SCHEDULING OPTIMAL MAINTENANCE TIMES FOR A SYSTEM BASED ON COMPONENT RELIABILITIES

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(ABSTRACT)

This dissertation extends the work done on single component maintenance planning to a multi-component series system. An attempt is made to develop a function which represents the expected cost rate (cost per unit time) of any maintenance plan. Three increasingly complex cases are considered.

The first and simplest case assumes that the component is restored to an "as good as new" condition after a maintenance operation. The second case assumes that an occasional imperfect maintenance operation may occur. During this period of time, the failure rate of the component is higher. Hence, the likelihood of a failure is greater until the component is properly maintained in a subsequent maintenance operation. The final case assumes that there is some deterioration in the component behavior even after a maintenance operation. Therefore, it is necessary to replace the system at some point in time.

Models for all three cases are developed. Based on these models, cost rate functions are constructed. The cost rate functions reflect the cost rates of maintaining a component at a particular time. In addition, the savings obtained through the simultaneous maintenance of components is also accounted for in the cost rate functions. A series of approximations are made in order to make the cost rate functions mathematically tractable. Finally, an algorithmic procedure for optimizing the cost rate functions for all three cases is given.

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CHAPTER 1 INTRODUCTION

1.1 PROBLEM DESCRIPTION

Modern man has come to depend on machines (systems) of considerable complexity. Examples of these systems are power stations, computer systems, and aircraft. The unexpected failure of a system could result in huge delays, loss of the system itself, and even loss of life. In order to reduce the likelihood of a failure with such severe consequences, systems are frequently inspected and maintained. Such activities which are aimed at reducing the chance of a randomly occuring failure are refered to as Preventive Maintenance (PM).

A preventive maintenance plan is used when the costs of several planned interruptions of system operation are much smaller than the expense incurred due to a random system failure. The maintenance plan for a system is determined largely by the characteristics of the system and includes the times at which maintenance is performed. For a multi-component system, the maintenance plan specifies the times at which each of the components is to be maintained. The maintenance plan often provides for more frequent maintenance of the less reliable components. In addition, a "good" maintenance plan should ensure that the risk of failure is reduced to manageable levels while keeping the cost of maintenance operations within acceptable limits.

Typically, the maintenance cost includes the cost of the actual maintenance (such as replacement of the component) and a setup cost. Setup cost refers to the cost incurred every time maintenance is performed and includes the costs of system startup and system shutdown, the cost of bringing in a maintenance crew, and other similar activity initiation costs. The setup cost is typically assumed to be constant and is independent of the number of components and the type of components being maintained at a given time.

The following example is intended to give the reader an idea of the type of systems that have motivated this research.

Aircraft Maintenance: In the case of commercial aircraft, certain components such as the engine blades and landing gear are subject to great stress and, are more likely to be damaged than other components. These components are also easy to inspect and so are maintained after a few flights. The cockpit instrumentation is very reliable and relatively easy to check and maintain, and are also maintained after a few flights. The more complex hydraulic system is well protected from environmental stress and is quite reliable. It is usually overhauled after a moderate number of flights. The fuselage of a commercial plane is designed with a large factor of safety and is very reliable. A thorough maintenance operation on the fuselage is performed only a few times in a year. However, due to the size of the fuselage, the maintenance cost is very high.

From the example given above, it is seen that a realistic maintenance schedule for a multi-component system may have a distinct maintenance time for each component. The intervals between maintenance for the more reliable components, such as the fuselage, are usually longer. However, as seen in the case of the instrumentation, it may be preferable to maintain even reliable components after relatively short intervals if the cost of maintenance is low. In addition, it may be possible to group components together and perform several maintenance operations at one time. This would result in fewer setup operations and avoid the costs associated with setup.

Most of the available literature on system maintenance treats the entire system as a single unit. Therefore, only a single maintenance time is determined for the entire system. In the case of a single-unit system, it is relatively easy to obtain an expression for the expected cost per unit time of a particular maintenance policy by viewing each maintenance operation as a renewal epoch (refer to the review of renewal theory given below).

An efficient maintenance plan for a multi-component system is comprised of schedules of the maintenance times for various groups of components. All components in a group have the same maintenance times. In the case of maintenance plans for multiple groups, it is no longer correct to view the time of a maintenance operation as a renewal epoch because the renewal is only for a certain component(s) rather than for the entire system.

1.2 RESEARCH OBJECTIVES

This aim of this research to develop a method of generating maintenance plans for multi-component systems such as the one described above. The maintenance plans are based on the failure rates of the individual components, as well as the costs of maintenance and failure. In addition, grouping of components for simultaneous maintenance is considered in order to determine the feasibility of reducing the number of setup operations.

1.3 MODEL DEVELOPMENT

In this document a series system is considered. The components are mutually independent and fail according to a Non-Homogeneous Poisson Process (NHPP). The preventive maintenance policy is assumed to be an age-replacement policy in which components are replaced on reaching a specified age or upon failure.

First, a mathematical model of the system is developed to show the effect of the individual component failure rates and the consequences of an age-replacement maintenance policy. A cost-rate function is constructed and analyzed in order to obtain efficient maintenance schedules.

A sequence of three increasingly complex cases is considered. Mathematical models for all three are developed and analyzed. Finally, algorithms for obtaining maintenance schedules for all three cases are given.

1.4 A BRIEF REVIEW OF RENEWAL THEORY

Renewal theory is typically used in modeling the behavior of processes that are repetitive in the sense that their realizations are stochastically identical. The study of renewal models has led to the definitions of many results which are useful in the computation of time-averages.

Definition: A renewal process is defined as a non-negative integer-valued stochastic process, $[N(t), t \geq 0]$, that registers the successive occurances of an event during the time interval (0,t], where the time durations between events are independent, identically distributed (i.i.d.) random variables (Cox [12], Wolff [35]). Renewal processes are used to model queues, inventory demands, and many other physical systems.

One of the quantities of interest in a renewal process is the expected number of renewals in a time interval. Let $M(t)$ denote the expected number of renewals by time t, then $M(t) = E[N(t)]$. By conditioning on the time of the first renewal, it is possible to obtain the following equation.

$$M(t) = F(t) + \int_0^t M(t-x)dF(x) \qquad [1.4.1]$$

The above equation is known as the Fundamental Renewal Equation and can sometimes be solved for $M(t)$ (Ross [31]). Applying the strong law of large numbers, it is possible to show that

$$\lim_{t \to \infty} \frac{M(t)}{t} = \frac{1}{\mu} \qquad [1.4.2]$$

where μ is the average length of a renewal interval. A result frequently used in maintenance modeling is stated as the Renewal-Reward theorem which is based on the strong law of large numbers.

Renewal-Reward Theorem: Let X_i define the length of the i^{th} renewal interval and R_i define the reward in that interval. Assume $\{R_i\}$ is a sequence of i.i.d. random variables where R_i may depend on X_i but is independent of $\{X_j, j \neq i\}$. Finally, assume $E[|R_i|] < \infty$ and $E[X_i] < \infty$. Then if $C(t)$ is the cumulative reward until time t, the average reward rate is given by:

$$\lim_{t \to \infty} \frac{C(t)}{t} = \frac{E(R_i)}{E(X_i)} \qquad [1.4.3]$$

This result does not depend on whether the rewards are collected at the beginning of each renewal interval, at the end of the interval, or whether they accumulate in some more complicated way in the interval. It will be seen later that the cost rate expressions for most maintenance policies are derived using the above theorem.

1.5 PREVENTIVE MAINTENANCE POLICIES

Preventive Maintenance (PM) policies for units which are continuously monitored are broadly classified as follows (Berg and Epstein [5]):

1. **Age-Replacement Policy:**

   Under an age-replacement policy, a unit is replaced either on reaching an age T_a or on failure. The time of replacement can be considered to be a renewal point. Then, using the Renewal-Reward Theorem, the average cost, $E[C_{T_a}]$, of an age-replacement policy is:

$$E[C_{T_a}] = \frac{C_f P(T \leq T_a) + C_m P(T > T_a)}{\int_0^{T_a} x f(x) dx + T_a P(T > T_a)} \qquad [1.5.4]$$

where C_f is the cost of component failure, C_m is the cost of component maintenance (or replacement), T is the time to failure, $P(T > Y)$ is the survival probability of the component beyond age Y, and $P(T \leq Y)$ is the probability that the component fails before age Y.

The numerator represents the expected cost incurred during the renewal interval. The denominator represents the expected length of the renewal interval. Cox [12] shows that $\int_0^{T_a} x f(x) dx + T_a P(T > T_a) = \int_0^{T_a} P(T > x) dx$. Thus, the following expression is obtained.

$$E[C_{T_a}] = \frac{C_f P(T \leq T_a) + C_m P(T > T_a)}{\int_0^{T_a} P(T > x) dx} \qquad [1.5.5]$$

2. **Block-Replacement Policy:**

In block replacement, a unit is replaced at multiples of time T_b independent of its failure history. The renewal points are the times of planned replacement, namely multiples of T_b. Hence, it can be shown that:

$$E[C_{T_b}] = \frac{C_f M[T_b] + C_m}{T_b} \qquad [1.5.6]$$

where $M[T_b] = E[N[T_b]]$ is the expected number of failures in an interval of length T_b.

3. **Opportunistic-Replacement Policy:**

In an opportunistic-replacement policy, replacement of a functioning unit is allowed only at some specific points in time and only if some specified conditions are met. An example of a typical Opportunistic Replacement Policy is given by Haurie and L'Ecuyer [18] in which replacement of functioning components may be replaced if an " opportunity" arises. The policy specifies that a functioning component is replaced only if when its age exceeds some specified value and a and a failed component is being replaced. Such policies are aimed at improving system performance measures by taking advantage of unplanned "opportunities" for replacement.

Note that in the single component case, it is possible to view the replacement cost as the cost of actually replacing the component as well as the cost of setup.

1.6 OUTLINE OF THE DISSERTATION

Chapter 2 provides a review of the relevant literature. Specifically, two topics are discussed; age-replacement policies and failure models. The basic age-replacement policy and subsequent modifications are considered. In addition, methods for obtaining optimal replacement times are also discussed. Finally, some failure models are reviewed and their limitations discussed.

The most basic system model is described in Chapter 3. A cost-rate function (CM1) is developed for this model. This cost function is a modification of the single component age-replacement cost function given by equation 1.3.5 and is based on the fact that until the time of failure of the system, each component group is independent of the others. Finally, the disadvantages of using CM1 are pointed out.

In Chapter 4, the system being considered is identical to the one in Chapter 3. Prior to any analysis, a general form for the system cost rate function is proposed. This is used to obtain expressions for the system cost rate for all subsequent models. The behavior of the system is analyzed and a characterization of the hazard rate is obtained. Based on this characterization, Cost Model 2 (CM2) is developed using the general form of the cost function. Using CM2, it is shown that if the failure rate of the components is approximately constant, then the system replacement issue need not be considered. In addition, algorithmic procedures for grouping components and determining group maintenance times are given. Finally, the results concerning the structure of the cost-rate function and conditions for grouping are obtained.

In Chapter 5, it is assumed that maintenance operations are not necessarily perfect and the component may not be restored to an "as good as new" condition. CM2 is suitably modified to include the possibility of a component having a higher failure rate due to an imperfect maintenance operation The new cost rate function is referred to as CM3. The distribution function for the system life is obtained and from this an expression for the expected system life is developed. As this expression is not mathematically tractable, approximations for the expected system life are developed. Since the structure of the CM3 is similar to CM2, the optimizing procedure and the related analyses are only briefly mentioned.

A slightly different system model is considered in Chapter 6. This model (CM4) allows for the possibility of permanent damage to the components. After every maintenance operation, the expected failure rate is assumed to increase. Consequently, the system failure rate increases and at some point the system has to be replaced. The distribution for the system life is obtained and is used to obtain the system hazard rate and the expected life of the system. These are used to obtain an expression for the system cost rate. Approximations for the system cost rate are then developed. An optimization procedure for this approximation is provided.

Chapter 7 contains the results of the numerical analyses performed on the various models. A few simple illustrative examples of the optimization procedure for models CM2 and CM4 are given. The procedure is used to solve for the grouping and maintenance times of components in a small system (5 and fewer components). These solutions are compared with the optimal groupings and times obtained through an exhaustive enumeration of the possible groupings. It is observed that in most cases the solution obtained by using the procedure gives the global optimal. Another interesting observation is that the LP relaxation of the assignment problem gives integer solutions even for systems with a 100 components. Note, however, that integer solutions are not guaranteed if only the LP relaxation is solved. Some of systems are simulated in order to validate the numerical results.

Conclusions and suggestions for future work are contained in Chapter 8. A discussion of the theoretical and numerical results obtained in the earlier chapters is provided. Based on this discussion, topics for future research are suggested.

CHAPTER 2 LITERATURE REVIEW

2.1 INTRODUCTION

This chapter provides a brief review of the literature relevant to the development and analysis of the cost-rate functions presented in the later chapters. This study includes a description of the various models of system failure and of the preventive maintenance policies which pertain to this research effort.

2.2 PREVENTIVE MAINTENANCE POLICIES

In the previous chapter, the three basic types of preventive maintenance policies are described. However, as the maintenance plans developed in the subsequent chapters are based on age-replacement policies for multi-component systems, the focus of this review will be age replacement policies and maintenance policies for multi-component systems. Single component maintenance policies are reviewed in detail by Valdez-Flores and Feldman [34]. Literature dealing with block-replacement or opportunistic replacement policies may be found in the review by Pierskalla and Voelker [29].

2.2.1 Age Replacement Policies:

The basic age-replacement policy as described by Barlow and Proschan [3] is defined in Chapter 1. Many researchers have modified the basic age-replacement policy in an effort to deal with specific types of systems. The focus of most literature on such age-replacement related maintenance policies is the search for optimal policies. In the case of systems which are to be used for long periods of time, the aim is typically to determine an expression for the long run cost per unit time and then to minimize it.

Block, Borges, and Savits [7] discuss some modified age-replacement policies seen in recent literature and present the corresponding expressions for the cost per unit time (cost rate). Among the more interesting modifications to age-replacement policies are maintenance policies in which a failed component may be restored to the condition it was in just prior to failure. Such maintenance operations are known as minimal repairs. If there is no failure, the component is replaced at age T. An example of minimal repair policy is given by Cleroux, Dubuc, and Tilquin [11]. They assume the repair cost is random and is denoted by C, the replacement cost at failure is c_1, and the cost of a planned replacement cost is c_2. They also assume that a replacement is only performed if $C < \delta c_1$ for some $0 < \delta < 1$. They then proceed to obtain an expression for the cost per unit time.

Block, Borges, and Savits then develop a general form for the cost rate of age-replacement policies which allow for minimal repairs of the failed component. The expression allows for the possibility that the probability and the cost of minimally restoring a component may vary with time.

Aven and Bergman [2] show that the general form of the expressions for the cost rates of all age-replacement policies is:

$$\frac{E[\int_0^T a(x)h(x)dx + c(0)]}{E[\int_0^T h(x)dx + p(0)]} \qquad [2.2.1]$$

where T is a stopping time, $\{a(x)\}$ is a non-decreasing stochastic process, $\{b(x)\}$ is a non-negative stochastic process, and $c(0)$ and $p(0)$ are non-negative random variables. The control variable is T. Hence, the optimal policy is determined by the value of T which minimizes the above expression.

It can be shown that if the failure rate is constant then the optimal time of maintenance is $T = \infty$ (Barlow and Proschan [3]). However, Okumoto and Elsayed [25] show that it is possible for a system of components in parallel, each with a constant failure rate, to have a finite optimal time of maintenance. The intuition is that after components fail, the rate of failure of the group changes as there are fewer components failing.

Bergman [6] develops methods to optimize the cost rate function of a single component which fails according to a non-homogeneous Poisson process (NHPP). Aven and Bergman [2] setup the problem in the form seen in equation 2.2.1 in order to use a fractional programming method which they refer to as λ optimization. They show that minimizing the above expression is equivalent to determining the optimal T and λ which minimizes the expression given below.

$$C^T{}_\lambda = E[\int_0^T a(x)h(x)dx + c(0)] - \lambda E[\int_0^T h(x)dx + p(0)] \qquad [2.2.2]$$

This method is shown to converge very quickly when there is only a single optimal variable T to be determined. However, it is not possible to use this method to determine the optimal component groups and the optimal maintenance times for groups of components.

The above models assume that a maintenance operation makes the component "as good as new" (except in the case of minimal repair). In an effort to develop more realistic system models, repairs are assumed to be imperfect. Imperfect repairs are repair operations which do not make the component "as good as new". These models make it a point to distinguish between replacements (which renew the component) and repairs (where the expected component life may be reduced).

Recent work by Yeh [37] assumes that the expected life of a component after a repair reduces in a geometric fashion while the expected duration of a repair operation increases geometrically. He suggests two distinct maintenance policies. The first is a replacement policy in which the component is replaced at a particular age. The second is a replacement policy in which the component is replaced after a certain number of repairs. Expressions for the cost rates for both policies are obtained. Rangan and Grace [30] modify the first policy to allow repairs which make the component "as good as new" (with probability p) and repairs after which the component has a reduced expected life (with probability $(1-p)$). They show that if $p = 0$, their cost function is the same as Yeh's. In another paper, Yeh [38] obtains a more general cost function in which he assumes the expected component life-lengths after a repair form a non-increasing sequence and the repair times form a non-decreasing sequence. He also gives a procedure for obtaining the optimal number of failures after which the component should be replaced.

2.2.2 Multi-Component Systems:

When components are maintained separately, many of the assumptions made in the single component models are no longer be valid. For example, the times of component maintenance and replacement are no longer renewal points for the system. In addition, single component cost rate expressions typically assume that the cost of setup is included for in the cost of maintenance. This would not be possible in the case of multi-component systems where components are often maintained simultaneously.

Cho and Parlar [9] give a detailed review of multi-component systems. The survey indicates that a large portion of the literature on multi-component systems deals with systems containing identical components and constant failure rate. The authors point out that independence between components can be of two types, economic and stochastic. Economic independence implies that maintenance expenses for a component does not affect the maintenance expenses for other components. Similarly, stochastic independence implies that the improper functioning of a component does not affect the performance of other components. It is shown that if a system has components which are stochastically independent and economically independent, the maintenance schedule for the components can be determined separately.

Haurie and L'Ecuyer [18] also assume a system consisting of identical components but allow the rate of component failure to depend on the age of the component. They suggest a maintenance policy in which where maintenance is performed only if a component has failed. They allow only opportunistic maintenance. A characterization of the optimal value of the average availability is given. This turns out to be too difficult to implement. An alternative procedure which is typically suboptimal is suggested.

Ozekici [26] also assumes a multi-component system with identical components where maintenance is performed on a component at a particular age or at failure. He then proceeds to prove that it may be optimal to replace a functioning component even if it has not reached the age of replacement. In other words, the optimal policy may be an opportunistic maintenance policy. However, while the optimal solution has been characterized, neither Haurie and L'Ecuyer nor Ozekici suggests a method for an optimal grouping of components and obtaining optimal group maintenance times.

2.3 FAILURE MODELS

Failure models are used to describe stochastic component failure processes. A majority of the models use either the non-homogeneous Poisson process (NHPP) or the shot-noise process to model the failure process. It is possible to increase the complexity of the failure models but it also becomes correspondingly more difficult to obtain solutions to these models. In addition, optimization of these models becomes very difficult and it is often necessary to approximate these models with a simpler form.

Many papers assume a system failure process to be a NHPP since these processes are relatively easy to analyze. It is often possible to obtain explicit characterizations of optimal policies when the failure process is a NHPP. Bergman [6] assumes a NHPP failure process and proceeds to prove the existence of an optimal single-component maintenance policy. He then proves that the optimal policy is a control-limit policy. Control-limit policies are policies in which the system is replaced either at the time of failure or when a particular system parameter such as the system age or system wear exceeds a threshold value.

Aven [1] also assumes a NHPP failure process to develop a discounted cost rate function for a minimal repair maintenance strategy. The cost function is then put in form of equation 2.2.1 and the λ optimization technique is used. A description of the iterative procedure used to obtain the optimal solution is given.

Another failure process which is frequently discussed is the shot-noise failure process. The shot-noise failure process assumes jolts (or shocks) of magnitude D occur at random points in time (Lemoine and Wencour [20]). Each jolt causes some stress to the system. The stress is additive and when the total stress in the system exceeds a certain limit, the system fails. The arrival distribution of the shocks in the shot-noise model is usually assumed to be a homogeneous Poisson distribution. However many of the current models allow the arrival process to be a non-homogeneous Poisson process as well. When the arrival distribution is Poisson, the models are known as cumulative damage models (CDM) (Barlow and Proschan [3]).

Selection of a shot-noise failure process allows for a system model where the system has different levels (states) of operation. Such systems are known as multi-state systems (Natvig [23]) where each state is identified by the amount of damage (or stress) there is in the system.

A model similar to the shot-noise model in which there is a degradation in the system behavior as a result of shocks is developed by Neuts and Bhattacharjee [24]. They assume that the rate of arrival of the shocks depends on the state of the system. The system state changes after every shock. The distribution of the time to failure and rate of failure of the system are obtained by conditioning on the state of the Markov process.

A more general form of the shot-noise process is the diffusion process. In a diffusion process, it is assumed that the levels of operation are continuous as opposed to the shot-noise process in which where the levels of operation are discrete. Typically the differential equations associated with these processes are very difficult to solve. However, there are many classes of diffusion processes that permit mathematically tractable solutions. The best known of these processes which are the time homogeneous Markov Processes. Lemoine and Wencour [21] describe in detail the characteristics of some of these classes. In a very thorough paper by Bendell and Humble [4], the failure process of the system is described in terms of both a shot-noise process and a diffusion process. Performance measures such as average time to failure and expected level of operation are obtained and a numerical example is provided.

A major failing in many models using shot-noise and diffusion processes is that while elegant methods are used to obtain the distribution of time to failure, they do not account for maintenance operations renewing the component (or system). However, Lemoine and Wencour [21] derive a class of failure distributions which allows for the system to heal itself. This model can be suitably modified to account for system maintenance by assuming that the system "heals" itself at fixed points in time which would correspond to the times of maintenance.

While the shot-noise and diffusion processes give a stronger characterization of the system failure process, they are very difficult to optimize. Typically, models of systems which are optimized assume the NHPP as the failure process due to the simpler form.

2.4 REMARKS

It is indicated in Chapter 1 that in most practical systems, not all components are maintained at the same time. It is not possible to directly extend the models discussed here to the type of system discussed in Chapter 1. When there are numerous component groups being maintained at different times, it is difficult to define a cost rate function in a form similar to that suggested by Aven and Bergman [2]. Likewise, characterizing the system failure probabilities is not simple when components are maintained separately. Other system performance measures such as availability and average system life are also difficult to obtain.

It is also observed that most of the age-replacement policies determine a single optimal time of maintenance. Thus, many of the optimization methods discussed in literature would require significant modifications in order to treat the multi-component system which has component groups being maintained at different times.

CHAPTER 3 DEVELOPMENT OF COST MODEL 1 (CM1)

3.1 INTRODUCTION

In Chapter 1, the basic age-replacement policy for a single unit system is discussed. In this chapter, a maintenance policy for a multi-component series system is developed. Each component is independent of the other components and fails according to a non-homogeneous Poisson Process (NHPP). The maintenance schedule for the system is similar to the basic age-replacement policy where maintenance is performed on a component when that component reaches a certain age. Every maintenance operation is assumed to "renew" the component. In addition, every time a maintenance operation is performed, the system is shut down and a setup cost is incurred. It is possible that by grouping the components there is a saving in the setup costs and in system down-time. The aim is to determine whether the grouping of components results in any savings in cost. If there are any savings by grouping, it is desired to determine the component groups and the times of maintenance of the different groups which minimizes the system cost rate.

If there are several groups of components, the cost-rate equation (1.3.1) for the age-replacement policy is no longer valid. In this chapter, equation 1.3.1 is extended to the case of multiple groups of components where each group of components has its own distinct time of maintenance. The cost function developed in this chapter is refered to as CM1. CM1 is based on the fact that until the time of failure of the system, the renewal behavior of each group is independent of the other groups. Hence, the stochastic process describing system behavior can be approximated by considering it to consist of several independent renewal processes. The limitations of CM1 are discussed at the end of the chapter.

3.2 ASSUMPTIONS

A statement of the assumptions is given below. Some of these assumptions are implicit in the description of of the system.

1. The system is a series system.
2. The failure process for each of the components is an increasing failure rate (IFR) process (Barlow and Proschan [3]).
3. The failure rate of a component does not affect the failure rate of any other component.
4. The system failure rate is the sum of the failure rates of the individual components.
5. A maintenance action on a component makes the component " as good as new ".
6. System operation is suspended during any maintenance operation. Thus, no component aging occurs during any maintenance operation.
7. The setup cost incurred during a maintenance action is the same irrespective of the number of components being maintained and the type of components being maintained.
8. Each component has a specific maintenance cost.
9. Failure of the system results in the entire system being replaced, i.e. every component is renewed.

3.3 NOTATION AND TERMINOLOGY

The following notation is used for the remainder of this document. Additional notation is provided in later chapters where necessary.

1. m - number of components in the system.
2. N - number of components groups.
3. n_i - number of components in group i.
4. T_i - time of maintenance of component group i.
5. $E[T_i]$ - expected value of T_i.
6. T_{sys} - time of system replacement.
7. $E[t]$ - expected time for system renewal.
8. S - set of times at which maintenance can be performed. Every group of components is maintained at one of the times in S.
9. C_f - cost of system failure.
10. C_{mi} - cost of maintenance of component i.
11. C_s - cost of setup.
12. C_{sys} - expected cost rate of system maintenance.
13. $E[C_{T_j}]$ expected cost of a renewal of length T_j.
14. $R_i(t)$ - probability of component i functioning at time t.
15. $R_{sys}(t)$ - probability of the system functioning at time t.
16. $F_{sys}(t) = 1 - R_{sys}(t)$ - probability of the system having failed by time t.
17. $f_{sys}(t)$ - density function of time to system failure
18. $\lambda_i(t)$ - failure rate of component i at component age t.
19. $\lambda_{sys}(t)$ - failure rate of the system at system age t.
20. $\Lambda_i(t)$ - integral of the failure rate of component i from 0 to t. $\Lambda_i(t) = \int_0^t \lambda_i(x)dx$.

3.4 COST MODEL 1 (CM1)

CM1 is based on the premise that the expected cost per unit time (cost rate) for the system is the sum of the expected cost rates of the separate groups of components. To illustrate, consider a two component system. Suppose the optimal maintenance times for components 1 and 2 are 10 and 20 respectively (ignoring setup costs). Then, in a time period of 20 units the total cost consists of the cost of one maintenance operation for component 2 and the cost of two maintenance operations for component 1. The cost of system maintenance per unit time is then the above sum divided by 20. It can be easily seen that this is the cost of maintenance for component 1 divided by 10 (the time to maintenance) plus cost of maintenance for component 2 divided by 20 (the time to maintenance).

Consider N groups of components being maintained at N distinct times. A setup cost, C_s, is incurred every time a group is maintained. Each maintenance operation on a group results in a renewal for that group but not for the system. By assuming that the groups are independent, it is possible to obtain an approximation for the cost rate.

For each group j consisting of n_j components, and being maintained at time T_j, the expected cost of a renewal is:

$$E[C_{T_j}] = \sum_{i=1}^{n_j} [C_{mi}R_i(T_j) + C_f(1 - R_i(T_j))] + C_s \qquad [3.4.1]$$

The system cost rate is the sum of $\frac{E[C_{T_j}]}{E[T_j]}$ over all groups, where $E[T_j]$ is the expected length of a renewal for group j. Therefore,

$$C_{sys} = \sum_{j} \frac{\sum_{i=1}^{n_j} [C_{mi} R_i(T_j) + C_f (1 - R_i(T_j))] + C_s}{E[T_j]} \qquad [3.4.2]$$

3.5 OPTIMIZATION PROCEDURE FOR CM1

As pointed out earlier, the problems of determining the optimal component groupings and the optimal maintenance times have to be treated jointly. The cost function given by equation 3.4.2 is a non-linear and non-convex function which is difficult to optimize. In order to deal with this situation, only a few points in time are considered and the optimal grouping of components over this set of times is determined. The set of candidate maintenance times is denoted by S . It is observed that the optimal maintenance time for a group of components seldom coincides with the optimal time of maintenance for a single component. Therefore, after the groups are formed, a search is performed for the best maintenance time for each group. The new times are added to S and the process is repeated. The steps in the optimization procedure are enumerated below.

1) The optimal time of maintenance for each component is determined. The set S initially contains these times. This set of times is optimal (or near-optimal) if C_s is negligible relative to the other costs. This set of times is chosen initially because it covers the worst case situation in which all components are maintained separately.

2) A integer programming (IP) formulation which reflects the consequences of any assignment of the components to the times in S is developed. The objective function is:

$$\min \sum_{j \in S} \frac{\sum_{i=1}^{m} [C_{mi}R_i(T_j) + C_f(1 - R_i(T_j))]X_{ij} + C_s Y_j}{E[T_j]} \qquad [3.5.3]$$

Both X_{ij} and Y_j are binary variables where:

$X_{ij} = 1$ if component i is maintained at time T_j and 0 otherwise.

$Y_j = 1$ if any component is maintained at time T_j and 0 otherwise.

The coefficient of X_{ij} reflects the expected cost of maintaining the component i at time T_j . The cost of setup at time T_j is the coefficient of Y_j.

The constraint equations are:

$$\sum_{j \in S} X_{ij} = 1 \quad (\forall i) \qquad [3.5.4]$$

$$Y_j \geq X_{ij} \quad (\forall i, j) \qquad [3.5.5]$$

The first set of constraints assigns each component to one of the times in S . For example, component 1 has to be maintained at one of the times in S. Selection of one of the times, T_2 (say), implies $X_{12} = 1$. The first constraint set requires that all other X_{1j} be assigned a value of zero. The second set of constraints requires Y_j to be the maximum of all X_{ij} for a fixed j. Thus, whenever a component i is assigned to be maintained at a time T_j, (i.e. $X_{ij} = 1$), the second set of constraints ensures $Y_j = 1$.

Typically, the LP relaxation of this formulation is solved first. If a non-integer solution is obtained, tighter relaxations such as those developed by Sherali and Adams [32], and by Glover and Woolsey [17] can be used. The solution to this formulation gives an initial grouping of components. The next step is to determine the optimal maintenance times for each individual group.

3) Searches are performed to determine the optimal time of maintenance for each group formed in step 2. Consider each group to be a single unit and determine a single optimal time of maintenance. For a group j, consisting of n_j components, the expression to be minimized is:

$$\frac{\sum_{i=1}^{n_j}[C_{mi}R_i(T_j) + C_f(1 - R_i(T_j))] + C_s}{E[T_j]} \qquad [3.5.6]$$

This is an optimization problem involving continuous functions in a single variable. The Newton-Raphson method (Kreyszig [19]) and other line-search methods are all very fast in determining the optimal solution. The new times are added to the set S and the LP formulation is repeated with the new set of times.

A line search method is chosen to obtain the optimal value of time T_j instead of setting the derivative of equation 3.5.6 equal to zero. Often it is difficult to obtain the real positive root of the resulting derivative expression without obtaining some of the other roots as well. It may also be pointed out that in order to solve for the roots, procedures based on the Newton method are often used.

4) Steps 2 and 3 are repeated until no further improvement is possible.

A flow chart of this procedure is given in Figure 1.

3.6 LIMITATIONS OF CM1

The primary problem in trying to characterize the cost-rate of a multi-component series system lies in the fact that failure of any component in any group results in the failure of the system. Thus, the assumption that the sum of the cost-rates of the individual groups is the cost-rate of the system

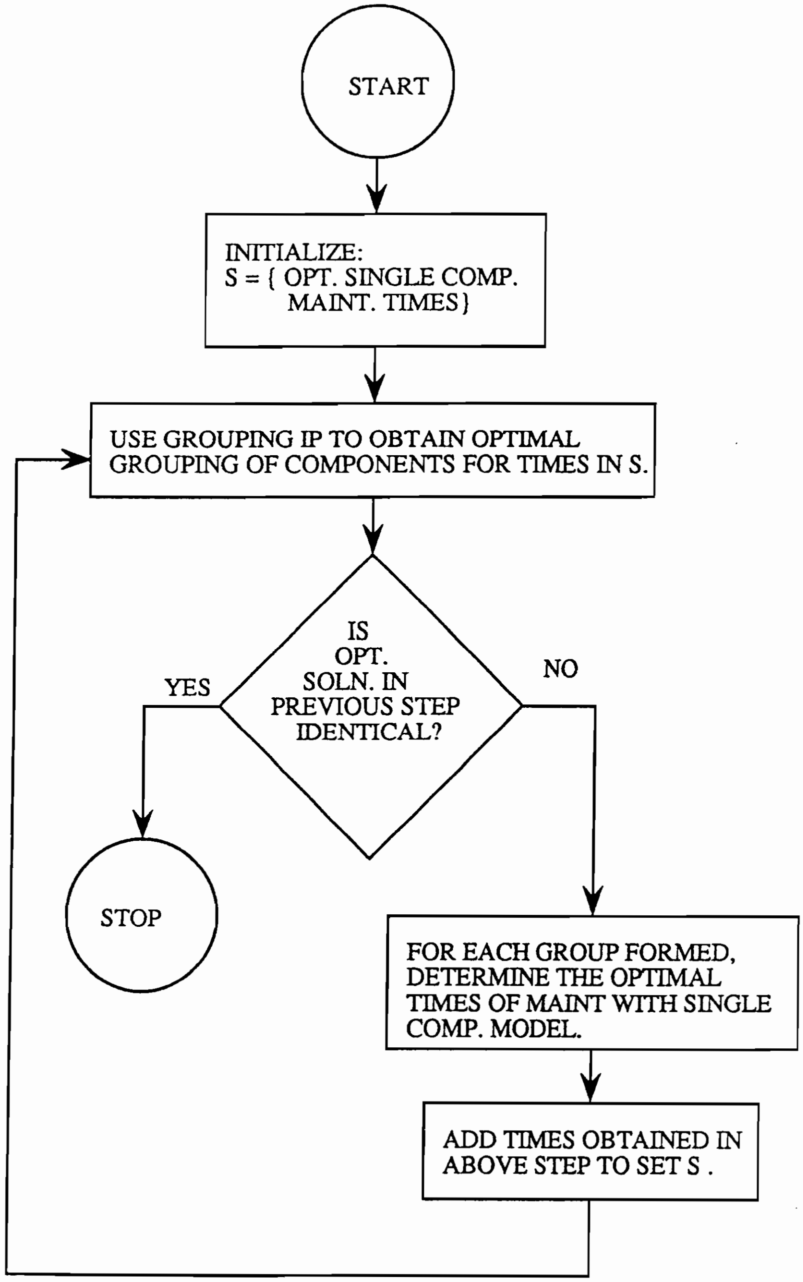

Figure 1. FLOWCHART FOR CM1 OPTIMIZATION PROCEDURE

is invalid. This assumption is valid only if the renewal process for a particular group is independent of all other groups. This is a good approximation if the system being examined is a highly reliable system since the dependence would be very weak. A simulation study is suggested to determine the validity of this approximation in the case of system subject to frequent breakdowns (low reliability).

In addition, the structure of CM1 is not conducive to analyzing the effect of the failure rate on the time of maintenance. CM1 is also very difficult to modify. When analyzing system models which consider partial component failures or systems in which the components deteriorate in time, a new model has to be developed. In order to deal with such shortcomings, a new model is proposed in the next chapter.

CHAPTER 4 DEVELOPMENT OF COST MODEL 2 (CM2)

4.1 INTRODUCTION

In the previous chapter, it is pointed out that the renewals of groups of components are not independent. This invalidates the assumption that the system cost rate is the sum of the group cost rates. A general structure for the system cost rate based on system renewal instead of renewals of groups is suggested below. This form is valid under all situations as the point of system renewal is common for all groups.

Typically, the cost of failure, C_f, is very large relative to the costs of component maintenance and setup, C_{mi} and C_s respectively. Consequently, optimal intervals between maintenance operations are relatively small and the corresponding failure rates of the components are approximately constant. Based on this observation, Cost Model 2 (CM2) is developed from the general form of the cost function.

The system being considered in this chapter is identical to the one in Chapter 3. Therefore, the notation and terminology remain the same.

4.2 COST RATE : A GENERAL FRAMEWORK

After identifying a renewal process, it is possible to use the Renewal-Reward theorem to obtain the average cost per unit time (cost rate). For the type of systems under consideration, the only renewal points are the times at which the entire system is "renewed" either due to failure or due to an age-replacement. The general expression for the system cost rate can be written as follows:

$$cost\ rate = \frac{Expected\ maintenance\ cost\ +\ Cost\ incurred\ at\ the\ end\ of\ the\ life}{Expected\ life\ of\ system} \quad [4.2.1]$$

The expected maintenance cost refers to the cost incurred due to all maintenance operations on all groups of components during the life of the system. For a component group j, the expected maintenance cost is given by:

$$Exp.\ Maint.\ Cost\ of\ group\ j = (Maint.\ cost\ of\ group\ j) \times \frac{Expected\ life\ of\ system}{T_j} \quad [4.2.2]$$

The maintenance cost of group j refers to the cost of a single maintenance operation on that group. Note that the expected number of maintenance operations is obtained by dividing the life of the system by the maintenance interval for group j. This is essentially a time-scaling i.e. number of units of length T_j in the expected system life. The total maintenance cost is the sum of the above expression for all the groups. Therefore, the expected system maintenance cost is given by:

$$Expected\ maintenance\ cost = \sum_j (Maint.\ cost\ of\ group\ j) \times \frac{Expected\ life\ of\ system}{T_j} \quad [4.2.3]$$

Thus, the expected cost rate of maintenance is given by:

$$Expected\ maintenance\ cost\ rate = \sum_j \frac{(Maint.\ cost\ of\ group\ j)}{T_j} \qquad [4.2.4]$$

Therefore, the expected cost rate for the system is given by:

$$cost\ rate = \sum_j \frac{(Maint.\ cost\ of\ group\ j)}{T_j} + \frac{Cost\ incurred\ at\ the\ end\ of\ the\ life}{Expected\ life\ of\ system} \qquad [4.2.5]$$

All of the quantities in the above expression are easily determined with the exception of the expected life of the system. One of the primary tasks in the analysis of the models is to construct an expression for expected life of the system in order that the above representation of the cost rate can be used.

4.3 CHARACTERIZATION OF THE SYSTEM HAZARD RATE

An expression for the expected system life as a function of the component reliabilities may be defined by expressing the system hazard rate at any point in time in terms of component hazard rates. Consider a single component i being maintained at regular intervals of length T_i. The probability that the component still functions at time t is denoted by $R_i(t)$ and is given by:

$$R_i(t) = \left(\exp\left(-\int_0^{T_i} \lambda_i(x)dx\right)\right)^{l_i} \left(\exp\left(-\int_0^{t-l_iT_i} \lambda_i(x)dx\right)\right) \qquad [4.3.6]$$

where $l_i = \max\{k;\ kT_i < t\}$.

The first term represents the probability that the component has not failed until the time of the last maintenance operation. The second term indicates the probability that the component has not failed from the time of last maintenance until time t. This can be written as follows:

$$R_i(t) = (\exp(-\Lambda_i(T_i)))^{l_i}(\exp(-\Lambda_i(t - l_i T_i))) \qquad [4.3.7]$$

It is fairly simple to extend this concept to the case of a series system with m components. As all components operate independently, the probability of the system functioning at time t is the product of the individual probabilities. Thus:

$$\begin{aligned} R_{sys}(t) = &(\exp(-\Lambda_1(T_1)))^{l_1}(\exp(-\Lambda_1(t - l_1 T_1))) \\ &(\exp(-\Lambda_2(T_2)))^{l_2}(\exp(-\Lambda_2(t - l_2 T_2))) \qquad [4.3.8] \\ &\ldots\ldots\ldots \end{aligned}$$

This expression may be simplified as shown below.

$$R_{sys}(t) = \prod_{i=1}^{m} (\exp(-\Lambda_i(T_i)))^{l_i}(\exp(-\Lambda_i(t - l_i T_i))) \qquad [4.3.9]$$

Grouping the terms, the following expression is obtained.

$$R_{sys}(t) = (\exp(\sum_{i=1}^{m} - l_i \Lambda_i(T_i)))(\exp(\sum_{i=1}^{m} - \Lambda_i(t - l_i T_i))) \qquad [4.3.10]$$

It is important to show that $R_{sys}(t)$ as defined by 4.3.10 corresponds to a distribution function. It is sufficient to show that it has the properties of a distribution function, $F(x)$ as suggested by Cinlar [8]. These are:

1. $F(x)$ is non-decreasing in x.
2. $F(x)$ is right-continuous.

3. $\lim_{b \to \infty} F(b) = 1$

4. $\lim_{b \to -\infty} F(b) = 0$

It is obvious that $R_{sys}(t)$ is a non-increasing function in t as it is a product of negative exponentials which are non-increasing functions. Thus, the complementary function $F_{sys}(t)$ is a non-decreasing function in t . Property 2 is also clear from the structure of the function. The probability of system survival at $t = \infty$ is 0. This is because l_i is ∞ for all i at $t = \infty$. Therefore, the second term in all component survival probabilities is 0. Hence, $F_{sys}(t) = 1$ at $t = \infty$. Property 4 is again evident from the structure of the function. Since all four conditions are satisfied, $F_{sys}(t)$ is a distribution function.

By differentiating the above expression for $F_{sys}(t)$ with respect to t, it is possible to obtain an expression for the probability density of the time to system failure. Note that l_i is a step function with unit increments at T_i. Step functions do not have a derivative at the point of increase. Therefore, the derivative of the distribution function does not exist at times of maintenance (multiples of T_i). However, it is still possible to obtain derivatives at points in time which are not multiples of T_i. It can be shown that the density function is given by the following expression:

$$f_{sys}(t) = (\sum_{i=1}^{m} \lambda_i(t - l_i T_i))(\exp(\sum_{i=1}^{m} - (l_i)\Lambda_i(T_i)))(\exp(\sum_{i=1}^{m} - \Lambda_i(t - l_i T_i))) \quad [4.3.11]$$

The system failure rate at time t is thus shown to be:

$$\lambda_{sys}(t) = \sum_{i=1}^{m} \lambda_i(t - l_i T_i) \quad [4.3.12]$$

This is what one would intuitively expect.

4.4 EXPECTED SYSTEM LIFE

The expected life of the system, $E[t]$, is obtained by integration of the survival probability obtained above from 0 to T_{sys} (the time of system replacement).

$$E[t] = \int_0^{T_{sys}} (\exp(\sum_{i=1}^{m} - l_i\Lambda_i(T_i)))(\exp(\sum_{i=1}^{m} - \Lambda_i(t - l_iT_i))) \qquad [4.4.13]$$

While it is possible to numerically evaluate the above expression, it is not convenient to deal with this expression when trying to determine the optimal group maintenance times. A simpler form is suggested below. This approximation is found to give results which closely approximate the above integral.

Consider a "large number" of failure events of the system. The total number of failures of the system is equal to the sum of individual component failures. (The Poisson assumption precludes the possibility of two components failing simultaneously.) Thus:

$$N_{sys}(t) = N_1(t) + N_2(t) + \ldots\ldots\ldots\ldots + N_m(t) \qquad [4.4.14]$$

Dividing by t and taking the limit as $t \to \infty$, the strong law of large numbers yields:

$$\frac{[N_{sys}(t)]}{t} = \lambda_1 + \lambda_2 + \ldots\ldots\ldots\ldots + \lambda_m \qquad [4.4.15]$$

where λ_i is the average failure rate of component i . Note that the expression on the left is average failure rate of the system. It may also be pointed out that until now no approximation has been made. The above expression can be written as:

$$\lambda_{sys} = \sum_{j=1}^{m} \lambda_i \qquad [4.4.16]$$

A component failure usually occurs at some point in the middle of its operation cycle and as a result all other components cease to function at that point in time. It is difficult to determine the average failure rate of each component exactly due to this dependence among components. In order to simplify the computation process, the following expression is used to obtain average component failure rate.

$$\lambda_i \cong \frac{\int_0^{T_i} \lambda_i(t)dt}{T_i} \qquad [4.4.17]$$

This can be expressed as:

$$\lambda_i \cong \frac{\Lambda_i(T_i)}{T_i} \qquad [4.4.18]$$

It may be observed that as the component reliabilities increase, the error from this assumption decreases very rapidly. Thus, the following approximation for the survival function, $R_{sys}(t)$, may be used:

$$R_{sys}(t) \cong (\exp(\sum_{i=1}^{m} - \frac{\Lambda_i(T_i)}{T_i} t)) \qquad [4.4.19]$$

This approximation is used below to obtain the optimal system replacement age.

In order to determine the expected system life, it is necessary to determine the age at which the system should be replaced. Consider the general expression for cost rate obtained above. If group j is maintained at time T_j, then the cost per maintenance operation of group j is given by:

$$\sum_{i=1}^{n_j} C_{mi} + C_s \qquad [4.4.20]$$

The above cost consists of the setup cost which is incurred whenever a group is maintained and the individual maintenance costs of components in the group. At the end of the renewal period, the expected cost incurred is the system replacement cost C_r with probability $R_{sys}(T_{sys})$ and system failure cost C_f with probability $F_{sys}(T_{sys})$. Thus, from the general form of the cost function, the cost rate C_{sys} is given by:

$$C_{sys} = \sum_{j} \frac{\sum_{i=1}^{n_j} C_{mi} + C_s}{T_j} + \frac{C_f F_{sys}(T_{sys}) + C_r R_{sys}(T_{sys})}{E[t]} \quad [4.4.21]$$

For fixed T_j, the above expression is similar to the single component age replacement cost function with an added constant term. Assume that the groupings and the corresponding times of maintenance are optimal at T_j^*. Since the above expression for $R_{sys}(t)$ is identical to the survival function of a component with a constant hazard rate, the optimal value of such a function would be at $T_{sys} = \infty$. Thus, the system under consideration will be replaced only on failure.

Using the approximation for the failure rate and assuming that the system is replaced only on failure, the expected life of the system can be approximated as follows:

$$E[t] \cong \int_0^{\infty} (\exp(\sum_{i=1}^{m} -\frac{\Lambda_i(T_i)}{T_i} t)) \quad [4.4.22]$$

This can be shown to be:

$$E[t] \cong \frac{1}{\sum_{i=1}^{m} \frac{\Lambda_i(T_i)}{T_i}} = \frac{1}{\sum_{j=1}^{m} \lambda_i} \quad [4.4.23]$$

Thus, the expected life of the system can be approximated by the reciprocal of the average failure rate of the system. It may be noted that this result is asymptotically true for superimposed renewal processes due to the fact that in the limit superimposed renewal processes behave like concurrent Poisson processes (Thompson [33]). The above approximation is used in CM2 and results in special structures which can be optimized easily as compared to the first cost model.

It is also possible to obtain the proportion of failures due to a particular component in the limit as $t \rightarrow \infty$. This is essentially the probability of failure due to component i given that the system has failed. The result is obtained below.

The proportion of failures due to component i is given by:

$$P(i \ failed \mid system \ failed) = \frac{N_i(t)}{N_1(t) + N_2(t) + \dots\dots\dots\dots + N_m(t)} \quad [4.4.24]$$

Dividing the numerator and denominator by t and taking the limit as $t \rightarrow \infty$,

$$P(i \ failed \mid system \ failed) = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \dots\dots\dots\dots + \lambda_m} \quad [4.4.25]$$

The above expression is similar to the result obtained by Park [27]. The approximation for λ_i in equation 4.4.16 is suggested when trying to compute the above probability.

4.5 COST MODEL 2 (CM2)

While CM1 is based on the probabilities of failure, CM2 is based on the rate of failure. CM2 is obtained by using the general form for the system cost rate given by equation 4.2.3. If λ_i is the average failure rate of component i, then the average failure rate of the system is given by equation 4.4.16.

$$\lambda_{sys} = \sum_{j=1}^{m} \lambda_i \qquad [4.4.16.]$$

Using equation 4.4.18, the average rate of failure of the system, λ_{sys}, can be expressed as:

$$\lambda_{sys} = \sum_{i=1}^{m} \frac{\Lambda_i(T_i)}{T_i} \qquad [4.5.26]$$

This is the reciprocal of the expected system life, $E[t]$.

As discussed in the previous section, the only point of renewal for the system is the time of failure. At this renewal point, a cost of C_f is incurred (once every renewal).

The maintenance cost for each group of components consists of a setup cost and a maintenance cost. Every maintenance operation for component group j costs C_s plus the sum of maintenance costs for all components in group j. Therefore, the cost per maintenance operation on group j during the life of the system is:

$$\sum_{i=1}^{n_j} C_{mi} + C_s \qquad [4.5.27]$$

Thus, the expression for C_{sys} is given by:

$$C_{sys} = \sum_{j} \frac{\sum_{i=1}^{n_j} C_{mi} + C_s}{T_j} + \frac{C_f}{E[t]} \qquad [4.5.28]$$

Since the average life of the system is the reciprocal of average failure rate, the average cost rate can be written as:

$$C_{sys} = \sum_{j} \frac{\sum_{i=1}^{n_j} C_{mi} + C_s}{T_j} + C_f \lambda_{sys} \qquad [4.5.29]$$

It is now possible to construct an LP model which can be used to group components.

4.6 OPTIMIZATION PROCEDURE FOR CM2

The optimization procedure is similar to the procedure used for CM1. The constraints are identical and the only difference is in the objective function. Again, only a few points in time are considered. The set of candidate maintenance times is denoted by S . Once the groups are formed, a search is carried out for the best time of maintenance for each group. The new times are added to S and the process is repeated. The steps in the procedure are enumerated below.

Initialization Step: The optimal time of maintenance for each component is determined. The set S initially contains these times. This set of times is optimal if C_s is negligible relative to the other costs. As in the case of CM1, this set of times is chosen initially because it covers the worst case situation in which all components are maintained separately.

Grouping Step: An IP objective function which reflects the consequences of any component grouping is given below:

$$\min \sum_{j \in S_g} \frac{\sum_{i=1}^{m} C_{mi} X_{ij} + C_s Y_j}{T_j} + C_f \frac{\sum_{i=1}^{m} \Lambda_i(T_j) X_{ij}}{T_j} \qquad [4.6.30]$$

Both X_{ij} and Y_j are binary variables where

$X_{ij} = 1$ if component i is maintained at time T_j and 0 otherwise.

$Y_j = 1$ if any component is maintained at time T_j and 0 otherwise.

The first term reflects the cost rate of maintaining a group at time T_j where there is a cost of maintaining a component i and a setup cost C_s if there is any component maintained at that time. The second term reflects the cost rate for failure where $\frac{\Lambda_i(T_j)}{T_j}$ is the average failure rate when component i is maintained at time T_j

The constraint equations are identical to equations in 3.6.3 and 3.6.4. Again, the LP relaxation is solved before attempting to solve the IP problem.

Determining Group Maintenance Times: In this part of the procedure, searches are performed to determine the optimal time of maintenance of each group. Consider each group to be a single component and determine a single optimal time of maintenance. For a group j, consisting of n_j components, the expression to be minimized is :

$$\min \frac{\sum_{i=1}^{n_j} C_{mi} + C_s}{T_j} + C_f \sum_{i=1}^{n_j} \frac{\Lambda_i(T_j)}{T_j} \qquad [4.6.31]$$

Again this is an optimization problem involving continuous functions of a single variable. The new times are added to the set S and the IP formulation is repeated with the new set of times.

4) Steps 2 and 3 are repeated until no further improvement is possible. Again, an integer solution is not guaranteed.

It may be pointed out that the procedure described is finite as in the limiting case all the component groupings will be considered.

4.7 ANALYSIS OF CM2

An analysis of CM2 is performed in order to give some insights into optimization of maintenance times and the grouping of components.

4.7.1 CONDITIONS FOR CONVEXITY OF THE SINGLE COMPONENT COST FUNCTION

While Bergman [5] and others have given methods for optimization of the single unit cost function, they have often failed to study the cost function adequately for convexity. Conditions for convexity of the single-component cost function need to be enforced when claiming global optimality. These conditions are especially important when determining the optimal group maintenance times. If a component in a group has a non-convex cost function, it is possible that standard line-search methods may not give the optimal group maintenance time. Sufficient conditions for convexity of the single component cost function are obtained below.

Consider a component being maintained at time T. Let the average failure rate of the component be denoted by $\lambda(T)$. Let E[C(T)] represent the cost function for component maintenance. Then,

$$E[C(T)] = \frac{C_m + C_s}{T} + C_f \frac{\Lambda(T)}{T} \qquad [4.7.32]$$

Applying the condition for convexity, i.e., $\frac{d^2E[C(T)]}{dT^2} \geq 0$.

$$\frac{2(C_m + C_s)}{T^3} + C_f \frac{T\frac{d\lambda(T)}{dT} - 2T^2\lambda(T) + + 2T\int_0^T \lambda(x)dx}{T^3} \geq 0 \qquad [4.7.33]$$

Rearranging the terms.

$$T^2 \frac{d\lambda(T)}{dT} - 2T\lambda(T) + 2\int_0^T \lambda(x)dx \geq -2\frac{(C_m + C_s)}{C_f} \qquad [4.7.34]$$

If the inequality were replaced by an equality, this is a second order differential equation in $\Lambda(T)$. The solution to this differential equation is $y(T) = c_1 T + c_2 T^2 - 2\frac{(C_m + C_s)}{C_f}$ where c_1 and c_2 are constants. It may be pointed out that the first two terms are the solutions to the homogeneous differential equation which in this case is the Cauchy equation (Kreyszig [19]).

On further analysis, if C_f is much larger than C_m and C_s, convexity is guaranteed when $\lambda(t)$ can be expressed as a polynomial in t with terms having positive coefficients and exponents greater than or equal to 1.

4.7.2 ANALYSIS OF THE OPTIMAL MAINTENANCE TIME OF A GROUP

When trying to determine the minima of a function, it is very useful if a range within which the search should be performed can be established. Ordinarily, it would be expected that the optimal maintenance time for a group of components would lie somewhere between the smallest and largest optimal maintenance times of components in the group. The next result proves that the optimal maintenance time for a group of components may lie before the minimum single component maintenance time. Consider two components 1 and 2 having single-component optimal maintenance times of T_1 and T_2 with $T_1 \leq T_2$. Define the cost functions for components 1 and 2 as $E[C_1(t)]$ and $E[C_2(t)]$ respectively. where

$$E[C_1(t)] = \frac{C_{m1} + C_s}{t} + C_f \frac{\Lambda_1(t)}{t} \qquad [4.7.35]$$

$$E[C_2(t)] = \frac{C_{m2} + C_s}{t} + C_f \frac{\Lambda_2(t)}{t} \quad [4.7.36]$$

Assume the cost functions satisfy the conditions of convexity and existence of the first derivative. Let the first derivatives be represented as $E'[C_1(t)]$ and $E'[C_2(t)]$ respectively. At T_1 , $E[C_1(T_1)]$ is optimal and $E'[C_1(T_1)] = 0$. Differentiating $E[C_1(t)]$, and evaluating the expression at T_1, the following relationship is obtained.

$$-\frac{C_{m1} + C_s}{T_1^{\,2}} + C_f \frac{T_1\lambda_1(T_1) - \Lambda_1(T_1)}{T_1^{\,2}} = 0 \quad [4.7.37]$$

This can be written as

$$-(C_{m1} + C_s) + C_f(T_1\lambda_1(T_1) - \Lambda_1(T_1)) = 0 \quad [4.7.38]$$

As both the cost functions are convex with $T_1 \leq T_2$, the following relationships hold.

$$E'[C_1(T_2)] \geq 0 \quad [4.7.39]$$

and

$$E'[C_2(T_1)] \leq 0 \quad [4.7.40]$$

If components 1 and 2 were to be maintained at the same time, t, the new cost function $E[C(t)]$ would be defined as:

$$E[C(t)] = \frac{C_{m1} + C_{m2} + C_s}{t} + C_f(\Lambda_1(t) + \Lambda_2(t)) \quad [4.7.41]$$

Differentiating the above expression,

$$E'[C(t)] = -\frac{C_{m1} + C_{m2} + C_s}{t^2} + C_f \frac{t(\lambda_1(t) + \lambda_2(t)) - (\Lambda_1(t) + \Lambda_2(t))}{t^2} \quad [4.7.42]$$

Evaluating this function at T_1 and rearranging the terms, the following expression is obtained:

$$\frac{[-(C_{m1}+C_s)+C_f(T_1\lambda_1(T_1)-\Lambda_1(T_1))]+[-C_{m2}+C_f(T_1\lambda_2(T_1)-\Lambda_2(T_1))]}{T_1^{\ 2}} \quad [4.7.43]$$

The first set of terms is equal to 0. Since $E'[C_2(T_1)] \leq 0$:

$$\frac{-(C_{m2}+C_s)+C_f(T_1\lambda_2(T_1)-\Lambda_2(T_1))}{T_1^{\ 2}} \leq 0 \quad [4.7.44]$$

Rearranging the terms:

$$\frac{-C_{m2}+C_f(T_1\lambda_2(T_1)-\Lambda_2(T_1))}{T_1^{\ 2}} \leq \frac{C_s}{T_1^{\ 2}} \quad [4.7.45]$$

Thus, it is possible for the gradient evaluated at T_1 to be positive. This implies that the optimal time of maintenance for the group could be at a time before T_1. A practical interpretation of this result is that the saving in setup cost when grouping components can be used to pay for increased maintenance.

4.7.3 CONDITION FOR THE GROUPING OF COMPONENTS:

Intuitively, it can be seen that there exists a range of times beyond which a component should not be grouped with other components. While it is difficult to obtain a range explicitly, it is possible to obtain a condition on the times at which a component may be grouped with other components. If the condition is not satisfied for a component i and a time T_j, then the corresponding variable X_{ij} is set to 0. Hence, this variable can be ignored in the IP formulation. Therefore, this condition can be used to reduce the number of variables in the IP formulation.

Consider a component i which is optimally maintained at time T_i. Equation 4.6.38 states the following condition:

$$-(C_{mi}+C_s)+C_f(T_i\lambda_i(T_i)-\Lambda_i(T_i))=0 \quad [4.6.38.]$$

This can rewritten as:

$$\frac{C_{mi} + C_s}{T_i} + C_f \frac{\Lambda_i(T_i)}{T_i} = C_f \lambda_i(T_i) \qquad [4.7.46]$$

When two components are maintained simultaneously, the setup cost per component is reduced. A component i will be grouped with other components at time T only if the single component cost rate with reduced setup cost is less than the optimal single component cost rate at time T_i. In other words, a component will be maintained at time T only if:

$$\frac{C_{mi} + C_s}{T_i} + C_f \frac{\Lambda_i(T_i)}{T_i} \geq \frac{C_{mi} + kC_s}{T} + C_f \frac{\Lambda_i(T)}{T} \qquad [4.7.47]$$

where $0 \leq k < 1$.

The above equation is based on the assumption that a component should not be grouped with other components at a non-optimal time if the reduction in the cost of setup is not greater than the increase in maintenance and failure costs. From the above two expressions, the following relation is obtained.

$$\frac{C_{mi}}{T} + C_f \frac{\Lambda_i(T)}{T} - C_f \lambda_i(T_i) + \frac{kC_s}{T} \leq 0 \qquad [4.7.48]$$

Rearranging the terms:

$$\frac{C_{mi}}{T} + C_f \frac{\Lambda_i(T)}{T} \leq C_f \lambda_i(T_i) - \frac{kC_s}{T} \qquad [4.7.49]$$

This implies:

$$\frac{C_{mi}}{T} + C_f \frac{\Lambda_i(T)}{T} \leq C_f \lambda_i(T_i) \qquad [4.7.50]$$

The above expression is a necessary condition for a component to be grouped with another components at time T. In addition, this condition can be used to develop quick heuristics which do not involve any LP.

CHAPTER 5 IMPERFECT MAINTENANCE MODEL (CM3)

5.1 INTRODUCTION

In the earlier chapters, it is assumed that a maintenance operation restores the component to an "as good as new" condition. However, factors such as human error or faulty components may cause the component to be in a "worse than new" condition. Therefore, a more realistic model includes the possibility of an imperfect maintenance operation where the component may not be restored to an "as good as new" condition after maintenance. Note that it may still be possible for a component to be restored from an imperfect state to an "as good as new" condition in a subsequent maintenance operation. Such situations arise when the maintenance personnel perform a faulty maintenance operation which is detected and rectified in a subsequent maintenance operation.

Cost Model 3 (CM3) is developed in the same fashion as CM2. The system being considered in this chapter is similar to the system examined in the earlier models. However, after every maintenance operation on a component, the component failure rate is $\lambda_{iq}(t)$ with probability p_{iq}. It is arbitrarily assumed that $q = 0$ corresponds to the "as good as new" condition. Hence, $\lambda_{i0}(t)$ is the smallest failure rate for component i. As in the case of the earlier models, the aim is to determine maintenance plans which minimize the long run expected cost rate.

5.2 ASSUMPTIONS

Assumptions 1 through 4 and Assumption 6 given in section 3.4 still apply to this model of the system. Assumption 5 is replaced with Assumption 5a which is given below.

5a. A maintenance action on component i results in the failure rate of the component being $\lambda_{iq}(t)$ with probability p_{iq} where each $\lambda_{iq}(t)$ is an IFR process.

5.3 NOTATION AND TERMINOLOGY

Additional notation is provided below.

1. $\lambda_{iq}(t)$ - rate of failure of component i at age t with probability p_{iq}.
2. p_{iq} - probability that the post-maintenance failure rate of component i is $\lambda_{iq}(t)$. For a system in which there is no imperfect repair $p_{i0} = 1$ and 0 for all other q. $\sum_q p_{iq} = 1$.
3. $\Lambda_{iq}(t) - = \int_0^t \lambda_{iq}(x)dx$.
4. $E[\lambda_i(t)]$ - expected failure rate of component i at component age t. $E[\lambda_i(t)] = \sum_q \lambda_{iq}(t)p_{iq}$.
5. $E[\lambda_{sys}(t)]$ - expected failure rate of the system at system age t.
6. $E[\Lambda_i(t)] - = \int_0^t E[\lambda_i(x)]dx$.

5.4 DISTRIBUTION OF TIME TO FAILURE

Consider a component i being maintained at regular intervals of length T_i. The probability that the component still functions at time t, $t \leq T_i$, is:

$$R_i(t) = \exp(-\int_0^t \lambda_{i0}(x)dx) \qquad [5.4.1]$$

This can be written as:

$$R_i(t) = \exp(-\Lambda_{i0}(t)) \qquad [5.4.2]$$

The probability that the component still functions at time t, $t > T_i$, is:

$$R_i(t) = \exp(-\int_0^{T_i} \lambda_{i0}(x)dx)(\exp(-\int_0^{T_i} E[\lambda_i(x)]dx))^{l_i - 1} \exp(-\int_0^{t - l_i T_i} E[\lambda_i(x)]dx) \qquad [5.4.3]$$

where $l_i = \max\{k; kT_i < t\}$. This can be written as follows:

$$R_i(t) = \exp(-\Lambda_{i0}(T_i))(\exp(-E[\Lambda_i(T_i)]))^{l_i - 1} \exp(-E[\Lambda_i(t - l_i T_i)]) \qquad [5.4.4]$$

The first term represents the probability that the component has not failed before the first maintenance period. The second term represents the probability that the component has not failed in the time interval between the time of the first maintenance operation and the last maintenance operation before time t. The third term indicates the probability that the system has not failed from the time of last maintenance until time t.

It is fairly simple to extend this concept to the case of a system with m components. As all components operate independently, the probability of the system functioning at time t, $t > \max\{T_i \ (\forall\ i)\}$, is the product of the individual probabilities. Thus:

$$R_{sys}(t) = \exp(-\Lambda_{10}(T_1))(\exp(-E[\Lambda_1(T_1)]))^{l_1 - 1}\exp(-E[\Lambda_1(t - l_1T_1)])$$
$$\exp(-\Lambda_{20}(T_2))(\exp(-E[\Lambda_2(T_2)]))^{l_2 - 1}\exp(-E[\Lambda_2(t - l_2T_2)]) \qquad [5.4.5]$$
$$\cdots\cdots\cdots$$

where l_i is defined above. This expression may be simplified as shown below.

$$R_{sys}(t) = \prod_{i=1}^{m}\exp(-\Lambda_{i0}(T_i))(\exp(-E[\Lambda_i(T_i)]))^{l_i - 1}\exp(-E[\Lambda_i(t - l_iT_i)]) \qquad [5.4.6]$$

Grouping the terms, the following expression is obtained.

$$R_{sys}(t) = \exp(\sum_{i=1}^{m} -\Lambda_{i0}(T_i))\exp(\sum_{i=1}^{m} -(l_i - 1)E[\Lambda_i(T_i)])\exp(\sum_{i=1}^{m} -E[\Lambda_i(t - l_iT_i)]) \quad [5.4.7]$$

It is desired to prove that $F_{sys}(t) = 1 - R_{sys}(t)$ is a distribution function. The reader is referred to the previous chapter for a list of the properties of a distribution function.

Again, $R_{sys}(t)$ is expressed in terms of negative exponentials which are non-increasing functions. Hence, the complementary function $F_{sys}(t)$ is a non-decreasing function in t. Property 2 is also clear from the structure of the function. It may be shown that the probability of system survival at $t = \infty$ is 0. This is because l_i is ∞ for all i at $t = \infty$. Hence, the second term in all component survival probabilities is 0. Thus, $F_{sys}(t) = 1$ at $t = \infty$. Property 4 is again evident from the structure of the function. Therefore, $F_{sys}(t)$ is a distribution function.

It is also possible to construct an expression for $R_{sys}(t)$ for t, $t \leq \max T_i$. Let $M = \{i; T_i \geq t\}$. In other words, M is the set of all components which have not been maintained before time t. The other components belong to M^c. Then, $R_{sys}(t)$ can be expressed as:

$$R_{sys}(t) = \exp(\sum_{i \in M} - \Lambda_{i0}(t)) \exp(\sum_{i \in M^c} - \Lambda_{i0}(T_i))$$
$$(\exp(\sum_{i \in M^c} - (l_i - 1)E[\Lambda_i(T_i)]))(\exp(\sum_{i \in M^c} - E[\Lambda_i(t - l_i T_i)])) \qquad [5.4.8]$$

The first term represents the survival probability of components belonging to M and the remaining terms represent the survival probability components belonging to M^c

By differentiating the expressions for $F_{sys}(t)$ with respect to t, it is possible to obtain an expression for the probability density. Again, since l_i is a step function with unit increments at T_i, the derivative of the distribution function does not exist at times of maintenance (multiples of T_i). However, it is possible to obtain derivatives at points in time which are not multiples of T_i. For $t > \max\{T_i \ (\forall\ i)\}$, the density function is given by the following expression:

$$f_{sys}(t) = (\sum_{i=1}^{m} E[\lambda_i(t - l_i T_i)]) \exp(\sum_{i=1}^{m} - \Lambda_{i0}(T_i))$$
$$\exp(\sum_{i=1}^{m} - (l_i - 1)E[\Lambda_i(T_i)]) \exp(\sum_{i=1}^{m} - E[\Lambda_i(t - l_i T_i)]) \qquad [5.4.9]$$

For $t < \max\{T_i\}$, the density function is given by:

$$f_{sys}(t) = (\sum_{i \in M} \lambda_{i0}(t) + \sum_{i \in M^c} E[\lambda_i(t - l_i T_i)])$$
$$\exp(\sum_{i \in M} - \Lambda_{i0}(t)) \exp(\sum_{i \in M^c} - \Lambda_{i0}(T_i)) \qquad [5.4.10]$$
$$\exp(\sum_{i \in M^c} - (l_i - 1)E[\Lambda_i(T_i)]) \exp(\sum_{i \in M^c} - E[\Lambda_i(t - l_i T_i)])$$

The system failure rate for $t > \max\{T_i \ (\forall\ i)\}$ is thus shown to be:

$$\lambda_{sys}(t) = \sum_{i=1}^{m} E[\lambda_i(t - l_i T_i)] \qquad [5.4.11]$$

This is what one would intuitively expect. The corresponding expression for $t \leq \max\{T_i\}$ is given by:

$$\lambda_{sys}(t) = \sum_{i \in M^c} E[\lambda_i(t - l_i T_i)] + \sum_{i \in M} \lambda_{i0}(t) \qquad [5.4.12]$$

From this point on, the system reliability for $t \leq \max\{T_i\}$ will not be considered. The following approximation for the survival function, $R_{sys}(t)$, ($t > \max\{T_i \ (\forall\ i)\}$) is used:

$$R_{sys}(t) \cong (\exp(\sum_{i=1}^{m} - \Lambda_{i0}(T_i)))(\exp(\sum_{i=1}^{m} - \frac{E[\Lambda_i(T_i)]}{T_i}(t - T_i))) \qquad [5.4.13]$$

This can be rewritten as:

$$R_{sys}(t) \cong \frac{(\exp(\sum_{i=1}^{m} - \Lambda_{i0}(T_i)))}{(\exp(\sum_{i=1}^{m} - E[\Lambda_i(T_i)]))}(\exp(\sum_{i=1}^{m} - \frac{E[\Lambda_i(T_i)]}{T_i}(t))) \qquad [5.4.14]$$

The first term is denoted by k and the final expression is given below:

$$R_{sys}(t) \cong k(\exp(\sum_{i=1}^{m} - \frac{E[\Lambda_i(T_i)]}{T_i}(t))) \qquad [5.4.15]$$

This approximation is used below to obtain optimal age of system replacement.

5.5 EXPECTED SYSTEM LIFE

In order to determine the expected system life, the optimal system replacement age has to be known. As seen in Chapter 4, the expression for the system cost function is given by:

$$C_{sys} = \sum_{j} \frac{\sum_{i=1}^{n_j} C_{mi} + C_s}{T_j} + \frac{C_f F_{sys}(T) + C_r R_{sys}(T)}{E[t]} \qquad [5.5.16]$$

Assume that the groupings and the corresponding times of maintenance are optimal. Then, for fixed $T_j^{\times}$, the above expression is similar to the cost rate for a single component with constant hazard. Hence, the optimal value of T_{sys} is ∞. Thus, the expected life of the system (expected time to failure), $E[t]$, is obtained by integration of the survival probability obtained above from 0 to infinity.

$$E[t] \cong \int_0^{\infty} \exp\left(\sum_{i=1}^{m} -\Lambda_{i0}(T_i)\right) \exp\left(\sum_{i=1}^{m} -(l_i - 1)E[\Lambda_i(T_i)]\right) \exp\left(\sum_{i=1}^{m} -E[\Lambda_i(t - l_i T_i)]\right) [5.5.17]$$

While it is possible to numerically evaluate the above expression, it is not convenient to deal with this expression when trying to determine the optimal group maintenance times. An approximation having a simpler form is developed below.

Since the rate of failure is approximately constant, the expected life can be approximated as follows:

$$E[t] \cong \int_0^{\infty} \exp(\sum_{i=1}^{m} -\frac{E[\Lambda_i(T_i)]}{T_i} t) \qquad [5.5.18]$$

This can be written as:

$$E[t] \cong \frac{1}{\sum_{i=1}^{m} \frac{E[\Lambda_i(T_i)]}{T_i}} \qquad [5.5.19]$$

The above approximation for $E[t]$ is used below to obtain the average cost-rate.

5.6 COST RATE EQUATION

The expression for group maintenance costs are identical to those obtained in the previous chapter and is given by:

$$(\sum_{i=1}^{n_j} C_{mi} + C_s) \qquad [5.6.20]$$

As it is shown that the optimal replacement time for the system is $T_{sys} = \infty$, the cost at the end of the renewal is the C_f.

Using the general form of the expression for the system cost rate, the following expression for C_{sys} is obtained.

$$C_{sys} = \sum_{j} \frac{\sum_{i=1}^{n_j} C_{mi} + C_s}{T_j} + C_f \sum_{i} \frac{E[\Lambda_i(T_j)]}{T_j} \qquad [5.6.21]$$

This expression is identical in form to the equivalent expression obtained in Chapter 4. It is now possible to construct an LP model which can be used to group components. The optimal groupings and times of maintenance are determined using the procedure given in Chapter 4. However, the new objective function is:

$$\min \sum_{j \in S} \frac{\sum_{i=1}^{m} C_{mi} X_{ij} + C_s Y_j}{T_j} + C_f \sum_{i=1}^{m} \frac{E[\Lambda_i(T_j)]}{T_j} X_{ij} \qquad [5.6.22]$$

The results obtained from the analysis of CM2 are also valid.

CHAPTER 6 DETERIORATING COMPONENT MODEL (CM4)

6.1 INTRODUCTION

Components are often irreversibly damaged while functioning. Hence, despite frequent maintenance operations, the condition of the component continues to deteriorate over time. This results in the deterioration of the entire system. At some point in time, it becomes necessary to replace the entire system. An example of such deterioration behavior is seen in the case of an automobile where the condition of the automobile deteriorates in time inspite of regular maintenance. Finally, at some point in time, the automobile has to be replaced.

Component-level deterioration can be modeled by assuming that the failure rate increases in a random manner after every maintenance operation. For example, if the component failure rate is given by $A + Bt$, component deterioration can be represented by increases in A and/or B. In this chapter, the increase in the failure rate is assumed to be due to increases in A only. A sample path showing the random increases in A is given in Figure 2a.

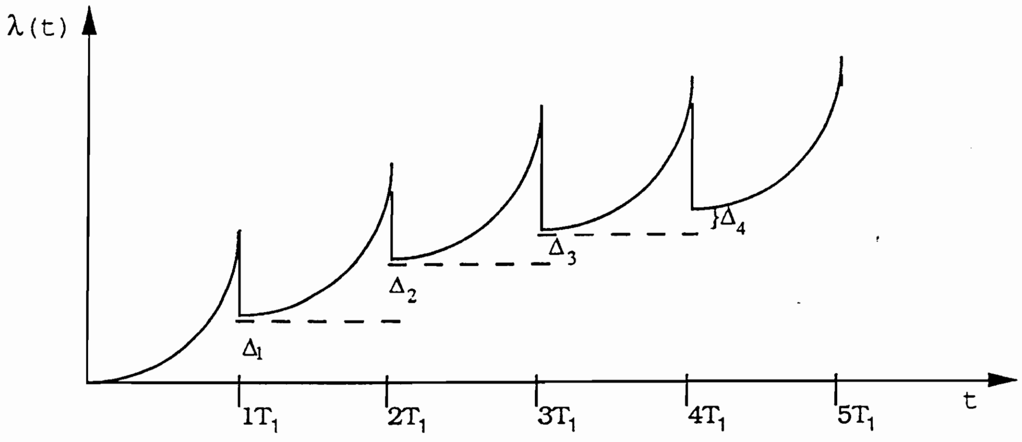

FIG.2a EXAMPLE OF A DETERIORATING COMPONENT

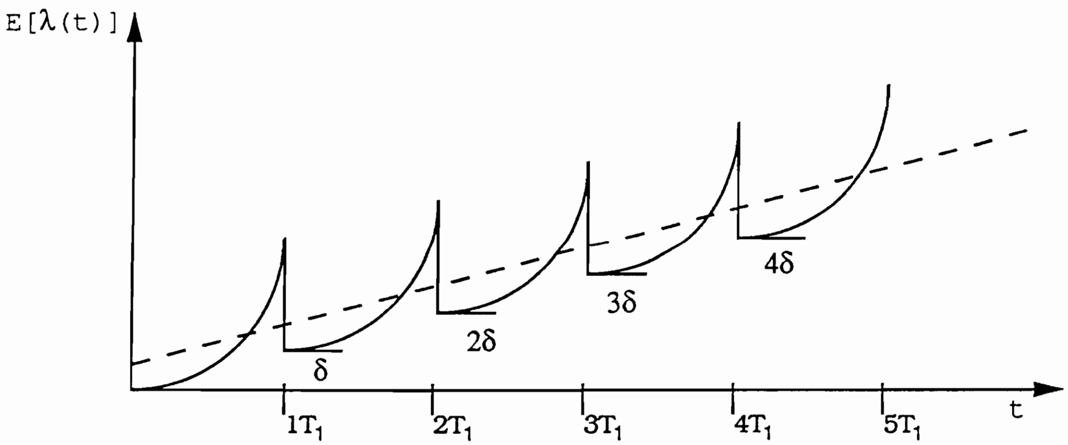

FIG.2b APPROXIMATION OF FAILURE RATE

Figure 2. COMPONENT WITH INCREASING FAILURE RATE

The increases are represented by Δ_1, Δ_2,..... etc. in the figure where the Δ_is are independent, identically distributed random variables. Since the system is a serial system, the increasing failure rate of the components causes the system failure rate to also increase. Hence, it is necessary to replace the entire system before the system failure rate becomes too high.

In this chapter, the expected component failure rates are approximated by linear failure rates (Figure 2b). Since the system failure rate is the sum of component failure rates, it can be approximated by a linear failure rate as well. It is then possible to determine the optimal system replacement age by viewing the entire system as a single unit with a linear failure rate.

Cost Model 4 (CM4) is constructed using the general form of the cost rate function given in Chapter 4. Unlike the expressions for the system cost rate in the previous chapters, it is necessary to include the expected cost due to system replacement. An iterative procedure for grouping components, determining maintenance times of component groups and system replacement age is given.

6.2 ASSUMPTIONS

Assumptions 1 through 4 and Assumption 6 given in section 3.2 still hold for this model of the system. Assumption 5 is no longer valid. A statement of the additional assumption is given below.

5b) A maintenance action on component i results in the failure rate of the component increasing by δ_{im} with probability q_{im}.

6.3 NOTATION AND TERMINOLOGY

The following additional notation is provided.

1. q_{im} - probability that the increase in the failure rate of component i after maintenance is δ_{im}. $\sum_m q_{im} = 1$.
2. δ_{im} - increase in the failure rate of component i with probability q_{im}. $\delta_{im} \geq 0$. $\delta_{im} = 0$ implies that the component i is restored to the same condition it was in at the end of the previous maintenance operation.
3. δ_i - expected increase in failure rate of component i. $\delta_i = \sum_m \delta_{im} q_{im}$.

6.4 DISTRIBUTION OF TIME TO FAILURE

Consider a component i being maintained at regular intervals of length T_i. The expected increase in the failure rate after every maintenance interval is δ_i. Therefore, the expected failure rate of component i in the interval $(nT_i,(n+1)T_i)$ is:

$$E[\lambda_i(x)] = \lambda_i(x - nT_i) + n\delta_i \qquad nT_i \leq x < (n+1)T_i \qquad [6.4.2]$$

Since the failure rate is a random variable, the survival probabilities are computed using the expected failure rates. The probability that the component does not fail in the interval $(nT_i,(n+1)T_i)$ is given by:

$$\exp(-\int_0^{T_i} (\lambda_i(x) + n\delta_i)dx) \qquad [6.4.3]$$

This can be simplified to:

$$\exp(-\Lambda_i(T_i) + n\delta_i T_i) \qquad [6.4.4]$$

Then the survival function for component i at time t, $R_i(t)$, is:

$$R_i(t) = \exp(-\sum_{n=0}^{l_i} \Lambda_i(T_i) + n\delta_i T_i) \times \exp(-\int_0^{t-l_iT_i} \lambda_i(x)dx) \times \exp(-l_i\delta_i(t - l_iT_i)) \qquad [6.4.5]$$

where $l_i = \max\{k; kT_i < t\}$. The first term represents the survival probability of the component until the time of the last maintenance operation before time t. The subsequent terms represent the probability that the system does not fail in the time interval (l_iT_i, t). The above expression can be rewritten as:

$$\begin{aligned} R_i(t) &= \exp(-l_i\Lambda_i(T_i)) \times \exp(-\frac{l_i(l_i+1)}{2}\delta_i T_i) \\ &\times \exp(-\Lambda_i(t - l_iT_i)) \times \exp(-l_i\delta_i(t - l_iT_i)) \end{aligned} \qquad [6.4.6]$$

Let $s_i = \frac{l_i(l_i+1)}{2}$. Then:

$$\begin{aligned} R_i(t) &= \exp(-l_i\Lambda_i(T_i)) \times \exp(-s_i\delta_i T_i) \\ &\times \exp(-\Lambda_i(t - l_iT_i)) \times \exp(-l_i\delta_i(t - l_iT_i)) \end{aligned} \qquad [6.4.7]$$

As all components operate independently, the probability of the system functioning at time t is the product of the component survival probabilities. Thus:

$$\begin{aligned} R_{sys}(t) &= \exp(-l_1\Lambda_1(T_1)) \exp(-s_1\delta_1 T_1) \exp(-\Lambda_1(t - l_1T_1)) \exp(-l_1\delta_1(t - l_1T_1)) \\ &\times \exp(-l_2\Lambda_2(T_2)) \exp(-s_2\delta_2 T_2) \exp(-\Lambda_2(t - l_2T_2)) \times \exp(-l_2\delta_2(t - l_2T_2)) \\ &\quad \ldots\ldots\ldots \end{aligned} \qquad [6.4.8]$$

where l_i is defined above. Simplifying the expression:

$$R_{sys}(t) = \prod_{i=1}^{m} \exp(-l_i\Lambda_i(T_i)) \exp(-s_i\delta_i T_i) \exp(-\Lambda_i(t - l_iT_i)) \exp(-l_i\delta_i(t - l_iT_i)) \qquad [6.4.9]$$

Grouping the terms, the following expression is obtained.

$$R_{sys}(t) = \exp(-\sum_{i=1}^{m} l_i\Lambda_i(T_i)) \times \exp(-\sum_{i=1}^{m} s_i\delta_i T_i) \times \exp(-\sum_{i=1}^{m} \Lambda_i(t - l_iT_i)) \times \exp(-\sum_{i=1}^{m} (l_i\delta_i(t - l_iT_i))) \quad [6.4.10]$$

In order for $F_{sys}(t) = 1 - R_{sys}(t)$ to be a distribution function, it has to satisfy the properties which are given in Chapter 4.

Again, as $R_{sys}(t)$ is expressed in terms of negative exponentials, it is a non-increasing function in t. Therefore, the complementary function $F_{sys}(t)$ is a non-decreasing function in t. Property 2 is also clear from the structure of the function. As $l_i = \infty$ for all i at $t = \infty$, the first and second terms in all component survival probabilities are 0. Hence, the probability of system survival at $t = \infty$ is 0. Therefore, $F_{sys}(t) = 1$ at $t = \infty$. Property 4 is again evident from the structure of the function. Thus, $F_{sys}(t)$ is a distribution function.

The probability density is obtained by differentiating $F_{sys}(t)$ with respect to t. It is seen in the earlier chapters that the derivatives of such distribution functions do not exist at the times of maintenance (multiples of T_i) but exist at points in time which are not multiples of T_i. The density function is given by the following expression:

$$f_{sys}(t) = (\sum_{i=1}^{m} (\lambda_i(t - l_iT_i) + l_i\delta_i)) \times \exp(\sum_{i=1}^{m} - l_i\Lambda_i(T_i)) \times \exp(\sum_{i=1}^{m} - \frac{l_i(l_i+1)}{2}\delta_iT_i) \times \exp(\sum_{i=1}^{m} - \Lambda_i(t - l_iT_i)) \times \exp(\sum_{i=1}^{m} (-l_i\delta_i(t - l_iT_i))) \quad [6.4.11]$$

The system failure rate is given by:

$$\lambda_{sys}(t) = \sum_{i=1}^{m} (\lambda_i(t - l_iT_i) + l_i\delta_i) \quad [6.4.12]$$

The expressions obtained above are too cumbersome to use in the procedure for obtaining the optimal groups and times. Hence, to facilitate ease of computation, the following expressions are developed.

Consider a component i with an expected failure rate given by $E[\lambda_i(t)]$. The first step in the approximation process involves replacing $E[\lambda_i(t)]$ by a linear failure rate $a_i(T_i) + b_i(T_i)t$ where $a_i(T_i)$ and $b_i(T_i)$ are functions of the length of the maintenance interval T_i. The survival probability, $R_{sys}(t)$ is thus approximated by:

$$R_i(t) \cong \exp(-(a_i(T_i)t + \frac{b_i(T_i)}{2}t^2)) \quad [6.4.13]$$

$a_i(T_i)$ and $b_i(T_i)$ are chosen such that approximation matches the exact survival probability of component i at multiples of time T_i. In order to obtain the expressions for $a_i(T_i)$ and $b_i(T_i)$, consider the survival probability at times T_i and $2T_i$. It is pointed out that the expressions for $a_i(T_i)$ and $b_i(T_i)$ obtained below will be the same irrespective of the choice of times mT_i and nT_i (m, n integer) at which the approximate function is evaluated. The exact expressions for the survival probabilities are given below.

$$R_i(T_i) = \exp(-\Lambda_i(T_i)) \quad [6.4.14]$$

$$R_i(2T_i) = \exp(-2\Lambda_i(T_i) + \delta_i T_i) \quad [6.4.15]$$

The approximations are:

$$R_i(T_i) \cong \exp(-(a_i(T_i)T_i + \frac{b_i(T_i)}{2}T_i^2)) \quad [6.4.16]$$

$$R_i(2T_i) \cong \exp(-(a_i(T_i)2T_i + b_i(T_i)2T_i^2)) \quad [6.4.17]$$

Equating the corresponding expressions yields:

$$\Lambda_i(T_i) = a_i(T_i)T_i + \frac{b_i(T_i)}{2} T_i^2 \qquad [6.4.18]$$

$$2\Lambda_i(T_i) + \delta_i = 2a_i(T_i)T_i + 2b_i(T_i)T_i^2 \qquad [6.4.19]$$

Solving for $a_i(T_i)$ and $b_i(T_i)$,

$$b_i(T_i) = \frac{\delta_i}{T_i} \qquad [6.4.20]$$

$$a_i(T_i) = \frac{\Lambda_i(T_i)}{T_i} - \frac{\delta_i}{2} \qquad [6.4.21]$$

Note that when $\delta_i = 0$, the model CM4 is identical to CM2 where the component failure rate is approximated with a constant failure rate. After computing $a_i(T_i)$ and $b_i(T_i)$ for all the components, it is possible to approximate the survival probability of the system at time t by the expression given below.

$$R_{sys}(t) \cong \exp(-\sum_{i=1}^{m} a_i(T_i)t + \frac{b_i(T_i)}{2} t^2) \qquad [6.4.22]$$

This approximation is used below when determining the age of system replacement.

6.5 COST RATE EQUATION

As with the earlier models, the general form of the cost rate equation given in chapter 4 is used. The term corresponding to the maintenance and setup costs of the groups is:

$$\sum_{j} \frac{(\sum_{i=1}^{n_j} C_{mi} + C_s)}{T_j} \qquad [6.5.23]$$

Since the system is replaced at time T_{sys}, the cost incurred at the end of the renewal interval is C_r with probability $R_{sys}(T_{sys})$ and C_f with probability $F_{sys}(T_{sys})$. The expected length of the renewal interval is obtained by the integration of the survival probability. Using the approximation for the survival probability, the second term in the expression for the system cost rate is:

$$\frac{C_r \exp(-\sum_{i=1}^{m} a_i(T_i)T_{sys} + \frac{b_i(T_i)}{2} T_{sys}^2) + C_f(1 - \exp(-\sum_{i=1}^{m} a_i(T_i)T_{sys} + \frac{b_i(T_i)}{2} T_{sys}^2))}{\int_0^{T_{sys}} \exp(-\sum_{i=1}^{m} a_i(T_i)t + \frac{b_i(T_i)}{2} t^2)dt} \qquad [6.5.24]$$

Note that this expression is very similar to the basic age-replacement cost rate equation seen in Chapter 1.

Any function which has terms which are the products of functions of several variables (terms which are of the form $f(T_1)g(T_2)$....) is very difficult to optimize. A Taylor series expansion of the exponential functions in the above expression indicates that there are a large number of such terms. In order to deal with this function more effectively, an approximation for the cost function is developed below. In the approximation, terms which are the products of multiple variables are dropped. The justification for this is based on the fact that for highly reliable systems $a_i(T_i)$s and $b_i(T_i)$s are very small and higher order terms will be negligible as compared to the first order terms.

Rearranging the terms in 6.5.24,

$$\frac{(C_r - C_f)\exp(-\sum_{i=1}^{m} a_i(T_i)T_{sys} + \frac{b_i(T_i)}{2}T_{sys}^2)}{\int_0^{T_{sys}} \exp(-\sum_{i=1}^{m} a_i(T_i)t + \frac{b_i(T_i)}{2}t^2)dt} + \frac{C_f}{\int_0^{T_{sys}} \exp(-\sum_{i=1}^{m} a_i(T_i)t + \frac{b_i(T_i)}{2}t^2)dt} \qquad [6.5.25]$$

Using the Taylor series expansion for the exponential terms, the first term of Eq. 6.5.25 is:

$$\frac{(C_r - C_f)\sum_{k=0}^{\infty}\frac{(\sum_{i=1}^{m} - a_i(T_i)T_{sys} + \frac{b_i(T_i)}{2}T_{sys}^2)^k}{k!}}{\int_0^{T_{sys}}\sum_{k=0}^{\infty}\frac{(\sum_{i=1}^{m} - a_i(T_i)t + \frac{b_i(T_i)}{2}t^2)^k}{k!}dt} \qquad [6.5.26]$$

Ignoring terms higher than the first order, the above expression reduces to:

$$\frac{(C_r - C_f)(1 - \sum_{i=1}^{m} a_i(T_i)T_{sys} + \frac{b_i(T_i)}{2}T_{sys}^2)}{\int_0^{T_{sys}}(1 - \sum_{i=1}^{m} a_i(T_i)t + \frac{b_i(T_i)}{2}t^2)dt} \qquad [6.5.27]$$

Simplifying,

$$\frac{(C_r - C_f)(1 - \sum_{i=1}^{m} a_i(T_i)T_{sys} + \frac{b_i(T_i)}{2} T_{sys}^2)}{(T_{sys} - \sum_{i=1}^{m} (a_i(T_i)\frac{T_{sys}^2}{2} + b_i(T_i)\frac{T_{sys}^3}{6}))} \qquad [6.5.28]$$

This can be rewritten as:

$$\frac{(C_r - C_f)(1 - \sum_{i=1}^{m} a_i(T_i)T_{sys} + \frac{b_i(T_i)}{2} T_{sys}^2)}{T_{sys}(1 - \sum_{i=1}^{m} (a_i(T_i)\frac{T_{sys}}{2} + b_i(T_i)\frac{T_{sys}^2}{6}))} \qquad [6.5.29]$$

Using the approximation $\frac{1}{(1-\alpha)} \cong 1 + \alpha$ (for small α), the above expression is modified to:

$$\frac{(C_r - C_f)(1 - \sum_{i=1}^{m} a_i(T_i)T_{sys} + \frac{b_i(T_i)}{2} T_{sys}^2)(1 + \sum_{i=1}^{m} (a_i(T_i)\frac{T_{sys}}{2} + b_i(T_i)\frac{T_{sys}^2}{6}))}{T_{sys}} \qquad [6.5.30]$$

Again, ignoring terms which are products of $a_i(T_i)$s and/or $b_i(T_i)$s:

$$\frac{(C_r - C_f)(1 - \sum_{i=1}^{m} (a_i(T_i)\frac{T_{sys}}{2} + b_i(T_i)\frac{T_{sys}^2}{3}))}{T_{sys}} \qquad [6.5.31]$$

Using similar approximations, the second term reduces to:

$$\frac{C_f\left(1+\sum_{i=1}^{m}\left(a_i\frac{T_{sys}}{2}+b_i\frac{T_{sys}^{2}}{6}\right)\right)}{T_{sys}} \qquad [6.5.32]$$

Adding the two approximations, the cost rate due to system replacement is:

$$\frac{C_f}{T_{sys}}\left(\sum_{i=1}^{m}\left(a_i(T_i)T_{sys}+b_i(T_i)\frac{T_{sys}^{2}}{2}\right)\right)+\frac{C_r}{T_{sys}}\left(1-\sum_{i=1}^{m}\left(a_i(T_i)\frac{T_{sys}}{2}+b_i(T_i)\frac{T_{sys}^{2}}{3}\right)\right) [6.5.33]$$

This is the final approximation which will be used in the optimization procedure. A discussion on the accuracy of this approximation is given in the Appendix A where a series of examples is given illustrating the validity of the approximation. Note that this approximation has the form $f(T_1)+g(T_2)+h(T_3)\ldots$. The advantage of dealing with such functions lies in being able to optimize each function separately.

The above approximation has several convenient properties. At the optimal point, $\frac{dC_s}{dT_{sys}}=0$. Therefore,

$$C_f\left(\sum_{i=1}^{m}\frac{b_i(T_i)}{2}\right)+C_r\left(\frac{-1}{T_{sys}^{2}}+\sum_{i=1}^{m}\frac{b_i(T_i)}{3}\right)=0 \qquad [6.5.34]$$

Solving for T_{sys},

$$T_{sys}=\left(\frac{C_r}{\left(\frac{C_f}{2}-\frac{C_r}{3}\right)\left(\sum_{i=1}^{m}b_i(T_i)\right)}\right)^{\frac{1}{2}} \qquad [6.5.35]$$

This expression is very convenient as it is now possible to obtain the optimal value for T_{sys} without resorting to any search procedure. Note that when $\delta_i = 0$, $T_{sys} = \infty$, since $b_i(T_i) = 0$. This result is consistent with the optimal system replacement time for CM2. In addition, for fixed $a_i(T_i)$ and $b_i(T_i)$, $\frac{d^2C_s}{dT_{sys}^2} = \frac{2C_r}{T_{sys}^3}$ Since all the quantities in the expression are positive, the function is strictly convex in T_{sys} (for fixed $a_i(T_i)$ and $b_i(T_i)$).

It can be seen from the above expression that T_{sys} and the T_is are mutually dependent. Hence, it will not be possible to determine the optimal values of the T_is and T_{sys} separately. A procedure similar to the ones described in the earlier chapters is given below.

6.6 OPTIMIZATION PROCEDURE FOR CM4

The approximation given above is intended to make the optimization process simpler. However, the function still remains non-convex. As in the case of the earlier models, only a few points in time are considered. The set of candidate maintenance times is denoted by S . Maintenance of any component can be scheduled at one of the times in S only. Once the groups are formed, a search is carried out for the best time of maintenance for each group. These times are used to obtain a new value of T_{sys}. The new times are added to S and the process is repeated. When the groupings do not change, the procedure is terminated. The steps in the procedure are:

1) Initialization Step: The set S initially contains the optimal single component maintenance times of all components. It is pointed out earlier that this set of times is chosen initially because it covers the worst case situation in which all components are maintained separately. The starting feasible solution assumes that each component is assigned to its corresponding optimal time of maintenance in S.

2) Computation of $a_i(T_j)$ and $b_i(T_j)$: For each component i and each $T_j \in S$, determine $a_i(T_j)$ and $b_i(T_j)$ from equations 6.4.20 and 6.4.21. Using equation 6.5.35, obtain T_{sys}.

3) Grouping Step: Using the values of $a_i(T_j)$ and $b_i(T_j)$ obtained in Step 2, an IP model is formulated reflecting the cost of maintaining a component at a certain time. The objective function to be minimized is given below.

$$\sum_{j \in S} \frac{\sum_{i=1}^{n_j} C_{mi} X_{ij} + C_s Y_j}{T_j} - \frac{C_r}{T_{sys}} \left(\sum_{i=1}^{n_j} \left(\frac{a_i(T_j)}{2} T_{sys} + \frac{b_i(T_j)}{3} T_{sys}^2 \right) X_{ij} \right) + \frac{C_f}{T_{sys}} \left(\sum_{i=1}^{n_j} \left(a_i(T_j) T_{sys} + \frac{b_i(T_j)}{2} T_{sys}^2 \right) X_{ij} \right) + \frac{C_r}{T_{sys}} \qquad [6.6.36]$$

where X_{ij} and Y_j are defined as before.

The first term is the cost rate for maintaining a group at time T_j where there is a cost of maintaining a component i and a setup cost C_s if there is any component is assigned to be maintained at that time. The second term is the cost rate for replacing the system at T_{sys} while the third term represents the cost rate due to failure of the system before time T_{sys}.

The constraint equations are identical to the equations given by 3.5.4 and 3.5.5. The solution to the IP formulation gives component groupings.

4) Optimizing Group Maintenance Times: The cost rate of maintaining a group j at time T_j is:

$$\frac{\sum_{i=1}^{n_j} C_{mi} + C_s}{T_j} - \frac{C_r}{T_{sys}} \sum_{i=1}^{n_j} \left(\frac{a_i(T_j)}{2} T_{sys} + \frac{b_i(T_j)}{3} T_{sys}^2 \right) + \frac{C_f}{T_{sys}} \sum_{i=1}^{n_j} \left(a_i(T_j) T_{sys} + \frac{b_i(T_j)}{2} T_{sys}^2 \right) \qquad [6.6.37]$$

This can be rewritten as:

$$\frac{\sum_{i=1}^{n_j} C_{mi} + C_s}{T_j} + \sum_{i=1}^{n_j} a_i(T_j)(C_f - \frac{C_r}{2}) + b_i(T_j)T_{sys}(\frac{C_f}{2} - \frac{C_r}{3}) \qquad [6.6.38]$$

The optimal group maintenance time is obtained by determining the T_j which minimizes the above function. This is again a minimization problem involving continuous functions in a single variable. The new times are then added to the set S

5) Steps 2 through 6 are repeated until there is no change in the solution of the IP.

A flow chart for the procedure is given in Figure 3.

This solution procedure is much slower than the methods suggested in the earlier chapters. The procedure is terminated when the IP solution does not change from one iteration to the next. It is pointed out that even if the IP solution remains the same from one iteration to another, the values of T_{sys} and T_j may change from iteration to the next. To limit the number of iterations in such a situation, it may be prudent to terminate the process if the improvement in the objective function is below a pre-specified value ε.

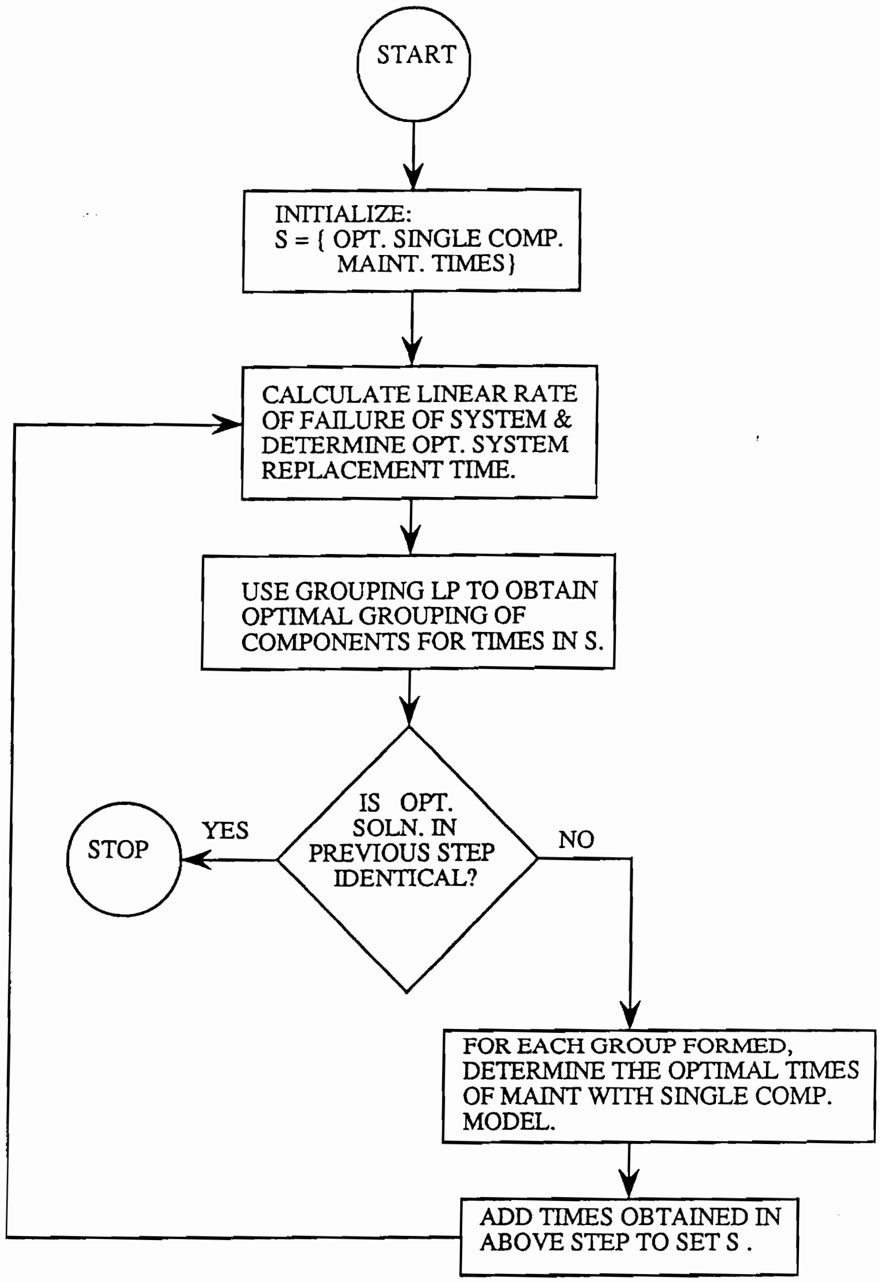

Figure 3. FLOW CHART FOR CM4 OPTIMIZATION PROCEDURE

CHAPTER 7 NUMERICAL RESULTS

7.1 INTRODUCTION

Predicting the behavior of the models in specific situations is a very difficult task. In order to obtain a better understanding of the models, a numerical analysis is performed. Such studies are also useful in pinpointing specific weaknesses in the models. The analysis also makes it possible to determine if the optimization procedures are sufficiently fast and accurate to make their use worthwhile.

A large number of software packages are used in the numerical analysis of the models. The minimization of the LP relaxation is done using various available optimization packages, namely GAMS 2.21 [8], MINOS 5.0 [22], and STORM 2.0 [14]. The single variable optimization is done using MINCT which is a code written in C (Refer Appendix B). Verification of the accuracy of the minimum given by MINCT is done using EUREKA 1.00 [15] and Mathematica 1.2.2 [34].

7.2 ILLUSTRATIVE EXAMPLES

Examples illustrating the optimization procedures for models CM2 and CM4 are given below. An example of the optimization procedure for model CM3 is omitted as it is identical to the procedure used for model CM2.

Model CM2

A 5 component system is considered in the example. The costs of maintenance, the failure rates, and the optimal single component maintenance times of the 5 components are enumerated in Table 1. In addition, $C_s = 150$ and $C_f = 20000$. The steps in the optimization procedure are given below.

Initialization Step: The set S contains the optimal single component maintenance times. Hence, $S = [0.147, 0.170, 1.140, 1.200, 0.403]$.

Grouping Step: The IP problem is formulated. X_{ij} is the variable indicating that component i is assigned to time T_j. Y_j is the variable indicating that a component has been assigned to be maintained at time T_j. The objective function is:

$$\min \sum_{j=1}^{5} C_j Y_j + \sum_{i=1}^{5} C_{ij} X_{ij}$$

The coefficients of X_{ij} and Y_j are computed using equation 4.6.30 and are given in Table 2. The constraints are obtained using equations 3.5.4 and 3.5.5.

The LP relaxation of the above IP problem is solved first and the following optimal solution is obtained. $X_{11} = 1$, $X_{21} = 1$, $X_{33} = 1$, $X_{43} = 1$, $X_{55} = 1$, $Y_1 = 1$, $Y_3 = 1$, $Y_5 = 1$. Since the solution is an integer solution, the IP problem is not attempted. From the above solution, the component groupings are {1,2}, {3,4}, and {5}.

Table 1. CHARACTERISTICS OF COMPONENT IN EXAMPLE (CM2)

component i	cost of maint.of i	$\lambda_i(t)$	λ_i	opt.maint. time of comp. i
1	500	3t	1.5t	.147
2	1000	4t	2t	.170
3	500	.05t	.025t	.490
4	1000	.08t	.04t	.524
5	500	.4t	.2t	.173

Setup Cost = 150, Failure Cost = 20000

Table 2 COEFFICIENTS OF IP VARIABLES IN EXAMPLE (CM2)

i	X_{1i}	X_{2i}	X_{3i}	X_{4i}	X_{5i}	Y_i
1	7811.4	12682.7	3474.9	6920.3	3989.4	1020.4
2	8036.1	12680.2	3032.9	6031.9	3626.5	884.4
3	34628.6	46477.2	1008.6	1789.2	4998.6	131.6
4	36387.0	48794.0	1016.5	1793.2	5213.0	125.1
5	13330.7	18601.4	1442.8	2803.8	2852.7	372.2

Determining Group Maintenance Times: Using equation 4.6.31, the cost rate function for each group is constructed and minimized. The optimal times of maintenance for the groups {1,2}, {3,4}, and {5} are determined to be 0.1535, 1.126, and 0.40 respectively.

The new times are added to S and a new IP problem is formulated by adding the corresponding new variables to the original IP problem. Solving the new IP problem, the component groups and maintenance times determined above prove to be optimal. Therefore, no further iterations are required. By performing an exhaustive enumeration of all possible groupings, it is verified that the above solution is globally optimal.

Note that the optimal maintenance time of group {3,4} is 1.126. This time does not lie in between the optimal single component maintenance times of components 3 and 4. This solution is consistent with the observation made in section 4.7.2.

Model CM4:

A system containing 5 components is considered. The components deteriorate from one maintenance operation to the next. The component characteristics are listed in Table 3. Also, $C_f = 100000$, $C_r = 15000$, and $C_s = 100$. The steps in the optimization procedure are enumerated below.

Initialization Step: The set S initially contains the individual optimal component maintenance times. Hence, $S = [.0632, .0742, .4906, .5243, .1732]$. The function $b_i(T_i)$ is computed for all i using the expressions in equation 6.4.20. Using 6.5.35, the time of system replacement is determined to be 4.69.

Grouping of Components: The IP problem is formulated using the last computed value of T_{sys}. X_{ij} and Y_j are defined as before. The objective function is similar to the one seen in the earlier example and is given by:

Table 3 CHARACTERISTICS OF COMPONENTS IN EXAMPLE (CM4)

component i	cost of maint.of i	$\lambda_i(t)$	λ_i	δ_i
1	500	3t	1.5t	.001
2	1000	4t	2t	.001
3	500	.05t	.025t	.010
4	1000	.08t	.04t	.010
5	500	.4t	.2t	.050

Setup Cost = 100, Failure Cost = 100000, Replacement Cost = 15000

$$\min \sum_{j=1}^{5} C_j Y_j + \sum_{i=1}^{5} C_{ij} X_{ij}$$

The coefficients of X_{ij} and Y_j are computed using equation 6.6.36 and are given in Table 4. The constraints are identical to the constraints in the earlier example.

The LP relaxation is solved first. The following optimal solution is obtained. $X_{12} = 1$, $X_{22} = 1$, $X_{34} = 1$, $X_{44} = 1$, $X_{53} = 1$, $Y_2 = 1$, $Y_4 = 1$, $Y_3 = 1$. Since an integer solution is obtained, the IP problem is not attempted. Thus, the component groupings are {1,2}, {3,4}, and {5}.

Determining Group Maintenance Times: Again, the cost function of each group obtained in the previous step is constructed using equation 6.6.37. The optimal maintenance times for the groups {1,2}, {3,4}, and {5} are determined to be 0.071, 0.597, and 0.241 respectively. The function $b_i(t)$ is evaluated at the new times and the new system replacement time is determined to be 5.48.

The new times are added to S and the IP problem is reformulated by adding the corresponding new variables to the original IP problem. Note that all the coefficients for X_{ij} and Y_j have to be re-computed since the time of system replacement T_{sys} has changed.

The next iteration gives the same grouping of components. Since the grouping does not change, the procedure is terminated.

7.3 ANALYSIS OF CM2

In this section, the consequences of varying certain parameters in CM2 such as the setup cost, the maintenance costs, and the number of components, are studied. The results of the analysis can also be extended to model CM3 as the structure of cost rate function for both models is identical.

Table 4 COEFFICIENTS OF IP VARIABLES IN EXAMPLE (CM4)

i	X_{1i}	X_{2i}	X_{3i}	X_{4i}	X_{5i}	Y_i
1	19973.5	19831.9	69473.8	74056.6	1582.3	1020.4
2	30807.9	30002.2	93183.2	99259.1	38988.0	1347.7
3	40989.0	34899.0	5993.0	5729.0	15010.2	204.1
4	48988.1	41732.5	7692.1	7410.1	18137.3	190.8
5	25546.3	22101.6	12014.9	12434.6	11952.4	578.0

Tables 5 through 7 summarize the results of the analyses on systems with 3, 4, and 5 components respectively. In each case, three different types of systems are considered; a system with components having low costs of maintenance, a system with components having both low and high maintenance costs, and a system with components having high costs of maintenance. The cost of failure is assumed to be 100000. The failure rate function of each component is assumed to be $0.1t$. The maintenance costs of the individual components are given in the tables. The setup cost for each case is expressed as a fraction of the failure cost and is also given in the table.

Table 5 summarizes the results of the optimization procedure as applied to 3 component systems. It can be seen that as the setup cost increases, the number of groups decreases. In addition, when the difference between component maintenance costs increases, the components are less likely to be grouped together. Finally, the number of iterations to get a solution is small.

The results of the analysis of 4 component systems are given in Table 6. Again, as the setup cost increases, the number of component groups decreases. Also, when the difference between the maintenance costs of the components decreases, the number of groups decreases.

Table 7 enumerates the results of the optimization of 5 component systems. The effects of varying the costs are similar to those seen in the case of the 4 component system discussed above.

The results of optimizing larger systems, each having 25 components, are summarized in Table 8. The costs of maintenance and failure rate functions of the individual components are given in Appendix D. Three different types of systems are considered. In the first system, all components have similar (but not identical) costs of maintenance but the failure rates of some components are significantly higher than the failure rates of the rest. The second system also has two types of components; components having a high cost of maintenance and components having a moderate cost of maintenance. All components in the system have similar rates of failure. There are three types of components in the third system; components having a high maintenance cost and a high failure

Table 5 SOLUTION OF 3 COMPONENT SYSTEM (CM2)

maint.cost of components in system	$\frac{C_s}{C_f}$	Optimal Solution	Number of Groups	Number of Iterations
10000, 15000, 20000	.01	52548.3	1	2
	.05	54772.3	1	2
	.10	57445.8	1	2
1000, 2000, 20000	.01	33143.0	2	2
	.05	40249.3	2	2
	.10	44497.3	1	2
1000, 2000, 3000	.01	20493.9	1	2
	.05	25690.5	1	2
	.10	30983.9	1	2

TABLE 6 SOLUTION OF 4 COMPONENT SYSTEM (CM2)

maint. cost of components in system	$\frac{C_s}{C_f}$	Optimal Solution	Number of Groups	Number of Iterations
10000 15000 20000 25000	.01	75144.4	2	2
	.05	77459.8	1	2
	.10	80000.0	1	2
10000 15000 1000 2000	.01	44898.2	2	2
	.05	51381.3	1	2
	.10	55136.2	1	2
1000 2000 3000 4000	.01	29664.8	1	2
	.05	34641.0	1	2
	.10	40000.0	1	2

Table 7 SOLUTION OF 5 COMPONENT SYSTEM (CM2)

maint. cost of components in system	$\frac{C_s}{C_f}$	Optimal Solution	Number of Groups	Number of Iterations
10000 20000 30000 40000 50000	.01	99776.8	2	2
	.05	102469.5	1	2
	.10	104881.6	1	2
1000 2000 3000 10000 20000	.01	65184.8	2	2
	.05	72863.3	1	2
	.10	76157.7	1	2
1000 2000 3000 4000 5000	.01	40000.0	1	2
	.05	44721.4	1	2
	.10	50000.0	1	2

rate, components having a moderate maintenance cost and a moderate failure rate, and components having a moderate maintenance cost and a high failure rate.

The above examples are used to illustrate how the costs of maintenance affect the grouping of components while the failure rate affects the component groupings only when the setup cost is low. In the first system, two distinct groups are formed where components in one group have a high failure rate and components in the other have a low or moderate failure rate. As the setup cost increases, all components are grouped together. In the second and third systems, there are at least two groups of components. It is observed that components with a high maintenance cost are never grouped with components having a moderate maintenance cost even when the setup cost is large. In the third system, the components having a moderate cost are grouped together even when the failure rates differ significantly.

In addition, the number of groups decreases as the setup cost increases. In spite of the number of components being much larger, the number of iterations required to obtain the optimal solution does not increase significantly. In addition, the number of groups is also surprisingly small even when the setup cost is low. By examining Tables 5 through 8, it is seen that systems with components having a small maintenance cost typically have fewer groups than systems having components with a large maintenance cost.

Table 9 gives the number of iterations required to get a solution for systems containing a large number of components. The costs of maintenance, setup, and failure, and the component failure rates for the systems are given in Appendix D. It is seen that the number of iterations required to obtain a solution does not increase drastically even when the number of components is very large. However, the computational effort required to get a solution for the IP formulation during each iteration increases as the number of components increases.

TABLE 8 SOLUTION OF 25 COMPONENT SYSTEM (CM2)

components in system	$\frac{C_s}{C_f}$	Optimal Solution	Number of Groups	Number of Iterations
8 M,h 17 M,m	.001	77791.1	2	2
	.005	80328.2	1	2
	.010	82895.1	1	2
9 M, l 16 H, l	.001	71707.4	4	3
	.005	73471.0	2	2
	.010	75114.5	2	3
13 L, h 9 M, m 3 M, h	.001	68961.5	2	2
	.005	89056.8	2	2
	.010	93300.2	2	2

L-large maint. cost
M-medium maint. cost
S-small maint. cost

h-high failure rate
m-medium failure rate
l-low failure rate

(Refer costs and failure rates in Appendix D)

Table 9 OPTIMIZATION OF LARGE SYSTEMS

nc no. of comp.s	Problem no.	no. of iterations $\frac{C_s}{C_f}$ =.001	$\frac{C_s}{C_f}$ = .005	$\frac{C_s}{C_f}$ = .010
25	1	2	2	2
	2	4	2	3
	3	2	2	2
50	1	3	3	3
	2	4	4	3
	3	3	3	4
100	1	6	6	5

(Refer Appendix D for costs and failure rates)

7.4 ANALYSIS OF CM4

Model CM4 is analyzed in the same fashion as CM2. Systems of different sizes are considered and the different costs (such as systems replacement cost, setup cost) are varied in an attempt to determine their effect on the optimal solution.

Tables 10 through 12 list the results of optimizing systems with 3, 4, and 5 components respectively. The cost of failure is again considered to be 100000 and is kept constant for all systems under consideration. The setup cost and the cost of replacement are expressed as fractions of the failure cost. The failure rate function of the components is assumed to be 0.1t and $\delta_i = .001$ for all components.

Table 10 lists the results of optimizing 3 component systems. An increases in the setup cost results in an increase in the system replacement age. Similarly, an increase in the replacement cost results in an increase in the age of system replacement. The optimization procedure converges after only 2 iterations for all systems.

The solutions for systems containing 4 and 5 components are listed in Tables 11 and 12 respectively. The effects of varying the costs are similar to those seen in the case of the 3 component systems.

Table 13 lists the results of the optimization of systems containing 25 components. The replacement cost is assumed to be 15000 for all systems. The component maintenance costs and failure rate functions are given in Appendix D. Although there are many types of components in each system, there are relatively few groups. Again, as the setup cost increases, the number of groups decreases. The number of iterations required to terminate the optimization procedure is still relatively small.

Table 10 SOLUTION OF 3 COMPONENT SYSTEM (CM4)

maint. cost of comp.s	$\frac{C_s}{C_f}$	$\frac{C_r}{C_f}$	Optimal Solution	T_{sys}	No.of Groups	No.of Iter.
100 200 300	.001	.09	10603.6	4.339	1	2
		.15	11522.7	5.935	1	2
		.21	12440.1	7.351	1	2
100 200 300	.005	.09	11821.7	4.699	1	2
		.15	12645.2	6.446	1	2
		.21	13190.5	8.069	1	2
100 200 300	.010	.09	13158.6	5.471	1	2
		.15	13892.7	7.219	1	2
		.21	14370.7	8.738	1	2

Table 11 SOLUTION OF 4 COMPONENT SYSTEM (CM4)

maint. cost of comp.s in system	$\frac{C_s}{C_f}$	$\frac{C_r}{C_f}$	Optimal Solution	T_{sys}	No.of Groups	No.of Iter.
100 200 500 1000	.001	.09	16176.8	4.210	1	2
		.15	17103.9	5.555	1	2
		.21	17689.3	6.973	1	2
100 200 500 1000	.005	.09	17314.4	4.231	1	2
		.15	18131.3	6.103	1	2
		.21	18653.0	7.388	1	2
100 200 500 1000	.010	.09	18504.2	4.843	1	2
		.15	19257.6	6.252	1	2
		.21	17817.6	7.567	1	2

Table 12 SOLUTION OF 5 COMPONENT SYSTEM (CM4)

maint. cost of comp.s	$\frac{C_s}{C_f}$	$\frac{C_r}{C_f}$	Optimal Solution	T_{sys}	No.of Groups	No.of Iter.
100 200 300 1000 2000	.001	.09	22036.3	3.740	2	2
		.15	22967.4	4.935	2	2
		.21	23554.7	5.973	2	3
100 200 300 1000 2000	.005	.09	23937.6	4.098	2	2
		.15	24747.4	5.407	2	2
		.21	25261.1	6.709	2	2
100 200 300 1000 2000	.010	.09	25002.1	4.376	1	2
		.15	25726.2	5.773	1	2
		.21	26151.6	6.988	1	2

Table 13 SOLUTION OF 25 COMPONENT SYSTEM (CM4)

components in system	$\frac{C_s}{C_f}$	Optimal Solution	Number of Groups	Number of Iterations
5 H, m 5 H, l 5 M, m 5 M, m 5 L, l	.001	63982.6	3	2
	.005	68857.9	2	2
	.010	70275.5	2	2
7 M, h 8 M, l 4 L, h 6 L, l	.001	82504.3	4	3
	.005	91615.9	2	2
	.010	117497.5	1	2
5 H, h 3 H, m 2 H, l 9 M, h 4 M, m 2 M, l	.001	76936.1	2	2
	.005	77515.6	1	2
	.010	88346.0	1	2

Refer Legend in Table 8

7.5 SIMULATION OF THE MODELS

The expressions for the system cost rate in the different models are all constructed using a series of approximations. Under certain conditions, the error due to the approximations may get very large. It is important to identify the conditions under which such a situation may arise.

Discontinuous failure rates make it difficult to get a strong bound on the approximation error. However, it is possible to get a good estimate of the accuracy of the approximation by comparing it with the cost rate of a simulated system.

Due to the difficulties involved in inverting the distribution functions, a continuous simulation of the system is preferred over a discrete event simulation. A continuous simulation is much slower than a discrete event simulation. Thus, the simulation takes a long time to reach steady state. A large number of components results in the simulation taking an even longer time to reach steady state. Hence, only systems containing a few components are simulated.

The simulation model for CM2 assumes a series system where every component in the system fails according to a NHPP. When a component reaches a specified age, maintenance is performed on the component. A maintenance operation on a component is assumed to make the component "as good as new". In addition to the cost of maintaining a component, a setup cost is incurred whenever a maintenance operation is performed. When a component fails, a failure cost is incurred and the entire system is replaced. Therefore, all components are assumed to be new and the above process is repeated.

At the end of the simulation, the average system cost rate is obtained by dividing the total cost incurred by the duration of the simulation. If the length of the simulation run is large enough, the cost rate of the simulated system should be very close to the cost rate of the actual system. All simulation runs are for 500 time units with increments at .001 time units.

The systems described in Tables 5, 6, and 7 are simulated. Type 1 systems are systems which contain components with low maintenance costs. Type 3 systems are systems which contain components with the high maintenance cost while Type 2 systems are those containing components with both high and low maintenance cost. Since a low cost of maintenance usually implies more frequent maintenance, components in Type 1 systems are maintained after shorter intervals of time. As the failure rate functions of the components are identical, it can be assumed that the reliability of components in Type 1 systems is higher than that of components in Type 2 systems. Using the same reasoning, Type 2 systems are expected to be more reliable than Type 3 systems.

The simulation of model CM3 is similar to that of CM2. However, the failure rate function of each component is random. Again, the systems given in Table 5, 6, and 7 are chosen for simulation. The failure rate function of all the components is assumed to be $\lambda(t) = .05t$ with probability $\frac{1}{2}$ and $\lambda(t) = .15t$ with probability $\frac{1}{2}$. Then, the expected failure rate is $E[\lambda(t)] = .1t$. Again, the systems given in Tables 5, 6, and 7 are simulated.

For model CM4, the failure rate function of component i is incremented by a constant amount δ_i after every maintenance operation. In addition, when the system age is equal to age of replacement, all components are replaced. Hence, the system age and all component ages are reset to 0. The systems described in Tables 10, 11, and 12 are simulated.

Table 14 gives the ratio of the cost rate for the simulated system to the optimal solution for model CM2. It is evident that the Type 1 systems give the most accurate results for systems of all sizes. Type 2 systems are slightly less accurate while Type 3 systems give the least accurate results. For the systems and costs considered the average ratio of the cost rate for the simulated system to optimal solution is 0.861.

Table 15 compares the simulation results for CM3 to the optimal solution. Again, Type 1 systems are the most accurate, followed by Type 2 and Type 3 systems in that order. A nominal increase in the number of components does not appear to significantly change the accuracy. The average

TABLE 14 SIMULATION RESULTS (CM2)

NO.OF COMP. S	TYPE	RATIO OF SIMULATION RESULT TO PREDICTED VALUE		
		$\frac{C_s}{C_f} = .01$	$\frac{C_s}{C_f} = .05$	$\frac{C_s}{C_f} = .10$
3	1	0.935	1.012	0.850
	2	0.858	0.892	0.806
	3	0.804	0.793	0.871
4	1	1.001	0.983	0.971
	2	0.865	0.933	0.901
	3	0.792	0.753	0.723
5	1	0.967	0.971	0.976
	2	0.852	0.839	0.776
	3	0.785	0.690	0.654

ratio of the cost rates is 0.840. Thus, model CM3 is slightly less accurate than model CM2. Similarly, an increase in the setup cost does not appear to affect the accuracy of the simulation results.

Table 16 gives the results of the simulation of CM4. The average ratio of the cost rates is 0.806. While a direct comparison between the CM4 and the earlier models is not possible, model CM4 appears to be less accurate than the earlier models. Again, Type 3 systems are very inaccurate. Neither changes in the setup cost nor the replacement cost cause any significant change in the accuracy of model.

TABLE15 SIMULATION RESULTS (CM3)

NO.OF COMPS	TYPE	RATIO OF SIMULATION RESULT TO PREDICTED VALUE		
		$\frac{C_s}{C_f} = .01$	$\frac{C_s}{C_f} = .05$	$\frac{C_s}{C_f} = .10$
3	1	0.934	0.909	0.953
	2	0.849	0.816	0.822
	3	0.796	0.790	0.777
4	1	0.964	0.976	0.917
	2	0.832	0.908	0.873
	3	0.803	0.762	0.726
5	1	0.906	0.987	0.924
	2	0.821	0.837	0.792
	3	0.689	0.655	0.662

TABLE 16 SIMULATION RESULTS (CM4)

$\frac{C_s}{C_f}$	TYPE	RATIO OF SIMULATION RESULT TO PREDICTED VALUE		
		$\frac{C_r}{C_f}$ = .09	$\frac{C_r}{C_f}$ = .15	$\frac{C_r}{C_f}$ = .21
.001	1	0.717	1.005	0.999
	2	0.716	0.878	0.895
	3	0.726	0.711	0.802
.005	1	0.725	0.805	0.893
	2	0.803	0.817	0.782
	3	0.691	0.788	0.751
.010	1	0.870	0.851	0.973
	2	0.878	0.757	0.860
	3	0.654	0.697	0.739

CHAPTER 8 CONCLUSIONS AND FUTURE WORK

8.1 INTRODUCTION

The analysis of the models carried out in the previous chapters gives several insights into the behavior of the models. These insights are useful in identifying situations in which use of the model is appropriate (or inappropriate). In addition, these insights may aid in pinpointing specific weaknesses in the models, thereby suggesting future areas of research.

8.2 CONCLUSIONS

8.2.1 Description of the Procedures for Developing Maintenance Plans

A method for obtaining maintenance plans for multi-component systems is developed. First, a system cost-rate function is constructed. The function is based on the failure rates of the individual components. In addition, the function reflects the savings in cost obtained by performing maintenance on several components simultaneously.

Then, an optimization procedure is given from which the components groupings and the group maintenance times are obtained. The first step in the optimization procedure involves the selection of a few candidate times for component maintenance. An objective function is formulated from the system cost-rate function. The component groupings which minimize this function are obtained. By solving for the optimal maintenance times for each group of components, additional candidate maintenance times are obtained. The objective function is reformulated with these times and again minimized. The optimization procedure is terminated when the component groupings do not change from one iteration to the next.

Three different models are developed. The first model (CM2) assumes that a maintenance operation on a component restores the component to an "as good as new" condition. The cost-rate function for CM2 is given by 4.5.29. From this function, an objective function is formulated (equation 4.6.30). The component groupings and times of maintenance are obtained by using the optimization procedure given above.

The next model (CM3) is described in chapter 5. In this model, the maintenance operations are no longer assumed to restore the components to an "as good as new" condition. The cost-rate function is constructed (equation 5.6.21). and the corresponding objective function (equation 5.6.22) is obtained. The optimization procedure described above gives the component groupings and component maintenance times which minimize the objective function.

The final model (CM4) assumes that the components deteriorate from one maintenance operation to the next. As a result, the system also deteriorates and at some point in time has to be replaced. A cost-rate function which accounts for system replacement costs is constructed. In order to make the function tractable, a series of approximations are carried out. Finally, an optimization procedure for grouping the components is given.

8.2.2 Behavior of the Models

From the numerical analysis in the previous chapter, it is clear that the maintenance plan for a system is largely determined by the costs associated with maintenance, setup, and system failure. The setup cost is a major factor in determining how components are grouped. An increase in the setup cost usually results in a reduction in the number of component groups. Thus, a large setup cost makes grouping of components for simultaneous maintenance economically attractive.

Even when the number of components in a system is large and the setup cost is small, there are only a few groups of components. In such situations, the number of groups is expected to be large. Thus, grouping components for simultaneous maintenance may possibly be more cost effective than originally expected.

Components having approximately similar rates of failure and costs of maintenance are usually grouped together. However, when the maintenance costs are large, there may be more than one group of similar (but not identical) components. Such a situation may arise when the cost savings resulting from grouping components is not enough to offset the cost increases that result from maintenance at a non-optimal time.

Often the optimal maintenance time for a group of components is less than the minimum single component maintenance time of the components in the group. A possible reason is that the cost savings due to the reduction in the number of setup operations is used to perform maintenance more frequently.

In the case of systems which deteriorate in time, the number of groups is usually small. However, when there is more than a single group, an increase in the setup cost results in a reduction in the number of groups. The replacement cost does not appear to have much effect on the number of component groups. However, the age of system replacement does increase when the replacement cost is increased.

8.2.3 Optimization Procedure

The optimization procedure for CM2 has typically given the global optimal solution when the system has only a few components (less than 5 components). However, none of the optimization procedures described in the earlier chapters guarantees a global optimal solution.

The optimization procedure converges relatively fast, even when the system has a large number of components. For all systems considered in the analyses, less than 4 iterations were required to obtain a solution for systems with up to 50 components for CM2. The optimization procedure for CM4 requires up to 4 iterations to converge when dealing with systems having 25 components. The need for only a few iterations could be due to the large number of candidate maintenance times in S when the system has a large number of components. This may result in components being assigned to optimal or near-optimal times of maintenance during the very first iteration of the optimization procedure.

Although an integer solution is not guaranteed, the LP relaxation of the IP problem consistently gives an integer solution for both CM2 and CM4. Therefore, it has not been necessary to solve the more difficult IP problem.

8.2.4 Accuracy of Models

From the simulation runs, it can be seen that the optimal cost rate closely matches the simulation results (within 5 % typically) when the system has components with a low maintenance cost. However, the accuracy falls when the system has components with a high maintenance cost.

During the construction of the cost rate functions for the different models, the component failure rate is approximated by either a constant failure rate (as in CM2/CM3) or a linear failure rate (in CM4). The approximation is a function of the length of the time interval between maintenance operations. The error due to the approximation becomes larger when the time interval between maintenance operations increases. Since components with a low maintenance cost are likely to be maintained at shorter intervals, the approximation error is relatively small. Hence, the accuracy for systems with a low maintenance cost is high.

The accuracy of model CM2 is better than that of model CM3 which in turn in better than that of CM4. This is expected as the number of approximations made in obtaining the cost rate function in CM2 is less the number of approximations required for CM3. The cost rate function of CM4 had the largest number of approximations and hence, is the least accurate.

8.2.5 Summary

In summary, grouping components together for simultaneous maintenance is shown to be economically viable if the setup cost is not negligible relative to the cost of failure. It is also demonstrated that it is necessary to consider replacement of the entire system when maintenance operations on components do not restore the components to an ″as good as new state″.

While the optimization procedure does not guarantee a global optimal solution, it converges very fast. In the case of systems having only a few components, a global optimal is usually obtained. In addition, the optimal solution to the LP relaxation is typically an integer solution. This implies that the computational effort required is far less than would be expected.

The models discussed are expected to be very accurate when the cost of component maintenance and cost of setup are small relative to the cost of failure. However, the accuracy drops significantly when the cost of maintenance is high. This is possibly due to the fact that approximations for the component failure rates are weaker in such situations.

When these models are used to represent real systems, it may be prudent to use model CM4. It provides a more realistic representation of the system as it allows for system replacement.

If the component deterioration is very low, CM2 is recommended as the optimization procedure for CM2 typically converges faster than the optimization procedure for CM4. In addition, the number of computations per iteration are less as the age of system replacement does not have to be determined.

8.3 EXTENSIONS AND FUTURE WORK

The models developed in the earlier chapters assume a certain type of system behavior. There exist several situations where the models are no longer valid. For example, the models developed in the earlier chapters may not be valid for systems in which an occasional failure is tolerated. Maintenance for such systems could be based on a block-replacement policy instead of an age-replacement policy. However, it may be possible to analyze some systems by using a modified form of the models discussed in the earlier chapters. A few extensions are discussed below.

8.3.1 Systems with Independent Sub-Systems

Many systems are made up of several independent sub-systems operating simultaneously. In such systems, failure of a component typically requires replacement of only the sub-system to which it belongs. Therefore, only some of the components are new after a failure. An example of a system with many sub-systems is the automobile. Components such as the transmission and the engine can be considered to be sub-systems in the automobile. The failure of a component such as a coupling may require replacement of the entire transmission. However, the engine, electrical and other systems are not replaced and hence only some of the components in the system are new after a failure.

The maintenance of components from different sub-systems can be scheduled simultaneously in order to reduce the number of setup operations. This is often done with the automobile where preventive maintenance is often performed on the engine and electrical system simultaneously.

If sub-systems are replaced only on failure, it should be possible to modify the cost rate function in CM2 to represent the system cost rate. Each sub-system is considered as a separate system with its own failure cost. However, components from different sub-systems can be maintained simultaneously. Hence, the set of candidated times, S, is common for all sub-systems. After having constructed the expression for the system cost rate, it should be possible to use the optimization procedure given in chapter 4 to get an optimal maintenance plan.

Modifying this model to include the possibility of a system replacement (as in CM4) is a non-trivial task. Since a subsystem may be replaced at any time prior to system replacement, it is difficult to predict the rate of system failure. Therefore, an expression for the cost rate which is similar in form to that obtained in chapter 6 is not expected.

8.3.2 System with Dependent Components

In many systems, a deteriorating component affects the failure rate of the other components as well. This type of behavior is usually observed in power generation stations where the load is shared by a number of generators. When a generator begins to malfunction, a portion of its load is transfered to the other generators. This increases the likelihood of other generators malfunctioning.

It may be possible to represent dependent failure rates by assuming that the failure rate of a component is governed by a Phase-type process. Neuts and Bhattacharjee [24] examine such behavior but fail to consider the effect of maintenance. However, by deriving a distribution for the time to failure, it may be possible to obtain explicit characterizations of the survival probabilities and approximate failure rates in a given time period. Given the approximate failure rate of the components, it should be fairly straight-forward to obtain an appropriate expression for the cost rate using the general form suggested in chapter 4.

8.3.3 System with Multi-State Components

Often component deterioration is a gradual and observable phenomenon. While the deteriorating component may not cause the system to fail, it may significantly affect the performance of the system. In addition, the maintenance cost of a component may depend on its condition at the time of maintenance. Hence, developing a system model in which the maintenance costs and rates of failure depend on the condition of the components may lead to many interesting results.

It is possible to get a strong representation of the deterioration process of the components by assuming that the condition (or deterioration) of each component is indicated by the state of the component. The state of a component can be assumed to change according to a NHPP. By solving the state transition equations, it is possible to determine the probability of being in a particular state at a given time. The expected failure rate of the component can be determined by conditioning on the state of the component.

It is possible to model the deterioration process of a component as a death-process where the final state corresponds to the failed state. Note that the rate of component failure is zero at all states except the penultimate state (the one prior to the failed state). Hence, the expected rate of component failure is the probability of being in the penultimate state multiplied by the transition rate to the final state.

By obtaining the expected failure rates for all the components, the expected failure rate of the system can be determined. It should then be possible to construct an expression for the system cost rate using the quantities obtained above. An optimization procedure similar to the one described in Chapter 4 can again be used when attempting to determine the optimal maintenance plan.

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Appendix A. APPENDIX A BEHAVIOR OF THE APPROXIMATION FOR CM4

The cost rate function discussed in Chapter 6 was obtained after a series of approximations. A strong analytical bound on the error due to the approximations is not possible. Hence, a series of comparisons between the actual cost rate function and its approximation is made.

The terms corresponding to the costs of component maintenance (equation 6.5.23) are not approximated. Hence, only a comparison between the terms corresponding to system replacement and system failure are considered.

The exact function is given by:

$$\frac{C_r \exp(-\sum_{i=1}^{m} a_i(T_i)T_{sys} + \frac{b_i(T_i)}{2} T_{sys}^2) + C_f(1 - \exp(-\sum_{i=1}^{m} a_i(T_i)T_{sys} + \frac{b_i(T_i)}{2} T_{sys}^2))}{\int_0^{T_{sys}} \exp(-\sum_{i=1}^{m} a_i(T_i)t + \frac{b_i(T_i)}{2} t^2)dt}$$

The approximation is given by:

$$\frac{C_f}{T_{sys}}\left(\sum_{i=1}^{m}(a_i(T_i)T_{sys} + b_i(T_i)\frac{T_{sys}^{2}}{2})\right)$$
$$+\frac{C_r}{T_{sys}}\left(1-\sum_{i=1}^{m}(a_i(T_i)\frac{T_{sys}}{2} + b_i(T_i)\frac{T_{sys}^{2}}{3})\right)$$

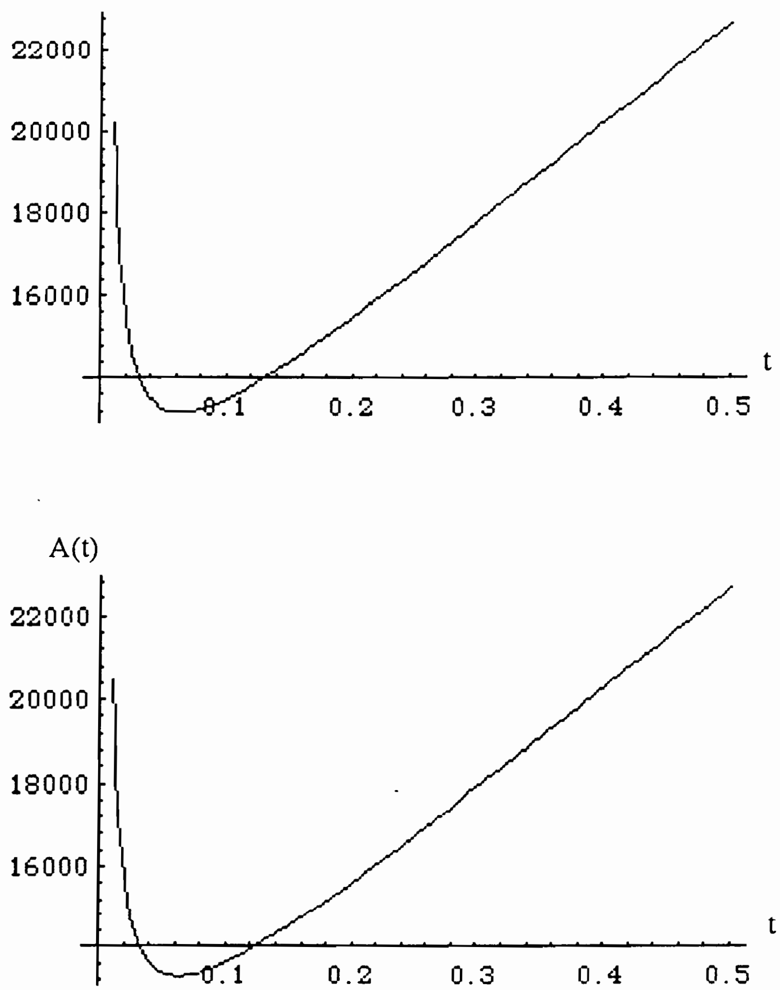
The accuracy of the approximation depends on the values of $\sum_{i=1}^{m} a_i(T_i)$ and $\sum_{i=1}^{m} b_i(T_i)$. Let $a=\sum_{i=1}^{m} a_i(T_i)$ and $b=\sum_{i=1}^{m} b_i(T_i)$. In addition, let $E[C(T_{sys})]$ represent the exact cost rate function and let $A(<T\ sub\ sys>)$ represent the approximate cost rate function. Then,

$$E[C(T_{sys})]=\frac{C_r \exp(-\sum_{i=1}^{m} aT_{sys}+\frac{b}{2}T_{sys}^{2}) + C_f(1-\exp(-\sum_{i=1}^{m} aT_{sys}+\frac{b}{2}T_{sys}^{2}))}{\int_0^{T_{sys}} \exp(-\sum_{i=1}^{m} a+\frac{b}{2}t^{2})dt}$$

and

$$A(T_{sys})=\frac{C_f}{T_{sys}}\left(\sum_{i=1}^{m}(aT_{sys}+b\frac{T_{sys}^{2}}{2})\right)+\frac{C_r}{T_{sys}}\left(1-\sum_{i=1}^{m}(a\frac{T_{sys}}{2}+b\frac{T_{sys}^{2}}{3})\right)$$

A comparison is made by plotting the two functions over T_{sys} for different values of a and b. It is important to note that a is an approximation for the expected number of failures in a maintenance interval. Hence, the functions are compared for relatively small values of $a(\leq .1)$. By observing these plots, it is possible to get an idea of the accuracy of the approximation. Some of the more significant plots are given in order to give the reader of how the approximation behaves in different situations.



a=.1, b=.5

Figure 4. E[C(t)] vs. A(t) - I

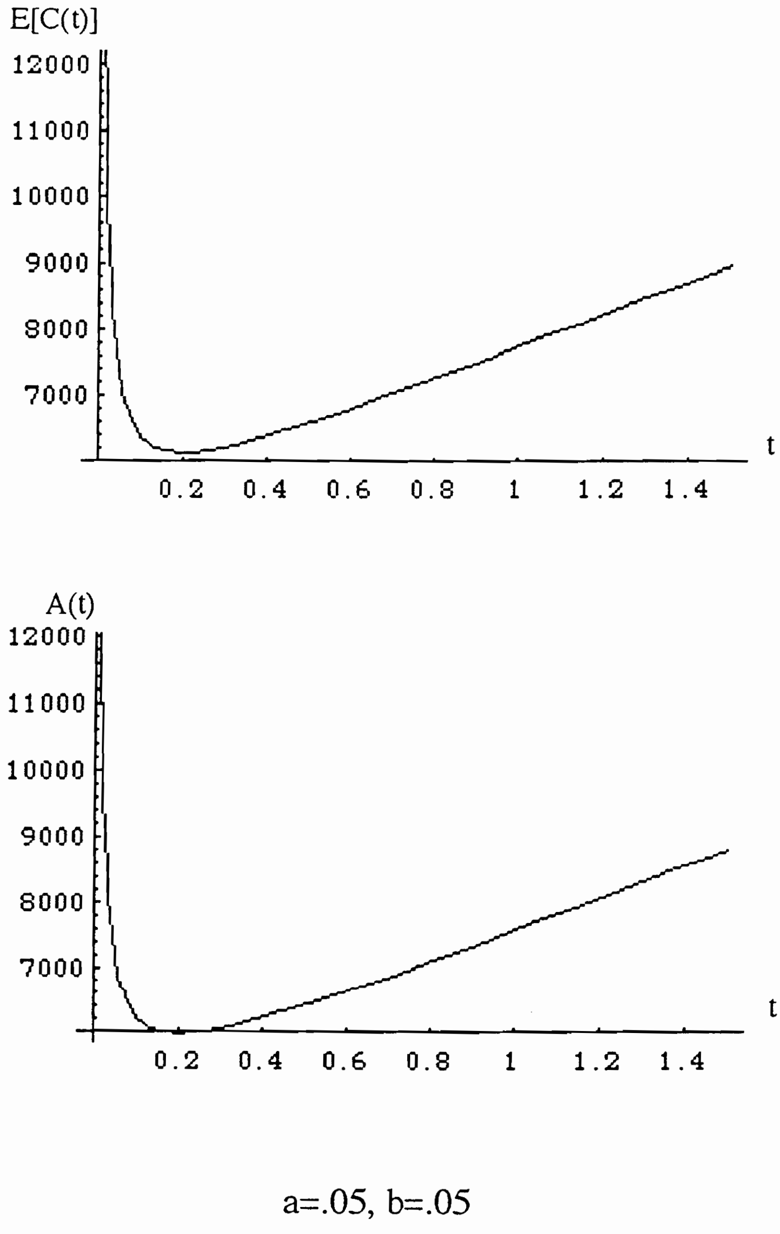
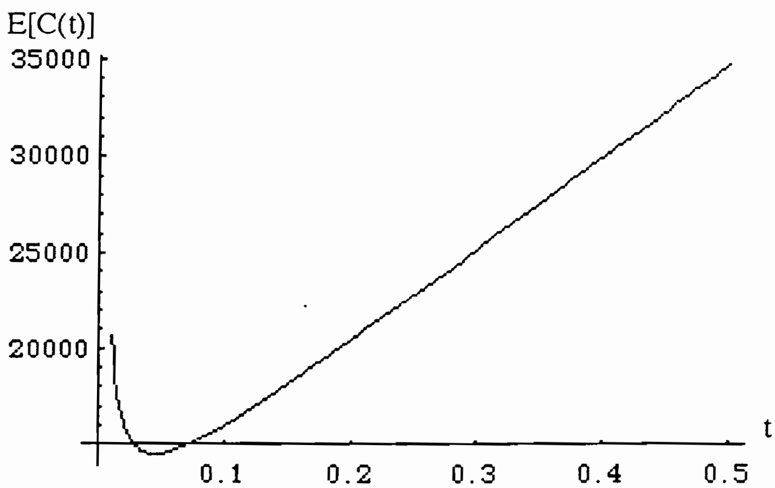
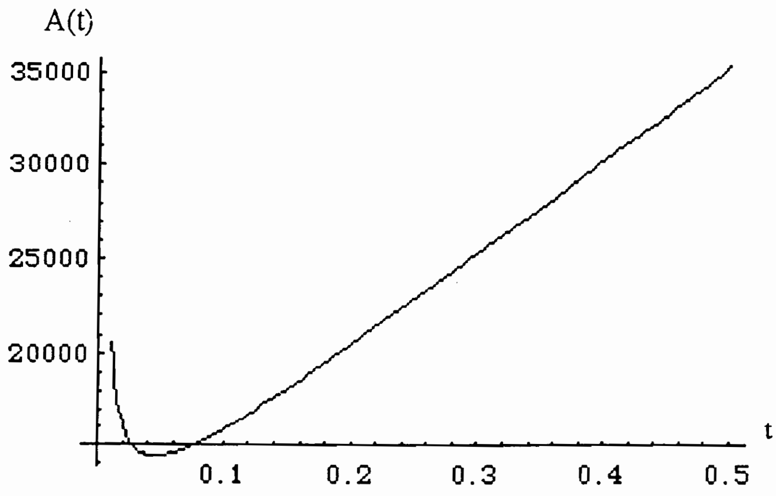


Figure 5. **E[C(t)] vs. A(t) - II**





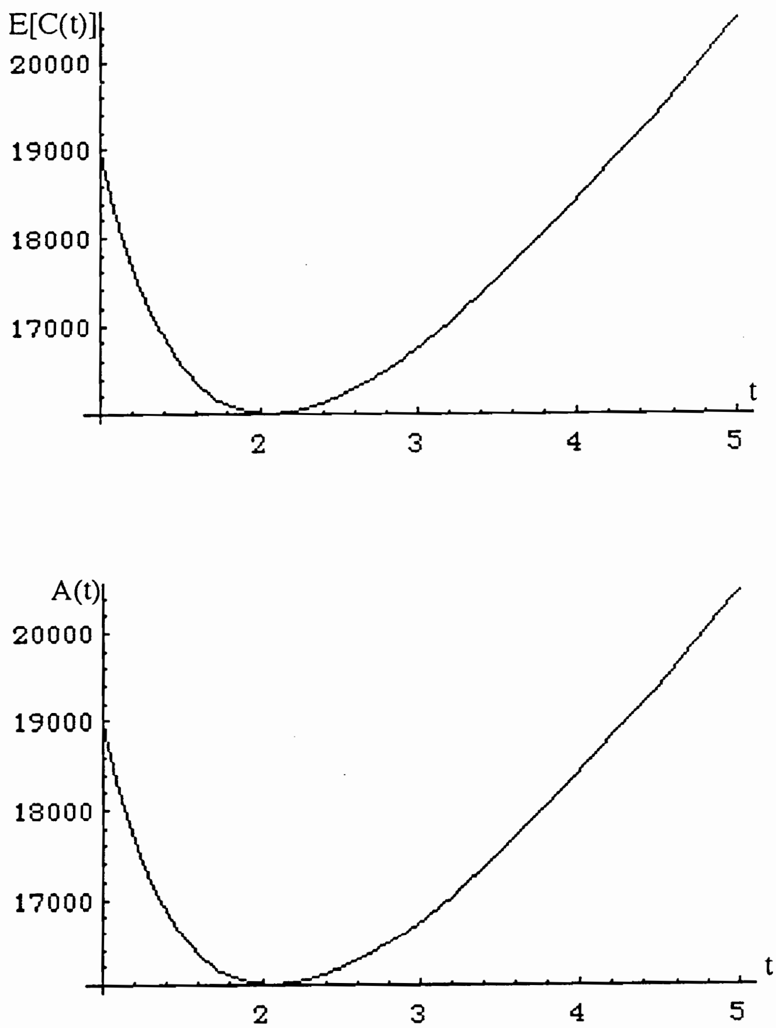
a=.1, b=1

Figure 6. E[C(t)] vs. A(t) - III

Figures 4 through 6 compare the cost rate function to the approximation for C_r:C_f = 1:10. The accuracy of the approximation seems to be very high in this case as the plots are almost identical in all situations.

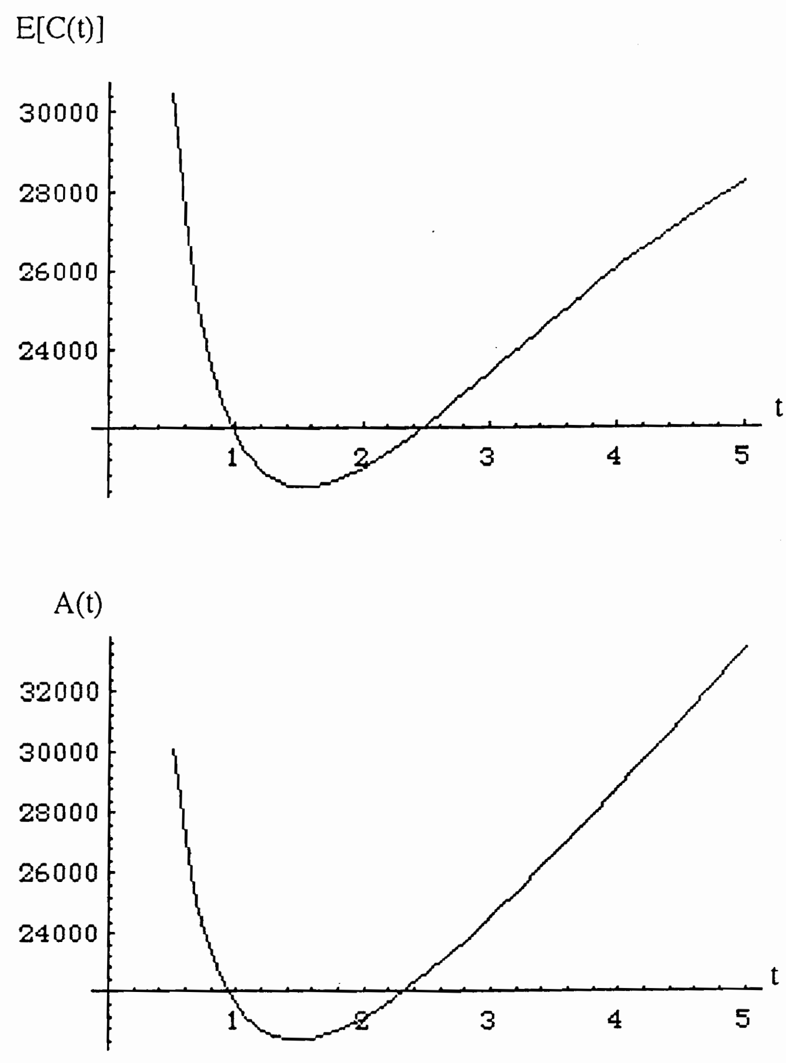
Figures 7 through 10 compare the cost rate function to the approximation for C_r:C_f = 1:100. The accuracy of the approximation changes only marginally for different values of *b*. The accuracy of the approximation changes significantly with changes in the value of *a*. It can be seen from the plots that when the value of *a* is relatively large ($\geq .1$), the error can get large. However, for small values of *a* and *b*, the error is typically below 5%.

It is important to note that *a* is an approximation for the expected number of failures in a maintenance interval and is typically be very small for highly reliable systems. Hence, the approximation is expected to be very accurate in the case of highly reliable systems.



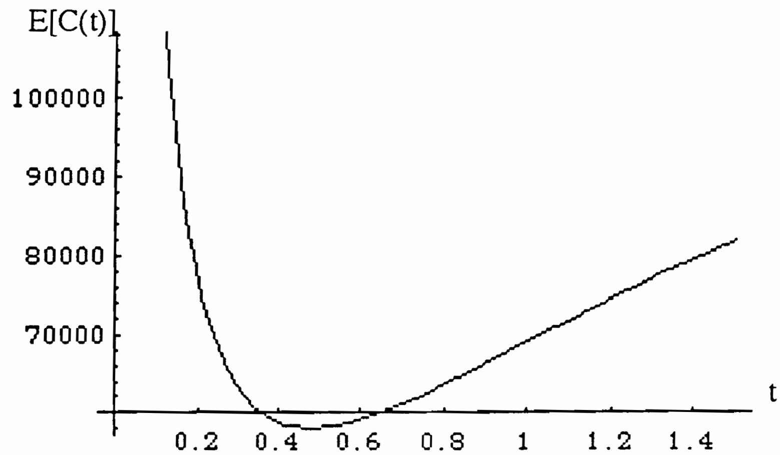
a = .05, b= .05

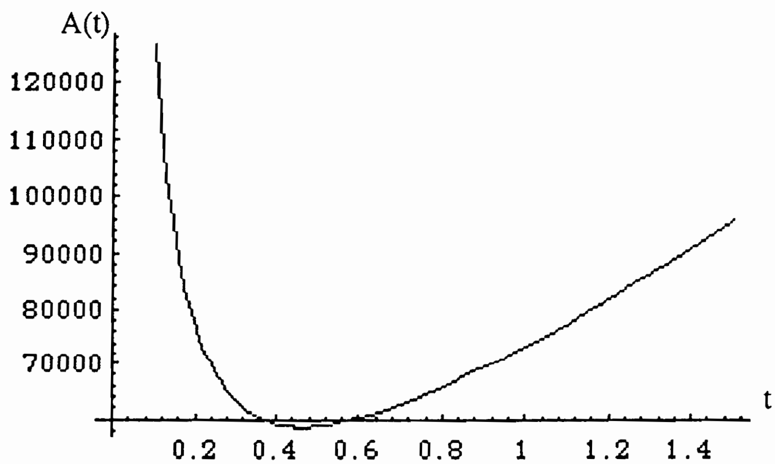
Figure 7. E[C(t)] vs. A(t) - IV



a=.05, b=.1

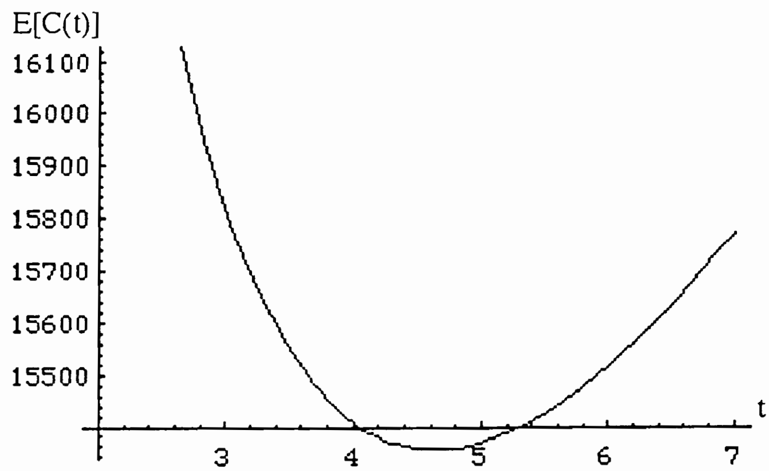
Figure 8. **E[C(t)] vs. A(t) - V**

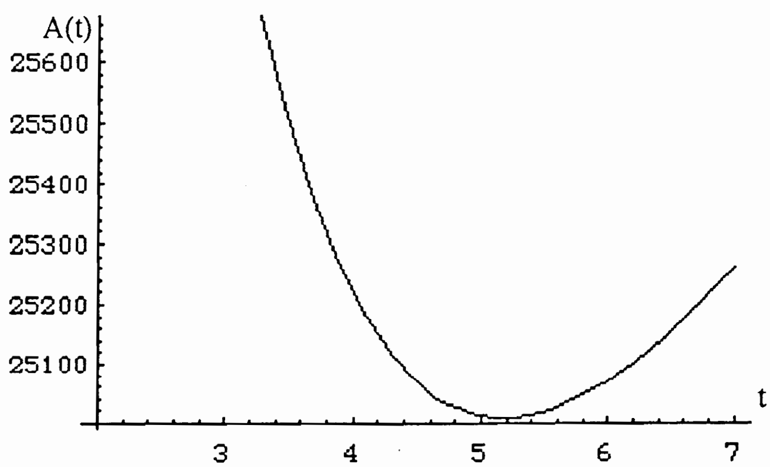




a=.1, b=1

Figure 9. E[C(t)] vs. A(t) - VI





a=.1, b=.01

Figure 10. E[C(t)] vs. A(t) - VII

Appendix B. 'C' PROGRAM FOR COMPUTING THE MINIMUM OF THE COST RATE FUNCTION

The program given below computes the minimum single component cost rate for all components in a system. In addition, it also computes the coefficients of the variables in IP formulation given in Chapter 4 given the set of candidate times.

The user is prompted through a series of menus which ask the user to specify the number of components, the failure rate coefficient, and the various costs.

The proram uses modified versions of the functions 'mbrak' and 'golden' given by Press, Flannery, Teukolsky, and Vettering [28]. The first function brackets the minimum of a function while the second function uses the golden section search to determine the minimum of a function. The accuracy of the solution can changed by changing the value of the constant TOL in the code.

```
#include <stdio.h>
#include <math.h>
#include <alloc.h>
```

```
#include <ctype.h>
#include <stdlib.h>

/*
   The primary input to this program is the set of costs
   cs - cost of setup
   cf - cost of failure
   nc - number of components
   cmi- cost of maintenance of component i
   rci- coeff. of integral of rate of failure for comp. i
   rpi- exponent of integral of rate of failure of comp. i
   The current code allows the user to define the exponent
   and the coefficient of the failure rate.
   The user will also be prompted for a range within which
   the minima of the function is expected.
*/

#define GOLD 1.618034
#define GLIMIT 100.0
#define TINY 1.0e-20
#define TOL 1.0e-6
#define MAX(a,b) ((a) > (b) ? (a) : (b))
#define SIGN(a,b) ((b) > 0.0 ? fabs(a) : -fabs(a))
#define SHFT(a,b,c,d) (a)=(b); (b)=(c); (c)=(d);
#define SHFT2(a,b,c) (a)=(b); (b)=(c);
#define R 0.61803399
#define C (1.0-R)


typedef struct
{
  float rc, cm;
  int rp;
} infoi;


float cs, cf;
int nc;
char menu(int *);
float func(float, infoi *);
float golden(float, float, float, infoi *,
float (*)(float, infoi), float, float *);
float rate(float);
int cont();
int tolower(int);
void coxij(float *, infoi *, float *, int, int);
void coyj(float *, float *, int);
void mnbrak(float *, float *, float *, float *, float *, float *,
infoi *, float (*)(float, float));


/* A menu of choices is offered to user who may wish to
   evaluate the functions at the ind. minima or at user
   specified points (or both).
*/
```

```
char menu(int *ntm)
{
  char ch;
  *ntm = 0;
  do
  {
   printf("MENU\n");
   printf("1. Det. opt. times of the comp. maint. & coeff.s of Xij and Yj\n");
   printf("2. Det. coeff.s of Xij and Yj for a given set of times\n");
   printf("3. Both of the above\n");
   printf(" Enter your choice: ");
   ch = getchar(); printf("menu ch %c ", ch);
  } while (ch != '1' && ch != '2' && ch != '3');
  switch(ch)
  {
    case '1': *ntm = nc; break;
    case '2':
      puts("\n Type the number of times to consider : ntm = ");
      scanf("%d", ntm);  break;
    case '3':
      puts("\n Type the number of additional times to consider : ntm = ");
      scanf("%d", ntm);
      *ntm + = nc;
  }
  printf("nc %d, ntm %d, ch %c\n", nc, *ntm, ch);
  return ch;
}


float func(float y, infoi *inf)
{
  float v, x;
  x = fabs(y);
  printf("func: value of x, y %f %f\n", x, y);
  v = ((inf->cm) + cs ) / x + cf * (inf->rc) * (pow(x, inf->rp)) / x ;
  printf("func: value of cm, %f , rc, %f , rp %d\n",(inf->cm), (inf->rc),
  (inf->rp) );
  printf("value of the function %f\n",v);
  return v;
}


float golden(float ax, float bx, float cx, infoi *ing,
float (*f)(float, infoi *), float tol, float *xmin)
{
  float  f1, f2, x0, x1, x2, x3;
  x0 = ax;
  x3 = cx;
  if (fabs(cx -bx) > fabs(bx - ax))
  {
    x1 = bx;
    x2 = bx + C*(cx - bx);
  }
  else
  {
```

```
    x2 = bx;
    x1 = bx - C * (cx - bx);
  }
  f1 = (*f)(x1, ing);
  f2 = (*f)(x2, ing);
  while (fabs(x3-x0) > tol * (fabs(x1) + fabs(x2)))
  {
    if (f2 < f1)
    {
      SHFT(x0, x1, x2, (R * x1 + C * x3) )
      SHFT2(f1, f2, (*f)(x1, ing) )
    }
    else
    {
      SHFT(x3, x2, x1, (R * x2 + C * x0) )
      SHFT2(f2, f1, (*f)(x1, ing) )
    }
  }
  if (f1 < f2)
  {
    *xmin = x1;
    return f1;
  }
  else
  {
    *xmin = x2;
    return f2;
  }
}


/* function inquires whether user wishes to proceed */
int cont()
{
  char ch;
  int yi, ni;
  yi = 'y';
  ni = 'n';
  puts("\n Do you want to continue?(y/n)");
  ch = tolower(getchar());
  while ((ch != yi) && (ch != ni))
  {
    puts("\n You must respond with a 'y' or 'n'.");
    ch = tolower(getchar());
  }
  return ch;
}


/*
   function coxij determines the coeff.s of Xij for all i and j
*/

void coxij(float *tj, infoi *inc, float *xij, int n, int ntx)
{
```

```
  short i, j;
  for (i=0; i<ntx; i++)
  {
   for (j=0; j<n; j++)
   {
     printf("i = %d, j = %d, cmj = %f, rpj = %d, tj = %f\n",
     i, j, (inc + j)->cm, (inc + j)->rp, *(tj + i));
     *(xij + i*n +j) = ((inc + j)->cm) / *(tj + i) +
     cf * ((inc + j)->rc) * pow(*(tj + i), (inc + j)->rp ) / *(tj + i);
   }
  }
  return;
}


/*
  function coyj determines the coeff.s of Yj for all times.
*/

void coyj(float *t, float *yj, int nty)
{
  short i;
  for (i=0; i<nty; i++)
  {
   *(yj + i) = cs / *(t + i);
   printf("j = %d, cs = %f, t = %f, y = %f\n",
   i, cs, *(t+i), *(yj+i) );
  }
  return;
}


/*
  The function 'mnbrak' is a function which gives the
  range in which the search for the minima should be made.
  This function is described in 'Numerical Recipes in C'.
  The function returns a triplet of points a, b, c, which
  bracket the minima.
  The function uses a parabolic (i.e quadratic) fit to
  obtain c and verifies it.
  If it fails to obtain a good fit, it uses a default
  magnification to obtain c.
*/

void mnbrak(float *ax, float *bx, float *cx, float *fa,
float *fb, float *fc, infoi *infm, float (*fun)(float, infoi *))
{
  float ulim, u, r, q, fu, dum, nu, mx, du ;
  *fa = (*fun)(*ax, infm);
  *fb = (*fun)(*bx, infm);
  if (*fb > *fa)
  {
   SHFT(dum, *ax, *bx, dum)
   SHFT(dum, *fa, *fb, dum)
  }
```

```
  *cx = (*bx) + GOLD * (*bx - *ax);
  *fc = (*fun)(*cx, infm);
  while (*fb > *fc)
  {
   r = (*bx - *ax) * (*fb - *fc);
   q = (*bx - *cx) * (*fb - *fa);
   nu = (*bx - *cx) * q - (*bx - *ax) * r;
   mx = MAX(fabs(q - r), TINY);
   du = SIGN( mx , (q - r));
   u = (*bx) - nu / (2.0 * du);
   ulim = (*bx) + GLIMIT * (*cx - *bx);
   if ((*bx-u)*(u-*cx) > 0.0)
   {
    fu = (*fun)(u, infm);
    if (fu < *fc)
    {
     *ax = (*bx);
     *bx = u;
     *fa = (*fb);
     *fb = fu;
     return;
    }
    else if (fu > *fb)
    {
     *cx = u;
     *fc = fu;
     return;
    }
    u = (*cx) + GOLD*(*cx - *bx);
    fu = (*fun)(u, infm);
   }
   else if ((*cx - u)*(u - ulim) > 0.0)
   {
    fu = (*fun)(u, infm);
    if (fu < *fc)
    {
     SHFT(*bx, *cx, u, *cx + GOLD*(*cx - *bx))
     SHFT(*fb, *fc, fu, (*fun)(u, infm) )
    }
   }
   else if ((u - ulim)*(ulim-*cx) > = 0.0)
   {
    u = ulim;
    fu = (*fun)(u, infm);
   }
   else
   {
    u = (*cx) + GOLD*(*cx - *bx);
    fu = (*fun)(u, infm);
   }
   SHFT(*ax, *bx, *cx, u)
   SHFT(*fa, *fb, *fc, fu)
  }
}
```

```
/*
   tabc determines a triplet of points a, b, c which
   bracket a minimum. A choice is offered to the user of
   using mnbrak or typing in the triplet.
*/

void tabc(float *ta, float *tb, float *tc, infoi *tinfi,
float (*tf)(float, infoi *))
{
  char td;
  float tfa, tfb, tfc;
  puts("\n Do you want mnbrak to give a bracketing triplet of points?(y/n)");
  td = 'x';
  while ((td != 'y') && (td != 'n'))
  {
    puts("\n You must type in a 'y' or 'n'.");
    td = tolower(getchar());
  }
  if (td == 'n')
  {
    puts("\n Type in points a, b, c such that f(a) > f(b) < f(c).");
    puts("\n a = ");
    scanf("%f", ta);
    puts("\n b = ");
    scanf("%f", tb);
    puts("\n c = ");
    scanf("%f", tc);
    return;
  }
  puts("\n Type in a suitable range in which the minima is expected.");
  puts("\n Lower value b =");
  scanf("%f", tb);
  puts("\n Upper value a =");
  scanf("%f", ta);
  mnbrak(ta, tb, tc, &tfa, &tfb, &tfc, tinfi, tf);
  printf("%14s %12s %12s \n", "a", "b", "c");
  printf("%3s %14.6f %12.6f %12.6f\n", "x", *ta, *tb, *tc);
  printf("%3s %14.6f %12.6f %12.6f\n", "fx", tfa, tfb, tfc);
  return;
}


main()
{
  char md;
  float a, b, c;
  float *cxij, *cyj, *gval, *xmin;
  int ni, nt;
  infoi *infi;
  short i, j;
/* yi = 'y'; */
  ni = 'n';
  puts(" \n Type the cost of failure: cf = ");
  scanf("%f", &cf);
```

```
  puts("\n Type the cost of setup: cs = ");
  scanf("%f", &cs);
  puts("\n Type the number of components: nc = ");
  scanf("%d", &nc);
  nt = 0;
  md = menu(&nt);
  if (nt == 0) exit(0);
  printf("nt = %d\n", nt);
  cxij = (float *)calloc(nt*nc, sizeof(float));
  cyj = (float *)calloc(nt, sizeof(float));
  gval = (float *)calloc(nc, sizeof(float));
  infi = (infoi *)calloc(nc, sizeof(infoi));
  xmin = (float *)calloc(nt, sizeof(float));
  for (i=0; i < nc; i++)
  {
    printf("\n Type the cost of maintenance for component %d: cm = ", i);
    scanf("%f", &(infi+i)->cm);
    printf("\n Type the coeff. of the rate function %d: rc = ", i);
    scanf("%f", &(infi+i)->rc);
    printf("\n Type the power of the rate function %d: rp = ", i);
    scanf("%d", &(infi+i)->rp);
  }
  if (md == '1' || md == '3')
  {
    for (i=0; i<nc; i++)
    {
      if (cont() == ni) exit(0);
      tabc(&a, &b, &c, (infi + i), func);
      if (cont() == ni) exit(0);
      *(gval + i) = golden(a, b, c, (infi + i), func, TOL, (xmin + i));
      printf("\n %15.6f %12.6f", *(xmin + i), *(gval + i));
    }
  }
  if (md == '2' || md == '3')
  {
    i = (md == '2') ? 0 : nc;
    puts("\n Type in the times at which the coeffs. are to be evaluated.");
    for (;i < nt; i++)
    {
      printf("\n time %d ", i);
      scanf("%f", (xmin + i));
    }
  }
  if (cont() == ni) exit(0);
  coyj(xmin, cyj, nt);
  coxij(xmin, infi, cxij, nc, nt);
  for (j=0; j < nt; j++)
  {
    printf("\n coeff. of Yj for j = %d is %f", j , *(cyj + j));
    for (i=0; i < nc; i++)
    printf("\n coeff. of Xij for i = %d is %f", i, *(cxij + j*nc + i));
  }
  return 0;
}
```

Appendix C. FORTRAN PROGRAM FOR SIMULATING CM2

The program given below was used for a continuous simulation of CM2. Slight modifications are required to simulate systems CM3 and CM4.

The inputs to the program are the times of maintenance, components, the costs of maintenance, and the costs of setup and failure. It is also necessary to specify one of the components in a group G(J). If there are no components in the group, enter a '0'.

The program computes the average cost per unit time of the system. In addition, the program also computes a number of other system measures which may be useful in obtaining an insight into the behavior of the system. Examples of such measures are fraction of system failures due to a particular component, average life of the system, and average life of a component.

```
      INTEGER I
      REAL CLK, SA, ANM(5)
      DIMENSION NF(6), TLF(6), TMAINT(5), EL(5), FL(5), TEL(6), COST(6)
      DIMENSION CM(5), NM(6), G(5), CSJ(5)
C**********************************************************************
C     THE PURPOSE OF THIS PROGRAM IS TO RUN A CONTINUOUS SIMULATION
C     FOR THE SYSTEM FAILURE MODEL. THE SYSTEM IS A SERIES SYSTEM.
C     THUS FAILURE OF A COMPONENT LEADS TO SYSTEM FAILURE.
C
```

```
C   INPUT
C   TMAINT(J) - MAINTENANCE TIME OF GROUP J
C   CM(I)     - MAINTENANCE COST OF COMPONENT I
C   CF        - COST OF FAILURE OF THE SYSTEM
C   CS        - SETUP COST PER GROUP MAINTENANCE
C
C   G(J)      - CONTAINS A COMPONENT WHICH BELONGS TO GROUP J
C
C   CLK       - CURRENT TIME
C   SA        - THE AGE OF THE SYSTEM
C   TLF(I)    - TIME SINCE LAST FAILURE OF COMPONENT I
C   NF(I)     - NUMBER OF FAILURES OF COMPONENT I AT THE END
C   NM(I)     - NUMBER OF MAINTENANCE OPERATIONS OF COMPONENT I
C   ANM(I)    - AV.NO. OF MAINT. OP.S OF COMP. I BEFORE SYSTEM FAILURE
C   COST(I)   - MAINTENANCE COST OF COMPONENT I
C   FC        - FINAL COST OF SYSTEM
C   EL(I)     - LIFE OF COMPONENT I WHEN SYSTEM FAILS
C   TEL(I)    - SUM OF EL(I) AT THE END OF THE SIMULATION
C   FL(I)     - RATIO OF EL(I)/TMAINT(I)
C
C   COMPONENT 6 REFERS TO THE SYSTEM
C
C   THE CLOCK RUNS AT INCREMENTS OF .001 TIME UNITS. THE INSTANTANEOUS
C   PROBABILITY OF FAILURE OF COMPONENT I IS GIVEN BY .001*LAM(I)
C   THE PROCESS IS RESTARTED AFTER THE FAILURE OF ANY COMPONENT I.E.
C   FAILURE OF THE SYSTEM.
C*********************************************************************
      READ(11,*)(TMAINT(I), I = 1,5)
      READ(11,*)(CM(I), I = 1,5)
      READ(11,*)CF, CS
      READ(11,*)(G(J), J = 1,5)
      DO 20 I = 1, 5, 1
        NF(I) = 0
        NM(I) = 0
        TLF(I) = 0.0
        TEL(I) = 0.0
        COST(I) = 0.0
 20   CONTINUE
      NF(6) = 0
      TLF(6) = 0.0
      CLK = 0.001
      SA = 0.001
      FC = 0.0
      DO 10 I = 1, 1000, 1
        CALL RF1(SA, TMAINT(1), NP1, EL(1))
        NF(1) = NF(1) + (1- NP1)
        CALL RF2(SA, TMAINT(2), NP2, EL(2))
        NF(2) = NF(2) + (1 - NP2)
        CALL RF3(SA, TMAINT(3), NP3, EL(3))
        NF(3) = NF(3) + (1 - NP3)
        CALL RF4(SA, TMAINT(4), NP4, EL(4))
        NF(4) = NF(4) + (1 - NP4)
        CALL RF5(SA, TMAINT(5), NP5, EL(5))
        NF(5) = NF(5) + (1 - NP5)
        NP = NP1*NP2*NP3*NP4*NP5
        IF (NP. EQ. 0) THEN
```

```
        NF(6) = NF(6) + 1
        WRITE (21,*) (NF(J), J=6,1,-1)
        WRITE (21,*) SA, CLK
        FC = FC + CF
        DO 30 J = 1,5,1
          TEL(J) = EL(J) + TEL(J)
          FM = SA/TMAINT(J)
          IFM = INT(FM)
          NM(J) = IFM + NM(J)
          COST(J) = COST(J) + IFM*CM(J)
 30       CONTINUE
        DO 60 J = 1,5,1
          IF ( G(J).EQ.0) THEN
            CSJ(J) = 0
          ELSE
            K = G(J)
            CSJ(J) = CS*NM(K)
          ENDIF
 60       CONTINUE
        SA = .0001
      ELSE
       SA = SA + .001
      ENDIF
      CLK  = CLK + .001
 10   CONTINUE
     DO 40 J = 1,5,1
      FL(J) = TEL(J) / TMAINT(J)
 40   CONTINUE
     WRITE (21,*) NF(6), CLK
     AL = CLK / NF(6)
     WRITE (21,*) AL
     DO 50 J = 1,5,1
      ANM(J) = NM(J) / NF(6)
      FC = FC + COST(J) + CSJ(J)
 50   CONTINUE
     WRITE (21,*) NF(6), CLK
     AL = CLK / NF(6)
     WRITE (21,*) AL
     ASC = FC / CLK
     WRITE (21,*) (ANM(I), I = 1, 3, 1)
     WRITE (21,*) (NM(I), I = 1, 3, 1)
     WRITE (21,*) FC
     WRITE (21,*) ASC
     WRITE (21,*) (ANM(I), I = 1, 5, 1)
     WRITE (21,*) (NM(I), I = 1, 5, 1)
     STOP
     END
C*********************************************************************
C   THE FOLLOWING SUBROUTINES ARE CALLED AT EVERY TIME INCREMENT
C   AND DETERMINE THE PROBABILITY OF FAILING IN THAT TIME INTERVAL
C   IT IS ASSUMED ONLY LINEAR TERMS NEED TO BE CONSIDERED AS THE TIME
C   INCREMENT IS VERY SMALL
C*********************************************************************
     SUBROUTINE RF1(T, TM, NP, TLM)
     REAL T, TM
     INTEGER NP
```

```
U01R = RNUNF()
R = T/TM
MN = INT(R)
TLM = T - MN*TM
FPT = 3*TLM*0.001
IF (U01R.LT.FPT) THEN
  NP=0
ELSE
  NP=1
ENDIF
RETURN
END
SUBROUTINE RF2(T, TM, NP, TLM)
REAL T, TM
INTEGER NP
U01R = RNUNF()
R = T/TM
MN = INT(R)
TLM = T - MN*TM
FPT = 4*TLM*0.001
IF (U01R.LT.FPT) THEN
  NP=0
ELSE
  NP=1
ENDIF
RETURN
END
SUBROUTINE RF3(T, TM, NP, TLM)
REAL T, TM
INTEGER NP
U01R = RNUNF()
R = T/TM
MN = INT(R)
TLM = T - MN*TM
FPT = .05*TLM*0.001
IF (U01R.LT.FPT) THEN
  NP=0
ELSE
  NP=1
ENDIF
RETURN
END
SUBROUTINE RF4(T, TM, NP, TLM)
REAL T, TM
INTEGER NP
U01R = RNUNF()
R = T/TM
MN = INT(R)
TLM = T - MN*TM
FPT = .08*TLM*0.001
IF (U01R.LT.FPT) THEN
  NP=0
ELSE
  NP=1
ENDIF
RETURN
```

```
END
SUBROUTINE RF5(T, TM, NP, TLM)
REAL T, TM
INTEGER NP
U01R = RNUNF()
R = T/TM
MN = INT(R)
TLM = T - MN*TM
FPT = .4*TLM*0.001
IF (U01R.LT.FPT) THEN
  NP=0
ELSE
  NP=1
ENDIF
RETURN
END
```

Appendix D. COMPONENT DATA FOR LARGE SYSTEMS

The following tables give the component characteristics for the large component problems in Chapter 7.

Table 17. CHARACTERISTICS OF COMPONENTS IN 25 COMPONENT SYSTEM-I (CM2)

MC	A(t)	MC	A(t)	MC	A(t)	MC	A(t)
200	.115t	100	.120t	200	.120t	300	.120t
400	.125t	100	.050t	200	.055t	300	.055t
400	.055t	200	.060t	300	.060t	400	.060t
200	.065t	300	.065t	300	.070t	400	.070t
200	.075t	300	.080t	300	.085t	300	.090t
400	.090t	400	.095t	300	.100t	300	.105t
400	.110t						

(MC - maintenance cost of component : A(t) - failure rate function)

Table 18. CHARACTERISTICS OF COMPONENTS IN 25 COMPONENT SYSTEM-II (CM2)

MC	A(t)	MC	A(t)	MC	A(t)	MC	A(t)
700	.005t	800	.005t	700	.010t	800	.010
700	.015t	800	.015t	900	.015t	800	.020
900	.020t	1200	.025t	900	.025t	500	.025
600	.025t	600	.030t	1000	.030t	1100	.030
600	.035t	700	.035t	800	.035t	800	.040
900	.040t	700	.045t	800	.045t	900	.045
1000	.045t						

(MC - maintenance cost of component : A(t) - failure rate function)

Table 19. CHARACTERISTICS OF COMPONENTS IN 25 COMPONENT SYSTEM-III (CM2)

MC	A(t)	MC	A(t)	MC	A(t)	MC	A(t)
700	.005t	800	.005t	700	.010t	800	.010
700	.015t	800	.015t	900	.015t	800	.020
900	.020t	1200	.025t	900	.025t	500	.025
600	.025t	600	.030t	1000	.030t	1100	.030
600	.035t	700	.035t	800	.035t	800	.040
900	.040t	700	.045t	800	.045t	900	.045
1000	.045t						

(MC - maintenance cost of component : A(t) - failure rate function)

Table 20. CHARACTERISTICS OF COMPONENTS IN 50 COMPONENT SYSTEM-I (CM2)

MC	A(t)	MC	A(t)	MC	A(t)	MC	A(t)
700	.005t	800	.005t	700	.010t	800	.010t
700	.015t	800	.015t	900	.015t	800	.020t
900	.020t	1200	.025t	900	.025t	500	.025t
600	.025t	600	.030t	1000	.030t	1100	.030t
600	.035t	700	.035t	800	.035t	800	.040t
900	.040t	700	.045t	800	.045t	900	.045t
1000	.045t	10	.250t	20	.250t	30	.250t
40	.250t	10	.300t	20	.300t	20	.350t
30	.350t	50	.350t	20	.400t	30	.400t
40	.400t	50	.400t	10	.450t	20	.450t
30	.450t	40	.450t	50	.450t	10	.500t
20	.500t	30	.500t	40	.500t	50	.500t
10	.550t	20	.550t				

(MC - maintenance cost of component : A(t) - failure rate function)

Table 21. CHARACTERISTICS OF COMPONENTS IN 50 COMPONENT SYSTEM-II (CM2)

MC	A(t)	MC	A(t)	MC	A(t)	MC	A(t)
10	.250t	20	.250t	30	.250t	40	.250t
10	.300t	20	.300t	20	.350t	30	.350t
50	.350t	20	.400t	30	.400t	40	.400t
50	.400t	10	.450t	20	.450t	30	.450t
40	.450t	50	.450t	10	.500t	20	.500t
30	.500t	40	.500t	50	.500t	10	.550t
20	.550t	100	.050t	200	.055t	300	.055t
400	.055t	200	.060t	300	.060t	400	.060t
200	.065t	300	.065t	300	.070t	400	.070t
200	.075t	300	.080t	300	.085t	300	.090t
400	.090t	400	.095t	300	.100t	300	.105t
400	.110t	200	.115t	100	.120t	200	.120t
300	.120t	400	.125t				

(MC - maintenance cost of component : A(t) - failure rate function)

Table 22. CHARACTERISTICS OF COMPONENTS IN 50 COMPONENT SYSTEM-III (CM2)

MC	A(t)	MC	A(t)	MC	A(t)	MC	A(t)
700	.005t	800	.005t	700	.010t	800	.010t
700	.015t	800	.015t	900	.015t	800	.020t
900	.020t	1200	.025t	900	.025t	500	.025t
600	.025t	600	.030t	1000	.030t	1100	.030t
600	.035t	700	.035t	800	.035t	800	.040t
900	.040t	700	.045t	800	.045t	900	.045t
1000	.045t	100	.050t	200	.055t	300	.055t
400	.055t	200	.060t	300	.060t	400	.060t
200	.065t	300	.065t	300	.070t	400	.070t
200	.075t	300	.080t	300	.085t	300	.090t
400	.090t	400	.095t	300	.100t	300	.105t
400	.110t	200	.115t	100	.120t	200	.120t
300	.120t	400	.125t				

(MC - maintenance cost of component : A(t) - failure rate function)

Table 23. CHARACTERISTICS OF COMPONENTS IN 100 COMPONENT SYSTEM (CM2)

MC	A(t)	MC	A(t)	MC	A(t)	MC	A(t)
700	.005t	800	.005t	700	.010t	800	.010t
700	.015t	800	.015t	900	.015t	800	.020t
900	.020t	1200	.025t	900	.025t	500	.025t
600	.025t	600	.030t	1000	.030t	1100	.030t
600	.035t	700	.035t	800	.035t	800	.040t
900	.040t	700	.045t	800	.045t	900	.045t
1000	.045t	100	.050t	200	.055t	300	.055t
400	.055t	200	.060t	300	.060t	400	.060t
200	.065t	300	.065t	300	.070t	400	.070t
200	.075t	300	.080t	300	.085t	300	.090t
400	.090t	400	.095t	300	.100t	300	.105t
400	.110t	200	.115t	100	.120t	200	.120t
300	.120t	400	.125t	10	.250t	20	.250t
30	.250t	40	.250t	10	.300t	20	.300t
20	.350t	30	.350t	50	.350t	20	.400t
30	.400t	40	.400t	50	.400t	10	.450t
20	.450t	30	.450t	40	.450t	50	.450t
10	.500t	20	.500t	30	.500t	40	.500t
50	.500t	10	.550t	20	.550t	10	.005t
20	.010t	30	.015t	40	.015t	200	.015t
20	.020t	20	.025t	30	.025t	50	.065t
20	.070t	30	.070t	40	.075t	50	.080t
10	.085t	20	.090t	30	.090t	400	.450t
1000	.450t	300	.500t	400	.500t	700	.500t
900	.500t	1000	.500t	1100	.500t	1200	.500t

(MC - maintenance cost of component : A(t) - failure rate function)

Table 24. CHARACTERISTICS OF COMPONENTS IN 25 COMPONENT SYSTEM-I (CM4)

MC	A(t)	MC	A(t)	MC	A(t)	MC	A(t)
10	.250t	20	.250t	30	.250t	40	.250t
10	.300t	20	.300t	20	.350t	30	.350t
50	.350t	20	.400t	30	.400t	40	.400t
50	.400t	300	.065t	300	.070t	400	.070t
200	.075t	300	.080t	300	.085t	300	.090t
400	.090t	400	.095t	300	.100t	300	.105t
400	.110t						

(MC - maintenance cost of component : A(t) - failure rate function)

Table 25. CHARACTERISTICS OF COMPONENTS IN 25 COMPONENT SYSTEM-II (CM4)

MC	A(t)	MC	A(T)	MC	A(T)	MC	A(T)
200	.115t	100	.120t	200	.120t	300	.120t
400	.125t	100	.050t	200	.055t	300	.055t
400	.055t	200	.060t	300	.060t	400	.060t
200	.065t	300	.065t	300	.070t	400	.070t
200	.075t	300	.080t	300	.085t	300	.090t
400	.090t	400	.095t	300	.100t	300	.105t
400	.110t						

(MC - maintenance cost of component ; A(t) - failure rate function)

Table 26. CHARACTERISTICS OF COMPONENTS IN 25 COMPONENT SYSTEM-III (CM4)

MC	A(t)	MC	A(T)	MC	A(T)	MC	A(T)
700	.005t	800	.005t	700	.010t	800	.010
700	.015t	800	.015t	900	.015t	800	.020
900	.020t	1200	.025t	900	.025t	500	.025
600	.025t	600	.030t	1000	.030t	1100	.030
600	.035t	700	.035t	800	.035t	800	.040
900	.040t	700	.045t	800	.045t	900	.045
1000	.045t						

(MC - maintenance cost of component ; A(t) - failure rate function)

Vita

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