}) + \mathcal{L}(F_2) \\
&= 10^{97} \cdot 10^{-100} + 3 \cdot 10^{97} \cdot 10^{-100} \\
&= 4 \cdot 10^{-3}.
\end{aligned}$$

Generally in k, we will need to consider $2^k + 1$ scenarios where two of those scenarios are subcases of $\widetilde{G_k}$ and the other subcases we lump into F_k. That is, F_k will consist of $2^k - 1$ "kinds" of intervals, each of length 10^{-10^k}, and there will be $10^{10^k-(k+1)}$ intervals of the same kind. Therefore we obtain the bound

$$\begin{aligned}
\mathcal{L}(\Delta(\widetilde{G_k})) &\leq \mathcal{L}(\widetilde{G_k}) + \mathcal{L}(F_k) \\
&= 10^{10^k-(k+1)} \cdot 10^{-10^k} + (2^k - 1) \cdot 10^{10^k-(k+1)} \cdot 10^{-10^k} \\
&= 2^k \cdot 10^{-(k+1)} \\
&\to 0
\end{aligned}$$

as $k \to \infty$. Next, we wish to apply the monotone convergence theorem for sets, but we just need to check that $\left(\Delta(\widetilde{G_k}\right)_{k\geq 0}$ is a decreasing sequence. Fix k and take $x \in \Delta(\widetilde{G_k})$ so that $x = y - z$ with $y, z \in \widetilde{G_k}$. Then recall that it is possible for the 10^kth decimal is 1 since we include the right endpoints of intervals in $\widetilde{G_k}$, however we still require the 10^ith decimal place to be 0 for all for all $i \in \{1, 2, \cdots, k-1\}$. Thus, $y, z \in \widetilde{G_{k-1}}$ and therefore, $x = y - z \in \Delta(\widetilde{G_{k-1}})$. Thus, $\Delta(\widetilde{G_0}) \supset \Delta(\widetilde{G_1}) \supset \cdots$. So by the monotone convergence theorem,

$$\begin{aligned}
\mathcal{L}(\Delta(\widetilde{A})) &\leq \lim_{k\to\infty} \mathcal{L}(\Delta(\widetilde{G_k})) \\
&\leq \lim_{k\to\infty} 2^k \cdot 10^{-(k+1)} \\
&= 0.
\end{aligned}$$

Therefore, $\widetilde{A}$ is a compact set with $\dim \widetilde{A} = 1$ and $\mathcal{L}(\Delta(\widetilde{A})) = 0$. $\square$

Chapter 5

Distance Set and Thickness of a Class of Self-Similar Sets

In this chapter, we will calculate the distance set then thickness for a special class of self-similar sets.

5.1 Distance Set Calculation

In 1917, Hugo Steinhaus proved that the sum set of the ternary Cantor set, C, is the interval $[0,2]$, i.e. $C+C=[0,2]$ where $C+C:=\{x+y : x,y \in C\}$. This also shows that $C-C=[-1,1]$ where $C-C$ is defined analogously. He did this by noticing a connection between elements in $C+C$ and elements in $C\times C$ which he then proceeded to use a satisfying geometric proof. While this approach is elegant, it is difficult to see how this approach could be used on more complicated Cantor-type sets. The proof of the theorem below uses not a geometric proof, but utilizes the self-similarity of Cantor sets.

Theorem 5.1. *Let $k \in \mathbb{N}\backslash\{1\}$ and $\lambda = \frac{1}{2k-1}$. Let η with $0 \leq \eta < 1/k$ and let $C_{k,\eta}$ be a symmetric Cantor set on $[0,1]$ generated by $\mathcal{I} = \{S_1, \cdots, S_k\}$.*

(i) If η is such that $\lambda \leq \eta < 1/k$, then $D_0(C_{k,\eta}) = [-1,1]$.

(ii) If η is such that $0 \leq \eta < \lambda$, then $\mathcal{L}\left(D_0(C_{k,\eta})\right) = 0$.

where the definition for a symmetric Cantor set and the definition for D_0 is given below.

We want to understand the distance set of Cantor sets where we now allow each interval in a Cantor set to produce more than just two intervals per iteration. That is, given $\lambda < 1/2$, then C_λ is the unique compact set that satisfies

$$C_\lambda = S_1(C_\lambda) \cup S_2(C_\lambda)$$

where $x \in [0,1]$ and the S_i are similarities that satisfy $S_1(x) = \lambda x$, $S_2(x) = (1-\lambda) + \lambda x$ so that each interval that is produced in $C_\lambda^N = \bigcap_{i=1}^N C_\lambda^i$ produces two intervals where these intervals follow the structure of S_1 and S_2. Before generalizing, we start with an example. If we allow the sets to produce three intervals rather than two, then where should these intervals be placed and how large should the gaps be between them? First, we take $\lambda < 1/3$ and we desire each of the three intervals to be of length λ and be equally spaced, that is, we wish the gap between the first and second interval to be the same as the gap between the second and third interval. Then because we start with the interval $[0,1]$ and because we now desire $[0,1]$ to produce three intervals of length λ, we will have two gaps of length say ℓ. So we should have

$$1 = 3\lambda + 2\ell$$

which suggests that $\ell = \frac{1-3\lambda}{2}$. Now we consider how each interval should look like. The first and third intervals are easy, these are just the intervals $[0,\lambda]$ and $[1-\lambda,1]$. Thus, the second interval should be

$$\left[\lambda + \frac{1-3\lambda}{2}, \lambda + \frac{1-3\lambda}{2}\right] = \left[\frac{1-\lambda}{2}, \frac{1+\lambda}{2}\right].$$

Thus, our intervals are $[0,\lambda]$, $[(1-\lambda)/2, (1+\lambda)/2]$, and $[1-\lambda,1]$. This implies that our

similarity functions should be defined as

$$S_1(x) = \lambda x,$$

$$S_2(x) = \lambda x + \frac{1-\lambda}{2},$$

$$S_3(x) = \lambda x + (1-\lambda).$$

Then note that these similarity functions also have the following structure:

$$S_1(x) = \lambda x + 0 \cdot \left(\lambda + \frac{1-3\lambda}{2}\right),$$

$$S_2(x) = \lambda x + 1 \cdot \left(\lambda + \frac{1-3\lambda}{2}\right) = S_1(x) + \left(\lambda + \frac{1-3\lambda}{2}\right),$$

$$S_3(x) = \lambda x + 2 \cdot \left(\lambda + \frac{1-3\lambda}{2}\right) = S_2(x) + \left(\lambda + \frac{1-3\lambda}{2}\right)$$

where the $\lambda + \frac{1-3\lambda}{2}$ is not simplified so as to better show the structure of the similarity function. Viewing it in this way shows that S_i jumps over one interval and one gap since the length of each interval is λ and the length of each gap is $\frac{1-3\lambda}{2}$. This is the way the similarities will be stated in upcoming definition. Firstly though, we first note that if we allow the set to produce k intervals in the same fashion as previous, then we need $\lambda < 1/k$ and we need to satisfy the equation

$$1 = k\lambda + (k-1)\ell$$

where ℓ represents the gap length between the intervals. Thus, $\ell = \frac{1-k\lambda}{k-1}$. Now we state the definition of these sorts of Cantor sets.

Definition 5.2. Let $k \in \mathbb{N}\backslash\{1\}$ and $\lambda < 1/k$. Define a set of similarities on $[0,1]$ by $S_1(x) = \lambda x$, $S_2(x) = \lambda x + 1 \cdot \left(\lambda + \frac{1-k\lambda}{k-1}\right), \cdots, S_k(x) = \lambda x + (k-1) \cdot \left(\lambda + \frac{1-k\lambda}{k-1}\right)$. Let $C_{k,\lambda}$

denote the attractor of these similarities, i.e. $C_{k,\lambda}$ is the unique compact set such that

$$C_{k,\lambda} = \bigcup_{i=1}^{k} S_i(C_{k,\lambda}).$$

Then $C_{k,\lambda}$ is called a **symmetric Cantor set** generated by the iterated function system $\mathcal{I} = \{S_1, \cdots, S_k\}$.

Note that the standard ternary Cantor set, $C_{2,\frac{1}{3}}$, is a symmetric Cantor set generated by similarities $S_1(x) = x/3$ and $S_2(x) = 2/3 + x/3$ defined on $[0,1]$. Also notice that $S_i(x) = \lambda x + (i-1)\cdot\left(\frac{1-\lambda}{k-1}\right)$ and for $i \geq 2$, we also have the recursive formula, $S_i(x) = S_{i-1}(x) + \frac{1-\lambda}{k-1}$. This also leads to a characterization of the difference between two of these similarities which will soon be useful; if we have i and j where $0 \leq j \leq i$, then there exists $\ell \in \{0, 1, \cdots, k-1\}$ such that

$$S_i(x) - S_j(y) = S_i(x) - S_{i-\ell}(y) = \lambda(x-y) + \ell \cdot \frac{1-\lambda}{k-1}. \tag{5.1}$$

Similarly, if $j \geq i$ then there exists $\ell \in \{0, 1, \cdots, k-1\}$ such that

$$S_i(x) - S_j(y) = S_i(x) - S_{i+\ell}(y) = \lambda(x-y) - \ell \cdot \frac{1-\lambda}{k-1}. \tag{5.2}$$

Next, define the signed distance set of a measurable set $E \in \mathbb{R}^d$ by $D_0(E) := \{x - y : x, y \in E\}$. Then note that we allow negative distances to occur which is not the case for Δ. This is because we want the distance set of a self-similar set to also be self-similar, which does not necessarily occur if we allow only positive distances.

The distance set operation on $C_{k,\lambda}$ induces a new set of similarities by the following procedure. Since we start with the unit interval $[0,1]$, take $x, y \in [0,1]$. Then for $i \in \{1, \cdots, k\}$ we have

$$S_i(x) - S_i(y) = \lambda(x-y),$$

so let $S_1'(z) = \lambda z$ where $z \in [-1, 1]$. Now take i, j where $1 \leq j < i$. Then from (4.1), we have that $\ell \in \{1, \cdots, k-1\}$ and also by (4.1) we have

$$S_i(x) - S_j(y) = \lambda(x - y) + \ell \cdot \frac{1-\lambda}{k-1}.$$

From this, because $\ell \in \{1, \cdots, k-1\}$ we generate $k-1$ unique similarity functions of the form

$$S_n'(z) = \lambda z + n \cdot \frac{1-\lambda}{k-1}$$

where $n \in \{1, \cdots, k-1\}$ and $z \in [-1, 1]$. Then taking i, j where $i < j \leq k$, from (4.2) we similarly get $\ell \in \{1, \cdots, k-1\}$ so that

$$S_i(x) - S_j(x) = \lambda(x - y) - \ell \cdot \frac{1-\lambda}{k-1}.$$

Again, because $\ell \in \{1, \cdots, k-1\}$ we generate $k-1$ unique similarity functions of the form

$$S_n'(z) = \lambda z - n \cdot \frac{1-\lambda}{k-1}$$

where $n \in \{1, \cdots, k-1\}$ and $z \in [-1, 1]$. Therefore, we generate a new iterated function system $\mathcal{I}' = \{S'_{-(k-1)}, \cdots, S'_{-1}, S'_0, S'_1, \cdots, S'_{k-1}\}$ where $S_i'(z) = \lambda z + i \cdot \frac{1-\lambda}{k-1}$ for $i \in \{-(k-1), \cdots, -1, 0, 1, \cdots, k-1\}$, $z \in [-1, 1]$. Then note that $\#\mathcal{I}' = 2k-1$ where $\#(\cdot)$ denotes the counting measure. Then set

$$D_0(C_{k,\lambda}) = \bigcup_{i=-(k-1)}^{k-1} S_i'(D_0(C_{k,\lambda})).$$

For the following results we also establish an equivalent way of creating $D_0(C_{k,\lambda})$ which will

give us useful language. By setting $D_0^0\left(C_{k,\lambda}\right) = [-1,1]$, define

$$D_0^m(C_{k,\lambda}) := \bigcup_{i=-(k-1)}^{k-1} \left(\lambda \cdot D_0^{m-1}\left(C_{k,\lambda}\right) + i \cdot \frac{1-\lambda}{k-1}\right) = \bigcup_{i=-(k-1)}^{k-1} S_i'\left(D_0^{m-1}(C_{k,\lambda})\right).$$

Then we get

$$D_0(C_{k,\lambda}) = \bigcap_{m=0}^{\infty} D_0^m(C_{k,\lambda})$$

and we call $D_0^m(C_{k,\lambda})$ the mth iteration of $D_0(C_{k,\lambda})$. Using these tools allows us to easily establish the results in this chapter.

Lemma 5.3. *Let $k \in \mathbb{N}\backslash\{1\}$ and $\lambda = \frac{1}{2k-1}$. Let η be such that $0 < \eta < \lambda$ and let $C_{k,\eta}$ be a symmetric Cantor set on $[0,1]$. Then every interval in $D_0^m(C_{k,\eta})$ is disjoint. Thus, every interval in $D_0(C_{k,\eta})$ is disjoint.*

Proof. We prove this my induction on m. So first let $m = 0$. So $D_0^0(C_{k,\eta}) = [-1,1]$. Take i such that $-(k-1) \leq i \leq k-2$, we will then calculate the gap between S_i' and S_{i+1}'. Since $S_n'(z) = \eta z + n \cdot \frac{1-\eta}{k-1}$ this yields

$$S_i'([-1,1]) = \left[-\eta + i \cdot \frac{1-\eta}{k-1}, \eta + i \cdot \frac{1-\eta}{k-1}\right],$$

$$S_{i+1}'([-1,1]) = \left[-\eta + (i+1) \cdot \frac{1-\eta}{k-1}, \eta + (i+1) \cdot \frac{1-\eta}{k-1}\right].$$

Then the gap between these two intervals is the distance between the left endpoint of S_{i+1}' and the right endpoint of S_i'. Thus, the gap is

$$-\eta + (i+1) \cdot \frac{1-\eta}{k-1} - \eta - i \cdot \frac{1-\eta}{k-1} = \frac{-2\eta(k-1) + 1 - \eta}{k-1} = \frac{\eta(1-2k)+1}{k-1}.$$

Then since $k \in \mathbb{N}$, we have $\eta(1-2k) < 0$ and because $\eta < \lambda$ this yields $\eta(1-2k) > \lambda(1-2k)$.

Thus,

$$\begin{aligned}\frac{\eta(1-2k)+1}{k-1} &> \frac{\lambda(1-2k)+1}{k-1}\\ &= \frac{\frac{1}{2k-1}(1-2k)+1}{k-1}\\ &= \frac{-1+1}{k-1}\\ &= 0.\end{aligned}$$

Thus, every interval in the first iteration of the distance set is disjoint.

So now suppose the claim is true up to the mth iteration of the distance set and first note that, by the base case, the similarities map $[-1,1]$ to an interval, say $[p,q]$ such that $[p,q]$ is a strict subset of $[-1,1]$. Thus, we will calculate the image of the interval $[a,b]$ where $[a,b] \in D_0^m(C_{k,\eta})$ since every interval in $D_0^{m+1}(C_{k,\eta})$ is produced in this manner. By the base case, $[a,b]$ is disjoint from all other intervals in mth iteration of distance set of $C_{\eta,k}$. Now

$$S_i'([a,b]) = \left[-\eta a + i\cdot\frac{1-\eta}{k-1}, \eta b + i\cdot\frac{1-\eta}{k-1}\right],$$

$$S_{i+1}'([a,b]) = \left[-\eta a + (i+1)\cdot\frac{1-\eta}{k-1}, \eta b + (i+1)\cdot\frac{1-\eta}{k-1}\right]$$

so the gap between these two intervals is

$$\begin{aligned}
-\eta a + (i+1)\cdot\frac{1-\eta}{k-1} - \eta b - i\cdot\frac{1-\eta}{k-1} &= -\eta(a+b) + \frac{1-\eta}{k-1} \\
&= \frac{-\eta(a+b)(k-1)+1-\eta}{k-1} \\
&> \frac{-\lambda(a+b)(k-1)+1-\lambda}{k-1} \\
&= \frac{-\frac{1}{2k-1}(a+b)(k-1)+1-\frac{1}{2k-1}}{k-1} \\
&\geq \frac{\frac{(a+b)(1-k)+2k-2}{2k-1}}{k-1} \\
&= \frac{(a+b)(1-k)+2k-2}{(2k-1)(k-1)}.
\end{aligned}$$

Because $-1 \leq a, b \leq 1$ and $k \geq 2$ this yields $-2(1-k) \geq (1-k)(a+b) \geq 2(1-k)$. Thus,

$$\frac{(a+b)(1-k)+2k-2}{(2k-1)(k-1)} \geq \frac{2(1-k)+2k-2}{(2k-1)(k-1)} = 0.$$

Thus, the gap is a positive number implying all intervals in $D_0^{m+1}(C_{k,\eta})$ are disjoint. Thus, all intervals in $D_0(C_{k,\eta})$ are disjoint. □

Now we prove Theorem 5.1

Proof of Theorem 5.1. As before, the distance set operation induces a new set of similarities so that $D_0(C_{k,\eta})$ can be defined as before. So assume $\lambda \leq \eta$ and first focus on $D_0(C_{k,\lambda})$. Take any i where $-(k-1) \leq i \leq k-2$ and recall that $D_0^0(C_{k,\lambda}) = [-1,1]$ and $D_0(C_{k,\lambda}) = \bigcap_{m=0}^{\infty} C_0^m(C_{k,\lambda})$. As done in the proof of Lemma 1, we will calculate the gap between S_i' and S_{i+1}'. For any $z \in [-1,1]$ we have

$$S_i'(z) = \frac{z}{2k-1} + i\cdot\frac{1-\frac{1}{2k-1}}{k-1} = \frac{z}{2k-1} + \frac{2i}{2k-1}$$

and similarly,

$$S'_{i+1}(z) = \frac{z}{2k-1} + \frac{2i+2}{2k-1}.$$

Therefore

$$S'_i([-1,1]) = \left[-\frac{1}{2k-1} + \frac{2i}{2k-1}, \frac{1}{2k-1} + \frac{2i}{2k-1}\right] = \left[\frac{2i-1}{2k-1}, \frac{2i+1}{2k-1}\right]$$

and

$$S'_{i+1}([-1,1]) = \left[-\frac{1}{2k-1} + \frac{2i+2}{2k-1}, \frac{1}{2k-1} + \frac{2i+2}{2k-1}\right] = \left[\frac{2i+1}{2k-1}, \frac{2i+3}{2k-1}\right].$$

So the gap between these two intervals is

$$\frac{2i+1}{2k-1} - \frac{2i+1}{2k-1} = 0.$$

Therefore, there are no gaps between any of the intervals generated by the similarities which this shows $D_0(C_{k,\lambda}) = [-1,1]$. Then we also get $D_0(C_{k,\eta}) = [-1,1]$ since we simply have more elements to choose distances from and because $D_0(C_{k,\delta}) \subseteq [-1,1]$ for any $\delta < 1/k$. So this establishes the first part of the theorem.

For the second part of the theorem, assume $0 \leq \eta < \lambda$. By Lemma 1, every interval in $D_0^m(C_{k,\eta})$ is disjoint. Using similar notation as before, we set $C_{k,\eta}^0 = [0,1]$ and $C_{k,\eta}^m = \bigcup_{i=1}^k S_i(C_{k,\eta}^{m-1})$. Then every interval in $D_{k,\eta}^m$ is of the form $[\ell, \ell+\eta^m]$ and they are all disjoint. So for any two intervals in $D_{k,\eta}^m$, say $[\ell_i, \ell_i+\eta^m]$ and $[\ell_j, \ell_j+\eta^m]$ where $\ell_i < \ell_j$, we get that the

$$[\ell_i, \ell_i+\eta^m] - [\ell_j, \ell_j+\eta^m] = [\ell_i - \ell_j - \eta^m, \ell_i + \eta^m - \ell_j]$$

and thus, the length of this interval is $2 \cdot \eta^m$. Therefore, every interval in $D_0^m(C_{k,\eta})$ is of

length $2 \cdot \eta^m$. Thus, because every interval in $D_0^m(C_{k,\eta})$ is disjoint, $\#\mathcal{I}' = 2k-1$, and because $\eta < 1/(2k-1)$, we get

$$\lim_{m\to\infty} \mathcal{L}\left(D_0^m(C_{k,\eta})\right) = \lim_{m\to\infty} 2 \cdot \eta^m (2k-1)^m = 0.$$

Furthermore, it is clear that $D_0^0(C_{k,\eta}) \supset D_0^1(C_{k,\eta}) \supset \cdots$ and because $\mathcal{L}(D_0^0(C_{k,\eta})) < \infty$, by the monotone convergence theorem for sets we get

$$\mathcal{L}\left(D_0(C_{k,\eta})\right) = \mathcal{L}\left(\bigcap_{m=0}^{\infty} D_0^m(C_{k,\eta})\right) = \lim_{m\to\infty} \mathcal{L}\left(D_0^m(C_{k,\eta})\right) = 0$$

which proves the theorem. □

To end this chapter, we make some quick notes. First, this generalizes the result of Steinhaus. As stated before, the ternary Cantor set, $C_{2,\frac{1}{3}}$ is a symmetric Cantor set. Then by Theorem 5.1, $D_0(C_{2,\frac{1}{3}}) = [-1,1]$ because here, $k = 2$ so $\lambda = \frac{1}{3}$. Also, the theorem shows a strange behavior of Cantor sets which is that they have a hard drop-off in terms of the size of their distance sets. Lastly, it is important here that similarity ratio λ is the same for each similarity in $\mathcal{I}$ because this was directly used to generate the recurrence relation between each similarity and it is not clear on how to use this approach when the similarity ratios are different.

5.2 Thickness Calculation

We now calculate the thickness of symmetric Cantor sets.

Theorem 5.4. *Let $\lambda < 1/k$ and $C_{k,\lambda}$ be a symmetric Cantor set. Then*

$$\tau(C_{k,\lambda}) = \frac{\lambda k - \lambda}{1 - k\lambda}$$

Proof. We will show that the thickness at each iteration of $C_{k,\lambda}$ is $(\lambda k - \lambda)/(1 - k\lambda)$ which will then imply $\tau(C) = (\lambda k - \lambda)/(1 - k\lambda)$. We will do this by induction. At the first iteration of $C_{k,\lambda}$, we will have $k - 1$ bounded gaps, each of length $(1 - k\lambda)/(k - 1)$ where they each is a distance of λ away from one of the other bounded gaps or the unbounded gaps since each bounded gap is separated by an interval of length λ. Thus, the thickness at the first iteration is

$$\frac{\lambda}{(1 - k\lambda)/(k - 1)} = \frac{\lambda k - \lambda}{1 - k\lambda}$$

and this shows the base case. So now assume we are at the nth iteration of C. Then each interval in the $(n-1)$th iteration will be of length λ^{n-1} and will produce k child intervals each of length λ^n. Furthermore, the gap between these child intervals will have diameter strictly smaller than any gap in the $(n - 1)$th iteration and since, by the inductive hypothesis, the thickness at the $(n-1)$th iteration is $(\lambda k - \lambda)/(1 - k\lambda)$, we only need to check the thickness at G where G is one of the bounded gaps between two of these new child intervals. The distance from G to any gap which has larger diameter will be λ^n since λ^n is the length of the new child intervals and because the new child intervals border gaps from the kth iteration for some $k \leq n$. So now we only need to find the length of G. Let I be the parent

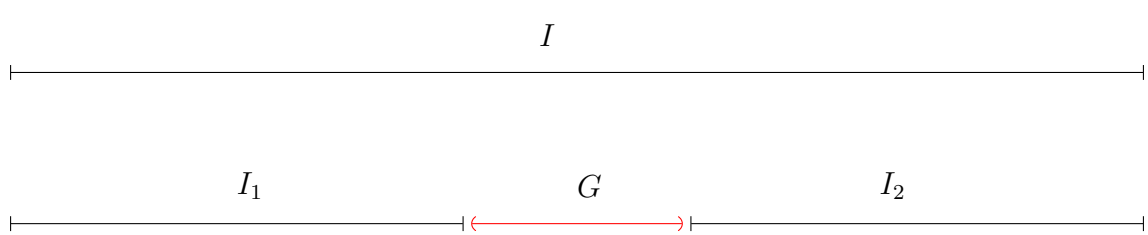

Figure 5.1: An example of the scenario being described when $k = 2$.

interval to the child intervals $I_1, \cdots, I_k$ and denote the gaps between these child intervals by $G_1, \cdots, G_{k-1}$. Then $\operatorname{diam}(I_i) = \operatorname{diam}(I_j)$ for all i and j and $\operatorname{diam}(G_i) = \operatorname{diam}(G_j)$ for all i and j. Furthermore, we also have that

$$k\lambda^n + (k-1)\operatorname{diam}(G_1) = k\operatorname{diam}(I_1) + (k-1)\operatorname{diam}(G_1) = \operatorname{diam}(I) = \lambda^{n-1}$$

which shows that

$$\operatorname{diam}(G) = \operatorname{diam}(G_1) = \frac{\lambda^{n-1} - k\lambda^n}{k-1}.$$

So the thickness at G is

$$\frac{\lambda^n}{(\lambda^{n-1} - k\lambda^n)/(k-1)} = \frac{\lambda k - \lambda}{1 - k\lambda}$$

which shows the claim. □

To finish the chapter, we make one quick observation. From Theorem 5.1, the smallest possibly λ for which we get $D_0(C_{k,\lambda}) = [-1, 1]$ is when $\lambda = 1/(2k-1)$. So taking this λ we have that

$$\tau(C_{k,\lambda}) = \frac{\frac{k}{2k-1} - \frac{1}{2k-1}}{1 - \frac{k}{2k-1}} = \frac{\frac{1}{2k-1}(k-1)}{\frac{1}{2k-1}(2k-1+k)} = \frac{k-1}{k-1} = 1.$$

Thus, to ensure that the distance set of $C_{k,\lambda}$ is large, the smallest thickness that will guarantee this is thickness equal to one.

Chapter 6

Conclusion

The main focus of this work was on distance sets and finding conditions on a set that guarantees when its distance set is large and/or small. The part of this thesis was Theorem 3.5 which generalizes Theorem 1.8 in [14]. Along these lines, thickness provides a way of ensuring that the distance set of a product set in $\mathbb{R}^2$ will be large in the sense of Lebesgue measure. Continuing with the theme of distance sets, in Chapter 4 we gave an example of a set with large Hausdorff dimension, with respect to $\mathbb{R}^1$, but with a small distance set and in Chapter 5, we showed a way of calculating the distance set and thickness of a class of self-similar sets. Also, there are certainly future projects that could be pursued. Two future projects have been briefly spoken about in Section 3.3 and at the end of Section 3.1.

The end of Section 3.1 talked about issues with distance function $g_{x,t}$ when trying to prove similar results in $\mathbb{R}^2$. Specifically, a different way to prove Lemma 3.10 would be needed as you need to now view $g_{x,t}$ as a multivariable function, and working with the norm of its derivative is not as pretty as it is with absolute values. However, the rest of the proof should follow fairly similarly to the $\mathbb{R}^1$ case. A possibly way around this is to consider compact sets $C_1 \in \mathbb{R}^1$ and $C_2 \in \mathbb{R}^2$ and have $g_{x,t}$ act only on C_1. This would ensure that our norm is still the absolute value and we can simply reuse Lemma 3.10. We would however need to change the mapping $g_{x,t}$. A possibly way of changing is by taking $x \in \mathbb{R}^3$ (in $\mathbb{R}^3$ because we are now

considering $C_1 \times C_2$) and define $g_{x,t}$ as

$$g_{x,t} := \sqrt{\frac{1}{2}t^2 - \frac{1}{2}(x_1 - y_1)^2}.$$

Now take $\pi_1 : X \times Y \to X$ defined by $\pi_1(x, y) := x$ and $\pi_2 : X \times Y \to Y$ defined by $\pi_2(x, y) := y$. Then we would have $t \in \Delta_x(C_1 \times C_2)$ provided that

$$g_{x,t}(C_1) \cap (\pi_1(C_2) - x_2) \cap (\pi_2(C_2) - x_3) \neq \emptyset.$$

We could then simply try and find assumptions compact sets that

$$\tau\left(g_{x,t}(C_1)\right) \cdot \tau\left((\pi_1(C_2) - x_2) \cap (\pi_2(C_2) - x_3)\right) > 1.$$

However this is unsatisfying as both $\pi_1(C_2) - x_2$ and $\pi_2(C_2) - x_3$ could have large thickness, but their intersection could have small thickness. So we would now like to be able to talk about thickness results involving three sets. This leads to the next future project.

Section 3.3 gives a brief discussion concerning thickness results when now looking at the intersection of three sets rather than two. An issue that crops up when trying to determine when three sets have non-empty intersection is the loss of control of elements on the boundary of gaps. A very general idea of the proof of the gap lemma is as follows: Assume we have two compact sets C_1 and C_2 in $\mathbb{R}^d$. The idea is to assume that they have empty intersection and then proceed by contradiction. You reduce to the case when are looking at bounded gaps U_1 of C_1 and V_1 of C_2 which satisfy a property called linked. That is, $U_1 \cap V_1 \neq \emptyset$, $\partial U_1 \backslash V_1 \neq \emptyset$, and $\partial V_1 \backslash U_1 \neq \emptyset$. You can then show that you are able to produce a new set of linked bounded gaps of either the form (U_1, V_1') or (U_1', V_1) of C_1 and C_2 respectively such that either $\operatorname{diam}(U_1') < \operatorname{diam}(U_1)$ or $\operatorname{diam}(V_1') < \operatorname{diam}(V_1)$. You can then keep repeating this

process so that you can find elements on the boundary of this sequence of bounded gaps and since one of their diameters shrinks to zero, you get that there will be a point that is on the boundary of both bounded gaps, which then must be an element in C_1 and C_2. The issue that arises when you now consider three compact sets is that if you start with (U_1, V_1, W_1) of linked bounded gaps of C_1, C_2, C_3 (whatever linked would mean in the context of three sets) you can easily lose control of the diameter of one of these bounded gaps, and therefore, you lose control of the elements on the boundary of that set. So the same proof strategy of creating a sequence of linked gaps with decreasing diameter wouldn't apply.

A possible work-around to this issue is to impose more conditions on the gaps of C_1, C_2, C_3. If you can control where the boundary elements of all bounded gaps lie, then this could help to come to a satisfactory conclusion. A point that comes in the proof of the gap lemma is that when working with two compact sets C_1 and C_2, elements on the boundary of a gap of C_1 must be an element of C_1. Therefore, since we assume $C_1 \cap C_2 = \emptyset$, you can then assume that elements on the boundary of this gap must then be elements in another gap of C_2. This is partially what is able to allow you to find a new set of bounded gaps where one of the diameters has shrunk. Thus, when thinking about three sets C_1, C_2, C_3 if you can assume that boundary of elements of a bounded gap of C_1 are in both a bounded gap of C_2 and C_3 (and similarly for C_2 and C_3), then this could result in more control of the diameters of these bounded gaps.

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