A Modal Analysis Method

for a Lumped Parameter Model

of a Dynamic Fluid System

by Matthew L. Wicks

A thesis submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

Master of Science

in

Mechanical Engineering

APPROVED:

Dr. F.J. Pierce, Chairman

Dr. C.L. Dancey

Dr. D.J. Nelson

May, 1993

Blacksburg, Virginia

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Thesis Director: Dr. Felix J. Pierce

(ABSTRACT)

A lumped parameter model is developed for the analysis of dynamic fluid systems and the techniques of modal analysis are applied. An introduction to the lumped parameter modeling approach is accomplished by a thorough review of the dynamic mechanical system. This review of mechanical system analysis introduces terms such as the natural frequency, damping ratio and the frequency response function. For the analysis of more complex mechanical systems the topic of modal analysis is introduced. Proceeding in a manner analogous to that of the review of the mechanical system, the lumped parameter fluid model is introduced. This introduction includes the definition of the dynamic fluid properties and two relatively simple examples of how these properties may be used in the modeling of fluid systems. As an example of this method an analytical model is developed for a compressor system and the techniques of modal analysis are applied in a fluid sense.

ACKNOWLEDGMENTS

I would like to express my gratitude to Dr. Felix J. Pierce for his instruction, guidance and support during this work. His patience and enthusiasm are greatly appreciated. I would also like to thank Dr. Clint L. Dancey and Dr. Douglas J. Nelson for serving as members of my graduate committee and for their contributions and suggestions toward this effort.

I would also like to thank my father, Dr. Alfred L. Wicks, for his behind the scenes guidance and support. His knowledge of modal analysis and engineering as a whole proved instrumental in the completion of this work , as did the use of his computer.

I would also like to thank my family and friends for their patience and encouragement during my many years of study. A special thanks to my wife, Elin, who managed to put up with me as I struggled along. They all have made the hard work worthwhile and have my heartfelt thanks.

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Chapter 1 Introduction

The purpose of this study is to develop an approach to analyze the transient response of what may be broadly referred to as fluid systems. The methodology to be introduced uses a lumped parameter approach to model the system, and applies techniques borrowed from the analysis of dynamic mechanical systems to the lumped parameter fluid system model. Specifically, the field of modal analysis, as applicable to mechanical systems, will be introduced to analyze fluid systems. From this application, certain design parameters common to mechanical system analysis will be defined for a fluid system. These parameters may be useful in reducing or eliminating undesirable transient effects.

1.1 Historic Unsteady Fluid System Analysis

Historically, the analysis of unsteady flow systems has been difficult. The governing equations for fluid systems, i.e., the mass, momentum, and energy equations, are a set of non-linear partial differential equations. For this reason, analytic solutions are difficult to obtain. By making certain assumptions, some solutions are possible for certain time varying flow systems. One such set of solutions exists for flows with moving boundaries, as presented by White [1]. These are commonly referred to as Stokes' first and second problems. Two cases are discussed; one where there is a sudden acceleration of the boundary to some constant velocity, and a second where the boundary oscillates at a constant frequency. Each of these cases is based on the laminar parallel-flow assumption. Due to the assumptions made, the momentum equation becomes an ordinary differential equation for which solutions may be readily obtained. These solutions relate time varying velocities in a single spatial coordinate. Ideally, because the non-linear partial differential equations can be expressed as linear ordinary equations, the principle of superposition could be applied. Superposition would allow solutions to be obtained for even the most complex boundary conditions, by combining simpler solutions. Although these solutions may be obtained, they are limited by the assumptions that made them possible and their application to real fluid systems is questionable.

With the introduction of the modern computer and continuing advances in computing capability, it has been possible to relax many of the assumptions necessary to obtain solutions for more complex unsteady flow systems. Computational fluid dynamics (CFD) codes typically use a finite element, finite volume or finite difference approach to solve the non-linear partial differential governing set of equations. These solutions tend to be time consuming and some difficulties still exist. One such difficulty is dealing with turbulent flows. The added complexity of turbulence increases the computational difficulty and hence the solution time. Computational fluid dynamics codes, although time consuming, are a very powerful tool for the design and analysis of fluid systems. These have generally been applied to external aerodynamic flows, with some recent successful applications to internal flows in turbomachinery blade passages.

1.2 Proposed Method

An alternate method of obtaining useful information regarding the transient behavior of complex fluid systems is to model the flow system by a lumped parameter method. Such an approach forms the basis of the proposed analogy with mechanical systems and allows for a similar analytical approach to fluid system analysis. Specifically, the lumped parameter model for the fluid system could be used in conjunction with CFD codes in a fashion analogous to the relationship that exists between modal analysis and mechanical finite element codes. It is the analogy to dynamic mechanical system analysis that provides much of the background and motivation for this alternative means of unsteady fluid system analysis.

In many cases, the focus of unsteady fluid system analysis is on the bulk system quantities, such as mass flow and pressure. CFD codes tend to deal with the local primitive quantities, i.e., velocity components and pressures. The proposed alternative approach "lumps" the local quantities to form a model that relates bulk system quantities. This model can then be used to find the time response of these quantities. Such an approach would yield a powerful tool in the analysis of complex flow systems. Furthermore, this method could be used in conjunction with or as a validation of the results obtained from computational fluid dynamics codes.

Fluid systems are non-linear in nature, thus the difficulty in formulating solutions. To apply the proposed method, it is necessary to formulate the system model in such a way as to view the model as a linear model. To achieve this condition, it is necessary to linearize the system parameters about some operating condition. The assumptions required to do this are similar to those applied to mechanical systems. Reducing the system to a set of linear equations greatly reduces the complexity of the solution techniques involved. Forming a linear system also introduces analytical methods that have been extensively used in mechanical system analysis, but have not been fully developed for fluid systems.

This text is offered as a tutorial, using the analogy between mechanical and fluid system analysis. Specifically, elementary mechanical system analysis, including element definitions and simple system formulation and analysis, are thoroughly reviewed. From the review of simple mechanical systems, the introduction of experimental modal analysis and its application follows. An analogy is then made between the well-established mechanical system analysis and a lumped parameter approach to fluid system analysis, including an application of the modal analysis techniques.

Chapter 2 Motivation and Background

A brief review of system modeling and analysis will provide general information on the motivation behind formulating the lumped parameter fluid model. This will be accomplished by first defining the concept of a system, how a real system is modeled, and the results that such a model will yield. A review of the relevant literature will discuss lumped parameter modeling of fluid systems and some experimental results obtained from the utilization of such models.

2.1 System Dynamics and Modeling

The word system is broadly used to refer to a set of components acting synergistically to meet a certain objective. Dynamics refers to situations that change with time. Thus, the study of system dynamics is concerned with the analysis of systems with respect to time. A variety of dynamic systems exist in engineering studies. These engineering systems may be categorized as mechanical, electrical, fluid, thermal or any combination thereof. Often in the study of these systems, the system may be considered as a "black box", i.e., the details within the system are not the concern of the analysis. The focus of such an analysis is the time response of the system to some external disturbance. This establishes the input/output relationship, which is the goal of dynamic system analysis [2].

In order to study any of these systems it is necessary to have an appropriate model. The model can be either a physical or mathematical construct. A physical model may be a small scale clone of the actual physical system constructed under the appropriate laws of scaling to accurately reproduce the behavior of the real physical system of interest. For complex physical systems accurate physical models may be difficult to create or be of significant expense. For these reasons much attention is given to the formulation of mathematical models.

A mathematical model is based on appropriate governing physical laws. The goal of such modeling is to develop a set of equations that can predict the dynamic response of a system. In reality, the processes of formulating these equations and finding solutions can be extremely difficult. For example, in analyzing flow systems, the Navier-Stokes and energy equations can be applied almost universally, but are extremely difficult to solve in complex flow cases. Laminar flows are an example where physical laws are clearly understood and quantified, but where solutions can be very difficult to generate. In this case it is necessary to simplify the governing equations to the point where the mathematics required to obtain solutions is manageable.

Real systems are said to be continuous, i.e., the properties of the system vary continuously with space throughout the system. Mathematical expressions for the laws governing these systems often are non-linear partial differential equations for which solutions are difficult to obtain. Often such expressions are formidable to utilize in system analysis. Therefore it is necessary to apply various mathematical techniques to model real systems in such a way that solutions are more readily obtained. These mathematical models often fall within two groups; distributed parameter or lumped parameter models.

Distributed parameter models assume that the continuous system may be discretized into multiple small but discrete parts. Such is the basis of finite element mechanical design codes. The set of non-linear partial different equations can then be solved by stepping through each discrete element within the system. Although not formally labeled as distributed parameter models, thermal/fluid science often uses models of this type. An example of this approach to modeling is seen in boundary layer solutions where the continuity and momentum equations must be satisfied for each individual element and boundary conditions must be specified. Such an approach usually requires state of the art computing capability combined with extensive numerical techniques, and often require substantial running time to analyze complex systems.

The nature of the discrete parameter approach requires an analysis of the entire flow field, hence the necessity of extensive computational ability. If the focus of the analysis is on macroscopic system quantities, rather than the flow field specifics, the system may be discretized into fewer, larger elements forming a lumped parameter model. Such modeling efforts attempt to take a real continuous system and describe the system in relatively few discrete parts or "lumps." The objective of this modeling technique is to replace the governing partial differential equations of the real system, with a relatively simple set of ordinary linear differential equations. In forming such a model there is usually a loss of detail and accuracy. However, in many cases there is a reasonable trade off between this loss and a significant decrease in the computational complexity in obtaining a solution.

The focus of this research is to adapt the lumped parameter approach of modeling dynamic mechanical systems and apply this approach to the modeling and analysis of fluid systems. A significant result of this modeling approach is the development of the frequency response function, FRF. The FRF refers to a particular form of a transfer function relating the system inputs and outputs, or the system's response. The ability to formulate such a function should be valuable in the prediction of the fluid system's transient response to any input.

2.2 Literature Review

This review will be accomplished by considering two areas of published research utilizing this technique; the definition of the fluid systems and the parameters that will serve to provide an analytical model for analysis and actual experimentation on such models in predicting the response of physical fluid devices.

2.2.1 System and Element Definition

In order to utilize the lumped parameter technique to analyze any system, it is necessary to consider the system itself and the relevant inputs and outputs for the system. It is necessary to consider what characteristics can be "lumped" and how such terms can be formed to construct an accurate model of the system to be analyzed.

Doebelin [3] considers a variety of representative engineering systems and the components in modeling such systems. As an introduction, Doebelin discusses a general system analysis and a variety of mathematical modeling techniques. These techniques are classified based on the nature of the system model. The most realistic and most difficult to solve, are continuous field problems, which attempt to solve the governing equations directly. At the other end of the spectrum of modeling techniques are discrete network problems, which may lack some realism, but solution techniques are well-defined and easier to perform. The lumped parameter model falls into the latter category.

Doebelin [3] then defines the system parameters for modeling mechanical, electrical and fluid systems. He stresses that lumped parameter modeling has been successfully used in the design and analysis of mechanical and electrical systems, but is relatively new in modeling fluid or thermal systems. Doebelin defines the fluid parameters of inertance, resistance and compliance. Since the lumped parameter method is more established in the analysis of mechanical and electrical systems, Doebelin draws an analogy between fluid system parameters and mechanical and electrical system parameters. Each component of the fluid model is considered individually in both a real physical manner and a linearized element approach. This consideration is further developed by offering experimental means of quantifying the ideal fluid parameters through the measurement of common flow properties, pressure drop and flow rate, and relating them. Such experimental measures prove useful in determining the degree of non-linearity existing in real systems. Doebelin stresses the importance of the nature of the flow in determining the nature of each of the dynamic fluid system properties and the assumptions necessary to formulate lumped parameter models.

In subsequent chapters, Doebelin [3] utilizes these parameters to formulate models of simple, real fluid systems such as open and closed tank flows and fluid accumulators. These models are offered more as examples of the modeling techniques and the relevant mathematics, rather than demonstrating the validity of such an approach. However, Doebelin's discussion forms a solid basis for the development of the analysis of fluid systems through a lumped parameter approach.

A similar analytic approach is taken in the field of fluidics and is presented by Foster and Parker [4] and in part by Karnopp and Rosenberg [5]. Each of these sources draws an analogy between fluid system and electric circuit analysis. This analogy relates flow rate to current and the pressure drop to the driving voltage. Thus, each of the fluid system terms has corresponding electrical terms. This analogy is useful in formulating fluid models in a form similar to electric circuits and with analysis using the principles of electric theory. Although the focus of this text is to obtain a model analogous to the mechanical system, such an electric circuit analogy yields insight into the formulation of the model components and the analysis of the lumped parameter fluid model.

2.2.2 Experimental Model Verification

This review of the existing literature also revealed several attempts by researchers to model real fluid systems experimentally using the lumped parameter model, and to, formulate frequency response functions to predict system response, in a manner analogous to that for mechanical systems.

A study by Krejsa, et. al., [6,7] at NASA's Lewis Research Center attempted to model a forced flow, hollow, single tube boiler by formulating an analytical form of the frequency response function relating boiler inlet pressure to the inlet flow rate. This study was performed in an attempt to predict boiler instabilities due to unsteady flow with pressure-flow rate coupling. Modeling such a flow was difficult because of the two-phase flow in part of the flow regime. In spite of this difficulty, experimental measurements, based on swept-sine testing of the real physical system supported the analytical model.

A more generalized study was performed by Goldschmeid [8]. Goldschmeid formed frequency response functions through an analytical model, and experimentally testing this model using air, carbon dioxide and helium flows in rigid tubes feeding into plenums at one end. The frequency response functions of this study relate the time variant pressure response of the system and are based on the Stokes number and a dimensionless frequency. The experimental data was shown to validate the analytical model being tested.

A subsequent model developed by Goldschmeid and Wormley [9] considers a system consisting of a blower, duct and plenum. In analyzing this system Goldschmeid and Wormley developed an analytical function for the frequency response function based on a pressure amplitude ratio. They were then able to develop functions for the FRF's of each of the system's components, i.e., the blower, duct and plenum. The contributions of each of these component FRF's to the total system's FRF as initially found could then be seen. The experimental data collected by testing the actual physical apparatus of the complete system and each individual component showed general agreement with the analytically developed models.

Modeling of fluid systems through a lumped parameter approach shows a great deal of promise for models that accurately represent unsteady flows. As seen, some effort has been made to utilize this approach. This text will serve as an introduction to lumped parameter modeling as seen in mechanical system analysis and these ideas will then be applied to fluid systems.

Chapter 3 Mechanical Systems

A well-established example of the use of lumped parameter modeling to achieve accurate results is seen in the mechanical analysis of vibrating systems. Lumped models form the basis of much of vibration theory and have become of practical use in the field of modal analysis. Dynamic mechanical system analysis considers the time response of structures to forcing functions that vary in time. An analytic model of the structure can be used to accurately predict the response of the structure. Responses include accelerations, velocities and/or displacements due to any given input or forcing function.

This chapter reviews lumped parameter modeling of mechanical systems. For a mechanical system, the review will introduce the elements of a mechanical model, develop the model to introduce the methodology and parameters of interest, and present some results. These background ideas will then be used in a brief introduction to the subject of modal analysis.

3.1 System Elements

A system is a group of interacting elements. In general, mechanical systems can include either or both translational or rotational motions. For the purpose of simplicity the elements and discussion will consider only the translational case. These elements, masses, springs and dampers, are the building blocks of models for any system and can be combined to create models for even the most complex systems.

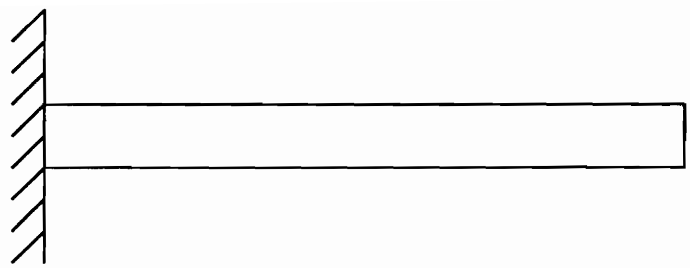
3.1.1 The Mass Element

Every mechanical system contains some quantity of mass. The quantity depends upon the structure and the materials of which it is comprised. In real structures this mass is distributed throughout the entire structure. Under the lumped parameter model method, the true physical mass is divided into a finite number of "mass lumps" and these lumps are positioned at points of interest throughout the structure. An example is a simple cantilevered beam shown in Figure 3-1. Each of these lumped masses will undergo only rigid body motion, i.e., they will move as a solid mass without deformation. Under this rigid body motion assumption each mass would move as a unit, hence any increase in the number of the mass lumps will improve the model. It has been shown that in many cases a limited number of masses are sufficient for reasonable accuracy.

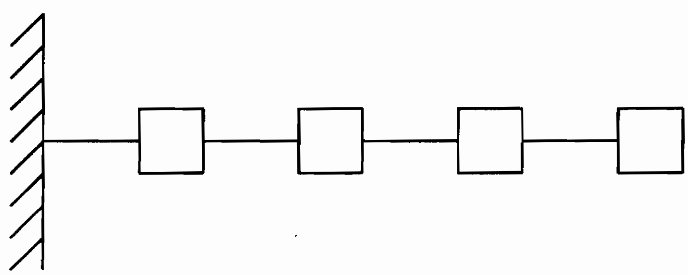
3.1.2 The Spring Element

The spring element demonstrates the "springiness" or elasticity of a structure. The spring serves as means of energy storage. Given a physical spring, there is a length at which the spring has no energy, its equilibrium position. Once the spring is placed in tension or compression, it stores potential energy. In an attempt to return to its equilibrium position, the potential energy of the spring is transferred to the mass where the energy appears as kinetic energy. This energy storage and transfer in the spring-mass system results in the oscillatory nature of a vibrating spring-mass system.

The ideal spring used in the mechanical system model is considered to have no mass (all structural mass is placed in the mass element). It is also assumed to be a perfect energy storage unit, i.e., the potential energy stored by the spring can be converted to kinetic energy of the mass without dissipation of energy and vice versa. The ideal spring



(a) Real, Continuous Cantilevered Beam



(b) Lumped Parameter Representation of Cantilevered Beam

Figure 3-1 Cantilevered Beam: Continuous and Lumped Representations is assumed to maintain a linear relation between the force and the spring deflection resulting from that force, where:

$$F_{spring} = kx \tag{3.1}$$

The parameter, k, is defined as the spring constant or stiffness. It is a property of the spring and is a measure of the elastic properties of the material and spring geometry. The inverse of the stiffness, (1/k), is defined as the compliance of the spring.

In real springs this linear relationship is usually a valid approximation for small displacements of the spring about its equilibrium position. Outside this range, physical springs will often demonstrate non-linear behavior due to the physical properties of the material or the geometry of the structure. The assumption of a linear spring usually requires a small deflection restriction in the analysis.

3.1.3 The Damper Element

Since neither the spring nor the mass elements have any means of dissipating energy, the damper element is introduced for that purpose. For vibrating structures, energy is dissipated by being converted to thermal energy. The primary means of this conversion is friction. Physically, friction can be described in three forms; static, Coulomb and viscous [10]. Static and Coulomb friction are closely related. Static friction occurs in attempting to initiate movement of a stationary object. Coulomb friction occurs in maintaining the movement of an object already in motion. Each of these is due to the contact resistance between surfaces and each has its own non-linear nature. Viscous friction occurs as an object is moved through a viscous fluid. In reality, some combination of all three of these types of friction usually occurs. The non-linear nature of static and Coulomb friction makes it difficult to formulate a mathematical model of damped behavior.

Because of the difficulties with the non-linearities introduced by static and Coulomb friction, typical mechanical models utilize only a linear model for viscous friction. Physically, viscous friction can be represented by a dashpot mechanism (e.g., the shock absorber of a car), and mathematically the force in the dashpot is given by the linear equation:

$$F_{damper} = cv \tag{3.2}$$

where v is the mass velocity and c is a proportionality constant called the damping coefficient.

Like the spring element, the viscous damper is a convenient idealization valid for small deflections. Unlike springs however, the physical understanding of the mechanism of damping, involving material properties and geometric configurations, is less clearly understood.

3.2 The Single Degree of Freedom Mechanical Model

The system elements described can be combined in a variety of ways to model complex mechanical systems. As an introduction, a simple single degree of freedom system will be used to demonstrate the technique utilized to model the system and review the kinds of results that can be obtained from this lumped parameter approach.

The number of degrees of freedom of a system is defined to be the number of independent coordinates necessary to fully described the motion of the physical system. A rigid body in free motion will have six degrees of freedom; three defining translational motion and three defining rotational motion. A continuous elastic body would have an infinite number of degrees of freedom, since one must describe the motion of each of the infinite number of individual points describing the body. Through the lumped parameter approach certain parts of such a continuous body would be lumped together and treated as rigid mass lumps, thus reducing the number of degrees of freedom of the model. In order to determine the number of degrees of freedom of a system, rather than simply counting the number of mass elements within the system, it is necessary to consider the number of equations needed to describe the motion of each of the masses and the number of constraint equations on the system [10]. The number of degrees of freedom is found by subtracting the number of constraints from the number of motion equations.

As an introduction, consider a system containing a single mass, spring and damper shown in Fig. 3-2a. The motion of the mass will be restricted to translation in only the x-direction, thus forming a single degree of freedom system. The free body diagram of the mass element in Fig. 3-2b shows the external forcing function and the forces due to the spring and damper elements. By applying Newton's second law of motion to the mass, all system parameters can be related by a single equation:

$$\sum F_{external} = ma \qquad (3.3)$$

$$F(t) - F_{damper} - F_{spring} = ma \qquad (3.4)$$

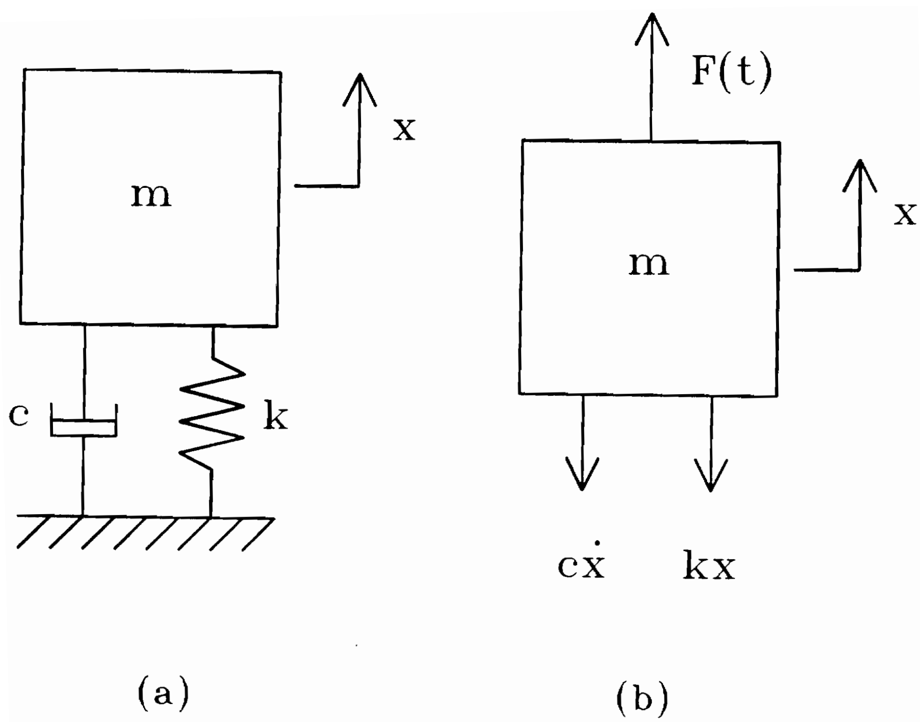


Figure 3-2 Single Degree of Freedom Mechanical System

$$F(t) - cv - kx = ma \tag{3.5}$$

Writing the velocity and acceleration as the first and second time derivative of displacement respectively, and by rearranging, the result is the familiar second order ordinary linear differential equation:

$$F(t) = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx \tag{3.6}$$

It is this equation and its solution for particular boundary conditions that formulate the basis of the lumped parameter model of mechanical systems for all ranges of complexity.

3.2.1 General Time Response

The solution of the second order ordinary linear differential ,Eq. 3.6, gives the displacement, x, as a function of time, t. For a simple, known forcing function the solution of Eq. 3.6 is straight forward. It is first necessary to formulate the complimentary solution of the homogeneous differential equation. This is accomplished by assuming no forcing function with a solution of the form:

$$x(t) = e^{st} \tag{3.7}$$

Such a solution form accounts for the constant coefficient and satisfies the condition that the derivative terms differ only by multiplicative factors.

Taking the time derivative of the assumed solution form, and substituting back into the original equation, Eq. 3.6, results in:

$$(ms^2 + cs + k)\, e^{st} = 0 \tag{3.8}$$

Solving for the roots of the quadratic term in the parenthesis yields solutions for the unknown, s, which are of the form:

$$s = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{3.9}$$

These roots are then used in the general solution form given by:

$$x_c(t) = e^{-\frac{c}{2m}t}\left[C_1 e^{\sqrt{(\frac{c}{2m})^2 - \frac{k}{m}}\,t} + C_2 e^{-\sqrt{(\frac{c}{2m})^2 - \frac{k}{m}}\,t}\right] \tag{3.10}$$

Under the principle of superposition this general solution is then added to the particular solution to formulate the entire time response function x(t). The particular solution is obtained by assuming that the response will be of the same form as the forcing function, e.g., a sinusoidal force will yield a sinusoidal response. The particular solution is called the steady state response in that it describes the dominant motion for increasing time. Once the complete solution is found, the unknown coefficients can be found by applying the boundary conditions unique to a given problem. For the mechanical system these are usually initial conditions of the displacement and velocity.

Of particular interest in the field of vibrations is the complimentary, or transient solution. Since this portion of the total solution was found by assuming no external forcing function, it reveals important characteristics of the system that are inherent and exist regardless of the nature of the external excitation.

Considering the transient response, it is apparent that the first exponential term shows the decaying effect on motion due to the energy dissipation caused by the damping element. However, it is the bracketed term which will determine the nature of the specific behavior of the system response to various inputs. Specifically, the response of the system depends on whether the radicand in the exponents is positive, zero or negative. When $(c/2m)^2$ is greater than k/m the exponents are real and no oscillatory behavior will occur. This condition is referred to as an overdamped system. When the two terms are equal, the exponents are equal to zero, and the system response is governed by the initial decaying exponential term. This condition is called critical damping. The more usual case of interest in vibrations is when $(c/2m)^2$ is less than k/m, and the radicand is negative. In this case the exponents are imaginary and the response will be characterized by oscillatory motion. This is referred to as the underdamped case and will be the focus in this review of mechanical systems, because in this case the system response is often very significant in terms of magnitude.

It is convenient to introduce the natural frequency and damping ratio as two new parameters that combine the mass, spring and damper properties. If we remove the damper from the simple mechanical system being considered, the homogeneous solution found in Eq. 3.10 reduces to:

$$x_c(t) = C_1 e^{\sqrt{\frac{k}{m}}t} + C_2 e^{-\sqrt{\frac{k}{m}}t} \qquad (3.11)$$

The exponents seen here are imaginary, hence the response of the displacement over time is governed by a combination of harmonic components. Each of these harmonic terms has a frequency term given by $(k/m)^{1/2}$. This term is defined as the natural frequency of the system:

$$\omega_n = \sqrt{\frac{k}{m}} \qquad (3.12)$$

This is a system parameter and is independent of the applied forcing function.

Consider the critically damped case previously discussed, this will occur when $(c/2m)^2 = (k/m)$. Defining the damping coefficient for this case as the critical damping, c_c, the damping ratio is defined as the ratio of the damping coefficient to the critical value:

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{mk}} \qquad (3.13)$$

Thus oscillations will occur for ζ values less than one. The importance of these two parameters will become more apparent as the system analysis is more thoroughly developed.

3.2.2 Impulse Response

A transient solution of particular importance is the response to an impulsive force. In the physical world an impulsive force is one where the magnitude of the force is large in comparison to its time duration. Mathematically, an impulse is defined as the time integral of force and is designated by $\hat{F}$ where:

$$\hat{F} = \int F(t)\,dt \tag{3.14}$$

Using Newton's second law one can write:

$$F(t)\,dt = m\,dv \tag{3.15}$$

Integrating and using the definition of an impulse, applying such a force to a mass yields a sudden change in velocity without an appreciable change in displacement. This instantaneous change in velocity is given by $\hat{F}/m$, and this becomes the initial velocity condition in the solution of the transient response equation.

Returning to the general equation for the transient response:

$$x_c(t) = e^{-\frac{c}{2m}t}\left[C_1 e^{\sqrt{(\frac{c}{2m})^2 - \frac{k}{m}}\,t} + C_2 e^{-\sqrt{(\frac{c}{2m})^2 - \frac{k}{m}}\,t}\right] \tag{3.16}$$

For simplicity let A = c/2m and B = (c/2m) - k/m

$$x_c(t) = C_1 e^{(B-A)t} + C_2 e^{-(A+B)t} \tag{3.17}$$

and take the first time derivative:

$$\dot{x}_c(t) = C_1(B-A)\, e^{(B-A)t} - C_2(A+B)\, e^{-(A+B)t} \tag{3.18}$$

These two equations can be solved for the two unknown coefficients, C_1 and C_2, by applying two known conditions, either initial or boundary conditions. For the case of an impulse forcing function these two conditions are given by x(t=0) = 0 no initial displacement and dx/dt (t=0) = $\hat{F}$/m, the instantaneous initial velocity resulting from the impulsive force. Applying these two conditions to solve for the coefficients, C_1 and C_2, and introducing the undamped natural frequency and damping ratio as parameters, results in the scaled impulse response function given by:

$$x(t) = \hat{F}\, [\, \frac{1}{2m\omega_n\sqrt{\zeta^2-1}} \left(e^{(-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1})\,t} - e^{(-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1})\,t}\right)\,] \tag{3.19}$$

The bracketed term defines an important system characteristic called the unit impulse response function, h(t):

$$h(t) = \frac{1}{2m\omega_n\sqrt{\zeta^2-1}}\, [\, e^{(-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1})\,t} - e^{(-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1})\,t}\,] \tag{3.20}$$

The response of the linear time invariant system to any impulse can be found by scaling the unit function by the magnitude of F.

The unit impulse response function is very important in describing the response to more general forcing functions by means of the convolution integral. For linear time invariant systems the principle of convolution states that any arbitrary forcing function may be represented as a series of impulses [11]. Consider a forcing function, F(t), acting over some time interval as shown in Fig. 3-3. At some particular time τ the contribution of the forcing function can be thought of as an impulse of magnitude $F(\tau)$ and infinitesimal duration $\Delta\tau$. Applying this idea across the entire time interval, the forcing function may be represented by a series of unit impulses weighted by the magnitude of the forcing function, F(t), at each individual time, τ. The system response to such a force may also be discretized into the summation of each of the impulse response functions $h(\tau)$ for each assumed implied impulse over the entire time interval. Mathematically, this is shown by the convolution integral as the continuous summation of the unit impulse response function $h(\tau)$ weighted by the true forcing function for each time, τ, and given by:

$$x(t) = \int_{-\infty}^{\infty} h(\tau)\, F(t-\tau)\, d\tau \tag{3.21}$$

Equation 3.21 can be simplified by considering only physically realizable or causal systems. A causal system is defined as a system which responds to only past inputs [2]. This means that the system will respond only after a force is applied, i.e. for $F(t) = 0$ for $t < 0$, with the response $x(t)=0$ for $t < 0$. Thus in considering the time response of the system, one need only consider inputs occurring from the time of the initial force input, and the lower limit on the integration becomes zero.

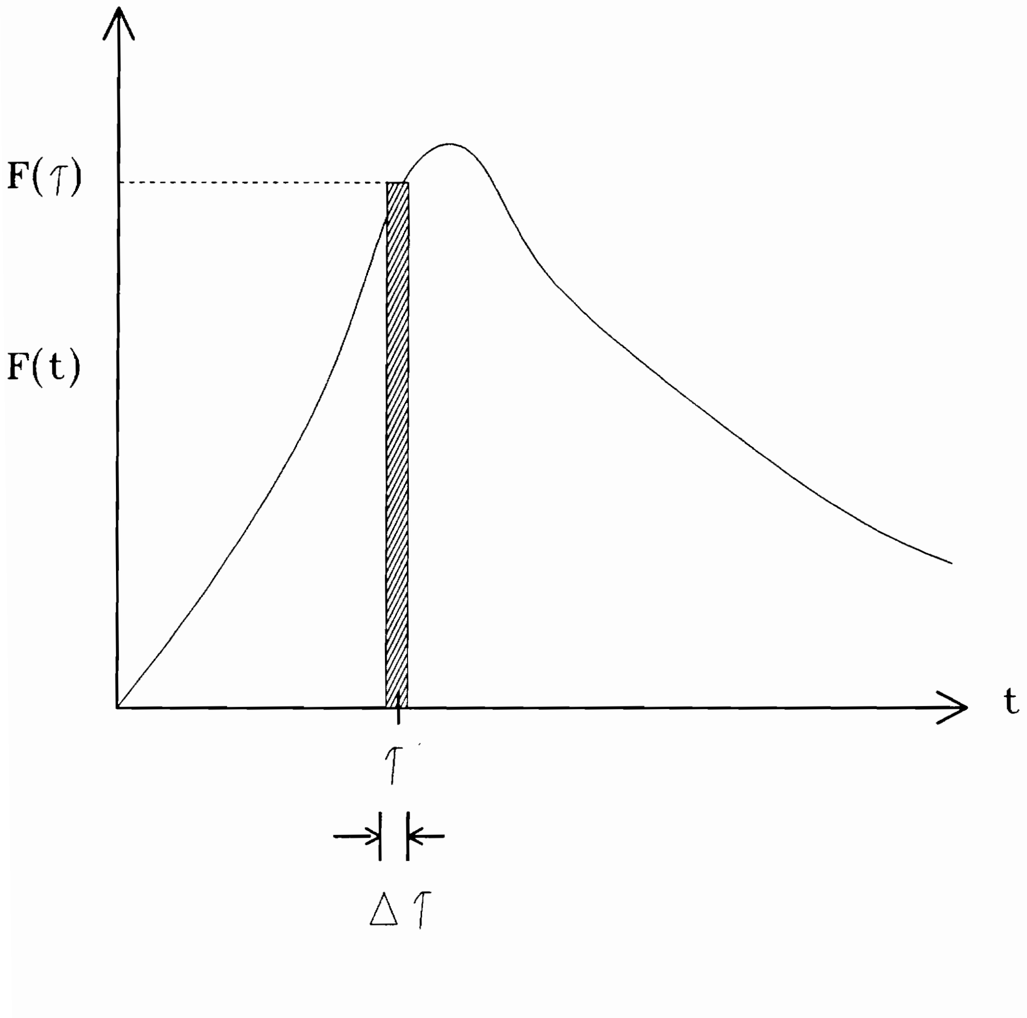


Figure 3-3 Impulse Representation of Arbitrary Forcing Fuction, F(t)

For easily integrated impulse response and forcing functions the convolution integral is very useful. In practice many real world input functions, even though they can be described by a mathematical function, may not lend themselves to integration.

3.2.3 The Frequency Response Function

The time response of the mechanical system to an external forcing function can be represented through the convolution integral. In many cases, this integration may be difficult or impossible to perform and the true system response functions remain unknown. However, where convolution in the time domain is accomplished through integration, the same convolution can be accomplished by simple multiplication in the frequency domain. Often, the system response may be more readily obtained using a particular transformation from the time domain to the frequency domain. This transformation is given by the Fourier transform [12], a particular form of the Laplace transform. Mathematically, the general form of the Fourier transform for causal systems is given by:

$$H(\omega) = \int_0^\infty h(t)\, e^{-i\omega t}\, dt \tag{3.22}$$

Consider the unit impulse response function, h(t), in Eq. 3.20. Performing the Fourier transform on h(t) is accomplished by first substituting the impulse response function, Eq. 3.20, into the Fourier transform, Eq. 3.22, yielding:

$$H(\omega) = \int_0^\infty \frac{1}{2m\omega_n\sqrt{\zeta^2-1}}\left(e^{\left(-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}\right)t} - e^{\left(-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}\right)t}\right) e^{i\omega t}\, dt \tag{3.23}$$

Rearranging the constant term outside the integral and multiplying through the parentheses yields:

$$H(\omega) = \frac{1}{2m\omega_n\sqrt{\zeta^2-1}}\int_0^{\infty}\left(e^{\left(-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}\ -\ i\omega\right)t} - e^{\left(-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}\ -\ i\omega\right)t}\right)dt \tag{3.24}$$

Performing the integration one gets:

$$H(\omega) = \frac{1}{2m\omega_n\sqrt{\zeta^2-1}}\left[\frac{e^{\left(-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}\ -\ i\omega\right)t}}{-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}\ -i\omega} - \frac{e^{\left(-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}\ -\ i\omega\right)t}}{-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}\ -\ i\omega}\right]\Bigg|_0^{\infty} \tag{3.25}$$

Applying the limits of the integral, then

$$H(\omega) = \frac{1}{2m\omega_n\sqrt{\zeta^2-1}}\left[-\left(\frac{1}{-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}-i\omega} - \frac{1}{-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}-i\omega}\right)\right] \tag{3.26}$$

or

$$H(\omega) = \frac{1}{2m\omega_n\sqrt{\zeta^2-1}}\left(\frac{-2\omega_n\sqrt{\zeta^2-1}}{\left(-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}-i\omega\right)\left(-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}-i\omega\right)}\right) \tag{3.27}$$

and finally

$$H(\omega) = -\frac{1}{m}\left[\frac{1}{\left(-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1} - i\omega\right)\left(-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1} - i\omega\right)}\right] \tag{3.28}$$

Multiplying out the denominator and combining the appropriate terms yields the following complex function in frequency:

$$H(\omega) = -\frac{1}{m}\left[\frac{1}{\omega^2 - \omega_n^2 - 2i\zeta\omega\omega_n}\right] \tag{3.29}$$

Dividing an ω_n out of the denominator results in the final form of the complex function called the frequency response function, FRF. Thus the FRF, here $H(\omega)$, is the Fourier transform of the impulse response function, usually written in the following form.

$$H(\omega) = \frac{1/k}{\left(1 - \left(\omega/\omega_n\right)^2\right) + 2i\zeta\,\omega/\omega_n} \tag{3.30}$$

By transforming the input function into the frequency domain, the convolution integral can be written as a simple algebraic expression [2].

$$X(\omega) = H(\omega)\ F(\omega) \tag{3.31}$$

Using this expression, the FRF can be written as the ratio of the output to the input, relating the displacement of the system to any given forcing function, but in the frequency domain rather than the time domain.

$$\frac{X(\omega)}{F(\omega)} = \frac{1/k}{1-\left(\omega/\omega_n\right)^2 + 2i\zeta\,\omega/\omega_n} \tag{3.32}$$

For any known forcing function the displacement can be found by relatively simple mathematics, instead of attempting the integration necessary in the time domain.

Another advantage of working in the frequency domain is that differentiation and integration in the time domain reduce to multiplication and division in the frequency domain. For example, this ratio of displacement to force can be multiplied by $i\omega$ to yield a velocity to force ratio, and by $-\omega^2$ to give an acceleration to force ratio. Not only is the FRF a useful tool for displacement information, it can easily be used to provide information about velocities and accelerations.

Since the frequency response function is complex valued it can be treated as a vector in the complex plane with a magnitude and phase angle. Each of these quantities are given as functions of frequency by the following equations:

$$\text{Magnitude: } \left|\frac{X}{F}\right| = \frac{1/k}{\sqrt{\left(1-\left(\omega/\omega_n\right)^2\right)^2 + \left(2\xi\,\omega/\omega_n\right)^2}} \tag{3.33}$$

Phase: $$\phi = \tan^{-1}\left[\frac{2\zeta \,\omega/\omega_n}{1-\left(\omega/\omega_n\right)^2}\right] \qquad (3.34)$$

Each of these terms can be plotted against the frequency ratio, as in Figs. 3-4 and 3-5, and reveal interesting behavior dependent on the forcing frequency and damping ratio. One important characteristic apparent in the plots is the resonant frequency. The resonant frequency occurs at the point of the peak magnitude and the 90 degree phase shift point. In real structures the resonance condition is extremely important, as the peak magnitude shows a large displacement to force ratio. Forcing a structure at its resonant frequency may have serious consequences, including structural failure. The resonant frequency can be found analytically by minimizing the denominator of the magnitude of the FRF, Eq. 3.33, and is given by:

$$\omega_r = \omega_n \sqrt{1-2\zeta} \qquad (3.35)$$

From the phase plot, at resonance the phase angle between the displacement and force is 90 degrees. This corresponds to the force being in phase with the velocity of the mass.

The figures also show some interesting trends. On the magnitude plot, at zero frequency there is a one-to-one ratio between the displacement and force. This ratio increases as the resonant frequency is approached, then decreases beyond this point. At higher frequencies the system will tend to a motionless state. The phase plot shows an increasing phase angle from 0 to 90 degrees at resonance, and at this point the force is in

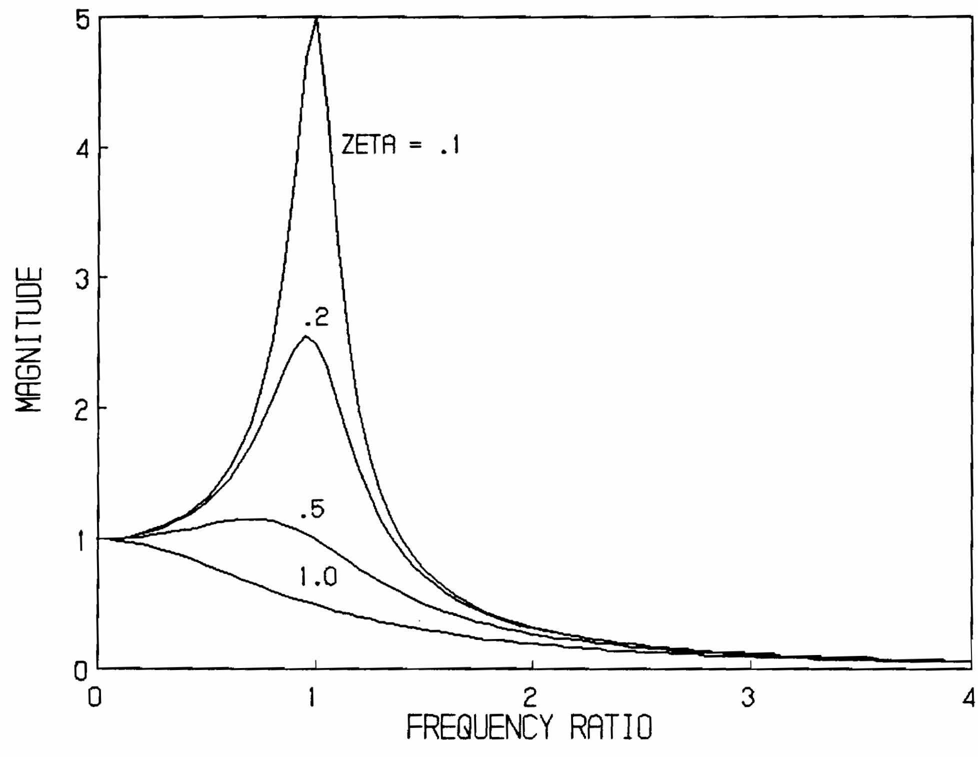


Figure 3-4 Magnitude of the Frequency Response Function for a Single Degree of Freedom Mechanical System

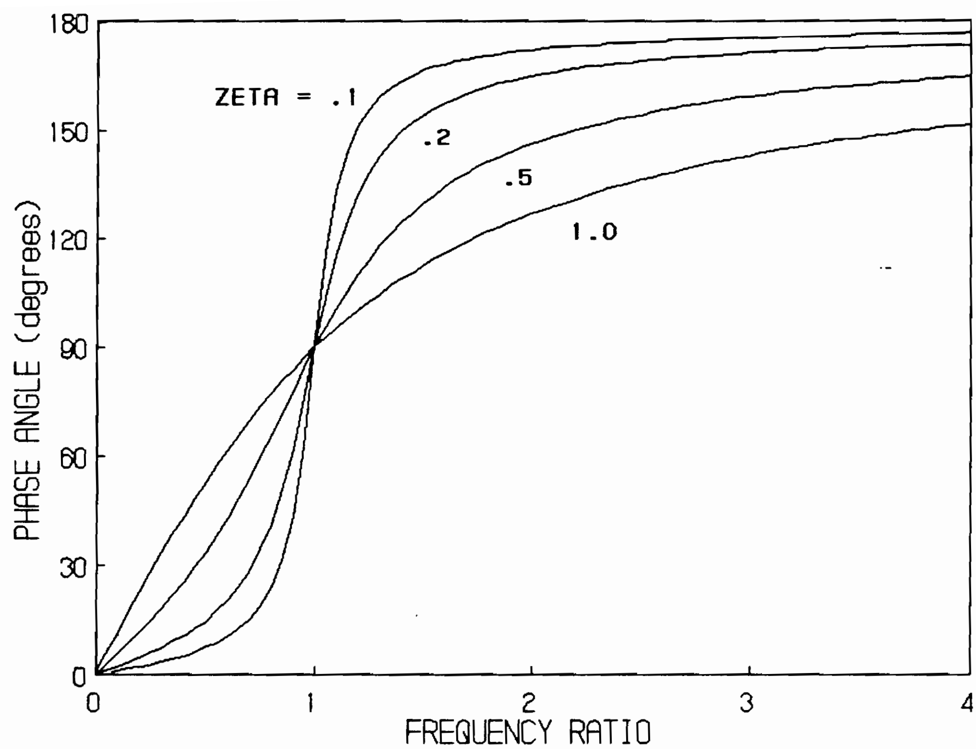


Figure 3-5 Phase Angle of the Frequency Response Function for a Single Degree of Freedom Mechanical System

phase with the velocity. Beyond resonance the phase angle continues to increase toward 180 degrees, where it is in phase with the acceleration of the mass. The rates of these changes over the frequency range are dependent on the damping ratio as shown.

3.2.4 Summary

For the single degree of freedom system, analytical functions for both the time and frequency response are relatively simple in their derivation. Most real structures, however, are not accurately modeled as a single degree of freedom system. None the less, the single degree of freedom model does provide useful insight into how lumped parameters may be utilized, and demonstrates both the approach and the results provided by this technique.

The principles and terminology introduced in considering the single degree of freedom mechanical system can be applied to more general models containing multiple elements. As with the single degree of freedom system, the analysis of more complex systems of multiple degree of freedom systems focuses on the system response. Important to this analysis is the determination of the parameters defined earlier in this chapter, i.e. the natural frequency and damping ratio. These parameters are used to predict the resonant frequencies to be avoided in the design process.

3.3 Multiple Degree of Freedom Systems

Using the lumped parameter modeling technique to represent real mechanical systems requires a trade off between accuracy in the modeling and computational complexity dependent upon the number of degrees of freedom being considered. The analytical model describing the response of a single degree of freedom system was demonstrated to provide a basis for the lumped parameter approach. This same single degree of freedom model approach can be applied to systems containing multiple degrees of freedom, though the equations describing the model dynamics are usually not derived as easily.

3.3.1 Equations of Motion

A two degree of freedom system will demonstrate the increased analytic difficulties. The two degree of freedom system can be represented physically by adding another mass, spring and damper in series with the single degree of freedom model considered previously in Fig. 3-6a . A second translational coordinate is added to describe the motion of the second mass. Two separate equations of motion corresponding to the motion of each individual mass are required. With this additional degree of freedom and additional equation of motion, there are multiple natural frequencies and damping ratios, and the frequency response function becomes much more difficult to define and derive .

As with the single degree of freedom system, the equations of motion follow from Newton's second law of motion applied to each of the masses. Figure 3-6b shows the free body diagrams with the appropriate coordinates and relevant forces applied to each mass. In considering the first mass, m_1, the first spring and damper combination will act in a similar fashion to that seen in the single mass example, acting solely on m_1. With the addition of the second mass, m_2, a second independent translational coordinate system has been added to describe the motion of the second mass. The spring and damper added to connect the two masses will require both coordinates to define the forces

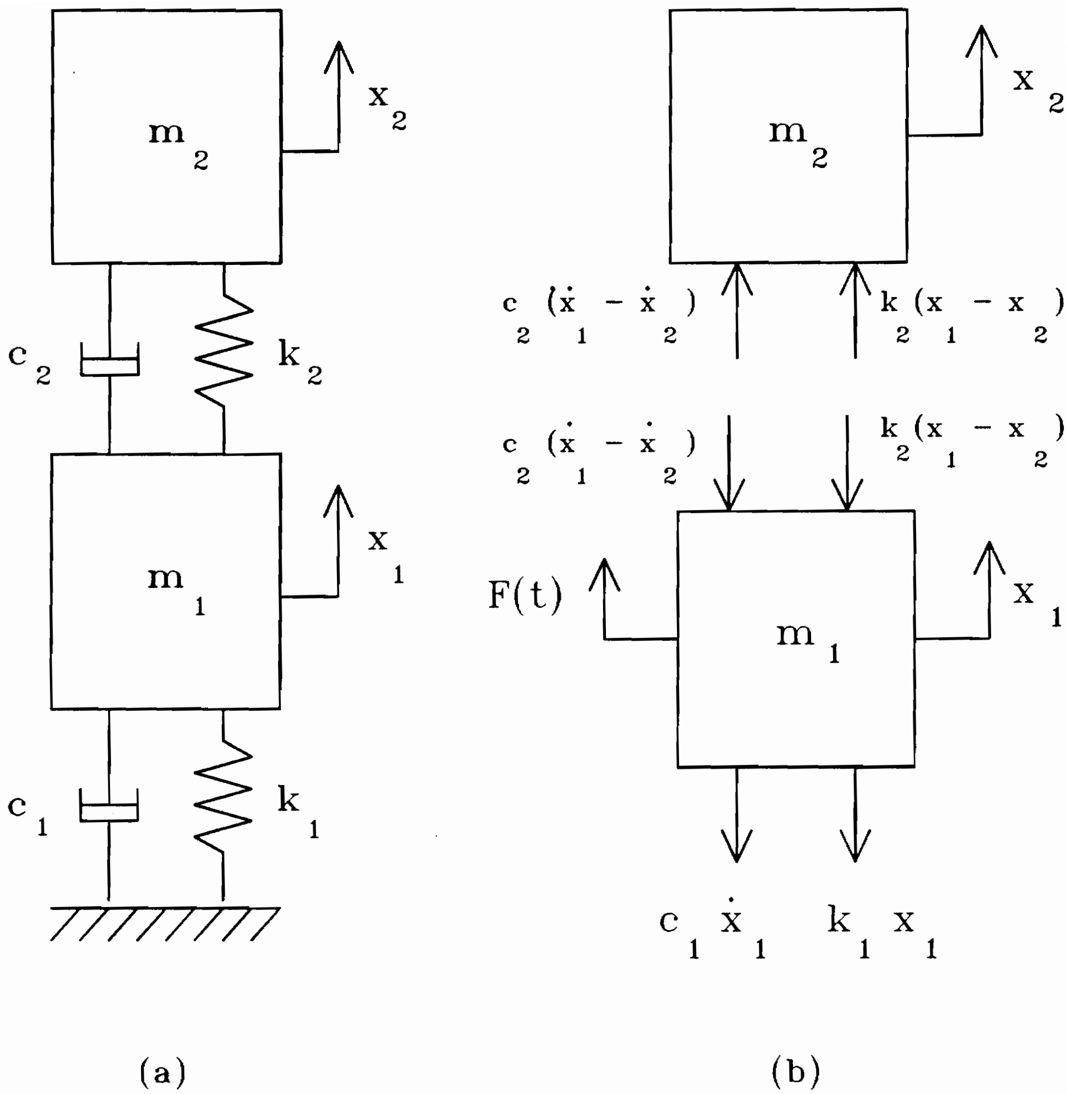


Figure 3-6 Two Degree of Freedom Mechanical System

associated with those elements. Summing the forces acting on each of the masses, as seen in the free body diagram, and relating them through Newton's second law of motion to each of the masses, results in two equations of motion for the system given by:

Mass 1: $$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - c_1\dot{x}_1 - c_2(\dot{x}_1 - \dot{x}_2) \quad (3.36)$$

Mass 2: $$m_2\ddot{x}_2 = k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) \quad (3.37)$$

3.3.2 Frequency Response Function

In the analysis of the single degree of freedom system, the frequency response function was derived by considering the impulse response function and forming the Fourier transform of the result. The additional mass of the two degree of freedom system significantly increases the complexity of deriving the frequency response function in a similar manner. As an alternative method to forming the frequency response function for this system, assume that a forcing function, harmonic in nature and given by $f(t)= Fe^{i\omega t}$, is applied to the first mass, m_1. The harmonic forcing function will yield a harmonic response in both masses, and by the characteristic of frequency preservation for the constant coefficient linear system, this response will be at the same frequency. This method of arriving at the frequency response function is common to vibration textbooks and is somewhat easier than utilizing the impulse response function. Returning to the derivation, the response of the two masses to the harmonic excitation is given by $x_1(t)= X_1 e^{i\omega t}$ and $x_2(t) = X_2 e^{i\omega t}$. Using these two relations, their respective time derivatives and with the assumed harmonic forcing function acting only on the first mass, the equations of motion can be rewritten as:

Mass 1: $$\left[-m_1\omega^2 + i\omega(c_1 + c_2) + (k_1 + k_2)\right]X_1 + \left[-i\omega c_2 - k_2\right]X_2 = F \qquad (3.38)$$

Mass 2: $$\left[-m_2\omega^2 + i\omega c_2 + k_2\right]X_2 + \left[-i\omega c_1 - k_1\right]X_1 = 0 \qquad (3.39)$$

Solving for X_2 in terms of X_1 in the equation for the second mass, and substituting back into the first equation, results in an equation that relates the forcing function to the response of the first mass. This is a frequency response function. Manipulating the equation into the FRF form yields:

$$\frac{X_1}{F} = \frac{1/m_1}{\left[\dfrac{k_1 + k_2}{m_1} - \omega - \dfrac{k_2}{m_1}\left(\dfrac{\dfrac{k_2}{m_2} + i\omega\dfrac{c_2}{m_2}}{\dfrac{k_2}{m_2} - \omega^2 + i\omega\dfrac{c_2}{m_2}}\right) + i\omega\left(\dfrac{c_1 + c_2}{m_1} - \dfrac{c_2}{m_1}\left(\dfrac{\dfrac{k_2}{m_2} + i\omega\dfrac{c_2}{m_2}}{\dfrac{k_2}{m_2} - \omega^2 + \dfrac{c_2}{m_2}}\right)\right)\right]} \qquad (3.40)$$

The response of the second mass is obtained by substituting back into Eq. 3.38:

$$\frac{X_2}{F} = \frac{1/m_1}{\left[\dfrac{\dfrac{k_2}{m_2} + i\omega\dfrac{c_2}{m_2}}{\dfrac{k_2}{m_2} - \omega^2 + i\omega\dfrac{c_2}{m_2}}\right]\left[\dfrac{k_1 + k_2}{m_1} - \omega - \dfrac{k_2}{m_1}\left(\dfrac{\dfrac{k_2}{m_2} + i\omega\dfrac{c_2}{m_2}}{\dfrac{k_2}{m_2} - \omega^2 + i\omega\dfrac{c_2}{m_2}}\right) + i\omega\left(\dfrac{c_1 + c_2}{m_1} - \dfrac{c_2}{m_1}\left(\dfrac{\dfrac{k_2}{m_2} + i\omega\dfrac{c_2}{m_2}}{\dfrac{k_2}{m_2} - \omega^2 + \dfrac{c_2}{m_2}}\right)\right)\right]} \qquad (3.41)$$

These two equations are the frequency response functions for the two degree of freedom system in Fig. 3-6a. Each predicts the response or motion of the particular mass for the forcing function applied to m_1. The equations are somewhat awkward to utilize, but their ability to reduce a difficult time response problem to algebraic manipulation should be noted.

Though not obvious from the derivation, this system will have two natural frequencies, one corresponding to each degree of freedom. To extract the natural frequencies of this system, we first assume zero damping, in either Eq. 3.40 or 3.41, as with the single degree of freedom case. Considering Eq. 3.40, for zero damping:

$$\frac{X_1}{F} = \frac{1/m_1}{\left(\frac{k_1 + k_2}{m_1} - \omega - \frac{k_2}{m_1} \left(\frac{\frac{k_2}{m_2}}{\frac{k_2}{m_2} - \omega^2} \right) \right)} \tag{3.42}$$

The natural frequencies can be found by setting the denominator of Eq. 3.42 equal to zero and solving for ω. Since the solution results in a quadratic in ω^2, there will be two possible solutions for the natural frequency given by:

$$\omega_n^2 = \frac{1}{2}\left[\frac{k_2}{m_2} + \frac{k_1 + k_2}{m_1}\right] \pm \frac{1}{2}\left[\left(\frac{k_2}{m_2}\right)^2 + \left(\frac{k_1}{m_1}\right)^2 + 2\left(\frac{k_2^2 - k_1 k_2}{m_1 m_2} + \frac{k_1 k_2}{m_1^2}\right)\right]^{1/2} \tag{3.43}$$

Deriving the impulse response function for this system would be a tedious and formidable task, as would be the development of analytical forms for the damping ratios. The difficulty of approaching multiple degree of freedom systems in the "classical" sense serves as motivation to introduce a method that requires much less mathematical manipulation, and still yields pertinent parametric information regarding the system. This is the modal analysis method.

Chapter 4 Modal Analysis

The multiple degree of freedom system of the previous chapter demonstrated the difficulty of pursuing an analysis of the system through "classical" techniques, and the need for an easier method of extracting the pertinent parameters. Such is the reasoning behind the introduction of modal analysis. Modal analysis is very useful in the analysis of multiple degree of freedom mechanical systems and shall be shown to be applicable to other lumped parameter system models.

With the addition of more degrees of freedom to a system model, the analytical model grew in complexity. For the mechanical system, there were also additional natural frequencies and damping ratios for each new degree of freedom. The method of modal analysis performs a coordinate transformation with the result that the entire system response may be considered as a sum of single degree of freedom system responses, each containing one of the original system's natural frequencies and damping ratios. This idea of the superposition of single degree of freedom system solutions greatly reduces the difficulty in formulating an expression for the system response.

An additional benefit of the application of the modal analysis techniques is that they very readily lend themselves to physical experiments. The results of the methods to be described can be used to experimentally determine the system's frequency response function, as well as to extract the important parameters of natural frequencies and damping ratios.

4.1 Introduction

Consider the two degree of freedom mechanical system shown in Fig. 3-6. Examining the equations of motion, Eqs. 3.36 and 3.37, it is apparent that coupling is present in these differential equations. Coupling occurs physically when the motion of one mass in the model is dependent upon the motion of the other. This is reflected in the equations of motion with both coordinates appearing in each of the equations, showing the interdependency of the motion of the two masses. It is this coupling of the equations of motion that caused the difficulties seen in the previous chapter in forming an analytical expression for the frequency response functions and the parameters of natural frequency and damping ratio.

Equations 3.36 and 3.37 can be formulated in a matrix form. Matrices are formed for the constant parameter terms of mass, damping and stiffness. Multiplying these matrices by the respective vector terms for acceleration, velocity and displacement, these are summed and set equal to a vector form of the forcing functions. The matrix form of the two degree of freedom system given in Fig. 3-6 is:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F(t) \\ 0 \end{Bmatrix} \quad (4.1)$$

or in a generalized form, given by:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\} \quad (4.2)$$

The purpose of modal analysis is to take the matrix form of these coupled equations and perform a coordinate transformation to decouple the equations. This is accomplished through the modal transformation to what is called the system's principle coordinate system. In this principle coordinate system, the equations of motion are independent and can be considered separately. For the two degree of freedom system in Eq. 4.1, the transformation will yield two linear, independent equations of motion which represent two single degree of freedom systems. The solution to these equations may be combined through the principle of superposition to construct the system response. In general, an n-degree of freedom system will yield n independent equations, with n natural frequencies and n damping ratios, all of which contribute to the entire system response.

This chapter will introduce two models frequently used in modal analysis, the proportional damping and state space models. Each of these models reduce a coupled set of equations to an uncoupled set of equations, where the solution to each equation represents the response of a single degree of freedom system. These results can be combined to yield the system response. Each of these methods makes use of the modal coordinate transformation to decouple the coupled equations. The specifics of each of these techniques will be discussed and the results they produce demonstrated in the following.

4.2 Proportional Damping Model

Mechanical models may include modeling assumptions to quantify the damping properties of a physical system. In the previous chapter, viscous damping was used to model the damper element, even in those cases when Coulomb or static friction was present or even dominant. For the proportional damping model used in modal analysis, the damping is assumed to be proportional to the mass and the stiffness [10]. In other words, the damping matrix of a multiple degree of freedom system may be written as a linear combination of the mass and stiffness matrices

$$[C] = \alpha[M] + \beta[K] \tag{4.3}$$

where one of the coefficients, α or β, may be zero. Under this assumption, the coordinate transformation used to uncouple the equations should provide a damping matrix form that results in uncoupled component equations.

4.2.1 Model Formulation

Using the proportional damping assumption, the matrix equation of the two degree of freedom system given by Eq. 4.2 can be solved in a manner similar to that used for the equation of the single degree of freedom system, Eq. 3.6. Consider the undamped, unforced two degree of freedom system that results by removing the damping matrix from the homogeneous matrix equation, with:

$$[M]\{\ddot{x}\} + [K]\{x\} = 0 \tag{4.4}$$

First, premultiply by the inverse of the mass matrix

$$[M]^{-1}[M]\{\ddot{x}\} + [M]^{-1}[K]\{x\} = 0 \tag{4.5}$$

For the harmonic response of a free vibration of an undamped system, one can write:

$$\{\ddot{x}\} = -\omega_n^2\{x\} = 0 \tag{4.6}$$

Therefore

$$\left[-[I]\omega_n^2 + [M]^{-1}[K]\right]\{x\} = 0 \tag{4.7}$$

If we let $\lambda = \omega_n^2$, this becomes the familiar eigenvalue problem

$$\left|[M]^{-1}[K] - [I]\lambda\right| = 0 \tag{4.8}$$

Returning to the previously defined mass and stiffness matrices for the system, Eq. 4.8 becomes:

$$\begin{vmatrix} \frac{k_1 + k_2}{m_1} - \lambda & -\frac{k_2}{m_2} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} - \lambda \end{vmatrix} = 0 \tag{4.9}$$

Solving the determinant of Eq. 4.9 yields a quadratic equation in λ, given by:

$$\left(\frac{k_1 + k_2}{m_1} - \lambda\right)\left(\frac{k_2}{m_2} - \lambda\right) - \left(\frac{k_2}{m_2}\right)^2 = 0 \tag{4.10}$$

The roots of Eq. 4.10 correspond to the natural frequencies of the system. These are in agreement with those found previously for the same system given by Eq. 3.43 :

$$\omega_n^2 = \frac{1}{2}\left[\frac{k_2}{m_2} + \frac{k_1 + k_2}{m_1}\right] \pm \frac{1}{2}\left[\left(\frac{k_2}{m_2}\right)^2 + \left(\frac{k_1}{m_1}\right)^2 + 2\left(\frac{k_2^2 - k_1 k_2}{m_1 m_2} + \frac{k_1 k_2}{m_1^2}\right)\right]^{1/2} \tag{3.43}$$

The eigenvalues, which are the natural frequencies of the system, can be used to obtain the eigenvectors, ϕ_i. The eigenvectors are non-unique and scalable vectors, defined by the system's natural frequencies. They are a significant result in system analysis. The eigenvectors are scalable solutions for the undamped, unforced system response at each natural frequency, and these solutions are referred to as the normal modes of the system. Although the eigenvector solutions are non-unique, they yield insight about the response of the system at each of the natural frequencies.

An important characteristic of the normal modes or eigenvectors of the mechanical system is their orthogonality with respect to the mass and stiffness matrices [10]. Orthogonality in this respect results in the condition where the transpose of the eigenvector matrix is equal to the inverse of this same matrix. Due to this characteristic, the matrix of the eigenvectors, Φ, diagonalizes both the mass and stiffness matrices and the result is uncoupled component equations. The matrix of the eigenvectors is defined to be the modal matrix and is the basis of the modal transformation.

Since the modal matrix is the basis of the transformation to the principal or modal coordinates, we can introduce the new coordinate system as follows:

$$\{x\} = [\Phi]\{z\} \tag{4.11}$$

This transformation between physical spatial coordinates and modal space is the initial step to decoupling the equations of motion. Applying this coordinate transformation and using the diagonalization property of the modal matrix will decouple the equations. The response of the system will be given by the solution of the n independent equations for the n degree of freedom system.

Because of the difficulty in providing a generalized analysis, a numerical example is offered to demonstrate this method. For the two degree of freedom system of Fig. 3-6 , assume mass 1 has a value of four lbm, mass 2 has a value of 10 lbm, all spring constants are 100 lbf/in and the damping constants are each one lbf-s/in, Eq. 4.1 becomes

$$\begin{bmatrix} 4 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 200 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \tag{4.12}$$

Forming the undamped matrix equation as given by Eq. 4.4, and developing the eigenvalue problem yields:

$$\begin{vmatrix} 50-\lambda & -25 \\ -10 & 10-\lambda \end{vmatrix} = 0 \tag{4.13}$$

Solving the determinant of this equation, yields a quadratic equation in λ, given by:

$$\lambda^2 - 60\lambda + 250 = 0 \tag{4.14}$$

Solving for λ,, the natural frequencies are the positive square roots of λ with values of 2.12 and 7.45 radians/second, respectively. Using these values to form the eigenvectors yields:

$$\phi_1 = \begin{Bmatrix} 10 - \lambda_1 \\ 10 \end{Bmatrix} = \begin{Bmatrix} 5.5 \\ 10 \end{Bmatrix} \qquad \text{Normalized}\,\phi_1 = \begin{Bmatrix} 0.55 \\ 1 \end{Bmatrix} \tag{4.15}$$

$$\phi_2 = \begin{Bmatrix} 10 - \lambda_2 \\ 10 \end{Bmatrix} = \begin{Bmatrix} -45.5 \\ 10 \end{Bmatrix} \qquad \text{Normalized}\,\phi_2 = \begin{Bmatrix} 1 \\ -0.22 \end{Bmatrix} \tag{4.16}$$

Plotting these normalized eigenvectors will graphically show the nature of the two mode shapes of the system, as shown in Fig. 4-1.

Forming the eigenvector matrix (modal matrix) and its transpose will provide the means of diagonalization:

$$\Phi = \begin{bmatrix} 0.55 & 1 \\ 1 & -0.22 \end{bmatrix} \tag{4.17}$$

$$\Phi^T = \begin{bmatrix} 0.55 & 1 \\ 1 & -0.22 \end{bmatrix} \tag{4.18}$$

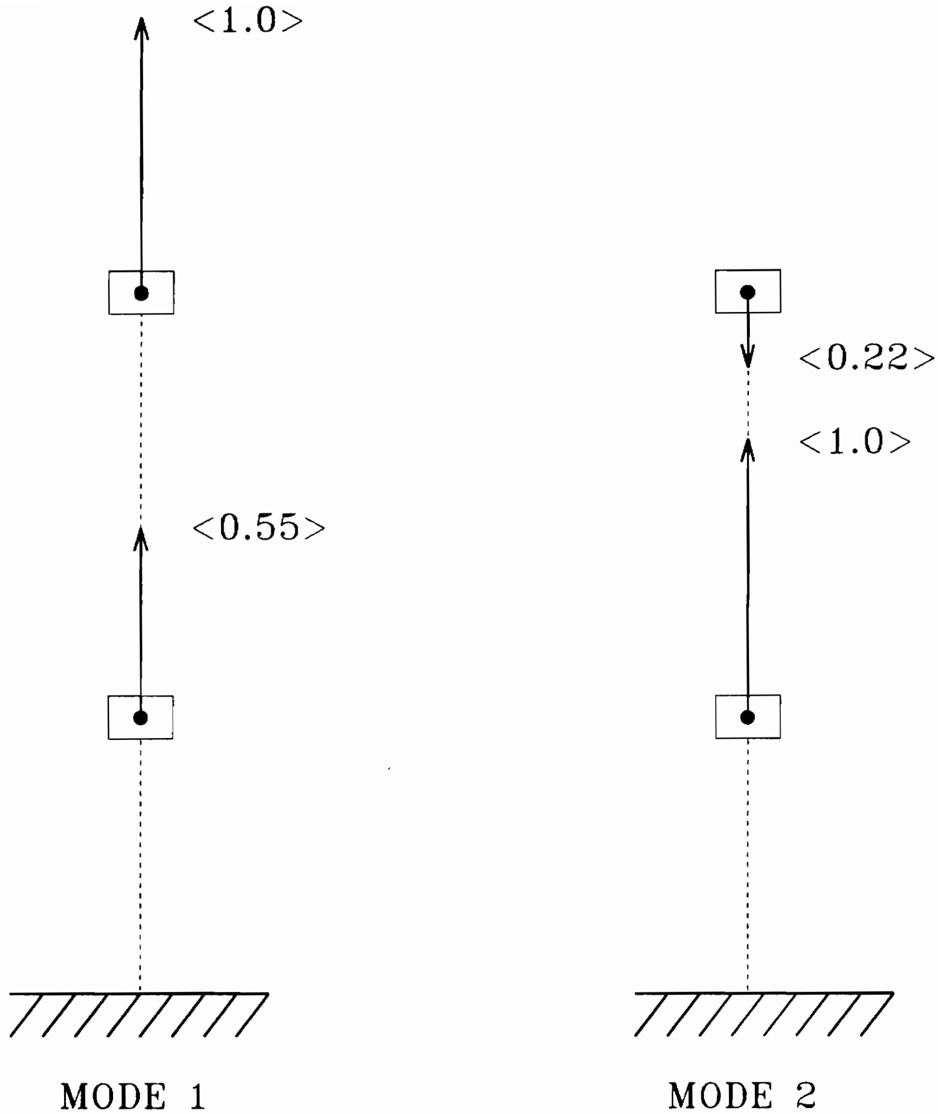


Figure 4-1 Mode Shapes for Two Degree of Freedom System as Determined by Normalized Eigenvectors

Multiplying each of the parameter matrices [M], [K] and [C], as follows will diagonalize the matrices for use in modal space, and define the modal mass matrix, the modal stiffness matrix and the modal damping matrix:

$$[\Phi]^T[M][\Phi]=\begin{bmatrix}11.21 & 0\\ 0 & 4.484\end{bmatrix} \tag{4.19}$$

$$[\Phi]^T[K][\Phi]=\begin{bmatrix}50.5 & 0\\ 0 & 248.8\end{bmatrix} \tag{4.20}$$

$$[\Phi]^T[C][\Phi]=\begin{bmatrix}0.505 & 0\\ 0 & 2.488\end{bmatrix} \tag{4.21}$$

Using these newly defined matrices, Eq. 3.12 defines the values of the natural frequencies in the modal coordinate:

$$\omega_{n1}=\sqrt{\frac{k_{1m}}{m_{1m}}}=\left(\frac{50.5}{11.21}\right)^{1/2}=2.12\ \mathrm{rad/s} \tag{4.22}$$

$$\omega_{n2}=\sqrt{\frac{k_{2m}}{m_{2m}}}=\left(\frac{248.8}{4.484}\right)^{1/2}=7.45\ \mathrm{rad/s} \tag{4.23}$$

Using the definition of the damping ratio for the single degree of freedom case, Eq. 3.13, gives the two damping ratios in the modal coordinates:

$$\zeta_1 = \frac{c_{1m}}{2\sqrt{k_{1m}m_{1m}}} = \frac{0.505}{2((50.5)(11.21))^{1/2}} = 0.0106 \quad (4.24)$$

$$\zeta_2 = \frac{c_{2m}}{2\sqrt{k_{2m}m_{2m}}} = \frac{2.488}{2((248.8)(4.484))^{1/2}} = 0.0372 \quad (4.25)$$

The system properties of natural frequency and damping ratio are global, i.e., spatially invariant [10].

Using the modal coordinates defined by Eq. 4.11 and their first two derivatives with respect to time in the general matrix form, Eq. 4.2, yields:

$$[M][\Phi]\{\ddot{z}\}+[C][\Phi]\{\dot{z}\}+[K][\Phi]\{z\}=\{F\} \quad (4.26)$$

Premultiplying by the transpose of the modal matrix yields:

$$[\Phi]^T[M][\Phi]\{\ddot{z}\}+[\Phi]^T[C][\Phi]\{\dot{z}\}+[\Phi]^T[K][\Phi]\{z\}=[\Phi]^T\{F\} \quad (4.27)$$

The diagonalization properties of the modal matrix transform the previously coupled parametric matrices in physical space to the corresponding diagonalized modal space parametric matrices. For the general two degree of freedom case, using the subscript m to denote the modal properties, these can be written as:

$$\begin{bmatrix} m_{1m} & 0 \\ 0 & m_{2m} \end{bmatrix}\begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix}+\begin{bmatrix} c_{1m} & 0 \\ 0 & c_{2m} \end{bmatrix}\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix}+\begin{bmatrix} k_{1m} & 0 \\ 0 & k_{2m} \end{bmatrix}\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}=\begin{bmatrix} \Phi_{11} & \Phi_{21} \\ \Phi_{12} & \Phi_{22} \end{bmatrix}\begin{Bmatrix} F(t) \\ 0 \end{Bmatrix} \quad (4.28)$$

The component equations can be written in modal space coordinates as:

$$m_{1m}\ddot{z}_1 + c_{1m}\dot{z}_1 + k_{1m}z_1 = \Phi_{11}F$$
$$m_{2m}\ddot{z}_2 + c_{2m}\dot{z}_2 + k_{2m}z_2 = \Phi_{12}F \qquad (4.29)$$

For simplicity, it is assumed that a harmonic forcing function acts only on the first mass. The response of each mass in the modal coordinate system will also be harmonic and at the frequency of the forcing function. Following the procedure used in the single degree of freedom system analysis, i.e., assuming harmonic motion given by $z_1 = Z_1e^{i\omega t}$ and $z_2 = Z_2e^{i\omega t}$ with the definition of natural frequency in Eqs. 4.22 and 4.23 and damping ratio in Eqs. 4.24 and 4.25, Eq. 4.29 can be written as a function of frequency as :

$$\left(-\omega^2 + \omega_{n1}^2 + 2i\zeta_1\omega\omega_{n1}\right)Z_1 = \Phi_{11}F$$
$$\left(-\omega^2 + \omega_{n2}^2 + 2i\zeta_2\omega\omega_{n2}\right)Z_2 = \Phi_{12}F \qquad (4.30)$$

The frequency response function can be formed in modal space for Z_1 and Z_2, as:

$$\frac{Z_1}{F} = \frac{\Phi_{11}}{-\omega^2 + \omega_{n1}^2 + 2i\zeta_1\omega\omega_{n1}}$$

$$\frac{Z_2}{F} = \frac{\Phi_{12}}{-\omega^2 + \omega_{n2}^2 + 2i\zeta_2\omega\omega_{n2}} \tag{4.31}$$

One can now return to the physical spatial coordinate system, by using the modal transformation, Eq. 4.11, to form the frequency response function for the response of each coordinate in physical space. Note that the FRF for each physical spatial coordinate is the sum of two contributions; one from each of the normal modes, and as before is a complex valued function:

$$\frac{X_1}{F} = \frac{\Phi_{11}\Phi_{11}}{-\omega^2 + \omega_{n1}^2 + 2i\zeta_1\omega\omega_{n1}} + \frac{\Phi_{11}\Phi_{12}}{-\omega + \omega_{n2}^2 + 2i\zeta_2\omega\omega_{n2}}$$

$$\frac{X_2}{F} = \frac{\Phi_{12}\Phi_{21}}{-\omega^2 + \omega_{n1}^2 + 2i\zeta_1\omega\omega_{n1}} + \frac{\Phi_{12}\Phi_{22}}{-\omega + \omega_{n2}^2 + 2i\zeta_2\omega\omega_{n2}} \tag{4.32}$$

These results are demonstrated by returning to the numerical example. Substituting the values for the natural frequencies, damping ratios and modal matrix terms, the FRF for each mass, are given by:

$$\frac{X_1}{F} = \frac{0.3025}{-\omega^2 + 4.94 + 0.045\omega} + \frac{.55}{-\omega^2 + 55.5 + .554\omega}$$

$$\frac{X_2}{F} = \frac{1}{-\omega^2 + 4.94 + 0.045\omega} + \frac{-.22}{-\omega^2 + 55.5 + .554\omega} \tag{4.33}$$

Plotting the magnitude and phase of each of these complex valued functions over a frequency range of 0 to 10 rad/sec, Fig. 4-2,4-3,4-4 and 4-5, shows the details of the FRFs expected for a two degree of freedom system. The figures show the resonant conditions and the normal modes' contributions to the system response. Note that one of the magnitude contributions is negative, thus it subtracts from the overall FRF magnitude. This occurs due to the phase difference found in the second mode shown by the negative sign in the eigenvector.

4.2.2 Proportional Damping Model Summary

The two degree of freedom system has been used to (1) form a general model and (2) to provide a specific example of the application and results for the proportionally damped model. The power of the modal transformation has been demonstrated to show how the equations of motion can be decoupled and a relatively simple form of the frequency response function can be formulated.

From the two degree of freedom system described, it is possible to extrapolate a general form of the frequency response function for the proportionally damped model. Assuming a general, n-degree of freedom system and letting i denote the particular degree of freedom in the physical spatial coordinates, j denote the physical location of

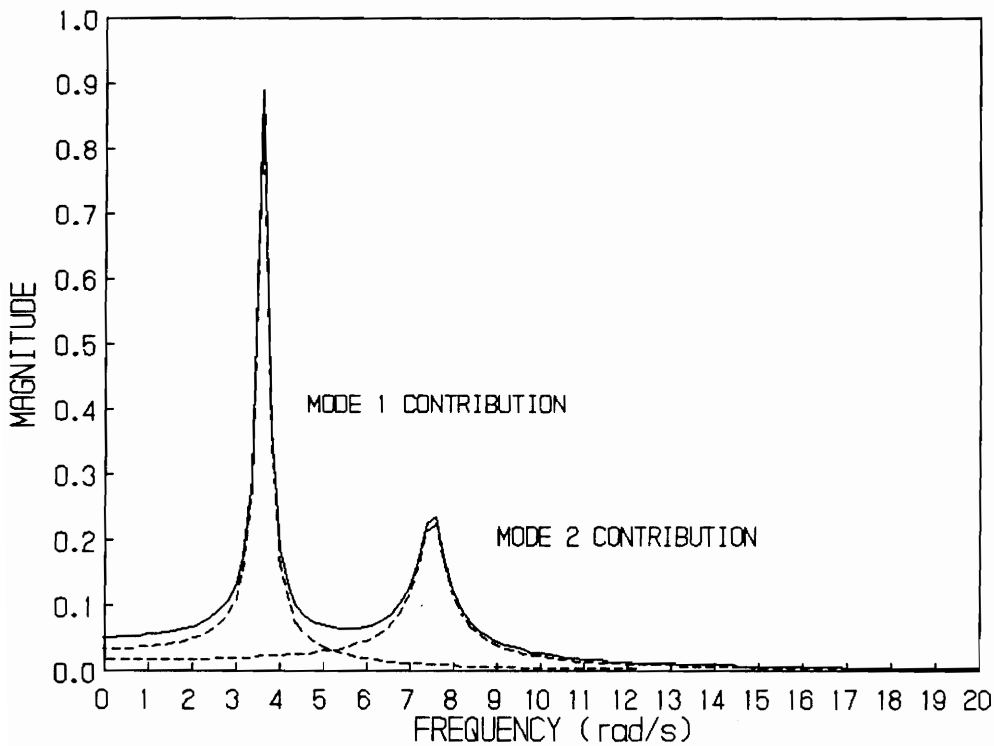


Figure 4-2 Magnitude of the FRF, X_1 / F, for Mass 1

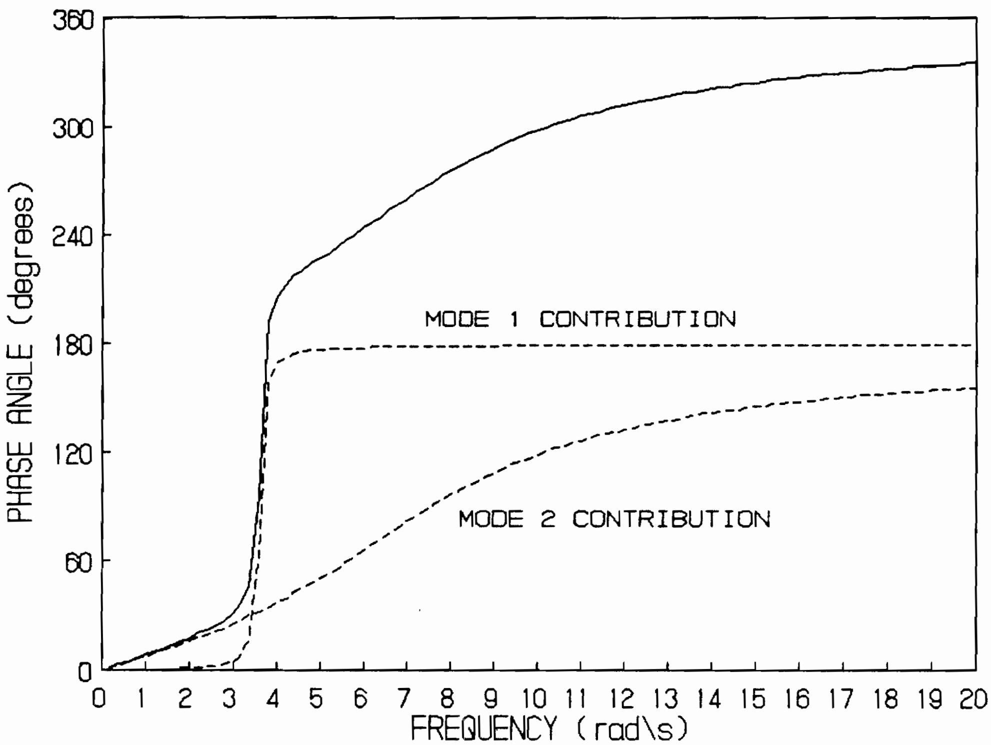


Figure 4-3 Phase Angle of the FRF for Mass 1

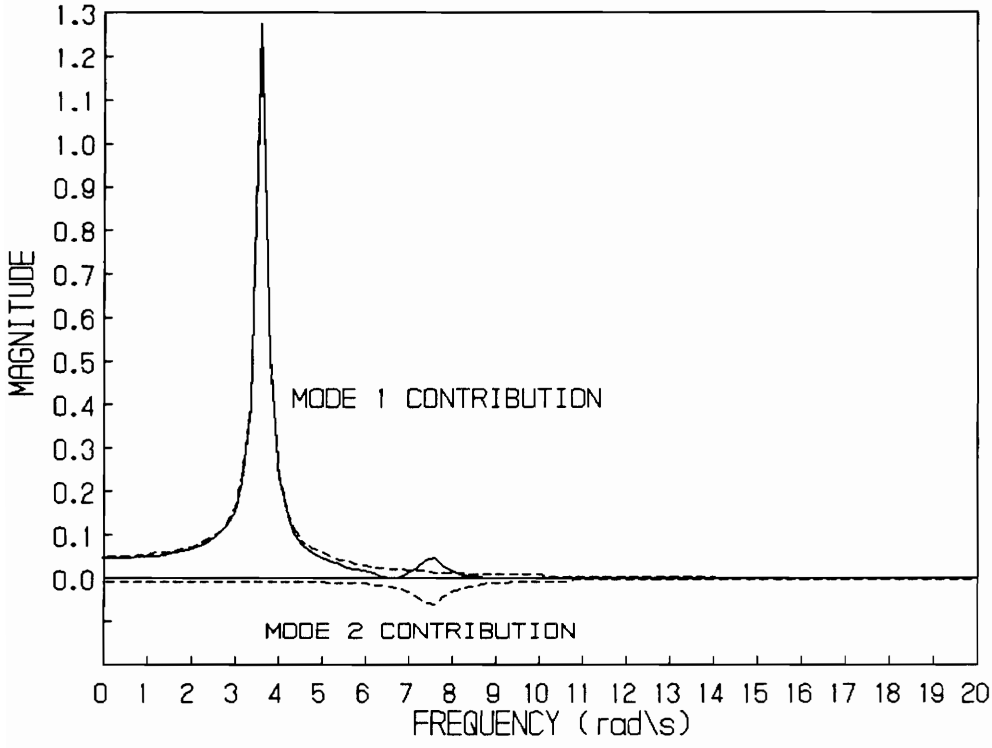


Figure 4-4 Magnitude of the FRF, X_2 / F, for Mass 2

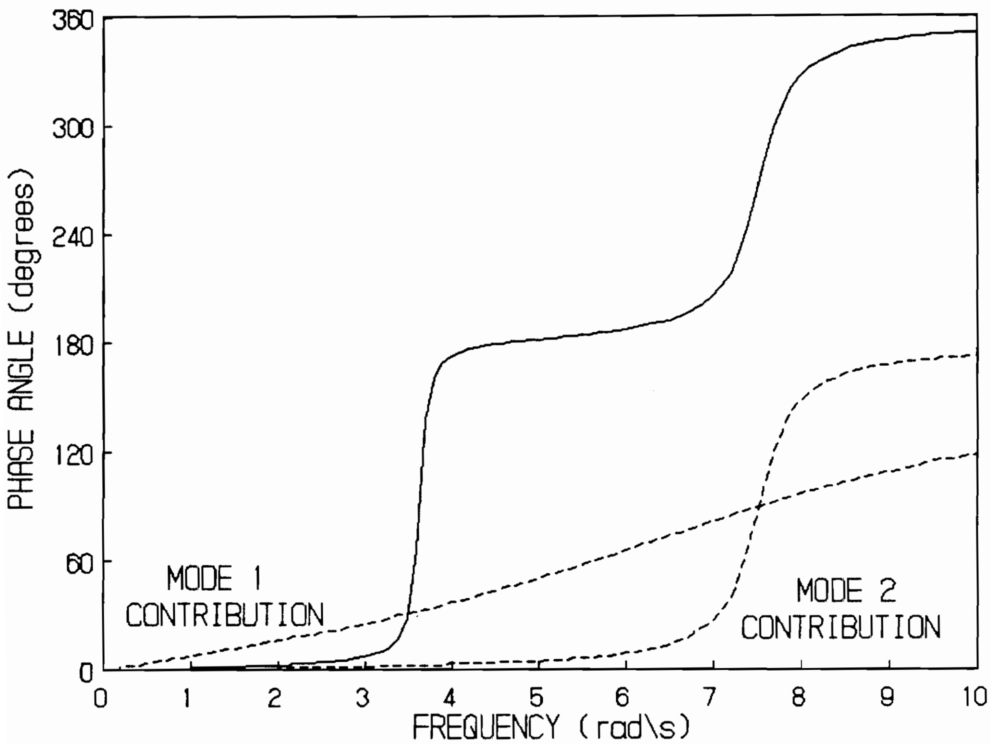


Figure 4-5 Phase Angle of the FRF for Mass 2

the forcing function and k denote the z-coordinate or the degree of freedom in modal space, the general form of the frequency response function is given by:

$$\frac{X_i}{F_j} = \sum_{k=1}^{N} \frac{\Phi_{jk}\Phi_{ik}}{-\omega^2 + \omega_{nk}^2 + 2i\zeta_k \omega\omega_{nk}} \tag{4.34}$$

This result shows the contribution of each of the normal modes to the system response, with the FRF being the summation of each of the single degree of freedom responses for each of the individual forcing function.

The assumption of proportional damping resulted in a diagonaliized form for the damping, mass and stiffness matrices. This resulted in the uncoupling of the component equations and these uncoupled equations can be solved independently. The system FRF is then made up of a linear combination of the normal mode solutions. Since the proportional damping assumption may not be universally applicable, it is necessary to introduce a more general model form, which will relax this assumption while not significantly increasing the complexity of the solution.

4.3 The State Space Model

The proportionally damped model results in a powerful tool for formulating the frequency response function of a multi-degree of freedom system. Physically, however, the assumption of proportional damping may be questionable. The state space model will be introduced to relax this assumption and create a more broadly applicable model.

Combining the original matrix form of the equations of motion, Eq. 4.2:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F\} \tag{4.2}$$

with the identity:

$$[M]\{\dot{x}\}-[M]\{\dot{x}\}=0 \tag{4.35}$$

gives a new matrix equation:

$$\begin{bmatrix}[0] & [M]\\ [M] & [C]\end{bmatrix}\begin{Bmatrix}\{\ddot{x}\}\\ \{\dot{x}\}\end{Bmatrix}+\begin{bmatrix}[-M] & [0]\\ [0] & [K]\end{bmatrix}\begin{Bmatrix}\{\dot{x}\}\\ \{x\}\end{Bmatrix}=\begin{Bmatrix}\{0\}\\ \{F\}\end{Bmatrix} \tag{4.36}$$

Equation 4.36 contains two 2N x 2N square matrices, where N is the number of degrees of freedom of the modeled system.

It is convenient to define a new variable called a state space variable to replace the vector combining the displacement and velocity terms. This new variable is defined as:

$$\{q\}=\begin{Bmatrix}\{\dot{x}\}\\ \{x\}\end{Bmatrix} \tag{4.37}$$

Using this new variable and designating the two 2N x 2N matrices as [A] and [B] respectively, yields a reduced order form of the matrix equation, given by:

$$[A]\{\dot{q}\}+[B]\{q\}=\{f\} \quad \text{where } \{f\}=\begin{Bmatrix}\{0\}\\ \{F\}\end{Bmatrix} \tag{4.38}$$

Consider the homogeneous solution of this first order matrix equation by letting {f}=0. Assuming the solution form q(t)= $e^{\lambda t}$ yields:

$$\lambda[A]\{q\}+[B]\{q\}=0 \tag{4.39}$$

Forming the eigenvalue problem, by extracting the {q} term, yields:

$$[\lambda[A]+[B]]\{q\}=0 \tag{4.40}$$

The solution of the eigenvalue problem yields a set of 2N eigenvalues. In general the eigenvalues will be complex and occur in conjugate pairs. The eigenvectors resulting from these complex eigenvalues will also be complex. As in the proportional damping case, these eigenvectors will be orthogonal with respect to both [A] and [B]. Recall from Section 4.2, that this orthogonality condition can be used to diagonalize the two matrices [A] and [B] [10].

The eigenvalues obtained from the homogeneous solution yield useful information regarding the system parameters. The complex eigenvalues are found to be of the form:

$$\lambda=-\zeta\omega_n \pm i\omega_n\sqrt{\zeta^2-1} \tag{4.41}$$

For this case of complex valued eigenvalues which are complex conjugates, the natural frequencies and damping ratios can be found by considering their real and imaginary parts, and solving for the corresponding natural frequencies and damping ratios.

The particular solution of Eq. 4.38 will yield the forced response. Assuming harmonic forcing $\{F(t)\}=\{F\}e^{i\omega t}$ and utilizing a transformation, similar to that relating physical space to modal space in the proportionally damped discussion, then $\{q\} = [\Psi]\{Z\}$, where $[\Psi]$ is the modal matrix. Substituting into Eq. 4.38 yields:

$$i\omega[A][\Psi]\{Z\}+[B][\Psi]\{Z\}=\{F\} \tag{4.42}$$

As before, the modal matrix is the matrix of the eigenvectors. Premultiplying by the transpose of the modal matrix yields:

$$i\omega[\Psi]^T[A][\Psi]\{Z\}+[\Psi]^T[B][\Psi]\{Z\}=\{F\} \tag{4.43}$$

The orthogonality property of the modal matrix diagonalizes [A] and [B], and using the diagonalized matrices Eq. 4.42 can be rewritten as:

$$\left[i\omega\begin{bmatrix}\ddots & & \\ & a & \\ & & \ddots\end{bmatrix}+\begin{bmatrix}\ddots & & \\ & b & \\ & & \ddots\end{bmatrix}\right]\{Z\}=[\Psi]^T\{F\} \tag{4.44}$$

Forming Eq. 4.44 in terms of the modal coordinate, yields:

$$\{Z\} = \frac{[\Psi]^T \{F\}}{i\omega[a]+[b]} \tag{4.45}$$

Recalling that $\{q\}=[\Psi]\{z\}$ and substituting into Eq. 4.45 yields:

$$\{q\} = \frac{[\Psi]^T \{F\}[\Psi]}{i\omega[a]+[b]} \tag{4.46}$$

Recalling the eigenvalue solution from Eq. 4.40,

$$[B]+\lambda[A]=0 \tag{4.47}$$

it is possible to write a relationship between the two diagonalized matrices as:

$$[b]=-\lambda[a] \tag{4.48}$$

Substituting Eq. 4.48 into Eq. 4.47, makes use of the eigenvalue solution, λ in the response function, yielding:

$$\{q\} = \frac{[\Psi]^T \{F\}[\Psi]}{[a](i\omega - \lambda)} \tag{4.49}$$

Using the eigenvalue solution in Eq. 4.42 and substituting it into Eq. 4.49, yields:

$$\{q\} = \frac{[\Psi]^T \{F\}[\Psi]}{[a]\left(i\omega + \zeta\omega_n \mp i\omega_n\sqrt{1-\zeta^2}\right)} \tag{4.50}$$

Maintaining the subscript notation for each of the physical spatial coordinates, the general form of the complexed valued frequency response function in the original physical space may be written as a summation of the normal modes given by:

$$\frac{X_i}{F_j} = \sum_{k=1}^{N}\left[\frac{\Psi_{jk}\Psi_{ik}}{a_k\left(i\omega + \zeta_k\omega_{nk} - i\omega_{nk}\sqrt{1-\zeta_k^2}\right)} + \frac{\Psi^*{}_{jk}\Psi^*{}_{ik}}{a^*{}_k\left(i\omega + \zeta_k\omega_{nk} + i\omega_{nk}\sqrt{1-\zeta_k^2}\right)}\right] \tag{4.51}$$

The two terms of the FRF represent the eigenvalues occurring in conjugate pairs. This result is slightly different from that seen in the proportional damping model, but is effectively the same since the FRF has been reduced to a summation of mode contributions. The relaxation of the proportional damping, assumption provides results that are more broadly applicable. However, the resulting FRF for the state space model is somewhat more complicated in form and application.

Modal analysis is a relatively recent addition to the field of vibrations. It is most useful in that it utilizes the lumped parameter approach of modeling to form a frequency response function as a summation of single degree of freedom modal contributions in order to predict system response. The illustration of these approaches is important in understanding what such models will yield in terms of results.

Chapter 5 Fluid Systems

The purpose of this research is to adopt the previously reviewed methods for the analysis of the time response of mechanical systems to fluid flow systems. The development will follow that presented for mechanical systems, including the application of the modal method.

5.1 System Parameters

As with the mechanical systems, the building blocks of the fluid flow model are the fluid parameters; fluid inertance, fluid compliance and fluid damping. These terms are analogous to those of the mechanical model, and are based on the physical laws governing fluid motion. Where the constant parameter mechanical element definitions were based on an assumption of small deflections, the fluid parameters proposed here are derived around the assumption that the unsteady behavior occurs as small perturbations about some quasi-steady operating point.

5.1.1 Fluid Inertance

Fluids, both liquids and gases, have the property of mass. For fluids in motion the mass is not as easily quantifiable as with the mechanical system. Fluid flow problems are usually analyzed from a control volume view rather than as a clearly identifiable mass. It is not convenient to label a particular quantity of mass and analyze it, as the identity of the mass within a control volume will vary in time. A more convenient method of describing the mass is to consider a particular location within the flow system and consider the flow characteristics for the mass flowing through this location (a local control volume type of analysis).

As an example, consider flow in a pipe. For a pipe of constant area, A, and length, L, at an instant in time, the mass of the fluid in the section is given by $m=\rho AL$. Using this construct, it is possible to formulate lumped masses within the fluid flow system, with various lumped masses representing various flow quantities distributed throughout and making up the entire flow system. As in mechanical systems, each of the lumped masses are assumed to exhibit rigid body motion. Thus, the fluid lump is assumed to have a uniform velocity profile. This uniform flow assumption may require a corrective factor. For example, in laminar, fully developed, steady pipe flows, the actual velocity profile is parabolic. To account for this difference in velocity profiles, the momentum flux correction factor is used. For low frequency perturbations in laminar flows, the momentum flux correction factor is 4/3 the ρAL value. For turbulent and high frequency laminar flows, the velocity profile effect is not as significant and the original uncorrected term is usually used.

The fluid inertance is defined using this term. Given a lump of fluid, motion of the lump is caused by a pressure force difference across it. Using Newton's 2^{nd} law this motion is described by:

$$F = A\Delta P = M\frac{dv}{dt} \tag{5.1}$$

Assuming that the flow is incompressible, the velocity can be written as:

$$v = \frac{Q}{A} \tag{5.2}$$

where Q is the volume flow rate. Using this velocity term and the equivalent mass, the following can be written:

$$A\Delta P = \rho AL\left(\frac{1}{A}\frac{dQ}{dt}\right) \tag{5.3}$$

$$\Delta P = \frac{\rho L}{A}\frac{dQ}{dt} \tag{5.4}$$

The ρ L/A term is defined as the fluid inertance, I. Thus utilizing Newton's law on the fluid lump, the dynamic property of fluid inertance relates two important flow properties, pressure and flow rate.

5.1.2 Fluid Compliance

The compliance parameter for the mechanical system was defined as the inverse of the stiffness, k. Physically, this is related to the amount of "give" or elasticity within the system. For fluid systems, the compliance is related to the amount compressibility demonstrated by the fluid. Real fluids, including liquids, will demonstrate some amount of compressibility. This compressibility will allow for mass storage in the control volume defining the fluid lump. The compliance of fluids can be represented by the isothermal compressibility of the fluid, or its reciprocal, the isothermal bulk modulus. Experimentally, the bulk modulus can be estimated by taking a volume of fluid V, applying an incremental pressure ΔP to it and measuring the resulting change in volume ΔV. Thus the experimental definition of the bulk modulus is given by:

$$B = -\frac{\Delta P}{\left(\Delta V / V\right)}\Bigg|_T \tag{5.5}$$

Using this definition and the conservation of mass law, one can derive a term for the fluid compliance parameter. In this formulation of the fluid compliance, the density of the fluid is considered to be nominally constant and the mass storage within the control volume occurs because of the difference in inlet and outlet flow rates. Writing the law of mass conservation, and including the mass storage term yields:

$$\dot{m}_{in} - \dot{m}_{out} = \frac{\Delta M}{\Delta t} \tag{5.6}$$

Introducing the bulk modulus and manipulating the terms, one can define the fluid compliance, C, as

$$(\rho A v)_{in} - (\rho A v)_{out} = \rho \frac{\Delta V}{\Delta t} \tag{5.7}$$

$$Q_{in} - Q_{out} = \frac{\Delta V}{\Delta t} \tag{5.8}$$

$$Q_{in} - Q_{out} = -\frac{V}{B}\frac{\Delta P}{\Delta t} \tag{5.9}$$

$$Q_{out} - Q_{in} = \frac{AL}{B}\frac{\Delta P}{\Delta t} \tag{5.10}$$

$$\Delta Q = \frac{AL}{B}\frac{\Delta P}{\Delta t} \tag{5.11}$$

$$\Delta P = \frac{B}{AL}\Delta Q \Delta t \tag{5.12}$$

The coefficient term, B/AL, defines the inverse of the fluid compliance, C, therefore:

$$C = \frac{AL}{B} = \frac{V}{B} = \frac{B\Delta V}{\Delta P}\left(\frac{1}{B}\right) = \frac{\Delta V}{\Delta P} \tag{5.13}$$

In addition to the compliance of the fluid itself, the device through which the fluid flows may also introduce some amount of compliance. Devices such as pipes or tubing may expand with increases in the fluid pressure, thus introducing compliance even with truly incompressible flows. The amount of compliance introduced by the elastic deformation of the physical device is dictated by the geometry and the material properties of the device.

The contributions to the compliance of the fluid and the physical device in which it is contained can be combined to form a single compliance term describing the fluid lump, given by the volume change for an applied pressure, as in Eq. 5.13:

$$C = \frac{\Delta V}{\Delta P} \tag{5.14}$$

As an application of this definition, consider a tank of volume, V, containing a gas at pressure, P_0 , [3]. Using the ideal gas law, one can write:

$$P_0 V = MRT \tag{5.15}$$

Assuming isothermal changes and nominally constant density, one can introduce an incremental amount of additional mass, $dM = \rho\, dV$, into the tank. This change in mass will cause a change in pressure, dP, given by:

$$dP = \frac{RT}{V} dM \tag{5.16}$$

Substituting for the incremental mass term yields:

$$dP = \frac{RT}{V} \rho dV \tag{5.17}$$

Substituting for the density with the ideal gas law, yields:

$$dP = \frac{RT}{V} \frac{P_o}{RT} dV = \frac{P_o}{V} dV \tag{5.18}$$

Using the definition for the compliance parameter from Eq. 5.13, one can write:

$$C = \frac{\Delta V}{\Delta P} = \left.\frac{dV}{dP}\right|_{P=P_o} = \frac{V}{P_0} \tag{5.19}$$

This compliance term will be useful for small changes around some operating pressure, P_0, where perturbations are assumed to occur approximately isothermally.

For more rapid, yet still small, changes in pressure the isothermal assumption may be replaced by an adiabatic assumption. If the small changes are further assumed to be reversible, one can write the isentropic process relation

$$P V^k = \text{constant} \tag{5.20}$$

where k is the ratio of specific heats. Differentiating this term yields:

$$\frac{dP}{P} + k\frac{dV}{V} = 0 \tag{5.21}$$

Omitting the sign term to consider only magnitudes, yields a compliance parameter, C, for reversible adiabatic changes about a steady mean pressure P_0, as

$$C = \frac{\Delta V}{\Delta P} = \frac{dV}{dP} = \frac{V}{kP_o} \tag{5.22}$$

The compliance term is used as an approximation for fluid behavior just as in the mechanical system where the stiffness term is used as an approximation for elastic behavior. As with the mechanical stiffness parameter, the model may only be valid only around a limited operating range, yet still be used in the lumped parameter model to yield accurate results.

5.1.3 Fluid Resistance

In the mechanical model, the damping element was introduced as a means of energy dissipation. For vibration reduction, damping is a desired property. In fluid systems, energy dissipation is more often than not a parasitic effect which exists in real flows. Fluid resistance is introduced to provide a means of modeling this dissipative property.

Physically, resistance in a flow system usually occurs as a drop in pressure as the fluid flows through the device. This pressure drop is due to viscous shear or friction in straight ducts, bends, elbows, valves, etc. For simplicity, consider only the losses due to frictional effects in straight sections. Without considering the physical details and mechanisms of the friction losses, a general form of the resistance property is given by:

$$R = \frac{\Delta P}{Q} \tag{5.23}$$

In order to quantify the fluid resistance parameter additional information concerning the flow must be known. For steady flows, the inertance and compliance properties will not manifest themselves as they are defined only for unsteady flows. However, a steady flow solution can be used to estimate a fluid resistance parameter. Analytic solutions exist for steady flows in various geometries, relating the flow rate and pressure drop due to viscous friction. Different relations exist for laminar and turbulent flow regimes, thus the nature of the flow, determined by the Reynolds number, is important.

As an example of how the steady flow solution for the pressure drop can be used to form a resistance parameter, consider the Hagen-Poiseuille solution for laminar fully-developed flow in a circular pipe of diameter D and length L [13]:

$$Q = \left(\frac{\pi D^4}{128\ \mu L}\right)\Delta P \tag{5.24}$$

Forming the resistance parameter yields:

$$R = \frac{\Delta P}{Q} = \frac{128\,\mu L}{\pi D^4} \tag{5.25}$$

This resistance parameter, based on a steady flow solution, is utilized in an unsteady model.

Analytical solutions are known and provide accurate results for simple geometries. For more complex devices a resistance parameter may be estimated by experimentally measuring the pressure drop and flow rate for a steady flow.

The dynamic fluid parameters, inertance, compliance and resistance have been formulated and are analogous to the mass, stiffness and damping for a mechanical model. As with the mechanical parameters, each of the fluid parameters are considered constant. For the fluid model, the relations in which these dynamic fluid parameters are used are assumed to be linear over a small range around a steady operating condition.

5.2 Simple Fluid System Models

As an introduction to the use of the dynamic fluid parameters in forming a linear lumped parameter model, consider an example utilizing two of these three parameters. This example of an oscillating flow will demonstrate how these elements are utilized in a model. Analysis will show the type of results to be expected from such approaches. A second example, the Helmholtz resonator, will use all three of these parameters to provide the basis for the analysis of more complex flow devices.

5.2.1 Oscillating Flow System

Consider a flow device consisting of a rigid length of pipe with tanks at both ends connected by some pressure regulating device as in Fig. 5-1 [3]. The pipe is assumed rigid and will not introduce any compliance. The assumption of an absolutely incompressible fluid yields a model of the fluid in the pipe based on only the inertance and resistance properties. The fluid lump is taken to be all the fluid in the pipe. For the equation of motion for the fluid lump, the forces will include the pressure difference across the pipe length and the friction force effect expressed in terms of the resistance property. The force sum is equated to the acceleration of the fluid lump, with:

$$\Delta P - RQ = I\frac{dQ}{dt} \tag{5.26}$$

If the flow is assumed to be laminar, with low frequency oscillations, the inertance can be described by the effective mass term (4/3 the physical mass as noted in Section 5.1.1) and the resistance will be assumed to be given by the Hagen-Pouiseuille result. Substituting these terms into the equation of motion yields:

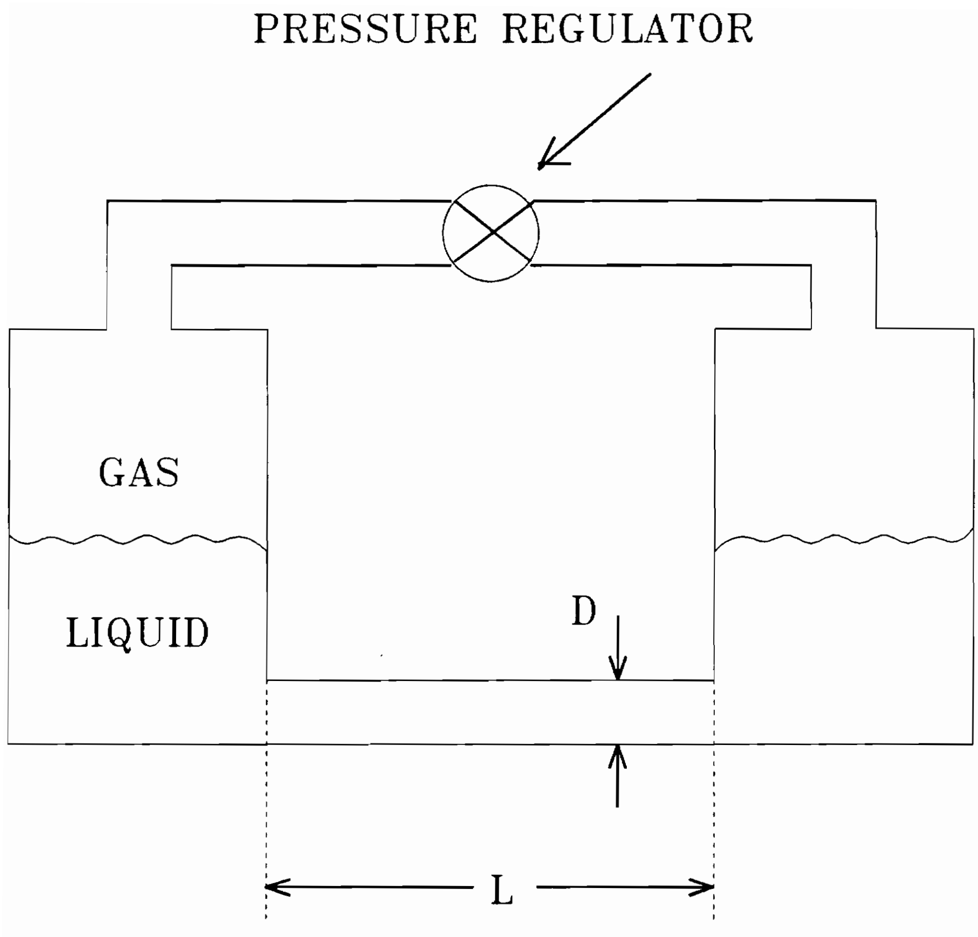


Figure 5-1 Physical Device for Oscillating Flow System

$$\frac{4}{3}\frac{\rho L}{A}\frac{dQ}{dt} + \frac{128\mu L}{\pi D^4}Q = \Delta P(t) \tag{5.27}$$

This is a first order, ordinary linear differential equation, where the pressure difference is analogous to the forcing function of the mechanical model.

The solution of this equation will yield a function of the flow rate with respect to time for a given variation in the pressure. Forming the complimentary solution by setting the pressure term to zero and assuming an exponential solution as in the mechanical system given in equation Eq. 3.7, yields:

$$(Is + R)e^{st} = 0 \tag{5.29}$$

Solving for s yields:

$$s = -\frac{R}{I} \tag{5.30}$$

Returning to the assumed solution and using the solution of Eq. 5.30 yields:

$$Q(t) = C_1 e^{\left(-\frac{R}{I}\right)t} \tag{5.31}$$

Since Eq. 5.28 is a first order equation, there is only a single unknown coefficient. Unlike the second order mechanical equation, there is no possibility of an imaginary exponent, and thus no oscillatory transient response. The entire solution would also require the particular solution, which would follow from the particular form of the pressure function or input, together with the boundary conditions on the system.

A simple formulation of the frequency response function can be obtained by assuming that the pressure regulator provides a harmonic pressure function across the length of pipe given by:

$$P(t) = \Delta P\, e^{i\omega t} \tag{5.32}$$

Writing the particular form of Eq. 5.28 using the harmonic pressure function and the assumed harmonic response for the flow rate yields:

$$(I\, i\omega + R)\, Qe^{i\omega t} = \Delta P\, e^{i\omega t} \tag{5.33}$$

Forming the frequency response function as the ratio of the output Q to the input P results in:

$$\frac{Q(\omega)}{P(\omega)} = \frac{1}{I\, i\omega + R} \tag{5.34}$$

The magnitude and phase of this complex valued function are given by:

$$\text{Magnitude:} \quad \left|\frac{Q}{P}\right| = \frac{1}{\sqrt{(I\omega)^2 + R^2}} \tag{5.35}$$

$$\text{Phase:} \quad \Phi = \tan^{-1}\left(\frac{-I\omega}{R}\right) \tag{5.36}$$

In order to quantify these terms, consider a numerical example [3]. Assume that the fluid is an oil with a density of 7.95 x 10E-5 lbf s^2/in^4 and a viscosity of 4.0 x 10E-6 lbf s/in^2 . The pipe geometry is assumed to be 20 in. in length and .05 in. in diameter. The resistance coefficient, R, can be found to be 522 psi/(in^3 /s), and for the assumed effective mass the inertance given by I = 1.079 lbf s^2/in^5. For the actual mass I = 0.8093 lbf s^2/in^5.

Plotting the magnitude and phase of the FRF in Figs. 5-2 and 5-3 demonstrates the trends of a general first order linear system. Unlike the second order mechanical system shown previously in Figs. 3-4 and 3-5, this system will not demonstrate resonance. The assumption of no compliance has formed a fluid system equivalent to an overdamped mechanical system. Both the magnitude and phase plots show a gradual decreasing trend at relatively low frequencies and exhibit asymptotic behavior as frequency increases, with the magnitude going to zero and the phase to -90 degrees.

As discussed in section 5.1.1, the value of the inertance parameter is dependent on the velocity profile correction assumed for the flow. The effect of the inertance value on the frequency response function is demonstrated by plotting the FRF for both the corrected and uncorrected inertance terms in Figs. 5-2 and 5-3. It is apparent that the choice of the correction factor for the inertance value has a significant effect. Therefore, to avoid errors, it is important to understand the nature of the system and make a reasonable estimate of the correction factor for the inertance term.

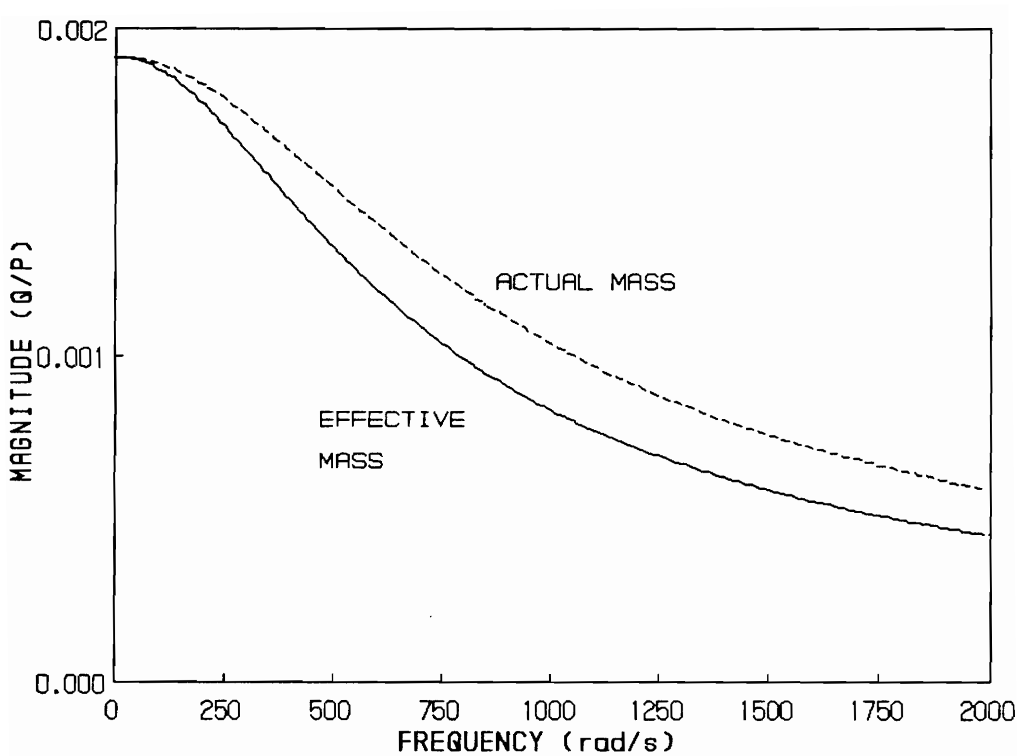


Figure 5-2 Magnitude of the FRF for Oscillating Flow System

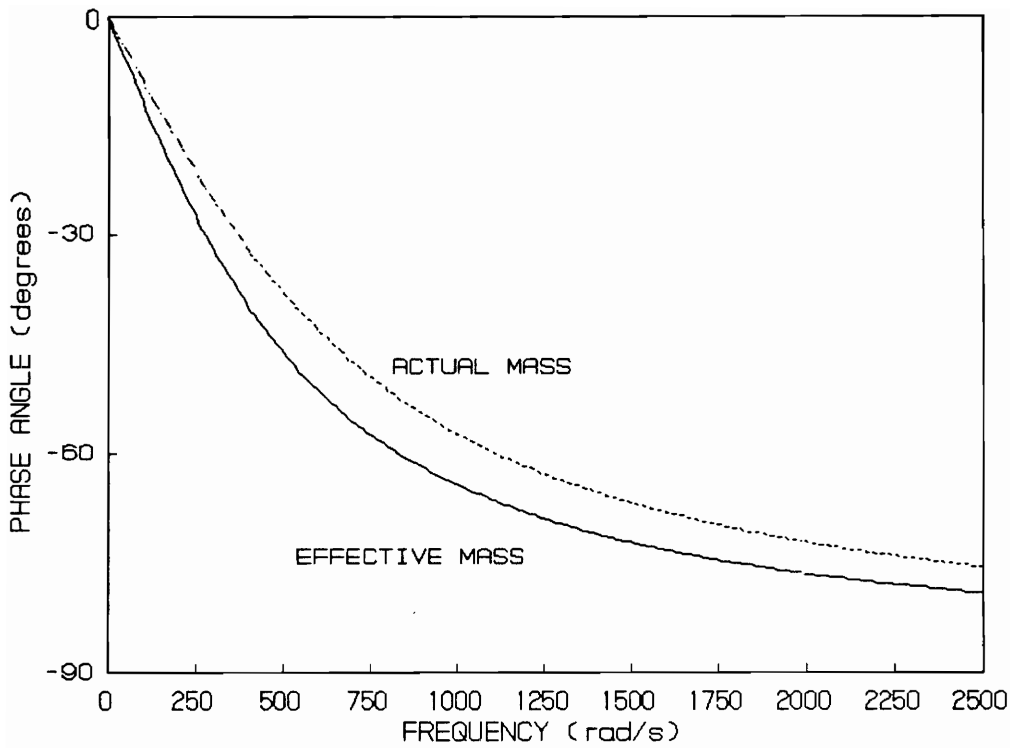


Figure 5-3 Phase Angle of the FRF for Oscillating Flow System

5.2.2 The Helmholtz Resonator

As an example of a second order fluid system modeled by a linear lumped parameter technique utilizing each of the three fluid parameters, consider the Helmholtz resonator [3]. This device, shown in Fig. 5-4, consists of a pipe attached to a closed tank. The pipe and tank walls are assumed to be rigid and the volume of the tank is large in comparison to the pipe. Hence the compliance effects may be assumed to be only those due to the fluid within the tank volume. The fluid lump which will introduce inertance and resistance to the system is characterized by the fluid in the pipe length.

5.2.2.1 Governing Equation Derivation

To derive the equation relating these fluid properties, assume that the pressure into the tube and within the tank are steady at some value P_m. The unsteady behavior of interest is introduced by some perturbation in the pressure at the pipe inlet, P_i. This perturbation is assumed to be small relative to the steady pressure value. From the first order model the resistance and inertance elements can be related by:

$$\Delta P(t) - RQ(t) = I\frac{dQ}{dt} \tag{5.37}$$

The pressure difference is given by the inlet pressure perturbation $P_i(t)$, and the response of the pressure within the tank $P_o(t)$, where P_i and P_o are the pressure differences between the inlet and outlet perturbations and the initial steady pressure P_m, i.e., $P_i(t) = P_{inlet} - P_m$. Introducing the compliance of the fluid within the tank, the flow into the tank and the pressure within the tank can be related by:

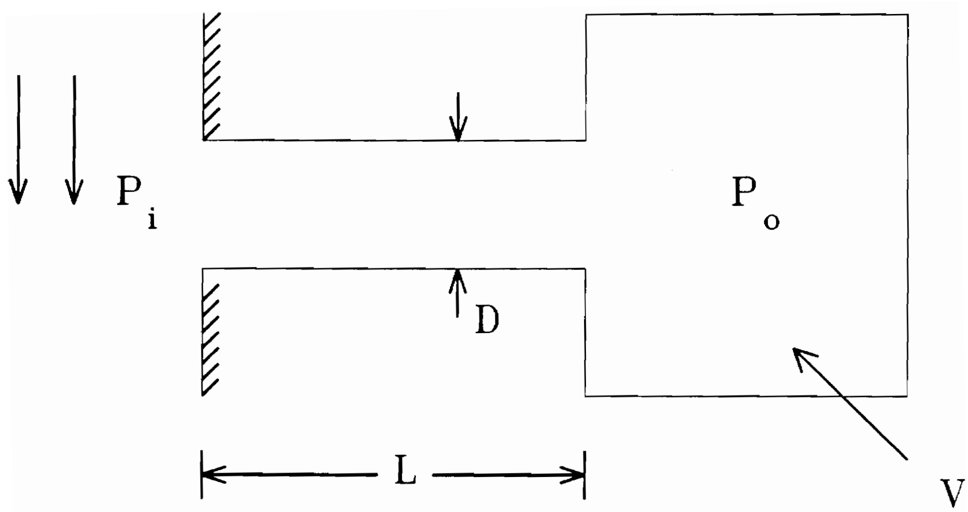


Figure 5-4 Helmholtz Resonator

$$\int Q(t)\, dt = C P_o \tag{5.38}$$

Differentiating to write this as a function of flow rate yields:

$$Q(t) = C \frac{dP_o}{dt} \tag{5.39}$$

Using Eq. 5.39 for the flow rate and substituting into Eq. 5.37, yields a second order ordinary linear equation relating the inlet pressure perturbation to the resulting pressure within the tank. This relation is given by:

$$P_i = P_o + RC \frac{dP_o}{dt} + IC \frac{d^2 P_o}{dt^2} \tag{5.40}$$

This equation is very similar to that derived for the mechanical system that was analyzed in section 3.2. The same analysis will be applied to this system to derive an analogous impulse response function and frequency response function and to investigate the results that the lumped parameter model will yield.

It is important to review the assumptions made in forming Eq. 5.40 . The compliance of the fluid in the tank is dependent upon the pressure within the tank, which is varying in time. For this analysis, it is assumed that (1) the value of the compliance, C, can be found by using the initial steady pressure, P_m, and thereafter assumed to be constant. With the variation of the pressure in time, pressure dependent quantities such as density will also vary, introducing a non-linearity into Eq. 5.40. To avoid this non-linearity, (2) the density, and other such quantities are found for the steady flow case, at the steady flow pressure, P_m and are assumed to be constant. Lastly, (3) temperature changes within the fluid are neglected, as these changes also introduce a non-linearity into Eq. 5.40.

5.2.2.2 Time Response

The solution of Eq. 5.40 results in an expression for the tank pressure as a function of time. The complimentary portion of the solution is found by setting the input pressure disturbance equal to zero, and assuming an exponential solution form. With the assumed solution $P_i(t) = P\, e^{st}$ and the first and second time derivatives, substituted back into Eq. 5.40, one has:

$$(IC\, s^2 + RC\, s + 1)\, e^{st} = 0 \tag{5.41}$$

Solving for the roots of the term in the parenthesis, the unknown, s, is given by:

$$s = -\frac{R}{2I} \pm \sqrt{\frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{1}{CI}} \tag{5.42}$$

Substituting this value for s back into the assumed general form of the complimentary solution yields:

$$P_o(t) = e^{\left(-\frac{R}{2I}\right)t}\left[C_1 e^{\sqrt{\frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{1}{CI}}\, t} + C_2 e^{-\sqrt{\frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{1}{CI}}\, t}\right] \tag{5.43}$$

This complimentary solution yields the transient response of the system. For the mechanical system analysis the transient system response was shown to be an important characteristic of the system, dependent on the values of the system parameters, I, C, and R.

From this transient response, one defines two parameters analogous to the natural frequency and damping ratio of the mechanical model, but here based on the fluid parameters. First, assuming that the flow is not acted on by friction, the resistance term is assumed to be zero. For this case, the exponents on the exponential terms in Eq. 5.43 become imaginary, and thus can be thought of as harmonic components with a frequency given by $(1/CI)^{1/2}$. This term will be considered the natural frequency of the fluid flow system:

$$\omega_{n_{fluid}} = \sqrt{\frac{1}{CI}} \tag{5.44}$$

Next, consider the case where the radicand of the exponents is equal to zero, as with the critical damped case of the mechanical model. Setting the terms of the radicand equal, $1/4\ (R/I)^2 = 1/CI$ and solving for the resistance yields what can be referred to as a critical resistance for the fluid flow system and defined as:

$$R_{critical} = 2\sqrt{\frac{I}{C}} \tag{5.45}$$

The ratio of this critical value to the actual system resistance term, a fluid resistance ratio analogous the damping ratio can be defined as:

$$\zeta_{fluid} = \frac{R}{R_{critical}} = \frac{R}{2\sqrt{I/C}} \tag{5.46}$$

Each of the analogous fluid terms for the natural frequency and damping ratio serve to emphasize important system characteristics. In formulating analytical models these terms reduce the unknown coefficients from three (resistance, inertance and compliance) to the two terms, combining the parameter values, and yet still yielding useful information.

5.2.2.3 Impulse Response

As with the mechanical system analysis, the transient solution of the linear differential equation for the fluid model can be used to determine the fluid system response to an impulsive pressure perturbation. Using the mathematical representation of the impulse for the mechanical system, and dividing through by the constant area, A, a similar term for the pressure perturbation impulse is given by:

$$\frac{\hat{F}}{A} = \int \frac{F(t)}{A} dt = \int P(t)\, dt = \hat{P} \tag{5.47}$$

From the definition of the fluid inertance, one can write:

$$P(t)\, dt = I\, dQ \tag{5.48}$$

Substituting this term back into the impulse definition, Eq. 3.14 yields:

$$\hat{P} = \int I \, dQ \tag{5.49}$$

Integrating for the flow rate and substituting the relation between the flow rate and the compliance parameter, from Eq. 5.39, into Eq. 5.49 yields:

$$\frac{\hat{P}}{IC} = \frac{dP_o}{dt} \tag{5.50}$$

This becomes the initial condition on the first derivative of the output pressure for the comparable impulse solution for the fluid system analysis.

Recalling the solution form of the transient response given by:

$$P_o(t) = e^{\left(-\frac{R}{2I}\right)t}\left[C_1 e^{\sqrt{\frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{1}{CI}}\,t} + C_2 e^{-\sqrt{\frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{1}{CI}}\,t}\right] \tag{5.43}$$

the first derivative is given by:

$$\frac{dP_o}{dt} = \left(-\frac{R}{2I} + \sqrt{\frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{1}{CI}}\right)e^{\left(-\frac{R}{2I}\right)t}\left[C_1 e^{\sqrt{\frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{1}{CI}}\,t} + C_2 e^{-\sqrt{\frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{1}{CI}}\,t}\right] \tag{5.51}$$

Applying the two known initial conditions, P_o (t=0)=0 and dP_o/dt (t=0) = $\hat{P}/IC$, the two constants C_1 and C_2 can be found.

Using the values of these constants in the general transient response equation yields:

$$P_o(t) = \hat{P}\left[\frac{1}{2IC\sqrt{\frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{1}{CI}}}\, e^{-\frac{R}{2I}t}\left[e^{\sqrt{\frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{1}{CI}}\,t} - e^{-\sqrt{\frac{1}{4}\left(\frac{R}{I}\right)^2 - \frac{1}{CI}}\,t}\right]\right] \qquad (5.52)$$

As with the mechanical system the bracketed term is defined as the unit impulse response function. Here it is scaled by the magnitude of the pressure perturbation to yield the fluid system response function. Using the previously defined fluid natural frequency and resistance ratio, Eq. 5.44 and 5.46, the unit impulse response function can be written as:

$$h(t) = \frac{1}{2IC\,\omega_n\sqrt{\zeta^2 - 1}}\, e^{-\omega_n \zeta t}\left(e^{\omega_n\sqrt{\zeta^2-1}\,t} - e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right) \qquad (5.53)$$

For the assumed linear constant parameter fluid system and assuming a casual system (as previously defined in section 3.2.2), the principle of convolution is applicable. This principle can be used to represent the input pressure perturbation function as a series of impulses. Likewise, the response of the system can be thought of as the series of unit impulse response functions weighted by the applied pressure function. As seen previously, convolution for this system is represented mathematically by:

$$P_o(t) = \int_0^t h(\tau)\, P_i(t-\tau)\, d\tau \tag{5.54}$$

The principle of convolution is useful for easily integratable functions of the input pressure function. It is also a motivation for the use of the frequency domain to analyze such systems.

5.2.2.4 Frequency Response

The convolution principle, introduced in Section 3.2.2, was shown to be a useful tool in determining the time response functions of linear systems. Convolution allowed the time response of a system to be represented as the superposition of a series of unit impulse response functions, weighted by the input forcing function. The integration involved in utilizing the convolution principle is often difficult or impossible to perform. Therefore, the transformation from the time domain to the frequency domain was performed through the Fourier transform for the mechanical system to alleviate this difficulty.

Performing the Fourier transform on the unit impulse response function derived for the fluid model will yield the frequency response function. This FRF is identical in utility to that obtained for the single degree of freedom mechanical model, and has the same basic form, given by:

$$H(\omega) = \frac{1}{\omega_n^2 - \omega^2 + 2i\,\zeta\omega\omega_n} \tag{5.55}$$

When written using the analogous natural frequency and resistance ratio, the magnitude and phase terms will be the same as those for the mechanical model. The effects of the individual fluid parameters of natural frequency and resistance ratio can be demonstrated using the magnitude and phase expressions written in terms of these originally defined fluid parameters as follows:

$$\text{Magnitude:} \quad |H(\omega)| = \frac{1}{\sqrt{\left(1-\omega^2(CI)\right)^2 + (\omega RC)^2}} \tag{5.56}$$

$$\text{Phase:} \quad \Phi = \tan^{-1}\left(\frac{\omega RC}{1-\omega^2(CI)}\right) \tag{5.57}$$

As an example of the FRF for a fluid system [3], consider air flow through a Helmholtz resonator. The resonator tank volume is 2 in^3. with a 0.2 in. diameter pipe of 3 in. length. Assuming a temperature of 75 F and pressure of 14.7 psia for the mean flow, the density and viscosity are 0.00231 $lbf\text{-}s^2/ft^4$ and 3.8 x10 E-7 $lbf\text{-}s/ft^2$ respectively. Assuming laminar Poiseuille-Hagen flow, one can compute each of the fluid elements values as:

$$I \text{ (based on effective mass)} = \frac{4}{3}\frac{\rho L}{A} = 0.0000142 \text{ lbf-s}^2/\text{in}^5$$

$$I \text{ (based on physical mass)} = \frac{\rho L}{A} = 0.0000107 \text{ lbf-s}^2/\text{in}^5$$

$$R = \frac{128\,\mu L}{\pi D^4} = 0.0000672 \text{ lbf-s/in}^5$$

$$C = \frac{V}{kP_o} = 0.09718 \text{ in}^5/\text{lbf}$$

Using these values, plots of the FRF magnitude and phase are shown in Figs. 5-5 and 5-6. As seen in the magnitude plot, this second order system demonstrates resonance. The frequency at which this resonance occurs and the magnitude of the peak is dependent upon the choice of inertance modeling. This further demonstrates the importance of the choice of the inertance model assumed for this analytical model.

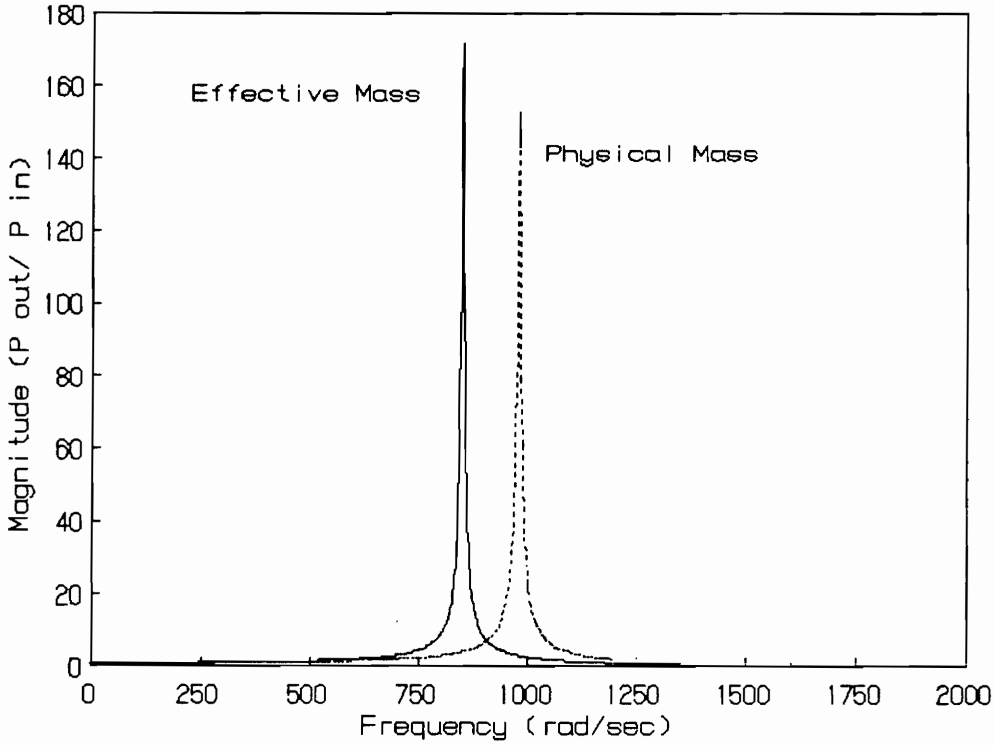


Figure 5-5 Magnitude of the FRF for the Helmholtz Model

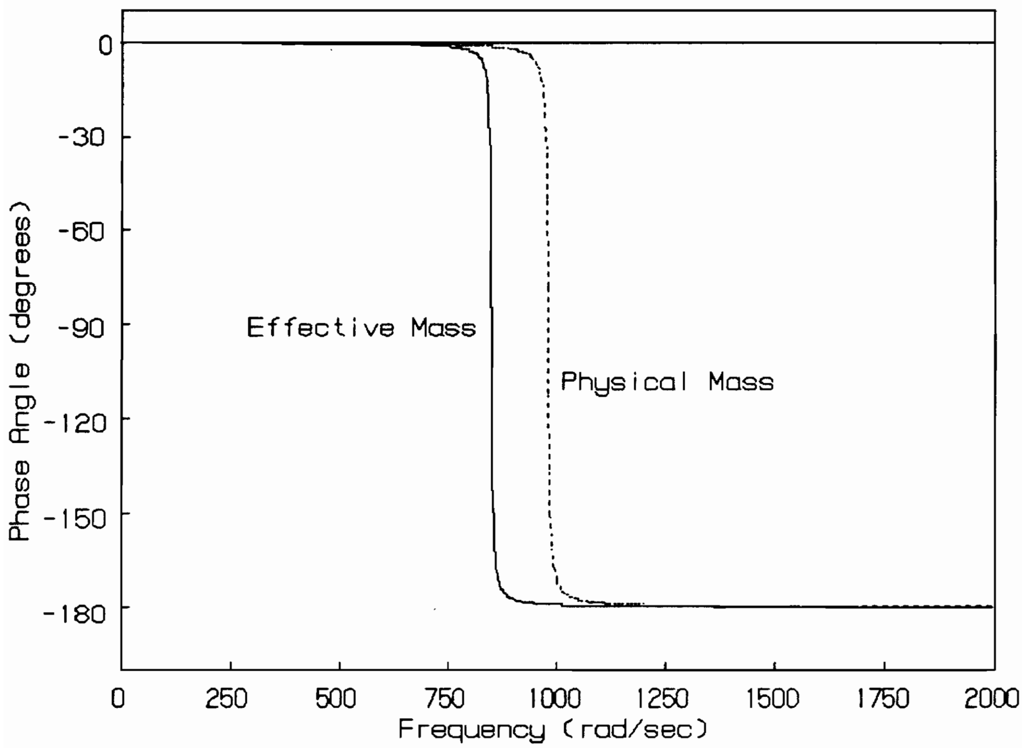


Figure 5-6 Phase Angle of the FRF for the Helmholtz Model

Chapter 6 Multiple Degree of Freedom Fluid Systems

Introducing the lumped fluid parameters of inertance, resistance and compliance allows for the analysis of more complex fluid systems using the more advanced mechanical methods of modal analysis analogous to their application in mechanical system analysis. The discussion of the application of the lumped parameter approach to more complex fluid systems will be accomplished through the use of an example. The fluid system to be used a the compressor/duct/plenum system presented by Greitzer [14].

The analysis by Greitzer is based on a set of non-dimensional, non-linear equations used to predict the dynamic response of a compressor and which he suggests can be used to predict the onset of either surge or rotating stall. The goal of the analysis here is not as ambitious as to utilize the lumped parameter technique to predict surge or stalled behavior. The goal is to take the fluid equations describing this complex system and form a lumped parameter model. From the lumped parameter model, this example will yield some insight into the transient behavior of the fluid system as determined in a manner analogous to that used in mechanical system analysis.

6.1 Equations of Motion

The system, presented by Greitzer [14], to be considered is shown in Fig. 6-1. It consists of a compressor connected to a duct of length L_c and of area A_c , feeding to a plenum of volume V_p. From the plenum there is an exit flow through a throttle of length L_t and area A_t. Both the compressor inlet and throttle exit are at an ambient pressure P.

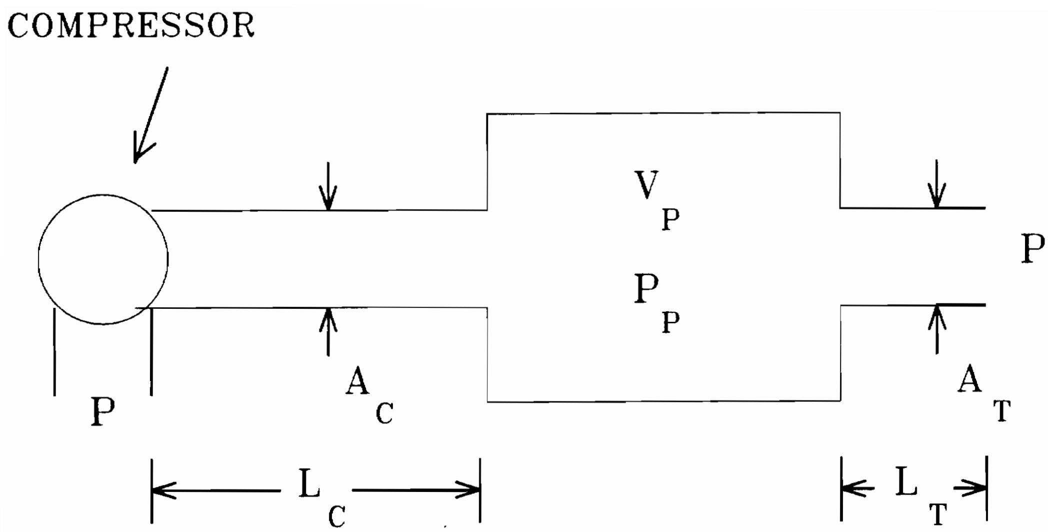


Figure 6-1 Greitzer's Compressor / Duct / Plenum System

Beginning with the inlet to the system, an equation can be written relating the flow velocity through the duct (c_x) and the pressure drop over the duct length:

$$P - P_p + \Delta P_c = \rho L_c \frac{dc_x}{dt} \tag{6.1}$$

The ΔP_c term represents the increase in fluid pressure due to the compressor pressure rise. The other pressure term, P - P_p, represents the pressure drop across the length of duct from the ambient pressure to that in the plenum. The term on the right hand side of Eq. 6.1 can be written in terms of the mass flow rate through the compressor, resulting in:

$$P - P_p + \Delta P_c = \frac{L_c}{A_c} \frac{d\dot{m}_c}{dt} \tag{6.2}$$

The throttle section of the system can be treated in a similar way with the same type of equation written for the flow exiting through the throttle:

$$P_p - P - \Delta P_t = \frac{L_t}{A_t} \frac{d\dot{m}_t}{dt} \tag{6.3}$$

The ΔP_t term represents the pressure loss due to friction in the throttle section. These two equations, relating pressure and mass flow, are similar to those seen in the analysis of the Helmholtz resonator (Section 5.2.2)

The mass flows in the compressor and the throttle sections can be related by applying the conservation of mass to the plenum. It is assumed that the velocities within the plenum are negligible. Hence the static pressure within the plenum can be assumed to be uniform throughout at any instant in time. Thus the continuity equation can be written as:

$$\dot{m}_c - \dot{m}_t = V_p \frac{d\rho_p}{dt} \tag{6.4}$$

where the subscript p denotes properties of the fluid within the plenum. If the process in the plenum is assumed to be isentropic, the plenum density changes can be related to changes in the plenum pressure and the following can be written for the continuity equation:

$$\dot{m}_c - \dot{m}_t = \frac{\rho_p V_p}{\gamma P_p} \frac{dP_p}{dt} \tag{6.5}$$

To this point the development follows that given by the Greitzer analysis [14]. The assumptions made were experimentally verified and shown by Greitzer to yield accurate results. At this point, the derivation of the lumped parameter model varies from that of the non-linear study of Greitzer and yields expressions analogous to those seen in mechanical system analysis.

From Eqs. 6.2 and 6.3, one can assume that the pressure rise through the compressor and the loss through the throttle are linearly proportional to the respective mass flow through each appropriate section and given by:

$$\Delta P_c = R_c \dot{m}_c \tag{6.6}$$

$$\Delta P_t = R_t \dot{m}_t \tag{6.7}$$

This assumption is valid for the loss through the throttle (P_t) as shown in the previous chapter in the discussion of friction losses (Section 5.1.2). For the pressure rise across the compressor, the "resistance" relationship between mass flow and pressure rise can be approximated from the compressor characteristic curves. Although such characteristic curves are usually non-linear, these curves may be treated as linear over a small range where the local slope may be assumed to be constant. For small operating ranges, the assumption of local linearity should be valid. Substituting these terms into the Eqs. 6.2 and 6.3 yields:

$$P - P_p + R_c \dot{m}_c = \frac{L_c}{A_c} \frac{d\dot{m}_c}{dt} \tag{6.8}$$

$$P_p - P - R_t \dot{m}_t = \frac{L_t}{A_t} \frac{d\dot{m}_t}{dt} \tag{6.9}$$

Taking the derivative, with respect to time, of each of these linear, first order differential equations results in:

$$\frac{dP}{dt} - \frac{dP_p}{dt} + R_c \frac{d\dot{m}_c}{dt} = \frac{L_c}{A_c} \frac{d^2\dot{m}_c}{dt^2} \tag{6.10}$$

$$\frac{dP_p}{dt} - \frac{dP}{dt} - R_t \frac{d\dot{m}_t}{dt} = \frac{L_t}{A_t}\frac{d^2\dot{m}_t}{dt^2} \qquad (6.11)$$

Substituting for the dP_p/dt term from the continuity equation, Eq. 6.5, yields two second order, linear, coupled equations in the mass flow rate given by:

$$\frac{L_c}{A_c}\frac{d^2\dot{m}_c}{dt^2} - R_c \frac{d\dot{m}_c}{dt} + \frac{\gamma P_p}{\rho V_p}\left(\dot{m}_c - \dot{m}_t\right) = \frac{dP}{dt} \qquad (6.12)$$

$$\frac{L_t}{A_t}\frac{d^2\dot{m}_t}{dt^2} + R_t \frac{d\dot{m}_t}{dt} + \frac{\gamma P_p}{\rho V_p}\left(\dot{m}_t - \dot{m}_c\right) = \frac{dP}{dt} \qquad (6.13)$$

These two equations will be shown to be similar to the equations of motion that could be written for the two degree of freedom mechanical system shown in Fig. 6-2. This is an important result. Using the basic equations for the fluid system and some simplifying assumptions, a set of equations analogous to those for a mechanical system was developed. This mathematical representation for the fluid system allows the techniques of mechanical system analysis to be applied to the fluid system.

A further simplifying assumption would be to set the right hand side of the second equation, Eq. 6.13, equal to zero. Effectively, this restricts the pressure perturbations to be only those at the compressor inlet.

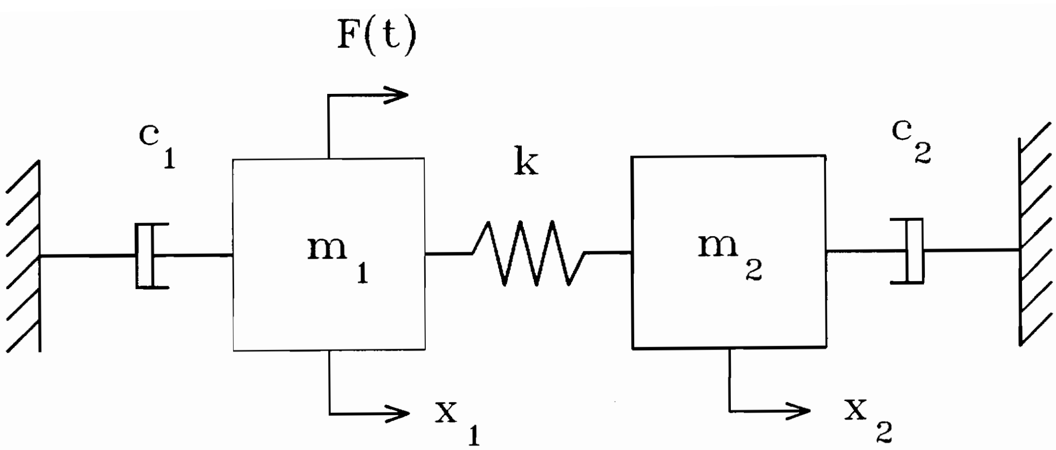


Figure 6-2 Mechanical System Analogy for Greitzer's Compressor / Duct / Plenum System

Using terminology introduced in the review of mechanical system analysis, this system has been reduced to a linear, two degree of freedom system with a single forcing function. It was shown in Chapter 4 that such a system can readily be analyzed through the techniques of modal analysis. Applying these techniques will yield frequency response functions for the transient fluid system, relating compressor inlet pressure perturbations and the corresponding changes in mass flow.

Equations 6.12 and 6.13 are important results as they are the basis of the lumped parameter model for this particular system. Before continuing with the analysis, a review of some of the ideas introduced in Chapter 5 and used in Eqs. 6.12 and 6.13 should be useful. In particular, the parameters of fluid inertance, compliance and resistance will be identified in Eqs. 6.12 and 6.13.

Equations 6.6 and 6.7 introduce the resistance parameter discussed in Section 5.1.3. For the pressure loss in the throttle this resistance term is a reasonable assumption. The actual nature of this term is not discussed as the nature of the flow is not known. As discussed previously, several steady flow solutions, such as the Hagen-Pouseuille solution, exist and may be adequate to describe the resistance behavior.

The pressure rise through the compressor is a much more complex behavior than the pressure loss through the throttle. In terms of the mechanical system, the rise of pressure through this section would be analogous to negative damping, or an energy input to the system, comparable to an internal forcing function. The resistance parameter, R_c , is a convenient linearization used under the assumption of small changes around some mean operating condition. As with the throttle resistance term, the particulars of the compressor resistance require knowledge of the particular compressor being modeled.

The other fluid parameters of inertance and compliance, defined in the previous chapter (Sections 5.1.1 and 5.1.2 respectively), are also realized in this development of a lumped parameter model for the Greitzer system. Their forms are slightly different due to the use of mass flows instead of the volumetric flow rates. In the inertance definition, the density term remains in the mass flow rate thus the inertance for this system may be shown to be, in general:

$$I = \frac{L}{A} \tag{6.14}$$

The compliance term physically manifests itself in the plenum, similar to the Helmholtz resonator. As with the inertance, the form of the compliance is slightly different due to the use of mass flow rate causing the addition of a density term. The compliance term is seen in the mass balance for the plenum section, Eq. 6.5, and is given by:

$$C = \frac{\rho V_p}{\gamma P_p} \tag{6.15}$$

where γ is the ratio of specific heats. As in the case of the Helmholtz resonator, the properties determining the compliance for the Greitzer system are assumed to be found at the initial steady operating condition and assumed to be constant.

Substituting the fluid parameter terms into equations 6.12 and 6.13 yields:

$$I_c \frac{d^2\dot{m}_c}{dt^2} - R_c \frac{d\dot{m}_c}{dt} + \frac{1}{C}(\dot{m}_c - \dot{m}_t) = \frac{dP}{dt} \tag{6.16}$$

$$I_t \frac{d^2\dot{m}_t}{dt^2} + R_t \frac{d\dot{m}_t}{dt} + \frac{1}{C}(\dot{m}_t - \dot{m}_c) = 0 \tag{6.17}$$

These two equations, Eqs. 6.16 and 6.17, are in terms of the defined fluid parameters of inertance, resistance and compliance. As such, they will be the basis of the general fluid model for the Greitzer system [14].

6.2 Proportional Damping Modal Model

The linear equations, given by Eqs. 6.17 and 6.18, form the basis of the lumped parameter model of the Greitzer fluid system. For this linear model, it is possible to apply the techniques of modal analysis introduced in Chapter 4. The first modal model discussed in Chapter 4 was the proportional damping model. Following the procedure described in Section 4.2 for the mechanical system model, the proportional damping model will be formed for the Greitzer fluid system.

The fluid equations of motion , Eqs. 6.17 and 6.18, can be written in matrix form as:

$$\begin{bmatrix} I_c & 0 \\ 0 & I_t \end{bmatrix} \begin{Bmatrix} \dfrac{d^2\dot{m}_c}{dt^2} \\ \dfrac{d^2\dot{m}_t}{dt^2} \end{Bmatrix} + \begin{bmatrix} -R_c & 0 \\ 0 & R_t \end{bmatrix} \begin{Bmatrix} \dfrac{d\dot{m}_c}{dt} \\ \dfrac{d\dot{m}_t}{dt} \end{Bmatrix} + \begin{bmatrix} 1/C & -1/C \\ -1/C & 1/C \end{bmatrix} \begin{Bmatrix} \dot{m}_c \\ \dot{m}_t \end{Bmatrix} = \begin{Bmatrix} \dfrac{dP}{dt} \\ 0 \end{Bmatrix} \tag{6.20}$$

Or in general symbolic form as:

$$[I]\left\{\frac{d^2\dot{m}}{dt^2}\right\}+[R]\left\{\frac{d\dot{m}}{dt}\right\}+[C]\{\dot{m}\}=\{P\} \tag{6.21}$$

As described in Chapter 4, coupling is seen to exist in the compliance matrix in Eq. 6.20. Applying the methods of modal analysis, the two fluid equations of motion may be uncoupled to form two linear equations that could be solved independently.

The equations can be decoupled using a transformation of coordinates to the principle coordinate system as discussed in Chapter 4. To achieve this transformation, we begin with the matrix form of the fluid equations of motion, assuming zero resistance and no pressure perturbation:

$$[I]\left\{\frac{d^2\dot{m}}{dt^2}\right\}+[C]\{\dot{m}\}=0 \tag{6.22}$$

Premultiplying by the inverse of the inertance matrix yields:

$$\left\{\frac{d^2\dot{m}}{dt^2}\right\}+[I]^{-1}[C]\{\dot{m}\}=0 \tag{6.23}$$

Assuming that the response of the flow rate in an unforced, fluid system modeled as only inertance and compliance parameters with no dissipative element, is harmonic, it is possible to write a function for the mass flow rate, given by $\dot{m}(t) = \dot{M}e^{i\omega t}$. Substituting this term into Eq. 6.23 yields.

$$\left(-\omega^2 + [I]^{-1}[C]\right)\dot{M} = 0 \tag{6.24}$$

Letting $\lambda = \omega^2$ forms Eq. 6.24 into an eigenvalue problem. The eigenvalues are found from the determinant of the matrix formed by the product of the inverse inertance and compliance matrices. The eigenvalues determined in this manner are the fluid natural frequencies of the fluid system.

Performing the mathematics for the Greitzer system, yields an inverse inertance matrix given by:

$$[I]^{-1} = \begin{bmatrix} 1/I_c & 0 \\ 0 & 1/I_t \end{bmatrix} \tag{6.25}$$

The product of the inverse inertance matrix and the compliance matrix yields:

$$[I]^{-1}[C] = \begin{bmatrix} 1/I_cC & -1/I_cc \\ -1/I_tC & 1/I_tC \end{bmatrix} \tag{6.26}$$

Solving the eigenvalue problem formed by Eq. 6.24 for the Greitzer system, results in a quadratic in λ given by:

$$\lambda^2 - \lambda\left(\frac{1}{I_t C} + \frac{1}{I_c C}\right) = 0 \quad (6.27)$$

The two roots of the quadratic in Eq. 6.27 yield solutions for the two natural frequencies of the Greitzer fluid system:

$$\lambda_1 = \omega_{n_1}^2 = 0 \quad (6.28)$$

$$\lambda_2 = \omega_{n_2}^2 = \left(\frac{1}{I_t C} + \frac{1}{I_c C}\right) \quad (6.29)$$

Before continuing with the analysis, it is important to explain the nature of the zero term found for the first natural frequency. This zero term implies rigid body motion, i.e., in the analogous mechanical system the spring (compliance) element could be replaced with a rigid connection, reducing the system to a single mass (inertance) acted on by the two dampers (resistances) only.

In the corresponding fluid sense, this is a reasonable result in that physically, it relates to flow straight through the system with the corresponding pressure additions and losses as previously derived in the fluid equations of motion.

As seen in the previous chapter, a system consisting of only inertance and resistance does not experience oscillatory motion. Thus, it would appear that only the second, non-zero, natural frequency term, Eq. 6.29, would be of interest in a transient flow analysis.

Returning to the formulation of the proportionally damped model for the Greitzer fluid system, using the eigenvalues, Eqs. 6.26 and 6.27, one can find the corresponding eigenvectors: These eigenvalues, denoted by ϕ, are given by:

$$\text{for}\,\lambda_1 = 0 \qquad \phi_1 = \begin{Bmatrix} 1/I_t C \\ 1/I_t C \end{Bmatrix} \tag{6.30}$$

$$\text{for}\ \lambda_2 = \left(\frac{1}{I_t C} + \frac{1}{I_c C}\right) \qquad \phi_2 = \begin{Bmatrix} -1/I_c C \\ 1/I_t C \end{Bmatrix} \tag{6.31}$$

For the mechanical system, the eigenvectors were shown to be scalable solutions for the free response of the system called the mode shapes. The idea of the mode shapes is not directly applicable to fluid system analysis. However, the eigenvectors are scalable solutions for the free response of the fluid system and may yield pertinent information. In particular, the eigenvectors will yield information regarding the phase relationship between the mass flows through the compressor and throttle respectively.

Forming a matrix of the eigenvectors (the modal matrix) and its transpose will serve to diagonalize or uncouple the matrices of the original equations of motion. Performing this operation on the inertance, compliance and resistance matrices yields the following results:

$$[\Phi]^T[I][\Phi] = \begin{bmatrix} I_c/(I_t C)^2 + 1/I_t C^2 & 0 \\ 0 & 1/I_c C^2 + 1/I_t C^2 \end{bmatrix} \tag{6.32}$$

$$[\Phi]^T[C][\Phi] = \begin{bmatrix} 0 & 0 \\ 0 & 1/I_c^2 C^3 + 1/I_t^2 C^3 + 1/2I_c I_t C^3 \end{bmatrix} \tag{6.33}$$

$$[\Phi]^T[R][\Phi] = \begin{bmatrix} -R_c + R_t/(I_t C)^2 & R_c/I_c I_t C^2 + R_t/(I_t C)^2 \\ R_c/I_c I_t C^2 + R_t/(I_t C)^2 & -R_c/(I_c C)^2 + R_t/(I_t C)^2 \end{bmatrix} \tag{6.34}$$

As seen in Eq. 6.34, the modal resistance matrix has not been diagonalized by the transformation. The assumption of proportional damping (resistance) states that this resistance matrix may be considered proportional to the inertance and compliance matrices. For the proportional damping model to be valid, the off-diagonal terms must be zero. One can determine the necessary constraint on the fluid system for the proportional damping (resistance) assumption by setting the off-diagonal terms to zero.

Performing the mathematics, the constraint on the resistance to allow for the proportional damping model is found to be:

$$R_c = -R_t\left(\frac{I_c}{I_t}\right) \tag{6.35}$$

In words, Eq. 6.35 says that the proportional damping model will be valid when the pressure rise through the compressor is equal to the friction loss through the throttle multiplied by the ratio of the inertance terms. Recalling that the inertances are simply based on the duct geometries and considering how the resistance terms were originally modelled, such a constraint seems unreasonably rigid. In addition, by examining the original inertance and resistance matrices in Eq. 6.20, there is no single coefficient that when multiplied by the inertance matrix will yield the proper resistance matrix. This problem is caused by the negative term required for the pressure increase of the compressor. For this system the proportional damping model appears inappropriate due to the constraint on the resistance terms. Because of this inadequacy of the proportionally damped model, it is necessary to form a more general lumped parameter model to describe the system. This is accomplished by utilizing the state space modal model.

6.3 The State Space Model

The proportional damping model required an unacceptable constraint be placed upon the resistance terms to account for the proportionality. In the following a more general model formulation that relieves this unacceptable constraint condition will be developed through a state space formulation.

As described in Chapter 4, the state space formulation results in a reduced order of the original matrix equation. This is accomplished by introducing a state variable, q, given for this fluid system by Eq. 6.36 and introducing an identity equation for the inertance matrix, similar to that for the mass matrix in Eq. 4.35.

$$\{q\} = \left\{ \begin{matrix} \left\{ \dfrac{d\dot{m}}{dt} \right\} \\ \{\dot{m}\} \end{matrix} \right\} \quad \text{and} \quad \{\dot{q}\} = \left\{ \begin{matrix} \left\{ \dfrac{d^2\dot{m}}{dt^2} \right\} \\ \left\{ \dfrac{d\dot{m}}{dt} \right\} \end{matrix} \right\} \tag{6.36}$$

Thus the state space formulation for the previously derived model of the Greitzer system in matrix form is given by :

$$\begin{bmatrix} 0 & 0 & I_c & 0 \\ 0 & 0 & 0 & I_t \\ I_c & 0 & -R_c & 0 \\ 0 & I_t & 0 & R_t \end{bmatrix} \begin{Bmatrix} d^2\dot{m}_c/dt^2 \\ d^2\dot{m}_t/dt^2 \\ d\dot{m}_c/dt \\ d\dot{m}_t/dt \end{Bmatrix} + \begin{bmatrix} -I_c & 0 & 0 & 0 \\ 0 & -I_t & 0 & 0 \\ 0 & 0 & 1/C & -1/C \\ 0 & 0 & -1/C & 1/C \end{bmatrix} \begin{Bmatrix} d\dot{m}_c/dt \\ d\dot{m}_t/dt \\ \dot{m}_c \\ \dot{m}_t \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ dP/dt \\ 0 \end{Bmatrix} \tag{6.37}$$

Proceeding with the analysis in an a manner analogous to that followed for the mechanical system will yield a form of the frequency response function for this fluid system, with no additional constraint placed on the resistance terms.

Consider the general form of the state space model matrix formulation given by Eq. 4.38:

$$[A]\{\dot{q}\} + [B]\{q\} = \{F\} \tag{4.38}$$

The particular forms of [A], [B] and {F} for the Greitzer fluid system have been given in Eq. 6.37. Using the general form of Eq. 4.38, one assumes that the free response of the system may be written as a harmonic with the assumed form $q(t) = Q\,e^{\lambda t}$.The equation for the free response of the fluid system may then be written as:

$$[A]\lambda + [B] = 0 \tag{6.38}$$

Pre-multiplying by the inverse of the A matrix results in an eigenvalue problem given by:

$$[I]\lambda + [A]^{-1}[B] = 0 \tag{6.39}$$

For the particular system under consideration here, $[A]^{-1}$ is given by :

$$[A]^{-1} = \begin{bmatrix} R_c/I_c^2 & 0 & 1/I_c & 0 \\ 0 & -R_t/I_t^2 & 0 & -1/I_t \\ 1/I_c & 0 & 0 & 0 \\ 0 & 1/I_t & 0 & 0 \end{bmatrix} \tag{6.40}$$

Thus, the product, $[A]^{-1}$ [B] is the matrix:

$$[A]^{-1}[B] = \begin{bmatrix} -R_c/I_c & 0 & 1/I_cC & -1/I_cC \\ 0 & R_t/I_t & -1/I_tC & -1/I_tC \\ -1.0 & 0 & 0 & 0 \\ 0 & -1.0 & 0 & 0 \end{bmatrix} \tag{6.41}$$

In general, the solution for the eigenvalues will yield a fourth order polynomial in λ . For the Greitzer fluid system, the polynomial was found to be:

$$\lambda^4 + \left(\frac{R_c}{I_c} - \frac{R_t}{I_t}\right)\lambda^3 + \left(\frac{1}{I_c C} + \frac{1}{I_t C} - \frac{R_c R_t}{I_c I_t}\right)\lambda^2 + \left(\frac{R_c - R_t}{I_c I_t C}\right)\lambda = 0 \tag{6.42}$$

Since a lambda can be factored out of Eq. 6.42, one root is zero, and the polynomial is reduced to a cubic equation. The roots of a cubic equation can be found through a complicated closed form solution. The particulars of such a solution technique will not be presented here, but the interested reader may, for example, consider [15]. However some pertinent generalizations can be made regarding the roots.

The solution for the roots is written in terms of the parameter D, which is given by:

$$D^2 = \frac{\alpha^2}{4} + \frac{\beta^3}{27} \tag{6.43}$$

where

$$\alpha = \left(2a_1^3 \; 27a_0^2\, a_3 - 9a_0 a_1 a_2\right)/27a_0^3$$

$$\beta = \left(3a_0 a_2 - a_1^2\right)/3a_0^2$$

and the a's are the coefficients of the cubic equation general form given by:

$$a_o x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

The value of D, as determined by the physical properties of the system, will determine the nature of the roots. When $D < 0$, there are three unequal real roots. For $D = 0$, three real roots exist, two of which are equal (repeated root). In a system dynamics sense, such solutions marked by real roots demonstrate system stability. Since the focus of this analysis is on transient unstable behavior, such stable solutions (analogous to the overdamped type of response for the mechanical system) are important in so far as knowing how and when they will exist, but are "uninteresting" beyond that in the scope of this text.

The case of interest for this analysis occurs when D > 0. This solution produces one real root, and one pair of complex conjugate roots. For this case, an instability in the system exists. Applying the mechanical analogy, such an instability relates to the underdamped case where resonant frequencies are a design concern. For the Greitzer model, depending upon the fluid properties, for the case of an instability, one would expect the single real root of the cubic to be zero. For the imaginary roots, the lambda is equivalent to the frequency portion of the exponential term seen in the impulse response function, as discussed in Section 4.3 and is given by:

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} \tag{4.41}$$

Relating the real and imaginary components yields two equations in the two unknowns; the natural frequency and damping ratio. Therefore using the eigenvalue terms it is possible to extract values for these two important parameters. These eigenvalues may in turn be used to extract the eigenvectors, from which the modal matrix may be formed.

Under the assumption that an instability exists, the terms such as natural frequency and resonance have relevance, and the eigenvalues can be utilized to formulate a frequency response function for the fluid system as demonstrated in the discussion of the state space model for the mechanical system.

Returning to the general state space matrix form of the system equations for the forced response due to the pressure perturbation, this was written as:

$$[A]\{\dot{q}\}+[B]\{q\}=\{F\} \tag{4.38}$$

If it is now assumed that the pressure perturbation is harmonic in nature, the forced response of the system, i.e., the response of each of the mass flows, will also be harmonic. This harmonic response will occur at the forced frequency. It is therefore possible to write the response, using the state space variable, as $\{q\}=\{Q\}\,e^{i\omega t}$. The matrix equation now becomes:

$$(i\omega[A]+[B])\{Q\}=\{P\} \tag{6.44}$$

Applying the assumed modal transformation, and substituting for $\{Q\}$ with $[\Psi]\{Z\}$, where $[\Psi]$ is the matrix of the eigenvectors, i.e., the modal matrix, into Eq. 6.44 yields:

$$i\omega[A][\Psi]\{Z\}+[B][\Psi]\{Z\}=\{P\} \tag{6.45}$$

Premultiplying by the transpose of the eigenvector matrix will diagonalize both the [A] and [B] matrices as discussed in Chapter 4. Performing this operation yields:

$$\left(i\omega\begin{bmatrix}\ddots & & \\ & a & \\ & & \ddots\end{bmatrix}+\begin{bmatrix}\ddots & & \\ & b & \\ & & \ddots\end{bmatrix}\right)\{Z\}=[\Psi]^T\{P\} \tag{6.46}$$

Solving for the modal coordinate {Z} in terms of the forcing pressure vector yields:

$$\{Z\} = \frac{[\Psi]^{T}\{P\}}{i\omega[a]+[b]} \tag{6.47}$$

Returning to the free response equation, Eq. 6.35, one can utilize the eigenvector matrix and its transpose to rewrite the equation in terms of the diagonalized matrices [a] and [b]:

$$[A]\lambda + [B] = 0 = [a]\lambda + [b] \tag{6.48}$$

It is then possible to write a relation for [b] in terms of [a], given by:

$$[b] = -\lambda[a] \tag{6.49}$$

Equation 6.49 can be substituted back into the forced response equation 6.47 resulting in:

$$\{Z\} = \frac{[\Psi]^{T}\{P\}}{[a]\left(i\omega - \{\lambda\}\right)} \tag{6.50}$$

Replacing the modal coordinates {Z} with the state space vector {q} in Eq. 6.50 and substituting the form of the eigenvalue, λ, as assumed in Eq. 4.41, yields:

$$\{q\} = \frac{\Psi^{T}\{P\}\Psi}{[a]\left(i\omega + \zeta\omega_{n} \mp i\omega_{n}\sqrt{1-\zeta^{2}}\right)} \tag{6.51}$$

By bringing the perturbing pressure vector {P} over to the left hand side of Eq. 6.51, the frequency response function for the fluid system is formed in a manner analogous to that used in Chapter 4 for the mechanical system. The expanded form of the resulting FRF is shown in Chapter 4 by Eq. 4.51:

$$\frac{X_i}{F_j} = \sum_{k=1}^{N} \left[\frac{\Psi_{jk}\Psi_{ik}}{a_k\left(i\omega + \zeta_k\omega_{nk} - i\omega_{nk}\sqrt{1-\zeta_k^2} \right)} + \frac{\Psi^*{}_{jk}\Psi^*{}_{ik}}{a^*{}_k\left(i\omega + \zeta_k\omega_{nk} + i\omega_{nk}\sqrt{1-\zeta_k^2} \right)} \right] \tag{4.51}$$

Unlike the results of the proportionally damped model, the formulation of the FRF, through the state space model, has no restriction on the resistance. Therefore, it is a much more general and a more broadly applicable method of modelling a fluid system.

It would be convenient if the resulting frequency response function were directly comparable to the results obtained by Greitzer [14]. However, there is some difficulty in comparing the two models. The non-linear model of Greitzer yields parameters whose values may not easily be used to define the modal model's inertance, resistance and compliance parameters. However, some generalized comparisons may be possible. The manner in which these comparisons could be made is discussed in Section 7.2.

Chapter 7 Summary and Recommendations

This study developed a lumped parameter model for the analysis of dynamic fluid systems. This was accomplished by reviewing the methods used in the classical analysis of mechanical systems, followed by an introduction to the modal method. Using these ideas and methods analogous models were developed for fluid systems. The lumped parameter approach allowed the fluid system to be modeled as a linear system. It was then possible to utilize many of the well-established techniques for linear system analysis.

7-1 Conclusions

As noted in the review of literature, Section 2.2, some attention has been given to the analysis of dynamic fluid systems through the use of lumped parameter models. The results of these efforts have offered some promise to the use of lumped parameter models to model the transient effects in some flow systems with reasonable accuracy. The ultimate goal of these efforts is to develop an analytical model that could be used to accurately predict transient effects and the response of a dynamic flow system to various inputs.

An important step toward attaining the desired goal is the introduction of the modal method to fluid system analysis. Modal analysis is a relatively new development in the analysis of dynamic mechanical systems. With respect to mechanical systems, modal analysis was shown to reduce the difficulties in dealing with multiple degree of freedom systems. This was accomplished through a coordinate transformation that decoupled the equations of motion for the multiple degree of freedom model. Once the equations are decoupled, the system response, given analytically by the frequency response function, reduces to the summation of single degree of freedom system contributions.

Developing the mechanical system example and using it to introduce the modal method allowed an analogous development for the lumped parameter fluid model. The development of the lumped parameter fluid model as a linear system made it possible to apply the modal method. Applying the modal method to a multiple degree of freedom fluid system, the Greitzer compressor/ duct/ plenum system, demonstrated the ability of the modal analysis techniques to be utilized in the development of a frequency response function for a dynamic fluid system.

Previous attempts at the formulation of frequency response functions for the prediction of transient effects in fluid systems have experimentally been shown to yield accurate results. The introduction of the modal method to fluid system analysis will greatly reduce the complexity of the analytical forms of the frequency response functions and yield these expressions in terms of real measurable flow quantities. Most importantly, the modal analysis techniques as applied to dynamic flow systems will yield relatively simple frequency response functions that will greatly enhance the ability to predict the transient effects of the dynamic fluid system.

7.2 Future Research Recommendations

The analytical model developed has introduced several possible areas of further research. These can be classified in two categories; model validation and model application.

Experimental model validation precludes the application of the lumped parameter model to other fluid systems. The validation of the lumped model is two fold: (1) the assumption of constant parameter terms needs to be investigated, and (2) the range of linearity for the models needs to be determined.

Both the Helmholtz resonator and Greitzer system models, used as examples here, have been built upon the defined fluid parameters of inertance, compliance and resistance. An important research objective should be the validation of these parameters, and the assumptions made to form them. This would require the determination of the range over which these parameters may be considered constant. This constant parameter assumption is crucial to the analysis.

It will also be necessary to experimentally determine the range over which the assumption of linearity is valid. The analytical models utilized constant parameter, linear equations. Both the models for the Helmholtz resonator and the Greitzer system assume that the pressure perturbations entering the system are small changes about some operating condition. It is necessary to determine the magnitude of these perturbations that will retain the linear assumption for the entire model.

It will also be necessary to determine the range of linearity of each of the individual fluid parameter relations. In Chapter 5, relations were formed for the fluid inertance, resistance and compliance. These were based on pressure and flow rate. For each of these relations it will be necessary to test the linear range of the relation. This will yield insight into the sensitivity of each of the parametric terms on the entire system model.

Another area of research is the practical application and utilization of this method to analyze fluid systems. Two such systems have been offered here as examples and will lend themselves quite easily to experimental investigations.

The frequency response function can be estimated through experimental methods. This would be accomplished by simultaneously measuring the input and resulting output quantities in time and forming their respective frequency domain representations. From these experimental estimations of the FRF, curve fits to the general forms of the FRF provided by the discussion in Chapter 4 can be obtained. The curve fits of the experimental data would allow for the determination of the fluid system's natural frequencies and damping ratios.

It would be of significant interest to utilize the two examples provided by this text. For either real system, the fluid properties could be used to analytically form a frequency response function. Experimentation on this system could then be used to verify the analytical FRF. The experimental results could then be compared to results obtained through a model developed using computational fluid dynamics.

Of particular interest would be the comparison of the model developed through the modal method for the compressor/ duct/ plenum system and the results achieved by Greitzer [14]. As discussed, some difficulties exist due to a mismatch of terms between the two models. It may be possible however, to compare the types of results achieved by both models and verify that the general predicted behavior is comparable. The Greitzer model yields responses of the mass flows in both the duct and throttle sections and the pressure changes at the inlet and through the compressor as a function of a dimensionless time variable. Transforming these "time" responses into the frequency domain and forming a ratio of the mass flow terms to the inlet pressure frequency information will yield frequency response functions of the same general form as those developed using the modal method. Therefore it would be possible make some general comparisons between the two models.

In the future, it is the hope of the author that this analysis may find its application in the analysis of real world systems, such as compressor testing or pumping system analysis. Further development of the lumped parameter model would yield a powerful tool that could be utilized in conjunction with the computational methods of analyzing unsteady flow systems.

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VITA

Matthew Lauder Wicks was born on September 25, 1970, and raised in Pittsford, New York. He left Pittsford Sutherland High School after three years to attend Virginia Polytechnic Institute and State University. Matthew received his B.S. in Mechanical Engineering from VPI in May, 1991. He entered the graduate program at VPI in August, 1991 and received his M.S. in Mechanical Engineering in May, 1993. Matthew intends to pursue a Ph.D. after spending some time working in industry.