Solving Pure Yang-Mills Theory in 2 + 1 Dimensions

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We analytically compute the spectrum of the spin zero glueballs in the planar limit of pure Yang-Mills theory in 2 + 1 dimensions. The new ingredient is provided by our computation of a new nontrivial form of the ground state wave functional. The mass spectrum of the theory is determined by the zeroes of Bessel functions, and the agreement with large N lattice data is excellent.

The understanding of the nonperturbative dynamics of Yang-Mills theory is one of the grand problems of theoretical physics. In this Letter we announce new analytical results pertaining to the spectrum of the spin zero glueballs of 2 + 1-dimensional Yang-Mills theory [1]. This theory is expected on many grounds to share the essential features of its 3 + 1-dimensional cousin, such as asymptotic freedom and confinement, yet is distinguished by the existence of a dimensionful coupling constant. Here, we determine the ground state wave functional in the planar limit, the knowledge of which enables us to do the necessary computations regarding the mass gap, string tension and the glueball spectrum. The results are in excellent agreement with the lattice data in the planar limit [2]. Full technical details will appear in a longer publication [3].

Our approach, as in our previous work [4], is based on the remarkable work of Karabali and Nair [5]. The Karabali-Nair approach can be summarized as follows. Consider an $SU(N)$ YM$_{2+1}$ in the Hamiltonian gauge $A_0 = 0$. Write the gauge potentials as $A_i = -it^a A_i^a$, for $i = 1, 2$, where t^a are the Hermitian $N \times N$ matrices in the $SU(N)$ Lie algebra $[t^a, t^b] = if^{abc}t^c$ with the normalization $2\operatorname{Tr}(t^a t^b) = \delta^{ab}$. Define complex coordinates $z = x_1 - ix_2$ and $\bar{z} = x_1 + ix_2$, and furthermore $2A^a = A_1^a + iA_2^a$, $2\bar{A}^a = A_1^a - iA_2^a$.

The Karabali-Nair parametrization is

$$A = -\partial_z M M^{-1}, \qquad \bar{A} = +M^{\dagger-1}\partial_{\bar{z}}M^{\dagger}, \qquad (1)$$

where M is a general element of $SL(N, \mathbb{C})$. Note that a (time independent) gauge transformation $A \to gAg^{-1} - \partial g g^{-1}$, $\bar{A} \to g\bar{A}g^{-1} - \bar{\partial}g g^{-1}$, where $g \in SU(N)$, becomes simply $M \to gM$. Correspondingly, a local gauge-invariant variable is $H \equiv M^{\dagger}M$. The standard Wilson loop operator may be written

$$\Phi(C) = \operatorname{Tr}P\exp\left\{\oint_C dz\partial_z HH^{-1}\right\}. \qquad (2)$$

The definition of M implies a holomorphic invariance

$$M(z, \bar{z}) \to M(z, \bar{z})h^{\dagger}(\bar{z}), \qquad M^{\dagger}(z, \bar{z}) \to h(z)M^{\dagger}(z, \bar{z}), \qquad (3)$$

where $h(z)$ is an arbitrary unimodular complex matrix whose matrix elements are independent of $\bar{z}$. This is distinct from the original gauge transformation, since it acts as right multiplication rather than left and is holomorphic. Under the holomorphic transformation, the gauge-invariant variable H transforms homogeneously

$$H(z, \bar{z}) \to h(z)H(z, \bar{z})h^{\dagger}(\bar{z}). \qquad (4)$$

The theory written in terms of the gauge-invariant H fields will have its own local (holomorphic) invariance. The gauge fields, and the Wilson loop variables, know nothing about this extra invariance. We will deal with this by requiring that the physical state wave functionals be holomorphically invariant.

One of the most extraordinary properties of this parametrization is that the Jacobian relating the measures on the space of connections C and on the space of gauge-invariant variables H can be explicitly computed [5]

$$d\mu[C] = \sigma d\mu[H]e^{2c_A S_{\mathrm{WZW}}[H]} \qquad (5)$$

where c_A is the quadratic Casimir in the adjoint representation of $SU(N)$ ($c_A = N$), σ is a constant determinant factor and

$$S_{\mathrm{WZW}}(H) = -\frac{1}{2\pi}\int d^2z \operatorname{Tr}H^{-1}\partial HH^{-1}\bar{\partial}H + \frac{i}{12\pi}$$
$$\times \int d^3x\epsilon^{\mu\nu\lambda}\operatorname{Tr}H^{-1}\partial_{\mu}HH^{-1}\partial_{\nu}HH^{-1}\partial_{\lambda}H \qquad (6)$$

is the level $-c_A$ $SU(N)$ Wess-Zumino-Witten action, which is both gauge and holomorphic invariant. Thus the inner product may be written as an overlap integral of gauge-invariant wave functionals with nontrivial measure

$$\langle 1|2\rangle = \int d\mu[H]e^{2c_A S_{WZW}(H)}\Psi_1^*\Psi_2. \qquad (7)$$

We note that in this norm, $\Psi = 1$ is normalizable.

From these expressions it is clear that a useful gauge-invariant variable is the current

$$J = \frac{c_A}{\pi}\partial_z H H^{-1} \qquad (8)$$

which transforms as a holomorphic connection

$$J \mapsto hJh^{-1} + \frac{c_A}{\pi}\partial_z h h^{-1}. \qquad (9)$$

Note that $\bar{\partial}J$ transforms homogeneously, and a holomorphic-covariant derivative is given by

$$D^{ab} = \delta^{ab}\partial_z + i\frac{\pi}{c_A}f^{abc}J^c.$$

The standard YM_{2+1} Hamiltonian

$$\mathcal{H}_{YM} = \int \text{Tr}\left(g_{YM}^2 E_i^2 + \frac{1}{g_{YM}^2}B^2\right) \qquad (10)$$

can be also explicitly rewritten in terms of gauge-invariant variables. The collective field form [6] of this Hamiltonian [which we refer to as the Karabali-Nair (KN) Hamiltonian] can be easily appreciated from its explicit form in terms of the currents as follows:

$$\mathcal{H}_{KN}[J] = T + V = m\left(\int_x J^a(x)\frac{\delta}{\delta J^a(x)}\right.$$
$$+ \int_{x,y}\Omega_{ab}(x,y)\frac{\delta}{\delta J^a(x)}\frac{\delta}{\delta J^b(y)}\right)$$
$$+ \frac{\pi}{mc_A}\int_x \bar{\partial}J^a\bar{\partial}J^a \qquad (11)$$

where

$$m = \frac{g_{YM}^2 c_A}{2\pi},$$
$$\Omega_{ab}(\vec{x},\vec{y}) = \frac{c_A}{\pi^2}\frac{\delta_{ab}}{(x-y)^2} - \frac{i}{\pi}\frac{f_{abc}J^c(\vec{x})}{(x-y)}. \qquad (12)$$

Interpreted as a collective field theory, one can expect to compute, at large N, correlators of gauge-invariant operators. Note that the magnetic field is

$$B = -2M^{\dagger-1}\bar{\partial}(\partial H H^{-1})M^\dagger = -\frac{2\pi}{c_A}M^{\dagger-1}\bar{\partial}JM^\dagger. \qquad (13)$$

The derivation of this Hamiltonian involves carefully regulating certain divergent expressions in a gauge-invariant manner [4,5]. We note that the scale m is essentially the 't Hooft coupling.

The purpose of this Letter is to determine masses of some of the lowest lying glueball states. To do so, we wish to determine the form of the vacuum wave functional and make use of the planar limit. Accordingly, we take the following ansatz for the vacuum wave functional:

$$\Psi_0 = \exp\left(-\frac{\pi}{2c_A m^2}\int \bar{\partial}J K(L)\bar{\partial}J + \cdots\right). \qquad (14)$$

This form of the wave functional is explicitly gauge and holomorphic invariant. The kernel K is a formal Taylor expansion of $L = (D\bar{\partial} + \bar{\partial}D)/2m^2$, while the ellipsis contains terms higher order in $\bar{\partial}J$ (or B). This wave functional has the form of a "generalized coherent state" appropriate to large N [6], but its form is not completely dictated by large N counting. The form of the ansatz, as we shall see, is sufficient to capture the mass spectrum of gauge-invariant states, which we will probe using local operators. The large N limit ensures that these states are noninteracting, but we are also neglecting the size of the states by using local probes. (For further details on these points, see Ref. [3].)

In order to be physically sensible, K should have certain properties at long and short distances. We derive these properties below. In particular, the low momentum (large 't Hooft coupling) limit, $p^2 \ll m^2$, of the vacuum wave functional is easily determined to be of the form

$$\Psi_0 = \exp\left(-\frac{1}{2g_{YM}^2 m}\int \text{Tr}B^2\right). \qquad (15)$$

(Equivalently, at low momentum, we should have $K \to 1$.) This wave functional provides a probability measure $\Psi_0^*\Psi_0$ equivalent to the partition function of the Euclidean two-dimensional Yang-Mills theory with an effective Yang-Mills coupling $g_{2D}^2 \equiv mg_{YM}^2$. Using the results from Ref. [7], Karabali, Kim, and Nair deduced the area law for the expectation value of the Wilson loop operator

$$\langle \Phi \rangle \sim \exp(-\sigma A) \qquad (16)$$

with the string tension following from the results of [7]

$$\sigma = g_{YM}^4\frac{N^2-1}{8\pi}. \qquad (17)$$

This formula agrees nicely with extensive lattice simulations [2], and is consistent with the appearance of a mass gap as well as the large N 't Hooft scaling.

Coming back to the derivation of the vacuum wave functional, we argue in Ref. [3] that operators $\mathcal{O}_n \equiv \int \bar{\partial}J L^n \bar{\partial}J$, which would appear in a series expansion of $K(L)$, satisfy

$$T\mathcal{O}_n = (2+n)m\mathcal{O}_n + \cdots. \qquad (18)$$

In Ref. [3] (see Sec. 3 and App. A), we have presented a series of calculations supporting this important result. Further evidence is provided by lattice considerations. Given this, we can formally write

$$TK(L) \longrightarrow \frac{1}{L}\frac{d}{dL}[L^2 K(L)]. \qquad (19)$$

The full vacuum Schrödinger equation, combining all contributions self-consistently to quadratic order in $\bar{\partial}J$

$$\mathcal{H}_{KN}\Psi_0 = E_0\Psi_0 = [\cdots + \int \mathrm{tr}\bar{\partial}J\mathcal{R}\bar{\partial}J + \cdots]\Psi_0,$$

$$(20)$$

with suitable subtractions, then formally leads to the following differential equation for K:

$$\frac{c_A m}{\pi}\mathcal{R} = -K - \frac{L}{2}\frac{d}{dL}[K(L)] + LK^2 + 1 = 0. \quad (21)$$

In this equation, the final term is the contribution of the potential B^2 term of the KN Hamiltonian, while the second to last term arises from the Ω term in the kinetic energy. Equation (21) comes by consistently keeping all terms quadratic in $\bar{\partial}J$ in the Schrödinger equation.

Although this equation is nonlinear, it is easily solved by substituting $K = -y'/2y$; the resulting equation may be recast as a Bessel equation. The only normalizable solution has the correct physical asymptotics for large and small L and is given by

$$K(L) = \frac{1}{\sqrt{L}}\frac{J_2(4\sqrt{L})}{J_1(4\sqrt{L})} \quad (22)$$

where J_n denotes the Bessel function of the first kind. This remarkable formula encodes information on the spectrum of the theory, as we show below. We note that this kernel has the following asymptotics (where $L \sim -p^2/4m^2$):

$$p \to 0, \qquad K \to 1; \qquad p \to \infty, \qquad K \to 2m/p \quad (23)$$

consistent with confinement and asymptotic freedom, respectively.

In order to determine the spectrum, we factorize suitable correlation functions at large distances. The operators appearing in the correlation functions will have definite J^{PC} quantum numbers, which will be inherited by the single particle poles contributing to the correlation function.

As a first example, we consider the 0^{++} states which may be probed by the operator $\mathrm{Tr}\bar{\partial}J\bar{\partial}J$. We have

$$\langle \mathrm{Tr}(\bar{\partial}J\bar{\partial}J)_x \mathrm{Tr}(\bar{\partial}J\bar{\partial}J)_y \rangle \sim [K^{-1}(|x-y|)]^2. \quad (24)$$

Here, we have computed the correlation function in the planar limit given our knowledge of the vacuum wave functional.

To proceed further, we note the identity

$$\frac{J_{\nu-1}(z)}{J_\nu(z)} = \frac{2\nu}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2 - j_{\nu,n}^2} \quad (25)$$

where $j_{\nu,n}$ are ordered zeros of the Bessel functions. For example, the first few zeros [8] of $J_2(z)$ are $j_{2,1} = 5.14$, $j_{2,2} = 8.42$, $j_{2,3} = 11.62$, $j_{2,4} = 14.80$, etc. Apart from additive constants, we then deduce

$$K^{-1}(k) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{M_n^2}{M_n^2 + k^2} \quad (26)$$

where $M_n = j_{2,n}m/2$. The Fourier transform at large $|x - y|$ is

$$K^{-1}(|x-y|) = -\frac{1}{4\sqrt{2\pi|x-y|}}\sum_{n=1}^{\infty}(M_n)^{3/2}e^{-M_n|x-y|}.$$

$$(27)$$

In particular the 0^{++} correlator mentioned above is

$$\approx \frac{1}{32\pi|x-y|}\sum_{n,m=1}^{\infty}(M_nM_m)^{3/2}e^{-(M_n+M_m)|x-y|}. \quad (28)$$

Note that each term here has the correct $|\vec{x} - \vec{y}|$ dependence for a single particle pole of mass $M_n + M_m$ in $2 + 1$ dimensions. The 0^{++} glueball masses are

$$M_{0^{++}} = M_1 + M_1 = 5.14m,$$
$$M_{0^{++*}} = M_1 + M_2 = 6.78m,$$
$$M_{0^{++**}} = M_2 + M_2 = 8.42m, \quad (29)$$
$$M_{0^{++**'}} = M_1 + M_3 = 8.38m,$$
$$M_{0^{++***}} = M_2 + M_3 = 10.02m.$$

Since m is not a physical scale, we should rewrite these results in terms of the string tension. Given equations presented above, at large N we have $\sqrt{\sigma} \simeq \sqrt{\frac{\pi}{2}}m$. Our results are given in Table I. Several comments are now in order. First, note that we have been able to predict masses of the 0^{++} resonances, as well as the lowest lying member, in contrast to the original results of Karabali and Nair (which differ significantly numerically). The supergravity results listed in the table are a result of calculations [9] using the anti-de Sitter/conformal field theories (AdS/CFT) correspondence [11]; in that case, the overall normalization was not predicted but was determined by fitting to the lattice data, for example, to the mass of the lowest 0^{++} glueball. Our results for the excited state masses differ at the 10–15% level from the lattice results. We note that precisely these masses are more difficult to compute on the lattice [12], and thus the apparent 10–15% discrepancy may be illusory [13]. Finally, we note that there is some interesting approximate degeneracies in the spectrum.

TABLE I. 0^{++} glueball masses in YM_{2+1}. All masses are in units of $\sqrt{\sigma}$. AdS/CFT computations [9] are also given for comparison. The percent difference between our prediction and lattice data is given in the last column.

State	Lattice, $N \to \infty$	Sugra	Our prediction	Diff,%
0^{++}	4.065 ± 0.055	4.07(input)	4.10	0.8
0^{++*}	6.18 ± 0.13	7.02	5.41	12.5
0^{++**}	7.99 ± 0.22	9.92	6.72	16
0^{++***}	9.44 ± 0.38^a	12.80	7.99	15

[a]Mass of 0^{++***} state was computed on the lattice for $SU(2)$ only [10]. The number quoted here was obtained by a simple rescaling of $SU(2)$ result.

P H Y S I C A L   R E V I E W   L E T T E R S

TABLE II. 0^{--} glueball masses in YM_{2+1}. Columns are as in Table I.

State	Lattice, $N \to \infty$	Sugra	Our prediction	Diff,%
0^{--}	5.91 ± 0.25	6.10	6.15	4
0^{--*}	7.63 ± 0.37	9.34	7.46	2.3
0^{--**}	8.96 ± 0.65	12.37	8.77	2.2

Let us move on to a discussion of the 0^{--} glueball resonances. In this case, our predicted masses are much closer to the lattice data, which we believe to be more reliable in this case. We may probe these states with the operator $\mathrm{Tr}\bar{\partial}J\bar{\partial}J\bar{\partial}J$. We are thus interested in the correlation function [14]

$$\langle \mathrm{Tr}(\bar{\partial}J\bar{\partial}J\bar{\partial}J)_x \, \mathrm{Tr}(\bar{\partial}J\bar{\partial}J\bar{\partial}J)_y \rangle \sim [K^{-1}(|x-y|)]^3. \quad (30)$$

Using the results given above, we obtain glueball masses which are the sum of three M_n's.

$$\begin{aligned} M_{0^{--}} &= M_1 + M_1 + M_1 = 7.70m, \\ M_{0^{--*}} &= M_1 + M_1 + M_2 = 9.34m, \quad (31) \\ M_{0^{--**}} &= M_1 + M_2 + M_2 = 10.99m. \end{aligned}$$

These results are compared to lattice and supergravity data in Table II. We see that the resulting masses are within a few percent of the lattice data, and are much better than the supergravity predictions.

Our results suggest that there exist hidden constituent as well as integrable structures in $2 + 1$-dimensional Yang-Mills theory. Note that the full integrability of $2 + 1$-dimensional pure Yang-Mills has been suspected for some time [15]. In a longer paper [3], we will more carefully explain our techniques and results and will investigate other J^{PC} glueball states and the corresponding Regge trajectories. It has not escaped our attention that a similar parametrization may be used in $3 + 1$ dimensions in a variational sense and preliminary numerical results are encouraging.

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