DYNAMIC STABILITY OF SHEAR DEFORMABLE VISCOELASTIC COMPOSITE PLATES

by

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Thesis submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

ENGINEERING SCIENCE AND MECHANICS

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7 DECEMBER 1987

Blacksburg, Virginia

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(ABSTRACT)

Linear viscoelasticity theory is used to analyze the dynamic stability of composite, viscoelastic flat plates subjected to in-plane, biaxial edge loads. In deriving the associated governing equations, a hereditary constitutive law is assumed. In addition, having in view that composite-type structures exhibit weak rigidity in transverse shear, the associated governing equations account for the transverse shear deformations, as well as the transverse normal stress effect. The integro-differential equations governing the stability are solved for simply-supported boundary conditions by using the Laplace transform technique, thus yielding the characteristic equation of the system.

In order to predict the effective time-dependent properties of the orthotropic plate, an elastic behavior is assumed for the fiber, whereas the matrix is considered as linearly viscoelastic.

In order to evaluate the nine independent properties of the orthotropic viscoelastic material in terms of its isotropic constituents, the micromechanical relations developed by Aboudi [24] are considered in conjunction with the correspondence principle for linear viscoelasticity. The stability behavior analyzed here concerns the determination of the critical in-plane normal edge loads yielding asymptotic stability of the plate. The problem is studied as an eigenvalue problem.

The general dynamic stability solutions are compared with their quasi-static counterparts. Comparisons of the various solutions obtained in the framework of the Third Order Transverse Shear Deformation Theory (TTSD) are made with its first order counterpart. Se-

veral special cases are considered and pertinent numerical results are compared with the very few ones available in the field literature.

Acknowledgements

I owe many thanks to my principal advisor, Dr. L. Librescu, without whose guidance and encouragement this work would have never been possible. The training he has provided will stand me in good stead in my future pursuits.

I would also like to express my sincerest gratitude to Dr. R. H. Plaut and Dr. E. R. Johnson for their useful suggestions throughout the period of my research work. Thanks also to Dr. H. F. Brinson for reviewing the manuscript and providing useful suggestions.

My special thanks to Dr. J. Aboudi without whom this work would never have gotten started.

My thanks are also due to all my friends at Virginia Tech who have helped me throughout my stay here.

Thanks to for her patience and excellent typing of the manuscript.

Last but not least, I would like to thank my parents whose moral support and encouragement shall remain unparalleled.

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CHAPTER 1. INTRODUCTION

Advanced fiber-reinforced composites have gained increasing attention due to their widespread use in the design of primary and secondary load-bearing members where the strength/stiffness to weight ratio is of paramount importance. Such applications include, e.g., the aircraft and aerospace structures in which elevated temperature gradients arise due to high-speed flight; rocket engines; nuclear reactors where thermal insulation is the governing criterion during design, etc. Under the influence of such high temperatures, these composite materials exhibit time-dependent properties which could be modelled by linear (or nonlinear) viscoelastic constitutive laws.

For such advanced composite structures exhibiting viscoelastic properties, it is essential to determine their stability behavior under compressive load systems.

1.1 Background

The analysis of viscoelastic composite plates requires a complete knowledge of their time-dependent material behavior. The relaxation moduli (and creep compliances) for a transversely isotropic laminate are given by McQuillen [1]. However, in this analysis which uses micromechanical relations developed by Hashin [2], a quasi-elastic (also referred to as quasi-static) approximation is implied. In this manner, the micromechanical relations for the viscoelastic body in a 2-D state of stress were derived from their elastic counterparts by merely replacing elastic constants with time-dependent material properties. Similar quasi-elastic analyses were also performed by Wilson [3] in which the Halpin-Tsai relations were employed, and by Sims [4] who used the method developed by McQuillan [1]. Both Sims [4] and Wilson [3] assume an elastic behavior in dilatation for the viscoelastic matrix.

Stability of viscoelastic composite panels undergoing cylindrical bending was studied by Malmeister et al. [5] for the case when the composite exhibits viscoelastic properties in transverse shear only. However, it is evident that this is a restriction being imposed on the material behavior since the extensional moduli in the direction perpendicular to the fibers are also time-dependent due to the viscoelastic behavior of the matrix.

Wilson [3] analyzes the stability of rectangular, viscoelastic, orthotropic plates subjected to biaxial compression. In deriving the equations governing the stability, he uses the principle of stationarity of the potential energy functional. The foregoing variational principle may be valid for a quasi-elastic approximation of the problem. Strictly speaking, however, the use of the aforementioned variational principle is questionable due to the presence of internal damping in viscoelastic systems which results in a non-conservative problem. Both Sims [4] and Wilson [3], by using the quasi-elastic approximation, analyze the system as an instantaneous time-dependent elastic system, thereby not implying the hereditary material behavior.

In all these methods, linearized sytems were considered and stability was analyzed over an infinitely long period of time (i.e., the asymptotic behavior of motion was studied). However, another interesting problem concerning viscoelastic systems is the determination of a critical time defined by that instant when the deflection or its rate of change become infinite. Grigoliuk and Lipovtsev [6] have studied the snap-through buckling of a viscoelastic shell and have pointed out that introduction of either geometric or physical non-linearities may result in the existence of such a critical time, $t = t_{cr}$ (the time at which snap-through buckling occurs).

Szyszkowski and Glockner [7] analyze the stability of a viscoelastic column using a geometrically non-linear formulation of the problem, thus allowing them to determine a critical time. They point out that the problem of critical time evaluation becomes important for loads lying between the safe-load-limit, P_v, (defined by asymptotic instability of the viscoelastic structure) and the instantaneous Euler buckling load, P_E. Thus, when $P_V < P < P_E$, the system "may" become unstable at a finite time defined as the "critical time". However, such a study goes beyond the scope of the present work.

1.2 Scope

In this study, a method of analyzing the linearized stability of viscoelastic composite plates has been developed. To this end, an "exact" dynamic approach has been used in the formulation and solution of the problem. The composite material was modelled through a 3-D linearly viscoelastic, hereditary, constitutive law. Effects of transverse shear deformations (which are highly pronounced for materials exhibiting high degrees of anisotropy, and/or the plate thickness-to-length aspect ratio is high) have also been incorporated in the analysis. Emphasis has also been given to the effect of transverse normal stress, σ_{33} , which was neglected by Wilson [3].

The analysis is done for an orthotropic, viscoelastic plate in the framework of a third-order transverse shear deformation theory (TTSD) and its first-order counterpart (FSDT). Then, making use of the single equation representing the interior solution (discussed in Sec. 2.5), it has been shown that for an isotropic plate, the results obtained by solving the exact system of three coupled equations agree very well with the approximate solutions obtained via the "single equation'. This result constitutes an extension for the viscoelastic case of the elastic counterpart obtained by Librescu [8, 9].

Comparison studies between TTSD, FSDT and the classical Kirchhoff theory of plates are also presented.

CHAPTER 2. PROBLEM FORMULATION

2.1 Introduction and Problem Statement

The increased interest of the utilization in aeronautical and aerospace structures of composite material systems requires the development of adequate analytical methods for their rational design. This demand becomes more stringent in the case of materials exhibiting time-dependent properties as is the case for high-speed aircraft structures operating at elevated temperatures. The behavior of such time-dependent materials may be described by a linear viscoelastic model.

Another aspect of equal importance is the effect of transverse shear deformations which is highly prevalent in composite structures. The importance of this becomes more prominent when the material exhibits high degrees of anisotropy or when the plate (or shell) is thick enough. The classical theory does not account for this effect, thereby implying infinite rigidity in transverse shear. Moreover, the assumption involved in the classical theory of plates does not allow the fulfillment of the boundary conditions on the external bounding planes of the panel. In addition, the simultaneous consideration of zero transverse normal stress and zero transverse normal strain implies a contradiction which is to be eliminated in the framework of the refined theories of plates. This calls for the use of refined theories when analyzing beams, plates, etc., in which the contradictory assumptions as well as the shortcomings of the classical theory, stated earlier, are removed.

In this study an anisotropic rectangular plate is analyzed for dynamic stability. The composite material considered herein consists of a linearly viscoelastic matrix reinforced by elastic fibers. This, however, is not a restriction on the behaviour of the fibers which may also be treated as linearly viscoelastic if desired, as will be seen in Chapter 3. The plate is subject to constant inplane edge loads and assumed to be simply supported along its edges. The plate is considered as a moderately thick one with h/L ratio of about 5.

2.2 *Viscoelasticity Preliminaries*

The time-dependent behavior of a typical epoxy resin material is shown in Figs. 1 and 4 and was used by Schapery [10] in his analysis. This requires the use of a constitutive law different from the elastic Hooke's law. In the following there are two methods of describing the constitutive law, which are entirely equivalent.

2.2.1 Differential Form of Stress-Strain Relations

The differential equation relating stresses to strains for a linearly viscoelastic material (ref. Christensen [11]) may be expressed as:

$$p_o\, \sigma_{ij}[t] + p_1\, \dot{\sigma}_{ij}[t] + p_2\, \ddot{\sigma}_{ij}[t] + ... = q_o\, e_{ij}[t] + q_1\, \dot{e}_{ij}[t] + q_2\, \ddot{e}_{ij}[t] + ...$$

or in a compact form as,

$$P[D]\,\sigma_{ij} = Q[D]\,e_{ij} \tag{2.1}$$

where,

$$P[D] = \sum_{k=0}^{M} p_k\, D^k$$

$$Q[D] = \sum_{k=0}^{M} q_k\, D^k$$

$$D^k = \frac{d^k}{dt^k},$$

p_k *and* q_k being the viscoelastic characteristics of the material. Relation (2.1) corresponds to a Voigt model represented as a combination of springs and dashpots (in series and parallel). The resulting coefficients p_k *and* q_k are algebraic functions of the spring and dashpot constants (see, e.g., Flugge [12]).

2.2.2 Creep Compliance and Relaxation Modulus

The Laplace transform of Eq. (2.1) yields

$$P[s]\,\bar{\sigma}_{ij}[s] - \sum_{k=1}^{N} p_k \sum_{r=1}^{k} s^{r-1}\,\sigma_{ij}^{(k-r)}[0] = Q[s]\,\bar{e}_{ij}[s] - \sum_{k=1}^{N} q_i \sum_{r=1}^{k} s^{r-1}\,e_{ij}^{(k-r)}[0] \tag{2.2}$$

where,

$$P[s] = \sum_{k=0}^{N} p_k [s]^k$$

$$Q[s] = \sum_{k=0}^{N} q_k [s]^k$$

and

$$\sigma_{ij}^{(k-r)}[0] = \frac{d^{(k-r)} \sigma_{ij}}{dt^{(k-r)}} \mid t = 0$$

with a similar definition for $e_{ij}^{(k-r)}[0]$. The overbars denote the Laplace transform (L.T.) with s as the L.T. variable. Now the coefficients of various powers of s must be equal on either side of (2.2). For arbitrary initial conditions, this requires that the coefficients of powers of s be equal in the terms containing these initial conditions on either side of (2.2). Equating the coefficients of like powers in the terms involving the initial conditions results in,

$$\sum_{r=k}^{N} p_r \, \sigma_{ij}^{(r-k)}[0] = \sum_{r=k}^{N} q_r \, e_{ij}^{(r-k)}[0] , \qquad k = 1,2,\ldots,N$$

Hence (2.2) reduces to

$$P[s] \, \bar{\sigma}_{ij}[s] = Q[s] \, \bar{e}_{ij}[s]$$

from which we obtain

$$\bar{\sigma}_{ij}[s] = \frac{Q[s]}{P[s]} \, \bar{e}_{ij}[s]$$

or

$$\bar{\sigma}_{ij}[s] = \mathfrak{E}[s]\, \bar{e}_{ij}[s] \tag{2.3}$$

where,

$$\mathfrak{E}[s] \triangleq \frac{Q[s]}{P[s]}$$

Let the input be a constant strain represented as,

$$e_{ij}[t] = H[t]\, e_{ij}$$

where H[t] is the Heaviside distribution. Thus after inverting into the time domain, (2.3) yields

$$\sigma_{ij}[t] = E[t]\, e_{ij} \tag{2.4}$$

where,

$$E[t] = \mathscr{L}^{-1}\left(\frac{\mathfrak{E}[s]}{s}\right) = \mathscr{L}^{-1}\left(\frac{1}{s}\frac{Q[s]}{P[s]}\right);$$

$\mathscr{L}^{-1}$ denoting the inverse Laplace transform.

Thus E[t], the relaxation modulus, is the resulting stress for a unit input strain. The inverse of (2.3) would yield a strain-stress law involving D[t], the creep compliance, which is the resulting strain for unit input stress.

2.2.3 Integral Form of Stress-Strain Constitutive Relations---Boltzmann Constitutive Law and Stieltjes Convolutions

Using the superposition principle for a linear viscoelastic material results in the constitutive law involving the Stieltjes convolution (ref. Pipkin [13]). This relates the time-dependent stresses to the time-varying input strains (or vice-versa). The derivation considers the input strains (or stresses) to be a superposition of Heaviside functions with various step sizes. The output stresses (or strains) are then related to the input quantity by using (2.4) (or its inverse) and summing over all the step intervals.

The resulting constitutive law is,

$$\sigma_{ij}[t] = \int_{0^-}^{t} E[t-\tau]\, \dot{e}_{ij}\, d\tau$$

and the inverse law is,

$$e_{ij}[t] = \int_{0^-}^{t} D[t-\tau]\, \dot{\sigma}_{ij}\, d\tau$$

Generalizing to an anisotropic material for a 3-dimensional state of stress we obtain, (ref. [12])

$$\sigma_{ij}[t] = \int_{0^-}^{t} E_{ijmn}[t-\tau]\, \dot{e}_{mn}[\tau]\, d\tau \tag{2.5a}$$

$$e_{ij}[t] = \int_{0^-}^{t} F_{ijmn}[t-\tau]\, \dot{\sigma}_{mn}[\tau]\, d\tau \tag{2.5b}$$

Here $E_{ijmn}[t]$ and $F_{ijmn}[t]$ are the tensors of relaxation moduli and creep compliances which exhibit the following symmetry properties:

$$E_{ijmn}[t] = E_{jimn}[t] = E_{ijnm}[t]$$

and

$$E_{ijmn}[t] = E_{mnij}[t]$$

the former arising due to the symmetry of the stress and strain tensors while the latter can be proved by using Onsager's principle in irreversible thermodynamcis (see Biot [14]). The same symmetry properties result for the creep compliance tensor (F_{ijmn}).

Now for the sake of generality let the initial condition for the input strains be represented by

$$e_{mn}[t] = e_{mn}[0]\, H[t]\,, \qquad\qquad 0^- \leq t \leq 0^+$$

Substituting this in (2.5a) yields,

$$\sigma_{ij}[t] = \int_{0^+}^{t} E_{ijmn}[t-\tau]\, \dot{e}_{mn}[\tau]\, d\tau + \int_{0^-}^{0^+} E_{ijmn}[t-\tau]\, e_{mn}[0]\, \delta[\tau]\, d\tau$$

where we have used the fact that the derivative of the Heaviside function is the Dirac delta function. The above equation thus becomes,

$$\sigma_{ij}[t] = \int_{0^+}^{t} E_{ijmn}[t-\tau]\, \dot{e}_{mn}[\tau]\, d\tau + E_{ijmn}[t]\, e_{mn}[0] \tag{2.6}$$

where the second term in (2.6) represents the response due to initial strain $e_{mn}[0]$. Integration of (2.6) by parts yields,

$$\sigma_{ij}[t] = E_{ijmn}[t]\, e_{mn}[0] + \left[e_{mn}[\tau]\, E_{ijmn}[t-\tau]\right]_{0^+}^{t}$$

$$- \int_{0^+}^{t} \frac{d\, E_{ijmn}[t-\tau]}{d[t-\tau]} \frac{d[t-\tau]}{d\tau}\, e_{mn}[\tau]\, d\tau$$

$$\sigma_{ij}[t] = e_{mn}[t]\, E_{ijmn}[0] + \int_{0^+}^{t} \frac{d\, E_{ijmn}[t-\tau]}{d[t-\tau]}\, e_{mn}[\tau]\, d\tau$$

where we have used the fact that

$$\lim_{t \to 0} \; e_{mn}[t] = e_{mn}[0^+] \tag{2.7}$$

On replacing [t - τ] by s we obtain, (see Malmeister et al., [5])

$$\sigma_{ij}[t] = e_{mn}[t]\, E_{ijmn}[0] + \int_0^t \dot{E}_{ijmn}[s]\, e_{mn}[t-s]\, ds \tag{2.8}$$

where the first term in (2.8) represents the elastic part of the response. Again making use of (2.7) we can change the lower limit 0^+ in (2.6) to 0. Equations (2.6), (2.8) are equivalent forms of (2.5) which is the integral constitutive law for an anisotropic viscoelastic body in a 3-dimensional state of stress, and is also referred to as the Boltzmann hereditary constitutive law.

Taking the Laplace transform (2.6) (which is the form of the constitutive law used henceforth) we obtain,

$$\bar{\sigma}_{ij} = \bar{E}_{ijmn}[s](s\, \bar{e}_{mn} - e_{mn}[0^+]) + \bar{E}_{ijmn}[s]\, e_{mn}[0]$$

Introducing (2.7) in the above yields

$$\bar{\sigma}_{ij} = s\,\bar{E}_{ijmn}[s]\,\bar{e}_{mn}[s] \qquad (2.9a)$$

and

$$\bar{e}_{ij} = s\,\bar{F}_{ijmn}[s]\,\bar{\sigma}_{mn}[s] \qquad (2.9b)$$

which is similar to (2.3), obtained by the differential form of the constitutive law.

As is readily seen, Eqs. (2.9) take into account the initial conditions of the input variable ($e_{mn}[t]$) or ($\sigma_{mn}[t]$).

2.3 Equations Governing the Stability of Viscoelastic Flat Plates Using a Third Order Refined Theory (TTSD)

As noted earlier for anisotropic composite plates, the assumptions involved in the classical theory disregarding the influence of transverse shear deformations and transverse normal stresses are to be eliminated by use of a refined theory of plates. In the following development we extend a method developed by Librescu [8] (see also Librescu and Reddy [15], and Librescu, Khdeir and Reddy [16]) to the viscoelastic case. A geometrically nonlinear theory for a viscoelastic flat plate is developed and the governing equations are derived in the Laplace transformed space. These are then inverted to the time domain.

We shall consider the case of a flat plate of uniform thickness h. By $S^{\pm}$ we denote the upper and lower bounding planes of the plate, symmetrically disposed with respect to its mid-plane σ.

We denote the edge boundary surface (see Fig. 2) by Ω. It is assumed that $S^{\pm}$ and σ (defined by $x_3 = \pm h/2$ and $x_3 = 0$, respectively) are sufficiently smooth without singularities. The points of the 3-D space of the plate will be referred to a rectangular Cartesian system of coordinates x_i, where $x_\alpha(\alpha = 1,2)$ denote the in-plane coordinates, x_3 being the coordinate normal to the plane $x_3 = 0$.

2.3.1 Strain-Displacement Equations

The higher-order theory of plates is developed by using the following representation of the displacement field across the thickness of the plate:

$$V_\alpha[x_\omega, x_3, t] = \sum_{n=0}^{N} (x_3)^n \overset{(n)}{V_\alpha} [x_\omega, t] \tag{2.10a}$$

$$V_3[x_\alpha, x_3, t] = \sum_{n=0}^{R} (x_3)^n \overset{(n)}{V_3}[x_\omega, t] \tag{2.10b}$$

where the Greek indices range from 1 to 2, while the Latin indices range from 1 to 3, with t being the time variable. The numbers N and R denote two natural numbers identifying the truncation levels in the displacement expansion. At this point it is worthwhile to note that the terms corresponding to the stretching state of stress are $\overset{(2r)}{V_\alpha}$, $\overset{(2r+1)}{V_3}$ and those corresponding to the bending state are $\overset{(2r+1)}{V_\alpha}$ *and* $\overset{(2r)}{V_3}$.

For a third-order bending theory which retains the assumption of the inextensiblity of the transverse normal elements, we may write the following expansions for the displacement components:

$$V_\alpha = x_3 \overset{(1)}{V_\alpha} + (x_3)^3 \overset{(3)}{V_\alpha} \tag{2.11a}$$

$$V_3 = \overset{(0)}{V_3} \tag{2.11b}$$

where

$$\overset{(n)}{V}_i = \overset{(n)}{V}_i [x_\alpha, t] \tag{2.11c}$$

We now introduce the nonlinear Lagrangian strain tensor as,

$$e_{ij}[x_\alpha, x_3, t] = \frac{1}{2}(V_{i,j} + V_{j,i} + V_{r,i} V_{r,j})$$

where a comma denotes differentiation with respect to the index following it. Using the Von-Karman approximation which neglects the nonlinearities involving the inplane displacement components or their gradients (i.e., $V_{\alpha,\lambda} V_{\beta,\pi}$, etc.), we obtain,

$$e_{11} = V_{1,1} + \frac{1}{2}(V_{3,1})^2 \tag{2.12a}$$

$$e_{22} = V_{2,2} + \frac{1}{2}(V_{3,2})^2 \tag{2.12b}$$

$$e_{33} = \frac{1}{2}\left[(V_{1,3})^2 + (V_{2,3})^2 + (V_{3,3})^2\right] + V_{3,3} \tag{2.12c}$$

$$e_{12} = \frac{1}{2}[V_{1,2} + V_{2,1} + V_{3,1} V_{3,2}] \tag{2.12d}$$

$$e_{13} = \frac{1}{2}[V_{1,3} + V_{3,1} + V_{1,1} V_{1,3} + V_{2,1} V_{2,3} + V_{3,1} V_{3,3}] \tag{2.12e}$$

$$e_{23} = \frac{1}{2}[V_{2,3} + V_{3,2} + V_{2,2} V_{2,3} + V_{1,2} V_{1,3} + V_{3,2} V_{3,3}] \tag{2.12f}$$

Introducing the displacement expansions (2.11) into (2.12) and neglecting the resulting non-linearities associated with $\overset{(n)}{V}_\alpha$ and their derivatives, we obtain

$$e_{11} = x_3 \overset{(1)}{V}_{1,1} + (x_3)^3 \overset{(3)}{V}_{1,1} + \frac{1}{2}\left(\overset{(0)}{V}_{3,1}\right)^2 \tag{2.13a}$$

$$e_{22} = x_3 \overset{(1)}{V}_{2,2} + (x_3)^3 \overset{(3)}{V}_{2,2} + \frac{1}{2}\left(\overset{(0)}{V}_{3,2}\right)^2 \tag{2.13b}$$

$$e_{12} = \frac{1}{2}\left(x_3 \overset{(1)}{V}_{1,2} + (x_3)^3 \overset{(3)}{V}_{1,2} + x_3 \overset{(1)}{V}_{2,1} + (x_3)^3 \overset{(3)}{V}_{2,1} + \overset{(0)}{V}_{3,1} \overset{(0)}{V}_{3,2} \right) \tag{2.13c}$$

$$e_{13} = \frac{1}{2}\left(\overset{(1)}{V}_1 + 3(x_3)^2 \overset{(3)}{V}_1 + \overset{(0)}{V}_{3,1} \right) \tag{2.13d}$$

$$e_{23} = \frac{1}{2}\left(\overset{(1)}{V}_2 + 3(x_3)^2 \overset{(3)}{V}_2 + \overset{(0)}{V}_{3,2} \right) \tag{2.13e}$$

$$e_{33} = 0 \tag{2.13f}$$

Examining (2.13) we see that only e_{11}, e_{22} and e_{12} are affected by the Von Karman type nonlinearities. Taking the L.T. of equation (2.13) yields

$$\bar{e}_{11} = x_3 \overset{\overline{(1)}}{V}_{1,1} + (x_3)^3 \overset{\overline{(3)}}{V}_{1,1} + \frac{1}{2}\mathscr{L}\left\{ \left(\overset{(0)}{V}_{3,1} \right)^2 \right\} \tag{2.14a}$$

$$\bar{e}_{22} = x_3 \overset{\overline{(1)}}{V}_{2,2} + (x_3)^3 \overset{\overline{(3)}}{V}_{2,2} + \frac{1}{2}\mathscr{L}\left\{ \left(\overset{(0)}{V}_{3,2} \right)^2 \right\} \tag{2.14b}$$

$$\bar{e}_{12} = \frac{1}{2}\left(x_3 \overset{\overline{(1)}}{V}_{1,2} + (x_3)^3 \overset{\overline{(3)}}{V}_{1,2} + x_3 \overset{\overline{(1)}}{V}_{2,1} + (x_3)^3 \overset{\overline{(3)}}{V}_{2,1} + \mathscr{L}\left\{ \left(\overset{(0)}{V}_{3,1} \overset{(0)}{V}_{3,2} \right) \right\} \right) \tag{2.14c}$$

$$\bar{e}_{13} = \frac{1}{2}\left(\overset{\overline{(1)}}{V}_1 + 3(x_3)^2 \overset{\overline{(3)}}{V}_1 + \overset{\overline{(0)}}{V}_{3,1} \right) \tag{2.14d}$$

$$\bar{e}_{23} = \frac{1}{2}\left(\overset{\overline{(1)}}{V}_2 + 3(x_3)^2 \overset{\overline{(3)}}{V}_2 + \overset{\overline{(0)}}{V}_{3,2} \right) \tag{2.14e}$$

where the overbars denote the Laplace transform (L.T.) of the quantity free of overbars,

$$\bar{e}_{ij} = \bar{e}_{ij}[x_\alpha, x_3, s]$$

$$\overset{\overline{(n)}}{V}_i = \overset{\overline{(n)}}{V}_i[x_\alpha, s]$$

and $\mathscr{L}$ is the L.T. operator defined as,

$$\mathscr{L}\{f[x_\alpha,\ x_3,\ t]\} = \int_0^\infty e^{-st} f[x_\alpha,\ x_3,\ t]dt \equiv \bar{f}[x_\alpha, x_3, s]$$

with s being the L.T. variable.

2.3.2 Constitutive Equations

Reproducing the constitutive law in the form as expressed in (2.6), we have,

$$\sigma_{ij}[x_\alpha, x_3, t] = \int_{0^+}^{t} E_{ijmn}[t-\tau]\, \dot{e}_{mn}[x_\alpha, x_3, \tau]\, d\tau + E_{ijmn}[t]\, e_{mn}[x_\alpha, x_3, 0]$$

Taking the L.T. of the above in conjunction with (2.8) yields (2.9), which is reproduced as follows:

$$\bar{\sigma}_{ij} = s\, \bar{E}_{ijmn}\, \bar{e}_{mn} \tag{2.15}$$

Writing (2.15) in its full form for an orthotropic body (see, e.g., Librescu [8]), for which the components of $\bar{E}_{ijmn}[s]$ involving indices repeated once or thrice are to be considered zero, we obtain

$$\bar{\sigma}_{11} = s\, \bar{E}_{1111}\, \bar{e}_{11} + s\, \bar{E}_{1122}\, \bar{e}_{22} + s\, \bar{E}_{1133}\, \bar{e}_{33} \tag{2.16a}$$

$$\bar{\sigma}_{22} = s\, \bar{E}_{2211}\, \bar{e}_{11} + s\, \bar{E}_{2222}\, \bar{e}_{22} + s\, \bar{E}_{2233}\, \bar{e}_{33} \tag{2.16b}$$

$$\bar{\sigma}_{33} = s\, \bar{E}_{3311}\, \bar{e}_{11} + s\, \bar{E}_{3322}\, \bar{e}_{22} + s\, \bar{E}_{3333}\, \bar{e}_{33} \tag{2.16c}$$

$$\bar{\sigma}_{12} = 2s\, \bar{E}_{1212}\, \bar{e}_{12} \tag{2.16d}$$

$$\bar{\sigma}_{13} = 2s\, \bar{E}_{1313}\, \bar{e}_{13} \tag{2.16e}$$

$$\bar{\sigma}_{23} = 2s\,\bar{E}_{2323}\,\bar{e}_{23} \tag{2.16f}$$

Now we introduce the following notation which is helpful in further developments:

$$s\,\bar{E}_{ijmn} = \bar{E}^{*}_{ijmn}$$

where the star (*) along with the overbar (-) identifies the Carson transform which is the L.T. multiplied by s. Thus $\bar{E}^{*}_{ijmn}$ is the Carson transform of the tensor of relaxation moduli. Thus, introducing $\bar{e}_{33}$ from (2.16c) into (2.16a) and (2.16b), we obtain,

$$\bar{\sigma}_{11} = \tilde{\bar{E}}^{*}_{1111}\,\bar{e}_{11} + \tilde{\bar{E}}^{*}_{1122}\,\bar{e}_{22} + \delta_A\,\hat{\bar{E}}^{*}_{1133}\,\bar{\sigma}_{33} \tag{2.17a}$$

$$\bar{\sigma}_{22} = \tilde{\bar{E}}^{*}_{2211}\,\bar{e}_{11} + \tilde{\bar{E}}^{*}_{2222}\,\bar{e}_{22} + \delta_A\,\hat{\bar{E}}^{*}_{2233}\,\bar{\sigma}_{33} \tag{2.17b}$$

$$\bar{\sigma}_{12} = 2s\,\bar{E}_{1212}\,\bar{e}_{12} = 2\,\bar{E}^{*}_{1212}\,\bar{e}_{12} \tag{2.17c}$$

$$\bar{\sigma}_{13} = 2s\,\bar{E}_{1313}\,\bar{e}_{13} = 2\,\bar{E}^{*}_{1313}\,\bar{e}_{13} \tag{2.17d}$$

$$\bar{\sigma}_{23} = 2s\,\bar{E}_{2323}\,\bar{e}_{23} = 2\,\bar{E}^{*}_{2323}\,\bar{e}_{23} \tag{2.17e}$$

where,

$$\tilde{\bar{E}}^{*}_{\alpha\beta\omega\pi} \triangleq s\,\tilde{\bar{E}}_{\alpha\beta\omega\pi} = s\,\bar{E}_{\alpha\beta\omega\pi} - \frac{s\,\bar{E}_{\alpha\beta33}\,s\,\bar{E}_{33\omega\pi}}{s\,\bar{E}_{3333}} \triangleq \bar{E}^{*}_{\alpha\beta\omega\pi} - \frac{\bar{E}^{*}_{\alpha\beta33}\,\bar{E}^{*}_{33\omega\pi}}{\bar{E}^{*}_{3333}} \tag{2.18a}$$

and

$$\hat{\bar{E}}^{*}_{\alpha\beta33} \triangleq s\,\frac{\bar{E}_{\alpha\beta33}}{s\,\bar{E}_{3333}} = s\,\frac{\bar{E}_{\alpha\beta33}}{\bar{E}^{*}_{3333}} = \frac{\bar{E}^{*}_{\alpha\beta33}}{\bar{E}^{*}_{3333}} \triangleq s\,\hat{\bar{E}}_{\alpha\beta33} \tag{2.18b}$$

In (2.18a) $\bar{\tilde{E}}^{*}_{\alpha\beta\omega\pi}$ represents the Carson transform of the reduced stiffness $\tilde{E}_{\alpha\beta\omega\pi}$ (analogous to its elastic counterpart) (see Librescu [8]). Also the tracer δ_A identifying the presence of σ_{33} will take the value 0 or 1, according to whether this influence is ignored or included. Thus we can write (2.17) in compact form as,

$$\bar{\sigma}_{\alpha\beta} = \bar{\tilde{E}}^{*}_{\alpha\beta\omega\pi}\, \bar{e}_{\omega\pi} + \delta_A\, \bar{\hat{E}}^{*}_{\alpha\beta33}\, \bar{\sigma}_{33} \tag{2.19a}$$

$$\bar{\sigma}_{\omega3} = 2s\, \bar{E}_{\omega3\lambda3}\, \bar{e}_{\lambda3} = 2\, \bar{E}^{*}_{\omega3\lambda3}\, \bar{e}_{\lambda3} \tag{2.19b}$$

Inverting (2.19) back to the time domain, we get by using Borel's Theorem (see Appendix [A]) and Eqns. (2.18)

$$\sigma_{\alpha\beta}[t] = \int_0^t (\dot{\tilde{E}}_{\alpha\beta\omega\pi}[t-\tau] + \tilde{E}_{\alpha\beta\omega\pi}[0]\, \delta[t-\tau])\, e_{\omega\pi}[\tau]\, d\tau$$

$$+ \delta_A \int_0^t (\dot{\hat{E}}_{\alpha\beta33}[t-\tau] + \hat{E}_{\alpha\beta33}[0]\, \delta[t-\tau])\, \sigma_{33}[\tau] d\tau$$

$$\therefore\ \sigma_{\alpha\beta}[t] = \int_0^t \dot{\tilde{E}}_{\alpha\beta\omega\pi}[t-\tau]\, e_{\omega\pi}[\tau]\, d\tau + \tilde{E}_{\alpha\beta\omega\pi}[0]\, e_{\omega\pi}[t] + \int_0^t \dot{\hat{E}}_{\alpha\beta33}[t-\tau]\, \sigma_{33}[\tau]\, d\tau$$

$$+ \hat{E}_{\alpha\beta33}[0]\, \sigma_{33}[t] \tag{2.20a}$$

and

$$\sigma_{\omega 3}[t] = 2\int_0^t \dot{E}_{\omega 3\lambda 3}[t-\tau]\, e_{\lambda 3}[\tau]\, d\tau + 2\, E_{\omega 3\lambda 3}[0]\, e_{\lambda 3}[t] \tag{2.20b}$$

where: (as shown in Appendix [A])

$$\tilde{E}_{\alpha\beta\omega\pi}[0] = E_{\alpha\beta\omega\pi}[0] - \frac{E_{\alpha\beta 33}[0]\, E_{33\omega\pi}[0]}{E_{3333}[0]} \tag{2.20c}$$

$$\hat{E}_{\alpha\beta 33}[0] = \frac{E_{\alpha\beta 33}[0]}{E_{3333}[0]} \tag{2.20d}$$

Eqs. (2.20c,d) allow one to infer that all these time-dependent moduli attain their corresponding elastic values at t = 0. We also note that in (2.20a) and (2.20b)

$$\tilde{E}_{\alpha\beta\omega\pi}[t] \triangleq \mathscr{L}^{-1}\left\{\frac{1}{s}\left(s\,\bar{E}_{\alpha\beta\omega\pi} - \frac{s\,\bar{E}_{\alpha\beta 33}\, s\,\bar{E}_{33\omega\pi}}{s\,\bar{E}_{3333}}\right)\right\} = \mathscr{L}^{-1}\left\{\bar{E}_{\alpha\beta\omega\pi} - \frac{\bar{E}_{\alpha\beta 33}\,\bar{E}_{33\omega\pi}}{\bar{E}_{3333}}\right\} \tag{2.20e}$$

and

$$\hat{E}_{\alpha\beta 33}[t] = \mathscr{L}^{-1}\left\{\frac{1}{s}\left(\frac{s\,\bar{E}_{\alpha\beta 33}}{s\,\bar{E}_{3333}}\right)\right\} = \hat{E}_{\alpha\beta 33}[t] = \mathscr{L}^{-1}\left\{\frac{\bar{E}_{\alpha\beta 33}}{s\,\bar{E}_{3333}}\right\} \tag{2.20f}$$

and the overdots (.) denote time derivatives.

2.3.3 Equations of Motion

The equations of motion in Lagrangian description for a 3-D continuum undergoing finite deformations have been derived by Green and Adkins [17] (see also Amenzade [18] and Fung [19]). They are:

$$[\sigma_{jk}(\delta_{ik} + V_{i,k})]_{,j} + \rho H_i = \rho \ddot{V}_i \tag{2.21}$$

where σ_{jk} is the second Piola-Kirchhoff stress tensor (symmetric), δ_{ik} is the Kronecker delta, ρ is the mass density of the medium and H_i are the body forces per unit mass. We now define the unsymmetric tensor S_{ij} (referred to as the first Piola-Kirchhoff stress tensor) as follows:

$$S_{ij} = \sigma_{jk}(\delta_{ik} + V_{i,k}).$$

Introduction of the above definition into (2.21) yields,

$$S_{ij,j} + \rho H_i = \rho \ddot{V}_i$$

The above equation is another form of the equation of motion (in Lagrangian description) of a continuum undergoing finite deformations.

Writing (2.21) in full form, we obtain (see Librescu [20] for the FSDT counterpart),

$$\sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} + (\sigma_{1k} V_{1,k})_{,1} + (\sigma_{2k} V_{1,k})_{,2} + (\sigma_{3k} V_{1,k})_{,3} + \rho H_1 = \rho \ddot{V}_1 \tag{2.22a}$$

$$\sigma_{12,1} + \sigma_{22,2} + \sigma_{32,3} + (\sigma_{1k} V_{2,k})_{,1} + (\sigma_{2k} V_{2,k})_{,2} + (\sigma_{3k} V_{2,k})_{,3} + \rho H_2 = \rho \ddot{V}_2 \tag{2.22b}$$

$$\sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} + (\sigma_{1k} V_{3,k})_{,1} + (\sigma_{2k} V_{3,k})_{,2} + (\sigma_{3k} V_{3,k})_{,3} + \rho H_3 = \rho \ddot{V}_3 \tag{2.22c}$$

Neglecting the nonlinearities associated with gradients of inplane displacements and stresses, we may write (2.22) as follows:

$$\sigma_{11,1} + \sigma_{21,2} + \sigma_{31,3} + \rho H_1 = \rho \ddot{V}_1 \tag{2.23a}$$

$$\sigma_{12,1} + \sigma_{22,2} + \sigma_{32,3} + \rho H_2 = \rho \ddot{V}_2 \tag{2.23b}$$

$$\begin{aligned}\sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} + (\sigma_{11} V_{3,1})_{,1} + (\sigma_{12} V_{3,2})_{,1} + (\sigma_{21} V_{3,1})_{,2} + (\sigma_{22} V_{3,2})_{,2} \\ + (\sigma_{31} V_{3,1})_{,3} + (\sigma_{32} V_{3,2})_{,3} + \rho H_3 = \rho \ddot{V}_3\end{aligned} \tag{2.23c}$$

where in (2.23) the fact that $V_{3,3}=0$ was also used (see equation (2.11) concerning displacement expansions).

The stability problem when formulated in terms of the displacement field may be reduced to a system of equations in 5 unknown quantities, $\overset{(1)}{V}_{\alpha}$, $\overset{(3)}{V}_{\alpha}$ and $\overset{(0)}{V}_{3}$. In order to obtain the governing equations in terms of these displacement quantities we need five equations of equilibrium. To this end we consider the moment of order zero and one of the first two equations of motion (2.23(a) and (b)) and the moment of order zero of the third equation (2.23c). Also we note that henceforth we neglect all body forces.

Taking the moment of order one of the first two equations of motion (2.23(a) and (b)), we obtain:

$$\overset{(1)}{L}_{11,1}+\overset{(1)}{L}_{21,2}+\int_{-h/2}^{+h/2}\sigma_{31,3}\,x_3\,dx_3=\delta_c\,\overset{(1)}{f_1} \tag{2.24a}$$

$$\overset{(1)}{L}_{12,1}+\overset{(1)}{L}_{22,2}+\int_{-h/2}^{+h/2}\sigma_{32,3}\,x_3\,dx_3=\delta_c\,\overset{(1)}{f_2} \tag{2.24b}$$

where, (in a general form)

$$\overset{(n)}{L}_{\alpha\beta}[x_1,x_2;t]=\int_{-h/2}^{+h/2}\sigma_{\alpha\beta}(x_3)^n\,dx_3 \tag{2.25a}$$

and

$$\overset{(n)}{f}_{\alpha}[x_1,x_2;t]=\int_{-h/2}^{+h/2}\rho\,\ddot{V}_{\alpha}(x_3)^n\,dx_3 \tag{2.25b}$$

and δ_c is a tracer which identifies the presence or absence of rotary inertia terms (i.e., $\overset{(n)}{f_\alpha}$) by taking values 1 and 0, respectively. Now consider the term $\int_{-h/2}^{+h/2}\sigma_{3\alpha,3}\, x_3\, dx_3$ appearing in (2.24). An integration by parts yields,

$$\int_{-h/2}^{h/2} \sigma_{3\alpha,3}\, x_3\, dx_3 = (\sigma_{3\alpha}\, x_3)\Big|_{-h/2}^{+h/2} - \int_{-h/2}^{+h/2} \sigma_{3\alpha}\, dx_3$$

$$\therefore \int_{-h/2}^{+h/2} \sigma_{3\alpha,3}\, x_3\, dx_3 = \overset{(1)}{p_\alpha} - \overset{(0)}{L_{\alpha 3}} \tag{2.26}$$

where (in a general form),

$$\overset{(n)}{p_\alpha}[x_1, x_2;t] = (\sigma_{\alpha 3}(x_3)^n)\Big|_{-h/2}^{+h/2} \tag{2.27a}$$

and

$$\overset{(n)}{L_{\alpha 3}}[x_1, x_2;t] = \int_{-h/2}^{+h/2} \sigma_{\alpha 3}(x_3)^n\, dx_3 \tag{2.27b}$$

Introducing (2.26) into (2.24), we obtain,

$$\overset{(1)}{L_{11,1}} + \overset{(1)}{L_{21,2}} + \overset{(1)}{p_1} - \overset{(0)}{L_{13}} = \delta_c \overset{(1)}{f_1}$$

$$\overset{(1)}{L_{12,2}} + \overset{(1)}{L_{22,2}} + \overset{(1)}{p_2} - \overset{(0)}{L_{23}} = \delta_c \overset{(1)}{f_2}$$

The above when written in compact form yield,

$$\overset{(1)}{L_{\alpha\beta,\,\beta}} + \overset{(1)}{p_\alpha} - \overset{(0)}{L_{\alpha 3}} = \delta_c \overset{(1)}{f_\alpha} \tag{2.28}$$

Equation (2.28) represents the first two equations governing the motion of a flat plate. The Laplace transforms of (2.28), (2.25), (2.27) yield

$$\overset{\overline{(1)}}{L}_{\alpha\beta,\,\beta} + \overset{\overline{(1)}}{p}_{\alpha} - \overset{\overline{(0)}}{L}_{\alpha 3} = \delta_c \overset{\overline{(1)}}{f}_{\alpha} \tag{2.29}$$

where,

$$\overset{\overline{(n)}}{L}_{\alpha\beta}[x_\omega,\, s] = \int_{-h/2}^{+h/2} \overline{\sigma}_{\alpha\beta}[x_\omega,\; x_3,\; s](x_3)^n\, dx_3 \tag{2.30a}$$

$$\overset{\overline{(n)}}{f}_{\alpha}[x_\omega,\, s] = \int_{-h/2}^{+h/2} \rho\, \overline{\ddot{V}}_{\alpha}[x_\omega,\, x_3,\, s]\, (x_3)^n\, dx_3 \tag{2.30b}$$

$$\overset{\overline{(n)}}{p}_{\alpha}[x_\omega,\, s] = \sigma_{\alpha 3}[x_\omega,\, x_3, s](x_3)^n \Big|_{-h/2}^{+h/2} \tag{2.30c}$$

$$\overset{\overline{(n)}}{L}_{\alpha 3}[x_\omega, s] = \int_{-h/2}^{+h/2} \overline{\sigma}_{\alpha 3}[x_\omega,\, x_3, s](x_3)^n\, dx_3 \tag{2.30d}$$

In deriving (2.29) and (2.30) we have used the fact that the order of integration performed over x_3 and t can be interchanged.

Now we represent $\overset{\overline{(1)}}{L}_{\alpha\beta,\,\beta}$ and $\overset{\overline{(0)}}{L}_{\alpha 3}$ in terms of $\overset{\overline{(1)}}{V}_{\alpha}$ and $\overset{\overline{(0)}}{V}_{3}$. To this end we introduce (2.14d) and (2.14e) into (2.19b) to obtain,

$$\overline{\sigma}_{\omega 3} = \vec{\overline{E}}_{\omega 3\lambda 3}\left(\overset{\overline{(1)}}{V}_{\lambda} + 3(x_3)^2\, \overset{\overline{(3)}}{V}_{\lambda} + \overset{\overline{(0)}}{V}_{3,\,\lambda}\right) \tag{2.31}$$

Writing (2.14) in compact form yields,

$$\overline{e}_{\omega\pi} = \frac{1}{2}\left[(x_3)\left(\overset{\overline{(1)}}{V}_{\omega,\,\pi} + \overset{\overline{(1)}}{V}_{\pi,\,\omega}\right) + (x_3)^3\left(\overset{\overline{(3)}}{V}_{\omega,\,\pi} + \overset{\overline{(3)}}{V}_{\pi,\,\omega}\right) + \mathscr{L}\left\{\overset{(0)}{V}_{3,\,\omega}\, \overset{(0)}{V}_{3,\,\pi}\right\}\right] \tag{2.32a}$$

$$\overline{e}_{\omega 3} = \frac{1}{2}\left[\overset{\overline{(1)}}{V}_{\omega} + 3(x_3)^2\, \overset{\overline{(3)}}{V}_{\omega} + \overset{\overline{(0)}}{V}_{3,\,\omega}\right] \tag{2.32b}$$

where,

$$\bar{e}_{ij} = \bar{e}_{ij}[x_\alpha, x_3, s] \tag{2.33a}$$

$$\overline{\overset{(n)}{V}}_i = \overline{\overset{(n)}{V}}_i[x_\alpha, s] \tag{2.33b}$$

as defined previously.

The introduction of (2.32b) into (2.19a) considered in conjunction with (2.33) yields,

$$\bar{\sigma}_{\alpha\beta} = \frac{1}{2}\bar{\bar{E}}^{*}_{\alpha\beta\omega\pi}\left[(x_3)\left(\overline{\overset{(1)}{V}}_{\omega,\pi} + \overline{\overset{(1)}{V}}_{\pi,\omega}\right) + (x_3)^3\left(\overline{\overset{(3)}{V}}_{\omega,\pi} + \overline{\overset{(3)}{V}}_{\pi,\omega}\right) + \mathscr{L}\left(\overset{(0)}{V}_{3,\omega}\overset{(0)}{V}_{3,\pi}\right) + \delta_A\,\bar{\hat{E}}^{*}_{\alpha\beta33}\,\bar{\sigma}_{33}\right] \tag{2.34}$$

where $\bar{\bar{E}}^{*}_{\alpha\beta\omega\pi}$ and $\bar{\hat{E}}^{*}_{\alpha\beta33}$ have been defined in (2.18).

Determination of $\bar{\sigma}_{33}(x_\alpha, x_3, s)$

The third equation of motion represented by (2.23c) when written in compact form yields,

$$\sigma_{\omega3,\omega} + \sigma_{33,3} + (\sigma_{\mu\delta}V_{3,\delta})_{,\mu} + (\sigma_{3\mu}V_{3,\mu})_{,3} = \rho\ddot{V}_3 \tag{2.35}$$

The Laplace transform of (2.35) yields,

$$\begin{aligned}\bar{\sigma}_{\omega3,\omega} + \bar{\sigma}_{33,3} + \mathscr{L}\{(\sigma_{\mu\delta}V_{3,\delta})_{,\mu}\} + \mathscr{L}\{(\sigma_{3\mu}V_{3,\mu})_{,3}\} \\ = \rho\left(s^2\,\overline{\overset{(0)}{V}}_3 - s\,\overset{(0)}{V}_3[0] - \overset{(\dot{0})}{V}_3[0]\right)\end{aligned} \tag{2.36}$$

We shall first deal with the two nonlinear terms in the above equation. Consider the term $\mathscr{L}\{(\sigma_{\mu\delta}V_{3,\delta})_{,\mu}\}$.

Introducing (2.20a) for $\sigma_{\mu\delta}[x_\alpha, x_3, t]$, (2.13) for $e_{\omega\pi}[x_\alpha, x_3, t]$ in conjunction with (2.11), we obtain for the above term,

$$\mathscr{L}\{(\sigma_{\mu\delta} V_{3,\delta}),_{\mu}\} = \mathscr{L}\Bigg\{\Bigg[\Bigg(\int_0^t \{\frac{1}{2}\dot{\tilde{E}}_{\mu\delta\omega\pi}[t-\tau][(x_3)\overset{(1)}{V}_{\omega,\pi}[\tau] + (x_3)^3\overset{(3)}{V}_{\omega,\pi}[\tau] + (x_3)\overset{(1)}{V}_{\pi,\omega}[\tau] \tag{2.37}$$

$$+ (x_3)^3\overset{(3)}{V}_{\pi,\omega}[\tau] + \overset{(0)}{V}_{3,\omega}[\tau]\overset{(0)}{V}_{3,\pi}[\tau]] + \dot{\hat{E}}_{\mu\delta33}[t-\tau]\,\sigma_{33}[\tau]\}d\tau$$

$$+\frac{1}{2}\tilde{E}_{\mu\delta\omega\pi}[0][(x_3)\overset{(1)}{V}_{\omega,\pi}[t] + (x_3)^3\overset{(3)}{V}_{\omega,\pi}[t] + (x_3)\overset{(1)}{V}_{\pi,\omega}[t]$$

$$+ (x_3)^3\overset{(3)}{V}_{\pi,\omega}[t] + \overset{(0)}{V}_{3,\omega}[t]\overset{(0)}{V}_{3,\pi}[t]] + \hat{E}_{\mu\delta33}[0]\,\sigma_{33}[t]\Bigg)\Bigg(\overset{(0)}{V}_{3,\delta}\Bigg)\Bigg]_{,\mu}\Bigg\}$$

Consider the term $\mathscr{L}\ \{(\sigma_{3\mu} V_{3,\mu}),_3\}$. Similarly introducing (2.20b) for $\sigma_{3\mu}$, (2.13) for $e_{\lambda 3}$ in conjunction with (2.11), we obtain for this term,

$$\begin{aligned}\mathscr{L}\{(\sigma_{3\mu} V_{3,\mu}),_3\} = \mathscr{L}\Bigg\{\Bigg[\Bigg(2\int_0^t \dot{E}_{\mu3\lambda3}[t-\tau][\overset{(1)}{V}_{\lambda}[\tau] + 3(x_3)^2\overset{(3)}{V}_{\lambda}[\tau] + \overset{(0)}{V}_{3,\lambda}[\tau]]\,d\tau \\ + 2E_{\mu3\lambda3}[0][\overset{(1)}{V}_{\lambda}[t] + 3(x_3)^2\overset{(3)}{V}_{\lambda}[t] + \overset{(0)}{V}_{3,\lambda}[t]]\Bigg)\Bigg(\overset{(0)}{V}_{3,\mu}\Bigg)\Bigg]_{,3}\Bigg\}\end{aligned} \tag{2.38}$$

Introducing (2.37), (2.38) into (2.36) and then integrating (2.36) through the segment $[0, x_3)$ in conjunction with (2.11), (2.33), we obtain for $\bar{\sigma}_{33}$,

$$\bar{\sigma}_{33} = \delta_B\,\rho\,x_3\Bigg(s^2\overline{\overset{(0)}{V}}_3 - s\overset{(0)}{V}_3[0] - \overset{(\dot{0})}{V}_3[0]\Bigg) - s\,\bar{E}_{\omega3\lambda3}\Bigg[(x_3)\overline{\overset{(1)}{V}}_{\lambda,\omega} + (x_3)^3\overline{\overset{(3)}{V}}_{\lambda,\omega} + (x_3)\overline{\overset{(0)}{V}}_{3,\lambda\omega}\Bigg]$$

$$-\mathscr{L}\Bigg\{\Bigg[\Bigg(\int_0^t \{\frac{1}{2}\dot{\tilde{E}}_{\mu\delta\omega\pi}[t-\tau][\frac{(x_3)^2}{2}\overset{(1)}{V}_{\omega,\pi}[\tau] + \frac{(x_3)^4}{4}\overset{(3)}{V}_{\omega,\pi}[\tau] + \frac{(x_3)^2}{2}\overset{(1)}{V}_{\pi,\omega}[\tau]$$

$$+\frac{(x_3)^4}{4}\overset{(3)}{V}_{\pi,\,\omega}[\tau]+(x_3)\overset{(0)}{V}_{3,\,\omega}[\tau]\overset{(0)}{V}_{3,\,\pi}[\tau]]$$

$$+\dot{\hat{E}}_{\mu\delta 33}[t-\tau]\int_{-h/2}^{h/2}\sigma_{33}[\tau]\,dx_3\}\,d\tau$$

$$+\frac{1}{2}\tilde{E}_{\mu\delta\omega\pi}[0][\frac{(x_3)^2}{2}\overset{(1)}{V}_{\omega,\,\pi}[t]+\frac{(x_3)^4}{4}\overset{(3)}{V}_{\omega,\,\pi}[t]+\frac{(x_3)^2}{2}\overset{(1)}{V}_{\pi,\,\omega}[t]$$

$$+\frac{(x_3)^4}{4}\overset{(3)}{V}_{\pi,\,\omega}[t]+(x_3)\overset{(0)}{V}_{3,\,\omega}[t]\overset{(0)}{V}_{3,\,\pi}[t]]+\hat{E}_{\mu\delta 33}[0]\int_{-h/2}^{h/2}\sigma_{33}[t]\,dx_3\Big)\Big(\overset{(0)}{V}_{3,\,\delta}\Big)\Big]_{,\,\mu}\Big\}$$

$$-\mathscr{L}\Big\{\Big[\Big(2\int_0^t\dot{E}_{\mu 3\lambda 3}[t-\tau][3(x_3)^2\overset{(3)}{V}_{\lambda}[\tau]]\,d\tau+2E_{\mu 3\lambda 3}[0][3(x_3)^2\overset{(3)}{V}_{\lambda}[t]]\Big)\Big(\overset{(0)}{V}_{3,\,\mu}\Big)\Big]\Big\} \qquad (2.39)$$

Introducing $\overline{\sigma}_{33}$ from (2.39) into (2.34), we obtain,

$$\overline{\sigma}_{\alpha\beta}=\frac{1}{2}\overline{\tilde{E}}^{*}_{\alpha\beta\omega\pi}\Big[x_3(\overline{\overset{(1)}{V}}_{\omega,\,\pi}+\overline{\overset{(1)}{V}}_{\pi,\,\omega})+(x_3)^3(\overline{\overset{(3)}{V}}_{\omega,\,\pi}+\overline{\overset{(3)}{V}}_{\pi,\,\omega})+\mathscr{L}\Big(\overset{(0)}{V}_{3,\,\omega}\overset{(0)}{V}_{3,\,\pi}\Big)\Big]$$

$$+\delta_A\overline{\hat{\tilde{E}}}^{*}_{\alpha\beta 33}\Big[\delta_B\,\rho\,x_3\Big(s^2\overline{\overset{(0)}{V}}_3-s\overset{(0)}{V}_3[0]-\overset{(0)}{\dot{V}}_3[0]\Big)-s\,\overline{E}_{\omega 3\lambda 3}\Big[(x_3)\overline{\overset{(1)}{V}}_{\lambda,\,\omega}+(x_3)^3\overline{\overset{(3)}{V}}_{\lambda,\,\omega}+(x_3)\overline{\overset{(0)}{V}}_{3,\,\lambda\omega}\Big]$$

$$-\mathscr{L}\Big\{\Big[\Big(\int_0^t\{\frac{1}{2}\dot{\tilde{E}}_{\mu\delta\omega\pi}[t-\tau][\frac{(x_3)^2}{2}\overset{(1)}{V}_{\omega,\,\pi}[\tau]+\frac{(x_3)^4}{4}\overset{(3)}{V}_{\omega,\,\pi}[\tau]+\frac{(x_3)^2}{2}\overset{(1)}{V}_{\pi,\,\omega}[\tau]$$

$$+\frac{(x_3)^4}{4}\overset{(3)}{V}_{\pi,\,\omega}[\tau]+(x_3)\overset{(0)}{V}_{3,\,\omega}[\tau]\overset{(0)}{V}_{3,\,\pi}[\tau]]$$

$$+ \dot{\hat{E}}_{\mu\delta 33}[t-\tau] \int_{-h/2}^{h/2} \sigma_{33}[\tau]\, dx_3 \} \, d\tau$$

$$+ \frac{1}{2} \tilde{E}_{\mu\delta\omega\pi}[0][\frac{(x_3)^2}{2} \overset{(1)}{V}_{\omega,\pi}[t] + \frac{(x_3)^4}{4} \overset{(3)}{V}_{\omega,\pi}[t] + \frac{(x_3)^2}{2} \overset{(1)}{V}_{\pi,\omega}[t]$$

$$+ \frac{(x_3)^4}{4} \overset{(3)}{V}_{\pi,\omega}[t] + (x_3) \overset{(0)}{V}_{3,\omega}[t] \overset{(0)}{V}_{3,\pi}[t]] + \hat{E}_{\mu\delta 33}[0] \int_{-h/2}^{h/2} \sigma_{33}[t]\, dx_3 \Big) \Big(\overset{(0)}{V}_{3,\delta} \Big) \Big]_{,\mu} \Big\}$$

$$- \mathscr{L} \Big\{ \Big[\Big(2 \int_0^t \dot{E}_{\mu 3\lambda 3}[t-\tau][3(x_3)^2 \overset{(3)}{V}_{\lambda}[\tau]]\, d\tau + 2E_{\mu 3\lambda 3}[0][3(x_3)^2 \overset{(3)}{V}_{\lambda}[t]] \Big) \Big(\overset{(0)}{V}_{3,\mu} \Big) \Big] \Big\} \Big] \tag{2.40}$$

Introducing (2.40) in (2.30a) in conjunction with (2.11c), (2.33b) and (2.40), we obtain, for $\overline{\overset{(1)}{L}}_{\alpha\beta}$,

$$\overline{\overset{(1)}{L}}_{\alpha\beta}[x_\alpha, s] = \frac{1}{2} \overline{\tilde{E}}_{\alpha\beta\omega\pi} \Big[\frac{h^3}{12} (\overline{\overset{(1)}{V}}_{\omega,\pi} + \overline{\overset{(1)}{V}}_{\pi,\omega}) + \frac{h^5}{80} (\overline{\overset{(3)}{V}}_{\omega,\pi} + \overline{\overset{(3)}{V}}_{\pi,\omega}) \Big] \tag{2.41}$$

$$+ \delta_A \overline{\hat{E}}_{\alpha\beta 33} \Big[\frac{h^3}{12} \Big\{ \delta_B\, \rho \Big(s^2 \overline{\overset{(0)}{V}}_3 - s \overset{(0)}{V}_3[0] - \dot{\overset{(0)}{V}}_3[0] \Big) - s\, \bar{E}_{\omega 3\lambda 3} \Big(\overline{\overset{(1)}{V}}_{\lambda,\omega} + \overline{\overset{(0)}{V}}_{3,\lambda\omega} \Big) \Big\}$$

$$- \frac{h^5}{80} \Big\{ s\, \bar{E}_{\omega 3\lambda 3} \overline{\overset{(3)}{V}}_{\lambda,\omega} \Big\}$$

$$- \frac{h^3}{24} \mathscr{L} \Big\{ \Big[\Big(\int_0^t \dot{\tilde{E}}_{\mu\delta\omega\pi}[t-\tau]\, [\overset{(0)}{V}_{3,\omega}[\tau] \overset{(0)}{V}_{3,\pi}[\tau]]\, d\tau + \tilde{E}_{\mu\delta\omega\pi}[0]\, [\overset{(0)}{V}_{3,\omega}[t] \overset{(0)}{V}_{3,\pi}[t]] \Big) \Big(\overset{(0)}{V}_{3,\delta} \Big) \Big]_{,\mu} \Big\} \Big]$$

We see that the last term in the expression of $\overline{\overset{(1)}{L}}_{\alpha\beta}$, i.e., the one under the Laplace transform operator ($\mathscr{L}$) contains the cubic nonlinearity of $\overset{(0)}{V}_{3,\omega} \overset{(0)}{V}_{3,\pi} \overset{(0)}{V}_{3,\delta}$. This represents the product of small disturbances and can thus be neglected for a linear analysis of the problem. Thus on linearizing $\overline{\overset{(1)}{L}}_{\alpha\beta}$ we obtain,

$$\overset{\overline{(1)}}{L}_{\alpha\beta}[x_\alpha, s] = \frac{1}{2} \tilde{\bar{E}}_{\alpha\beta\omega\pi} \left[\frac{h^3}{12} \left(\overset{\overline{(1)}}{V}_{\omega,\,\pi} + \overset{\overline{(1)}}{V}_{\pi,\,\omega} \right) + \frac{h^5}{80} \left(\overset{\overline{(3)}}{V}_{\omega,\,\pi} + \overset{\overline{(3)}}{V}_{\pi,\,\omega} \right) \right] + \tag{2.42}$$

$$\delta_A \hat{\bar{E}}_{\alpha\beta 33} \left[\frac{h^3}{12} \{ \delta_B \, \rho (s^2 \overset{\overline{(0)}}{V}_3 - s \overset{(0)}{V}_3[0] - \overset{(0)}{V}_3[0]) - s \bar{E}_{\omega 3 \lambda 3} (\overset{\overline{(1)}}{V}_{\lambda,\,\omega} + \overset{\overline{(0)}}{V}_{3,\,\lambda\omega}) \} - \frac{h^5}{80} \{ s \bar{E}_{\omega 3 \lambda 3} \overset{\overline{(3)}}{V}_{\lambda,\,\omega} \} \right]$$

We now solve for $\overset{\overline{(3)}}{V}_\pi$ in order to eliminate it from the governing equations of stability. Towards this end we recall the constitutive equations (2.8) which are as follows:

$$\sigma_{ij}[t] = \int_0^t \left[\dot{E}_{ijkl}[t-\tau] + E_{ijkl}[0] \, \delta[t-\tau] \right] e_{kl}[\tau] \, d\tau \tag{2.43}$$

with the inverse relation being,

$$e_{kl}[\tau] = \int_0^\tau \left[\dot{F}_{klmn}[\tau-\tau'] + F_{klmn}[0] \, \delta[\tau-\tau'] \right] \sigma_{mn}[\tau'] \, d\tau' \tag{2.44}$$

Introduction of $e_{kl}[\tau]$ in $\sigma_{ij}[t]$ above yields:

$$\sigma_{ij}[t] = \int_0^t \left(\left[\dot{E}_{ijkl}[t-\tau] + E_{ijkl}[0] \, \delta[t-\tau] \right] \int_0^\tau \left[\dot{F}_{klmn}[\tau-\tau'] + F_{klmn}[0] \, \delta[\tau-\tau'] \right] \sigma_{mn}[\tau'] \, d\tau' \right) d\tau \tag{2.45}$$

Interchanging the order of integration in the expression above yields:

$$\sigma_{ij}[t] = \int_0^t \left(\int_{\tau'}^t \left[\dot{E}_{ijkl}[t-\tau] + E_{ijkl}[0] \, \delta[t-\tau] \right] \left[\dot{F}_{klmn}[\tau-\tau'] + F_{klmn}[0] \, \delta[\tau-\tau'] \right] d\tau \right) \sigma_{mn}[\tau'] \, d\tau' \tag{2.46}$$

Due to the symmetry property of the stress tensor $\sigma_{ij}[t]$, we may write the following expression (see Illyushin and Pobedria [21]):

$$\sigma_{ij}[t]=\frac{1}{2}[\delta_{im}\,\delta_{jn}+\delta_{in}\,\delta_{jm}]\,\sigma_{mn}[t]=\int_0^t\frac{1}{2}\,\delta[t-\tau'][\delta_{im}\,\delta_{jn}+\delta_{in}\,\delta_{jm}]\,\sigma_{mn}[\tau']\,d\tau' \quad (2.47)$$

By virtue of (2.46) and (2.47), we obtain,

$$\int_{\tau'}^t[E_{ijkl}[t-\tau]+E_{ijkl}[0]\,\delta[t-\tau]]\left[\dot{F}_{klmn}[\tau-\tau']+F_{klmn}[0]\,\delta[\tau-\tau']\right]d\tau \quad (2.48)$$

$$=\frac{1}{2}\,\delta[t-\tau'][\delta_{im}\,\delta_{jn}+\delta_{in}\,\delta_{jm}]$$

Setting $\tau'=0$ in (2.48) and then taking the L.T. of the resulting expression yields,

$$s^2\,\bar{E}_{ijkl}\,\bar{F}_{klmn}=\frac{1}{2}\,[\delta_{im}\,\delta_{jn}+\delta_{in}\,\delta_{jm}] \quad (2.49)$$

From (2.49) we thus obtain the following result for an orthotropic body:

$$4s\,\bar{E}_{\omega 3\lambda 3}s\,\bar{F}_{\omega 3\pi 3}=\delta_{\pi\lambda} \quad (2.50)$$

In writing (2.50), we have also assumed that the relaxation moduli $E_{ijmn}[t]$ and creep compliances $F_{ijmn}[t]$ are continuous functions and thus possess unique Laplace transforms $\bar{E}_{ijmn}[s]$ and $\bar{F}_{ijmn}(s)$ exhibiting the same symmetry properties as their originals. Making use of (2.50) in (2.31) yields,

$$4s\,\bar{F}_{\omega 3\pi 3}\,\bar{\sigma}_{\omega 3}=\delta_{\pi\lambda}\left[\overset{(\bar{1})}{V}_{\lambda}+3(x_3)^2\,\overset{(\bar{3})}{V}_{\lambda}+\overset{(\bar{0})}{V}_{3,\,\lambda}\right] \quad (2.51)$$

In order to satisfy the boundary conditions on the boundary planes, we get by introducing (2.51) in (2.30c) the result:

$$4s\,\bar{F}_{\omega 3\pi 3}\,\overset{(\bar{1})}{p}_{\omega}=h\,\overset{(\bar{1})}{V}_{\pi}+\frac{3}{4}\,h^3\,\overset{(\bar{3})}{V}_{\pi}+h\,\overset{(\bar{0})}{V}_{3,\,\pi} \quad (2.52)$$

Solving for $\overset{(\bar{3})}{V}_\pi$ from (2.52), we get,

$$\overset{(\bar{3})}{V}_\pi = \frac{4}{3h^2}\left[\frac{4s}{h}\bar{F}_{\omega 3\pi 3}\overset{(\bar{1})}{p}_\omega - \overset{(\bar{1})}{V}_\pi - \overset{(\bar{0})}{V}_{3,\,\pi}\right] \tag{2.53}$$

Introducing (2.53) into (2.42), in conjunction with (2.50) and the symmetry property

$\bar{\bar{E}}^*_{\alpha\beta\omega\pi} = \bar{\bar{E}}^*_{\alpha\beta\pi\omega}$, we obtain for $\overset{(\bar{1})}{L}_{\alpha\beta}$ the following:

$$\begin{aligned}\overset{(\bar{1})}{L}_{\alpha\beta} = {} & \bar{\bar{E}}^*_{\alpha\beta\omega\pi}\left[\frac{h^3}{15}\overset{(\bar{1})}{V}_{\omega,\,\pi} + \frac{h^2}{30}\left(s\,\bar{F}_{\rho 3\omega 3}\overset{(\bar{1})}{p}_{\rho,\,\pi} + s\,\bar{F}_{\rho 3\pi 3}\overset{(\bar{1})}{p}_{\rho,\,\omega}\right) - \frac{h^3}{60}\overset{(\bar{0})}{V}_{3,\,\pi\omega}\right] \\ & + \delta_A\,\bar{\hat{E}}^*_{\alpha\beta 33}\left[\frac{h^3}{12}\left\{\delta_B\,\rho\left(s^2\overset{(\bar{0})}{V}_3 - s\overset{(0)}{V}_3[0] - \overset{(0)}{\dot{V}}_3[0]\right) - s\,\bar{E}_{\omega 3\lambda 3}\left(\overset{(\bar{1})}{V}_{\lambda,\,\omega} + \overset{(\bar{0})}{V}_{3,\,\lambda\omega}\right)\right\}\right. \\ & \left. - \frac{h^2}{60}\overset{(\bar{1})}{p}_{\rho,\,\rho} + \frac{h^3}{60}s\,\bar{E}_{\omega 3\lambda 3}\left(\overset{(\bar{1})}{V}_{\lambda,\,\omega} + \overset{(\bar{0})}{V}_{3,\,\lambda\omega}\right)\right]\end{aligned} \tag{2.54}$$

Now introducing (2.31) into (2.30d), and then making use of (2.53) in conjunction with (2.50), yields for $\overset{(\bar{0})}{L}_{\omega 3}$ the following:

$$\overset{(\bar{0})}{L}_{\omega 3} = \frac{2h}{3}s\,\bar{E}_{\omega 3\lambda 3}\left[\overset{(\bar{1})}{V}_\lambda + \overset{(\bar{0})}{V}_{3,\,\lambda}\right] + \frac{1}{3}\overset{(\bar{1})}{p}_\omega \tag{2.55}$$

Upon introducing (2.54), (2.55) into the Laplace transformed equations of motion (i.e., (2.29)) we obtain, after multiplying throughout by $\frac{60}{h^3}$,

$$\begin{aligned}& \bar{\bar{E}}^*_{\alpha\beta\omega\pi}\overset{(\bar{0})}{V}_{3,\,\pi\omega\beta} - \frac{4}{h}\bar{\bar{E}}^*_{\alpha\beta\omega\pi}s\,\bar{F}_{\rho 3\omega 3}\overset{(\bar{1})}{p}_{\rho,\,\pi\beta} - 4\,\bar{\bar{E}}^*_{\alpha\beta\omega\pi}\overset{(\bar{1})}{V}_{\omega,\,\pi\beta} + 4\,\delta_A\,\bar{\hat{E}}^*_{\alpha\beta 33}s\,\bar{E}_{\omega 3\lambda 3}\left(\overset{(\bar{1})}{V}_{\lambda,\,\omega\beta} + \overset{(\bar{0})}{V}_{3,\,\lambda\omega\beta}\right) \\ & - 5\,\delta_A\,\delta_B\,\rho\,\bar{\hat{E}}^*_{\alpha\beta 33}\left(s^2\overset{(\bar{0})}{V}_{3,\,\beta} - s\overset{(0)}{V}_{3,\,\beta}[0] - \overset{(0)}{\dot{V}}_{3,\,\beta}[0]\right) + \frac{\delta_A}{h}\bar{\hat{E}}^*_{\alpha\beta 33}\overset{(\bar{1})}{p}_{\rho,\,\rho\beta} \\ & - \frac{40}{h^3}\overset{(\bar{1})}{p}_\alpha + \frac{40}{h^2}s\,\bar{E}_{\alpha 3\lambda 3}\overset{(\bar{1})}{V}_\lambda + \frac{40}{h^2}s\,\bar{E}_{\alpha 3\lambda 3}\overset{(\bar{0})}{V}_{3,\,\lambda} + \delta_C\frac{60}{h^3}\overset{(\bar{1})}{f}_\alpha = 0\end{aligned} \tag{2.56}$$

In (2.56), $\overset{(\bar{1})}{f}_\alpha$ is obtained by first evaluating $\overset{(1)}{f}_\alpha$ and then taking its Laplace transform. Introducing (2.11a) into (2.25b), we obtain for $\overset{(1)}{f}_\alpha$,

$$\overset{(1)}{f_\alpha} = \rho\left[\frac{h^3}{12}\overset{(\ddot{1})}{V_\alpha} + \frac{h^5}{80}\overset{(\ddot{3})}{V_\alpha}\right] \tag{2.57}$$

Inverting (2.53) into the time domain, we obtain,

$$\overset{(3)}{V_\pi}[t] = \frac{4}{3h^2}\left[\frac{4}{h}\int_0^t\left[\dot{F}_{\omega3\pi3}[t-\tau] + F_{\omega3\pi3}[0]\,\delta[t-\tau]\right]\overset{(1)}{p_\omega}[\tau]\,d\tau - \overset{(1)}{V_\pi} - \overset{(0)}{V_{3,\pi}}\right] \tag{2.58}$$

Differentiating twice w.r.t. 't' we obtain (using Leibnitz's rule),

$$\begin{aligned}\overset{(\ddot{3})}{V_\pi}[t] = \frac{4}{3h^2}\Bigg[&\frac{4}{h}\int_0^t \dddot{F}_{\omega3\pi3}[t-\tau]\overset{(1)}{p_\omega}[\tau]\,d\tau + \frac{4}{h}\ddot{F}_{\omega3\pi3}[0]\,\overset{(1)}{p_\omega}[t] + \frac{4}{h}\dot{F}_{\omega3\pi3}[0]\,\overset{(\dot{1})}{p_\omega} \\ &+ \frac{4}{h}F_{\omega3\pi3}[0]\,\overset{(\ddot{1})}{p_\omega}[t] - \overset{(\ddot{1})}{V_\pi} - \overset{(\ddot{0})}{V_{3,\pi}}\Bigg]\end{aligned} \tag{2.59}$$

Introducing (2.59) in (2.57) yields, for zero surface tractions,

$$\overset{(1)}{f_\alpha} = \rho\left(\frac{h^3}{15}\overset{(\ddot{1})}{V_\alpha} - \frac{h^3}{60}\overset{(\ddot{0})}{V_{3,\alpha}}\right) \tag{2.60}$$

Its L.T. yields,

$$\overline{\overset{(1)}{f_\alpha}} = \rho\left(\frac{h^3}{15}\left[s^2\,\overline{\overset{(1)}{V_\alpha}} - s\,\overset{(1)}{V_\alpha}[0] - \overset{(\dot{1})}{V_\alpha}[0]\right] - \frac{h^3}{60}\left[s^2\,\overline{\overset{(0)}{V_{3,\alpha}}} - s\,\overset{(0)}{V_{3,\alpha}}[0] - \overset{(\dot{0})}{V_{3,\alpha}}[0]\right]\right) \tag{2.61}$$

Introduce the notation

$$\overline{\hat{E}}^{*}_{\alpha\beta33}\, s\,\overline{E}_{\omega3\lambda3} = s\left(\overline{E}_{\omega3\lambda3}\frac{\overline{E}_{\alpha\beta33}}{\overline{E}_{3333}}\right) = s\,\overline{\hat{E}}_{\alpha\beta\omega\lambda} = \overline{\hat{E}}^{*}_{\alpha\beta\omega\lambda} \tag{2.62a}$$

and

$$\hat{E}_{\alpha\beta\omega\lambda} = \mathscr{L}^{-1}\left(\overline{E}_{\omega3\lambda3}\frac{\overline{E}_{\alpha\beta33}}{\overline{E}_{3333}}\right) \tag{2.62b}$$

Also, $\hat{E}_{\alpha\beta\omega\lambda}[0] = \dfrac{E_{\omega 3\lambda 3}[0]\, E_{\alpha\beta 33}[0]}{E_{3333}[0]}$ (see Appendix [A]). Substituting (2.61), (2.62a) into (2.56), we obtain for zero surface tractions,

$$\overline{\vec{\tilde{E}}}^{*}_{\alpha\beta\omega\pi} \overline{\overset{(0)}{V}}_{3,\,\pi\omega\beta} - 4\, \overline{\vec{\tilde{E}}}^{*}_{\alpha\beta\omega\pi} \overline{\overset{(1)}{V}}_{\omega,\,\pi\beta} + 4\,\delta_A\, \overline{\hat{E}}^{*}_{\alpha\beta\omega\lambda} \left(\overline{\overset{(1)}{V}}_{\lambda,\,\omega\beta} + \overline{\overset{(0)}{V}}_{3,\,\lambda\omega\beta} \right) \tag{2.63}$$

$$- 5\delta_A\,\delta_B\,\rho\, \overline{\hat{E}}^{*}_{\alpha\beta 33} \left(s^2 \overline{\overset{(0)}{V}}_{3,\,\beta} - s\, \overset{(0)}{V}_{3,\,\beta}[0] - \overset{(\dot{0})}{V}_{3,\,\beta}[0] \right) + \frac{40}{h^2} \overline{\vec{E}}_{\alpha 3\lambda 3} \overline{\overset{(1)}{V}}_{\lambda} + \frac{40}{h^2} \overline{\vec{E}}_{\alpha 3\lambda 3} \overline{\overset{(0)}{V}}_{3,\,\lambda}$$

$$+ \delta_C \frac{60}{h^3} \rho \left(\frac{h^3}{15} \left[s^2 \overline{\overset{(1)}{V}}_{\alpha} - s\, \overset{(1)}{V}_{\alpha}[0] - \overset{(\dot{1})}{V}_{\alpha}[0] \right] - \frac{h^3}{60} \left[s^2 \overline{\overset{(0)}{V}}_{3,\,\alpha} - s\, \overset{(0)}{V}_{3,\,\alpha}[0] - \overset{(\dot{0})}{V}_{3,\,\alpha}[0] \right] \right) = 0$$

Inverting the above in the time domain, we have, using Borel's theorem for convolution integrals,

$$\begin{aligned}
&\int_0^t \dot{\tilde{E}}_{\alpha\beta\omega\pi}(t-\tau)\, V_{3,\,\pi\omega\beta}(\tau)\, d\tau + \tilde{E}_{\alpha\beta\omega\pi}(0)\, V_{3,\,\pi\omega\beta}[t] - 4 \int_0^t \dot{\tilde{E}}_{\alpha\beta\omega\pi}(t-\tau)\, V_{\omega,\,\pi\beta}(\tau) d\tau \\
&- 4\, E_{\alpha\beta\omega\pi}(0)\, \overset{(1)}{V}_{\omega,\,\pi\beta}[t] + 4\,\delta_A \int_0^t \dot{\hat{E}}_{\alpha\beta\omega\lambda}(t-\tau) \left[\overset{(1)}{V}_{\lambda,\,\omega\beta}(\tau) + \overset{(0)}{V}_{3,\,\lambda\omega\beta}(\tau) \right] d\tau \\
&+ 4\,\delta_A\, \hat{E}_{\alpha\beta\omega\lambda}(0) \left[\overset{(1)}{V}_{\lambda,\,\omega\beta}[t] + \overset{(0)}{V}_{3,\,\lambda\omega\beta}[t] \right] - 5\,\delta_A\,\delta_B\,\rho \int_0^t \dot{\hat{E}}_{\alpha\beta 33}(t-\tau)\, \overset{(\ddot{0})}{V}_{3,\,\beta}(\tau)\, d\tau \\
&- 5\,\delta_A\,\delta_B\,\rho\, \hat{E}_{\alpha\beta 33}(0)\, \ddot{V}_{3,\,\beta}[t] + \frac{40}{h^2} \int_0^t \dot{E}_{\alpha 3\lambda 3}(t-\tau)\, \overset{(1)}{V}_{\lambda}(\tau)\, d\tau + \frac{40}{h^2} E_{\alpha 3\lambda 3}(0)\, \overset{(1)}{V}_{\lambda}[t] \\
&+ \frac{40}{h^2} \int_0^t \dot{E}_{\alpha 3\lambda 3}(t-\tau)\, \overset{(0)}{V}_{3,\,\lambda}(\tau)\, d\tau + \frac{40}{h^2} E_{\alpha 3\lambda 3}(0)\, \overset{(0)}{V}_{3,\,\lambda}[t] + \delta_C \frac{60}{h^3} \rho \left(\frac{h^3}{15} \overset{(\ddot{1})}{V}_{\alpha} - \frac{h^3}{60} \overset{(\ddot{0})}{V}_{3,\,\alpha} \right) = 0
\end{aligned} \tag{2.64}$$

Equations (2.64) are the first two (of three) governing equations for a geometrically nonlinear theory of an orthotropic, linearly viscoelastic flat plate subject to zero inplane surface tractions $\left(\overset{(n)}{p}_{\alpha} = 0\right)$, using a third order transverse shear deformation theory.

Now the Laplace transform of the third equation of motion (2.23c) yields,

$$\left[\bar{\sigma}_{\alpha3} + \mathscr{L}\left(\sigma_{\alpha\beta} V_{3,\beta}\right)\right]_{,\alpha} + \left[\bar{\sigma}_{33} + \mathscr{L}(\sigma_{31} V_{3,1}) + \mathscr{L}(\sigma_{32} V_{3,2})\right]_{,3} = \delta_D\, \rho\, \ddot{V}_3 \tag{2.65}$$

Here the tracer δ_D identifies the transverse inertia term. Taking the moment of order zero of equation (2.65), we obtain,

$$\left[\overset{\overline{(0)}}{L}_{\alpha3} + \mathscr{L}\left(\overset{(0)}{L}_{\alpha\beta} V_{3,\beta}\right)\right]_{,\alpha} + \left[\bar{\sigma}_{33} + \mathscr{L}(\sigma_{31} V_{3,1}) + \mathscr{L}(\sigma_{32} V_{3,2})\right]_{-h/2}^{+h/2}$$

$$- \delta_D\, \rho\, h \left(s^2 \overset{\bar{(0)}}{V}_3 - s\, \overset{(0)}{V}_3[0] - \overset{(\dot{0})}{V}_3[0]\right) = 0$$

From the above we obtain,

$$\overset{\overline{(0)}}{L}_{13,1} + \overset{\overline{(0)}}{L}_{23,2} + \mathscr{L}\left(\overset{(0)}{L}_{11} \overset{(0)}{V}_{3,11} + \overset{(0)}{L}_{22} \overset{(0)}{V}_{3,22} + 2\, \overset{(0)}{L}_{12} \overset{(0)}{V}_{3,12} + \left[\overset{(0)}{L}_{11,1} + \overset{(0)}{L}_{21,2}\right] \overset{(0)}{V}_{3,1}\right. \tag{2.66}$$

$$\left. + \left[\overset{(0)}{L}_{12,1} + \overset{(0)}{L}_{22,2}\right] \overset{(0)}{V}_{3,2}\right)$$

$$+ \left[\bar{\sigma}_{33} + \mathscr{L}(\sigma_{31} \overset{(0)}{V}_{3,1}) + \mathscr{L}(\sigma_{32} \overset{(0)}{V}_{3,2})\right]_{-h/2}^{+h/2} - \delta_D\, \rho\, h \left(s^2 \overset{\bar{(0)}}{V}_3 - s\, \overset{(0)}{V}_3[0] - \overset{(\dot{0})}{V}_3[0]\right) = 0$$

The zeroth order moments of (2.23b) and (2.23c) yield,

$$\overset{(0)}{L}_{11,1} + \overset{(0)}{L}_{12,2} + \overset{(0)}{p}_1 = 0 \tag{2.67a}$$

$$\overset{(0)}{L}_{12,1} + \overset{(0)}{L}_{22,2} + \overset{(0)}{p}_2 = 0 \tag{2.67b}$$

Substituting (2.67) in (2.66) and neglecting surface tractions (which implies $\overset{(0)}{p}_1 = \overset{(0)}{p}_2 = 0$ and $\overset{(0)}{p}_3 = 0$), we obtain,

$$\overset{\overline{(0)}}{L}_{13,1} + \overset{\overline{(0)}}{L}_{23,2} + \mathscr{L}\left(\overset{(0)}{L}_{11} \overset{(0)}{V}_{3,11} + \overset{(0)}{L}_{22} \overset{(0)}{V}_{3,22} + 2\, \overset{(0)}{L}_{12} \overset{(0)}{V}_{3,12}\right) - \delta_D\, \rho\, h \left(s^2 \overset{\overline{(0)}}{V}_3 - s\, \overset{(0)}{V}_3[0] - \overset{(\dot{0})}{V}_3[0]\right) = 0 \tag{2.68}$$

For constant edge loads, equation (2.68) converts to a linear form which reads,

$$\overset{\overline{(0)}}{L}_{13,1}+\overset{\overline{(0)}}{L}_{23,2}+\overset{(0)}{L}_{11}\overset{\overline{(0)}}{V}_{3,11}+\overset{(0)}{L}_{22}\overset{\overline{(0)}}{V}_{3,22}+2\overset{(0)}{L}_{12}\overset{\overline{(0)}}{V}_{3,12}-\delta_D\,\rho\,h\left(s^2\overset{\overline{(0)}}{V}_3-s\overset{(0)}{V}_3[0]-\overset{(\dot{0})}{V}_3[0]\right)=0 \qquad (2.69)$$

Introducing (2.55) into (2.69) yields,

$$\begin{aligned}\frac{2}{3}h\,s\,\bar{E}_{\omega3\lambda3}\left[\overset{\overline{(1)}}{V}_{\lambda,\,\omega}+\overset{\overline{(0)}}{V}_{3,\,\lambda\omega}\right]+\overset{(0)}{L}_{11}\overset{\overline{(0)}}{V}_{3,11}+\overset{(0)}{L}_{22}\overset{\overline{(0)}}{V}_{3,22}+2\overset{(0)}{L}_{12}\overset{\overline{(0)}}{V}_{3,12}&\\ -\delta_D\,\rho\,h\left(s^2\overset{\overline{(0)}}{V}_3-s\overset{(0)}{V}_3[0]-\overset{(\dot{0})}{V}_3[0]\right)=0&\end{aligned} \qquad (2.70)$$

Inverting (2.70) into the time domain, we obtain,

$$\begin{aligned}\frac{2}{3}h\int_0^t\dot{E}_{\omega3\lambda3}[t-\tau]\left[\overset{(1)}{V}_{\lambda,\,\omega}[\tau]+\overset{(0)}{V}_{3,\,\lambda\omega}[\tau]\right]d\tau+\frac{2}{3}h\,E_{\omega3\lambda3}[0]\left[\overset{(1)}{V}_{\lambda,\,\omega}[t]+\overset{(0)}{V}_{3,\,\lambda\omega}[t]\right]&\\ +\overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11}[t]+\overset{(0)}{L}_{22}\overset{(0)}{V}_{3,22}[t]+2\overset{(0)}{L}_{12}\overset{(0)}{V}_{3,12}[t]-\delta_D\,\rho\,h\overset{(\ddot{0})}{V}_3=0&\end{aligned} \qquad (2.71)$$

Equation (2.71) represents the third equation governing the stability of an orthotropic, linearly viscoelastic, shear deformable flat plate.

In the next section we introduce the elastic-viscoelastic correspondence principle.

Boundary conditions

Equations (2.64) and (2.71) represent a sixth-order governing equation system. Its solution must be determined in conjunction with the prescribed boundary conditions (which are to be three at each edge). For a simply supported plate, i.e., hinged-free in the normal direction, we have the following boundary conditions:

$$\overset{(1)}{V}_2=\overset{(0)}{V}_3=\overset{(1)}{L}_{11}=0 \quad \text{at } x_1=0,\, l_1$$

$$\overset{(1)}{V}_1=\overset{(0)}{V}_3=\overset{(1)}{L}_{22}=0 \quad \text{at } x_2=0,\, l_2$$

The above equations represent the conditions of zero tangential displacement, zero transverse displacement and zero moment resultant along the edges of the plate.

2.3.4 The Elastic-Viscoelastic Correspondence Principle

Consider the equations of motion for infinitesimal strain theory which write as:

$$\sigma_{ij,j} + \rho\, H_i = \rho\, \ddot{V}_i \tag{2.72a}$$

The linearized strain-displacement relations are:

$$e_{ij} = \frac{1}{2}\left(V_{i,j} + V_{j,i}\right) \tag{2.72b}$$

Also the constitutive equations for an elastic continuum write as:

$$\sigma_{ij} = E_{ijmn}\, e_{mn} \tag{2.72c}$$

and the boundary conditions are,

$$\sigma_{ij}\, n_j = \underset{\sim}{F}_i \qquad |S_\sigma \tag{2.72d}$$

$$V_i = \underset{\sim}{V}_i \qquad |S_v \tag{2.72e}$$

where S_σ denotes that part of the boundary S on which the components of the stress vector are prescribed, while S_v is that part of S over which the components of the displacement vector are prescribed. S_σ and S_v are required to remain invariant with time. For a viscoelastic continuum the equations of motion, strain-displacement equations and boundary conditions are identical to (2.72a, b, d and e), respectively, the only change being in the constitutive equations. For a linearly viscoelastic anisotropic material, these were given in (2.5) and are reproduced here as,

$$\sigma_{ij}[t] = \int_{0^-}^{t} E_{ijmn}[t-\tau]\, \dot{e}_{mn}[\tau]\, d\tau \tag{2.72d}$$

Taking the L.T. of (2.72c) yields (for the elastic body):

$$\bar{\sigma}_{ij}[s] = E_{ijmn}\,\bar{e}_{mn}[s] \tag{2.73a}$$

The corresponding L.T. of the viscoelastic constitutive equation (2.72d) yields,

$$\bar{\sigma}_{ij}[s] = s\,\bar{E}_{ijmn}[s]\,\bar{e}_{mn}[s] \tag{2.73b}$$

Now the L.T. of (2.72a, b, d, e) would be identical for the elastic and viscoelastic continua. This fact when considered in conjunction with (2.73a) and (2.73b) gives rise to the elastic-viscoelastic correspondence principle (see Christensen [11]). The principle states that the L.T. of the governing equations for a viscoelastic continuum can be obtained by taking the Laplace transform of the corresponding governing equations of an elastic continuum and then replacing the moduli and compliances by their Carson transforms. This means that E_{ijmn} (or F_{ijmn}) which are constant for the elastic body (and are thus unaffected by taking the Laplace transform of the governing equations pertaining to the elastic body) should be replaced by their Carson transforms in order to obtain the corresponding transformed governing equations for the viscoelastic body.

We note that the equations of motion and strain-displacement relations used in establishing the correspondence principle (C. P. hereafter) are those pertaining to the infinitesimal displacement theory. However we shall see in the following section that the C.P. can be used for deriving the governing equations of the linear stability problem.

2.3.5 Derivation of the governing stability equations using the correspondence principle (TTSD)

Making use of the general procedure followed in Sec. 2.3.3 for an orthotropic viscoelastic flat plate, we may derive the equations governing the stability of an orthotropic, elastic, flat plate. Neglecting the inplane tractions and the transverse loads on the bounding surfaces (i.e., $\overset{(n)}{p}_\alpha = \overset{(n)}{p}_3 = 0$), we obtain the following system of equations governing the stability of an orthotropic, elastic, flat plate:

$$\tilde{E}_{\alpha\beta\omega\pi} \overset{(0)}{V}_{3,\,\omega\pi\beta} - 4\,\tilde{E}_{\alpha\beta\omega\pi} \overset{(1)}{V}_{\omega,\,\pi\beta} + 4\,\delta_A \frac{E_{\alpha\beta 33}}{E_{3333}} E_{\omega 3\lambda 3}\left(\overset{(1)}{V}_{\lambda,\,\omega\beta} + \overset{(0)}{V}_{3,\,\omega\lambda\beta}\right)$$

$$- 5\,\delta_A\,\delta_B \frac{E_{\alpha\beta 33}}{E_{3333}} \rho\, \overset{(\ddot{0})}{V}_{3,\,\beta} + \frac{40}{h^2} E_{\alpha 3\lambda 3} \overset{(1)}{V}_{\lambda} + \frac{40}{h^2} E_{\alpha 3\lambda 3} \overset{(0)}{V}_{3,\,\lambda} + \frac{60}{h^3}\,\delta_C \overset{(1)}{f}_\alpha = 0 \tag{2.74}$$

where,

$$\overset{(1)}{f}_\alpha = \rho\left(\frac{h^3}{15} \overset{(\ddot{1})}{V}_\alpha - \frac{h^3}{60} \overset{(\ddot{0})}{V}_{3,\,\alpha}\right) \tag{2.75}$$

and,

$$\frac{2}{3} E_{\omega 3\lambda 3}\, h\left(\overset{(1)}{V}_{\lambda,\,\omega} + \overset{(0)}{V}_{3,\,\lambda\omega}\right) + \overset{(0)}{L}_{11} \overset{(0)}{V}_{3,11} + \overset{(0)}{L}_{22} \overset{(0)}{V}_{3,12} + 2\,\overset{(0)}{L}_{12} \overset{(0)}{V}_{3,12} - \rho_0\, h\, \overset{(\ddot{0})}{V}_3 = 0 \tag{2.76}$$

At this stage we note that (2.74), which represents the first two governing equations when finite displacements are considered, remains unaltered when compared with its counterpart obtained for infinitesimal displacement theory (see Librescu and Reddy [15]). The third governing equation (2.76) may be obtained, for the case when $\overset{(1)}{p}_\alpha = 0$, by replacing $\overset{(0)}{p}_3$ by $\overset{(0)}{p}_3 + \overset{(0)}{L}_{11} \overset{(0)}{V}_{3,11} + \overset{(0)}{L}_{22} \overset{(0)}{V}_{3,22} + 2\,L_{12} \overset{(0)}{V}_{3,12}$ in the third governing equation derived in conjunction with the infinitesimal displacement theory (see Librescu and Reddy [15]). In equations (2.74) and (2.76) the quantities $\tilde{E}_{\alpha\beta\omega\pi}$, $E_{\alpha\beta 33}$, $E_{\omega 3\lambda 3}$, E_{3333} and $\frac{E_{\alpha\beta 33}}{E_{3333}}$ are constants defined by (2.20c), (2.20d). They are the elastic constants which coincide with their viscoelastic counterparts at time t = 0 (see equations (2.20) and (2.62)).

Furthermore we also observe that setting the time derivatives of material properties to zero in equations (2.64) and (2.71) in conjunction with (2.20), (2.62) yields equations (2.74) and (2.76), respectively. Also note that in (2.71) the edge loads are assumed constant and hence are equal to $\overset{(0)}{L}_{11}, \overset{(0)}{L}_{22}$ and $\overset{(0)}{L}_{12}$, whereas in (2.76) this restriction is not necessarily present so that $\overset{(0)}{L}_{11}, \overset{(0)}{L}_{22}, \overset{(0)}{L}_{12}$ are functions of $[x_\alpha, x_3; t]$ and don't take on the meaning of edge loads. However for the following analysis we assume constant edge loads.

Now using the correspondence principle (as stated in Sec. 2.3.4) in equations (2.74), (2.75), (2.76) we obtain,

$$
\begin{aligned}
&\vec{E}_{\alpha\beta\omega\pi}\, \overline{\overset{(0)}{V}}_{3,\,\omega\pi\beta} - 4\, \vec{E}_{\alpha\beta\omega\pi}\, \overline{\overset{(1)}{V}}_{\omega,\,\pi\beta} + 4\, \delta_A \frac{\vec{E}_{\alpha\beta 33}}{\vec{E}_{3333}} \vec{E}_{\omega 3\lambda 3} \left(\overline{\overset{(1)}{V}}_{\lambda,\,\omega\beta} + \overline{\overset{(0)}{V}}_{3,\,\omega\lambda\beta} \right) \qquad (2.77) \\
&- 5\, \delta_A\, \delta_B \frac{\vec{E}_{\alpha\beta 33}}{\vec{E}_{3333}} \rho \left(s^2\, \overline{\overset{(0)}{V}}_{3,\,\beta} - s\, \overset{(0)}{V}_{3,\,\beta}[0] - \overset{(\dot{0})}{V}_{3,\,\beta}[0] \right) + \frac{40}{h^2} \vec{E}_{\alpha 3\lambda 3}\, \overline{\overset{(1)}{V}}_{\lambda} \\
&+ \frac{40}{h^2} \vec{E}_{\alpha 3\lambda 3}\, \overline{\overset{(0)}{V}}_{3,\,\lambda} + \frac{60}{h^3} \delta_C\, \overline{\overset{(1)}{f}}_{\alpha} = 0
\end{aligned}
$$

where,

$$
\overline{\overset{(1)}{f}}_{\alpha} = \rho \left(\frac{h^3}{15} \left[s^2\, \overline{\overset{(1)}{V}}_{\alpha} - s\, \overset{(1)}{V}_{\alpha}[0] - \overset{(\dot{1})}{V}_{\alpha}[0] \right] - \frac{h^3}{60} \left[s^2\, \overline{\overset{(0)}{V}}_{3,\,\alpha} - s\, \overset{(0)}{V}_{3,\,\alpha}[0] - \overset{(\dot{0})}{V}_{3,\,\alpha}[0] \right] \right) \qquad (2.78)
$$

$$
\begin{aligned}
&\frac{2}{3} h\, \vec{E}_{\omega 3\lambda 3} \left(\overline{\overset{(1)}{V}}_{\lambda,\,\omega} + \overline{\overset{(0)}{V}}_{3,\,\lambda\omega} \right) + \overset{(0)}{L}_{11}\, \overline{\overset{(0)}{V}}_{3,11} + \overset{(0)}{L}_{22}\, \overline{\overset{(0)}{V}}_{3,22} + 2\, \overset{(0)}{L}_{12}\, \overline{\overset{(0)}{V}}_{3,12} \\
&\qquad - \rho_0 h \left(s^2\, \overline{\overset{(0)}{V}}_{3} - s\, \overset{(0)}{V}_{3}[0] - \overset{(\dot{0})}{V}_{3}[0] \right) = 0
\end{aligned}
\qquad (2.79)
$$

where in writing (2.79) the stress resultants $\overset{(0)}{L}_{11}, \overset{(0)}{L}_{22}, \overset{(0)}{L}_{12}$ are constant and could be interpreted as the edge loads as noted above.

Now inverting (2.77), (2.78), (2.79) into the time domain in conjunction with (2.18), (2.20), (2.62) and Borel's theorem, we obtain:

$$
\begin{aligned}
&\int_0^t \dot{\tilde{E}}_{\alpha\beta\omega\pi}[t-\tau]\, \overset{(0)}{V}_{3,\,\omega\pi\beta}[\tau]d\tau + \tilde{E}_{\alpha\beta\omega\pi}[0]\, \overset{(0)}{V}_{3,\,\omega\pi\beta}[t] - 4\int_0^t \dot{\tilde{E}}_{\alpha\beta\omega\pi}[t-\tau]\, \overset{(1)}{V}_{\omega,\,\pi\beta}[\tau]\, d\tau \\
&- 4\tilde{E}_{\alpha\beta\omega\pi}[0]\, \overset{(1)}{V}_{\omega,\,\pi\beta}[t] + 4\,\delta_A \int_0^t \dot{\hat{E}}_{\alpha\beta\omega\lambda}[t-\tau]\left[\overset{(1)}{V}_{\lambda,\,\omega\beta}[\tau] + \overset{(0)}{V}_{3,\,\omega\lambda\beta}[\tau]\right] d\tau \\
&+ 4\,\delta_A\, \hat{E}_{\alpha\beta\omega\pi}[0]\left[\overset{(1)}{V}_{\lambda,\,\omega\beta}[t] + \overset{(0)}{V}_{3,\,\omega\lambda\beta}[t]\right] - 5\,\delta_A\,\delta_B\,\rho \int_0^t \dot{\hat{E}}_{\alpha\beta 33}[t-\tau]\, \overset{(\ddot{0})}{V}_{3,\,\beta}[\tau]\, d\tau \qquad (2.80) \\
&- 5\,\delta_A\,\delta_B\,\rho\, \hat{E}_{\alpha\beta 33}[0]\, \overset{(\ddot{0})}{V}_{3,\,\beta}[t] + \frac{40}{h^2}\int_0^t \dot{E}_{\alpha 3\lambda 3}[t-\tau]\left[\overset{(1)}{V}_{\lambda}[\tau] + \overset{(0)}{V}_{3,\,\lambda}[\tau]\right] d\tau \\
&+ \frac{40}{h^2} E_{\alpha 3\lambda 3}[0]\left[\overset{(1)}{V}_{\lambda}[t] + \overset{(0)}{V}_{3,\,\lambda}[t]\right] + \frac{60}{h^3}\,\delta_C\, \overset{(1)}{f}_{\alpha} = 0
\end{aligned}
$$

and,

$$
\begin{aligned}
&\frac{2}{3} h \int_0^t \dot{E}_{\omega 3\lambda 3}[t-\tau]\left[\overset{(1)}{V}_{\lambda,\,\omega}[\tau] + \overset{(0)}{V}_{3,\,\lambda\omega}[\tau]\right] d\tau \qquad (2.81) \\
&+ \overset{(0)}{L}_{11}\, \overset{(0)}{V}_{3,11}[t] + \overset{(0)}{L}_{22} V_{3,22}[t] + 2\,\overset{(0)}{L}_{12}\, \overset{(0)}{V}_{3,12}[t] - \delta_D\, \rho_0\, h\, \overset{(\ddot{0})}{V}_3 = 0
\end{aligned}
$$

In eqn. (2.80) $\overset{(1)}{f}_{\alpha}$ is as defined in (2.75). Equations (2.80), (2.81) are the governing equations of stability for the orthotropic, linearly viscoelastic, flat plate subject to constant inplane edge loads and obtained by using the correspondence principle. The comparison of (2.80) and (2.81) with (2.64) and (2.71) shows that the two sets of governing equations coincide. This proves again that the C.P. is a powerful tool for analyzing problems in linear viscoelasticity, and will be used in developing the stability problem in the framework of the first order transverse shear deformation theory (FSDT).

2.4 Equations Governing the Stability of Viscoelastic Flat Plates Using a First Order Transverse Shear Deformation Theory (FSDT)

In the framework of this theory, the following representation of the displacement field is postulated:

$$V_\alpha = x_3 \overset{(1)}{V}_\alpha \tag{2.82a}$$

$$V_3 = \overset{(0)}{V}_3 \tag{2.82b}$$

Disregarding the influence of the transverse normal stress σ_{33} in the constitutive equations (2.17) and making use of the general procedure developed in Sec. 2.3.1 - 2.3.3, the governing equations of stability may be derived for a viscoelastic flat plate. However, employment of the procedure developed in Sec. 2.3.5 yields an identical system of governing equations, as was shown in Sec. 2.3.5. Thus, using the latter method, we proceed in the following manner. The governing equations for the elastic plate using the FSDT theory and the infinitesimal strain assumption were obtained by Librescu [8] (see also Librescu and Reddy [15]) and are written as,

$$\frac{h^3}{12}\tilde{E}_{\alpha\beta\mu\rho}\overset{(1)}{V}_{\mu,\rho\alpha} - K^2 h E_{\beta 3\lambda 3}\left(\overset{(1)}{V}_\lambda + \overset{(0)}{V}_{3,\lambda}\right) + \overset{(1)}{p}_\beta - \delta_c m_1 \overset{(\ddot{1})}{V}_\beta = 0 \tag{2.83a}$$

$$K^2 h E_{\beta 3\lambda 3}\left(\overset{(1)}{V}_{\lambda,\beta} + \overset{(0)}{V}_{3,\lambda\beta}\right) + \overset{(0)}{p}_3 - \delta_D m_0 \overset{(\ddot{0})}{V}_3 = 0 \tag{2.83b}$$

where K^2 is the transverse shear correction factor associated with the transverse shear moduli $E_{\beta 3\lambda 3}$; $m_0 = \rho h$ and $m_1 = \dfrac{\rho h^3}{12}$.

For the case when the inplane surface tractions are neglected (i.e., $\overset{(1)}{p}_\beta = 0$), we obtain the equations corresponding to (2.83) for finite displacements by replacing $\overset{(0)}{p}_3$ by $\overset{(0)}{p}_3 + \overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11} + \overset{(0)}{L}_{22}\overset{(0)}{V}_{3,22} + 2\,\overset{(0)}{L}_{12}\overset{(0)}{V}_{3,12}$. Furthermore, neglecting transverse normal loads (i.e., $\overset{(0)}{p}_3 = 0$), we obtain the following equations governing the stability of the orthotropic, elastic plate:

$$\frac{h^3}{12}\tilde{E}_{\alpha\beta\mu\rho}\overset{(1)}{V}_{\mu,\rho\alpha} - K^2\,h\,E_{\beta3\lambda3}\left(\overset{(1)}{V}_\lambda + \overset{(0)}{V}_{3,\lambda}\right) - \delta_c\,m_1\,\overset{(\ddot{1})}{V}_\beta = 0 \tag{2.84a}$$

and

$$K^2\,h\,E_{\beta3\lambda3}\left(\overset{(1)}{V}_{\lambda,\beta} + \overset{(0)}{V}_{3,\lambda\beta}\right) + \overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11} + \overset{(0)}{L}_{22}\overset{(0)}{V}_{3,12} + 2\overset{(0)}{L}_{12}\overset{(0)}{V}_{3,12} - \delta_D\,m_0\,\overset{(\ddot{0})}{V}_3 = 0 \tag{2.84b}$$

Using the correspondence principle in (2.84) for the case of constant edge loads, we have:

$$\frac{h^3}{12}\vec{\tilde{E}}_{\alpha\beta\mu\rho}\overline{\overset{(1)}{V}}_{\mu,\rho\alpha} - K^2\,h\,\vec{E}_{\beta3\lambda3}\left(\overline{\overset{(1)}{V}}_\lambda + \overline{\overset{(0)}{V}}_{3,\lambda}\right) - \delta_c\,m_1\left(s^2\,\overline{\overset{(1)}{V}}_\beta - s\,\overset{(1)}{V}_\beta[0] - \overset{(\dot{1})}{V}_\beta[0]\right) = 0 \tag{2.85a}$$

and

$$\begin{aligned} K^2\,h\,\vec{E}_{\beta3\lambda3}\left(\overline{\overset{(1)}{V}}_{\lambda,\beta} + \overline{\overset{(0)}{V}}_{3,\lambda\beta}\right) &+ \overset{(0)}{L}_{11}\overline{\overset{(0)}{V}}_{3,11} + \overset{(0)}{L}_{22}\overline{\overset{(0)}{V}}_{3,22} + 2\,\overset{(0)}{L}_{12}\overline{\overset{(0)}{V}}_{3,12} \\ &- \delta_D\,m_0\left(s^2\,\overline{\overset{(0)}{V}}_3 - s\,\overset{(0)}{V}_3[0] - \overset{(\dot{0})}{V}_3[0]\right) = 0 \end{aligned} \tag{2.85b}$$

Equations (2.85) represent the Laplace transformed equations governing the stability of an orthotropic, viscoelastic, flat plate subject to constant inplane edge loads and no surface loads present. As noted earlier, in (2.85b) the stress resultants represent the edge loads as the latter are assumed constant.

Inverting (2.85) into the time domain in conjunction with (2.18), (2.20) and Borel's theorem, we obtain:

$$\frac{h^3}{12}\int_0^t \dot{\tilde{E}}_{\alpha\beta\mu\rho}[t-\tau]\,\overset{(1)}{V}_{\mu,\,\rho\alpha}[\tau]\,d\tau + \frac{h^3}{12}\,\tilde{E}_{\alpha\beta\mu\rho}[0]\,\overset{(1)}{V}_{\mu,\,\rho\alpha}[t]$$

$$- K^2 h\int_0^t \dot{E}_{\beta3\lambda3}[t-\tau]\left[\overset{(1)}{V}_{\lambda}[\tau] + \overset{(0)}{V}_{3,\,\lambda}[\tau]\right]d\tau - K^2 h\, E_{\beta3\lambda3}[0]\left[\overset{(1)}{V}_{\lambda}[\tau] + \overset{(0)}{V}_{3,\,\lambda}[t]\right] - \delta_c\, m_1\, \overset{(\ddot{1})}{V}_{\beta} = 0 \tag{2.86a}$$

and

$$K^2 h\int_0^t \dot{E}_{\beta3\lambda3}[t-\tau]\left[\overset{(1)}{V}_{\lambda,\,\beta}[\tau] + \overset{(0)}{V}_{3,\,\lambda\beta}[\tau]\right]d\tau + K^2 h\, E_{\beta3\lambda3}[0]\left[\overset{(1)}{V}_{\lambda,\,\beta}[t] + \overset{(0)}{V}_{3,\,\lambda\beta}[t]\right]$$

$$+ \overset{(0)}{L}_{11}\,\overset{(0)}{V}_{3,11} + \overset{(0)}{L}_{22}\,\overset{(0)}{V}_{3,22} + 2\,\overset{(0)}{L}_{12}\,\overset{(0)}{V}_{3,12} - \delta_D\, m_0\, \overset{(\ddot{0})}{V}_{3} = 0 \tag{2.86b}$$

Equations (2.86a and b) are the equations governing the stability of an orthotropic viscoelastic flat plate subject to constant inplane edge loads . The problem of determining the transverse shear correction factor K^2 is discussed later in Sec. 2.6.

2.5 Alternative Representation of the Governing Equations (2.86) for the FSDT Theory

We now make use of the procedure developed by Librescu (see Librescu [8, 22, 9]) allowing one to recast the stability problem governed by (2.86a and b) in terms of two independent equations, for a transversely isotropic body. For a transversely isotropic, elastic material we have (see Librescu [8], pp. 402-403),

$$\tilde{E}_{\alpha\beta\mu\rho} = \frac{E}{1+\nu}\left[\frac{1}{2}\left(\delta_{\alpha\mu}\,\delta_{\beta\rho} + \delta_{\mu\beta}\,\delta_{\alpha\rho}\right) + \frac{\nu}{1-\nu}\,\delta_{\mu\rho}\,\delta_{\alpha\beta}\right] \tag{2.87a}$$

$$E_{\beta 3\lambda 3} = G' \, \delta_{\beta\lambda} \tag{2.87b}$$

where E, ν denote the Young's modulus and Poisson's ratio corresponding to the plane of isotropy and E', ν', G' are the Young's modulus, Poisson's ratio and transverse shear modulus for the plane normal to the plane of isotropy.

Using the correspondence principle in (2.87), we obtain:

$$\tilde{\vec{E}}_{\alpha\beta\mu\rho} = \frac{\vec{E}}{1+\bar{\nu}^*}\left[\frac{1}{2}\left(\delta_{\alpha\mu}\,\delta_{\beta\rho} + \delta_{\mu\beta}\,\delta_{\alpha\rho}\right) + \frac{\bar{\nu}^*}{1-\bar{\nu}^*}\,\delta_{\mu\rho}\,\delta_{\alpha\beta}\right] \tag{2.88a}$$

$$\vec{E}_{\beta 3\lambda 3} = \overline{G'}^{*} \, \delta_{\beta\lambda} \tag{2.88b}$$

Introducing (2.88) for a transversely isotropic body in (2.85), we obtain,

$$\begin{aligned} &\frac{h^3}{24}\,\frac{\vec{E}}{1+\bar{\nu}^*}\left[\overset{\overline{(1)}}{V}_{\beta,\,\mu\mu} - \overset{\overline{(1)}}{V}_{\mu,\,\mu\beta}\right] + \frac{h^3}{12}\,\frac{\vec{E}}{1-(\bar{\nu}^*)^2}\,\overset{\overline{(1)}}{V}_{\mu,\,\mu\beta} - K^2 h\,\overline{G'}^{*}\left(\overset{\overline{(1)}}{V}_{\beta} + \overset{\overline{(0)}}{V}_{3,\,\beta}\right) \\ &\qquad - \delta_C\, m_1\left(s^2\,\overset{\overline{(1)}}{V}_{\beta} - s\,\overset{(1)}{V}_{\beta}[0] - \overset{(\dot{1})}{V}_{\beta}[0]\right) = 0 \end{aligned} \tag{2.89a}$$

$$\begin{aligned} \overset{\overline{(1)}}{V}_{\lambda,\,\lambda} = &\frac{1}{K^2\,\overline{G'}^{*}\,h}\left(\delta_D\, m_0\left[s^2\,\overset{\overline{(0)}}{V}_3 - s\,\overset{(0)}{V}_3[0] - \overset{(\dot{0})}{V}_3[0]\right] - \overset{(0)}{L}_{11}\,\overset{\overline{(0)}}{V}_{3,11} - \overset{(0)}{L}_{22}\,\overset{\overline{(0)}}{V}_{3,22}\right. \\ &\qquad \left. - 2\,\overset{(0)}{L}_{12}\,\overset{\overline{(0)}}{V}_{3,12}\right) - \overset{\overline{(0)}}{V}_{3,\,\lambda\lambda} = 0 \end{aligned} \tag{2.89b}$$

Introducing the 2-D permutation symbol $\varepsilon_{\alpha\beta}$ (where $\varepsilon_{12} = -\varepsilon_{21} = 1$, $\varepsilon_{11} = \varepsilon_{22} = 0$), we may write the following result:

$$\overset{\overline{(1)}}{V}_{\beta,\,\mu\mu} - \overset{\overline{(1)}}{V}_{\mu,\,\mu\beta} = \varepsilon_{\beta\gamma}\,\varepsilon_{\mu\lambda}\,\overset{\overline{(1)}}{V}_{\mu,\,\lambda\gamma} \tag{2.90}$$

Introducing (2.89b) in (2.89a) in conjunction with (2.90), we obtain:

$$\frac{h^3}{24}\,\frac{\vec{E}}{1+\bar{\nu}^*}\,\varepsilon_{\beta\gamma}\,\varepsilon_{\mu\lambda}\,\overset{\overline{(1)}}{V}_{\mu,\,\lambda\gamma} + \frac{h^3}{12}\,\frac{\vec{E}}{1-(\bar{\nu}^*)^2}\left[\frac{1}{K^2\,h\,\overline{G'}^{*}}\left(\delta_D\, m_0\left[s^2\,\overset{\overline{(0)}}{V}_{3,\,\beta} - s\,\overset{(0)}{V}_{3,\,\beta}[0] - \overset{(\dot{0})}{V}_{3,\,\beta}[0]\right]\right.\right.$$

$$
- \overset{(0)}{L}_{11} \overline{\overset{(0)}{V}}_{3,11\beta} - \overset{(0)}{L}_{22} \overline{\overset{(0)}{V}}_{3,22\beta} - 2 \overset{(0)}{L}_{12} \overline{\overset{(0)}{V}}_{3,12\beta} \Bigg) - \overline{\overset{(0)}{V}}_{3,\lambda\lambda\beta} \Bigg]
$$

$$
- K^2 h \overline{G'^*} \left(\overline{\overset{(1)}{V}}_\beta + \overline{\overset{(0)}{V}}_{3,\beta} \right) - \delta_c m_1 \left(s^2 \overline{\overset{(1)}{V}}_\beta - s \overset{(1)}{V}_\beta[0] - \dot{\overset{(1)}{V}}_\beta[0] \right) = 0 \tag{2.91}
$$

Equation (2.91) represents the L.T. of the first two equations governing the stability of a transversely isotropic plate. Solving for $\overline{\overset{(1)}{V}}_\beta$ from (2.91), we obtain,

$$
\overline{\overset{(1)}{V}}_\beta (K^2 h \overline{G'^*} + \delta_c m_1 s^2) = \frac{h^3}{24} \frac{\overline{E}^*}{1 + \overline{\nu}^*} \varepsilon_{\beta\gamma} \varepsilon_{\mu\lambda} \overline{\overset{(1)}{V}}_{\mu,\lambda\gamma} \tag{2.92}
$$

$$
+ \frac{h^3}{12} \frac{\overline{E}^*}{1 - (\overline{\nu}^*)^2} \left[\frac{1}{K^2 h \overline{G'^*}} \left(\delta_D m_0 \left[s^2 \overline{\overset{(0)}{V}}_{3,\beta} - s \overset{(0)}{V}_{3,\beta}[0] - \dot{\overset{(0)}{V}}_{3,\beta}[0] \right] \right. \right.
$$

$$
\left. \left. - \overset{(0)}{L}_{11} \overline{\overset{(0)}{V}}_{3,11\beta} - \overset{(0)}{L}_{22} \overline{\overset{(0)}{V}}_{3,22\beta} - 2 \overset{(0)}{L}_{12} \overline{\overset{(0)}{V}}_{3,12\beta} \right) - \overline{\overset{(0)}{V}}_{3,\lambda\lambda\beta} \right]
$$

$$
- K^2 h \overline{G'^*} \, \overline{\overset{(0)}{V}}_{3,\beta} + \delta_c m_1 (s \overset{(1)}{V}_\beta[0] + \dot{\overset{(1)}{V}}_\beta[0])
$$

Now we define the potential function $\phi = \phi\,[x_1, x_2, t]$ as follows:

$$
- \varepsilon_{\omega\beta} \overline{\phi}_{,\omega} \triangleq \frac{h^3}{24} \frac{\overline{E}^*}{1 + \overline{\nu}^*} \varepsilon_{\beta\gamma} \varepsilon_{\mu\lambda} \overline{\overset{(1)}{V}}_{\mu,\lambda\gamma} \tag{2.93}
$$

Inverting (2.92) into the time domain, we obtain:

$$
-\varepsilon_{\omega\beta} \phi_{,\omega} = \frac{h^3}{24} \varepsilon_{\beta\gamma} \varepsilon_{\mu\lambda} \int_0^t 2\dot{G}[t-\tau] \overset{(1)}{V}_{\mu,\lambda\gamma}[\tau] d\tau + \frac{h^3}{24} \varepsilon_{\beta\gamma} \varepsilon_{\mu\lambda} \, 2G[0] \overset{(1)}{V}_{\mu,\lambda\gamma}[t] \tag{2.94a}
$$

where we define

$$
2G[t-\tau] \triangleq \mathscr{L}^{-1} \left\{ \frac{1}{s} \left(\frac{\overline{E}^*}{1 + \overline{\nu}^*} \right) \right\} \quad \text{and} \quad 2G[0] = \frac{E[0]}{1 + \nu[0]} \tag{2.94b}
$$

Introducing (2.93) into (2.92), we obtain the following result:

$$\overline{\overset{(1)}{V}}_{\beta}(K^2h\overline{G'}^{*} + \delta_c m_1 s^2) = -\varepsilon_{\omega\beta}\overline{\phi}_{,\omega} \tag{2.95}$$

$$+\frac{h^3}{12}\frac{\vec{E}^{*}}{1-(\overline{v}^{*})^2}\Bigg[\frac{1}{K^2h\overline{G'}^{*}}\Bigg(\delta_D m_O\Big[s^2\overline{\overset{(0)}{V}}_{3,\beta} - s\overset{(0)}{V}_{3,\beta}[0] - \overset{(\dot{0})}{V}_{3,\beta}[0]\Big]$$

$$-\overset{(0)}{L}_{11}\overline{\overset{(0)}{V}}_{3,11\beta} - \overset{(0)}{L}_{22}\overline{\overset{(0)}{V}}_{3,22\beta} - 2\overset{(0)}{L}_{12}\overline{\overset{(0)}{V}}_{3,12\beta}\Bigg) - \overline{\overset{(0)}{V}}_{3,\lambda\lambda\beta}\Bigg]$$

$$-K^2h\overline{G'}^{*}\overline{\overset{(0)}{V}}_{3,\beta} + \delta_c m_1\Big(s\overset{(1)}{V}_{\beta}[0] + \overset{(\dot{1})}{V}_{\beta}[0]\Big)$$

Introducing (2.95) in (2.93) and using the properties of the permutation symbol $\varepsilon_{\alpha\beta}$ results in:

$$\frac{1}{(K^2h\overline{G'}^{*} + \delta_c m_1 s^2)}\frac{h^3}{24}\frac{\vec{E}^{*}}{1+\overline{v}^{*}}[\varepsilon_{\beta\gamma}\varepsilon_{\mu\lambda}\delta_c m_1(s\overset{(1)}{V}_{\mu,\lambda\gamma}[0] + \overset{(\dot{1})}{V}_{\mu,\lambda\gamma}[0]) + \varepsilon_{\beta\gamma}\overline{\phi}_{,\omega\omega\gamma}] \tag{2.96}$$

$$+\varepsilon_{\omega\beta}\overline{\phi}_{,\omega} = 0$$

Multiplying throughout by $(K^2h\overline{G'}^{*} + \delta_c m_1 s^2)$, we obtain,

$$\frac{h^3}{24}\frac{\vec{E}^{*}}{1+\overline{v}^{*}}[\delta_c m_1(s\overset{(1)}{V}_{\beta,\gamma\gamma}[0] - s\overset{(1)}{V}_{\gamma,\beta\gamma}[0] + \overset{(\dot{1})}{V}_{\beta,\gamma\gamma}[0] - \overset{(\dot{1})}{V}_{\gamma,\beta\gamma}[0]) + \varepsilon_{\beta\gamma}\overline{\phi}_{,\omega\omega\gamma}] \tag{2.97}$$

$$+K^2h\overline{G'}^{*}\varepsilon_{\omega\beta}\overline{\phi}_{,\omega} + \delta_c m_1 s^2\varepsilon_{\omega\beta}\overline{\phi}_{,\omega} = 0$$

Inverting (2.97) to the time domain and assuming zero initial conditions, i.e., $\overset{(1)}{V}_{\beta}[0] = \overset{(\dot{1})}{V}_{\beta}[0] = 0$ in conjunction with (2.94a) and (2.94b), we obtain:

$$\frac{h^3}{24}\varepsilon_{\beta\gamma}\int_0^t 2\dot{G}[t-\tau]\phi_{,\omega\omega\gamma}[\tau]d\tau + \frac{h^3}{24}\varepsilon_{\beta\gamma}2G[0]\phi_{,\omega\omega\gamma}[t] \tag{2.98}$$

$$+ K^2 h \varepsilon_{\omega\beta} \int_0^t \dot{G}'[t-\tau]\phi_{,\omega}[\tau] + K^2 h \varepsilon_{\omega\beta} G'[0]\phi_{,\omega}[t] + \delta_c m_1 \varepsilon_{\omega\beta} \ddot{\phi}_{,\omega} = 0$$

Using the property $\varepsilon_{\gamma\beta} = -\varepsilon_{\beta\gamma}$ in (2.98), we obtain

$$-\frac{h^3}{24}\int_0^t 2\dot{G}[t-\tau]\phi_{,\pi\pi\omega}[\tau]d\tau - \frac{h^3}{24}2G[0]\phi_{,\pi\pi\omega}[t] + K^2 h \int_0^t \dot{G}'[t-\tau]\phi_{,\omega}[\tau]d\tau \tag{2.99}$$

$$+ K^2 h G'[0]\phi_{,\omega}[t] + \delta_c m_1 \ddot{\phi}_{,\omega} = 0$$

Integrating (2.99) w.r.t x_ω yields:

$$-\frac{h^3}{24}\int_0^t 2\dot{G}[t-\tau]\phi_{,\pi\pi}[\tau]d\tau - \frac{h^3}{24}2G[0]\phi_{,\pi\pi}[t] + K^2 h \int_0^t \dot{G}'[t-\tau]\phi[\tau]d\tau \tag{2.100}$$

$$+ K^2 h G'[0]\phi[t] + \delta_c m_1 \ddot{\phi} = 0$$

Equation (2.100) represents the first equation governing the stability of the transversely isotropic plate. We observe that it is an equation expressed in terms of the potential function $\phi[x_1, x_2, t]$ only.

Introducing (2.95) in (2.89b) and assuming zero initial conditions for the displacement quantities, we obtain (also note that $\varepsilon_{\omega\beta}\overline{\phi}_{,\omega\beta} = 0$):

$$\frac{1}{(K^2 h\overline{G'^*} + \delta_c m_1 s^2)} \frac{h^3}{12} \frac{\overline{E}^*}{1-(\overline{v}^*)^2}\left[\frac{1}{K^2\overline{G'^*}h}\left(\delta_D m_0 s^2 \overline{V}_{3,\beta\beta}\right.\right.$$

$$\left.\left. - \overset{(0)}{L}_{11}\overset{\overline{(0)}}{V}_{3,11\beta\beta} - \overset{(0)}{L}_{22}\overset{\overline{(0)}}{V}_{3,22\beta\beta} - 2\overset{(0)}{L}_{12}\overset{\overline{(0)}}{V}_{3,12\beta\beta}\right) - \overset{\overline{(0)}}{V}_{3,\lambda\lambda\beta\beta}\right]$$

$$-\frac{K^2 h\overline{G'^*}\,\overset{\overline{(0)}}{V}_{3,\beta\beta}}{(K^2 h\overline{G'^*}+\delta_c m_1 s^2)}+\overset{\overline{(0)}}{V}_{3,\beta\beta}-\frac{1}{K^2 h\overline{G'^*}}\delta_D m_0 s^2\overset{\overline{(0)}}{V}_3$$

$$+\frac{1}{K^2 h\overline{G'^*}}(\overset{(0)}{L}_{11}\overset{\overline{(0)}}{V}_{3,11}+L_{22}\overset{\overline{(0)}}{V}_{3,22}+2L_{12}\overset{\overline{(0)}}{V}_{3,12})=0$$

Multiplying throughout by $(K^2h\overline{G'^*}+\delta_c m_1 s^2)$, we obtain:

$$\frac{h^3}{12}\,\frac{\vec{E^*}}{1-(\overset{*}{v})^2}\left[\frac{1}{K^2\overline{G'^*}h}\left(\delta_D m_0 s^2\overline{V}_{3,\beta\beta}-\overset{(0)}{L}_{11}\overset{\overline{(0)}}{V}_{3,11\beta\beta}-\overset{(0)}{L}_{22}\overset{\overline{(0)}}{V}_{3,22\beta\beta}-2\overset{(0)}{L}_{12}\overset{\overline{(0)}}{V}_{3,12\beta\beta}\right)-\overset{(0)}{V}_{3,\lambda\lambda\beta\beta}\right]$$

$$+\delta_c m_1 s^2\overset{\overline{(0)}}{V}_{3,\beta\beta}-\delta_D m_0 s^2\overset{\overline{(0)}}{V}_3-\frac{\delta_c\delta_D m_1 m_0 s^4}{K^2h\overline{G'^*}}\overset{\overline{(0)}}{V}_3+\overset{(0)}{L}_{11}\overset{\overline{(0)}}{V}_{3,11}+\overset{(0)}{L}_{22}\overset{\overline{(0)}}{V}_{3,22}+2\overset{(0)}{L}_{12}\overset{\overline{(0)}}{V}_{3,12}$$

$$+\frac{1}{K^2h\overline{G'^*}}\delta_c m_1 s^2(\overset{(0)}{L}_{11}\overset{\overline{(0)}}{V}_{3,11}+\overset{(0)}{L}_{22}\overset{\overline{(0)}}{V}_{3,22}+2\overset{(0)}{L}_{12}\overset{\overline{(0)}}{V}_{3,12})=0 \qquad (2.101)$$

Inverting (2.101) into the time domain for zero initial conditions, we obtain:

$$\int_0^t \dot{D}[t-\tau]\overset{(0)}{V}_{3,\lambda\lambda\beta\beta}[\tau]d\tau+D[0]\overset{(0)}{V}_{3,\lambda\lambda\beta\beta}[t]-(\overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11}[t]+\overset{(0)}{L}_{22}\overset{(0)}{V}_{3,22}[t]+2\overset{(0)}{L}_{12}\overset{(0)}{V}_{3,12}[t]) \qquad (2.102a)$$

$$+\frac{2}{3K^2}\int_0^t \dot{C}_3[t-\tau](\overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11}[\tau]+\overset{(0)}{L}_{22}\overset{(0)}{V}_{3,22}[\tau]+2\overset{(0)}{L}_{12}\overset{(0)}{V}_{3,12}[\tau])_{,\beta\beta}d\tau$$

$$+\frac{2}{3K^2}C_3[0](\overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11}[t]+L_{22}\overset{(0)}{V}_{3,22}[t]+2\overset{(0)}{L}_{12}\overset{(0)}{V}_{3,12}[t])_{,\beta\beta}$$

$$+\delta_D m_0[\ddot{V}_3(0)-\frac{2}{3K^2}\int_0^t\dot{C}_3[t-\tau]\ddot{V}_{3,\beta\beta}[\tau]d\tau-\frac{2}{3K^2}C_3(0)\ddot{V}_{3,\beta\beta}[t]]$$

$$- \delta_c \frac{m_1}{2K^2}[\int_0^t \dot{C}_2[t-\tau](\overset{(0)}{L}_{11}\overset{(\ddot{0})}{V}_{3,\,11}[t] + \overset{(0)}{L}_{22}\overset{(\ddot{0})}{V}_{3,\,22}[\tau] + 2\overset{(0)}{L}_{12}\overset{(\ddot{0})}{V}_{3,\,12}[\tau])d\tau$$

$$+ C_2\,(0)(\overset{(0)}{L}_{11}\overset{(\ddot{0})}{V}_{3,\,11}[t] + \overset{(0)}{L}_{22}\overset{(0)}{V}_{3,\,22}[t] + 2\overset{(0)}{L}_{12}\overset{(\ddot{0})}{V}_{3,\,12}\,[t])]$$

$$+ \delta_c\delta_D \frac{m_1 m_0}{2K^2}[\int_0^t \dot{C}_2[t-\tau]\overset{(\ddot{0})}{V}_3\,[\tau]d\tau + C_2\,(0)\overset{(\ddot{0})}{V}_3[t]] - \delta_c m_1 \overset{(\ddot{0})}{V}_{3,\,\beta\beta} = 0$$

where we have defined,

$$D[t] \triangleq \mathcal{L}^{-1}\{\frac{1}{s}(\frac{h^3}{12}\,\frac{\overline{E}^*}{1-(\overline{v}^*)^2})\} \quad \text{with} \quad D[0] = \frac{h^3}{12}(\frac{E(0)}{1-v^2[0]}) \tag{2.102b}$$

$$C_3\,[t] = \mathcal{L}^{-1}\left\{\frac{h^2}{8}\frac{1}{s}\left(\frac{\overline{E}^*}{\overline{G'}^*(1-(\overline{v}^*)^2)}\right)\right\} = \frac{h^2}{4}\mathcal{L}\left\{\frac{1}{s}\left(\frac{\overline{G}^*}{\overline{G'}^*(1-\overline{v}^*)}\right)\right\} \tag{2.102c}$$

with $\quad C_3\,[0] = \dfrac{h^2}{4}\,\dfrac{G[0]}{G'[0](1-v[0])}$ $\quad$ (see Eqn. 2.94b)

$$C_2 = \mathcal{L}^{-1}\{\frac{2}{h}\frac{1}{s}(\frac{1}{\overline{G'}^*})\} \tag{2.102d}$$

Equation (2.102) represents the second equation governing the stability of the transversely isotropic plate subject to constant inplane edge loads. We observe that (2.102) is expressed in terms of the transverse displacement quantity (i.e., $\overset{(0)}{V}_3$) only. Thus for zero initial conditions, the coupled form of the equations (2.86) governing the stability of a transversely isotropic plate (transversal isotropy being a special case of orthotropy) can be recast into two independent equations, (a) one governing the basic state of stress, i.e., the interior solution, (2.100), and (b) the other one governing the boundary layer solution, (2.102). We also observe that setting the time derivatives of material properties to zero, we obtain the corresponding elasticity equations from (2.100) and (2.102) which coincide with those obtained by Librescu [8, 9] (see also Librescu and Reddy [15]).

2.6 Determination of the Transverse Shear Correction Factor K^2

Following a procedure similar to that described in Sec. 2.5, we can derive the counterparts of (2.100) and (2.102) for a transversely isotropic, viscoelastic plate in the framework of the TTSD. These equations were obtained by Librescu and Reddy [15] for the elastic case. Here, they point out that in the absence of inplane surface tractions (i.e., $\overset{(1)}{p}_{\alpha} = 0$) the de-coupled system of equations for the FSDT coincide with those for the TTSD when the transverse shear correction factor $K^2 \rightarrow \frac{5}{6}$ in the FSDT and the transverse normal stress is neglected in the TTSD (i.e., $\delta_A = \delta_B = 0$). The value of K^2 does not change for the viscoelastic case as it is a constant.

CHAPTER 3. DETERMINATION OF MATERIAL PROPERTIES OF A FIBER-REINFORCED COMPOSITE WITH VISCOELASTIC MATERIAL BEHAVIOR

3.1 Introduction

In order to render explicitly the constitutive law for an anisotropic viscoelastic body in a 3-D state of stress (see equation (2.5)), we require the relaxation moduli (E_{ijmn}) as functions of time. Towards this end we seek a suitable micromechanical model that predicts the overall behavior (i.e., effective properties E_{ijmn}) of the unidirectional fiber-reinforced composite in terms of the properties of its constituents (i.e., fiber and matrix).

Wilson [3] determines the effective properties by using the Halpin-Tsai equations, themselves formulated by a strength of materials approach. Other, more rigorous methods have been developed by Aboudi (see [23]. [24], [25]) and Hashin (see Hashin and Rosen [26]). The model of Aboudi [24] was adopted in the treatment of the problem. In the following section, a brief description of this model is presented (see also Appendix[B]).

3.2 Use of Micromechanical Model to Determine Material Properties

By using the results presented in Appendix [B] along with the elastic-viscoelastic correspondence principle (hereafter C.P.), we obtain the relevant micromechanical equations pertaining to a viscoelastic fiber-reinforced composite. Using Eqn. (B-10) and the C.P. for a viscoelastic transversely isotropic matrix yields

$$\overline{E}^{*(m)}_{1111} = \overline{E}^{*(m)}_{A} + 4\overline{K}^{*(m)}(\overline{\nu}^{*(m)}_{A})^2 \tag{3.1a}$$

$$\overline{E}^{*(m)}_{1122} = \overline{E}^{*(m)}_{1133} = 2\overline{K}^{*(m)}\overline{\nu}^{*(m)}_{A} \tag{3.1b}$$

$$\overline{E}^{*(m)}_{2222} = \overline{K}^{*(m)} + \frac{0.5\overline{E}^{*(m)}_{T}}{1+\overline{\nu}^{*(m)}_{T}} \tag{3.1c}$$

$$\overline{E}^{*(m)}_{2233} = \overline{K}^{*(m)} - \frac{0.5\overline{E}^{*(m)}_{T}}{1+\overline{\nu}^{*(m)}_{T}} \tag{3.1d}$$

$$\overline{E}^{*(m)}_{1212} = \overline{G}^{*(m)}_{A} \tag{3.1e}$$

$$\overline{E}^{*(m)}_{2323} = \frac{\overline{E}^{*(m)}_{2222} - \overline{E}^{*(m)}_{2233}}{2} \tag{3.1f}$$

The relevant expressions for viscoelastic, transversely isotropic fibers are obtained from (3.1) by replacing the superscript (m) with (f). Now the resulting constitutive law for an orthotropic body relating average stresses ($\overset{v}{\sigma}_{ij}$) to average strains ($\overset{v}{e}_{ij}$) are from [24]. After using the C.P. for an orthotropic viscoelastic body, we obtain the relevant constitutive law in the L.T. domain. These relations write as,

$$\overset{\bar{v}}{\sigma}_{11} = \vec{E}_{1111}\overset{\bar{v}}{e}_{11} + \vec{E}_{1122}\overset{\bar{v}}{e}_{22} + \vec{E}_{1133}\overset{\bar{v}}{e}_{33} \tag{3.2a}$$

$$\overset{\bar{v}}{\sigma}_{22} = \vec{E}_{1122}\overset{\bar{v}}{e}_{11} + \vec{E}_{2222}\overset{\bar{v}}{e}_{22} + \vec{E}_{2233}\overset{\bar{v}}{e}_{33} \tag{3.2b}$$

$$\overset{\bar{v}}{\sigma}_{33} = \vec{E}_{1133}\overset{\bar{v}}{e}_{11} + \vec{E}_{2233}\overset{\bar{v}}{e}_{22} + \vec{E}_{3333}\overset{\bar{v}}{e}_{33} \tag{3.2c}$$

$$\overset{\bar{v}}{\sigma}_{12} = 2\vec{E}_{1212}\overset{\bar{v}}{e}_{12} \tag{3.2d}$$

$$\overset{\bar{v}}{\sigma}_{13} = 2\vec{E}_{1313}\overset{\bar{v}}{e}_{13} \tag{3.2e}$$

$$\overset{\bar{v}}{\sigma}_{23} = 2\vec{E}_{2323}\overset{\bar{v}}{e}_{23} \tag{3.2f}$$

where (see [26] for the elastic counterpart)

$$\vec{E}_{1111} = \frac{1}{V}\Big[v_{11}\vec{E}^{(f)}_{1111} + \vec{E}^{(m)}_{1111}(v_{12} + v_{21} + v_{22}) + (\vec{E}^{(m)}_{1122} - \vec{E}^{(f)}_{1122})(\vec{Q}_2 + \vec{Q}_3)\Big] \tag{3.3a}$$

$$\vec{E}_{1122} = \frac{1}{V}\left[\frac{h}{h_1}(\vec{E}^{(m)}_{1122}\, v_{12} + \vec{Q}_1\vec{E}^{(m)}_{2222} + \vec{Q}_3\vec{E}^{(m)}_{2233}) + \frac{h}{h_2}(\vec{E}^{(m)}_{1122}\, v_{21} + \vec{Q}_2\vec{E}^{(m)}_{2222} + \vec{Q}_4\vec{E}^{(m)}_{2233})\right] \tag{3.3b}$$

$$\vec{E}_{1133} = \frac{1}{V}\left[\frac{l}{l_1}(\vec{E}^{(m)}_{1122}\, v_{21} + \vec{Q}_2\vec{E}^{(m)}_{2233} + \vec{Q}_4\vec{E}^{(m)}_{2222}) + \frac{l}{l_2}(\vec{E}^{(m)}_{1122}\, v_{12} + \vec{Q}_1\vec{E}^{(m)}_{2233} + \vec{Q}_3\vec{E}^{(m)}_{2222})\right] \tag{3.3c}$$

$$\vec{E}_{2222} = \frac{1}{V}\left[\frac{h}{h_1}(\vec{E}^{(m)}_{2222}\, [v_{12} + \bar{Q}'^{*}_{1}] + \bar{Q}'^{*}_{3}\vec{E}^{(m)}_{2233}) + \frac{h}{h_2}(\vec{E}^{(m)}_{2222}\, [v_{21} + \bar{Q}'^{*}_{2}] + \bar{Q}'^{*}_{4}\vec{E}^{(m)}_{2233})\right] \tag{3.3d}$$

$$\vec{E}_{2233} = \frac{1}{V}\left[\frac{l}{l_1}(\vec{E}^{(m)}_{2233}\, [v_{21} + \bar{Q}'^{*}_{2}] + \bar{Q}'^{*}_{4}\vec{E}^{(m)}_{2222}) + \frac{l}{l_2}(\vec{E}^{(m)}_{2233}\, [v_{12} + \bar{Q}'^{*}_{1}] + \bar{Q}'^{*}_{3}\vec{E}^{(m)}_{2222})\right] \tag{3.3e}$$

$$\vec{E}_{3333} = \frac{1}{V}\left[\frac{l}{l_1}(\vec{E}^{(m)}_{2222}\,[v_{21} + \overline{Q}''^{*}_{4}] + \overline{Q}''^{*}_{2}\vec{E}^{(m)}_{2233}) + \frac{l}{l_2}(\vec{E}^{(m)}_{2222}\,[v_{12} + \overline{Q}''^{*}_{3}] + \overline{Q}''^{*}_{1}\vec{E}^{(m)}_{2233}) \right] \tag{3.3f}$$

$$\vec{E}_{1212} = \frac{1}{V\vec{\Delta}}\left[\vec{E}^{(m)}_{1212}\,[\vec{E}^{(f)}_{1212}\,(h[v_{11} + v_{21}] + h_2[v_{12} + v_{22}]) + \vec{E}^{(m)}_{1212}(v_{12} + v_{22})h_1] \right] \tag{3.3g}$$

$$\vec{E}_{1313} = \frac{1}{V\vec{\Delta}}\left[\vec{E}^{(m)}_{1212}\,[\vec{E}^{(f)}_{1212}\,(l[v_{11} + v_{12}] + l_2[v_{21} + v_{22}]) + \vec{E}^{(m)}_{1212}(v_{21} + v_{22})l_1] \right] \tag{3.3h}$$

$$\vec{E}_{2323} = \frac{1}{\vec{\Delta}_2}\left[\vec{E}^{(f)}_{2323}\,\vec{E}^{(m)}_{2323}\,hl \right] \tag{3.3i}$$

We have the following relations for the terms appearing in (3.3):

$$\begin{aligned} \overline{Q}^{*}_{1} = {} & v_{11}\vec{E}^{(f)}_{1122}\,(\vec{T}_1 + \vec{T}_9) - v_{12}\vec{E}^{(m)}_{1122}\,(\frac{h_2}{h_1}\,\vec{T}_5 + \frac{l_1}{l_2}\vec{T}_9) - v_{21}\vec{E}^{(m)}_{1122}\,(\frac{h_1}{h_2}\,\vec{T}_1 + \frac{l_2}{l_1}\vec{T}_{13}) \\ & + v_{22}\vec{E}^{(m)}_{1122}\,(\vec{T}_5 + \vec{T}_{13}) \end{aligned} \tag{3.4a}$$

$$\begin{aligned} \overline{Q}'^{*}_{1} = {} & v_{11}(\vec{E}^{(f)}_{2222}\,\vec{T}_1 + \vec{E}^{(f)}_{2233}\vec{T}_9) - v_{12}(\frac{h_2}{h_1}\,\vec{E}^{(m)}_{2222}\,\vec{T}_5 + \frac{l_1}{l_2}\vec{E}^{(m)}_{2222}\,\vec{T}_9) \\ & - v_{21}(\frac{h_1}{h_2}\vec{E}^{(m)}_{2222}\,\vec{T}_1 + \frac{l_2}{l_1}\vec{E}^{(m)}_{2233}\,\vec{T}_{13}) + v_{22}(\vec{E}^{(m)}_{2222}\,\vec{T}_5 + \vec{E}^{(m)}_{2233}\,\vec{T}_{13}) \end{aligned} \tag{3.4b}$$

$$\begin{aligned} \overline{Q}''^{*}_{1} = {} & v_{11}(\vec{E}^{(f)}_{2233}\,\vec{T}_1 + \vec{E}^{(f)}_{2222}\vec{T}_9) - v_{12}(\frac{h_2}{h_1}\,\vec{E}^{(m)}_{2233}\,\vec{T}_5 + \frac{l_1}{l_2}\vec{E}^{(m)}_{2222}\,\vec{T}_9) \\ & - v_{21}(\frac{h_1}{h_2}\vec{E}^{(m)}_{2233}\,\vec{T}_1 + \frac{l_2}{l_1}\vec{E}^{(m)}_{2222}\,\vec{T}_{13}) + v_{22}(\vec{E}^{(m)}_{2233}\,\vec{T}_5 + \vec{E}^{(m)}_{2222}\,\vec{T}_{13}) \end{aligned} \tag{3.4c}$$

$$\vec{\Delta} = l_1\vec{E}^{(m)}_{1212} + l_1\vec{E}^{(f)}_{1212} \tag{3.4d}$$

$$\Delta_1 = h_1 l_1 + h_2 l_1 + h_2 l_2 \tag{3.4e}$$

$$\vec{\Delta}_2 = h_1 l_1 \vec{E}^{(m)}_{2323} + \Delta_1 \vec{E}^{(f)}_{2323} \tag{3.4f}$$

In addition $\overline{Q}_2^*, \overline{Q}_3^*, \overline{Q}_4^*$ can be obtained from $\overline{Q}_1^*$ (given by 3.3a) by replacing T_i in the latter by $T_{i+1}, T_{i+2}, T_{i+3}$, respectively. Following the same procedure, we obtain $\overline{Q}''^*_2, \overline{Q}''^*_3, \overline{Q}''^*_4$, from $\overline{Q}''^*_1$, (given by 3.3b) and $\overline{Q}'''^*_2, \overline{Q}'''^*_3, \overline{Q}'''^*_4$, from $\overline{Q}'''^*_1$, (given by 3.3c).

We have the following relations for the terms appearing in (3.4):

$$\vec{T}_1 = -\frac{1}{\vec{D}}(\vec{A}_5\,\vec{A}_8\,\vec{A}_{12} + \vec{A}_6\,\vec{A}_9\,\vec{A}_{11}) \tag{3.5a}$$

$$\vec{T}_2 = -\frac{1}{\vec{D}}(\vec{A}_2\,\vec{A}_8\,\vec{A}_{12} + \vec{A}_3\,\vec{A}_9\,\vec{A}_{11} - \vec{A}_1\,\vec{A}_9\,\vec{A}_{12}) \tag{3.5b}$$

$$\vec{T}_3 = \frac{1}{\vec{D}}(\vec{A}_1\,\vec{A}_5\,\vec{A}_{12} + \vec{A}_2\,\vec{A}_6\,\vec{A}_{11} - \vec{A}_3\,\vec{A}_5\,\vec{A}_{11}) \tag{3.5c}$$

$$\vec{T}_4 = \frac{1}{\vec{D}}(\vec{A}_1\,\vec{A}_6\,\vec{A}_9 + \vec{A}_8\,[\vec{A}_3\,\vec{A}_5 - \vec{A}_2\,\vec{A}_6]) \tag{3.5d}$$

$$\vec{T}_5 = \frac{1}{\vec{D}}(\vec{A}_6\,\vec{A}_9\,\vec{A}_{10} + \vec{A}_{12}\,[\vec{A}_5\,\vec{A}_7 - \vec{A}_4\,\vec{A}_9]) \tag{3.5e}$$

$$\vec{T}_6 = -\frac{1}{\vec{D}}(\vec{A}_2\,\vec{A}_7\,\vec{A}_{12} + \vec{A}_3\,\vec{A}_9\,\vec{A}_{10}) \tag{3.5f}$$

$$\vec{T}_7 = \frac{1}{\vec{D}}(\vec{A}_3\,\vec{A}_5\,\vec{A}_{10} + \vec{A}_2\,[\vec{A}_4\,\vec{A}_{12} - \vec{A}_6\,\vec{A}_{10}]) \tag{3.5g}$$

$$\vec{T}_8 = \frac{1}{\vec{D}}(\vec{A}_2\,\vec{A}_6\,\vec{A}_7 + \vec{A}_3\,[\vec{A}_4\,\vec{A}_9 - \vec{A}_5\,\vec{A}_7]) \tag{3.5h}$$

$$\vec{T}_9 = \frac{1}{\vec{D}}(\vec{A}_4\,\vec{A}_8\,\vec{A}_{12} + \vec{A}_6\,[\vec{A}_7\,\vec{A}_{11} - \vec{A}_8\,\vec{A}_{10}]) \tag{3.5i}$$

$$\vec{T}_{10} = \frac{1}{\vec{D}}(\vec{A}_1\,\vec{A}_7\,\vec{A}_{12} + \vec{A}_3\,[\vec{A}_8\,\vec{A}_{10} - \vec{A}_7\,\vec{A}_{11}]) \tag{3.5j}$$

$$\vec{T}_{11} = \frac{1}{\vec{D}}(\vec{A}_3\,\vec{A}_4\,\vec{A}_{11} + \vec{A}_1\,[\vec{A}_6\,\vec{A}_{10} - \vec{A}_4\,\vec{A}_{12}]) \tag{3.5k}$$

$$\vec{T}_{12} = -\frac{1}{\vec{D}}(\vec{A}_1\,\vec{A}_6\,\vec{A}_7 + \vec{A}_3\,\vec{A}_4\,\vec{A}_8) \tag{3.5l}$$

$$\vec{T}_{13} = \frac{1}{\vec{D}}(\vec{A}_4\,\vec{A}_9\,\vec{A}_{11} + \vec{A}_5\,[\vec{A}_8\,\vec{A}_{10} - \vec{A}_7\,\vec{A}_{11}]) \tag{3.5m}$$

$$\vec{T}_{14} = \frac{1}{\vec{D}}(\vec{A}_1\,\vec{A}_9\,\vec{A}_{10} + \vec{A}_2\,[\vec{A}_7\,\vec{A}_{11} - \vec{A}_8\,\vec{A}_{10}]) \tag{3.5n}$$

$$\vec{T}_{15} = -\frac{1}{\vec{D}}(\vec{A}_1\,\vec{A}_5\,\vec{A}_{10} + \vec{A}_2\,\vec{A}_4\,\vec{A}_{11}) \tag{3.5o}$$

$$\vec{T}_{16} = \frac{1}{\vec{D}}(\vec{A}_1[\vec{A}_5\,\vec{A}_7 - \vec{A}_4\,\vec{A}_9] + \vec{A}_2\;\vec{A}_4\,\vec{A}_8) \tag{3.5p}$$

where (see [24]),

$$\vec{D} = \det\begin{bmatrix} 0 & \vec{A}_1 & \vec{A}_2 & \vec{A}_3 \\ \vec{A}_4 & 0 & \vec{A}_5 & \vec{A}_6 \\ \vec{A}_7 & \vec{A}_8 & \vec{A}_9 & 0 \\ \vec{A}_{10} & \vec{A}_{11} & 0 & \vec{A}_{12} \end{bmatrix} \tag{3.6}$$

and

$$\vec{A}_1 = \vec{E}^{(m)}_{2222}(1 + \frac{h_2}{h_1}) \tag{3.7a}$$

$$\vec{A}_2 = \vec{E}^{(m)}_{2233}(\frac{l_1}{l_2}) \tag{3.7b}$$

$$\vec{A}_3 = \vec{E}^{(m)}_{2233} = \vec{A}_{11} \tag{3.7c}$$

$$\vec{A_4} = \vec{E}^{(m)}_{2222}(\frac{h_1}{h_2}) + \vec{E}^{(f)}_{2222} \tag{3.7d}$$

$$\vec{A_5} = \vec{E}^{(f)}_{2233} \tag{3.7e}$$

$$\vec{A_6} = \vec{E}^{(m)}_{2233}(\frac{l_2}{l_1}) \tag{3.7f}$$

$$\vec{A_7} = \vec{E}^{(f)}_{2233} \tag{3.7g}$$

$$\vec{A_8} = \vec{E}^{(m)}_{2233}(\frac{h_2}{h_1}) \tag{3.7h}$$

$$\vec{A_9} = \vec{E}^{(f)}_{2222} + \vec{E}^{(m)}_{2222}(\frac{l_1}{l_2}) \tag{3.7i}$$

$$\vec{A}_{10} = \vec{E}^{(m)}_{2233}(\frac{h_1}{h_2}) \tag{3.7j}$$

$$\vec{A}_{12} = \vec{E}^{(m)}_{2222}(\frac{l_2}{l_1} + 1) \tag{3.7k}$$

3.2.1 Use of the 3-parameter solid (or standard linear solid) to model viscoelastic material behavior

The 3-parameter solid (see Flugge [12]) can be viewed as a Kelvin element and a spring in series (see Fig. 3). The differential equation of the material behavior (e.g., in a state of uniaxial tension) writes as:

$$\sigma + p_1\dot{\sigma} = q_0 e + q_1 \dot{e} \tag{3.8a}$$

where

$$p_1 = \frac{\eta_1}{E_0 + E_1}; \quad q_0 = \frac{E_0 E_1}{E_0 + E_1}; \quad q_1 = \frac{E_0 \eta_1}{E_0 + E_1} \tag{3.8b}$$

By taking the L.T. of (3.8) and introducing (2.3) and (2.4), we obtain the relaxation modulus, E(t), as:

$$E(t) = q_0 + \left(\frac{q_1 - p_1 q_0}{p_1}\right) e^{-t/p_1} \tag{3.9}$$

Re-writing (3.9), we have,

$$E(t) = \overset{(0)}{E} + \overset{(1)}{E} e^{-\overset{(2)}{E} t} \tag{3.10a}$$

where we have defined,

$$\overset{(0)}{E} \triangleq q_0 \tag{3.10b}$$

$$\overset{(1)}{E} \triangleq \frac{q_1 - p_1 q_0}{p_1} \tag{3.10c}$$

$$\overset{(2)}{E} \triangleq \frac{1}{p_1} \tag{3.10d}$$

Henceforth, we shall consider the form of the relaxation modulus given by (3.10a). Representing the material properties of the transversely isotropic matrix (i.e., the 5 independent constants $E_A^{(m)}$, $\nu_A^{(m)}$, $G_A^{(m)}$, $E_T^{(m)}$, $\nu_T^{(m)}$) as a 3-parameter solid, we write,

$$E_A^{(m)}[t] = \overset{(0)}{E}{}_A^{(m)} + \overset{(1)}{E}{}_A^{(m)} e^{-\overset{(2)}{E}{}_A^{(m)} t} \tag{3.11a}$$

$$E_T^{(m)}[t] = \overset{(0)}{E}{}_T^{(m)} + \overset{(1)}{E}{}_T^{(m)} e^{-\overset{(2)}{E}{}_T^{(m)} t} \tag{3.11b}$$

$$\nu_A^{(m)}[t] = \overset{(0)}{\nu}{}_A^{(m)} - \overset{(1)}{\nu}{}_A^{(m)} e^{-\overset{(2)}{\nu}{}_A^{(m)} t} \tag{3.11c}$$

$$\nu_T^{(m)}[t] = \overset{(0)}{\nu}{}_T^{(m)} - \overset{(1)}{\nu}{}_T^{(m)} e^{-\overset{(2)}{\nu}{}_T^{(m)} t} \tag{3.11d}$$

$$G_A^{(m)}[t] = \overset{(0)}{G}{}_A^{(m)} + \overset{(1)}{G}{}_A^{(m)} e^{-\overset{(2)}{G}{}_A^{(m)} t} \tag{3.11e}$$

Upon taking the Carson transform of (3.11), we obtain,

$$\overline{E}_A^{*(m)} = \overset{(0)}{E}{}_A^{(m)} + \frac{\overset{(1)}{E}{}_A^{(m)} s}{s + \overset{(2)}{E}{}_A^{(m)}} \tag{3.12a}$$

$$\overline{E}_T^{*(m)} = \overset{(0)}{E}{}_T^{(m)} + \frac{\overset{(1)}{E}{}_T^{(m)} s}{s + \overset{(2)}{E}{}_T^{(m)}} \tag{3.12b}$$

$$\overline{\nu}_A^{*(m)} = \overset{(0)}{\nu}{}_A^{(m)} - \frac{\overset{(1)}{\nu}{}_A^{(m)} s}{s + \overset{(2)}{\nu}{}_A^{(m)}} \tag{3.12c}$$

$$\overline{\nu}_T^{*(m)} = \overset{(0)}{\nu}{}_T^{(m)} - \frac{\overset{(1)}{\nu}{}_T^{(m)} s}{s + \overset{(2)}{\nu}{}_T^{(m)}} \tag{3.12d}$$

$$\overline{G}_A^{*(m)} = \overset{(0)}{G}{}_A^{(m)} + \frac{\overset{(1)}{G}{}_A^{(m)} s}{s + \overset{(2)}{G}{}_A^{(m)}} \tag{3.12e}$$

Now using the C.P. in (B.11) for the transversely isotropic matrix yields

$$\overline{K}^{*(m)} = \frac{0.25\overline{E}_A^{*(m)}}{\left[0.5(1 - \overline{\nu}_T^{*(m)})\left(\dfrac{\overline{E}_A^{*(m)}}{\overline{E}_T^{*(m)}}\right) - (\overline{\nu}_A^{*(m)})^2\right]} \tag{3.13}$$

For an isotropic matrix, we make use of the following simplifications:

$$\bar{E}_A^{*(m)} = \bar{E}_T^{*(m)} = \bar{E}^{*(m)} \tag{3.14a}$$

$$\bar{\nu}_A^{*(m)} = \bar{\nu}_T^{*(m)} = \bar{\nu}^{*(m)} \tag{3.14b}$$

$$\bar{G}_A^{*(m)} = \frac{\bar{E}^{*(m)}}{2(1 + \bar{\nu}^{*(m)})} \tag{3.14c}$$

The relevant equations pertaining to the transversely isotropic or isotropic fibers can be obtained by replacing the superscript (m) with (f) in (3.11), (3.12), (3.13) and (3.14).

Now introducing (3.12), (3.13) [and (3.14) if the constituents are isotropic] into (3.1), we obtain the relevant Carson transformed relaxation moduli of the constituents, i.e., $\bar{E}_{ijmn}^{*(m)}$ and $\bar{E}_{ijmn}^{*(f)}$. Then upon introducing these along with (3.4), (3.5), (3.6), (3.7) into (3.3), we obtain the Carson transformed relaxation moduli of the orthotropic fiber-matrix composite, i.e., $\bar{E}_{ijmn}^{*}[s]$. Then upon inverting $\frac{\bar{E}_{ijmn}^{*}[s]}{s}$ into the time domain, we obtain the relaxation moduli, i.e., $\bar{E}_{ijmn}^{*}[t]$ for the orthotropic composite. At this stage, it is worth noting that for square fibers (i.e., $h_1 = l_1$) arranged at equal spacing (i.e., $h_2 = l_2$) we have the following result [after comparing 3.3(d) with 3.3(f), 3.3(g) with 3.3(h) , 3.3(b) with 3.3(c) and using the fact that continuous functions possess unique inverse Laplace transforms]:

$$E_{1122}[t] = E_{1133}[t] \tag{3.15a}$$

$$E_{2222}[t] = E_{3333}[t] \tag{3.15b}$$

$$E_{1212}[t] = E_{1313}[t] \tag{3.15c}$$

Thus the 9 independent constants of the orthotropic composite reduce to 6 independent constants in the case of equally-spaced square fibers.

The material properties for the isotropic, elastic, boron fibers were considered from Aboudi et al. [27] while the properties of the isotropic, visoelastic, epoxy were taken from Mohlenpah et al. [28] (see fig. 4) and Schapery [10] (see fig. 1).

Fiber properties:

The properties of the isotropic, elastic, boron fibers considered from Aboudi et al. [27] are as follows:

$G^{(f)} = 25 \times 10^6$p.s.i.

$K^{(f)} = 33.2 \times 10^6$p.s.i.

Using the above properties, we obtain the following:

$$\lambda^{(f)} = K^{(f)} - \frac{2}{3}G^{(f)} = 16.53 \times 10^6 \text{p.s.i.} \tag{3.16a}$$

$$E^{(f)} = \frac{G^{(f)}(3\lambda^{(f)} + 2G^{(f)})}{\lambda^{(f)} + G^{(f)}} = 6.426 \times 10^7 \text{p.s.i.} \tag{3.16b}$$

$$\nu^{(f)} = \frac{(K^{(f)} - \frac{2}{3}G^{(f)})}{2(K^{(f)} + \frac{1}{3}G^{(f)})} = 0.1990 \tag{3.16c}$$

Matrix properties

Using the properties for the isotropic, visoelastic matrix in Figures 4 and 1, we can represent them as a 3-parameter solid in the following manner:

$$E^{(m)}[t] = 0.8 \times 10^5 + 0.18 \times 10^6 e^{-0.4115\times 10^{-3}t} \qquad \text{for } 0 \le t \le 2000 \text{ hrs} \tag{3.17a}$$

$$\nu^{(m)}[t] = 0.372 - 0.007e^{-0.2403\times 10^{-2}t} \qquad \text{for } 0 \le t \le 2000 \text{ hrs} \tag{3.17b}$$

where the time t is in minutes.

The material properties given by (3.16) and (3.17) are shown in Table 1, which is used in Eqn. (3.12) (and its counterpart for the fibers) in order to obtain the relaxation moduli $E_{ijmn}(t)$ for the orthotropic plate.

3.3 Numerical Inversion of the Laplace Transform

In order to obtain the time-dependent relaxation moduli, $E_{ijmn}[t]$, for the viscoelastic orthotropic composite plate, we must invert the corresponding Laplace transformed moduli given by $(s)^{-1}\bar{E}^{*}_{ijmn}[s]$.

A number of approximate techniques for Laplace transform inversions are discussed by Cost [29] and Swanson [30]. However, a lot of these methods are applicable only for specific problems which impose restrictions on the nature of the desired inverse L.T. For example, Schapery's direct method assumes that the function f(t) (which is the desired inverse L.T. of $\bar{f}[s]$) has a linear variation with log t. This means that $\dot{f}$ is proportional to $\frac{1}{t}$. This is an unnecessary restriction on the nature of f[t] and, moreover, since we desire an exponential variation of f[t] with time (t), it is inconsistent to use this method of L.T. inversion for our problem. A second example is the general inversion formula developed by Post and Widder (see Cost [29] and Bellman [31]). This represents the general case of a set of inversion methods that have been considered, namely, the methods of Alfrey, ter Haar and Schapery (see [29]). Widder's general inversion formula is written as (see [29]),

$$f[t] = \lim_{n\to\infty} [\frac{(-1)^n s^{n+1}}{n!} \frac{d^n}{ds^n} (\bar{f}[s])]_{s=n/t} \tag{3.18}$$

Table 1. Material properties of boron fibers and epoxy matrix.

	$\overset{(0)}{E}$ p.s.i.	$\overset{(1)}{E}$ p.s.i.	$\overset{(2)}{E}$ (mins^{-1})	$\overset{(0)}{\nu}$	$\overset{(1)}{\nu}$	$\overset{(2)}{\nu}$ (mins^{-1})
Boron fibers (elastic)	0.6426×10^{8}	0	0	0.1990	0	0
Epoxy matrix at temp = 25° C (viscoelastic)	0.8000×10^{5}	0.1800×10^{6}	0.4115×10^{-3}	0.3720	0.0070	0.2403×10^{-2}

$V_f = 0.45$

As shown by Bellman [31], this method relies on the fact that the derivatives of $\bar{f}[s]$ are evaluated accurately since small inaccuracies in the derivatives produce large errors in f [t], especially for $t \to 0$.

A rigorous treatment of the problem of L.T. inversion is given by Bellman and Kalaba [31] and this method, which was chosen for our problem, has been effectively used by Swanson [30] for dynamic viscoelastic problems. In this method, known as Bellman's technique, the definition of the L.T. is used to invert the L.T. by means of a Gaussian quadrature using orthogonal polynomials. Due to their excellent convergence properties, Legendre polynomials were mainly used by Bellman [31]. It was observed that for the order of the polynominal N = 10, the convergence had been attained to provide accurate results. For N = 15, the results were almost the same as for N = 10. A brief description of Bellman's technique is provided in Appendix C.

3.3.1 Use of Bellman's Technique to Obtain E_{ijmn} (t)

Using (C.7) with $g[x] = E_{ijmn}[-\ln x]$, we can write,

$$E_{ijmn}[-\ln x_k] = [M_{pk}](\frac{\vec{\bar{E}}_{ijmn}[s]}{s})|_{s=k} = [M_{pk}](\bar{E}_{ijmn}[s])|_{s=k}, \quad \text{p, k} = 1,2,....\text{N} \tag{3.19}$$

Thus, we can evaluate $E_{ijmn}[t]$ for values of t given by,

$$t_k = -\ln x_k \tag{3.20}$$

where x_k are the roots of the N^{th} order shifted Legendre polynominal $P_N^*[r]$ and the coefficients of M_{pk} are tabulated in [31] for N ranging from 3 to 15.

Using N=15, we can obtain values of $E_{ijmn}[t]$ for 15 discrete values of t. Using (3.20) and the tabulated values of x_i from [31], we see that we can obtain values of $E_{ijmn}[t]$ in the interval

$0 < t < 5$ minutes. In order to obtain $E_{ijmn}[t]$ for larger values of time, we make use of the following property of the L.T. (known as the change of time scale property), i.e.,

If

$$\bar{f}[s] = \mathscr{L}\{f[t]\}$$

then

$$\frac{\bar{f}[s/a]}{a} = \mathscr{L}\{f[at]\} \tag{3.21}$$

Using (3.21) in conjunction with (3.19), we can write,

$$E_{ijmn}[-a \ln x_k] = [M_{pk}]\left(\frac{\vec{\bar{E}}_{ijmn}[s/a]}{s}\right)\Big|_{s=k}, \qquad p, k = 1,2,\ldots N$$

Thus by changing the value of a, we obtain values of f[t] for larger times [t].

The results obtained for $E_{ijmn}[t]$ versus time [t] are shown in Figures 5-9 for the case of equally-spaced square fibers. When these plots are fitted by an exponential series of the form (3.10(a)) (which represents a 3-parameter solid), we obtain the following results:

$$E_{1111} = 0.2903 \times 10^8 + 0.2500 \times 10^6 e^{-(0.3746\times 10^{-3})t} \tag{3.22a}$$

$$E_{2222} = E_{3333} = 0.3212 \times 10^6 + 0.6769 \times 10^6 e^{-(0.3986\times 10^{-3})t} \tag{3.22b}$$

$$E_{1122} = E_{1133} = 0.1294 \times 10^6 + 0.2633 \times 10^6 e^{-(0.3785\times 10^{-3})t} \tag{3.22c}$$

$$E_{2233} = 0.1304 \times 10^6 + 0.2609 \times 10^6 e^{-(0.3687\times 10^{-4})t} \tag{3.22d}$$

$$E_{1212} = E_{1313} = 0.6921 \times 10^5 + 0.1548 \times 10^6 e^{-(0.4356\times 10^{-3})t} \tag{3.22e}$$

$$E_{2323} = 0.5321 \times 10^5 + 0.1194 \times 10^6 e^{-(0.4365\times 10^{-3})t} \tag{3.22f}$$

Equations (3.22) represent the desired time-dependent relaxation moduli for the orthotropic plate with equally-spaced, square elastic fibers (boron) and a viscoelastic matrix (epoxy).

CHAPTER 4. SOLUTION OF THE STABILITY PROBLEM

4.1 Definition of Stability

The equations governing the stability of viscoelastic composite plates in biaxial compression, as derived in Chapter 2 (see eq. (2.64 and 2.71)), represent a system of linear integro-differential equations which must be solved for the displacement field represented by $\overset{(1)}{V}_{\alpha}(x_{\omega};t)$ and $\overset{(0)}{V}_{3}(x_{\omega};t)$ in order to yield the stress solution of the problem. However, a stability analysis does not require the explicit solution of the governing equations. Therefore, instead of determining the response of the system to a given input (i.e., edge loads in this case), we merely seek the conditions on the edge loads that would lead to instability of the system (represented in the present case by the plate).

We define instability as the phenomenon characterized by displacements increasing unboundedly as time unfolds. In doing so, we consider the stability (or instability) of a certain equilibrium configuration of the system referred to as the undisturbed equilibrium state. In addition, we also consider disturbed forms of motion, close to the undisturbed equilibrium state. We now define stable equilibrium (or stability) as that state in which small disturbances yield small deviations from the undisturbed equilibrium state. If these deviations tend asymptotically to zero as time unfolds, then the equilibrium configuration is known as asymptotically stable. However, if these disturbances, no matter how small, cause a finite deviation from the undisturbed state of equilibrium, then this undisturbed state of equilibrium of the system is unstable (in the small, according to the Liapunov definition of stability for dynamical systems). Thus, we consider the plate to be deformed by inplane edge loads which are small enough such that the flat configuration of the plate is the only possible equilibrium state, and this is a stable equilibrium state. If these edge loads are increased, the flat configuration of the plate may become unstable, i.e., the plate may pass, under the effect of negligibly small perturbations, to a new equilibrium state with a curved configuration. Therefore, in order to determine the stability of the plate, we need to analyze the behavior of the disturbed configuration, i.e., the stability equations (ref. Ambartsumian [32]. Also ref. Meirovitch [33] and Porter [34] for other equivalent definitions of stability for linear dynamical systems).

4.2 Stability Analysis Using the Third Order Transverse Shear Deformation Theory (TSDT)

It was noted in Chapter 2 that the solution of the equation governing the stability of the plate requires the fulfillment of the boundary conditions given in sec. 2.3.3. To this end, the following representation of the displacement field $\overset{(1)}{V}_\alpha(x_\omega;t)$ and $\overset{(0)}{V}_3(x_\omega;t)$ which satisfies the boundary conditions of the plate is postulated:

$$\overset{(1)}{V_1} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos[\lambda_m x_1] \sin[\lambda_n x_2] f_{mn}[t] \tag{4.1a}$$

$$\overset{(1)}{V_2} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin[\lambda_m x_1] \cos[\lambda_n x_2] f_{mn}[t] \tag{4.1b}$$

$$\overset{(0)}{V_3} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin[\lambda_m x_1] \sin[\lambda_n x_2] f_{mn}[t] \tag{4.1c}$$

where $\lambda_m = \frac{m\pi}{L_1}$, $\lambda_n = \frac{n\pi}{L_2}$ and A_{mn}, B_{mn}, C_{mn} are constants representing the amplitudes of the displacement quantities. Now the L.T. of equations (2.64) and (2.71) are (2.63) and (2.70), respectively. Thus, introducing (4.1) into (2.64) for the free index $\alpha = 1$ in conjunction with Eqn. (2.16) yields the following equation:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\bar{Y}_{mn}[s]\bar{f}_{mn} + \bar{I}_{mn}[s]) \cos[\lambda_m x_1] \sin[\lambda_n x_2] = 0 \tag{4.2a}$$

where,

$$\begin{aligned}
\bar{Y}_{mn}[s] \triangleq & - C_{mn}\left(\overline{\tilde{\vec{E}}}^{*}_{1111}\lambda_m^3 + \overline{\tilde{\vec{E}}}^{*}_{1122}\lambda_m\lambda_n^2 + 2\overline{\tilde{E}}^{*}_{1212}\lambda_m\lambda_n^2\right) + 4A_{mn}\left(\overline{\tilde{\vec{E}}}^{*}_{1111}\lambda_m^2 + \overline{\tilde{\vec{E}}}^{*}_{1212}\lambda_n^2\right) \\
& + 4B_{mn}\left(\overline{\tilde{\vec{E}}}^{*}_{1122}\lambda_m\lambda_n + \overline{\tilde{\vec{E}}}^{*}_{1212}\lambda_m\lambda_n\right) - A_{mn}\left(4\delta_A\overline{\hat{\vec{E}}}^{*}_{1111}\lambda_m^2\right) - B_{mn}\left(4\delta_A\overline{\hat{\vec{E}}}^{*}_{1122}\lambda_m\lambda_n\right) \\
& - C_{mn}\left(4\delta_A\overline{\hat{\vec{E}}}^{*}_{1111}\lambda_m^3 + 4\delta_A\overline{\hat{\vec{E}}}^{*}_{1122}\lambda_m\lambda_n^2\right) - C_{mn}\left(5\delta_A\delta_B\rho\overline{\hat{\vec{E}}}^{*}_{1133}\lambda_m\right)s^2 + A_{mn}\left(\frac{40}{h^2}\vec{E}_{1313}\right) \\
& + C_{mn}\left(\frac{40}{h^2}\lambda_m\vec{E}_{1313}\right) + A_{mn}(4\delta_C\rho)s^2 - C_{mn}(\delta_C\,\rho\,\lambda_m)s^2
\end{aligned} \tag{4.2b}$$

and

$$\bar{I}_{mn}[s] \triangleq \left\{C_{mn}\left(-5\delta_A\delta_B\rho\overline{\hat{\vec{E}}}^{*}_{1133}\lambda_m - \delta_C\rho\,\lambda_m\right) + A_{mn}(4\delta_C\,\rho\,)\right\}\left\{sf_{mn}[0] + \dot{f}_{mn}[0]\right\} \tag{4.2c}$$

The equation corresponding to the free index $\alpha = 2$ can be obtained from equation (4.2) by replacing the index 1 with 2 (and vice versa). We also note that repeated indices in equation (4.2) do not imply a summation over their range. Examining (4.2) (and its counterpart for the index $\alpha = 2$), we see that due to the orthogonality of the sine and cosine functions, we have the following result:

$$\bar{Y}_{mn}[s]\,\bar{f}_{mn}[s] + \bar{l}_{mn}[s] = 0 \tag{4.3}$$

where $\bar{Y}_{mn}[s]$ and $\bar{l}_{mn}[s]$ are polynomials with s as the variable. From the above, we observe that theoretically it is possible to obtain $f_{mn}[t]$ as a sum of exponentials e^{st} by inverting (4.3) into the time domain. Now for the initially undisturbed system, i.e., $\bar{l}_{mn}[s] = 0$, we obtain,

$$\bar{Y}_{mn}[s]\bar{f}_{mn}[s] = 0 \tag{4.4}$$

Thus, for non-trivial solutions of $f_{mn}[t]$ (i.e., $\bar{f}_{mn}[s] \neq 0$) we have,

$$\bar{Y}_{mn}[s] = 0 \qquad \text{(and its counterpart for the index } \alpha = 2) \tag{4.5}$$

Now introducing (4.1) into (2.71) for the case of uniform biaxial compression (i.e., $\overset{(0)}{L}_{11} = \underset{\sim}{\sigma}_{11}\,h$, $\overset{(0)}{L}_{22} = \underset{\sim}{\sigma}_{22}\,h$, $\overset{(0)}{L}_{12} = 0$) we obtain,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (\bar{W}_{mn}[s]\bar{f}_{mn}[s] + \bar{J}_{mn}[s]) \sin[\lambda_m x_1] \cos[\lambda_n x_2] = 0 \tag{4.6a}$$

where

$$\begin{aligned}\bar{W}_{mn}[s] \triangleq\ & A_{mn}(\tfrac{2}{3}\,h\,\vec{E}_{1313}\lambda_m) + B_{mn}(\tfrac{2}{3}\,h\,\vec{E}_{2323}\lambda_n) + C_{mn}(\tfrac{2}{3}\,h\,\vec{E}_{1313}\lambda_m^2 + \tfrac{2}{3}\,h\,\vec{E}_{2323}\lambda_n^2) \\ & + C_{mn}(h[\underset{\sim}{\sigma}_{11}\lambda_m^2 + \underset{\sim}{\sigma}_{22}\lambda_n^2]) + C_{mn}(\delta_D\,\rho h)s^2\end{aligned} \tag{4.6b}$$

and

$$\bar{J}_{mn}[s] \triangleq C_{mn}(\delta_D\,\rho\,h)(sf_{mn}[0] + \dot{f}_{mn}[0]) \tag{4.6c}$$

Now for an initially undisturbed body, i.e., $\bar{J}_{mn}[s] = 0$ we obtain (by using the orthogonality property of the sine and cosine functions),

$$\overline{W}_{mn}[s]\bar{f}_{mn}[s] = 0 \tag{4.7}$$

and for nontrivial solutions of $f_{mn}[t]$, we have,

$$\overline{W}_{mn}[s] = 0 \tag{4.8}$$

Equations (4.5) (and its counterpart for index $\alpha = 2$) represent the first two equations governing the stability of the plate, whereas equation (4.8) represents the third equation governing the stability. This set of three equations represents a homogeneous system of equations in terms of the unknown amplitudes A_{mn}, B_{mn}, C_{mn} (which represent the eigenvector). Thus, using (4.5) and (4.8) in conjunction with (4.2) and (4.6), we can write this system of homogeneous equations in the following form:

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \begin{bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \end{bmatrix} = 0 \tag{4.9a}$$

where,

$$Z_{11} = [(4\bar{\tilde{\vec{E}}}_{1111}\lambda_m^2 + 4\bar{\tilde{\vec{E}}}_{1212}\lambda_n^2 - 4\delta_A\,\bar{\hat{\vec{E}}}_{1111}\lambda_m^2 + \frac{40}{h^2}\,\vec{E}_{1313}) + (4\delta_c\,\rho)s^2] \tag{4.9b}$$

$$Z_{12} = [(4\bar{\tilde{\vec{E}}}_{1122}\lambda_m\lambda_n + 4\bar{\tilde{\vec{E}}}_{1212}\lambda_m\lambda_n - 4\delta_A\,\bar{\hat{\vec{E}}}_{1122}\lambda_m\lambda_n)] \tag{4.9c}$$

$$\begin{aligned} Z_{13} = [(&- \bar{\tilde{\vec{E}}}_{1111}\lambda_m^3 - \bar{\tilde{\vec{E}}}_{1122}\lambda_m\lambda_n^2 - 2\bar{\tilde{\vec{E}}}_{1122}\lambda_m\lambda_n^2 - 4\delta_A\,\bar{\hat{\vec{E}}}_{1111}\lambda_m^3 - 4\delta_A\,\bar{\hat{\vec{E}}}_{1122}\lambda_m\lambda_n^2 + \frac{40}{h^2}\,\lambda_m\,\vec{E}_{1313}) \\ &- (5\delta_A\delta_B\,\rho\,\bar{\hat{\vec{E}}}_{1133}\lambda_m + \delta_c\,\rho\,\lambda_m)s^2] \end{aligned} \tag{4.9d}$$

$$Z_{21} = [(4\overline{\overline{E}}^{*}_{1122}\lambda_m\lambda_n + 4\overline{\overline{E}}^{*}_{1212}\lambda_m\lambda_n + 4\delta_A \overline{\hat{E}}^{*}_{2211}\lambda_m\lambda_n)] \tag{4.9e}$$

$$Z_{22} = [(4\overline{\overline{E}}^{*}_{2222}\lambda_n^2 + 4\overline{\overline{E}}^{*}_{1212}\lambda_m^2 - 4\delta_A \overline{\hat{E}}^{*}_{2222}\lambda_n^2 + \frac{40}{h^2}\vec{E}_{2323}) + (4\delta_c \rho\,)s^2] \tag{4.9f}$$

$$Z_{23} = [(- \overline{\overline{E}}^{*}_{2222}\lambda_n^3 - \overline{\overline{E}}^{*}_{1122}\lambda_m^2\, \lambda_n - 2\overline{\overline{E}}^{*}_{1212}\lambda_m^2\, \lambda_n - 4\delta_A \overline{\hat{E}}^{*}_{2222}\lambda_n^3 - 4\delta_A \overline{\hat{E}}^{*}_{2211}\lambda_m^2\, \lambda_n + \frac{40}{h^2}\lambda_n \vec{E}_{2323}) \tag{4.9g}$$

$$- (5\delta_A\, \delta_B\, \rho\, \overline{\hat{E}}^{*}_{2233}\lambda_n + \delta_c\, \rho\, \lambda_n)s^2]$$

$$Z_{31} = [(\frac{2}{3} h\, \vec{E}_{1313}\lambda_m)] \tag{4.9h}$$

$$Z_{32} = [(\frac{2}{3} h\, \vec{E}_{2323}\lambda_n)] \tag{4.9i}$$

$$Z_{33} = [(\frac{2}{3} h\, \vec{E}_{1313}\lambda_m^2 + \frac{2}{3} h\, \vec{E}_{2323}\lambda_n^2 + h[g_{11}\lambda_m^2 + g_{22}\lambda_n^2]) + (\delta_D\, \rho\, h)s^2] \tag{4.9j}$$

Now from (4.9a) we see that for non-trivial solutions of A_{mn}, B_{mn}, C_{mn} , we must have the following identity:

$$\det[z_{ij}] \equiv 0 \tag{4.10}$$

Equation (4.10) yields a characteristic equation of the form,

$$\frac{P_{mn}[s]}{Q_{mn}[s]} = 0 \tag{4.11}$$

where $P_{mn}[s]$ and $Q_{mn}[s]$ are polynomials in s. Thus, the zeros of equation (4.11) are determined by writing,

$$P_{mn}[s] = 0 \tag{4.12}$$

Equation (4.12) is the characteristic equation of the system (i.e., the plate subjected to uniform biaxial compression). The zeros of this equation, i.e., the roots s_i of $P_{mn}[s]$, are the eigenvalues of the system, which are complex quantities in general.

Now we recall that the time dependency of the displacement field is given by $f_{mn}[t]$ which has the general solution of the form e^{st} . Thus, we see that the eigenvalues [s] of the system decide the nature of $f_{mn}[t]$ and hence the stability of the system. Representing the eigenvalues in the form $s = a + ib$, a and b being real numbers, we see that when $a(\equiv Re[s])$ is greater than zero, $f_{mn}[t]$ becomes unbounded as time unfolds and hence the system becomes unstable. In general, we may have the following cases arising as a result of the nature of s:

1. $(Re[s] = a) > 0; (Im[s] = b) = 0$

   Thus, we have $f_{mn}[t] = e^{at}$. Therefore, $f_{mn}[t]$ grows exponentially with time, and we have instability by divergence.

2. $(Re[s] = a) > 0 ; (Im[s] = b) \neq 0$

   Thus, we have $f_{mn}[t] = e^{at}e^{ibt}$. Therefore, $f_{mn}[t]$ has an oscillatory growth with time and the amplitude of oscillations is given by e^{at}. This leads to instability by flutter.

   When $a \leq 0$, we have asymptotic stability or marginal stability according to whether $a < 0$ or $a = 0$, respectively. However, when we have a double root lying on the imaginary axis, i.e., $a = 0$, the following cases will arise:

3. $(Re[s] = a) = 0; (I\,m\,[s] = b) = 0$; double root

   Thus, we have $f_{mn}[t] = t$. Therefore, $f_{mn}[t]$ grows linearly with time, and we have divergence instability.

4. $(Re[s] = a) = 0; (Im[s] = b) \neq 0$; double root

   Thus, we have $f_{mn}[t] = te^{ibt}$. Therefore, $f_{mn}[t]$ has an oscillatory growth with time and the amplitude of oscillations is given by t. This leads to instability by flutter.

   Instabilities of type (3) and (4) are also referred to as t-type instability and correspond to the situation when a double root lies on the imaginary axis (ref. Porter [34]).

Thus, we see that the stability problem is reduced to one in which we examine the nature of the zeros of the characteristic equation of the system given by (4.12). The coefficients of the characteristic polynominal, $P_{mn}[s]$, in equation (4.12) can be varied by suitably varying the inplane edge loads σ_{11} and σ_{22} in order to yield the stability boundaries of the system.

<u>Initial conditions</u>

In the preceding analysis, we have assumed, in Eqs. (4.2a) and (4.6a), zero initial conditions ($\bar{I}_{mn}[s] = \bar{J}_{mn}[s] = 0$) for an initially undisturbed system. For non-zero initial conditions, the counterparts of Eqs. (4.4) and (4.7) would write as,

$$\overline{Y}_{mn}[s]\,\bar{f}_{mn}[s] + \bar{I}_{mn}[s] = 0$$

$$\overline{W}_{mn}[s]\,\bar{f}_{mn}[s] + \bar{J}_{mn}[s] = 0$$

In the above equations, the terms $1/\overline{Y}_{mn}$ and $1/\overline{W}_{mn}$ represent the L.T. of the response $f_{mn}[t]$ of the system (i.e., $\bar{f}_{mn}[s]$) for a unit impulse input (i.e., the Dirac Delta function as defined in Appendix [A]). We also note that the response of a linear system (of the type considered above) to an arbitrary input may be represented as a superposition of impulse responses (see Meirovitch [35]). In view of this fact, we may conclude that the stability of the system is governed by whether its impulse response (i.e., $\mathscr{L}^{-1}[1/\overline{Y}_{mn}]$ and $\mathscr{L}^{-1}[1/\overline{W}_{mn}]$ in the present case) remains bounded as $t \rightarrow \infty$ (ref. Porter [34]). Therefore, the initial conditions, represented by $\bar{I}_{mn}$ and $\bar{J}_{mn}$, are not required in a stability analysis of a linear dynamical system and may be assumed to be zero without any loss of generality. However, it is obvious that the initial conditions will play an important role in a response analysis of the problem.

4.3 Stability Analysis Using a First Order Transverse Shear Deformation Theory (FSDT)

Examining the equations governing the stability of the plate in the framework of the FSDT (i.e., equation (2.85)), we see that this represents a sixth-order governing equation system, the solution of which requires that three boundary conditions be prescribed at each edge of the plate. Thus, for a hinged-free (in the normal direction) set of boundary conditions, we may represent the boundary conditions in exactly the same form as done for the TTSD in Sec. 2.3.3.

Following a procedure analogous to that in Sec. 4.2 and considering the L.T. of equation (2.85), we obtain the characteristic equation of the system in exactly the same form as given by (4.12) but with different coefficients. These coefficients are determined by eqn. (4.10) in which, for the FSDT, we have,

$$Z_{11} = [\bar{\bar{\vec{E}}}_{1111}(\frac{h^3}{12}\lambda_m^2) + \bar{\bar{\vec{E}}}_{1212}(\frac{h^3}{12}\lambda_n^2) + \vec{E}_{1313}(k^2 h) + s^2\,\delta_c\, m_1] \tag{4.13a}$$

$$Z_{12} = [\bar{\bar{\vec{E}}}_{1122}(\frac{h^3}{12}\lambda_m\lambda_n) + \bar{\bar{\vec{E}}}_{1212}(\frac{h^3}{12}\lambda_m\lambda_n)] \tag{4.13b}$$

$$Z_{13} = [\vec{E}_{1313}(k^2 h\,\lambda_m)] \tag{4.13c}$$

$$Z_{21} = Z_{12} \tag{4.13d}$$

$$Z_{22} = [\bar{\bar{\vec{E}}}_{2222}(\frac{h^3}{12}\lambda_n^2) + \bar{\bar{\vec{E}}}_{1212}(\frac{h^3}{12}\lambda_m^2) + \vec{E}_{2323}(k^2 h) + s^2\,\delta_c\, m_1] \tag{4.13e}$$

$$Z_{23} = [\vec{E}_{2323}(k^2 h\,\lambda_n)] \tag{4.13f}$$

$$Z_{31} = Z_{13} \tag{4.13g}$$

$$Z_{32} = Z_{23} \tag{4.13h}$$

$$Z_{33} = [\vec{E}_{1313}(k^2\, h\, \lambda_m^2) + \vec{E}_{2323}(k^2\, h\, \lambda_n^2) + h(\underset{\sim}{g}_{11}\, \lambda_m^2 + \underset{\sim}{g}_{22}\, \lambda_n^2) + (\delta_D\, m_0)s^2] \tag{4.13i}$$

Proceeding in exactly the same manner as described in Sec. 4.2, we obtain the stability boundaries of the orthotropic plate subjected to uniform inplane edge loads.

4.4 Stability Analysis Using the Equations Representing the Interior Solution in the Framework of the FSDT

As was shown in Sec. 2.5 (see Librescu [8, 22, 9]), the coupled equations governing the stability of a transversely isotropic plate (i.e., (2.86)) can be recast into two independent equations, (2.100) and (2.102). We now solve (2.102) independent of (2.100) with the aim of showing, through numerical comparison with the solution obtained in Sec. 4.3, that (2.102) represents the interior solution of stability and is thus by itself sufficient to analyze the stability of transversely isotropic plates. To this end, we consider the following representation for the transverse displacement $\overset{(0)}{V}_3$ given as:

$$\overset{(0)}{V}_3\,[t] = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin[\lambda_m\, x_1]\, \sin[\lambda_n x_2]\, f_{mn}[t] \tag{4.14}$$

In equation (4.14) λ_m and λ_n are as defined in Sec. 4.2, while $f_{mn}[t]$ represents the time dependent amplitude of $\overset{(0)}{V}_3$. The representation of $\overset{(0)}{V}_3$ given by (4.14) satisfies the two boundary conditions, given below, for each edge of the plate:

$$\overset{(0)}{V}_3 = 0\ ;\ \overset{(0)}{V}_{3,\,11} = 0, \quad \text{at}\ \ x_2 = 0, L_2$$

$$\overset{(0)}{V_3} = 0 \;;\; \overset{(0)}{V}_{3,22} = 0, \quad \text{at} \quad x_1 = 0, L_1$$

By introducing (4.14) into the L.T. of equation (2.102) [i.e., (2.101)], we obtain:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\overline{R}_{mn}[s]\, \bar{f}_{mn}[s] + \overline{Q}_{mn}[s]) \sin[\lambda_m x_1] \sin[\lambda_n x_2] = 0 \tag{4.15a}$$

where

$$\begin{aligned} \overline{R}_{mn}[s] = {} & \overline{D}^{*}(\lambda_m^2 + \lambda_n^2)^2 + h(g_{11}\, \lambda_m^2 + g_{22}\, \lambda_n^2) + \frac{2h}{3k^2}\, \overrightarrow{C}_3(g_{11}[\lambda_m^4 + \lambda_m^2\, \lambda_n^2] + g_{22}[\lambda_n^4 + \lambda_m^2\, \lambda_n^2]) \\ & + \delta_D\, m_0\, s^2 + \delta_D\, m_0 \frac{2}{3k^2}\, \overrightarrow{C}_3\, s^2(\lambda_m^2 + \lambda_n^2) + \delta_c \frac{m_1 m_0}{2K^2}\, \overrightarrow{C}_2\, s^4 \\ & + \delta_c \frac{m_1}{2k^2} \overrightarrow{C}_2\, s^2\left(h[g_{11}\, \lambda_m^2 + g_{22}\, \lambda_n^2]\right) + \delta_c\, m_1\, s^2(\lambda_m^2 + \lambda_n^2) \end{aligned} \tag{4.15b}$$

and,

$$\begin{aligned} \overline{Q}_{mn}[s] = {} & \delta_D\, m_0\, f_{mn}[0]s + \delta_D\, m_0\, (\lambda_m^2 + \lambda_n^2) \frac{2}{3k^2} \overrightarrow{C}_3(s f_{mn}[0] + \dot{f}_{mn}[0]) + \delta_c \frac{m_1 m_0}{2k^2} \overrightarrow{C}_2[s] \\ & (s^3 f_{mn}[0] + s^2\, \dot{f}_{mn}[0] + s\, \ddot{f}_{mn}[0] + \dddot{f}_{mn}[0]) \\ & + \delta_c\, m_1\, f_{mn}[0](\lambda_m^2 + \lambda_n^2)s + (\delta_D\, m_0\, \dot{f}_{mn}[0] + \delta_c\, m_1\, \dot{f}_{mn}[\lambda_m^2 + \lambda_n^2]) \\ & + \delta_c \frac{m_1}{2k^2}(h[g_{11}\, \lambda_m^2 + g_{22}\, \lambda_n^2]) \overrightarrow{C}_2(s f_{mn}[0] + \dot{f}_{mn}[0]) \end{aligned} \tag{4.15c}$$

The terms $\overline{D}^{*}$, $\overline{C}^{*}$ and $\overline{C}^{*}$ appearing in equation (4.15) are defined in equation (2.102). Using the orthogonality property of the sine and cosine functions and following a procedure similar to that in Sec. 4.2, we obtain the characteristic equation of the initially undisturbed system as,

$$\overline{R}_{mn}[s] = 0 \tag{4.16}$$

Now proceeding in the same manner as described in Sec. 4.2, we can establish the stability boundaries of the transversely isotropic plate subject to uniform inplane edge loads.

4.5 Stability of a Transversely Isotropic Viscoelastic Plate Undergoing Cylindrical Bending

Consider a transversely isotropic plate with an infinitely large aspect ratio (i.e., $L_2/L_1 \to \infty$) and simply supported along its edges $x_1 = 0,\ L_1$. The plate is subjected to a uniform compressive force applied along its edges. Thus, the plate undergoes cylindrical bending and in this case the operator $\frac{\partial}{\partial x_2} \to 0$. In the following developments, we analyze the stability of the plate in the framework of the FSDT by making use of the correspondence principle. To this end, we consider the elastic counterpart of equation (2.102) in conjunction with the condition that $\frac{\partial}{\partial x_2} \to 0$, and we obtain,

$$D\overset{(0)}{V}_{3,1111}[t] - \overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11}[t] + \frac{2}{3k^2}C_3\overset{(0)}{L}_{11}\overset{(0)}{V}_{3,1111}[t] + \delta_D m_0 \overset{(\ddot{0})}{V}_3[0] - \frac{2}{3k^2}C_3\overset{(\ddot{0})}{V}_{3,11}[t]$$

$$-\delta_C\frac{m_1}{2k^2}C_2\overset{(0)}{L}_{11}\overset{(\ddot{0})}{V}_{3,11}[t] + \delta_C\delta_D\frac{m_1 m_0}{2k^2}C_2\overset{(\ddot{\ddot{0}})}{V}_3[t] - \delta_C m_1\overset{(\ddot{0})}{V}_{3,\beta\beta} = 0 \tag{4.17}$$

In equation (4.17), D, C_2, C_3 are the elastic counterparts (defined at t = 0) of the quantities D[t], $C_2[t]$, $C_3[t]$ defined in equation (2.102).

Now we represent the engineering material constants given in (4.17) in terms of the tensorial constants by making use of equation (2.87). Thus, we obtain the following results for a transversely isotropic body:

$$\tilde{E}_{1111} = \frac{E}{1-\nu^2} \tag{4.18a}$$

$$G' = E_{1313} \tag{4.18b}$$

$$G = \frac{E}{2(1+\nu)} \tag{4.18c}$$

Introducing Eqs. (4.18) into eqs. (2.102b,c,d), evaluated at t = 0, we obtain,

$$D = \tilde{E}_{1111}\frac{h^3}{12} \tag{4.19a}$$

$$C_3 = \frac{h^2}{8}\,\frac{\tilde{E}_{1111}}{E_{1313}} \tag{4.19b}$$

Now introducing (4.19) into (4.17), neglecting the effect of rotary and transverse inertias (i.e., $\delta_c = \delta_D = 0$), and postulating that $K^2 = 5/6$, we obtain:

$$\tilde{E}_{1111}\frac{h^3}{12}\overset{(0)}{V}_{3,1111} - \overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11} + \frac{h^2}{10}\,\frac{\tilde{E}_{1111}}{E_{1313}}\overset{(0)}{L}_{11}\overset{(0)}{V}_{3,1111} = 0 \tag{4.20}$$

Now we consider the constitutive equations for a transversely isotropic plate undergoing cylindrical bending and having viscoelastic properties in shear only. These equations write as,

$$\sigma_{11}[t] = \tilde{E}_{1111}e_{11}[t] \tag{4.21a}$$

$$\sigma_{13}[s] = 2\bar{E}_{1313}[s]\bar{e}_{13}[s] \tag{4.21b}$$

Making use of the C.P. in equation (4.20) in conjunction with (4.21) yields

$$-\frac{1}{10}\,\frac{\tilde{E}_{1111}}{s\bar{E}_{1313}}h^2\overset{(0)}{L}_{11}\overline{\overset{(0)}{V}}_{3,1111} - \tilde{E}_{1111}\frac{h^3}{12}\overline{\overset{(0)}{V}}_{3,\,1111} + \overset{(0)}{L}_{11}\overline{\overset{(0)}{V}}_{3,\,1111} = 0 \tag{4.22}$$

Introducing (2.50) into (4.22), we can write,

$$-\frac{1}{10}\tilde{E}_{1111}h^2\overset{(0)}{L}_{11}(\mathscr{L}[\dot{F}_{1313}] + F_{1313}[0])\overline{\overset{(0)}{V}}_{3,\,1111} - \tilde{E}_{1111}\frac{h^3}{12}\overline{\overset{(0)}{V}}_{3,\,1111} + \overset{(0)}{L}_{11}\overline{\overset{(0)}{V}}_{3,\,11} = 0 \tag{4.23}$$

Inverting (4.23) into the time domain, we obtain,

$$
\begin{aligned}
&-\frac{h^2}{10}\overset{(0)}{L_{11}}\tilde{E}_{1111}\int_0^t \dot{F}_{1313}[t-\tau]\overset{(0)}{V}_{3,\,1111}[\tau]d\tau-\frac{h^2}{10}\overset{(0)}{L_{11}}\tilde{E}_{1111}\,F_{1313}[0]\overset{(0)}{V}_{3,\,1111}\\
&-\tilde{E}_{1111}\,\frac{h^3}{12}\overset{(0)}{V}_{3,\,1111}+\overset{(0)}{L_{11}}\,\overset{(0)}{V}_{3,\,11}=0
\end{aligned}
\tag{4.24}
$$

Equation (4.24) is the equation governing the stability of a viscoelastic transversely isotropic plate of infinite aspect ratio, exhibiting viscoelastic properties in transverse shear only.

Solution of stability equation:

Now we consider the following representation of the creep compliance in transverse shear which corresponds to a 3-Parameter solid:

$$
F_{1313}[t]=\overset{(1)}{F}_{1313}-\overset{(2)}{F}_{1313}e^{-\overset{(3)}{F}_{1313}t}
\tag{4.25}
$$

We also assume the following representation of the transverse displacement (valid for the simply supported boundary conditions at $x_1=0, L_1$):

$$
\overset{(0)}{V}_3=\sum_{m=1}^{\infty} f_m[t]\sin[\lambda_m x_1]\ , \qquad \lambda_m=\frac{m\pi}{L_1}
\tag{4.26}
$$

Introducing (4.25) and (4.26) into the L.T. of (4.24), we obtain,

$$
\sum_{m=1}^{\infty}\left[\frac{h^2}{10}\overset{(0)}{L_{11}}\,\tilde{E}_{1111}\left[\overset{(1)}{F}_{1313}-\frac{s\overset{(2)}{F}_{1313}}{s+\overset{(3)}{F}_{1313}}\right]\lambda_m^4-\tilde{E}_{1111}\,\frac{h^3}{12}\,\lambda_m^4-\overset{(0)}{L_{11}}\,\lambda_m^2\right]\bar{f}_m\ \sin[\lambda_m x_1]=0
\tag{4.27}
$$

Invoking the orthogonality property of the sine function along with the argument of non-trivial solutions (i.e., $\bar{f}_m\neq 0$) in (4.27) yields the following characteristic equation of the system:

$$\frac{h^2}{10}\overset{(0)}{L}_{11}\,\widetilde{E}_{1111}[(\overset{(1)}{F}_{1313}-\overset{(2)}{F}_{1313})s]\lambda_m^4+\frac{h^2}{10}\overset{(0)}{L}_{11}\,\widetilde{E}_{1111}[\overset{(1)}{F}_{1313}\,\overset{(3)}{F}_{1313}]\lambda_m^4$$
$$+[s+\overset{(3)}{F}_{1313}][\widetilde{E}_{1111}\,\frac{h^3}{12}\lambda_m^4+\overset{(0)}{L}_{11}\,\lambda_m^2]=0 \tag{4.28}$$

Equation (4.28) is a linear equation in the unknown s (i.e., the eigenvalue), therefore possessing only real solutions (since the coefficients are real). Thus, we conclude that instability could occur by divergence only. Having in view the fact that for divergence instability, s = 0 in (4.28), we obtain,

$$\overset{(0)}{L}_{11}=\frac{-\widetilde{E}_{1111}\,\frac{h^3}{12}\lambda_m^2}{[\frac{h^2}{10}\widetilde{E}_{1111}\,\overset{(1)}{F}_{1313}\,\lambda_m^2\;+1]} \tag{4.29}$$

Equation (4.29) gives us the value of the applied inplane edge load when divergence occurs. It is easily seen that the lowest value of this load corresponds to m = 1 for which we obtain,

$$\overset{(0)}{L}_{11}=\frac{-\widetilde{E}_{1111}\,\frac{h^3}{12}(\frac{\pi}{L_1})^2}{[\frac{h^2}{10}\widetilde{E}_{1111}\overset{(1)}{F}_{1313}(\frac{\pi}{L_1})^2\;+1]} \tag{4.30}$$

Considering (4.28) with $\overset{(3)}{F}_{1313}=0$ (i.e., the elastic case), it is easy to verify that the static buckling load for the elastic transversely isotropic plate would be,

$$\overset{(0)}{L}_{11}=\frac{-\widetilde{E}_{1111}\,\frac{h^3}{12}(\frac{\pi}{L_1})^2}{[\frac{h^2}{10}\widetilde{E}_{1111}(\overset{(1)}{F}_{1313}-\overset{(2)}{F}_{1313})(\frac{\pi}{L_1})^2\;+1]} \tag{4.31}$$

Since $\overset{(1)}{F}_{1313}$, $\overset{(2)}{F}_{1313}$ and $\overset{(3)}{F}_{1313}$ in (4.25) are all greater than zero, we see, upon comparing (4.30) and (4.31), that the instability (divergence) load for the viscoelastic case is lower than that for the elastic case. Equations (4.30) and (4.31) coincide with the solutions obtained by Malmeister et al. [5] (where the stability problem is solved as a special case of the response problem using a time domain analysis). We notice that for the classical Kirchhoff theory $F_{1313} \to 0$ and hence equations (4.30) and (4.31) yield the same divergence loads for the elastic and viscoelastic case, which is the Euler load for a wide column given by,

$$\overset{(0)}{L}_{11} = -\tilde{E}_{1111}\frac{h^3}{12}\left(\frac{\pi}{L_1}\right)^2 \tag{4.32}$$

This evident result is a direct consequence of the assumption implied in eqs. (4.21) which considers the viscoelasticity in the transverse shear direction only.

<u>Alternative method of deriving the equation governing the stability of viscoelastic transversely isotropic plate undergoing cylindrical bending</u>

In the absence of inplane surface loads and rotary inertia effects, the first order moments of the first two equations of motion (i.e., (2.29)) when specialized for the case of cylindrical bending (yielding $\frac{\partial}{\partial x_2} \to 0$) results in:

$$\overset{(0)}{L}_{11} - \overset{(0)}{L}_{13} = 0 \tag{4.33a}$$

$$\overset{(1)}{L}_{21,1} - \overset{(0)}{L}_{23} = 0 \tag{4.33b}$$

In the absence of transverse inertia effects (i.e., $\delta_0 = 0$), the L.T. of the third equation of motion (2.62) when specialized for cylindrical bending writes as,

$$\overset{(1)}{L}_{13,1} + \overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11} = 0 \tag{4.34}$$

Introducing (4.33a) into (4.34), we obtain,

$$\overset{(1)}{L}_{11,11} + \overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11} = 0 \tag{4.35}$$

Equation (4.35) is the desired equation of motion of the plate undergoing cylindrical bending. We shall now solve for $\overset{(1)}{L}_{11,11}$ in terms of $\overset{(0)}{V}_3$ in order to obtain the governing equation expressed in terms of the displacement field. To this end, we postulate the following representation for the transverse shear stress σ_{13} in the framework of the first order transverse shear stress theory (S.T.) [which yields identical results as the FSDT (see Librescu [8, 9])]:

$$\sigma_{13} = f[x_3]\psi[x_1, t] \tag{4.36a}$$

$$f[z] = \frac{1}{2}[(x_3)^2 - \frac{h^2}{4}] \tag{4.36b}$$

The constitutive equations for the 2-D elastic body (i.e., the elastic counterpart of (4.21)) write as follows:

$$e_{11} = \frac{\sigma_{11}}{\tilde{E}_{1111}} \tag{4.37a}$$

$$e_{13} = \frac{\sigma_{13}}{2E_{1313}} \tag{4.37b}$$

We now see that in this case we must consider the following functional dependence for the displacement field:

$$V_1 = V_1[x_1, x_3] \tag{4.38a}$$

$$V_3 = \overset{(0)}{V}_3[x_1] \tag{4.38b}$$

The substitution of the strain-displacement relation (4.12e) considered in conjunction with (4.38b) and (4.36) into (4.37b) yields,

$$V_{1,3} = \frac{1}{2E_{1313}}[(x_3)^2 - \frac{h^2}{4}]\psi - V_{3,1} \tag{4.39}$$

Integrating (4.39) across the interval $[0\,,\ x_3)$ in conjunction with (4.38), we obtain:

$$V_1 = \frac{1}{2E_{1313}}\left[\frac{(x_3)^3}{3} - x_3 \frac{h^2}{4}\right]\psi - x_3 \overset{(0)}{V}_{3,1} \tag{4.40}$$

Introducing (4.40) into (4.37a) in conjunction with (2.12a) results in,

$$\sigma_{11} = \frac{\tilde{E}_{1111}}{2E_{1313}}\left[\frac{(x_3)^3}{3} - x_3 \frac{h^2}{4}\right]\psi_{,1} - \tilde{E}_{1111}\, x_3 \overset{(0)}{V}_{3,11} + \frac{1}{2}\tilde{E}_{1111}(\overset{(0)}{V}_{3,1})^2 \tag{4.41}$$

Replacement of (4.41) into (2.25a) in conjunction with (4.38) yields,

$$\overset{(1)}{L}_{11} = \frac{-\tilde{E}_{1111}}{E_{1313}} \frac{h^5}{120}\psi_{,1} - \tilde{E}_{1111} \frac{h^3}{12} \overset{(0)}{V}_{3,11} \tag{4.42}$$

Now in order to obtain the equation governing the stability in terms of the displacement field, we need to express $\psi_{,1}$ in terms of the displacement field. To this end, by introducing (4.36) into (2.27b), we obtain

$$\overset{(0)}{L}_{13} = \frac{-h^3}{12}\psi$$

which yields,

$$\psi_{,1} = -\frac{12}{h^3}\overset{(0)}{L}_{13,1} \tag{4.43}$$

Upon introducing (4.33) and (4.35) into (4.43), we obtain,

$$\psi_{,1} = -\frac{12}{h^3}\overset{(1)}{L}_{11,11} = \frac{12}{h^3}\overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11} \tag{4.44}$$

Substitution of (4.44) into (4.42) results in

$$\overset{(1)}{L}_{11} = \frac{-\tilde{E}_{1111}}{E_{1313}} \frac{h^2}{10}\overset{(0)}{L}_{11}\overset{(0)}{V}_{3,11} - \tilde{E}_{1111} \frac{h^3}{12} \overset{(0)}{V}_{3,11} \tag{4.45}$$

Upon introducing (4.45) into (4.35), we obtain the desired governing equation of an elastic transversely isotropic plate undergoing cylindrical bending. This equation coincides with Eqn. (4.20), i.e., the one obtained by considering the single equation which represents the interior solution (i.e., (2.102)). From here onwards, the procedure for obtaining the viscoelastic counterpart of the equations governing the stability is the same as that used at the beginning of this section.

CHAPTER 5. NUMERICAL RESULTS AND CONCLUSIONS

5.1 Numerical Results

The stability boundary was obtained by solving the characteristic polynomial (e.g., Eqn. (4.12)) using the IMSL subroutine ZPOLR. The following problems were considered in this context:

Problem (1). The TTSD (also referred to as the higher order shear deformation theory - HSDT) represented by Eqn. (4.12).

Problem (2). The FSDT counterpart of Eqn. (4.12).

Problem (3). The "single equation" representing the interior solution of stability in the framework of the FSDT, represented by (4.15b).

Problems (1) and (2) above were considered for an orthotropic viscoelastic plate and were specialized for the orthotropic elastic, isotropic viscoelastic, and isotropic elastic cases.

Problem (3) was considered for an isotropic viscoelastic plate and was specialized for its elastic counterpart.

All cases above were considered so as to obtain an "exact" dynamic solution, i.e., for $\delta_A = \delta_B = \delta_C = \delta_D = 1$ where δ_B, δ_C, δ_D are tracers identifying the dynamic effect of σ_{33} , rotary inertia and transverse inertia, respectively, and δ_A is a tracer identifying the overall (i.e., static and dynamic) effect of σ_{33}. It was observed that the inclusion or exclusion of the inertia terms does not affect the results.

All elastic counterparts were solved as special cases of the corresponding viscoelastic problem by considering the initial value theorem (ref. Appendix [A]) for the Laplace transformed material properties appearing in Eqns. (4.9), (4.13) and (4.15b). Comparisons were made for the orthotropic elastic plate with Ambartsumian [32] and Ashton and Whitney [36] (ref. Figures 38-41) and the results show an excellent agreement.

The results associated with the classical Kirchhoff theory were obtained as a special case of the FSDT by considering $K^2 \to \infty$. The results obtained in this study are not universal since a non-dimensional analysis was not possible due to the inherent complexity of the problem.

<u>Discussion of Figs. 13-41</u>

The stability boundaries for the following cases are shown in Figs.13-41.

(i) Orthotropic, viscoelastic, flat plate

(ii) Orthotropic, elastic, flat plate

(iii) Isotropic, viscoelastic, flat plate

(iv) Isotropic, elastic, flat plate.

Cases (i)-(iv) are considered for thick plates (L/H = 4.8) as well as thin plates (L/H = 24). In addition, the following sub-cases were considered:

(1) Biaxial compression: For this case, the aspect ratio (A.R. = L_1/L_2) of the plate was taken as unity. The values of the inplane, normal edge loads $\underset{\sim}{\sigma}_{11}$ versus $\underset{\sim}{\sigma}_{22}$ are plotted to obtain the stability boundaries.

(2) Uniaxial compression: In this case, the aspect ratio, A.R., was varied and the corresponding value of $\underset{\sim}{\sigma}_{11}$ was plotted in order to obtain the stability boundaries. The inplane normal edge load, $\underset{\sim}{\sigma}_{11}$, is applied in the direction of the fibers, at the edges $x_1 = 0, L_1$.

For all plots shown, M and N denote the mode numbers in the x_1 and x_2 direction, respectively (see Eqn. (4.1)). It was observed that for biaxial compression, the stability boundaries corresponding to M=1 were the lowest ones, whereas for uniaxial compression, those corresponding to N=1 were the lowest ones. Therefore, in each of these two sub-cases, only the lowest stability boundaries were displayed. For all the cases, unless otherwise indicated, instability occurs by divergence only. Flutter boundaries are indicated on the figures. For uniaxial compression cases, flutter instability occurs to the right of the arrow indicated in the figures.

Fig. 13: This plot depicts the stability boundaries for the case of biaxial compression of an orthotropic, viscoelastic, thick plate (L/H = 4.8). The solid line corresponds to the result obtained in the framework of the HSDT with $\delta_A = 1$, whereas the dotted line corresponds to its FSDT counterpart. It is observed that the inclusion of the transverse normal stress (σ_{33}) in the HSDT has a beneficial effect on the stability by yielding higher stability boundaries as compared to the FSDT counter-

parts. Moreover, the inclusion of σ_{33} could yield a change in the character of the instability boundaries, i.e., the conversion of flutter (for FSDT) into divergence (for HSDT) instability boundaries.

Fig. 14: The results shown here are for a similar case as those shown in Fig. 13, except that $\delta_A = 0$. Thus, we observe that when the effect of σ_{33} is neglected in the HSDT, the stability boundaries are lowered and coincide with those obtained in the framework of the FSDT.

Fig. 15: Here the results obtained for uniaxial compression of an orthotropic, viscoelastic, thick plate are displayed. On comparing the results obtained for the HSDT with $\delta_A = 1$ (solid line) with those obtained for the FSDT (dotted line), we observe once again that the former yields higher stability boundaries. The undulating nature of these plots could not be interpreted.

Fig. 16: This plot shows that for the case considered in Fig. 15, when σ_{33} is neglected in the HSDT (i.e., $\delta_A = 0$), the stability boundaries for the HSDT are lowered and coincide with those obtained in the framework of the FSDT.

Figs. 17, 18: These are results for the elastic counterparts of those displayed in Figs. 13 and 14. From these plots, we infer that σ_{33} has only a minimal but beneficial effect on the stability boundaries as compared to its effect for the viscoelastic case considered in Figs. 13 and 14.

Figs. 19, 20: Fig. 19 depicts the results obtained for biaxial compression of an isotropic, elastic, thick plate in the framework of the HSDT with $\delta_A = 1$ (solid line), the FSDT (dotted line) and the FSDT "single equation" (broken line). The results for the two FSDT cases coincide, whereas those for the HSDT represent higher stability boundaries when compared to their FSDT counterparts. However, for the critical stability boundary (corresponding to $M = N = 1$), the three cases yield identical results. In Fig. 20, the only difference is that $\sigma_{33} = 0$ for the HSDT, and we observe that for all modes, the three cases yield identical stability boundaries.

Fig. 21: This plot shows the results for uniaxial loading of an isotropic, viscoelastic, thick plate. The stability boundaries for the HSDT with $\delta_A = 1$ are slightly higher than those for the FSDT and FSDT "single equation". The latter two cases coincide.

Figs. 22, 23: Fig. 22 displays the results for the biaxial compression of an isotropic, elastic, thick plate, and Fig. 23 shows the results for the corresponding uniaxial compression case. In both these plots, we observe that the HSDT (with or without the effect of σ_{33}), the FSDT, and the FSDT "single equation" yield identical stability boundaries.

Fig. 24: This plot shows the comparison between results obtained by using the FSDT and the classical theory of plates for the biaxial compression of an orthotropic, viscoelastic, thick plate. It is evident from this figure that the classical theory of plates yields much higher stability boundaries than the FSDT and, hence, we conclude that the transverse shear deformation effects play a very important role for the case considered in this plot.

Fig. 25: Here the results for the uniaxial compression of an orthotropic, viscoelastic, thick plate obtained in the framework of the FSDT are compared with those obtained by using the classical theory.

It is evident from this plot that the classical theory over-predicts the stability boundaries. However, we observe that for large aspect ratios (L_1/L_2) the two theories tend to yield similar results. This is because for large aspect ratios, the quantity, L_1/H, increases which means that in fact the plate tends to become a thin plate.

Fig. 26: This plot compares the results obtained by the FSDT and the classical theory for the biaxial compression of an orthotropic, elastic, thick plate. It is evident that the classical theory over-predicts the stability boundaries.

Figs. 27, 28: In Fig. 27, comparison between the FSDT, FSDT "single equation", and classical theory are shown for the biaxial compression of an isotropic, viscoelastic, thick plate. Here it is observed that the two FSDT results coincide with each other. We observe that the classical theory over-predicts the stability boundaries. Fig. 28 shows the corresponding results for the case of uniaxial compression. It can be observed that for low aspect ratios, the classical theory over-predicts the stability boundary, but as the A.R. increases, the classical theory results tend to coincide with the FSDT ones.

Figs. 29, 30: These results correspond to the elastic counterpart of the results obtained in Figs. 27, 28, in which similar trends are observed. However, for the biaxial compression case, the classical theory yields the same critical boundary (i.e., for $M=N=1$) as its FSDT counterpart.

Figs. 31, 32, 33: These plots correspond to those obtained for the cases considered in Figs. 15, 17 and 18. Here a thin plate ($L/H = 24$) was considered. it is observed that for the thin plate (orthotropic, elastic or viscoelastic), the effect of σ_{33} is minimal.

Figs. 34, 35, 36, 37: These results correspond to those obtained for the cases displayed in Figs. 19, 21, 22, 23, but restricted here for a thin plate. The effect of σ_{33} when included in the HSDT is negligible, as can be seen from these figures.

Figs. 38, 39: In these plots, comparisons are made between the results for the HSDT (where the effect of σ_{33} is included) and the results obtained by Ambartsumian [32] (for which the effect of σ_{33} is neglected). However, for the case when the effect of σ_{33} is included, the stability boundaries are slightly higher. The cases considered in these figures pertain to orthotropic, elastic, thick plates.

Figs. 40, 41: In these figures, the results obtained by the classical theory are compared with those obtained by Ashton et al. [36]. The cases considered in these figures are those for an orthotropic, elastic, thin plate. It may be seen that these results compare extremely well.

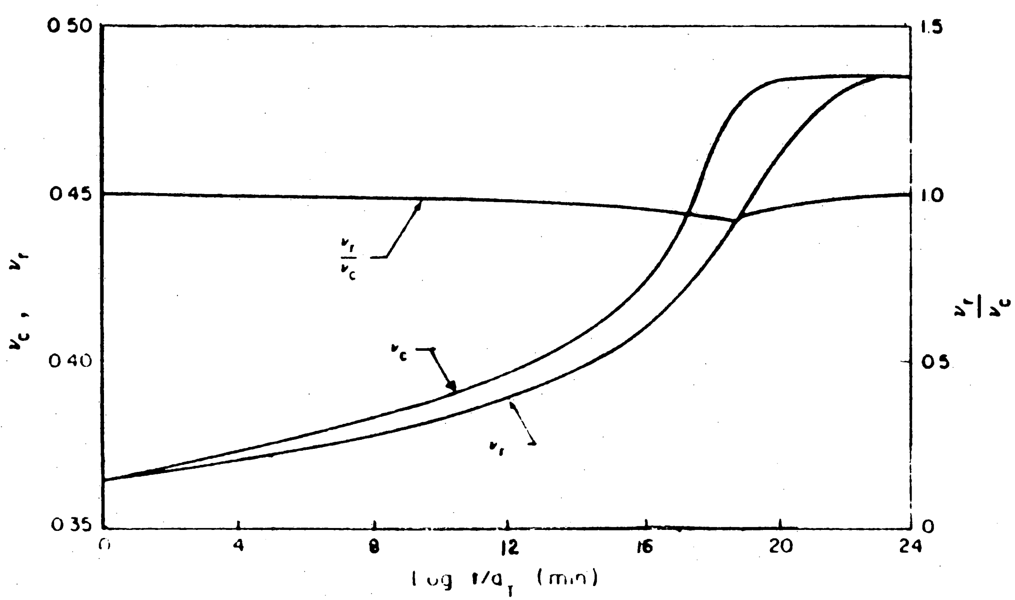
5.2 Conclusions

In this study, a stability analysis of orthotropic and transversely isotropic, viscoelastic rectangular plates has been done. The equations governing the stability were derived by using a direct approach as well as the correspondence principle technique. The material properties were obtained by considering the micromechanical relations developed by Aboudi [24]. In the modeling of the problem, the Boltzmann hereditary constitutive law has been used for a 3-D viscoelastic medium. The stability problem was analyzed in the Laplace transformed space in order to determine the asymptotic stability behavior.

The special cases considered in the numerical applications allow one to conclude the following:

1. The stability boundary determined for a viscoelastic plate is lower than that for its elastic counterpart.
2. Incorporation of transverse shear deformation effects yields a stability boundary which is much lower than in the case of its transversely rigid (classical) counterpart. This effect is more pronounced in the case of an orthotropic viscoelastic plate than in its isotropic counterpart, for which the effect is minimal.

3. The above conclusion remains valid in the case of their elastic counterparts.
4. The results show that σ_{33} may influence the viscoelastic stability boundary in a strong and beneficial way, especially in the case of an orthotropic viscoelastic plate.
5. In addition, we may conclude that transverse shear deformation effects are more pronounced in viscoelastic plates than in their elastic counterparts. In this regard, we observe that for the special case when viscoelasticity is considered in the transverse shear direction only, the exclusion of transverse shear deformation effects results in identical solutions for the viscoelastic and elastic plates (see Sec. 4.5).
6. The analysis performed here allows one to obtain the nature of loss of stability, i.e., either by divergence or by flutter. It was observed that for an isotropic, viscoelastic plate the instability occurs by divergence only. However, for an orthotropic viscoelastic plate, the instability may result both by divergence and by flutter.
7. It was observed that the boundary layer effect considered for an isotropic viscoelastic plate has no effect on the stability boundary.
8. It is observed that for large aspect ratios (L_1/L_2) the stability boundary tends to coincide with the classical Kirchhoff theory of plates.



$\nu_c \rightarrow$ Poisson's ratio for creep test (constant stress)
$\nu_r \rightarrow$ Poisson's ratio for relaxation test (constant strain)
$a_T \rightarrow$ Shift factor

Figure 1. **Variation of Poisson's ratio for the epoxy matrix.**

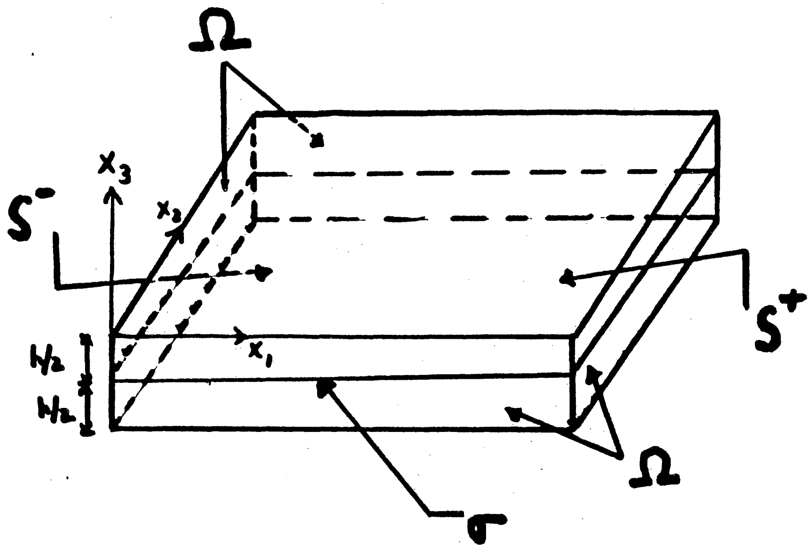


Figure 2. Layup of plate and coordinate system.

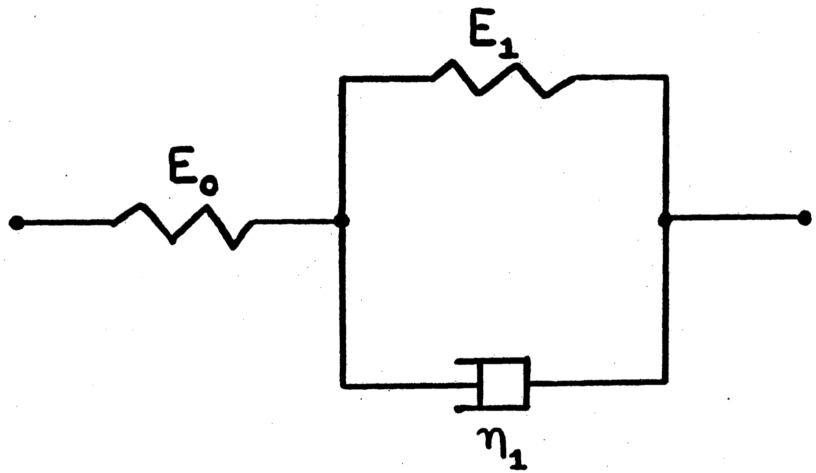


Figure 3. 3-Parameter solid element.

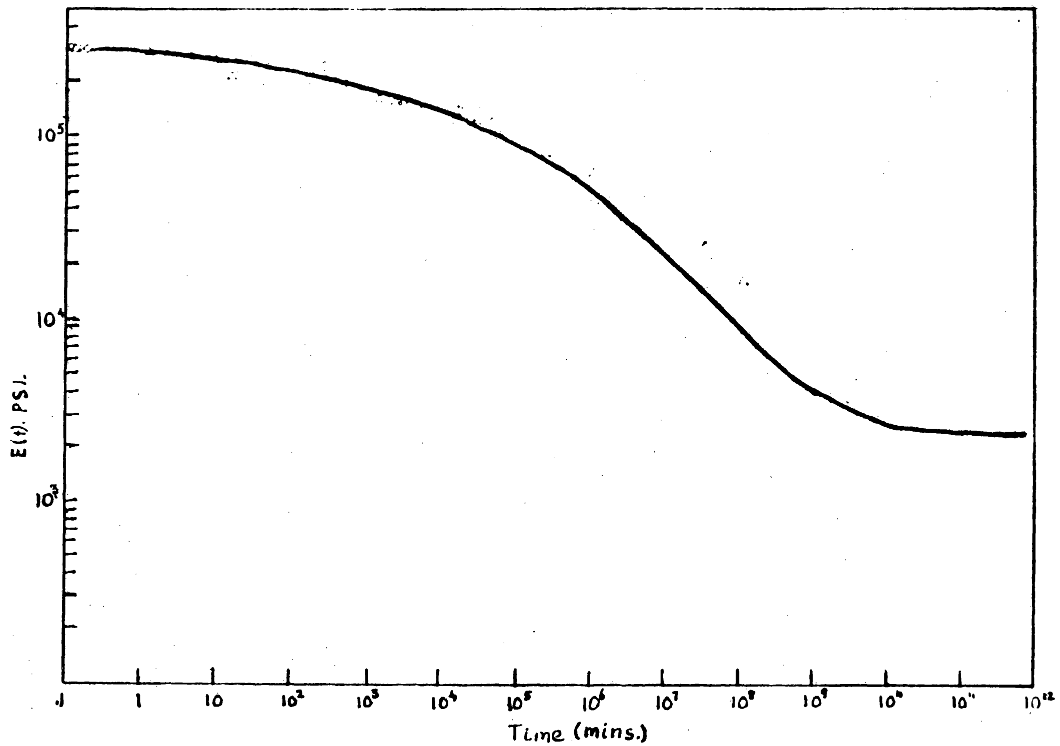


Figure 4. Variation of Young's modulus for the epoxy matrix.

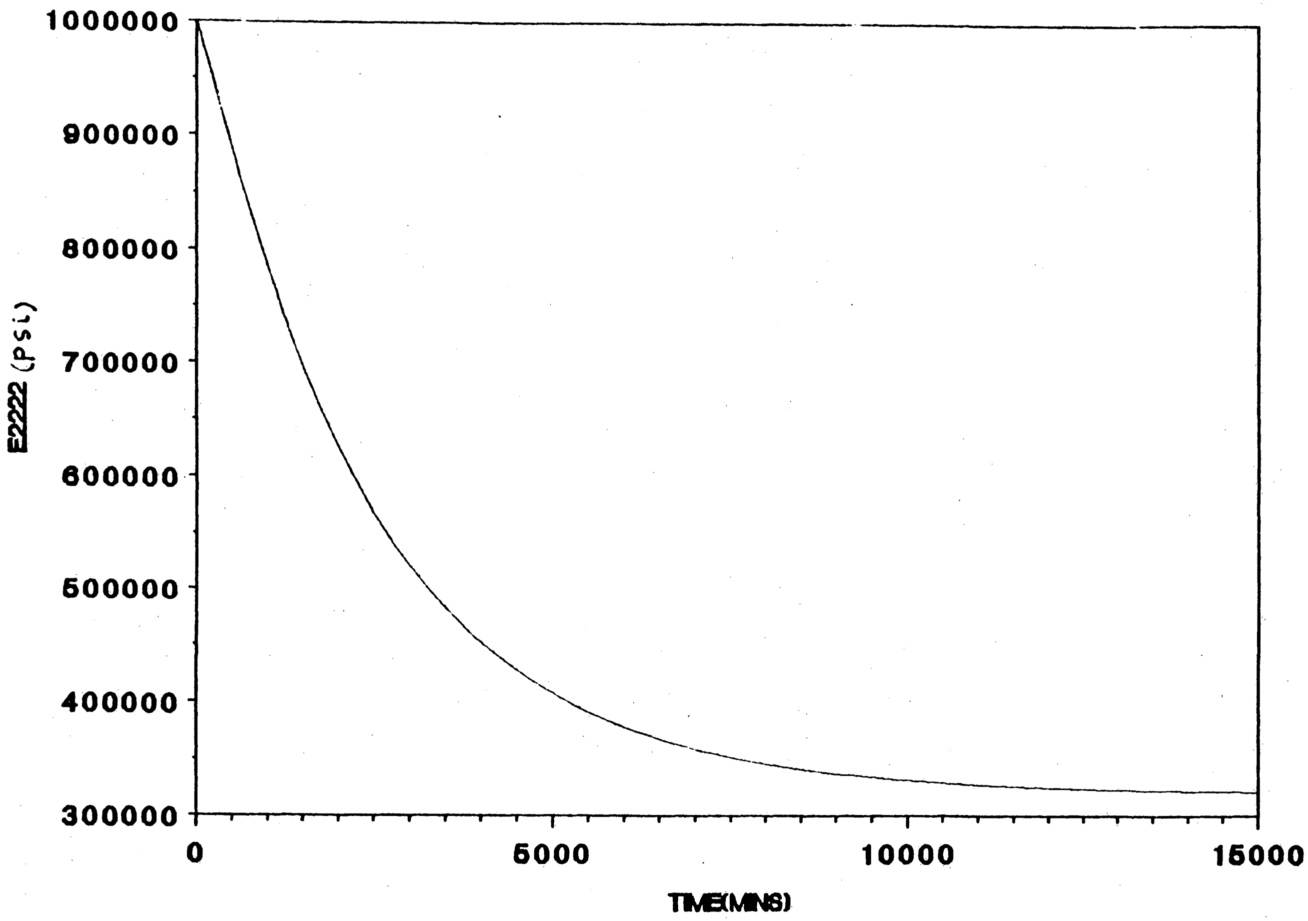


Figure 5. Material property for the orthotropic plate - $E_{2222} = E_{3333}$

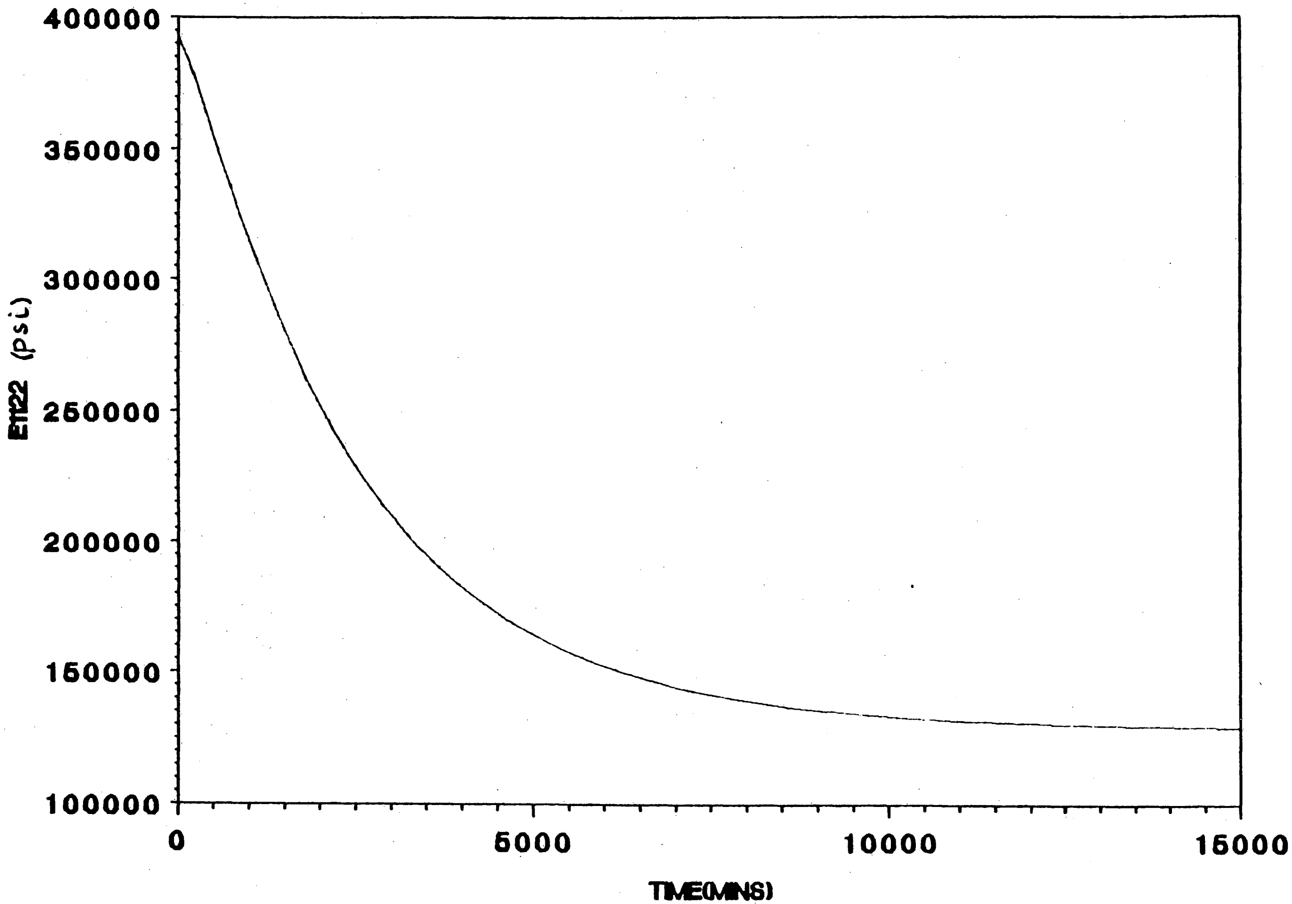


Figure 6. Material property for the orthotropic plate - $E_{1122} = E_{1133}$

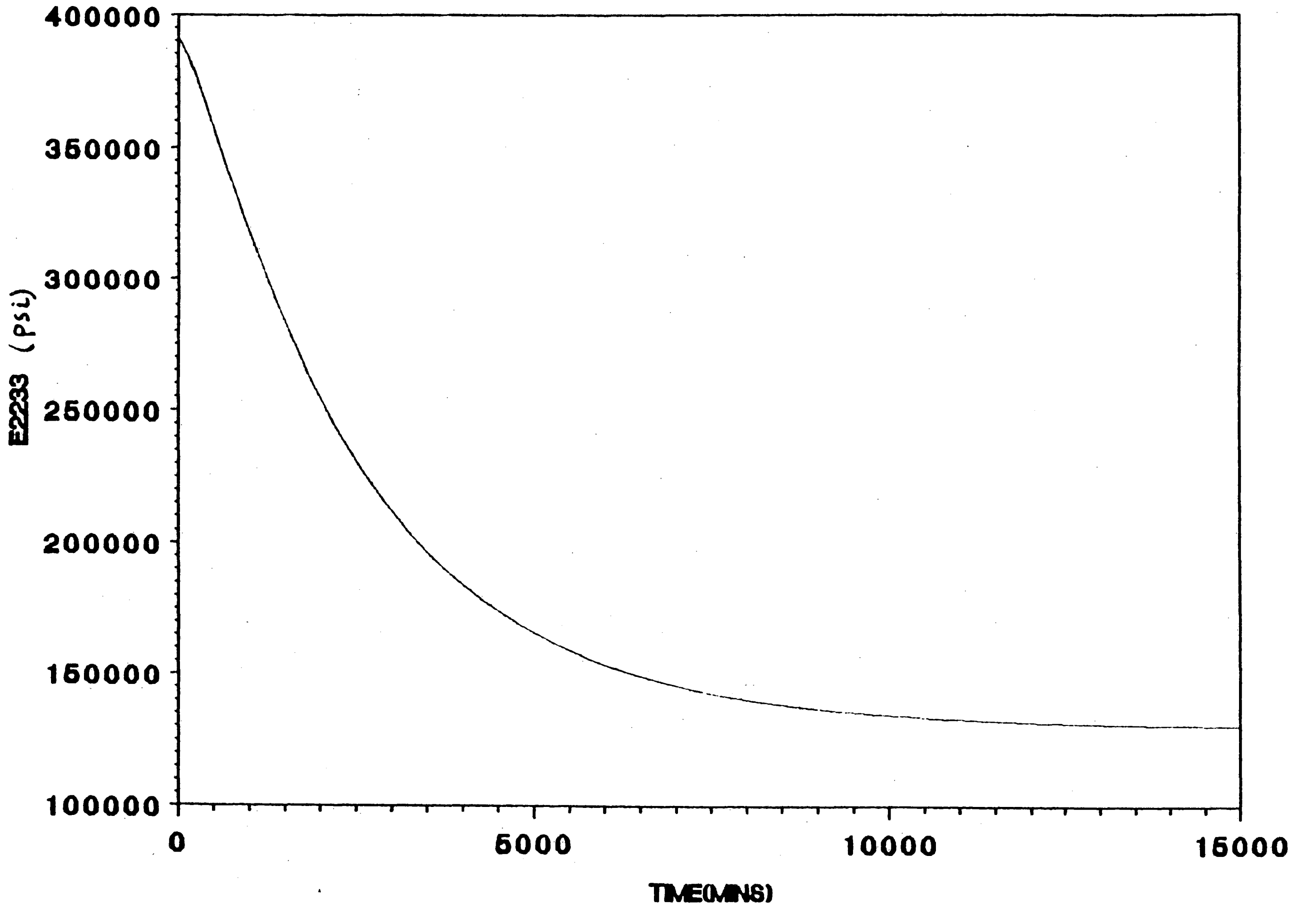


Figure 7. Material property for the orthotropic plate - E_{2233}

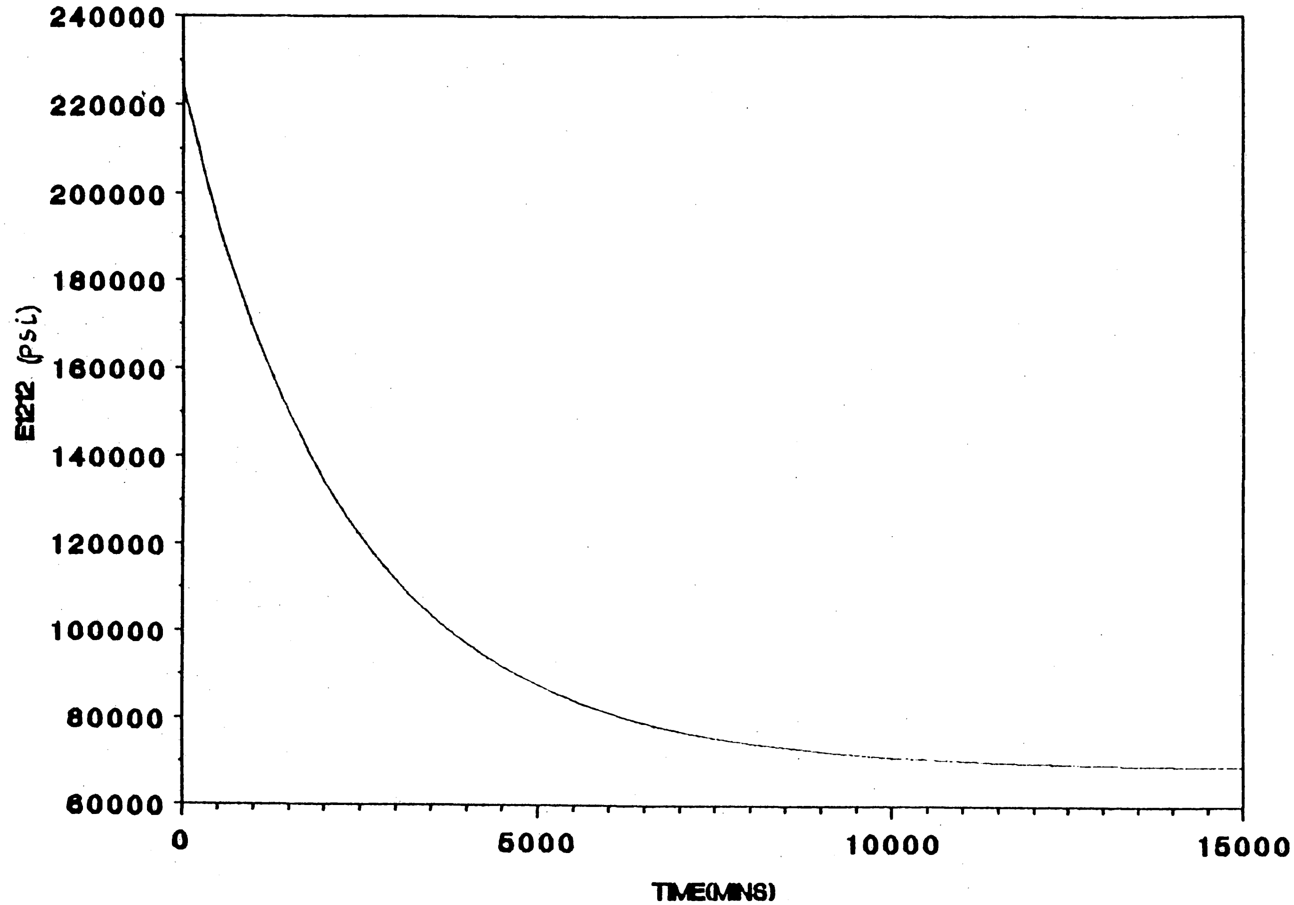


Figure 8. Material property for the orthotropic plate - $E_{1212} = E_{1313}$

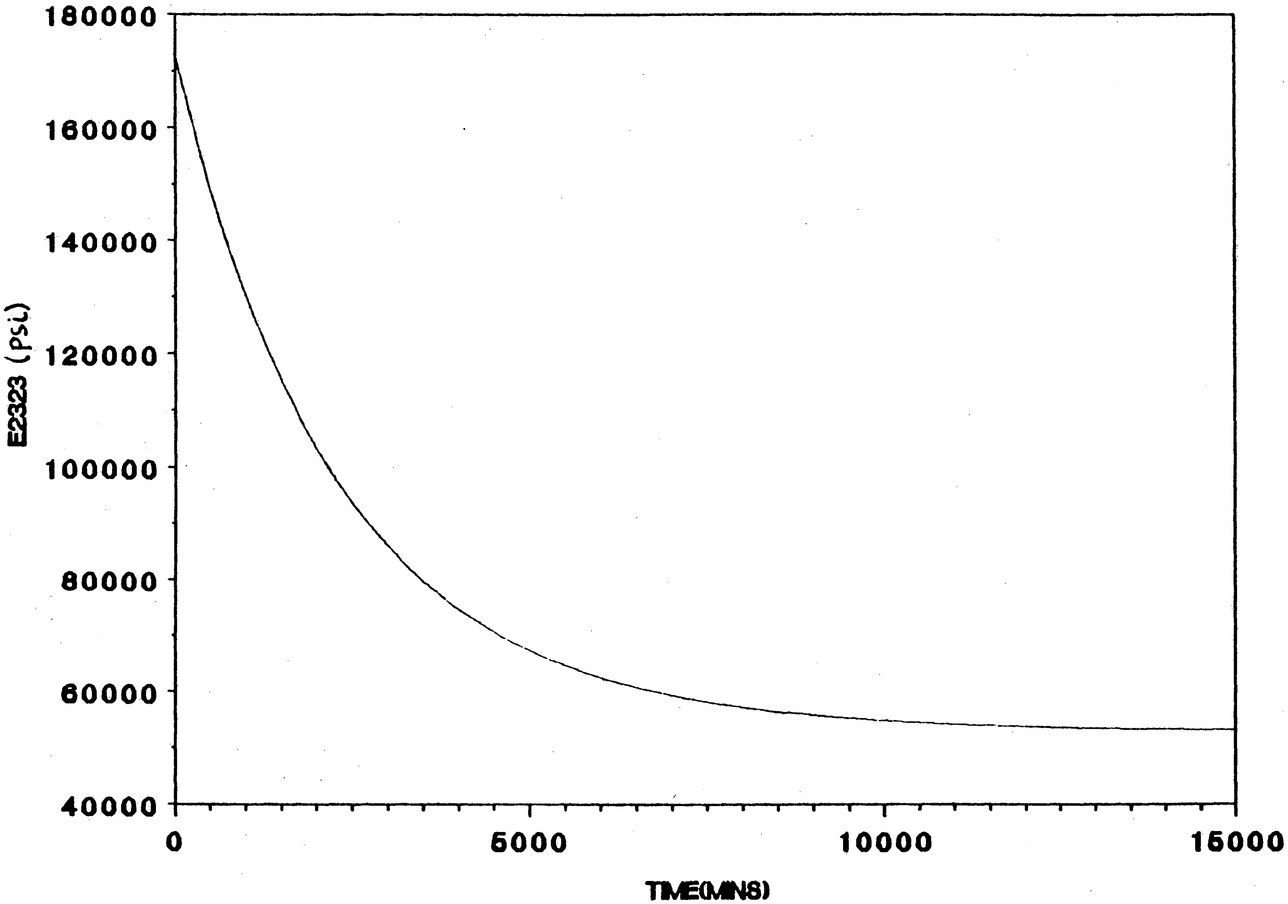


Figure 9. Material property for the orthotropic plate - E_{2323}

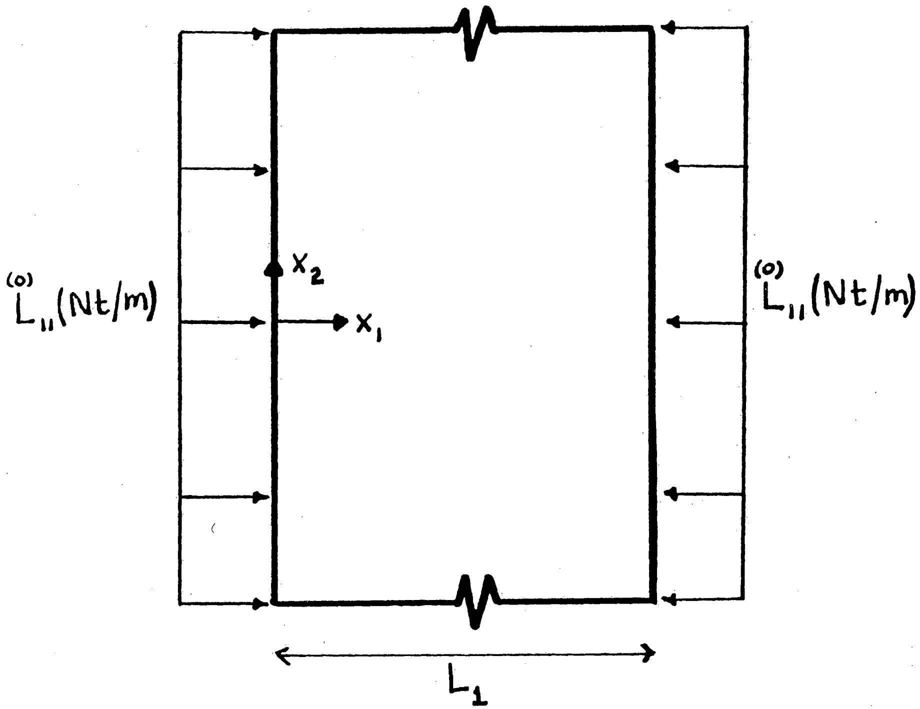


Figure 10. Plate in cylindrical bending

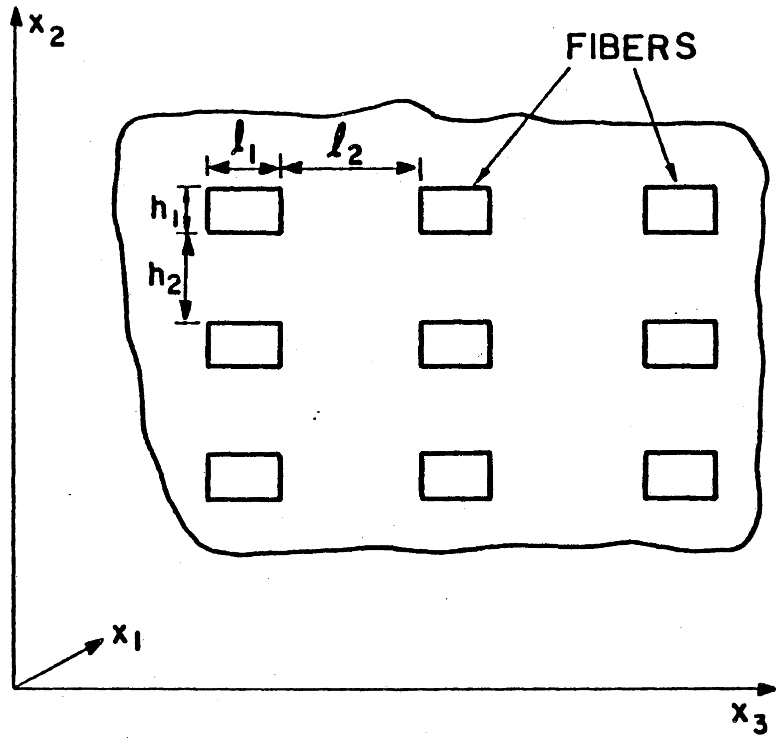


Figure 11. Arrangement of fibers in matrix.

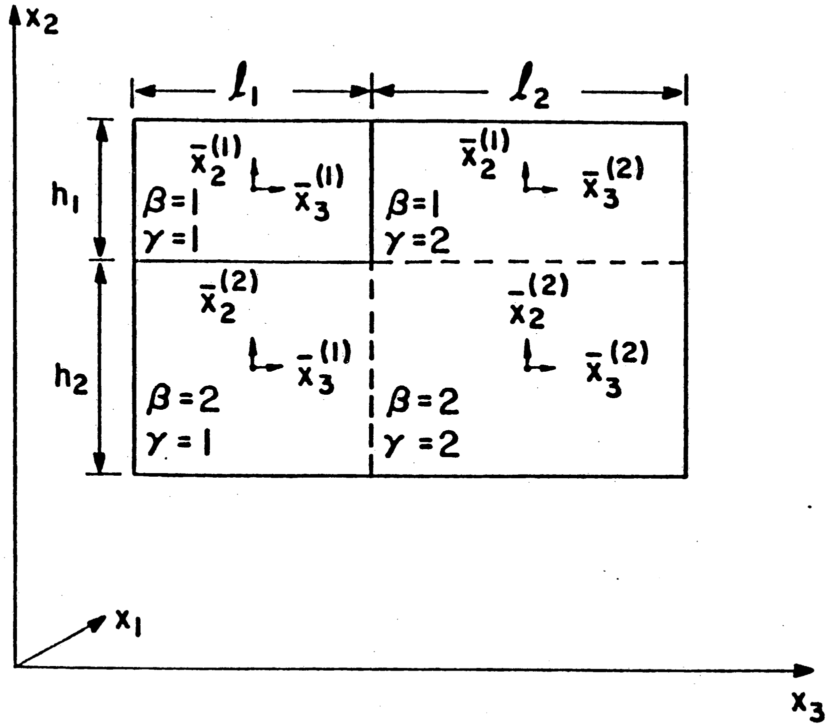


Figure 12. Representative volume element for a fiber-matrix composite.

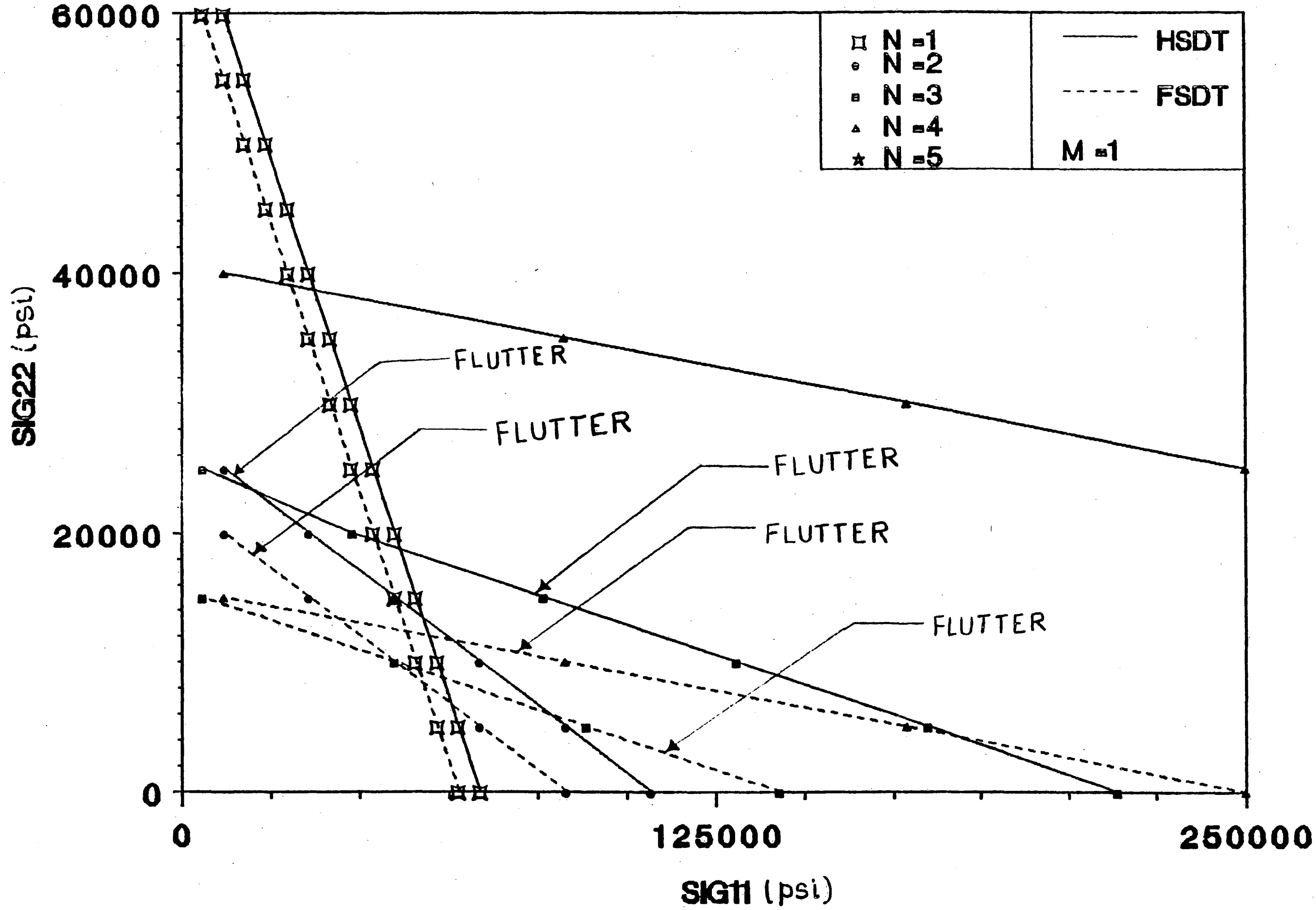


Figure 13. Stability boundary for orthotropic viscoelastic plate; L/h = 4.8; biaxial compression; δ_A = 1

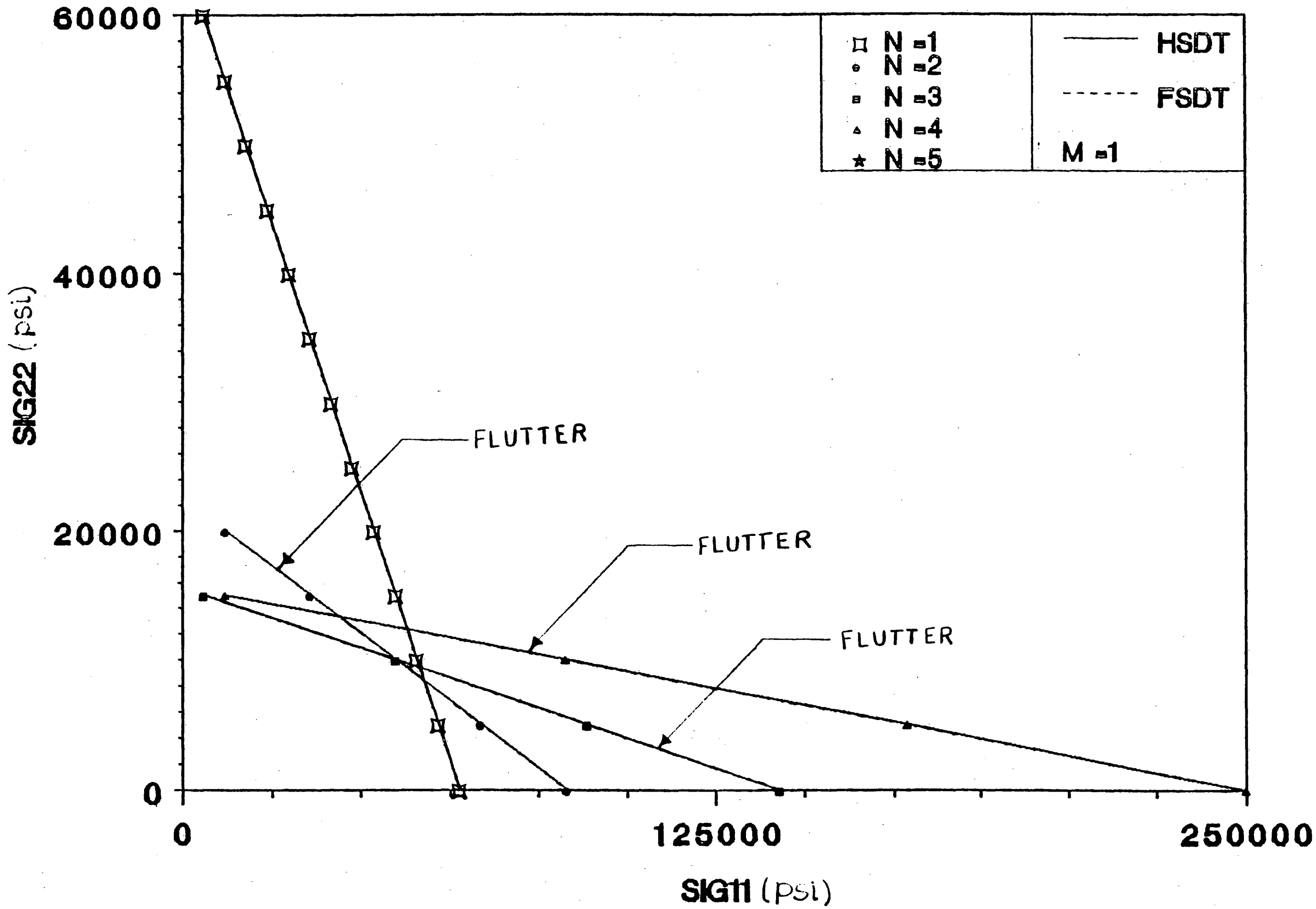


Figure 14. Stability boundary for orthotropic viscoelastic plate; L/h = 4.8; biaxial compression; δ_A = 0

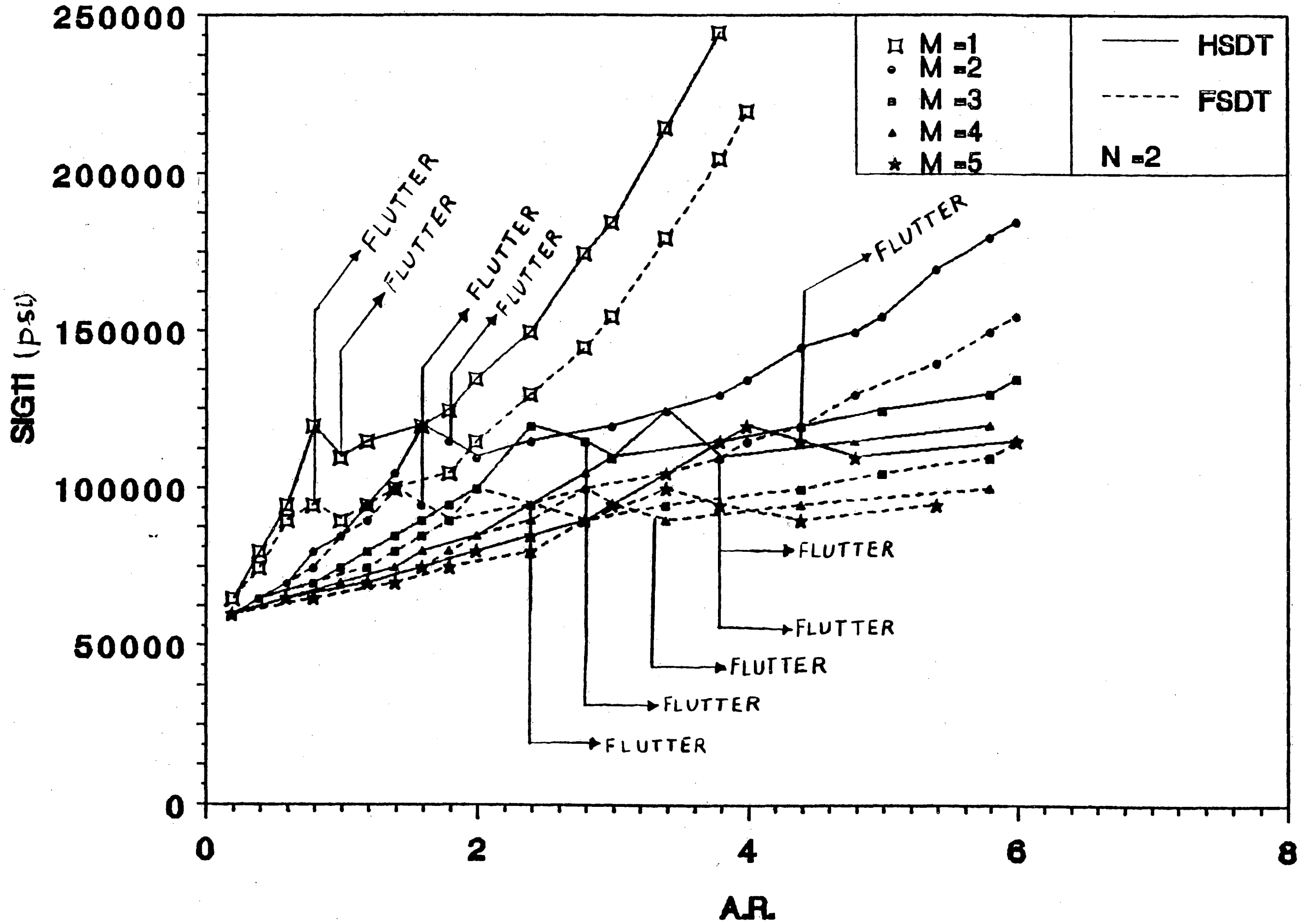


Figure 15. Stability boundary for orthotropic viscoelastic plate; L/h = 4.8; uniaxial compression; δ_A = 1

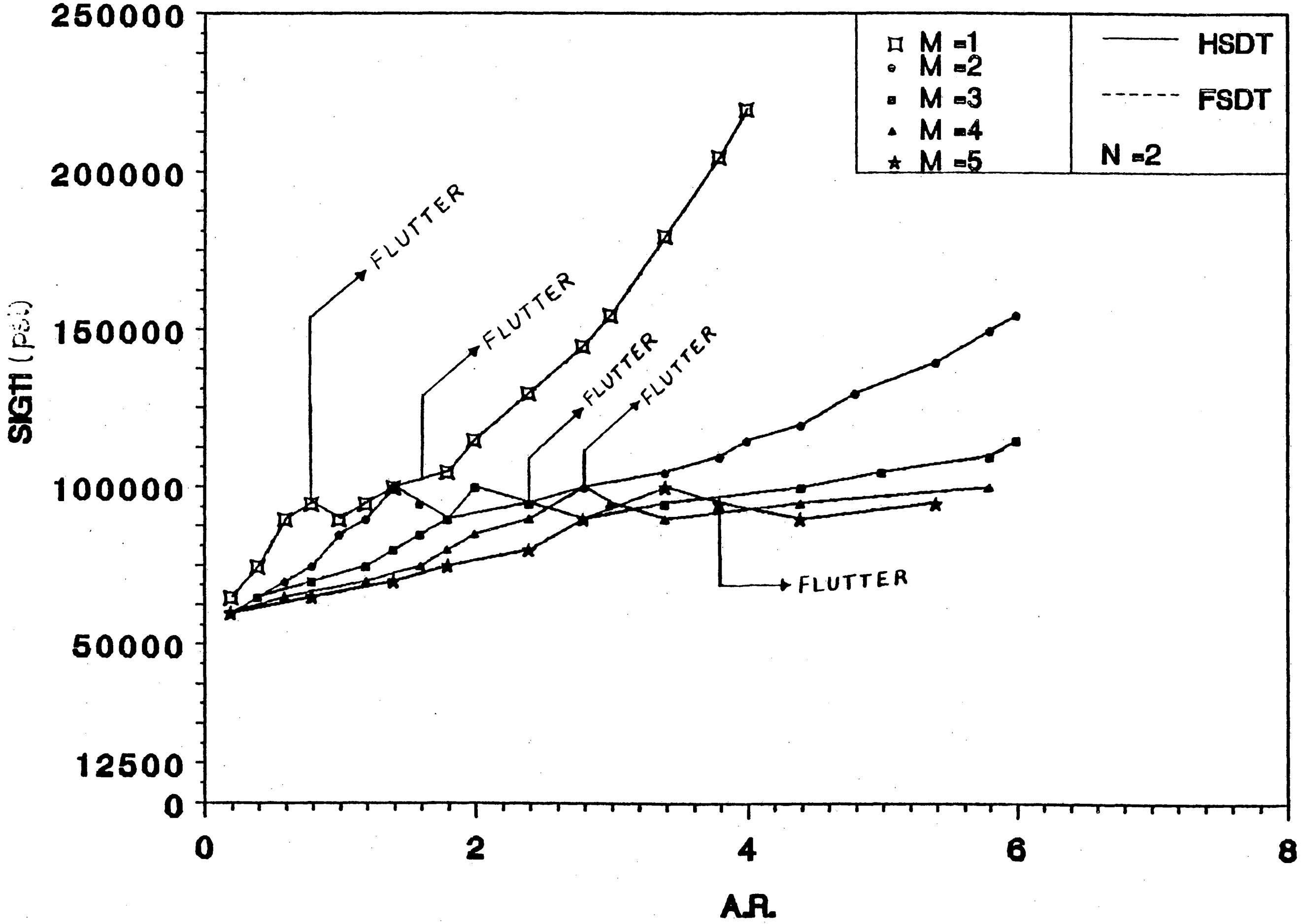


Figure 16. Stability boundary for orthotropic viscoelastic plate; L/h = 4.8; uniaxial compression; $\delta_A = 0$

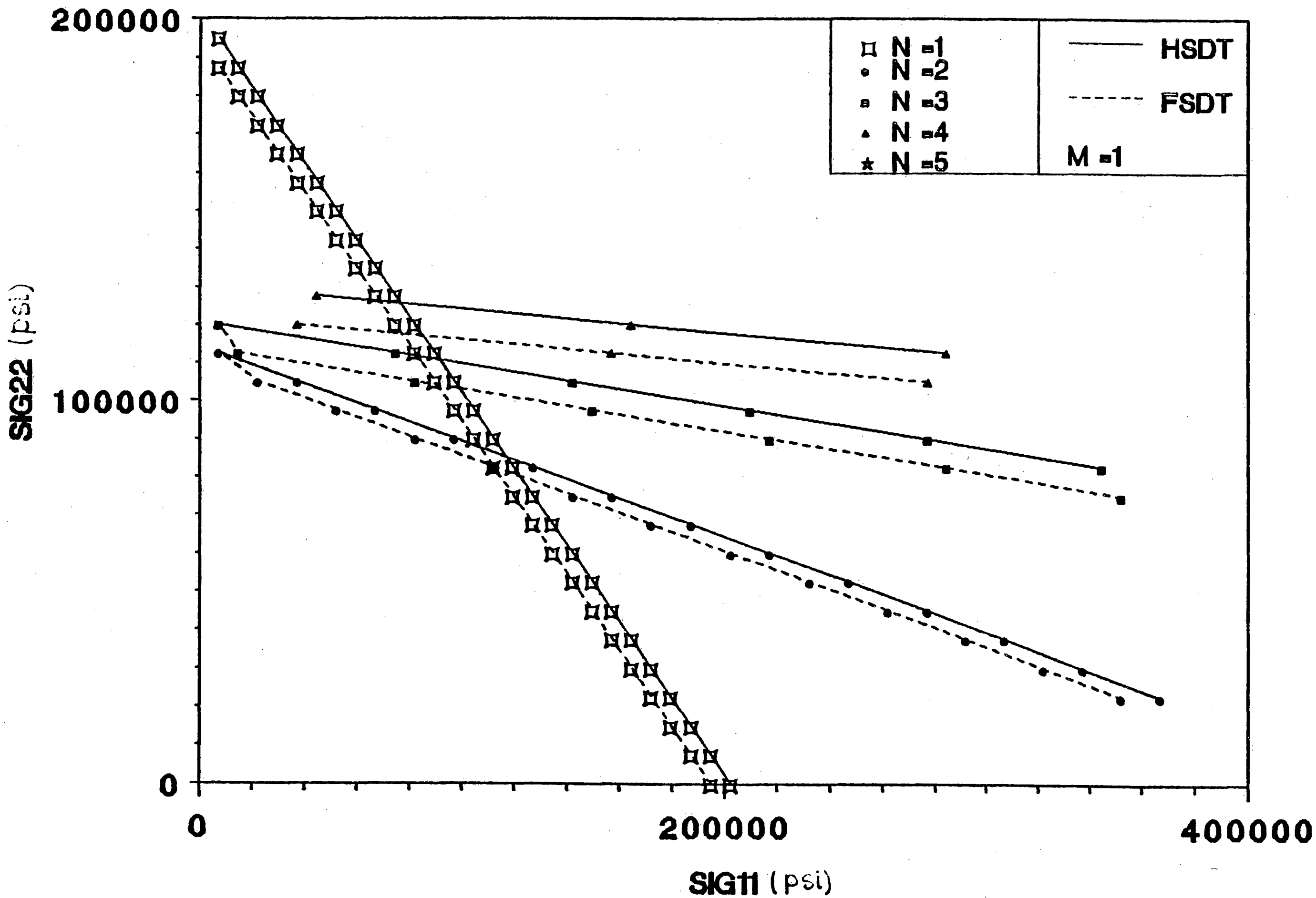


Figure 17. Stability boundary for orthotropic elastic plate; L/h = 4.8; biaxial compression; $\delta_A = 1$ or $\delta_A = 0$

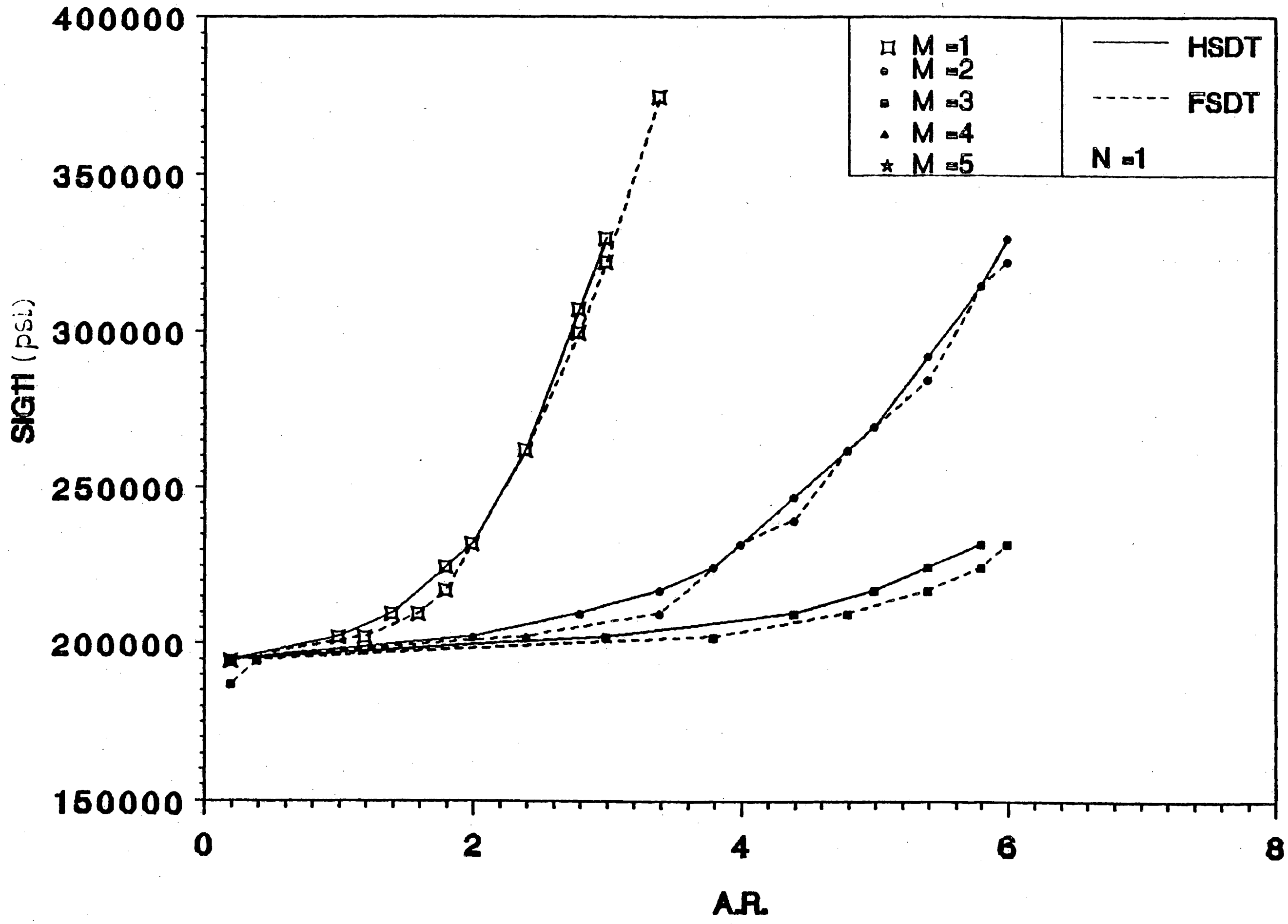


Figure 18. Stability boundary for orthotropic elastic plate; L/h = 4.8; uniaxial compression; $\delta_A = 1$ or $\delta_A = 0$

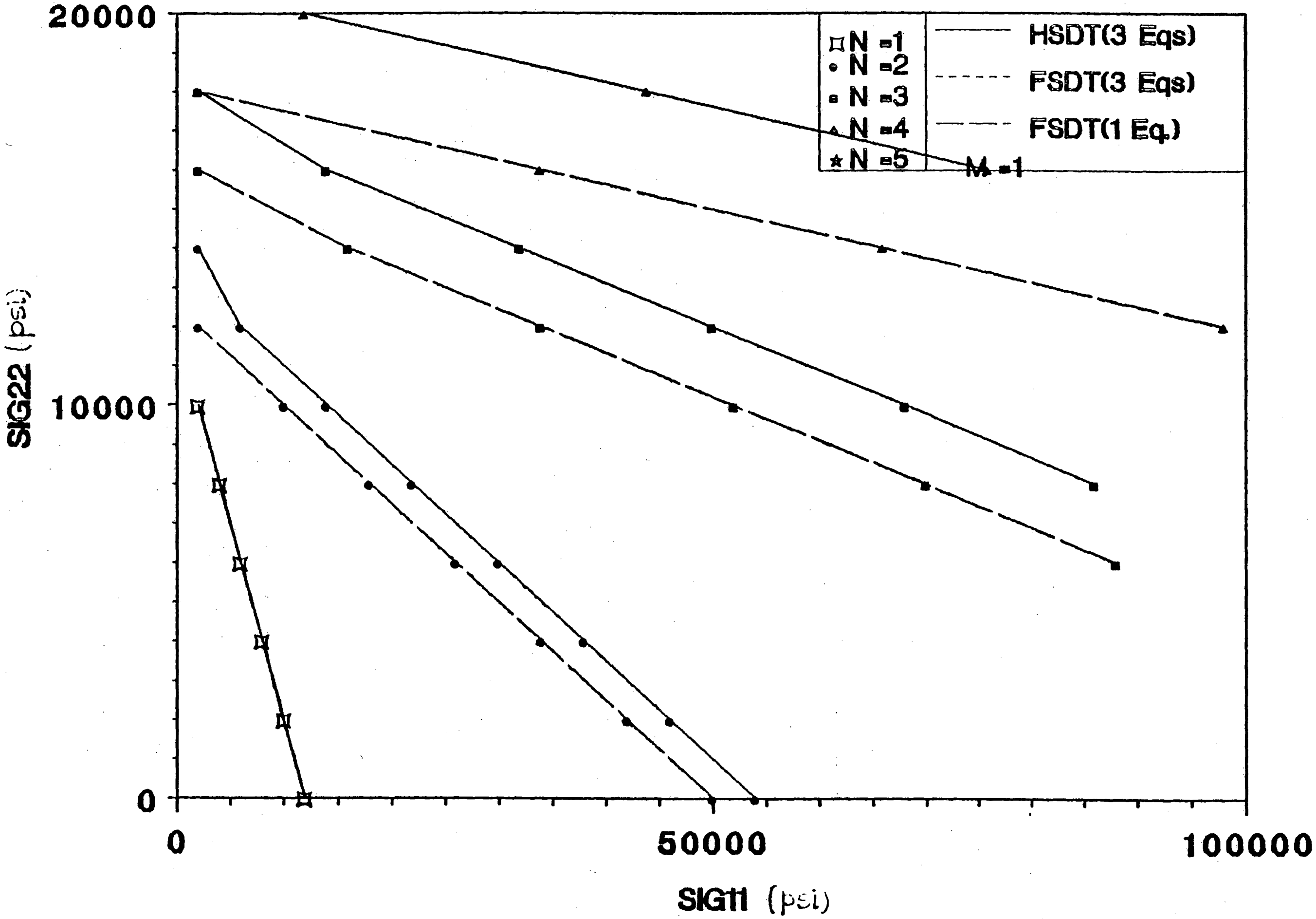


Figure 19. Stability boundary for isotropic viscoelastic plate; L/h = 4.8; biaxial compression; $\delta_A = 1$

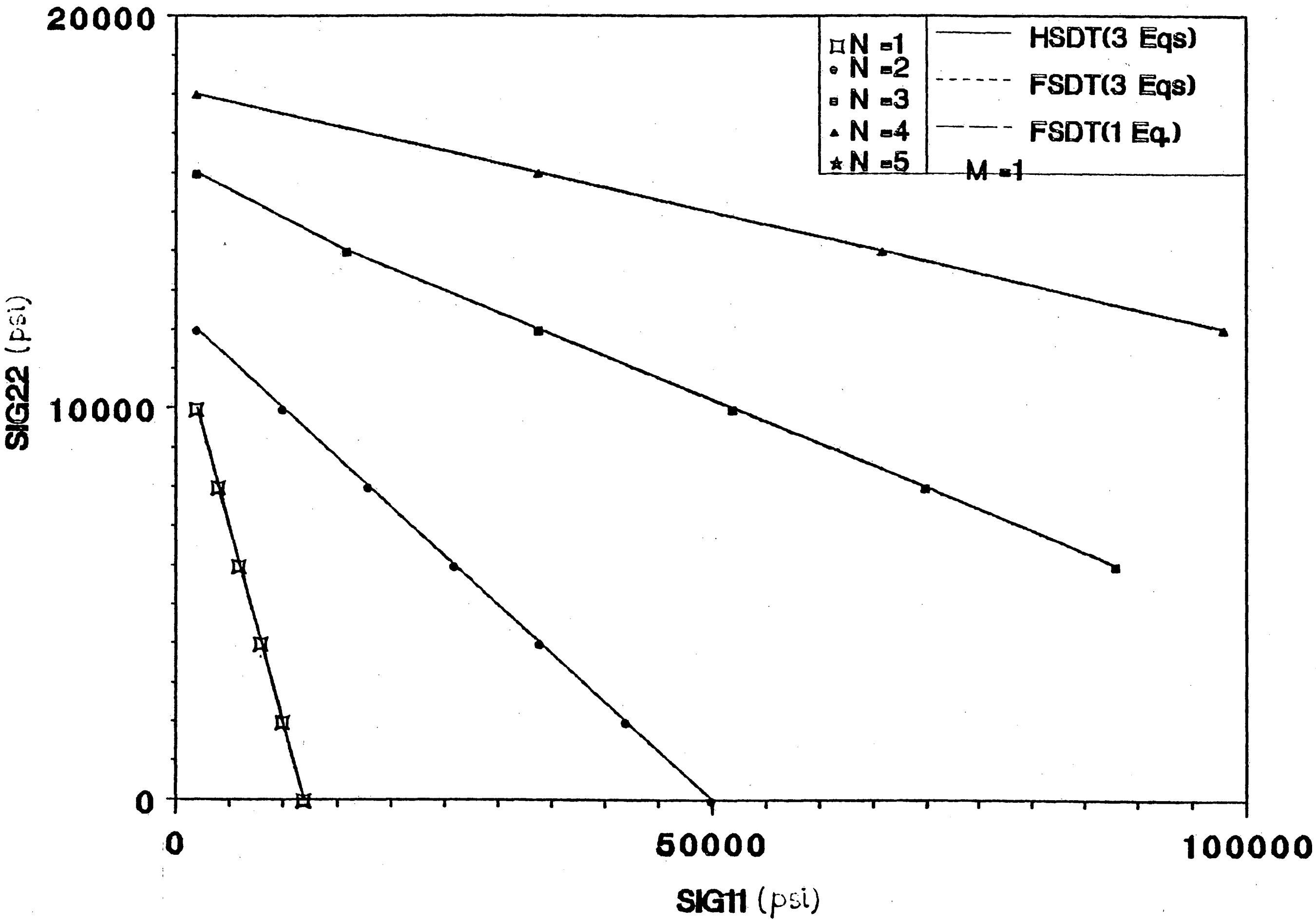


Figure 20. Stability boundary for isotropic viscoelastic plate; L/h = 4.8; biaxial compression; δ_A = 0

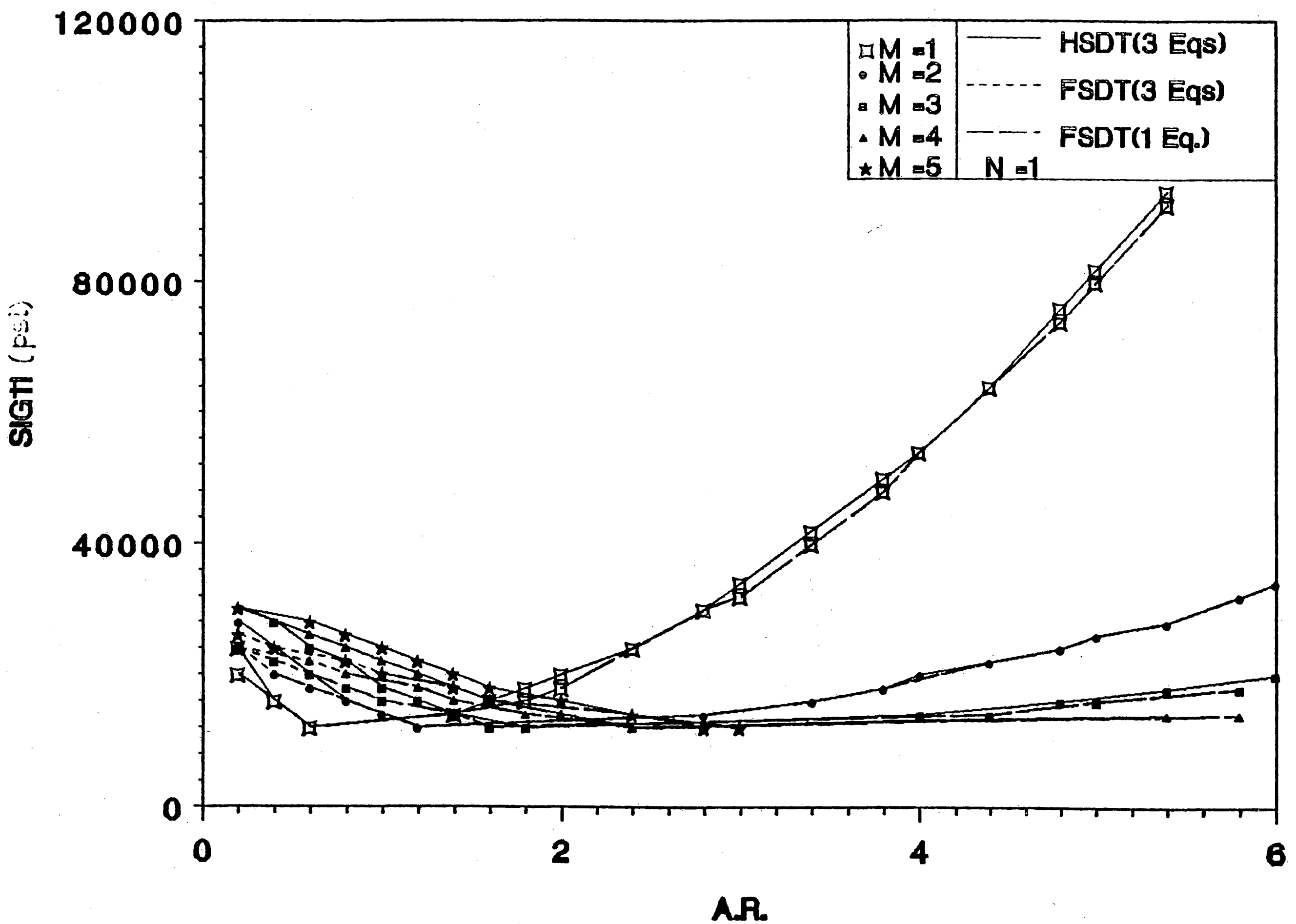


Figure 21. Stability boundary for isotropic viscoelastic plate; L/h = 4.8; uniaxial compression; $\delta_A = 1$ or $\delta_A = 0$

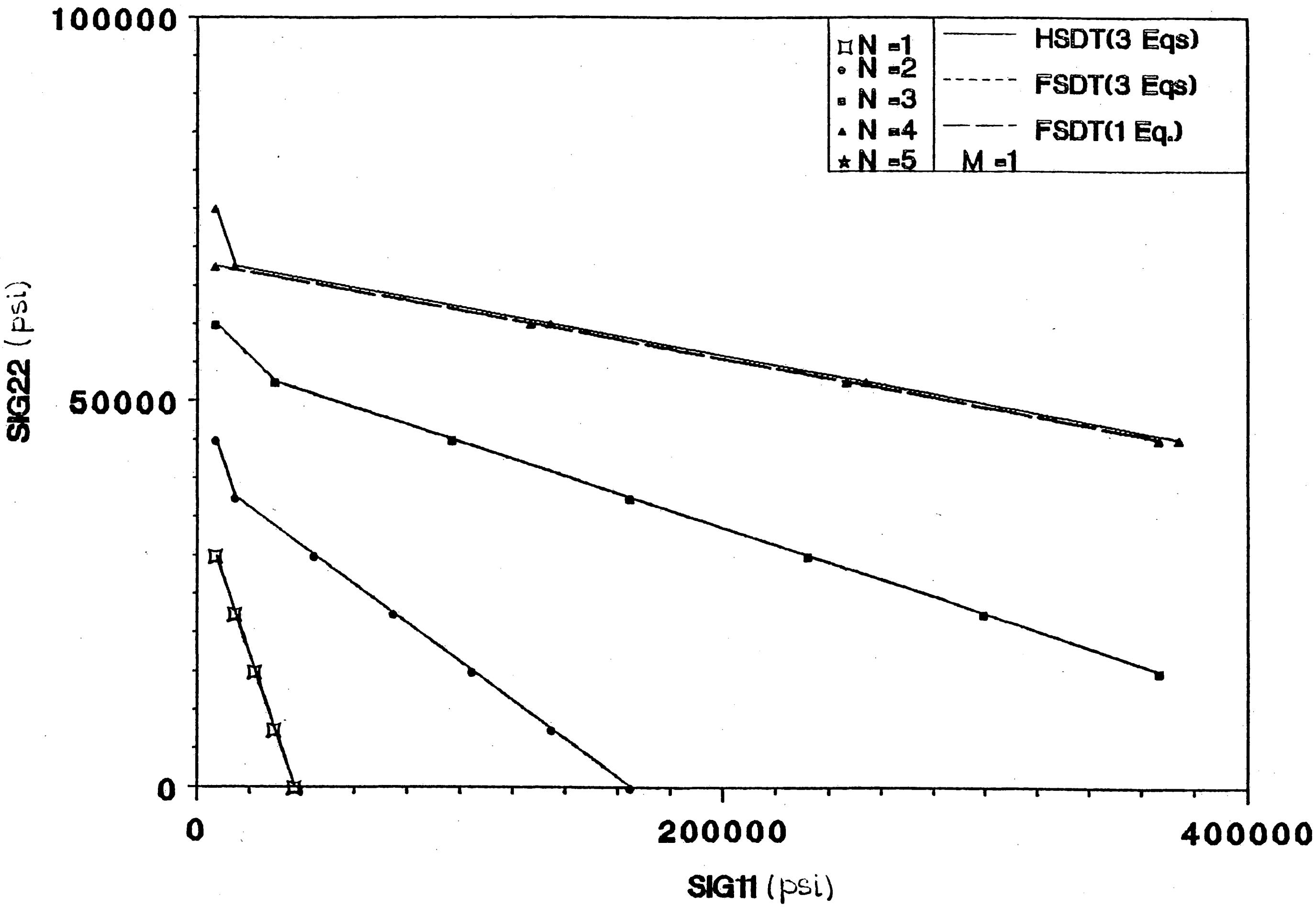


Figure 22. Stability boundary for isotropic elastic plate; L/h = 4.8; biaxial compression; δ_A = 1 or δ_A = 0

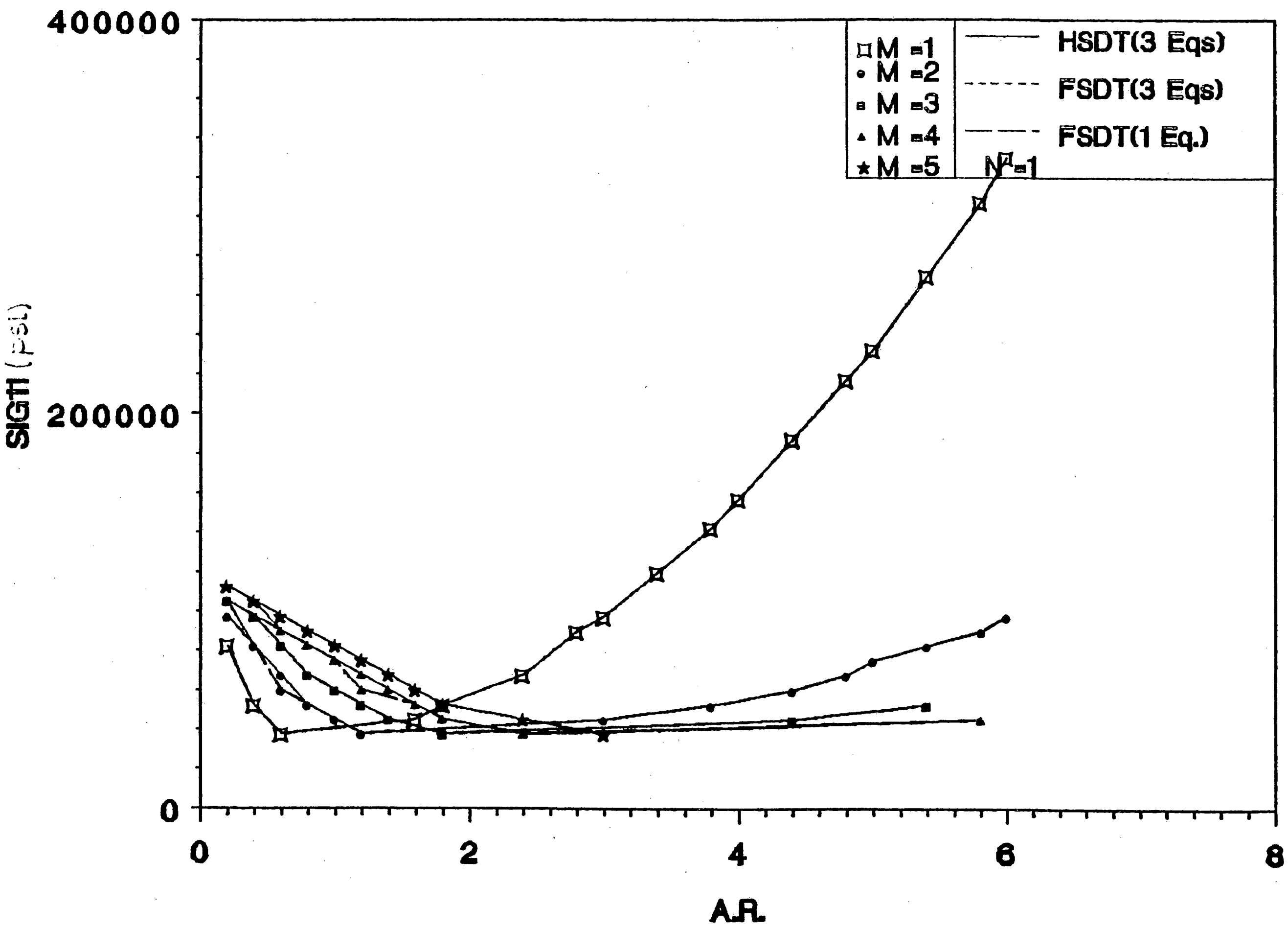


Figure 23. Stability boundary for isotropic elastic plate; L/h = 4.8; uniaxial compression; $\delta_A = 1$ or $\delta_A = 0$

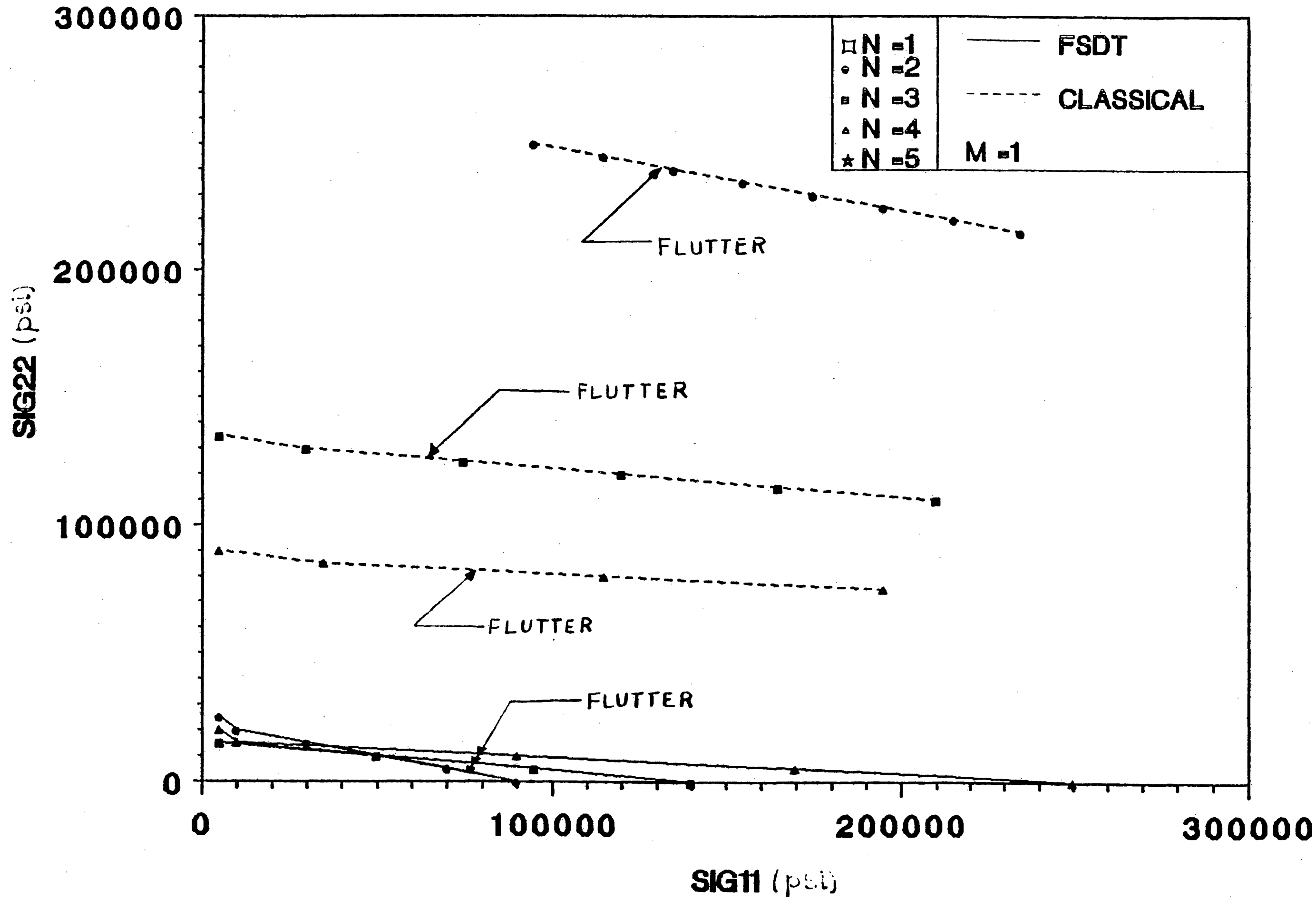


Figure 24. Comparison of stability boundaries for orthotropic viscoelastic plate; L/h = 4.8; biaxial compression

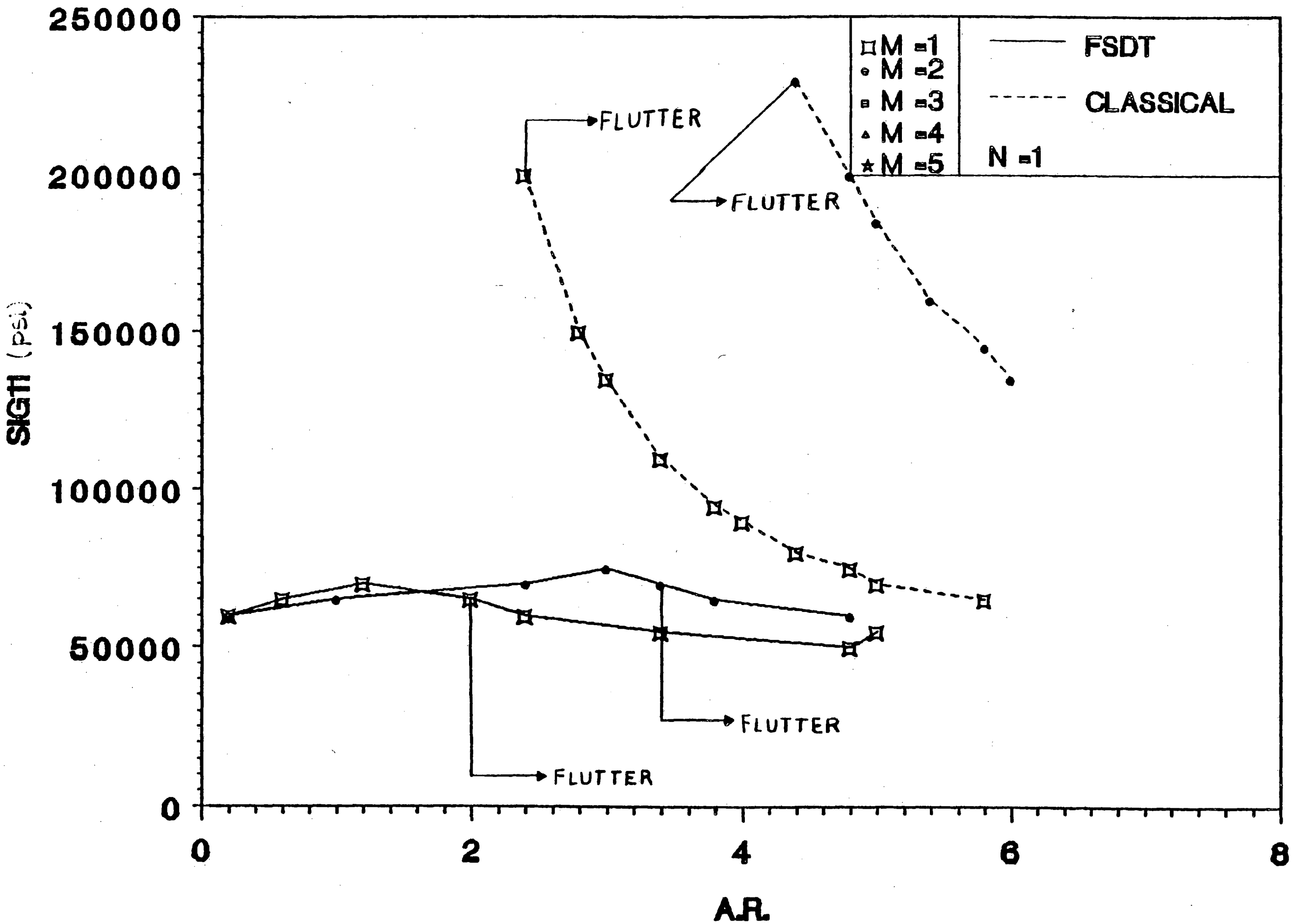


Figure 25. Comparison of stability boundaries for orthotropic viscoelastic plate; L/h = 4.8; uniaxial compression

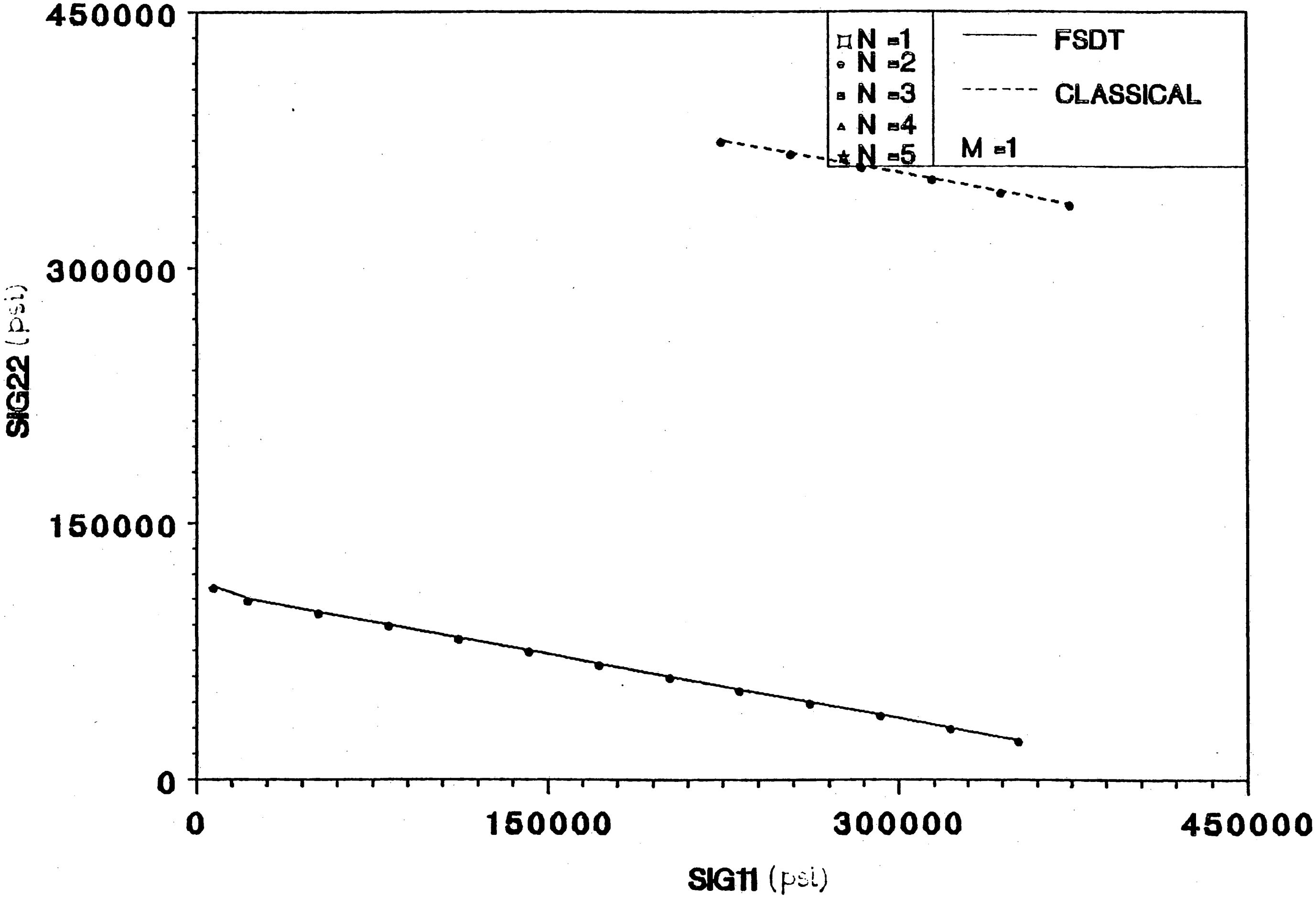


Figure 26. Comparison of stability boundaries for orthotropic elastic plate; L/h = 4.8; biaxial compression

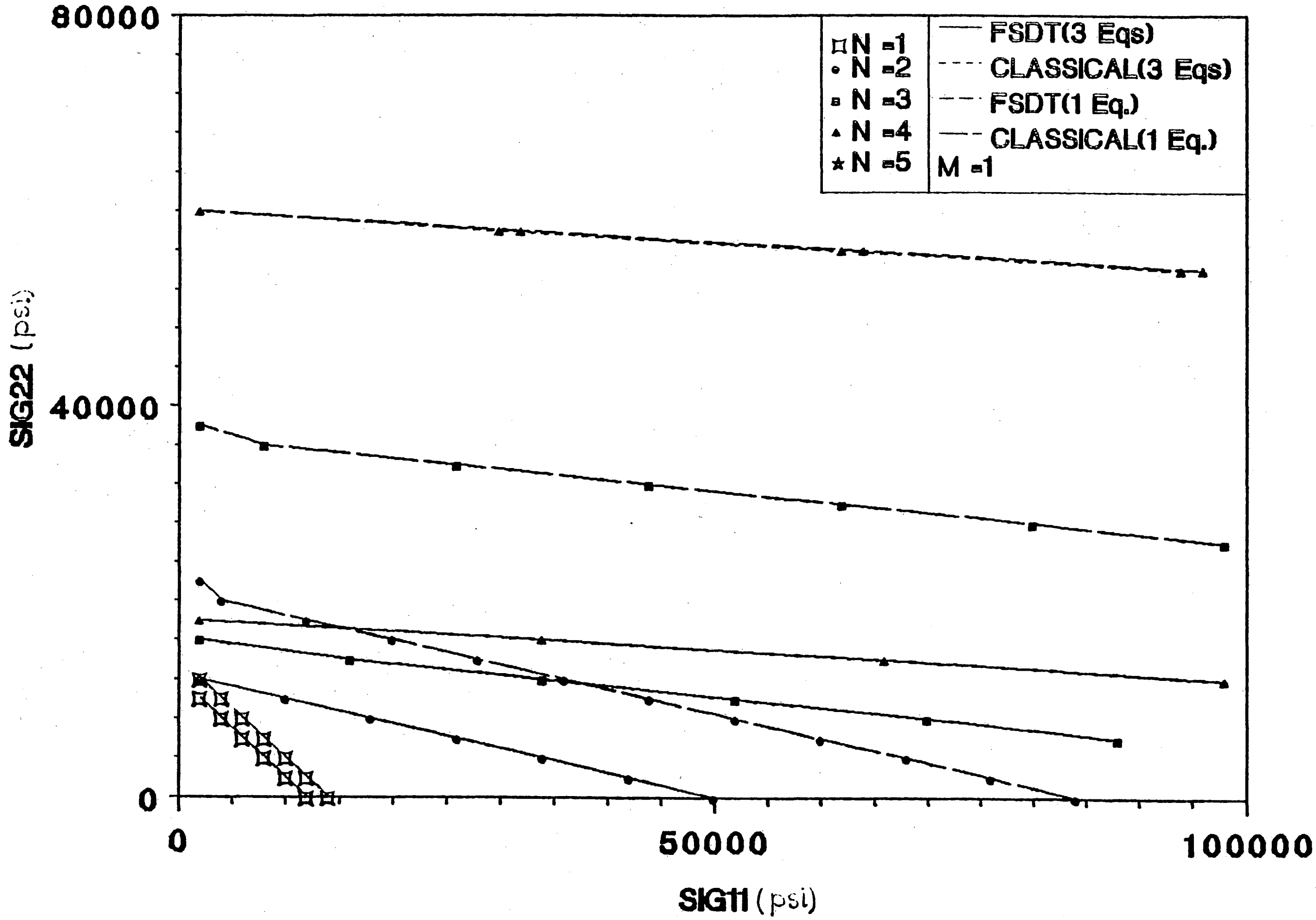


Figure 27. Comparison of stability boundaries for isotropic viscoelastic plate; L/h = 4.8; biaxial compression

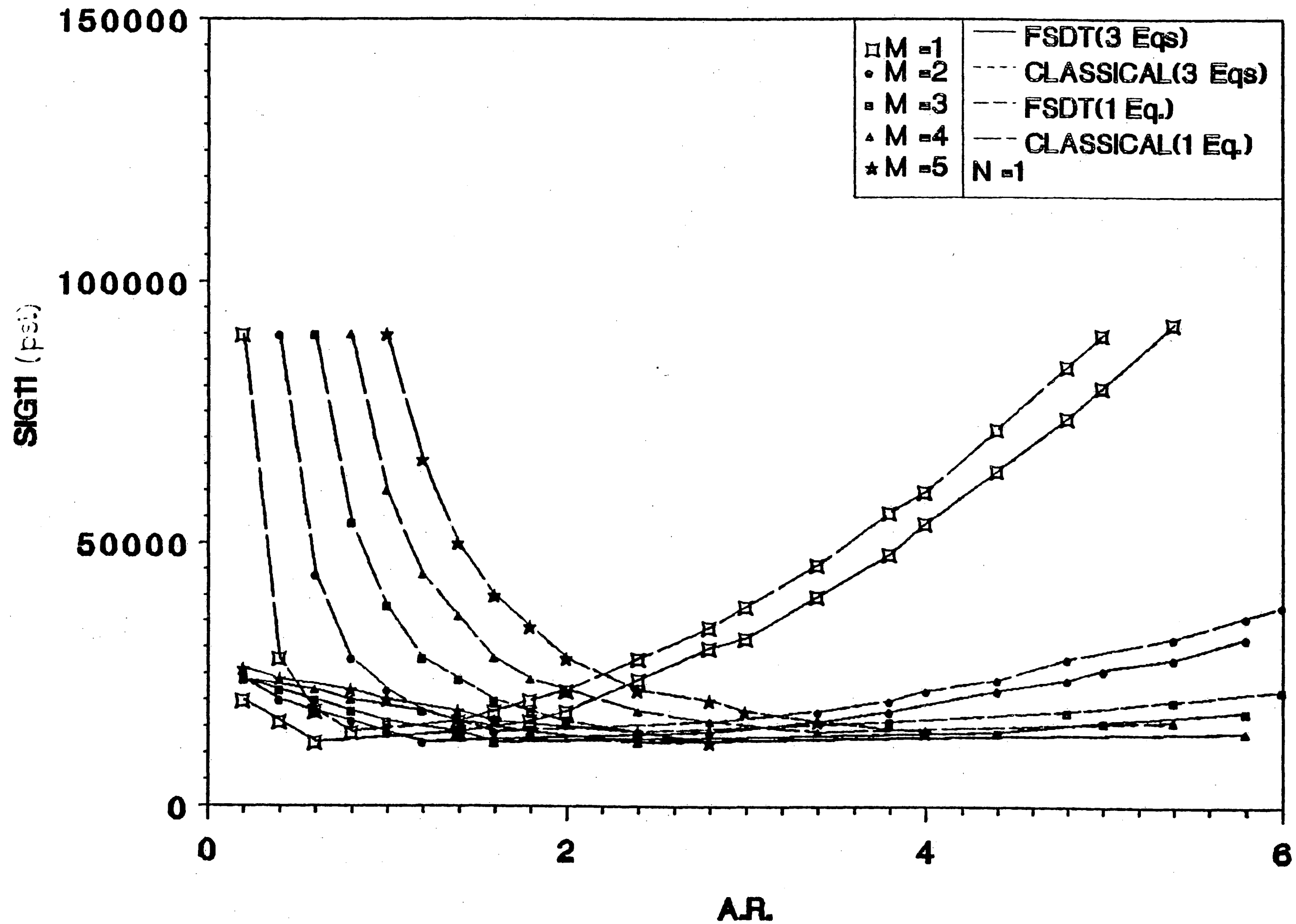


Figure 28. Comparison of stability boundaries for isotropic viscoelastic plate; L/h = 4.8; uniaxial compression

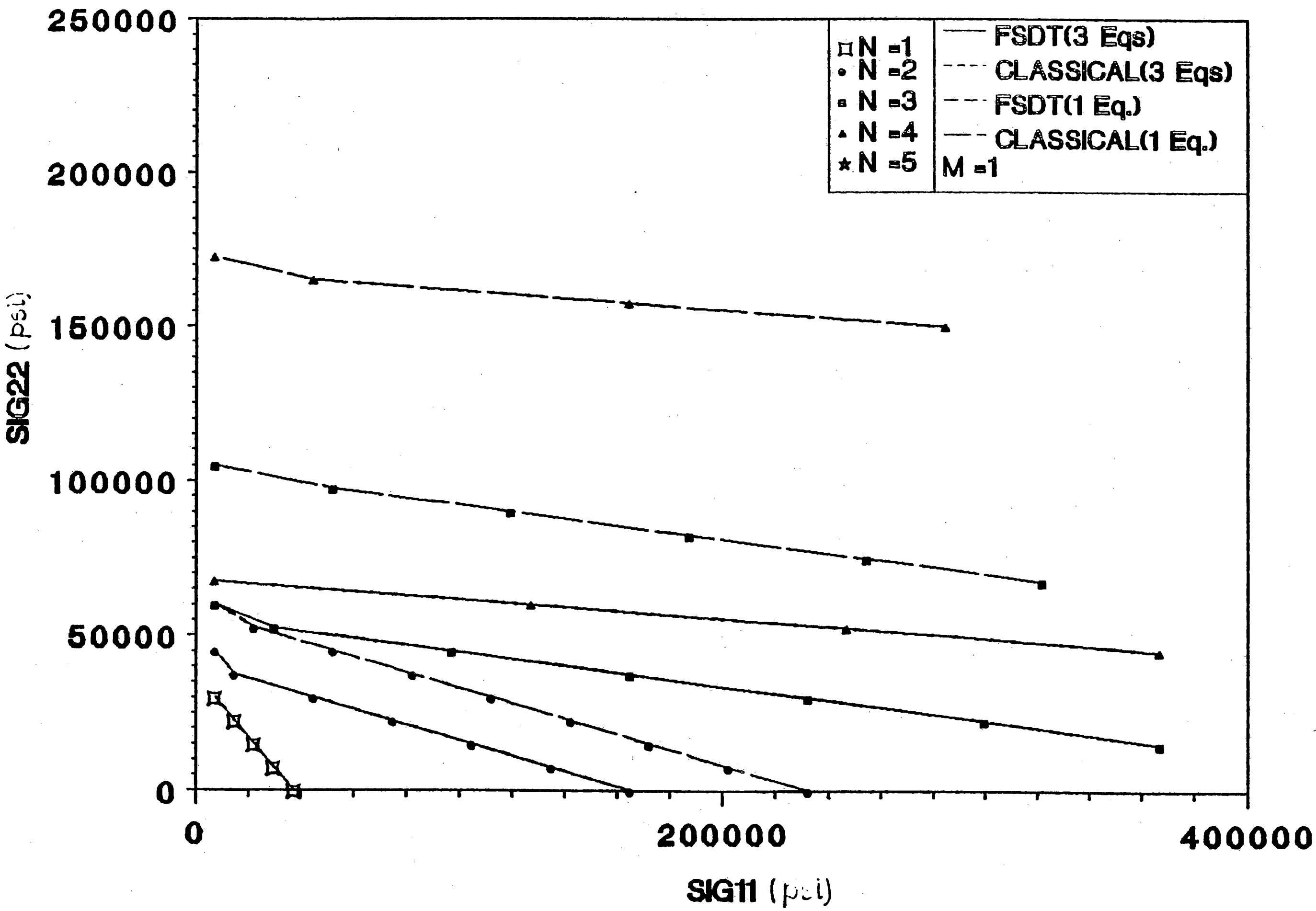


Figure 29. Comparison of stability boundaries for isotropic elastic plate; L/h = 4.8; biaxial compression

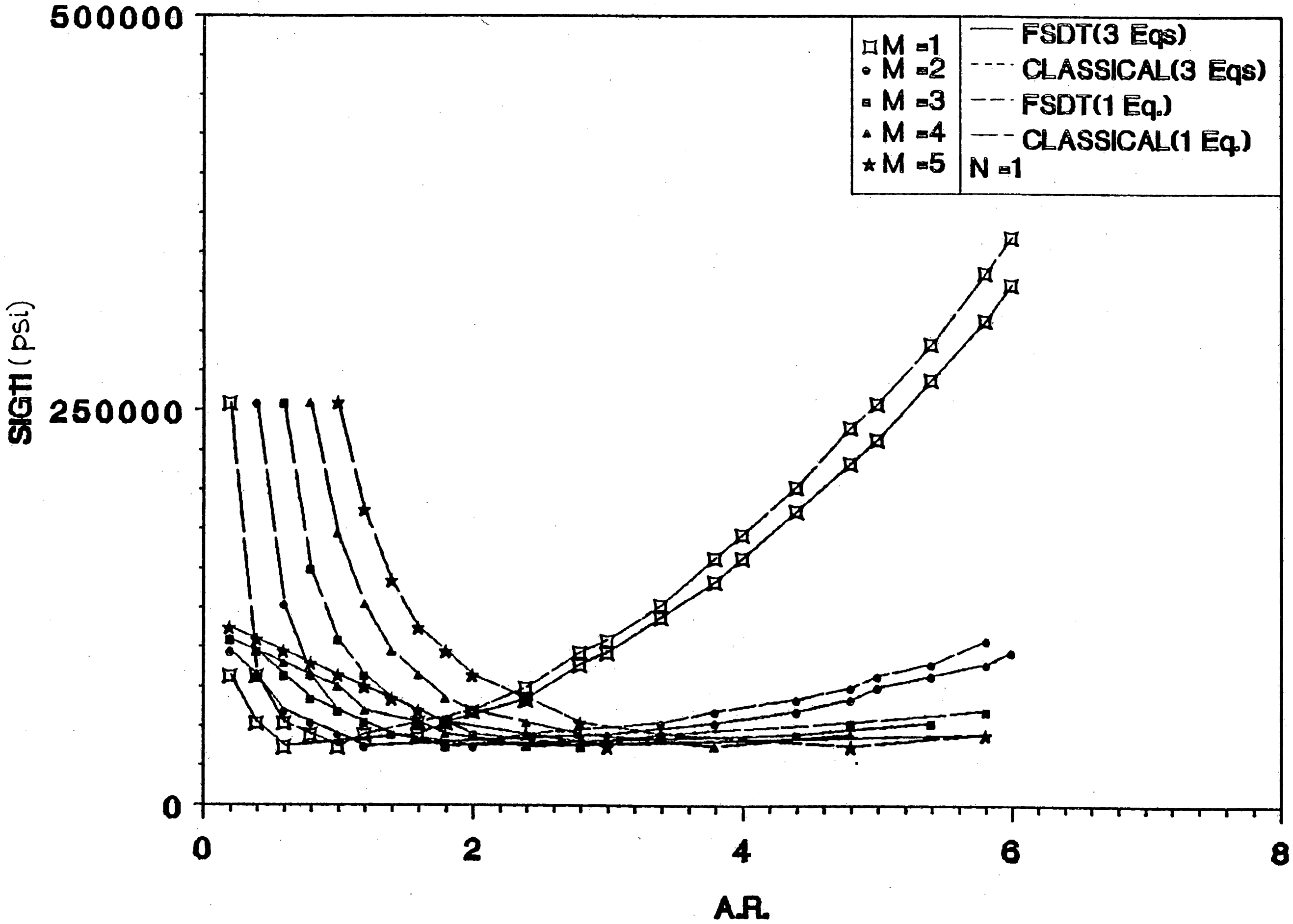


Figure 30. Comparison of stability boundaries for isotropic elastic plate; L/h = 4.8; uniaxial compression

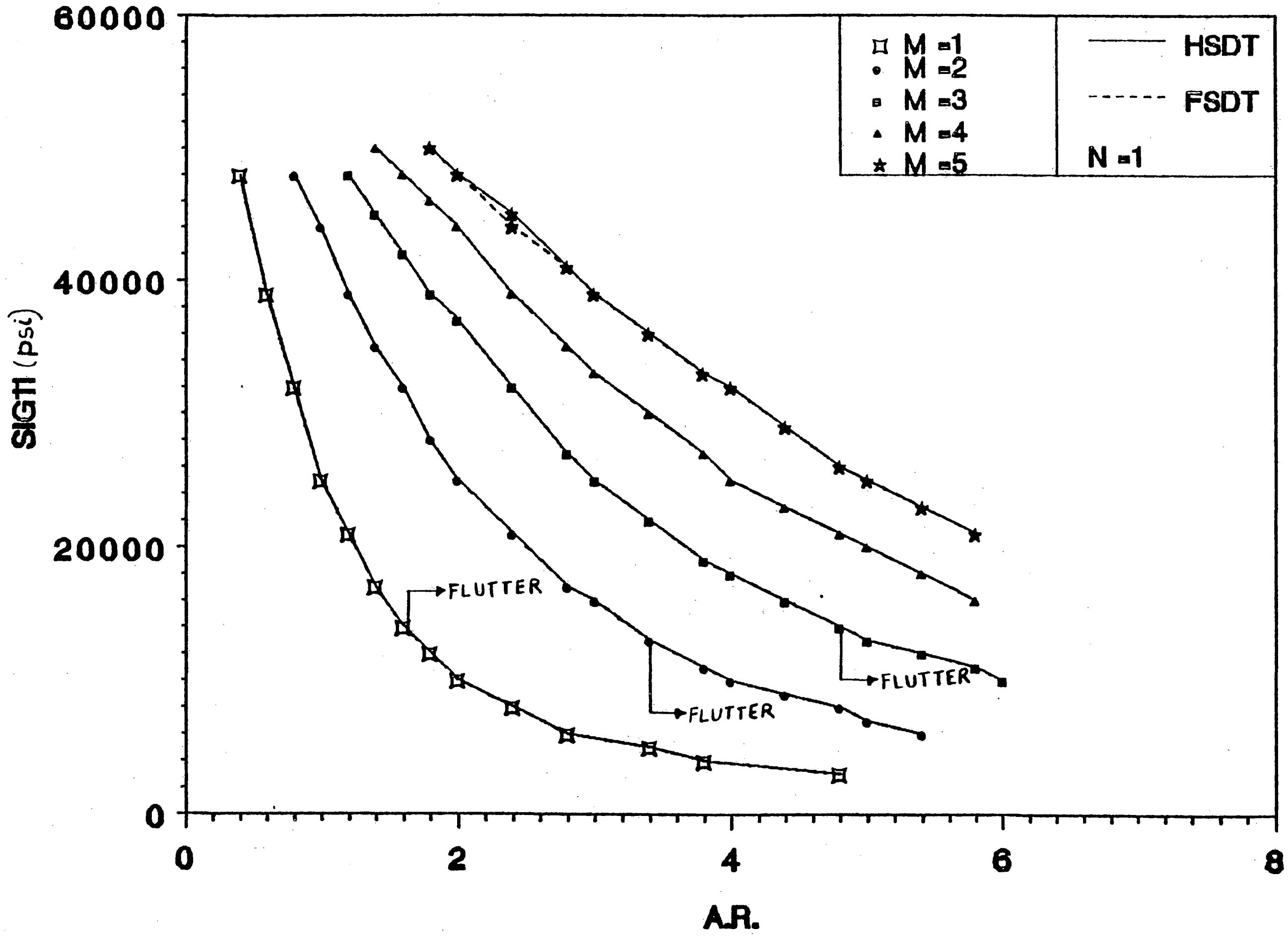


Figure 31. Stability boundary for orthotropic viscoelastic plate; L/h = 24; uniaxial compression; $\delta_A = 1$ or $\delta_A = 0$

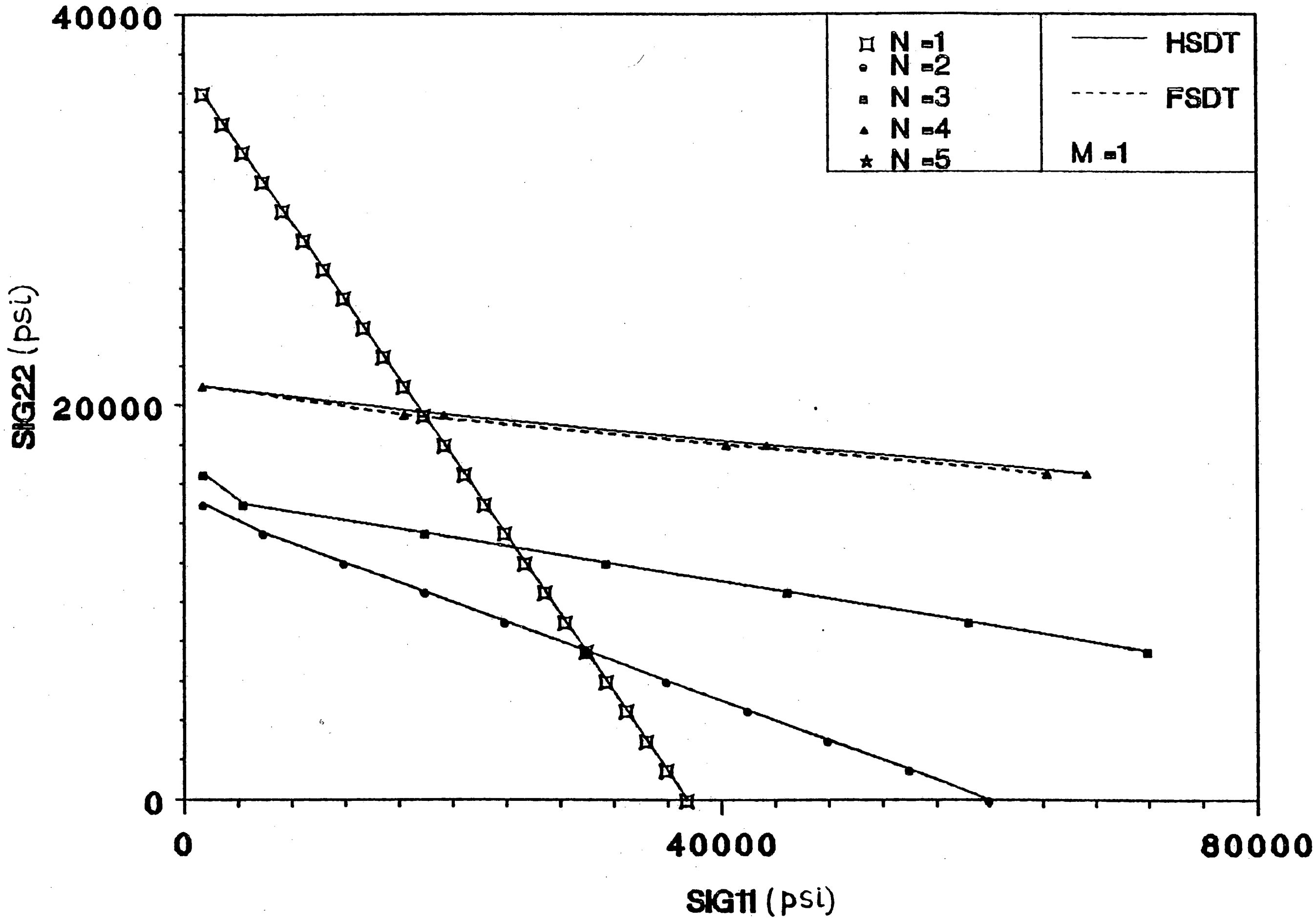


Figure 32. Stability boundary for orthotropic elastic plate; L/h = 24; biaxial compression; $\delta_A = 1$ or $\delta_A = 0$

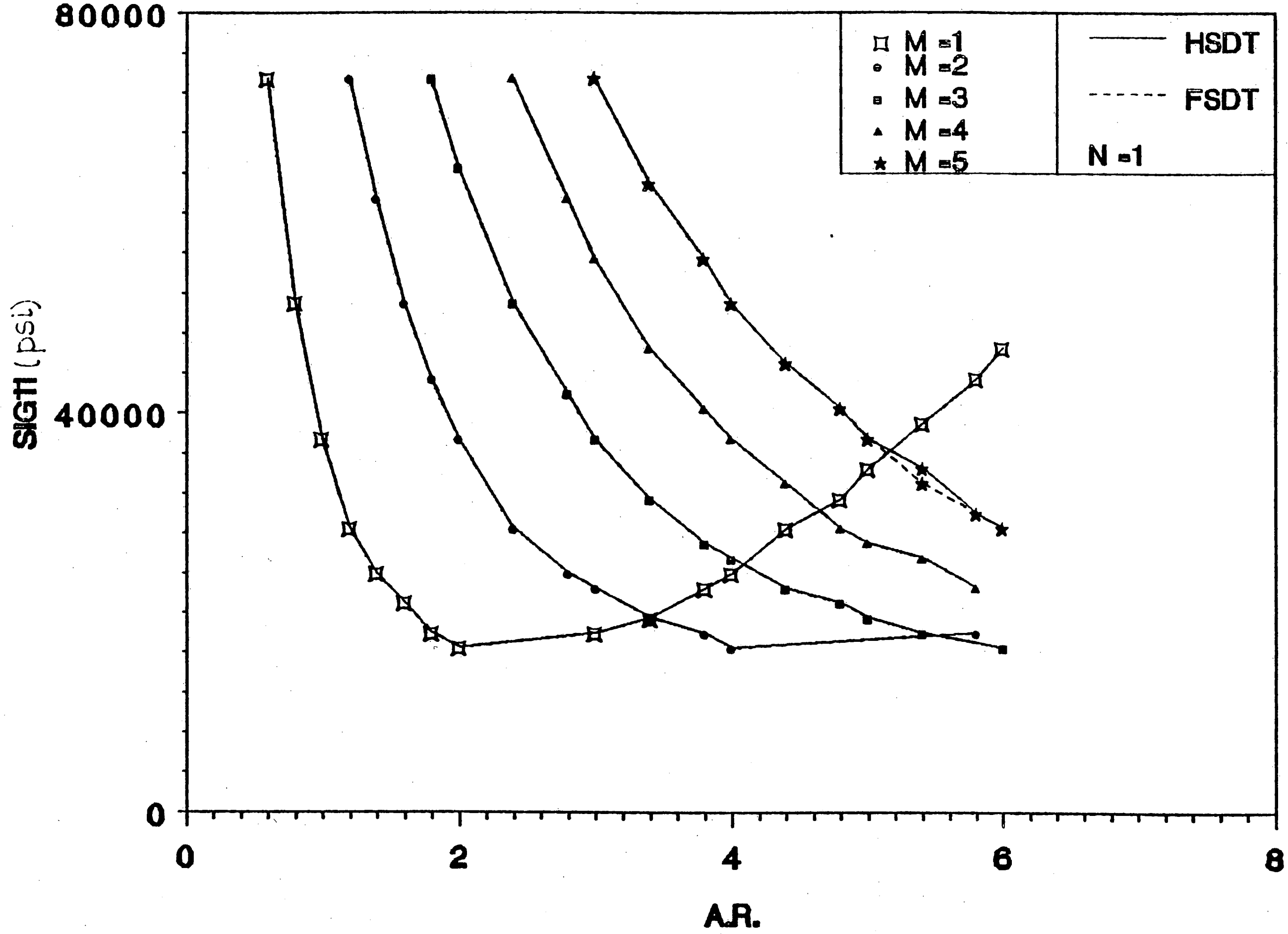


Figure 33. Stability boundary for orthotropic elastic plate; L/h = 24; uniaxial compression; δ_A = 1 or δ_A = 0

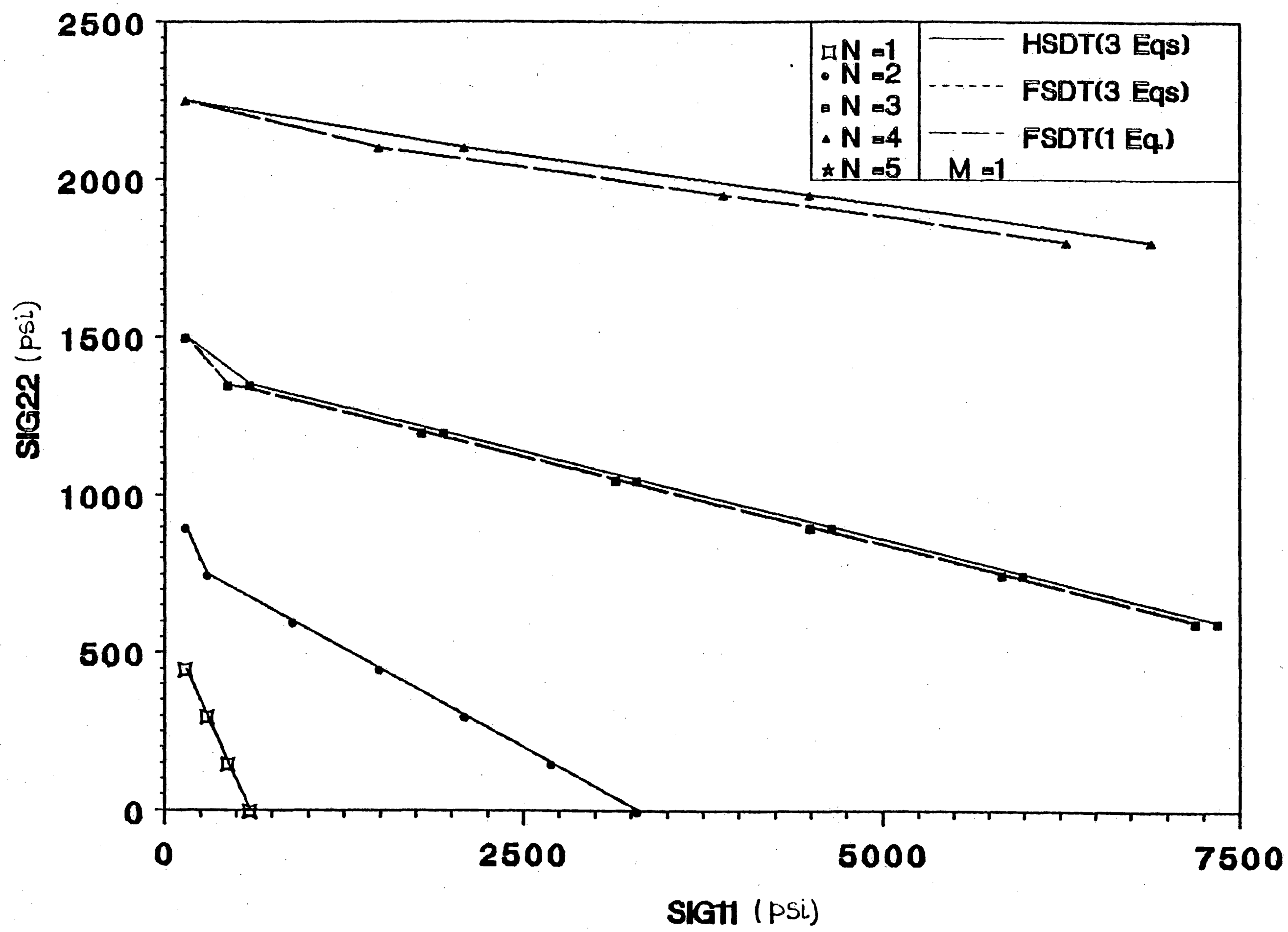


Figure 34. Stability boundary for isotropic viscoelastic plate; L/h = 24; biaxial compression; δ_A = 1 or δ_A = 0

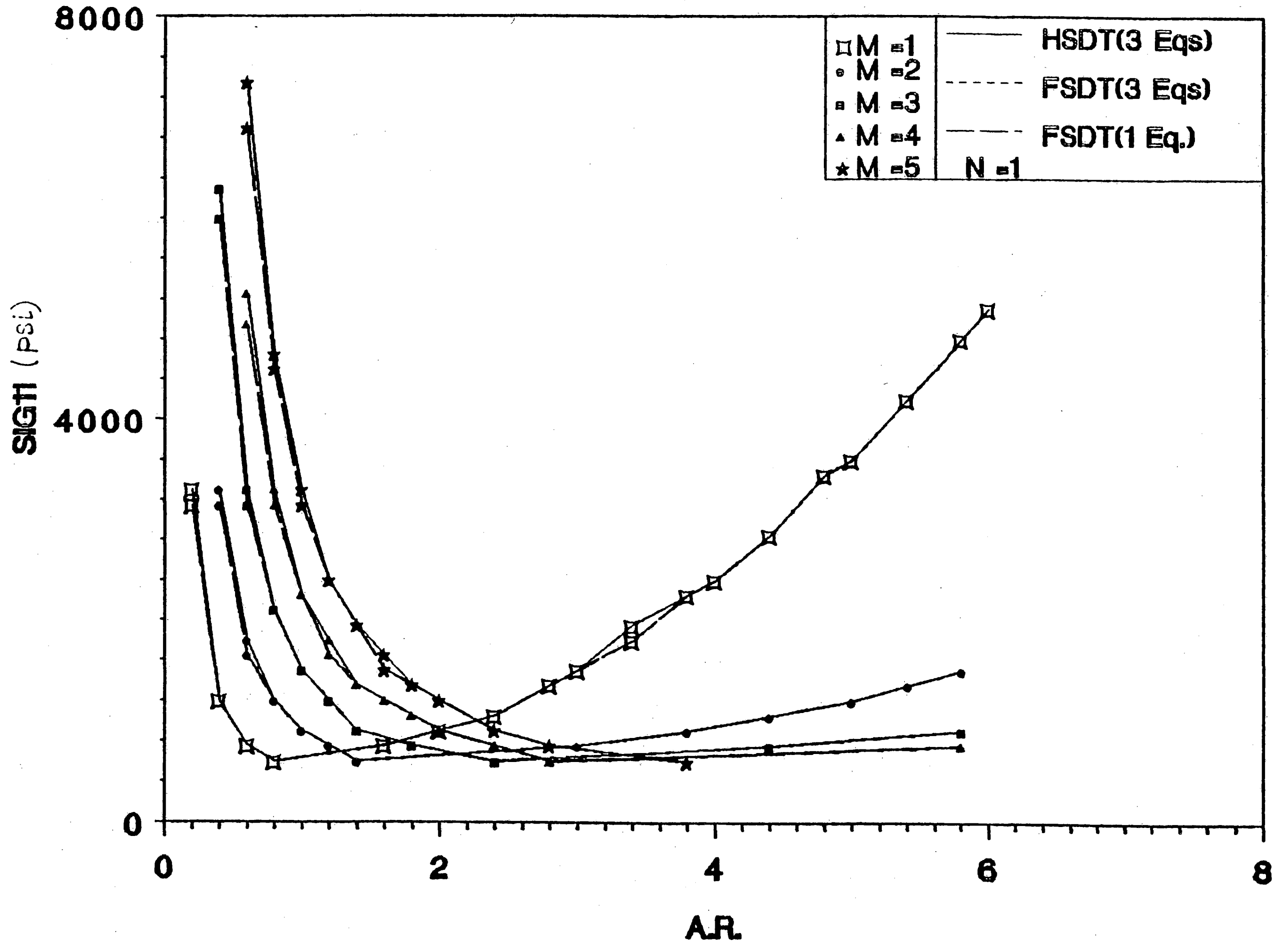


Figure 35. Stability boundary for isotropic viscoelastic plate; L/h = 24; uniaxial compression; $\delta_A = 1$ or $\delta_A = 0$

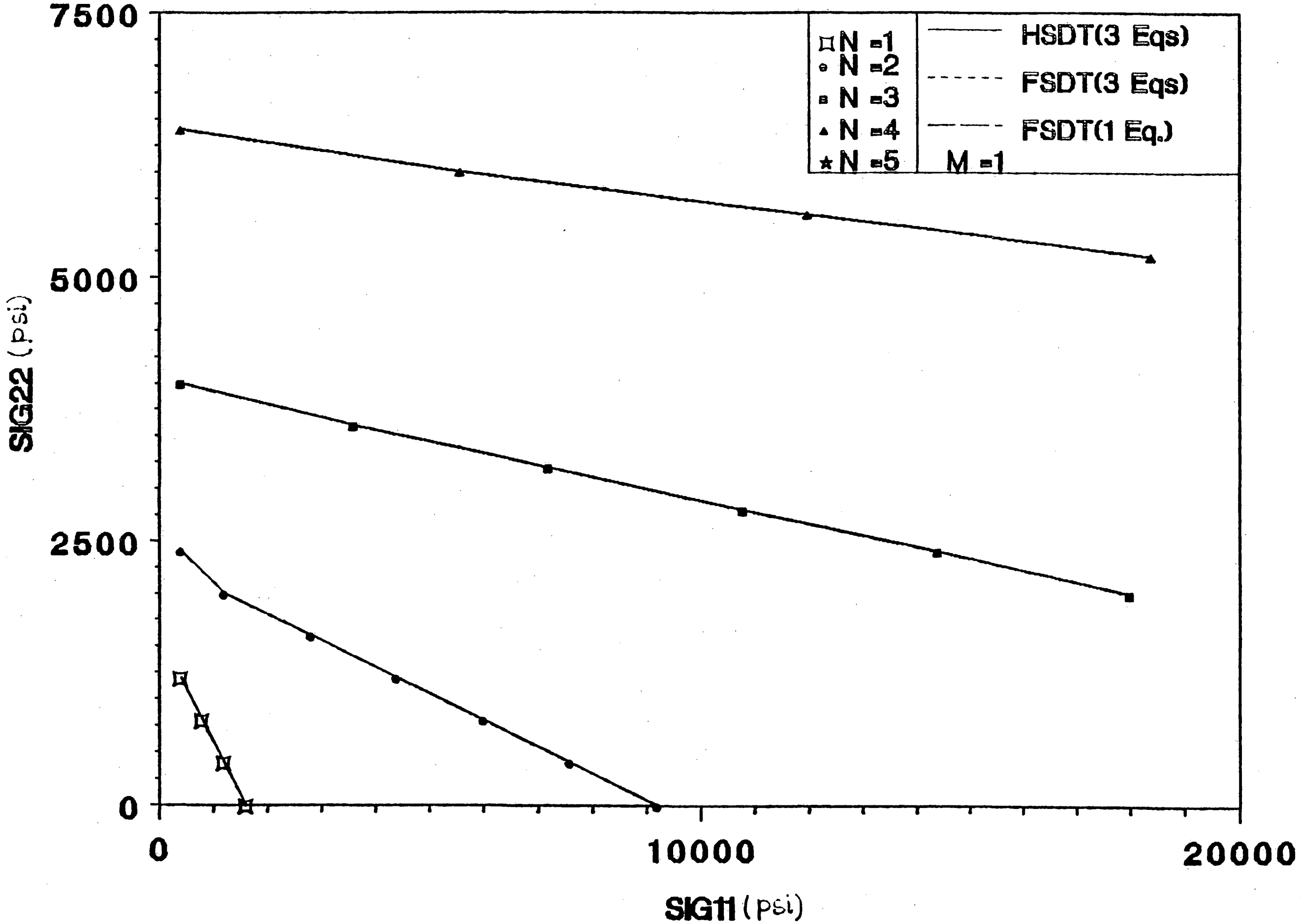


Figure 36. Stability boundary for isotropic elastic plate; L/h = 24; biaxial compression; $\delta_A = 1$ or $\delta_A = 0$

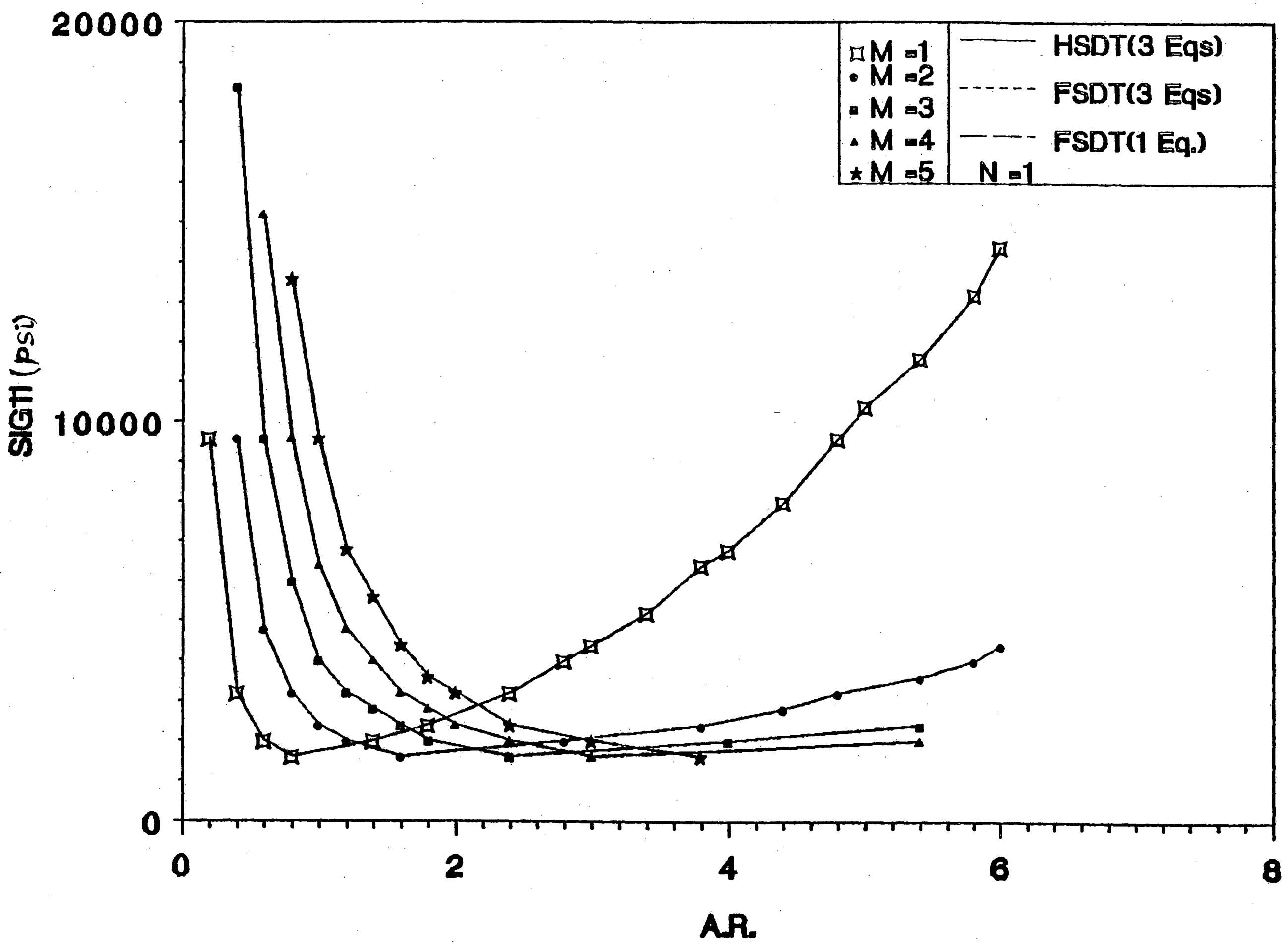


Figure 37. Stability boundary for isotropic elastic plate; L/h = 24; uniaxial compression; $\delta_A = 1$ or $\delta_A = 0$

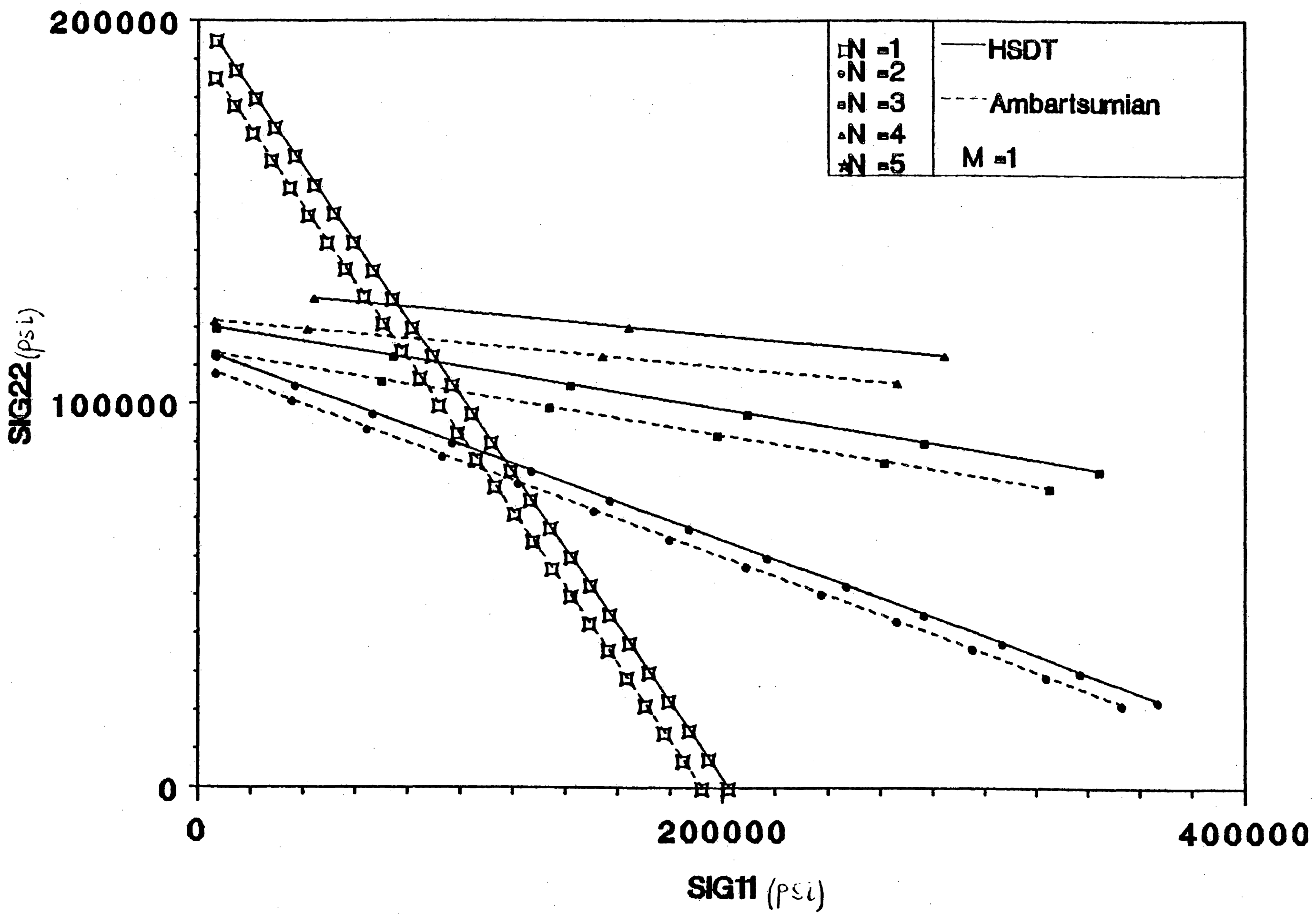


Figure 38. Comparison of stability boundaries for orthotropic elastic plate; L/h = 4.8; biaxial compression

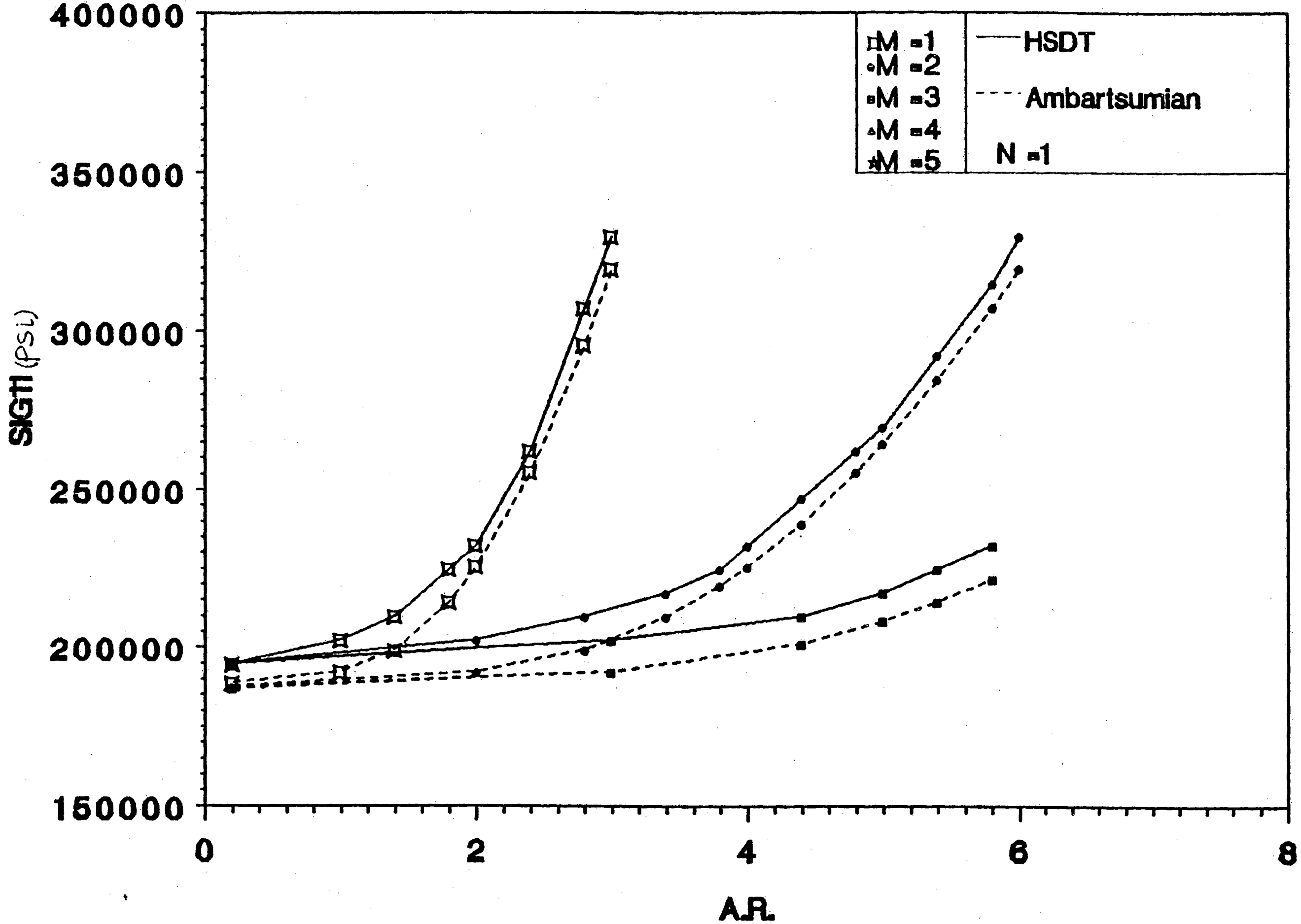


Figure 39. Comparison of stability boundaries for orthotropic elastic plate; L/h = 4.8; uniaxial compression

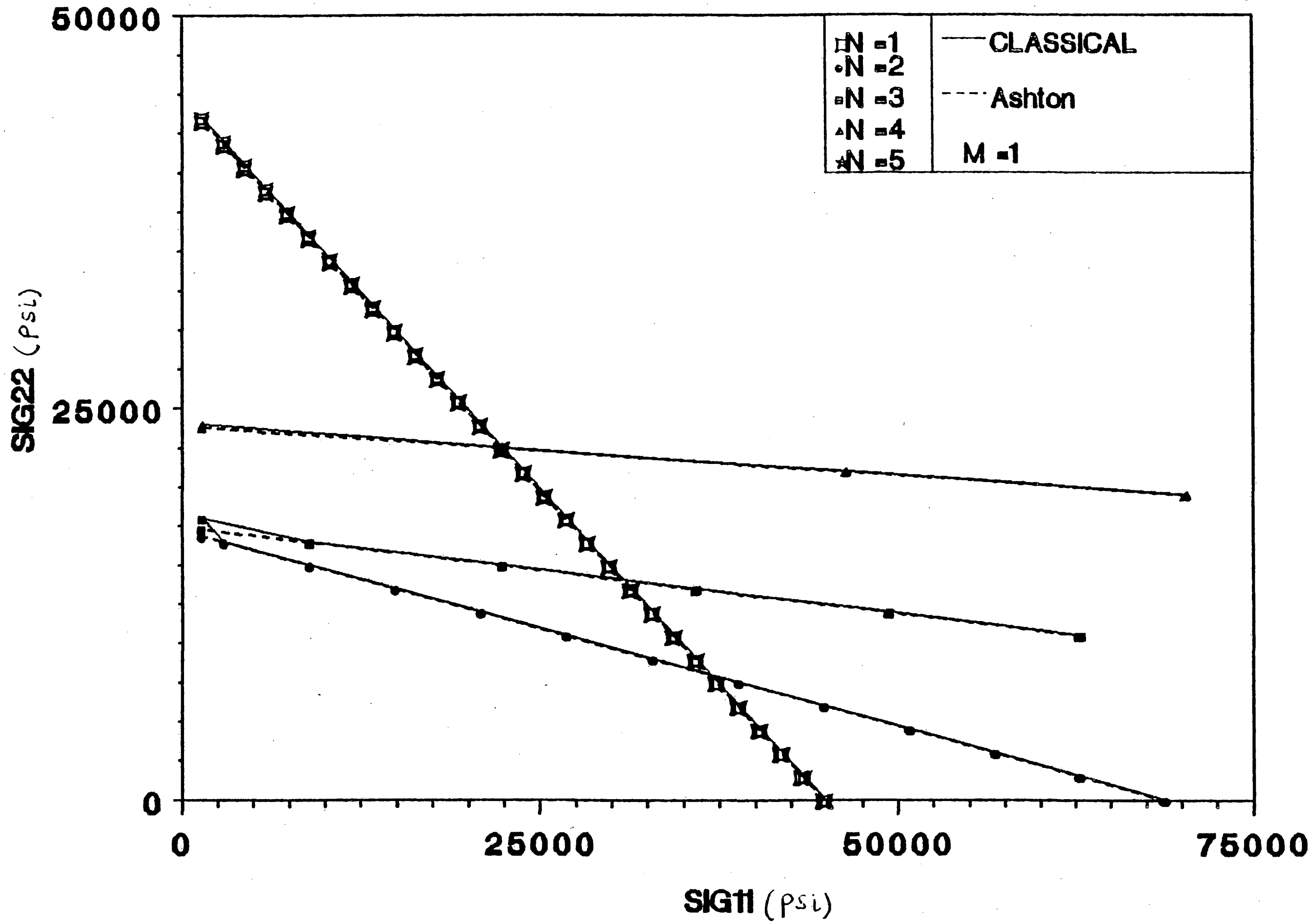


Figure 40. Comparison of stability boundaries for orthotropic elastic plate; L/h = 24; biaxial compression

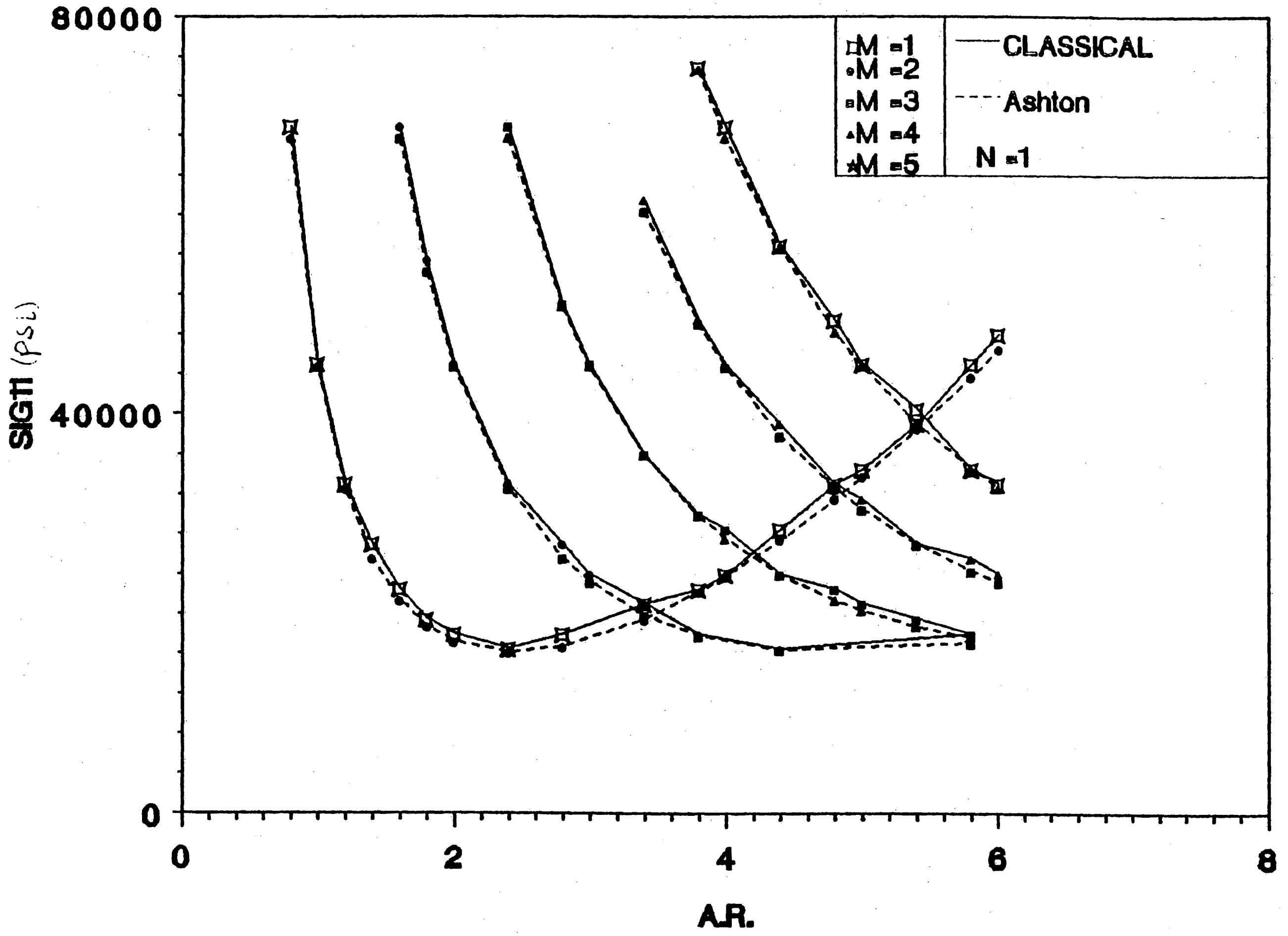


Figure 41. Comparison of stability boundaries for orthotropic elastic plate; L/h = 24; uniaxial compression

Appendix A. THE LAPLACE TRANSFORM AND ASSOCIATED THEOREMS

The Laplace Transform (L.T.) of f(t) is defined as follows:

$$\mathscr{L}[f[t]] = \bar{f}[s] = \int_0^{\infty} e^{-st} f[t] dt \qquad (A.1)$$

The L.T. of f[t] is said to exist if the integral in Eqn. (A.1) converges for some value of s.

<u>Borel's Theorem and the convolution integral</u>

If $\mathscr{L}^{-1}[\bar{f}[s]] = f(t)$ and $\mathscr{L}^{-1}[\bar{g}[s]] = g[t]$, then

$$\mathscr{L}^{-1}[\bar{f}[s]\bar{g}[s]] = \int_0^{\infty} f[\tau]g[t-\tau]d\tau = f{*}g \qquad (A.2)$$

and

$$f{*}g = g{*}f \qquad (A.3)$$

Initial value theorem

If the limit indicated below exists, then

$$\lim_{t \to 0} f[t] = \lim_{s \to \infty} s\bar{f}[s] \tag{A.4}$$

L.T. of derivatives

If $\mathscr{L}[f[t]] = \bar{f}[s]$, then

$$\mathscr{L}[f^{(n)}[t]] = s^n \bar{f}[s] - s^{n-1} f[0] - s^{n-2} \dot{f}[0] - \ldots.. - s f^{(n-2)}[0] - f^{(n-1)}[0] \tag{A.5}$$

The inverse L.T.

If $\mathscr{L}[f[t]] = \bar{f}[s]$, then $f[t] = \mathscr{L}^{-1}[\bar{f}[s]]$ where $\mathscr{L}^{-1}$ is the inverse L.T. operator.

Now if $f[0] = 0$, then

$$\mathscr{L}^{-1}[s\bar{f}[s]] = \dot{f}[t] \tag{A.6}$$

However, if f [0] $\neq$ 0, then

$$\mathscr{L}^{-1}[s\bar{f}[s]] = \dot{f}[t] + f[0]\delta[t] \tag{A.7}$$

where $\delta[t]$ is the Dirac delta or unit impulse function.

Use of initial value theorem in determining elastic values of material properties

Introducing Eqn. (A.4) into Eqns. (2.18a), (2.18b) and (2.62b) and noting that for a physically real material, the limit of $s\bar{E}_{ijmn}[s]$ as $s \to \infty$ must exist, we obtain the required values of $\tilde{E}_{\alpha\beta\mu\pi}[t]$, $\hat{E}_{\alpha\beta 33}[t]$ and $\hat{E}_{\alpha\beta\omega\lambda}[t]$ as $t \to 0$.

Appendix B. MICROMECHANICAL MODEL FOR UNIDIRECTIONAL FIBER-REINFORCED COMPOSITES

The continuum model for fiber-reinforced composites developed by Aboudi [23, 24, 25] assumes that continuous fibers extend in the x_1 direction and are arranged in a doubly periodic array in the x_2 and x_3 direction (see Fig. 11). The cross-section of the rectangular fibers is $h_1\, l_1$, and h_2, l_2 represent their spacing in the matrix. Due to this periodic arrangement, we need to analyze only a representative element as shown in Fig. 12. This representative cell contains four sub-cells identified by β, γ = 1, 2. Four local coordinate systems defined by x_1, $\bar{x}_2^{(\beta)}$, $\bar{x}_3^{(\gamma)}$, and having their origins at the center of each sub-cell, are shown in Fig. 12. The following first-order displacement expansion in each sub-cell is considered:

$$V_i^{(\beta\gamma)} = w_i^{(\beta\gamma)} + \bar{x}_2^{(\beta)}\phi_i^{(\beta\gamma)} + \bar{x}_3^{(\gamma)}\psi_i^{(\beta\gamma)} \qquad (B.1)$$

where $w_i^{(\beta\gamma)}$ are the displacement components of the center of each subcell with $\phi_i^{(\beta\gamma)}$, $\psi_i^{(\beta\gamma)}$ characterizing the linear dependence of the displacements on the local coordinates $\bar{x}_2^{(\beta)}$, $\bar{x}_3^{(\gamma)}$.

We note that in (B.1) and for the remainder of this appendix repeated greek indices do not imply a summation. The infinitesimal strain tensor writes as:

$$e_{ij}^{(\beta\gamma)} = \frac{1}{2}[v_{j,i}^{(\beta\gamma)} + v_{i,j}^{(\beta\gamma)}] \tag{B.2}$$

where in (B.2) for indices 2, 3 the differential is w.r.t. the respective local coordinates. At this point, we note that for the field variables (or microvariables) w_1, ϕ_i, ψ_i we have,

$$w_i = w_i[x_1, x_2] \tag{B.3a}$$

$$\phi_i = \phi_i[x_1, x_2] \tag{B.3b}$$

$$\psi_i = \psi_i[x_1, x_2] \tag{B.3c}$$

with $w_i^{(\beta\gamma)}$, $\phi_i^{(\beta\gamma)}$, $\psi_i^{(\beta\gamma)}$ being the values of w_i, ϕ_i, ψ_i evaluated at the center of each subcell.

The displacement continuity at the interfaces between sub-cells requires that the following relations be satisfied (see [23] for a complete derivation):

$$w_i^{(11)} = w_i^{(12)} = w_i^{(21)} = w_i^{(22)} = w_i \tag{B.4a}$$

$$h_1\phi_i^{(1\gamma)} = h_2\phi_i^{(2\gamma)} = (h_1 + h_2)\frac{\partial w_i}{\partial x_2} \tag{B.4b}$$

$$l_1\psi_i^{(\beta 1)} + l_2\psi_i^{(\beta 2)} = (l_1 + l_2)\frac{\partial w_i}{\partial x_3} \tag{B.4c}$$

Now the average strain in the composite is written as,

$$\overset{v}{e}_{ij} = \frac{1}{V}\sum_{\beta,\gamma=1}^{2} V_{\beta\gamma}e_{ij}^{(\beta\gamma)} \tag{B.5a}$$

where

$$v_{\beta\gamma} = h_\beta l_\gamma \tag{B.5b}$$

and

$$V = (h_1 + h_2)(l_1 + l_2) \tag{B.5c}$$

Introducing (B.2) into (B.5) in conjunction with (B.1) and (B.4), we may obtain (see [23] for a detailed derivation) the following:

$$\overset{v}{e}_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) \tag{B.6}$$

Now the average stress in the composite is written as,

$$\overset{v}{\sigma}_{ij} = \frac{1}{V} \sum_{\beta,\,\gamma=1}^{2} v_{\beta\gamma}\, s_{ij}^{(\beta\gamma)} \tag{B.7}$$

where $s_{ij}^{(\beta\gamma)}$ is the average stress in each sub-cell, given by,

$$s_{ij}^{(\beta\gamma)} = \frac{1}{v_{\beta\gamma}} \int_{-h_\beta/2}^{h_\beta/2} \int_{-l_\gamma/2}^{l_\gamma/2} \sigma_{ij}^{(\beta\gamma)} d\bar{x}_2^{(\beta)} d\bar{x}_3^{(\gamma)} \tag{B.8}$$

For transversely isotropic, elastic constituents, the constitutive law for each subcell writes as,

$$\sigma_{ij}^{(\beta\gamma)} = E_{ijmn}^{(\beta\gamma)} e_{mn}^{(\beta\gamma)} \tag{B.9}$$

where omitting the superindices $(\beta\gamma)$ for simplicity,

$$E_{1111} = E_A + 4k\nu_A^2 \tag{B.10a}$$

$$E_{1122} = E_{1133} = 2k\nu_A \tag{B.10b}$$

$$E_{2222} = E_{3333} = k + 0.5E_T/(1 + \nu_T) \tag{B.10c}$$

$$E_{2233} = k - 0.5E_T/(1 + \nu_T) \tag{B.10d}$$

$$E_{1212} = E_{1313} = G_A \tag{B.10e}$$

$$E_{2323} = (E_{2222} - E_{2233})/2 \tag{B.10f}$$

In (B.10) E_A, ν_A, G_A are the axial Young's modulus, Poisson's ratio and shear modulus and E_T, ν_T are the transverse Young's modulus and Poisson's ratio of the material identified by the subcell $(\beta\gamma)$. In (B.10), k is given by,

$$k = 0.25E_A/[0.5(1 - \nu_T)(E_A/E_T) - \nu_A^2] \tag{B.11}$$

Now upon introducing (B.9) for a transversely isotropic body into (B.8) in conjunction with (B.5), (B.6) and (B.1), we may obtain (see [23] for details),

$$s_{11}^{(\beta\gamma)} = E_{1111}^{(\beta\gamma)}\overset{v}{e}_{11} + E_{1122}^{(\beta\gamma)}(\phi_2^{(\beta\gamma)} + \psi_3^{(\beta\gamma)}) \tag{B.12a}$$

$$s_{22}^{(\beta\gamma)} = E_{1122}^{(\beta\gamma)}\overset{v}{e}_{11} + E_{2222}^{(\beta\gamma)}\phi_2^{(\beta\gamma)} + E_{2233}^{(\beta\gamma)}\psi_3^{(\beta\gamma)} \tag{B.12b}$$

$$s_{33}^{(\beta\gamma)} = E_{1122}^{(\beta\gamma)}\overset{v}{e}_{11} + E_{2233}^{(\beta\gamma)}\phi_2^{(\beta\gamma)} + E_{2222}^{(\beta\gamma)}\psi_3^{(\beta\gamma)} \tag{B.12c}$$

$$s_{12}^{(\beta\gamma)} = E_{1212}^{(\beta\gamma)}(\frac{\partial w_2}{\partial x_1} + \phi_1^{(\beta\gamma)}) \tag{B.12d}$$

$$s_{13}^{(\beta\gamma)} = E_{1212}^{(\beta\gamma)}(\frac{\partial w_3}{\partial x_1} + \psi_1^{(\beta\gamma)}) \tag{B.12e}$$

$$s_{23}^{(\beta\gamma)} = E_{2323}^{(\beta\gamma)}(\phi_3^{(\beta\gamma)} + \psi_2^{(\beta\gamma)}) \tag{B.12f}$$

Upon imposing the conditions for the continuity of tractions along the interfaces of the subcells, we can obtain (see [23] for a complete derivation) the following:

$$s_{2i}^{(1\gamma)} = s_{2i}^{(2\gamma)} \tag{B.13a}$$

$$s_{3i}^{(\beta 1)} = s_{3i}^{(\beta 2)} \tag{B.13b}$$

Now using the above expressions, we can derive the constitutive equations for a unidirectional composite in explicit form (see [23] for details). The brief outline of the procedure is as follows:

1. Using the displacement continuity equations (B.4) and the traction continuity relations (B.13), we can obtain a set of equations in terms of the microvariables $\phi_i^{(\beta\gamma)}$, $\psi_i^{(\beta\gamma)}$.
2. Solving for the microvariables above, we can obtain the explicit constitutive law relating average stresses ($\check{\sigma}_{ij}$) to average strains $\check{e}_{ij}$ by introducing the microvariables $\phi_i^{(\beta\gamma)}$, $\psi_i^{(\beta\gamma)}$ into (B.7).

The detailed expressions for the constitutive law of the elastic body can be found in [23]. In Chapter 3, the viscoelastic counterparts of these equations are obtained using the correspondence principle. In order to obtain the corresponding equations pertaining to the elastic body, we only need to replace Carson transformed quantities by their elastic constants.

Appendix C. NUMERICAL INVERSION OF THE LAPLACE TRANSFORM USING BELLMAN'S TECHNIQUE

Consider the L.T. of a function f(t) defined by,

$$\bar{f}[s] = \int_0^\infty e^{-st} f[t] dt \qquad (C.1)$$

Now we introduce the change of variable $x = e^{-t}$ in (C.1), yielding

$$\bar{f}[s] = \int_0^1 x^{s-1} f[-\ln\ x] dx \qquad (C.2)$$

Writing $g[x] = f[-\ln\ x]$ we have from (C.2),

$$\bar{f}[s] = \int_0^1 x^{s-1} g[x] dx. \qquad (C.3)$$

Now consider the following representation of the integral defined by

$$\int_0^1 f[r]dr \cong \sum_{i=1}^{N} w_i f[r_i] \tag{C.4}$$

where r_i are the points at which f is evaluated and w_i are weighting functions. Using a standard Gaussian quadrature to evaluate the integral in (C.4), it can be shown that (see Bellman [31]) r_i, w_i are the roots and weights, respectively, of the shifted Legendre polynomial $P_N^*[r]$ defined by,

$$P_N^*[r] = P_N[1-2r] \tag{C.5a}$$

where $P_N(r)$ is the n^{th} order Legendre polynomial defined by,

$$P_N[r] = \sum_{R=0}^{m} (-1)^k \frac{(2n-2k)!r^{n-2k}}{2^n k!(n-k)!(n-2k)!} \tag{C.5b}$$

and w_i are shown to be,

$$w_i = \frac{1}{2}\int_{-1}^{1} \frac{P_N[r]dr}{(r-r_i)P'_N[r_i]} \quad , i = 1, 2, \ldots N \tag{C.5c}$$

Introducing (C.4) into (C.3) we obtain,

$$\bar{f}[s] = \sum_{i=1}^{N} w_i x_i^{s-1} g[x_i] \tag{C.6}$$

Now letting s in (C.6) assume N different values, say s = 1, 2,...,N, yields a linear system of N equations in the N unknowns, $g[x_i]$, $i = 1, 2, \ldots, N$, given by,

$$\bar{f}[k] = \sum_{i=1}^{N} w_i x_i^{k-1} g[x_i] \quad , k = 1,...,N \tag{C.7}$$

where w_i, x_i are the weights and roots of $P_N^*[r]$ given by (C.5). Equation (C.7) can be inverted to obtain the solution of $g[x_i]$ and we can write,

$$g[x_i] = [M_{ik}]\bar{f}_i \qquad , i, k = 1,...,N \tag{C.8}$$

where $\bar{f}_i = \bar{f}[s]|_{s=i}$ and M_{ij} is the inverse of the NxN matrix given by $(w_i x_i^{k-1})$. Bellman [31] has tabulated the weights (w_i) and roots (r_i) of $P_N^*[r]$ and also the coefficients of the matrix M_{ik} for values of N ranging from 3 to 15.

Appendix D. EXPLICIT FORM OF THE CHARACTERISTIC EQUATION.

The characteristic equation given by Eqn. (4.11) may be written as:

$$\frac{P_{mn}[s]}{Q_{mn}[s]} = \sum_{i=1}^{n} T_i \, E_i[s] = 0$$

For the TSDT, n = 92 and we have the following relations for T_i and $E_i[s]$:

$$T_1 = \frac{40}{3} h \lambda_m^4 \lambda_n^2 \quad ; \quad E_1 = \overline{\tilde{\vec{E}}}_{1111} \overline{\tilde{\vec{E}}}_{2222} \overline{\tilde{\vec{E}}}_{1212}$$

$$T_2 = \frac{40}{3} h \lambda_m^6 \quad ; \quad E_2 = \overline{\tilde{\vec{E}}}_{1111} (\overline{\tilde{\vec{E}}}_{1212})^2$$

$$T_3 = -\frac{40}{3} h \delta_A \lambda_m^4 \lambda_n^2 \quad ; \quad E_3 = \overline{\tilde{\vec{E}}}_{1111} \overline{\tilde{\vec{E}}}_{1212} \overline{\hat{\vec{E}}}_{2222}$$

$$T_4 = \frac{400}{3h} \lambda_m^4 + T_1 \quad ; \quad E_4 = \overline{\tilde{\vec{E}}}_{1111} \vec{E}_{2323} \overline{\tilde{\vec{E}}}_{1212}$$

$$T_5 = \frac{40}{3} h \lambda_m^2 \lambda_n^4 \quad ; \quad E_5 = \overline{\tilde{\vec{E}}}_{2222} (\overline{\tilde{\vec{E}}}_{1212})^2$$

$$T_6 = \frac{-80}{3} h\delta_A \lambda_m^2 \lambda_n^4 \quad ; \quad E_6 = \bar{\hat{\vec{E}}}_{2222}(\bar{\tilde{\vec{E}}}_{1212})^2$$

$$T_7 = \frac{1600}{3h} \lambda_m^2 \lambda_n^2 \quad ; \quad E_7 = \vec{E}_{2323}(\bar{\tilde{\vec{E}}}_{1212})^2$$

$$T_8 = T_5 \quad ; \quad E_8 = \bar{\tilde{\vec{E}}}_{1111}\bar{\tilde{\vec{E}}}_{2222}\vec{E}_{2323}$$

$$T_9 = \frac{40}{3} h\lambda_n^6 + \frac{400}{3h}\lambda_n^4 \quad ; \quad E_9 = \bar{\tilde{\vec{E}}}_{1212}\bar{\tilde{\vec{E}}}_{2222}\vec{E}_{2323}$$

$$T_{10} = \frac{T_6}{2} \quad E_{10} = \bar{\hat{\vec{E}}}_{1111}\bar{\tilde{\vec{E}}}_{2222}\vec{E}_{2323}$$

$$T_{11} = 2T_3 \quad ; \quad E_{11} = \bar{\hat{\vec{E}}}_{1111}\bar{\tilde{\vec{E}}}_{1212}\vec{E}_{2323}$$

$$T_{12} = -T_1 \quad ; \quad E_{12} = \bar{\tilde{\vec{E}}}_{1212}(\bar{\tilde{\vec{E}}}_{1122})^2$$

$$T_{13} = 2T_{12} \quad ; \quad E_{13} = (\bar{\tilde{\vec{E}}}_{1212})^2\bar{\tilde{\vec{E}}}_{1122}$$

$$T_{14} = \frac{T_6}{2} \quad ; \quad E_{14} = \bar{\tilde{\vec{E}}}_{1122}\bar{\hat{\vec{E}}}_{2222}\bar{\tilde{\vec{E}}}_{1212}$$

$$T_{15} = \frac{T_7}{2} - 2T_5 \quad ; \quad E_{15} = \bar{\tilde{\vec{E}}}_{1122}\vec{E}_{2323}\bar{\tilde{\vec{E}}}_{1212}$$

$$T_{16} = -\frac{40}{3} h\delta_A\delta_B\rho\lambda_m^2\lambda_n^2 \quad ; \quad E_{16} = s^2\bar{\tilde{\vec{E}}}_{1122}\bar{\hat{\vec{E}}}_{2233}\bar{\tilde{\vec{E}}}_{1212}$$

$$T_{17} = T_{16} \quad ; \quad E_{17} = s^2(\bar{\tilde{\vec{E}}}_{1212})^2\bar{\hat{\vec{E}}}_{2233}$$

$$T_{18} = \frac{-T_6}{2} \quad ; \quad E_{18} = \bar{\tilde{\vec{E}}}_{1212}\bar{\hat{\vec{E}}}_{1122}\bar{\tilde{\vec{E}}}_{2222}$$

$$T_{19} = -T_3 \quad ; \quad E_{19} = \bar{\tilde{\vec{E}}}_{1212}\bar{\tilde{\vec{E}}}_{1122}\bar{\hat{\vec{E}}}_{1122}$$

$$T_{20} = 2T_{19} \quad ; \quad E_{20} = \bar{\hat{E}}^{*}_{1122}(\bar{\tilde{\vec{E}}}_{1212})^2$$

$$T_{21} = \frac{40}{3} h\delta_A^2\delta_B\rho\lambda_m^2\lambda_n^2 \quad ; \quad E_{21} = s^2\bar{\hat{E}}^{*}_{1122}\bar{\hat{E}}^{*}_{2233}\bar{\tilde{\vec{E}}}_{1212}$$

$$T_{22} = -T_5 \quad ; \quad E_{22} = \vec{E}_{2323}(\bar{\tilde{\vec{E}}}_{1122})^2$$

$$T_{23} = T_3 \quad ; \quad E_{23} = \bar{\tilde{\vec{E}}}_{1122}\bar{\hat{E}}^{*}_{1111}\vec{E}_{2323}$$

$$T_{24} = T_{16} \quad ; \quad E_{24} = s^2\,\bar{\tilde{\vec{E}}}_{1122}\bar{\hat{E}}^{*}_{1133}\vec{E}_{2323}$$

$$T_{25} = T_{16} + \frac{400}{3h}\,\delta_A\delta_B\rho\lambda_m^2 \quad ; \quad E_{25} = s^2\bar{\tilde{\vec{E}}}_{1212}\bar{\hat{E}}^{*}_{1133}\vec{E}_{2323}$$

$$T_{26} = -T_3 \quad ; \quad E_{26} = \vec{E}_{2323}\bar{\hat{E}}^{*}_{2211}\bar{\tilde{\vec{E}}}_{1111}$$

$$T_{27} = T_{18} \quad ; \quad E_{27} = \vec{E}_{2323}\bar{\tilde{\vec{E}}}_{1122}\bar{\hat{E}}^{*}_{2211}$$

$$T_{28} = 2T_{18} \quad ; \quad E_{28} = \bar{\hat{E}}^{*}_{2211}\bar{\tilde{\vec{E}}}_{1212}\vec{E}_{2323}$$

$$T_{29} = T_{21} \quad ; \quad E_{29} = s^2\,\bar{\hat{E}}^{*}_{2211}\bar{\hat{E}}^{*}_{1133}\vec{E}_{2323}$$

$$T_{30} = -T_{16} \quad ; \quad E_{30} = s^2\,\bar{\tilde{\vec{E}}}_{1212}\bar{\hat{E}}^{*}_{1133}\bar{\tilde{\vec{E}}}_{2222}$$

$$T_{31} = \frac{40}{3} h\,\delta_A\delta_B\rho\lambda_m^4 \quad ; \quad E_{31} = s^2(\bar{\tilde{\vec{E}}}_{1212})^2\bar{\hat{E}}^{*}_{1133}$$

$$T_{32} = -T_{21} \quad ; \quad E_{32} = s^2\,\bar{\tilde{\vec{E}}}_{1212}\bar{\hat{E}}^{*}_{1133}\bar{\hat{E}}^{*}_{2222}$$

$$T_{33} = -T_{16} \quad ; \quad E_{33} = s^2\,\vec{E}_{2323}\bar{\hat{E}}^{*}_{2233}\bar{\tilde{\vec{E}}}_{1111}$$

$$T_{34} = \frac{40}{3} h\,\delta_A\delta_B\rho\lambda_n^4 + \frac{400}{3h}\delta_A\delta_B\rho\lambda_n^2 \quad ; \quad E_{34} = s^2\vec{E}_{2323}\bar{\hat{E}}^{*}_{2233}\bar{\tilde{\vec{E}}}_{1212}$$

$$T_{35} = -T_{21} \quad ; \quad E_{35} = s^2\, \vec{E}_{2323} \overline{\hat{E}^*}_{2233} \overline{\hat{E}^*}_{1111}$$

$$T_{36} = 16hq_{mn}\lambda_m^4 \quad ; \quad E_{36} = \overline{\tilde{E}^*}_{1111} \overline{\tilde{E}^*}_{1212}$$

$$T_{37} = \frac{40}{3} h\delta_C \rho \lambda_m^4 + 16h\delta_D \rho \lambda_m^4 \quad ; \quad E_{37} = s^2 E_{36}$$

$$T_{38} = \frac{160}{h} q_{mn}\lambda_n^2 + 16hq_{mn}\lambda_n^4 \quad ; \quad E_{38} = \overline{\tilde{E}^*}_{1212} \overline{\tilde{E}^*}_{2222}$$

$$T_{39} = \frac{40}{3} h\, \delta_C \rho \lambda_n^2 \lambda_m^2 + \frac{160}{h} \delta_D \rho \lambda_n^2 + 16h\delta_D \rho \lambda_n^4 \quad ; \quad E_{39} = s^2 E_{38}$$

$$T_{40} = \frac{160}{h} q_{mn}\lambda_m^2 \quad ; \quad E_{40} = (\overline{\tilde{E}^*}_{1212})^2$$

$$T_{41} = \frac{40}{3} h\, \delta_C \rho \{\lambda_m^4 + \lambda_m^2 \lambda_n^2\} + \frac{160}{h} \delta_D\, \rho\, \lambda_m^2 \quad ; \quad E_{41} = s^2 E_{40}$$

$$T_{42} = -\,\delta_A\, T_{38} \quad ; \quad E_{42} = \overline{\hat{E}^*}_{2222} \overline{\tilde{E}^*}_{1212}$$

$$T_{43} = -\,\delta_A\, T_{39} \quad ; \quad E_{43} = s^2 E_{42}$$

$$T_{44} = \frac{1600}{h^3} q_{mn} + \frac{160}{h} q_{mn}\lambda_n^2 \quad ; \quad E_{44} = \overline{\tilde{E}^*}_{1212} \vec{E}_{2323}$$

$$T_{45} = \frac{400}{3h} \delta_C \rho \{\lambda_m^2 + \lambda_n^2\} + \frac{1600}{h^3} \delta_D \rho + \frac{40}{3} h\, \delta_C \rho \{\lambda_m^2 \lambda_n^2 + \lambda_n^4\} + \frac{160}{h} \delta_D\, \rho \lambda_n^2 \; ; \quad E_{45} = s^2 E_{44}$$

$$T_{46} = T_{40} \quad ; \quad E_{46} = \overline{\tilde{E}^*}_{1111} \vec{E}_{2323}$$

$$T_{47} = \frac{40}{3} h\delta_C \rho \lambda_m^2 \lambda_n^2 + \frac{160}{h} \delta_D \rho \lambda_m^2 \quad ; \quad E_{47} = s^2 E_{46}$$

$$T_{48} = \frac{40}{3} h\delta_C \rho \lambda_m^4 \quad ; \quad E_{48} = s^2 \overline{\tilde{E}^*}_{2222} \vec{E}_{2323}$$

$$T_{49} = -\,\delta_A T_{40} \quad ; \quad E_{49} = \overline{\hat{E}}^{*}_{1111}\overline{E}^{*}_{2323}$$

$$T_{50} = -\,\delta_A T_{47} \quad ; \quad E_{50} = s^2 E_{49}$$

$$T_{51} = 16 h q_{mn} \lambda_m^2 \lambda_n^2 \quad ; \quad E_{51} = \overline{\tilde{E}}^{*}_{1111}\overline{\tilde{E}}^{*}_{2222}$$

$$T_{52} = 16 h \delta_D \rho \lambda_m^2 \lambda_n^2 \quad ; \quad E_{52} = s^2 E_{51}$$

$$T_{53} = -\delta_A T_{51} \quad ; \quad E_{53} = \overline{\tilde{E}}^{*}_{1111}\overline{\hat{E}}^{*}_{2222}$$

$$T_{54} = -\delta_A T_{52} \quad ; \quad E_{54} = s^2 E_{53}$$

$$T_{55} = T_{53} \quad ; \quad E_{55} = \overline{\hat{E}}^{*}_{1111}\overline{\tilde{E}}^{*}_{2222}$$

$$T_{56} = T_{54} \quad ; \quad E_{56} = s^2 E_{55}$$

$$T_{57} = -\,\delta_A T_{36} \quad ; \quad E_{57} = \overline{\hat{E}}^{*}_{1111}\overline{\tilde{E}}^{*}_{1212}$$

$$T_{58} = -16 h \delta_A \delta_D \rho \lambda_m^4 \quad ; \quad E_{58} = s^2 E_{57}$$

$$T_{59} = -\delta_A T_{53} \quad ; \quad E_{59} = \overline{\hat{E}}^{*}_{1111}\overline{\hat{E}}^{*}_{2222}$$

$$T_{60} = -\delta_A T_{54} \quad ; \quad E_{60} = s^2 E_{59}$$

$$T_{61} = -T_{53} \quad ; \quad E_{61} = \overline{\tilde{E}}^{*}_{1212}\overline{\hat{E}}^{*}_{1122}$$

$$T_{62} = \frac{40}{3} h \delta_A \delta_C \rho \lambda_m^2 \lambda_n^2 + 16 h \delta_A \delta_D \rho \lambda_m^2 \lambda_n^2 \quad ; \quad E_{62} = s^2 E_{61}$$

$$T_{63} = \frac{40}{3} h \delta_A \delta_C \rho \lambda_m^2 \lambda_n^2 \quad ; \quad E_{63} = s^2 \overline{E}^{*}_{2323}\overline{\hat{E}}^{*}_{2211}$$

$$T_{64} = \frac{40}{3} h \delta_A \delta_B \delta_C \rho^2 \lambda_m^2 \quad ; \quad E_{64} = s^4 \overline{\tilde{E}}^{*}_{1212}\overline{\hat{E}}^{*}_{1133}$$

$$T_{65} = \frac{40}{3} h \delta_A \delta_B \delta_C \rho^2 \lambda_n^2 \quad ; \quad E_{65} = s^4 \vec{E}_{2323} \overline{\hat{\vec{E}}}_{2233}$$

$$T_{66} = -T_{51} \quad ; \quad E_{66} = (\overline{\tilde{\vec{E}}}_{1122})^2$$

$$T_{67} = -T_{52} \quad ; \quad E_{67} = s^2 E_{66}$$

$$T_{68} = -2T_{51} \quad ; \quad E_{68} = \overline{\tilde{\vec{E}}}_{1212} \overline{\tilde{\vec{E}}}_{1122}$$

$$T_{69} = -2T_{52} \quad ; \quad E_{69} = s^2 E_{68}$$

$$T_{70} = \delta_A T_{51} \quad ; \quad E_{70} = \overline{\tilde{\vec{E}}}_{1122} \overline{\hat{\vec{E}}}_{2211}$$

$$T_{71} = \delta_A T_{52} \quad ; \quad E_{71} = s^2 E_{70}$$

$$T_{72} = \delta_A T_{51} \quad ; \quad E_{72} = \overline{\tilde{\vec{E}}}_{1122} \overline{\hat{\vec{E}}}_{1122}$$

$$T_{73} = \delta_A T_{52} \quad ; \quad E_{73} = s^2 E_{72}$$

$$T_{74} = T_{72} \quad ; \quad E_{74} = \overline{\tilde{\vec{E}}}_{1212} \overline{\hat{\vec{E}}}_{2211}$$

$$T_{75} = T_{73} \quad ; \quad E_{75} = s^2 E_{74}$$

$$T_{76} = -\delta_A T_{72} \quad ; \quad E_{76} = \overline{\hat{\vec{E}}}_{1122} \overline{\hat{\vec{E}}}_{2211}$$

$$T_{77} = -\delta_A T_{73} \quad ; \quad E_{77} = s^2 E_{76}$$

$$T_{78} = 16 \delta_c \rho h q_{mn} \{\lambda_m^2 + \lambda_n^2\} + \frac{160}{h} \delta_c \rho q_{mn} \quad ; \quad E_{78} = s^2 \overline{\tilde{\vec{E}}}_{1212}$$

$$T_{79} = \frac{40}{3} h \delta_c^2 \rho^2 \lambda_m^2 + 16 \delta_c \delta_D \rho^2 h \{\lambda_m^2 + \lambda_n^2\} + \frac{160}{h} \delta_c \delta_D \rho^2 \quad ; \quad E_{79} = s^2 E_{78}$$

$$T_{80} = \frac{40}{3} h \delta_c^2 \rho^2 \lambda_n^2 \quad ; \quad E_{80} = s^4 \vec{E}_{2323}$$

$$T_{81} = 16 \delta_c \rho h q_{mn} \lambda_m^2 \quad ; \quad E_{81} = s^2 \overline{\tilde{\vec{E}}}_{1111}$$

$$T_{82} = 16 \delta_c \delta_D \rho^2 h \lambda_m^2 \quad ; \quad E_{82} = s^2 E_{81}$$

$$T_{83} = 16 \delta_c \rho h q_{mn} \lambda_n^2 \quad ; \quad E_{83} = s^2 \overline{\tilde{\vec{E}}}_{2222}$$

$$T_{84} = 16 \delta_c \delta_D \rho^2 h \lambda_n^2 \quad ; \quad E_{84} = s^2 E_{83}$$

$$T_{85} = -\delta_A T_{81} \quad ; \quad E_{85} = s^2 \overline{\hat{\vec{E}}}_{1111}$$

$$T_{86} = -\delta_A T_{82} \quad ; \quad E_{86} = s^2 E_{85}$$

$$T_{87} = -\delta_A T_{83} \quad ; \quad E_{87} = s^2 \overline{\hat{\vec{E}}}_{2222}$$

$$T_{88} = -\delta_A T_{84} \quad ; \quad E_{88} = s^2 E_{87}$$

$$T_{89} = \frac{160}{h} \delta_c \rho q_{mn} \quad ; \quad E_{89} = s^2 \vec{E}_{2323}$$

$$T_{90} = \frac{160}{h} \delta_c \delta_D \rho^2 \quad ; \quad E_{90} = s^2 E_{89}$$

$$T_{91} = 16 \delta_c^2 \rho^2 h q_{mn} \quad ; \quad E_{91} = s^4$$

$$T_{92} = 16 \delta_c^2 \delta_D \rho^3 h \quad ; \quad E_{92} = s^6$$

where,

$$q_{mn} = g_{11} \lambda_m^2 + g_{22} \lambda_n^2$$

For the FSDT, n = 38, and we have the following relations for T_i and $E_i[s]$:

$$T_1 = K^2 \frac{h^7}{144} \lambda_m^4 \lambda_n^2 \quad ; \quad E_1 = \bar{\tilde{\vec{E}}}_{1212} \bar{\tilde{\vec{E}}}_{1111} \bar{\tilde{\vec{E}}}_{2222}$$

$$T_2 = K^2 \frac{h^7}{144} \lambda_m^2 \lambda_n^4 \quad ; \quad E_2 = \bar{\tilde{\vec{E}}}_{1111} \bar{\tilde{\vec{E}}}_{2222} \vec{E}_{2323}$$

$$T_3 = K^2 \frac{h^7}{144} \lambda_m^6 \quad ; \quad E_3 = \bar{\tilde{\vec{E}}}_{1111} (\bar{\tilde{\vec{E}}}_{1212})^2$$

$$T_4 = T_1 + K^4 \frac{h^5}{12} \lambda_m^4 \quad ; \quad E_4 = \bar{\tilde{\vec{E}}}_{1111} \bar{\tilde{\vec{E}}}_{1212} \vec{E}_{2323}$$

$$T_5 = T_2 \quad ; \quad E_5 = \bar{\tilde{\vec{E}}}_{2222} (\bar{\tilde{\vec{E}}}_{1212})^2$$

$$T_6 = K^2 \frac{h^7}{144} \lambda_n^6 + K^4 \frac{h^5}{12} \lambda_n^4 \quad ; \quad E_6 = \bar{\tilde{\vec{E}}}_{2222} \bar{\tilde{\vec{E}}}_{1212} \vec{E}_{2323}$$

$$T_7 = K^4 \frac{h^5}{3} \lambda_m^2 \lambda_m^2 \quad ; \quad E_7 = (\bar{\tilde{\vec{E}}}_{1212})^2 \vec{E}_{2323}$$

$$T_8 = \frac{T_7}{2} - 2T_2 \quad ; \quad E_8 = \bar{\tilde{\vec{E}}}_{1122} \bar{\tilde{\vec{E}}}_{1212} \vec{E}_{2323}$$

$$T_9 = -2T_1 \quad ; \quad E_9 = \bar{\tilde{\vec{E}}}_{1122} (\bar{\tilde{\vec{E}}}_{1212})^2$$

$$T_{10} = -T_1 \quad ; \quad E_{10} = (\bar{\tilde{\vec{E}}}_{1122})^2 \bar{\tilde{\vec{E}}}_{1212}$$

$$T_{11} = -T_2 \quad ; \quad E_{11} = (\bar{\tilde{\vec{E}}}_{1122})^2 \vec{E}_{2323}$$

$$T_{12} = \frac{h^7}{144} q_{mn} \lambda_m^4 \quad ; \quad E_{12} = \bar{\tilde{\vec{E}}}_{1111} \bar{\tilde{\vec{E}}}_{1212}$$

$$T_{13} = \delta_c m_1 K^2 \frac{h^4}{12} \lambda_m^4 + \delta_D m_0 \frac{h^6}{144} \lambda_m^4 \quad ; \quad E_{13} = s^2 E_{12}$$

$$T_{14} = \delta_c m_1 K^2 \frac{h^4}{12} \lambda_n^4 \quad ; \quad E_{14} = s^2 \bar{\tilde{\vec{E}}}_{2222} \vec{E}_{2323}$$

$$T_{15} = K^2 \frac{h^5}{12} q_{mn} \lambda_m^2 \quad ; \quad E_{15} = \bar{\tilde{\vec{E}}}_{1111} \vec{E}_{2323}$$

$$T_{16} = \delta_c m_1 K^2 \frac{h^4}{12} \lambda_m^2 \lambda_n^2 + \delta_D m_0 K^2 \frac{h^4}{12} \lambda_m^2 \quad ; \quad E_{16} = s^2 E_{15}$$

$$T_{17} = \frac{h^7}{144} q_{mn} \lambda_n^4 + K^2 \frac{h^5}{12} q_{mn} \lambda_n^2 \quad ; \quad E_{17} = \bar{\tilde{\vec{E}}}_{2222} \bar{\tilde{\vec{E}}}_{1212}$$

$$T_{18} = \delta_c m_1 K^2 \frac{h^4}{12} \lambda_n^2 \lambda_m^2 + \delta_D m_0 \frac{h^6}{144} \lambda_n^4 + \delta_D m_0 k^2 \frac{h^4}{12} \lambda_n^2 \quad ; \quad E_{18} = s^2 E_{17}$$

$$T_{19} = T_{15} \quad ; \quad E_{19} = (\bar{\tilde{\vec{E}}}_{1212})^2$$

$$T_{20} = \delta_c m_1 K^2 \frac{h^4}{12} \{\lambda_n^2 \lambda_m^2 + \lambda_m^4\} + \delta_D m_0 k^2 \frac{h^4}{12} \lambda_n^2 \quad ; \quad E_{20} = s^2 E_{19}$$

$$T_{21} = K^2 \frac{h^5}{12} q_{mn} \lambda_n^2 + K^4 h^3 q_{mn} \quad ; \quad E_{21} = \bar{\tilde{\vec{E}}}_{1212} \vec{E}_{2323}$$

$$T_{22} = \delta_c m_1 K^4 h^2 \{\lambda_m^2 + \lambda_n^2\} + \delta_c m_1 K^2 \frac{h^4}{12} \{\lambda_n^4 + \lambda_n^2 \lambda_m^2\} + \delta_D m_0 K^2 \frac{h^4}{12} \lambda_n^2 + \delta_D m_0 K^4 h^2 \quad ; \quad E_{22} = s^2 E_{21}$$

$$T_{23} = \frac{h^7}{144} q_{mn} \lambda_m^2 \lambda_n^2 \quad ; \quad E_{23} = \bar{\tilde{\vec{E}}}_{1111} \bar{\tilde{\vec{E}}}_{2222}$$

$$T_{24} = \delta_D m_0 \frac{h^6}{144} \lambda_m^2 \lambda_n^2 \quad ; \quad E_{24} = s^2 E_{23}$$

$$T_{25} = -2 T_{23} \quad ; \quad E_{25} = \bar{\tilde{\vec{E}}}_{1122} \bar{\tilde{\vec{E}}}_{1212}$$

$$T_{26} = -2 T_{24} \quad ; \quad E_{26} = s^2 E_{25}$$

$$T_{27} = \delta_c m_1 \frac{h^4}{12} q_{mn} \lambda_m^2 \quad ; \quad E_{27} = s^2 \bar{\tilde{\vec{E}}}_{1111}$$

$$T_{28} = \delta_c \delta_D m_1 m_0 \frac{h^3}{12} \lambda_m^2 \quad ; \quad E_{28} = s^2 E_{27}$$

$$T_{29} = \delta_c m_1 \frac{h^4}{12} q_{mn} \lambda_n^2 \quad ; \quad E_{29} = s^2 \bar{\vec{\tilde{E}}}_{2222}$$

$$T_{30} = \delta_c \delta_D m_1 m_0 \frac{h^3}{12} \lambda_n^2 \quad ; \quad E_{30} = s^2 E_{29}$$

$$T_{31} = \delta_c m_1 \frac{h^4}{12} q_{mn} \{\lambda_n^2 + \lambda_m^2\} + \delta_c m_1 K^2 h^2 q_{mn} \quad ; \quad E_{31} = s^2 \bar{\vec{\tilde{E}}}_{1212}$$

$$T_{32} = \delta_c m_1^2 K^2 h \lambda_m^2 + \delta_c \delta_D m_1 m_0 \frac{h^3}{12} \{\lambda_n^2 + \lambda_m^2\} + \delta_c \delta_D m_1 m_0 K^2 h \quad ; \quad E_{32} = s^2 E_{31}$$

$$T_{33} = \delta_c m_1 K^2 h^2 q_{mn} \quad ; \quad E_{33} = s^2 \bar{\vec{\tilde{E}}}_{2323}$$

$$T_{34} = \delta_c^2 m_1^2 K^2 h \lambda_n^2 + \delta_c \delta_D m_1 m_0 K^2 h \quad ; \quad E_{34} = s^2 E_{33}$$

$$T_{35} = \delta_c^2 m_1^2 h q_{mn} \quad ; \quad E_{35} = s^4$$

$$T_{36} = \delta_c^2 \delta_D m_1^2 m_0 \quad ; \quad E_{36} = s^6$$

$$T_{37} = -T_{23} \quad ; \quad E_{37} = (\bar{\vec{\tilde{E}}}_{1122})^2$$

$$T_{38} = -T_{24} \quad ; \quad E_{38} = s^2 E_{37}$$

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