

**OBSERVABILITY ANALYSIS IN NAVIGATION SYSTEMS WITH AN
UNDERWATER VEHICLE APPLICATION**

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ABSTRACT

Precise navigation of autonomous underwater vehicles (AUV) is one of the most important challenges in the realization of distributed and cooperative algorithms for marine applications. We investigate an underwater navigation technology that enables an AUV to compute its trajectory in the presence of unknown currents in real time and simultaneously estimate the currents, using range or distance measurements from a single known location. This approach is potentially useful for small AUVs which have severe volume and power constraints.

The main contribution of this work is observability analysis of the proposed navigation system using novel approaches towards uniform observability of linear time-varying (LTV) systems. We utilize the notion of limiting systems in order to address uniform observability of LTV systems. Uniform observability of an LTV system can be studied by assessing finite time observability of its limiting systems. A new definition of uniform observability over a finite interval is introduced in order to address existence of an observer whose estimation error is bounded by an exponentially decaying function on the finite interval. We also show that for a class of LTV systems, uniform observability of a lower dimensional subsystem derived from an LTV system is sufficient for uniform observability of the LTV system.

To My Grandmother

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Chapter 1

Introduction

1.1 Contribution

An acoustic underwater navigation system is proposed which is suitable for small autonomous underwater vehicles (AUVs). This navigation system utilizes distance (or range) of an AUV from a single acoustic beacon at a known location, along with velocity (heading and speed) of the AUV, to estimate position of the AUV. It enables an AUV to compute its trajectory in real time and in the presence of unknown currents, which are estimated simultaneously with the position of the AUV. This navigation algorithm is referred to as *range navigation* in this work.

The main contribution of this work is the observability analysis of the proposed navigation system using novel approaches to assess uniform observability of a linear time-varying (LTV) system. We utilize the notion of limiting systems to study uniform observability of LTV systems. We show that for a class of LTV systems, uniform observability of a lower dimensional subsystem derived from the LTV system is sufficient for uniform observability of the LTV system. We introduce a new definition of

uniform observability over a finite time interval which facilitates assessment of sufficient conditions for the existence of an observer whose estimation error is bounded by an exponentially decaying function.

The range navigation system is mathematically characterized and observability rigorously studied. We characterize AUV trajectories for which the navigation system is observable guaranteeing sufficient conditions for the existence of a stable observer.

The proposed range navigation algorithm has been successfully deployed on the VT Miniature AUV using an acoustic modem on the AUV and an external acoustic transponder. The system can be extended to use more than one acoustic transponder and also provides a framework to implement synchronous navigation in which AUVs navigate in a group by knowing positions of and ranges from other AUVs.

1.2 Motivation

To decrease costs and deployment complexity associated with AUVs, there have been several efforts to design a very small AUV, including the AUVs reported in [1], [2], and [3]. At this very small size, standard navigation technology may not be suitable due to volume and/or power constraints. These constraints have motivated us to consider development of underwater navigation algorithms that require low volume and low power instrumentation that can be fitted on small AUVs.

Range navigation provides a low volume and low power alternative to underwater navigation for small AUVs. It could potentially result in reduced hardware complexity as well as reduced deployment complexity. It would also enable deployment of a fleet of AUVs from a mothership and enable leader-follower formation control of multiple

AUVs [4].

In practice, navigation systems are implemented as observers, with the extended Kalman filter being the most common. For such an observer with stable error dynamics to exist, the navigation system must satisfy certain uniform observability conditions. However, uniform observability of navigation systems is almost never assessed in literature. This is mainly due to the fact that assessment of uniform observability using the observability grammian can be prohibitively difficult.

Instead of directly assessing observability, it is more common to identify conditions under which the estimation error covariance does not diverge [5, 6]. This is the classic design by simulation approach. The system is simulated and the estimation error covariance matrix is examined. If the estimation error covariance matrix diverges, then either new sensors are added to the system to achieve observability or trajectories that result in divergence of the covariance matrix are avoided. We seek to provide a sound mathematical basis to assess uniform observability of the range navigation algorithm and towards this end we introduce new approaches to study uniform observability of LTV systems.

1.3 Background

There are two general classes of technology commonly utilized for underwater navigation: acoustic navigation and inertial navigation. Acoustic navigation is exemplified by a long-base-line (LBL) navigation system, which consists of a grid of acoustic transponders pre-surveyed onto the sea floor. An underwater vehicle can triangulate its position using range measurements from these transponders. The distance between two transponders in the LBL system, known as the baseline length, varies

from 100 meters to more than 6000 meters. LBL is not dissimilar from terrestrial GPS and provides very good positioning accuracy (positioning error is less than 1m) for long range navigation [7]. However, deployment complexity of an LBL system is high due to the requirement of pre-surveying a grid of acoustic transponders on the sea-floor and recovery of transponders after the mission. Some of these issues have been addressed by using acoustic transponders on the sea surface, which use GPS to get position information, thus removing the need for surveying the ocean floor. However this system is not as accurate as the LBL [8].

Other acoustic navigation technologies are the short-base-line (SBL) and the ultra-short-base-line (USBL) navigation systems. In SBL, the AUV calculates its bearing and range from multiple transponders. These transponders are normally deployed on the ship, thus removing the need to deploy transponders in the ocean. The baseline for SBL is a few tens of meters. The USBL system extracts the arrival phase information of an acoustic ping transmitted by a single acoustic beacon, using a transducer with multiple acoustic receivers on the AUV. Phase differences of the acoustic ping arriving at multiple receivers are used to calculate the bearing of the AUV with respect to the beacon. Normal baseline for USBL is about 10cm. Both SBL and USBL provide position with respect to the transponders and hence additional sensors are required to get absolute position of the AUV.

Inertial navigation systems rely on precision instrumentation, such as accelerometers and gyros, to measure the AUV attitude (pitch, roll and yaw angles and corresponding angular rates) and linear accelerations in order to estimate the AUV position. High accuracy inertial measurement units are very expensive and are not suitable for small AUVs due to volume and power constraints. Navigation based on dead-reckoning uses heading and pitch angle and forward speed information to esti-

mate the position of the AUV. However due to sensor drift, dead-reckoning alone is not sufficient, as localization errors accumulate over time.

1.4 Related Work

Navigation based on range measurements from a single location is related to the problem of target motion analysis (TMA), an analysis technique developed for missile guidance applications. The goal of TMA is to estimate the position and velocity of a moving target using various measurements. Extensive work has been reported for bearings-only tracking, including [9, 10, 11, 12], in which position and velocity of the target are determined from measurements of the angle of the line of sight to the target. However, the literature for range-only tracking, which is closely related to our proposed underwater navigation system, is very limited. Local observability for range-only TMA is addressed in [5] by the analysis of the Fisher information matrix. In [13], similar analysis is performed for a restricted case when the observing vehicle is moving in a circular trajectory and the target is moving along a straight line with constant velocity. Effects of seeker vehicle maneuvers on observability in bearings-only tracking are studied in [14] and optimal maneuvers that maximize the performance of the estimator are identified in [15].

The underwater navigation system that uses range measurements from a single known location, along with the AUV velocity information, is discussed in [4], [8] and [16, 17, 18].

In [8], navigation system based on range measurements is termed as the synthetic long baseline (SLBL) navigation. A variant of SLBL uses range rate as an additional measurement. SLBL is shown to be a good candidate navigation system for off-shore

surveys. It is shown that for line surveys, SLBL has significantly better accuracy than dead-reckoning.

A navigation system that integrates range measurements from a single acoustic transponder with inertial sensor measurements is implemented as a least-squares algorithm in [18]. This system enables an AUV to determine its position relative to the acoustic beacon in the presence of unknown currents. It is shown that addition of range measurements improves the navigation performance over dead-reckoning.

Various applications of underwater navigation using range measurements from a single acoustic transponder are explored in [4, 16, 17]. These applications include homing, feature following and large area surveys using multiple vehicles that share the same acoustic beacon.

1.5 Organization

The dissertation is organized as follows. Mathematical tools used throughout the dissertation are introduced in Chapter 2. New approaches to uniform observability analysis are presented in Chapter 3, where we use the notion of limiting systems to address uniform observability and show that for a class of LTV systems, uniform observability of a lower dimensional subsystem derived from the LTV system is sufficient for uniform observability of the original LTV system. Also we introduce a new definition of uniform observability over a finite interval, in order to address existence of an observer whose estimation error is bounded by an exponentially decaying function on the finite interval. The Range navigation algorithm is introduced with a discussion of uniform observability for the case of known currents in Chapter 4. Chapter 5 presents extension of the range navigation algorithm to the case of un-

known currents. Range navigation algorithm has been successfully implemented on VT Miniature AUV. Hardware and software aspects of this implementation discussed in Chapter 6. In chapter 7 we present results from field trials of the range navigation algorithm. Finally we present conclusions in Chapter 8.

Chapter 2

Mathematical Preliminaries

2.1 Notation

Notation used throughout this dissertation is standard and any special notation is defined before its use.

The set of real numbers is denoted $\mathbb{R}$. The set of $n \times m$ real matrices is denoted $\mathbb{R}^{n \times m}$. The $n \times n$ identity matrix is denoted by I_n . The Euclidean norm of a vector is denoted by $\|\cdot\|$, and the spectral norm of a matrix is denoted by $\|\cdot\|$.

2.2 Concepts from Linear Systems Theory

In this section we provide basic introduction to the linear systems theory. For in depth discussion of concepts introduced here, the reader is referred to [\[19, 20, 21\]](#).

2.2.1 Linearization

We consider systems represented by a nonlinear ordinary differential equation

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t), t), \quad t \geq t_o, \quad x(t_o) = x_o \\ y(t) &= g(x(t), u(t))\end{aligned}\tag{2.1}$$

where $x(t) \in \mathbb{R}^{n \times 1}$, $u(t) \in \mathbb{R}^{m \times 1}$, $y(t) \in \mathbb{R}^{p \times 1}$. A linearized model can be used to approximate behavior of the system about a nominal state trajectory $x_o(t)$ and the corresponding nominal input $u_o(t)$. The linearized system is

$$\begin{aligned}\dot{x}_\delta(t) &= A_o(t)x_\delta(t) + B_o(t)u_\delta(t), \quad t \geq t_o, \quad x_\delta(t_o) = x_{\delta o} \\ y_\delta(t) &= C_o(t)x_\delta(t) + D_o(t)u_\delta(t)\end{aligned}\tag{2.2}$$

where

$$\begin{aligned}A_o(t) &= \left. \frac{\partial f(x(t), u(t), t)}{\partial x(t)} \right|_{x_o(t), u_o(t)} \\ B_o(t) &= \left. \frac{\partial f(x(t), u(t), t)}{\partial u(t)} \right|_{x_o(t), u_o(t)} \\ C_o(t) &= \left. \frac{\partial y(x(t), u(t), t)}{\partial x(t)} \right|_{x_o(t), u_o(t)} \\ D_o(t) &= \left. \frac{\partial y(x(t), u(t), t)}{\partial u(t)} \right|_{x_o(t), u_o(t)}\end{aligned}\tag{2.3}$$

and $x_\delta(t) = x(t) - x_o(t)$, $u_\delta(t) = u(t) - u_o(t)$ and $y_\delta(t) = y(t) - y_o(t)$.

2.2.2 Linear State Equation

A linear system is represented by

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_o) = x_o \\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned}\tag{2.4}$$

where $A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times m}$, $C(t) \in \mathbb{R}^{p \times n}$ and $D(t) \in \mathbb{R}^{p \times m}$. All coefficient matrices are assumed to be bounded and continuous.

2.2.3 Solution of Linear State Equation

For any initial time t_o and initial state x_o , the complete solution to (2.4) is given by

$$x(t) = \Phi(t, t_o)x_o + \int_{t_o}^t \Phi(t, \sigma)B(\sigma)u(\sigma)d\sigma \quad (2.5)$$

where $\Phi(t, t_o)$ is the transition matrix corresponding to $A(t)$. The transition matrix $\Phi(t, t_o)$ is the unique solution of the matrix differential equation

$$\dot{\Phi}(t, t_o) = A(t)\Phi(t, t_o), \quad \Phi(t_o, t_o) = I, \quad t \geq t_o \quad (2.6)$$

Some of the transition matrix properties used in this work are

$$\frac{d}{dt}\Phi(t, \tau) = A(t)\Phi(t, \tau) \quad (2.7)$$

$$\frac{d}{d\tau}\Phi(t, \tau) = -\Phi(t, \tau)A(\tau) \quad (2.8)$$

2.2.4 Stability of a Linear System

Definition 2.2.1 (Uniform Exponential Stability). *The linear system (2.4) is uniformly exponentially stable if there exist finite positive constants $\gamma \geq 1$ and $\lambda > 0$, such that for any t_o and x_o ,*

$$\|x(t)\| \leq \gamma e^{-\lambda(t-t_o)}\|x_o\|, \quad t \geq t_o$$

Lemma 2.2.2 (Uniform Exponential Stability). *The linear system (2.4) is uniformly exponentially stable if there exists a continuously differentiable and symmetric matrix $Q(t)$ and positive constants η , ρ and ν such that*

$$\eta I \leq Q(t) \leq \rho(t)$$

$$A^T(t)Q(t) + Q(t)A(t) + \dot{Q}(t) \leq -\nu I$$

Definition 2.2.3 (Uniform Asymptotic Stability). *The linear system (2.4) is uniformly asymptotically stable if it is uniformly stable and for any positive constant δ there exist positive T , such that for any t_o and x_o ,*

$$\|x(t)\| \leq \delta \|x_o\|, \quad t \geq t_o + T$$

A system is uniformly asymptotically stable if and only if it is uniformly exponentially stable.

2.2.5 Observability

Definition 2.2.4 (Observability). *The linear system (2.4) is observable over a finite interval $[t_o, t_f]$ if the initial state $x(t_o)$ can be uniquely determined from the input $u(t)$ and the output $y(t)$ for all t in $[t_o, t_f]$.*

Lemma 2.2.5 (Observability). *The linear system (2.4) is observable over a finite interval $[t_o, t_f]$ if and only if the observability grammian*

$$M(t_o, t_f) = \int_{t_o}^{t_f} \Phi^T(t, t_o) C^T(t) C(t) \Phi(t, t_o) dt \quad (2.9)$$

is invertible.

A stronger form of observability is defined by uniform observability.

Definition 2.2.6 (Uniform Observability). *The linear system (2.4) is uniformly observable if there exist positive constants δ , α_1 and α_2 such that*

$$0 < \alpha_1 I \leq M(t, t + \delta) \leq \alpha_2 I \quad (2.10)$$

for all $t \geq t_o$.

Definition 2.2.7 (Controllability). *The linear system (2.4) is controllable over a finite interval $[t_o, t_f]$ if given any initial state $x(t_o) = x_o$, there exists a continuous input signal $u(t)$ such that $x(t_f) = 0$.*

Lemma 2.2.8 (Controllability). *The linear system (2.4) is controllable over a finite interval $[t_o, t_f]$ if and only if the controllability grammian*

$$W(t_o, t_f) = \int_{t_o}^{t_f} \Phi(t_o, t) B(t) B^T(t) \Phi^T(t_o, t) dt \quad (2.11)$$

is invertible.

Definition 2.2.9 (Uniform Controllability). *The linear system (2.4) is uniformly controllable if there exist positive constants δ , α_1 and α_2 such that*

$$0 < \alpha_1 I \leq W(t, t + \delta) \leq \alpha_2 I \quad (2.12)$$

for all $t \geq t_o$.

2.3 State Observation and Observer Design

An observer is described by the linear state equation

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + H(t)[y(t) - C(t)\hat{x}(t)], \quad t \geq t_o, \quad \hat{x}(t_o) = \hat{x}_o \quad (2.13)$$

where $\hat{x}(t) \in \mathbb{R}^{n \times 1}$ is the estimated state and $H(t) \in \mathbb{R}^{n \times p}$ is the observer gain. The observer error $e(t) = x(t) - \hat{x}(t)$ satisfies

$$\dot{e}(t) = [A(t) - H(t)C(t)]e(t), \quad t \geq t_o, \quad e(t_o) = e_o \quad (2.14)$$

The observer gain $H(t)$ is chosen such that regardless of the initial error e_o , the observer error asymptotically reaches zero, that is, $\hat{x}(t) \rightarrow x(t)$ as $t \rightarrow \infty$. If the LTV system is uniformly observable, then such an observer gain exists (see e.g. [19], Theorem 15.2).

2.4 Stochastic Estimation and the Kalman Filter

The Kalman filter, also known as the Kalman-Bucy filter, is an optimal recursive algorithm for the state reconstruction of a linear stochastic system of the form

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + G(t)\omega(t), \quad t \geq t_o, \quad x(t_o) = x_o \\ y(t) &= C(t)x(t) + D(t)\nu(t)\end{aligned}\tag{2.15}$$

where $G(t) \in \mathbb{R}^{n \times m}$, $D(t) \in \mathbb{R}^{p \times q}$ are coefficient matrices and $\omega(t) \in \mathbb{R}^{m \times 1}$ and $\nu(t) \in \mathbb{R}^{q \times 1}$ are zero mean, white Gaussian processes such that

$$\text{cov} [\omega(t), \omega(\tau)] = Q(t)\delta(t - \tau)$$

$$\text{cov} [\nu(t), \nu(\tau)] = R(t)\delta(t - \tau)$$

$$\text{cov} [\omega(t), \nu(\tau)] = 0$$

The estimation error covariance is

$$P(t) = \mathbb{E} \left\{ [x(t) - \hat{x}(t)] [x(t) - \hat{x}(t)]^T \right\}\tag{2.16}$$

and the Kalman gain is

$$K(t) = P(t)C^T(t)R^{-1}(t)\tag{2.17}$$

The Kalman filter consists of a differential equation for the conditional mean of the state and another for the state estimation error covariance matrix [22].

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + P(t)C^T(t)R^{-1}(t)[y(t) - C(t)\hat{x}(t)]\tag{2.18}$$

$$\begin{aligned}\dot{P}(t) &= A(t)P(t) + P(t)A^T(t) + G(t)Q(t)G^T(t) \\ &\quad - P(t)C^T(t)R^{-1}(t)C(t)P(t)\end{aligned}\tag{2.19}$$

The error state equation is written

$$\dot{e}(t) = [A(t) - P(t)C^T(t)R^{-1}(t)C(t)] e(t), \quad t \geq t_o, \quad e(t_o) = e_o\tag{2.20}$$

The Kalman filter requires that (2.15) is uniformly controllable and uniformly observable [23].

Definition 2.4.1 (Uniform observability of a stochastic linear system). *The stochastic LTV system (2.15) is uniformly observable if and only if there exist positive constants δ , α_1 and α_2 such that*

$$0 < \alpha_1 I_p \leq \int_{t-\delta}^t \Phi^T(\tau, t - \delta) C^T(\tau) R^{-1}(\tau) C(\tau) \Phi(\tau, t - \delta) d\tau \leq \alpha_2 I_p \quad (2.21)$$

for all t .

Definition 2.4.2 (Uniform controllability of a stochastic linear system). *The stochastic LTV system (2.15) is uniformly controllable if and only if there exist positive constants δ , α_1 and α_2 such that*

$$0 < \alpha_1 I_n \leq \int_{t-\delta}^t \Phi(t - \delta, \tau) G(\tau) Q(\tau) G^T(\tau) \Phi^T(t - \delta, \tau) d\tau \leq \alpha_2 I_n \quad (2.22)$$

for all t .

Under the assumption of positive definiteness of $Q(t)$ for all t , (2.12) implies (2.22). Similarly, under the assumption of positive definiteness of $R(t)$ for all t , (2.10) implies (2.21).

Chapter 3

Observability of LTV Systems

3.1 Introduction

In this chapter we introduce mathematical tools that are used for observability analysis of the range navigation algorithm. These tools help address the two main problems in assessment of uniform observability - computation of the observability grammian and study of asymptotic properties of the observability grammian. We present new notions of observability of an LTV system and introduce relationship between uniform observability and limiting systems of an LTV system. The notion of limiting systems allows us to study finite time observability of limiting systems of an LTV system in order to assess uniform observability of the LTV system. We show that for a class of LTV systems uniform observability can be assessed by studying uniform observability of a lower dimensional system derived from the original LTV system. We also introduce a new definition of uniform observability over a finite interval.

We restrict observability analysis to LTV systems of the form

$$\begin{aligned}\dot{x}(t) &= A(t)x(t), \quad t \geq t_o, \quad x(t_o) = x_o \\ y(t) &= C(t)x(t)\end{aligned}\tag{3.1}$$

where $A(t) \in \mathbb{R}^{n \times n}$ and $C(t) \in \mathbb{R}^{p \times n}$ are uniformly continuous and bounded.

3.2 Observability of an LTV System

Observability of an LTV system is related to how the state of the system affects the output of the system. In addition to invertibility of the grammian in Lemma 2.2.5, sufficient conditions for observability of an LTV system are given by a rank test (see e.g [19]).

Lemma 3.2.1 (Observability). *Suppose $q \leq (n - 1)$ is a positive integer such that $C(t)$ is at least q times continuously differentiable, and $A(t)$ is $q-1$ times continuously differentiable. Then the linear system is observable on $[t_0, t_f]$, if for some $t \in [t_0, t_f]$, $\text{rank } O(t) = n$, where*

$$O(t) = \begin{bmatrix} L_0(t) \\ \vdots \\ L_q(t) \end{bmatrix}\tag{3.2}$$

$$L_0(t) = C(t),$$

$$L_j(t) = L_{j-1}(t)A(t) + \dot{L}_{j-1}(t)$$

By definition, observability of an LTV system is tied to a particular finite time interval. Observability over one finite interval does not necessarily imply observability over another finite interval. However, if an LTV system is observable on a finite interval $[t_o, t_f]$, it remains observable for any finite interval that contains $[t_o, t_f]$.

3.3 Uniform Observability of an LTV System

According to Definition 2.2.6, uniform observability bounds the observability gramian away from being singular for all t . Uniform observability implies observability, however the converse need not be true.

Remark 3.3.1. *Since $A(t)$ and $C(t)$ are bounded, there always exists α_2 such that the upper bound in (2.10) is satisfied. Thus in the sequel we consider only the lower bound.*

In this work, we consider a specific δ to assess uniform observability of an LTV system, since as is shown in Section 3.7, smaller values of δ are preferred for improved estimator performance.

Definition 3.3.2 (Uniform Observability for δ). *The linear system in (3.1) is uniformly observable for a given $\delta > 0$ if and only if there exist positive constants α_1 and α_2 such that*

$$0 < \alpha_1 I \leq M(t, t + \delta) \leq \alpha_2 I \quad (3.3)$$

for all t .

By construction, if an LTV system is uniformly observable for some δ then it is uniformly observable for any $\delta_1 > \delta$. Under this definition of uniform observability for δ , an LTV system that is not uniformly observable for some δ_1 may still be uniformly observable for some $\delta_2 > \delta_1$.

To study uniform observability of an LTV system for a δ , we must assess asymptotic properties of the observability grammian. The notion of limiting systems provides a tool that lets us use the standard observability rank test from Lemma 3.2.1, to assess uniform observability of LTV systems. In the next section we introduce the notion of limiting systems of an LTV system.

3.4 Limiting Systems

The notion of limiting systems of an LTV system in the analysis of controllability and observability appears in early work [24], although our treatment of the subject borrows heavily from [25]. More recently, limiting systems have been used for analysis of non-autonomous nonlinear systems in [26, 27].

In this section, we first show how limiting systems are constructed, and then show that they exist under suitable hypothesis. Given a sequence of positive scalars $\{t_i\}$, $i = 1 \dots, \infty$, we define a sequence $\{A^{t_i}(t)\}$ of translations of the matrix coefficients in (3.1) by

$$A^{t_i}(t) := A(t + t_i)$$

and the sequence $\{C^{t_i}(t)\}$ is defined similarly.

Definition 3.4.1 (Limiting Systems). *If $\{A^{t_i}(t)\}$ and $\{C^{t_i}(t)\}$ converge to $A_o(t)$ and $C_o(t)$ respectively, then the system*

$$\dot{x}(t) = A_o(t)x(t)$$

$$y(t) = C_o(t)x(t)$$

is a limiting system of (3.1) induced by $\{t_i\}$.

We first show that the sequences in question have convergent subsequences.

Lemma 3.4.2 (Existence of Limiting System). *Given bounded and uniformly continuous $S(t) : [t_o, \infty) \mapsto \mathbb{R}^{n \times m}$ and an infinite sequence of positive scalars $\{t_i\}$, there exists a subsequence $\{\tau_j\}$ of $\{t_i\}$ such that the corresponding sequence of translations $\{S^{\tau_j}(t)\}$ converges to some $S_o(t)$.*

Proof. Let $E = \{e_1, e_2, \dots\}$ be a countable dense subset of $[t_o, \infty)$. First, we show that $\{S^{t_i}(t)\}$ converges on E using a standard argument (see for example [28], Theorem 7.23). Since $S(t)$ is bounded, the sequence of matrices $\{S^{t_i}(e_1)\}$ has a convergent subsequence $\{S^{\sigma_{1,i}}(e_1)\}$ indexed by $\{\sigma_{1,i}\} \subset \{t_i\}$ that converges to some S_1 and thus $\lim_{i \rightarrow \infty} S^{\sigma_{1,i}} = S_1$. Note that the sequence of matrices $\{S^{\sigma_{1,i}}(e_2)\}$ is bounded and thus has a convergent subsequence indexed by $\{\sigma_{2,i}\} \subset \{\sigma_{1,i}\}$ such that $\lim_{i \rightarrow \infty} S^{\sigma_{2,i}}(e_2) = S_2$. Moreover, since $\{\sigma_{2,i}\} \subset \{\sigma_{1,i}\}$, $\lim_{i \rightarrow \infty} S^{\sigma_{2,i}}(e_1) = S_1$. We continue this process, resulting in a set of nested sequences

$$\{t_i\} \supset \{\sigma_{1,i}\} \supset \dots \supset \{\sigma_{n,i}\} \supset \dots$$

with the property that

$$\lim_{i \rightarrow \infty} S^{\sigma_{n,i}}(e_j) = S_j$$

for all $1 \leq j \leq n$. Let $\{\tau_j\}$ be the sequence obtained from the set of sequence $\{\sigma_{n,i}\}$ as $n \rightarrow \infty$, then $\{S^{\tau_j}(e_j)\}$ converges to S_j for all $e_j \in E$. Since $S(t)$ is uniformly continuous, there exists a continuous extension $S_o(t)$ (see for example [28], p. 99). $\square$

Lemma 3.4.3. *If the sequences $\{A^{\tau_i}(t)\}$ and $\{C^{\tau_i}(t)\}$ converge to $A_o(t)$ and $C_o(t)$, respectively, then the corresponding sequence of observability grammians $M^{\tau_i}(t, t + \delta)$ converges to the observability grammian $M_o(t, t + \delta)$ of the limiting system.*

Proof. Note that

$$\begin{aligned} M^{\tau_i}(t, t + \delta) &= M(t + \tau_i, t + \tau_i + \delta) \\ &= \int_{t+\tau_i}^{t+\delta+\tau_i} [C(\sigma)\Phi(\sigma, t + \tau_i)]^T [C(\sigma)\Phi(\sigma, t + \tau_i)] d\sigma \\ &= \int_t^{t+\delta} [C(\sigma + \tau_i)\Phi(\sigma + \tau_i, t + \tau_i)]^T [C(\sigma + \tau_i)\Phi(\sigma + \tau_i, t + \tau_i)] d\sigma \end{aligned}$$

Since $\lim_{i \rightarrow \infty} A^{\tau_i}(t) = A_o(t)$ and $\lim_{i \rightarrow \infty} C^{\tau_i}(t) = C_o(t)$, we can write

$$\begin{aligned}
\lim_{i \rightarrow \infty} M^{\tau_i}(t, t + \delta) &= \\
&= \lim_{i \rightarrow \infty} \int_t^{t+\delta} [C(\sigma + \tau_i) \Phi(\sigma + \tau_i, t + \tau_i)]^T [C(\sigma + \tau_i) \Phi(\sigma + \tau_i, t + \tau_i)] d\sigma \\
&= \int_t^{t+\delta} \lim_{i \rightarrow \infty} [C^{\tau_i}(\sigma) \Phi^{\tau_i}(\sigma, t)]^T [C^{\tau_i}(\sigma) \Phi^{\tau_i}(\sigma, t)] d\sigma \\
&= \int_t^{t+\delta} [C_o(\sigma) \Phi_o(\sigma, t)]^T [C_o(\sigma) \Phi_o(\sigma, t)] d\sigma \\
&= M_o(t, t + \delta)
\end{aligned}$$

□

In the next section, we present the principal conclusion regarding uniform observability for δ of an LTV system from our use of limiting systems.

3.5 Limiting Systems and Uniform Observability for δ

Lemma 3.5.1. *The linear system (3.1) is uniformly observable for $\delta > 0$ if and only if all of its limiting systems are observable on $[t, t + \delta]$ for all $t \geq t_o$.*

Proof. Suppose that (3.1) is not uniformly observable for δ . Then there exist a monotonically decreasing sequence of positive scalars $\{\alpha_i\}$ that converges to zero and a sequence of times $\{t_i\}$ and corresponding unit-norm vectors $\{x_i\}$ such that

$$x_i^T M(t_i, t_i + \delta) x_i \leq \alpha_i, \quad i = 1, 2, \dots, \infty. \quad (3.4)$$

Then by Lemma 3.4.2, there exists a subsequence $\{\sigma_{1,i}\} \subset \{t_i\}$ for which $\{A^{\sigma_{1,i}}(t)\}$ converges. Applying Lemma 3.4.2 to $\{C^{\sigma_{1,i}}(t)\}$ yields a subsequence $\{\sigma_{2,i}\} \subset \{\sigma_{1,i}\}$

such that $\{C^{\sigma_{2,i}}(t)\}$ converges. Let the subsequence of unity-norm vectors $\{x_{2,i}\} \subset \{x_i\}$ correspond to the subsequence $\{\sigma_{2,i}\}$, then $\{x_{2,i}\}$ is bounded and there is a subsequence $\{\tau_i\} \subset \{\sigma_{2,i}\}$ for which x_{τ_i} converges. Since

$$\{\tau_i\} \subset \{\sigma_{2,i}\} \subset \{\sigma_{1,i}\}$$

the limits

$$\lim_{i \rightarrow \infty} A^{\tau_i}(t) = A_o(t)$$

$$\lim_{i \rightarrow \infty} C^{\tau_i}(t) = C_o(t)$$

$$\lim_{i \rightarrow \infty} x_{\tau_i}(t) = x_o(t)$$

exist. Thus

$$\lim_{i \rightarrow \infty} x_{\tau_i}^T M^{\tau_i}(t, t + \delta) x_{\tau_i} = 0$$

implies that

$$x_o^T M_o(t, t + \delta) x_o = 0$$

and that the limiting system

$$\dot{x}(t) = A_o(t)x(t)$$

$$y(t) = C_o(t)x(t)$$

is not observable on $[t, t + \delta]$.

Now suppose that (3.1) is uniformly observable for δ . Then there exists $\alpha > 0$ such that the corresponding observability grammian satisfies

$$M(t, t + \delta) \geq \alpha I$$

for all $t \geq t_o$. By construction, the observability grammian of any limiting system of (3.1) must satisfy the same bound and must be observable for δ on $[t, t + \delta]$ for all $t \geq t_o$. $\square$

3.6 Uniform Observability for δ Using Reduced Order System

In this section, we use the notion of limiting systems of an LTV system to facilitate assessment of uniform observability for δ .

Lemma 3.6.1. *For $x(t) = [x_1^T(t), x_2^T(t)]^T$, where $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, the LTV system*

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} x(t), & x(t_o) &= x_o \\ y(t) &= \begin{bmatrix} I_{n_1} & 0 \end{bmatrix} x(t) \end{aligned} \quad (3.5)$$

where matrix partitions match partitioning of $x(t)$ and all coefficient matrices are uniformly continuous, is uniformly observable for $\delta > 0$ if the LTV system

$$\begin{aligned} \dot{z}(t) &= A_{22}(t)z(t), & z(t_o) &= z_o \\ y_z(t) &= A_{12}(t)z(t) \end{aligned} \quad (3.6)$$

is uniformly observable for δ .

Proof. Suppose the (3.5) is not uniformly observable for δ . Then it possesses a limiting system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} A_{o,11}(t) & A_{o,12}(t) \\ A_{o,21}(t) & A_{o,22}(t) \end{bmatrix} x(t), & x(t_o) &= x_o \\ y(t) &= \begin{bmatrix} I_{n_1} & 0 \end{bmatrix} x(t) \end{aligned} \quad (3.7)$$

that is not observable on $[t, t + \delta]$ for some t . Thus the observability grammian

$$M_o(t, t + \delta) := \int_t^{t+\delta} [C_o(\sigma)\Phi_o(\sigma, t)]^T [C_o(\sigma)\Phi_o(\sigma, t)] d\sigma \quad (3.8)$$

is not invertible, where $C_o(\sigma) = \begin{bmatrix} I_{n_1} & 0 \end{bmatrix}$ and $\Phi_o(\sigma, t)$ is the transition matrix corresponding to (3.7). Thus there exists a vector $\nu \neq 0$ such that

$$\nu^T M_o(t, t + \delta) \nu = 0$$

By non-negativity of the integrand in (3.8),

$$C_o(\sigma) \Phi_o(\sigma, t) \nu = 0, \quad \sigma \in [t, t + \delta]$$

Define

$$z(\sigma) := \Phi_o(\sigma, t) \nu = \begin{bmatrix} z_1(\sigma) \\ z_2(\sigma) \end{bmatrix}$$

where, $z_1(\sigma) \in \mathbb{R}^{n_1}$ and $z_2(\sigma) \in \mathbb{R}^{n_2}$. Then from $C_o = [I_{n_1}, 0]$,

$$z_1(\sigma) = 0, \quad \sigma \in [t, t + \delta]$$

We can write,

$$\begin{aligned} \frac{d}{d\sigma} z(\sigma) &= \frac{d}{d\sigma} \Phi_o(\sigma, t) \nu \\ &= A_o(t) \Phi_o(\sigma, t) \nu \\ &= A_o(t) \begin{bmatrix} 0 \\ z_2(\sigma) \end{bmatrix} \end{aligned}$$

where $A_o(t)$ is the coefficient matrix in (3.7). Thus we can write

$$\frac{d}{d\sigma} z_2(\sigma) = A_{0,22}(\sigma) z_2(\sigma)$$

which implies,

$$z_2(\sigma) = \Phi_{A_{0,22}}(\sigma, t) \nu_2, \quad \nu_2 \neq 0$$

Also,

$$A_{o,12}(\sigma) z_2(\sigma) = 0, \quad \sigma \in [t, t + \delta]$$

Consider the observability grammian for (3.6)

$$M_{o,22}(t, t + \delta) := \int_t^{t+\delta} [A_{o,12}(\sigma)\Phi_{A_{o,22}}(\sigma, t)]^T [A_{o,12}(\sigma)\Phi_{A_{o,22}}(\sigma, t)] d\sigma \quad (3.9)$$

Multiplying by ν_2 yields,

$$M_{o,22}(t, t + \delta)\nu_2 = \int_t^{t+\delta} [A_{o,12}(\sigma)\Phi_{A_{o,22}}(\sigma, t)]^T [A_{o,12}(\sigma)\Phi_{A_{o,22}}(\sigma, t)\nu_2] d\sigma \quad (3.10)$$

which implies,

$$M_{o,22}(t, t + \delta)\nu_2 = \int_t^{t+\delta} [A_{o,12}(\sigma)\Phi_{A_{o,22}}(\sigma, t)]^T [A_{o,12}(\sigma)z_2(\sigma)] d\sigma = 0 \quad (3.11)$$

Thus there exists a vector $\nu_2 \neq 0$ such that

$$\nu_2^T M_{o,22}(t, t + \delta)\nu_2 = 0$$

Thus the limiting system of (3.6) is not observable for over the interval $[t, t + \delta]$ for $\delta > 0$ and hence (3.6) is not uniformly observable for δ . $\square$

3.7 Uniform Observability for δ over a Finite Interval

A stable observer exists for an LTV system if it is uniformly observable. In general, it is difficult to assess uniform observability by computing the observability grammian and then studying its asymptotic properties. In practical systems, however, it can be acceptable to consider finite time intervals $[t_o, t_f]$. We seek to show that there exists a Luenberger observer over the finite interval with observer error bounded by an exponentially decaying function. For this, we require observability to be satisfied over a sliding interval contained in $[t_o, t_f]$. We formally define this requirement as uniform observability for δ over a finite interval.

Definition 3.7.1. An LTV system is **uniformly observable for δ over a finite interval** $[t_o, t_f]$ if the LTV system is observable over intervals $[t - \delta, t]$ for all t in $[t_o + \delta, t_f]$.

We next show that uniform observability for δ over a finite interval is sufficient for the existence of a Luenberger observer with observer error bounded by an exponentially decaying function on the finite interval.

Theorem 3.7.2. Suppose the LTV system (3.1) is uniformly observable for δ over a finite interval $[t_o, t_f]$ and let

$$Q(t) = \int_{t-\delta}^t \exp(\tau - t) \Phi^T(\tau, t) C^T(\tau) C(\tau) \Phi(\tau, t) d\tau \quad (3.12)$$

where $C(t)$ is the matrix coefficient of (3.1) and $\Phi(\tau, t)$ is the state transition matrix of (3.1). Then the observer gain $H(t) = Q^{-1}(t) C^T(t)$ is such that the error-state is bounded by an exponentially decaying function for all t in $[t_o + \delta, t_f]$.

Proof. Since the system (3.1) is uniformly observable for δ over the finite interval $[t_o + \delta, t_f]$, the matrix

$$\begin{aligned} N(t - \delta, t) &= \Phi^T(t - \delta, t) M(t - \delta, t) \Phi(t - \delta, t) \\ &= \int_{t-\delta}^t \Phi^T(\tau, t) C^T(\tau) C(\tau) \Phi(\tau, t) d\tau \end{aligned} \quad (3.13)$$

is invertible for all $t \in [t_o + \delta, t_f]$, where $M(t - \delta, t)$ is the observability grammian of (3.1). Since we consider t only in a finite interval and since all coefficient matrices of (3.1) are bounded, there exist positive constants ϵ_1 and ϵ_2 such that

$$0 < \epsilon_1 I \leq N(t - \delta, t) \leq \epsilon_2 I \quad (3.14)$$

Hence we can write

$$0 < \exp(-\delta) \epsilon_1 I \leq Q(t) \leq \epsilon_2 I \quad \forall t \in [t_o + \delta, t_f] \quad (3.15)$$

Differentiating $Q(t)$ with respect to t ,

$$\begin{aligned}
\dot{Q}(t) &= C^T(t)C(t) - \exp(-\delta)\Phi^T(t-\delta, t)C^T(t-\delta)C(t-\delta)\Phi(t-\delta, t) \\
&\quad - \int_{t-\delta}^t \exp(\tau-t)\Phi^T(\tau, t)C^T(\tau)C(\tau)\Phi(\tau, t)d\tau \\
&\quad - A^T(t) \int_{t-\delta}^t \exp(\tau-t)\Phi^T(\tau, t)C^T(\tau)C(\tau)\Phi(\tau, t)d\tau \\
&\quad - \int_{t-\delta}^t \exp(\tau-t)\Phi^T(\tau, t)C^T(\tau)C(\tau)\Phi(\tau, t)d\tau A(t)
\end{aligned} \tag{3.16}$$

Substituting (3.12) in (3.16) yields

$$\begin{aligned}
\dot{Q}(t) &= C^T(t)C(t) - \exp(-\delta)\Phi^T(t-\delta, t)C^T(t-\delta)C(t-\delta)\Phi(t-\delta, t) \\
&\quad - Q(t) - A^T(t)Q(t) - Q(t)A(t)
\end{aligned} \tag{3.17}$$

The observer-error state equation

$$\dot{e}(t) = [A(t) - H(t)C(t)]e(t), \quad t \geq t_o, \quad e(t_o) = e_o$$

can be written as

$$\dot{e}(t) = [A(t) - Q^{-1}(t)C^T(t)C(t)]e(t), \quad t \geq t_o, \quad e(t_o) = e_o = x_0 - \hat{x}_0 \tag{3.18}$$

Define $\tilde{A}(t) = A(t) - Q^{-1}(t)C^T(t)C(t)$. Hence we can write

$$\begin{aligned}
\tilde{A}^T(t)Q(t) + Q(t)\tilde{A}(t) + \dot{Q}(t) &= [A(t) - Q^{-1}(t)C^T(t)C(t)]^T Q(t) \\
&\quad + Q(t) [A(t) - Q^{-1}(t)C^T(t)C(t)] + \dot{Q}(t) \\
&= A^T(t)Q(t) + Q(t)A(t) - 2C^T(t)C(t) + \dot{Q}(t)
\end{aligned} \tag{3.19}$$

Substituting (3.17) in (3.19) yields

$$\begin{aligned}
&\tilde{A}^T(t)Q(t) + Q(t)\tilde{A}(t) + \dot{Q}(t) \\
&= -C^T(t)C(t) - \exp(\delta)\Phi^T(t-\delta, t)C^T(t-\delta)C(t-\delta)\Phi(t-\delta, t) - Q(t)
\end{aligned} \tag{3.20}$$

Noting that the first two terms on the right hand side in (3.20) are negative semi-definite, we can write

$$\tilde{A}^T(t)Q(t) + Q(t)\tilde{A}(t) + \dot{Q}(t) \leq -Q(t) \quad (3.21)$$

From (3.15) we can write

$$\tilde{A}^T(t)Q(t) + Q(t)\tilde{A}(t) + \dot{Q}(t) \leq -\exp(-\delta)\epsilon_1 I \quad (3.22)$$

Thus there exists a bounded symmetric matrix $Q(t)$ that is continuously differentiable and satisfies

$$\exp(-\delta)\epsilon_1 I \leq Q(t) \leq \epsilon_2 I \quad (3.23)$$

$$\tilde{A}^T(t)Q(t) + Q(t)\tilde{A}(t) + \dot{Q}(t) \leq -\exp(-\delta)\epsilon_1 I, \quad \forall t \in [t_0 + \delta, t_f] \quad (3.24)$$

where $\tilde{A}(t)$ is the observer system matrix. Using (3.23) we can write

$$\exp(-\delta)\epsilon_1 \|e(t)\|^2 \leq e^T(t)Q(t)e(t) \leq \epsilon_2 \|e(t)\|^2, \quad \forall t \in [t_0 + \delta, t_f] \quad (3.25)$$

Similarly, for all $t \geq t_0 + \delta$,

$$\begin{aligned} \frac{d}{dt} [e^T(t)Q(t)e(t)] &= e^T(t) [\tilde{A}^T(t)Q(t) + Q(t)\tilde{A}(t) + \dot{Q}(t)] e(t) \\ &\leq \exp(-\delta)\epsilon_1 \|e(t)\|^2 \\ &\leq \exp(-\delta)\frac{\epsilon_1}{\epsilon_2} [e^T(t)Q(t)e(t)] \end{aligned} \quad (3.26)$$

Hence for any $e_0 = e(t_0)$ and $Q_0 = Q(t_0)$ we can write,

$$e^T(t)Q(t)e(t) \leq \exp \left[\exp(-\delta)\frac{\epsilon_1}{\epsilon_2} (t - t_0) \right] e_0^T Q_0 e_0, \quad t \geq t_0 + \delta \quad (3.27)$$

Using (3.25) we get,

$$\begin{aligned} \|e(t)\|^2 &\leq \frac{1}{\epsilon_1 \exp(-\delta)} \exp \left[-\exp(-\delta)\frac{\epsilon_1}{\epsilon_2} (t - t_0) \right] e_0^T Q_0 e_0 \\ &\leq \frac{\epsilon_2}{\epsilon_1 \exp(-\delta)} \exp \left[-\exp(-\delta)\frac{\epsilon_1}{\epsilon_2} (t - t_0) \right] \|e_0\|^2 \end{aligned} \quad (3.28)$$

Defining

$$\gamma = \exp(\delta) \frac{\epsilon_2}{\epsilon_1} \quad (3.29)$$

we can write

$$\|e(t)\|^2 \leq \gamma \exp \left[-\frac{1}{\gamma} (t - t_0) \right] \|e_o\|^2 \quad \forall t \geq t_o + \delta \quad (3.30)$$

Here $\gamma \geq 1$ since $\epsilon_2 \geq \epsilon_1$ and $\delta > 0$. Thus the observer error is bounded by uniformly asymptotically decaying function.

Thus there exists an observer gain $H(t) = Q^{-1}(t)C^T(t)$ such that the error-state is bounded by an exponentially decaying curve for all t in $[t_0 + \delta, t_f]$. $\square$

As can be seen from (3.30), smaller values of δ are preferred for improved estimator performance.

3.8 Existence of the Kalman filter

In this section we show that uniform observability for δ over a finite interval is sufficient to guarantee Kalman gain that is stabilizing over a finite interval $[t_o, t_f]$.

For the Kalman filter implementation, the dynamical system must be excited by white noise [29]. The LTV system (3.1) with process and measurement noise can be written as

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + G(t)\omega(t) \\ y(t) &= C(t)x(t) + \nu(t) \end{aligned} \quad (3.31)$$

where $\omega(t)$ and $\nu(t)$ are zero-mean, white Gaussian processes such that

$$\begin{aligned} \text{cov} [\omega(t), \omega(\tau)] &= Q(t)\delta(t - \tau) \\ \text{cov} [\nu(t), \nu(\tau)] &= R(t)\delta(t - \tau) \\ \text{cov} [\omega(t), \nu(\tau)] &= 0 \end{aligned}$$

and $Q(t)$ and $R(t)$ are assumed to be bounded, symmetric, positive definite and continuously differentiable for all t .

Assuming matrix functions $A(t)$, $G(t)$ and $C(t)$ to be bounded in (3.31), the Kalman filter (2.18) is uniformly asymptotically stable if the pair $\{A(t), G(t)\}$ is uniformly controllable and the pair $\{A(t), C(t)\}$ is uniformly observable [29]. Without loss of generality, we assume that the pair $\{A(t), G(t)\}$ is uniformly controllable and assume that the pair $\{A(t), C(t)\}$ is uniformly observable for δ over the finite interval $[t_o, t_f]$. We next show that under these assumptions, the Kalman gain is stabilizing on $[t_o + \delta, t_f]$.

Lemma 3.8.1 (Existence of Stabilizing Kalman Gain). *Suppose (3.31) is uniformly controllable and uniformly observable for δ over the finite interval $[t_o, t_f]$. Then the Kalman gain $K(t) = P(t)C^T(t)R^{-1}(t)$ is such that the state estimation error is bounded by an exponentially decaying function for all t in $[t_o + \delta, t_f]$.*

Proof. This proof is based on the approach taken in [30]. Since (3.31) is uniformly observable for δ over the finite interval $[t_o, t_f]$, there exist positive constants α_1 and α_2 such that

$$0 < \alpha_1 I \leq \int_{t-\delta}^t \Phi^T(\tau, t) C^T(\tau) R^{-1}(t) C(\tau) \Phi(\tau, t) d\tau \leq \alpha_2 I$$

for all t in $[t_o + \delta, t_f]$. Under the assumptions of uniform controllability and uniform observability for δ over the finite interval $[t_o, t_f]$, along with positive semi-definiteness of the initial error covariance $P(0)$, solution $P(t)$ of the matrix Riccati equation exists for all t in $[t_o + \delta, t_f]$ [23] and $P(t)$ is bounded such that there exist positive constants β_1 and β_2 such that,

$$0 < \beta_1 I \leq P(t) \leq \beta_2 I, \quad \forall t \in [t_o + \delta, t_f]$$

Hence, $P^{-1}(t)$ exists for all t in $[t_o + \delta, t_f]$. Since $P(t)$ is positive definite, so is $P^{-1}(t)$, and there exist positive constants β_3 and β_4 such that,

$$0 < \beta_3 I \leq P^{-1}(t) \leq \beta_4 I, \quad \forall t \in [t_o + \delta, t_f]$$

Also we can write,

$$\begin{aligned} \dot{P}^{-1}(t) = & -P^{-1}(t)A(t) - A^T(t)P^{-1}(t) \\ & - P^{-1}(t)G(t)Q(t)G^T(t)P^{-1}(t) + C^T(t)R^{-1}(t)C(t) \end{aligned} \quad (3.32)$$

Define a candidate Lyapunov function by

$$V(e(t), t) = e^T(t)P^{-1}(t)e(t) \quad (3.33)$$

where $e(t) = x(t) - \hat{x}(t)$ is the estimation error. Since $P^{-1}(t)$ is bounded from below and above, there exist positive constants β_5 and β_6 such that,

$$0 < \beta_5 I \leq V(e(t), t) \leq \beta_6 I, \quad \forall t \in [t_o + \delta, t_f]$$

Differentiating $V(e(t), t)$ with respect to time, we can write

$$\begin{aligned} \dot{V}(e(t), t) = & e^T(t) \{ [A(t) - P(t)C^T(t)R^{-1}(t)C(t)]^T P^{-1}(t) \\ & + P^{-1}(t) [A(t) - P(t)C^T(t)R^{-1}(t)C(t)] + \dot{P}^{-1}(t) \} e(t) \end{aligned}$$

Substituting for $\dot{P}^{-1}(t)$ from (3.32) and simplifying,

$$\dot{V}(e(t), t) = -e^T(t) [P^{-1}(t)G(t)Q(t)G^T(t)P^{-1}(t) + C^T(t)R^{-1}(t)C(t)] e(t) \quad (3.34)$$

The right hand side of (3.34) is in general negative semi-definite and hence $V(e(t), t)$ is non-increasing for all t in $[t_o + \delta, t_f]$, and hence the error $e(t)$ is bounded. That is, there exists some $\gamma \geq 1$, such that

$$\|e(t)\| \leq \gamma \|e_o\|, \quad \forall t \in [t_o + \delta, t_f]$$

To show that $e(t)$ is bounded by a uniformly exponentially decaying function, we need to show that $\dot{V}(e(t), t)$ is negative definite for all t in $[t_o + \delta, t_f]$. It is sufficient to show that

$$V(e(t), t) - V(e(t - \delta), t - \delta) < 0, \quad \forall t \in [t_o + \delta, t_f]$$

or equivalently show that

$$\int_{t-\delta}^t \dot{V}(e(\tau), \tau) < 0, \quad \forall t \in [t_o + \delta, t_f]$$

This follows from (3.34) and the conditions of observability over sliding intervals $[t - \delta, t]$ in $[t_o + \delta, t_f]$ and uniform complete controllability of (3.31). Detailed steps that show $V(e(t), t) - V(e(t - \delta), t - \delta) < 0$ are presented in [23] and are not reproduced here.

Thus Kalman gain $K(t) = P(t)C^T(t)R^{-1}(t)$ is such that the state estimation error is bounded by an exponentially decaying function for all t in $[t_o + \delta, t_f]$. $\square$

In the next chapter, we introduce the range navigation algorithm.

Chapter 4

Range Navigation

4.1 Introduction

In this chapter we present the problem statement of the range navigation algorithm and introduce the system dynamics therein. We identify three cases under which we study observability of the range navigation system. These three cases are - when no currents are present, when the currents present are known and when the currents present are unknown. We present observability analysis of the range navigation algorithm for cases when no currents are present and when the currents present are known. Analysis of range navigation in the presence of unknown currents is presented in Chapter [5](#).

4.2 System Dynamics

We use the Cartesian coordinate system for defining system dynamics. Figure [4.1](#) shows the coordinate frame along with the nomenclature used for defining the system

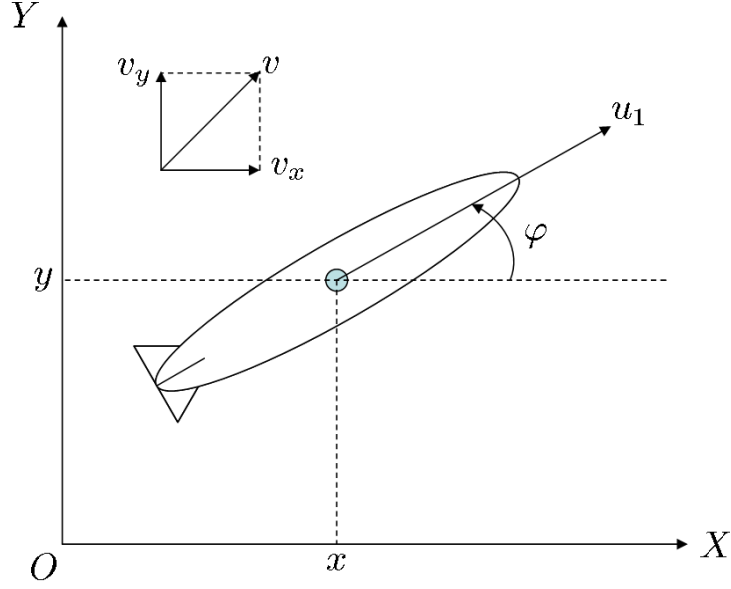


Figure 4.1: Coordinate system and AUV dynamics.

dynamics. Since the vertical position of an AUV is easily instrumented using a depth sensor, we simplify the notation and consider motion only in the horizontal $x - y$ plane. Without loss of generality, the absolute position of the acoustic beacon is assumed to be $(0, 0)$. The position of the AUV is at time t is $[x(t), y(t)]^T$. The yaw (heading) angle of the AUV is $\varphi(t)$ and it is measured counter-clockwise from the x -axis. Water-relative speed of the AUV is $u_1(t)$. The currents are denoted by $v = [v_x, v_y]^T$, where v_x and v_y are components of currents in x and y directions respectively. We assume that the currents are constant.

As is standard for navigation algorithms, the system dynamics utilized herein are based on a kinematic model of the AUV. Dynamic models are difficult to use since they would vary with AUVs and AUV configurations, and they are often not available in sufficient fidelity.

The system dynamics are

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\varphi}(t) \\ \dot{v}_x(t) \\ \dot{v}_y(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \cos \varphi(t) \\ u_1(t) \sin \varphi(t) \\ u_2(t) \\ 0 \\ 0 \end{bmatrix}, \quad t \geq t_o, \quad s(t_o) = s_o \quad (4.1)$$

The AUV measures its range from the origin and its heading angle. The measurement equations are

$$h(t) = \begin{bmatrix} \sqrt{x^2(t) + y^2(t)} \\ \varphi(t) \end{bmatrix} \quad (4.2)$$

4.2.1 Implications of Modeling Errors

Since we are using a kinematic model to describe motion of an AUV, there will be a mismatch between the estimated and true motion of the AUV for certain maneuvers. For example, a kinematic model does not capture the side-slip motion during turning when there is a non-zero angle-of-attack $\alpha(t) = \varphi(t) - \rho(t)$, where $\rho(t)$ is the instantaneous direction of the AUV's velocity $\dot{p}(t)$. Figure 4.2 shows the angle-of-attack of an AUV. Angle-of-attack is small for typical straight line maneuvers in practice and it is difficult to instrument on a small AUV. In the kinematic model, the angle-of-attack is always zero so that $\varphi(t)$ and $\rho(t)$ are the same. To demonstrate the performance of the navigation algorithm when the angle-of-attack is non-zero, we utilize a full six degree-of-freedom dynamic model of a representative AUV in Section 4.6. We empirically show that bounded position estimation errors occur when the vehicle is turning and experiences a non-zero angle of attack, yet the position estimation errors converge to zero when the vehicle returns to a straight line trajectory.

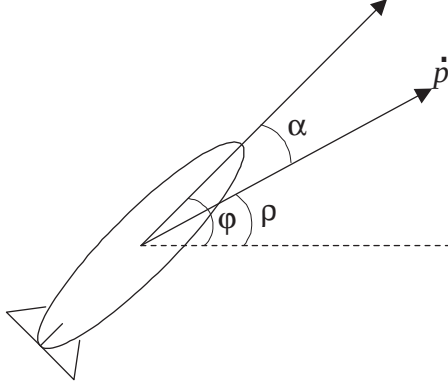


Figure 4.2: Angle of attack

4.3 The Three Cases

In this section we identify the three cases under which observability properties of the range navigation algorithm are studied. We also introduce system dynamics in polar coordinates for all the three cases. Polar coordinates are used for the observability analysis in the sequel.

4.3.1 Case 1: No Currents

When no currents are present, the state of the system is $s(t) = [x(t), y(t), \varphi(t)]^T$ and the system dynamics are

$$\dot{s}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \cos \varphi(t) \\ u_1(t) \sin \varphi(t) \\ u_2(t) \end{bmatrix}, \quad t \geq t_o, \quad s(t_o) = s_o \quad (4.3)$$

$$h(t) = \begin{bmatrix} \sqrt{x^2(t) + y^2(t)} \\ \varphi(t) \end{bmatrix} \quad (4.4)$$

In the polar coordinates, we define the state of the system $s_p(t) = [r(t), \theta(t), \varphi(t)]^T$. Any point (x, y) in the Cartesian coordinates is equivalent to its polar form (r, θ) through the relations

$$x = r \cos \theta$$

$$y = r \sin \theta$$

The system dynamics and the measurement equation in the polar coordinates are

$$\dot{s}_p(t) = \begin{bmatrix} \dot{r}(t) \\ \dot{\theta}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \cos [\varphi(t) - \theta(t)] \\ \frac{u_1(t)}{r(t)} \sin [\varphi(t) - \theta(t)] \\ u_2(t) \end{bmatrix}, \quad t \geq t_o, \quad s_p(t_o) = s_{po} \quad (4.5)$$

$$h_p(t) = \begin{bmatrix} r(t) \\ \varphi(t) \end{bmatrix} \quad (4.6)$$

We assume that $r(t) > 0$ for all t .

4.3.2 Case 2: Known Currents

When currents present are known, we define the state of the system by $s(t) = [x(t), y(t), \varphi(t)]^T$ and the system dynamics are

$$\dot{s}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \cos \varphi(t) + v_x \\ u_1(t) \sin \varphi(t) + v_y \\ u_2(t) \end{bmatrix}, \quad t \geq t_o, \quad s(t_o) = s_o \quad (4.7)$$

$$h(t) = \begin{bmatrix} \sqrt{x^2(t) + y^2(t)} \\ \varphi(t) \end{bmatrix} \quad (4.8)$$

The system dynamics and the measurement equation in the polar coordinates are

$$\dot{s}_p(t) = \begin{bmatrix} \dot{r}(t) \\ \dot{\theta}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \cos(\varphi(t) - \theta(t)) + v_x \cos \theta(t) + v_y \sin \theta(t) \\ \frac{u_1(t)}{r(t)} \sin(\varphi(t) - \theta(t)) - \frac{v_x}{r(t)} \sin \theta(t) + \frac{v_y}{r(t)} \cos \theta(t) \\ u_2(t) \end{bmatrix} \quad (4.9)$$

$$h_p(t) = \begin{bmatrix} r(t) \\ \varphi(t) \end{bmatrix} \quad (4.10)$$

4.3.3 Case 3: Unknown Currents

When currents present are unknown, the range navigation algorithm estimates the currents along with the position of the AUV. We define the state of the system by $s(t) = [x(t), y(t), \varphi(t), v_x(t), v_y(t)]^T$ and the system dynamics are

$$\dot{s}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\varphi}(t) \\ \dot{v}_x(t) \\ \dot{v}_y(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \cos \varphi(t) + v_x \\ u_1(t) \sin \varphi(t) + v_y \\ u_2(t) \\ 0 \\ 0 \end{bmatrix}, \quad t \geq t_o, \quad s(t_o) = s_o \quad (4.11)$$

$$h(t) = \begin{bmatrix} \sqrt{x^2(t) + y^2(t)} \\ \varphi(t) \end{bmatrix} \quad (4.12)$$

The system dynamics and the measurement equation in the polar coordinates are

$$\dot{s}_p(t) = \begin{bmatrix} \dot{r}(t) \\ \dot{\theta}(t) \\ \dot{\varphi}(t) \\ \dot{v}_x(t) \\ \dot{v}_y(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \cos(\varphi(t) - \theta(t)) + v_x \cos \theta(t) + v_y \sin \theta(t) \\ \frac{u_1(t)}{r(t)} \sin(\varphi(t) - \theta(t)) - \frac{v_x}{r(t)} \sin \theta(t) + \frac{v_y}{r(t)} \cos \theta(t) \\ u_2(t) \\ 0 \\ 0 \end{bmatrix} \quad (4.13)$$

$$h_p(t) = \begin{bmatrix} r(t) \\ \varphi(t) \end{bmatrix} \quad (4.14)$$

4.4 Range Navigation with No Currents

In this section, we examine the case when there are no currents present. We linearize (4.5) around any arbitrary trajectory and study observability of the LTV system thus obtained.

4.4.1 The Linearized Model

The LTV system obtained by linearizing (4.5) around a nominal state trajectory $s_{po}(t)$ and corresponding nominal input $u_o(t)$ is

$$\begin{aligned} \dot{s}_{p\delta}(t) &= A_o(t)s_{p\delta}(t) + B_o(t)u_\delta(t), \quad s_{p\delta}(t_0) = s_{p\delta o}, \quad t \geq t_o \\ h_{p\delta}(t) &= C_o(t)s_{p\delta}(t) \end{aligned} \quad (4.15)$$

where $s_{p\delta}(t) = s_p(t) - s_{po}(t) = [r_\delta(t), \theta_\delta(t), \varphi_\delta(t)]^T$ is the perturbation from the nominal state $s_{po}(t)$, $h_{p\delta}(t) = h_p(t) - h_{po}(t)$ is the perturbation from the nominal output $h_{po}(t)$, and $u_\delta(t) = u(t) - u_o(t)$ is the perturbation from the nominal input $u_o(t)$.

Defining $\beta_o(t) = \varphi_o(t) - \theta_o(t)$, the coefficient matrices can be written as

$$\begin{aligned} A_o(t) &= \begin{bmatrix} 0 & u_{1o}(t) \sin \beta_o(t) & -u_{1o}(t) \sin \beta_o(t) \\ -\frac{u_{1o}(t)}{r_o^2(t)} \sin \beta_o(t) & -\frac{u_{1o}(t)}{r_o(t)} \cos \beta_o(t) & \frac{u_{1o}(t)}{r_o(t)} \cos \beta_o(t) \\ 0 & 0 & 0 \end{bmatrix} \\ B_o(t) &= \begin{bmatrix} \cos \beta_o(t) & 0 \\ \frac{1}{r_o(t)} \sin \beta_o(t) & 0 \\ 0 & 1 \end{bmatrix}, \quad C_o(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (4.16)$$

Using the relationships in (4.5), $A_o(t)$ can be expressed as

$$A_o(t) = \begin{bmatrix} 0 & \dot{\theta}_o(t)r_o(t) & -\dot{\theta}_o(t)r_o(t) \\ -\frac{\dot{\theta}_o(t)}{r_o(t)} & -\frac{\dot{r}_o(t)}{r_o(t)} & \frac{\dot{r}_o(t)}{r_o(t)} \\ 0 & 0 & 0 \end{bmatrix} \quad (4.17)$$

Using similarity transform, we transform (4.15) into

$$\begin{aligned} \begin{bmatrix} \dot{z}_a(t) \\ \dot{z}_b(t) \end{bmatrix} &= \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{bmatrix} z_a(t) \\ z_b(t) \end{bmatrix} + \begin{bmatrix} B_{11}(t) \\ B_{21}(t) \end{bmatrix} u_\delta(t) \\ y(t) &= \begin{bmatrix} I_2 & 0 \end{bmatrix} \begin{bmatrix} z_a(t) \\ z_b(t) \end{bmatrix} = z_a(t) \end{aligned} \quad (4.18)$$

where

$$\begin{aligned} z(t) &= \begin{bmatrix} z_a^T(t) & \vdots & z_b^T(t) \end{bmatrix}^T = \begin{bmatrix} r_\delta(t) & \varphi_\delta(t) & \vdots & \theta_\delta(t) \end{bmatrix}^T \\ A_{11}(t) &= \begin{bmatrix} 0 & -\dot{\theta}_o(t)r_o(t) \\ 0 & 0 \end{bmatrix}, \quad A_{12}(t) = \begin{bmatrix} \dot{\theta}_o(t)r_o(t) \\ 0 \end{bmatrix} \\ A_{21}(t) &= \begin{bmatrix} -\frac{\dot{\theta}_o(t)}{r_o(t)} & \frac{\dot{r}_o(t)}{r_o(t)} \end{bmatrix}, \quad A_{22}(t) = \begin{bmatrix} -\frac{\dot{r}_o(t)}{r_o(t)} \end{bmatrix} \\ B_{11}(t) &= \begin{bmatrix} \cos \beta_o(t) & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{21}(t) = \begin{bmatrix} \frac{1}{r_o(t)} \sin \beta_o(t) & 0 \end{bmatrix} \end{aligned} \quad (4.19)$$

4.4.2 Conditions for Uniform Observability for δ

We seek sufficient conditions on $s_{po}(t)$ for which (4.18) is uniformly observable for δ . Under Lemma (3.6.1), uniform observability for δ of (4.18) is equivalent to uniform observability for δ of the pair $[A_{22}(t), A_{12}(t)]$ from (4.19).

Let the observability grammian for the pair $[A_{22}(t), A_{12}(t)]$ from (4.19) over an interval $[t - \delta, t]$ be

$$M_r(t - \delta, t) = \int_{t-\delta}^t \phi_{22}^T(\tau, t - \delta) A_{12}^T(\tau) A_{12}(\tau) \phi_{22}(\tau, t - \delta) d\tau \quad (4.20)$$

where $\phi_{22}(\tau, t - \delta)$ is the state transition matrix corresponding to $A_{22}(t)$. We can express the state transition matrix as

$$\begin{aligned}\phi_{22}(\tau, t - \delta) &= \exp \left[\int_{t-\delta}^{\tau} A_{22}(\sigma) d\sigma \right] \\ &= \exp \left[\int_{t-\delta}^{\tau} -\frac{\dot{r}_o(\sigma)}{r_o(\sigma)} d\sigma \right] \\ &= \exp [-\ln r_o(\sigma)]_{t-\delta}^{\tau}\end{aligned}\tag{4.21}$$

Simplifying, and noting that $r_o(t) > 0$ for all t , we can write

$$\phi_{22}(\tau, t - \delta) = \frac{r_o(t - \delta)}{r_o(\tau)}\tag{4.22}$$

Substituting for $\phi_{22}(\tau, t - \delta)$ and $A_{12}(\tau)$ in (4.20) and simplifying

$$M_r(t - \delta, t) = r_o^2(t - \delta) \int_{t-\delta}^t \dot{\theta}_o^2(\tau) d\tau\tag{4.23}$$

Note that $M_r(t - \delta, t)$ is a scalar.

Theorem 4.4.1. *Suppose there exists scalar $\rho > 0$ such that $r(t) \geq \rho$ for all t . The pair $[A_{22}(t), A_{12}(t)]$ in (4.19) is uniformly observable for δ if there exists $\epsilon > 0$ and at least one $\tau \in [t - \delta, t]$, for every $t \geq t_o + \delta$, such that $|\dot{\theta}(\tau)| > \epsilon$.*

Proof. Since $|\dot{\theta}_o(\tau)| \geq \epsilon$, we can write $\dot{\theta}_o^2(\tau) \geq \epsilon^2$. There exist positive constants K_1 and K_2 such that $|\dot{\theta}_o(t)| < K_1$ and $|\ddot{\theta}_o(t)| < K_2$ for all t . Therefore

$$\frac{d}{dt} \dot{\theta}_o^2(t) = 2\dot{\theta}_o(t)\ddot{\theta}_o(t) < 2K_1K_2 := K\tag{4.24}$$

For any t_a in $[t - \delta, t]$

$$|\epsilon^2 - \dot{\theta}_o^2(t_a)| \leq K|\tau - t_a|\tag{4.25}$$

It follows that

$$\dot{\theta}_o^2(t_a) \geq \epsilon^2 - K|\tau - t_a| \quad (4.26)$$

There exists an interval T of length $l = \min\left(\frac{\delta}{2}, \frac{\epsilon}{K}\right)$, on one side of τ or the other, such that

$$\dot{\theta}_o^2(t_T) \geq \epsilon^2 - K|\tau - t_T| > 0, \quad \forall t_T \in T \quad (4.27)$$

Integrating over T ,

$$\int_T \dot{\theta}_o^2(t_1) dt_1 \geq \epsilon^2 l - K \int_0^l z dz > 0 \quad (4.28)$$

which results in

$$\int_T \dot{\theta}_o^2(t_1) dt_1 \geq \epsilon^2 l - K \frac{l^2}{2} > 0 \quad (4.29)$$

Define

$$\alpha = \epsilon^2 l - K \frac{l^2}{2}$$

Thus $\alpha > 0$. Since $T \subset [t - \delta, t]$,

$$\int_{t-\delta}^t \dot{\theta}_o^2(t_1) dt_1 \geq \int_T \dot{\theta}_o^2(t_1) dt_1 \geq \alpha \quad (4.30)$$

and

$$r^2(t - \delta) \int_{t-\delta}^t \dot{\theta}_o^2(t_1) dt_1 \geq r^2(t - \delta) \alpha := \alpha_1 \quad (4.31)$$

The observability grammian for the pair $[A_{22}(t), A_{12}(t)]$ can be written as

$$M_r(t - \delta, t) = r^2(t - \delta) \int_{t-\delta}^t \dot{\theta}_o^2(t_1) dt_1 \geq \alpha_1, \quad \forall t \geq t_o + \delta \quad (4.32)$$

Thus the pair $[A_{22}(t), A_{12}(t)]$ is uniformly observable. $\square$

The nominal trajectory described by arbitrarily small $\dot{\theta}_o(\tau)$ for all τ in some interval $[t - \delta, t]$ describes a straight line segment whose extension would pass through the origin.

Remark 4.4.2. *Suppose the AUV travels along straight line segments whose extensions pass through the origin for finite intervals and let $[t_1, t_2]$ be the longest of such intervals. We can choose $\delta \geq t_2 - t_1$ so as to ensure that the system is uniformly observable for δ .*

Remark 4.4.3. *Local observability of the nonlinear system does not imply global observability. Indeed, there exist AUV trajectories that are not globally observable even though they are locally observable. For the nonlinear system (4.18), we can identify trajectories that are not distinguishable but about which the linearized system is locally observable. Specifically, any straight line trajectory is not globally observable. For any straight line trajectory, there is a parallel trajectory with equal range to the origin, as shown in Figure 4.3. The original trajectory and its mirror image are indistinguishable. While straight line trajectories are not globally observable, they are locally observable so long as they are not part of a line segment that passes through the origin. Thus straight line trajectories are permissible if the estimate of the initial state of the vehicle is sufficiently close to the true initial state.*

4.5 Range Navigation in Known Non-Zero Currents

In this section, we consider the case that known constant currents act on the AUV.

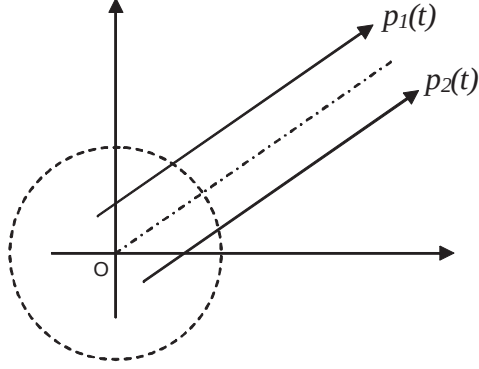


Figure 4.3: Indistinguishable trajectories

4.5.1 The Linearized Model

The linearized system obtained from (4.9) is

$$\begin{aligned} \dot{s}_{p\delta}(t) &= A_o(t)s_{p\delta}(t) + B_o(t)u_\delta(t), \quad s_{p\delta}(t_0) = s_{p\delta o}, \quad t \in [t_o, t_f] \\ h_{p\delta}(t) &= C_o(t)s_{p\delta}(t) \end{aligned} \quad (4.33)$$

where, defining $\beta_o(t) = \varphi_o(t) - \theta_o(t)$ and $\gamma_o(t) = \varphi_c - \theta_o(t)$,

$$\begin{aligned} A_o(t) &= \begin{bmatrix} 0 & A_{12}(t) & -u_{1o}(t) \sin \beta_o(t) \\ A_{21}(t) & A_{22}(t) & \frac{u_{1o}(t)}{r_o(t)} \cos \beta_o(t) \\ 0 & 0 & 0 \end{bmatrix} \\ B_o(t) &= \begin{bmatrix} \cos \beta_o(t) & 0 \\ \frac{1}{r_o(t)} \sin \beta_o(t) & 0 \\ 0 & 1 \end{bmatrix}, \quad C_o(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (4.34)$$

where

$$\begin{aligned}
A_{12}(t) &= u_{1o}(t) \sin \beta_o(t) - v_x \sin \theta_o(t) + v_y \cos \theta_o(t) \\
A_{21}(t) &= -\frac{u_{1o}(t)}{r_o^2(t)} \sin \beta_o(t) - \frac{v_x}{r_o^2(t)} \sin \theta_o(t) + \frac{v_y}{r_o^2(t)} \cos \theta_o(t) \\
A_{22}(t) &= -\frac{u_{1o}(t)}{r(t)} \cos \beta_o(t) - \frac{v_x}{r_o(t)} \cos \theta_o(t) - \frac{v_y}{r_o(t)} \sin \theta_o(t)
\end{aligned} \tag{4.35}$$

From the relationships in (4.9), $A_o(t)$ can be written as

$$A_o(t) = \begin{bmatrix} 0 & \dot{\theta}_o(t)r_o(t) & -u_{1o}(t) \sin \beta_o(t) \\ -\frac{\dot{\theta}_o(t)}{r_o(t)} & -\frac{\dot{r}_o(t)}{r_o(t)} & \frac{u_{1o}(t)}{r_o(t)} \cos \beta_o(t) \\ 0 & 0 & 0 \end{bmatrix} \tag{4.36}$$

As in Section 4.4, we express the state equation (4.33) as

$$\begin{aligned}
\dot{z}(t) &= \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} z(t) + \begin{bmatrix} B_{11}(t) \\ B_{21}(t) \end{bmatrix} u_\delta(t) \\
y(t) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} z(t)
\end{aligned} \tag{4.37}$$

where

$$\begin{aligned}
z(t) &= \begin{bmatrix} r_\delta(t) & \varphi_\delta(t) & \theta_\delta(t) \end{bmatrix}^T \\
A_{11}(t) &= \begin{bmatrix} 0 & -u_{1o}(t) \sin \beta_o(t) \\ 0 & 0 \end{bmatrix}, \quad A_{12}(t) = \begin{bmatrix} \dot{\theta}_o(t)r_o(t) \\ 0 \end{bmatrix} \\
A_{21}(t) &= \begin{bmatrix} -\frac{\dot{\theta}_o(t)}{r_o(t)} & \frac{u_{1o}(t)}{r_o(t)} \cos \beta_o(t) \end{bmatrix}, \quad A_{22}(t) = \begin{bmatrix} -\frac{\dot{r}_o(t)}{r_o(t)} \end{bmatrix} \\
B_{11}(t) &= \begin{bmatrix} \cos \beta_o(t) & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{21}(t) = \begin{bmatrix} \frac{1}{r_o(t)} \sin \beta_o(t) & 0 \end{bmatrix}
\end{aligned} \tag{4.38}$$

We see that the expressions for the pair $[A_{22}(t), A_{12}(t)]$ in (4.38) are identical to the expressions in (4.19).

4.5.2 Conditions for Uniform Observability

We seek conditions under which the pair $[A_{22}(t), A_{12}(t)]$ from (4.38) is uniformly observable for δ .

Theorem 4.5.1. *Let there exist some $\rho > 0$ such that $r(t) \geq \rho$ for all t . The pair $[A_{22}(t), A_{12}(t)]$ in (4.38) is uniformly observable for δ if there exists some $\epsilon > 0$ and at least one $\tau \in [t - \delta, t]$ for every $t \geq t_o + \delta$, such that $|\dot{\theta}(\tau)| > \epsilon$.*

The proof of Theorem 4.5.1 is identical to the proof of Theorem 4.4.1 and hence it is omitted here.

The condition obtained in Theorem 4.5.1 is exactly the same as in the case of no currents. It is worth emphasizing that in the context of the original non-linear system describing AUV motion, Theorem 4.5.1 is a local result, as is Theorem 4.4.1. An observer for the nonlinear system will converge only if the estimate of the initial state is sufficiently close to the true initial state.

4.6 Effect of Non-Zero Angle of Attack on AUV Trajectory Estimation

In this section we show the effects of angle-of-attack on the performance of the observer using a 6-DOF nonlinear model of an AUV. We show through simulation that bounded estimation errors occur during turning maneuvers when the angle-of-attack is non-zero, but the estimation error converges to zero when the AUV starts moving along straight line trajectory after the turn.

A 6-DOF dynamic model of an AUV is used to generate reference trajectories in the horizontal plane. The 6-DOF model captures the side-slip of the AUV during

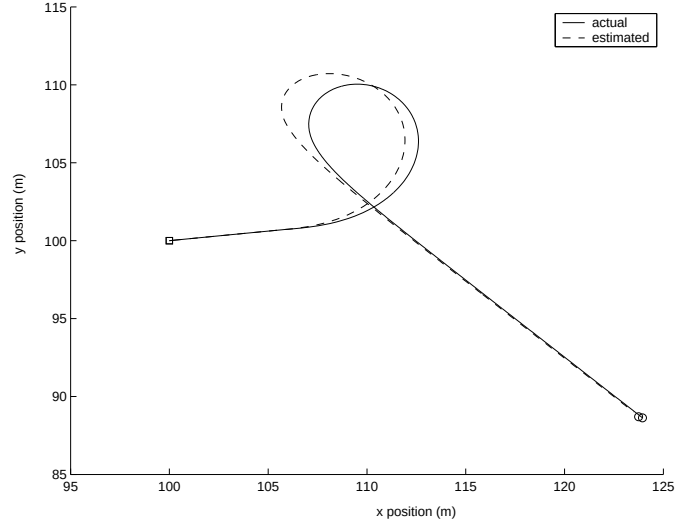


Figure 4.4: Effect of the angle-of-attack on the estimation. Squares and circles denote start and end points respectively.

turning maneuvers. The angle-of-attack is non-zero during turning maneuvers, but negligible during straight line trajectories. This results in an increase in the AUV position estimation error during turning maneuvers, but the estimation error converges to zero during the subsequent straight-line trajectory. This fact is illustrated using the trajectory shown in Figure 4.4. The true trajectory is generated using a 6-DOF dynamic model, while the estimated trajectory is computed using an extended Kalman filter based on the kinematic model of AUV motion. For the same trajectory, Figure 4.5 shows the rate of change of heading of the AUV, the angle-of-attack, and the position estimation error.

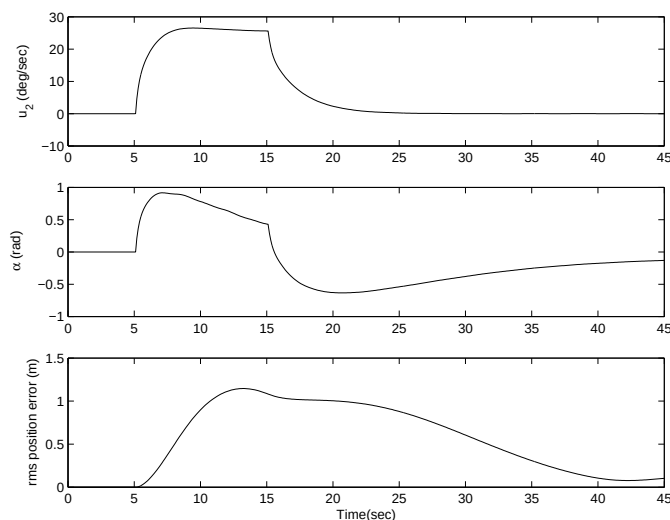


Figure 4.5: AUV angular rate, angle-of-attack and position error

4.7 Demonstration

The navigation algorithm was demonstrated on 21 November 2003, using representative hardware but without building or otherwise acquiring the acoustic hardware required to compute underwater range from a known location. The VT Miniature AUV was deployed in a freshwater pond that had no significant currents. The AUV was run on the surface so that the GPS signals were available continuously. Position information from the GPS was used to compute the ranges of the AUV from a fixed location and to provide the ground truth for the algorithm. The AUV was instrumented with an electronic compass for measuring the AUV heading $\varphi(t)$ and a MEMS gyro for measuring the angular rate $u_2(t)$. Propeller position of the AUV was instrumented, enabling estimates of the AUV speed $u_1(t)$ from the steady-state propeller rotation rate.

Position data from the GPS is converted into an Earth-Fixed (EF) reference frame.

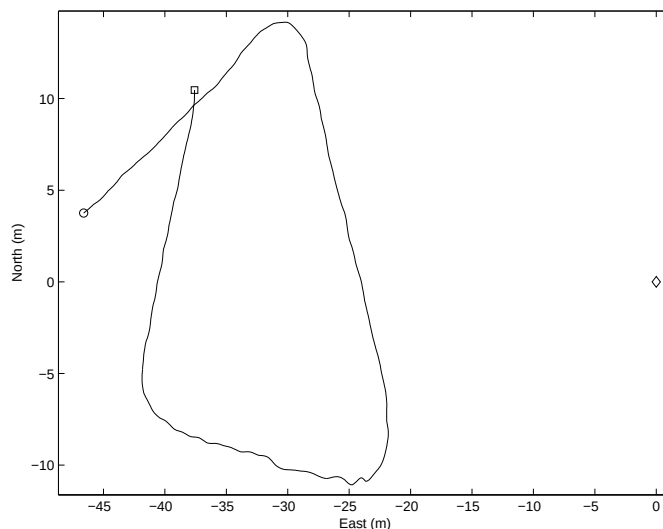


Figure 4.6: Actual AUV trajectory - square and circle denote start and end point, respectively, and diamond is the origin set at a known location.

The EF frame has its origin fixed to an arbitrary point on the surface of the earth, and its x -axis points due North, y -axis points due East and z -axis points towards the center of the earth. At its origin, the $x - y$ plane of the EF frame is locally tangential to the earth's surface. In our experiments, the origin of this frame is set to a known location near the pond where the experiment was conducted. The trajectory of the AUV in the EF frame, as determined from the GPS measurements, is shown in Figure 4.6. The AUV heading $\varphi(t)$ obtained from the electronic compass is shown in Figure 4.7. Figure 4.8 shows the AUV angular rate $u_2(t)$ obtained from an MEMS gyro. Peaks in Figure 4.8 indicate turns made by the AUV.

A discrete time extended Kalman filter (EKF), based on the kinematic model (4.7), was implemented to estimate the AUV position. Implementation of the extended Kalman filter is standard (see e.g. [31], pp. 496), except we assume that the process noise covariance matrix $Q(k)$ varies with time. When the AUV makes a turn we

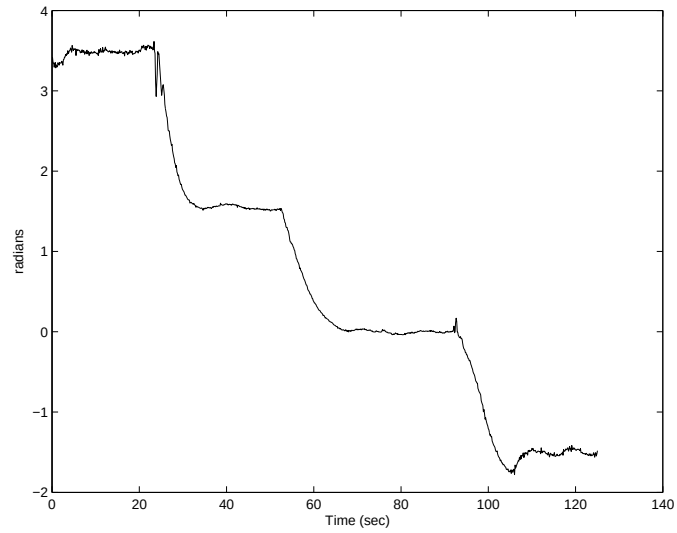


Figure 4.7: AUV heading $\varphi(t)$ from compass.

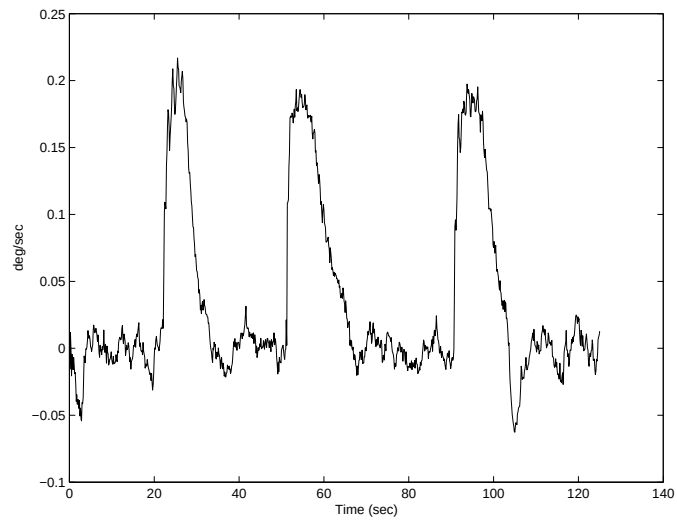


Figure 4.8: AUV rate of change of heading $u_2(t)$.

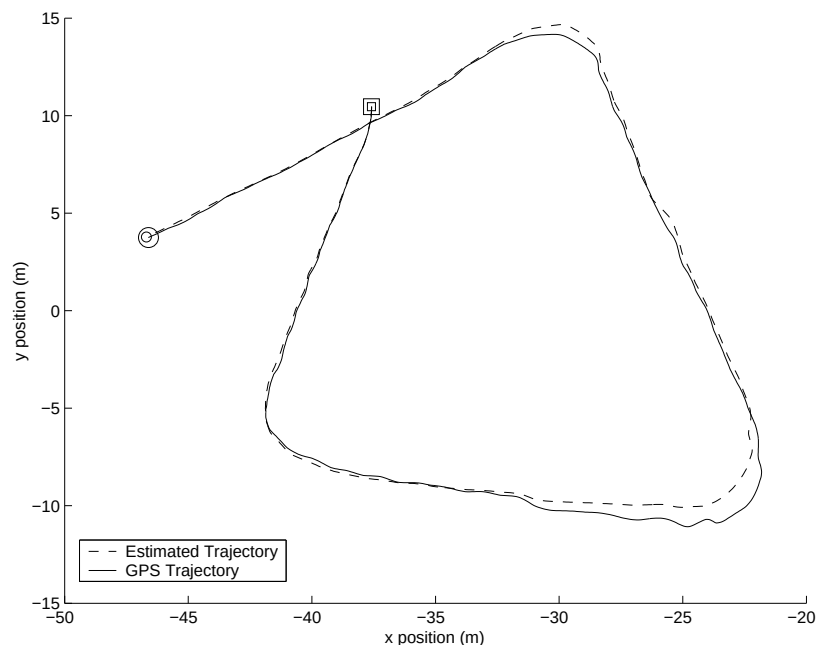


Figure 4.9: Actual and estimated trajectories. Squares and circles denote start and end point, respectively.

assume that $Q(k)$ is larger than when the AUV is traveling along a straight line. This matches the fact that along a turn the kinematic model of the AUV is less accurate due to non-zero angle-of-attack.

Figure 4.9 shows the GPS track of the AUV and the estimated AUV trajectory determined by the EKF. The estimated trajectory agrees well with the GPS track except along turns. As predicted, the estimated trajectory converges toward the actual trajectory after each turning maneuver. The turn centered near $(-25, -10)$ shows modest divergence of the estimated trajectory before the turning maneuver. We believe this is due to oscillations in the GPS track and corresponding range measurement that do not reflect actual motion of the AUV.

Chapter 5

Range Navigation in the Presence of Unknown Currents

5.1 Introduction

In this chapter, we consider the case when the currents present are unknown and must be estimated along with the position of the AUV. For this case the reduced order system does not lead to easy expression for the observability grammian, as it did for the cases of no currents and known currents. Here we use the notion of uniform observability for δ over a finite interval $[t_o, t_f]$ to study the range navigation algorithm.

5.2 The Linearized Model

The systems dynamics in polar coordinates for the case of unknown currents are introduced in Section [4.3.3](#). A linear time-varying system is obtained by linearizing

(4.13) around a nominal trajectory

$$\begin{aligned} \dot{s}_{p\delta}(t) &= A_o(t)s_{p\delta}(t) + B_o(t)u_{p\delta}(t), \quad s_{p\delta}(t_o) = s_{\delta 0}, \quad t \in [t_0, t_f] \\ h_{p\delta}(t) &= C_o(t)s_{p\delta}(t) \end{aligned} \quad (5.1)$$

where, defining $\beta_o(t) = \varphi_o(t) - \theta_o(t)$,

$$A_o(t) = \begin{bmatrix} 0 & A_{12}(t) & -u_{1o}(t) \sin \beta_o(t) & \cos \theta_o(t) & \sin \theta_o(t) \\ A_{21}(t) & A_{22}(t) & \frac{u_{1o}(t)}{r_o(t)} \cos \beta_o(t) & -\frac{1}{r_o(t)} \sin \theta_o(t) & \frac{1}{r_o(t)} \cos \theta_o(t) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.2)$$

$$\begin{aligned} A_{12}(t) &= u_{1o}(t) \sin \beta_o(t) - v_x \sin \theta_o(t) + v_y \cos \theta_o(t) \\ A_{21}(t) &= -\frac{u_{1o}(t)}{r_o^2(t)} \sin \beta_o(t) - \frac{v_x}{r_o^2(t)} \sin \theta_o(t) + \frac{v_y}{r_o^2(t)} \cos \theta_o(t) \\ A_{22}(t) &= -\frac{u_{1o}(t)}{r_o(t)} \cos \beta_o(t) - \frac{v_x}{r_o(t)} \cos \theta_o(t) - \frac{v_y}{r_o(t)} \sin \theta_o(t) \end{aligned} \quad (5.3)$$

and

$$B_o(t) = \begin{bmatrix} \cos \beta_o(t) & 0 \\ \frac{1}{r_o(t)} \sin \beta_o(t) & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_o(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (5.4)$$

Using relationships in (4.13), $A_o(t)$ can be expressed as

$$A_o(t) = \begin{bmatrix} 0 & \dot{\theta}_o(t)r_o(t) & -u_{1o}(t)\sin\beta_o(t) & \cos\theta_o(t) & \sin\theta_o(t) \\ -\frac{\dot{\theta}_o(t)}{r_o(t)} & -\frac{\dot{r}_o(t)}{r_o(t)} & \frac{u_{1o}(t)}{r_o(t)}\cos\beta_o(t) & -\frac{1}{r_o(t)}\sin\theta_o(t) & \frac{1}{r_o(t)}\cos\theta_o(t) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.5)$$

In the next section we find conditions under which (5.1) is observable over a finite interval and then combine these conditions with the requirement that (5.1) must be observable over all sliding intervals $[t - \delta, t]$ in $[t_o + \delta, t_f]$. This combination yields the conditions under which (5.1) is uniformly observable for δ over the finite interval $[t_o, t_f]$.

5.3 Observability Analysis

Theorem 5.3.1. *Assume $u_{1o}(t) \neq 0$ for all $t \in [t_0, t_f]$. Then the system (5.1) is observable over the finite interval $[t_0, t_f]$ if*

$$\dot{\theta}_o(t) \neq \frac{4a^3}{4a^2 - 4abt + (4a^4 + b^2)t^2} \quad (5.6)$$

for some t in $[t_0, t_f]$, where $a = \dot{\theta}_o(t_0)$ and $b = \ddot{\theta}_o(t_0)$, such that a and b are not both zero.

Proof. The observability matrix $O(t)$ for the linear system (5.1) is

$$O(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & O_{32}(t) & O_{33}(t) & O_{34}(t) & O_{35}(t) \\ 0 & 0 & 0 & 0 & 0 \\ O_{51}(t) & O_{52}(t) & O_{53}(t) & O_{54}(t) & O_{55}(t) \\ 0 & 0 & 0 & 0 & 0 \\ O_{71}(t) & O_{72}(t) & O_{73}(t) & O_{74}(t) & O_{75}(t) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.7)$$

where

$$O_{32}(t) = \dot{\theta}_o(t)r_o(t)$$

$$O_{33}(t) = -u_{1o}(t) \sin \beta_o(t)$$

$$O_{34}(t) = \cos \theta_o(t)$$

$$O_{34}(t) = \cos \theta_o(t)$$

$$O_{35}(t) = \sin \theta_o(t)$$

$$O_{51}(t) = -\dot{\theta}_o^2(t)$$

$$O_{52}(t) = \ddot{\theta}_o(t)r_o(t)$$

$$O_{54}(t) = -2\dot{\theta}_o(t) \sin \theta_o(t)$$

$$O_{55}(t) = 2\dot{\theta}_o(t) \cos \theta_o(t)$$

$$O_{71}(t) = -3\dot{\theta}_o(t)\ddot{\theta}_o(t)$$

$$O_{72}(t) = -\dot{\theta}_o^3(t)r_o(t) + \ddot{\theta}_o(t)r_o(t)$$

$$O_{74}(t) = -3\dot{\theta}_o^2(t) \cos \theta_o(t) - 3\ddot{\theta}_o(t) \sin \theta_o(t)$$

$$O_{75}(t) = -3\dot{\theta}_o^2(t) \sin \theta_o(t) + 3\ddot{\theta}_o(t) \cos \theta_o(t)$$

Assume that the linearized system (5.1) is not observable over the interval $[t_0, t_f]$. Then by Lemma 3.2.1, $O(t)$ is not full rank for all $t \in [t_0, t_f]$. Since columns 1 and 3 are independent of each other, the 3x3 submatrix

$$\bar{O}(t) = \begin{bmatrix} O_{32}(t) & O_{34}(t) & O_{35}(t) \\ O_{52}(t) & O_{54}(t) & O_{55}(t) \\ O_{72}(t) & O_{74}(t) & O_{75}(t) \end{bmatrix} \quad (5.8)$$

obtained from $O(t)$ by removing columns 1 and 3 and all all-zero rows, must be rank deficient for all $t \in [t_0, t_f]$. That is,

$$\det \bar{O}(t) = 0 \quad \forall t \in [t_0, t_f]$$

We can therefore write, for all $t \in [t_0, t_f]$,

$$4r_o(t)\dot{\theta}_o^4(t) - 3r_o(t)\ddot{\theta}_o^2(t) + 2r(t)\dot{\theta}_o(t)\ddot{\theta}_o(t) = 0 \quad (5.9)$$

Since $r_o(t) > 0$ for all t in $[t_0, t_f]$, (5.9) is equivalent to

$$4\dot{\theta}_o^4(t) - 3\ddot{\theta}_o^2(t) + 2\dot{\theta}_o(t)\ddot{\theta}_o(t) = 0 \quad (5.10)$$

Solving the above differential equation for $\dot{\theta}_o(t)$, we can write,

$$\dot{\theta}_o(t) = \frac{4a^3}{4a^2 - 4abt + (4a^4 + b^2)t^2} \quad (5.11)$$

where $a = \dot{\theta}_o(t_0)$ and $b = \ddot{\theta}_o(t_0)$.

Thus if the linearized system (5.1) is not observable then

$$\dot{\theta}(t) = \frac{4a^3}{4a^2 - 4abt + (4a^4 + b^2)t^2} \quad \text{for all } t \in [t_0, t_f] \quad (5.12)$$

Invoking the contrapositive yields the desired result. $\square$

Remark 5.3.2. *It is important to note that for real numbers a, b and real time t , the denominator in (5.12) is zero if and only if both a and b are zero. Knowing a and b , if we solve for t such that the denominator in (5.12) is zero, the solution to the quadratic equation in t is an imaginary number.*

The case where a and b are both zero is a special case which results in a straight line trajectory whose extension passes through the origin.

Theorem 5.3.3. *Straight line trajectories whose extension passes through the origin are not locally observable.*

Proof. If (5.1) is not observable, then from (5.10) we can write

$$4\dot{\theta}_o^4(t) - 3\ddot{\theta}_o^2(t) + 2\dot{\theta}_o(t)\ddot{\theta}_o(t) = 0$$

for all t in $[t_o, t_f]$. A trivial solution to this differential equation is given by $\dot{\theta}_o(t) = 0$ for all t in $[t_o, t_f]$. For straight line trajectories whose extensions pass through origin, bearing angle $\theta_o(t)$ is constant for all t in $[t_o, t_f]$, and hence $\dot{\theta}_o(t) = 0$ for all t in $[t_o, t_f]$. Thus straight line trajectories passing through the origin are not locally observable. $\square$

5.4 Uniform Observability for δ over a Finite Interval

In this section we find conditions under which (5.1) is uniformly observable for δ over the finite interval $[t_o, t_f]$. According to Theorem 3.7.2, combining the specific observability requirements from Theorem 5.3.1 with the requirements for uniform observability for δ over the finite interval $[t_o, t_f]$ yields specific conditions on the

trajectory of an AUV for which a Luenberger observer exists, at least on the interval $[t_o + \delta, t_f]$.

Theorem 5.4.1. *Suppose $u_{1o}(t) \neq 0$ for all t in $[t_o, t_f]$. Then (5.1) is uniformly observable for δ over the finite interval $[t_o, t_f]$ if*

$$\dot{\theta}_o(t) \neq \frac{4\dot{\theta}_o^3(\tau)}{4\dot{\theta}_o^2(\tau) - 4\dot{\theta}_o(\tau)\ddot{\theta}_o(\tau)\delta + \left[4\dot{\theta}_o^4(\tau) + \ddot{\theta}_o^2(\tau)\right]\delta^2} \quad (5.13)$$

for all t in $[t_o + \delta, t_f]$, where $\tau = t - \delta$.

Remark 5.4.2. *Even though it is difficult to find trajectories for which the condition (5.13) is not satisfied, we can show that for simple trajectories, such as a straight line trajectory whose extension does not pass through the origin, (5.13) is satisfied. Condition in (5.13) can be checked by an AUV in real time to avoid unobservable trajectories.*

5.5 Illustrative Experiment

The VT miniature AUV was deployed in the Lafayette river near Norfolk, Virginia, on 30 August 2004. Tidal flow produces significant currents in the river. The AUV was run on the surface so that GPS signals were available continuously. Position information from the GPS (without differential corrections) was used to compute the range of the AUV from a fixed known location and to provide the ground truth for the algorithm. The AUV was instrumented with the sensors described in Section 4.7. To measure currents, three drifters, designed by Dr. Jay Austin of the Old Dominion University [32], were deployed. The GPS data from the drifters was used to compute the currents. Trajectories of the AUV and the drifters are shown in Figure 5.1. The

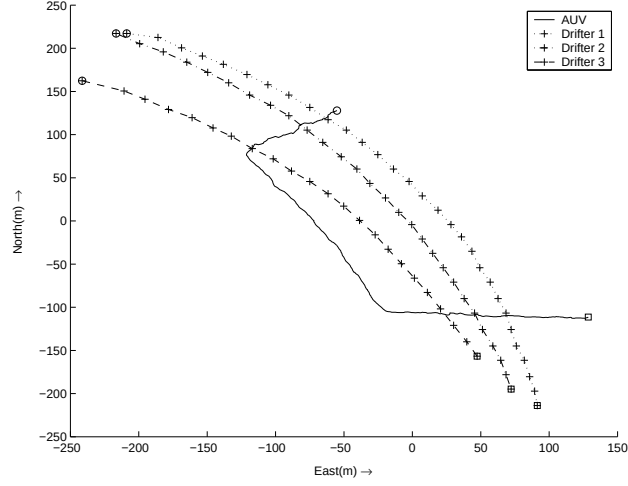


Figure 5.1: AUV and drifter trajectories. Squares and circles denote start and end points respectively. '+' indicates points where each drifter collected GPS data.

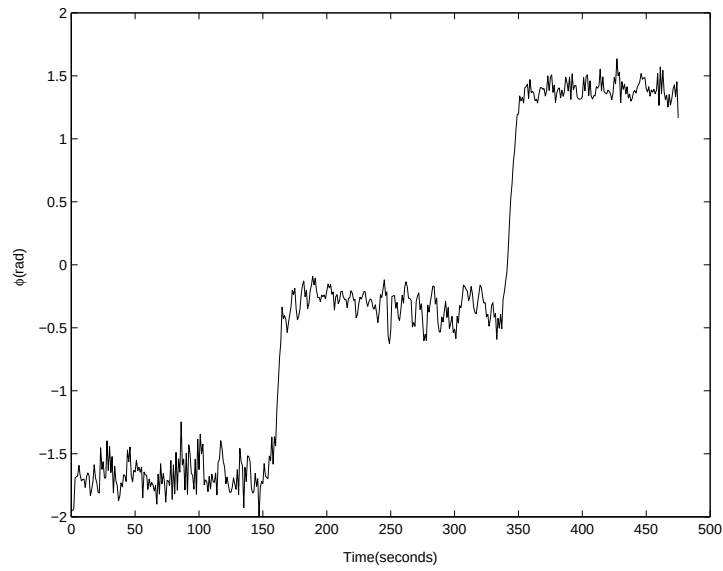


Figure 5.2: AUV heading $\varphi(t)$ obtained from the compass.

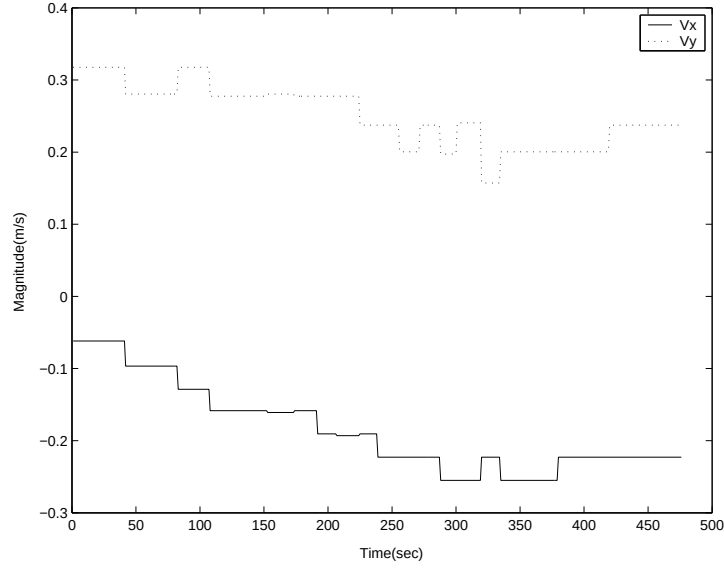


Figure 5.3: Current components v_x and v_y obtained by the nearest drifter point along the AUV trajectory.

AUV heading $\varphi(t)$ obtained from an electronic compass is shown in Figure 5.2. At any point (x, y) on the AUV trajectory, components v_x and v_y are approximated by v_x and v_y at the nearest drifter data point. This approximation to v_x and v_y is used as the ground truth for the current estimation. Figure 5.3 shows v_x and v_y along the AUV trajectory. The AUV experiences changing currents along its trajectory and hence from the AUVs perspective, the currents vary with time. Figure 5.4 shows the actual GPS trajectory and the trajectory estimated by the EKF. The estimated trajectory closely follows the actual GPS trajectory except along the turns. This is due to the mismatch between the kinematic model of the AUV motion and the actual AUV dynamics during turning maneuvers [33]. Figure 5.5 shows the actual and the estimated currents, v_x and v_y , along the AUV trajectory. The estimate of currents by the EKF is close to the actual currents calculated from the drifters position data. The

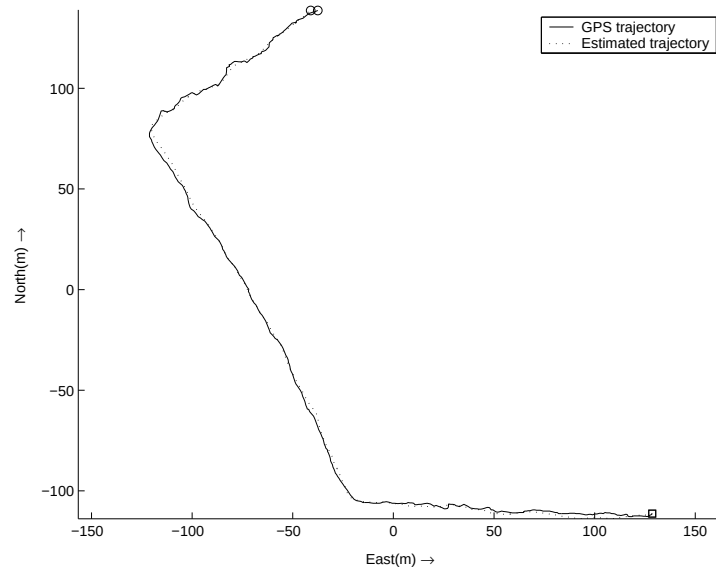


Figure 5.4: Actual and estimated trajectories. Squares and circles indicate the start and the end points respectively.

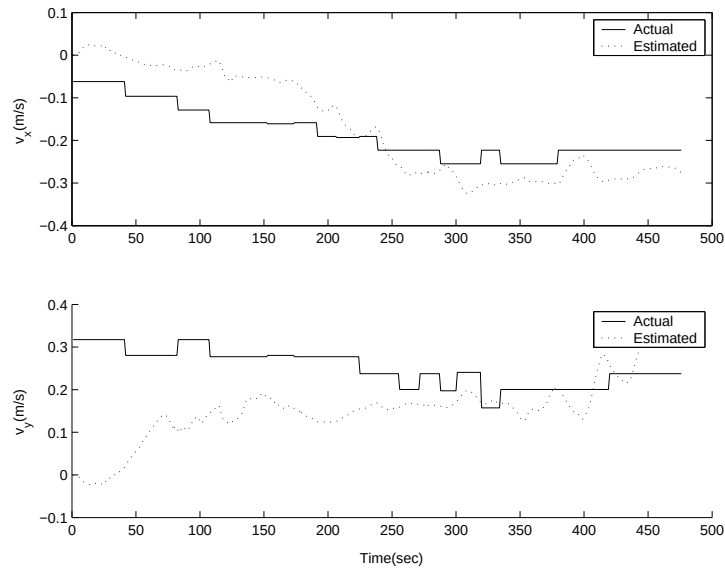


Figure 5.5: Actual and estimated current components v_x and v_y

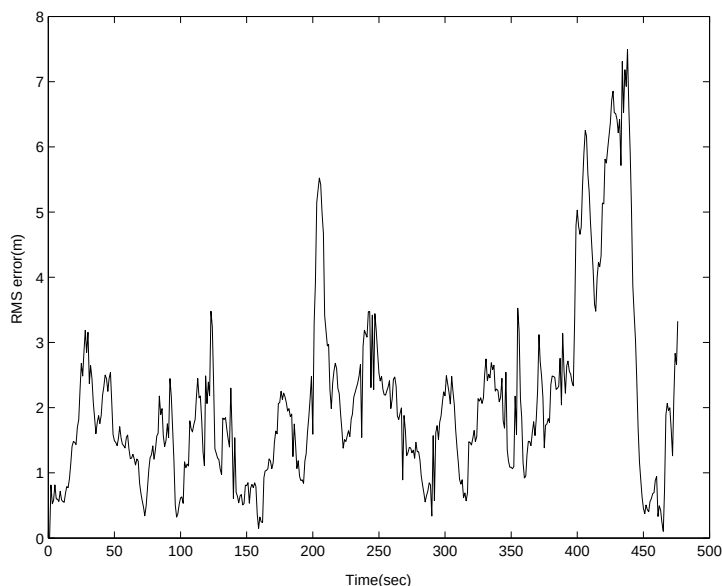


Figure 5.6: Position estimation error

currents along the AUV trajectory are not constant, however the kinematic model assumes the currents are constant. Due to the constant currents assumption, errors are introduced in the position estimation whenever currents change. Despite this, the EKF could track changes in the currents. We believe if currents change slowly, the position estimation errors induced by changing currents are negligible. Figure 5.6 shows the position error in the EKF position estimation. The median position estimation error is about 2.5m over the entire run. Figure 5.6 shows large position estimation errors just after the AUV makes the turns.

The range navigation algorithm has been successfully deployed on the VT Miniature AUV using acoustic hardware. We present details of this implementation in the next chapter.

Chapter 6

Implementation of Range Navigation System

In this chapter we present implementation of the proposed single range underwater navigation system on the VT Miniature AUV. We discuss hardware as well as software aspects of this implementation.

The single range navigation system is implemented using acoustic hardware consisting of an acoustic transponder anchored at a known location and an acoustic modem on the AUV. The acoustic modem pings the acoustic transponder and the one-way time-of-flight of the ping from the AUV to the transponder is converted into a range measurement using the speed of sound.

6.1 The Hydroid Acoustic Transponder

A transponder is a listening device that responds to a specific incoming signal with a predefined output signal. The acoustic transponder used in the VT Miniature AUV



Figure 6.1: Hydroid digital transponders

range navigation system is the Hydroid REMUS Digital Transponder [34]. It responds to a navigation ping from the acoustic modem thus enabling the AUV to compute one-way time-of-flight of the ping.

The Hydroid acoustic transponder is a 4-channel transponder based on a multi-channel receiver and operates at 25kHz. It allows up to four different AUVs to simultaneously ping the transponder without any explicit coordination or time-slicing. The transponder consists of an electronics package that includes batteries, power amplifier and a micro-controller board in a waterproof case and a transducer outside the waterproof case.

Before the start of an AUV mission, a transponder is moored at a known location using an anchor. The transponder is buoyant and holds the transducer upright (± 20 degrees). Figure 6.1 shows two Hydroid digital transponders used in the range navigation system.

6.2 The WHOI Micro-Modem

The VT Miniature AUV carries the WHOI Micro-Modem, an acoustic modem developed by Woods Hole Oceanographic Institution (WHOI) [35]. The Micro-Modem can be used for navigation as well as underwater communication.

The Micro-Modem has the capability for low-rate frequency shift keying with frequency hopping (FH-FSK) and variable rate phase-coherent shift keying (PSK) modulation. In the configuration used on the VT Miniature AUV, the navigation pings are transmitted using PSK codes while data packets are sent using FH-FSK which allows data transmission in shallow waters [35]. However data rate achieved by FH-FSK in the current configuration is restricted to 80 bits per second.

The Micro-Modem is configured to operate at 25kHz with a bandwidth of 4kHz to allow operation with Hydroid digital transponders. The Micro-Modem uses four different PSK codes thus providing unique code for each transponder channel and thus can simultaneously ping to maximum of four transponders. The two-way time-of-flight measurement precision of the Micro-Modem is $\pm 125\mu\text{S}$ which translates into one-way range estimate accuracy of 9.375cm, assuming speed of sound in water to be 1500 m/s [35].

Figure 6.2 shows the Micro-Modem hardware and assembly in a custom-designed housing. Picture on the left shows micro-modem hardware stack and the transducer. Picture on the right shows micro-modem assembly in a custom housing which is strapped underneath the VT Miniature AUV (see Figure 6.3).

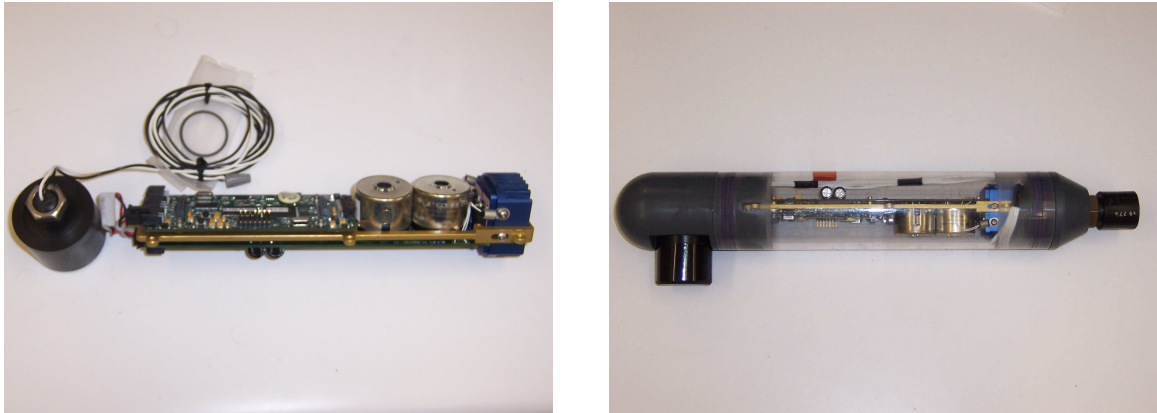


Figure 6.2: The WHOI Micro-modem assembly.

6.3 The Micro-Modem Interface

The Micro-modem communicates with the host computer through NMEA compatible interface that defines a set of ASCII messages [36]. Micro-modem messages can be classified into command, data, status and error messages. Micro-modem command messages are used for micro-modem configuration and task execution. Different tasks that can be executed by the micro-modem include active LBL navigation using REMUS digital transponders, data transmission and reception using an acoustic network protocol, pinging to monitor acoustic network connectivity, diagnostics and status update.

The Micro-modem cannot be preprogrammed to carry out certain tasks on its own. For every task that needs to be executed, the host computer needs to initiate the task and if required needs to interact with the micro-modem during the task execution to provide additional commands and/or data depending on the task.

6.3.1 Micro-Modem Setup for Active Navigation

Performing active navigation using a micro-modem and digital transponders requires setting up certain micro-modem parameters, using micro-modem configuration commands. Navigation performance (computing range of the micro-modem from the transponder) depends on correct selection of these parameters.

The micro-modem parameter configuration command has the following format [\[36\]](#)

\$CCCCFG,NNN,VV*CS

NNN Name of the parameter to set

VV Value of the parameter to set

***CS** Optional checksum

Micro-modem echoes back configuration command using

\$CACFG,NNN,VV*CS

Following list shows micro-modem parameters that need to be configured. The numbers in the parentheses are the optimal parameter values.

1. AGN - Analog gain (50 - 6dB)
2. NPT - Power Threshold for Navigation Detector (50)
3. NDT - Detection Threshold for Navigation Detector (120)
4. DTH - Matched Filter Signal Threshold (108)
5. PTH - Matched Filter Detector Power Threshold (50)
6. TAT - Transponder Turn-Around-Time (50ms)

6.3.2 Acquiring Range from a Transponder

The AUV initiates the process of acquiring range from the transponder by sending the digital ping command to the micro-modem. The command has the format

\$CCPDT,GRP,CH,SF,STO,TO,AF,BF,CF,DF*CS

GRP	Transponder group identifier
CH	Channel number (1,2,3,4)
SF	Hardware synchronization flag
STO	Hardware synchronization timeout (ms)
TO	Maximum time to listen for echo (ms)
AF,BF,CF,DF	Transponder use flag
*CS	Optional checksum

Micro-modem echoes back ping command with

\$SNPDT,GRP,CH,SF,STO,AF,BF,CF,DF*CS

Micro-modem then sends the acoustic ping and listens for the echo back from the transponders. One way time-of-flight data and navigation matched filter data is returned in \$SNTTA and \$SNMFD messages respectively.

The one-way time of flight data message has the format

\$SNTTA,TA,TB,TC,TD,hhmmss.ss*CS

TA	One-way travel time from Transponder A
TB	One-way travel time from Transponder B
TC	One-way travel time from Transponder C
TD	One-way travel time from Transponder D
hhmmss.ss	Time of ping (from micro-modem real time clock)
*CS	Checksum

The navigation matched filter data message has the format

\$SNMFD,NN,MFPK,MFPWR,MFRATIO*CS

NN	Transponder channel
MFPK	Matched filter peak value
MFPWR	Incoherent broadband power at the peak
MFRATIO	MFPK/MFPWR ratio used for detector test
*CS	Checksum

Following listing shows transactions that take place between the host and the micro-modem while acquiring ranges from two transponders simultaneously. In this transaction, the micro-modem pings transponders DT4A and DT4B in group 1, over channel 1, without hardware synchronization and with listening timeout of 1 second.

```
Host to Modem → $CCPDT,1,1,0,0,1000,1,1,0,0
Modem to Host → $SNPDT,1,1,0,0,1000,1,1,0,0*70
— Micro-modem then pings the transponders
Modem to Host → $SNMFD,01,4499,1350,0333*57
Modem to Host → $SNMFD,02,4669,1205,0387*57
Modem to Host → $SNTTA,0.0599,0.1688,,,182420.00*51
```

There may be situations when range measurements from all transponders are not obtained. In such a case, no \$SNMFD message is generated for transponders from which no range information is available and the corresponding travel time fields are left blank in \$SNTTA message. Following listing shows such a scenario where the micro-modem failed to get range from Transponder A but got range from transponder B.

```
Host to Modem → $CCPDT,1,1,0,0,1000,1,1,0,0
Modem to Host → $SNPDT,1,1,0,0,1000,1,1,0,0*70
Modem to Host → $SNMFD,02,1691,0979,0172*5C
Modem to Host → $SNTTA,,0.1942,,,182632.00*42
```



Figure 6.3: VT Miniature AUV equipped with range navigation hardware

The navigation algorithm is implemented as a module in the AUV software. Following sections describe AUV hardware, software and implementation details of the extended Kalman filter used in the range navigation system implementation.

6.4 VT Miniature AUV

The VT Miniature AUV was developed with the goal to build an inexpensive and highly flexible research platform which can be used for testing distributed and cooperative control algorithms. Figure 6.3 shows a picture of VT Miniature AUV equipped with the range navigation hardware.

The AUV has a streamlined shape with one propeller and three independent control surfaces in the stern. It is a fully field-deployable AUV and it has successfully supported a number of field experiments. Table 6.1 lists the AUV's specifications.

Table 6.1: The VT Miniature AUV specifications

Parameter	Specification
Length	78cm
Outer/Inner diameter	3.75"/3.5"
Mass	4.5Kg
Stability	Passively stable in roll and pitch
Power	25.2V Li-Ion battery stack
Propulsion	24V, 40W Brushless DC motor
Control surfaces	3 Independent digital servos
Computer	32-bit x86 compatible @ 266MHz
Operating System	Linux 2.6.17
Connectivity	Wi-Fi and RF Modem (RS232)
Navigation	GPS on surface, Inertially aided dead-reckoning/Range navigation/Synchronous navigation underwater

6.5 The VT Miniature AUV Hardware

The main objective of VT Miniature AUV is to provide an inexpensive and flexible research platform that can carry various payloads for various types of research work. This objective requires a modular design that enables flexible configurations and provides an easy upgrade path. The AUV's hardware design reflects this philosophy.

VT Miniature AUV does not use any embedded devices. The main AUV computer is an x86 compatible microprocessor. All devices communicate using standard RS232

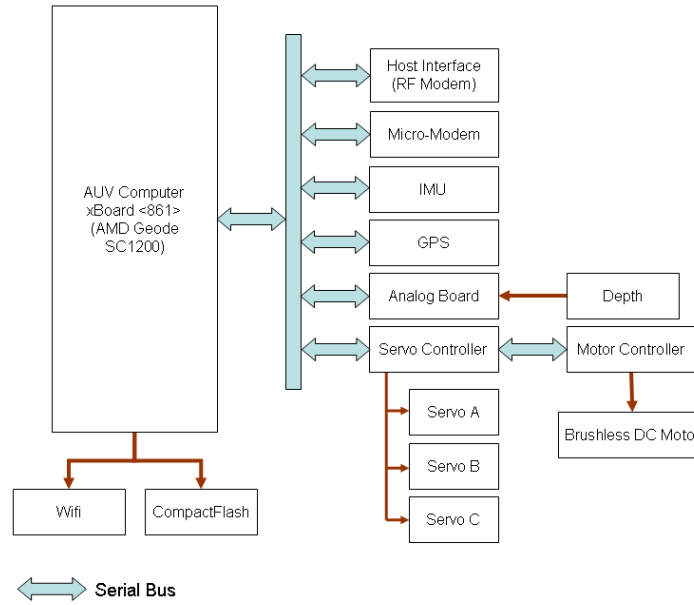


Figure 6.4: VT Miniature AUV hardware architecture

serial communication. This enables a flexible architecture where devices can be easily interchanged or upgraded without affecting functionality of the system. All scientific payloads necessary for a particular mission are strap-on external payloads. The VT Miniature AUV has flown with a YSI probe to measure conductivity, temperature and dissolved oxygen and a moving mass actuator for experiments in low speed control[37]. Figure 6.4 shows the AUV hardware architecture details.

The main AUV computer is Kontron xBoard < 861 > module which combines a 32-bit x86 compatible microprocessor and peripheral interfaces. The module is based on AMD Geode SC1200 microprocessor running at 266MHz with 33MHz bus. The microprocessor has a 16KB integrated cache and 128MB SDRAM and includes hardware watchdog functionality. The xBoard < 861 > module sits on a custom

designed motherboard which provides access to all peripheral functions, that include three OHCI USB 1.1 host controllers, two native serial ports, a PCI bus and two IDE channels. One IDE channel is used for an onboard 128MB flash and the other IDE channel is used for CompactFlash on the motherboard. The motherboard provides Wi-Fi connectivity using the PCI interface.

A number of sensors are used to measure the AUV attitude, depth and position. All sensors communicate via serial interface. Table 6.2 lists the sensor suite on board the AUV.

Table 6.2: The VT Miniature AUV sensor suite

AUV Variable	Sensor
AUV Attitude	Microstrain 3DM-GX1 AHRS - provides roll, pitch, yaw, angular rates and linear accelerations
Absolute Position	Falcom Navi-S GPS - provides absolute position (Lat-Lon-Alt), CEP 10m
Depth	Strain gauge pressure sensor - provides analog output which is processed by the analog sensors board
Payload sensor	Variable; watertight bulkhead connector is provided underneath the AUV to mount external payloads, any payload that can communicate over serial interface can be used

The AUV actuators include a drive motor for the propeller and three digital servos for the three control surfaces. The main drive motor is a 24 volt, 40 watt three-phase brushless DC motor with hall effect and encoder feedback. A custom designed controller board based on the 16-bit dsPIC30F3011 microcontroller provides the PWM outputs for the three servos and sends the propeller speed commands to a dedicated motor controller (EZServo EZSV23).

Power is provided by a lithium-ion battery stack providing 25.2 volts at 4.2Ah. A custom power distribution board provides +5V and +12V regulated power. The power board uses Vicor quarter-brick DC-DC converters for power regulation and a magnetic reed switch to turn the AUV on and off.

6.6 The VT Miniature AUV Software

The application software is a C/C++ Posix compliant multi-threaded application written for Unix/Linux platform. Most of the software modules are written in C; however newer modules, such as the micro-modem driver and the extended Kalman filter used in range navigation, are implemented as C++ objects.

The software is divided into various tasks - initialization, communication, data acquisition and logging, navigation, control and watchdog. Each task is accomplished by one or more dedicated threads. Data is exchanged among various threads using data structures and thread-safe circular buffers. Figure 6.5 shows the AUV software architecture.

The application is initialized by reading a configuration file in the XML format that contains information about control frequency, various safety thresholds, serial port assignments for various devices (sensors and actuators) and controller gains.

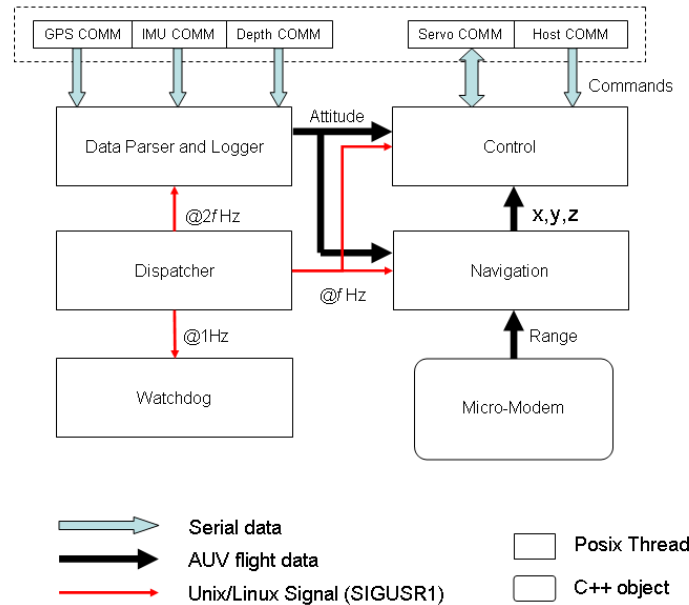


Figure 6.5: VT Miniature AUV software architecture

After reading the configuration file the initialization routine sets up data structures, opens serial ports, configures the AHRS and the micro-modem drivers, and then quits.

Once all initialization is completed successfully, the application instantiates the micro-modem driver and spawns all serial communication threads. It then creates the dispatcher, data parsing, navigation, control and watchdog threads.

The micro-modem driver is implemented as a C++ object. It is responsible for configuring the micro-modem by reading an XML configuration file. The configuration file contains information about various parameters that must be set for proper operation as well as information about tasks that micro-modem must perform during an AUV flight. The micro-modem driver spawns a private thread that is respon-

sible for micro-modem serial communications as well as execution of micro-modem tasks that include pinging to digital transponders and additional tasks such as data transmission and synchronous navigation transmissions.

For each serial device a dedicated serial communication thread is spawned. These threads are created for AHRS, GPS, analog board, RF modem and servo controller. All serial communication threads are configured such that they sleep until data is available on respective serial ports. Any serial data read is stored into thread-safe circular buffers to be used by the data parser and logger thread.

The dispatcher thread is the thread that keeps track of timings of all timed threads. Data parser and logger, navigation, control and watchdog threads are timed threads and each runs at a specific frequency. Control and navigation threads run at the control frequency, data parser and logger runs at twice the control frequency whereas watchdog runs at 1Hz. Nominal control frequency is 20Hz which can be changed through the configuration file. Dispatcher thread internally runs a system timer at twice the control frequency and uses Unix/Linux signal mechanism to run timed threads at specified frequencies.

The data parser and logger thread is responsible for extracting data received from all serial devices and logging it so that it can be used later for AUV flight analysis. The data parser and logger thread stores extracted data in various data structures dedicated to different devices. This thread runs at twice the control frequency thus ensuring availability of the latest data for the navigation and control threads.

The navigation thread uses an extended Kalman filter, implemented as a C++ object, to run the range navigation algorithm. Currently the navigation thread can implement range navigation using one or two transponders. It can also implement the synchronous navigation algorithm [38]. The navigation thread receives the AUV

attitude data from the data parser and logger thread and receives the range data from the micro-modem driver.

The control thread is responsible for executing AUV missions as well as performing low level AUV flight control which includes heading and depth control. The control thread reads commands sent to the AUV over the RF modem and can queue up any number of such commands. The control thread computes the actuator commands and sends them to the servo controller over a serial interface. The control thread is also responsible for implementing various safety features which include maximum run-time for a mission queue, maximum depth threshold, minimum and maximum rates of change of depth, sensor and actuator communication failure and propeller speed feedback.

The watchdog thread keeps track of all threads and communications. If any thread stops responding, the watchdog thread kills the entire application process to avoid any catastrophe. The watchdog thread also updates a console display with the latest flight information.

6.7 The Extended Kalman Filter for Range Navigation

The range navigation algorithm is implemented in the VT Miniature AUV software as a discrete-time extended Kalman filter, using system dynamics in the Cartesian coordinates as describe in (4.11). The EKF implementation is standard (see e.g. [31], pp. 496) and we summarize it below.

The discrete-time kinematic equations of motion of an AUV are

$$s(k+1) = f(s(k), k) + \omega(k) \quad (6.1)$$

$$h(k) = \bar{h}(s(k), k) + \nu(k) \quad (6.2)$$

where $s(k) = [x(k), y(k), \phi(k), v_x(k), v_y(k)]^T$, $\omega(k)$ is the process noise, $\nu(k)$ is the measurement noise and

$$\begin{aligned} f(s(k), k) &= \begin{bmatrix} x(k) \\ y(k) \\ \phi(k) \\ v_x(k) \\ v_y(k) \end{bmatrix} + \Delta T \begin{bmatrix} u_1(k) \cos \phi(k) + v_x(k) \\ u_1(k) \sin \phi(k) + v_y(k) \\ u_2(k) \\ 0 \\ 0 \end{bmatrix} \\ \bar{h}(s(k), k) &= \begin{bmatrix} \sqrt{x^2(k) + y^2(k)} \\ \phi(k) \end{bmatrix} \end{aligned} \quad (6.3)$$

where ΔT is the sampling time. Let $Q(k)$ be the process noise covariance and $R(k)$ be the measurement noise covariance. We define

$$\begin{aligned} F(k+1, k) &= \left. \frac{\partial f(s, k)}{\partial s} \right|_{s=\hat{s}(k|k)} \\ H(k) &= \left. \frac{\partial \bar{h}(s, k)}{\partial s} \right|_{s=\hat{s}(k|k-1)} \end{aligned}$$

The extended Kalman filter algorithm can be summarized as follows.

Initialization:

$$\hat{s}(0|0) = E[s(0)] \quad (6.4)$$

$$P(0|0) = E[(s(0) - E[s(0)])(s(0) - E[s(0)])^T] \quad (6.5)$$

For $k = 1, 2, 3, \dots$

Prediction:

$$\hat{s}(k|k-1) = f(s(k-1), k-1) \quad (6.6)$$

$$P(k|k-1) = F(k, k-1)P(k-1|k-1)F^T(k, k-1) + Q(k-1) \quad (6.7)$$

Correction:

$$K(k) = P(k|k-1)H^T(k) [H(k)P(k|k-1)H^T(k) + R(k)]^{-1} \quad (6.8)$$

$$\hat{s}(k|k) = \hat{s}(k|k-1) + K(k) [h(k) - \bar{h}(\hat{s}(k|k-1), k)] \quad (6.9)$$

Chapter 7

Field Trials of Range Navigation System

In this chapter we present results from the field trials of the proposed single range underwater navigation system implemented on the VT Miniature AUV. We discuss important aspects of practical implementation of the range navigation algorithm and present results obtained. We present an example that shows how the AUV trajectory affects the performance of the range navigation algorithm.

7.1 Data Collection and Processing

Field trials of the range navigation algorithm were conducted on 20 December 2006 in Chesapeake bay near Solomons, MD. Tidal cycles produce significant currents in the bay. Two Hydroid digital transponders, DT4A and DT4B, were deployed at locations $(38.330672^{\circ}\text{N}, 76.404912^{\circ}\text{W})$ and $(38.327097^{\circ}\text{N}, 76.404675^{\circ}\text{W})$ respectively. The GPS receiver used to mark the positions of both the transponders has a circular

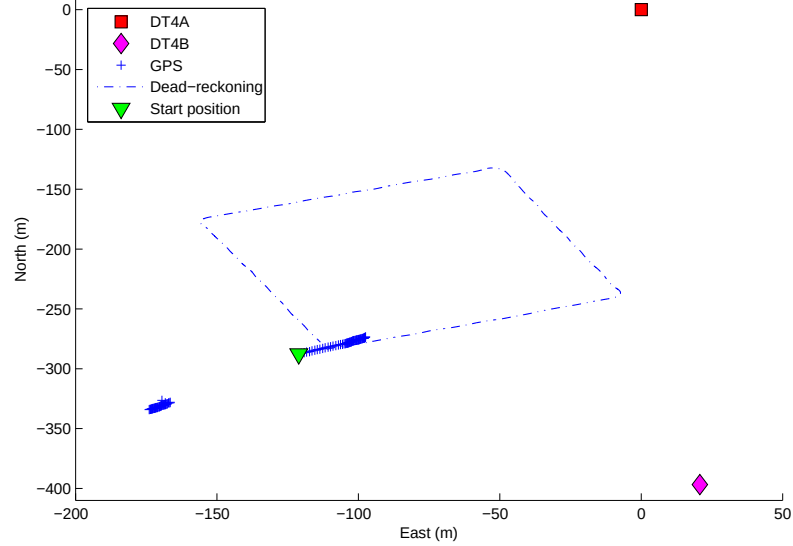


Figure 7.1: GPS and the dead-reckoning trajectory of the AUV flight

error probable (CEP) of 10m and hence the actual transponder positions can be anywhere within a circle of 10m radius with centers at the respective GPS positions. Transponders were anchored to the bottom and they were at the depths of 2.7m and 3.6m from the surface respectively. Deploying two transponders enables computation of the ground truth for AUV positions using the range measurements simultaneously obtained from both the transponders. Care was taken to make sure that the AUV did not cross the line of symmetry joining the two transponders. Transponder DT4A was designated as the origin of the local coordinate system. Figure 7.1 shows the AUV trajectory obtained from the GPS and the dead-reckoning trajectory. The GPS data is available only when the AUV is on the surface.

The AUV uses a GPS receiver to get its absolute position when it is on the surface. The GPS receiver on the AUV has a circular error probable of 10m. During

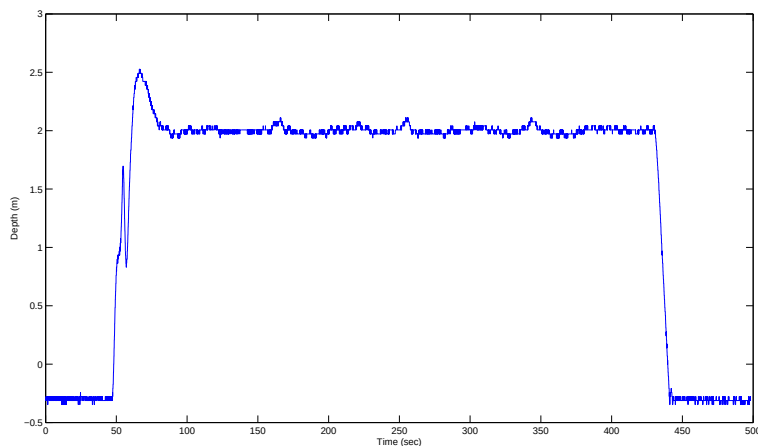


Figure 7.2: AUV depth during the flight

a flight, the AUV uses the last known GPS position as the initial position estimate for the range navigation algorithm. The AUV position and the transponder positions obtained by GPS receivers are in the latitude-longitude-altitude (LLA) format. These positions are converted into the Earth-Fixed (EF) coordinate frame whose center is located at the position of transponder DT4A. The EF coordinate frame is a planar $x - y$ frame which is tangential to Earth's surface at the origin of the coordinate frame. The coordinate transformation from LLA to EF frame is valid only over short distances close to the EF frame origin. Typically this distance is in the range of 1000m.

The AUV was commanded to run at a constant depth of 2m during the entire flight (see Figure 7.2). The depth sensor resolution is 0.05m. The actual AUV depth oscillates around the commanded depth with the worst case peak-to-peak oscillation at 0.15m.

Heading $\varphi(t)$ and the rate of change of heading $\dot{\varphi}(t) = u_2(t)$ were measured by an

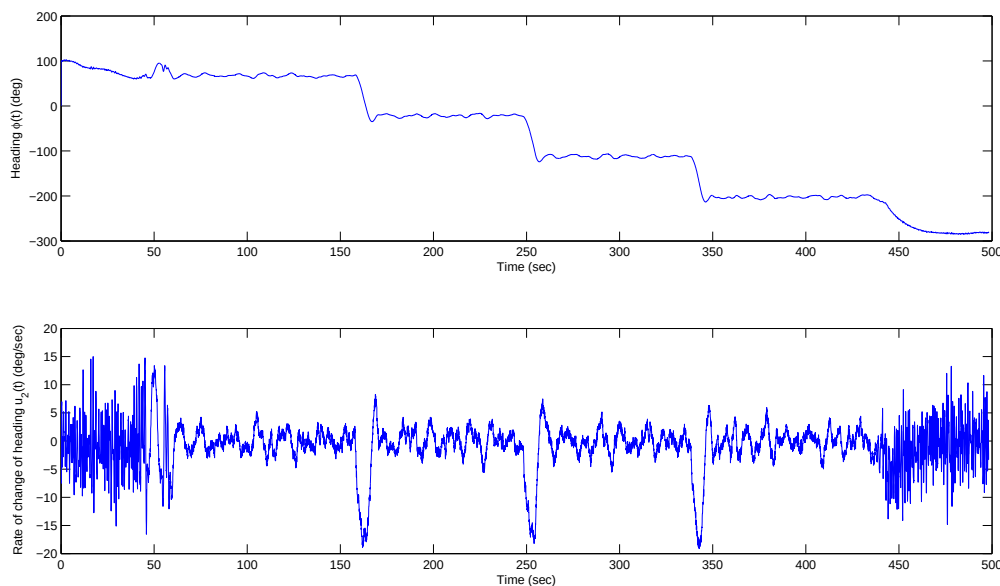


Figure 7.3: AUV heading and rate of change of heading during the flight

AHRS. The AHRS measures heading in degrees clockwise from the North. Figure 7.3 shows $\varphi(t)$ and $u_2(t)$ measured during the AUV flight. In this particular flight, the AUV heading commands were 60° , $330^\circ(-30^\circ)$, $240^\circ(-120^\circ)$ and $150^\circ(-210^\circ)$. AUV thus made three left turns during the flight which result in the three negative peaks in $u_2(t)$. Rapid variations in $u_2(t)$ at the start and the end correspond to times before the start and after the end of the flight when the AUV was sitting on the surface.

The propeller speed of the AUV is instrumented. For this experiment, the propeller operated at 1200 rpm and the corresponding speed of the AUV was approximately 1.28m/s. Figure 7.4 shows the speed of the AUV. The negative speed indicates the backward dive of the AUV during the flight.

Ranges from transponders were measured using the micro-modem. The micro-

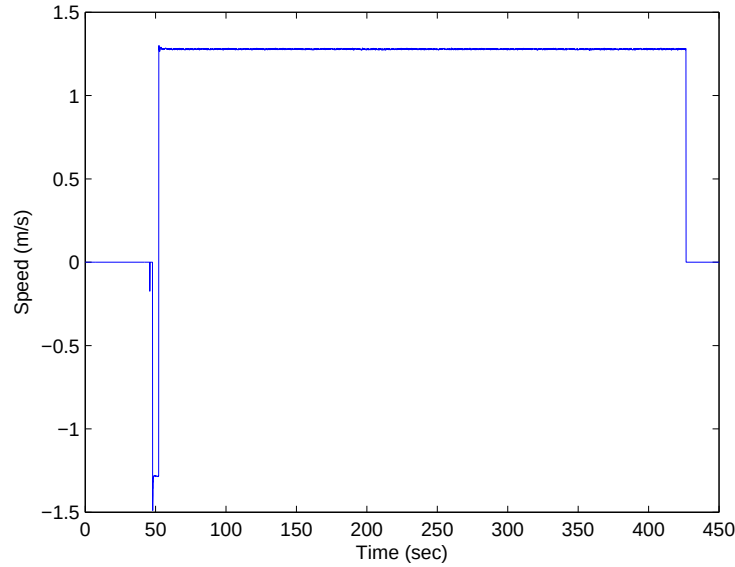


Figure 7.4: AUV speed during the flight

modem was set up to ping both transponders every 2 seconds. Field trials were conducted in shallow water (maximum depth of around 12m) which affects range measurements due to noise. Figure 7.5 shows a set of range measurements obtained from the micro-modem. The micro-modem sent navigation pings to both transponders 174 times during the AUV flight. No range measurements were obtained for 31 pings from DT4A and for 58 pings from DT4B. These void data points can be seen in Figure 7.5 as blank spaces. The AUV was commanded to fly in a rectangular pattern. The first leg in the AUV flight was a straight line trajectory with the commanded heading of 60° . Subsequent legs in the AUV flight were straight line trajectories with commanded headings of 330° , 240° and 150° degrees respectively. The first vertical line in Figure 7.5 represents the start of the AUV flight. It also marks the start of the first leg in the flight. Subsequent vertical lines indicate starts of new legs and the

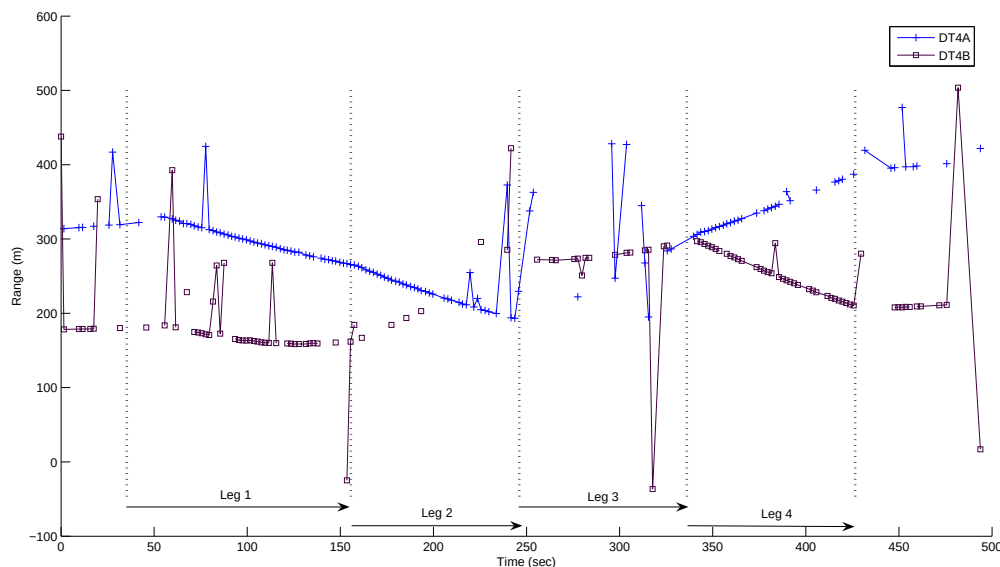


Figure 7.5: Range measurements during the flight

last vertical line indicates the end of the AUV flight.

The AUV trajectory was computed using the range measurements obtained simultaneously from both the transponders. From here onwards, we call this trajectory the *acoustic trajectory*. For this particular flight, there are 61 range measurements that were obtained simultaneously from both the transponders. Figure 7.6 shows the AUV GPS positions, the acoustic trajectory and the dead-reckoning trajectory. The GPS positions are available only before the start and after the end of the AUV flight. The triangle pointing down indicates the last good GPS position just before the start of the AUV flight. The dead-reckoning trajectory in Figure 7.6 clearly shows the four different legs in the AUV flight. These legs are straight line trajectories with commanded headings of 60° , $330^\circ(-30^\circ)$, $240^\circ(-120^\circ)$ and $150^\circ(-210^\circ)$. We can see that the acoustic trajectory does not have any positions along the second leg of the

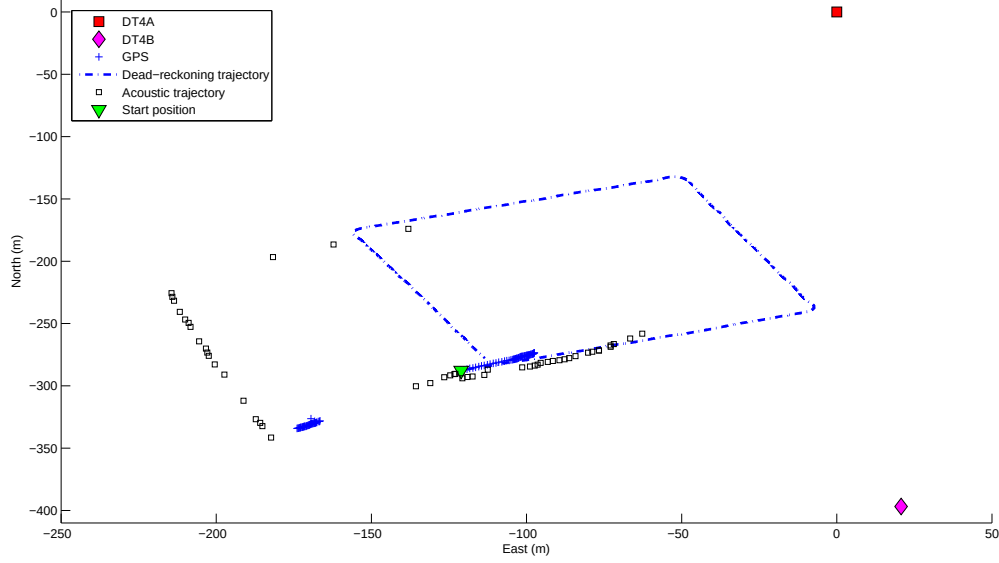


Figure 7.6: AUV GPS positions and the acoustic trajectory

flight and just three positions along the third leg of the acoustic trajectory.

Before range measurements can be processed by the extended Kalman filter, it is important to discard bad range measurements. In order to decide if a range measurement $r(k)$ obtained at instant k is good, we use state estimation error covariance matrix $P(k|k-1)$. Diagonal terms of $P(k|k-1)$ are variances of errors in estimates of the state variables. For the state vector $s(k) = [x(k), y(k), \varphi(k), v_x, v_y]^T$, the first two terms on the principal diagonal of $P(k|k-1)$ are the variances σ_{xx}^2 and σ_{yy}^2 of the errors in the estimates of $x(k)$ and $y(k)$ respectively. Using these variances, we define a circular region of position uncertainty around the position estimate at instant k . The radius of this region is

$$r_e(k) = \sqrt{\max(\sigma_{xx}^2, \sigma_{yy}^2)} \quad (7.1)$$

Given the position estimate at k , the expected range measurement is

$$\hat{r}(k) = \sqrt{\hat{x}^2(k) + \hat{y}^2(k)} \quad (7.2)$$

A range measurement $r(k)$ is considered good if it satisfies

$$r_e(k) \leq |r(k) - \hat{r}(k)| \quad (7.3)$$

Any range measurement $r(k)$ that does not satisfy (7.3) is discarded.

An Acoustic Doppler Current Profiler (ADCP) on a Doppler Velocity Log (DVL) mounted on the Autonomous Surface Vehicle (ASV) measures currents in North and East directions. In the EF coordinate frame, they correspond to v_x and v_y respectively. For every current measurement, the ADCP records the GPS position and the time of that measurement. The ADCP measures currents at various water column depths through the entire water column. For these experiments, the ADCP measured currents at the depths of 0.25m, 0.5m, ..., 2m, 2.25m, ... Ground truth for the current estimation was obtained by post-processing the ADCP data. For every point on the AUV trajectory estimated by the range navigation algorithm, the closest point on the ADCP trajectory was obtained. Currents were then averaged using 50 ensembles on either side of that ADCP data point, thus giving a time average of currents at every point along the estimated AUV trajectory.

7.2 Results

To assess performance of the range navigation algorithm we must compare the AUV trajectory estimated by the extended Kalman filter against a ground truth AUV trajectory computed by triangulation of ranges from both the transponders. We refer to this ground truth as the acoustic trajectory. However the acoustic trajectory

computed for this AUV flight does not have enough data points to be able to compare it against the range navigation estimate. Due to this limitation, we compute a least squares solution and use that to compare the performance of the range navigation algorithm against.

7.2.1 The Least Squares Solution

Let the position of the AUV on the acoustic trajectory at the time when the AUV dives be $s_0 = [x_0, y_0]^T$ and let the time of this estimate be t_0 . At the end of the flight, let the position of the AUV on the acoustic trajectory at time t_1 be $s_1 = [x_1, y_1]^T$. Using s_0 and s_1 as the start and the end positions for the AUV flight, we compute the least squares solution to estimate the currents.

From the AUV kinematics we have

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \cos(\varphi(t)) + v_x \\ u_1(t) \sin(\varphi(t)) + v_y \end{bmatrix}, \quad t \in [t_0, t_1] \quad (7.4)$$

which can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \cos(\varphi(t)) \\ u_1(t) \sin(\varphi(t)) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix}, \quad t \in [t_0, t_1] \quad (7.5)$$

Thus,

$$\begin{bmatrix} x_1 \\ y_1 \\ \varphi_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ \varphi_0 \end{bmatrix} + \int_{t_0}^{t_1} \begin{bmatrix} u_1(t) \cos(\varphi(t)) \\ u_1(t) \sin(\varphi(t)) \end{bmatrix} dt + \begin{bmatrix} t_1 - t_0 & 0 \\ 0 & t_1 - t_0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} \quad (7.6)$$

We can solve for v_x and v_y

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \frac{1}{t_1 - t_0} \left\{ \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} - \int_{t_0}^{t_1} \begin{bmatrix} u_1(t) \cos(\varphi(t)) \\ u_1(t) \sin(\varphi(t)) \end{bmatrix} dt \right\} \quad (7.7)$$

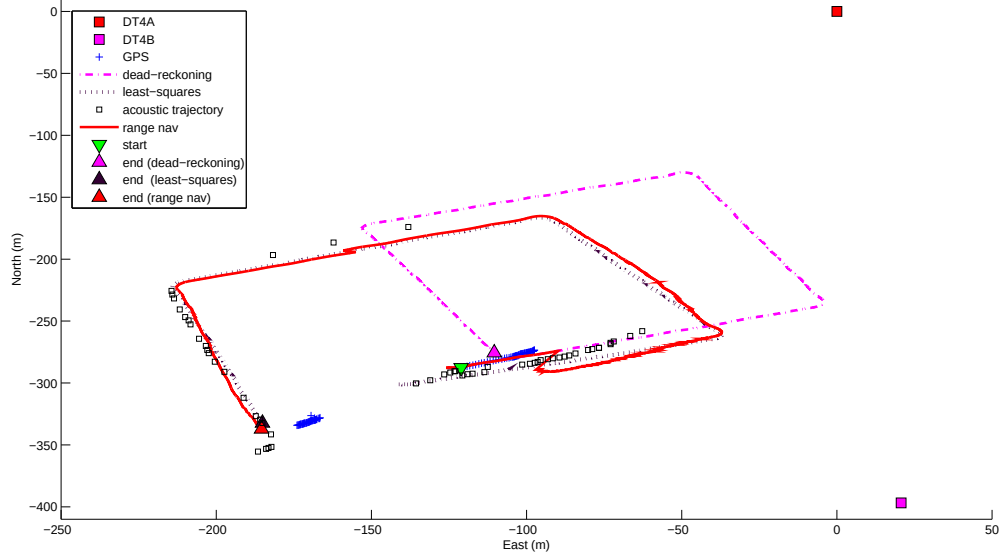


Figure 7.7: Range navigation performance - AUV trajectory estimation

The currents computed using the least squares are

$$v_x = -0.1159\text{m/s}$$

$$v_y = -0.1580\text{m/s}$$

7.2.2 Performance of the Range Navigation Algorithm

Figure 7.7 shows the AUV trajectory estimated by the range navigation algorithm along with the dead reckoning trajectory, the acoustic trajectory and the trajectory computed using the least squares solution. The dead-reckoning trajectory is computed by using $v_x = 0\text{m/s}$ and $v_y = 0\text{m/s}$ and thus does not take currents into consideration. The trajectory computed using the least squares solution uses $v_x = -0.1159\text{m/s}$ and $v_y = -0.1580\text{m/s}$. At the start of the flight, the AUV position estimate computed

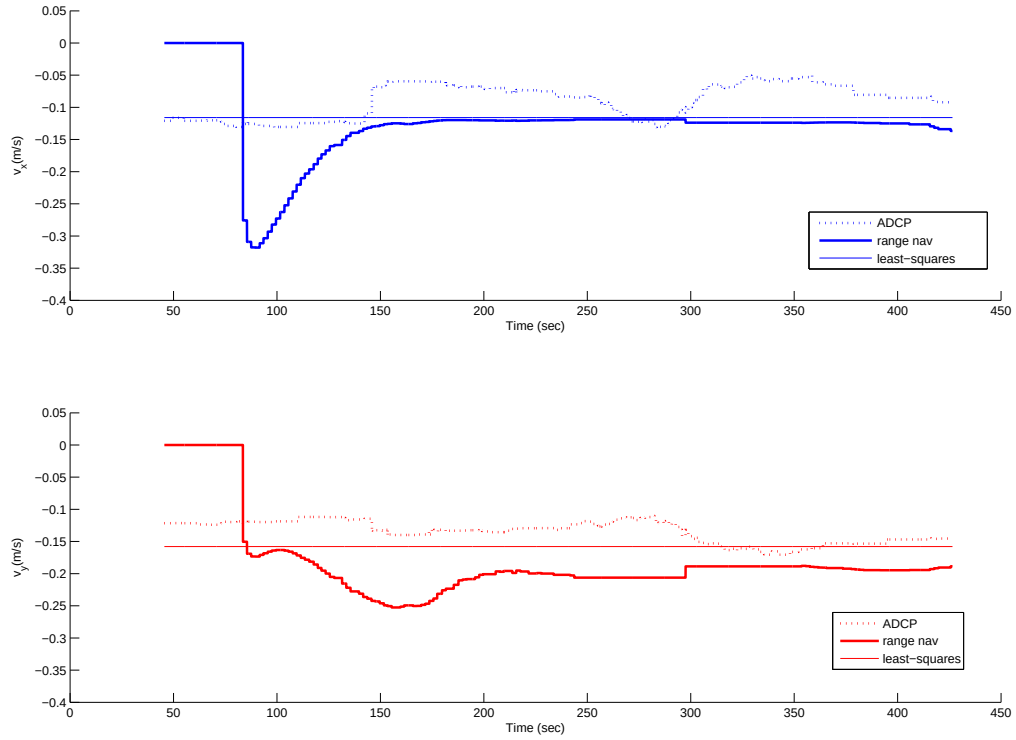


Figure 7.8: Range navigation performance - currents estimation

by the extended Kalman filter is significantly away from the least-squares solution. However as the EKF converges, the trajectory estimated by the EKF stays close to the least-squares solution.

Figure 7.8 shows the currents estimated by the range navigation algorithm along with the currents measured by the ADCP and the currents computed by the least squares solution. The initial change in the EKF estimate of currents is due to the GPS error in the initial AUV position estimate and the transponder position. As mentioned earlier, when the AUV dives it uses the last known GPS fix as the initial

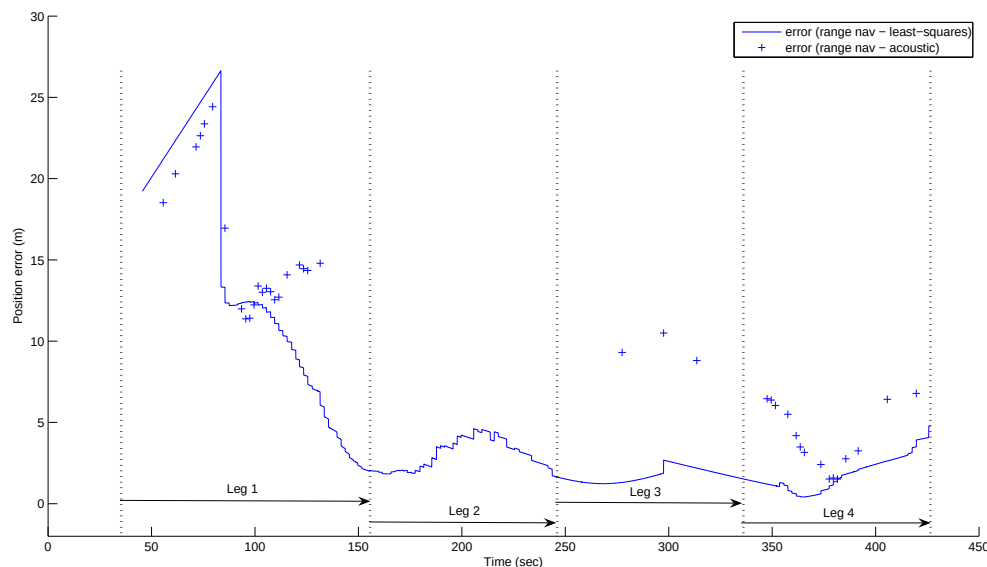


Figure 7.9: Range navigation performance - position estimation errors

position estimate for EKF. This position estimate carries the GPS error which has a CEP of 10m. When a valid range measurement is available, the EKF adjusts the currents estimate to compensate for the GPS error. As more range measurements become available, estimated currents converge to true currents.

Figure 7.9 shows the error between the range navigation EKF position estimate and the least-squares solution. It also shows the error between the EKF position estimate and the acoustic trajectory. In this figure, the first vertical line indicates the start of AUV flight and the start of the first leg, with subsequent vertical lines indicating starts of subsequent legs in the flight. The last vertical line indicates the end of the AUV flight. During the first leg of the flight, the estimate of currents has not converged and thus the error between the EKF position estimate and the least-squares estimate is large. As the currents estimate starts converging, the error starts

decreasing. From the end of the second leg and through most of the third leg, very few good range measurements are available. When no good range measurements are available, EKF does not run the correction step and hence the estimate of currents does not change in this period (see Figure 7.8). EKF however continues to run the time update step using the last estimate of currents. The trajectory estimated by the EKF shows a jump during the third leg of the trajectory. This again is due to the fact that very few good measurements are available in this leg which results in EKF dead-reckoning for most part of the third leg and thus the position estimate starts going away from the actual position. A good range measurement during this leg allows the EKF to correct the position estimate after a long period of no measurements, thus resulting in a jump. Again at the end of the fourth leg there are no range measurements available which results in the error starting to grow.

7.3 Effects of Radial Trajectory

According to the observability analysis of the range navigation algorithm, any radial trajectory is not observable. The performance of the range navigation algorithm deteriorates when a trajectory is close to being radial. In this section, we experimentally show how a radial trajectory affects the performance of the range navigation algorithm. For this experimental demonstration, we use the data from an AUV flight in which the first leg is very close to being radial with the transponder DT4B as the origin.

Figure 7.10 shows the range measurements obtained for this particular flight. The micro-modem sent 328 pings to both the transponders during the flight. Out of these pings, no measurements were available from DT4A for 128 pings and from DT4B for

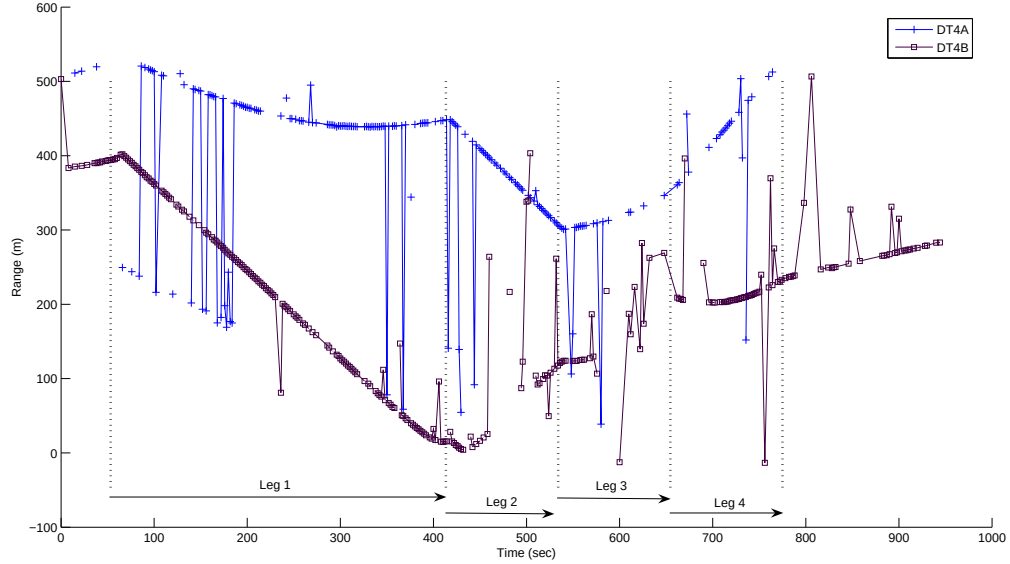


Figure 7.10: Range measurements during the flight with a radial leg

44 pings. The range measurements available are very noisy which severely affects the range navigation performance. During the first leg, there are a large number of good range measurements from the transponder DT4B whereas the number of range measurements from DT4A is small and the measurements are noisy.

Figure 7.11 shows the trajectory estimated by the range navigation algorithm using transponder DT4A as the origin, along with the dead-reckoning trajectory and the acoustic trajectory. Although the performance of range navigation is affected by highly noisy range measurements, the EKF could follow the acoustic trajectory.

Figure 7.12 shows the trajectory estimated by the range navigation algorithm using transponder DT4B as the origin, along with the dead-reckoning trajectory, the least-squares solution and the acoustic trajectory. When the range navigation algorithm is run with transponder DT4B as the origin, the first leg of the flight is

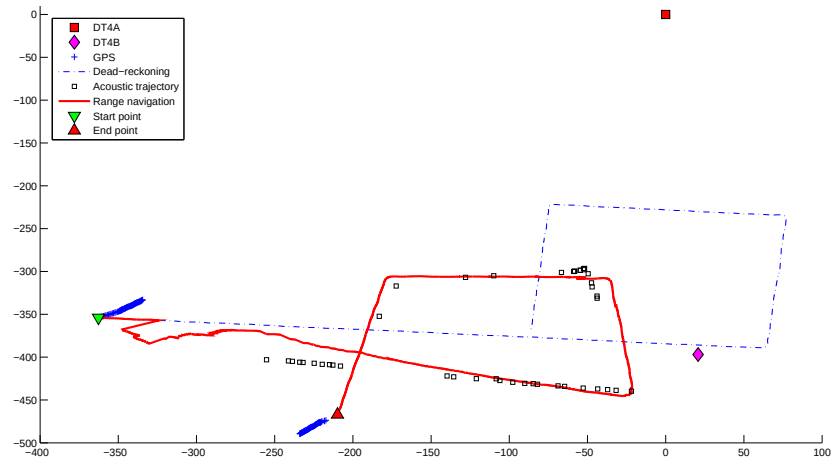


Figure 7.11: AUV trajectory estimation with the origin at DT4A

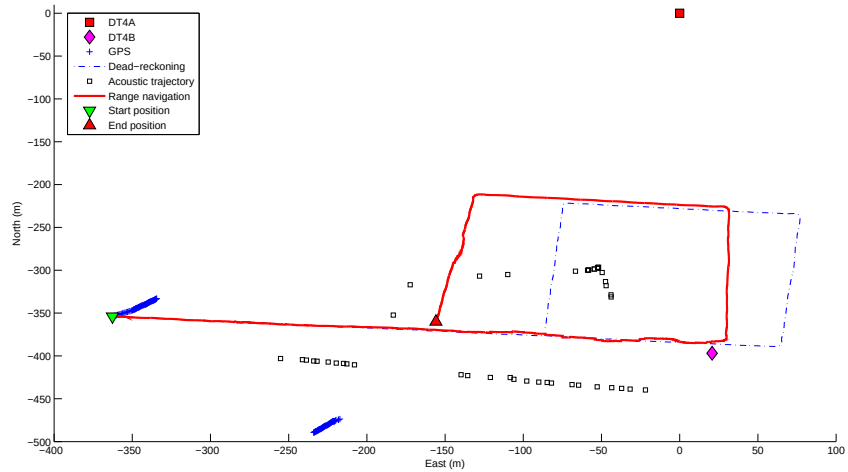


Figure 7.12: AUV trajectory estimation with the origin at DT4B

close to being radial and this severely affects the performance, even though there is a large number of good range measurements available. Coupled with the fact that there are not very many good range measurements during subsequent legs in the AUV flight, the trajectory estimated by the range navigation algorithm tends to be close to the dead-reckoning trajectory.

Chapter 8

Conclusions

In this work, an underwater navigation algorithm has been proposed that is potentially suitable for very small AUVs. The principal contribution of this work is the precise observability analysis that characterizes observable AUV trajectories in the presence of unknown currents and towards this end, we propose new notions that facilitate assessment of uniform observability of LTV systems.

To address the difficulty in studying asymptotic properties of the observability grammian, we define finite-time uniform observability of an LTV system which retains all properties of uniform observability over a finite interval. It is shown that if an LTV system is uniformly observable over a finite interval then an observer exists whose estimation error is bounded by an exponentially decaying function on that interval. The notion of limiting systems is used to assess uniform observability of an LTV system by studying finite time observability of its limiting systems. This approach thus eliminates the need to study asymptotic properties of the observability grammian altogether.

In order to simplify computation of the observability grammian, we show that

uniform observability of a reduced order system derived from an LTV system is sufficient for uniform observability of that LTV system. This approach is applicable to LTV systems where a subset of the state variables are directly measured.

These new notions are employed to study observability of the range navigation algorithm. Most of the AUV trajectories, with the exception of straight line trajectories whose extensions pass through the origin, are observable. The range navigation algorithm was successfully deployed on VT Miniature AUV using acoustic hardware.

Implementation of the range navigation algorithm as a discrete time extended Kalman filter raises the important question of preservation of observability under sampling. It is shown in [39], under the assumption of analytic output function, that if a system is observable, then there exists a sampling rate τ_o such that for any sampling rate $\tau \geq \tau_o$, the sampled discrete-time system remains observable. If the range navigation algorithm runs at a sufficiently fast sampling rate, then all the conditions that guarantee observability of the continuous-time system are inherited by the sampled-data implementation.

The conditions on AUV trajectories that guarantee observability can be evaluated by an AUV in real-time and the AUV can take appropriate action to change its course to achieve observability if it determines that it is traveling on an unobservable trajectory.

In future work, the range navigation algorithm will be extended to 3-dimensional implementation. It can also be extended to use more than one transponder and to synchronous navigation [38]. These extensions have been successfully tested on the VT Miniature AUV; however mathematical analysis of these extensions remains an open question.

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