# SYNTHESIS OF ELECTRIC NETWORKS INTERCONNECTING PZT ACTUATORS TO EFFICIENTLY DAMP MECHANICAL VIBRATIONS.

**BY**

**M.PORFIRI**

**A Dissertation submitted to the Graduate School
in partial fulfillment of the requirements
for the Degree**

**Master of Science
in
Engineering Science and Mechanics**

**Dr. Edmund G. Henneke**

**Dr. Francesco dell'Isola**

**Dr. Romesh Batra**

**Dr. Dean Mook**

**Dr. Don H.Morris**

**December 12, 2000
Blacksburg, Virginia**

**Keywords: Vibration control, Black box approach, Passive networks,
Piezoelectric actuators, Distributed control, Electrical analogs**

# SYNTHESIS OF ELECTRIC NETWORKS INTERCONNECTING PZT ACTUATORS TO EFFICIENTLY DAMP MECHANICAL VIBRATIONS.

**M.PORFIRI**

**(ABSTRACT)**

**The aim of this thesis is to show that it is possible to damp mechanical vibrations in a given frame, constituted by Euler beam governed by the equations of an elastica, by means of piezoelectric actuators glued on every beam and interconnected each other via electrical networks.Since we believe that the most efficient way to damp mechanical vibrations by means of electrical networks, is to realize a strong modal coupling between the electrical and the mechanical motion, we will synthesize a distributed circuit analog to the Euler beam.We will approach this synthesis problem following the black box approach to mechanical systems, studied by many engineers and scientists during the 1940's in an attempt to design analog computers.It will be shown that it is possible to obtain a quick energy exchange between its mechanical and electrical forms, using available piezoelectric actuators.Finally we will study a numerical simulation for the damping of transverse vibrations of a beam clamped at both ends.**

*To my Teacher Francesco dell'Isola.*

# Contents

# List of Figures

# Chapter 1

# Theory of networks: a brief introduction.

### Systems

The aim of this chapter is to introduce concepts from the theory of networks that will be needed in syntesizing an electrical *parallel* of the Euler beam. The first step of our discussion is the presentation of the model *black box* for a wide class of electrical devices those communicating with the outer world by a finite number of access points, called *terminals.*

It must be underlined that, although generally the described concepts are used in the special context of the theory of networks, they can be fruitfully applied to different branches of Engineering Science, where different physical devices may be involved. Indeed different physical objects can be described by the black box model. Generalizing the discussion given by Molly (1958) [1], we will do so for an element of a beam.

The state of every terminal $\mathcal{T}_i$ is characterized by a pair of $l-$tuples $(\alpha_i, \tau_i) = \left(\left(\alpha_i^1, ..., \alpha_i^l\right), \left(\tau_i^1, ..., \tau_i^l\right)\right)$. The set of the pairs $(\alpha_i, \tau_i) \in \mathbb{R}^l \times \mathbb{R}^l$ characterizes completely the state of the device. Generally speaking $\boldsymbol{\tau}_i^j$ is referred to as a *through* variable, while $\alpha_i^j$ is called an *across* variable.

The evolution of the system in the time interval $[0, \infty)$ will be described by *motion*, i.e. a function $\mathfrak{M} : t \mapsto (\alpha_i(t), \tau_i(t))$, $i = 1...l$. The motion is a real vector valued function of the time variable $t$, and all its scalar components, i.e. all the scalar functions $\alpha_i^j : t \mapsto \alpha_i^j(t)$, $\tau_i^j : t \mapsto \tau_i^j(t)$, belong to a *Signal Space* $\mathcal{D}_+$ which needs to be rigorously defined.

**Definition 1** *The Signal Space $\mathcal{D}_+$ is the Linear Space of infinitely continuously differentiable real valued functions of a real variable, whose support is in $I\!R^+$.*[1]

The Signal Space $\mathcal{D}_+$ is neither an inner product space nor a normed space, nevertheless we introduce whenever it is possible an inner product and a norm, extending the notion of inner product in $L^2$.

Note that for a generic black box the $\delta-$Dirac distributions are not allowed as signals of the system, nevertheless we will need these distributions to describe the input-output relationship for a particular class of networks.

We explicitly remark that the following definitions do not apply to every function of $\mathcal{D}_+$, but only to the functions belonging to the intersection $\mathcal{D}_+ \cap L^2$.

**Definition 2** *Let $x$ and $y$ be in $\mathcal{D}_+$. The inner product of $x$ and $y$ is given by:*

$$\langle x, y \rangle = \int_{-\infty}^{\infty} x(t)y(t)dt \tag{1.1}$$

*if it exists.*

**Definition 3** *Let $x$ be in $\mathcal{D}_+$. The norm of $x$ is:*

$$\|x\| = \langle x, x \rangle^{1/2} \tag{1.2}$$

[1] The support of a function $\varphi$ of a real variable is defined as the smallest closed set outside which $\varphi$ vanishes and denoted by $Supp\varphi$

*if it exists.*

The only topological concept we require in all $\mathcal{D}_+$ is the notion of convergence:

**Definition 4** *Let $\langle x_n \rangle$ be a sequence of functions in $\mathcal{D}_+$, that is $\langle x_n \rangle : \mathbb{N} \to \mathcal{D}_+$. Then this sequence is said to converge to $x \in \mathcal{D}_+$ if the sequence $\left\langle \dfrac{d^\alpha x_n}{dt^\alpha} \right\rangle$ converges uniformly on $\mathcal{D}_+$ to $\dfrac{d^\alpha x}{dt^\alpha}$ on any compact set, $\forall \alpha \in \mathbb{N}$.*

The variables $\alpha_i$, $\tau_i$ in the pair relative to $\mathcal{T}_i$ are generally called conjugate variables since we assume their inner product $(\sum_j^l \alpha_i^j \tau_i^j)$ to have the dimension of a power. Typical conjugate variables are current and voltage, force and velocity, moment and angular velocity and temperature and entropy change.

Usually in the theory of networks the state at every terminal is assumed to be fully characterized only by a pair of scalar variables.

However we intend to generalize such treatment in order to be allowed to regard a plane-beam element as a "network". Indeed the *kinematic state* of such an element at its terminals is characterized by two displacements and a variation of attitude, while the conjugate quantities will be two force components and one bending moment.

Let us now label consecutively the terminals of our device $\mathcal{T}_i$, $i = 1, ..., k$, where $k$ is the number of such terminals; and arrange all the across variables in a $k \times l$ matrix $\boldsymbol{\alpha}$, and the through variable in a $k \times l$ matrix $\boldsymbol{\tau}$.

**Definition 5** *We will group the state variables of our device following the convention:*

$$
\begin{aligned}
\boldsymbol{\alpha} &= \begin{pmatrix} \alpha_1 \\ ... \\ \alpha_k \end{pmatrix} = \begin{pmatrix} \alpha_1^1 & ... & \alpha_1^l \\ ... & ... & ... \\ \alpha_k^1 & ... & \alpha_k^l \end{pmatrix} \\
\tau &= \begin{pmatrix} \tau_1 \\ ... \\ \tau_k \end{pmatrix} = \begin{pmatrix} \tau_1^1 & ... & \tau_1^l \\ ... & ... & ... \\ \tau_k^1 & ... & \tau_k^l \end{pmatrix} \\
(\boldsymbol{\alpha},\tau) &= \begin{pmatrix} \alpha_1 & ... & \alpha_1^l & \tau_1^1 & ... & \tau_1^l \\ ... & ... & ... & ... & ... & ... \\ \alpha_k & ... & \alpha_k^l & \tau_k^1 & ... & \tau_k^l \end{pmatrix}
\end{aligned}
$$

The space of $k \times l$ matrices of entries functions of $\mathcal{D}_+$ will be called $\mathcal{D}_+^{k\times l}$: it is natural to generalize the definitions of inner product and norm given in $\mathcal{D}_+$ to the space $\mathcal{D}_+^{k\times l}$, which is clearly a Linear Space.

**Definition 6** *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be in* $\mathcal{D}_+^{k\times l}$*. The inner product of* $x$ *and* $y$ *is given by:*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \int_{-\infty}^{\infty} \mathbf{x}(t) : \mathbf{y}(t) dt \tag{1.3}$$

*if it exists. Where* $:$ *denotes the standard inner product of matrices, i.e.* $\mathbf{x} : \mathbf{y} = \sum_{i=1}^k \sum_{j=1}^l x_{ij} y_{ij}$.

**Definition 7** *Let* $\mathbf{x}$ *be in* $\mathcal{D}_+^{k\times l}$*. The norm of* $\mathbf{x}$ *is:*

$$\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2} \tag{1.4}$$

*if it exists.*

Each physical device is determined by the relation that it places upon the variables at the terminals, that is, different devices are characterized by different relations.

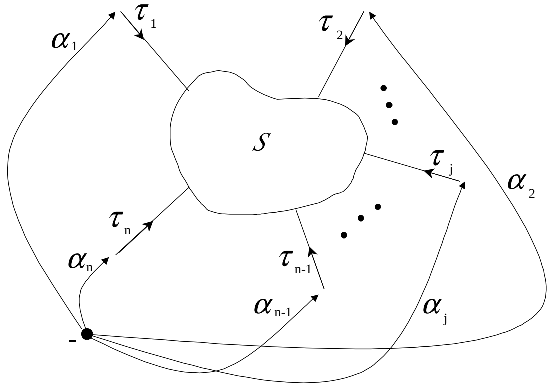


Figure 1.1: Representation of a system

A relation $C_{\mathcal{S}}$ is a binary relation on the set of all pairs $(\boldsymbol{\alpha}, \boldsymbol{\tau})$: its action is to select the set of admissible pairs $(\boldsymbol{\alpha}, \boldsymbol{\tau})$.

**Definition 8** *Given a binary relation $C_{\mathcal{S}}$ on $\mathcal{D}_+^{k\times l} \times \mathcal{D}_+^{k\times l}$, a System $\mathcal{S}$ is:*

$$\mathcal{S} = \left\{ (\boldsymbol{\alpha}, \boldsymbol{\tau}) \in \mathcal{D}_+^{k\times l} \times \mathcal{D}_+^{k\times l},\ \boldsymbol{\alpha} C_{\mathcal{S}}\ \boldsymbol{\tau} \right\} \tag{1.5}$$

**Remark 9** *When the black box is interpreted as a mechanical device the binary relation $C_{\mathcal{S}}$ just prescribes the relationships among the state variables induced by:*

- *balance equations*
- *constitutive relations*

**Remark 10** *Not all possible binary relations $C_{\mathcal{S}}$ model actual physical devices. The set of physically admissible constraints has to be restricted, e.g. by introducing the axiom of causality. For a formulation of this axiom see axiom (64).*

When we limit our attention to the electrical case, where $\boldsymbol{\tau}$ is a current vector and $\boldsymbol{\alpha}$ is a voltage vector a system is called a *network*. In this particular case, $l = 1$ and then $\boldsymbol{\tau}$ and $\boldsymbol{\alpha}$ belong to the space $\mathcal{D}_+^{k\times 1}$ of columns $k-$vector with elements in $\mathcal{D}_+$.

One of the most significant and useful concepts in the theory of networks is that of *port.*

A pair of terminals is a port relative to the subset $\mathcal{P} \subset \mathcal{S}$ if for every pair $(\boldsymbol{\alpha}, \boldsymbol{\tau}) \in \mathcal{P}$ the current entering one terminal is equal to the current leaving the other terminal. If all terminals occur as ports, we can relabel the terminals, calling those for the $j$-th port $\mathcal{T}_j^+$ and $\mathcal{T}_j^-$, $j = 1, ..., n = \frac{k}{2}$ and then call the network an $n-$port $\mathcal{N}$.

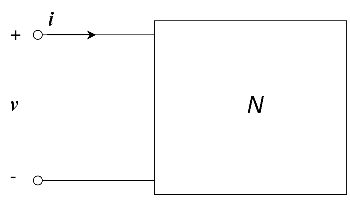


Figure 1.2: N-port Network

For an $n-$port $\mathcal{N}$ the meaningful variables are only the $n-$vectors $\mathbf{v}$ and $\mathbf{i}$ of port voltages (i.e. the voltage between $\mathcal{T}_j^+$ and $\mathcal{T}_j^-$) and currents, then given a binary relation

$C_{\mathcal{N}}$ we define a $n-$port $\mathcal{N}$ by:

$$\mathcal{N} = \{(\mathbf{v}, \mathbf{i}) \in \mathcal{D}_+^n \times \mathcal{D}_+^n, \mathbf{v} C_{\mathcal{N}}\ \mathbf{i}\} \tag{1.6}$$

In the following we will use the convention that assumes as positive the currents entering the networks, and as positive the voltages between $\mathfrak{T}_i^-$ and $\mathfrak{T}_i^+$.

**Remark 11** *Generally speaking whether or not a pair of terminals forms a port is dramatically dependent not only on the topology of the circuits, but also on how it is interconnected to the outer world; thus a pair of terminals can behave like a port only under the particular circumstances specified by $\mathcal{P}$. Some networks behave like $n-$port networks under every circumstance (for instance a resistor can be always regarded as a $1-$port network), while others exhibit this behavior only in extremely particular cases (for instance when two given terminals are interconnected by means of external resistors, and/or sources).*

**Definition 12** *A circuit $\mathcal{C}$ is a <u>specific interconnection-topology</u> of networks. Any constituent of a given circuit is called subnetwork.*

**Remark 13** *A given network, i.e. a given system as specified by the binary relation $C_{\mathcal{N}}$, can be realized with several different circuits.*

Therefore the concept of circuit is well distinguished from the concept of network. In fact we can think of a network as the equivalence class of all those circuits having the same behavior at their terminals: a circuit allows for a particular realization of a network.

**Example 14** *The "elementary" network inductor can be also realized by means of the interconnection of nullors, resistors and capacitor (see next section for more details).*

In the definition of a network we are not interested in its internal structure, but only in the properties that the network shows to the outer world through its terminals. In the following sections we will be interested in the synthesis of a given network: it will necessarily involve the choice of a circuit among those realizing the given network; this choice will depend on several conditions that will become clearer in the next sections.

**Definition 15** *The total instantaneous power expended into $\mathcal{N}$ is*

$$p(t) = \mathbf{v}(t)^T \mathbf{i}(t) \tag{1.7}$$

*and the net energy*[2] *delivered to the network to time $\tilde{t}$ is*

$$\mathcal{E}(\tilde{t}) = \int_{-\infty}^{\tilde{t}} p(t)dt \tag{1.8}$$

If $\mathcal{E}(\tilde{t})$ is positive, then at time $\tilde{t}$ the network $\mathcal{N}$ is said to have absorbed net positive energy, while if $\mathcal{E}(\tilde{t})$ is negative then the network $\mathcal{N}$ is said to have delivered net positive energy.

The connections of different subnetworks can be mathematically represented as operations on the subnetworks dictated by the well-known Kirchhoff's laws, that we need to state again in the particular context of black boxes.

**Definition 16** *Given a circuit, we call node the set of terminals of constituting networks interconnected in the given topology. Every terminal of the subnetworks must belong to exactly one node.*

[2] The definition of energy is well posed, in fact the integral always exists since $(\mathbf{v}, \mathbf{i}) \in \mathcal{D}_+^k \times \mathcal{D}_+^k$, and then their supports are bounded on the left. ($\mathbf{v}$ and $\mathbf{i}$ are $\mathbf{0}$ until the time $0$)

**Remark 17** *The previous definition allows for a node constituted by a single terminal, which is open circuited.*

**Axiom 18** *Given a circuit $\mathcal{C}$, let denote by $\{\mathfrak{n}_i\}$ the set of its nodes*

- *The sum of the currents entering $\mathfrak{n}_i$ must vanish.*
- *Let $(\mathfrak{n}_0, \mathfrak{n}_1, ..., \mathfrak{n}_k, \mathfrak{n}_{k+1} = \mathfrak{n}_0)$ be a closed loop of nodes in $\mathcal{C}$, $\alpha_i$ be their voltages with respect to the same reference and $\Delta_i = \alpha_{i+1} - \alpha_i$, then*

$$\sum_{i=1}^{k} \Delta_i = 0 \tag{1.9}$$

## Network analysis

### Building blocks

In this section we will introduce the *elementary* networks. They will be the building blocks of the circuits that further we will need to synthesize; the following definitions are not the most general that one can conceive, nevertheless they seem to be general enough for our purposes.

**Definition 19** *The resistor $\mathcal{N}_R$ is a $1-$port network defined by:*

$$\mathcal{N}_R = \{(v,i) \in \mathcal{D}_+ \times \mathcal{D}_+ : v = Ri,\ R \in \mathrm{I\!R}\} \tag{1.10}$$

*$R$ is its resistance.*

**Definition 20** *The capacitor $\mathcal{N}_C$ is a $1-$port network defined by:*

$$\mathcal{N}_C = \left\{(v,i) \in \mathcal{D}_+ \times \mathcal{D}_+ : i = C\frac{dv}{dt},\ C \in \mathrm{I\!R}\right\} \tag{1.11}$$

*$C$ is its capacitance.*

**Definition 21** *The inductor $\mathcal{N}_L$ is a $1-$port network defined by:*

$$\mathcal{N}_L = \left\{ (v,i) \in \mathcal{D}_+ \times \mathcal{D}_+ : v = L\frac{di}{dt},\ L \in \mathbb{R} \right\} \tag{1.12}$$

*$L$ is its inductance.*

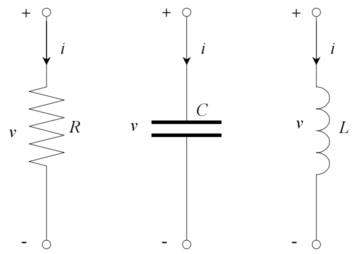


Figure 1.3: Representations of a resistor, capacitor, and inductor

**Definition 22** *The nullator $\mathcal{N}_0$ is $1-$port network defined by:*

$$\mathcal{N}_0 = \{(v,i) \in \mathcal{D}_+ \times \mathcal{D}_+ : v = 0,\ i = 0\} \tag{1.13}$$

**Definition 23** *The norator $\mathcal{N}_\infty$ is $1-$port network defined by:*

$$\mathcal{N}_\infty = \mathcal{D}_+ \times \mathcal{D}_+ \tag{1.14}$$

**Definition 24** *The nullor $\mathcal{N}_N$ is a $2-$port network defined by:*

$$\mathcal{N}_0 = \left\{ (\mathbf{v}, \mathbf{i}) \in \mathcal{D}_+^2 \times \mathcal{D}_+^2 : (v_1, i_1) = (0,0) \right\} \tag{1.15}$$

**Remark 25** *The nullor is an elementary network of capital importance in the synthesis of active filters and simulated capacitors and inductors.*

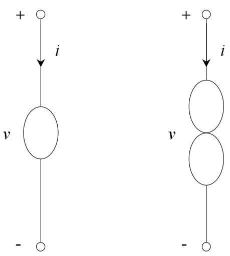


Figure 1.4: Representations of the nullator and norator

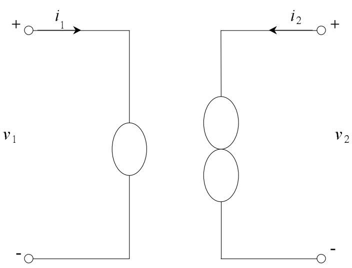


Figure 1.5: Representation of the nullor

**Definition 26** *The ideal transformer $\mathcal{N}_T$ is an $(l+m)$-port network with*

$$\mathbf{v} \in \mathcal{D}_+^{l+m} := \mathcal{D}_+^l \oplus \mathcal{D}_+^m; \ i.e.\ \mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \\ \mathbf{v}_2 \end{pmatrix},\ \mathbf{v}_1 \in \mathcal{D}_+^l \text{ and } \mathbf{v}_2 \in \mathcal{D}_+^m \tag{1.16}$$

$$\mathbf{i} \in \mathcal{D}_+^{l+m} = \mathcal{D}_+^l \oplus \mathcal{D}_+^m; \ i.e.\ \mathbf{i} = \begin{pmatrix} \mathbf{i}_1 \\ \\ \mathbf{i}_2 \end{pmatrix},\ \mathbf{i}_1 \in \mathcal{D}_+^l \text{ and } \mathbf{i}_2 \in \mathcal{D}_+^m$$

*defined by:*

$$\mathcal{N}_T = \left\{ (\mathbf{v}, \mathbf{i}) \in \mathcal{D}_+^{l+m} \times \mathcal{D}_+^{l+m} : \left\{ \begin{array}{c} \mathbf{v}_1 = \mathbf{T}^T \mathbf{v}_2 \\ \\ \mathbf{i}_2 = -\mathbf{T}\,\mathbf{i}_1 \end{array} \right. ,\ \mathbf{T} \in Lin\left( I\!R^m, I\!R^l \right) \right\} \tag{1.17}$$

$\mathbf{T}$ *is its turns-ratio matrix.*

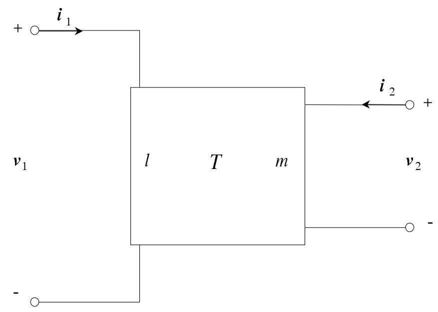


Figure 1.6: Representation of the ideal transformer

**Remark 27** *We will see in what follows that the constitutive assumption (1.17) assures that the ideal transformer is lossless.*

**Definition 28** *The voltage source $\mathcal{N}_V$ is a $1-$port network defined by:*

$$\mathcal{N}_V = \{(v,i) \in \mathcal{D}_+ \times \mathcal{D}_+ : v = V,\ V \in \mathcal{D}_+\} \tag{1.18}$$

*$V$ is its applied voltage.*

**Definition 29** *The current source $\mathcal{N}_I$ is a $1-$port network defined by:*

$$\mathcal{N}_I = \{(v,i) \in \mathcal{D}_+ \times \mathcal{D}_+ : i = I,\ I \in \mathcal{D}_+\} \tag{1.19}$$

*$I$ is its applied current.*

## Fundamental properties of networks

In this section we will discuss some of the main properties of networks, and will specify the class of networks to be taken into account for our issues.

As a preliminary, let us specify the linear structure of the space $\mathcal{V}$ consisting of the pairs of column $k-$vectors $(\mathbf{v}, \mathbf{i}) \in \mathcal{D}_+^k \times \mathcal{D}_+^k$ introducing the simple operation of sum of

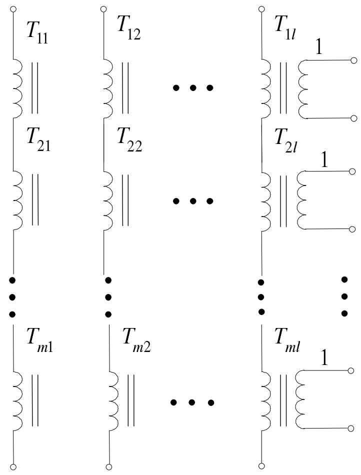


Figure 1.7: Belevitch's detailed representation of the ideal transformer

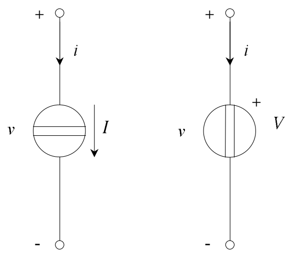


Figure 1.8: Representation of the current source and voltage source.

pairs and multiplication of a pair by a real number:

$$\begin{cases} (\mathbf{v}_1, \mathbf{i}_1) + (\mathbf{v}_2, \mathbf{i}_2) := (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{i}_1 + \mathbf{i}_2) \\ \alpha\,(\mathbf{v}, \mathbf{i}) = (\alpha\mathbf{v}, \boldsymbol{\alpha}\,\mathbf{i}) \end{cases} \tag{1.20}$$

As a topological concept we require only the notion of convergence, this is an immediate extension of what we have done in $\mathcal{D}_+$.

**Definition 30** *A sequence* $\langle(\mathbf{v}_n, \mathbf{i}_n)\rangle : \mathbb{N} \to \mathcal{D}_+^k$ *is said to converge to* $(\mathbf{v}, \mathbf{i}) \in \mathcal{D}_+^k$ *if*

$$\forall\ i \in \{1, ..., k\}\ ,\ v_{n_i}\ \in \mathcal{D}_+ \text{ converges to } v_l \in \mathcal{D}_+, \text{ and } i_{n_i}\ \in \mathcal{D}_+ \text{ converges to } i_l \in \mathcal{D}_+ \tag{1.21}$$

Now we can define one of the most important properties of networks:

**Definition 31** *An* $n-$*port network* $\mathcal{N}$ *is* <u>*linear*</u> *if* $\mathcal{N}$ *is a subspace of* $\mathcal{V}$*, that is*

$$\forall\ (\mathbf{v}_1, \mathbf{i}_1), (\mathbf{v}_2, \mathbf{i}_2) \in \mathcal{N}\ ,\ \alpha \in \mathbb{R} \begin{cases} (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{i}_1 + \mathbf{i}_2) \in \mathcal{N} \\ (\alpha\mathbf{v}, \mathbf{i}\,\alpha) \in \mathcal{N} \end{cases} \tag{1.22}$$

The requirement of linearity leads straightforward by to the *principle of superposition*; note that in general this principle does not hold for non-linear networks.

**Example 32** *The resistor, capacitor, inductor and ideal transformer are all linear networks; this statement follows directly from the linearity of the differentiation.*

**Example 33** *The nullator is linear since* $\mathcal{N}_0$ *is the singleton of the null vector of* $\mathcal{V}$*. The norator is also linear, since it allows all conceivable pairs. Clearly the nullor is linear.*

**Example 34** *The voltage and current source are both non-linear networks, since they violate both the requirement of the definition.*

Now we introduce two very tricky concepts, the notion of solvability and complete solvability; as a preliminary we have to introduce a particular network, called *series augmented network* $\mathcal{N}_a$ associated with $\mathcal{N}$.

**Definition 35** *Let $\mathcal{N}$ be an $n-$port network, $\mathcal{N}_a$ is said to be the series augmented network if:*

$$(\mathbf{v}, \mathbf{i}) \in \mathcal{N} \Rightarrow (1\Omega\mathbf{i} + \mathbf{v}, \mathbf{i}) \in \mathcal{N}_a \tag{1.23}$$

The series augmented network can be thought of as the the network obtained by connecting one unit resistor to each of the ports of the given network $\mathcal{N}$.

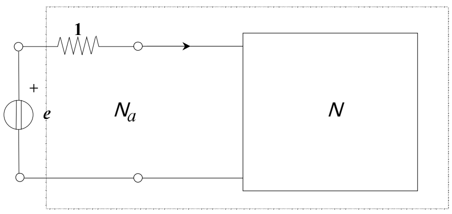


Figure 1.9: Augmented network

**Remark 36** *With an abuse of notation in what follows we will write* $\mathbf{i} + \mathbf{v}$ *instead of* $1\Omega\,\mathbf{i} + \mathbf{v}$.

**Definition 37** $\mathcal{N}$ *is solvable if* $\forall\ \mathbf{e} \in \mathcal{D}_+^n\ \exists!\ (\mathbf{v}, \mathbf{i}) \in \mathcal{N}$ *such that* $\mathbf{e} = \mathbf{i} + \mathbf{v}$

**Remark 38** $\mathcal{N}$ *is solvable if and only if there is a unique pair* $(\mathbf{v}, \mathbf{i}) \in \mathcal{N}$ *verifying Kirchhoff's laws in the circuit represented in the previous figure.*

**Definition 39** $\mathcal{N}$ *is completely solvable if it is solvable and if*

$$\forall\ \langle \mathbf{e}_n \rangle : \mathbb{N} \to \mathcal{D}_+^n \text{ convergent to } \mathbf{e} = \mathbf{i} + \mathbf{v}\ ,\ \langle (\mathbf{v}_n, \mathbf{i}_n) \rangle \text{ converges to } (\mathbf{v}, \mathbf{i})\,,\ \text{with } \mathbf{e}_n = \mathbf{i}_n + \mathbf{v}_n \tag{1.24}$$

**Remark 40** *$\mathcal{N}$ is completely solvable if the currents and voltages generated by the voltage source* $\mathbf{e}$ *depend continuously on* $\mathbf{e}$.

**Example 41** *The capacitor and the inductor are completely solvable; the resistor is completely solvable if the resistance is different from* $-1$.

**Example 42** *The nullator is not solvable since it forces* $\mathbf{e} = \mathbf{0}$*, while the nullator is not solvable since* $\forall\ \mathbf{e} \in \mathcal{D}_{+}^{n}\ \exists$ *infinite* $(\mathbf{v}, \mathbf{i}) \in \mathcal{N}_{\infty}$ *such that* $\mathbf{e} = \mathbf{i} + \mathbf{v}$.

**Definition 43** *$\mathcal{N}$ is time invariant if*

$$\forall\ (\mathbf{v}, \mathbf{i}) \in \mathcal{N},\ \forall \tau \in \mathbb{R}\ \exists\ (\mathbf{v}_{\tau}, \mathbf{i}_{\tau}) \in \mathcal{N}\ :\ (\mathbf{v}_{\tau}(t), \mathbf{i}_{\tau}(t)) = (\mathbf{v}(t+\tau), \mathbf{i}(t+\tau)) \tag{1.25}$$

The property of time invariance essentially restricts the class of circuits represented by $\mathcal{N}$, to those circuits constituted by building blocks which do not vary with time.

**Example 44** *The resistor, the capacitor, the inductor and the transformer are time-invariant.*

**Example 45** *The norator and the nullator are time invariant.*

**Example 46** *The voltage source and the current source are not in general time invariant, since they constrain one conjugate variable to vary with time in a prescribed non constant way.*

**Definition 47** *$\mathcal{N}$ is passive if*

$$\begin{aligned} \forall\ (\mathbf{v}, \mathbf{i}) &\in \mathcal{N}, \forall t \in \mathbb{R} \\ \mathcal{E}(t) &= \int_{-\infty}^{t} \mathbf{v}^{T}(\tau)\, \mathbf{i}(\tau)\, d\tau \geqslant 0 \end{aligned} \tag{1.26}$$

*otherwise* $\mathcal{N}$ *is active.*

Thus if $\mathcal{N}$ absorbs net positive energy for every $(\mathbf{v}, \mathbf{i})$ then it is called passive, while if it is able to deliver energy it is called active

**Example 48** *The instantaneous power for the ideal transformer is always zero, in fact:*

$$p(t) = \mathbf{v}_1(\tau)^T \mathbf{i}_1(\tau) + \mathbf{v}_2(\tau)^T \mathbf{i}_2(\tau) = \mathbf{T}^T \mathbf{v}_2(\tau)\,\mathbf{i}_1(\tau) + (-\mathbf{v}_2(\tau)\,\mathbf{T}\,\mathbf{i}_1(\tau)) = 0 \quad (1.27)$$

*then, trivially* $\forall\ (\mathbf{v}, \mathbf{i}) \in \mathcal{N}$, $\forall t \in \mathbb{R}\ \mathcal{E}_{\mathbf{T}}(t) = 0$, *thus the ideal transformer is passive.*

**Example 49** *Trivially, the instantaneous power for a nullator is zero, and then the nullator is passive. On the contrary the norator is active since every* $(\mathbf{v}, \mathbf{i})$ *belongs to* $\mathcal{N}_\infty$.

**Example 50** *The energy delivered to a capacitor is:*

$$\mathcal{E}_C(t) = \int_{-\infty}^{t} v(\tau)\,i(\tau)\,dt = C\int_{-\infty}^{t} v(\tau)\,\frac{dv(\tau)}{d\tau}dt = \frac{1}{2}C\,v(t)^2 \quad (1.28)$$

*then a capacitor is passive only if* $C \geqslant 0$, *otherwise it is active.*

**Example 51** *The energy delivered to an inductor is:*

$$\mathcal{E}_L(t) = \int_{-\infty}^{t} v(\tau)\,i(\tau)\,dt = L\int_{-\infty}^{t} i(\tau)\,\frac{di(\tau)}{d\tau}dt = \frac{1}{2}L\,i(t)^2 \quad (1.29)$$

*then an inductor is passive only if* $L \geqslant 0$, *otherwise it is active.*

**Example 52** *The energy delivered to a resistor is:*

$$\mathcal{E}_R(t) = \int_{-\infty}^{t} v(\tau)\,i(\tau)\,d\tau = R\int_{-\infty}^{t} i(\tau)^2\,d\tau \quad (1.30)$$

*then a resistance is passive only if* $R \geqslant 0$, *otherwise it is active.*

**Lemma 53** *If $\mathcal{N}$ is passive, solvable and $\mathbf{e} \in \mathcal{D}_+^n \cap L_2^n$, then $\mathbf{v}$ and $\mathbf{i}$ belong to $\mathcal{D}_+^n \cap L_2^n$ too, and $\mathcal{E}(\infty) \in \mathrm{I\!R}^+$.*

***Proof.*** *Since the network is solvable then for every $\mathbf{e} \in \mathcal{D}_+^n$ there is a unique pair $(\mathbf{v}, \mathbf{i}) \in \mathcal{N}$ such that $\mathbf{e} = \mathbf{v} + \mathbf{i}$; thus*

$$\int_{-\infty}^{t} \mathbf{e}^T(\tau)\,\mathbf{e}(\tau)\,d\tau = \int_{-\infty}^{t} \mathbf{v}^T(\tau)\,\mathbf{v}(\tau)\,d\tau + \int_{-\infty}^{t} \mathbf{i}^T(\tau)\,\mathbf{i}(\tau)\,d\tau + 2\int_{-\infty}^{t} \mathbf{v}^T(\tau)\,\mathbf{i}(\tau)\,d\tau \tag{1.31}$$

*Supposing that $\mathbf{e} \in \mathcal{D}_+^n \cap L_2^n$ we have that $\|\mathbf{e}\| \in \mathrm{I\!R}^+$, but since $\mathcal{N}$ is passive, the previous equality implies that both $\mathbf{v}$ and $\mathbf{i}$ belong to $L_2^n$ and that $\mathcal{E}(\infty) = \int_{-\infty}^{t} \mathbf{v}^T(\tau)\,\mathbf{i}(\tau)\,d\tau \in \mathrm{I\!R}^+$.* ■

For the following critical result we will omit the proof, see Newcomb (1966) [3].

**Theorem 54** *A passive and solvable network $\mathcal{N}$ is completely solvable.*

These last considerations allow us to define lossless networks; the main idea of the definition is that the energy fed to the augmented network of a lossless network by a square integrable voltage source $\mathbf{e}$, can be dissipated only by the unit resistors. We do not require that the energy absorbed by a lossless network should be zero in every time interval, but only that this energy should vanish at infinity.

We will base our definition upon the solvability and passivity of the network, and in that way, because of the Lemma (53), we guarantee the existence of $\mathcal{E}(\infty)$.

**Definition 55** *$\mathcal{N}$ is lossless if:*

- *$\mathcal{N}$ is passive*

- *$\mathcal{N}$ is solvable*

- *for every* $\mathbf{e} \in \mathcal{D}_+^n \cap L_2^n$, $\mathcal{E}(\infty) = 0$

**Example 56** *The transformer is lossless since it is passive and solvable and $\mathcal{E}(t)$ is always zero.*

**Example 57** *The inductor and the capacitor are both lossless if $C$ and $L$ are non negative.*

**Example 58** *The nullator is not lossless since it is not solvable, while the norator is not lossless since it is not passive.*

**Definition 59** *$\mathcal{N}$ is reciprocal if*

$$\forall (\mathbf{v}_1, \mathbf{i}_1), \ (\mathbf{v}_2, \mathbf{i}_2) \in \mathcal{N}, \qquad \mathbf{v}_1^T * \mathbf{i}_2 = \mathbf{v}_2^T * \mathbf{i}_1 \tag{1.32}$$

*where $*$ means the convolution*[3].

**Remark 60** *Consider two measurements, denoted by the subscripts 1 and 2 a network $\mathcal{N}$ is said to be reciprocal if the voltage responses are independent of an interchange of response and excitation points.*

**Example 61** *The transformer is reciprocal.*

---

[3] Given $(\mathbf{f}, \mathbf{g}) \in \mathcal{D}_+^n \times \mathcal{D}_+^n$, $\mathbf{f} * \mathbf{g} \in \mathcal{D}_+^n$ and is defined by:

$$\mathbf{f} * \mathbf{g}\ (t) = \int_{-\infty}^{\infty} \mathbf{f}^T(t-\tau)\, \mathbf{g}(\tau)\, d\tau$$

whenever it exists.

For a rigorous definition of the convolution see the section below.

**Proposition 62** *A network constituted by resistors, inductors, capacitors and ideal transformers is reciprocal.*

***Proof.*** *For a detailed proof of this proposition see Martinelli (1986)[16]* ■

**Example 63** *The nullator is reciprocal, while the norator is not.*

**Axiom 64** *Every network is causal, i.e. if the application of some variable at time $t_0$ causes other variables to appear at time $t \geqslant t_0$.*

***Remark 65*** *If the network $\mathcal{N}$ is time-invariant, then requiring that the $(\mathbf{v}, \mathbf{i}) \in \mathcal{D}_+^n \times \mathcal{D}_+^n$, we automatically satisfy the axiom of causality. In fact supposing that $\mathbf{v}$ is zero before an instant $\tilde{t}$, then if, by contradiction, $\mathbf{i}$ is non zero before $\tilde{t}$ the time-invariance would be violated by the pair $\left(\mathbf{v}\left(t+\tilde{t}\right), \mathbf{i}\left(t+\tilde{t}\right)\right)$, as $\mathbf{i}\left(t+\tilde{t}\right) \notin \mathcal{D}_+^n$.*

As a final comment to this section we list some relationships between $\mathcal{N}$ and $\mathcal{N}_a$:

- $\mathcal{N}$ is linear $\Rightarrow$ $\mathcal{N}_a$ is linear
- $\mathcal{N}$ is passive $\Rightarrow$ $\mathcal{N}_a$ is passive
- $\mathcal{N}$ is time invariant $\Rightarrow$ $\mathcal{N}_a$ is time invariant
- $\mathcal{N}$ is reciprocal $\Rightarrow$ $\mathcal{N}_a$ is reciprocal
- $\mathcal{N}$ is solvable $\nRightarrow$ $\mathcal{N}_a$ is solvable

## A deeper insight in the Signal Space

In the preceding section an $n-$port network $\mathcal{N}$ was defined as a binary relation on $\mathcal{D}_+^n$. As such a definition is too general, it is not convenient for practical applications.

Thus in this section we will restrict to a particular class of networks, i.e. those linear and completely solvable, characterized as linear and continuous mapping on $\mathcal{D}_{+}^{n}$.

Further we will require our network to fulfil the property of time invariance, so as to represent the previous mapping by a simpler operation: the convolution.

### Mathematical preliminaries

As a preliminary to further discussions, we will now introduce some mathematical elements of the theory of distributions. (An exhaustive development of the theory is given in Cristescu (1973) [4] and Friedlander (1998) [5])

**Definition 66** *We shall denote by $\mathcal{D}(\mathbb{R})$ the space of all real valued functions $\varphi$ of a real variable which have the following property:*

- *$\varphi$ is infinitely continuously differentiable, i.e. $\varphi \in C^{\infty}(\mathbb{R})$*
- *$\varphi$ has compact support.*

These functions will be called *test functions* and from now on the space $\mathcal{D}(\mathbb{R})$ will be denoted by $\mathcal{D}$.

**Example 67** *$\mathcal{D}$ is not empty: indeed consider the function*

$$\varphi(t) = \begin{cases} \exp\left(\dfrac{1}{(t+1)^2}\right)\exp\left(\dfrac{1}{(t-1)^2}\right) & t \in [-1,1] \\ 0 & t \notin [-1,1] \end{cases} \tag{1.33}$$

*This function vanishes outside $[-1,1]$ and it is infinitely continuously differentiable on this interval.*

The space $\mathcal{D}$ is obviously a linear space for the usual operation of addition of functions and multiplication of a function by a scalar; the only topological concept that we will define is the notion of convergence:

**Definition 68** *Let $\langle \varphi_n \rangle$ be a sequence of functions in $\mathcal{D}$, that is $\langle \varphi_n \rangle : \mathbb{N} \to \mathcal{D}$. This sequence is said to converge to $\varphi \in \mathcal{D}$ , and we will write $\varphi_n \to \varphi$, if all the functions $\varphi_n$ have the support in the same bounded set $\Omega$ and if all the derivatives of the functions $\varphi_n$ converge uniformly to the corresponding derivative of $\varphi$ on $\Omega$.*

We now introduce the concept of distribution:

**Definition 69** *A distribution is a generic functional* $\mathrm{F}$: $\mathcal{D} \to \mathbb{R}$, *that satisfies the following conditions:*

- *Linearity:* $\mathrm{F}(\alpha\varphi_1 + \beta\varphi_2) = \alpha\mathrm{F}(\varphi_1) + \beta\mathrm{F}(\varphi_2)$
- *Continuity: if $\varphi_n \to \varphi$ then* $\mathrm{F}(\varphi_n) \to \mathrm{F}(\varphi)$

The space of distributions will be denoted by $\mathcal{D}'$ and obviously it is a linear space for the usual operation of addition of functionals and multiplication of a functional by a real number.

An important notion, which allows for the determination of some classes of distributions which appear in applications, is that of the support of a distribution.

**Definition 70** *A distribution* $\mathrm{F}$ *is said to <u>vanish</u> on an open interval $I$ if* $\mathrm{F}\varphi$ *vanishes for all the $\varphi$'s belonging to $\mathcal{D}$, having support included in $I$. The complement of the union of all such open intervals is called the <u>support</u> of* $\mathrm{F}$ *and denoted by Supp*$\mathrm{F}$.

Sometimes a distribution is formally represented as a generalized function $\mathsf{f}(t)$ introducing the symbolic *scalar product*

$$\mathrm{F}\,(\varphi) = \int_{-\infty}^{\infty} \mathsf{f}(t)\varphi(t)dt \tag{1.34}$$

also denoted by $\langle \mathsf{f}, \varphi \rangle$ .

**Remark 71** *We explicitly remark that equation (1.34) is meaningful only when* $\mathsf{f}(t)$ *is a locally integrable function. Indeed given such a function equation (1.34) rigorously defines a functional belonging to* $\mathcal{D}'$*. However there are elements of* $\mathcal{D}'$ *which cannot be represented in this way, e.g. the* $\delta$ *Dirac distribution. The somehow misleading tradition (stemming from the papers of P.A.M. Dirac) in distribution theory allows for the use of (1.34) also in the more general case.*

**Definition 72** *In what follows we will call* $\mathcal{D}'_+$ *the subspace of* $\mathcal{D}'$ *consisting of distributions having support included in* $\mathbb{R}^+$ *and* $\mathcal{E}'$ *the subspace of* $\mathcal{D}'$ *consisting of distributions having compact supports. Using the identification between functions and distributions given by (1.34) it is easy to see:* $\mathcal{D}_+ \subset \mathcal{D}'_+$ *and* $\mathcal{D} \subset \mathcal{E}'$.

**Example 73** *Consider the distribution of Heaviside* $\mathrm{U}$ *(called in Engineering Literature step distribution) defined as*

$$\mathrm{U}\,(\varphi) = \int_{0}^{\infty} \varphi(t)dt \tag{1.35}$$

*This distribution is representable as the locally integrable function* $\mathsf{u}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geqslant 0 \end{cases}$ *, since* $\mathrm{U}(\varphi) = \langle \mathsf{u}, \varphi \rangle$

**Example 74** *Consider the distribution of Dirac $\delta$ defined as:*

$$\delta(\varphi) = \varphi(0) = \langle \delta, \varphi \rangle \tag{1.36}$$

*Clearly this distribution cannot be written as a scalar product of two functions, i.e. $\delta(t)$ is not a function.*

**Definition 75** *A sequence of distribution $\langle \mathrm{F}_n \rangle$ is said to converge to a distribution $\mathrm{F}$, written $\mathrm{F}_n \to \mathrm{F}$ or $\mathsf{f}_n \to \mathsf{f}$ if:*

$$\mathrm{F}_n(\varphi) \to \mathrm{F}(\varphi) \quad \text{or } \langle \mathsf{f}_n, \varphi \rangle \to \langle \mathsf{f}, \varphi \rangle \qquad \forall \varphi \in \mathcal{D} \tag{1.37}$$

Now we introduce a fundamental theorem which enables us to define distributions as limits in a distributional sense of functions belonging to $\mathcal{D}$.

**Theorem 76** *For every distribution $\mathrm{F}$ there exists a sequence $\langle \varphi_n \rangle : \mathbb{N} \to \mathcal{D}$ such that, distributionally:*

$$\mathrm{F}_n \to \varphi \qquad \text{or} \qquad \mathsf{f} = \lim_{n\to\infty} \varphi_n$$

**Definition 77** *Let $\mathrm{F}$ be a distribution then we define* <u>*weak*</u> *derivative of $\mathrm{F}$ the distribution $\mathrm{F}'$ such that:*

$$\forall\ \varphi \in \mathcal{D} \qquad \mathrm{F}'(\varphi) = -\mathrm{F}\left(\frac{d\varphi}{dt}\right) \tag{1.38}$$

**Remark 78** *Note that for every $\mathrm{F}$ represented by a differentiable function we have:*

$$\mathrm{F}'(\varphi) = -\mathrm{F}\left(\frac{d\varphi}{dt}\right) = -\left\langle \mathsf{f}, \varphi' \right\rangle = -\int_{-\infty}^{\infty} \mathsf{f}(t)\,\varphi'(t)\,dt \tag{1.39}$$

*Integrating by parts, when $\mathsf{f}(t)$ is a (strongly) differentiable function, we get:*

$$-\int_{-\infty}^{\infty} \mathsf{f}(t)\,\varphi'(t)\,dt = \mathsf{f}\varphi|_{-\infty}^{+\infty} + \int_{-\infty}^{\infty} \mathsf{f}'(t)\,\varphi(t)\,dt = \left\langle \mathsf{f}', \varphi \right\rangle \tag{1.40}$$

*so that the distributional derivative of* $\mathrm{F}$ *coincides with the first (strong) derivative of* $\mathsf{f}$*. Roughly speaking the definition of derivative for a distribution is the natural extension of the definition of derivative for a "true" function.*

**Example 79** *Consider the derivative of the Heaviside distribution:*

$$\mathrm{U}\prime(\varphi) = -\mathrm{F}\left(\frac{d\varphi}{dt}\right) = -\int_{-\infty}^{\infty} \mathsf{u}(t)\,\varphi'(t)\,dt = -\int_{0}^{\infty} \varphi'(t)\,dt = \varphi(0) = \delta(\varphi) \tag{1.41}$$

*then the derivative of the Heaviside function is the distribution of Dirac* $\delta$*.*

**Definition 80** *Let* $\mathrm{F}$ *be a distribution then we define the* $l-th$ *<u>weak</u> derivative of* $\mathrm{F}$ *the distribution* $D^l\mathrm{F}$ *such that:*

$$\forall\ \varphi \in \mathcal{D} \qquad D^l\mathrm{F}(\varphi) = (-1)^l\,\mathrm{F}\left(\frac{d^l\varphi}{dt^l}\right) \tag{1.42}$$

Up to now we have considered test functions of only one real variable, but these concepts can be easily generalized to real functions on $\mathbb{R}^n$ and thus define a linear space $\mathcal{D}(\mathbb{R}^n)$ of test functions $\varphi : \mathbb{R}^n \to \mathbb{R}$.

In this generalized framework a distribution will be a continuous linear functional of $\mathcal{D}(\mathbb{R}^n)$ and the space of distributions is a linear space denoted by $\mathcal{D}(\mathbb{R}^n)'$

Further it is again possible to introduce the formal scalar product defined as:

$$\mathrm{F}(\varphi) = \int_{\mathbb{R}^n} \mathsf{f}(t_{1,}...,t_n)\,\varphi(t_{1,}...,t_n)\,dt_1...dt_n = \langle \mathsf{f}(t_1,...,t_n)\,,\ \varphi(t_1,...,t_n)\rangle \tag{1.43}$$

where $\mathsf{f}(t_{1,}...,t_n)$ is again called generalized function.

This extension will become now very useful in the discussion of mapping of the space of test functions $\mathcal{D}$ into the space of distributions $\mathcal{D}'$ since we will discover that such a mapping is represented, under particular hypothesis, by a distribution of two variables.

**Definition 81** *Given two functions $f$ and $g$ belonging to $\mathcal{D}$ the tensor product $f \otimes g$ is a function of $\mathcal{D}\left(\mathbb{R}^2\right)$ defined as follows:*

$$f \otimes g\ (x, y) = f\,(x)\, g\ (y) \qquad (x, y) \in \mathbb{R}^2 \tag{1.44}$$

Before stating an amazing theorem called *Kernel Theorem* due to L.Schwartz, let we define a continuous linear mapping of $\mathcal{D}$ into $\mathcal{D}'$:

**Definition 82** $\mathfrak{G} : \mathcal{D} \to \mathcal{D}'$ *is a continuous linear mapping if:*

- $\mathfrak{G}\left(\alpha\varphi_1 + \beta\varphi_2\right) = \alpha\mathfrak{G}\left(\varphi_1\right) + \beta\mathfrak{G}\left(\varphi_2\right)$
- $\varphi_n \to \varphi$ *then* $\mathfrak{G}\left(\varphi_n\right) \to \mathfrak{G}\left(\varphi\right)$

**Theorem 83** *If $\mathfrak{G}$ is a continuous linear mapping of $\mathcal{D}$ into $\mathcal{D}'$, then there exists a distribution $\mathrm{G} \in \mathcal{D}\left(\mathbb{R}^2\right)'$ such that:*

$$\forall\varphi, \psi \in \mathcal{D} \quad \left(\mathfrak{G}\left(\varphi\right)\right)\psi = \mathrm{G}\left(\varphi \otimes \psi\right), \qquad i.e.\ \forall\varphi \in \mathcal{D} \qquad \mathfrak{G}\left(\varphi\right) = \left\langle \mathsf{g}\left(t, \tau\right),\ \varphi\left(\tau\right)\right\rangle \tag{1.45}$$

Generally the distribution $\mathrm{G}$ is called *distributional Kernel* and the scalar product $\left\langle \mathsf{g}\left(t, \tau\right),\ \varphi\left(\tau\right)\right\rangle$ denoted by $\mathrm{G}{\cdot}\varphi$.

This theorem establishes an *isomorphism* between the space of distributions in $\mathcal{D}\left(\mathbb{R}^2\right)'$ and the space of linear and continuous mapping of $\mathcal{D}$ into $\mathcal{D}'$.

This theorem has a number of additional variations, wherein the domain and the codomain for the mapping may be varied; usually these space are required to contain $\mathcal{D}$ as a subset and be subset of $\mathcal{D}'$, for more details see Treves (1967) [6].

One of the basic operation of the theory of distributions is the tensor product, which is the natural extension of the tensor product of "true" functions.

All results mentioned in what follows about tensor product and convolution of pairs of distributions can be found in Friedlander (1998) [5] and Schwartz (1978) [8].

**Definition 84** *Given two distributions* $\mathrm{G}$ *and* $\mathrm{F}$ *belonging to* $\mathcal{D}'$*, then the tensor product* $\mathrm{G}\otimes\mathrm{F}$ *is a distribution of* $\mathcal{D}\left(\mathbb{R}^2\right)'$ *such that:*

$$\forall\varphi,\psi\in\mathcal{D}\qquad \mathrm{G}\otimes\mathrm{F}\ (\varphi\otimes\psi)=\mathrm{G}\,(\varphi)\ \mathrm{F}\,(\psi) \tag{1.46}$$

**Theorem 85** *For every pair of distributions* $\mathrm{G}$ *and* $\mathrm{F}$ *in* $\mathcal{D}'$ *there exists a unique tensor product* $\mathrm{G}\otimes\mathrm{F}$*. It can be computed as:*

$$\forall\varphi\in\mathcal{D}\left(\mathbb{R}^2\right)\qquad \mathrm{G}\otimes\mathrm{F}\ (\varphi\,(t,\tau))=\mathrm{G}_t\ (\mathrm{F}_\tau\,(\varphi\,(t,\tau))) \tag{1.47}$$

*where* $\mathrm{F}_\tau$ *means that* $\mathrm{F}$ *operates on* $\varphi$ *as a function of* $\tau$*, while* $\mathrm{G}_t$ *means that* $\mathrm{G}$ *operates on* $\mathrm{F}_\tau\,(\varphi\,(t,\tau))$ *as a function of* $t$*. Further:*

$$Supp\mathrm{G}\otimes\mathrm{F}=Supp\mathrm{G}\times Supp\mathrm{F} \tag{1.48}$$

*and the tensor product is separately continuous bilinear form on* $\mathcal{D}'\times\mathcal{D}'$*.*

Now we digress to an application that will become very useful in the theory of networks:

**Definition 86** *Let* $\mathrm{F}\in\mathcal{D}'$*, and* $\tau\in\mathbb{R}$*. The translate* $\mathfrak{T}_\tau\mathrm{F}$ *of* $\mathrm{F}$ *is the distribution such that*

$$\forall\varphi\in\ \mathcal{D}\qquad \mathfrak{T}_\tau\,(\mathrm{F}\ (\varphi\,(t)))=\mathrm{F}_t\ (\varphi\,(t+\tau)) \tag{1.49}$$

*where* $\mathrm{F}_t$ *means that* $\mathrm{F}$ *operates on* $\varphi$ *as a function of* $t$*.*

**Remark 87** *Note that $\varphi(t+\tau)$ is the translate function $\varphi$ translated by a quantity $-\tau$.*

***Definition 88*** *A linear mapping $\mathfrak{G}$ of $\mathcal{D}'$ into $\mathcal{D}'$ is called translation invariant if:*

$$\forall \mathrm{F} \in \mathcal{D}', \ \forall \tau \in \mathbb{R} \qquad \mathfrak{G}\left(\mathfrak{T}_\tau(\mathrm{F})\right) = \mathfrak{T}_\tau\left(\mathfrak{G}(\mathrm{F})\right) \tag{1.50}$$

Another important concept that we will need in the following is the notion of convolution.

**Definition 89** *Let $\mathrm{G} \in \mathcal{E}'$ and $\mathrm{F} \in \mathcal{D}'$, the convolution $\mathrm{G}*\mathrm{F}$ is defined by:*

$$\forall \varphi \in \mathcal{D} \qquad \mathrm{G} * \mathrm{F}\ (\varphi) = \mathrm{G} \otimes \mathrm{F}\ \left(\rho(t)\,\varphi(t+\tau)\right) \tag{1.51}$$

*where $\rho \in \mathcal{D}$ is a cut-off function such that $\rho = 1$ on a neighborhood of the support of $\mathrm{G}$.*

**Remark 90** *(Lemma of Uryson) If $K \subset \mathbb{R}^n$ is a compact set, and $X$ is an open neighborhood of $K$. Then there $\rho \in \mathcal{D}$, such that $Supp\rho \subset X$, $\rho \in [0,1]$ and $\rho = 1$ on a neighborhood of $K$. Such function will be called cut-off function.*

**Remark 91** *The definition of convolution is independent of the choice of cut-off function. For, if $\sigma \in \mathcal{D}$ and $\sigma = 1$ on a neighborhood of the support of $\mathrm{G}$ then:*

$$\mathrm{G} \otimes \mathrm{F}\ \left(\rho(t)\,\varphi(t+\tau)\right) - \mathrm{G} \otimes \mathrm{F}\ \left(\sigma(t)\,\varphi(t+\tau)\right) = \mathrm{G} \otimes \mathrm{F}\ \left(\left(\rho(t)-\sigma(t)\right)\varphi(t+\tau)\right) = 0 \tag{1.52}$$

*since*

$$Supp\left(\left(\rho(t)-\sigma(t)\right)\varphi(t+\tau)\right) \subset \mathbb{R} - Supp\mathrm{G} \tag{1.53}$$

*which implies that*

$$Supp\left(\left(\rho(t)-\sigma(t)\right)\varphi(t+\tau)\right) \subset \mathbb{R}^2 - Supp\mathrm{G} \times Supp\mathrm{F} \tag{1.54}$$

**Definition 92** ***Remark 93*** *The introduction of the cut-off function is necessary to assure that* $\mathrm{F}_\tau\left(\rho(t)\varphi(t+\tau)\right)$ *is a test function.*

**Theorem 94** *Let* G$\in \mathcal{E}'$ *and* F$\in \mathcal{D}'$. *Then the convolution* G$*$F *defines a distribution in* $\mathcal{D}'$ *that can be computed as follows:*

$$\forall\varphi\in\ \mathcal{D} \qquad \mathrm{G}*\mathrm{F}\ (\varphi)=\mathrm{G}_t\ \left(\rho(t)\,\mathrm{F}_\tau\ (\varphi(t+\tau))\right) \tag{1.55}$$

*Furthermore convolution is commutative:* G$*$F=F$*$G.

Convolution is well defined for pairs of distributions G, F other than G$\in \mathcal{E}'$ and F$\in \mathcal{D}'$. In particular it can be extended to distributions G, F$\in \mathcal{D}'_+$. This extension will be dealt with the following considerations, since in the theory of networks we need to make the convolution between distributions in $\mathcal{D}'_+$.

In order to ensure the existence of the convolution of two distributions, we can require these distributions to fulfil the following condition.

**Condition 95** *Let* G$\in \mathcal{D}'$ *and* F$\in \mathcal{D}'$. *Then the condition can be stated as follows:*

$$\forall\delta>0 \qquad \begin{cases} (t,\tau)\in Supp\mathrm{G}\times Supp\mathrm{F} \\ \qquad |t+\tau|<\delta \end{cases} \Rightarrow \exists\delta'>0:|t|<\delta',\ \ |\tau|<\delta' \tag{1.56}$$

*that is:*

$$\forall\delta>0 \qquad (Supp\mathrm{G}\times Supp\mathrm{F})\cap\left\{(t,\tau)\in\mathbb{R}^2:|t+\tau|<\delta\right\}\ \textit{is a compact set.} \tag{1.57}$$

**Theorem 96** *Let* G$\in \mathcal{D}'$ *and* F$\in \mathcal{D}'$, *and let their supports satisfy condition (95) then the convolution* G$*$F *is the distribution defined by:*

$$\forall\varphi\in\ \mathcal{D} \qquad \mathrm{G}*\mathrm{F}\ (\varphi)=\mathrm{G}\otimes\mathrm{F}\ \left(\rho_\varphi(t,\tau)\,\varphi(t+\tau)\right) \tag{1.58}$$

*where* $\rho_\varphi \in \mathcal{D}(\mathbb{R}^2)$ *is a cut-off function such that* $\rho = 1$ *on a neighborhood of* $I$*, defined as:*

$$I = \{(t, \tau) \in Supp\mathrm{G} \times Supp\mathrm{F} : t + \tau \in Supp\varphi\}. \tag{1.59}$$

***Remark 97*** *The previous extension of the convolution is independent of the choice of cut-off function.*

**Theorem 98** *Let* G$\in \mathcal{D}'$ *and* F$\in \mathcal{D}'$ *satisfying condition (95) then the convolution* G$*$F *can be computed by:*

$$\forall\varphi \in \ \mathcal{D} \qquad \mathrm{G} * \mathrm{F}\ (\varphi) = \mathrm{G}_t\ \left(\mathrm{F}_\tau\ \left(\rho_\varphi\,(t, \tau)\,\varphi\,(t + \tau)\right)\right) \tag{1.60}$$

*where* $\rho_\varphi$ *is defined as in Theorem (96). Further the convolution is commutative and*

$$Supp\mathrm{G} * \mathrm{F} \subset Supp\mathrm{G} \cup Supp\mathrm{F} \tag{1.61}$$

***Remark 99*** *If* G$\in \mathcal{D}'_+$ *and* F$\in \mathcal{D}'_+$ *then they satisfy (95). For, since Supp*G$\subset \mathbb{R}^+$*, Supp*F$\subset \mathbb{R}^+$ *then:*

$$\begin{cases} (t, \tau) \in Supp\mathrm{G} \times Suppf \\ |t + \tau| < \delta \end{cases} \Longrightarrow \begin{cases} (t, \tau) \in Supp\mathrm{G} \times Supp\mathrm{F} \\ t + \tau < \delta \end{cases} \Rightarrow |t| < \delta,\ \ |\tau| < \delta. \tag{1.62}$$

*then the convolution of two distributions in* $\mathcal{D}'_+$ *is defined, and since Supp*G$*$F$\subset$ *Supp*G$\cup$*Supp*F *then it is a distribution in* $\mathcal{D}'_+$.

***Remark 100*** *The convolution of a* $\delta$ *Dirac with a generic distribution leaves this last unchanged:*

$$\forall\mathrm{F} \in \mathcal{D}' \qquad \delta * \mathrm{F} = \mathrm{F} \tag{1.63}$$

Now we are ready to introduce a very powerful result due to Schwartz, which will enable us to characterize the behavior of a particular class of network by the well known *impulsive response.*

**Theorem 101** *If $\mathfrak{G}$ is a continuous translation invariant, linear mapping of $\mathcal{D}$ into $\mathcal{D}'$ then:*

$$\forall \varphi \in \mathcal{D} \qquad \mathfrak{G}(\varphi) = \mathfrak{G}(\delta) * \varphi \tag{1.64}$$

**Remark 102** *Eq. (1.64) is generally used for representing the solutions of differential equations in terms of generalized "Green" function.*

**Corollary 103** *If $\mathfrak{G}$ is a continuous translation invariant linear mapping of $\mathcal{D}'_+$ into $\mathcal{D}'_+$ then there exists a unique distribution* G *in $\mathcal{D}'_+$, such that:*

$$\forall \mathrm{F} \in \mathcal{D}'_+ \qquad \mathfrak{G}(\mathrm{F}) = \mathrm{G} * \mathrm{F} \tag{1.65}$$

The proof of this theorem and of the corollary can be found in Beltrami (1966) [7].

Up to now we have restricted our study to real valued functions, anyway these results can be generalized to column $n-$vector function in a quite straightforward way; we will use these results dealing with $n-$port networks.

We will also use the previous theorem in studying mechanical systems.

### Time domain representation of linear, completely solvable and time-invariant network.

In this section we will exploit the fundamental results of the theory of distribution mentioned before so as to characterize a linear, completely solvable and time-invariant $n-$port network by a representation amenable to convenient and fruitful analysis.

**Definition 104** *Since $\mathcal{N}$ is solvable then* $\forall\ \mathbf{e} \in \mathcal{D}_{+}^{n}\ \exists!\ (\mathbf{v}, \mathbf{i}) \in \mathcal{N}$ *such that* $\mathbf{e} = \mathbf{v} + \mathbf{i}$*; let me call* $\mathfrak{Y}_a$ *the mapping of* $\mathcal{D}_{+}^{n}$ *into* $\mathcal{D}_{+}^{n}$ *such that* $\mathbf{i} = \mathfrak{Y}_a(\mathbf{e})$.

**Proposition 105** *Since $\mathcal{N}$ is linear then $\mathcal{N}_a$ is linear, and $\mathfrak{Y}_a$ is a linear mapping of $\mathcal{D}_{+}^{n}$ into $\mathcal{D}_{+}^{n}$.*

***Proof.*** *In fact, thanks to the linearity of $\mathcal{N}_a$:*

$$\forall \mathbf{e}_1,\ \mathbf{e}_2 \in \mathcal{D}_{+}^{n},\ \alpha,\ \beta \in \mathbb{R} \qquad (\alpha(\mathbf{v}_1+\mathbf{i}_1) + \beta(\mathbf{v}_2+\mathbf{i}_2), \alpha\mathbf{i}_1 + \beta\mathbf{i}_2) \in \mathcal{N}_a \tag{1.66}$$

*thus*

$$\mathfrak{Y}_a(\alpha\mathbf{e}_1 + \beta\mathbf{e}_2) = \alpha\mathbf{i}_1 + \beta\mathbf{i}_2 = \alpha\mathfrak{Y}_a(\mathbf{e}_1) + \beta\mathfrak{Y}_a(\mathbf{e}_2) \tag{1.67}$$

■

**Proposition 106** *Further, since $\mathcal{N}$ is completely solvable then $\mathcal{N}_a$ is completely solvable and $\mathfrak{Y}_a$ is continuous.*

***Proof.*** *In fact:*

$$\mathbf{e}_n \to \mathbf{e} \Rightarrow (\mathbf{v}_n+\mathbf{i}_n, \mathbf{i}_n) \to (\mathbf{v} + \mathbf{i}, \mathbf{i}) \tag{1.68}$$

*thus*

$$\mathbf{e}_n \to \mathbf{e} \Rightarrow \mathbf{i}_n \to \mathbf{i} \Rightarrow \mathfrak{Y}_a(\mathbf{e}_n) \to \mathfrak{Y}_a(\mathbf{e}) \tag{1.69}$$

■

Thus a linear completely solvable network $\mathcal{N}$ defines a linear continuous mapping of $\mathcal{D}_{+}^{n}$ into $\mathcal{D}_{+}^{n}$.

**Proposition 107** *Further, since the network $\mathcal{N}$ is time-invariant then $\mathcal{N}_a$ is time-invariant and $\mathfrak{Y}_a$ is a translation invariant mapping.*

***Proof.*** *In fact:*

$$\forall \mathbf{e} \in \mathcal{D}_+^n \qquad \mathfrak{Y}_a\left(\mathfrak{T}_T\left(\mathbf{e}\right)\right) = \mathfrak{T}_T\left(\mathfrak{Y}_a\left(\mathbf{e}\right)\right) \tag{1.70}$$

■

Thus $\mathfrak{Y}_a$ is a linear, continuous translation invariant mapping of $\mathcal{D}_+^n$ into $\mathcal{D}_+^n$; since $\mathcal{D}_+^n$ is a subset of $\mathcal{D}_+^{n\prime}$ we can use an analog of Corollary (103) extended to $\mathcal{D}_+^{n\prime}$,to state that:

**Proposition 108** *For a linear, completely solvable and time invariant $n-$port network there exists a unique distribution $\mathrm{Y}_a$ in $\mathcal{D}_+^{n\prime}$, such that:*

$$\forall \mathbf{e} \in \mathcal{D}_+^n \qquad \mathbf{i} = \mathrm{Y}_a * \mathbf{e} \tag{1.71}$$

*and since $\mathbf{e} = \mathbf{i} + \mathbf{v}$, then:*

$$\mathbf{v} = -\mathrm{Y}_a * \mathbf{e} + \mathbf{e} = \left(\delta - \mathrm{Y}_a\right) * \mathbf{e} \tag{1.72}$$

*where $\delta$ is the Dirac distribution in $\mathcal{D}_+^{n\prime}$.*

**Definition 109** $\mathrm{Y}_a$ *will be called the* <u>*admittance*</u> *of the augmented network $\mathcal{N}_a$.*

In the particular case of a network $\mathcal{N}$ linear, passive and completely solvable the current $\mathbf{i}$ of the network is related to the voltage source $\mathbf{e}$ by a convolution between a distribution $\mathrm{Y}_a$ ,depending on the characteristics of $\mathcal{N}$, and the voltage source $\mathbf{e}$.

Roughly speaking we can have a physical interpretation of $\mathrm{Y}_a$ as the matrix $n \times n$ whose generic entry $\mathrm{Y}_a\left(t\right)\big|_{i,j}$ is the current response at port $i$ of the augmented network at time $t$ to a $\delta$ voltage applied at port $k$ at the instant zero. This interpretation is clearly not rigorous since we have restricted the space of our admissible input and output to $\mathcal{D}_+^n$, and $\delta$ is clearly outside this space.

**Remark 110** *As a remark we claim that, because of prop. (108) and Theorem (54) the representation (1.71) holds also for a linear, solvable, passive and time invariant network because of Theorem (54).*

### Frequency domain representation of linear, completely solvable, time invariant $n$-port networks

From now on we will restrict our interest to the particular class of $n-$port networks linear, passive and completely solvable.

In order to characterize in the so called "frequency domain" the response of networks, we need to introduce Laplace bilateral transform.

However it is known (see Beltrami (1966)[7]) that the Laplace transform cannot be defined in all $\mathcal{D}'_+$. Therefore when a Laplace transform will be needed we will restrict the consideration to the signals belonging to a subset of $\mathcal{D}'_+$ to be specified .

**Definition 111** *Let $\mathcal{S}$ be the subset of $C^\infty$ of tempered functions:*

$$\mathcal{S} = \left\{ f \in C^\infty : \exists\, (\alpha, \beta) \in \mathbb{N}^2 \Rightarrow \sup_{t \in \mathbb{R}} \left| t^\alpha \frac{d^\beta f}{dt^\beta} \right| \in \mathbb{R} \right\} \tag{1.73}$$

**Definition 112** *The sequence $\langle f_n \rangle$ of elements in $\mathcal{S}$ is said to converge to $f \in \mathcal{S}$ if and only if:*

$$\forall\, (\alpha, \beta) \in \mathbb{N}^2 \qquad \lim_{n \to +\infty} \left| t^\alpha \frac{d^\beta (f - f_n)}{dt^\beta} \right| = 0 \tag{1.74}$$

**Definition 113** *Let $C_0^{slow}$ be the set of continuous functions of polynomial growth, i.e.*

$$C_0^{slow} := \left\{ \lambda \in C^0 : \exists \left(M, C\right) \in \mathbb{R}^+ \times \mathbb{R}^+ \Rightarrow \left|\lambda\left(t\right)\right| < C\left(1+t^2\right)^M \right\} \tag{1.75}$$

*then the set of tempered distributions is defined as:*

$$\mathcal{S}' := \left\{ \mathrm{F} \in \mathcal{D}' : \exists \lambda \in C_0^{slow},\ l \in \mathbb{N} \Rightarrow D^l \lambda = \mathrm{F} \right\} \tag{1.76}$$

**Remark 114** *There are distributions which do not belong to $\mathcal{S}'$. For instance the function $e^t$, when regarded as a distribution, does not belong to $\mathcal{S}'$. Indeed all its primitives do not belong to $C_0^{slow}$.*

The following Theorem is due to Schwartz:

**Theorem 115** *(Theorem of Structure) $\mathcal{S}'$ coincides with the set of linear continuous functionals defined on $\mathcal{S}$.*

Now on we will limit our attention to a particular class of distribution, those for which a bilateral Laplace transform can be defined.

In what follows we will need to regard a distribution as a functional defined on the set of complex valued test functions $\mathcal{D}_c$.

**Definition 116** *Let $\mathrm{F} \in \mathcal{D}'$, $\varphi \in \mathcal{D}_c$ then*

$$\mathrm{F}\left(\varphi\right) = \mathrm{F}\left(\mathrm{Re}\left[\varphi\right]\right) + i\mathrm{F}\left(\mathrm{Im}\left[\varphi\right]\right) \tag{1.77}$$

**Definition 117** *Let $\mathrm{F} \in \mathcal{D}'$, $g \in C^\infty$ then $\mathrm{F}\, g$ is the distribution in $\mathcal{D}'$ defined by:*

$$\forall \varphi \in \mathcal{D} \qquad \mathrm{F}\, g\ \left(\varphi\right) = \mathrm{F}\ \left(g\, \varphi\right) \tag{1.78}$$

The following considerations allow for the definition of Laplace transform:

**Definition 118** *Let* $\mathrm{F} \in \mathcal{D}'$ *and let* $s = \sigma + i\omega$. *We will call damping region, the set:*

$$\Gamma_{\mathrm{F}} := \left\{\sigma \in \mathbb{R} : \mathrm{F}\, e^{-\sigma t} \in \mathcal{S}'\right\} \tag{1.79}$$

**Lemma 119** $\Gamma_{\mathrm{F}}$ *is an interval. If* $\mathrm{F} \in \mathcal{D}'_{+}$ *then* $\Gamma_{\mathrm{F}}$ *is a semiinfinite interval included in* $\mathbb{R}^{+}$.

**Definition 120** $\Gamma^{c}_{\mathrm{F}} = \{s \in \mathbb{C} : \mathrm{Re}\,[s] \in \Gamma_{\mathrm{F}}\}$

**Theorem 121** *Let* $\mathrm{F} \in \mathcal{D}'_{+}$, $\Gamma_{\mathrm{F}} = (\sigma_0, \infty)$ *and* $\rho \in \mathcal{D}_{+}$ *such that* $\rho\,(t) = 1$ *if* $t \in Supp\mathrm{F}$. *Then for every* $s \in \Gamma^{c}_{\mathrm{F}}$, $s = \sigma + i\omega$:

- $\exists \sigma_1 \in \mathbb{R}^{+}$, $\sigma > \sigma_1 > \sigma_0 \Rightarrow \mathrm{F}e^{-\sigma_1 t} \in \mathcal{S}'$, $\rho\,(t)\, e^{-(s-\sigma_1)t} \in \mathcal{S}$
- $\left(\mathrm{F}e^{-\sigma_1 t}\right)\left(\rho\,(t)\, e^{-(s-\sigma_1)t}\right) =: \mathcal{L}\,[\mathrm{F}]\,(s)$ *independently of the choice of* $\sigma_1$ *and* $\rho$.
- *The function defined in* $\Gamma^{c}_{\mathrm{F}}$, $s \longmapsto \mathcal{L}\,[\mathrm{F}]\,(s)$ *is holomorphic and is called Laplace transform of* $\mathrm{F}$.
- *For every* $m \in \mathbb{R}$ $\qquad \mathcal{L}\left[(-t)^m\, \mathrm{F}\right] = \dfrac{d^m \mathcal{L}\,[\mathrm{F}]}{ds^m}$

Now we introduce some fundamental results in the theory of Laplace transform distributions, for a detailed analysis of these properties see Schwartz (1966) [9]:

**Theorem 122** *Let* $\mathrm{F}$ *and* $\mathrm{G}$ *be two distributions in* $\mathcal{D}'_{+}$, *and let* $\Gamma_{\mathrm{F}}$ *and* $\Gamma_{\mathrm{C}}$ *be their damping region. Then* $\mathrm{F}*\mathrm{G}$ *has a Laplace transform in* $\Gamma_{\mathrm{F}} \cap \Gamma_{\mathrm{C}}$ *and:*

$$\mathcal{L}\,[\mathrm{F} * \mathrm{G}] = \mathcal{L}\,[\mathrm{F}]\,\mathcal{L}\,[\mathrm{G}] \tag{1.80}$$

***Corollary 123*** $\mathcal{L}\left[D^l \mathrm{F}\right] = s^l \mathcal{L}\left[\mathrm{F}\right]$

Let us turn again our attention to the linear, completely solvable and time invariant $n-$port network considered in prop.(108).

Laplace transform of eq.(1.71) leads to:

$$\mathbf{I}(s) = \mathbf{Y}_a(s)\,\mathbf{E}(s) \tag{1.81}$$

where $\mathbf{I}(s) = \mathcal{L}[\mathbf{i}](s)$, $\mathbf{Y}_a(s) = \mathcal{L}[\mathrm{Y}_a](s)$, $\mathbf{E}(s) = \mathcal{L}[\mathbf{e}](s)$.

Repeating the same reasonings as eq.(1.71), we get:

$$\mathbf{V}(s) = (\mathbf{1} - \mathbf{Y}_a(s))\,\mathbf{E}(s) \tag{1.82}$$

where $\mathbf{V}(s) = \mathcal{L}[\mathbf{v}](s)$.

**Remark 124** *It has been decided to characterize a network $\mathcal{N}$ by means of the admittance of its augmented network. This is done to guarantee the universal frequency domain representation of the network. Instead the alternative possible choices presented in what follows, i.e. to introduce for characterizing $\mathcal{N}$, its own immittance or transmission matrices do not have general validity. Some networks which deserve attention could be singular, i.e. do not admit one of the aforementioned representations.*

Thus in the frequency domain a network is represented by the admittance matrix $\mathbf{Y}_a(s)$; in the following when it will be possible without misunderstandings we will drop the variables when indicating $\mathbf{I}$ and $\mathbf{V}$, i.e. the (Laplace transform) current and voltage respectively.

Now we are ready to introduce some of the most used and fruitful matrix descriptions of $\mathcal{N}$ in the frequency domain.

**Definition 125** *Let us start with the impedance Matrix* $\mathbf{Z}(s)$ :

$$\begin{aligned} \mathbf{V} &= \mathbf{Z}(s)\,\mathbf{I} \\ \mathbf{Z}(s) &= (\mathbf{1} - \mathbf{Y}_a(s))\,\mathbf{Y}_a(s)^{-1} \end{aligned} \tag{1.83}$$

*The generic entry* $\mathbf{Z}(s)|_{i,j}$ *represents the ratio between the voltage at port* $i$ *and the current applied at port* $j$ *when all the other ports are open-circuited:* $\mathbf{Z}(s)|_{i,j} = \left.\frac{V_i}{I_j}\right|_{I_k=0,\ k\neq j}$.

The terms on the diagonal of $\mathbf{Z}$ will be called *driving impedances*, while the others will be called *transfer impedance.*

**Example 126** *The impedance of an inductor is*

$$Z_L(s) = sL$$

**Example 127** *The impedance of a resistor is*

$$Z_R(s) = R$$

**Example 128** *The impedance of a capacitor is*

$$Z_C(s) = \frac{1}{sC}$$

**Remark 129** *Note that it is not always possible to describe a network by an impedance matrix, since* $\mathbf{Y}_a(s)$ *can be singular.*

**Definition 130** *Another common matrix used to describe a network is the Admittance Matrix* $\mathbf{Y}(s)$ *:*

$$\begin{aligned} \mathbf{I} &= \mathbf{Y}(s)\,\mathbf{V} \\ \mathbf{Y}(s) &= \mathbf{Y}_a(s)\,(\mathbf{1}-\mathbf{Y}_a(s))^{-1} \end{aligned} \tag{1.84}$$

*The generic entry* $\left.\mathbf{Y}(s)\right|_{i,j}$ *represents the ratio between the current at port* $i$ *and the voltage applied at port* $j$ *when all the other ports are short-circuited:* $\left.\mathbf{Y}(s)\right|_{i,j} = \left.\frac{I_i}{V_j}\right|_{I_k=0,\ k\neq j}$.

**Remark 131** *Clearly* $\mathbf{Y}$ *is the inverse of* $\mathbf{Z}$*, and its existence is related to the invertibility of* $(\mathbf{1}-\mathbf{Y}_a(s))^{-1}$*, thus it may happen that a network admits the* $\mathbf{Y}$ *representation without admitting the* $\mathbf{Z}$ *representation and viceversa. It can happen also that no one of these two representation is admissible, this being the case of the ideal transformer. Generally* $\mathbf{Y}$ *and* $\mathbf{Z}$ *are both called <u>immitance matrices</u>.*

**Example 132** *The admittance of an inductor is:*

$$Y_L(s) = \frac{1}{sL}$$

**Example 133** *The admittance of a capacitor is:*

$$Y_C(s) = sC$$

**Example 134** *The admittance of a resistor is:*

$$Y_R(s) = \frac{1}{R}$$

The last description matrix that will be used all over this work is the *Transmission Matrix* $\mathbf{T}$.

The use of this description for $\mathcal{N}$ will become very useful in the discussion of the transfer properties of the network, and in development of the so called *cascade connection.*

Assume that the network has an even number of ports $n = 2m$ then we can partition the ports into two sets 1 and 2 such that 1 contains the first $m$ ports and 2 the others, according to this partition we can write:

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_1 \\ \\ \mathbf{V}_2 \end{pmatrix} \text{ and } \mathbf{I} = \begin{pmatrix} \mathbf{I}_1 \\ \\ \mathbf{I}_2 \end{pmatrix} \tag{1.85}$$

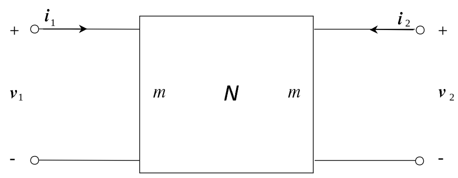


Figure 1.10: Partitioned network

**Definition 135** *The transmission matrix associates to the "input" vector the "output"* *vector, i.e.*[4]*:*

$$\begin{pmatrix} \mathbf{V}_2 \\ \\ -\mathbf{I}_2 \end{pmatrix} = \Upsilon \begin{pmatrix} \mathbf{V}_1 \\ \\ \mathbf{I}_1 \end{pmatrix} \tag{1.86}$$

**Example 136** *The* $(m+m)-$*port transformer has a transmission matrix:*

$$\Upsilon = \begin{pmatrix} \mathbf{T}^T & 0 \\ \\ 0 & \mathbf{T}^{-1} \end{pmatrix}$$

The transmission matrix, sometimes called *Chain Matrix*, can be related to $\mathbf{Y}$ and $\mathbf{Z}$, whenever it is possible, by the following reasoning.

Let me partition the impedance matrix and the transmission matrix into four submatrices $m \times m$ each:

$$\Upsilon = \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} \\ \\ \Upsilon_{21} & \Upsilon_{22} \end{pmatrix} \qquad \mathbf{Z} = \begin{pmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{pmatrix} \tag{1.87}$$

then substituting these two expressions into 125 and 135 we get:

$$\begin{cases} \mathbf{V}_1 = \mathbf{Z}_{11}\mathbf{I}_1 + \mathbf{Z}_{12}\mathbf{I}_2 \\ \\ \mathbf{V}_2 = \mathbf{Z}_{21}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2 \\ \\ \mathbf{V}_2 = \Upsilon_{11}\mathbf{V}_1 + \Upsilon_{12}\mathbf{I}_1 \\ \\ \mathbf{I}_2 = -\Upsilon_{21}\mathbf{V}_1 - \Upsilon_{22}\mathbf{I}_1 \end{cases} \tag{1.88}$$

This system yields:

$$\Upsilon = \begin{pmatrix} -\mathbf{Z}_{12}^{-1} & \mathbf{Z}_{12}^{-1}\mathbf{Z}_{11} \\ \\ \mathbf{Z}_{22}\mathbf{Z}_{12}^{-1} & \mathbf{Z}_{12} - \mathbf{Z}_{22}\mathbf{Z}_{12}^{-1}\mathbf{Z}_{11} \end{pmatrix} \qquad \mathbf{Z} = \begin{pmatrix} -\Upsilon_{21}^{-1}\Upsilon_{22} & -\Upsilon_{21}^{-1} \\ \\ -\Upsilon_{11}\Upsilon_{21}^{-1}\Upsilon_{22} + \Upsilon_{12} & -\Upsilon_{11}\Upsilon_{21}^{-1} \end{pmatrix} \tag{1.89}$$

[4]The minus sign in the following equation is needed in order to account for the different conventions used when introducing immittance and trasmission matrices.

In the first case the current is positive when it enters the network,while in the second case the current is positive when it leaves the network.

Analogous relations can be found between the impedance matrix and the transmission matrices.

**Balanced networks** We have developed all the previous matrix terminology for multiport networks, nevertheless we can extend these ideas also to networks in which pairs of terminal cannot be grouped into ports.

Consider a $n+1$ terminal network $\mathcal{N}$:

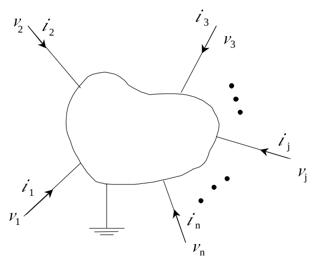


Figure 1.11: N+1-terminal network

where the $(n+1)$th terminal is assumed to act as a reference point for the voltages of the other terminals, and all the currents are assumed to enter the terminals.

In the framework of linear,time invariant and completely solvable networks we can go again through all the steps mentioned dealing with $n-$port networks and establish an isomorphism between $(n+1)$ $-$terminal networks and $n-$port networks.

Sometimes in literature an $(n+1)$ $-$terminal network is called an $n-$port balanced network.

### Interconnection of networks

Usually the problem of the synthesis of a networks is solved regarding it as decomposed into several subnetworks, each of them performing a specific task.

That is, once we have found the matrix representation of the network, instead of trying the immediate synthesis of a corresponding circuit, we decompose the global matrix in terms of simpler matrices and then we turn to the problem of designing a circuit for a set of component subnetworks.

In this section we want to examine the most common techniques of network interconnection, and to understand the relationship between connections and matrix descriptions.

**Series connection** Let us start with the *Series* connection of two $n-$ports $\mathcal{N}_1$ and $\mathcal{N}_2$:

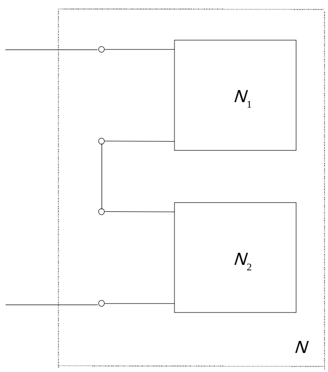


Figure 1.12: Series

In the series connection the terminal $\mathcal{T}_i'$ of $\mathcal{N}_1$ is connected to $\mathcal{T}_i$ of $\mathcal{N}_2$,i.e. all the currents leaving the first networks, enter the other network:

$$\mathbf{I}_1 = \mathbf{I}_2 \tag{1.90}$$

Clearly the impedance matrix of the series connections is equal to the sum of the two impedances matrices:

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 \tag{1.91}$$

**Example 137** *The impedance of the series between an inductor and a capacitor is:*

$$Z(s) = sL + \frac{1}{sC} = \frac{s^2LC + 1}{sC}$$

*while the admittance is:*

$$Y(s) = (Z(s))^{-1} = \frac{sC}{s^2LC + 1}$$

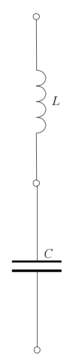


Figure 1.13: Series between an inductor and a capacitor

**Parallel connection** In the *Parallel* connection the terminal $\mathcal{T}_i$ of $\mathcal{N}_1$ is connected to $\mathcal{T}_i$ of $\mathcal{N}_2$, i.e. the two networks have the same voltages:

$$\mathbf{V}_1 = \mathbf{V}_2 \tag{1.92}$$

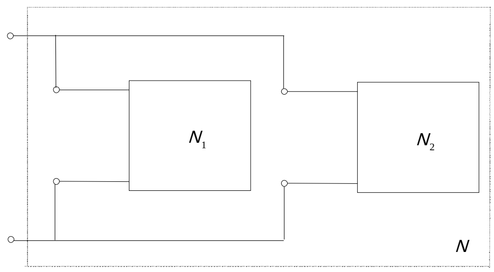


Figure 1.14: Parallel

Obviously the admittance matrix of the parallel connection is equal to the sum of the two admittance matrices:

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 \tag{1.93}$$

**Example 138** *The admittance of the parallel between an inductor and a capacitor is:*

$$Y(s) = sC + \frac{1}{sL} = \frac{s^2LC + 1}{sL}$$

*while the impedance is:*

$$Z(s) = (Y(s))^{-1} = \frac{sL}{s^2LC + 1}$$

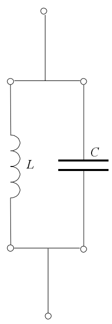


Figure 1.15: Parallel between an inductor and a capacitor

**Cascade connection** Now we have to discuss the *Cascade* connection; this connection represents the interaction of two networks where the second one accepts as "input" the "output" of the first one.

In fact, consider two $2m$ ports $\mathcal{N}_1$ and $\mathcal{N}_2$, and connect the $(m+j)$th port of $\mathcal{N}_1$ to the $j$th of $\mathcal{N}_2$:

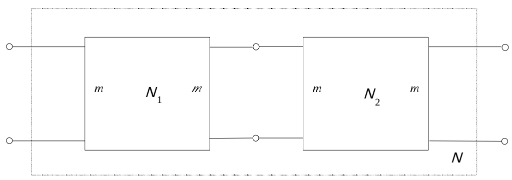


Figure 1.16: Cascade

Clearly the best matrix representation of the cascade connection is the transmission matrix, and that matrix is equal to the product of the transmission matrices of the subnetworks:

$$\Upsilon = \Upsilon_2 \Upsilon_1 \tag{1.94}$$

**Cascade-loaded transformer** Finally we have to discuss the cascade-loaded transformer, this connection will become very useful in the synthesis of finite networks with more than 2 ports. Indeed it will enables us to synthesize a given $\mathbf{Z}$ using a simple circuit (e.g. with diagonal or spherical impedance) to cascade load a multiport transformer.

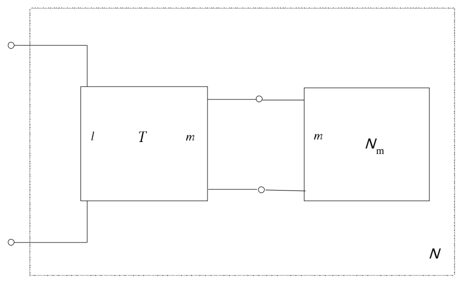


Figure 1.17: Cascade-loaded transformer

In particular given an $(l+m)-$port transformer one can terminate it with an $m-$port network described by an impedance matrix $\mathbf{Z}_m$. The thus obtained loaded transformer is an $l-$port network the $l \times l$ impedance matrix $\mathbf{Z}$ of which is equal to:

$$\mathbf{Z} = \mathbf{T}^T \mathbf{Z}_m \mathbf{T} \tag{1.95}$$

## Realizability considerations

We will characterize the set of impedance and admittance matrices of linear, time invariant, completely solvable and passive $n-$port network.

Then we will discuss the problem of the realizability of a given network, under further specified constraints such as reciprocity, finiteness and losslessness.

The proofs of all theorems mentioned in this subsections can be found in Newcomb (1966)[3].

**Definition 139** *An $n \times n$ matrix $\mathbf{A}(s)$, $s = \sigma + i\omega$, is called positive-real if: (the $*$ denotes complex variable conjugation)*

- $\mathbf{A}(s)$ *is holomorphic in the half plane* $\Theta = \{s \in \mathbb{C} : \sigma > 0\}$
- $\mathbf{A}^*(s) = \mathbf{A}(s^*)$ *in $\Theta$, this implies that* $\mathbf{A}(\sigma + i0)$ *is real in* $\Theta$
- $\mathbf{A}_H(s) = \frac{1}{2}\left(\mathbf{A}(s) + \mathbf{A}^{*T}(s)\right)$ *semidefinite positive in $\Theta$,* $\mathbf{A}_H(s)$ *is called the Hermitian part of* $\mathbf{A}(s)$.

**Theorem 140** *The immittance matrices of a linear solvable, time invariant, passive $n-$port network are positive real, whenever they exist.*

As a remark, we have to underline that for a $1-$port network, usually called *bipole,* the theorem can be stated as follows:

**Corollary 141** *The immittance $A(s)$ of a linear solvable, time invariant, passive one-port network is positive real (whenever it exists), that is:*

- $A(s)$ *is holomorphic in* $\Theta$
- $A^*(s) = A(s^*)$ *in $\Theta$, this implies that* $A(\sigma + i0)$ *is real in* $\Theta$
- $\mathrm{Re}\,[A(s)] \geqslant 0$, *in* $\Theta$

Now we will exhibit and discuss some examples of immittances, which will be used in the synthesis of the networks involved in the electro-mechanical parallel we will consider later.

**Example 142** *Consider the trascendental immittance:*

$$A(s) = A_0 \tanh \gamma s \tag{1.96}$$

*where $A_0$ and $\gamma$ are positive real constants, in particular $A_0$ has the dimension of a resistance if the immittance is an impedance and it has the dimension of the inverse of a resistance if the immittance is an admittance, and $\gamma$ has the dimension of a time. Now let us investigate the conditions of Theorem (141):*

- $A(s) = A_0 \tanh \gamma s = A_0 \dfrac{\sinh \gamma s}{\cosh \gamma s}$ *is singular at the points $p_\alpha$ where $\cosh \gamma s$ vanishes, i.e.:*

$$\cosh \gamma p_\alpha = 0 \Rightarrow \begin{cases} \cos \gamma \omega_\alpha \cosh \gamma \sigma_\alpha = 0 \\ \sin \gamma \omega_\alpha \sinh \gamma \sigma_\alpha = 0 \end{cases} \Rightarrow p_\alpha = 0 + i \frac{\pi}{2\gamma}(2k+1) \qquad k \in \not{Z} \tag{1.97}$$

  *Thus $A(s)$ is holomorphic in $\Theta$, and it is meromorphic in every bounded region of the complex plane.*[5] *In the following $A(s)$ will be called meromorphic.*

- $A^*(s) = A_0 (\tanh \gamma s)^* = A_0 \tanh \gamma s^* = A(s^*)$

- $\mathrm{Re}[A(s)] = \dfrac{A(s) + A(s)^*}{2} = A_0 \dfrac{\sinh \gamma s (\cosh \gamma s)^* + \cosh \gamma s (\sinh \gamma s)^*}{2 |\cosh \gamma s|^2} = A_0 \dfrac{\sinh 2\gamma\sigma}{2 |\cosh \gamma s|^2} \geqslant 0$, *in $\Theta$. Where the last equality holds because of:*

$$\sinh \alpha \cosh \beta + \sinh \beta \cosh \alpha = \sinh(\alpha + \beta)$$

[5] A function of a complex variable is said to be *meromorphic* in a region of the complex plane, if it is holomorphic in the region exept at a finite number of *poles*.

***Remark 143*** *On the imaginary axis*

$$A\left(i\omega\right) = iA_0 \tanh \gamma\omega \tag{1.98}$$

**Example 144** *Consider the trascendental immittance function:*

$$A(s) = s\frac{A_0}{k^3}\frac{\sin k \cosh k - \cos k \sinh k}{(1 + \cos k \cosh k)}$$

*where* $k = \sqrt{\alpha s}\, e^{-\frac{i\pi}{4}}$ *and* $A_0$ *is real positive constant. Let us investigate the conditions of Theorem (141):*

- $A(s)$ *is singular at the points* $p_\beta$*, where* $\left(1 + \cos\left(\sqrt{\alpha s}e^{-\frac{i\pi}{4}}\right)\cosh\left(\sqrt{\alpha s}e^{-\frac{i\pi}{4}}\right)\right)$ *vanishes, since*

$$\lim_{s\to 0} A(s) = 0 \tag{1.99}$$

*It is easy to show that*

$$\left(1 + \cos\left(\sqrt{\alpha s}e^{-\frac{i\pi}{4}}\right)\cosh\left(\sqrt{\alpha s}e^{-\frac{i\pi}{4}}\right)\right) = \frac{\cosh\left(\sqrt{2}\sqrt{\alpha s}\right) + \cos\left(\sqrt{2}\sqrt{\alpha s}\right)}{2} + 1 \tag{1.100}$$

*Thus the points* $p_\beta$ *can be found from:*

$$\cosh\left(\sqrt{2}\sqrt{\alpha s}\right) + \cos\left(\sqrt{2}\sqrt{\alpha s}\right) + 2 = 0 \tag{1.101}$$

*Let us introduce the auxiliary variable* $z = \sqrt{2}\sqrt{\alpha s} := a + ib$*, then:*

$$\cosh z + \cos z + 2 = 0 \Rightarrow \begin{cases} \cos b \cosh a + \cos a \cosh b = 2 \\ \sin b \sinh a = \sin a \sinh b \end{cases} \tag{1.102}$$

*since:*

$$\begin{aligned} \cosh z &= \cos b \cosh a + i \sin b \sinh a \\ \cos z &= \cos a \cosh b - i \sin a \sinh b \end{aligned} \tag{1.103}$$

*The system (1.102) is invariant under an exchange of the imaginary part of $z$ into the real part of $z$, and viceversa. Thus:*

$$z = a(1+i) \tag{1.104}$$

*Finally:*

$$a(1+i) = \sqrt{2}\sqrt{\alpha p_\beta} \Rightarrow \mathrm{Re}\left[p_\beta\right] = 0 \tag{1.105}$$

*Thus $A\left(s\right)$ is holomorphic in $\Theta$, further it is meromorphic. The poles are in complex conjugate pairs and only on the imaginary axis. They can be found solving the following trascendental equation:*

$$\cos r = \frac{-1}{\cosh r} \tag{1.106}$$

*with*$r = \sqrt{\alpha\omega}$*. The first five roots of this equation are:*

$$\begin{aligned}
r_1 &= \sqrt{\alpha\omega_1} \simeq 1.875 \\
r_2 &= \sqrt{\alpha\omega_2} \simeq 4.694 \\
r_3 &= \sqrt{\alpha\omega_3} \simeq 7.853 \\
r_4 &= \sqrt{\alpha\omega_4} \simeq 10.996 \\
r_5 &= \sqrt{\alpha\omega_5} \simeq 14.137
\end{aligned}$$

*When $\sqrt{\alpha\omega} > \sqrt{\alpha\omega_5}$ eq.(1.106) can be written in the approximate form:*

$$\cos\sqrt{\alpha\omega} \simeq 0 \tag{1.107}$$

*so*

$$\sqrt{\alpha\omega_n} \simeq \frac{(2n-1)}{2}\pi \qquad n \geqslant 5 \tag{1.108}$$

- *Since:*

$$\begin{aligned}
\sin k \cosh k &= \frac{\sin\sqrt{2\alpha s}}{2} + \frac{\sinh\sqrt{2\alpha s}}{2i} \\
\cos k \sinh k &= \frac{\sin\sqrt{2\alpha s}}{2i} + \frac{\sinh\sqrt{2\alpha s}}{2}
\end{aligned}$$

*Then:*

$$\begin{aligned}
A(s) &= -s\frac{A_0}{\left(\sqrt{\alpha s}\right)^3 e^{-\frac{i3\pi}{4}}}\frac{(1+i)\left(\sinh\sqrt{2}\sqrt{(\alpha s)} - \sin\sqrt{2}\sqrt{(\alpha s)}\right)}{\cosh\sqrt{2}\sqrt{(\alpha s)} + \cos\sqrt{2}\sqrt{(\alpha s)} + 2} = \quad (1.109)\\
A(s) &= s\frac{A_0\sqrt{2}}{\left(\sqrt{\alpha s}\right)^3}\frac{\left(\sinh\sqrt{2}\sqrt{(\alpha s)} - \sin\sqrt{2}\sqrt{(\alpha s)}\right)}{\cosh\sqrt{2}\sqrt{(\alpha s)} + \cos\sqrt{2}\sqrt{(\alpha s)} + 2}
\end{aligned}$$

*Clearly* $A(s)^* = A(s^*)$ *since*

$$\left(\sqrt{s}\right)^* = \sqrt{s^*} \tag{1.110}$$

- *Further it can be proved that:*

$$\mathrm{Re}\left[A(s)\right] = \frac{A(s) + A(s)^*}{2} \geqslant 0, \ \ in\ \Theta. \tag{1.111}$$

***Remark 145*** *On the imaginary axis:*

$$A(i\omega) = \frac{A_0}{\left(\sqrt{\alpha}\right)^3}\frac{i}{\sqrt{\omega}}\frac{\sin\sqrt{\alpha\omega}\cosh\sqrt{\alpha\omega} - \cos\sqrt{\alpha\omega}\sinh\sqrt{\alpha\omega}}{(1+\cos\sqrt{\alpha\omega}\cosh\sqrt{\alpha\omega})} \tag{1.112}$$

**Reciprocal, lossless, finite networks.**

In synthesis it can be needed to impose further properties, in addition to passivity, such as finiteness, reciprocity and losslessness. Consequently, we have to understand the effects of these additional assumptions upon the matrix descriptions of the networks.

**Reciprocity** Let us start our analysis with the assumption of reciprocity.

**Proposition 146** *The immittance matrices of a linear solvable, time invariant, passive and reciprocal $n-$port network are symmetric.*

Sometimes this statement is taken as definition of reciprocity for an $n-$port network.

We believe, however, that the reciprocity is a concept the definition of which is more fundamental than the symmetry of immittance matrices.

**Finiteness** Let us detail now the assumption of finiteness. A network is said to be finite if it is obtained by interconnecting a finite number of elementary networks: resistors, inductors, capacitors and ideal transformers.

If the number of subnetwork is infinite, the network is called infinite or *distributed.*

**Remark 147** *By the definition a finite network is reciprocal. For proving this simply recall prop.(62)*

Before we go further, consider the following definition:

**Definition 148** *An $n \times n$ matrix $\mathbf{A}(s)$ is called real-rational if all its entries are rational functions with real coefficients, i.e.:*

$$A(s)_{i,j} = \frac{a_n^{i,j} s^n + a_{n-1}^{i,j} s^{n-1} + ... + a_0^{i,j}}{b_m^{i,j} s^m + b_{m-1}^{i,j} s^{m-1} + ... + b_0^{i,j}} \tag{1.113}$$

*where all the coefficients are real numbers.*

**Remark 149** *In the examples given dealing with immittance representation, are found the immittances for elementary networks. They are all real-rational. Their interconnection produces networks with immittance matrices which are obtained by rational operations. Therefore finite networks have real-rational immittances.*

Thus we can state the following proposition:

**Proposition 150** *The immittance matrices of a linear solvable, finite, time invariant, passive $n-$port network are positive-real, real-rational and symmetric, whenever they exist.*

**Remark 151** *The immittance $A(s)$ of a linear solvable, finite, time invariant, passive one-port network is positive-real and real-rational (whenever it exists).*

**Example 152** *The immittance of examples (142) and (144) cannot represent a finite one-port network, i.e. it does not exist a circuit represented by that immittance constituted by the interconnection of a finite number of resistors, capacitors, inductors and ideal transformers.*

**Losslessness** Now we are ready to understand which restrictions on immittance matrices are implied by the assumption of losslessness, which we will require in the synthesis of some of our electrical analogs for mechanical devices.

**Definition 153** *Given an $n \times n$ matrix $\mathbf{A}(s)$, the Hurwitz conjugate $\mathbf{A}_*(s)$ is defined as:*

$$\mathbf{A}_*(s) = \mathbf{A}(-s) \tag{1.114}$$

***Proposition 154*** *If the immittance matrix of a lossless, solvable, linear, time invariant $n-$port network is meromorphic[6] then:*

$$\mathbf{A}(s) = -\mathbf{A}_*^T(s) \tag{1.115}$$

[6] A matrix is meromorphic if all its entrys are meromorphic.

**Remark 155** *As a remark, note that a rational matrix is a particular meromorphic matrix. Then we can state: the immittances matrices of a linear, solvable, lossless, time invariant, finite $n-$port network are positive-real, real-rational and symmetric and*

$$\mathbf{A}(s) = -\mathbf{A}(-s) \tag{1.116}$$

*whenever they exist.*

**Corollary 156** *If the immittance of a linear lossless, time invariant one-port network is meromorphic then:*

$$A(s) = -A(-s) \tag{1.117}$$

*i.e., $A(s)$ is an odd function.*

**Example 157** *Consider again the immittance of example (142):*

$$A\left(s\right) = A_0 \tanh \gamma s \tag{1.118}$$

*clearly $A\left(s\right)$ is an odd function, then this immittance can be realized by a lossless circuit.*

**Example 158** *Consider again the immittance of example (144):*

$$A\left(s\right) = s\frac{A_0}{k^3}\frac{\sin k \cosh k - \cos k \sinh k}{\left(1 + \cos k \cosh k\right)} \tag{1.119}$$

*where $k = \sqrt{\alpha s}\, e^{-\frac{i\pi}{4}}$. Clearly $A\left(s\right)$ is an odd function, then this immittance can be realized by a lossless circuit.*

### Fundamental properties of the meromorphic immittance function for a linear, lossless, time invariant one-port network.

This section deals with the analysis of the immittance of a linear, lossless, time invariant one-port network in the particular case in which its immittance is meromorphic function, in order to achieve a different form for them amenable for practical synthesis.

**Remark 159** *Assuming that the immittance matrices are meromorphic restricts the subsequent analysis to practical applications where the evolutionary operators have discrete spectrum.*

As a preliminary, let us summarize the properties that we have so far found for the considered particular one-port networks:

**Summary 160** *The immittance $A(s)$ of a linear solvable, time invariant, passive one-port network (whenever it exists) is*

- *positive real*
- *if it is meromorphic then it is an odd function of $s$.*

**Remark 161** *At the moment we are not focusing our attention on the particular case of finite one-port network. The reason for this generality will become clearer in what follows, in particular when we will synthesize one-port networks simulating mechanical impedances or mobilities.*

By the corollary (141) we know that $A(s)$ is holomorphic in the right plane, but since $A(s)$ has to be an odd function of $s$, then $A(s)$ is also holomorphic in the left plane.

Thus the poles of $A(s)$ lie on the imaginary axis; further they are complex conjugate, by virtue of $A(s) = A^*(s^*)$.

Now we will prove that all the poles of $A(s)$ are simple, including the pole $s = 0$.

In fact, consider a pole $p_i$ of order $m$, remembering that $A(s)$ is meromorphic then $A(s)$ is holomorphic in a ring-shaped neighborhood of $p_i$, $\mathcal{R}_\varepsilon(p_i)$ of external radius $\varepsilon$.

Then in that region it can be expanded in Laurent series:

$$A(s) \simeq \frac{k_i}{(s-p_i)^m} \qquad \forall s \in \mathcal{R}_\varepsilon(p_i) \tag{1.120}$$

where $k_i$ is the residue of $A(s)$ at $p_i$.

On the external contour of $\mathcal{R}_\varepsilon(p_i)$, i.e. on the circumference of center $p_i$ and radius $\varepsilon$:

$$A(s) \simeq k_i \varepsilon e^{im\vartheta} \tag{1.121}$$

By means of the corollary (141) we require also $\mathrm{Re}\left[A(s)\right]$ to be greater or equal to zero in the right half plane.

Then:

$$\mathrm{Re}\left[k_i \varepsilon e^{im\vartheta}\right] > 0 \qquad \vartheta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{1.122}$$

This implies:

$$\mathrm{Re}\left[k_i\right]\cos(m\vartheta) + \mathrm{Im}\left[k_i\right]\sin(m\vartheta) > 0 \qquad \vartheta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{1.123}$$

This inequality holds only if $m = 1$, $\mathrm{Re}\left[k_i\right] > 0$ and $\mathrm{Im}\left[k_i\right] = 0$; that is the pole $p_i$ is simple and the residue at $p_i$ is real positive.

Furthermore $s = 0$ is a pole or a zero of $A(s)$. For, consider the Laurent series expansion in the neighborhood of $s = 0$ then ,by contradiction, if $s$ is nor a pole neither a zero then:

$$A(s) \simeq A(0) \qquad \forall s \in \mathcal{R}_\varepsilon(0) \tag{1.124}$$

Thus in $\mathcal{R}_\varepsilon(0)$ $A(s)$ is not an odd function of $s$.

We have thus proved that:

**Proposition 162** *If the immittance $A(s)$ of a linear lossless, time invariant one-port network is meromorphic then $A(s)$ has simple complex conjugate poles on the $i\omega$ axis (so they include the pole at zero) and the residues at every pole are real positive. Further $s = 0$ is a either a pole or a zero.*

**Remark 163** *The previous properties which we have proven for the networks verifying the properties listed in prop.(162) in Zinn (1951)[10] are assumed as hypothesis for developing "network representation of trascendental functions" while in Dah-You Maa (1943)[11] are accepted without proof.*

**Example 164** *Consider again the immittance of example (142):*

$$A(s) = A_0 \tanh \gamma s \tag{1.125}$$

*As we have shown in the previous section $A(s)$ is meromorphic and all its poles are simple purely imaginary:*

$$p_k = i\frac{\pi}{2\gamma}(2k+1) \qquad k \in \mathrm{Z} \tag{1.126}$$

*Besides $s = 0$ is a zero of $A(s)$.*

**Example 165** *Consider again the immittance of example (144):*

$$A(s) = s\frac{A_0}{k^3}\frac{\sin k \cosh k - \cos k \sinh k}{(1 + \cos k \cosh k)} \tag{1.127}$$

*where $k = \sqrt{\alpha s}\, e^{-\frac{i\pi}{4}}$. As we have shown in the previous section $A(s)$ is meromorphic and all its poles are simple purely imaginary. Besides $s = 0$ is a zero of $A(s)$.*

The following Theorem provides a tool for breaking up, under quite general restrictions, meromorphic functions into an infinite series of simple fractions.

**Theorem 166** *(Mittag-Leffler) Let $A(s)$ be a function satisfying the following hypothesis:*

- *Let $p_1, p_2, ...$ be the sequence of its poles, with $|p_i| < |p_{i+1}|$ $\forall i \in \mathbb{N}$, then*

$$\forall i \in \mathbb{N} \qquad 0 < |p_i| < +\infty \qquad \text{and } p_i \text{ is a simple pole} \tag{1.128}$$

- *there is a sequence of closed contours $\mathcal{C}_n$, such that $\mathcal{C}_n$ encloses $p_1, p_2, ..., p_n$ but no other poles.*

- *the minimum distance of $\mathcal{C}_n$ from the origin tends to infinity with $n$;*

- $A(s) = A^*(s^*)$

- *$A(s) = o(R_n)$ as $n \to \infty$, i.e.*

$$\lim_{n\to\infty} \frac{A(s)|_{s\in\mathcal{C}_n}}{R_n} = 0 \tag{1.129}$$

*Then:*

$$A(s) = A(0) + \sum_{n=-\infty}^{\infty} k_n \left( \frac{1}{s - p_n} + \frac{1}{p_n} \right) \qquad \forall s \in \mathbb{C},\ s \neq p_n \tag{1.130}$$

*where $k_1, k_2, ...$ are the residues at the poles $p_1, p_2, ...$ respectively. And*

$$p_{-n} = p_n^*,\ k_{-n} = k_n^* \tag{1.131}$$

*since by virtue of $A(s) = A^*(s^*)$ the poles occur in conjugate complex pairs.*

***Proof.*** *For a detailed proof of the theorem see Titchmarsh (1939)[2]* ■

**Remark 167** *The residues are determined by:*

$$k_n = \lim_{s \to p_n} (s - p_n) A(s) \tag{1.132}$$

Further if $A(s)$ has all the poles on the imaginary axis and the residues are positive, then the Mittlag-Leffler's expansion states:

$$A(s) = A(0) + \sum_{n=1}^{\infty} 2k_n \left( \frac{s}{s^2 + \omega_n^2} \right) \tag{1.133}$$

**Remark 168** *Given a meromorphic immittance function $A(s)$ of a linear, lossless, time invariant one-port network then the only condition that we have to check before applying the Mittag-Leffler's expansion are:*

- *there is not a pole at $s = 0$.*
- *there is a sequence of closed contours $\mathcal{C}_n$, such that $\mathcal{C}_n$ encloses $p_1, p_2, ..., p_m$ but no other poles.*
- *the minimum distance $R_n$ of $\mathcal{C}_n$ from the origin tends to infinity with $n$;*
- *$A(s) = o(R_n)$ as $n \to \infty$.*

**Remark 169** *The first condition of the previous remark is not stringent, since given an immittance function $A(s)$ having a pole at $s = 0$, it is always possible to consider the inverse of $A(s)$ which is still an immittance, but with a zero at $s = 0$. The third condition is verified if for example $A(s)$ is bounded on $\cup_n \mathcal{C}_n$.*

**Example 170** *Consider the immittance of example (142)*

$$A(s) = A_0 \tanh \gamma s \tag{1.134}$$

*Clearly we can choose $\mathcal{C}_n$ as the sphere of radius $\dfrac{(k+1)}{\gamma}\pi$. Further the immittance function has not a pole at infinity, since:*

$$\lim_{|s|\to+\infty} A_0 \tanh\gamma s = A_0 \tag{1.135}$$

*Then:*

$$A\left(s\right) = o(R_n) \;\; as \; n \to \infty. \tag{1.136}$$

*The residues of $A\left(s\right)$ are:*

$$\begin{aligned}
&k_n = \lim_{s\to p_n}\left(s-p_n\right)A\left(s\right) = \lim_{s\to p_n}\left(s-p_n\right)A_0\frac{\sinh\gamma s}{\cosh\gamma s} = A_0\lim_{s\to p_n}\frac{\left(s-p_n\right)}{\frac{\cosh\gamma s}{\sinh\gamma s}}\\
&= A_0\lim_{s\to p_n}\frac{1}{\frac{d}{ds}\left(\frac{\cosh\gamma s}{\sinh\gamma s}\right)} = \left.\frac{A_0}{-\frac{\gamma}{\cosh^2\gamma s-1}}\right|_{i\frac{\pi}{2\gamma}(2n+1)} = -\frac{A_0}{\gamma}\sinh^2\left(i\pi\left(n+\tfrac{1}{2}\right)\right)\Big|_{i\frac{\pi}{2\gamma}(2n+1)} =\\
&= \frac{A_0}{\gamma}\sin^2\left(\pi\left(n+\tfrac{1}{2}\right)\right)\Big|_{i\frac{\pi}{2\gamma}(2n+1)} = \frac{A_0}{\gamma}
\end{aligned} \tag{1.137}$$

*and:*

$$A\left(0\right) = 0 \tag{1.138}$$

*Then:*

$$A\left(s\right) = A_0\tanh\gamma s = \frac{2A_0}{\gamma}\sum_{n=1}^{\infty}\left(\frac{s}{s^2+\omega_n^2}\right) \tag{1.139}$$

*where* $\omega_n = \dfrac{\pi}{2\gamma}\left(2n+1\right).$

**Example 171** *Consider the immittance of example (144)*

$$A\left(s\right) = s\frac{A_0}{k^3}\frac{\sin k\cosh k - \cos k\sinh k}{\left(1+\cos k\cosh k\right)} \tag{1.140}$$

*where* $k = \sqrt{\alpha s}\, e^{-\frac{i\pi}{4}}$. *The immittance can also be written as follows:*

$$A(s) = s\frac{A_0\sqrt{2}}{(\sqrt{\alpha s})^3}\frac{\left(\sinh\sqrt{2}\sqrt{(\alpha s)} - \sin\sqrt{2}\sqrt{(\alpha s)}\right)}{\cosh\sqrt{2}\sqrt{(\alpha s)} + \cos\sqrt{2}\sqrt{(\alpha s)} + 2} \tag{1.141}$$

*Clearly*

$$A(s) \to 0 \text{ as } |s| \to \infty \tag{1.142}$$

*and it is possible to choose a sequence of circles* $\mathcal{C}_n$ *whose radius is between* $\sqrt{\alpha\omega_{n-1}}$ *and* $\sqrt{\alpha\omega_n}$. *The residues of* $A(s)$ *are:*

$$k_n = \lim_{s\to p_n}(s-p_n)A(s) = \lim_{r\to i\omega_n}(i\omega - i\omega_n)\left(i\omega\frac{A_0}{\sqrt{\alpha\omega}^3}\frac{\sin\sqrt{\alpha\omega}\cosh\sqrt{\alpha\omega} - \cos\sqrt{\alpha\omega}\sinh\sqrt{\alpha\omega}}{(1+\cos\sqrt{\alpha\omega}\cosh\sqrt{\alpha\omega})}\right) \tag{1.143}$$

*and:*

$$A(0) = 0 \tag{1.144}$$

*Then:*

$$A(s) = \sum_{n=1}^{\infty} 2s\left(\frac{k_n}{s^2+\omega_n^2}\right) \tag{1.145}$$

*where* $\omega_n$ *are the imaginary part of the poles.*

### Properties of realizable matrices

For the various decomposition needed for synthesis of networks, it is necessary to introduce the algebraic properties of real-positive, real-rational and symmetric matrices.

**Remark 172** *Now we will restrict to the case of finite networks.*

A fundamental necessary and sufficient condition for a real-rational matrix to be positive-real is established by the following direct test:

**Theorem 173** *An $n \times n$ real rational matrix $\mathbf{A}(s)$ is positive-real if and only if:*

- $\mathbf{A}(s)$ *has no poles in* $\Theta$
- *Poles of* $\mathbf{A}(s)$ *on* $\sigma = 0$ *are simple*
- *For each pole on* $\sigma = 0$*, the residue* $\mathbf{K}$ *is Hermitian, i.e.*

$$\mathbf{K} = \mathbf{K}_H := \frac{\mathbf{K} + \mathbf{K}^{*T}}{2} \tag{1.146}$$

*, and semidefinite positive.*

***Definition 174*** • $\mathbf{A}_H(i\omega)$ *semidefinite positive whenever it is defined.*

***Proof.*** *For a detailed proof of the theorem see Newcomb(1966) [3]. The proof is conceptually identical to the one we have given dealing with non-finite one-port networks.*

■

**Corollary 175** *A real rational function $A(s)$ is positive-real if and only if:*

- $A(s)$ *has no poles in* $\Theta$
- *Poles of* $A(s)$ *on* $\sigma = 0$ *are simple*
- *For each pole on* $\sigma = 0$*, the residue* $K$ *is real and positive*
- $\mathrm{Re}\,[A(i\omega)] \geqslant 0$ *whenever it is defined.*

**Remark 176** *The above result allows us to check the positive-real character of a function by examining the behavior of its real-part on the imaginary axis $\sigma = 0$ only, instead of over the entire right half plane.*

We recall that the immittance matrices of a linear, solvable, lossless, time invariant, finite $n-$port network are positive-real, real-rational and symmetric and

$$\mathbf{A}(s) = -\mathbf{A}(-s) \tag{1.147}$$

whenever they exist.

Then, by the holomorphy of $\mathbf{A}(s)$ in $\Theta$, it follows that all the poles are complex conjugate on the imaginary axis. To prove this assumes that $p_i$ is a pole of $\mathbf{A}(s)$, then $-p_i$ is a pole of $\mathbf{A}(-s)$, but any pole of $\mathbf{A}(-s)$ is also a pole of $\mathbf{A}(s)$, showing that $p_i$ and $-p_i$ are both poles of $\mathbf{A}(s)$; by the holomorphic property it follows that $p_i = i\omega_i$.

By means of Theorem (173) $p_i$ is a simple pole and the residue matrix $\mathbf{K}_i$ is Hermitian and positive semidefinite. Further $p_i^*$ is also a pole and its residue is $\mathbf{K}_i^*$ which is clearly Hermitian and positive semidefinite.

Since $\mathbf{A}(s)$ is symmetric, all the residues are symmetric matrices; further we know that these matrices are Hermitian then:

$$\mathbf{K}_i = \mathbf{K}_i^T \qquad \mathbf{K}_i = \frac{\mathbf{K}_i + \mathbf{K}_i^{*T}}{2} \Rightarrow \mathbf{K}_i = \mathbf{K}_i^* \Rightarrow \mathbf{K}_i \text{ are matrices of real constants} \tag{1.148}$$

Combining these two conjugates poles in the fraction expansion we get:

$$\frac{\mathbf{K}_i}{s - i\omega_i} + \frac{\mathbf{K}_i^*}{s + i\omega_i} = \frac{i\omega_i\left(\mathbf{K}_i - \mathbf{K}_i^*\right)}{s^2 + \omega_i^2} + \frac{s\left(\mathbf{K}_i + \mathbf{K}_i^*\right)}{s^2 + \omega_i^2} = 2\frac{s}{s^2 + \omega_i^2}\mathbf{K}_i \tag{1.149}$$

Hence the partial-fraction expansion for $\mathbf{A}(s)$ states:

$$\mathbf{A}(s) = s\mathbf{K}_\infty + \sum_{i=1}^{n} 2\frac{s}{s^2 + \omega_i^2}\mathbf{K}_i + \frac{1}{s}\mathbf{K}_0 \tag{1.150}$$

Usually these expression is called Foster's form of the immittance.

Let us resume these fundamental results in the following proposition:

**Proposition 177** *The immittance matrices of a linear, solvable, lossless, time invariant, finite $n-$port network are positive-real, real-rational, symmetric and odd functions of $s$. They can be expressed in the generalized Foster's form, whenever they exist:*

$$\mathbf{A}(s) = s\mathbf{K}_\infty + \sum_{i=1}^{n} 2\frac{s}{s^2+\omega_i^2}\mathbf{K}_i + \frac{1}{s}\mathbf{K}_0 \tag{1.151}$$

*where the residues $\mathbf{K}_i$ are real constant matrices, symmetric and positive semidefinite.*

**Proposition 178** *The immittances of a linear, solvable, lossless, time-invariant, finite one-port network are positive-real, real rational and odd functions of $s$. They can be expressed in the Foster's form, whenever they exist:*

$$A(s) = sk_\infty + \sum_{i=1}^{n} 2\frac{s}{s^2+\omega_i^2}k_i + \frac{1}{s}k_0 \tag{1.152}$$

*where the residues $k_i \in \mathbb{R}^+$.*

# Chapter 2

# Network synthesis

In this chapter we will give an outline of the approaches used in the synthesis of networks, for a deeper discussion about this topic see Baher (1984)[15].

### Synthesis of linear, solvable, time-invariant, lossless, finite one-port networks.

From Foster's form of the immittance, we can get two different techniques for design depending on the kind of immittance we are dealing with. In fact if the given network is represented as an impedance $Z(s)$ we get Foster's first form:

$$Z(s) = sk_\infty + \sum_{i=1}^{n} 2\frac{s}{s^2+\omega_i^2}k_i + \frac{1}{s}k_0 \tag{2.1}$$

It is clear that $Z$ is a series coonection of $n+2$ one-port sub-networks, in particular:

- $sK_\infty$ represents an inductor, the inductance of which is $L_\infty = k_\infty$
- $2\frac{s}{s^2+\omega_i^2}k_i$ represents the parallel connection between an inductor and a capacitor, with inductance $L_i = \frac{2k_i}{\omega_i^2}$ and capacitance $C_i = \frac{1}{2k_i}$ respectively
- $\frac{1}{s}k_0$ represents a capacitor, the capacitance of which is $C_0 = \frac{1}{2k_0}$

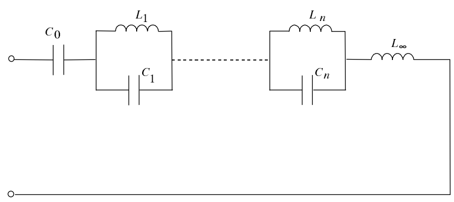


Figure 2.1: Foster's first form

If the one-port network is given in term of its admittance we get Foster's second form:

$$Y(s) = sk_\infty + \sum_{i=1}^{n} 2\frac{s}{s^2+\omega_i^2}k_i + \frac{1}{s}k_0 \tag{2.2}$$

It is clear that $Y$ is the parallel connection of $n+2$ one-port subnetworks, in particular:

- $sk_\infty$ represents a capacitor, the capacitance of which is $C_\infty = k_\infty$
- $2\frac{s}{s^2+\omega_i^2}k_i$ represents the series connection between an inductor and a capacitor, with inductance $L_i = \frac{1}{2k_i}$ and capacitance $C_i = \frac{2k_i}{\omega_i^2}$ respectively
- $\frac{1}{s}k_0$ represents an inductor, the inductance of which is $L_0 = \frac{1}{k_0}$

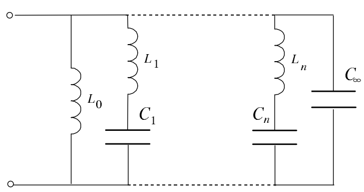


Figure 2.2: Foster's second form

## Synthesis of linear, solvable, time-invariant, lossless, infinite one-port networks.

In general the immittance of a one port network satisfying the conditions of the Mittag-Leffler's Theorem can be expanded as:

$$A(s) = A(0) + \sum_{n=1}^{\infty} 2k_n \left( \frac{s}{s^2 + \omega_n^2} \right) \tag{2.3}$$

Then if $A(0) = 0$, $A(s)$ can be represented by the connection of infinite one-port networks.

Indeed if $A(s)$ is an impedance, then it can be designed as a series connection of infinite one-port networks, each of them representing the parallel connection of a capacitor and an inductor, such as:

$$L_n = \frac{2k_n}{\omega_i^2} \qquad C_n = \frac{1}{2k_n} \tag{2.4}$$

While if $A(s)$ is an admittance, then it can be designed as the parallel of infinite one-port networks, each of them being the series between a capacitor and an inductor, such as:

$$L_n = \frac{1}{2k_n} \qquad C_n = \frac{2k_n}{\omega_n^2} \tag{2.5}$$

**Remark 179** *The condition $A(0) = 0$ does not restrict the field of applicability of our technique, since as we have proven $s = 0$ is either a pole or a zero. Thus if $A(0) \neq 0$ then $A(0) = \infty$, and we can consider the synthesis of $\frac{1}{A(s)}$ instead of $A(s)$.*

## Synthesis of linear, solvable, time-invariant, lossless, finite two-port networks.

We have, so far, established that for a given $2 \times 2$ matrix $\mathbf{A}(s)$ to be realizable as the immittance matrix of a linear, solvable, time-invariant, lossless and reciprocal two-port network, it is necessary that $\mathbf{A}(s)$ be a generalized Foster's matrix, i.e.:

$$\mathbf{A}(s) = s\mathbf{K}_\infty + \sum_{i-1}^{n} 2\frac{s}{s^2+\omega_i^2}\mathbf{K}_i + \frac{1}{s}\mathbf{K}_0 \tag{2.6}$$

where the residue matrices are all real constants, symmetric and positive semidefinite.

Momentarily suppose that $\mathbf{A}(s)$ represents an impedance matrix, that is $\mathbf{Z}(s) := \mathbf{A}(s)$.

Let $\mathbf{K}$ be one of the residue matrices $\mathbf{K}_\infty$, $\mathbf{K}_i$, $\mathbf{K}_0$; also let $\psi(s)$ be one of the three functions $\frac{1}{s}$, $2\frac{s}{s^2+\omega_i^2}$, or $s$. Accordingly a *typical term* of the expansion (2.6) can be written in the form:

$$\mathbf{Z}_i(s) = \mathbf{K}\psi(s) = \begin{pmatrix} K_{1,1}\psi(s) & K_{1,2}\psi(s) \\ K_{1,2}\psi(s) & K_{2,2}\psi(s) \end{pmatrix} \tag{2.7}$$

where, in particular by virtue of the semidefinite positiveness:

$$\begin{aligned} &\det \mathbf{K} \geqslant 0 \\ &K_{1,1} \geqslant 0 \\ &K_{2,2} \geqslant 0 \end{aligned} \tag{2.8}$$

We now want to demonstrate that the above typical term $\mathbf{K}\psi(s)$, can be realized by the following network:

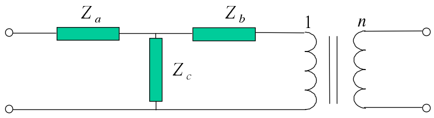


Figure 2.3: A two-port network capable of realizing any *typical term* in Foster's expansion.

The impedance matrix of the introduced network, sometimes called a $T-$network is (for the meaning of symbols see the previous figure):

$$\mathbf{Z}_T = \begin{pmatrix} Z_a + Z_c & nZ_c \\ nZ_c & n^2 (Z_b + Z_c) \end{pmatrix} \tag{2.9}$$

After this demonstration, it becomes obvious that the entire impedance matrix $\mathbf{Z}(s)$ is realizable as the series connection of $n + 2$ $T-$networks each representing a term in Foster's expansion.

Thus we want to show that it is always possible to find $Z_a$, $Z_b$, $Z_c$ and $n$ such that:

$$\mathbf{Z}_T = \mathbf{K}\psi(s) \tag{2.10}$$

Let us assume that $Z_a$, $Z_b$, $Z_c$ may be expressed in the form:

$$Z_a = a\psi(s) \qquad Z_b = b\psi(s) \qquad Z_c = c\psi(s) \tag{2.11}$$

where $a$, $b$, $c$ are real positive constants.

Thus from eq.(2.10), we get the following relations:

$$\begin{aligned} & a\psi(s) + c\psi(s) = K_{1,1}\psi(s) \\ & n\left(c\psi(s)\right) = K_{1,2}\psi(s) \\ & n^2\left(b\psi(s) + c\psi(s)\right) = K_{2,2}\psi(s) \end{aligned} \tag{2.12}$$

Hence:

$$\begin{aligned} & a + c = K_{1,1} \\ & nc = K_{1,2} \\ & n^2(b + c) = K_{2,2} \end{aligned} \tag{2.13}$$

From which, $n$ has the same sign of $K_{1,2}$ to guarantee that $c \in \mathbb{R}^+$, and:

$$\begin{aligned} & a = K_{1,1} - \frac{K_{1,2}}{n} = K_{1,1} - \frac{|K_{1,2}|}{|n|} \\ & c = \frac{K_{1,2}}{n} = \frac{|K_{1,2}|}{|n|} \\ & b = \frac{K_{2,2}}{n^2} - \frac{K_{1,2}}{n} = \frac{K_{2,2}}{n^2} - \frac{|K_{1,2}|}{|n|} \end{aligned} \tag{2.14}$$

Now we must require that both $a$ and $b$ are non-negative:

$$K_{1,1} - \frac{|K_{1,2}|}{|n|} \geqslant 0 \qquad \frac{K_{2,2}}{n^2} - \frac{|K_{1,2}|}{|n|} \geqslant 0 \tag{2.15}$$

thus:

$$K_{1,1} \geqslant \frac{|K_{1,2}|}{|n|} \qquad K_{2,2} \geqslant |K_{1,2}|\,|n| \tag{2.16}$$

or:

$$\frac{|K_{1,2}|}{K_{1,1}} \leqslant |n| \leqslant \frac{K_{2,2}}{|K_{1,2}|} \Rightarrow |n| \in I := \left\{ x \in \mathbb{R}^+ : \frac{|K_{1,2}|}{K_{1,1}} \leqslant x \leqslant \frac{K_{2,2}}{|K_{1,2}|} \right\} \tag{2.17}$$

the set $I$ is not empty, since:

$$\left\{ \begin{array}{l} \det \mathbf{K} \geqslant 0 \Rightarrow K_{1,1}K_{2,2} - K_{1,2}^2 \geqslant 0 \\ \\ K_{1,1} \geqslant 0,\ K_{2,2} \geqslant 0 \end{array} \right. \Rightarrow \frac{|K_{1,2}|}{K_{1,1}} \leqslant \frac{K_{2,2}}{|K_{1,2}|} \tag{2.18}$$

**Remark 180** *In general the synthesis of a lossless reciprocal two-port network requires the use of transformers, nevertheless if* $1 \in I$ *and* $K_{1,2} \geqslant 0$ *then it is possible to choose* $n = 1$ *and the transformer may be dispensed with.*

**Remark 181** *If* $\psi(s) = \dfrac{1}{s}$ *then the impedances of the* $T-$*network are capacitors, if* $\psi(s) = 2\dfrac{s}{s^2 + \omega_i^2}$ *then each of these impedances is the parallel of an inductor and a capacitor, while if* $\psi(s) = s$ *all these impedances are inductors.*

If the admittance $\mathbf{A}(s)$ represents an immittance $\mathbf{Y}(s)$ it is easy to show that it is realizable as the parallel connection of $n+2$ networks, of the form showed below, called a $\pi-$network.

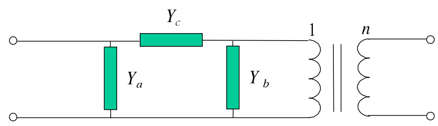


Figure 2.4: A two-port capable of realizing any *typical term* in the expansion of the admittance

## An alternative approach

If the impedance matrix is of the form:

$$\mathbf{Z}(s) = \mathbf{K}\psi(s) = \begin{pmatrix} K_{1,1}\psi(s) & K_{1,2}\psi(s) \\ K_{1,2}\psi(s) & K_{1,1}\psi(s) \end{pmatrix} \tag{2.19}$$

then it can be realized by a network called a *symmetric lattice:*

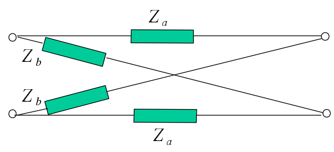


Figure 2.5: Symmetric lattice

The impedance matrix of this network is of the form:

$$\mathbf{Z}_S = \frac{1}{2}\begin{pmatrix} Z_b + Z_a & Z_b - Z_a \\ Z_b - Z_a & Z_b + Z_a \end{pmatrix} \tag{2.20}$$

Thus we want to show that it is always possible to find $Z_a$, $Z_b$ such that:

$$\mathbf{Z}_S = \mathbf{K}\psi\left(s\right) \tag{2.21}$$

Let us assume that $Z_a$, $Z_b$ may be expressed in the form:

$$Z_a = a\psi\left(s\right) \qquad Z_b = b\psi\left(s\right) \tag{2.22}$$

where $a$, $b$ are real positive constants.Then eq.(2.21) yields:

$$\begin{aligned} \tfrac{1}{2}\left(b\psi\left(s\right) + a\psi\left(s\right)\right) &= K_{1,1}\psi\left(s\right) \\ \tfrac{1}{2}\left(b\psi\left(s\right) - a\psi\left(s\right)\right) &= K_{1,2}\psi\left(s\right) \end{aligned} \tag{2.23}$$

Hence:

$$\begin{aligned} b &= K_{1,1} - K_{1,2} \\ a &= K_{1,1} + K_{1,2} \end{aligned} \tag{2.24}$$

Since $\det\mathbf{K} \geqslant 0$ then $K_{1,1} \geqslant |K_{1,2}|$ and the previous system is solvable with $a$, $b$ both positive real constants.

## Synthesis of multiport networks

The impedance matrix of a linear, solvable, time-invariant, lossless and reciprocal multiport network (suppose that the network has more than two ports), can be expressed in the generalized Foster's form as:

$$\mathbf{Z}(s) = s\mathbf{K}_\infty + \sum_{i-1}^{n} 2\frac{s}{s^2+\omega_i^2}\mathbf{K}_i + \frac{1}{s}\mathbf{K}_0 \tag{2.25}$$

where the residue matrices are all real constants, symmetric and positive semi-definite.

Let $\mathbf{K}$ be one of the residue matrices $\mathbf{K}_\infty$, $\mathbf{K}_i$, $\mathbf{K}_0$; also let $\psi(s)$ be one of the three functions $\frac{1}{s}$, $2\frac{s}{s^2+\omega_i^2}$, or $s$. Accordingly a generic term of this expansion can be written in the form:

$$\mathbf{Z}_i(s) = \mathbf{K}\psi(s) \tag{2.26}$$

Since the generic residue $\mathbf{K}$ is symmetric and semidefinite positive it can be diagonalized and all the eigenvalues are nonnegative.

The eigenvalues can be assembled in an $n \times n$ matrix $\Lambda$**:**

$$\Lambda = \begin{pmatrix} \lambda_1 & ... & 0 \\ ... & ... & ... \\ 0 & ... & \lambda_n \end{pmatrix} \tag{2.27}$$

Furthermore it is possible to find an orthonormal set of eigenvectors $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ following the Gram-Schimdt procedure. These eigenvalues can be assembled by rows in an orthonormal matrix $\mathbf{E}$**:**

$$\mathbf{E} = \begin{pmatrix} \mathbf{e}_1 \\ ... \\ \mathbf{e}_n \end{pmatrix} \tag{2.28}$$

such that:

$$\Lambda = \mathbf{E}\mathbf{K}\mathbf{E}^T \Rightarrow \mathbf{K} = \mathbf{E}^T \Lambda \mathbf{E} \tag{2.29}$$

For a proof of all the statements we are using see Pease (1965)[14].

Then a generic term of the expansion can be written as:

$$\mathbf{Z}_i(s) = \mathbf{K}\psi\,(s) = \mathbf{E}^T\,(\Lambda\psi\,(s))\,\mathbf{E} \tag{2.30}$$

Introducing a magnification factor $\kappa$ multiplying the eigenvectors it is possible to express $\mathbf{Z}_i(s)$ by:

$$\mathbf{Z}_i(s) = \mathbf{T}^T \left(\frac{1}{\kappa^2}\Lambda\psi\,(s)\right)\mathbf{T} \tag{2.31}$$

where $\mathbf{T} = \kappa\mathbf{E}$.

Then the generic term can be synthesized as $n$ uncoupled impedances of values $\frac{1}{\kappa^2}\lambda_i\psi\,(s)$ terminating a transformer with the $n \times n$ turns-ratio matrix $\mathbf{T}$.

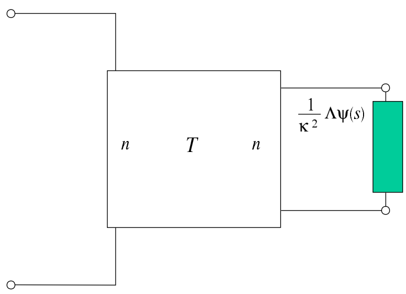


Figure 2.6: Network capable of realizing a generic term in the Foster's expansion

By a series connection of these $n+2$ cascade loaded transformers it is easy to realize the given impedance matrix.

**Remark 182** *If* $\psi(s) = \frac{1}{s}$ *then the impedances are capacitors; if* $\psi(s) = 2\frac{s}{s^2+\omega_i^2}$ *then they are the parallel snnection of an inductor and a capacitor; while if* $\psi(s) = s$ *they are inductors.*

# Chapter 3

# Timoshenko beam

### Kinematics

The beam theory of Bernoulli-Navier allows for a rough description of the mechanical behavior of an elastic body, under the following hypothesis:

- the reference configuration is a cylinder
- the maximum diameter of the cross sections is much smaller than the length of the body
- the deformation of the sections is negligible, that is the section can be treated as rigid bodies.

In the following we will be interested only in plane beams the reference configuration of which is a straight line $\gamma$, on which is fixed an abscissa $s \in [0, l]$. Furthermore, we will associate a section, regarded as rigid, to every point on the axis.

From now on we will always refer to an observer $\mathbf{O}$ characterized by an origin $\mathfrak{o}$ and a basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of the space of translations.

The kinematical descriptors of the beam will be the three scalar fields $w(s,t)$, $u(s,t)$, $\vartheta(s,t)$ depending on the abscissa $s$ and the time $t$, furthermore we will assume that they belong to the space $C^{\infty}([0,l] \times [0,\infty))$.

The evolution of the system in the time interval $[0, \infty)$ will be described by *motion*, i.e. a function $\mathfrak{M} : t \mapsto (w(s,t), u(s,t), \vartheta(s,t))\ \forall s \in [0,l]$.

Let us explain each of these fields: $w$ is the longitudinal displacement, $u$ is the transverse displacement and $\vartheta$ is the change of attitude of the section with respect to the reference configuration.

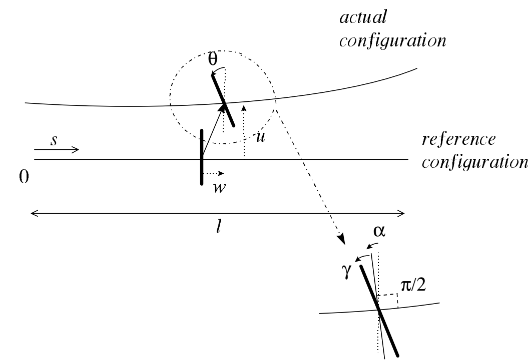


Figure 3.1: Kinematical descriptors of the beam

## Small deformations

Let us define $\alpha(s,t)$ the angle between the normal to the beam in the reference configuration and the normal to the beam in the actual configuration.

It is trivial to see that $\alpha(s,t)$ coincides with the angle between the centerline of the beam in the reference configuration and the tangent to the centerline in the actual configuration, i.e.:

$$\tan\alpha(s,t) = u'(s,t) \tag{3.1}$$

Dealing with small deformations, we can linearize the tangent in the previous equation to establish that:

$$\alpha\,(s,t) \simeq u'(s,t) \tag{3.2}$$

Let us now introduce, the functions describing the small deformation of the beam, in the so called *linearized kinematics*:

1. The *shear deformation*, supposing that all of the cross sections are orthogonal to the axis of the beam in the reference configuration, is defined as the angle between the section in the actual configuration and the normal to the axis in the actual configuration:

   $$\gamma\,(s,t) := u'(s,t) - \vartheta(s,t) \tag{3.3}$$

   thus $\gamma$ represents the lack of orthogonality of the cross sections to the centerline of the beam in the actual configuration.

2. Consider two sections at $s$ and $(s+ds)$, in the reference configuration. In the actual configuration these two sections will undergo two different axial displacements $w\,(s,t)$ and $w\,(s+ds,t)$. The *axial deformation* is described by:

   $$\varepsilon_w\,(s,t) := \frac{w\,(s+ds,t) - w\,(s,t)}{ds} = w'\,(s,t) \tag{3.4}$$

3. Consider two sections at $s$ and $(s+ds)$, in the reference configuration. In the actual configuration these two sections will undergo two different change of attitude $\vartheta\,(s,t)$ and $\vartheta\,(s+ds,t)$. The bending deformation is described by:

   $$\varepsilon_\vartheta\,(s,t) := \frac{\vartheta\,(s+ds,t) - \vartheta\,(s,t)}{ds} = \vartheta'\,(s,t) \tag{3.5}$$

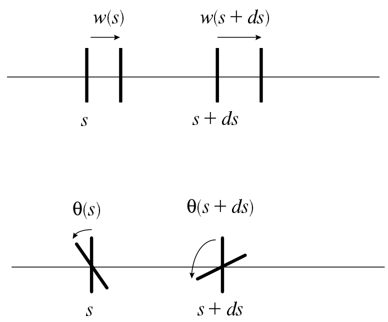


Figure 3.2: Axial and bending deformation

### Dynamics

Consider the beam characterized by the interval of the abscissa $[s_1, s_2]$, subjected to a force per unit length $\mathbf{b}(s) = b_N(s)\,\mathbf{e}_1 + b_T(s)\mathbf{e}_2$ and a couple per unit length $\boldsymbol{\mu}(s) = \mu(s)\mathbf{e}_3$ and introduce the potentials $U_{b_N}(s)$, $U_{b_T}(s)$ and $U_\mu(s)$ defined by:

$$\begin{aligned} U_{b_N}(w) &= -\int_{s_1}^{s_2} b_N(s)\,w(s,t)\,ds \\ U_{b_N}(u) &= -\int_{s_1}^{s_2} b_T(s)\,u(s,t)\,ds \\ U_\mu(u) &= -\int_{s_1}^{s_2} \mu(s)\,\vartheta(s,t)\,ds \end{aligned} \tag{3.6}$$

In the following we will assume that the cross section of the beam is constant, and that the beam is homogeneous.

**Axiom 183** *The evolution of a dynamical system from time $t = t_1$ to a later time $t = t_2$ is always such that the* <u>*action*</u> *is stationary. The action is the integral over the time domain $[t_1, t_2]$ of the Lagrangian which can be determined as the difference between kinetic and potential energy.*

**Axiom 184** *The kinetic energy of an element of the beam characterized by the interval*

$[s_1, s_2]$ *is:*

$$T\left(\dot{w}, \dot{u}, \dot{\vartheta}\right) = \int_{s_1}^{s_2} \frac{1}{2}\lambda\dot{w}^2 + \frac{1}{2}\lambda\dot{u}^2 + \frac{1}{2}I\dot{\vartheta}^2 ds \tag{3.7}$$

*where* $\lambda$ *is the density per unit length of the beam, and* $I$ *is the moment of inertia of the section. Both these quantities are positive.*

**Axiom 185** *The potential elastic energy of an element of the beam characterized by the interval* $[s_1, s_2]$ *is:*

$$U\left(\vartheta, w', u', \vartheta'\right) = \int_{s_1}^{s_2} \frac{1}{2}k_N\left(w'\right)^2 + \frac{1}{2}k_T\left(u' - \vartheta\right)^2 + \frac{1}{2}k_M\left(\vartheta'\right)^2 ds \tag{3.8}$$

*where* $k_N$ *is called axial stiffness,* $k_T$ *shear stiffness and* $k_M$ *rotational stiffness. All these quantities are positive.*

**Remark 186** *Sometimes it is convenient to introduce a kinetic energy density and a potential energy density, defined by:*

$$\begin{aligned} \varpi_T\left(\dot{w}, \dot{u}, \dot{\vartheta}, s\right) &= \tfrac{1}{2}\lambda\dot{w}^2 + \tfrac{1}{2}\lambda\dot{u}^2 + \tfrac{1}{2}I\dot{\vartheta}^2 \\ \varpi_U\left(\vartheta, w', u', \vartheta', s\right) &= \tfrac{1}{2}k_N\left(w'\right)^2 + \tfrac{1}{2}k_T\left(u' - \vartheta\right)^2 + \tfrac{1}{2}k_M\left(\vartheta'\right)^2 \end{aligned} \tag{3.9}$$

Once we have assumed expressions (3.7) and (3.8) to be the kinetic and potential energy of the beam, the Lagrangian $\mathfrak{L}$ of the beam becomes:

$$\mathcal{L}\left(w, u, \vartheta, w', u', \vartheta', \dot{w}, \dot{u}, \dot{\vartheta}, s\right) = T - U - U_{b_N} - U_{b_T} - U_\mu \tag{3.10}$$

And the action becomes:

$$\mathcal{A}\left(w\left(\cdot\right), u\left(\cdot\right), \vartheta\left(\cdot\right)\right) = \int_{t_1}^{t_2} \mathcal{L}\left(w, u, \vartheta, w', u', \vartheta', \dot{w}, \dot{u}, \dot{\vartheta}, s\right) dt \tag{3.11}$$

By virtue of the Hamilton's principle the motion $\mathcal{M}$ is determined by imposing the stationarity of the Action (3.11).

In order to have a well-posed variational problem we have to add a set of boundary conditions to the functions $w$, $u$, $\vartheta$, limiting the set of admissible variations to a defined set.

Initially we will not specify the boundary conditions at $s_1$ and $s_2$, so as to get a general expression for the first variation of the Action valid for different boundary conditions; nevertheless, we will immediately impose the boundary conditions at the time $t_1$ and $t_2$ given by:

$$\begin{aligned} &w\left(s,t_1\right)=w_1\left(s,t_1\right) \quad w\left(s,t_2\right)=w_2\left(s,t_2\right)\\ &u\left(s,t_1\right)=u_1\left(s,t_1\right) \quad u\left(s,t_2\right)=u_2\left(s,t_2\right)\\ &\vartheta\left(s,t_1\right)=\vartheta_1\left(s,t_1\right) \quad \vartheta\left(s,t_2\right)=\vartheta_2\left(s,t_2\right) \end{aligned} \tag{3.12}$$

then the variations $\delta w$, $\delta u$, $\delta\vartheta$ of the functions $w$, $u$, $\vartheta$ will be such that:

$$\begin{aligned} &\delta w\left(s,t_2\right)=\delta w\left(s,t_1\right)=0\\ &\delta u\left(s,t_2\right)=\delta u\left(s,t_1\right)=0\\ &\delta\vartheta\left(s,t_2\right)=\delta\vartheta\left(s,t_1\right)=0 \end{aligned} \tag{3.13}$$

these variations are called synchronous.

Consider the action corresponding to the functions $w$, $u$, $\vartheta$ variated by the quantities $\delta w$, $\delta u$, $\delta\vartheta$, respectively:

$$\begin{aligned} &\mathcal{A}\left(w+\delta w,u+\delta u,\vartheta+\delta\vartheta\right)=\int_{t_1}^{t_2}\int_{s_1}^{s_2}\varpi_T\left(\tfrac{\partial}{\partial t}\left(w+\delta w\right),\tfrac{\partial}{\partial t}\left(u+\delta u\right),\tfrac{\partial}{\partial t}\left(\vartheta+\delta\vartheta\right),s\right)-\\ &-\varpi_U\left(\vartheta+\delta\vartheta,\left(w+\delta w\right)',\left(u+\delta u\right)',\left(\vartheta+\delta\vartheta\right)',s\right)\\ &+b_N\left(w+\delta w\right)+b_T\left(u+\delta u\right)+\mu\left(\vartheta+\delta\vartheta\right)dsdt \end{aligned} \tag{3.14}$$

Since the differentiation with respect to the independent variables $t$ and $s$ is commutative with the $\delta$ operator, from the previous equation we get:

$$\mathcal{A}(w+\delta w, u+\delta u, \vartheta+\delta\vartheta) = \int_{t_1}^{t_2}\int_{s_1}^{s_2} \varpi_T\left((\dot{w}+\delta\dot{w}), (\dot{u}+\delta\dot{u}), \left(\dot{\vartheta}+\delta\dot{\vartheta}\right), s\right) -$$

$$-\varpi_U\left(\vartheta+\delta\vartheta, (w'+\delta w'), (u'+\delta u'), (\vartheta'+\delta\vartheta'), s\right)$$

$$+b_N(w+\delta w) + b_T(u+\delta u) + \mu(\vartheta+\delta\vartheta)\, dsdt \tag{3.15}$$

Consider now the first term in the expression(3.15):

$$\int_{t_1}^{t_2}\int_{s_1}^{s_2} \varpi_T\left((\dot{w}+\delta\dot{w}), (\dot{u}+\delta\dot{u}), \left(\dot{\vartheta}+\delta\dot{\vartheta}\right), s\right) dsdt \tag{3.16}$$

and substitute the first of (3.9):

$$\int_{t_1}^{t_2}\int_{s_1}^{s_2} \frac{1}{2}\lambda(\dot{w}+\delta\dot{w})^2 + \frac{1}{2}\lambda(\dot{u}+\delta\dot{u})^2 + \frac{1}{2}I\left(\dot{\vartheta}+\delta\dot{\vartheta}\right)^2 dsdt \tag{3.17}$$

that is:

$$\int_{t_1}^{t_2}\int_{s_1}^{s_2} \frac{1}{2}\lambda\dot{w}^2 + \lambda\dot{w}\delta\dot{w} + \frac{1}{2}\lambda\dot{u}^2 + \lambda\dot{u}\delta\dot{u} + \frac{1}{2}I\dot{\vartheta}^2 + I\dot{\vartheta}\delta\dot{\vartheta}\, dsdt + O(\delta^2) \tag{3.18}$$

considering again (3.9), we can write the previous expression as:

$$\int_{t_1}^{t_2}\int_{s_1}^{s_2} \varpi_T\left(\dot{w}, \dot{u}, \dot{\vartheta}, s\right) + \lambda\dot{w}\delta\dot{w} + \lambda\dot{u}\delta\dot{u} + I\dot{\vartheta}\delta\dot{\vartheta}\, dsdt + O(\delta^2) \tag{3.19}$$

Consider now the second term in the expression(3.15):

$$-\int_{t_1}^{t_2}\int_{s_1}^{s_2} \varpi_U\left(\vartheta+\delta\vartheta, (w'+\delta w'), (u'+\delta u'), (\vartheta'+\delta\vartheta'), s\right) dsdt \tag{3.20}$$

and substitute the second of (3.9):

$$-\int_{t_1}^{t_2}\int_{s_1}^{s_2} \frac{1}{2}k_N(w'+\delta w')^2 + \frac{1}{2}k_T\left((u'+\delta u') - (\vartheta+\delta\vartheta)\right)^2 + \frac{1}{2}k_M(\vartheta'+\delta\vartheta')^2\, dsdt \tag{3.21}$$

that is:

$$
\begin{aligned}
&-\int_{t_1}^{t_2}\int_{s_1}^{s_2} \tfrac{1}{2}k_N\left(w'\right)^2 + k_N w'\delta w' + \tfrac{1}{2}k_T\left(u'-\vartheta\right)^2 + k_T\left(u'-\vartheta\right)\left(\delta u'-\delta\vartheta\right) \\
&+\tfrac{1}{2}k_M\left(\vartheta'\right)^2 + k_M\vartheta'\delta\vartheta' dsdt + O(\delta^2)
\end{aligned}
\tag{3.22}
$$

considering again (3.9), we can write the previous equation as:

$$
-\int_{t_1}^{t_2}\int_{s_1}^{s_2} \varpi_U\left(\vartheta, w', u', \vartheta', s\right) + k_N w'\delta w' + k_T\left(u'-\vartheta\right)\left(\delta u'-\delta\vartheta\right) + k_M\vartheta'\delta\vartheta' dsdt + O(\delta^2)
\tag{3.23}
$$

Finally consider the remaining three terms in the expression (3.15):

$$
\int_{t_1}^{t_2}\int_{s_1}^{s_2} b_N\left(w+\delta w\right) + b_T\left(u+\delta u\right) + \mu\left(\vartheta+\delta\vartheta\right)\, dsdt
\tag{3.24}
$$

and expand the products to get:

$$
\int_{t_1}^{t_2}\int_{s_1}^{s_2} b_N w + b_T u + \mu\vartheta + b_N\delta w + b_T\delta u + \mu\delta\vartheta\, dsdt
\tag{3.25}
$$

Substituting these results into eq.(3.15) we get:

$$
\begin{aligned}
&\mathcal{A}\left(w+\delta w, u+\delta u, \vartheta+\delta\vartheta\right) = \mathcal{A}\left(w, u, \vartheta\right) \\
&\int_{t_1}^{t_2}\int_{s_1}^{s_2} \left(\lambda\dot{w}\delta\dot{w} + \lambda\dot{u}\delta\dot{u} + I\dot{\vartheta}\delta\dot{\vartheta}\right) - \left(k_N w'\delta w' + k_T\left(u'-\vartheta\right)\left(\delta u'-\delta\vartheta\right) + k_M\vartheta'\delta\vartheta'\right) + \\
&+\left(b_N\delta w + b_T\delta u + \mu\delta\vartheta\right)\, dsdt + O(\delta^2)
\end{aligned}
\tag{3.26}
$$

Hence the *first variation* of the action $\mathcal{A}$ is:

$$
\begin{aligned}
\delta\mathcal{A} = &\int_{t_1}^{t_2}\int_{s_1}^{s_2} \left(\lambda\dot{w}\delta\dot{w} + \lambda\dot{u}\delta\dot{u} + I\dot{\vartheta}\delta\dot{\vartheta}\right) - \left(k_N w'\delta w' + k_T\left(u'-\vartheta\right)\left(\delta u'-\delta\vartheta\right) + k_M\vartheta'\delta\vartheta'\right) + \\
&+\left(b_N\delta w + b_T\delta u + \mu\delta\vartheta\right)\, dsdt
\end{aligned}
\tag{3.27}
$$

Consider the first term in eq.(3.27):

$$\int_{t_1}^{t_2}\int_{s_1}^{s_2}\left(\lambda\dot{w}\delta\dot{w}+\lambda\dot{u}\delta\dot{u}+I\dot{\vartheta}\delta\dot{\vartheta}\right)dsdt \tag{3.28}$$

and integrate by parts with respect to $t$:

$$\int_{s_1}^{s_2}\left(\lambda\dot{w}\delta w+\lambda\dot{u}\delta u+I\dot{\vartheta}\delta\vartheta\right)\Big|_{t=t_1}^{t=t_2}ds-\int_{t_1}^{t_2}\int_{s_1}^{s_2}\left(\lambda\ddot{w}\delta w+\lambda\ddot{u}\delta u+I\ddot{\vartheta}\delta\vartheta\right)dsdt \tag{3.29}$$

Consider now the second term in eq.(3.27):

$$-\int_{t_1}^{t_2}\int_{s_1}^{s_2}\left(k_N w'\delta w'+k_T\left(u'-\vartheta\right)\left(\delta u'-\delta\vartheta\right)+k_M\vartheta'\delta\vartheta'\right)dsdt \tag{3.30}$$

and integrate by parts with respect to $s$:

$$\begin{aligned}&-\int_{t_1}^{t_2}\left(k_N w'\delta w+k_T\left(u'-\vartheta\right)\delta u+k_M\vartheta'\delta\vartheta\right)\Big|_{s=s_1}^{s=s_2}dt+\\&\int_{t_1}^{t_2}\int_{s_1}^{s_2}\left(k_N w''\delta w+k_T\left(\left(u''-\vartheta'\right)\delta u+\left(u'-\vartheta\right)\delta\vartheta\right)+k_M\vartheta''\delta\vartheta\right)dsdt\end{aligned} \tag{3.31}$$

Substituting these expressions into (3.27) we get:

$$\begin{aligned}\delta\mathcal{A}=&\int_{s_1}^{s_2}\left(\lambda\dot{w}\delta w+\lambda\dot{u}\delta u+I\dot{\vartheta}\delta\vartheta\right)\Big|_{t=t_1}^{t=t_2}ds-\int_{t_1}^{t_2}\left(k_N w'\delta w+k_T\left(u'-\vartheta\right)\delta u+k_M\vartheta'\delta\vartheta\right)\Big|_{s=s_1}^{s=s_2}dt+\\&+\int_{t_1}^{t_2}\int_{s_1}^{s_2}\left(k_N w''\delta w+k_T\left(\left(u''-\vartheta'\right)\delta u+\left(u'-\vartheta\right)\delta\vartheta\right)+k_M\vartheta''\delta\vartheta\right)-\left(\lambda\ddot{w}\delta w+\lambda\ddot{u}\delta u+I\ddot{\vartheta}\delta\vartheta\right)\\&+\left(b_N\delta w+b_T\delta u+\mu\delta\vartheta\right)dsdt\end{aligned} \tag{3.32}$$

Manipulating this expression we get:

$$\begin{aligned}\delta\mathcal{A}=&\int_{s_1}^{s_2}\left(\lambda\dot{w}\delta w+\lambda\dot{u}\delta u+I\dot{\vartheta}\delta\vartheta\right)\Big|_{t=t_1}^{t=t_2}ds-\int_{t_1}^{t_2}\left(k_N w'\delta w+k_T\left(u'-\vartheta\right)\delta u+k_M\vartheta'\delta\vartheta\right)\Big|_{s=s_1}^{s=s_2}dt+\\&+\int_{t_1}^{t_2}\int_{s_1}^{s_2}\left(k_N w''-\lambda\ddot{w}+b_N\right)\delta w+\left(k_T\left(u'-\vartheta\right)'-\lambda\ddot{u}+b_T\right)\delta u+\\&\left(k_T\left(u'-\vartheta\right)+k_M\vartheta''-I\ddot{\vartheta}+\mu\right)\delta\vartheta\, dsdt\end{aligned} \tag{3.33}$$

and now the problem is to find $(w, u, \vartheta)$ such that the first variation of $\mathcal{A}$ is zero, i.e. the stationary points of $\mathcal{A}$ :

$$\delta\mathcal{A} = 0 \tag{3.34}$$

The first integral in eq.(3.33) immediately vanishes since we are considering synchronous variations. In order to find the extremals of the action, we have to add the boundary condition at the edges of the beam-element to the variational problem.

These boundary conditions cannot be chosen arbitrarily, but they have to fulfill the transversality conditions:

$$\begin{aligned}
&k_N w'(s_2, t)\,\delta w(s_2, t) = 0 \\
&k_N w'(s_1, t)\,\delta w(s_1, t) = 0 \\
&k_T\left(u'(s_2, t) - \vartheta(s_2, t)\right)\delta u(s_2, t) = 0 \\
&k_T\left(u'(s_1, t) - \vartheta(s_1, t)\right)\delta u(s_1, t) = 0 \\
&k_M \vartheta'(s_2, t)\,\delta\vartheta(s_2, t) = 0 \\
&k_M \vartheta'(s_1, t)\,\delta\vartheta(s_1, t) = 0
\end{aligned} \tag{3.35}$$

in order to make it possible to find a solution of the variational problem.

**Example 187** *One of the possible choices of admissible boundary conditions, would be:*

$$\begin{aligned}
&w(s_1, t) = w_1(t) \quad && w(s_2, t) = w_2(t) \\
&u(s_1, t) = u_1(t) \quad && u(s_2, t) = u_2(t) \\
&\vartheta(s_1, t) = \vartheta_1(t) \quad && \vartheta(s_2, t) = \vartheta_2(t)
\end{aligned} \tag{3.36}$$

*which limit the set of admissible variations as follows:*

$$
\begin{aligned}
&\delta w\left(s_1,t\right)=0 \quad \delta w\left(s_2,t\right)=0\\
&\delta u\left(s_1,t\right)=0 \quad \delta u\left(s_2,t\right)=0\\
&\delta\vartheta\left(s_1,t\right)=0 \quad \delta\vartheta\left(s_2,t\right)=0
\end{aligned}
\tag{3.37}
$$

Introducing the functions $N$, $T$, $M$ defined by:

$$
\begin{aligned}
&N=k_N w'\\
&T=k_T\left(u'-\vartheta\right)\\
&M=k_M\vartheta'
\end{aligned}
\tag{3.38}
$$

the transversality conditions become:

$$
\begin{aligned}
&N\left(s_2,t\right)\delta w\left(s_2,t\right)=0\\
&N\left(s_1,t\right)\delta w\left(s_1,t\right)=0\\
&T\delta u\left(s_2,t\right)=0\\
&T\delta u\left(s_1,t\right)=0\\
&M\delta\vartheta\left(s_2,t\right)=0\\
&M\delta\vartheta\left(s_1,t\right)=0
\end{aligned}
\tag{3.39}
$$

The functions $N$, $T$, $M$ are called *contact actions*, in particular $N$ is called the generalized normal force, $T$ the generalized shear force and $M$ the bending moment. Furthermore the equations in (3.38) are called *constitutive equations* and they establish a relationship between the deformation parameters and the contact actions.

Imposing these conditions on the set of admissible variations, also the second integral vanishes and since the third should be zero for every admissible variation, we get the following Euler-Lagrange equations:

$$\begin{aligned} & k_N w'' - \lambda \ddot{w} + b_N = 0 \\ & k_T (u' - \vartheta)' - \lambda \ddot{u} + b_T = 0 \\ & k_T (u' - \vartheta) + k_M \vartheta'' - I \ddot{\vartheta} + \mu = 0 \end{aligned} \tag{3.40}$$

The equations in (3.40) are called the *equations of an elastica*, or Navier equations, for a Timoshenko beam.

$$\begin{aligned} & N' - \lambda \ddot{w} + b_N = 0 \\ & T' - \lambda \ddot{u} + b_T = 0 \\ & T + M' - I \ddot{\vartheta} + \mu = 0 \end{aligned} \tag{3.41}$$

The partial differential equations in (3.41) are called *balance equations.*

Substituting the constitutive equations (3.38) into the definition of the density of potential elastic energy (3.9) we obtain:

$$\varpi_U \left( \vartheta, w', u', \vartheta', s \right) = \frac{1}{2} N w' + \frac{1}{2} T \left( u' - \vartheta \right) + \frac{1}{2} M \vartheta' \tag{3.42}$$

## Energy of a Timoshenko beam

**Definition 188** *The Energy of an element of the beam characterized by the interval* $[s_1, s_2]$ *and subjected to the external actions per unit length detailed in the previous section is:*

$$\mathcal{E} \left( w, u, \vartheta, w', u', \vartheta', \dot{w}, \dot{u}, \dot{\vartheta}, s \right) = T + U + U_{b_N} + U_{b_T} + U_\mu \tag{3.43}$$

**Proposition 189** *The beam element is conservative.*

***Proof.*** *To prove that the beam element is conservative, we have to demonstrate:*

$$\frac{d}{dt} \mathcal{E} = 0 \tag{3.44}$$

*i.e. the energy is a first integral of the motion. Hence:*

$$\frac{d}{dt}\mathcal{E} = \frac{d}{dt}\left(\int_{s_1}^{s_2} (\varpi_T + \varpi_U)\, ds + U_{b_N} + U_{b_T} + U_\mu\right) \tag{3.45}$$

*Substituting (3.9) and (3.6) we get:*

$$\frac{d}{dt}\mathcal{E} = \frac{d}{dt}\int_{s_1}^{s_2} \tfrac{1}{2}\lambda\dot{w}^2 + \tfrac{1}{2}\lambda\dot{u}^2 + \tfrac{1}{2}I\dot{\vartheta}^2 + \tfrac{1}{2}k_N\,(w')^2 + \tfrac{1}{2}k_T\,(u'-\vartheta)^2 + \tfrac{1}{2}k_M\,(\vartheta')^2$$
$$-b_N\,(s)\,w - b_T\,(s)\,u - \mu\,(s)\,\vartheta\, ds \tag{3.46}$$

*Differentiating the argument of the integral with respect to time, we obtain:*

$$\frac{d}{dt}\mathcal{E} = \int_{s_1}^{s_2} \lambda\dot{w}\ddot{w} + \lambda\dot{u}\ddot{u} + I\dot{\vartheta}\ddot{\vartheta} + k_N w'\dot{w}' + k_T\left(u'-\vartheta\right)\left(\dot{u}'-\dot{\vartheta}\right) + k_M\vartheta'\dot{\vartheta}' - b_N\dot{w} - b_T\dot{u} - \mu\dot{\vartheta}ds \tag{3.47}$$

*Integrating by parts the 4-th, 5-th and 6-th terms of the previous equation we get:*

$$\frac{d}{dt}\mathcal{E} = \int_{s_1}^{s_2} \dot{w}\,(\lambda\ddot{w} - k_N w'' - b_N) + \dot{u}\left(\lambda\ddot{u} - b_T - k_T\left(u''-\vartheta'\right)\right) + \dot{\vartheta}\left(I\ddot{\vartheta} - k_T\,(u'-\vartheta) - \mu - k_M\vartheta''\right) ds$$
$$+ k_N w'\dot{w}\big|_{s=s_1}^{s=s_2} + k_T\,(u'-\vartheta)'\left(\dot{u}'-\dot{\vartheta}\right)\Big|_{s=s_1}^{s=s_2} + k_M\vartheta'\dot{\vartheta}\Big|_{s=s_1}^{s=s_2} \tag{3.48}$$

*By virtue of the equations of an elastica (3.40) and the transversality conditions (3.35),* $\frac{d}{dt}\mathcal{E} = 0.$ ■

## Euler beam

An Euler beam is a Timoshenko beam, with the additional constitutive assumption:

$$u' = \vartheta \tag{3.49}$$

i.e., the sections in the actual configuration are always orthogonal to the axis of the beam.

This assumption immediately leads to:

$$k_T \to \infty \tag{3.50}$$

since, even if the section is orthogonal to the centerline in the actual configuration, the shear contact action $T$ has to be finite.

Furthermore, the energy densities for an Euler beam become:

$$\begin{aligned} &\varpi_T\left(\dot{w},\dot{u},\dot{\vartheta},s\right) = \tfrac{1}{2}\lambda\dot{w}^2 + \tfrac{1}{2}\lambda\dot{u}^2 + \tfrac{1}{2}I\left(\dot{u}'\right)^2 \\ &\varpi_U\left(\vartheta,w',u',\vartheta',s\right) = \tfrac{1}{2}k_N\left(w'\right)^2 + \tfrac{1}{2}k_M\left(u''\right)^2 \end{aligned} \tag{3.51}$$

The equations of an elastica for an Euler beam become:

$$\begin{aligned} &k_N w'' - \lambda\ddot{w} + b_N = 0 \\ &k_M u'''' - I\,\ddot{u}'' + \lambda\ddot{u} = b_T - \mu' \end{aligned} \tag{3.52}$$

**Remark 190** *In the applications we are dealing with the kinetic energy due to the rotation of the section is negligible with respect of the kinetic energy due to the motion of the axis of the beam, then:*

$$T = \int_{s_1}^{s_2} \frac{1}{2}\lambda\dot{w}^2 + \frac{1}{2}\lambda\dot{u}^2 ds \tag{3.53}$$

*Furthermore we will neglect $I\,\ddot{u}''$ in the previous equations to get:*

$$\begin{aligned} &k_N w'' - \lambda\ddot{w} + b_N = 0 \\ &k_M u'''' + \lambda\ddot{u} = b_T - \mu' \end{aligned} \tag{3.54}$$

# Chapter 4

# Electrical analogs for simple mechanical structures

**Claim 191** *"An analogy is a recognized relationship of consistent mutual similarity between the equations and structures appearing within two or more fields of knowledge, and an identification and association of the quantities and structural elements which play mutually similar roles in these equations and structures, for the purpose of facilitating transfer of knowledge of mathematical procedures of analysis and behavior of the structures between these fields." Floyd A. Firestone (1956)[12]*

### History of the mechanical impedance methods for vibration problems

In the classical approach to vibration problems, the vibrating structure is studied as a unique mechanical system, i.e. the governing equations for the entire structure are written and any change in the topology of the structure leads dramatically to a new set of equations which needs to be solved anew.

During the 1940's a lot of effort was expended in attempts to approach the problem as a "black box". That is, rather than describing the complete behavior of the whole structure at every point, only the motion of a few of its points is considered, resulting from forces applied at these assigned points. This procedure recalls that based on the ideas of Maxwell, Castigliano, Mohr, and the method of Muller-Breslau for the study of indeterminate systems. Furthermore, with this procedure, it is possible to characterize the behavior of each structural member and then assemble them following well-established connection rules derived from the theory of networks. The first attempts were essentially focused on lumped parameter systems, disregarding distributed elements such as beams and plates (see Firestone (1956)[13]). Later the impedance method was applied to more complex systems, leading to fruitful results in industrial and civil applications such as torsional vibration of wings and vibration of floors.

Following this black box approach the response of the whole system may be built up from partial characteristics of its structural members, each of them modelled as a one-port network. Every one port network establishes a relation between the "force" at a point (chosen as the across variable) and the "velocity" at the same point (chosen as the through variable). Depending on the particular structural member the "force" can be either a shear force or a normal force, and the "velocity" can be either a deflection or an elongation. In this framework it is possible to define a so called *mechanical impedance* and a *mobility.* The mechanical impedance, for brevity impedance, is defined as the ratio of the Laplace transform of force to the Laplace transform of velocity, while the mobility is defined as the inverse of the impedance.

However this black box approach can not be applied to a generic structure, since not all the structural members can be fully described by one-port networks. In fact, only those structural members modeled at their terminals by one dynamic variable and one kinematic variable can be completely represented by one port networks. Indeed, as we will see in the following chapter, any frame, constituted by an Euler beam, needs to be modeled as a 6−port network, in order to consider all its vibrating properties.

In what follows we want to show two particular applications of the impedance analogy, and a simple structure analyzed by virtue of this black box approach.

### Simply supported beam loaded at the end by an axial force

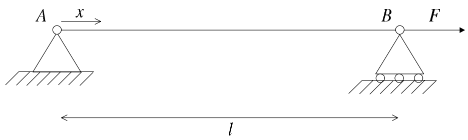


Figure 4.1: Simply supported beam

We assume that the constitutive equation for the beam is:

$$N = k_N w'$$

where $k_N$ is a positive real constant.

From the balance equation for $N$, taking into account the inertial bulk force, we get:

$$N' + \lambda \ddot{w} = 0 \tag{4.1}$$

The pivot at $A$ and the roller at $B$ set the following boundary conditions:

$$N(t,l) = F(t) \Rightarrow w'(t,l) = \frac{F(t)}{k_N} \qquad w(t,0) = 0 \tag{4.2}$$

The first step of our study is to non-dimensionalize the equations of the structural problem.

Let us assume as characteristic values:

$$\begin{cases} l \text{ is the lenght of the beam} \\ t_0 = \frac{2\pi}{\omega_0}, \text{ where } \omega_0 \text{ is the angular frequency of the first vibrational mode of the beam} \end{cases} \tag{4.3}$$

Let us introduce the following dimensionless variables:

$$\xi = \frac{x}{l} \qquad \tau = \frac{t}{t_0} \qquad \zeta = \frac{w}{l} \tag{4.4}$$

Then the constitutive equation becomes:

$$N = k_N \zeta' \tag{4.5}$$

Introducing a dimensionless normal action $F_N = \dfrac{N}{k_N}$, we get:

$$\begin{aligned} & F_N = \zeta' \\ & \frac{F(\tau)}{k_N} := F_F(\tau) \end{aligned} \tag{4.6}$$

Further, for the balance equation we get:

$$\frac{k_N}{l} F_N' - \frac{\lambda l}{t_0^2} \ddot{\zeta} = 0 \tag{4.7}$$

Substituting the constitutive equation into the previous equation, we obtain the equation of evolution of the structure:

$$\zeta'' = \frac{\lambda l^2}{k_N t_0^2} \ddot{\zeta} \tag{4.8}$$

Considering as kinematical descriptor of the system the velocity, instead of the displacement, we get in dimensionless variables:

$$\nu'' = \frac{\lambda l^2}{k_N t_0^2} \ddot{\nu} \tag{4.9}$$

The boundary conditions are, in term of the dimensionless velocity:

$$\nu(\tau, 0) = 0 \qquad \nu'(\tau, 1) = \dot{F}_F(\tau) \tag{4.10}$$

Transforming (4) and the boundary conditions by a bilateral Laplace transform we get:

$$\begin{aligned} \tilde{\nu}(\eta, \xi)'' &= \frac{\lambda l^2}{k_N t_0^2} \eta^2 \tilde{\nu}(\eta, \xi) \\ \tilde{\nu}(\eta, 0) &= 0 \qquad \tilde{\nu}'(\eta, 1) = \eta \tilde{F}_F(\eta) \end{aligned} \tag{4.11}$$

where the tilda denotes the Laplace transformation and $\eta$ is the dimensionless Laplace variable.

The solution of this boundary value problem is:

$$\tilde{\nu}(\eta, \xi) = \frac{\tilde{F}_F(\eta)}{\alpha} \tanh(\alpha \eta \xi)$$

where $\alpha = \dfrac{l}{t_0}\sqrt{\dfrac{\lambda}{k_N}}$.

The dimensionless mechanical impedance of the structure is:

$$Z_0(\eta) := \frac{\tilde{F}_F(\eta)}{\tilde{\nu}(\eta, 1)} = \frac{\alpha}{\tanh \alpha \eta} \tag{4.12}$$

while the mobility is:

$$\alpha_0(\eta) := \frac{\tilde{\nu}(\eta, 1)}{\tilde{F}_F(\eta)} = \frac{\tanh \alpha \eta}{\alpha} \tag{4.13}$$

Now let us find the mechanical impedance $Z(s)$ and the mobility $\alpha(s)$:

$$\begin{aligned} Z(s) &:= \frac{\mathcal{L}\left[F\left(t\right)\right]}{\mathcal{L}\left[\dot{w}\left(t,l\right)\right]} = \frac{\tilde{F}\left(s\right)}{s\,\tilde{w}\left(s,l\right)} \\ \alpha(s) &:= \frac{\mathcal{L}\left[\dot{w}\left(t,l\right)\right]}{\mathcal{L}\left[F\left(t\right)\right]} = \frac{s\,\tilde{w}\left(s,l\right)}{\tilde{F}\left(s\right)} \end{aligned} \tag{4.14}$$

They are related to the dimensionless values by:

$$\begin{aligned} Z(s) &= \frac{\tilde{F}\left(s\right)}{s\,\tilde{w}\left(s,l\right)} = \frac{k_N\tilde{F}_F\left(\eta\right)}{\frac{l}{t_0}\,\tilde{\nu}\left(\eta,1\right)} = \frac{k_N t_0}{l}Z_0(\eta) \Rightarrow Z(s) = \frac{\sqrt{\lambda k_N}}{\tanh\sqrt{\frac{\lambda}{k_N}}.ls} \\ \alpha(s) &= \frac{s\,\tilde{w}\left(s,l\right)}{\tilde{F}\left(s\right)} = \frac{\alpha_0(\eta)}{k_N t_0} \Rightarrow \alpha(s) = \frac{\tanh\sqrt{\frac{\lambda}{k_N}}ls}{\sqrt{\lambda k_N}} \end{aligned} \tag{4.15}$$

**Synthesis of the electric analog**

Given the mobility $\alpha(s)$ we can introduce an electrical admittance $Y(s)$ :

$$Y(s) = \frac{1}{R_0}\tanh\sqrt{\frac{\lambda}{k_N}}.ls = \frac{\sqrt{\lambda k_N}}{R_0}\alpha(s) \tag{4.16}$$

where $R_0$ is a constant that has the dimension of a resistance.

By virtue of the techniques developed in chapter (2) we can synthesize a one-port network, the admittance of which is $Y(s)$. The ratio of current and voltage for this one port-network is proportional to the ratio of velocity and force by the factor $\dfrac{\sqrt{\lambda k_N}}{R_0}$.

From the examples given in section (1), it is clear that:

$$A\left(s\right) = A_0\tanh\gamma s = \frac{2A_0}{\gamma}\sum_{n=1}^{\infty}\left(\frac{s}{s^2+\omega_n^2}\right)$$

where $\omega_n = \dfrac{\pi}{2\gamma}\left(2n+1\right)$ and $A_0$ and $\gamma$ are positive real constants.

Then:

$$Y(s) = \frac{1}{R_0}\tanh\left(\sqrt{\frac{\lambda}{k_N}}l\right)s = \frac{2}{R_0 l}\sqrt{\frac{k_N}{\lambda}}\sum_{n=1}^{\infty}\left(\frac{s}{s^2+\omega_n^2}\right) \tag{4.17}$$

where $\omega_n = \sqrt{\frac{k_N}{\lambda}}\frac{\pi}{2l}(2n+1)$.

Thus $Y(s)$ can be designed as infinite one-port networks in parallel, each of them being a capacitor and an inductor in series, such as:

$$L_n = \frac{R_0 l}{2}\sqrt{\frac{\lambda}{k_N}} \qquad C_n = \frac{8l}{R_0}\sqrt{\frac{\lambda}{k_N}}\frac{1}{(2n+1)^2} \tag{4.18}$$

We can explicitly write:

$$Y(s) = \sum_{n=1}^{\infty}\frac{sC_n}{s^2 L_n C_n + 1} \tag{4.19}$$

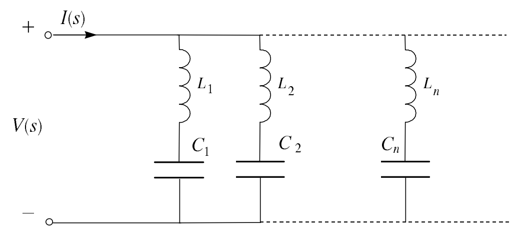


Figure 4.2: Circuit analog to the simply supported beam

### Cantilever beam

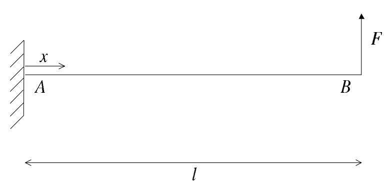


Figure 4.3: Cantilever beam

The constitutive equations for the Timoshenko beam are:

$$\begin{aligned} M &= k_M \vartheta' \\ T &= k_T \left(u' - \vartheta\right) \end{aligned} \tag{4.20}$$

where $k_N$ and $k_T$ are positive real constants.

The balance equations, neglecting the rotational inertia, for the shear action and the bending moment are:

$$\begin{aligned} M' + T &= 0 \\ T' &= \lambda \ddot{u} \end{aligned} \tag{4.21}$$

The clamping device at $A$ and the force at $B$ determine the following boundary conditions:

$$\begin{aligned} u(t,0) &= 0 \\ \vartheta(t,0) &= 0 \\ M(t,l) &= 0 \\ T(t,l) &= F(t) \end{aligned} \tag{4.22}$$

Now let us restrict to the Euler beam, thus supposing $k_T \to \infty$, then:

$$u' - \vartheta = 0 \tag{4.23}$$

Thus the constitutive equation for the bending moment becomes:

$$M = k_M u'' \tag{4.24}$$

And the boundary conditions:

$$\begin{aligned} u(t,0) &= 0 \\ u'(t,0) &= 0 \\ M(t,l) &= 0 \\ T(t,l) &= F(t) \end{aligned} \tag{4.25}$$

The first step of our study is to non-dimensionalize the equations of the structural problem.

Let us assume as characteristic values:

$$\begin{cases} l \text{ is the lenght of the beam} \\ t_0 = \frac{2\pi}{\omega_0}, \text{ where } \omega_0 \text{ is the angular frequency of the first mode of the beam} \\ r_0 \text{ is the radius of gyration of the section} \end{cases} \tag{4.26}$$

Let us introduce the following dimensionless variables:

$$\xi = \frac{x}{l} \qquad \tau = \frac{t}{t_0} \qquad \zeta = \frac{u}{r_0} \tag{4.27}$$

Then the constitutive equation for the bending moment becomes:

$$M = \frac{k_M r_0}{l^2}\zeta'' \tag{4.28}$$

Introducing a dimensionless bending moment $F_M = \dfrac{l^2}{(k_M r_0)}M$, we get:

$$F_M = \zeta'' \tag{4.29}$$

Further, for the balance equation of bending moment we get:

$$T + \frac{k_M r_0}{l^3}F_M' = 0 \tag{4.30}$$

Introducing a dimensionless shear action $F_T = \frac{l^3}{(k_M r_0)} T$, we get:

$$F_T + F_M' = 0 \tag{4.31}$$

$$F_F = \frac{l^3}{(k_M r_0)} F \tag{4.32}$$

Further from the balance equation for $T$ we obtain:

$$F_T' = \alpha^2 \ddot{\zeta} \tag{4.33}$$

where $\alpha^2 = \frac{\lambda l^4}{t_0^2 k_M}$.

Substituting (4.29) and (4.33) into (4.31) we get:

$$\zeta^{IV} + \alpha^2 \ddot{\zeta} = 0 \tag{4.34}$$

Considering the kinematical descriptor of the system to be the velocity, instead of the displacement, we get in dimensionless variables:

$$\nu^{IV} + \alpha^2 \ddot{\nu} = 0 \tag{4.35}$$

The boundary conditions are in terms of the dimensionless velocity:

$$\begin{aligned} \nu(\tau, 0) &= 0 \\ \nu'(\tau, 0) &= 0 \\ \nu''(\tau, 1) &= 0 \\ -\nu'''(\tau, 1) &= \dot{F}_F(\tau) \end{aligned} \tag{4.36}$$

Transforming (4.35) and the boundary conditions by a bilateral Laplace transform we get:

$$\begin{aligned}
&\tilde{\nu}\left(\eta,\xi\right)^{IV}+\alpha^2\eta^2\tilde{\nu}\left(\eta,\xi\right)=0\\
&\tilde{\nu}(\eta,0)=0\\
&\tilde{\nu}'\left(\eta,0\right)=0\\
&\tilde{\nu}''(\eta,1)=0\\
&\tilde{\nu}'''(\eta,1)=-\eta\tilde{F}\left(\eta\right)
\end{aligned}\tag{4.37}$$

The general solution of the differential equation is:

$$\tilde{\nu}\left(\eta,\xi\right)=A\sin k\xi+B\cos k\xi+C\sinh k\xi+D\cosh k\xi$$

where $k=\sqrt{\eta\alpha}e^{-i\frac{\pi}{4}}$.

Imposing the boundary conditions we can find the dimensionless mechanical impedance of the beam:

$$Z_0(\eta):=\frac{\tilde{F}_F\left(\eta\right)}{\tilde{\nu}(\eta,1)}=\frac{k^3}{\eta}\frac{\left(1+\cos k\cosh k\right)}{\sin k\cosh k-\cos k\sinh k}\tag{4.38}$$

while the mobility is:

$$\alpha_0(\eta):=\frac{\tilde{\nu}(\eta,1)}{\tilde{F}_F\left(\eta\right)}=\frac{\eta}{k^3}\frac{\sin k\cosh k-\cos k\sinh k}{\left(1+\cos k\cosh k\right)}\tag{4.39}$$

Now let us find the mechanical impedance $Z(s)$ and the mobility $\alpha(s)$:

$$\begin{aligned}
Z(s)&:=\frac{\mathcal{L}\left[F\left(t\right)\right]}{\mathcal{L}\left[\dot{u}\left(t,l\right)\right]}=\frac{\tilde{F}\left(s\right)}{s\,\tilde{u}\left(s,l\right)}\\
\alpha(s)&:=\frac{\mathcal{L}\left[\dot{u}\left(t,l\right)\right]}{\mathcal{L}\left[F\left(t\right)\right]}=\frac{s\,\tilde{u}\left(s,l\right)}{\tilde{F}\left(s\right)}
\end{aligned}\tag{4.40}$$

They are related to the dimensionless values by:

$$k=\sqrt{\eta\alpha}e^{-i\frac{\pi}{4}}=e^{-i\frac{\pi}{4}}\sqrt{st_0\frac{l^2}{t_0}\sqrt{\frac{\lambda}{k_M}}}=e^{-i\frac{\pi}{4}}l\sqrt{s}\sqrt[4]{\frac{\lambda}{k_M}}:=e^{-i\frac{\pi}{4}}\sqrt{\beta s}\tag{4.41}$$

with

$$\beta^2 = \frac{\lambda l^4}{k_M} \qquad k = k\left(s\right) = e^{-i\frac{\pi}{4}}\sqrt{\beta s} \tag{4.42}$$

thus:

$$Z(s) = \frac{\tilde{F}\left(s\right)}{s\,\tilde{u}\left(s,l\right)} = \frac{\frac{k_M r_0}{l^3}\tilde{F}_F\left(\eta\right)}{\frac{r_0}{t_0}\,\tilde{\nu}\left(\eta,1\right)} = \frac{k_M t_0}{l^3}Z_0(\eta) \Rightarrow Z(s) = \frac{k_M}{l^3}\frac{k^3}{s}\frac{\left(1+\cos k\cosh k\right)}{\sin k\cosh k - \cos k\sinh k}$$

$$\alpha(s) = \frac{s\,\tilde{u}\left(s,l\right)}{\tilde{F}\left(s\right)} = \frac{1}{Z(s)} \Rightarrow \alpha(s) = \frac{l^3}{k_M}\frac{s}{k^3}\frac{\sin k\cosh k - \cos k\sinh k}{\left(1+\cos k\cosh k\right)} \tag{4.43}$$

**Synthesis of the electric analog.**

Given the mobility $\alpha(s)$ we can introduce an electrical admittance $Y(s)$ :

$$Y(s) = A_0\frac{s}{k^3}\frac{\sin k\cosh k - \cos k\sinh k}{\left(1+\cos k\cosh k\right)} = A_0\frac{k_M}{l^3}\alpha(s) \tag{4.44}$$

where $A_0$ is a positive real constant.

By virtue of the techniques developed in chapter (2) we can synthesize a one-port network, the admittance of which is $Y(s)$. The ratio of current and voltage for this one port-network is proportional to the ratio of velocity and force by the factor $\frac{A_0}{l^3}k_M$.

From the examples given in section (1), it is clear that:

$$Y\left(s\right) = \sum_{n=1}^{\infty} 2s\left(\frac{k_n}{s^2+\omega_n^2}\right) \tag{4.45}$$

Thus $Y(s)$ can be designed as infinite one-port networks in parallel, each of them being a capacitor and an inductor in series, such as:

$$L_n = \frac{1}{2k_n} \qquad C_n = \frac{2k_n}{\omega_n^2} \tag{4.46}$$

We can explicitly write:

$$Y(s) = \sum_{n=1}^{\infty} \frac{sC_n}{s^2 L_n C_n + 1} \tag{4.47}$$

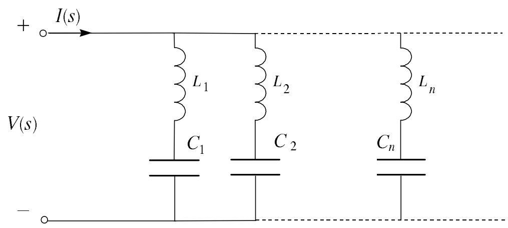


Figure 4.4: Circuit analog to cantilever beam

**Remark 192** *The natural frequencies of this admittance are not equally spaced on the imaginary axis.*

# Chapter 5

# A black box approach to the theory of vibrating structures

The purpose of this chapter is to extend the *black box* approach studied in the theory of networks to vibrations of plane beam-structures. (Where there will not be possible misunderstandings we will call a plane beam-structure, simply structure).

We will limit our observations to the mechanical devices which can be modelled as black boxes, communicating to the outer world by a finite number of access points called *terminals*.

Furthermore we will suppose that the state of each terminal $\mathcal{T}_i$ is completely characterized by a pair of 3−tuples $(\alpha_i, \tau_i) = \left(\left(v_1^i, v_2^i, \omega^i\right), \left(t_1^i, t_2^i, M^i\right)\right)$.

The pair $\left(v_1^i, v_2^i\right)$ represents the velocities and $\omega^i$ the angular velocity, at the terminal $\mathcal{T}_i$, with respect to a given observer $\mathbf{O}$ characterized by an origin $\mathfrak{o}$ and a basis $(\mathbf{e}_1, \mathbf{e}_2)$ of the space of translations, while $\left(t_1^i, t_2^i, M^i\right)$ represents the contact actions, force and bending moment, applied at a Cauchy cut at the terminal $\mathcal{T}_i$, with respect to $\mathbf{O}$.

**Notation 193** *For a* $n-$*terminal mechanical device, we will group the state variables*

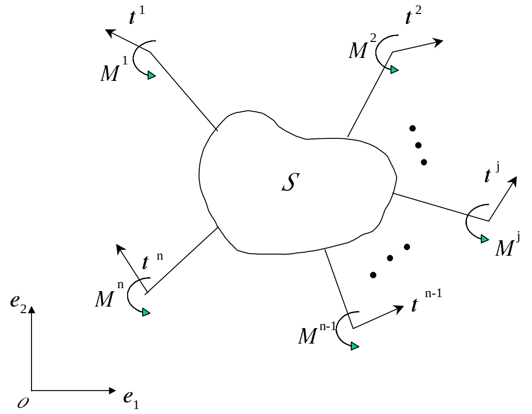


Figure 5.1: Representation of a structure

*with the following convention:*

$$\boldsymbol{\alpha} \;:\; = (\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}) := \begin{pmatrix} v_1^1 & v_2^1 & \omega^1 \\ ... & ... & ... \\ v_1^n & v_2^n & \omega^n \end{pmatrix} \tag{5.1}$$

$$\boldsymbol{\tau} \;:\; = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{M}) := \begin{pmatrix} t_1^1 & t_2^1 & M^1 \\ ... & ... & ... \\ t_1^n & t_2^n & M^n \end{pmatrix}$$

As we have done dealing with networks, we will suppose that the Signal Space is still $\mathcal{D}_+$.

**Definition 194** *Given a binary relation $\mathcal{C}_S$ on $\mathcal{D}_+^{n\times 3} \times \mathcal{D}_+^{n\times 3}$, a beam-structure $\mathcal{S}$ is:*

$$\mathcal{S} = \left\{ \left( (\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}), (\mathbf{t}_1, \mathbf{t}_2, \mathbf{M}) \right) \in \mathcal{D}_+^{n\times 3} \times \mathcal{D}_+^{n\times 3}, (\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega})\, \mathcal{C}_S\, (\mathbf{t}_1, \mathbf{t}_2, \mathbf{M}) \right\} \tag{5.2}$$

**Claim 195** *We have established a physical analogy between the model of a network and* *the model of a structure. In fact we can claim that an* $n-$ *terminal structure* $\mathcal{S}$ *is analogous to a* $3n-$*port network* $\mathcal{N}$.

**Definition 196** *A frame* $\mathcal{F}$ *is a* <u>*specific interconnection-topology*</u> *of structures. Any constituent of a given structure is called a structural member.*

**Definition 197** *The total instantaneous power expended into* $\mathcal{S}$ *is:*

$$p\left(t\right)=\left(\mathbf{t}_{1},\mathbf{t}_{2},\mathbf{M}\right):\left(\mathbf{v}_{1},\mathbf{v}_{2},\boldsymbol{\omega}\right) \tag{5.3}$$

*and the net energy delivered to the network at time* $\tilde{t}$ *is:*

$$\mathcal{E}\left(\tilde{t}\right)=\int_{-\infty}^{\tilde{t}}p\left(t\right)dt \tag{5.4}$$

The connection of different structural members can be mathematically represented as operations on the structural members dictated by the force and moment balance equations, and the congruence relation.

**Definition 198** *Given a frame, we call a node the set of terminals of constituting structures interconnected in the given topology.*

**Axiom 199** *Given a frame* $\mathcal{F}$*, let's* $\{\mathfrak{n}_i\}$ *denote the set of its nodes:*

- *The sums of forces and bending moment at every node must vanish*

- *Let* $(\mathfrak{n}_0,\mathfrak{n}_1,...,\mathfrak{n}_k,\mathfrak{n}_{k+1}=\mathfrak{n}_0)$ *be a closed loop of nodes in* $\mathcal{F}$*:*

$$\sum_{i=1}^{k}\left(v_i^1,v_i^2,\omega_i\right)=(0,0,0) \tag{5.5}$$

## Structure Analysis

### Building blocks

In this section we will give a brief outline of the most common structural members used in engineering applications. These will be the building blocks of the frames we are interested in.

### Constraints

**Definition 200** *A pivot $\mathcal{S}_P$ is a one-terminal structure defined by:*

$$\mathcal{S}_P = \left\{ \left( (v_1, v_2, \omega), (t_1, t_2, M) \right) \in \mathcal{D}_+^{1\times3} \times \mathcal{D}_+^{1\times3} : (v_1, v_2, M) = (0,0,0) \right\} \tag{5.6}$$

**Definition 201** *A clamping device $\mathcal{S}_C$ (or encastre) is a one-terminal structure defined by:*

$$\mathcal{S}_C = \left\{ \left( (v_1, v_2, \omega), (t_1, t_2, M) \right) \in \mathcal{D}_+^{1\times3} \times \mathcal{D}_+^{1\times3} : (v_1, v_2, \omega) = (0,0,0) \right\} \tag{5.7}$$

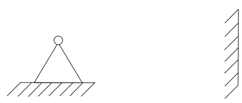

Figure 5.2: Representation of the pivot and clamping device

**Definition 202** *A roller $\mathcal{S}_R^{\varphi}$ is a one-terminal structure defined by:*

$$\mathcal{S}_R^{\varphi} = \left\{ \begin{array}{c} \left( (v_1, v_2, \omega), (t_1, t_2, M) \right) \in \mathcal{D}_+^{1\times3} \times \mathcal{D}_+^{1\times3} : -\sin\varphi v_1 + \cos\varphi v_2 = 0, \ M = 0, \\ \cos\varphi t_1 + \sin\varphi t_2 = 0 \end{array} \right\} \tag{5.8}$$

**Definition 203** *A link-block $\mathcal{S}^{\varphi}_{L-B}$ is a one-terminal structure defined by:*

$$\mathcal{S}^{\varphi}_{L-B} = \left\{ \begin{array}{c} ((v_1, v_2, \omega), (t_1, t_2, M)) \in \mathcal{D}^{1\times 3}_{+} \times \mathcal{D}^{1\times 3}_{+} : -\sin\varphi v_1 + \cos\varphi v_2 = 0,\ \omega = 0 \\ \cos\varphi t_1 + \sin\varphi t_2 = 0 \end{array} \right\} \tag{5.9}$$

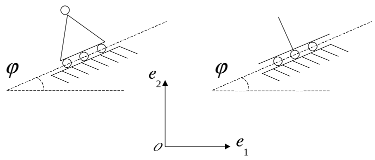


Figure 5.3: Representation of a roller and a link-block

**Lumped elements**

For a direct definition of lumped elements see Molly (1958)[1].

**Definition 204** *A damping device $\mathcal{S}_D$ is a two terminal structure such that:*

$$\mathcal{S}_D = \left\{ \begin{array}{c} ((\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}), (\mathbf{t}_1, \mathbf{t}_2, \mathbf{M})) \in \mathcal{D}^{2\times 3}_{+} \times \mathcal{D}^{2\times 3}_{+} : \left(t^1_1, t^1_2, M^1\right) = -\left(t^2_1, t^2_2, M^2\right), \\ \left(t^1_1, t^1_2, M^1\right) = r\left(v^2_1 - v^1_1, v^2_2 - v^1_2, \omega^2 - \omega^1\right) \end{array} \right\} \tag{5.10}$$

*where $r \in \mathbb{R}^+$ is the damping ratio.*

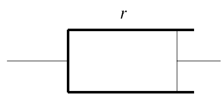


Figure 5.4: Representation of a damping device

**Definition 205** *A spring $\mathcal{S}_{Sp}^{\varphi}$ is a two terminal structure such that:*

$$\mathcal{S}_{Sp}^{\varphi} = \left\{ \begin{array}{l} \left(\left(\mathbf{v}_1, \mathbf{v}_2, \dot{\boldsymbol{\vartheta}}\right), \left(\mathbf{t}_1, \mathbf{t}_2, \mathbf{M}\right)\right) \in \mathcal{D}_{+}^{2\times 3} \times \mathcal{D}_{+}^{2\times 3} : \mathbf{M} = \mathbf{0},\ \left(t_1^1, t_2^1\right) = -\left(t_1^2, t_2^2\right), \\ -t_1^1 \sin\varphi + t_2^1 \cos\varphi = 0, \\ \left(\dot{t}_1^1 \cos\varphi + \dot{t}_2^1 \sin\varphi\right) = -k\left(\left(v_1^2 - v_1^1\right)\cos\varphi + \left(v_2^2 - v_2^1\right)\sin\varphi\right) \end{array} \right\} \tag{5.11}$$

*where $k \in \mathrm{I\!R}^{+}$ is the stiffness constant of the spring.*

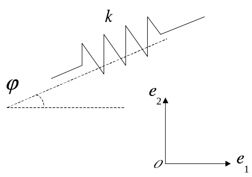


Figure 5.5: Representation of a spring

**Definition 206** *A mass $\mathcal{S}_m$ is a two terminal structure such that:*

$$\mathcal{S}_m = \left\{ \begin{array}{c} \left(\left(\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}\right), \left(\mathbf{t}_1, \mathbf{t}_2, \mathbf{M}\right)\right) \in \mathcal{D}_{+}^{2\times 3} \times \mathcal{D}_{+}^{2\times 3} : \mathbf{M} = \mathbf{0},\ \left(v_1^1, v_2^1\right) = \left(v_1^2, v_2^2\right) \\ \left(t_1^1 - t_1^2, t_2^1 - t_2^2\right) = m\left(\dot{v}_1^1 - \dot{v}_1^2, \dot{v}_2^1 - \dot{v}_2^2\right) \end{array} \right\} \tag{5.12}$$

*$m \in \mathrm{I\!R}^{+}$ is the mass of $\mathcal{S}_m^{\varphi}$.*

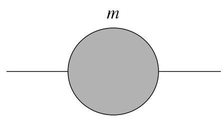


Figure 5.6: Representation of a mass

**Sources**

**Definition 207** *A shaker* $\mathcal{S}_{Sh}$ *is a one terminal structure such that:*

$$\mathcal{S}_S = \left\{ \left( \left( v_1, v_2, \omega \right), \left( t_1, t_2, M \right) \right) \in \mathcal{D}_+^{1\times 3} \times \mathcal{D}_+^{1\times 3} : \left( v_1, v_2, \omega \right) = \left( V_1\left(t\right), V_2\left(t\right), \Omega\left(t\right) \right) \right\} \tag{5.13}$$

**Definition 208** *A load* $\mathcal{S}_L$ *is a one terminal structure such that:*

$$\mathcal{S}_L = \left\{ \left( \left( v_1, v_2, \omega \right), \left( t_1, t_2, M \right) \right) \in \mathcal{D}_+^{1\times 3} \times \mathcal{D}_+^{1\times 3} : \left( t_1, t_2, M \right) = \left( T_1\left(t\right), T_2\left(t\right), M\left(t\right) \right) \right\} \tag{5.14}$$

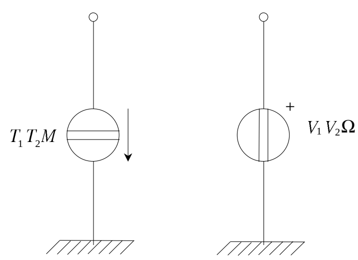


Figure 5.7: Representation of a load and a shaker

**Definition 209** *A beam is a two-terminal structure defined by:*

$$\mathcal{S}_B = \left\{ \left( \left( \mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega} \right), \left( \mathbf{t}_1, \mathbf{t}_2, \mathbf{M} \right) \right) \in \mathcal{D}_+^{2\times 3} \times \mathcal{D}_+^{2\times 3} : \left( \mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega} \right) \mathcal{C}_B \left( \mathbf{t}_1, \mathbf{t}_2, \mathbf{M} \right) \right\} \tag{5.15}$$

*where the binary relation* $\mathcal{C}_B$ *on* $\mathcal{D}_+^{2\times 3} \times \mathcal{D}_+^{2\times 3}$ *select those* $\left( \left( \mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega} \right), \left( \mathbf{t}_1, \mathbf{t}_2, \mathbf{M} \right) \right)$ *such that:*

$$\begin{cases} \left( \mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega} \right) = \begin{pmatrix} w_1 (0,t) & w_2 (0,t) & w_3 (0,t) \\ w_1 (l,t) & w_2 (l,t) & w_3 (l,t) \end{pmatrix} \\ \left( \mathbf{t}_1, \mathbf{t}_2, \mathbf{M} \right) = \begin{pmatrix} -s_1 (0,t) & -s_2 (0,t) & -c (0,t) \\ s_1 (l,t) & s_2 (l,t) & c (l,t) \end{pmatrix} \end{cases} \tag{5.16}$$

*where* $l$ *is the length of the beam and the fields* $w_1 (x,t)$ *,* $w_2 (x,t)$ *,* $w_3 (x,t)$ *,* $s_1 (x,t)$ *,* $s_2 (x,t)$ *,* $c (x,t)$ *verify:*

$$\begin{cases} \mathbf{s}' (x,t) + \mathbf{b} (x,t) = \mathbf{0} \\ \left( \mathbf{c}' (x,t) + \mathbf{s} (x,t) \times \mathbf{r}' (x) + \boldsymbol{\mu} (x,t) \right) \cdot \left( \mathbf{e}_1 \times \mathbf{e}_2 \right) = 0 \\ \begin{pmatrix} \dot{s}_1 (x,t) \\ \dot{s}_2 (x,t) \\ \dot{c} (x,t) \end{pmatrix} = \mathbf{R} \begin{pmatrix} w_1' (x,t) \\ w_2' (x,t) \\ w_3' (x,t) \end{pmatrix} \\ \mathbf{b} (x,t) = -\lambda \dot{\mathbf{v}} (x,t) \\ \boldsymbol{\mu} (x,t) = \mathbf{0} \end{cases} \tag{5.17}$$

**Remark 210** *The previous definition is well-posed since, by virtue of the definition of our signal space* $\mathcal{D}_+$*, at* $t = 0$ *the beam is at rest.*

**Remark 211** *The first two equations of the previous set stand for the balance equation of forces and torques for a plane beam of generic shape. The third equation is not exactly* *a constitutive relation, since the variables on the left hand side are not the strains. The remaining two equations specify the bulk actions, maintaining the negligibility of the rotational inertia due to the physical assumption of thin sections with respect to the vertical direction* $\mathbf{e}_2$.

Now we consider two particular kinds of beam, in particular we will assume that for these beams the undeformed shape is straight and parallel to the $\mathbf{e}_1$ vector.

**Definition 212** *A Timoshenko beam* $\mathcal{S}_T$ *is a particular beam defined by:*

$$\mathcal{S}_T = \left\{ \left( \left( \mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega} \right), \left( \mathbf{t}_1, \mathbf{t}_2, \mathbf{M} \right) \right) \in \mathcal{D}_+^{2\times 3} \times \mathcal{D}_+^{2\times 3} : \left( \mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega} \right) \mathcal{C}_T \left( \mathbf{t}_1, \mathbf{t}_2, \mathbf{M} \right) \right\} \tag{5.18}$$

*where the binary relation* $\mathcal{C}_T$ *on* $\mathcal{D}_+^{2\times 3} \times \mathcal{D}_+^{2\times 3}$ *select those* $\left( \left( \mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega} \right), \left( \mathbf{t}_1, \mathbf{t}_2, \mathbf{M} \right) \right)$ *such that:*

$$\begin{cases} \left( \mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega} \right) = \begin{pmatrix} \dot{w}\left(0,t\right) & \dot{u}\left(0,t\right) & \dot{\vartheta}\left(0,t\right) \\ \dot{w}\left(l,t\right) & \dot{u}\left(l,t\right) & \dot{\vartheta}\left(l,t\right) \end{pmatrix} \\ \left( \mathbf{t}_1, \mathbf{t}_2, \mathbf{M} \right) = \begin{pmatrix} -N\left(0,t\right) & -T\left(0,t\right) & -M\left(0,t\right) \\ N\left(l,t\right) & T\left(l,t\right) & M\left(l,t\right) \end{pmatrix} \end{cases} \tag{5.19}$$

*where* $l$ *is the length of the beam and the fields* $\dot{w}\left(x,t\right), \dot{u}\left(x,t\right), \dot{\vartheta}\left(x,t\right), N\left(x,t\right), T\left(x,t\right),$

*$M(x,t)$ verify:*

$$\begin{cases} N(x,t) + b_N(x,t) = 0 \\ T(x,t) + b_T(x,t) = 0 \\ M(x,t)' + T(x,t) + \mu(x,t) = 0 \\ \begin{pmatrix} \dot{N}(x,t) \\ \dot{T}(x,t) \\ \dot{M}(x,t) \end{pmatrix} = \begin{pmatrix} k_N & 0 & 0 \\ 0 & k_T & 0 \\ 0 & 0 & k_M \end{pmatrix} \begin{pmatrix} \dot{w}'(x,t) \\ \dot{u}'(x,t) - \dot{\vartheta}(x,t) \\ \dot{\vartheta}'(x,t) \end{pmatrix} \\ b_N(x,t) = -\lambda \dot{w}(x,t) \\ b_T(x,t) = -\lambda \dot{u}(x,t) \\ \mu(s,t) = 0 \end{cases} \tag{5.20}$$

**Definition 213** *An Euler beam $\mathcal{S}_E$ is a particular beam defined by:*

$$\mathcal{S}_E = \left\{ \left( (\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}), (\mathbf{t}_1, \mathbf{t}_2, \mathbf{M}) \right) \in \mathcal{D}_+^{2\times 3} \times \mathcal{D}_+^{2\times 3} : (\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}) \, \mathcal{C}_E \, (\mathbf{t}_1, \mathbf{t}_2, \mathbf{M}) \right\} \tag{5.21}$$

*where the binary relation $\mathcal{C}_E$ on $\mathcal{D}_+^{2\times 3} \times \mathcal{D}_+^{2\times 3}$ selects those $(\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}), (\mathbf{t}_1, \mathbf{t}_2, \mathbf{M})$ such that:*

$$\begin{cases} (\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}) = \begin{pmatrix} \dot{w}(0,t) & \dot{u}(0,t) & \dot{\vartheta}(0,t) \\ \dot{w}(l,t) & \dot{u}(l,t) & \dot{\vartheta}(l,t) \end{pmatrix} \\ (\mathbf{t}_1, \mathbf{t}_2, \mathbf{M}) = \begin{pmatrix} -N(0,t) & -T(0,t) & -M(0,t) \\ N(l,t) & T(l,t) & M(l,t) \end{pmatrix} \end{cases} \tag{5.22}$$

*where $l$ is the length of the beam and the fields $\dot{w}(x,t)$, $\dot{u}(x,t)$, $\dot{\vartheta}(x,t)$, $N(x,t)$, $T(x,t)$,*

*$M(x,t)$ verify:*

$$\begin{cases} N(x,t) + b_N(x,t) = 0 \\ T(x,t) + b_T(x,t) = 0 \\ M(x,t)' + T(x,t) + \mu(x,t) = 0 \\ \begin{pmatrix} \dot{N}(x,t) \\ \dot{M}(x,t) \end{pmatrix} = \begin{pmatrix} k_N & 0 \\ 0 & k_M \end{pmatrix} \begin{pmatrix} \dot{w}'(x,t) \\ \dot{\vartheta}'(x,t) \end{pmatrix} \\ \dot{u}'(x,t) - \dot{\vartheta}(x,t) = 0 \\ b_N(x,t) = -\lambda \dot{w}(x,t) \\ b_T(x,t) = -\lambda \dot{u}(x,t) \\ \mu(s,t) = 0 \end{cases} \tag{5.23}$$

*the assumption $\dot{u}'(x,t) - \dot{\vartheta}(x,t) = 0$ can be considered as a kinematical constraint, due to the physical hypothesis of shear indeformability, i.e. $k_T \to \infty$.*

### Fundamental properties of structures

As we have done dealing with networks, we will introduce briefly the most fundamental properties of an $n-$terminal structure $\mathcal{S}$.

**Definition 214** *A structure $\mathcal{S}$ is linear if $\mathcal{S}$ is a subspace of $\mathcal{V} := \mathcal{D}_+^{n\times 3} \times \mathcal{D}_+^{n\times 3}$, i.e.:*

$$\forall (\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1), (\boldsymbol{\alpha}_2, \boldsymbol{\tau}_2) \in \mathcal{S}, \beta \in \mathbb{R} \begin{cases} (\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2, \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2) \in \mathcal{S} \\ (\beta\boldsymbol{\alpha}, \boldsymbol{\beta\tau}) \in \mathcal{S} \end{cases} \tag{5.24}$$

**Remark 215** *$\mathcal{V}$ is trivially a vector space, once it is endowed with the simple operations of sum of two vectors $(\alpha, \boldsymbol{\tau})$, and multiplication of a pair by a real number. Furthermore we introduce the notion of convergence as we have done dealing with networks.*

**Example 216** *All the introduced constraints are linear.*

**Example 217** *All the introduced lumped elements are linear.*

**Example 218** *None of the introduced sources is linear.*

**Example 219** *All the introduced continuous elements are linear. In particular since* $\mathcal{C}_B$ *is defined by a set of linear ordinary differential equations and the constitutive equations are linear then the beam is linear.*

**Definition 220** *Let* $\mathcal{S}$ *be a structure,* $\mathcal{S}_a$ *is said to be the augmented structure if:*

$$(\boldsymbol{\alpha},\tau) \in \mathcal{S} \Rightarrow (\boldsymbol{\alpha}+\boldsymbol{\tau},\boldsymbol{\tau}) \in \mathcal{S}_a \tag{5.25}$$

The augmented structure can be thought as the structure obtained by connecting one unit damping device to each of the terminal of the given structure $\mathcal{S}$.

**Definition 221** $\mathcal{S}$ *is solvable if* $\forall\ (\mathbf{V}_1,\mathbf{V}_2,\Omega) \in \mathcal{D}_+^{n\times 3}\ \exists!\ (\boldsymbol{\alpha},\tau) \in \mathcal{S}$ *such that*

$$(\mathbf{V}_1,\mathbf{V}_2,\Omega) = \boldsymbol{\alpha}+\boldsymbol{\tau} = (\mathbf{v}_1,\mathbf{v}_2,\boldsymbol{\omega}) + (\mathbf{t}_1,\mathbf{t}_2,\mathbf{M}) \tag{5.26}$$

**Definition 222** $\mathcal{S}$ *is completely solvable if it is solvable and if*

$$\begin{aligned}
&\forall \langle(\mathbf{V}_1,\mathbf{V}_2,\Omega)_n\rangle : \mathbb{N} \to \mathcal{D}_+^{n\times 3} \text{ convergent to } (\mathbf{V}_1,\mathbf{V}_2,\Omega)\,, \\
&\qquad \langle((\mathbf{v}_1,\mathbf{v}_2,\boldsymbol{\omega})_n\,,(\mathbf{t}_1,\mathbf{t}_2,\mathbf{M})_n)\rangle \text{ converges to } ((\mathbf{v}_1,\mathbf{v}_2,\boldsymbol{\omega})\,,(\mathbf{t}_1,\mathbf{t}_2,\mathbf{M})) \\
&\qquad\qquad \text{with } (\mathbf{V}_1,\mathbf{V}_2,\Omega)_n = (\mathbf{v}_1,\mathbf{v}_2,\boldsymbol{\omega})_n + (\mathbf{t}_1,\mathbf{t}_2,\mathbf{M})_n
\end{aligned} \tag{5.27}$$

***Example 223*** *All the introduced constraints are completely solvable.*

***Example 224*** *All the introduced lumped elements are completely solvable.*

***Example 225*** *All the introduced sources are completely solvable.*

***Example 226*** *All the introduced continuous elements are completely solvable.*

**Definition 227** $\mathcal{S}$ *is time-invariant if:*

$$\forall (\boldsymbol{\alpha}, \boldsymbol{\tau}) \in \mathcal{S},\ \forall t_0 \in \mathrm{I\!R}\ \ \exists (\boldsymbol{\alpha}_{t_0}(t), \boldsymbol{\tau}_{t_0}(t)) = (\boldsymbol{\alpha}(t + t_0), \boldsymbol{\tau}(t + t_0)) \tag{5.28}$$

**Remark 228** *In mechanics the time-invariance property is called memoryless, and it essentially maintains that the structure is not able to store information about its load cycles.*

**Example 229** *All the constraints are time-invariant.*

**Example 230** *All the lumped elements are time-invariant.*

**Example 231** *No one of the sources is in general time invariant since they fix on set of variable, in a way that is generally varying with time.*

**Example 232** *All the continuous elements are time-invariant, since the set of differential equations and the constitutive relations are time invariant.*

**Definition 233** $\mathcal{S}$ *is passive if:*

$$\forall (\boldsymbol{\alpha}, \boldsymbol{\tau}) \in \mathcal{S},\ \forall t \in \mathrm{I\!R},\ \mathcal{E}(t) = \int_{-\infty}^{t} \boldsymbol{\alpha}(\xi) : \boldsymbol{\tau}(\xi)\, d\xi \geqslant 0 \tag{5.29}$$

*otherwise it is active.*

**Example 234** *All the constraints are passive, since for all of them:*

$$p(t) = \boldsymbol{\alpha}(t) : \boldsymbol{\tau}(t) = 0 \Rightarrow \mathcal{E}(t) = 0 \tag{5.30}$$

**Remark 235** *In general these constraints are called "perfect", since they expend zero power on all the possible displacements.*

**Example 236** *The damping element is passive, since:*

$$p(t) = \boldsymbol{\alpha}(t) : \boldsymbol{\tau}(t) = \sum_{i=1}^{2} t_1^i v_1^i + t_2^i v_2^i + M^i \dot{\vartheta}^i = r\left[\left(t_1^1(t)\right)^2 + \left(t_2^1(t)\right)^2 + \left(M^1(t)\right)^2\right] \Rightarrow \mathcal{E}(t) \geqslant 0 \tag{5.31}$$

**Example 237** *The spring is passive since:*

$$\mathcal{E}(t) = \frac{1}{2}k\left(\Delta l(t)\right)^2 \geqslant 0 \tag{5.32}$$

*where $\Delta l(t) = l(t) - l_0$ is the variation of length of the spring with respect to its initial length.*

**Example 238** *The mass is passive since:*

$$\mathcal{E}(t) = \frac{1}{2}m\left(\left(v_1^1(t)\right)^2 + \left(v_1^2(t)\right)^2\right) \geqslant 0 \tag{5.33}$$

**Example 239** *All the sources are active*

From now on for simplicity, we will restrict to Timoshenko and Euler beams.

**Example 240** *All the continuous elements are passive, since:*

$$p(t) = N\,\dot{w}\big|_0^l + T\,\dot{u}\big|_0^l + M\,\omega\big|_0^l \tag{5.34}$$

*and:*

$$\mathcal{E}(t) = \int_{s_1}^{s_2} \frac{1}{2}\lambda\dot{w}^2 + \frac{1}{2}\lambda\dot{u}^2 + \frac{1}{2}k_N\left(w'\right)^2 + \frac{1}{2}k_T\left(u' - \vartheta\right)^2 + \frac{1}{2}k_M\left(\vartheta'\right)^2 ds \geqslant 0 \tag{5.35}$$

As a preliminary to the definition of a lossless structure, we state the following lemma:

**Lemma 241** *If $\mathcal{S}$ is passive, solvable and* $\forall\ (\mathbf{V}_1, \mathbf{V}_2, \Omega) \in \mathcal{D}_+^{n\times 3} \cap L_2^{n\times 3}$, *then* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\tau}$ *belong to* $\mathcal{D}_+^{n\times 3} \cap L_2^{n\times 3}$ *too, and* $\mathcal{E}(\infty) \in \mathbb{R}^+$.

**Proof.** Since the network is solvable then for every $(\mathbf{V}_1, \mathbf{V}_2, \Omega) \in \mathcal{D}_+^{n\times 3}$ there is a unique pair $(\boldsymbol{\alpha}, \boldsymbol{\tau}) \in \mathcal{S}$ such that $(\mathbf{V}_1, \mathbf{V}_2, \Omega) = \boldsymbol{\alpha} + \boldsymbol{\tau}$; thus

$$\begin{aligned} &\int_{-\infty}^{t} (\mathbf{V}_1, \mathbf{V}_2, \Omega)^T(\tau)\ (\mathbf{V}_1, \mathbf{V}_2, \Omega)(\tau)\, d\tau \\ &= \int_{-\infty}^{t} \boldsymbol{\alpha}(\xi)^T \boldsymbol{\tau}(\xi)\, d\xi + \int_{-\infty}^{t} \boldsymbol{\tau}(\xi)^T \boldsymbol{\tau}(\xi)\, d\xi + 2\int_{-\infty}^{t} \boldsymbol{\alpha}^T(\xi)\, \boldsymbol{\tau}(\xi)\, d\xi \end{aligned} \tag{5.36}$$

Supposing that $(\mathbf{V}_1, \mathbf{V}_2, \Omega) \in \mathcal{D}_+^{n\times 3} \cap L_2^{n\times 3}$ we have that $\|(\mathbf{V}_1, \mathbf{V}_2, \Omega)\| \in \mathbb{R}^+$, but, since $\mathcal{S}$ is passive, the previous equality implies that both $\boldsymbol{\alpha}$ and $\boldsymbol{\tau}$ belong to $L_2^{n\times 3}$ and that $\mathcal{E}(\infty) = \int_{-\infty}^{t} \boldsymbol{\alpha}^T(\xi)\, \boldsymbol{\tau}(\xi)\, d\xi \in \mathbb{R}^+$. ■

**Definition 242** *A structure $\mathcal{S}$ is lossless if:*

- *$\mathcal{S}$ is passive*
- *$\mathcal{S}$ is solvable*
- $\forall\ (\mathbf{V}_1, \mathbf{V}_2, \Omega) \in \mathcal{D}_+^{n\times 3} \cap L_2^{n\times 3}$, $\mathcal{E}(\infty) = 0$

**Example 243** *All the introduced constraints are lossless since, in particular,* $p(t) = 0$.

**Example 244** *The introduced mass and the spring are lossless, while the damping device is not lossless.*

**Example 245** *The introduced continuous elements are lossless.*

**Notation 246** *In what follows, in order to emphasize the correspondence between the model of network and the model of structure, and to simplify the mathematics involved in the definition of reciprocity and in the time domain representation, we will use a "Voigt" representation for the state variable* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\tau}$*. That is, we will assemble these two* $n \times 3$ *matrix in two* $3n$ *column vectors in* $\mathcal{D}_+^{3n}$ *:*

$$\boldsymbol{\alpha} = \begin{pmatrix} v_1^1 \\ v_2^1 \\ \omega^1 \\ ... \\ v_1^n \\ v_2^n \\ \omega^n \end{pmatrix}, \qquad \boldsymbol{\tau} = \begin{pmatrix} t_1^1 \\ t_2^1 \\ M^1 \\ ... \\ t_1^n \\ t_2^n \\ M^n \end{pmatrix} \tag{5.37}$$

**Definition 247** $\mathcal{S}$ *is reciprocal if:*

$$\forall \left(\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1\right), \left(\boldsymbol{\alpha}_2, \boldsymbol{\tau}_2\right) \in \mathcal{S}, \qquad \boldsymbol{\alpha}_1^T * \boldsymbol{\tau}_1 = \boldsymbol{\alpha}_2^T * \boldsymbol{\tau}_1 \tag{5.38}$$

***Example 248*** *All the introduced lumped structures are reciprocal.*

**Remark 249** *If* $(\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1)$ *and* $(\boldsymbol{\alpha}_2, \boldsymbol{\tau}_2)$ *belong to* $\left(\mathcal{D}_+^{3n} \times \mathcal{D}_+^{3n}\right) \cap \left(\mathcal{S}'^{3n} \times \mathcal{S}'^{3n}\right)$ *then by the theorem of convolution, the previous equation states:*

$$\forall \left(\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1\right), \left(\boldsymbol{\alpha}_2, \boldsymbol{\tau}_2\right) \in \mathcal{S} \cap \left(\mathcal{S}'^{n\times 3} \times \mathcal{S}'^{n\times 3}\right) \qquad \left(\mathcal{L}\left[\boldsymbol{\alpha}_1\right]\right)^T \mathcal{L}\left[\boldsymbol{\tau}_2\right] = \left(\mathcal{L}\left[\boldsymbol{\alpha}_2\right]\right)^T \mathcal{L}\left[\boldsymbol{\tau}_1\right] \tag{5.39}$$

**Proposition 250** *(Dynamical Maxwell-Betti reciprocal theorem) All the introduced continuous structures are reciprocal.*

**Proof.** Suppose that $(\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1)$ and $(\boldsymbol{\alpha}_2, \boldsymbol{\tau}_2)$ belong to $\left(\mathcal{D}_+^{3n} \times \mathcal{D}_+^{3n}\right) \cap \left(\mathcal{S}'^{3n} \times \mathcal{S}'^{3n}\right)$, i.e. every entry is a tempered distribution. Then we can take Laplace transform of all the balance equations, to get:

$$\begin{aligned} N' - \lambda\, \dot{v}_1 &= 0 \Longrightarrow \tilde{N}' = s\lambda\, \tilde{v}_1 \\ M' + T &= 0 \Longrightarrow \tilde{M}' + \tilde{T} = 0 \\ T' - \lambda\, \dot{v}_2 &\Longrightarrow \tilde{T}' = s\lambda\, \tilde{v}_2 \end{aligned} \tag{5.40}$$

Where the tilde denotes the Laplace transform, i.e.

$$\mathbf{f} \in \left(\mathcal{D}_+^{3n} \times \mathcal{D}_+^{3n}\right) \cap \left(\mathcal{S}'^{3n} \times \mathcal{S}'^{3n}\right),\ \mathcal{L}\left[\mathbf{f}\,(x,t)\right](s) =: \tilde{\mathbf{f}}\,(x,s) \tag{5.41}$$

.Furthermore we can take Laplace tansform of the constitutive relations:

$$\begin{aligned} \dot{N} &= k_N v_1' \Longrightarrow s\tilde{N} = k_N \tilde{v}_1' \\ \dot{T} &= k_T v_2' \Longrightarrow s\tilde{T} = k_T \tilde{v}_2' \\ \dot{M} &= k_M \omega' \Longrightarrow s\tilde{M} = k_M \tilde{\omega}' \end{aligned} \tag{5.42}$$

Consider now the two given set of equations for the pair $(\boldsymbol{\alpha}_1, \boldsymbol{\tau}_1)$ and regard the other pair $(\boldsymbol{\alpha}_2, \boldsymbol{\tau}_2)$ as a test vector, i.e. multiply the first balance equation for the normal stresses by the velocity $\tilde{v}_1^2\,(x)$ and integrate over the domain:

$$\begin{aligned} \int_0^l \tilde{N}_1' \tilde{v}_1^2 dx &= \int_0^l s\lambda\, \tilde{v}_1^1 \tilde{v}_1^2 dx \Longrightarrow \frac{k_N}{s} \int_0^l \tilde{v}_1^{1\prime\prime} \tilde{v}_1^2 dx = \int_0^l s\lambda\, \tilde{v}_1^1 \tilde{v}_1^2 dx \Longrightarrow \\ N_1 \tilde{v}_1^2\big|_0^l &= \int_0^l s\lambda\, \tilde{v}_1^1 \tilde{v}_1^2 dx + \frac{k_N}{s} \int_0^l \tilde{v}_1^{1\prime} \tilde{v}_1^{2\prime} dx \end{aligned} \tag{5.43}$$

Now let us multiply the second balance equation by the angular velocity $\tilde{\omega}^2\,(x)$ and integrate over the domain:

$$\int_0^l \left(\tilde{M}_1' \tilde{\omega}^2 + \tilde{T}_1 \tilde{\omega}^2\right) dx = 0 \Longrightarrow \tilde{M}_1 \tilde{\omega}^2\Big|_0^l - \int_0^l \tilde{M}_1 \tilde{\omega}^{2\prime} dx + \int_0^l \tilde{T}_1 \tilde{\omega}^2 dx = 0 \tag{5.44}$$

Now consider the velocity $\tilde{v}_2^2(x)$ such that $\tilde{\omega}^2 = -s\dfrac{\tilde{T}_2}{k_T} + \tilde{v}_2^{2\prime}$:

$$\tilde{M}_1\tilde{\omega}^2\Big|_0^l - k_M\int_0^l \tilde{\omega}^{1\prime}\tilde{\omega}^{2\prime}dx + \int_0^l \tilde{T}_1\left(-s\frac{\tilde{T}_2}{k_T} + \tilde{v}_2^{2\prime}\right)dx = 0 \Longrightarrow$$

$$\tilde{M}_1\tilde{\omega}^2\Big|_0^l + \tilde{T}_1\tilde{v}_2^2\Big|_0^l = k_M\int_0^l \tilde{\omega}^{1\prime}\tilde{\omega}^{2\prime}dx + \frac{s}{k_T}\int_0^l \tilde{T}_1\tilde{T}_2dx + s\int_0^l \lambda\tilde{v}_2^1\tilde{v}_2^2dx$$

Hence:

$$N_1\tilde{v}_1^2\big|_0^l + \tilde{M}_1\tilde{\omega}^2\Big|_0^l + \tilde{T}_1\tilde{v}_2^2\Big|_0^l =$$
$$= \int_0^l s\lambda\,\tilde{v}_1^1\tilde{v}_1^2dx + \frac{k_N}{s}\int_0^l \tilde{v}_1^{1\prime}\tilde{v}_1^{2\prime}dx + k_M\int_0^l \tilde{\omega}^{1\prime}\tilde{\omega}^{2\prime}dx + \frac{s}{k_T}\int_0^l \tilde{T}_1\tilde{T}_2dx + s\int_0^l \lambda\tilde{v}_2^1\tilde{v}_2^2dx =$$
$$= N_2\tilde{v}_1^1\big|_0^l + \tilde{M}_2\tilde{\omega}^1\Big|_0^l + \tilde{T}_2\tilde{v}_2^1\Big|_0^l$$

■

## Time domain representation of linear, completely solvable and time-invariant structures.

As we have stated in Claim(195) an $n-$terminal structure $\mathcal{S}$ is completely analog to a $3n-$port network $\mathcal{N}$.

Hence we can exploit all the results obtained in the time domain representation for networks when dealing with structures.

**Corollary 251** *For a linear, completely solvable and time invariant $n-$terminal structure $\mathcal{S}$ there exists a unique distribution* $\Upsilon_a$ *in* $\mathcal{D}_+^{3n\prime}$ *such that:*

$$\forall \mathbf{S} \in \mathcal{D}_+^{3n} \qquad \boldsymbol{\tau} = \Upsilon_a * \mathbf{S} \tag{5.45}$$

*where*

$$\mathbf{S} := \begin{pmatrix} V_1^1 \\ V_2^1 \\ \Omega^1 \\ ... \\ V_1^n \\ V_2^n \\ \Omega^n \end{pmatrix} \tag{5.46}$$

*is the generic Shaker. Since* $\mathbf{S} = \boldsymbol{\tau} + \boldsymbol{\alpha}$ *then:*

$$\boldsymbol{\alpha} = -\mathrm{Y}_a * \mathbf{S} + \mathbf{S} = (\delta - \mathrm{Y}_a) * \mathbf{S} \tag{5.47}$$

*where* $\delta$ *is the Dirac distribution in* $\mathcal{D}_+^{3n}$*.*

## Frequency domain representation of linear, completely solvable and time-invariant structures.

In this section we will introduce the most useful frequency domain representation for linear, completely solvable and time invariant $n-$terminal structures.

Hence, from eq.(5.45) and eq.(5.47) we get:

$$\tilde{\boldsymbol{\tau}}(s) = \mathbf{Y}_a(s)\tilde{\mathbf{S}}(s) \tag{5.48}$$

$$\tilde{\boldsymbol{\alpha}}(\mathbf{s}) = (\mathbf{1} - \mathbf{Y}_a(s))\tilde{\mathbf{S}}(s) \tag{5.49}$$

**Definition 252** *The Mobility matrix* $\mathbf{M}(s)$ *is defined as:*

$$\tilde{\boldsymbol{\alpha}} = \mathbf{M}(s)\tilde{\boldsymbol{\tau}} \tag{5.50}$$

$$\mathbf{M}(s) = (\mathbf{1} - \mathbf{Y}_a(s))\mathbf{Y}_a^{-1}(s) \tag{5.51}$$

**Remark 253** *The Mobility matrix can be partitioned as:*

$$\mathbf{M} = \begin{pmatrix} \boldsymbol{\mu}^{1,1} & ... & \boldsymbol{\mu}^{1,n} \\ ... & ... & ... \\ \boldsymbol{\mu}^{n,1} & ... & \boldsymbol{\mu}^{1,n} \end{pmatrix} \tag{5.52}$$

*where* $\boldsymbol{\mu}^{i,j}$ *is a* $3 \times 3$ *matrix which needs a physical interpretation:*

- $\boldsymbol{\mu}^{i,j}_{1,1}$ *is the velocity* $v_1$ *at the terminal* $\mathcal{T}_i$ *when all the terminals* $\mathcal{T}_k$ *with* $k \neq j$ *are load free and at* $\mathcal{T}_j$ *a unit horizontal force is applied.*
- $\boldsymbol{\mu}^{i,j}_{1,2}$ *is the velocity* $v_1$ *at the terminal* $\mathcal{T}_i$ *when all the terminals* $\mathcal{T}_k$ *with* $k \neq j$ *are load free and at* $\mathcal{T}_j$ *a unit vertical force is applied.*
- $\boldsymbol{\mu}^{i,j}_{1,3}$ *is the velocity* $v_1$ *at the terminal* $\mathcal{T}_i$ *when all the terminals* $\mathcal{T}_k$ *with* $k \neq j$ *are load free and at* $\mathcal{T}_j$ *a unit couple is applied.*
- $\boldsymbol{\mu}^{i,j}_{2,1}$ *is the velocity* $v_2$ *at the terminal* $\mathcal{T}_i$ *when all the terminals* $\mathcal{T}_k$ *with* $k \neq j$ *are load free and at* $\mathcal{T}_j$ *a unit horizontal force is applied.*
- $\boldsymbol{\mu}^{i,j}_{2,2}$ *is the velocity* $v_2$ *at the terminal* $\mathcal{T}_i$ *when all the terminals* $\mathcal{T}_k$ *with* $k \neq j$ *are load free and at* $\mathcal{T}_j$ *a unit vertical force is applied.*
- $\boldsymbol{\mu}^{i,j}_{2,3}$ *is the velocity* $v_2$ *at the terminal* $\mathcal{T}_i$ *when all the terminals* $\mathcal{T}_k$ *with* $k \neq j$ *are load free and at* $\mathcal{T}_j$ *a unit couple is applied.*
- $\boldsymbol{\mu}^{i,j}_{3,1}$ *is the angular velocity* $\omega$ *at the terminal* $\mathcal{T}_i$ *when all the terminals* $\mathcal{T}_k$ *with* $k \neq j$ *are load free and at* $\mathcal{T}_j$ *a unit horizontal force is applied.*
- $\boldsymbol{\mu}^{i,j}_{3,2}$ *is the angular velocity* $\omega$ *at the terminal* $\mathcal{T}_i$ *when all the terminals* $\mathcal{T}_k$ *with* $k \neq j$ *are load free and at* $\mathcal{T}_j$ *a unit vertical force is applied.*

- $\boldsymbol{\mu}_{3,3}^{i,j}$ *is the angular velocity* $\omega$ *at the terminal* $\mathcal{T}_i$ *when all the terminals* $\mathcal{T}_k$ *with* $k \neq j$ *are load free and at* $\mathcal{T}_j$ *a unit couple is applied.*

**Remark 254** *The terms on the diagonal of* $\mathbf{M}$ *are called driving mobility:*

- $\boldsymbol{\mu}_{1,1}^{i,i}$ *is the velocity* $v_1$ *at the terminal* $\mathcal{T}_i$ *when all the terminals* $\mathcal{T}_k$ *with* $k \neq i$ *are load free and at* $\mathcal{T}_i$ *a unit horizontal force is applied.*

- $\boldsymbol{\mu}_{2,2}^{i,i}$ *is the velocity* $v_2$ *at the terminal* $\mathcal{T}_i$ *when all the terminals* $\mathcal{T}_k$ *with* $k \neq i$ *are load free and at* $\mathcal{T}_i$ *a unit vertical force is applied.*

- $\boldsymbol{\mu}_{3,3}^{i,i}$ *is the velocity* $v_2$ *at the terminal* $\mathcal{T}_i$ *when all the terminals* $\mathcal{T}_k$ *with* $k \neq i$ *are load free and at* $\mathcal{T}_i$ *a unit couple is applied.*

**Definition 255** *The Mechanical impedance matrix* $\mathbf{Z}(s)$ *is defined as:*

$$\tilde{\boldsymbol{\tau}} = \mathbf{Z}(s)\,\tilde{\boldsymbol{\alpha}} \tag{5.53}$$

$$\mathbf{M}(s) = \mathbf{Y}_a(s)\left(\mathbf{1} - \mathbf{Y}_a(s)\right)^{-1} \tag{5.54}$$

**Remark 256** *The Mechanical impedance matrix can be partitioned as:*

$$\mathbf{Z} = \begin{pmatrix} \boldsymbol{\zeta}^{1,1} & ... & \boldsymbol{\zeta}^{1,n} \\ ... & ... & ... \\ \boldsymbol{\zeta}^{n,1} & ... & \boldsymbol{\zeta}^{1,n} \end{pmatrix} \tag{5.55}$$

*where* $\boldsymbol{\zeta}^{i,j}$ *is a* $3 \times 3$ *matrix, the physical interpretation of each entry can be deduced from the detailed analysis of the mobility matrix. The elements on the diagonal of* $\mathbf{Z}$ *are called driving impedances.*

Assume that the structure has an even number of terminals $n = 2m$, then we can partition the terminals into two sets 1 and 2 such that 1 contains the first $m$ terminals and 2 the others. According to this partition we can write:

$$\tilde{\boldsymbol{\alpha}} = \begin{pmatrix} \tilde{\boldsymbol{\alpha}}_1 \\ \tilde{\boldsymbol{\alpha}}_2 \end{pmatrix} \text{ and } \tilde{\boldsymbol{\tau}} = \begin{pmatrix} \tilde{\boldsymbol{\tau}}_1 \\ \tilde{\boldsymbol{\tau}}_2 \end{pmatrix} \tag{5.56}$$

**Definition 257** *The transmission matrix is defined as:*

$$\begin{pmatrix} \tilde{\boldsymbol{\alpha}}_2 \\ -\tilde{\boldsymbol{\tau}}_2 \end{pmatrix} = \Upsilon(s) \begin{pmatrix} \tilde{\boldsymbol{\alpha}}_1 \\ \tilde{\boldsymbol{\tau}}_1 \end{pmatrix} \tag{5.57}$$

Sometimes it is possible to relate the transmission matrix to the mobility matrix, by the following formulas, equivalent to those stated dealing with networks:

$$\Upsilon(s) = \begin{pmatrix} -\mathbf{M}_{1,2}^{-1} & \mathbf{M}_{1,2}^{-1}\mathbf{M}_{1,1} \\ \mathbf{M}_{2,2}\mathbf{M}_{1,2}^{-1} & \mathbf{M}_{1,2} - \mathbf{M}_{2,2}\mathbf{M}_{1,2}^{-1}\mathbf{M}_{1,1} \end{pmatrix} \tag{5.58}$$

$$\mathbf{M}(s) = \begin{pmatrix} -\Upsilon_{1,2}^{-1}\Upsilon_{2,2} & -\Upsilon_{2,1}^{-1} \\ -\Upsilon_{1,1}\Upsilon_{2,1}^{-1}\Upsilon_{2,2} + \Upsilon_{1,2} & -\Upsilon_{1,1}\Upsilon_{2,1}^{-1} \end{pmatrix} \tag{5.59}$$

### Frequency domain representation of an Euler beam

A beam is a linear, completely solvable and time invariant two-terminal structure, thus we can study its frequency domain representation by means of the tools developed in the previous section.

In order to make use of this representation in the synthesis of the analog network, it is better to work with dimensionless variables and equations.

Hence let us introduce the following characteristic values:

$$\begin{cases} l \text{ is the lenght of the beam} \\ t_0 = \frac{2\pi}{\omega_0}, \text{ where } \omega_0 \text{ is a typical angular frequency that will be defined} \\ \qquad\qquad \text{according to the application we are dealing with.} \\ r_0 \text{ is the radius of gyration of the section} \end{cases} \tag{5.60}$$

and the following dimensionless variables:

$$\varepsilon = \frac{x}{l} \qquad \tau = \frac{t}{t_0} \qquad \xi = \frac{w}{l} \qquad \zeta = \frac{u}{r_0} \qquad \theta = \frac{l}{r_0}\vartheta \tag{5.61}$$

Thus the set of constitutive equations that specify an Euler beam becomes:

$$N = k_N \frac{\partial w}{\partial x} \Rightarrow N = k_N \frac{\partial \xi}{\partial \varepsilon} \tag{5.62}$$

$$\frac{\partial u}{\partial x} = \vartheta \Rightarrow \frac{r_0}{l}\frac{\partial \zeta}{\partial \varepsilon} = \frac{r_0}{l}\theta \Rightarrow \frac{\partial \zeta}{\partial \varepsilon} = \theta \tag{5.63}$$

$$M = k_M \frac{\partial \vartheta}{\partial x} \Rightarrow M = \frac{k_M r_0}{l^2}\frac{\partial \theta}{\partial \varepsilon} \tag{5.64}$$

Introducing the dimensionless contact actions:

$$F_N = \frac{1}{k_N} N \qquad F_T = \frac{l^3}{k_M r_0} T \qquad F_M = \frac{l^2}{k_M r_0} M \tag{5.65}$$

The set of constitutive relations becomes:

$$F_N = \frac{\partial \xi}{\partial \varepsilon} \tag{5.66}$$

$$\frac{\partial \zeta}{\partial \varepsilon} = \theta \tag{5.67}$$

$$F_M = \frac{\partial \theta}{\partial \varepsilon} \tag{5.68}$$

and the set of balance equations becomes:

$$\begin{aligned}
\frac{\partial N}{\partial x} - \lambda \frac{\partial^2 w}{\partial t^2} &= 0 \Rightarrow \frac{k_N}{l}\frac{\partial F_N}{\partial \varepsilon} - \lambda \frac{l}{t_0^2}\frac{\partial^2 \xi}{\partial \tau^2} = 0 \Rightarrow \frac{\partial F_N}{\partial \varepsilon} = \frac{\lambda l^2}{t_0^2 k_N}\frac{\partial^2 \xi}{\partial \tau^2} && (5.69)\\
\frac{\partial T}{\partial x} - \lambda \frac{\partial^2 u}{\partial t^2} &= 0 \Rightarrow \frac{k_M r_0}{l^4}\frac{\partial F_T}{\partial \varepsilon} - \lambda \frac{r_0}{t_0^2}\frac{\partial^2 \zeta}{\partial \tau^2} = 0 \Rightarrow \frac{\partial F_T}{\partial \varepsilon} = \frac{\lambda l^4}{t_0^2 k_M}\frac{\partial^2 \zeta}{\partial \tau^2} && (5.70)\\
\frac{\partial M}{\partial x} + T &= 0 \Rightarrow \frac{k_M r_0}{l^3}\frac{\partial F_M}{\partial \varepsilon} + \frac{k_M r_0}{l^3} F_T = 0 \Rightarrow \frac{\partial F_M}{\partial \varepsilon} + F_T = 0 && (5.71)
\end{aligned}$$

As far as we have done in the analysis of the structure $\mathcal{S}_E$ Euler beam, let us write all the constitutive and balance equations in terms of velocity fields, instead of displacement fields.

Hence, let us introduce the following dimensionless velocity fields:

$$v_\xi = \frac{\partial \xi}{\partial \tau} \qquad v_\zeta = \frac{\partial \zeta}{\partial \tau} \qquad v_\theta = \frac{\partial \theta}{\partial \tau} \tag{5.72}$$

and the two real positive constants:

$$\beta^2 = \frac{\lambda l^4}{t_0^2 k_M} \qquad \gamma^2 = \frac{\lambda l^2}{t_0^2 k_N} \tag{5.73}$$

**Summary 258** *Thus a "dimensionless" Euler beam can be specified by the following relations:*

$$\left\{ \begin{array}{l}
(\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}) = \begin{pmatrix} v_\xi(0,\tau) & v_\zeta(0,\tau) & v_\theta(0,\tau) \\ v_\xi(1,\tau) & v_\zeta(1,\tau) & v_\theta(1,\tau) \end{pmatrix} \\
(\mathbf{t}_1, \mathbf{t}_2, \mathbf{M}) = \begin{pmatrix} -F_N(0,\tau) & -F_T(0,\tau) & -F_M(0,\tau) \\ F_N(1,\tau) & F_T(1,\tau) & F_M(1,\tau) \end{pmatrix}
\end{array} \right. \tag{5.74}$$

*where the fields so far introduced satisfy:*

$$
\begin{cases}
\dfrac{\partial F_N}{\partial \varepsilon} = \gamma^2 \dfrac{\partial v_\xi}{\partial \tau}, \qquad \dfrac{\partial F_T}{\partial \varepsilon} = \beta^2 \dfrac{\partial v_\zeta}{\partial \tau} \\[2em]
\dfrac{\partial F_M}{\partial \varepsilon} + F_T = 0, \qquad \dfrac{\partial F_N}{\partial \tau} = \dfrac{\partial v_\xi}{\partial \varepsilon} \\[2em]
\dfrac{\partial v_\zeta}{\partial \varepsilon} = v_\theta, \qquad \dfrac{\partial F_M}{\partial \tau} = \dfrac{\partial v_\theta}{\partial \varepsilon}
\end{cases}
\tag{5.75}
$$

**Transmission matrix**

Let us find an expression for the dimensionless transmission matrix, defined as:

$$
\begin{pmatrix} \tilde{v}_1^2 \\ \tilde{v}_2^2 \\ \tilde{\omega}^2 \\ -\tilde{t}_1^2 \\ -\tilde{t}_2^2 \\ -\tilde{M}^2 \end{pmatrix} = \Upsilon(\eta) \begin{pmatrix} \tilde{v}_1^1 \\ \tilde{v}_2^1 \\ \tilde{\omega}^1 \\ \tilde{t}_1^1 \\ \tilde{t}_2^1 \\ \tilde{M}^1 \end{pmatrix}
\tag{5.76}
$$

i.e.:

$$
\begin{pmatrix} \tilde{v}_\xi(1,\eta) \\ \tilde{v}_\zeta(1,\eta) \\ \tilde{v}_\theta(1,\eta) \\ -\tilde{F}_N(1,\eta) \\ -\tilde{F}_T(1,\eta) \\ -\tilde{F}_M(1,\eta) \end{pmatrix} = \Upsilon(\eta) \begin{pmatrix} \tilde{v}_\xi(0,\eta) \\ \tilde{v}_\zeta(0,\eta) \\ \tilde{v}_\theta(0,\eta) \\ -\tilde{F}_N(0,\eta) \\ -\tilde{F}_T(0,\eta) \\ -\tilde{F}_M(0,\eta) \end{pmatrix}
\tag{5.77}
$$

To obtained the requested transmission matrix $\Upsilon(\eta)$ , let us consider the Laplace transformed set of balance and constitutive equations for the dimensionless Euler beam:

$$\begin{cases} \dfrac{\partial \tilde{F}_N}{\partial \varepsilon} = \gamma^2 \eta \tilde{v}_\xi, & \dfrac{\partial \tilde{F}_T}{\partial \varepsilon} = \beta^2 \eta \, \tilde{v}_\zeta \\ \\ \dfrac{\partial \tilde{F}_M}{\partial \varepsilon} = -\tilde{F}_T, & \dfrac{\partial \tilde{v}_\xi}{\partial \varepsilon} = \eta \tilde{F}_N \\ \\ \dfrac{\partial \tilde{v}_\zeta}{\partial \varepsilon} = \tilde{v}_\theta, & \dfrac{\partial \tilde{v}_\theta}{\partial \varepsilon} = \eta \tilde{F}_M \end{cases} \tag{5.78}$$

Let us assemble these equations in the so called "normal form", where the prime $'$ denotes the differentiation with respect to the variable $\varepsilon$:

$$\begin{pmatrix} \tilde{v}_\xi(\varepsilon,\eta) \\ \tilde{v}_\zeta(\varepsilon,\eta) \\ \tilde{v}_\theta(\varepsilon,\eta) \\ -\tilde{F}_N(\varepsilon,\eta) \\ -\tilde{F}_T(\varepsilon,\eta) \\ -\tilde{F}_M(\varepsilon,\eta) \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 0 & -\eta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\eta \\ -\gamma^2\eta & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta^2\eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{v}_\xi(\varepsilon,\eta) \\ \tilde{v}_\zeta(\varepsilon,\eta) \\ \tilde{v}_\theta(\varepsilon,\eta) \\ -\tilde{F}_N(\varepsilon,\eta) \\ -\tilde{F}_T(\varepsilon,\eta) \\ -\tilde{F}_M(\varepsilon,\eta) \end{pmatrix} \tag{5.79}$$

The set of equations can be arranged in the following normal form which immediately shows that bending and axial deformations are governed by uncoupled equations:

$$\begin{pmatrix} \tilde{v}_\zeta \\ \tilde{v}_\theta \\ -\tilde{F}_T \\ -\tilde{F}_M \\ \tilde{v}_\xi \\ -\tilde{F}_N \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\eta & 0 & 0 \\ -\beta^2\eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\eta \\ 0 & 0 & 0 & 0 & -\gamma^2\eta & 0 \end{pmatrix} \begin{pmatrix} \tilde{v}_\zeta \\ \tilde{v}_\theta \\ -\tilde{F}_T \\ -\tilde{F}_M \\ \tilde{v}_\xi \\ -\tilde{F}_N \end{pmatrix} \tag{5.80}$$

where the matrix and the columns can be partitioned as follows:

$$\begin{pmatrix} \tilde{\chi}_B(\varepsilon,\eta) \\ \tilde{\chi}_C(\varepsilon,\eta) \end{pmatrix}' = \begin{pmatrix} \mathbf{B} & \mathbf{0}_{4\times 2} \\ \mathbf{0}_{2\times 4} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \tilde{\chi}_B(\varepsilon,\eta) \\ \tilde{\chi}_C(\varepsilon,\eta) \end{pmatrix} \tag{5.81}$$

with:

$$\tilde{\chi}_B = \begin{pmatrix} \tilde{v}_\zeta \\ \tilde{v}_\theta \\ -\tilde{F}_T \\ -\tilde{F}_M \end{pmatrix}, \quad \tilde{\chi}_C = \begin{pmatrix} \tilde{v}_\xi \\ -\tilde{F}_N \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\eta \\ -\beta^2\eta & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & -\eta \\ -\gamma^2\eta & 0 \end{pmatrix} \tag{5.82}$$

**B** is called the *bending* matrix, while **C** is called the *compression* matrix since they relate respectively the bending and the compression of the beam.

**Remark 259** *In the considered model of the beam structure it is assumed that the normal stress does not spend any power on the vertical displacement of the beam. This is true because we limit our study to small deformations and linearize in the neighborhood of the undeformed straight line configuration. Then the bending and the compression modes are completely uncoupled, and it will be possible to synthesize them by means of two different circuits.*

By virtue of the previous remark it is possible to study the bending and the compression as two distinct phenomena, thus introducing two different matrix representations:

$$\begin{pmatrix} \tilde{v}_2^2 \\ \tilde{\omega}^2 \\ -\tilde{t}_2^2 \\ -\tilde{M}^2 \end{pmatrix} = \Upsilon_B(\eta) \begin{pmatrix} \tilde{v}_2^1 \\ \tilde{\omega}^1 \\ \tilde{t}_2^1 \\ \tilde{M}^1 \end{pmatrix} \tag{5.83}$$

$$\begin{pmatrix} \tilde{v}_1^2 \\ -\tilde{t}_1^2 \end{pmatrix} = \Upsilon_C(\eta) \begin{pmatrix} \tilde{v}_1^1 \\ \tilde{t}_1^1 \end{pmatrix} \tag{5.84}$$

where $\Upsilon_B(\eta)$ and $\Upsilon_C(\eta)$ will be called respectively the *bending* and the *compression transmission matrix.*

The solution of (5.81) can be expressed in terms of the value of $\begin{pmatrix} \tilde{\chi}_B \\ \tilde{\chi}_C \end{pmatrix}$ at $\varepsilon = 0$ in the following form, see Pease (1965)[14] for more details :

$$\begin{cases} \tilde{\chi}_B(\varepsilon, \eta) = e^{\mathbf{B}\varepsilon}\, \tilde{\chi}_B(0, \eta) \\ \tilde{\chi}_C(\varepsilon, \eta) = e^{\mathbf{C}\varepsilon}\, \tilde{\chi}_C(0, \eta) \end{cases} \tag{5.85}$$

where the matrices $e^{\mathbf{B}\varepsilon}$ and $e^{\mathbf{C}\varepsilon}$ can be computed as follows.

Let us start with the compression matrix and find the eigenvectors and eigenvalues of $\mathbf{C}$ :

$$\mathbf{C} = \begin{pmatrix} 0 & -\eta \\ -\gamma^2\eta & 0 \end{pmatrix} \tag{5.86}$$

The eigenvectors and eigenvalues of $\mathbf{C}$ are:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ -\gamma \end{pmatrix} \quad \lambda_1 = \eta\gamma, \qquad \mathbf{e}_2 = \begin{pmatrix} 1 \\ \gamma \end{pmatrix} \quad \lambda_2 = -\eta\gamma \tag{5.87}$$

Assembling the eigenvectors in a matrix we get:

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -\gamma & \gamma \end{pmatrix} \tag{5.88}$$

And assembling the eigenvalues in a diagonal matrix we obtain:

$$\Lambda = \begin{pmatrix} \eta\gamma & 0 \\ 0 & -\eta\gamma \end{pmatrix} \tag{5.89}$$

It is well known that:

$$\mathbf{C} = \mathbf{P}\Lambda\mathbf{P}^{-1} \tag{5.90}$$

and the matrix $e^{\mathbf{C}\varepsilon}$ can be evaluated as follows:

$$e^{\mathbf{C}\varepsilon} = \mathbf{P}e^{\Lambda\varepsilon}\mathbf{P}^{-1} = \mathbf{P}\begin{pmatrix} e^{\eta\gamma\varepsilon} & 0 \\ 0 & e^{-\eta\gamma\varepsilon} \end{pmatrix}\mathbf{P}^{-1} = \begin{pmatrix} \cosh\eta\gamma\varepsilon & -\frac{1}{\gamma}\sinh\eta\gamma\varepsilon \\ -\gamma\sinh\eta\gamma\varepsilon & \cosh\eta\gamma\varepsilon \end{pmatrix} \tag{5.91}$$

Thus the Compression Transmission matrix is:

$$\Upsilon_C(\eta) = e^{\mathbf{C}1} = \begin{pmatrix} \cosh\eta\gamma & -\frac{1}{\gamma}\sinh\eta\gamma \\ -\gamma\sinh\eta\gamma & \cosh\eta\gamma \end{pmatrix} \tag{5.92}$$

Now we have to evaluate the matrix $e^{\mathbf{B}\varepsilon}$, following the same steps as we have done in the compression analysis. The eigenvectors and eigenvalues of $\mathbf{B}$ are:

$$\begin{array}{ccccc}
\textit{Eigenvectors} & \begin{pmatrix} \frac{\sqrt{(-\beta^2\eta^2)}}{\beta^2\eta} \\ \frac{\left(\sqrt[4]{(-\beta^2\eta^2)}\right)^3}{\beta^2\eta} \\ -\sqrt[4]{(-\beta^2\eta^2)} \\ 1 \end{pmatrix} & \begin{pmatrix} -\frac{\sqrt{(-\beta^2\eta^2)}}{\beta^2\eta} \\ -i\frac{\left(\sqrt[4]{(-\beta^2\eta^2)}\right)^3}{\beta^2\eta} \\ -i\sqrt[4]{(-\beta^2\eta^2)} \\ 1 \end{pmatrix} & \begin{pmatrix} \frac{\sqrt{(-\beta^2\eta^2)}}{\beta^2\eta} \\ -\frac{\left(\sqrt[4]{(-\beta^2\eta^2)}\right)^3}{\beta^2\eta} \\ \sqrt[4]{(-\beta^2\eta^2)} \\ 1 \end{pmatrix} & \begin{pmatrix} -\frac{\sqrt{(-\beta^2\eta^2)}}{\beta^2\eta} \\ i\frac{\left(\sqrt[4]{(-\beta^2\eta^2)}\right)^3}{\beta^2\eta} \\ i\sqrt[4]{(-\beta^2\eta^2)} \\ 1 \end{pmatrix} \\
\textit{Eigenvalues} & \lambda_1 = \sqrt[4]{(-\beta^2\eta^2)} & \lambda_2 = i\sqrt[4]{(-\beta^2\eta^2)} & \lambda_3 = -\sqrt[4]{(-\beta^2\eta^2)} & \lambda_4 = -i\sqrt[4]{(-\beta^2\eta^2)}
\end{array} \tag{5.93}$$

Choosing

$$\sqrt[4]{(-\beta^2\eta^2)} = \sqrt{\beta}\sqrt{\eta}e^{i\frac{\pi}{4}} =: \sqrt{\beta}k \tag{5.94}$$

where $\sqrt{\eta}$ is such that $\operatorname{Im}\left[\sqrt{\eta}\right] \geqslant 0$ and $\sqrt{\beta} \in \mathbb{R}^{+}$, the set of eigenvectors and eigenvalues of $\mathbf{B}$ becomes:

$$\textit{Eigenvectors} \quad \begin{pmatrix} \frac{i}{\beta} \\ \\ i\frac{k}{\sqrt{\beta}} \\ \\ -\sqrt{\beta}k \\ 1 \end{pmatrix} \quad \begin{pmatrix} -\frac{i}{\beta} \\ \\ \frac{k}{\sqrt{\beta}} \\ \\ -i\sqrt{\beta}k \\ 1 \end{pmatrix} \quad \begin{pmatrix} \frac{i}{\beta} \\ \\ -i\frac{k}{\sqrt{\beta}} \\ \\ \sqrt{\beta}k \\ 1 \end{pmatrix} \quad \begin{pmatrix} -\frac{i}{\beta} \\ \\ -\frac{k}{\sqrt{\beta}} \\ \\ i\sqrt{\beta}k \\ 1 \end{pmatrix} \tag{5.95}$$

$$\textit{Eigenvalues} \qquad \lambda_1 = \sqrt{\beta}k \qquad \lambda_2 = i\sqrt{\beta}k \qquad \lambda_3 = -\sqrt{\beta}k \quad \lambda_4 = -i\sqrt{\beta}k$$

**Remark 260** *On the imaginary axis $\eta = i\varpi$, $\varpi \in \mathbb{R}$ and*

$$k = \sqrt{i\varpi}e^{i\frac{\pi}{4}} = i\sqrt{\varpi} \tag{5.96}$$

*where $\sqrt{\varpi} \in \mathbb{R}^{+}$.*

Assembling by columns the eigenvectors in a matrix we get:

$$\mathbf{P} := \begin{pmatrix} \frac{i}{\beta} & -\frac{i}{\beta} & \frac{i}{\beta} & -\frac{i}{\beta} \\ \\ i\frac{k}{\sqrt{\beta}} & \frac{k}{\sqrt{\beta}} & -i\frac{k}{\sqrt{\beta}} & -\frac{k}{\sqrt{\beta}} \\ \\ -\sqrt{\beta}k & -i\sqrt{\beta}k & \sqrt{\beta}k & i\sqrt{\beta}k \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{5.97}$$

And assembling the eigenvalues in a diagonal matrix we obtain:

$$\Lambda := \begin{pmatrix} \sqrt{\beta}k & 0 & 0 & 0 \\ 0 & i\sqrt{\beta}k & 0 & 0 \\ 0 & 0 & -\sqrt{\beta}k & 0 \\ 0 & 0 & 0 & -i\sqrt{\beta}k \end{pmatrix} \tag{5.98}$$

It is well known that:

$$\mathbf{B} = \mathbf{P}\Lambda\mathbf{P}^{-1} \tag{5.99}$$

and the matrix $e^{\mathbf{B}\varepsilon}$ can be evaluated as follows:

$$e^{\mathbf{B}\varepsilon} = \mathbf{P}e^{\Lambda\varepsilon}\mathbf{P}^{-1} = \mathbf{P}\begin{pmatrix} e^{\sqrt{\beta}k\varepsilon} & 0 & 0 & 0 \\ 0 & e^{i\sqrt{\beta}k\varepsilon} & 0 & 0 \\ 0 & 0 & e^{-\sqrt{\beta}k\varepsilon} & 0 \\ 0 & 0 & 0 & e^{-i\sqrt{\beta}k\varepsilon} \end{pmatrix}\mathbf{P}^{-1} \tag{5.100}$$

the expanded expression for $e^{\mathbf{B}\varepsilon}$ is cumbersome, because of the great amount of trascendental functions involved in each entry of the matrix. However in the sections below we will need this expression in terms of its submatrices. The Bending Transmission matrix is:

$$\Upsilon_B(\eta) = e^{\mathbf{B}1} \tag{5.101}$$

### Mobility matrix

In this subsection we want to find the mobility matrix of the Euler beam; to simplify the problem it is worthwhile to find separately the bending mobility matrix $\mathbf{M}_B$ and the compression mobility matrix $\mathbf{M}_C$ defined by:

$$\begin{pmatrix} \tilde{v}_2^1 \\ \tilde{\omega}^1 \\ \tilde{v}_2^2 \\ \tilde{\omega}^2 \end{pmatrix} = \mathbf{M}_B(\eta) \begin{pmatrix} \tilde{t}_2^1 \\ \tilde{M}^1 \\ \tilde{t}_2^2 \\ \tilde{M}^2 \end{pmatrix} \qquad \begin{pmatrix} \tilde{v}_1^1 \\ \tilde{v}_1^2 \end{pmatrix} = \mathbf{M}_C(\eta) \begin{pmatrix} \tilde{t}_1^1 \\ \tilde{t}_1^2 \end{pmatrix} \tag{5.102}$$

i.e.:

$$\begin{pmatrix} \tilde{v}_\zeta(0,\eta) \\ \tilde{v}_\theta(0,\eta) \\ \tilde{v}_\zeta(1,\eta) \\ \tilde{v}_\theta(1,\eta) \end{pmatrix} = \mathbf{M}_B \begin{pmatrix} -\tilde{F}_T(0,\eta) \\ -\tilde{F}_M(0,\eta) \\ \tilde{F}_T(1,\eta) \\ \tilde{F}_M(1,\eta) \end{pmatrix} \qquad \begin{pmatrix} \tilde{v}_\xi(0,\eta) \\ \tilde{v}_\xi(1,\eta) \end{pmatrix} = \mathbf{M}_C \begin{pmatrix} -\tilde{F}_N(0,\eta) \\ \tilde{F}_N(1,\eta) \end{pmatrix} \tag{5.103}$$

Let us start finding the compression mobility matrix $\mathbf{M}_C$ by the relation:

$$\mathbf{M}_C = \begin{pmatrix} -\Upsilon_{C_{2,1}}^{-1}\Upsilon_{C_{2,2}} & -\Upsilon_{C_{2,1}}^{-1} \\ -\Upsilon_{C_{1,1}}\Upsilon_{C_{2,1}}^{-1}\Upsilon_{C_{2,2}} + \Upsilon_{C_{1,2}} & -\Upsilon_{C_{1,1}}\Upsilon_{C_{2,1}}^{-1} \end{pmatrix} =$$
$$= \begin{pmatrix} -(-\gamma\sinh\eta\gamma)^{-1}\cosh\eta\gamma & -(-\gamma\sinh\eta\gamma)^{-1} \\ -\cosh\eta\gamma\,(-\gamma\sinh\eta\gamma)^{-1}\cosh\eta\gamma - \dfrac{1}{\gamma}\sinh(\eta\gamma\varepsilon) & -\cosh\eta\gamma\,(-\gamma\sinh\eta\gamma)^{-1} \end{pmatrix} \tag{5.104}$$

Thus:

$$\mathbf{M}_C = \begin{pmatrix} \dfrac{1}{\gamma}\coth\eta\gamma & \dfrac{1}{\gamma\sinh\eta\gamma} \\ \dfrac{1}{\gamma\sinh\eta\gamma} & \dfrac{1}{\gamma}\coth\eta\gamma \end{pmatrix} \tag{5.105}$$

**Remark 261** *The compression mobility matrix is not a rational matrix, thus it cannot be synthesized as a finite network. Indeed we will see in chapter(6) that an infinite* *network is needed which is the cascade of an infinite number of finite networks called moduli.*

Now let us turn our attention to the bending mobility matrix $\mathbf{M}_B$.

At first let us investigate the four blocks constituting the transmission bending matrix $\Upsilon_B$, defined by:

$$\Upsilon_B = \mathbf{P} \begin{pmatrix} e^{\sqrt{\beta}k} & 0 & 0 & 0 \\ 0 & e^{i\sqrt{\beta}k} & 0 & 0 \\ 0 & 0 & e^{-\sqrt{\beta}k} & 0 \\ 0 & 0 & 0 & e^{-i\sqrt{\beta}k} \end{pmatrix} \mathbf{P}^{-1} = \begin{pmatrix} \Upsilon_{B_{1,1}} & \Upsilon_{B_{1,2}} \\ \Upsilon_{B_{2,1}} & \Upsilon_{B_{2,2}} \end{pmatrix} \tag{5.106}$$

$$\Upsilon_{B_{1,1}} = \begin{pmatrix} \frac{1}{2}\left(\cosh\sqrt{\beta}k + \cos\sqrt{\beta}k\right) & \frac{1}{2\sqrt{\beta}k}\left(\sin\sqrt{\beta}k + \sinh\sqrt{\beta}k\right) \\ \frac{1}{2}\sqrt{\beta}k\left(\sinh\sqrt{\beta}k - \sin\sqrt{\beta}k\right) & \frac{1}{2}\left(\cosh\sqrt{\beta}k + \cos\sqrt{\beta}k\right) \end{pmatrix} \tag{5.107}$$

$$\Upsilon_{B_{1,2}} = \begin{pmatrix} \frac{1}{2}\frac{i}{k\left(\sqrt{\beta}\right)^3}\left(\sin\sqrt{\beta}k - \sinh\sqrt{\beta}k\right) & \frac{1}{2}\frac{i}{\beta}\left(\cosh\sqrt{\beta}k - \cos\sqrt{\beta}k\right) \\ \frac{1}{2}\frac{i}{\beta}\left(\cos\sqrt{\beta}k - \cosh\sqrt{\beta}k\right) & \frac{1}{2}\frac{i}{\sqrt{\beta}}k\left(\sin\sqrt{\beta}k + \sinh\sqrt{\beta}k\right) \end{pmatrix} \tag{5.108}$$

$$\Upsilon_{B_{2,1}} = \begin{pmatrix} \frac{1}{2}i\left(\sqrt{\beta}\right)^3 k\left(\sin\sqrt{\beta}k + \sinh\sqrt{\beta}k\right) & \frac{1}{2}i\beta\left(\cosh\sqrt{\beta}k - \cos\sqrt{\beta}k\right) \\ \frac{1}{2}i\beta\left(\cos\sqrt{\beta}k - \cosh\sqrt{\beta}k\right) & \frac{1}{2}\frac{i}{k}\sqrt{\beta}\left(\sin\sqrt{\beta}k - \sinh\sqrt{\beta}k\right) \end{pmatrix} \tag{5.109}$$

$$\Upsilon_{B_{2,2}} = \begin{pmatrix} \frac{1}{2}\left(\cosh\sqrt{\beta}k + \cos\sqrt{\beta}k\right) & \frac{1}{2}\sqrt{\beta}k\left(\sin\sqrt{\beta}k - \sinh\sqrt{\beta}k\right) \\ -\frac{1}{2\sqrt{\beta}k}\left(\sin\sqrt{\beta}k + \sinh\sqrt{\beta}k\right) & \frac{1}{2}\left(\cosh\sqrt{\beta}k + \cos\sqrt{\beta}k\right) \end{pmatrix} \tag{5.110}$$

The mobility matrix $\mathbf{M}_B$ can be expressed in terms of the previous submatrices as follows:

$$\mathbf{M}_B = \begin{pmatrix} -\Upsilon_{B_{2,1}}^{-1}\Upsilon_{B_{2,2}} & -\Upsilon_{B_{2,1}}^{-1} \\ \\ -\Upsilon_{B_{1,1}}\Upsilon_{B_{2,1}}^{-1}\Upsilon_{B_{2,2}}+\Upsilon_{B_{1,2}} & -\Upsilon_{B_{1,1}}\Upsilon_{B_{2,1}}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{B_{1,1}} & \mathbf{M}_{B_{1,2}} \\ \\ \mathbf{M}_{B_{2,1}} & \mathbf{M}_{B_{2,2}} \end{pmatrix} \tag{5.111}$$

where the submatrices are:

$$\mathbf{M}_{B_{1,1}} = \begin{pmatrix} -i\frac{\sin\sqrt{\beta}k\cosh\sqrt{\beta}k-\cos\sqrt{\beta}k\sinh\sqrt{\beta}k}{k\left(\sqrt{\beta}\right)^3\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} & i\frac{\sin\sqrt{\beta}k\sinh\sqrt{\beta}k}{\beta\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} \\ \\ i\frac{\sin\sqrt{\beta}k\sinh\sqrt{\beta}k}{\beta\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} & -ik\frac{\sin\sqrt{\beta}k\cosh\sqrt{\beta}k+\cos\sqrt{\beta}k\sinh\sqrt{\beta}k}{\sqrt{\beta}\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} \end{pmatrix} \tag{5.112}$$

$$\mathbf{M}_{B_{1,2}} = \begin{pmatrix} -i\frac{\sin\sqrt{\beta}k-\sinh\sqrt{\beta}k}{k\left(\sqrt{\beta}\right)^3\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} & i\frac{\cosh\sqrt{\beta}k-\cos\sqrt{\beta}k}{\beta\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} \\ \\ -i\frac{\cosh\sqrt{\beta}k-\cos\sqrt{\beta}k}{\beta\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} & -ik\frac{\sin\sqrt{\beta}k+\sinh\sqrt{\beta}k}{\sqrt{\beta}\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} \end{pmatrix} \tag{5.113}$$

$$\mathbf{M}_{B_{2,1}} = \begin{pmatrix} i\frac{\sinh\sqrt{\beta}k-\sin\sqrt{\beta}k}{k\left(\sqrt{\beta}\right)^3\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} & -i\frac{\cosh\sqrt{\beta}k-\cos\sqrt{\beta}k}{\beta\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} \\ \\ i\frac{\cosh\sqrt{\beta}k-\cos\sqrt{\beta}k}{\beta\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} & -ik\frac{\sin\sqrt{\beta}k+\sinh\sqrt{\beta}k}{\sqrt{\beta}\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} \end{pmatrix} \tag{5.114}$$

$$\mathbf{M}_{B_{2,2}} = \begin{pmatrix} -i\frac{\sin\sqrt{\beta}k\cosh\sqrt{\beta}k-\cos\sqrt{\beta}k\sinh\sqrt{\beta}k}{k\left(\sqrt{\beta}\right)^3\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} & -i\frac{\sin\sqrt{\beta}k\sinh\sqrt{\beta}k}{\beta\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} \\ \\ -i\frac{\sin\sqrt{\beta}k\sinh\sqrt{\beta}k}{\beta\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} & -ik\frac{\sin\sqrt{\beta}k\cosh\sqrt{\beta}k+\cos\sqrt{\beta}k\sinh\sqrt{\beta}k}{\sqrt{\beta}\left(-1+\cosh\sqrt{\beta}k\cos\sqrt{\beta}k\right)} \end{pmatrix} \tag{5.115}$$

**Remark 262** *The bending mobility matrix is symmetric, and it can be written as:*

$$\mathbf{M}_B = \begin{pmatrix} M_{B_{1,1}} & M_{B_{1,2}} & M_{B_{1,3}} & M_{B_{1,4}} \\ M_{B_{1,2}} & M_{B_{2,2}} & -M_{B_{1,4}} & M_{B_{2,4}} \\ M_{B_{1,3}} & -M_{B_{1,4}} & M_{B_{1,1}} & -M_{B_{1,2}} \\ M_{B_{1,4}} & M_{B_{2,4}} & -M_{B_{1,2}} & M_{B_{2,2}} \end{pmatrix} \tag{5.116}$$

**Remark 263** *The bending mobility matrix is not a rational matrix, thus it cannot be synthesized as a finite network. Indeed we will see in chapter(6) that an infinite network is needed, which can be designed as the cascade of an infinite number of finite networks called moduli.*

Once we have found the two matrices $\mathbf{M}_B$ and $\mathbf{M}_C$ it is easy to manipulate them in order to obtain the Mobility matrix $\mathbf{M}$ of the Euler beam.

**Cantilever beam** It is interesting to deduce the impedance of a cantilever beam by means of this powerful matrix formulation. Indeed we can think of the cantilever beam as a frame $\mathcal{F}_{C-b}$ constituted by two elements: a clamping device $\mathcal{S}_C$ and an Euler beam $\mathcal{S}_E$.

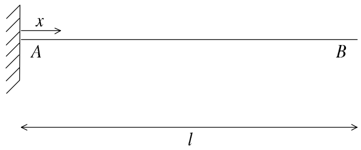


Figure 5.8: Cantilever beam

In particular $\mathcal{F}_{C-b}$ can be thought of as a representative of a class of $1-$terminal structure defined by a $3\times 3$ mobility matrix $\mathbf{M}$ which can be evaluated by the following steps.

Consider eq.(5.76), taking into account the action of the clamping device on the first terminal of the beam:

$$\begin{pmatrix} \tilde{v}_\xi\left(1,\eta\right) \\ \tilde{v}_\zeta\left(1,\eta\right) \\ \tilde{v}_\theta\left(1,\eta\right) \\ -\tilde{F}_N\left(1,\eta\right) \\ -\tilde{F}_T\left(1,\eta\right) \\ -\tilde{F}_M\left(1,\eta\right) \end{pmatrix} = \Upsilon\left(\eta\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\tilde{F}_N\left(0,\eta\right) \\ -\tilde{F}_T\left(0,\eta\right) \\ -\tilde{F}_M\left(0,\eta\right) \end{pmatrix} \tag{5.117}$$

i.e.:

$$\begin{pmatrix} \tilde{v}_\zeta\left(1,\eta\right) \\ \tilde{v}_\theta\left(1,\eta\right) \\ -\tilde{F}_T\left(1,\eta\right) \\ -\tilde{F}_M\left(1,\eta\right) \\ \tilde{v}_\xi\left(1,\eta\right) \\ -\tilde{F}_N\left(1,\eta\right) \end{pmatrix} = \begin{pmatrix} \Upsilon_{B_{1,1}} & \Upsilon_{B_{1,2}} & \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times 1} \\ \Upsilon_{B_{2,1}} & \Upsilon_{B_{2,2}} & \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times 1} \\ 0 & 0 & \Upsilon_{C_{1,1}} & \Upsilon_{C_{1,2}} \\ 0 & 0 & \Upsilon_{C_{2,1}} & \Upsilon_{C_{2,2}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\tilde{F}_T\left(0,\eta\right) \\ -\tilde{F}_M\left(0,\eta\right) \\ 0 \\ -\tilde{F}_N\left(0,\eta\right) \end{pmatrix} \tag{5.118}$$

Hence:

$$\begin{aligned}
&\begin{pmatrix} \tilde{v}_\zeta(1,\eta) \\ \tilde{v}_\theta(1,\eta) \\ -\tilde{F}_T(1,\eta) \\ -\tilde{F}_M(1,\eta) \end{pmatrix} = \begin{pmatrix} \Upsilon_{B_{1,1}} & \Upsilon_{B_{1,2}} \\ \Upsilon_{B_{2,1}} & \Upsilon_{B_{2,2}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\tilde{F}_T(0,\eta) \\ -\tilde{F}_M(0,\eta) \end{pmatrix} \Rightarrow \\
&\begin{cases} \begin{pmatrix} \tilde{v}_\zeta(1,\eta) \\ \tilde{v}_\theta(1,\eta) \end{pmatrix} = \Upsilon_{B_{1,2}} \begin{pmatrix} -\tilde{F}_T(0,\eta) \\ -\tilde{F}_M(0,\eta) \end{pmatrix} \\ \begin{pmatrix} -\tilde{F}_T(1,\eta) \\ -\tilde{F}_M(1,\eta) \end{pmatrix} = \Upsilon_{B_{2,2}} \begin{pmatrix} -\tilde{F}_T(0,\eta) \\ -\tilde{F}_M(0,\eta) \end{pmatrix} \end{cases}
\end{aligned} \tag{5.119}$$

$$\begin{aligned}
&\begin{pmatrix} \tilde{v}_\xi(1,\eta) \\ -\tilde{F}_N(1,\eta) \end{pmatrix} = \begin{pmatrix} \Upsilon_{C_{1,1}} & \Upsilon_{C_{1,2}} \\ \Upsilon_{C_{2,1}} & \Upsilon_{C_{2,2}} \end{pmatrix} \begin{pmatrix} 0 \\ -\tilde{F}_N(0,\eta) \end{pmatrix} \Rightarrow \\
&\begin{cases} \tilde{v}_\xi(1,\eta) = -\Upsilon_{C_{1,2}} \tilde{F}_N(0,\eta) \\ -\tilde{F}_N(1,\eta) = -\Upsilon_{C_{2,2}} \tilde{F}_N(0,\eta) \end{cases}
\end{aligned} \tag{5.120}$$

Finally:

$$\begin{pmatrix} \tilde{v}_\xi(1,\eta) \\ \tilde{v}_\zeta(1,\eta) \\ \tilde{v}_\theta(1,\eta) \end{pmatrix} = \begin{pmatrix} -\Upsilon_{C_{1,2}} \Upsilon_{C_{2,2}}^{-1} & \mathbf{0}_{1\times 2} \\ \mathbf{0}_{2\times 1} & -\Upsilon_{B_{1,2}} \Upsilon_{B_{2,2}}^{-1} \end{pmatrix} \begin{pmatrix} \tilde{F}_N(1,\eta) \\ \tilde{F}_T(1,\eta) \\ \tilde{F}_M(1,\eta) \end{pmatrix} \tag{5.121}$$

**Proposition 264** *The mobility matrix of a cantilever beam is:*

$$\mathbf{M} = \begin{pmatrix} -\Upsilon_{C_{1,2}} \Upsilon_{C_{2,2}}^{-1} & \mathbf{0}_{1\times 2} \\ \mathbf{0}_{2\times 1} & -\Upsilon_{B_{1,2}} \Upsilon_{B_{2,2}}^{-1} \end{pmatrix} \tag{5.122}$$

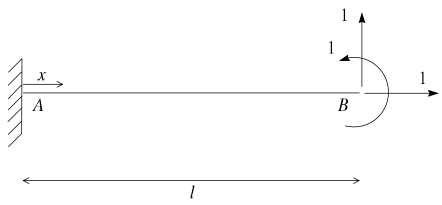


Figure 5.9: Loaded Cantilever beam

$$
\begin{array}{ll}
\textit{Entry} & \textit{Expression} \\
M_{1,1} & \dfrac{1}{\gamma}\tanh \eta\gamma \\
M_{2,2} & -i\dfrac{\sin\sqrt{\beta}k\cosh\sqrt{\beta}k-\cos\sqrt{\beta}k\sinh\sqrt{\beta}k}{k\left(\sqrt{\beta}\right)^3\left(\cosh\sqrt{\beta}k\cos\sqrt{\beta}k+1\right)} \\
M_{2,3}=M_{3,2} & -i\dfrac{\sinh\sqrt{\beta}k\sin\sqrt{\beta}k}{\beta\left(\cosh\sqrt{\beta}k\cos\sqrt{\beta}k+1\right)} \\
M_{3,3} & -ik\dfrac{\sin\sqrt{\beta}k\cosh\sqrt{\beta}k+\cos\sqrt{\beta}k\sinh\sqrt{\beta}k}{\sqrt{\beta}\left(\cosh\sqrt{\beta}k\cos\sqrt{\beta}k+1\right)}
\end{array}
\tag{5.123}
$$

**Remark 265** *All the terms on the diagonal of* $\mathbf{M}$ *represent a driving mobility:*

- *represents the axial velocity resulting from a unit axial load,*
- *represents the transverse velocity resulting from a unit shear load,*
- *represents the angular velocity resulting from a unit couple.*

**Remark 266** *All the results derived so far agree with the conclusions derived in chapter (4), but with the introduced theory of structures, we have saved many steps reaching more general and useful goals.*

# Chapter 6

# Synthesis of a circuit analog to the Euler beam

Now, it is time to merge the knowledge which we have acquired so far to solve the practical problem of the synthesis of a network analog to the Euler Beam. Let us detail this purpose:

**Problem 267** *Given the dimensionless mobility matrix* $\mathbf{M}$ *for an Euler beam, we have to find a circuit described by a dimensionless impedance* $\mathbf{z}$ *such that:*

$$\mathbf{z}(\eta) = \mathbf{M}(\eta) \tag{6.1}$$

*Such a circuit will be said to be the analog of the Euler beam.*

The immediate synthesis of the network is very difficult, since $\mathbf{M}$ is not a rational matrix.

In fact we want to realize a distributed circuit governed by the same set of equations that defines an Euler beam. We require the circuit to behave as a beam at every point, not only at its terminals.

This problem can be explicitly stated as:

**Problem 268** *Synthesize a distributed circuit* $\mathcal{C}_E$ *such that the* $6-$*port network obtained considering the circuit that lies from* $x = x_1$ *to* $x = x_2$ *is analog to the* $2-$*terminal structure obtained considering the beam-element from* $x = x_1$ *to* $x = x_2$ *for every choice of* $x_1$*and* $x_2$ *from* $0$ *to* $l$*.*

The way we will approach this problem will be detailed by the following steps, but as an anticipation we state that it will become necessary to partition the beam into a class of beam-elements and then synthesize a circuit simulating each element.

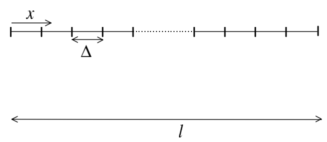


Figure 6.1: Mesh on the beam

1. Consider the Euler beam $\mathcal{S}_E$ of length $l$ as the cascade of $n$ Euler beams of size $\Delta = \dfrac{l}{n}$, then the transmission matrix $\Upsilon$ of $\mathcal{S}_E$ will be:

$$\Upsilon(\eta) = (\Upsilon_e(\eta))^n \tag{6.2}$$

where $\Upsilon_e$ is the transmission matrix of a generic structural member (since all the length of the beam-elements are equal, each of them is represented by the same transmission matrix $\Upsilon_e(\eta)$).

2. Supposing $\Delta$ to be "small", the mobility matrix $\mathbf{M}_e$ of a generic structural member is expanded as a Laurent series in terms of $\Delta$ in the neighborhood of $\Delta = 0$.

3. Truncate the expansion at a *suitable* degree of approximation.
4. Synthesize the resulting rational Mobility matrix $\mathbf{M}_e$ by a finite dimensionless subnetwork.
5. Cascade connect the subnetworks, obtaining a finite circuit which approximates the beam more closely as the number of structural members/subnetworks increases.

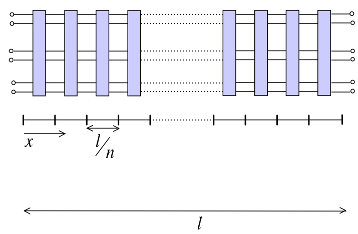


Figure 6.2: Circuit

**Remark 269** *For practical purposes, $\Delta$ cannot be as small as we want, for then the circuit analog will be well-behaved only on a limited range of frequencies. In fact as we will see, the expansion will involve a trascendental function whose argument is proportional to the product of frequency and size. Hence as the frequency increases, the size should become smaller and smaller in order to provide a good approximation.*

Initially let us find the dimensionless transmission matrix of a beam-element. Without lack of generality, we can consider the *first beam-element*, i.e. the one whose first terminal coincides with the first terminal of the whole beam.

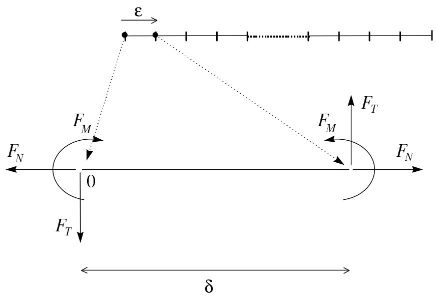


Figure 6.3: First beam-element

The dimensionless variables which will be used below are the same as the ones we have introduced in the previous chapter dealing with the frequency domain representation of the entire beam.

Going through all the steps detailed in the previous chapter and assuming $\delta := \dfrac{1}{n}$ we get:

$$\begin{pmatrix} \tilde{v}_1^2 \\ \tilde{v}_2^2 \\ \tilde{\omega}^2 \\ -\tilde{t}_1^2 \\ -\tilde{t}_2^2 \\ -\tilde{M}^2 \end{pmatrix} = \Upsilon^e(\eta) \begin{pmatrix} \tilde{v}_1^1 \\ \tilde{v}_2^1 \\ \tilde{\omega}^1 \\ \tilde{t}_1^1 \\ \tilde{t}_2^1 \\ \tilde{M}^1 \end{pmatrix} \tag{6.3}$$

i.e.:

$$\begin{pmatrix} \tilde{v}_\xi(\delta,\eta) \\ \tilde{v}_\zeta(\delta,\eta) \\ \tilde{v}_\theta(\delta,\eta) \\ -\tilde{F}_N(\delta,\eta) \\ -\tilde{F}_T(\delta,\eta) \\ -\tilde{F}_M(\delta,\eta) \end{pmatrix} = \Upsilon^e(\eta) \begin{pmatrix} \tilde{v}_\xi(0,\eta) \\ \tilde{v}_\zeta(0,\eta) \\ \tilde{v}_\theta(0,\eta) \\ -\tilde{F}_N(0,\eta) \\ -\tilde{F}_T(0,\eta) \\ -\tilde{F}_M(0,\eta) \end{pmatrix} \tag{6.4}$$

As we have noted in the previous chapter, it is possible to completely separate bending from compression, and to arrange the previous set of equations as follows:

$$\begin{pmatrix} \tilde{v}_2^2 \\ \tilde{\omega}^2 \\ -\tilde{t}_2^2 \\ -\tilde{M}^2 \end{pmatrix} = \Upsilon_B^e(\eta) \begin{pmatrix} \tilde{v}_2^1 \\ \tilde{\omega}^1 \\ \tilde{t}_2^1 \\ \tilde{M}^1 \end{pmatrix} \qquad \begin{pmatrix} \tilde{v}_1^2 \\ -\tilde{t}_1^2 \end{pmatrix} = \Upsilon_C^e(\eta) \begin{pmatrix} \tilde{v}_1^1 \\ \tilde{t}_1^1 \end{pmatrix} \tag{6.5}$$

It is convenient to split the synthesis problem of a circuit analog to the Euler beam into two easier problems:

- Synthesis of a circuit analog to the extending beam
- Synthesis of a circuit analog to the bending beam

### Synthesis of a circuit analog to the longitudinally vibrating beam

From the previous chapter it is easy to find the transmission compression matrix for this beam-element:

$$\Upsilon_C^e(\eta) = e^{\mathbf{C}\delta} = \begin{pmatrix} \cosh(\eta\gamma\delta) & -\frac{1}{\gamma}\sinh(\eta\gamma\delta) \\ -\gamma\sinh(\eta\gamma\delta) & \cosh(\eta\gamma\delta) \end{pmatrix} \tag{6.6}$$

Furthermore it is easy to find the mobility compression matrix for this beam-element:

$$\mathbf{M}_C(\eta) = \begin{pmatrix} \frac{1}{\gamma}\coth(\delta\eta\gamma) & \frac{1}{\gamma\sinh(\delta\eta\gamma)} \\ \\ \frac{1}{\gamma\sinh(\delta\eta\gamma)} & \frac{1}{\gamma}\coth(\delta\eta\gamma) \end{pmatrix} \tag{6.7}$$

**Remark 270** *As expected, $\mathbf{M}_C$ is symmetric, all its poles are simple and lie on the imaginary axis. Furthermore, it is an odd function of $\eta$. All these results are consequences of the reciprocity and losslessness of the Euler beam.*

Now let us expand each element of $\mathbf{M}_C$ as a Laurent series in the neighborhood of $\delta = 0$:

$$M_{C_{1,1}} = M_{C_{2,2}} \simeq \frac{1}{\gamma^2\eta}\delta^{-1} + \frac{1}{3}\eta\delta \tag{6.8}$$

$$M_{C_{1,2}} = M_{C_{2,1}} \simeq \frac{1}{\gamma^2\eta}\delta^{-1} + \left(-\frac{1}{6}\eta\right)\delta \tag{6.9}$$

Thus the approximate mobility matrix of the element is:

$$\hat{\mathbf{M}}_C = \begin{pmatrix} \frac{1}{\gamma^2\eta}\delta^{-1} + \frac{1}{3}\eta\delta & \frac{1}{\gamma^2\eta}\delta^{-1} + \left(-\frac{1}{6}\eta\right)\delta \\ \frac{1}{\gamma^2\eta}\delta^{-1} + \left(-\frac{1}{6}\eta\right)\delta & \frac{1}{\gamma^2\eta}\delta^{-1} + \frac{1}{3}\eta\delta \end{pmatrix} \tag{6.10}$$

**Remark 271** *The approximate mobility matrix $\hat{\mathbf{M}}_C$ fulfills all the requirements needed for it to represent a reciprocal and lossless network.*

In fact it is a real-rational, symmetric matrix expressible in Foster's canonic form as:

$$\hat{\mathbf{M}}_C = \frac{1}{\eta}\begin{pmatrix} \frac{1}{\gamma^2}\delta^{-1} & \frac{1}{\gamma^2}\delta^{-1} \\ \\ \frac{1}{\gamma^2}\delta^{-1} & \frac{1}{\gamma^2}\delta^{-1} \end{pmatrix} + \eta\begin{pmatrix} +\frac{1}{3}\delta & \left(-\frac{1}{6}\right)\delta \\ \\ \left(-\frac{1}{6}\right)\delta & +\frac{1}{3}\delta \end{pmatrix} \tag{6.11}$$

where the residues matrix are both symmetric and positive semidefinite:

$$\mathbf{K}_0 = \begin{pmatrix} \frac{1}{\gamma^2}\delta^{-1} & \frac{1}{\gamma^2}\delta^{-1} \\ \frac{1}{\gamma^2}\delta^{-1} & \frac{1}{\gamma^2}\delta^{-1} \end{pmatrix} \qquad \mathbf{K}_\infty = \begin{pmatrix} +\frac{1}{3}\delta & \left(-\frac{1}{6}\right)\delta \\ \left(-\frac{1}{6}\right)\delta & +\frac{1}{3}\delta \end{pmatrix} \tag{6.12}$$

$$\begin{aligned} &\mathbf{K}_{0_{1,1}} = \tfrac{1}{\gamma^2}\delta^{-1} > 0 \quad && \mathbf{K}_{\infty_{1,1}} = +\tfrac{1}{3}\delta \\ &\det \mathbf{K}_0 = 0 && \det \mathbf{K}_\infty = \tfrac{1}{12}\delta^2 \end{aligned} \tag{6.13}$$

Let us focus our attention to the matrix $\mathbf{z}_0(\eta)$ defined by:

$$\mathbf{z}_0(\eta) = \frac{1}{\eta}\mathbf{K}_0 = \frac{1}{\eta}\frac{1}{\gamma^2\delta}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{6.14}$$

Using the techniques developed in chapter (2), in particular the following condition on the turns-ratio of the transformer,

$$\frac{\left|K_{0_{1,2}}\right|}{K_{0_{1,1}}} \leqslant |n_0| \leqslant \frac{K_{0_{2,2}}}{\left|K_{0_{1,2}}\right|} \tag{6.15}$$

we find

$$|n_0| = 1 \tag{6.16}$$

Since $K_{0_{1,2}} > 0$ the transformer may be dispensed with, and $\mathbf{z}_0(\eta)$ may be realized as a shunt capacitor $c$ of capacitance $\gamma^2\delta$.

Now consider the impedance matrix $\mathbf{z}_\infty(\eta)$ defined by:

$$\mathbf{z}_\infty(\eta) = \eta\mathbf{K}_\infty = \eta\frac{\delta}{6}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{6.17}$$

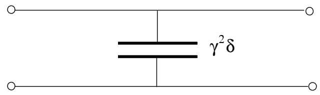


Figure 6.4: Realization of the first term in Foster' expansion

Condition (6.15) yields:

$$\frac{\left|K_{\infty_{1,2}}\right|}{K_{\infty_{1,1}}} \leqslant |n_\infty| \leqslant \frac{K_{\infty_{2,2}}}{\left|K_{\infty_{1,2}}\right|} \Rightarrow \frac{1}{2} \leqslant |n_\infty| \leqslant 2 \tag{6.18}$$

Even if 1 is a possible value for the absolute value of the turning ratio, here the transformer may be not dispensed with, since:

$$K_{\infty_{1,2}} < 0. \tag{6.19}$$

Thus we can choose $n_\infty = -1$.

Now we have to determine the inductances of the three inductors:

$$\begin{aligned} a_\infty &= K_{\infty_{1,1}} - \frac{\left|K_{\infty_{1,2}}\right|}{|n_\infty|} = \frac{\delta}{6}(2-1) = \frac{\delta}{6} \\ c_\infty &= \frac{\left|K_{\infty_{1,2}}\right|}{|n_\infty|} = \frac{\delta}{6} \\ b_\infty &= \frac{K_{\infty_{2,2}}}{n_\infty^2} - \frac{\left|K_{\infty_{1,2}}\right|}{|n_\infty|} = \frac{\delta}{3} - \frac{\delta}{6} = \frac{\delta}{6} \end{aligned} \tag{6.20}$$

Thus the three inductors have the same inductance, equal to $\dfrac{\delta}{6}$.

Connecting the two networks in series, we obtain the realization of the entire approximate mobility matrix $\hat{\mathbf{M}}_C$.

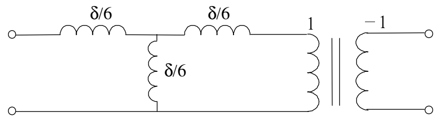


Figure 6.5: Realization of the second term

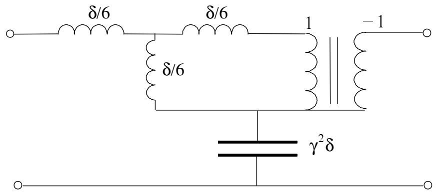


Figure 6.6: Realization of the approximate mobility matrix

In this particular case it is possible to dispense with the ideal transformer, if we synthesize the entire matrix by a symmetric lattice. In fact:

$$\hat{\mathbf{M}}_C = \begin{pmatrix} \frac{1}{\gamma^2\eta}\delta^{-1} + \frac{1}{3}\eta\delta & \frac{1}{\gamma^2\eta}\delta^{-1} + \left(-\frac{1}{6}\eta\right)\delta \\ \frac{1}{\gamma^2\eta}\delta^{-1} + \left(-\frac{1}{6}\eta\right)\delta & \frac{1}{\gamma^2\eta}\delta^{-1} + \frac{1}{3}\eta\delta \end{pmatrix} = \begin{pmatrix} \hat{M}_{C_{1,1}} & \hat{M}_{C_{1,2}} \\ \hat{M}_{C_{1,2}} & \hat{M}_{C_{1,1}} \end{pmatrix} \quad (6.21)$$

Thus the two impedances in the symmetric lattice are

$$\begin{aligned} z_a &= \hat{\mathbf{M}}_{C_{1,1}} - \hat{\mathbf{M}}_{C_{1,2}} = \frac{1}{\gamma^2\eta}\delta^{-1} + \frac{1}{3}\eta\delta - \left(\frac{1}{\gamma^2\eta}\delta^{-1} + \left(-\frac{1}{6}\eta\right)\delta\right) = \frac{1}{2}\eta\delta \\ z_b &= \hat{\mathbf{M}}_{C_{1,1}} + \hat{\mathbf{M}}_{C_{1,2}} = \frac{1}{\gamma^2\eta}\delta^{-1} + \frac{1}{3}\eta\delta + \left(\frac{1}{\gamma^2\eta}\delta^{-1} + \left(-\frac{1}{6}\eta\right)\delta\right) = \frac{2}{\gamma^2\eta\delta} + \frac{1}{6}\eta\delta \end{aligned} \quad (6.22)$$

Hence $z_a$ is an inductor of inductance $\frac{1}{2}\delta$ while $z_b$ is the series connection of an inductor of inductance $\frac{1}{6}\delta$ and a capacitor of capacitance $\frac{\gamma^2}{2}\delta$.

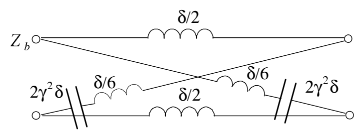


Figure 6.7: Realization of the approxiamte mobility matrix by a symmetric lattice

Now we have to find the analog dimensional circuit:

$$\mathbf{Z}(s) = R_0 \hat{\mathbf{M}}_C(\eta) \tag{6.23}$$

where $R_0$ has the dimension of a resistance.

Thus

$$\mathbf{Z}(s) = R_0 t_0 \begin{pmatrix} \frac{k_N}{x\lambda l}\frac{1}{s} + \frac{1}{3}s\frac{x}{l} & \frac{k_N}{x\lambda l}\frac{1}{s} + \left(-\frac{1}{6}s\right)\frac{x}{l} \\ \frac{k_N}{x\lambda l}\frac{1}{s} + \left(-\frac{1}{6}s\right)\frac{x}{l} & \frac{k_N}{x\lambda l}\frac{1}{s} + \frac{1}{3}s\frac{x}{l} \end{pmatrix}$$

And

$$\mathbf{Z}(s) = \begin{pmatrix} \frac{1}{Cs} + \frac{1}{3}sL & \frac{1}{Cs} + \left(-\frac{1}{6}sL\right) \\ \frac{1}{Cs} + \left(-\frac{1}{6}sL\right) & \frac{1}{Cs} + \frac{1}{3}sL \end{pmatrix} \tag{6.24}$$

with:

$$\begin{aligned} C &= \frac{x\lambda l}{k_N}\frac{R_0}{t_0} \\ R_0 L &= \frac{x}{l}t_0 \end{aligned} \tag{6.25}$$

and:

$$\frac{CL}{x^2} = \frac{\lambda}{k_N} \tag{6.26}$$

**Remark 272** *In the circuit simulating the beam, once we have fixed the size of the modulus $x$ and the value of the capacitance $C$ we are compelled to choose the inductance such that the previous equation holds.*

### Synthesis of a circuit analog to the transversally vibrating beam

We will follow the procedure sketched at the beginning of the chapter, but as a preliminary step we have to find the expression of the transmission matrix and the bending mobility matrix of the element.

The transmission bending matrix is given by:

$$\Upsilon_B^e(\eta) = e^{\mathbf{B}\delta} \tag{6.27}$$

with:

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\eta \\ -\beta^2\eta & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{6.28}$$

It can be evaluated as follows:

$$\Upsilon_B^e(\eta) = e^{\mathbf{B}\delta} = \mathbf{P}e^{\Lambda\delta}\mathbf{P}^{-1} = \mathbf{P}\begin{pmatrix} e^{\sqrt{\beta}k\delta} & 0 & 0 & 0 \\ 0 & e^{i\sqrt{\beta}k\delta} & 0 & 0 \\ 0 & 0 & e^{-\sqrt{\beta}k\delta} & 0 \\ 0 & 0 & 0 & e^{-i\sqrt{\beta}k\delta} \end{pmatrix}\mathbf{P}^{-1} \tag{6.29}$$

where:

$$k = \sqrt{\eta} e^{i\frac{\pi}{4}} \tag{6.30}$$

and $\sqrt{\eta}$ is such that $\mathrm{Im}\left[\sqrt{\eta}\right] \geqslant 0$, furthermore:

$$\mathbf{P} = \begin{pmatrix} \frac{i}{\beta} & -\frac{i}{\beta} & \frac{i}{\beta} & -\frac{i}{\beta} \\ i\frac{k}{\sqrt{\beta}} & \frac{k}{\sqrt{\beta}} & -i\frac{k}{\sqrt{\beta}} & -\frac{k}{\sqrt{\beta}} \\ -\sqrt{\beta}k & -i\sqrt{\beta}k & \sqrt{\beta}k & i\sqrt{\beta}k \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \sqrt{\beta}k & 0 & 0 & 0 \\ 0 & i\sqrt{\beta}k & 0 & 0 \\ 0 & 0 & -\sqrt{\beta}k & 0 \\ 0 & 0 & 0 & -i\sqrt{\beta}k \end{pmatrix} \tag{6.31}$$

The mobility matrix can be easily found to be:

$$\mathbf{M}_B = \begin{pmatrix} -\left(\Upsilon^e_{B_{2,1}}\right)^{-1}\Upsilon^e_{B_{2,2}} & -\left(\Upsilon^e_{B_{2,1}}\right)^{-1} \\ -\Upsilon^e_{B_{1,1}}\left(\Upsilon^e_{B_{2,1}}\right)^{-1}\Upsilon_{B_{2,2}}+\Upsilon_{B_{1,2}} & -\Upsilon_{B_{1,1}}\left(\Upsilon^e_{B_{2,1}}\right)^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{B_{1,1}} & \mathbf{M}_{B_{1,2}} \\ \mathbf{M}_{B_{2,1}} & \mathbf{M}_{B_{2,2}} \end{pmatrix} \tag{6.32}$$

where the submatrices can be expressed as follows:

$$\mathbf{M}_{B_{1,1}} = \begin{pmatrix} -i\frac{\sin\sqrt{\beta}k\delta\cosh\sqrt{\beta}k\delta-\cos\sqrt{\beta}k\delta\sinh\sqrt{\beta}k\delta}{k\left(\sqrt{\beta}\right)^3\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} & i\frac{\sin\sqrt{\beta}k\delta\sinh\sqrt{\beta}k\delta}{\beta\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} \\ i\frac{\sin\sqrt{\beta}k\delta\sinh\sqrt{\beta}k\delta}{\beta\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} & -ik\frac{\sin\sqrt{\beta}k\delta\cosh\sqrt{\beta}k\delta+\cos\sqrt{\beta}k\delta\sinh\sqrt{\beta}k\delta}{\sqrt{\beta}\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} \end{pmatrix} \tag{6.33}$$

$$\mathbf{M}_{B_{1,2}} = \begin{pmatrix} -i\frac{\sin\sqrt{\beta}k\delta-\sinh\sqrt{\beta}k\delta}{k\left(\sqrt{\beta}\right)^3\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} & i\frac{\cosh\sqrt{\beta}k\delta-\cos\sqrt{\beta}k\delta}{\beta\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} \\ -i\frac{\cosh\sqrt{\beta}k\delta-\cos\sqrt{\beta}k\delta}{\beta\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} & -ik\frac{\sin\sqrt{\beta}k\delta+\sinh\sqrt{\beta}k\delta}{\sqrt{\beta}\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} \end{pmatrix} \tag{6.34}$$

$$\mathbf{M}_{B_{2,1}} = \begin{pmatrix} i\frac{\sinh\sqrt{\beta}k\delta - \sin\sqrt{\beta}k\delta}{k\left(\sqrt{\beta}\right)^3\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} & -i\frac{\cosh\sqrt{\beta}k\delta - \cos\sqrt{\beta}k\delta}{\beta\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} \\ i\frac{\cosh\sqrt{\beta}k\delta - \cos\sqrt{\beta}k\delta}{\beta\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} & -ik\frac{\sin\sqrt{\beta}k\delta + \sinh\sqrt{\beta}k\delta}{\sqrt{\beta}\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} \end{pmatrix} \tag{6.35}$$

$$\mathbf{M}_{B_{2,2}} = \begin{pmatrix} -i\frac{\sin\sqrt{\beta}k\delta\cosh\sqrt{\beta}k\delta - \cos\sqrt{\beta}k\delta\sinh\sqrt{\beta}k\delta}{k\left(\sqrt{\beta}\right)^3\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} & -i\frac{\sin\sqrt{\beta}k\delta\sinh\sqrt{\beta}k\delta}{\beta\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} \\ -i\frac{\sin\sqrt{\beta}k\delta\sinh\sqrt{\beta}k\delta}{\beta\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} & -ik\frac{\sin\sqrt{\beta}k\delta\cosh\sqrt{\beta}k\delta + \cos\sqrt{\beta}k\delta\sinh\sqrt{\beta}k\delta}{\sqrt{\beta}\left(-1+\cosh\sqrt{\beta}k\delta\cos\sqrt{\beta}k\delta\right)} \end{pmatrix} \tag{6.36}$$

**Remark 273** *The mobility matrix* $\mathbf{M}_B$ *is clearly symmetric, and it can be proved that all its poles are simple and lie on the imaginary axis, and that it is an odd function of* $\eta$*. All these results are somehow clear from all the considerations we have made dealing with the properties of the Euler beam structure.*

**Remark 274** *Furthermore, from the previous expressions it is easy to see that the poles of the mobility matrix happen when:*

$$-1 + \cosh\sqrt{\beta}\sqrt{\varpi}\delta\cos\sqrt{\beta}\sqrt{\varpi}\delta = 0 \tag{6.37}$$

*with* $\eta = i\varpi$ *and* $k = i\sqrt{\varpi}$*. The first root of this equation is*

$$\sqrt{\beta}\sqrt{\varpi}\delta_1 = r_1 \simeq 4.73004 \tag{6.38}$$

From now on we will be only interested in an approximate form of this matrix, obtained by a Laurent expansion of each entry in the neighborhood of $\delta = 0$, the convergence of this expansion is guaranteed as long as $\delta$ is less than $\delta_{\max}$, defined by:

$$\delta_{\max} := \frac{r_1}{\sqrt{\beta}\sqrt{\varpi_{\max}}} \tag{6.39}$$

where $\varpi_{\max}$ is the maximum pulsation allowed for the beam.

$$
\begin{array}{ll}
\textit{Entry} & \textit{Approximate value} \\
M_{B_{1,1}} & \dfrac{4}{\eta\beta^2}\delta^{-1} + \dfrac{1}{105}\eta\delta^3 \\
M_{B_{1,2}} & -\dfrac{6}{\eta\beta^2}\delta^{-2} - \dfrac{11}{210}\eta\delta^2 \\
M_{B_{2,1}} & -\dfrac{6}{\eta\beta^2}\delta^{-2} - \dfrac{11}{210}\eta\delta^2 \\
M_{B_{2,2}} & \dfrac{12}{\eta\beta^2}\delta^{-3} + \dfrac{13}{35}\eta\delta \\
M_{B_{1,3}} & -\dfrac{2}{\beta^2\eta}\delta^{-1} + \dfrac{1}{140}\eta\delta^3 \\
M_{B_{1,4}} & -\dfrac{6}{\beta^2\eta}\delta^{-2} + \dfrac{13}{420}\eta\delta^2 \\
M_{B_{2,3}} & \dfrac{6}{\beta^2\eta}\delta^{-2} - \dfrac{13}{420}\eta\delta^2 \\
M_{B_{2,4}} & \dfrac{12}{\beta^2\eta}\delta^{-3} - \dfrac{9}{70}\eta\delta \\
M_{B_{3,1}} & -\dfrac{2}{\beta^2\eta}\delta^{-1} + \dfrac{1}{140}\eta\delta^3 \\
M_{B_{3,2}} & \dfrac{6}{\beta^2\eta}\delta^{-2} - \dfrac{13}{420}\eta\delta^2 \\
M_{B_{4,1}} & -\dfrac{6}{\beta^2\eta}\delta^{-2} + \dfrac{13}{420}\eta\delta^2 \\
M_{B_{4,2}} & \dfrac{12}{\beta^2\eta}\delta^{-3} - \dfrac{9}{70}\eta\delta \\
M_{B_{3,3}} & \dfrac{4}{\beta^2\eta}\delta^{-1} + \dfrac{1}{105}\eta\delta^3 \\
M_{B_{3,4}} & \dfrac{6}{\beta^2\eta}\delta^{-2} + \dfrac{11}{210}\eta\delta^2 \\
M_{B_{4,3}} & \dfrac{6}{\beta^2\eta}\delta^{-2} + \dfrac{11}{210}\eta\delta^2 \\
M_{B_{4,4}} & \dfrac{12}{\beta^2\eta}\delta^{-3} + \dfrac{13}{35}\eta\delta
\end{array}
\tag{6.40}
$$

Assembling these entries, we get:

$$\hat{\mathbf{M}}_B = \begin{pmatrix} \frac{4}{\eta\beta^2}\delta^{-1} + \frac{1}{105}\eta\delta^3 & -\frac{6}{\eta\beta^2}\delta^{-2} - \frac{11}{210}\eta\delta^2 & -\frac{2}{\beta^2\eta}\delta^{-1} + \frac{1}{140}\eta\delta^3 & -\frac{6}{\beta^2\eta}\delta^{-2} + \frac{13}{420}\eta\delta^2 \\ -\frac{6}{\eta\beta^2}\delta^{-2} - \frac{11}{210}\eta\delta^2 & \frac{12}{\eta\beta^2}\delta^{-3} + \frac{13}{35}\eta\delta & \frac{6}{\beta^2\eta}\delta^{-2} - \frac{13}{420}\eta\delta^2 & \frac{12}{\beta^2\eta}\delta^{-3} - \frac{9}{70}\eta\delta \\ -\frac{2}{\beta^2\eta}\delta^{-1} + \frac{1}{140}\eta\delta^3 & \frac{6}{\beta^2\eta}\delta^{-2} - \frac{13}{420}\eta\delta^2 & \frac{4}{\beta^2\eta}\delta^{-1} + \frac{1}{105}\eta\delta^3 & \frac{6}{\beta^2\eta}\delta^{-2} + \frac{11}{210}\eta\delta^2 \\ -\frac{6}{\beta^2\eta}\delta^{-2} + \frac{13}{420}\eta\delta^2 & \frac{12}{\beta^2\eta}\delta^{-3} - \frac{9}{70}\eta\delta & \frac{6}{\beta^2\eta}\delta^{-2} + \frac{11}{210}\eta\delta^2 & \frac{12}{\beta^2\eta}\delta^{-3} + \frac{13}{35}\eta\delta \end{pmatrix} \tag{6.41}$$

**Remark 275** *The matrix $\hat{\mathbf{M}}_B$ is a real rational, symmetric matrix expressible in the canonic Foster's form:*

$$\hat{\mathbf{M}}_B = \frac{1}{\eta}\mathbf{K}_0 + \eta\mathbf{K}_\infty \tag{6.42}$$

$$\begin{gathered} \mathbf{K}_0 = \begin{pmatrix} \frac{4}{\beta^2}\delta^{-1} & -\frac{6}{\beta^2}\delta^{-2} & -\frac{2}{\beta^2}\delta^{-1} & -\frac{6}{\beta^2}\delta^{-2} \\ -\frac{6}{\beta^2}\delta^{-2} & \frac{12}{\beta^2}\delta^{-3} & \frac{6}{\beta^2}\delta^{-2} & \frac{12}{\beta^2}\delta^{-3} \\ -\frac{2}{\beta^2}\delta^{-1} & \frac{6}{\beta^2}\delta^{-2} & \frac{4}{\beta^2}\delta^{-1} & \frac{6}{\beta^2}\delta^{-2} \\ -\frac{6}{\beta^2}\delta^{-2} & \frac{12}{\beta^2}\delta^{-3} & \frac{6}{\beta^2}\delta^{-2} & \frac{12}{\beta^2}\delta^{-3} \end{pmatrix} \\ \mathbf{K}_\infty = \begin{pmatrix} \frac{1}{105}\delta^3 & -\frac{11}{210}\delta^2 & +\frac{1}{140}\delta^3 & +\frac{13}{420}\delta^2 \\ -\frac{11}{210}\delta^2 & +\frac{13}{35}\delta & -\frac{13}{420}\delta^2 & -\frac{9}{70}\delta \\ +\frac{1}{140}\delta^3 & -\frac{13}{420}\delta^2 & +\frac{1}{105}\delta^3 & +\frac{11}{210}\delta^2 \\ +\frac{13}{420}\delta^2 & -\frac{9}{70}\delta & +\frac{11}{210}\delta^2 & +\frac{13}{35}\delta \end{pmatrix} \end{gathered} \tag{6.43}$$

*where both the residue matrices are positive semi-definite and symmetric.*

### Synthesis of the residue matrix $\mathbf{K}_0$

The eigenvectors and eigenvalues of $\mathbf{K}_0$ are:

$$\begin{array}{ccccc} \textit{Eigenvector} & \begin{pmatrix} -1 \\ -\delta \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} -\frac{1}{2}\delta \\ 1 \\ \frac{1}{2}\delta \\ 1 \end{pmatrix} \\ \textit{Eigenvalue} & 0 & 0 & \frac{2}{\beta^2\delta} & 6\frac{\delta^2+4}{\beta^2\delta^3} \end{array} \tag{6.44}$$

The eigenvalues can be arranged in the following diagonal matrix:

$$\Lambda_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\beta^2\delta} & 0 \\ 0 & 0 & 0 & 6\frac{\delta^2+4}{\beta^2\delta^3} \end{pmatrix} \tag{6.45}$$

an approximate form of $\Lambda_0$ is:

$$\hat{\Lambda}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\beta^2\delta} & 0 \\ 0 & 0 & 0 & \frac{24}{\beta^2}\delta^{-3} \end{pmatrix} \tag{6.46}$$

The system of eigenvectors obtained is not orthonormal.

Nevertheless by simple calculations, following the Gram-Schmidt algorithm, it is possible to find a set of orthonormal eigenvectors of $\mathbf{K}_0$:

$$
\begin{array}{ccccc}
\textit{Eigenvector} & \begin{pmatrix} 0 \\ -\frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{pmatrix} & \begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{(4+\delta^2)}} \\ -\frac{\sqrt{2}\delta}{2\sqrt{(4+\delta^2)}} \\ \frac{\sqrt{2}}{\sqrt{(4+\delta^2)}} \\ -\frac{\sqrt{2}\delta}{2\sqrt{(4+\delta^2)}} \end{pmatrix} & \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \\ 0 \end{pmatrix} & \begin{pmatrix} -\frac{\sqrt{2}}{2\sqrt{(4+\delta^2)}}\delta \\ \frac{\sqrt{2}}{\sqrt{(4+\delta^2)}} \\ \frac{\sqrt{2}}{2\sqrt{(4+\delta^2)}}\delta \\ \frac{\sqrt{2}}{\sqrt{(4+\delta^2)}} \end{pmatrix} \\
\textit{Eigenvalue} & 0 & 0 & \dfrac{2}{\beta^2\delta} & 6\dfrac{\delta^2+4}{\beta^2\delta^3}
\end{array}
\tag{6.47}
$$

The Eigenvectors of $\mathbf{K}_0$ can be assembled by rows in the matrix:

$$
\mathbf{E}_0 = \begin{pmatrix} 0 & -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ -\frac{\sqrt{2}}{\sqrt{(4+\delta^2)}} & -\frac{\sqrt{2}\delta}{2\sqrt{(4+\delta^2)}} & \frac{\sqrt{2}}{\sqrt{(4+\delta^2)}} & -\frac{\sqrt{2}\delta}{2\sqrt{(4+\delta^2)}} \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{\sqrt{2}\delta}{2\sqrt{(4+\delta^2)}} & \frac{\sqrt{2}}{\sqrt{(4+\delta^2)}} & \frac{\sqrt{2}\delta}{2\sqrt{(4+\delta^2)}} & \frac{\sqrt{2}}{\sqrt{(4+\delta^2)}} \end{pmatrix} \tag{6.48}
$$

Furthermore consider the matrix $\mathbf{T}_0$ obtained by multiplying $\mathbf{E}_0$ by a real positive number $\kappa_0$:

$$
\mathbf{T}_0 = \kappa_0\sqrt{2} \begin{pmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{\sqrt{(4+\delta^2)}} & -\frac{\delta}{2\sqrt{(4+\delta^2)}} & \frac{1}{\sqrt{(4+\delta^2)}} & -\frac{\delta}{2\sqrt{(4+\delta^2)}} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{\delta}{2\sqrt{(4+\delta^2)}} & \frac{1}{\sqrt{(4+\delta^2)}} & \frac{\delta}{2\sqrt{(4+\delta^2)}} & \frac{1}{\sqrt{(4+\delta^2)}} \end{pmatrix} \tag{6.49}
$$

A linearized form of $\mathbf{T}_0$ is:

$$\hat{\mathbf{T}}_0 = \kappa_0 \frac{\sqrt{2}}{4} \begin{pmatrix} 0 & -2 & 0 & 2 \\ -2 & -\delta & 2 & -\delta \\ 2 & 0 & 2 & 0 \\ -\delta & 2 & \delta & 2 \end{pmatrix} \tag{6.50}$$

It is well-known that:

$$\mathbf{K}_0 = \mathbf{E}_0^T \Lambda_0 \mathbf{E}_0 \tag{6.51}$$

$$\mathbf{K}_0 = \mathbf{T}_0^T \left( \frac{1}{\kappa_0^2} \Lambda_0 \right) \mathbf{T}_0 \tag{6.52}$$

Thus $\mathbf{K}_0$ can be realized by a multiport transformer $\mathcal{N}_T$ the turning ratio matrix of which is $\mathbf{T}_0$ cascade connected to a simple $4-$port network with impedance matrix $\frac{1}{\kappa_0^2}\Lambda_0$, i.e. "constituted" by two short-circuits and two capacitors which have capacitances equal to:

$$c_1 = \frac{\beta^2 \delta}{2\kappa_0^2} \qquad c_1 = \frac{\beta^2 \delta^3}{6\left(\delta^2 + 4\right)\kappa_0^2} \tag{6.53}$$

### Simplification of the capacitive circuit

In the previous section we have shown that:

$$\mathbf{K}_0 = \mathbf{T}_0^T \left( \frac{1}{\kappa_0^2} \Lambda_0 \right) \mathbf{T}_0 \tag{6.54}$$

with $\Lambda_0$ and $\mathbf{T}_0$ $4 \times 4$ matrices.

In this section we want to prove that it is possible to synthesize the capacitive residue, using a simpler circuit constituted by a multiport transformer whose turns-ratio

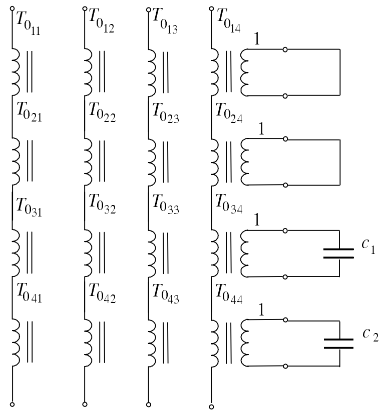


Figure 6.8: Realization of the residue matrix $\mathbf{K}_0$

matrix is $2 \times 4$ and a load which is constituted only by the two uncoupled capacitors previously introduced.

This simplification is due to the fact that the residue matrix is singular, in particular its rank is equal to 2.

Consider the equation (6.54), in the form:

$$\mathbf{K}_0 = \frac{1}{\kappa_0^2} \begin{pmatrix} \mathbf{T}_{0_{1,1}}^T & \mathbf{T}_{0_{2,1}}^T \\ \mathbf{T}_{0_{1,2}}^T & \mathbf{T}_{0_{2,2}}^T \end{pmatrix} \begin{pmatrix} \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} \\ \mathbf{0}_{2\times2} & \Lambda_{0_{2,2}} \end{pmatrix} \begin{pmatrix} \mathbf{T}_{0_{1,1}} & \mathbf{T}_{0_{1,2}} \\ \mathbf{T}_{0_{2,1}} & \mathbf{T}_{0_{2,2}} \end{pmatrix} \tag{6.55}$$

evaluating the product we get:

$$\mathbf{K}_0 = \frac{1}{\kappa_0^2} \begin{pmatrix} \mathbf{0}_{2\times2} & \mathbf{T}_{0_{2,1}}^T \Lambda_{0_{2,2}} \\ \mathbf{0}_{2\times2} & \mathbf{T}_{0_{2,2}}^T \Lambda_{0_{2,2}} \end{pmatrix} \begin{pmatrix} \mathbf{T}_{0_{1,1}} & \mathbf{T}_{0_{1,2}} \\ \mathbf{T}_{0_{2,1}} & \mathbf{T}_{0_{2,2}} \end{pmatrix} = \frac{1}{\kappa_0^2} \begin{pmatrix} \mathbf{T}_{0_{2,1}}^T \Lambda_{0_{2,2}} \mathbf{T}_{0_{2,1}} & \mathbf{T}_{0_{2,1}}^T \Lambda_{0_{2,2}} \mathbf{T}_{0_{2,2}} \\ \mathbf{T}_{0_{2,2}}^T \Lambda_{0_{2,2}} \mathbf{T}_{0_{2,1}} & \mathbf{T}_{0_{2,2}}^T \Lambda_{0_{2,2}} \mathbf{T}_{0_{2,2}} \end{pmatrix} \tag{6.56}$$

it is clear, that the only significant submatrices in the turns-ratio matrix of the transformer are $\mathbf{T}_{0_{2,1}}$ and $\mathbf{T}_{0_{2,2}}$.

Consider now the ideal transformer defined by the turns ratio matrix $\mathbf{S}_0$:

$$\mathbf{S}_0 = \begin{pmatrix} \mathbf{T}_{0_{2,1}} & \mathbf{T}_{0_{2,2}} \end{pmatrix} \tag{6.57}$$

and suppose to cascade load it with the two uncoupled capacitors represented by the matrix $\frac{1}{\kappa_0^2}\Lambda_{0_{2,2}}$, then we will obtain an impedance matrix defined by:

$$\frac{1}{\kappa_0^2}\mathbf{S}_0^T\Lambda_{0_{2,2}}\mathbf{S}_0 = \frac{1}{\kappa_0^2}\begin{pmatrix} \mathbf{T}_{0_{2,1}}^T \\ \mathbf{T}_{0_{2,2}}^T \end{pmatrix}\Lambda_{0_{2,2}}\begin{pmatrix} \mathbf{T}_{0_{2,1}} & \mathbf{T}_{0_{2,2}} \end{pmatrix} = \frac{1}{\kappa_0^2}\begin{pmatrix} \mathbf{T}_{0_{2,1}}^T\Lambda_{0_{2,2}}\mathbf{T}_{0_{2,1}} & \mathbf{T}_{0_{2,1}}^T\Lambda_{0_{2,2}}\mathbf{T}_{0_{2,2}} \\ \mathbf{T}_{0_{2,2}}^T\Lambda_{0_{2,2}}\mathbf{T}_{0_{2,1}} & \mathbf{T}_{0_{2,2}}^T\Lambda_{0_{2,2}}\mathbf{T}_{0_{2,2}} \end{pmatrix} \equiv \mathbf{K}_0 \tag{6.58}$$

Thus the residue matrix can be synthesized cascade connecting a transformer, the turns-ratio matrix of which is $\mathbf{S}_0$, with two uncoupled capacitors.

**Notation 276** *In the following, we will denote whenever it is possible the turns-ratio matrix* $\mathbf{S}_0$ *by* $\mathbf{T}_0$*.*

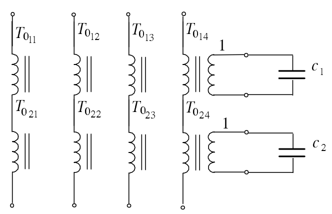


Figure 6.9: Simplified realization of the residue matrix

## Synthesis of the residue matrix $\mathbf{K}_\infty$

The eigenvectors and eigenvalue of $\mathbf{K}_\infty$ are:

$$
\begin{array}{cc}
\textit{Eigenvector} & \textit{Eigenvalue} \\
\begin{pmatrix}
1 \\
\frac{1}{5}\frac{\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2+\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta}{\delta^2} \\
1 \\
-\frac{1}{5}\frac{\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2+\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta}{\delta^2}
\end{pmatrix} & \left(\frac{1}{4}+\frac{1}{120}\delta^2+\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta \\
\begin{pmatrix}
1 \\
\frac{1}{5}\frac{\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2-\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta}{\delta^2} \\
1 \\
-\frac{1}{5}\frac{\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2-\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta}{\delta^2}
\end{pmatrix} & \left(\frac{1}{4}+\frac{1}{120}\delta^2-\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta \\
\begin{pmatrix}
\frac{2}{3}\frac{17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2+\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta}{\delta^2} \\
1 \\
-\frac{2}{3}\frac{17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2+\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta}{\delta^2} \\
1
\end{pmatrix} & \left(\frac{17}{140}+\frac{1}{840}\delta^2+\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta \\
\begin{pmatrix}
\frac{2}{3}\frac{17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2-\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta}{\delta^2} \\
1 \\
-\frac{2}{3}\frac{17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2-\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta}{\delta^2} \\
1
\end{pmatrix} & \left(\frac{17}{140}+\frac{1}{840}\delta^2-\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta
\end{array}
\tag{6.59}
$$

The eigenvalues can be arranged in a following diagonal matrix such that:

$$
\begin{aligned}
\Lambda_{\infty_{1,1}} &= \left(\tfrac{1}{4}+\tfrac{1}{120}\delta^2+\tfrac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta \\
\Lambda_{\infty_{2,2}} &= \left(\tfrac{1}{4}+\tfrac{1}{120}\delta^2-\tfrac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta \\
\Lambda_{\infty_{3,3}} &= \left(\tfrac{17}{140}+\tfrac{1}{840}\delta^2+\tfrac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta \\
\Lambda_{\infty_{4,4}} &= \left(\tfrac{17}{140}+\tfrac{1}{840}\delta^2-\tfrac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta .
\end{aligned}
\tag{6.60}
$$

An approximate form of $\Lambda_\infty$ is:

$$
\hat{\Lambda}_\infty = \begin{pmatrix} \frac{1}{2}\delta & 0 & 0 & 0 \\ 0 & \frac{1}{360}\delta^3 & 0 & 0 \\ 0 & 0 & \frac{17}{70}\delta & 0 \\ 0 & 0 & 0 & \frac{1}{2040}\delta^3 \end{pmatrix}
\tag{6.61}
$$

The norms of the eigenvectors so far obtained are not equal to 1.

Nevertheless by simple calculations it is possible to find a set of orthonormal eigenvectors of $\mathbf{K}_\infty$:

$$
\begin{array}{cc}
\textit{Eigenvector} & \textit{Linearized expression} \\
\begin{pmatrix}
\dfrac{5}{\sqrt{\left(50+2\dfrac{\left(\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2+\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}\right)}} \\
\dfrac{\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2+\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta}{\delta^2\sqrt{\left(50+2\dfrac{\left(\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2+\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}\right)}} \\
\dfrac{5}{\sqrt{\left(50+2\dfrac{\left(\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2+\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}\right)}} \\
-\dfrac{\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2+\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta}{\delta^2\sqrt{\left(50+2\dfrac{\left(\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2+\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}\right)}}
\end{pmatrix}
&
\begin{pmatrix}
\frac{1}{12}\sqrt{2}\delta \\
-\frac{1}{2}\sqrt{2} \\
\frac{1}{12}\sqrt{2}\delta \\
\frac{1}{2}\sqrt{2}
\end{pmatrix}
\end{array}
\tag{6.62}
$$

$$
\begin{array}{cc}
\textit{Eigenvector} & \textit{Linearized expression} \\
\begin{pmatrix}
\dfrac{5}{\sqrt{\left(50+2\dfrac{\left(\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2-\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}\right)}} \\
\dfrac{\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2-\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta}{\delta^2\sqrt{\left(50+2\dfrac{\left(\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2-\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}\right)}} \\
\dfrac{5}{\sqrt{\left(50+2\dfrac{\left(\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2-\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}\right)}} \\
-\dfrac{\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2-\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta}{\delta^2\sqrt{\left(50+2\dfrac{\left(\delta^3-60\left(\frac{1}{4}+\frac{1}{120}\delta^2-\frac{1}{120}\sqrt{\left(900+40\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}\right)}}
\end{pmatrix}
&
\begin{pmatrix}
\frac{1}{2}\sqrt{2} \\
\frac{1}{12}\sqrt{2}\delta \\
\frac{1}{2}\sqrt{2} \\
-\frac{1}{12}\sqrt{2}\delta
\end{pmatrix}
\end{array}
\tag{6.63}
$$

$$
\begin{array}{cc}
\textit{Eigenvector} & \textit{Linearized expression} \\
\begin{pmatrix}
2\dfrac{17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2+\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta}{\delta^2\sqrt{\left(8\dfrac{\left(17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2+\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}+18\right)}} \\
\dfrac{3}{\sqrt{\left(8\dfrac{\left(17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2+\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}+18\right)}} \\
-2\dfrac{17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2+\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta}{\delta^2\sqrt{\left(8\dfrac{\left(17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2+\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}+18\right)}} \\
\dfrac{3}{\sqrt{\left(8\dfrac{\left(17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2+\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}+18\right)}}
\end{pmatrix}
&
\begin{pmatrix}
-\frac{3}{68}\sqrt{2}\delta \\
\frac{1}{2}\sqrt{2} \\
\frac{3}{68}\sqrt{2}\delta \\
\frac{1}{2}\sqrt{2}
\end{pmatrix}
\end{array}
\tag{6.64}
$$

$$\begin{array}{cc} \textit{Eigenvector} & \textit{Linearized expression} \\ \begin{pmatrix} 2\dfrac{17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2-\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta}{\delta^2\sqrt{\left(8\dfrac{\left(17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2-\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}+18\right)}} \\ \dfrac{3}{\sqrt{\left(8\dfrac{\left(17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2-\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}+18\right)}} \\ -2\dfrac{17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2-\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta}{\delta^2\sqrt{\left(8\dfrac{\left(17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2-\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}+18\right)}} \\ \dfrac{3}{\sqrt{\left(8\dfrac{\left(17\delta-70\left(\frac{17}{140}+\frac{1}{840}\delta^2-\frac{1}{840}\sqrt{\left(10\,404+120\delta^2+\delta^4\right)}\right)\delta\right)^2}{\delta^4}+18\right)}} \end{pmatrix} & \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{3}{68}\sqrt{2}\delta \\ -\frac{1}{2}\sqrt{2} \\ \frac{3}{68}\sqrt{2}\delta \end{pmatrix} \end{array} \tag{6.65}$$

The Eigenvectors of $\mathbf{K}_\infty$ can be assembled by rows in the matrix $\mathbf{E}_\infty$. Nevertheless we will only be interested in the linearized matrix:

$$\hat{\mathbf{E}}_\infty = \begin{pmatrix} \frac{1}{12}\sqrt{2}\delta & -\frac{1}{2}\sqrt{2} & \frac{1}{12}\sqrt{2}\delta & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{12}\sqrt{2}\delta & \frac{1}{2}\sqrt{2} & -\frac{1}{12}\sqrt{2}\delta \\ -\frac{3}{68}\sqrt{2}\delta & \frac{1}{2}\sqrt{2} & \frac{3}{68}\sqrt{2}\delta & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{3}{68}\sqrt{2}\delta & -\frac{1}{2}\sqrt{2} & \frac{3}{68}\sqrt{2}\delta \end{pmatrix} \tag{6.66}$$

Furthermore, consider the matrix $\hat{\mathbf{T}}_\infty$ obtained through multiplying $\hat{\mathbf{E}}_\infty$ by a real positive number $\kappa_\infty$:

$$\hat{\mathbf{T}}_\infty = \kappa_\infty \frac{\sqrt{2}}{2} \begin{pmatrix} \frac{1}{6}\delta & -1 & \frac{1}{6}\delta & 1 \\ 1 & \frac{1}{6}\delta & 1 & -\frac{1}{6}\delta \\ -\frac{3}{34}\delta & 1 & \frac{3}{34}\delta & 1 \\ 1 & \frac{3}{34}\delta & -1 & \frac{3}{34}\delta \end{pmatrix} \tag{6.67}$$

It is well-known that:

$$\mathbf{K}_\infty = \mathbf{E}_\infty^T \Lambda_\infty \mathbf{E}_\infty = \mathbf{T}_\infty^T \left( \frac{1}{\kappa_\infty^2} \Lambda_\infty \right) \mathbf{T}_\infty \tag{6.68}$$

and:

$$\mathbf{K}_\infty \simeq \hat{\mathbf{E}}_\infty^T \hat{\Lambda}_\infty \hat{\mathbf{E}}_\infty = \hat{\mathbf{T}}_\infty^T \left( \frac{1}{\kappa_\infty^2} \hat{\Lambda}_\infty \right) \hat{\mathbf{T}}_\infty \tag{6.69}$$

Thus $\mathbf{K}_\infty$ can be realized by a multiport transformer $\mathcal{N}_T$ whose turning ratio matrix is $\mathbf{T}_\infty$ cascade connected to a simple 4−port network whose impedance matrix is $\frac{1}{\kappa_\infty^2}\Lambda_\infty$, i.e. "constituted" by four inductors whose inductances are equal to the eigenvalues of $\mathbf{K}_\infty$ divided by $\kappa_\infty^2$.

**Summary 277** *Given the mobility bending matrix $\mathbf{M}_B$ we have decomposed it into the sum of two matrices, one representing the capacitive effect and the other representing the inductive effect. We have realized both matrices as a cascade loaded transformer, where the loads are simple diagonal matrices. Below is given a table of the approximate values of the turning ratio matrices of the two transformers and the impedance matrices*

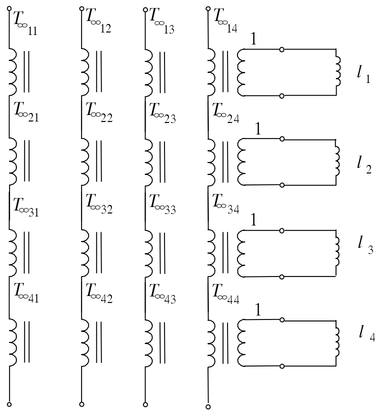


Figure 6.10: Realization of the residue $\mathbf{K}_\infty$

*of the loads:*

$$
\begin{array}{ccc}
\textit{matrix} & \mathbf{K}_0 & \mathbf{K}_\infty \\
\textit{turns-ratio} & \dfrac{\kappa_0\sqrt{2}}{4}\begin{pmatrix} 2 & 0 & 2 & 0 \\ -\delta & 2 & \delta & 2 \end{pmatrix} & \dfrac{\kappa_\infty\sqrt{2}}{2}\begin{pmatrix} \frac{1}{6}\delta & -1 & \frac{1}{6}\delta & 1 \\ 1 & \frac{1}{6}\delta & 1 & -\frac{1}{6}\delta \\ -\frac{3}{34}\delta & 1 & \frac{3}{34}\delta & 1 \\ 1 & \frac{3}{34}\delta & -1 & \frac{3}{34}\delta \end{pmatrix} \\
\textit{load} & \dfrac{1}{\kappa_0^2}\begin{pmatrix} \frac{2}{\beta^2\delta} & 0 \\ 0 & \frac{24}{\beta^2}\delta^{-3} \end{pmatrix} & \dfrac{1}{\kappa_\infty^2}\begin{pmatrix} \frac{1}{2}\delta & 0 & 0 & 0 \\ 0 & \frac{1}{360}\delta^3 & 0 & 0 \\ 0 & 0 & \frac{17}{70}\delta & 0 \\ 0 & 0 & 0 & \frac{1}{2040}\delta^3 \end{pmatrix}
\end{array}
\tag{6.70}
$$

**impedance matrix of the analog circuit**

In this section we want to find the numerical values of the inductances and the capacitances that constitute the cascade load of the so far discussed transformers.

The dimensionless inductances are:

$$l_1 = \frac{1}{\kappa_\infty^2}\frac{1}{2}\delta \qquad l_2 = \frac{1}{\kappa_\infty^2}\frac{1}{360}\delta^3 \qquad l_3 = \frac{1}{\kappa_\infty^2}\frac{17}{70}\delta \qquad l_4 = \frac{1}{\kappa_\infty^2}\frac{1}{2040}\delta^3 \tag{6.71}$$

while the dimensionless capacitances are:

$$c_1 = \kappa_0^2\frac{\beta^2\delta}{2} \qquad c_2 = \kappa_0^2\frac{\beta^2\delta^3}{24} \tag{6.72}$$

Then:

$$l_1 = \frac{1}{2\kappa_\infty^2}\frac{x}{l} \qquad l_2 = \frac{1}{360\kappa_\infty^2}\frac{x^3}{l^3} \qquad l_3 = \frac{17}{70\kappa_\infty^2}\frac{x}{l} \qquad l_4 = \frac{1}{2040\kappa_\infty^2}\frac{x^3}{l^3} \tag{6.73}$$

$$c_1 = \frac{\kappa_0^2}{2}\frac{\lambda x l^3}{t_0^2 k_M} \qquad c_2 = \frac{\kappa_0^2}{24}\frac{\lambda x^3 l}{t_0^2 k_M} \tag{6.74}$$

It is so possible to introduce the dimensionless impedance matrices:

$$\mathbf{z}_0(\eta) := \frac{1}{\eta}\begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & \frac{1}{c_2} \end{pmatrix} \qquad \mathbf{z}_\infty(\eta) := \eta\begin{pmatrix} l_1 & 0 & 0 & 0 \\ 0 & l_2 & 0 & 0 \\ 0 & 0 & l_3 & 0 \\ 0 & 0 & 0 & l_4 \end{pmatrix} \tag{6.75}$$

And the governing equation for the dimensionless circuit can be written as:

$$\tilde{\boldsymbol{\varphi}}(\eta) = \mathbf{M}_B\tilde{\boldsymbol{\iota}}(\eta) \simeq \left(\mathbf{T}_0^T\mathbf{z}_0(\eta)\,\mathbf{T}_0 + \mathbf{T}_\infty^T\mathbf{z}_\infty(\eta)\,\mathbf{T}_\infty\right)\tilde{\boldsymbol{\iota}}(\eta) \tag{6.76}$$

where $\boldsymbol{\varphi}(\tau)$ is the dimensionless voltage column vector and $\boldsymbol{\iota}(\tau)$ is the dimensionless current column vector defined by:

$$\begin{aligned}
&\boldsymbol{\varphi}(\tau) = \frac{1}{V_0}\mathbf{V}(t) \Rightarrow \tilde{\boldsymbol{\varphi}}(\eta) = \frac{1}{V_0 t_0}\tilde{\mathbf{V}}(s) \\
&\boldsymbol{\iota}(\tau) = \frac{1}{I_0}\mathbf{I}(t) \Rightarrow \tilde{\boldsymbol{\iota}}(\eta) = \frac{1}{I_0 t_0}\tilde{\mathbf{I}}(s) \\
&\frac{V_0}{I_0} = R_0 \Rightarrow \tilde{\boldsymbol{\varphi}}(\eta, \varepsilon) = \frac{1}{R_0 I_0 t_0}\tilde{\mathbf{V}}(s)
\end{aligned} \tag{6.77}$$

where $V_0$, $I_0$, $R_0$ are positive dimensional constants, and only one among them can be chosen arbitrarily. For instance, we can choose an arbitrary value for $V_0$.

Now we have to find the analog dimensional circuit. To reach this goal consider the dimensional load matrices:

$$\mathbf{Z}_0(s) = R_0\Lambda_0(\eta) \qquad \mathbf{Z}_\infty(s) = R_0\Lambda_\infty(\eta) \tag{6.78}$$

where $R_0$ has the dimension of a resistance.

Hence:

$$\mathbf{Z}_0(s) = R_0\frac{1}{st_0}\begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & \frac{1}{c_2} \end{pmatrix} \qquad \mathbf{Z}_\infty(s) = R_0 st_0\begin{pmatrix} l_1 & 0 & 0 & 0 \\ 0 & l_2 & 0 & 0 \\ 0 & 0 & l_3 & 0 \\ 0 & 0 & 0 & l_4 \end{pmatrix}$$

Thus:

$$\mathbf{Z}_0(s) = \frac{1}{s}\begin{pmatrix} \frac{1}{C_1} & 0 \\ 0 & \frac{1}{C_2} \end{pmatrix} \qquad \mathbf{Z}_\infty(s) = s\begin{pmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & L_4 \end{pmatrix} \tag{6.79}$$

with:

$$C_1 = \frac{c_1 t_0}{R_0}, \qquad C_2 = \frac{c_2 t_0}{R_0} \tag{6.80}$$

$$L_1 = l_1 R_0 t_0, \qquad L_2 = l_2 R_0 t_0, \qquad L_3 = l_3 R_0 t_0, \qquad L_4 = l_4 R_0 t_0 \tag{6.81}$$

i.e.:

$$C_1 = \frac{\kappa_0^2}{2} \frac{\lambda x l^3}{t_0 k_M R_0}, \qquad C_2 = \frac{\kappa_0^2}{24} \frac{\lambda x^3 l}{t_0 k_M R_0} \tag{6.82}$$

$$L_1 = \frac{1}{2\kappa_\infty^2} \frac{x R_0 t_0}{l}, \; L_2 = \frac{1}{360\kappa_\infty^2} \frac{x^3 R_0 t_0}{l^3}, \; L_3 = \frac{17}{70\kappa_\infty^2} \frac{x R_0 t_0}{l}, \; L_4 = \frac{1}{2040\kappa_\infty^2} \frac{x^3 R_0 t_0}{l^3} \tag{6.83}$$

This yields:

$$\begin{aligned} \frac{C_1}{C_2} &= 12 \frac{l^2}{x^2} \\ C_1 L_2 &= \frac{\kappa_0^2}{720\kappa_\infty^2} \frac{\lambda}{k_M} x^4 \\ C_1 L_4 &= \frac{\kappa_0^2}{4080\kappa_\infty^2} \frac{\lambda}{k_M} x^4 \\ C_2 L_1 &= \frac{\kappa_0^2}{48\kappa_\infty^2} \frac{\lambda}{k_M} x^4 \\ C_2 L_3 &= \frac{17}{1680} \frac{\kappa_0^2}{\kappa_\infty^2} \frac{\lambda}{k_M} x^4 \end{aligned} \tag{6.84}$$

The governing equation for the beam element can be written as follows:

$$\mathbf{V}(s) = \mathbf{Z}(s)\,\mathbf{I}(s) = \left(\mathbf{T}_0^T \mathbf{Z}_0(s) \mathbf{T}_0 + \mathbf{T}_\infty^T \mathbf{Z}_\infty(s) \mathbf{T}_\infty\right) \mathbf{I}(s) \tag{6.85}$$

**Remark 278** *Once we have chosen a particular value for $C_1$, for the size of the beam-element $x$ and for the magnification constants $\kappa_0$ and $\kappa_\infty$ all the subnetworks constituting the circuit analog to the transversally vibrating beam element are unequivocally determined. Furthermore the resistance $R_0$ is unequivocally determined, since $t_0$ is fixed.*

### Numerical example

In this subsection we will give a brief outline of the procedure, which leads to the evaluation of all the subnetworks of the previously described circuit.

Consider a beam of length $l$ and suppose the cross-section to have rectangular shape with breadth $b$ and depth $d$. The stiffness constant $k_M$ for this beam is:

$$k_M = E\frac{1}{12}bd^3 \tag{6.86}$$

where $E$ is the Young modulus of the material, and the radius of gyration $r_0$ is:

$$r_0 = \sqrt{\frac{\frac{1}{12}bd^3}{bd}} = \sqrt{\frac{1}{12}}d \tag{6.87}$$

Furthermore the density per unit length $\lambda$ is equal to:

$$\lambda = \rho bd \tag{6.88}$$

where $\rho$ is the density per unit volume of the material.

Assume the following parameters to be given:

$$\begin{aligned} &l = 1\,m \\ &b = 3\,cm \\ &d = 2\,mm \\ &E = 70\,GPa \\ &\rho = 2700\,Kg/m^3 \\ &r_0 = 0.57736\,mm \end{aligned} \tag{6.89}$$

Hence substituting these data into eq.(6.86) and eq.(6.88) we get:

$$k_M = 1.4\,Nm^2 \qquad \lambda = 0.162\,Kg/m \tag{6.90}$$

Furthermore suppose that the following electrical entities are fixed:

$$
\begin{aligned}
&C_1 = 0.6\,\mu F \\
&\kappa_0 = \tfrac{1}{\sqrt{2}} \\
&\kappa_\infty = 10\sqrt{2}
\end{aligned}
\tag{6.91}
$$

and to use 10 moduli in the circuit, i.e.:

$$
x = \frac{l}{10} = 10\,cm \tag{6.92}
$$

Then all the quantities in the subnetworks are determined by:

$$
\begin{aligned}
C_2 &= \frac{x^2}{12l^2}C_1 = 5.0\times 10^{-10}F = 50\,nF \\
L_2 &= \frac{\kappa_0^2}{720\kappa_\infty^2}\frac{\lambda}{k_M}x^4\frac{1}{C_1} = 6.6966\times 10^{-5}H \simeq 67\,\mu H \\
L_4 &= \frac{\kappa_0^2}{4080\kappa_\infty^2}\frac{\lambda}{k_M}x^4\frac{1}{C_1} = 1.1817\times 10^{-5}H \simeq 12\,\mu H \\
L_1 &= \frac{\kappa_0^2}{48\kappa_\infty^2}\frac{\lambda}{k_M}x^4\frac{1}{C_2} = 1.205\,4\times 10^{-2}H \simeq 12\,mH \\
L_3 &= \frac{17}{1680}\frac{\kappa_0^2}{\kappa_\infty^2}\frac{\lambda}{k_M}x^4\frac{1}{C_2} = 5.854\,7\times 10^{-3}H \simeq 5.8\,mH\ :
\end{aligned}
\tag{6.93}
$$

Finally consider the two transformers:

$$
\hat{\mathbf{T}}_0 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{40} & \frac{1}{2} & \frac{1}{40} & \frac{1}{2} \end{pmatrix} \qquad \hat{\mathbf{T}}_\infty = \begin{pmatrix} \frac{1}{6} & -10 & \frac{1}{6} & 10 \\ 10 & \frac{1}{6} & 10 & -\frac{1}{6} \\ -\frac{3}{34} & 10 & \frac{3}{34} & 10 \\ 10 & \frac{3}{34} & -10 & \frac{3}{34} \end{pmatrix} \tag{6.94}
$$

Furthermore, let us take $t_0 = 1\,s$, then the maximum frequency allowed for the circuit to simulate the trasversally vibrating beam given by eq.(6.39) is:

$$
\begin{aligned}
f_{\max} = \frac{1}{2\pi}\left(\frac{r_1}{\sqrt{\beta}\delta}\right)^2 = \frac{1}{2\pi}\frac{r_1^2}{\delta^2\beta} = \frac{1}{2\pi}\frac{r_1^2}{\delta^2 l^2}\sqrt{\frac{k_M}{\lambda}}t_0 \simeq \\
\frac{1}{2\pi}\frac{(4.7003)^2}{(0.1)^2(1)^2}\sqrt{\frac{1.4}{0.162}}\,(1)\ Hz = 1033.7\,Hz \simeq 1\,KHz
\end{aligned}
$$

## Governing equations for the distributed circuit analog to the transversally vibrating beam

Once we have found the circuit simulating the transversally vibrating beam element, we have to cascade connect a number of them to approximate the behavior of the entire beam.

Nevertheless it seems useless to study the resulting circuit as the finite cascade of a finite circuit, since it would be characterized by a great amount of simple equations, which would not lead to the physical understanding of the problem.

Thus we will go back to the differential equation, adopting a clever procedure called homogenization, which roughly consists of the substitution of many lumped circuits by a set of *cables* each of them characterized by distributed parameters.

Hence a further step of our analysis is the homogenization of the finite circuit approximating the beam, in order to get a distributed circuit, i.e. an infinite circuit governed by the same set of differential equations characterizing the entire beam.

As a preliminary step we have to introduce the variables defining the distributed circuit at every point $x$. Thus let us call $V_1(t,x)$ the voltage drop at the time $t$ and abscissa $x$ between the first pair of cables, $V_2(t,x)$ the voltage drop at the time $t$ and abscissa $x$ between the second pair of cables. Furthermore we will denote by $I_1(t,x)$ the current flowing in the positive $x$ direction in the first cable, which is equal to the current flowing in the negative $x$ direction in the second cable and we will denote by $I_2(t,x)$ the current flowing in the positive $x$ direction in the third cable, which is equal to the current flowing in the negative $x$ direction in the fourth cable.

The set of differential equations written, in dimensionless variables is clearly:

$$\begin{pmatrix} \tilde{\varphi}_1(\eta,\varepsilon) \\ \tilde{\varphi}_2(\eta,\varepsilon) \\ \tilde{\imath}_1(\eta,\varepsilon) \\ \tilde{\imath}_2(\eta,\varepsilon) \end{pmatrix}' = \mathbf{B} \begin{pmatrix} \tilde{\varphi}_1(\eta,\varepsilon) \\ \tilde{\varphi}_2(\eta,\varepsilon) \\ \tilde{\imath}_1(\eta,\varepsilon) \\ \tilde{\imath}_2(\eta,\varepsilon) \end{pmatrix} \tag{6.95}$$

where the derivation is with respect to the dimensionless variable $\varepsilon$.

Here $\varphi_1(\tau,\varepsilon)$ is the first dimensionless voltage drop at $\varepsilon$, $\varphi_2(\tau,\varepsilon)$ is the second dimensionless voltage drop at $\varepsilon$, and $i_1(\tau,\varepsilon)$ is the first dimensionless current and $i_2(\tau,\varepsilon)$ is the second dimensionless current.

The dimensionless variables are related to the dimensional ones as follows:

$$\begin{aligned}
&\varphi_1(\tau,\varepsilon) = \frac{1}{V_0} V_1(t,x) \Rightarrow \tilde{\varphi}_1(\eta,\varepsilon) = \frac{1}{V_0 t_0} \tilde{V}_1(s,x) \\
&\varphi_2(\tau,\varepsilon) = \frac{1}{V_0} V_2(t,x) \Rightarrow \tilde{\varphi}_2(\eta,\varepsilon) = \frac{1}{V_0 t_0} \tilde{V}_2(s,x) \\
&\iota_1(\tau,\varepsilon) = \frac{1}{I_0} I_1(t,x) \Rightarrow \tilde{\imath}_1(\eta,\varepsilon) = \frac{1}{I_0 t_0} \tilde{I}_1(s,x) \\
&\iota_2(\tau,\varepsilon) = \frac{1}{I_0} I_2(t,x) \Rightarrow \tilde{\imath}_2(\eta,\varepsilon) = \frac{1}{I_0 t_0} \tilde{I}_2(s,x) \\
&\frac{V_0}{I_0} = R_0 \Rightarrow \tilde{\varphi}_1(\eta,\varepsilon) = \frac{1}{R_0 I_0 t_0} \tilde{V}_1(s,x), \; \tilde{\varphi}_2(\eta,\varepsilon) = \frac{1}{R_0 I_0 t_0} \tilde{V}_2(s,x)
\end{aligned} \tag{6.96}$$

where $R_0$ is the resistance introduced in the previous section.

The dimensionless variables of the distributed circuit are related in the refined circuit by:

$$\begin{pmatrix} \tilde{\varphi}_1(\eta,m\delta) \\ \tilde{\varphi}_2(\eta,m\delta) \\ \tilde{\varphi}_1(\eta,(m+1)\delta) \\ \tilde{\varphi}_2(\eta,(m+1)\delta) \end{pmatrix} = \mathbf{M}_B(\eta) \begin{pmatrix} \tilde{\imath}_1(\eta,m\delta) \\ \tilde{\imath}_2(\eta,m\delta) \\ -\tilde{\imath}_1(\eta,(m+1)\delta) \\ -\tilde{\imath}_2(\eta,(m+1)\delta) \end{pmatrix} \tag{6.97}$$

where $m$ is an arbitrary natural number, specifying the $m-$th modulus.

**Notation 279** *Let us introduce a compact notation to better handle the following calculations:*

$$\tilde{\boldsymbol{\varphi}}_m(\eta) := \begin{pmatrix} \tilde{\varphi}_1(\eta, m\delta) \\ \tilde{\varphi}_2(\eta, m\delta) \\ \tilde{\varphi}_1(\eta, (m+1)\delta) \\ \tilde{\varphi}_2(\eta, (m+1)\delta) \end{pmatrix} := \begin{pmatrix} \tilde{\boldsymbol{\varphi}}_m^-(\eta) \\ \tilde{\boldsymbol{\varphi}}_m^+(\eta) \end{pmatrix}$$
$$\tilde{\boldsymbol{\iota}}_m(\eta) := \begin{pmatrix} \tilde{\iota}_1(\eta, m\delta) \\ \tilde{\iota}_2(\eta, m\delta) \\ -\tilde{\iota}_1(\eta, (m+1)\delta) \\ -\tilde{\iota}_2(\eta, (m+1)\delta) \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{\iota}}_m^-(\eta) \\ \tilde{\boldsymbol{\iota}}_m^+(\eta) \end{pmatrix} \tag{6.98}$$

**Modification of the governing equation of the distributed circuit dictated by the introduction of distributed current sources in parallel with the distributed capacitors in the cascade load.**

Consider the finite dimensionless circuit simulating the $m-$th beam-element, and introduce two dimensionless current sources $\iota_{g_1}(\tau, m\delta)\,\delta$ and $\iota_{g_2}(\tau, m\delta)\,\delta^3$ in parallel connection with $c_1$ and $c_2$ respectively.

Let us assemble these current sources into a current source column vector:

$$\tilde{\boldsymbol{\iota}}_{g_m}(\eta) := \begin{pmatrix} \tilde{\iota}_{g_1}(\eta, m\delta)\,\delta \\ \tilde{\iota}_{g_2}(\eta, m\delta)\,\delta^3 \end{pmatrix} \tag{6.99}$$

By the superposition principle the voltages at the ports of the network will be modified as follows:

$$\tilde{\boldsymbol{\varphi}}_m(\eta) = \mathbf{M}_B \tilde{\boldsymbol{\iota}}_m(\eta) + \tilde{\boldsymbol{\varphi}}_{g_m}(\eta) \tag{6.100}$$

where $\tilde{\boldsymbol{\varphi}}_{g_m}$ is temporarily an unknown.

Supposing that all the ports of the network are open circuited, i.e. $\tilde{\boldsymbol{\iota}}(\eta) = \mathbf{0}$, then:

$$\tilde{\boldsymbol{\varphi}}_m(\eta) = \tilde{\boldsymbol{\varphi}}_{g_m}(\eta) \tag{6.101}$$

If all the ports are open circuited, then the current through all the ports of the transformer is zero and the voltage at all the ports is determined only by the current sources by:

$$\tilde{\boldsymbol{\varphi}}_m(\eta) = \mathbf{Q}_0 \mathbf{z}_0 \tilde{\boldsymbol{\iota}}_{g_m} \Rightarrow \tilde{\boldsymbol{\varphi}}_{g_m}(\eta) = \mathbf{Q}_0 \mathbf{z}_0 \tilde{\boldsymbol{\iota}}_{g_m} = \begin{pmatrix} \mathbf{Q}_{0_1} \\ \mathbf{Q}_{0_2} \end{pmatrix} \mathbf{z}_0 \tilde{\boldsymbol{\iota}}_{g_m} = \begin{pmatrix} \mathbf{Q}_{0_1} \mathbf{z}_{0_{2,2}} \tilde{\boldsymbol{\iota}}_{g_m} \\ \mathbf{Q}_{0_2} \mathbf{z}_{0_{2,2}} \tilde{\boldsymbol{\iota}}_{g_m} \end{pmatrix} \tag{6.102}$$

with:

$$\mathbf{Q}_0 = \hat{\mathbf{T}}_0^T = \kappa_0 \frac{\sqrt{2}}{4} \begin{pmatrix} 2 & -\delta \\ 0 & 2 \\ 2 & \delta \\ 0 & 2 \end{pmatrix} \tag{6.103}$$

Now let us turn our attention back to the non-homogeneous set of differential equations defining the circuit with the distributed sources:

$$\begin{pmatrix} \tilde{\varphi}_1(\eta,\varepsilon) \\ \tilde{\varphi}_2(\eta,\varepsilon) \\ \tilde{\imath}_1(\eta,\varepsilon) \\ \tilde{\imath}_2(\eta,\varepsilon) \end{pmatrix}' = \mathbf{B} \begin{pmatrix} \tilde{\varphi}_1(\eta,\varepsilon) \\ \tilde{\varphi}_2(\eta,\varepsilon) \\ \tilde{\imath}_1(\eta,\varepsilon) \\ \tilde{\imath}_2(\eta,\varepsilon) \end{pmatrix} + \mathbf{b}(\eta,\varepsilon) \tag{6.104}$$

where $\mathbf{b}(\eta,\varepsilon)$ is a temporarily unknown that has to be determined.

The solution of the non-homogeneous system is given by, see Pease (1965)[14]:

$$\begin{pmatrix} \tilde{\varphi}_1(\eta,\varepsilon) \\ \tilde{\varphi}_2(\eta,\varepsilon) \\ \tilde{\imath}_1(\eta,\varepsilon) \\ \tilde{\imath}_2(\eta,\varepsilon) \end{pmatrix} = e^{\mathbf{B}(\varepsilon-\varepsilon_0)} \begin{pmatrix} \tilde{\varphi}_1(\eta,\varepsilon_0) \\ \tilde{\varphi}_2(\eta,\varepsilon_0) \\ \tilde{\imath}_1(\eta,\varepsilon_0) \\ \tilde{\imath}_2(\eta,\varepsilon_0) \end{pmatrix} + e^{\mathbf{B}\varepsilon} \int_{\varepsilon_0}^{\varepsilon} e^{-\mathbf{B}\alpha} \mathbf{b}(\eta,\alpha)\, d\alpha \tag{6.105}$$

Then referring eq.(6.105) to the $m-$th beam-element, we can state that the governing equation for that element is:

$$\begin{pmatrix} \tilde{\boldsymbol{\varphi}}_m^+(\eta) \\ \tilde{\boldsymbol{\iota}}_m^+(\eta) \end{pmatrix} = e^{\mathbf{B}\delta} \begin{pmatrix} \tilde{\boldsymbol{\varphi}}_m^-(\eta) \\ -\tilde{\boldsymbol{\iota}}_m^-(\eta) \end{pmatrix} + e^{\mathbf{B}(m+1)\delta} \int_{m\delta}^{(m+1)\delta} e^{-\mathbf{B}\alpha} \mathbf{b}(\eta,\alpha)\, d\alpha \tag{6.106}$$

now setting:

$$\bar{\alpha} := \alpha - m\delta \tag{6.107}$$

we obtain:

$$e^{\mathbf{B}(m+1)\delta} \int_{m\delta}^{(m+1)\delta} e^{-\mathbf{B}\alpha} \mathbf{b}(\eta,\alpha)\, d\alpha = e^{\mathbf{B}(m+1)\delta} \int_0^{\delta} e^{-\mathbf{B}(\bar{\alpha}+m\delta)} \mathbf{b}(\eta,\bar{\alpha})\, d\bar{\alpha} = e^{\mathbf{B}\delta} \int_0^{\delta} e^{-\mathbf{B}\bar{\alpha}} \mathbf{b}(\eta,\bar{\alpha})\, d\bar{\alpha} \tag{6.108}$$

Substituting the previous equation into eq.(6.106) we obtain:

$$\int_0^\delta e^{-\mathbf{B}\bar{\alpha}}\mathbf{b}\left(\eta,\bar{\alpha}\right)d\bar{\alpha} = -\begin{pmatrix} \tilde{\boldsymbol{\varphi}}_m^-\left(\eta\right) \\ -\tilde{\boldsymbol{\iota}}_m^-\left(\eta\right) \end{pmatrix} + e^{-\mathbf{B}\delta}\begin{pmatrix} \tilde{\boldsymbol{\varphi}}_m^+\left(\eta\right) \\ \tilde{\boldsymbol{\iota}}_m^+\left(\eta\right) \end{pmatrix} \tag{6.109}$$

If $\delta$ is small enough then:

$$\begin{aligned} &\textstyle\int_0^\delta e^{-\mathbf{B}\bar{\alpha}}\mathbf{b}\left(\eta,\bar{\alpha}\right)d\bar{\alpha} \simeq \delta\ \mathbf{b}\left(\eta,\bar{\alpha}\right)\big|_{\bar{\alpha}=0} = \delta\ \mathbf{b}\left(\eta,\alpha\right)\big|_{\alpha=m\delta} = \delta\mathbf{b}\left(\eta,m\delta\right) \\ &e^{-\mathbf{B}\delta} \simeq \left(\mathbf{1}-\mathbf{B}\delta\right) \end{aligned} \tag{6.110}$$

and eq.(6.109) yields:

$$\delta\mathbf{b}\left(\eta,m\delta\right) = -\begin{pmatrix} \tilde{\boldsymbol{\varphi}}_m^-\left(\eta\right) \\ -\tilde{\boldsymbol{\iota}}_m^-\left(\eta\right) \end{pmatrix} + \left(\mathbf{1}-\delta\mathbf{B}\right)\begin{pmatrix} \tilde{\boldsymbol{\varphi}}_m^+\left(\eta\right) \\ \tilde{\boldsymbol{\iota}}_m^+\left(\eta\right) \end{pmatrix} \tag{6.111}$$

Assuming that all the ports are open circuited we obtain:

$$\delta\mathbf{b}\left(\eta,m\delta\right) = -\begin{pmatrix} \tilde{\boldsymbol{\varphi}}_m^-\left(\eta\right) \\ \mathbf{0}_{2\times1} \end{pmatrix} + \left(\mathbf{1}-\delta\mathbf{B}\right)\begin{pmatrix} \tilde{\boldsymbol{\varphi}}_m^+\left(\eta\right) \\ \mathbf{0}_{2\times1} \end{pmatrix} \tag{6.112}$$

thus from eq.(6.102):

$$\delta\mathbf{b}\left(\eta,m\delta\right) = -\begin{pmatrix} \mathbf{Q}_{0_1}\mathbf{z}_0\tilde{\boldsymbol{\iota}}_{g_m} \\ \mathbf{0}_{2\times1} \end{pmatrix} + \left(\mathbf{1}-\delta\mathbf{B}\right)\begin{pmatrix} \mathbf{Q}_{0_2}\mathbf{z}_0\tilde{\boldsymbol{\iota}}_{g_m} \\ \mathbf{0}_{2\times1} \end{pmatrix} \tag{6.113}$$

manipulating the previous equation we get:

$$\delta\mathbf{b}\left(\eta,m\delta\right) = \begin{pmatrix} \left(\mathbf{Q}_{0_2}-\mathbf{Q}_{0_1}\right)\mathbf{z}_0\tilde{\boldsymbol{\iota}}_{g_m} \\ \mathbf{0}_{2\times1} \end{pmatrix} - \delta\mathbf{B}\begin{pmatrix} \mathbf{Q}_{0_2}\mathbf{z}_0\tilde{\boldsymbol{\iota}}_{g_m} \\ \mathbf{0}_{2\times1} \end{pmatrix} \tag{6.114}$$

Now let us focus our attention on all the terms in eq.(6.114)

$$\mathbf{Q}_{0_2} = \kappa_0 \tfrac{\sqrt{2}}{4} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \delta \kappa_0 \tfrac{\sqrt{2}}{4} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} := \mathbf{Q}^0_{0_2} + \delta \mathbf{Q}^1_{0_2}$$

$$\mathbf{Q}_{0_1} = \kappa_0 \tfrac{\sqrt{2}}{4} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \delta \kappa_0 \tfrac{\sqrt{2}}{4} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} := \mathbf{Q}^0_{0_1} + \delta \mathbf{Q}^1_{0_1}$$

$$\mathbf{z}_0 \tilde{\boldsymbol{\iota}}_{g_m} = \tfrac{1}{\eta} \begin{pmatrix} \frac{2}{\kappa_0^2 \beta^2 \delta} & 0 \\ 0 & \frac{24}{\kappa_0^2 \beta^2 \delta^3} \end{pmatrix} \begin{pmatrix} \tilde{\iota}_{g_1}(\eta, m\delta)\,\delta \\ \tilde{\iota}_{g_2}(\eta, m\delta)\,\delta^3 \end{pmatrix} = \frac{1}{\eta} \begin{pmatrix} \dfrac{2}{\kappa_0^2 \beta^2} \tilde{\iota}_{g_1}(\eta, m\delta) \\ \dfrac{24}{\kappa_0^2 \beta^2} \tilde{\iota}_{g_2}(\eta, m\delta) \end{pmatrix} \tag{6.115}$$

from the above set of expressions it is easy to see that:

$$\mathbf{Q}^0_{0_2} = \mathbf{Q}^0_{0_1} = \kappa_0 \frac{\sqrt{2}}{2} \mathbf{1}_{2\times 2}, \qquad \mathbf{Q}^1_{0_2} = -\mathbf{Q}^1_{0_1} \tag{6.116}$$

Hence the first term in eq.(6.114) becomes:

$$\begin{pmatrix} (\mathbf{Q}_{0_2} - \mathbf{Q}_{0_1})\, \mathbf{z}_0 \tilde{\boldsymbol{\iota}}_{g_m} \\ \mathbf{0}_{2\times 1} \end{pmatrix} = \begin{pmatrix} \delta \left( 2\mathbf{Q}^1_{0_2} \right) \mathbf{z}_0 \tilde{\boldsymbol{\iota}}_{g_m} \\ \mathbf{0}_{2\times 1} \end{pmatrix} = \delta \begin{pmatrix} \dfrac{1}{\eta} \dfrac{12\sqrt{2}}{\kappa_0 \beta^2} \tilde{\iota}_{g_2}(\eta, m\delta) \\ \mathbf{0}_{3\times 1} \end{pmatrix} \tag{6.117}$$

and the second becomes:

$$\delta \mathbf{B} \begin{pmatrix} \mathbf{Q}_{0_2} \mathbf{z}_0 \tilde{\boldsymbol{\iota}}_{g_m} \\ \mathbf{0}_{2\times 1} \end{pmatrix} = \delta \mathbf{B} \begin{pmatrix} \mathbf{Q}^0_{0_2} \mathbf{z}_0 \tilde{\boldsymbol{\iota}}_{g_m} \\ \mathbf{0}_{2\times 1} \end{pmatrix} + O(\delta^2) \simeq \delta \mathbf{B} \begin{pmatrix} \mathbf{Q}_{0_2} \mathbf{z}_{0_2} \tilde{\boldsymbol{\iota}}_{g_m} \\ \mathbf{0}_{2\times 1} \end{pmatrix} \tag{6.118}$$

with

$$\mathbf{B} \begin{pmatrix} \mathbf{Q}^0_{0_2} \mathbf{z}_0 \tilde{\boldsymbol{\iota}}_{g_m} \\ \mathbf{0}_{2\times 1} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\eta} \dfrac{12\sqrt{2}}{\kappa_0 \beta^2} \tilde{\iota}_{g_2}(\eta, m\delta) \\ 0 \\ -\dfrac{\sqrt{2}}{\kappa_0} \tilde{\iota}_{g_1}(\eta, m\delta) \\ 0 \end{pmatrix} \tag{6.119}$$

Finally, substituting eq.(6.117) and eq.(6.119) into the expression for $\mathbf{b}(\eta, m\delta)$ (6.114), we obtain:

$$\mathbf{b}(\eta, m\delta) = \begin{pmatrix} \frac{1}{\eta}\frac{12\sqrt{2}}{\kappa_0\beta^2}\tilde{\imath}_{g_2}(\eta, m\delta) \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{\eta}\frac{12\sqrt{2}}{\kappa_0\beta^2}\tilde{\imath}_{g_2}(\eta, m\delta) \\ 0 \\ -\frac{\sqrt{2}}{\kappa_0}\tilde{\imath}_{g_1}(\eta, m\delta) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ +\frac{\sqrt{2}}{\kappa_0}\tilde{\imath}_{g_1}(\eta, m\delta) \\ 0 \end{pmatrix} \tag{6.120}$$

If the mesh is refined enough, then the previous equation yields:

$$\mathbf{b}(\eta, \varepsilon) = \begin{pmatrix} 0 \\ 0 \\ +\frac{\sqrt{2}}{\kappa_0}\tilde{\imath}_{g_1}(\eta, \varepsilon) \\ 0 \end{pmatrix} \tag{6.121}$$

where $\tilde{\imath}_{g_1}(\eta, \varepsilon)$ is a current source acting on the unit of length, while $\tilde{\imath}_{g_2}(\eta, \varepsilon)$ is completely useless since it does not influence the non homogeneous term of the governing equation for the circuit. That is for every element of the circuit of dimensionless length $\delta$ a current source of total value $\delta\tilde{\imath}_{g_1}(\eta, \varepsilon)$ is applied.

Substituting eq.(6.28) and eq.(6.121) into eq.(6.105), we obtain

$$\begin{pmatrix} \tilde{\varphi}_1(\eta, \varepsilon) \\ \tilde{\varphi}_2(\eta, \varepsilon) \\ \tilde{\imath}_1(\eta, \varepsilon) \\ \tilde{\imath}_2(\eta, \varepsilon) \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\eta \\ -\beta^2\eta & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_1(\eta, \varepsilon) \\ \tilde{\varphi}_2(\eta, \varepsilon) \\ \tilde{\imath}_1(\eta, \varepsilon) \\ \tilde{\imath}_2(\eta, \varepsilon) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{2}}{\kappa_0}\tilde{\imath}_{g_1}(\eta, \varepsilon) \\ 0 \end{pmatrix} \tag{6.122}$$

Thus:

$$\begin{pmatrix} \tilde{\varphi}_1(\eta,\varepsilon) \\ \tilde{\varphi}_2(\eta,\varepsilon) \\ \tilde{\imath}_1(\eta,\varepsilon) \\ \tilde{\imath}_2(\eta,\varepsilon) \end{pmatrix}' = \begin{pmatrix} \tilde{\varphi}_2(\eta,\varepsilon) \\ -\eta\tilde{\imath}_2(\eta,\varepsilon) \\ -\beta^2\eta\tilde{\varphi}_1(\eta,\varepsilon) \\ -\tilde{\imath}_1(\eta,\varepsilon) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{2}}{\kappa_0}\tilde{\imath}_{g_1}(\eta,\varepsilon) \\ 0 \end{pmatrix} \tag{6.123}$$

Assembling the equations in a system, we get:

$$\begin{cases} \tilde{\varphi}_1'(\eta,\varepsilon) = \tilde{\varphi}_2(\eta,\varepsilon) \\ \tilde{\varphi}_2'(\eta,\varepsilon) = -\eta\tilde{\imath}_2(\eta,\varepsilon) \\ \tilde{\imath}_1'(\eta,\varepsilon) = -\beta^2\eta\tilde{\varphi}_1(\eta,\varepsilon) + \frac{\sqrt{2}}{\kappa_0}\tilde{\imath}_{g_1}(\eta,\varepsilon) \\ \tilde{\imath}_2'(\eta,\varepsilon) = -\tilde{\imath}_1(\eta,\varepsilon) \end{cases} \tag{6.124}$$

Differentiating the first equation of.(6.124) and substituting the second we get:

$$\tilde{\varphi}_1''(\eta,\varepsilon) = -\eta\tilde{\imath}_2(\eta,\varepsilon) \tag{6.125}$$

Differentiating eq.(6.125) and substituting the fourth equation of (6.124):

$$\tilde{\varphi}_1'''(\eta,\varepsilon) = \eta\tilde{\imath}_1(\eta,\varepsilon) \tag{6.126}$$

Differentiating eq.(6.126) and substituting the third equation of (6.124), we finally obtain:

$$\tilde{\varphi}_1''''(\eta,\varepsilon) = -\beta^2\eta^2\tilde{\varphi}_1(\eta,\varepsilon) + \eta\frac{\sqrt{2}}{\kappa_0}\tilde{\imath}_{g_1}(\eta,\varepsilon) \tag{6.127}$$

In the time domain this becomes

$$\varphi_1''''(\tau,\varepsilon) = -\beta^2\ddot{\varphi}_1(\tau,\varepsilon) + \frac{\sqrt{2}}{\kappa_0}i_{g_1}(\tau,\varepsilon) \tag{6.128}$$

and the dimensional equation becomes:

$$\frac{l^4}{V_0} V_1'''' (t, x) = -\frac{\lambda l^4}{t_0^2 k_M} \frac{t_0^2}{V_0} \ddot{V}_1 (t, x) + \frac{\sqrt{2} t_0}{\kappa_0 I_0} \dot{I}_{g_1} (t, x) \tag{6.129}$$

$$V_1'''' (t, x) = -\frac{\lambda}{k_M} \ddot{V}_1 (t, x) + \frac{R_0 t_0}{l^4} \frac{\sqrt{2}}{\kappa_0} \dot{I}_{g_1} (t, x) \tag{6.130}$$

Hence, finally:

$$V_1'''' (t, x) = -\frac{\lambda}{k_M} \ddot{V}_1 (t, x) + \frac{\kappa_0 \sqrt{2}}{2} \frac{\lambda}{k_M} \frac{x}{C_1 l} \dot{I}_{g_1} (t, x) \tag{6.131}$$

**Expression of the voltage across the first capacitor as a function of the voltage $V_1$**

Consider the m-th finite dimensionless circuit simulating the $m-$th beam element, and let us call $\tilde{\varphi}_P (\eta, m\delta)$ the voltage across the first capacitor $c_1$.

The voltage $\tilde{\varphi}_P (\eta, m\delta)$ can be expressed as the product between the current through the capacitor and the impedance of the capacitor:

$$\tilde{\varphi}_P (\eta, m\delta) = \tilde{\imath}_P (\eta, m\delta) \frac{1}{\eta c_1} \tag{6.132}$$

The current $\tilde{\imath}_P (\eta, m\delta)$ can be expressed in terms of the load current $\tilde{\imath}_L (\eta, m\delta)$ and the current source $\delta \tilde{\imath}_{g_1} (\eta, m\delta)$ by:

$$\tilde{\imath}_P (\eta, m\delta) = -\delta \tilde{\imath}_{g_1} (\eta, m\delta) + \tilde{\imath}_L (\eta, m\delta) \tag{6.133}$$

furthermore the load current $\tilde{\imath}_L (\eta, m\delta)$ can be expressed in terms of the port-current vector $\tilde{\boldsymbol{\imath}}_m (\eta)$ using the relation that the ideal transformer establishes between the currents at its ports:

$$\tilde{\imath}_L(\eta, m\delta) = -\begin{pmatrix} T_{0_{1,1}} \\ T_{0_{1,2}} \\ T_{0_{1,3}} \\ T_{0_{1,4}} \end{pmatrix}^T \boldsymbol{\tilde{\imath}}_m(\eta) \tag{6.134}$$

Substituting the expression(6.50) for the approximate turns-ratio matrix of the transformer into (6.134) we get:

$$\tilde{\imath}_L(\eta, m\delta) = -\kappa_0 \frac{\sqrt{2}}{4} \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}^T \begin{pmatrix} \tilde{\imath}_1(\eta, m\delta) \\ \tilde{\imath}_2(\eta, m\delta) \\ -\tilde{\imath}_1(\eta, (m+1)\delta) \\ -\tilde{\imath}_2(\eta, (m+1)\delta) \end{pmatrix} = -\kappa_0 \frac{\sqrt{2}}{2} \left(\tilde{\imath}_1(\eta, m\delta) - \tilde{\imath}_1(\eta, (m+1)\delta)\right) \tag{6.135}$$

Thus the current through the capacitor $c_1$ can be expressed as:

$$\tilde{\imath}_P(\eta, m\delta) = -\delta\tilde{\imath}_{g_1}(\eta, m\delta) - \kappa_0 \frac{\sqrt{2}}{2} \left(\tilde{\imath}_1(\eta, m\delta) - \tilde{\imath}_1(\eta, (m+1)\delta)\right) \tag{6.136}$$

and the voltage becomes:

$$\tilde{\varphi}_P(\eta, m\delta) = -\left(\delta\tilde{\imath}_{g_1}(\eta, m\delta) + \kappa_0 \frac{\sqrt{2}}{2} \left(\tilde{\imath}_1(\eta, m\delta) - \tilde{\imath}_1(\eta, (m+1)\delta)\right)\right) \frac{1}{\eta c_1} \tag{6.137}$$

Now we can substitute in the previous equation the expression of the capacitance $c_1 = \kappa_0^2 \frac{\beta^2\delta}{2}$, to obtain:

$$\tilde{\varphi}_P(\eta, m\delta) = -2\frac{\tilde{\imath}_{g_1}(\eta, m\delta)}{\eta\beta^2\kappa_0^2} - \left(\tilde{\imath}_1(\eta, m\delta) - \tilde{\imath}_1(\eta, (m+1)\delta)\right) \frac{\sqrt{2}}{\eta\kappa_0\beta^2\delta} \tag{6.138}$$

If $\delta$ is small enough then:

$$\tilde{\varphi}_P(\eta, \varepsilon) = -2\frac{\tilde{\imath}_{g_1}(\eta, \varepsilon)}{\eta\beta^2\kappa_0^2} + \frac{\sqrt{2}}{\eta\kappa_0\beta^2} \tilde{\imath}_1'(\eta, \varepsilon) \tag{6.139}$$

but from the system(6.124):

$$\tilde{\imath}_1' (\eta, \varepsilon) = -\beta^2 \eta \tilde{\varphi}_1 (\eta, \varepsilon) + \frac{\sqrt{2}}{\kappa_0} \tilde{\imath}_{g_1} (\eta, \varepsilon) \tag{6.140}$$

then:

$$\tilde{\varphi}_P (\eta, \varepsilon) = -2 \frac{\tilde{\imath}_{g_1} (\eta, \varepsilon)}{\eta \beta^2 \kappa_0^2} + \frac{\sqrt{2}}{\eta \kappa_0 \beta^2} \left( -\beta^2 \eta \tilde{\varphi}_1 (\eta, \varepsilon) + \frac{\sqrt{2}}{\kappa_0} \tilde{\imath}_{g_1} (\eta, \varepsilon) \right) = -\frac{\sqrt{2}}{\kappa_0} \tilde{\varphi}_1 (\eta, \varepsilon) \tag{6.141}$$

In the time domain the voltage $\varphi_P (\tau, \varepsilon)$ can be easily seen to be:

$$\varphi_P (\tau, \varepsilon) = -\frac{\sqrt{2}}{\kappa_0} \varphi_1 (\tau, \varepsilon) \tag{6.142}$$

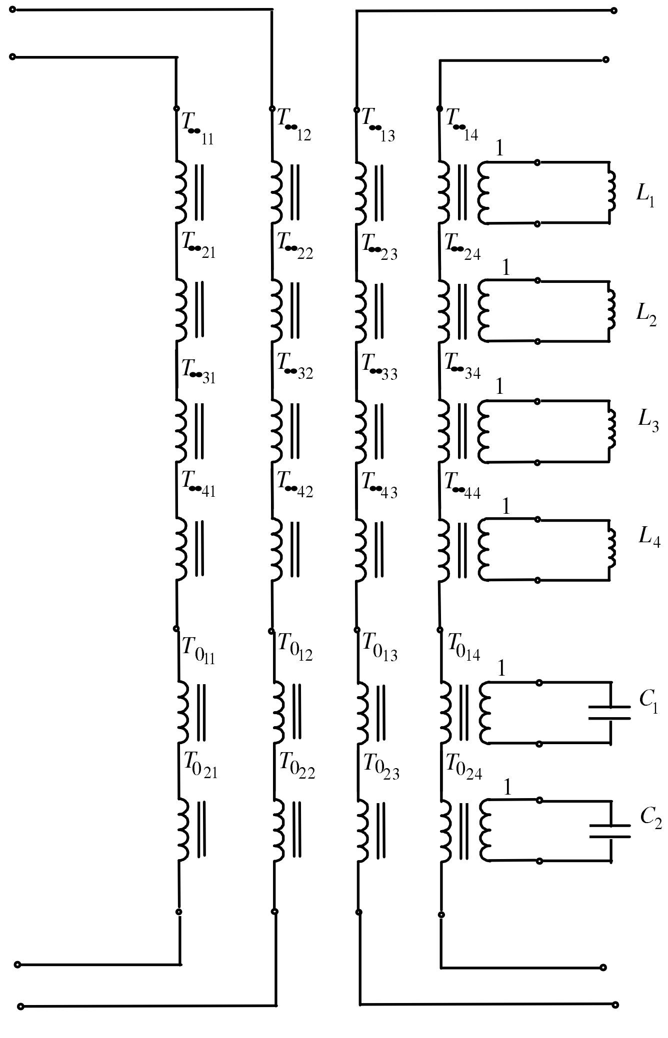


Figure 6.11: Circuit's modulus

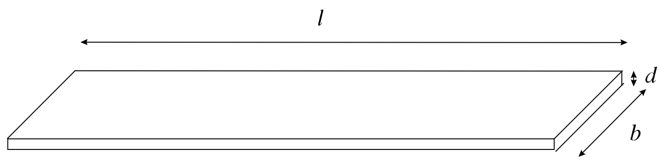


Figure 6.12: Three dimensional beam

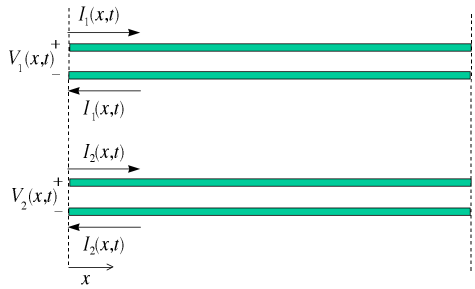


Figure 6.13: Imaginary cables, representing the distribuited circuit

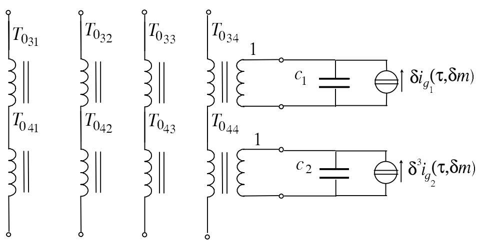


Figure 6.14: Sub-network of the $m-$th circuit' modulus

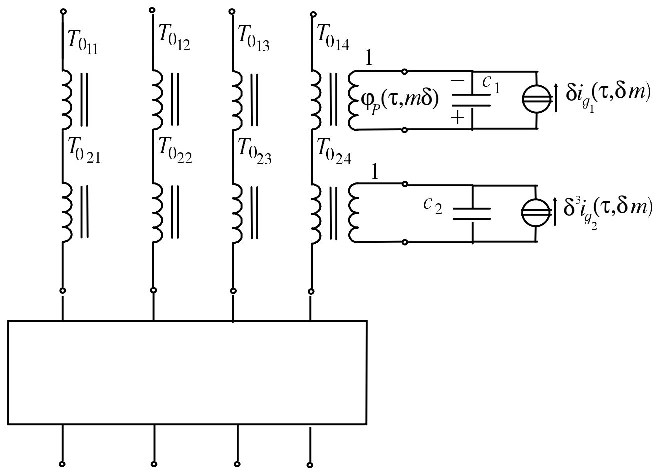


Figure 6.15: Circuit's modulus with applied current sources

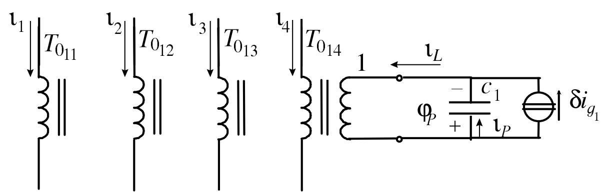


Figure 6.16: Subnetwork of the circuit modulus, with applied current sources

# Chapter 7

# Piezoelectromechanical beam

### Electro-mechanical structures

Once we have introduced the main concepts of the theory of networks and of the structural analysis it is time to merge this knowledge in the study of the piezoelectric actuators, which belong to a wider class of devices that we will call *electro-mechanical* devices.

As we have done for electric networks and structures, we will use again a black box approach in the description of the electro-mechanical devices we are interested in.

We will limit our considerations to the electro-mechanical devices which can be modelled as black boxes, communicating with the outer world by a finite number of terminals $\mathcal{T}_i$.

The state of each terminal $\mathcal{T}_i$ is completely characterized by a pair of 4−tuples $(\boldsymbol{\alpha}_i, \boldsymbol{\tau}_i) = \left(\left(v_1^i, v_2^i, \omega^i, V_i\right), \left(t_1^i, t_2^i, M^i, I_i\right)\right)$, where the pair $\left(v_1^i, v_2^i\right)$ represents the velocities and $\omega^i$ the angular velocity, at the terminal $\mathcal{T}_i$, with respect to a given observer $\mathbf{O}$ characterized by an origin $\mathfrak{o}$ and a basis $(\mathbf{e}_1, \mathbf{e}_2)$ of the space of translations, while $V_i$ represents the voltage at the terminal $\mathcal{T}_i$ with respect to a given potential reference.

The 3−tuple $\left(t_1^i, t_2^i, M^i\right)$ represents the contact actions, force and bending moment, applied at a Cauchy cut at the terminal $\mathcal{T}_i$, with respect to $\mathbf{O}$, while $I_i$ represents the current entering the terminal $\mathcal{T}_i$.

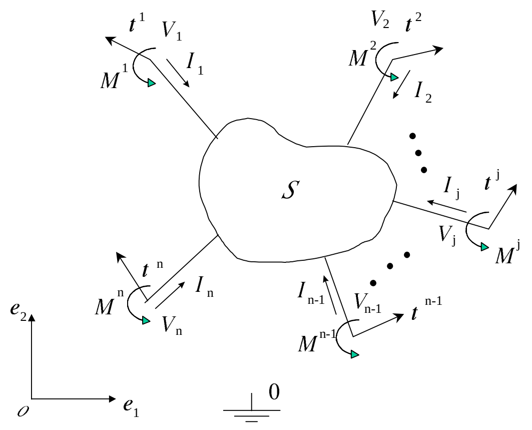


Figure 7.1: Representation of an electromechanical structure

**Notation 280** *For a $n-$terminal electro-mechanical device, we will group the state variables with the following convention:*

$$\boldsymbol{\alpha} \quad : \quad = (\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}, \mathbf{V}) := \begin{pmatrix} v_1^1 & v_2^1 & \omega^1 & V_1 \\ ... & ... & ... & ... \\ v_1^n & v_2^n & \omega^n & V_n \end{pmatrix} \tag{7.1}$$

$$\boldsymbol{\tau} \quad : \quad = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{M}, \mathbf{I}) := \begin{pmatrix} t_1^1 & t_2^1 & M^1 & I_1 \\ ... & ... & ... & ... \\ t_1^n & t_2^n & M^n & I_n \end{pmatrix}$$

As we have so far in done dealing with networks and structures, we will suppose that the signal space is still $\mathcal{D}_+$.

**Definition 281** *Given a binary relation $\mathcal{C}_{EM}$ on $\mathcal{D}_+^{n\times 4} \times \mathcal{D}_+^{n\times 4}$, an electro-mechanical* *structure $\mathcal{EM}$ is:*

$$\mathcal{EM} = \left\{ \left( (\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}, \mathbf{V}), (\mathbf{t}_1, \mathbf{t}_2, \mathbf{M}, \mathbf{I}) \right) \in \mathcal{D}_+^{n\times 4} \times \mathcal{D}_+^{n\times 4}, (\mathbf{v}_1, \mathbf{v}_2, \boldsymbol{\omega}, \mathbf{V})\, \mathcal{C}_{EM} \,(\mathbf{t}_1, \mathbf{t}_2, \mathbf{M}, \mathbf{I}) \right\} \tag{7.2}$$

**Remark 282** *All the beam-structures and the networks are electromechanical structures, since both $\mathcal{C}_S$ and $\mathcal{C}_N$ can be seen as a restriction of $\mathcal{C}_{EM}$ on a particular subspace of $\mathcal{D}_+^{n\times 4} \times \mathcal{D}_+^{n\times 4}$. In particular $\mathcal{C}_S$ can be the restriction of $\mathcal{C}_{EM}$ to $\mathcal{D}_+^{n} \times \mathcal{D}_+^{n}$ while $\mathcal{C}_N$ can be imagined as the restriction of $\mathcal{C}_{EM}$ to $\mathcal{D}_+^{n\times 3} \times \mathcal{D}_+^{n\times 3}$.*

## Piezoelectric actuators

In what follows we will not develop a general theory for the electro-mechanical structures, but will limit our observation to a particular class: the piezoelectric actuators.

**Definition 283** *A bending piezoelectric actuator, BPA for brevity, is an electro-mechanical structure defined by:*

$$\mathcal{C}_{BPA} = \left\{ \begin{array}{c} (\boldsymbol{\alpha}, \boldsymbol{\tau}) \in \mathcal{D}_+^{2\times 4} \times \mathcal{D}_+^{2\times 4} : (\mathbf{t}_1, \mathbf{t}_2) = (\mathbf{0}, \mathbf{0}),\ M^1 = -M^2,\ I_1 = -I_2 \\ \begin{pmatrix} \dot{M}^2 \\ I_2 \end{pmatrix} = \begin{pmatrix} K_{mm} & K_{me} \\ K_{em} & K_{ee} \end{pmatrix} \begin{pmatrix} \left(\omega^2 - \omega^1\right) \\ \dot{V}_2 - \dot{V}_1 \end{pmatrix} \end{array} \right\} \tag{7.3}$$

*where all the entries of the matrix are real numbers, and $l_P$ is the length of the actuator.*

**Remark 284** *The previous relations establish the constitutive relations of the actuators. Furthermore for the fourth matrix equation integrating both terms of the expression leads* *to:*

$$\begin{pmatrix} M \\ Q_P \end{pmatrix} = \begin{pmatrix} K_{mm} & K_{me} \\ K_{em} & K_{ee} \end{pmatrix} \begin{pmatrix} \chi \\ V_P \end{pmatrix} \tag{7.4}$$

*where*

$$\begin{aligned} M &:= \int_0^t \dot{M}^2 dt = M^2 \\ Q_P &:= \int_0^t I_2 dt \\ \chi &:= \int_0^t \left(\omega^2 - \omega^1\right) dt \\ V_P &: \int_0^t \left(V_2 - V_1\right) dt \end{aligned} \tag{7.5}$$

*in particular observe that* $Q_P$ *is the charge entrapped in the actuator and* $\chi$ *is the difference of attitude at the two terminals divided by the length of the actuator.*

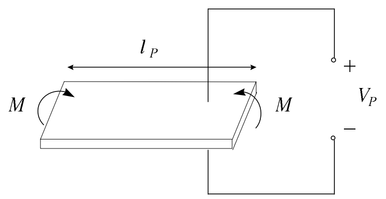


Figure 7.2: Schematic plot of a bending piezoelectric actuator

**Remark 285** *All the elements of the previous matrix have a precise physical meaning:*

- $K_{mm}$ *is the stiffness rigidity of the actuator if the terminals are short-circuited.*
- $K_{ee}$ *is the capacitance of the actuator if* $\omega^2 = \omega^1$
- $K_{me}$ *is the bending moment corresponding to a unit voltage applied if* $\omega^2 = \omega^1$.

- $K_{em}$ *is the current corresponding to* $\omega^2 - \omega^1 = 1$ *if the terminals are short-circuited.*

**Axiom 286** *The bending piezoelectric actuator is conservative*

**Proposition 287** *Requiring the actuator to be conservative leads to the following results:*

- $K_{mm} \in \mathbb{R}^+$
- $K_{ee} \in \mathbb{R}^+$
- $K_{em} = -K_{me}$

***Proof.*** *The work done on a virtual "displacement"* $(\delta\chi, \delta Q_P)$ *is assumed to be:*

$$\delta W = M\delta\chi + V_P \delta Q_P \tag{7.6}$$

*We require the existence of an energy* $\mathcal{E}(\chi, Q_P)$ *such that*

$$d\mathcal{E} = \delta W \tag{7.7}$$

*thus:*

$$\frac{\partial \mathcal{E}}{\partial \chi} = M \qquad \frac{\partial \mathcal{E}}{\partial Q_P} = V_P \tag{7.8}$$

*From (7.4) we get:*

$$\begin{cases} M = \left(K_{mm} - \dfrac{K_{em}K_{me}}{k_{ee}}\right)\chi + \dfrac{K_{me}}{K_{ee}} Q_P \\ V_P = \dfrac{1}{K_{ee}} Q_P - \dfrac{K_{em}}{K_{ee}}\chi \end{cases} \tag{7.9}$$

*Thus:*

$$\begin{cases} \dfrac{\partial \mathcal{E}}{\partial \chi} = \left(K_{mm} - \dfrac{K_{em}K_{me}}{K_{ee}}\right)\chi + \dfrac{K_{me}}{K_{ee}} Q_P \\ \dfrac{\partial \mathcal{E}}{\partial Q_P} = \dfrac{1}{K_{ee}} Q_P - \dfrac{K_{em}}{K_{ee}}\chi \end{cases} \tag{7.10}$$

*And*

$$
\begin{aligned}
\mathcal{E} =& \tfrac{1}{2}\left(K_{mm} - \frac{K_{em}K_{me}}{K_{ee}}\right)\chi^2 + \tfrac{1}{2}\frac{1}{K_{ee}}Q_P^2 + \frac{K_{me}}{K_{ee}}\chi Q_P \\
K_{me} =& -K_{em}
\end{aligned}
\tag{7.11}
$$

*Imposing $\mathcal{E}$ to be positive semidefinite we get the other two results.* ■

**Remark 288** *The coefficient $K_{mm}^{OC} := \left(K_{mm} - \frac{K_{em}K_{me}}{K_{ee}}\right)$ in the constitutive relation(7.9) expresses the stiffness rigidity of the actuator when the terminals are open-circuited.*

From an electrical point of view the actuator, by means of (7.4) can be considered as the parallel connection of a capacitor, the capacitance of which is equal to $K_{ee}$, and a current source equal to $K_{em}\left(\omega^2 - \omega^1\right) = K_{em}\dot{\chi}$.

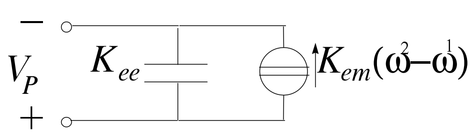


Figure 7.3: Electrical representation of a bending piezoelectric actuator

## Model of the piezoelectromechanic beam

Consider now an Euler beam, of length $l$, and suppose $n$ equally spaced bending actuators are glued on it; if $n$ is sufficiently big, we can imagine these actuators to form a thin piezoelectric layer on the beam. Furthermore consider the electric circuit analog to the transversally vibrating Euler beam that we have studied in chapter(6), and suppose the $m-$th actuator to be connected as the capacitor $C_1$ of the $m-$th modulus of the circuit.

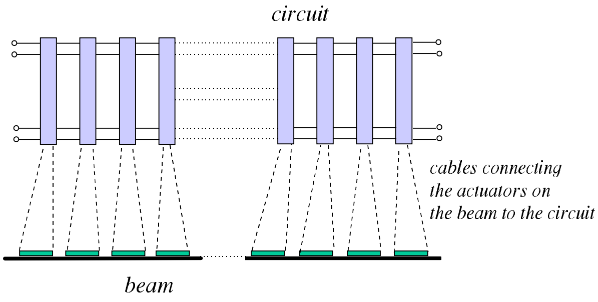


Figure 7.4: Piezoelectromechanical beam

The constitutive equations for the piezoelectromechanic beam, disregarding the extension, become:

$$\begin{aligned} &M(x,t) = (\gamma k_{mm} + k_M^m)\,\vartheta'(x,t) + \gamma K_{me} V_P(x,t) \\ &u'(x,t) = \vartheta(x,t) \end{aligned} \tag{7.12}$$

where $k_M^m$ is the rotational stiffness of the beam and $k_{mm}$ is the rotational stiffness per unit length of the actuators:

$$k_{mm} := K_{mm} l_P \tag{7.13}$$

and $\gamma$ is defined as the influence factor of each actuator, i.e. the ratio between the length of the actuator $l_P$ and the size of the modulus $x$, in symbols:

$$\gamma := \frac{l_P}{x} \tag{7.14}$$

Introducing a modified rotational stiffness

$$k_M := (\gamma k_{mm} + k_M^m) \tag{7.15}$$

representing the rotational stiffness of the piezoelectric beam when all the actuators are short-circuited, we can write the constitutive equation as:

$$\begin{aligned} &M(x,t) = k_M \vartheta'(x,t) + \gamma K_{me} V_P(x,t) \\ &u'(x,t) = \vartheta(x,t) \end{aligned} \tag{7.16}$$

While the balance equations are:

$$\begin{aligned} &M'(x,t) + T(x,t) = 0 \\ &T'(x,t) + \lambda\, \ddot{u}(x,t) = 0 \end{aligned} \tag{7.17}$$

After simple manipulations we get:

$$k_M u''''(x,t) + \gamma K_{me} V_P''(x,t) = -\lambda\, \ddot{u}(x,t) \tag{7.18}$$

Now non-dimensionalizing the previous equation we obtain:

$$\frac{r_0}{l^4} k_M \zeta''''(\varepsilon,\tau) + \gamma \frac{V_0}{l^2} K_{me} \varphi_P''(\varepsilon,\tau) = -\frac{r_0}{t_0^2} \lambda\, \ddot{\zeta}(\varepsilon,\tau) \tag{7.19}$$

Introducing again the constant $\beta^2 = \dfrac{\lambda l^4}{t_0^2 k_M}$ we get:

$$\zeta''''(\varepsilon,\tau) + \frac{\gamma V_0 l^4}{l^2 r_0 k_M} K_{me} \varphi_P''(\varepsilon,\tau) = -\beta^2 \ddot{\zeta}(\varepsilon,\tau) \tag{7.20}$$

Considering the velocity instead of the vertical displacement we get:

$$\nu_\zeta''''(\varepsilon,\tau) + \frac{\gamma V_0 l^2}{r_0 k_M} K_{me} \dot{\varphi}_P''(\varepsilon,\tau) = -\beta^2 \dot{\nu}_\zeta(\varepsilon,\tau) \tag{7.21}$$

Now let us turn our attention to the circuit analog of the Euler beam, where by means of the introduction of the actuators we have:

$$C_1 = K_{ee} \qquad \delta I_{g_1}(x,t) = k_{em} \dot{u}''(x,t) \tag{7.22}$$

where $k_{em}$ represents the electromechanical coupling effect per unit length:

$$k_{em} = K_{em} l_P \tag{7.23}$$

then:

$$K_{ee} = \kappa_0^2 \frac{\delta \beta^2}{2} \frac{t_0}{R_0} \qquad \iota_{g_1}(\varepsilon, \tau) = \frac{1}{I_0} I_{g_1}(x, t) = k_{em} \frac{1}{I_0} \frac{r_0}{t_0 l^2 \delta} \dot{\nu}_\zeta''(\varepsilon, \tau) \tag{7.24}$$

The dimensionless governing equation for the circuit becomes:

$$\varphi_1''''(\varepsilon, \tau) - k_{em} \frac{\sqrt{2}}{\kappa_0} \frac{1}{I_0} \frac{r_0}{t_0 l^2 \delta} \dot{\nu}_\zeta''(\varepsilon, \tau) = -\beta^2 \ddot{\varphi}_1(\varepsilon, \tau) \tag{7.25}$$

and the voltage across the actuator becomes:

$$\varphi_P(\varepsilon, \tau) = -\frac{\sqrt{2}}{\kappa_0} \varphi_1(\varepsilon, \tau) \tag{7.26}$$

Substituting the previous expression into eq.(7.21) we get:

$$\nu_\zeta''''(\varepsilon, \tau) + K_{me} \frac{\gamma V_0 l^2}{r_0 k_M} \left( -\frac{\sqrt{2}}{\kappa_0} \dot{\varphi}_1''(\varepsilon, \tau) \right) = -\beta^2 \ddot{\nu}_\zeta(\varepsilon, \tau) \tag{7.27}$$

remembering that $K_{me} = -K_{em}$ we obtain:

$$k_{em} = -K_{me} l_P \tag{7.28}$$

and:

$$\nu_\zeta''''(\varepsilon, \tau) + \frac{k_{em}}{l_P} \frac{\sqrt{2}}{\kappa_0} \frac{\gamma V_0 l^2}{r_0 k_M} \dot{\varphi}_1''(\varepsilon, \tau) = -\beta^2 \ddot{\nu}_\zeta(\varepsilon, \tau) \tag{7.29}$$

Grouping the two governing equations for the evolution of the piezoelectromechanical beam:

$$\begin{aligned} &\varphi_1''''(\varepsilon, \tau) - k_{em} \frac{\sqrt{2}}{\kappa_0} \frac{1}{I_0} \frac{r_0}{t_0 l^2 \delta} \dot{\nu}_\zeta''(\varepsilon, \tau) = -\beta^2 \ddot{\varphi}_1(\varepsilon, \tau) \\ &\nu_\zeta''''(\varepsilon, \tau) + \frac{k_{em}}{l_P} \frac{\sqrt{2}}{\kappa_0} \frac{\gamma V_0 l^2}{r_0 k_M} \dot{\varphi}_1''(\varepsilon, \tau) = -\beta^2 \ddot{\nu}_\zeta(\varepsilon, \tau) \end{aligned} \tag{7.30}$$

Now we want to choose $V_0$ such that the two coupling terms in the previous equations are equal:

$$k_{em}\frac{\sqrt{2}}{\kappa_0}\frac{1}{I_0}\frac{r_0}{t_0 l^2 \delta} = \frac{k_{em}}{l_P}\frac{\sqrt{2}}{\kappa_0}\frac{\gamma V_0 l^2}{r_0 k_M} \tag{7.31}$$

this yields:

$$V_0 = \frac{\kappa_0}{\sqrt{2}}\frac{r_0}{t_0}\sqrt{\frac{\lambda l_P}{\gamma K_{ee}}} \tag{7.32}$$

introducing a capacitance per unit length $k_{ee} := \dfrac{K_{ee}}{l_P}$ the previous equation yields:

$$V_0 = \frac{\kappa_0}{\sqrt{2}}\frac{r_0}{t_0}\sqrt{\frac{\lambda}{\gamma k_{ee}}} \tag{7.33}$$

Substituting this expression into the coupled equations we have:

$$\begin{aligned} &\varphi_1''''(\varepsilon,\tau) - \varrho\dot{\nu}_\zeta''(\varepsilon,\tau) = -\beta^2\ddot{\varphi}_1(\varepsilon,\tau) \\ &\nu_\zeta''''(\varepsilon,\tau) + \varrho\dot{\varphi}_1''(\varepsilon,\tau) = -\beta^2\ddot{\nu}_\zeta(\varepsilon,\tau) \end{aligned} \tag{7.34}$$

with

$$\begin{aligned} &\varrho = \frac{k_{em} l^2}{t_0 k_M l_P}\sqrt{\frac{\lambda}{k_{ee}}}\sqrt{\gamma} = \frac{K_{em} l^2}{t_0 k_M}\sqrt{\frac{\lambda}{k_{ee}}}\sqrt{\gamma} \\ &\beta^2 = \frac{\lambda l^4}{t_0^2 k_M} \end{aligned} \tag{7.35}$$

Consider the ratio of the two previously defined parameters as an index of the coupling effect:

$$\Gamma := \frac{\varrho}{\beta^2} = K_{em}\frac{t_0}{l^2}\sqrt{\frac{\gamma}{\lambda k_{ee}}} \tag{7.36}$$

which as we expected increases as the coefficient $K_{em}$ increases and as the density of the actuators over the beam represented by $\gamma$ increases.

Furthermore it increases as the capacitance per unit length $k_{ee}$ decreases, this effect can be understood thinking of the topology of the circuit we have synthesized. In fact as the capacitance decreases the impedance of the capacitor $C_1$ ,in the circuit's modulus, increases; hence the voltage drop across it increases since the current source is fixed, hence the voltage $\varphi_1$ increases.

## Energy considerations

### Energy of the dimensionless Euler beam

In section (3) we have proved that the energy of a free transversally vibrating beam is:

$$\mathcal{E}\left(u'', \dot{u}\right) = \int_0^l \frac{1}{2} k_M \left(u''\left(x,t\right)\right)^2 + \frac{1}{2}\lambda \dot{u}\left(x,t\right)^2 ds \tag{7.37}$$

Let us express this energy in terms of the dimensionless variables used all over this section:

$$\mathcal{E}\left(\zeta'', \dot{\zeta}\right) = \frac{k_M r_0^2}{l^3} \int_0^1 \frac{1}{2}\left(\zeta''\left(\varepsilon,\tau\right)\right)^2 d\varepsilon + \frac{\lambda r_0^2 l}{t_0^2} \int_0^1 \frac{1}{2}\dot{\zeta}\left(\varepsilon,\tau\right)^2 d\varepsilon \tag{7.38}$$

introducing a characteristic energy $\mathcal{E}_0 = \dfrac{r_0^2 k_M}{l^3}$ and a dimensionless energy $\epsilon_m\left(\zeta'', \dot{\zeta}\right) = \dfrac{\mathcal{E}}{\mathcal{E}_0}$ we get:

$$\epsilon_m\left(\zeta'', \dot{\zeta}\right) = \frac{1}{2}\int_0^1 \left(\zeta''\left(\varepsilon,\tau\right)\right)^2 d\varepsilon + \frac{1}{2}\beta^2 \int_0^1 \dot{\zeta}\left(\varepsilon,\tau\right)^2 d\varepsilon \tag{7.39}$$

or:

$$\epsilon_m\left(\zeta'', \dot{\zeta}\right) = \frac{1}{2}\int_0^1 F_M \zeta'' d\varepsilon + \frac{1}{2}\beta^2 \int_0^1 \dot{\zeta}^2 d\varepsilon \tag{7.40}$$

Furthermore let us introduce the dimensionless mechanical kinetic energy and potential mechanical energy:

$$\varkappa_m := \frac{1}{2}\beta^2 \int_0^1 \dot{\zeta}\left(\varepsilon,\tau\right)^2 d\varepsilon \qquad \upsilon_m := \frac{1}{2}\int_0^1 F_M\left(\zeta''\left(\varepsilon,\tau\right)\right) d\varepsilon \tag{7.41}$$

### Energy of the dimensionless circuit analog of the Euler beam

Since the circuit we have synthesized is completely analog to the Euler beam, the dimensionless energy of this circuit can be expressed by:

$$\epsilon_e\left(\psi'',\dot{\psi}\right)=\frac{1}{2}\int_0^1\left(\psi''\left(\varepsilon,\tau\right)\right)^2 d\varepsilon+\frac{1}{2}\beta^2\int_0^1\dot{\psi}\left(\varepsilon,\tau\right)^2 d\varepsilon \tag{7.42}$$

where $\psi$ is defined as the integral of the voltage $\varphi_1$:

$$\psi\left(\varepsilon,\tau\right)=\int_0^\tau\varphi_1\left(\varepsilon,\bar{\tau}\right)d\bar{\tau} \tag{7.43}$$

We need to introduce the integral of the voltage drop, because the voltage $\varphi_1$ is analogous to the dimensionless velocity $\nu_\zeta$ instead of being analogous to the dimensionless displacement $\zeta$.

Introducing the dimensionless current $\iota_2$ analogous to $F_M$ eq.(7.42) becomes:

$$\epsilon_e\left(\psi'',\dot{\psi}\right)=\frac{1}{2}\int_0^1\iota_2\psi'' d\varepsilon+\frac{1}{2}\beta^2\int_0^1\dot{\psi}^2 d\varepsilon \tag{7.44}$$

Furthermore let us introduce the dimensionless kinetic electrical energy and potential electrical energy:

$$\varkappa_e:=\frac{1}{2}\beta^2\int_0^1\dot{\psi}\left(\varepsilon,\tau\right)^2 d\varepsilon \qquad \upsilon_e:=\frac{1}{2}\int_0^1\iota_2\left(\psi''\left(\varepsilon,\tau\right)\right)d\varepsilon \tag{7.45}$$

### Energy of the piezoelectromechanical beam

In terms of the dimensionless displacement $\zeta$ and the dimensionless integral of the representative voltage $\psi$ eq.(7.34) becomes:

$$\begin{aligned}\psi''''\left(\varepsilon,\tau\right)-\varrho\dot{\zeta}''\left(\varepsilon,\tau\right)&=-\beta^2\ddot{\psi}\left(\varepsilon,\tau\right)\\ \zeta''''\left(\varepsilon,\tau\right)+\varrho\dot{\psi}''\left(\varepsilon,\tau\right)&=-\beta^2\ddot{\zeta}\left(\varepsilon,\tau\right)\end{aligned} \tag{7.46}$$

Now we want to find an expression for the energy of the piezoelectromechanical beam and show that the system is conservative.

The first step of our analysis is to heuristically seek for a Lagrangian of the system, which we will suppose to be the sum of the Lagrangian of the uncoupled system and a contribution representing the coupling:

$$\mathcal{L}\left(\dot{\psi}, \dot{\zeta}, \psi'', \zeta''\right) = \varkappa_e + \varkappa_m - \upsilon_m - \upsilon_e + \mathcal{L}_c \tag{7.47}$$

Furthermore we will assume the coupling Lagrangian to be constituted by two terms: $\mathcal{L}_{me}$ and $\mathcal{L}_{em}$ the first representing the *mechanical→electrical coupling*, and the latter the *electro→mechanical* coupling:

$$\mathcal{L}_c = \mathcal{L}_{me} + \mathcal{L}_{em} \tag{7.48}$$

Consider the constitutive equation for the bending moment of the piezoelectromecanical beam, neglecting the rigidity of the actuators with respect of the rigidity of the beam:

$$M(x,t) = k_M u''(x,t) + k_{me} V_P(x,t) \tag{7.49}$$

in dimensionless variables it becomes:

$$F_M(\varepsilon, \tau) = \zeta''(\varepsilon, \tau) + \frac{k_{me} V_0 l^2}{r_0 k_M} \varphi_P(\varepsilon, \tau) \tag{7.50}$$

substituting (7.26) we get:

$$F_M(\varepsilon, \tau) = \zeta''(\varepsilon, \tau) - \frac{\sqrt{2}}{\kappa_0} \frac{k_{me} V_0 l^2}{r_0 k_M} \varphi_1(\varepsilon, \tau) \tag{7.51}$$

considering the integral of the voltage and the definition of the constant $\varrho$ given in (7.35):

$$F_M(\varepsilon,\tau) = \zeta''(\varepsilon,\tau) + \varrho\,\dot{\psi}(\varepsilon,\tau) \tag{7.52}$$

Substituting (7.52) into (7.41) we get:

$$\frac{1}{2}\int_0^1 \left(\zeta''(\varepsilon,\tau) + \varrho\,\dot{\psi}(\varepsilon,\tau)\right)\zeta''(\varepsilon,\tau)\,d\varepsilon =: \upsilon_m + \mathcal{L}_{me} \tag{7.53}$$

then:

$$\mathcal{L}_{me}\left(\dot{\psi},\zeta''\right) := -\varrho\frac{1}{2}\int_0^1 \dot{\psi}(\varepsilon,\tau)\,\zeta''(\varepsilon,\tau)\,d\varepsilon \tag{7.54}$$

Now we will assume that

$$\mathcal{L}_{em}\left(\dot{\zeta},\psi''\right) := +\varrho\frac{1}{2}\int_0^1 \dot{\zeta}(\varepsilon,\tau)\,\psi''(\varepsilon,\tau)\,d\varepsilon \tag{7.55}$$

and we will show that the Lagrangian given in eq.(7.47) is such that the coupled set of equations (7.34) are the Euler-Lagrange equations for this Lagrangian.

Let us introduce the Action:

$$\mathcal{A}(\psi(\cdot),\zeta(\cdot)) = \int_{\tau_1}^{\tau_2} \left(\varkappa_e + \varkappa_m - \upsilon_e - \upsilon_m + \mathcal{L}_{me} + \mathcal{L}_{em}\right) d\tau \tag{7.56}$$

As we have done is chapter(3) consider the Action corresponding to the functions $\psi$ and $\zeta$ variated by the quantities $\delta\psi$ and $\delta\zeta$:

$$\mathcal{A}(\psi+\delta\psi,\zeta+\delta\zeta) = \int_{\tau_1}^{\tau_2}\int_0^1 \mathcal{J}\left(\frac{\partial}{\partial\tau}(\psi+\delta\psi),\frac{\partial}{\partial\tau}(\zeta+\delta\zeta),(\psi+\delta\psi)'',(\zeta+\delta\zeta)''\right) d\varepsilon d\tau \tag{7.57}$$

with $\mathcal{J}$ defined as a Lagrangian density per unit length:

$$\mathcal{J}\left(\dot{\psi},\dot{\zeta},\psi'',\zeta''\right) = \frac{1}{2}\beta^2\dot{\psi}^2 + \frac{1}{2}\beta^2\dot{\zeta}^2 - \frac{1}{2}(\psi'')^2 - \frac{1}{2}(\zeta'')^2 - \frac{1}{2}\varrho\dot{\psi}\zeta'' + \frac{1}{2}\varrho\dot{\zeta}\psi'' \tag{7.58}$$

since the differentiation with respect to the independent variables $\tau$ and $\varepsilon$ is commutative with the $\delta$ operator, from the previous equation we get:

$$\mathcal{A}(\psi+\delta\psi,\zeta+\delta\zeta)=\int_{\tau_1}^{\tau_2}\int_0^1 \mathcal{J}\left(\dot{\psi}+\delta\dot{\psi},\dot{\zeta}+\delta\dot{\zeta},\psi''+\delta\psi'',\zeta''+\delta\zeta''\right)d\varepsilon d\tau \tag{7.59}$$

but:

$$\mathcal{J}\left(\dot{\psi}+\delta\dot{\psi},\dot{\zeta}+\delta\dot{\zeta},\psi''+\delta\psi'',\zeta''+\delta\zeta''\right)=\mathcal{J}\left(\dot{\psi},\dot{\zeta},\psi'',\zeta''\right)+$$
$$\frac{\partial\mathcal{J}}{\partial\dot{\psi}}\delta\dot{\psi}+\frac{\partial\mathcal{J}}{\partial\dot{\zeta}}\delta\dot{\zeta}+\frac{\partial\mathcal{J}}{\partial\psi''}\delta\psi''+\frac{\partial\mathcal{J}}{\partial\zeta''}\delta\zeta''+O(\delta^2)$$

Then the first variation of the Action is:

$$\delta\mathcal{A}=\int_{\tau_1}^{\tau_2}\int_0^1 \frac{\partial\mathcal{J}}{\partial\dot{\psi}}\delta\dot{\psi}+\frac{\partial\mathcal{J}}{\partial\dot{\zeta}}\delta\dot{\zeta}+\frac{\partial\mathcal{J}}{\partial\psi''}\delta\psi''+\frac{\partial\mathcal{J}}{\partial\zeta''}\delta\zeta''\,d\varepsilon d\tau \tag{7.60}$$

Consider now the first term of the integral in (7.60) and integrate by parts with respect of the time variable:

$$\int_{\tau_1}^{\tau_2}\int_0^1 \frac{\partial\mathcal{J}}{\partial\dot{\psi}}\delta\dot{\psi}\,d\varepsilon d\tau=\int_0^1 \frac{\partial\mathcal{J}}{\partial\dot{\psi}}\delta\psi\bigg|_{\tau=\tau_1}^{\tau=\tau_2}d\varepsilon-\int_{\tau_1}^{\tau_2}\int_0^1\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{J}}{\partial\dot{\psi}}\right)\delta\psi\,d\varepsilon d\tau \tag{7.61}$$

Consider now the second term of the integral in (7.60) and integrate by parts with respect of the time variable:

$$\int_{\tau_1}^{\tau_2}\int_0^1 \frac{\partial\mathcal{J}}{\partial\dot{\zeta}}\delta\dot{\zeta}\,d\varepsilon d\tau=\int_0^1 \frac{\partial\mathcal{J}}{\partial\dot{\zeta}}\delta\zeta\bigg|_{\tau=\tau_1}^{\tau=\tau_2}d\varepsilon-\int_{\tau_1}^{\tau_2}\int_0^1\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{J}}{\partial\dot{\zeta}}\right)\delta\zeta\,d\varepsilon d\tau \tag{7.62}$$

Consider the third term of the integral in (7.60) and integrate twice by parts with respect of the space variable:

$$\int_{\tau_1}^{\tau_2}\int_0^1 \frac{\partial\mathcal{J}}{\partial\psi''}\delta\psi''\,d\varepsilon d\tau=\int_{\tau_1}^{\tau_2}\frac{\partial\mathcal{J}}{\partial\psi''}\delta\psi'\bigg|_{\varepsilon=0}^{\varepsilon=1}d\tau-\int_{\tau_1}^{\tau_2}\left(\frac{\partial\mathcal{J}}{\partial\psi''}\right)'\delta\psi\bigg|_{\varepsilon=0}^{\varepsilon=1}d\tau+\int_{\tau_1}^{\tau_2}\int_0^1\left(\frac{\partial\mathcal{J}}{\partial\psi''}\right)''\delta\psi\,d\varepsilon d\tau \tag{7.63}$$

Finally for the fourth term we get:

$$\int_{\tau_1}^{\tau_2}\int_0^1 \frac{\partial \mathcal{J}}{\partial \zeta''}\delta\zeta''\,d\varepsilon d\tau = \int_{\tau_1}^{\tau_2} \frac{\partial \mathcal{J}}{\partial \zeta''}\delta\zeta'\Big|_{\varepsilon=0}^{\varepsilon=1} d\tau - \int_{\tau_1}^{\tau_2}\left(\frac{\partial \mathcal{J}}{\partial \zeta''}\right)'\delta\zeta\Big|_{\varepsilon=0}^{\varepsilon=1} d\tau + \int_{\tau_1}^{\tau_2}\int_0^1 \left(\frac{\partial \mathcal{J}}{\partial \zeta''}\right)''\delta\zeta\,d\varepsilon d\tau \tag{7.64}$$

Substituting these results into eq.(7.60) we get:

$$\begin{aligned}
\delta\mathcal{A} = &\int_0^1 \left(\frac{\partial \mathcal{J}}{\partial \dot\psi}\delta\psi\Big|_{\tau=\tau_1}^{\tau=\tau_2} + \frac{\partial \mathcal{J}}{\partial \dot\zeta}\delta\zeta\Big|_{\tau=\tau_1}^{\tau=\tau_2}\right) d\varepsilon + \\
&\int_{\tau_1}^{\tau_2}\left(\frac{\partial \mathcal{J}}{\partial \psi''}\delta\psi'\Big|_{\varepsilon=0}^{\varepsilon=1} - \left(\frac{\partial \mathcal{J}}{\partial \psi''}\right)'\delta\psi\Big|_{\varepsilon=0}^{\varepsilon=1} + \frac{\partial \mathcal{J}}{\partial \zeta''}\delta\zeta'\Big|_{\varepsilon=0}^{\varepsilon=1} - \left(\frac{\partial \mathcal{J}}{\partial \zeta''}\right)'\delta\zeta\Big|_{\varepsilon=0}^{\varepsilon=1}\right) d\tau + \\
&\int_{\tau_1}^{\tau_2}\int_0^1\left(\left(\frac{\partial \mathcal{J}}{\partial \psi''}\right)'' - \frac{\partial}{\partial\tau}\left(\frac{\partial \mathcal{J}}{\partial \dot\psi}\right)\right)\delta\psi\,d\varepsilon d\tau + \int_{\tau_1}^{\tau_2}\int_0^1\left(\left(\frac{\partial \mathcal{J}}{\partial \zeta''}\right)'' - \frac{\partial}{\partial\tau}\left(\frac{\partial \mathcal{J}}{\partial \dot\zeta}\right)\right)\delta\zeta\,d\varepsilon d\tau
\end{aligned} \tag{7.65}$$

Considering synchronous variations the first term in (7.65) immediately vanishes, while from the second term we get the *transversality conditions*:

$$\begin{aligned}
&\frac{\partial \mathcal{J}}{\partial \psi''}\delta\psi'\Big|_{\varepsilon=0} = 0 \qquad &&\frac{\partial \mathcal{J}}{\partial \psi''}\delta\psi'\Big|_{\varepsilon=1} = 0\\
&\left(\frac{\partial \mathcal{J}}{\partial \psi''}\right)'\delta\psi\Big|_{\varepsilon=0} = 0 \qquad &&\left(\frac{\partial \mathcal{J}}{\partial \psi''}\right)'\delta\psi\Big|_{\varepsilon=1} = 0\\
&\frac{\partial \mathcal{J}}{\partial \zeta''}\delta\zeta'\Big|_{\varepsilon=0} = 0 \qquad &&\frac{\partial \mathcal{J}}{\partial \zeta''}\delta\zeta'\Big|_{\varepsilon=1} = 0\\
&\left(\frac{\partial \mathcal{J}}{\partial \zeta''}\right)'\delta\zeta\Big|_{\varepsilon=0} = 0 \qquad &&\left(\frac{\partial \mathcal{J}}{\partial \zeta''}\right)'\delta\zeta\Big|_{\varepsilon=1} = 0
\end{aligned} \tag{7.66}$$

Focusing our attention on the last two integrals in eq.(7.65) we get the Euler-Lagrange equations:

$$\begin{aligned}
&\left(\frac{\partial \mathcal{J}}{\partial \psi''}\right)'' - \frac{\partial}{\partial\tau}\left(\frac{\partial \mathcal{J}}{\partial \dot\psi}\right) = 0\\
&\left(\frac{\partial \mathcal{J}}{\partial \zeta''}\right)'' - \frac{\partial}{\partial\tau}\left(\frac{\partial \mathcal{J}}{\partial \dot\zeta}\right) = 0
\end{aligned} \tag{7.67}$$

Substituting the expression of the Lagrangian given in (7.58) into the Euler-Lagrange equations (7.67) we obtain:

$$\begin{aligned}\left(-\psi'' + \tfrac{1}{2}\varrho\dot{\zeta}\right)'' - \frac{\partial}{\partial\tau}\left(\beta^2\dot{\psi} - \tfrac{1}{2}\varrho\zeta''\right) = 0\\ \left(-\zeta'' - \tfrac{1}{2}\varrho\dot{\psi}\right)'' - \frac{\partial}{\partial\tau}\left(\beta^2\dot{\zeta} + \tfrac{1}{2}\varrho\psi''\right) = 0\end{aligned} \tag{7.68}$$

or:

$$\begin{aligned}\psi'''' - \varrho\dot{\zeta}'' + \beta^2\ddot{\psi} = 0\\ \zeta'''' + \varrho\dot{\psi}'' + \beta^2\ddot{\zeta} = 0\end{aligned} \tag{7.69}$$

Thus we have proven that the Euler-Lagrange equations of the assumed Action are the coupled equations (7.46).

The transversality conditions become:

$$\begin{aligned}\left(-\psi''(0,\tau) + \tfrac{1}{2}\varrho\dot{\zeta}(0,\tau)\right)\delta\psi'(0,\tau) = 0\\ \left(-\psi''(1,\tau) + \tfrac{1}{2}\varrho\dot{\zeta}(1,\tau)\right)\delta\psi'(1,\tau) = 0\\ \left(-\psi'''(0,\tau) + \tfrac{1}{2}\varrho\dot{\zeta}'(0,\tau)\right)\delta\psi(0,\tau) = 0\\ \left(-\psi'''(1,\tau) + \tfrac{1}{2}\varrho\dot{\zeta}'(1,\tau)\right)\delta\psi(1,\tau) = 0\\ \left(-\zeta''(0,\tau) - \tfrac{1}{2}\varrho\dot{\psi}(0,\tau)\right)\delta\zeta'(0,\tau) = 0\\ \left(-\zeta''(1,\tau) - \tfrac{1}{2}\varrho\dot{\psi}(1,\tau)\right)\delta\zeta'(1,\tau) = 0\\ \left(-\zeta'''(0,\tau) - \tfrac{1}{2}\varrho\dot{\psi}'(0,\tau)\right)\delta\zeta(0,\tau) = 0\\ \left(-\zeta'''(1,\tau) - \tfrac{1}{2}\varrho\dot{\psi}'(1,\tau)\right)\delta\zeta(1,\tau) = 0\end{aligned} \tag{7.70}$$

Now we want to prove that the piezoelectric beam is conservative, this is a requirement that should be absolutely satisfied since all the subnetworks of the circuit are lossless, the beam and the actuators are conservative.

We explicitly remark that the Lagrangian we have found before does not explicitly depend on the time variable $\tau$, thus the conservation of the energy is somehow intuitive.

Consider the dimensionless Energy $\epsilon$ defined by:

$$\epsilon\left(\dot{\psi},\dot{\zeta},\psi'',\zeta''\right)=\int_0^1\left(\dot{\zeta}\frac{\partial\mathcal{J}}{\partial\dot{\zeta}}+\dot{\psi}\frac{\partial\mathcal{J}}{\partial\dot{\psi}}-\mathcal{J}\right)d\varepsilon \tag{7.71}$$

then:

$$\begin{aligned}\epsilon\left(\dot{\psi},\dot{\zeta},\psi'',\zeta''\right)&=\textstyle\int_0^1\left(\dot{\zeta}\left(\beta^2\dot{\zeta}+\frac{1}{2}\varrho\psi''\right)+\dot{\psi}\left(\frac{1}{2}\beta^2\dot{\psi}-\frac{1}{2}\varrho\zeta''\right)\right)\\&-\left(\tfrac{1}{2}\beta^2\dot{\psi}^2+\tfrac{1}{2}\beta^2\dot{\zeta}^2-\tfrac{1}{2}\left(\psi''\right)^2-\tfrac{1}{2}\left(\zeta''\right)^2-\tfrac{1}{2}\varrho\dot{\psi}\zeta''+\tfrac{1}{2}\varrho\dot{\zeta}\psi''\right)d\varepsilon\end{aligned} \tag{7.72}$$

or:

$$\epsilon\left(\dot{\psi},\dot{\zeta},\psi'',\zeta''\right)=\int_0^1\frac{1}{2}\beta^2\dot{\psi}^2+\frac{1}{2}\beta^2\dot{\zeta}^2+\frac{1}{2}\left(\psi''\right)^2+\frac{1}{2}\left(\zeta''\right)^2d\varepsilon=\epsilon_e\left(\dot{\psi},\psi''\right)+\epsilon_m\left(\dot{\zeta},\zeta''\right) \tag{7.73}$$

as, we expected the energy of the piezoelectromechanical beam is equal to the sum of the electrical and the mechanical energy.

Now we want to show that $\dfrac{d}{d\tau}\epsilon=0$, hence consider the total derivative of the energy:

$$\frac{d}{d\tau}\epsilon=\frac{d}{d\tau}\int_0^1\left(\dot{\zeta}\frac{\partial\mathcal{J}}{\partial\dot{\zeta}}+\dot{\psi}\frac{\partial\mathcal{J}}{\partial\dot{\psi}}-\mathcal{J}\right)d\varepsilon \tag{7.74}$$

this yields:

$$\frac{d}{d\tau}\epsilon=\int_0^1\left(\ddot{\zeta}\frac{\partial\mathcal{J}}{\partial\dot{\zeta}}+\dot{\zeta}\frac{\partial}{\partial\tau}\frac{\partial\mathcal{J}}{\partial\dot{\zeta}}+\ddot{\psi}\frac{\partial\mathcal{J}}{\partial\dot{\psi}}+\dot{\psi}\frac{\partial}{\partial\tau}\frac{\partial\mathcal{J}}{\partial\dot{\psi}}-\frac{\partial}{\partial\tau}\mathcal{J}\right)d\varepsilon \tag{7.75}$$

thus, substituting the Euler-Lagrange equations and differentiating the Lagrangian density with respect of the time variable, we get:

$$\frac{d}{d\tau}\epsilon=\int_0^1\left(\ddot{\zeta}\frac{\partial\mathcal{J}}{\partial\dot{\zeta}}+\dot{\zeta}\left(\frac{\partial\mathcal{J}}{\partial\zeta''}\right)''+\ddot{\psi}\frac{\partial\mathcal{J}}{\partial\dot{\psi}}+\dot{\psi}\left(\frac{\partial\mathcal{J}}{\partial\psi''}\right)''-\left(\ddot{\psi}\frac{\partial\mathcal{J}}{\partial\dot{\psi}}+\ddot{\zeta}\frac{\partial\mathcal{J}}{\partial\dot{\zeta}}+\dot{\psi}''\left(\frac{\partial\mathcal{J}}{\partial\psi''}\right)+\dot{\zeta}''\left(\frac{\partial\mathcal{J}}{\partial\zeta''}\right)\right)\right)d\varepsilon \tag{7.76}$$

finally:

$$\frac{d}{d\tau}\epsilon = \int_0^1 \dot{\zeta}\left(\frac{\partial \mathcal{J}}{\partial \zeta''}\right)'' + \dot{\psi}\left(\frac{\partial \mathcal{J}}{\partial \psi''}\right)'' - \dot{\psi}''\left(\frac{\partial \mathcal{J}}{\partial \psi''}\right) - \dot{\zeta}''\left(\frac{\partial \mathcal{J}}{\partial \zeta''}\right)\, d\varepsilon \tag{7.77}$$

Substituting the expression (7.58):

$$\frac{d}{d\tau}\epsilon = \int_0^1 \dot{\zeta}\left(-\zeta'' - \frac{1}{2}\varrho\dot{\psi}\right)'' + \dot{\psi}\left(-\psi'' + \frac{1}{2}\varrho\dot{\zeta}\right)'' - \dot{\psi}''\left(-\psi'' + \frac{1}{2}\varrho\dot{\zeta}\right) - \dot{\zeta}''\left(-\zeta'' - \frac{1}{2}\varrho\dot{\psi}\right)\, d\varepsilon \tag{7.78}$$

hence:

$$\frac{d}{d\tau}\epsilon = \int_0^1 \dot{\zeta}\left(-\zeta'''' - \frac{1}{2}\varrho\dot{\psi}''\right) + \dot{\psi}\left(-\psi'''' + \frac{1}{2}\varrho\dot{\zeta}''\right) - \dot{\psi}''\left(-\psi'' + \frac{1}{2}\varrho\dot{\zeta}\right) - \dot{\zeta}''\left(-\zeta'' - \frac{1}{2}\varrho\dot{\psi}\right)\, d\varepsilon \tag{7.79}$$

integrating twice by parts the last two terms of the integral we get:

$$\begin{aligned}
\int_0^1 \dot{\psi}''\left(\psi'' - \tfrac{1}{2}\varrho\dot{\zeta}\right) d\varepsilon &= \left(\psi'' - \tfrac{1}{2}\varrho\dot{\zeta}\right)\dot{\psi}'\Big|_{\varepsilon=0}^{\varepsilon=1} - \left(\psi''' - \tfrac{1}{2}\varrho\dot{\zeta}'\right)\dot{\psi}\Big|_{\varepsilon=0}^{\varepsilon=1} + \int_0^1 \dot{\psi}\left(\psi'''' - \tfrac{1}{2}\varrho\dot{\zeta}''\right) d\varepsilon \\
\int_0^1 \dot{\zeta}''\left(\zeta'' + \tfrac{1}{2}\varrho\dot{\psi}\right) d\varepsilon &= \dot{\zeta}'\left(\zeta'' + \tfrac{1}{2}\varrho\dot{\psi}\right)\Big|_{\varepsilon=0}^{\varepsilon=1} - \dot{\zeta}'\left(\zeta''' + \tfrac{1}{2}\varrho\dot{\psi}'\right)\Big|_{\varepsilon=0}^{\varepsilon=1} + \int_0^1 \dot{\zeta}\left(\zeta'''' + \tfrac{1}{2}\varrho\dot{\psi}''\right) d\varepsilon
\end{aligned} \tag{7.80}$$

the first two terms of both the expressions are forced to vanish by the transversality conditions:

$$\begin{aligned}
\int_0^1 \dot{\psi}''\left(\psi'' - \tfrac{1}{2}\varrho\dot{\zeta}\right) d\varepsilon &= \int_0^1 \dot{\psi}\left(\psi'''' - \tfrac{1}{2}\varrho\dot{\zeta}''\right) d\varepsilon \\
\int_0^1 \dot{\zeta}''\left(\zeta'' + \tfrac{1}{2}\varrho\dot{\psi}\right) d\varepsilon &= \int_0^1 \dot{\zeta}\left(\zeta'''' + \tfrac{1}{2}\varrho\dot{\psi}''\right) d\varepsilon
\end{aligned} \tag{7.81}$$

thus the derivative of the energy becomes:

$$\frac{d}{d\tau}\epsilon = \int_0^1 \dot{\zeta}\left(-\zeta'''' - \frac{1}{2}\varrho\dot{\psi}''\right) + \dot{\psi}\left(-\psi'''' + \frac{1}{2}\varrho\dot{\zeta}''\right) + \dot{\psi}\left(\psi'''' - \frac{1}{2}\varrho\dot{\zeta}''\right) + \dot{\zeta}\left(\zeta'''' + \frac{1}{2}\varrho\dot{\psi}''\right)\, d\varepsilon \tag{7.82}$$

rearranging the terms, we can finally show that:

$$\frac{d}{d\tau}\epsilon = \int_0^1 \dot{\zeta}\left(-\zeta'''' - \frac{1}{2}\varrho\dot{\psi}'' + \zeta'''' + \frac{1}{2}\varrho\dot{\psi}''\right) + \dot{\psi}\left(-\psi'''' + \frac{1}{2}\varrho\dot{\zeta}'' + \psi'''' - \frac{1}{2}\varrho\dot{\zeta}''\right) d\varepsilon = 0 \tag{7.83}$$

**Free vibrations of a piezoelectromechanical beam clamped at both ends.**

In this section we want to study the free vibrations of the piezoelectromechanical beam, when we clamp the beam at both ends and short-circuit the ports at both side of the circuit.

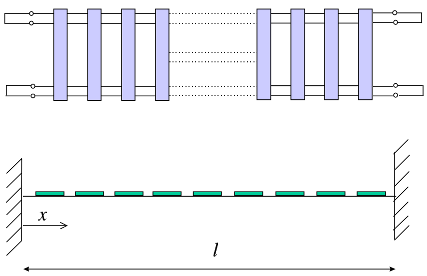


Figure 7.5: Piezoelectromechanical beam clamped at both ends

The boundary conditions for this problem are:

$$\begin{aligned} u(0,t) &= 0 \quad V_1(0,t) = 0 \\ \vartheta(0,t) &= 0 \quad V_2(0,t) = 0 \\ u(l,t) &= 0 \quad V_1(l,t) = 0 \\ \vartheta(l,t) &= 0 \quad V_2(l,t) = 0 \end{aligned} \tag{7.84}$$

in dimensionless variables and considering the dimensionless integrals of the voltage:

$$\begin{aligned} \psi_1(\varepsilon,\tau) &= \textstyle\int_0^{\tau} \varphi_1(\varepsilon,\bar{\tau})\, d\bar{\tau} \\ \psi_2(\varepsilon,\tau) &= \textstyle\int_0^{\tau} \varphi_2(\varepsilon,\bar{\tau})\, d\bar{\tau} \end{aligned} \tag{7.85}$$

we get:

$$\begin{array}{ll} \zeta(0,\tau)=0 & \psi_1(0,\tau)=0 \\ \theta(0,\tau)=0 & \psi_2(0,\tau)=0 \\ \zeta(1,\tau)=0 & \psi_1(1,\tau)=0 \\ \theta(1,\tau)=0 & \psi_2(1,\tau)=0 \end{array} \tag{7.86}$$

Taking into account that $\zeta' = \theta$ and $\psi_1' = \psi_2$ we finally get:

$$\begin{array}{ll} \zeta(0,\tau)=0 & \psi_1(0,\tau)=0 \\ \zeta'(0,\tau)=0 & \psi_1'(0,\tau)=0 \\ \zeta(1,\tau)=0 & \psi_1(1,\tau)=0 \\ \zeta'(1,\tau)=0 & \psi_1'(1,\tau)=0 \end{array} \tag{7.87}$$

Hence, we have to study the system of equations:

$$\begin{aligned} \psi_1''''(\varepsilon,\tau) - \varrho\,\dot{\zeta}''(\varepsilon,\tau) &= -\beta^2 \ddot{\psi}_1(\varepsilon,\tau) \\ \zeta''''(\varepsilon,\tau) + \varrho\,\dot{\psi}_1''(\varepsilon,\tau) &= -\beta^2 \ddot{\zeta}(\varepsilon,\tau) \end{aligned} \tag{7.88}$$

with the boundary conditions (7.87).

Now we have to consider the initial conditions in dimensionless variables:

$$\begin{array}{ll} \zeta(\varepsilon,0)=\zeta^0(\varepsilon) & \psi_1(\varepsilon,0)=\psi_1^0(\varepsilon) \\ \dot{\zeta}(\varepsilon,0)=\dot{\zeta}^0(\varepsilon) & \psi_1(\varepsilon,0)=\psi_1^0(\varepsilon) \end{array} \tag{7.89}$$

**Notation 289** *Furthermore we will denote the integral of the potential $\psi_1$ by $\psi$*

As a preliminary to the spectral analysis of the coupled equations (7.46), let us introduce some topics in Functional Analysis that we will need in the following.

## Basis functions

Let us consider the subset $\mathcal{M}$ of $L_2([0,1])$ defined by:

$$\mathcal{M}=\left\{\psi\in L_2([0,1]):\psi(0)=\psi(1)=0,\ \psi'(0)=\psi'(1)=0\text{ and }\frac{d^4}{dx^4}\psi\in L_2([0,1])\right\} \tag{7.90}$$

where $\frac{d^4}{dx^4}$ denotes the strong derivative,clearly $\mathcal{M}$ is a linear manifold of $L_2([0,1])$ and $C^\infty([0,1])\subset\mathcal{M}$.

Furthermore consider the subset $\mathcal{V}$ of $L_2([0,1])$ defined by:

$$\mathcal{V}=\left\{\psi\in L_2([0,1]):\psi(0)=\psi(1)=0,\ \psi'(0)=\psi'(1)\right\} \tag{7.91}$$

clearly $\mathcal{M}\subset\mathcal{V}$, and $\mathcal{V}$ is a closed linear manifold of $L_2([0,1])$.

Let us introduce the differential operator $\mathfrak{D}$ defined on $\mathcal{M}$ such that:

$$\forall\psi\in\mathcal{M}\qquad\mathfrak{D}\psi=\frac{d^4}{dx^4}\psi \tag{7.92}$$

**Proposition 290** *The differential operator $\mathfrak{D}$ is self-adjoint.*

***Proof.*** *Since $L_2([0,1])$ is an Hilbert space, then $\mathfrak{D}$ is self-adjoint if:*

$$\forall\psi\in\mathcal{M},\varphi\in\mathcal{M}\qquad\langle\mathfrak{D}\psi,\varphi\rangle=\langle\psi,\mathfrak{D}\varphi\rangle \tag{7.93}$$

*Thus, we have to prove that:*

$$\forall\psi\in\mathcal{M},\varphi\in\mathcal{M}\qquad\int_0^1\frac{d^4\psi}{dx^4}\varphi dx=\int_0^1\frac{d^4\varphi}{dx^4}\psi dx \tag{7.94}$$

*integrating four times by parts the left hand side of the previous equation we get:*

$$\int_0^1\frac{d^4\psi}{dx^4}\varphi dx=\left.\frac{d^3\psi}{dx^3}\varphi\right|_0^1-\left.\frac{d^2\psi}{dx^2}\frac{d\varphi}{dx}\right|_0^1+\left.\frac{d\psi}{dx}\frac{d^2\varphi}{d^2x}\right|_0^1-\left.\psi\frac{d^3\varphi}{dx^3}\right|_0^1+\int_0^1\psi\frac{d^4\varphi}{d^3x}dx \tag{7.95}$$

*Remembering that $\psi \in \mathcal{M}, \varphi \in \mathcal{M}$ the previous equation yields:*

$$\int_0^1 \frac{d^4\psi}{dx^4}\varphi dx = \int_0^1 \psi \frac{d^4\varphi}{d^3x} dx \tag{7.96}$$

■

**Proposition 291** *The differential operator $\mathfrak{D}$ is positive-definite.*

***Proof.*** *$\mathfrak{D}$ is positive-definite if:*

$$\forall \psi \in \mathcal{M},\ \psi \neq \mathbf{0} \qquad \langle \psi, \mathfrak{D}\psi \rangle > 0 \tag{7.97}$$

*Thus, we have to prove that:*

$$\forall \psi \in \mathcal{M},\ \psi \neq \mathbf{0} \qquad \int_0^1 \frac{d^4\psi}{dx^4}\psi > 0 \tag{7.98}$$

*integrating twice by parts, we get:*

$$\int_0^1 \frac{d^4\psi}{dx^4}\psi dx = \left.\frac{d^3\psi}{dx^3}\varphi\right|_0^1 - \left.\frac{d^2\psi}{dx^2}\frac{d\varphi}{dx}\right|_0^1 + \int_0^1 \left(\frac{d^2\psi}{dx^2}\right)^2 dx = \int_0^1 \left(\frac{d^2\psi}{dx^2}\right)^2 dx \geqslant 0 \qquad \forall \psi \in \mathcal{M} \tag{7.99}$$

*Furthermore, if $\langle \psi, \mathfrak{D}\psi \rangle = 0$ then*

$$\int_0^1 \left(\frac{d^2\psi}{dx^2}\right)^2 dx = 0 \Rightarrow \left(\frac{d^2\psi}{dx^2}\right) = 0 \Rightarrow \psi = ax + b \qquad a, b \in \mathbb{R} \tag{7.100}$$

*Since $\psi \in \mathcal{M}$ then both $a$ and $b$ should be equal to zero.* ■

Now let us find the spectrum and the eigenfunctions of $\mathfrak{D}$, since we have proven that $\mathfrak{D}$ is positive-definite and self-adjoint, then all the eigenvalues are real and strictly positive.

Hence, we have to find a real number $\lambda \in \mathbb{R}^+$ and a function $\psi \in \mathcal{M}$ such that:

$$\mathfrak{D}\psi = \lambda\psi \Rightarrow \frac{d^4\psi}{dx^4} = \lambda\psi \tag{7.101}$$

The solution of the previous differential equation can be written in the form:

$$\psi(x) = A_1 \cosh \sqrt[4]{\lambda} x + A_2 \sinh \sqrt[4]{\lambda} x + A_3 \cos \sqrt[4]{\lambda} x + A_4 \sin \sqrt[4]{\lambda} x \tag{7.102}$$

with $\sqrt[4]{\lambda} \in \mathbb{R}^+$.

Imposing the boundary conditions, i.e. $\psi \in \mathcal{M}$, we get:

$$\psi_m(x) = A_m \left( \cosh \sqrt[4]{\lambda_m} x - \cos \sqrt[4]{\lambda_m} x + \frac{\cosh \sqrt[4]{\lambda_m} - \cos \sqrt[4]{\lambda_m}}{\sinh \sqrt[4]{\lambda_m} - \sin \sqrt[4]{\lambda_m}} \left( \sin \sqrt[4]{\lambda_m} x - \sinh \sqrt[4]{\lambda_m} x \right) \right) \tag{7.103}$$

with $\lambda_m$ given by the following trascendental equation:

$$\cosh \sqrt[4]{\lambda_m} \cos \sqrt[4]{\lambda_m} = 1 \tag{7.104}$$

or

$$\cos r = \frac{1}{\cosh r} \tag{7.105}$$

with $r = \sqrt[4]{\lambda_m}$.

The first four solutions of this equation are:

$$r_1 \simeq 4.73004 \qquad r_2 \simeq 7.8352 \qquad r_3 \simeq 10.9956 \qquad r_4 \simeq 14.1372 \tag{7.106}$$

If $r > r_4$ then the trascendental equation can be approximated by:

$$\cos r \simeq 0 \tag{7.107}$$

giving the solutions:

$$r_k = \frac{\pi}{2} + k\pi \tag{7.108}$$

For instance the previous expression gives:

$$r_5 = \frac{\pi}{2} + 5\pi = 17.279 \tag{7.109}$$

while the right result is approximately:

$$r_5 = 17.27876 \tag{7.110}$$

this shows how good is the approximation (7.108).

Let us list the approximate values of the eigenvalues in the following table:

$$\begin{aligned}
&\lambda_1 \simeq 500.56 \\
&\lambda_2 \simeq 3768.8 \\
&\lambda_3 \simeq 14618 \\
&\lambda_4 \simeq 39944. \\
&\lambda_k \simeq \left(\frac{\pi}{2} + k\pi\right)^4 \text{ with } k > 4
\end{aligned} \tag{7.111}$$

Since $\mathfrak{D}$ is a self adjoint linear operator, then its eigenfunctions span all the space $\mathcal{V}$, i.e. every function $f(x) \in \mathcal{V}$ can be expanded in terms of these eigenfunctions; furthermore all the eigenvalues are distinct, thus the set $\mathcal{S} = \{\psi_m\}$ is orthogonal and it is possible to choose the constants $A_m$ such that $\mathcal{S}$ is orthonormal. More details can be found in Friedman (1966)[17].

By the previous considerations we can state:

$$\forall f \in \mathcal{V} \qquad f = \sum_m \langle f, \psi_m \rangle \psi_m \tag{7.112}$$

### Spectral analysis of the problem

Since both the functions $\psi$ and $\zeta$ are $C^\infty$ with respect of the variable $\varepsilon$ in the interval $[0, 1]$ and fulfill the given boundary conditions they belong to $\mathcal{M}$ (when regarded as function of $\varepsilon$): then both of them can be expanded in terms of the orthonormal set

$\{\psi_m\}$ as:

$$\begin{aligned} \psi(\varepsilon,\tau) &= \textstyle\sum_m p_m(\tau)\,\psi_m(\varepsilon) \\ \zeta(\varepsilon,\tau) &= \textstyle\sum_n q_n(\tau)\,\psi_n(\varepsilon) \end{aligned} \tag{7.113}$$

The same reasoning still holds for the initial conditions:

$$\begin{aligned} \psi^0(\varepsilon) &= \textstyle\sum_m \langle \varphi_1^0, \psi_m\rangle\,\psi_m(\varepsilon) \\ \dot{\psi}^0(\varepsilon) &= \textstyle\sum_m \langle \dot{\varphi}_1^0, \psi_m\rangle\,\psi_m(\varepsilon) \\ \zeta^0(\varepsilon) &= \textstyle\sum_m \langle \nu^0, \psi_m\rangle\,\psi_m(\varepsilon) \\ \dot{\zeta}^0(\varepsilon) &= \textstyle\sum_m \langle \dot{\nu}^0, \psi_m\rangle\,\psi_m(\varepsilon) \end{aligned} \tag{7.114}$$

Furthermore introducing the coupling operator $\mathfrak{C}(\cdot) := \dfrac{d^2}{d\varepsilon^2}(\cdot)$ defined on $\mathcal{M}$, the set of coupled differential equations(7.46) can be written in compact form as:

$$\begin{aligned} \mathfrak{D}\psi - \varrho\mathfrak{C}\dot{\zeta} &= -\beta^2\ddot{\psi} \\ \mathfrak{D}\zeta + \varrho\mathfrak{C}\dot{\psi} &= -\beta^2\ddot{\zeta} \end{aligned} \tag{7.115}$$

Substituting eqs.(7.113) into eqs.(7.115) we get:

$$\begin{aligned} &\textstyle\sum_m p_m(\tau)\,\mathfrak{D}\psi_m(\varepsilon) - \varrho\sum_n \dot{q}_n(\tau)\,\mathfrak{C}\psi_n(\varepsilon) = -\beta^2\sum_m \ddot{p}_m(\tau)\,\psi_m(\varepsilon) \\ &\textstyle\sum_{mn} q_n(\tau)\,\mathfrak{D}\psi_n(\varepsilon) + \varrho\sum_m \dot{q}_m(\tau)\,\mathfrak{C}\psi_m(\varepsilon) = -\beta^2\sum_n \ddot{q}_n(\tau)\,\psi_n(\varepsilon) \end{aligned} \tag{7.116}$$

Taking into account that $\psi_m$ is an eigenfunctions of $\mathfrak{D}$ with eigenvalue $\lambda_m$, we get:

$$\begin{aligned} &\textstyle\sum_m \lambda_m p_m(\tau)\,\psi_m(\varepsilon) - \varrho\sum_n \dot{q}_n(\tau)\,\mathfrak{C}\psi_n(\varepsilon) = -\beta^2\sum_m \ddot{p}_m(\tau)\,\psi_m(\varepsilon) \\ &\textstyle\sum_n \lambda_n q_n(\tau)\,\psi_n(\varepsilon) + \varrho\sum_m \dot{q}_m(\tau)\,\mathfrak{C}\psi_m(\varepsilon) = -\beta^2\sum_n \ddot{q}_n(\tau)\,\psi_n(\varepsilon) \end{aligned} \tag{7.117}$$

Taking the inner product of the two previous equations by the same eigenfunctions $\psi_k(\varepsilon)$ we get:

$$\begin{aligned} \lambda_k p_k(\tau) - \varrho\textstyle\sum_n C_{n,k}\,\dot{q}_n(\tau) &= -\beta^2\ddot{p}_k(\tau) \\ \lambda_k q_k(\tau) + \varrho\textstyle\sum_m C_{m,k}\,\dot{q}_m(\tau) &= -\beta^2\ddot{q}_n(\tau) \end{aligned} \tag{7.118}$$

with:

$$C_{i,j} := \left\langle \mathfrak{C}\psi_i, \psi_j \right\rangle \tag{7.119}$$

Let us investigate the generic term $C_{i,j}$:

$$C_{i,j} = \left\langle \mathfrak{C}\psi_i, \psi_j \right\rangle = \int_0^1 \frac{d^2\psi_i}{d\varepsilon^2}\psi_j d\varepsilon \tag{7.120}$$

integrating by parts, we get:

$$C_{i,j} = \left.\frac{d\psi_i}{d\varepsilon}\psi_j\right|_0^1 - \int_0^1 \frac{d\psi_i}{d\varepsilon}\frac{d\psi_j}{d\varepsilon}d\varepsilon = -\int_0^1 \frac{d\psi_i}{d\varepsilon}\frac{d\psi_j}{d\varepsilon}d\varepsilon \tag{7.121}$$

since $\psi_i$ and $\psi_j$ belong to $\mathcal{M}$, then $C_{i,j} = C_{j,i}$.

Now let us find an approximate solution of the problem, projecting the functions $\varphi_1(\varepsilon,\tau)$ and $\nu(\varepsilon,\tau)$ on a finite dimensional subspace of $\mathcal{M}$ which is spanned by the first $M$ eigenfunctions of $\mathfrak{D}$:

$$\begin{aligned} \hat{\psi}(\varepsilon,\tau) &= \textstyle\sum_m^M p_m(\tau)\,\psi_m(\varepsilon) \\ \hat{\zeta}(\varepsilon,\tau) &= \textstyle\sum_n^M q_n(\tau)\,\psi_n(\varepsilon) \end{aligned} \tag{7.122}$$

and for the initial conditions, we get:

$$\begin{aligned} \hat{\psi}^0(\varepsilon) &= \textstyle\sum_m \left\langle \psi^0, \psi_m \right\rangle \psi_m(\varepsilon) \\ \hat{\dot{\psi}}^0(\varepsilon) &= \textstyle\sum_m \left\langle \dot{\psi}^0, \psi_m \right\rangle \psi_m(\varepsilon) \\ \hat{\zeta}^0(\varepsilon) &= \textstyle\sum_m \left\langle \zeta^0, \psi_m \right\rangle \psi_m(\varepsilon) \\ \hat{\dot{\zeta}}^0(\varepsilon) &= \textstyle\sum_m \left\langle \dot{\zeta}^0, \psi_m \right\rangle \psi_m(\varepsilon) \end{aligned} \tag{7.123}$$

Going through all the steps detailed before, we finally get:

$$\begin{aligned} \lambda_k p_k(\tau) - \varrho \textstyle\sum_n^M C_{n,k}\,\dot{q}_n(\tau) &= -\beta^2 \ddot{p}_k(\tau) \\ \lambda_k q_k(\tau) + \varrho \textstyle\sum_m^M C_{m,k}\,\dot{q}_m(\tau) &= -\beta^2 \ddot{q}_k(\tau) \end{aligned} \tag{7.124}$$

This system can be written in matrix form as:

$$\begin{aligned} &\beta^2 \ddot{\mathbf{p}}(\tau) - \varrho \mathbf{C} \dot{\mathbf{q}}(\tau) + \Lambda \mathbf{p}(\tau) = \mathbf{0} \\ &\beta^2 \ddot{\mathbf{q}}(\tau) + \varrho \mathbf{C} \dot{\mathbf{p}}(\tau) + \Lambda \mathbf{q}(\tau) = \mathbf{0} \end{aligned} \tag{7.125}$$

or:

$$\beta^2 \begin{pmatrix} \ddot{\mathbf{p}} \\ \ddot{\mathbf{q}} \end{pmatrix} + \varrho \begin{pmatrix} \mathbf{0}_{M\times M} & -\mathbf{C} \\ \mathbf{C} & \mathbf{0}_{M\times M} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{q}} \end{pmatrix} + \begin{pmatrix} \Lambda & \mathbf{0}_{M\times M} \\ \mathbf{0}_{M\times M} & \Lambda \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{7.126}$$

with:

$$\begin{aligned} &\mathbf{p} = \begin{pmatrix} p_1 \\ ... \\ p_M \end{pmatrix} \qquad \mathbf{q} = \begin{pmatrix} q_1 \\ ... \\ q_M \end{pmatrix} \\ &\Lambda = \begin{pmatrix} \lambda_1 & ... & 0 \\ ... & ... & .. \\ 0 & .. & \lambda_M \end{pmatrix} \\ &C_{i,j} = -\int_0^1 \frac{d\psi_i}{d\varepsilon}\frac{d\psi_j}{d\varepsilon} d\varepsilon \end{aligned} \tag{7.127}$$

The initial conditions are, remembering that $\mathcal{S}$ is an orthonormal set:

$$\begin{aligned} &\mathbf{p}(0) = \begin{pmatrix} \langle \psi^0, \psi_1 \rangle \\ ... \\ \langle \psi^0, \psi_M \rangle \end{pmatrix} \qquad \dot{\mathbf{p}}(0) = \begin{pmatrix} \left\langle \dot{\psi}^0, \psi_1 \right\rangle \\ ... \\ \left\langle \dot{\psi}^0, \psi_M \right\rangle \end{pmatrix} \\ &\mathbf{q}(0) = \begin{pmatrix} \langle \zeta^0, \psi_1 \rangle \\ ... \\ \langle \zeta^0, \psi_M \rangle \end{pmatrix} \qquad \dot{\mathbf{q}}(0) = \begin{pmatrix} \left\langle \dot{\zeta}^0, \psi_1 \right\rangle \\ ... \\ \left\langle \dot{\zeta}^0, \psi_M \right\rangle \end{pmatrix} \end{aligned} \tag{7.128}$$

To solve the system of ordinary equations(7.126) it is better to write the system of equations in the so called normal form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{7.129}$$

with:

$$\mathbf{x} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ \dot{\mathbf{p}} \\ \dot{\mathbf{q}} \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ -\frac{1}{\beta^2}\Lambda & \mathbf{0} & \mathbf{0} & \frac{\varrho}{\beta^2}\mathbf{C} \\ \mathbf{0} & -\frac{1}{\beta^2}\Lambda & -\frac{\varrho}{\beta^2}\mathbf{C} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ \dot{\mathbf{p}} \\ \dot{\mathbf{q}} \end{pmatrix} \tag{7.130}$$

and the initial conditions:

$$\mathbf{x}(0) = \begin{pmatrix} \mathbf{p}(0) \\ \mathbf{q}(0) \\ \dot{\mathbf{p}}(0) \\ \dot{\mathbf{q}}(0) \end{pmatrix} \tag{7.131}$$

The solution of the system can be expressed by:

$$\mathbf{x}(\tau) = e^{A\tau}\mathbf{x}(0) \tag{7.132}$$

where $e^{A\tau}$ can be computed with the techniques developed in section(5).

**Energy considerations**

We have shown that the potential elastic energy and the kinetic energy for the dimensionless Euler beam are:

$$\upsilon_m := \frac{1}{2}\int_0^1 \left(\zeta''(\varepsilon,\tau)\right)^2 d\varepsilon \qquad \varkappa_m := \frac{1}{2}\beta^2 \int_0^1 \dot{\zeta}(\varepsilon,\tau)^2 d\varepsilon \tag{7.133}$$

while for the circuit, we have:

$$\upsilon_{el} := \frac{1}{2}\int_0^1 \left(\psi''(\varepsilon,\tau)\right)^2 d\varepsilon \qquad \varkappa_m := \frac{1}{2}\beta^2\int_0^1 \dot{\psi}(\varepsilon,\tau)^2 d\varepsilon \tag{7.134}$$

Now let us find approximate values for these energies, substituting into the previous equations the approximate expressions (7.122).

It is easy to show:

$$\begin{aligned} \hat{\upsilon}_m &= \frac{1}{2}\sum_{n=1}^{M}\lambda_n p_n^2(\tau) \quad & \hat{\varkappa}_m &= \frac{1}{2}\beta^2\sum_{n=1}^{M}\dot{p}_n^2(\tau) \\ \hat{\upsilon}_{el} &= \frac{1}{2}\sum_{n=1}^{M}\lambda_n q_n^2(\tau) \quad & \hat{\varkappa}_{el} &= \frac{1}{2}\beta^2\sum_{n=1}^{M}\dot{q}_n^2(\tau) \end{aligned} \tag{7.135}$$

## Uncoupled equations

In this section we want to study the set of equations(7.46) when $\varrho = 0$, i.e. the free vibrations of the beam and the free vibrations of the circuit, for more details on the vibrations of a beam fixed at both ends see McLachlan [18].

$$\begin{aligned} \psi''''(\varepsilon,\tau) &= -\beta^2\ddot{\psi}(\varepsilon,\tau) \\ \zeta''''(\varepsilon,\tau) &= -\beta^2\ddot{\zeta}(\varepsilon,\tau) \end{aligned} \tag{7.136}$$

Going through all the steps detailed in the previous section, we get by eq.(7.118):

$$\begin{aligned} \lambda_k p_k(\tau) &= -\beta^2\ddot{p}_k(\tau) \\ \lambda_k q_k(\tau) &= -\beta^2\ddot{q}_k(\tau) \end{aligned} \tag{7.137}$$

Since all these equations are uncoupled the general solution of the previous set of equation is:

$$\begin{aligned} p_k(\tau) &= A_k^p\cos\left(\frac{\sqrt{\lambda_k}}{\beta}\tau\right) + B_k^p\sin\left(\frac{\sqrt{\lambda_k}}{\beta}\tau\right) \\ q_k(\tau) &= A_k^q\cos\left(\frac{\sqrt{\lambda_k}}{\beta}\tau\right) + B_k^q\sin\left(\frac{\sqrt{\lambda_k}}{\beta}\tau\right) \end{aligned} \tag{7.138}$$

where the coefficients are determined by the initial conditions (7.114).

The dimensionless eigenpulsations of the system are:

$$\varpi_k = \frac{\sqrt{\lambda_k}}{\beta} \tag{7.139}$$

while the eigenfrequencies are:

$$f_k = \frac{1}{2\pi}\frac{\sqrt{\lambda_k}}{l^2}\sqrt{\frac{k_M}{\lambda}} \tag{7.140}$$

For the beam studied in chapter(6)

$$\begin{aligned} &k_M = 1.4\,Nm^2 \\ &\lambda = 0.162\,Kgm^{-1} \\ &l = 1\,m \end{aligned} \tag{7.141}$$

then the eigenfrequecies are:

$$f_k \simeq 0.46788\sqrt{\lambda_k}\,Hz \tag{7.142}$$

Let us list these eigenfrequencies in the following table, using table(7.111):

$$\begin{aligned} &f_1 \simeq 10.468\,Hz \\ &f_2 \simeq 28.724\,Hz \\ &f_3 \simeq 56.567\,Hz \\ &f_4 \simeq 93.51\,Hz \\ &f_k \simeq 0.46788\left(\frac{\pi}{2}+k\pi\right)^2\,Hz \text{ with } k > 4 \end{aligned} \tag{7.143}$$

Then, considering the restriction imposed by eq.(**??**) we can state that the Laurent expansion is valid until the mode $\bar{k}$ such that:

$$\left[0.46788\left(\frac{\pi}{2}+\bar{k}\pi\right)^2\right] = 1033.7 \tag{7.144}$$

that is:

$$\bar{k} = 14 \tag{7.145}$$

## Numerical Example

Consider the beam discussed in chapter (6) which has the following mechanical properties:

$$\begin{aligned} &l = 1\,m \\ &r_0 = 0.57736\,mm \\ &k_M^m = 1.4\,N\,m^2 \\ &\lambda = 0.162\,Kg/m \end{aligned} \tag{7.146}$$

Suppose to glue on the beam $n = 10$ actuators, each of them having the same constitutive parameters:

$$\begin{aligned} &K_{ee} = 0.6\,\mu F \\ &K_{em} = -2\,N\,mm\,V^{-1} \\ &K_{mm} = 20\,N\,m \\ &l_P = 2\,cm \end{aligned} \tag{7.147}$$

then the influence factor $\gamma$ become:

$$\gamma = \frac{x}{l_P} = \frac{l}{nl_P} = \frac{2\,cm}{10\,cm} = 0.2 \tag{7.148}$$

and the modified rotational stiffness $k_M$:

$$k_M = k_M^m + \gamma k_{mm} = k_M^m + \gamma K_{mm} l_P = (1.4 + 0.2 \times 20 \times 0.02)\,N\,m^2 = 1.48\,N\,m^2 \tag{7.149}$$

Let us choose the characteristic time $t_0$ to be the period of the first mode:

$$t_0 = \frac{1}{f_k} = 2\pi \frac{l^2}{\sqrt{\lambda_1}} \sqrt{\frac{\lambda}{k_M}} = 2\pi \frac{1}{\sqrt{500.56}} \sqrt{\frac{0.162}{1.48}} = 9.2916 \times 10^{-2}\,s = 92.916\,ms \tag{7.150}$$

then the two parameters $\varrho$ and $\beta^2$ become:

$$\varrho = \frac{K_{em} l^2}{t_0 k_M} \sqrt{\frac{\lambda}{k_{ee}}} \sqrt{\gamma} = \frac{(-0.002)\,(1)^2}{(9.2916 \times 10^{-2})\,(1.48)} \sqrt{\frac{0.162}{(0.6 * 10^{-6})\,/\,(0.02)}} \sqrt{0.2} = -0.47796$$

$$\beta^2 = \frac{\lambda l^4}{t_0^2 k_M} = \frac{(0.162)\,(1)^4}{(9.2916 \times 10^{-2})^2\,(1.48)} = 12.679 \tag{7.151}$$

and the ratio $\Gamma$ is:

$$\Gamma = \frac{\varrho}{\beta^2} = K_{em} \frac{t_0}{l^2} \sqrt{\frac{\gamma}{\lambda k_{ee}}} = (-0.002)\, \frac{\left(9.2916 \times 10^{-2}\right)}{(1)^2} \sqrt{\frac{0.2}{(0.162)\,(0.6 * 10^{-6})\,/\,(0.02)}} = -3.769\,8 \times 10^{-2} \tag{7.152}$$

then we have a coupling effect which can be estimated to by more or less equal to 4%.

Finally the characteristic voltage $V_0$ is, choosing $\kappa_0 = \frac{1}{\sqrt{2}}$:

$$V_0 = \frac{\kappa_0}{\sqrt{2}} \frac{r_0}{t_0} \sqrt{\frac{\lambda}{\gamma k_{ee}}} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{\left(0.57736 \times 10^{-3}\right)}{(8.6696 \times 10^{-2})} \sqrt{\frac{0.162}{(0.2)\,(0.6 * 10^{-6})\,/\,(0.02)}} = 0.54715V \tag{7.153}$$

# Bibliography

[1] Molly C.T. (1958) *Four Pole Parameters in Vibration Analysis* in *Mechanical impedance Methods for Mechanical Vibrations* edited by R. Plunkett.

[2] Titchmarsh E.C. (1939) *The Theory of Functions (second edition).*

[3] Newcomb Robert W. (1966) *Linear Multiport Synthesis.*

[4] Cristescu R. & Marinescu G. (1973) *Applications of the theory of distributions.*

[5] Friedlander F.G. (1998) *Introduction to the theory of distributions*

[6] Treves F. (1967) *Topological vector spaces, distributions and kernels.*

[7] Beltrami E.J. & Wohlers M.R. (1966) *Distributions and the Boundary Values of Analytic Functions.*

[8] Schwartz L. (1978) *Theorie des distributions*

[9] Schwartz L. (1966) *Mathematics for the Physical Sciences*

[10] Zinn M.K. (1951) *Network Representation of Trascendental impedance Functions* in *The Bell System Technical Journal.*

[11] Dah-You Maa (1943) *A General Reactance Theorem for Electrical, Mechanica, and Acoustical Systems* in *Proceedings of the I.R.E.*

[12] Firestone Floyd A. *Twixt Earth and Sky with Rod and Tube; the Mobility and Classical Impedance Analogies* edited in *The Journal of the Acoustical Society of America.*

[13] Firestone Floyd A. *The Mobility and Classical impedance Analyses* edited in *American Institute of Physics Handbook.*

[14] Pease M.C. III (1965) *Methods of Matrix Algebra.*

[15] Baher H. (1984) *Synthesis of Electrical Networks.*

[16] Martinelli & Salerno (1986) *Fondamenti di Elettrotecnica.*

[17] Friedman B. (1966) *Principles and techniques of applied mathematics.*

[18] McLachlan N.W. *Theory of Vibrations.*

# VITA

Maurizio Porfiri was born to Gianni Porfiri and AnnaMaria Carullo on November 19, 1976 in Rome, Italy. He graduated from high school in 1995 at Liceo Scientifico Cavour, Rome.

Upon graduation from high school, Mr. Porfiri enrolled at Universita' di Roma "La Sapienza", Rome. He graduated with a Bachelor degree in Electrical Engineering in 1998.

In the fall 1999, Mr. Porfiri enrolled at Virginia Tech Polytechnic Institute and State University for master work in Engineering Science and Mechanics.