TWO-DIMENSIONAL SUBSONIC COMPRESSIBLE FLOW ABOUT AN ARBITRARY JOUKOWSKY AIRFOIL

by

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III. NOMENCLATURE

The notations used in this thesis are summarized below:

Symbol	Definition
ϕ	potential function
ψ	stream function
x, y	Cartesian coordinates in physical plane
z	complex variable in physical plane
ζ	complex variable in auxilliary plane
w	complex variable in distorted hodograph plane
u, v	velocity components in x and y directions respectively
q	magnitude of velocity
θ	direction of velocity in physical plane
$\vec{q}$	velocity vector
$\mathfrak{q}$	magnitude of distorted velocity
q_{∞}	magnitude of velocity at infinity
$q_{i,\infty}$	velocity of undisturbed incompressible flow
p	pressure
ρ	density
a	speed of sound
γ	ratio of specific heats
M	Mach number
M_{∞}	freestream Mach number
$G(\zeta)$	complex potential of flow around circle in ζ -plane
Ω	complex potential of compressible fluid flow

Symbol	Definition
Γ	circulation of incompressible flow
α_i	angle of attack of incompressible flow around circle
α	angle of attack of compressible flow around airfoil
$f(\zeta), w(\zeta)$	analytic functions
$\omega(\zeta)$	conformal mapping of exterior of circle onto exterior of Joukowski profile
R	radius of circle in ζ-plane
δ	variable between 0 and 2π
C	constant in Joukowski mapping
$A_n, \beta_o, b_n, B_n, D_n$	auxilliary constants
k,K,P,Q,L	constants
μ	constant angle in ζ-plane
0,1	as subscripts, particular values of variables
$\vert\ \vert$	with symbol, absolute value
$\overline{}$	over term, complex conjugate

IV. INTRODUCTION

The fundamental equations of fluid motion are non-linear partial differential equations. As yet, a general method for the treatment of non-linear partial differential equations does not exist. Numerical methods of solution are generally rather laborious and hardly ever yield results of a general nature. The time required to compute numerical solutions is generally prohibitive to the engineer and, as a consequence, considerable attention has been given to approximate analytical methods.

In attacking the problem of the flow of particular fluids, certain simplifying assumptions can be made which will allow approximate solutions to be obtained. The forces which affect a fluid particle are forces due to pressure, inertia, and friction. Ludwig Prandtl pointed out in 1904 that for the flow of a gas, such as air, around a rigid body, the effects of friction are generally confined to a very thin layer next to the boundaries of the rigid body and outside the boundary layer the flow approximates very closely the flow of a perfect or non-viscous fluid. Therefore, the effects of friction can be isolated and the flow of the air outside of the boundary layer considered to be non-viscous. Another assumption which generally must be made to simplify the problem is that the flow be steady. This means that, at any fixed point in the flow field, no physical quantity such as velocity, temperature, or density shall vary with time. A consequence of this assumption is that the streamlines of the flow picture remain motionless in space.

In analyzing the flow of a gas, an additional complication enters the problem when the flow velocity is increased until its magnitude becomes appreciable when compared to the speed of propagation of sound waves in the gas. This complication is due to the effect of the compressibility of the gas. At very low speeds the flow of air can be considered to be incompressible which means that its density would not vary from point to point in the flow field. For the flow about an airplane wing under these conditions the pressure changes at any point on the wing are proportional to the square of the speed. However, at a sufficiently high speed, this relation would indicate a negative absolute pressure over the top of the wing, which of course, is impossible. Hence this simple relation connecting pressure difference and the square of the speed can no longer be true and a further investigation of the problem is necessary. In this investigation, the air must be regarded as compressible, with a density that varies from point to point round the profile.

Since the lift of a wing is obtained as a direct integration of the pressures which act over its surface, it would be very advantageous to be able to obtain the pressure distribution over an arbitrary airfoil shape in a subsonic compressible flow field. Two methods have recently been advanced to obtain the pressure distribution over airfoil shapes, but, so far, they have been applied only to the symmetrical type airfoil profile. The first method, proposed by Bers (reference 4), reduces the problem to a non-linear integral equation which can be solved numerically by an iteration method. This method of obtaining a solution is time consuming and extremely tedious. The second method, proposed by Gelbart (reference 7), is an approximate method of establishing the flow around a given body to a very high degree of accuracy. The method depends not on an integral equation, but on the transformation from the hodograph plane (in which the coordinates are the magnitude and inclination of the velocity vector) to the physical plane involving the determination of an arbitrary analytic function. The determination of the arbitrary analytic function by elementary means results in a close approximation of the given body.

It is the purpose of this thesis to apply the method proposed by Gelbart in the computation of the velocity distribution and over, not a symmetrical, but an arbitrary shaped Joukowsky type airfoil profile. It is assumed that the airflow over the profile is two-dimensional, non-viscous, steady, and compressible. The solution is restricted to the case where the flow velocity is everywhere less than the speed of sound by the introduction of the linearized pressure-density relationship proposed by Karman and Tsien (reference 13).

V. FUNDAMENTAL CONCEPTS AND RELATIONS

The four fundamental laws governing the steady, two-dimensional, irrotational flow of an ideal fluid are: The equation of state,

$$p = k\rho^{\gamma} \tag{1}$$

where p is the pressure, ρ the density, γ the ratio of the specific heats, and k a constant: Bernoulli's equation,

$$\frac{q^2}{2} + \int \frac{dp}{\rho} = \text{constant} \tag{2}$$

the continuity equation,

$$\text{div}\left(\frac{\rho}{\rho_o}\ \vec{q}\right) = 0 \tag{3}$$

and the circulation equation,

$$\text{curl}\ (\vec{q}) = 0 \tag{4}$$

where q is the magnitude of the velocity vector $\vec{q}$.

The general procedure used in attacking fluid flow problems of this type is to combine the above four fundamental laws in such a way as to give one equation in one unknown which governs the motion of the fluid. The velocity potential, ϕ, is introduced such that

$$\frac{\partial\phi}{\partial x} = u \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = v \tag{5}$$

where u and v are the velocity components in the x and y directions respectively. If equation (4) is written

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{6}$$

it is obvious that by substituting expressions (5) for u and v in equation (6) that the potential function satisfies the circulation equation.

The equation of continuity, equation (3), can be written

$$\frac{\partial}{\partial x}\left(\frac{\rho}{\rho_o}u\right) + \frac{\partial}{\partial y}\left(\frac{\rho}{\rho_o}v\right) = 0 \tag{7}$$

This equation will be satisfied if the stream function, ψ , is introduced such that,

$$\frac{\partial\psi}{\partial x} = -\frac{\rho}{\rho_o}v \quad \text{and} \quad \frac{\partial\psi}{\partial y} = \frac{\rho}{\rho_o}u \tag{8}$$

From the relations (5) and (8), by eliminating u and v, it is seen that

$$\frac{\partial\phi}{\partial x} = \frac{\rho_o}{\rho}\frac{\partial\psi}{\partial y} \quad \text{and} \quad \frac{\partial\phi}{\partial y} = -\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial x} \tag{9}$$

By using these relations and the fact that $a^2 = dp/d\rho$ where "a" is the velocity of sound, equations (1) to (4) may be combined to yield the single equation

$$\left(\frac{u^2}{a^2} - 1\right)\frac{\partial^2\phi}{\partial x^2} + \frac{2uv}{a^2}\frac{\partial^2\phi}{\partial x\partial y} + \left(\frac{v^2}{a^2} - 1\right)\frac{\partial^2\phi}{\partial y^2} = 0 \tag{10}$$

This equation is non-linear and a general solution does not exist.

The Russian mathematician, Chaplygin (reference 5), was the first to recognize (1904) that a change of the independent variables x and y to the hodograph coordinates q and θ transforms the non-linear equation to a linear one. It may appear that a method able to transform the equation of motion into a linear equation renders all previous approximate methods obsolete. This, however, is not the case. The hodograph method has one inherent difficulty: the boundary conditions are difficult to satisfy. This difficulty is basically due to the fact that the physical boundary conditions are given by equations involving only x and y. The shape of an airfoil, for example, is a geometrically defined region in the x,y system. It is generally difficult to find a solution in the hodograph plane which satisfies given boundary conditions in the physical plane. There exist exact solutions for certain simple boundary conditions such as the problem of flow around a sharp corner but, in general, approximate methods must be used to overcome the difficulty of satisfying given boundary conditions. An approximate method of solving the flow around given airfoil shapes, proposed by Gelbart (reference 7), appears to be very practical for engineering use and is used in this thesis.

From equations (1) and (2) the following fundamental relations are obtained:

$$a^2 = a_o^2 - \frac{\gamma-1}{2} q^2 \tag{11}$$

$$p = p_o \left(1 - \frac{\gamma-1}{2} \frac{q^2}{a_o^2}\right)^{\frac{\gamma}{\gamma-1}} = p_o \left(1 + \frac{\gamma-1}{2} \frac{q^2}{a^2}\right)^{-\frac{\gamma}{\gamma-1}} \tag{12}$$

$$\rho = \rho_o \left(1 - \frac{\gamma-1}{2} \frac{q^2}{a_o^2}\right)^{\frac{1}{\gamma-1}} = \rho_o \left(1 + \frac{\gamma-1}{2} \frac{q^2}{a^2}\right)^{-\frac{1}{\gamma-1}} \tag{13}$$

$$M^2 = \frac{q^2}{a^2} = -\frac{q}{\rho} \frac{d\rho}{dq} = \frac{q^2}{a_o^2 - \frac{\gamma-1}{2} q^2} \tag{14}$$

where M is the Mach number and the subscript o indicates values at a stagnation point where the velocity q is zero. It is convenient to normalize the constants so that $\rho_o = a_o = 1$. This is equivalent to introducing the dimensionless variables ρ/ρ_o and q/a_o.

VI. THE HODOGRAPH METHOD

It has been previously noted that the continuity equation gives rise to the potential function, ϕ, and that the circulation equation gives rise to the stream function, ψ.

Now, from equations (5),

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$
$$= u\, dx + v\, dy \tag{15}$$

and from equations (8),

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$
$$= -\rho v\, dx + \rho u\, dy \tag{16}$$

If the magnitude of the velocity at a point x,y of the flow is q and the angle of inclination with the x axis is θ, then $u = q \cos \theta$ and $v = q \sin \theta$ and equations (15) and (16) become

$$d\phi = q \cos \theta\, dx + q \sin \theta\, dy \tag{17}$$

$$d\psi = -\rho q \sin \theta\, dx + \rho q \cos \theta\, dy \tag{18}$$

Multiplying each side of equation (18) by $\frac{i}{\rho}$ and adding to equation (17)

$$dz = \frac{e^{i\theta}}{q} \left(d\phi + \frac{i}{\rho}\, d\psi\right) \tag{19}$$

where $z = x + iy$

Equation (19) is the mapping function from the hodograph plane to the physical plane.

The equations of motion in the hodograph plane are obtained from equation (19) as follows:

Equating real and imaginary parts of equation (19) yields

$$dx = \frac{\cos\theta}{q}\,d\phi - \frac{\sin\theta}{\rho q}\,d\psi$$

$$dy = \frac{\sin\theta}{q}\,d\phi + \frac{\cos\theta}{\rho q}\,d\psi \tag{20}$$

As long as the correspondence between the physical and hodograph plane is one to one, x and y can be expressed as functions of q,θ, and ϕ and ψ as functions of q,θ. Thus

$$d\phi = \phi_q\,dq + \phi_\theta\,d\theta$$

$$d\psi = \psi_q\,dq + \psi_\theta\,d\theta \tag{21}$$

where the partial derivatives are taken with respect to the independent variables indicated as subscripts. Substituting equations (21) into equations (20), the following expressions for dx and dy are obtained:

$$dx = \left(\frac{\cos\theta}{q}\,\phi_q - \frac{\sin\theta}{\rho q}\,\psi_q\right)dq + \left(\frac{\cos\theta}{q}\,\phi_\theta - \frac{\sin\theta}{\rho q}\,\psi_\theta\right)d\theta$$

$$dy = \left(\frac{\sin\theta}{q}\,\phi_q + \frac{\cos\theta}{\rho q}\,\psi_q\right)dq + \left(\frac{\sin\theta}{q}\,\phi_\theta + \frac{\cos\theta}{\rho q}\,\psi_\theta\right)d\theta \tag{22}$$

Since the left hand side of equations (22) are exact differentials, the reciprocity relation can be applied, and, therefore,

$$\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{q}\,\phi_q - \frac{\sin\theta}{\rho q}\,\psi_q\right) = \frac{\partial}{\partial q}\left(\frac{\cos\theta}{q}\,\phi_\theta - \frac{\sin\theta}{\rho q}\,\psi_\theta\right)$$

$$\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{q}\,\phi_q + \frac{\cos\theta}{\rho q}\,\psi_q\right) = \frac{\partial}{\partial q}\left(\frac{\sin\theta}{q}\,\phi_\theta + \frac{\cos\theta}{\rho q}\,\psi_\theta\right)$$

Performing the indicated differentiations yields

$$-\frac{\sin\theta}{q}\phi_q - \frac{\cos\theta}{\rho q}\psi_q = -\frac{\cos\theta}{q^2}\phi_\theta - \sin\theta\ \psi_\theta \frac{d}{dq}\left(\frac{1}{\rho q}\right)$$

$$\frac{\cos\theta}{q}\phi_q - \frac{\sin\theta}{\rho q}\psi_q = -\frac{\sin\theta}{q^2}\phi_\theta + \cos\theta\ \psi_\theta \frac{d}{dq}\left(\frac{1}{\rho q}\right) \qquad (23)$$

Multiplying the first of equations (23) by cos θ and the second by sin θ and adding yields

$$\frac{1}{\rho}\psi_q = \frac{1}{q}\phi_\theta \qquad (24)$$

Multiplying the first of equations (23) by sin θ and the second by cos θ and subtracting yields

$$\frac{1}{q}\phi_q = \psi_\theta \frac{d}{dq}\left(\frac{1}{\rho q}\right) \qquad (25)$$

Differentiating equation (2),

$$q\ dq + \frac{dp}{\rho} = 0$$

or

$$\frac{dp}{dq} = -\rho q \qquad (26)$$

Using equation (26) and the fact that $a^2 = dp/d\rho$, where "a" is the velocity of sound,

$$\frac{d\rho}{dq} = \frac{d\rho}{dp}\frac{dp}{dq} = -\frac{\rho q}{a^2} \qquad (27)$$

Using equation (27) and the definition of the Mach number, $M = q/a_o$

$$\frac{d}{dq}\left(\frac{1}{\rho q}\right) = -\frac{1}{\rho^2 q}\frac{d\rho}{dq} - \frac{1}{\rho q^2} = -\frac{1}{\rho q^2}(1 - M^2) \qquad (28)$$

By substituting equation (28) into equation (25), the equations of motion in the hodograph plane become

$$\phi_\theta = \frac{q}{\rho} \psi_q$$

$$\phi_q = -\frac{1}{\rho q} (1 - M^2) \psi_\theta \qquad (29)$$

Since the coefficients of the derivatives in equations (29) are functions of the independent variables only, these equations are linear. In fact, owing to the choice of θ and q, the coefficients are functions of only one independent variable, q.

VII. SYMMETRIZATION OF THE HODOGRAPH EQUATIONS

The hodograph equations (29) can be put into a more convenient form by an elementary transformation of variables. By a change of the independent variables from θ, q to $\theta, \tilde{q}$ where $\tilde{q}$ is defined as

$$d\tilde{q} = \sqrt{1 - M^2} \quad \frac{dq}{q} \tag{30}$$

the problem is removed from the original hodograph plane to a so-called "distorted" hodograph plane. The angle θ remains unchanged in this transformation and $\tilde{q}$ is the magnitude of the distorted velocity. Inserting equation (30) in the hodograph equations of a compressible flow, equations (29) become

$$\begin{aligned} \phi_\theta &= \frac{1}{\rho} \sqrt{1 - M^2} \; \psi_q \\ \phi_q &= -\frac{1}{\rho} \sqrt{1 - M^2} \; \psi_\theta \end{aligned} \tag{31}$$

These equations are symmetrical and simpler in form than the original hodograph equations.

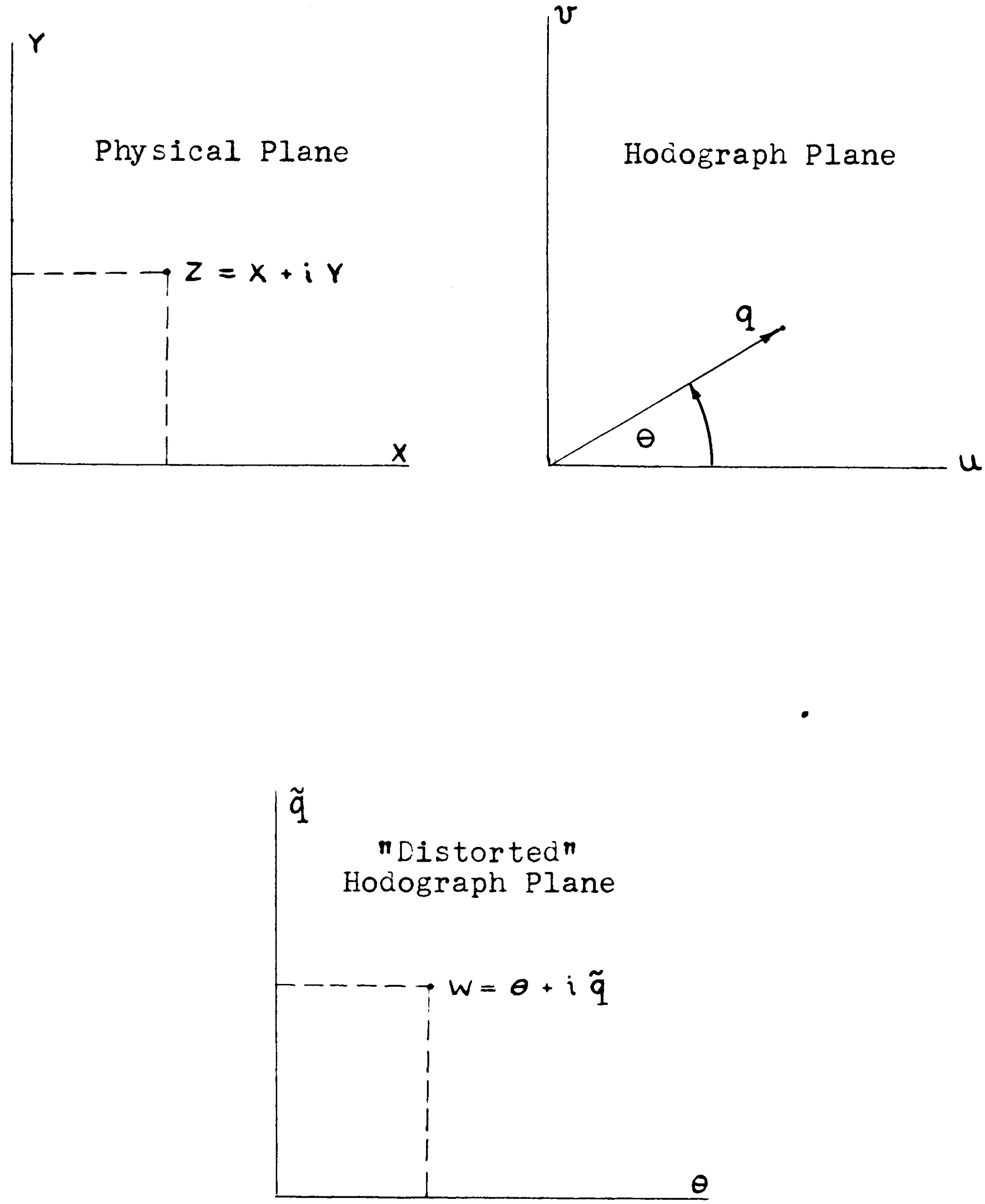

Figure 1. Physical, Hodograph, and Distorted Hodograph Planes

VIII. PHYSICAL INTERPRETATION OF A LINEARIZED PRESSURE-DENSITY RELATIONSHIP

The transformation of the equations of motion into the hodograph plane linearizes the non-linear equations. However, the resulting linear differential equations are rather complicated. The Russian mathematician Chaplygin gave the first extensive discussion of the hodograph equations and their applications to problems of gaseous jets in 1904. He introduced an approximation which will be discussed presently. Chaplygin's work remained relatively unnoticed until the early thirties when Demtchenko and Busemann used the method for certain problems in gas dynamics. Th. von Karman finally made an essential improvement in the method of approximation and thus made possible the application of Chaplygin's procedure to modern airfoil theory. The method of approximation based on von Karman's idea was worked out by H. S. Tsien (reference 13).

In equations (31), the factor

$$\frac{1}{\rho}\sqrt{1 - M^2}$$

becomes

$$\frac{\rho_o}{\rho}\sqrt{1 - M^2} \qquad (32)$$

if the constant, ρ_o, is not considered to be normalized, that is, ρ_o is not equal to unity. Chaplygin first noticed that the factor (32) differs little from unity for subsonic flow with M not too close to 1. For the incompressible flow case, $M = 0$, $\rho = \rho_o =$ constant, and the factor (32) is identically equal to unity. For the compressible flow case, by using equation (13), the factor (32) can be written

$$\frac{\rho_o}{\rho}\sqrt{1 - M^2} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma-1}} (1 - M^2)^{\frac{1}{2}} \qquad (33)$$

Developing in powers of M^2, equation (33) becomes

$$\frac{\rho_o}{\rho}\sqrt{1 - M^2} = \left(1 + \frac{M^2}{2} + (2 - \gamma)\frac{M^4}{8} + \dots\right)\left(1 - \frac{M^2}{2} - \frac{M^4}{8} + \dots\right)$$

and therefore

$$\frac{\rho_o}{\rho}\sqrt{1 - M^2} \doteq \left(1 - \frac{\gamma+1}{8} M^4 + \dots\right) \qquad (34)$$

Thus the factor (32) differs from unity only as a result of terms of the order of M^4 and higher. Taking $\gamma = 1.40$ and $M = 0.5$, the factor (32) differs from 1 only by about 2 per cent. Consequently, for relatively low speed compressible fluid flow the factor (32) can be replaced approximately by unity and the hodograph equations (31) become simply

$$\phi_\theta = \Psi_{\tilde{q}}$$

$$\phi_{\tilde{q}} = - \Psi_\theta \qquad (35)$$

That is, the hodograph equations reduce to the Cauchy-Riemann relations.

Replacing the factor (32) by unity can also be interpreted in a different way. It can be seen from equation (33) that if γ had the value $\gamma = -1$, the factor (32) would be exactly unity. Therefore, for $M < 1$, the assumption of $\gamma = -1$ satisfies the Chaplygin condition exactly.

Of course, a gas with this property, that is, with a negative ratio of the specific heats, does not exist. However, the true meaning of the condition $\gamma = -1$ can be arrived at by considering the isentropic relation between pressure and density for such an "imaginary" gas. For any perfect gas, the isentropes are given by equation (1). Therefore, for the "imaginary" gas,

$$p\rho = \text{constant} \tag{36}$$

In other words, $\gamma = -1$ corresponds to a straight-line isentrope (see Fig. 5) on the p versus $1/\rho$ plane. The general equation for a straight line can be written as

$$p = \frac{k}{\rho} + L \tag{37}$$

with k and L as constants. Now, by a proper choice of k and L, equation (37) can be made a tangent to the true isentropic curve. Hence, Chaplygin's condition that factor (32) equal unity is satisfied if the true isentrope is replaced by its tangent. Chaplygin, and later Busemann and Demtchenko, used the line tangent at the point (p_o, ρ_o), that is, the point corresponding to the stagnation conditions. Of course, this choice gives a good approximation only for comparatively low velocities for which p and ρ do not differ much from p_o and ρ_o respectively. Later, Karman and Tsien, on the other hand, chose as point of tangency the point (p_∞, ρ_∞) (the point corresponding to the undisturbed stream conditions), and thus were able to extend appreciably the range of applicability. By properly choosing the constants in equation (37), the tangent at the point (p_∞, ρ_∞) is represented by

$$p - p_\infty = \rho_\infty^2 a_\infty^2 \left(\frac{1}{\rho_\infty} - \frac{1}{\rho}\right) \tag{38}$$

If equation (1) is replaced by

$$p - p_1 = k\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) \tag{39}$$

equations (11) to (14) become,

$$a^2 = \frac{dp}{d\rho} = -\frac{k}{\rho^2} \tag{40}$$

$$p - p_1 = k\left(\sqrt{1 + q^2} - \frac{1}{\rho_1}\right) \tag{41}$$

$$\rho^2 = \frac{1}{1 + q^2} \tag{42}$$

$$M^2 = \frac{q^2}{1 + q^2} \tag{43}$$

equation (30) becomes,

$$\tilde{q} = \int_{q_\infty}^{q} \frac{dq}{q\sqrt{1 + q^2}} \tag{44}$$

and after the integration has been performed, equation (44) becomes,

$$\tilde{q} = \log \frac{Kq}{1 + \sqrt{1 + q^2}} \tag{45}$$

where

$$K = \frac{1 + \sqrt{1 + q_\infty^2}}{q_\infty} \tag{46}$$

The hodograph equations (31) become,

$$\frac{\partial \phi}{\partial \theta} = \frac{\partial \psi}{\partial \tilde{q}}$$

$$\frac{\partial \phi}{\partial \tilde{q}} = - \frac{\partial \psi}{\partial \theta} \qquad (47)$$

which are the Cauchy-Riemann relations.

IX. DISCUSSION OF THE PROBLEM

It has been seen that the equations of motion in the "distorted" hodograph plane reduce to the Cauchy-Riemann relations upon the assumption of a linearized pressure-density relationship. It is well known that any analytic function is a solution to the Cauchy-Riemann equations. The problem, therefore, is to find the analytic function which satisfies the boundary conditions. If the complex potential of a compressible flow in the physical plane is given by Ω where $\Omega = \phi + i\Psi$, then

$$\phi = \frac{\Omega + \bar{\Omega}}{2} \quad \text{and} \quad \Psi = \frac{\Omega - \bar{\Omega}}{2i} \tag{48}$$

where $\bar{\Omega}$ is the complex conjugate of Ω.

Using equations (48), equation (19) becomes

$$dz = \frac{e^{i\theta}}{q}\left[d\left(\frac{\Omega + \bar{\Omega}}{2}\right) + \frac{i}{\rho}\, d\left(\frac{\Omega - \bar{\Omega}}{2i}\right)\right]$$

$$= \frac{e^{i\theta}}{2}\left(\frac{1}{q} + \frac{1}{\rho q}\right) d\Omega + \frac{e^{i\theta}}{2}\left(\frac{1}{q} - \frac{1}{\rho q}\right) d\bar{\Omega} \tag{49}$$

Substituting equation (42) for ρ in equation (49) gives

$$dz = \frac{e^{i\theta}}{2}\left(\frac{1 + \sqrt{1 + q^2}}{q}\right) d\Omega + \frac{e^{i\theta}}{2}\left(\frac{1 - \sqrt{1 + q^2}}{q}\right) d\bar{\Omega} \tag{50}$$

From equation (45),

$$e^{\tilde{q}} = \frac{Kq}{1 + \sqrt{1 + q^2}} = -\frac{K\left(1 - \sqrt{1 + q^2}\right)}{q} \tag{51}$$

and

$$e^{-\tilde{q}} = \frac{1 + \sqrt{1 + q^2}}{Kq} \tag{52}$$

substituting equations (51) and (52) into equation (50) yields

$$dz = \frac{K}{2} e^{iw} d\Omega - \frac{1}{2K} e^{i\bar{w}} d\bar{\Omega} \quad (53)$$

where $w = \theta + i\tilde{q}$ and Ω is an analytic function of w.

Thus,

$$z = \frac{K}{2} \int e^{iw} d\Omega - \frac{1}{2K} \overline{\int e^{-iw} d\Omega} \quad (54)$$

The function Ω is the complex potential of a compressible flow in the physical plane or in the hodograph plane, depending on in which plane it is being considered. However, if Ω is considered as a function of w, it is analytic and represents the complex potential of an incompressible fluid flow in the w-plane.

Since the boundary in the physical plane is given in any problem, the left side of equation (54) is fixed. It would be desirable to find a relation between w and Ω which will allow the integrations to be performed on the right side of equation (54). If Ω represents the complex potential of an incompressible fluid flow in the w-plane, then the imaginary part of Ω must be a constant on the boundary in the w-plane in order to satisfy the boundary condition that the boundary of the given body be one of the streamlines in the physical plane. The remaining boundary condition is that the velocity component normal to the boundary in the physical plane must vanish at the surface of the body. This condition requires that the derivative of the real part of Ω, taken with respect to the normal to the boundary, vanish over the surface of the body. Since the velocity distribution over the given body in the physical plane is not know, then the boundary in the w-plane is also not known and the boundary conditions cannot be satisfied directly.

In order to overcome the difficulty that the boundary conditions cannot be satisfied directly in the w-plane an auxilliary ζ -plane is introduced, where $\zeta = \xi + i\eta$. There will now exist some conformal transformation $w = w(\zeta)$ under which equation (54) will remain invariant. If Ω is now identified as the complex potential of a uniform incompressible flow past a circle in the ζ -plane, there will be some relation between Ω and $w = w(\zeta)$. This relation is not known, however, since the exact form of the mapping function $w = w(\zeta)$ is not known. Since it is known that some relation between Ω and w must exist through the parameter ζ, it is permissible to set

$$\frac{K}{2}\int e^{iw}\, d\Omega = f(\zeta) \qquad (55)$$

in equation (54) and then determine the arbitrary analytic function, $f(\zeta)$, so that the boundary conditions are satisfied over the given boundary in the physical plane.

If $\Omega = G(\zeta)$ is the complex potential of a uniform incompressible flow in the ζ -plane then, by equation (55), equation (54) becomes

$$z = f(\zeta) - \frac{1}{4}\overline{\int \frac{[G'(\zeta)]^2}{f'(\zeta)}\, d\zeta} \qquad (56)$$

and

$$w(\zeta) = -i \log \frac{2f'(\zeta)}{KG'(\zeta)} \qquad (57)$$

Since $w = \theta + i\tilde{q}$, the magnitude of the velocity of the compressible fluid flow can be obtained from equations (57) and (45), for

$$e^{-i\theta}\, e^{\tilde{q}} = \frac{K}{2}\,\frac{G'(\zeta)}{f'(\zeta)}\,, \quad \tilde{q}(q_{\infty}) = 0 \tag{58}$$

and

$$q = \frac{|G'/f'|}{1 - |G'/2f'|^{2}} \tag{59}$$

It is clear from equation (59) that $|G'/2f'| < 1$.

X. DETERMINATION OF THE ARBITRARY ANALYTIC FUNCTION $f(\zeta)$

If mapping equation (56) is to keep the point at infinity fixed and is to be such that q_∞ is bounded, then $f'(\zeta)$ must have the form

$$f'(\zeta) = \sum_{n=0}^{\infty} \frac{b_n}{\zeta^n}, \quad b_o \neq 0$$

where the constants b_n are to be determined.

It will be found that the constant, b_o, determines the velocity of the flow at infinity and the proper choice of the constant b_1 insures that the mapping of the circle from the ζ-plane to the streamline in the z-plane will give a closed curve. The other coeﬀicients of $f'(\zeta)$, and thus $f(\zeta)$ may be so determined that the flow be past a preassigned obstacle.

If the function $f(\zeta)$ has been fixed in equation (56), then each point in the ζ-plane exterior to the circle R is mapped into a point in the z-plane such that the velocity of the compressible flow at the point z is given by equation (59) and the boundary of the circle $|\zeta| = R$ goes into a closed streamline in the z-plane (assuming b_1 has been suitable chosen). Therefore to solve the direct problem, that is, to obtain a uniform flow past a preassigned body, one need only determine $f(\zeta)$ such that the transformation in equation (56) transforms the circle $|\zeta| = R$ into a preassigned shape.

Because the left side of equation (56) is not an analytic function, the mapping of the circle onto the preassigned curve is not identical, point by point, with the conformal mapping of the circle onto the preassigned curve. Denote the conformal mapping by $\omega = \omega(\zeta)$. The assumption of Gelbart, in order to obtain a simple and highly accurate, though not exact, means of obtaining $f(\zeta)$, is that the inverse images of the two mappings, the conformal and the one given by equation (56), of the same point on the given curve subtend a small angle at the center of the circle $\zeta = Re^{i\delta}$, $0 \leqq \delta < 2\pi$. This has been verified for airfoil shapes and curves of circular shape by Gelbart.

On the basis of this assumption the maximum value of

$$\left| f(\zeta) - \frac{1}{4}\sqrt{\int \frac{[G'(\zeta)]^2}{f'(\zeta)}\, d\zeta} - \omega(\zeta) \right|$$

is small for $\zeta = Re^{i\delta}$. By equating coefficients of like powers of $e^{i\delta}$ in $z(\zeta)$ and in $\omega(\zeta)$, and thus determining $f(\zeta)$, it is assumed that the terms neglected in this process will be small.

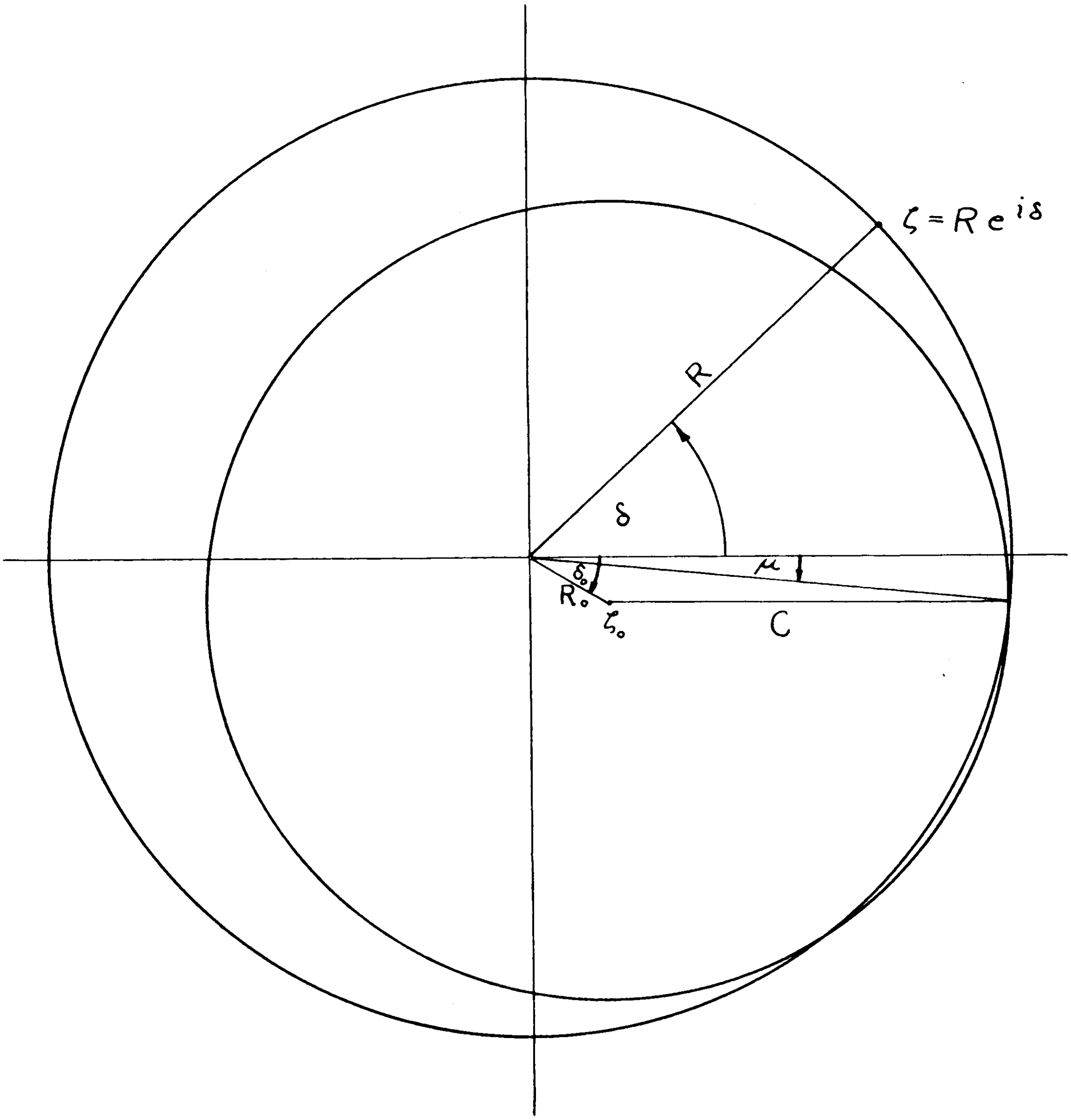

Figure 2. The Auxilliary ζ -Plane

XI. APPLICATION OF THE METHOD TO AN ARBITRARY JOUKOWSKY AIRFOIL

The method previously described in this thesis will now be applied to an arbitrary Joukowsky type airfoil for a given angle of attack, α, and a flow whose speed at infinity is q_∞.

To satisfy the Kutta-Joukowsky condition (that the trailing edge of the airfoil must be a stagnation point), the flow past the circle $\zeta = Re^{i\delta}$ with an angle of attack, α_i, or a circulation

$$\Gamma = -4\pi q_{i,\infty} R \sin(\alpha_i + \mu) \tag{60}$$

is given by

$$G(\zeta) = q_{i,\infty}\left(\zeta e^{-i\alpha_i} + \frac{R^2}{\zeta e^{-i\alpha_i}}\right) - \frac{i\Gamma}{2\pi} \log \frac{\zeta e^{-i\alpha_i}}{R} \tag{61}$$

where μ is a constant angle defined by the location of the center of the circle to be transformed (see Fig. 2). The stagnation points on the circle occur at $\delta = 0^\circ$ and $\delta = 180^\circ + 2\alpha_i + \mu$. Taking the derivative of $G(\zeta)$ and substituting equation (60) for Γ,

$$G'(\zeta) = q_{i,\infty} e^{-i\alpha_i}\left[1 + \frac{2i\, R \sin(\alpha_i + \mu)}{\zeta e^{-i\alpha_i}} - \frac{R^2}{\zeta^2 e^{-2i\alpha_i}}\right] \tag{62}$$

Squaring each side of equation (62),

$$[G'(\zeta)]^2 = q_{i,\infty}^2 e^{-2i\alpha_i}\left[A_0 + A_1\frac{1}{\zeta} + A_2\frac{1}{\zeta^2} + A_3\frac{1}{\zeta^3} + A_4\frac{1}{\zeta^4}\right] \tag{63}$$

where

$$\left.\begin{aligned} A_0 &= 1 \\ A_1 &= 4iRe^{i\alpha_i}\sin(\alpha_i + \mu) \\ A_2 &= -2R^2e^{2i\alpha_i}\left[1 + 2\sin^2(\alpha_i + \mu)\right] \\ A_3 &= -4iR^3e^{3i\alpha_i}\sin(\alpha_i + \mu) \\ A_4 &= R^4e^{4i\alpha_i} \end{aligned}\right\} \tag{64}$$

Recall that $f'(\zeta) = \sum_{n=0}^{\infty} \frac{b_n}{\zeta^n}$, Let

$$\frac{1}{f'(\zeta)} = \sum_{n=0}^{\infty} \frac{B_n}{\zeta^n} \tag{65}$$

Hence

$$1 = \left(\sum_{n=0}^{\infty} \frac{B_n}{\zeta^n}\right)\left(\sum_{n=0}^{\infty} \frac{b_n}{\zeta^n}\right) \tag{66}$$

so that

$$\left.\begin{aligned} 1 &= b_o\, B_o \\ 0 &= b_n B_o + \dots + b_o B_n, \quad n>0 \end{aligned}\right\} \tag{67}$$

Equations (67) can be solved for B_n:

$$\left.\begin{aligned} B_o &= \frac{1}{b_o} \\ B_n &= -\frac{1}{b_o}\,(b_n B_o + \dots + b_1 B_{n-1}) \end{aligned}\right\} \tag{68}$$

For later use B_1 and B_2 in terms of the unknown constants b_n are given:

$$\left.\begin{aligned} B_1 &= -\frac{b_1}{b_o^2} \\ B_2 &= -\frac{b_2}{b_o^2} + \frac{b_1^2}{b_o^3} \end{aligned}\right\} \tag{69}$$

Now it is possible to write

$$\frac{[G'(\zeta)]^2}{f'(\zeta)} = q_{i,\infty}^2\, e^{-2i\alpha_i} \sum_{n=0}^{\infty} D_n \frac{1}{\zeta^n} \tag{70}$$

where $D_n = A_o B_n + \dots + A_4 B_{n-4}$ (71)

Thus

$$-\overline{\int \frac{[G'(\zeta)]^2}{f'(\zeta)}\, d\zeta} = -q_{i,\infty}^2 e^{2i\alpha_i}\left(\bar{D}_o\bar{\zeta} + \bar{D}_1 \log \bar{\zeta} - \sum_{n=1}^{\infty} \frac{\bar{D}_{n+1}}{n\bar{\zeta}^n}\right) \qquad (72)$$

and

$$f(\zeta) = b_{-1} + b_o\zeta + b_1 \log \zeta - \sum_{n=1}^{\infty} \frac{b_{n+1}}{n\zeta^n} \qquad (73)$$

On the circle $\zeta = Re^{i\delta}$, equation (56) becomes

$$z = b_{-1} + b_o Re^{i\delta} + b_1 \log R + b_1 (i\delta) - \sum_{n=1}^{\infty} \frac{b_{n+1}}{nR^n} e^{-in\delta}$$

$$- \frac{q_{i,\infty}^2 e^{2i\alpha_i}}{4}\left(\bar{D}_o Re^{-i\delta} + \bar{D}_1 \log R - \bar{D}_1(i\delta) - \sum_{n=1}^{\infty} \frac{\bar{D}_{n+1}}{nR^n} e^{in\delta}\right) \qquad (74)$$

The Joukowski transformation that maps the exterior of a circle of radius R onto the exterior of an airfoil is given by

$$\omega(\zeta) = \zeta - \zeta_o + \frac{C^2}{\zeta - \zeta_o}$$

$$= \zeta - \zeta_o + \sum_{n=1}^{\infty} \frac{C^2 \zeta_o^{n-1}}{\zeta^n} \qquad (75)$$

where $\zeta_o = R_o e^{i\delta_o}$ is the center of the circle in the ζ-plane and C is a constant distance determined by the location of ζ_o in the ζ-plane (see Fig. 2). On the circle $\zeta = Re^{i\delta}$, equation (75) becomes

$$\omega(Re^{i\delta}) = -\zeta_o + Re^{i\delta} + \sum_{n=1}^{\infty} \frac{C^2 \zeta_o^{n-1}}{R^n} e^{-in\delta} \qquad (76)$$

Equating coefficients of like terms in equations (74) and (76), the following simultaneous equations for b_n are obtained:

$$-\zeta_o = b_{-1} + b_1 \log R - \frac{q_{i,\infty}^2 e^{2i\alpha_i}}{4} \bar{D}_1 \log R \qquad (77)$$

$$R = Rb_o + \frac{q_{i,\infty}^2 e^{2i\alpha_i}}{4R} \bar{D}_2 \qquad (78)$$

$$0 = b_1 + \frac{q_{i,\infty}^2 e^{2i\alpha_i}}{4} \bar{D}_1 \qquad (79)$$

$$\frac{C^2}{R} = -\frac{b_2}{R} - \frac{q_{i,\infty}^2 e^{2i\alpha_i}}{4} R\bar{D}_o \qquad (80)$$

and

$$\frac{C^2 \zeta_o^{n-1}}{R^n} = -\frac{b_{n+1}}{nR^n}, \quad n \geq 2 \qquad (81)$$

Since D_n is a function of $b_o, b_1, \ldots, b_n$ only, equations (78), (79), and (80) can be solved for b_o, b_1, and b_2. Once these are obtained, equation (77) can be trivially solved for b_{-1}. Equation (81) can also be trivially solved for b_n, $n \geq 3$. It remains then only to solve equations (78), (79), and (80) for b_o, b_1, and b_2.

Equation (79), it should be pointed out, is the condition on b_1 that the mapping of the circle $\zeta = Re^{i\delta}$ by equation (56) be a closed curve.

Equations (78), (79), and (80) are definitely not linear equations in b_o or $\bar{b}_o$. But if, from equation (58), the value of $q_{i,\infty}$ in terms of b_o is substituted into the three equations they become linear in β_o, b_1, and b_2, where β_o is the absolute value of b_o, that is, $\beta_o = |b_o|$.

For the point at infinity equation (58) gives rise to the relation

$$e^{-i\alpha} = \frac{K}{2} \frac{q_{i,\infty} e^{-i\alpha_i}}{b_o} \qquad (82)$$

or

$$q_{i,\infty} = \frac{2}{K} b_o e^{-i(\alpha-\alpha_i)} = 2P b_o e^{-i(\alpha-\alpha_i)} \tag{83}$$

where α is the angle of attack of the compressible fluid flow past the airfoil, and

$$P = \frac{1}{K} = \frac{q_\infty}{1 + \sqrt{1 + q_\infty^2}} \tag{84}$$

$$0 < P < 1 \tag{85}$$

Since $q_{i,\infty}$ and P are real, the argument of b_o must be $\alpha - \alpha_i$, or $b_o = \beta_o e^{i(\alpha-\alpha_i)}$. Equation (83) then becomes

$$q_{i,\infty} = 2P \beta_o \tag{86}$$

After substituting equation (86) and $\bar{D}_o = \frac{1}{\bar{b}_o}$ into equation (80),

$$b_2 = - C^2 - P^2\beta_o R^2 e^{i(\alpha+\alpha_i)} \tag{87}$$

Now

$$\begin{aligned} \bar{D}_1 &= \bar{A}_o\bar{B}_1 + \bar{A}_1\bar{B}_o \\ &= - \frac{\bar{b}_1}{\bar{b}_o^2} - \frac{4iRe^{-i\alpha_i} \sin(\alpha_i + \mu)}{\bar{b}_o} \end{aligned} \tag{88}$$

The substitution of this and equation (86) into equation (79) yields

$$b_1 = P^2 e^{2i\alpha} \bar{b}_1 - 4iP^2Re^{i\alpha} \beta_o \sin(\alpha_i + \mu) = 0 \tag{89}$$

The complex conjugate of equation (89) results in

$$\bar{b}_1 = P^2 e^{-2i\alpha} b_1 - 4iP^2Re^{-i\alpha} \beta_o \sin(\alpha_i + \mu) \tag{90}$$

and by substituting equation (90) into equation (89), equation (89) becomes

$$b_1 - P^4 b_1 + 4iP^4Re^{i\alpha}\, \beta_o \sin(\alpha_i + \mu) - 4iP^2Re^{i\alpha}\, \beta_o \sin(\alpha_i + \mu) = 0 \quad (91)$$

Hence

$$b_1 = \frac{4iP^2Re^{i\alpha} \sin(\alpha_i + \mu)}{1 + P^2}\, \beta_o \quad (92)$$

Proceeding as before,

$$\begin{aligned} \bar{D}_2 &= \bar{A}_o\bar{B}_2 + \bar{A}_1\bar{B}_1 + \bar{A}_2\bar{B}_o \\ &= -\frac{\bar{b}_2}{\bar{b}_o^2} + \frac{\bar{b}_1^2}{\bar{b}_o^3} + \frac{4iRe^{-i\alpha_i}\, \bar{b}_1 \sin(\alpha_i + \mu)}{\bar{b}_o^2} \\ &\quad - \frac{2R^2e^{-2i\alpha_i}\left[1 + 2\sin^2(\alpha_i + \mu)\right]}{\bar{b}_o} \end{aligned} \quad (93)$$

The substitution of this as well as equations (92) and (87) into equation (78) yields a linear equation in β_o:

$$\begin{aligned} R^2 &= \beta_o e^{i(\alpha-\alpha_i)}R^2 + P^2e^{2i\alpha}\, C^2 + P^4\beta_o R^2\, e^{i(\alpha-\alpha_i)} \\ &\quad - \frac{16P^6R^2e^{i(\alpha-\alpha_i)}\beta_o \sin^2(\alpha_i + \mu)}{(1 + P^2)^2} \\ &\quad + \frac{16P^4R^2e^{i(\alpha-\alpha_i)}\beta_o \sin^2(\alpha_i + \mu)}{1 + P^2} \\ &\quad - 2P^2R^2\, \beta_o\, e^{i(\alpha-\alpha_i)}\left[1 + 2\sin^2(\alpha_i + \mu)\right] \end{aligned} \quad (94)$$

Thus the solution for β_o can be written

$$\beta_o = \frac{R^2 - P^2\, e^{2i\alpha}\, C^2}{e^{i(\alpha-\alpha_i)}R^2(1 - P^2)^2\left\{1 - \left[\dfrac{2P \sin(\alpha_i + \mu)}{1 + P^2}\right]^2\right\}} \quad (95)$$

In determining the coefficients of $f(\zeta)$ it is desirable to do so in terms of q_∞ and α, which are to be considered as preassigned. The quantity α_i must then be considered as an unknown. The four equations for β_o, b_1, b_2, and α_i are then equations (78), (79), (80), and (83). Since β_o must be real, α_i can be obtained by setting the imaginary part of equation (95) equal to zero:

$$I_m \left\{ \left[R^2 - P^2 e^{2i\alpha} C^2\right] e^{-i(\alpha-\alpha_i)} \right\} = 0 \qquad (96)$$

or

$$-R^2 \sin(\alpha - \alpha_i) - P^2 C^2 \sin(\alpha + \alpha_i) = 0 \qquad (97)$$

Hence

$$\frac{\sin(\alpha - \alpha_i)}{\sin(\alpha + \alpha_i)} = -\frac{P^2 C^2}{R^2} = -Q \qquad (98)$$

It is seen from equation (98) that Q is positive. From equation (98)

$$\frac{\sin \alpha_i \cos \alpha - \cos \alpha_i \sin \alpha}{\sin \alpha_i \cos \alpha + \cos \alpha_i \sin \alpha} = Q \qquad (99)$$

and

$$1 - \frac{\tan \alpha}{\tan \alpha_i} = Q \left(1 + \frac{\tan \alpha}{\tan \alpha_i}\right) \qquad (100)$$

then for α_i:

$$\tan \alpha_i = \frac{1 + Q}{1 - Q} \tan \alpha \qquad (101)$$

where

$$Q = \left[\frac{P\,C}{R}\right]^2 < 1 \qquad (102)$$

Since $0 \leqq \alpha < 90^\circ$, α_i always has a solution between 0 and 90° and $\alpha_i \geqq \alpha$. When $\alpha = 0$, $\alpha_i = 0$.

To obtain b_n, $n \geqq 3$, equation (81) is solved:

$$b_n = -(n-1)\,C^2 \zeta_o^{n-2}, \quad n \geqq 3 \tag{103}$$

Finally,

$$f'(\zeta) = b_o + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + \sum_{n=3}^{\infty} \frac{b_n}{\zeta^n}$$

$$= b_o + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} - \sum_{n=3}^{\infty} \frac{C^2 (n-1) \zeta_o^{n-2}}{\zeta^n}$$

$$= b_o + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + C^2 \left[\frac{1}{\zeta^2} - \frac{1}{(\zeta - \zeta_o)^2}\right] \tag{104}$$

Also,

$$f(\zeta) = b_{-1} + b_o \zeta + b_1 \log \zeta - \frac{b_2 + C^2}{\zeta} + \frac{C^2}{\zeta - \zeta_o} \tag{105}$$

where b_{-1}, b_o, b_1, and b_2 are given by the equations (77), (95), (92), and (87).

The velocity at any point on the airfoil is given by

$$q = \frac{\left|\frac{G'(\zeta)}{f'(\zeta)}\right|}{1 - \left|\frac{G'(\zeta)}{2f'(\zeta)}\right|^2} \tag{106}$$

XII. RESULTS OF COMPUTATIONS

Computations have been carried out for an arbitrarily chosen Joukowsky airfoil (see Fig. 3) having the following geometric characteristics:

$\zeta_o = R_o\, e^{i\delta_o} = 0.18\, e^{i(-33^\circ 41')}$ and $R = 1.155$. The constants which are involved in the computations for a freestream Mach number of 0.5 are given in Table 1. The velocity distributions calculated for the actual profile for angles of attack of 0°, 5°, and 10° and the same Mach number are given in Table 2. The velocity distributions are also plotted for the three angles of attack in Figure 4.

The pressure distribution over the airfoil can be readily obtained once the velocity distribution is known and by a direct integration of the pressures which act over the airfoil the lift can be obtained.

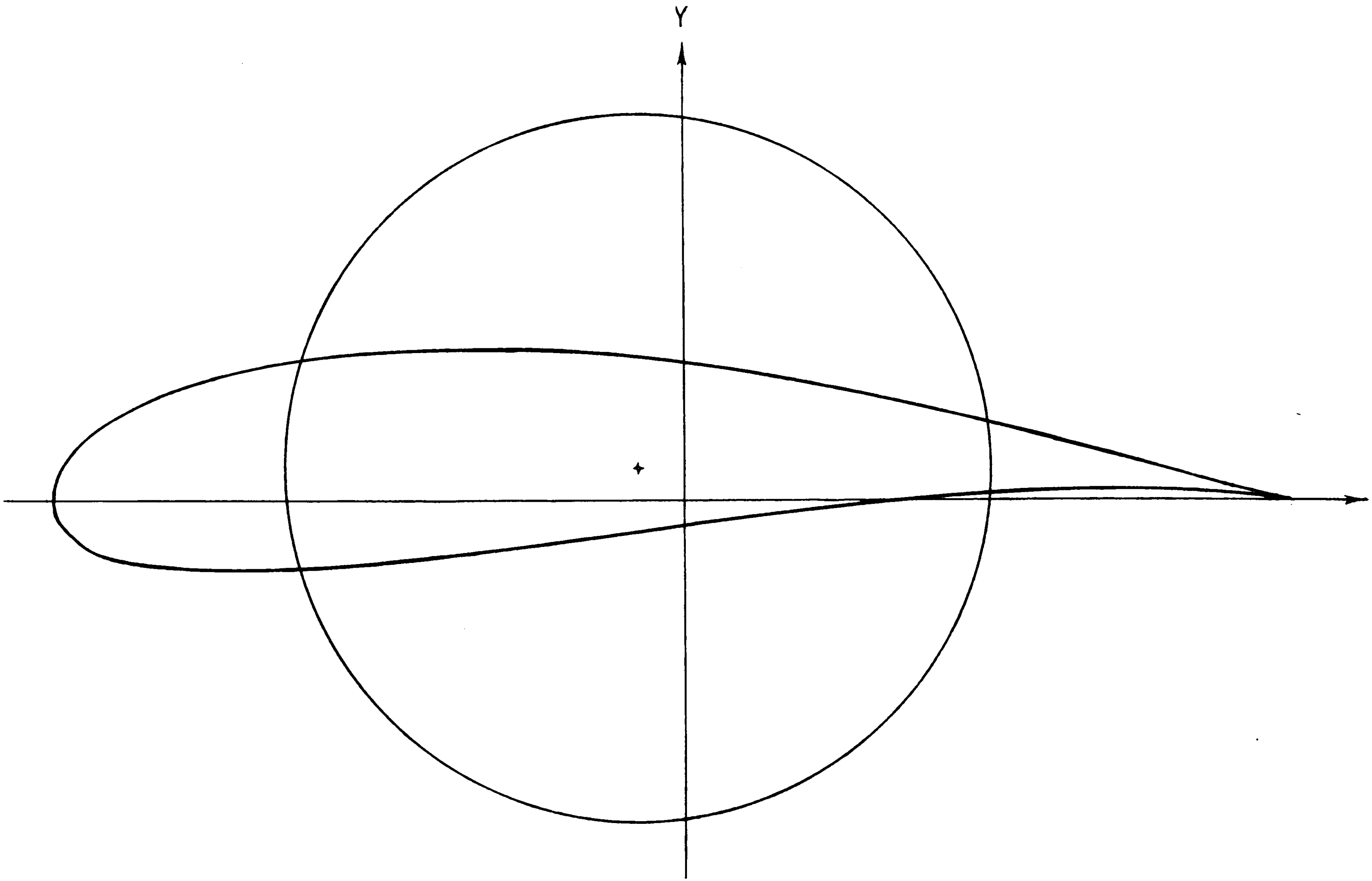

Figure 4. Joukowsky Airfoil Used and the Circle from which it came.

TABLE 1.

COMPUTATIONAL CONSTANTS

$M_\infty = 0.5$	$P = 0.2679$	$R_o = 0.18$	$C = 1.0$
$q_\infty = 0.5774$	$R = 1.155$	$\delta_o = -33^\circ\ 41'$	$\mu = 4^\circ\ 58'$

α_i	0°	5°	10°
α	0	$4^\circ\ 29'$	$9^\circ\ 0'$
β_o	1.1002	1.1072	1.1202
$q_{i,}$	0.5884	0.5932	0.6002
b_o	1.1002	1.1071 - i0.0010	1.1201 - i0.0196
B_1	i0.0297	-0.0047 + i0.0593	-0.0140 + i0.0886
b_2	-1.1054	1.1046 - i0.0175	-1.1015 - i0.0349

TABLE 2.

VELOCITY DISTRIBUTION, q/q_{∞}

($M_{\infty} = 0.5$; $R = 1.155$; $R_o = 0.18$; $\delta_o = -33^{\circ}\ 41'$)

δ \ α_i	0°	5°	10°
10	0.901	0.888	0.895
30	1.015	1.029	1.047
50	1.142	1.182	1.222
70	1.274	1.348	1.418
90	1.390	1.513	1.626
110	1.469	1.657	1.835
130	1.482	1.768	2.044
150	1.376	1.818	2.265
170	0.946	1.661	2.464
190	0.408	0.414	1.372
210	1.200	0.713	0.245
230	1.219	0.953	0.694
250	1.112	0.956	0.800
270	0.995	0.900	0.801
290	0.900	0.842	0.777
310	0.842	0.809	0.765
330	0.830	0.818	0.786
350	0.903	0.949	0.899

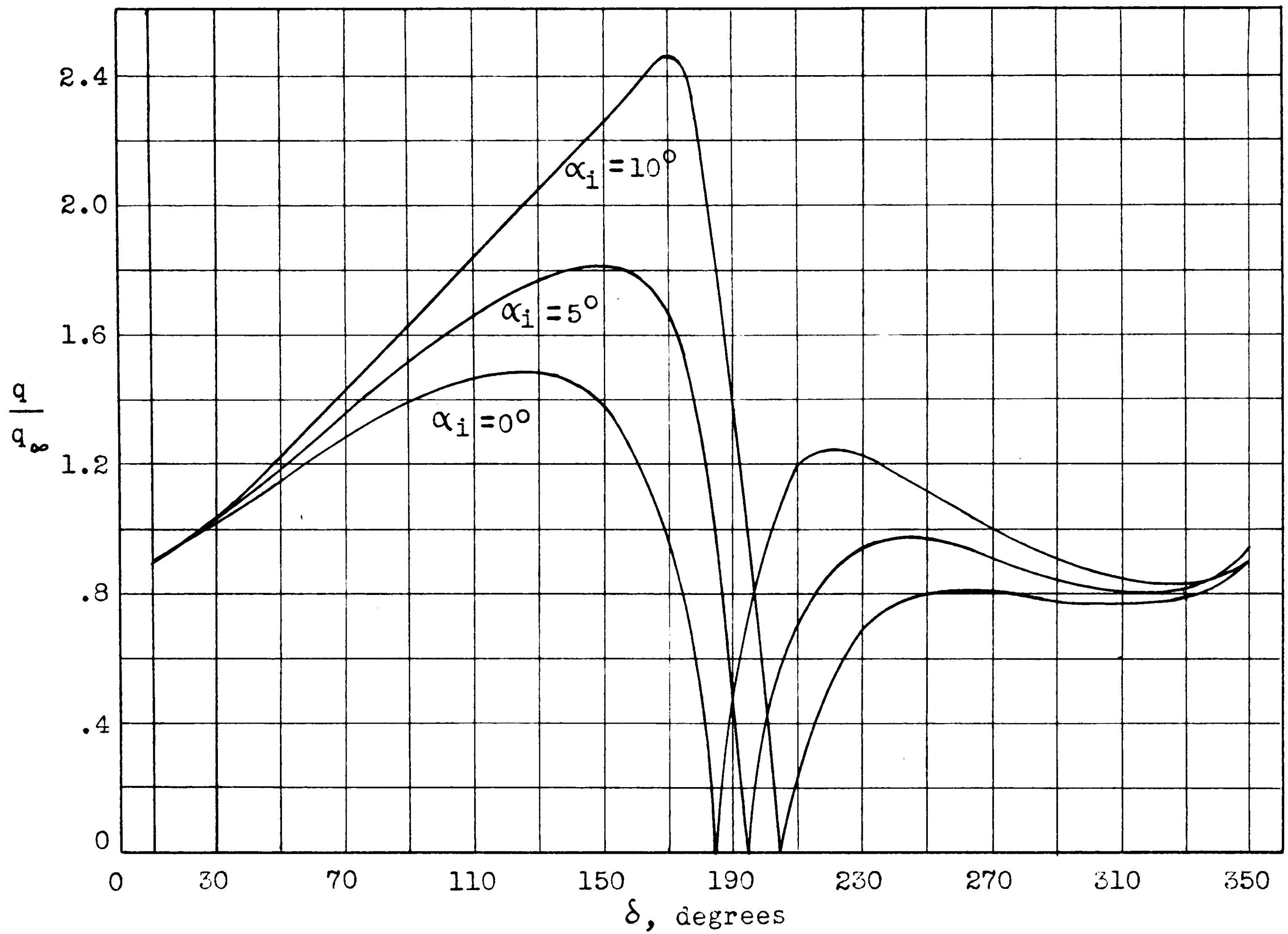

Figure 4. Velocity Distribution for various Angles of Attack

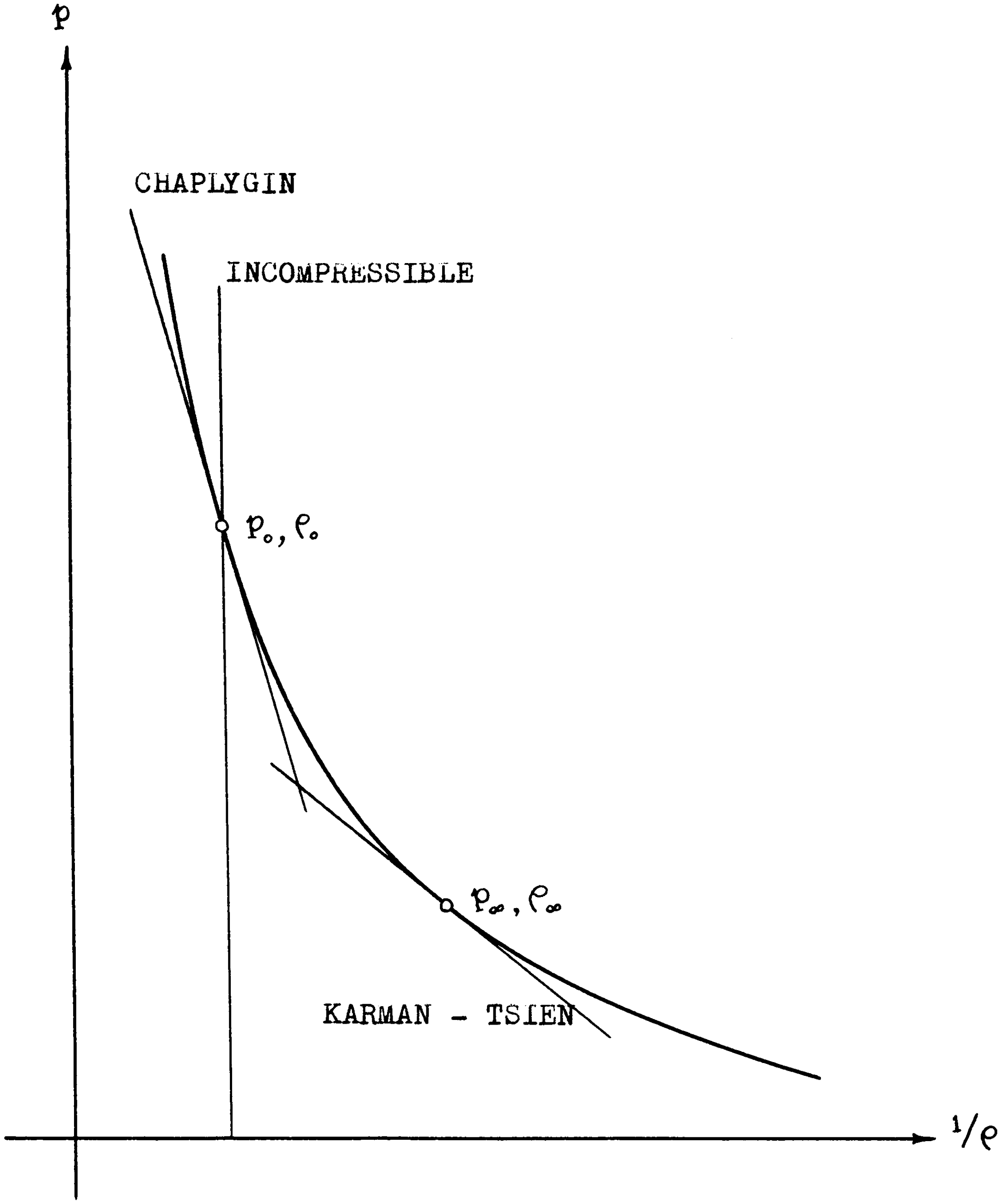

Figure 5. Approximations to the Isentrope

XIII. CONCLUSIONS

From the preceeding investigation, it may be concluded that the method of calculating subsonic compressible flows proposed by Gelbart is valid and practical for arbitrary airfoil shapes for which a convenient mapping function exists.

It was found that the method can be carried out with comparative ease as compared to numerical methods of solution and therefore lends itself to engineering use.

A particular advantage of the method is that it allows the approximate solution of the direct problem, that is, the flow about a given body at a prescribed angle of attack and freestream Mach number.

XIV. ACKNOWLEDGEMENTS

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