

ON COUPLE STRESS IN THE
MECHANICS OF MATERIALS

by

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Thesis submitted to the Graduate Faculty of the
Virginia Polytechnic Institute
in candidacy for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering Mechanics

January 5, 1966

Blacksburg, Virginia

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IV. INTRODUCTION

In the past century, classical continuum theory has been demonstrated to be applicable in numerous problems of solid and fluid mechanics. Classical elastic theory, for example, involving stresses, strains and displacements in isotropic, homogeneous bodies gives results which are quite consistent with experiment in a large number of problems for many types of real materials. More recently, metallurgists and physicists have applied anisotropic continuum theory to the study of the properties of single crystals.

Metals, it is well known, in the form of structural elements, plates, sheets, bars, etc., are polycrystalline in nature and, as such, can be considered homogeneous and isotropic only over long distances and large volumes compared to the average grain size or volume of the crystal. Thus the success of classical theory in metals would be expected when sections or characteristic sizes are large compared to grain size.

Classical theory, however, certainly would not be expected to tell the full story when we reduce characteristic dimensions to the order of magnitude of the microstructure of a metal or to some intermediate size where the effects of microstructure might be appreciable. Thus we observe phenomena called "size effects" in which we see deviations from classical theory for characteristically small dimensions. Such deviations would also be expected to be observed, as they are (1), when strain gradients are very large and stresses and strains vary greatly over short distances such as in notch effects in fatigue. As we might

imagine, these deviations from classical theory are more pronounced in materials with large grain size.

In 1936, Neuber (2) discussed such deviations from classical theory and proposed an empirical model to account for the effects in the theory of stress concentration. His analysis involved a new material constant, Neuber's constant, with units of length and showed a reduction in the classical stress concentration which depended on the magnitude of the constant compared to the radius or characteristic size of the notch. A great deal of data (1, 3, 4, 5) has been obtained using Neuber's model showing that variations from classical theory are most noticeable in soft, large grained metals.

Although Neuber's method of treating the problem of notch sensitivity was most useful from a practical point of view, it contained certain assumptions which are difficult to accept knowing the character of the microstructure of metals. Recently, Mindlin (6) and others (7, 8, 9, 10, 11, 12) have presented and discussed in detail a new continuum theory in which certain assumptions necessary in classical work have been omitted.

Basically, the new theory is a refinement of a more general theory introduced in 1909 by the brothers Cosserat (13). These investigators developed their work on the assumption that each material point in a continuum had six, not three, degrees of freedom. As a consequence of this formulation, a moment or couple per unit area became important in their equations and the force stress tensor was no longer necessarily symmetric. The original formulation has been treated by various authors

(see 12) and the present theory represents its simplest form.

In the present theory, we assume that we can divide our continuum into small parts but hold to the restriction, realistic in metals, that there exists some finite continuum limit at which the material no longer behaves as the macroscopic body. Naturally, as we approach the atomic level even in a single crystal or amorphous material, we would expect such a limit but we should certainly expect it sooner for a polycrystalline metal.

Thus, the smallest unit cube which behaves as the macroscopic body may have dimensions which are large enough that gradients of stress can not be considered negligible across a face. In order to insure equilibrium then, as we shall see later, we must consider that an element can apply couples as well as forces across its surface to its neighboring elements. This extension of the theory leads to a non-symmetric stress tensor as well as the inclusion of couple per unit area or couple stress terms in the elastic equations. New material constants for an isotropic body appear, in addition to those of ordinary elastic theory, one of which is prominent in plane strain and, like Neuber's constant, has units of length. Certain problems which have been presented (6, 12) using the new theory give results which qualitatively match observable phenomena in real materials. For example, Mindlin (6) presented the solution for the stress concentration factor (plane strain) for a round hole in an infinite plate showing a size effect similar, qualitatively, to experimental results.

Because of its apparent similarity on a qualitative basis to

observable phenomena and the similarity of this theory in certain parts to the empirical model of Neuber, a great deal of speculation has arisen (6, 10) as to the applicability of the new analysis in that grey area between classical continuum mechanics and the behavior of metals on a microstructural level.

The purpose of this investigation was to experimentally test the theory of Mindlin, Toupin, etc. to determine if it could be applied to analytically account for the discrepancies in the determination of elastic stress concentrations for small notches treated for so long by Neuber's empirical method and whether or not it could be applied to account for the effects of microstructure in a continuum theory. No experimental investigation had yet been reported in the literature to the best knowledge of the author.

V LIST OF SYMBOLS

a	radius of a circular stress raiser (inches)
D	flexural rigidity of a thin plate including the effect of couple stress (pound inches)
$\tilde{D}$	modified flexural rigidity parameter for a thin plate (inches ²)
E	modulus of elasticity
E_n	secant modulus based on nominal stress in a notched section
E_u	secant modulus at the ultimate strength of a material
f_i	body force per unit mass
g_i	body moment per unit mass
G	shear modulus (psi)
h	thickness of a thin plate
K_c	stress concentration factor determined by couple stress theory
K_n	stress concentration factor determined by Neuber's theory
K_f	stress concentration factor effective in fatigue
K	elastic stress concentration factor
L	gage width of a thin plate (inches)
ℓ	Mindlin's couple stress elastic constant (inches)
M_x, M_y, M_{xy}	moment stress resultants
m_{ij}	couple stress deviator tensor
N_i	unit normal vector to a surface
N_x, N_y, N_{xy}	normal stress resultants
Q_x, Q_y	shear stress resultants

S_t	torsional stiffness
u	strain energy per unit mass
u_1	displacement vector
α, β	scalar functions of y_1 and y_2
ϵ	width of a Neuber element
ϵ_{ij}	the small strain tensor
η	second couple stress elastic constant (dimensionless)
η_{ij}	curvature twist tensor
θ	angle of twist per unit length
μ_{ij}, μ_1	couple stress tensor and vector
ν	Poisson's ratio
ρ'	Neuber's constant
ρ_0	density of a continuum
ρ	radius of curvature of a notch
ψ	the couple stress function in plane strain
ϕ	the stress function in plane strain or the warping function in torsion
σ_{ij}, σ_1	the stress tensor and vector
σ_{ij}^S	the symmetric stress tensor
σ_{ij}^A	the anti-symmetric tensor
ω_1	the small rotation vector

VI THEORETICAL REVIEW

In classical continuum theory (14), as we consider the equilibrium of a region of a body, we shrink the size of this region small enough so that we can neglect the variation in stress across the face of elements and can replace the stress distribution over such a face by an equivalent force (see figure 6.1).

We then proceed to write moment and force equilibrium expressions which, in the absence of body forces and moments, yield the classical results:

$$\begin{aligned}\sigma_{ji,j} &= 0 \\ \sigma_{ij} &= \sigma_{ji}\end{aligned}\tag{6.1}$$

If, however, we yield to the restriction previously stated, that we can not shrink the size of any element beyond some finite continuum limit, we can no longer neglect, in general, the variation of stress across the face of such an element. Alternatively, even in classical theory, in a region of nearly infinite stress gradient, we would also have to consider the effect of stress gradient on the equilibrium of a region.

We can, however, still replace the stress distribution over a face of a small but finite element by its static equivalent (6). In the present theory of couple stress (6,7,9,10,11,12), it is assumed that the continuum limit is small enough that the stress variation across a face can be considered nearly linear. Thus, the stress distribution is replaced, in a manner of speaking, by a force and a couple (see figure 6.2).

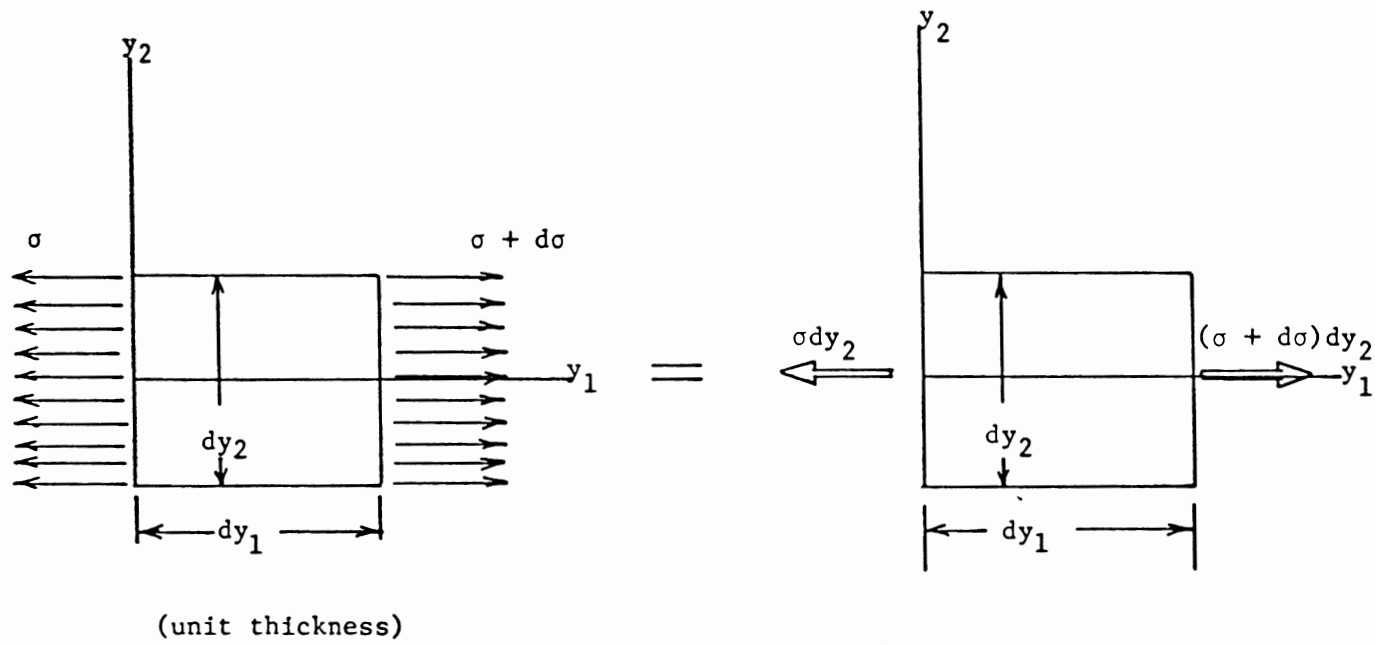


FIGURE 6.1 EQUIVALENT FORCE ON A SMALL ELEMENT

When considering the equilibrium of a region of a continuum then, we can not neglect the fact that one portion of a body may exert not only a force on its neighbors but also a couple. Thus we are lead to define a stress (force stress) and couple stress in the following manner:

$$\sigma_i = \lim_{\Delta S \rightarrow \Delta S'} \frac{F_i}{\Delta S} \quad 6.2$$

$$\mu_i = \lim_{\Delta S \rightarrow \Delta S'} \frac{C_i}{\Delta S} \quad 6.3$$

where σ_i is the stress vector
 μ_i is the couple stress vector
 F_i is a surface force
 C_i is a surface couple
 ΔS is a surface element, internal or external, oriented by the unit normal vector
 $\Delta S'$ is the smallest surface element to behave macroscopically

In addition, we shall define body moments per unit mass and body forces per unit mass as:

$$f_i = \lim_{\Delta V \rightarrow \Delta V'} \frac{B_i}{\rho \Delta V}$$

$$g_i = \lim_{\Delta V \rightarrow \Delta V'} \frac{M_i}{\rho_0 \Delta V}$$

where ΔV is a volume element
 ρ_0 is the average density of the continuum
 M_i is a body moment
 B_i is a body force
 $\Delta V'$ is the smallest volume to behave macroscopically

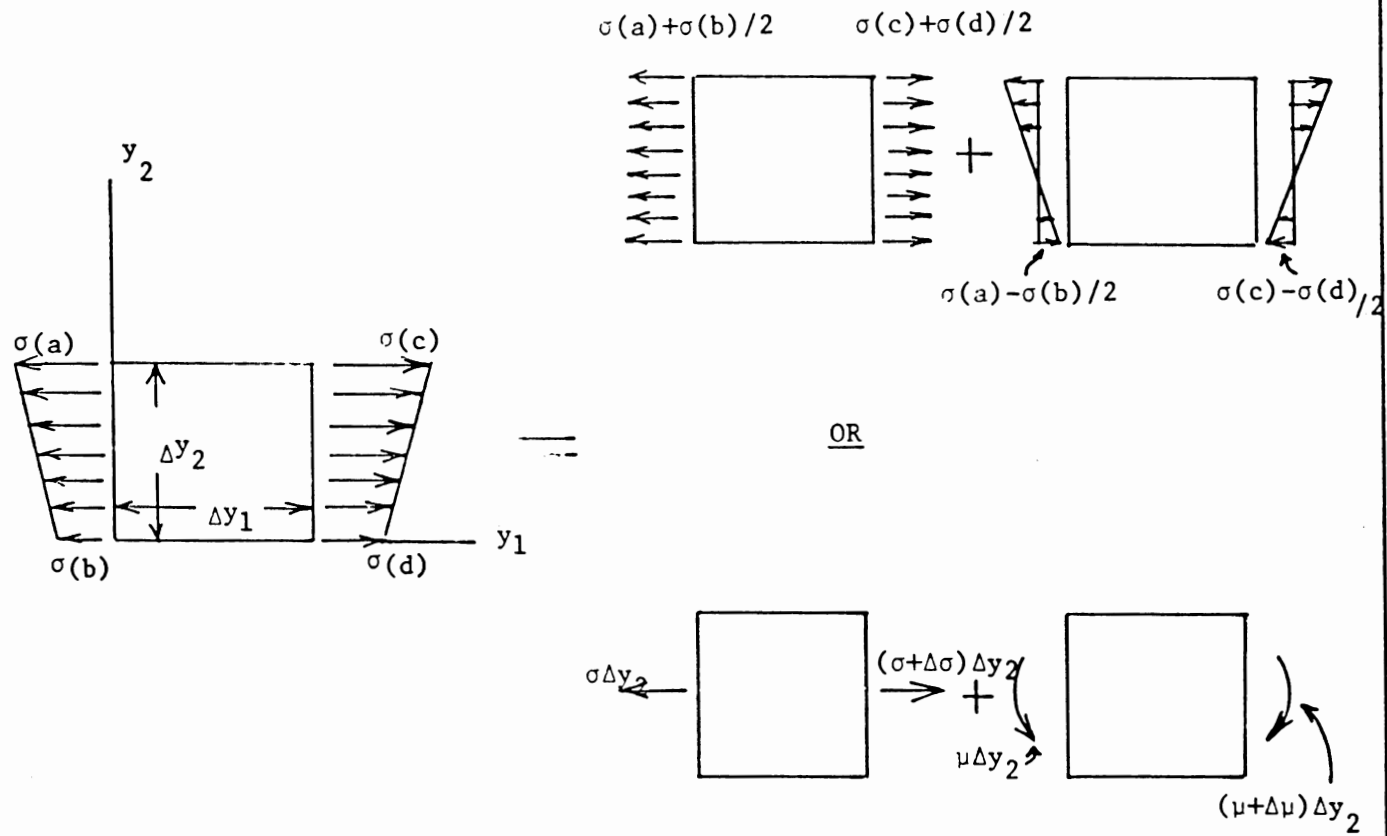


FIGURE 6.2 EQUIVALENT FORCE AND MOMENT ON A SMALL ELEMENT

Figure 6.3 shows a region of a body under the action of these quantities.

Upon writing moment and force equilibrium equations for this body (figure 6.3), we obtain the following expressions, omitting body forces and moments (7):

$$\sigma_{ji,j} = 0 \quad 6.4$$

$$\epsilon_{ijk} \sigma_{jk} + \mu_{li,l} = 0 \quad 6.5$$

where σ_{ji} is the stress tensor

μ_{ji} is the couple stress tensor (see appendix I)

Upon examination of equation 6.5, we can see that the stress tensor is no longer, in general, symmetric.

The two equilibrium equations can be combined (10) if we introduce the symmetric and anti-symmetric parts of the stress tensor as:

$$\sigma_{ij}^S = \frac{1}{2} (\sigma_{ij} + \sigma_{ji}) \quad 6.6$$

$$\sigma_{ij}^A = \frac{1}{2} (\sigma_{ij} - \sigma_{ji}) \quad 6.7$$

$$\sigma_{ij} = \sigma_{ij}^S + \sigma_{ij}^A \quad 6.8$$

After some algebra, 6.4 and 6.5 can be written, omitting body forces and moments:

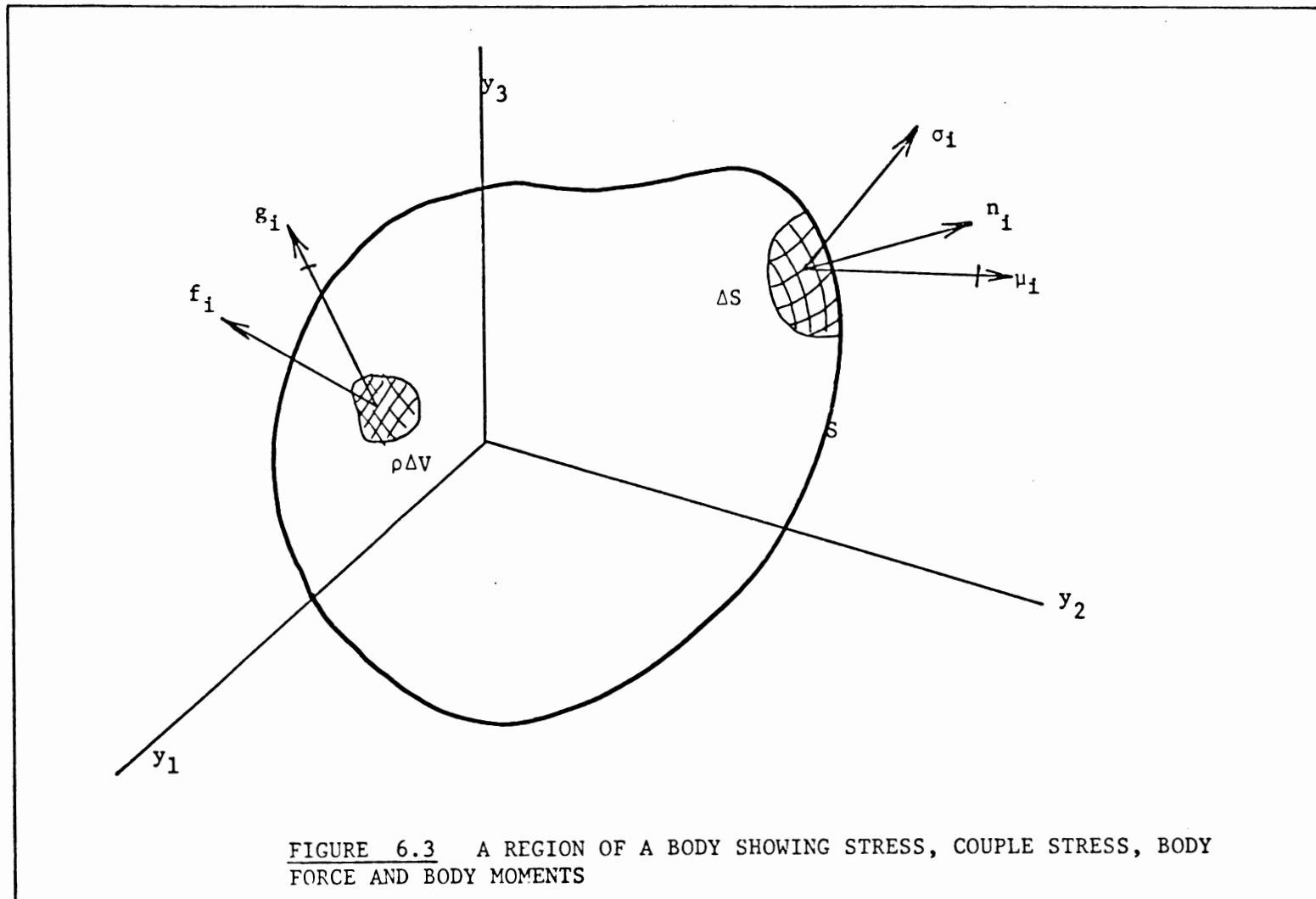
$$\sigma_{mn,m}^S - \frac{1}{2} \epsilon_{imn} m_{li,l} = 0 \quad 6.9$$

where: $m_{li} = \mu_{li} - \frac{1}{3} \mu_{kk} \delta_{li}$ is the couple stress deviator tensor

Detailed derivation and discussion of the elastic equations using couple stress theory is included in appendix I and the results stated here. The elastic equations become, for an isotropic body:

$$\sigma_{mn,m}^S - \frac{1}{2} \epsilon_{imn} m_{li,l} = 0 \quad 6.9$$

$$\omega_i = \frac{1}{2} \epsilon_{ijk} u_{k,j} \quad 6.10$$



$$\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad 6.11$$

$$\eta_{ij} = \omega_{i,j} \quad 6.12$$

$$\sigma_{ij}^S = 2G \left[\epsilon_{ij} + \frac{\nu}{1-2\nu} \epsilon_{kk} \delta_{ij} \right] \quad 6.13$$

$$m_{ji} = 4G\ell^2 (\eta_{ij} + \eta_{ji}) \quad 6.14$$

The unknowns involved in the problem are:

displacements	u_i	3	
strains	ϵ_{ij}	6	
rotations	ω_i	3	
curvatures	η_{ij}	8	$(\eta_{ii} \equiv 0)$
symmetric stress	σ_{ij}^S	6	
couple stress deviator	m_{ji}	8	$(m_{ii} \equiv 0)$

Thus, we have a total of 34 equations and 34 unknowns in the general, three dimensional couple stress elasticity problem compared to 15 in classical theory.

Stokes (7) and Koiter (10) have shown that the strain energy, displacements, rotations and boundary conditions in this formulation are independent of the trace of the couple stress tensor (μ_{kk}) and that this quantity can be chosen at will. In addition, Stokes (7), discussed in detail the role of the anti-symmetric stress tensor which remains indeterminate from the elastic equations.

Problems in the three dimensional theory have been solved by Mindlin (9), Toupin (8) and others dealing with wave propagation, centers of dilatation, concentrated forces, etc., but the most active research and thought provoking problems have been concerned with two dimensional (plane strain) formulations (6, 15, 16, 17).

With the restrictions of plane strain, i.e.

$$u_1 = u_1(y_1, y_2)$$

$$u_2 = u_2(y_1, y_2)$$

$$u_3 = \text{constant}$$

the elastic equations reduce (6, 7) to:

$$\nabla^4 \phi = 0 \quad 6.15$$

$$\nabla^2 \psi - \ell^2 \nabla^4 \psi = 0 \quad 6.16$$

where ϕ and ψ are stress functions such that:

$$m_{13} = \partial \psi / \partial y_1 \quad 6.17$$

$$m_{23} = \partial \psi / \partial y_2 \quad 6.18$$

$$\sigma_{11} = \partial^2 \phi / \partial y_2^2 - \partial^2 \psi / \partial y_1 \partial y_2 \quad 6.19$$

$$\sigma_{22} = \partial^2 \phi / \partial y_1^2 + \partial^2 \psi / \partial y_1 \partial y_2 \quad 6.20$$

$$\sigma_{12}^S = \partial^2 \phi / \partial y_1 \partial y_2 + \frac{1}{2} \left(\frac{\partial^2 \psi}{\partial y_1^2} - \frac{\partial^2 \psi}{\partial y_2^2} \right) \quad 6.21$$

A detailed derivation and discussion of these equations is given in Appendix II.

Using such a formulation, Dietrich (15) presented a numerical solution, assuming a value of the constant l , for the problem of concentrated forces on a semi-infinite half space (plane strain) and found that in general stress gradients are dispersed by couple stress effects. The magnitude of this dispersion depended on the magnitude of the constant l compared to the distance, y , from the loading. Figure 6.4 shows the difference he obtained in σ_y assuming a value of $l = .02$ inches for steel.

Weitsman (16) and Hartranft (17) presented a solution and discussion of the effect of couple stress on the stress concentration around a

circular inclusion in a field of uniaxial tension. They found that the effect of couple stress depended upon the ratio of the shear modulus of the inclusion to the shear modulus of the matrix. When the ratio was unity there was no effect since the classical matrix exerted no rotation on the particle. If the ratio was less than unity (soft inclusion) the classical theory was shown to be an upper bound and that the maximum stress was reduced in the matrix by an amount depending on the constant l for the matrix and the inclusion.

Perhaps the most important problem yet solved, was that of a round hole in an infinite plate (plane strain) for a field of uniaxial tension. It was presented by Mindlin (6,9) and by Stokes (7) using another method. For a hole of radius A in a field of tension S , it was found that the stress concentration factor K_c depended on the value of the characteristic length of the material (l) and also on Poisson's ratio. Classical theory, of course, shows a value of three regardless of the elastic constants. In addition, Mindlin's problem showed a variation in the stress concentration factor depending on the relationship of the size of the hole to the constant l . Figure 6.5 shows the results of Mindlin's calculation for uniaxial tension and figure 6.6 shows similar results, also from Mindlin, for the stress concentration in pure shear.

Observing the results for tension, we notice that for values of a and l near the same order of magnitude, couple stress has a significant lowering effect on the stress concentration factor. As a becomes large relative to l , however, the stress concentration factor approaches

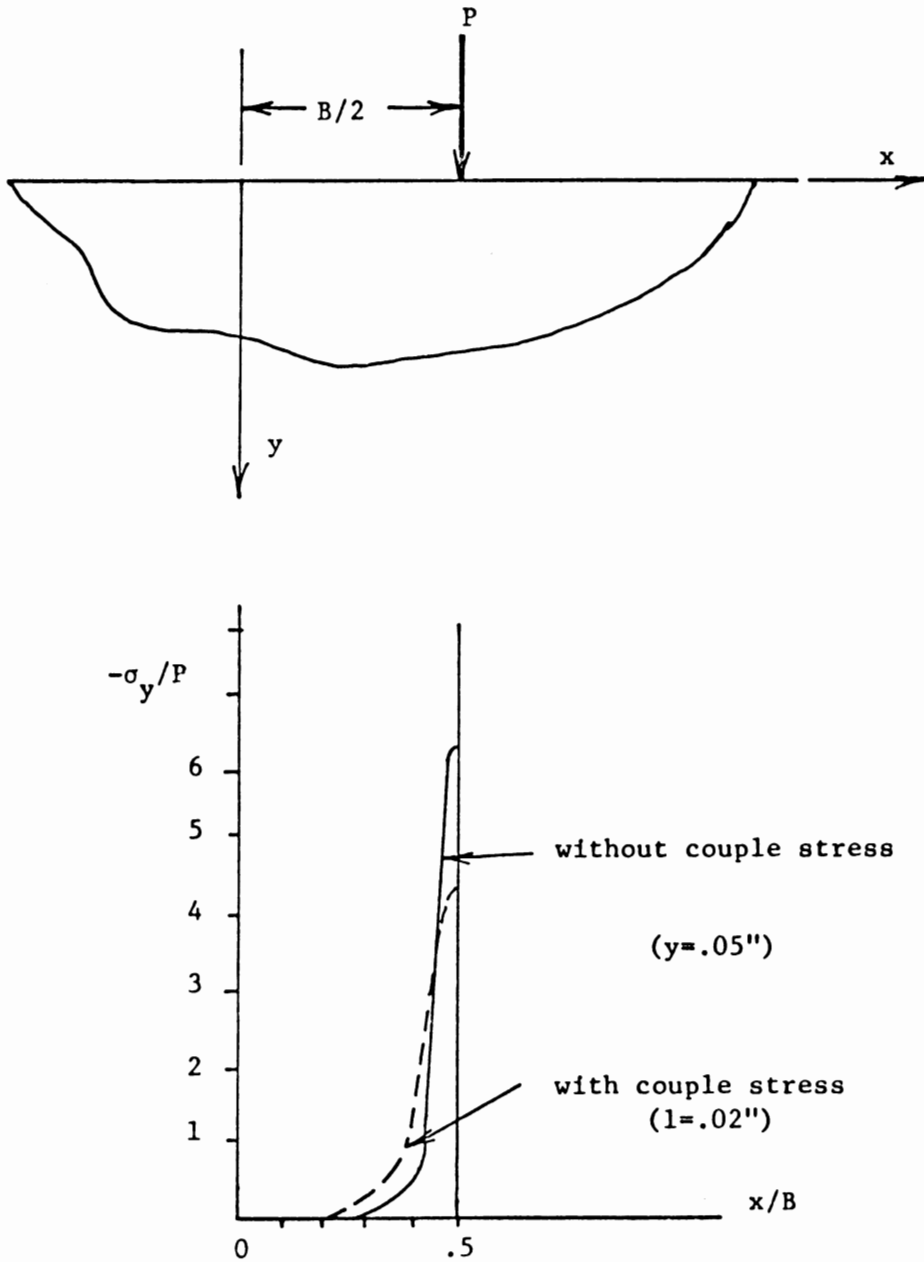


FIGURE 6.4 THE INFLUENCE OF COUPLE STRESS ON THE COMPRESSION OF A SEMI-INFINITE HALF SPACE (FROM 15)

the classical result. Here then is an analytical result which conforms, qualitatively, with an observed size effect.

We notice that as a/l approaches zero, the stress concentration factors, rather than approach unity as one would expect for the limiting case of no hole, approach a somewhat higher value. Mindlin (6,9) pointed out, however, that his solution is restricted to orders of magnitude of $a \geq 1$ by certain assumptions made in his analysis. His results then, can not be applied for $a \ll 1$. The effect can still be great, for example, for $a/l = 3$, the stress concentration factor can vary from 2.4 to 2.6 compared to 3 from classical theory.

Neuber (2), in his classical work on the theory of notch stresses, recognized that an analysis based only on classical elastic theory could not supply adequate models in the case of sharp or characteristically small notches. He pointed out that for the case of a shallow notch whose radius of curvature was 1/10000th of its depth, classical theory predicts a stress concentration factor of over 100 which, of course, is not consistent with a logical analysis of the problem. In addition, he noted that experimental results in such cases always gave results which were reduced from classical predictions. He chose to approach the problem from a semi-empirical point of view.

In Neuber's analysis, it is assumed that the material is made up of tiny, but finite "elements" across which the stress can not vary (3). Classical theory would give good results in regions of moderate stress gradient but where the gradient of stress was strong, such as near a pointed notch, or where the size of the discontinuity

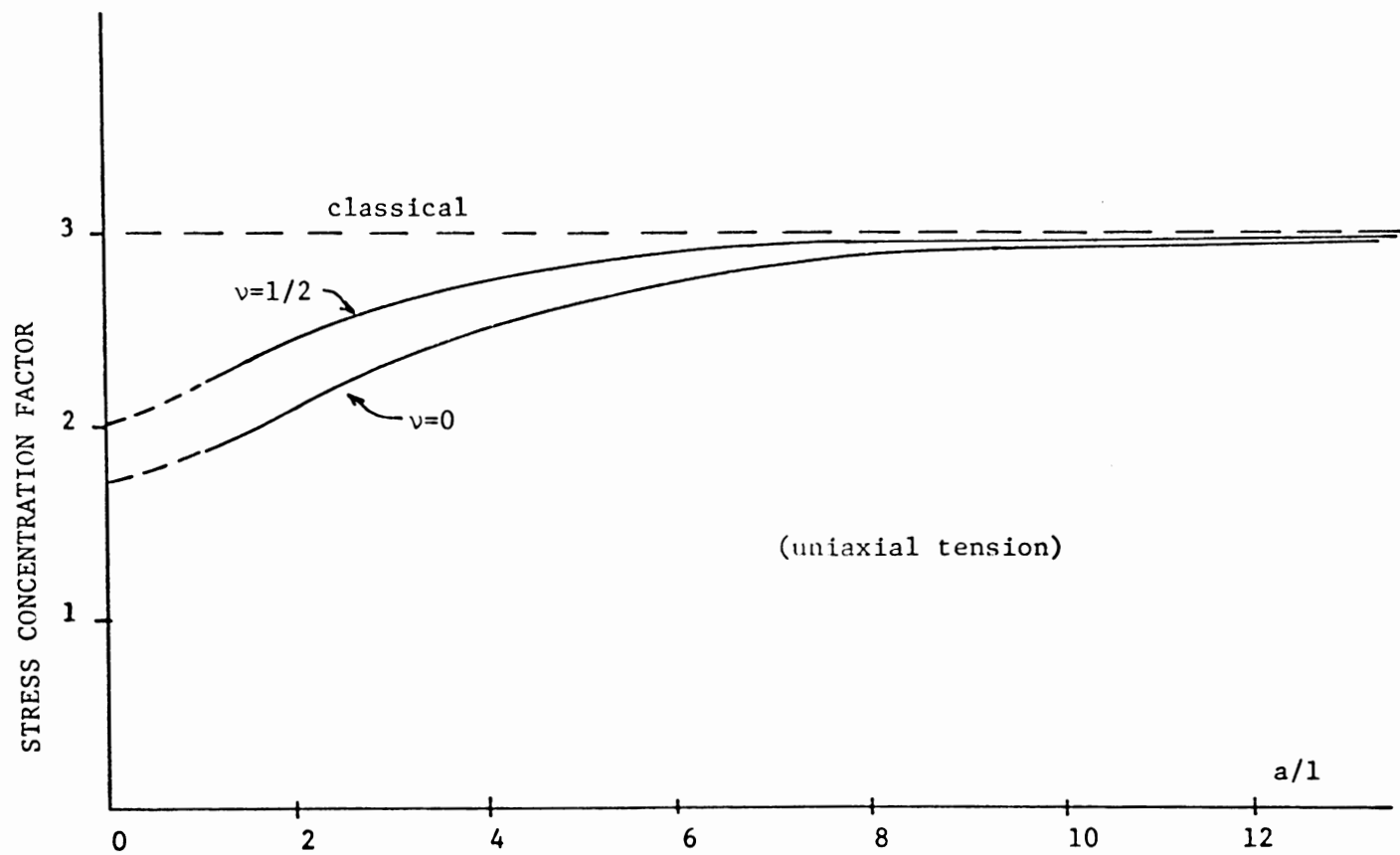


FIGURE 6.5 STRESS CONCENTRATION FACTOR FOR A ROUND HOLE IN AN INFINITE PLATE WITH COUPLE STRESS (FROM 6)

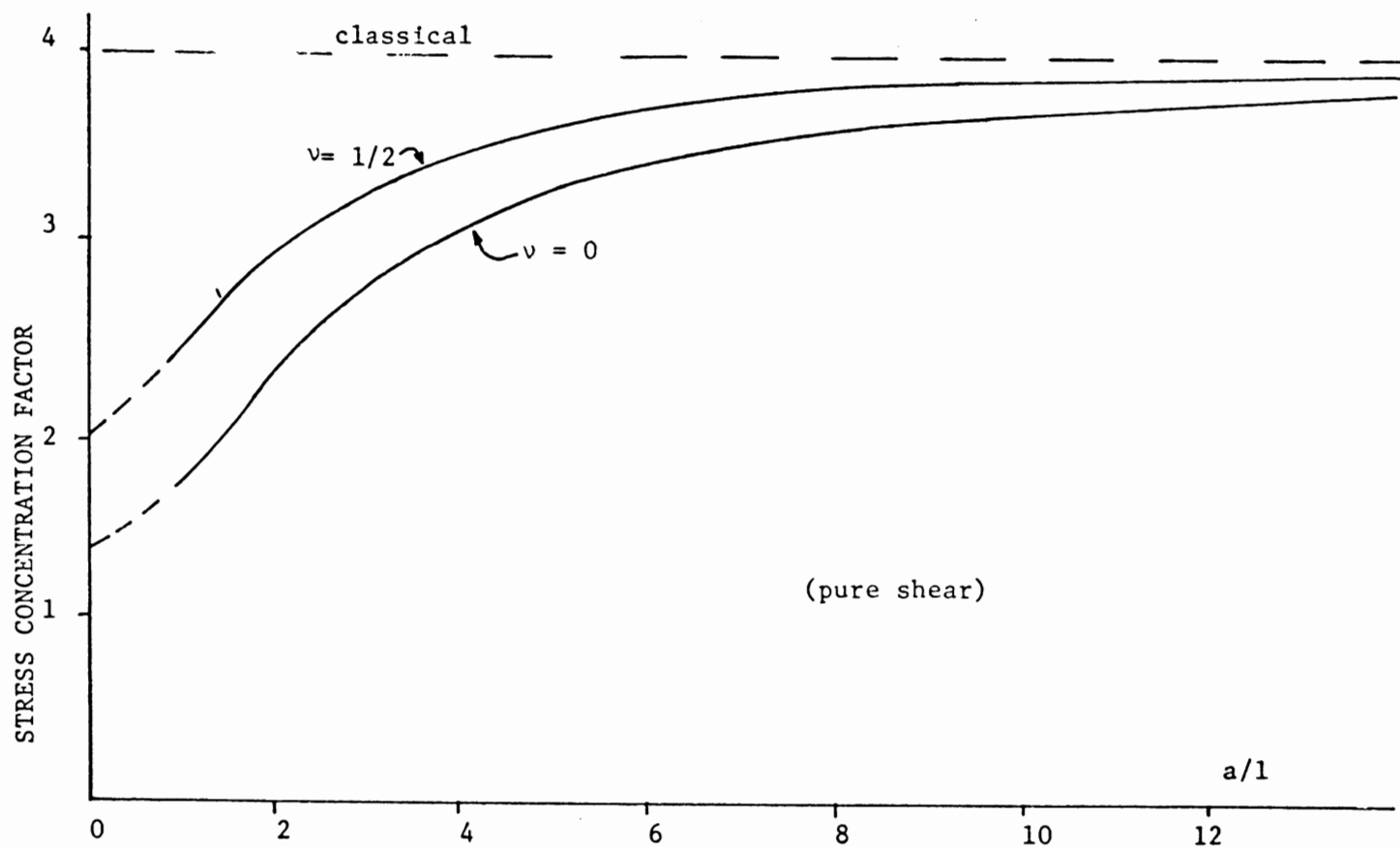


FIGURE 6.6 STRESS CONCENTRATION FACTOR FOR A ROUND HOLE IN AN INFINITE PLATE WITH COUPLE STRESS (FROM 6)

was near the size of the element, classical analysis would fail. These elements have no real relationship to the microstructure of the material (3) but are simply mathematical in nature.

In order to insure an equilibrium of forces in cases where classical theory fails then, it was necessary (2) to assume that an average of the stress predicted by classical theory acts across the elements in the region of severe stress gradient. Thus at the root of a notch or at the edge of a small stress raisor, we have:

$$\sigma_{\max} = \frac{1}{\epsilon} \int_0^{\epsilon} \sigma(\zeta) d\zeta \quad 6.22$$

where ϵ is a new material constant - the width of an elementary unit.

ζ is a variable which increases from the edge of the discontinuity (see figure 6.7).

As an example of such a calculation, the stress concentration factor for a round hole (plane strain) in an infinite plate under uni-axial tension will be presented (see figure 6.8).

From Timoshenko (18), the classical expression for σ_{θ} is:

$$\sigma_{\theta} = \frac{S}{2} \left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2} \left(1 + \frac{3a^4}{r^4}\right) \cos 2\theta \quad 6.23$$

The maximum stress thus occurs at $\theta = \pi/2$ and is given by:

$$\sigma_{\theta}|_{\pi/2} = \frac{S}{2} \left(2 + \frac{a^2}{r^2} + \frac{3a^4}{r^4}\right)$$

Following expression 6.22 we have, using Neuber's theory, letting $\zeta=r$:

$$(\sigma_{\theta})_{\max} = \frac{1}{\epsilon} \int_a^{\epsilon+a} \frac{S}{2} \left(2 + \frac{a^2}{r^2} + \frac{3a^4}{r^4}\right) dr$$

or:

$$\frac{(\sigma_{\theta})}{S} \max = K_n = \frac{2\epsilon + a[2 - (1+\frac{\epsilon}{a})^{-1} - (1+\frac{\epsilon}{a})^{-2}]}{2\epsilon}$$

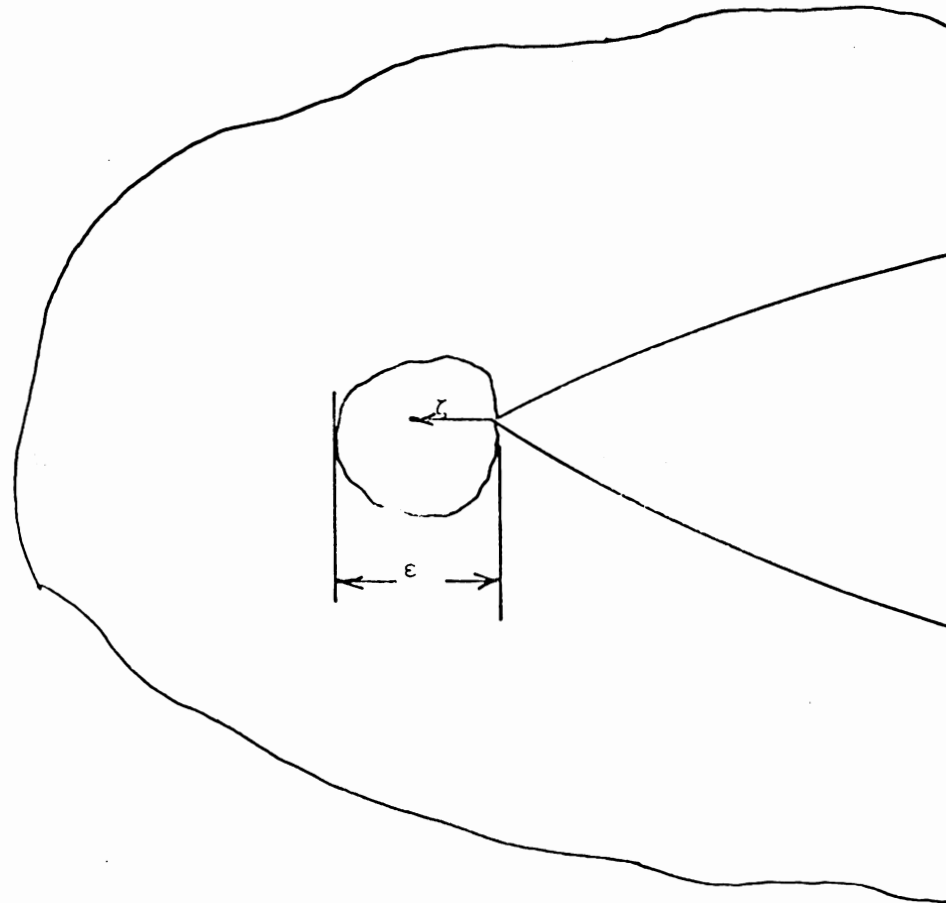


FIGURE 6.7 A NEUBER ELEMENT

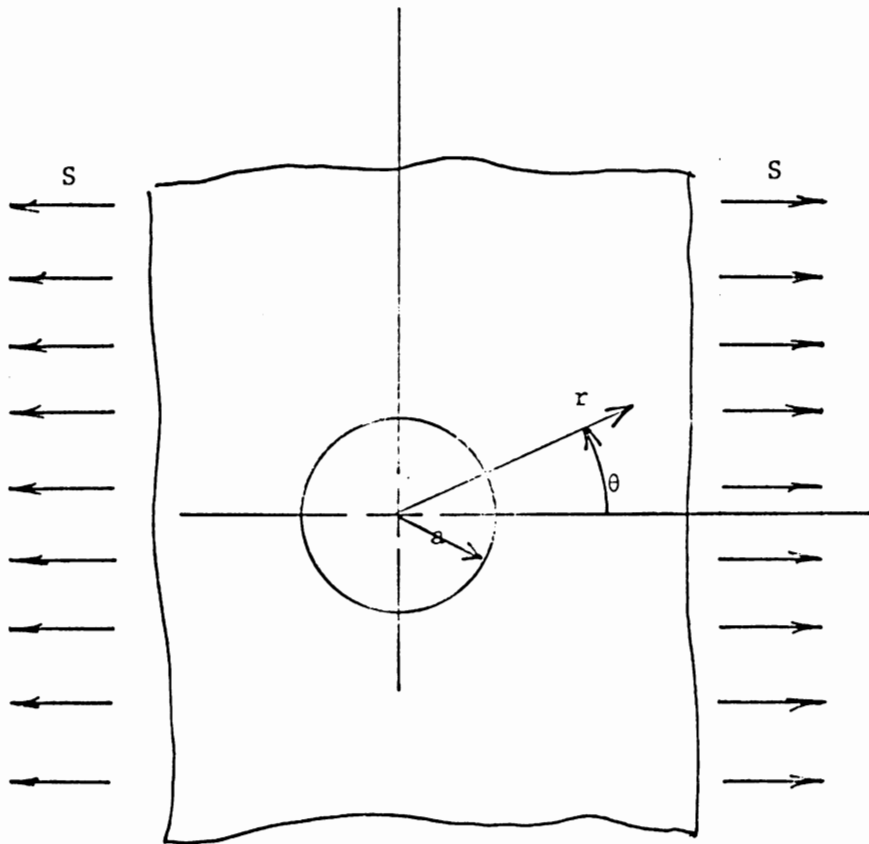


FIGURE 6.8 ROUND HOLE IN AN INFINITE PLATE UNDER
UNIAXIAL TENSION

This can be written:

$$K_n = 3 - \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n \left[\frac{3(4) \dots (2+n)}{n!} \right] \left(\frac{\epsilon}{a} \right)^{n-1} \quad 6.24$$

where we have used the relationships:

$$\left(1 + \frac{\epsilon}{a} \right)^{-1} = 1 + \sum_{n=1}^{\infty} (-1)^n \left(\frac{\epsilon}{a} \right)^n$$

$$\left(1 + \frac{\epsilon}{a} \right)^{-3} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(3)(3+1) \dots (3+n-1)}{n!} \left(\frac{\epsilon}{a} \right)^n$$

Upon examination of the limits of this expression, we find that as a/ϵ approaches zero, the stress concentration factor approaches one which we would logically assume. As ϵ approaches zero, K_n approaches the classical value of three. Figure 6.9 shows the variation of K_n with a/ϵ for a round hole in an infinite plate under uni-axial tension.

For more general pointed notches, Neuber (2) and others (3, 19) have presented the expression for stress concentration factor, based on the above analysis as:

$$K_n = 1 + \frac{K_E - 1}{1 + \sqrt{\rho'/\rho}} \quad 6.25$$

where K_E is the classical elasticity factor, ρ' is a material constant ($\epsilon/2$) and ρ is the notch radius.

More recent investigations (4,5) have included the effect of plastic action on the stress concentration factor. The final expression frequently used is:

$$K_u = 1 + (K_n - 1) \frac{E_u}{E_n} \quad 6.26$$

where K_u is the stress concentration factor for the ultimate tensile stress (assuming failure when the maximum stress reaches the ultimate strength of the material)

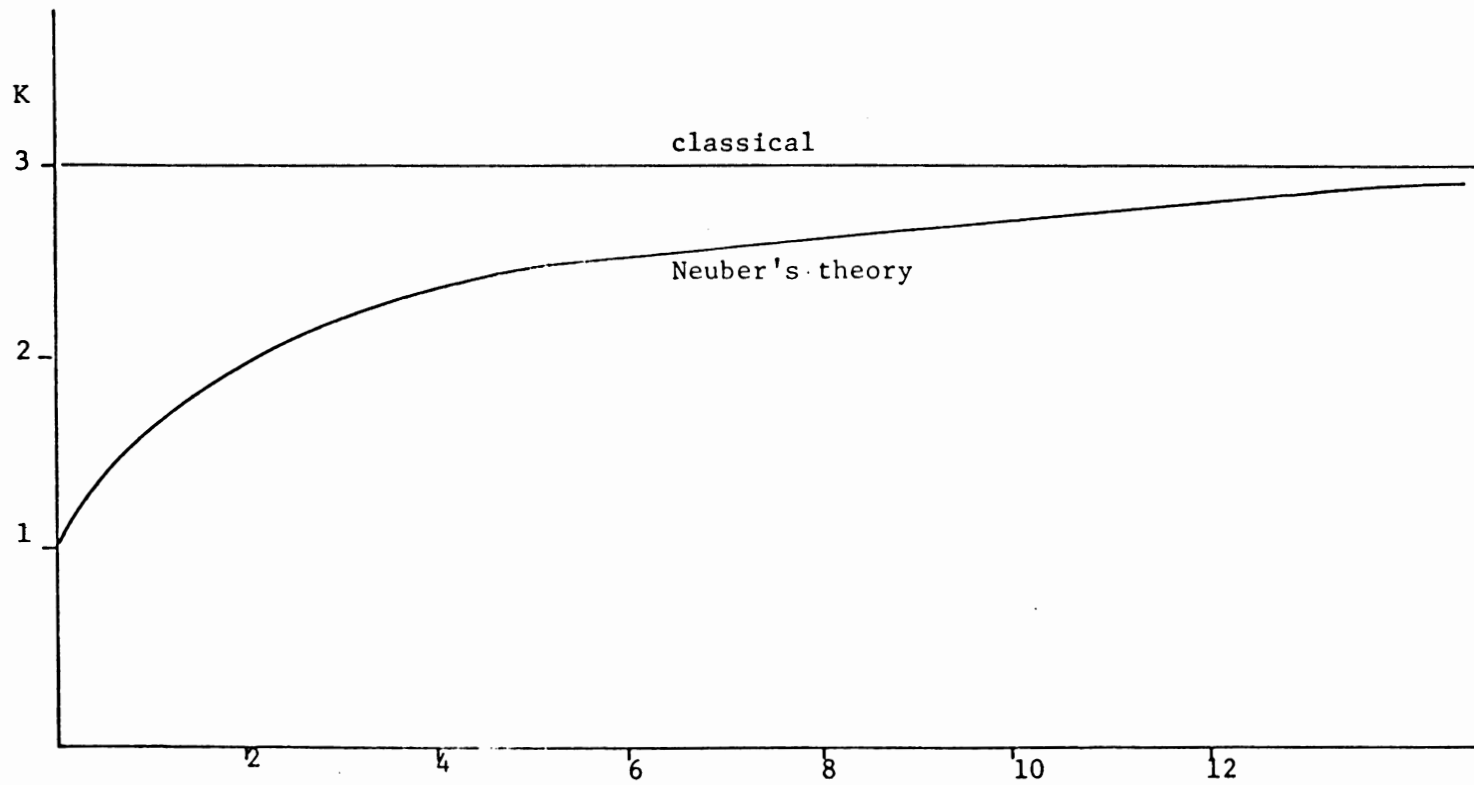


FIGURE 6.9 NEUBER'S CALCULATION OF STRESS CONCENTRATION FACTOR FOR A ROUND HOLE

E_u is the secant modulus at the ultimate tensile stress

E_n is the secant modulus for the nominal section stress

Using such an analysis, a great deal of experimental results (4,5,1) have been obtained for the static stress concentration factor including size effect and for the stress concentration factor active in fatigue. Kuhn and Figge (5) point out that, in light of data obtained for aluminum alloys, the classical stress concentration factor K_F is useful only for notches whose radii are in excess of .1 inches and that below that limit, for engineering purposes, Neuber's analysis is indicated. Radii smaller than .1 inches would appear on many actual parts.

Figure 6.10 shows a typical example of the value of Neuber's constant as a function of the ultimate strength of wrought aluminum alloys (from 5). It is interesting to note that the constant, and therefore the size effect, is greater for the softer alloys.

Figure 6.11 illustrates the agreement between Neuber's analysis and the stress concentration factor effective in fatigue (from 5) for a round hole in aluminum sheet under axial loading. It is of interest to note the statistical nature of the data shown in this figure and the fact that the agreement with Neuber's theory apparently is not as good for very small holes. Further comment will be made on this point in the discussion section to follow.

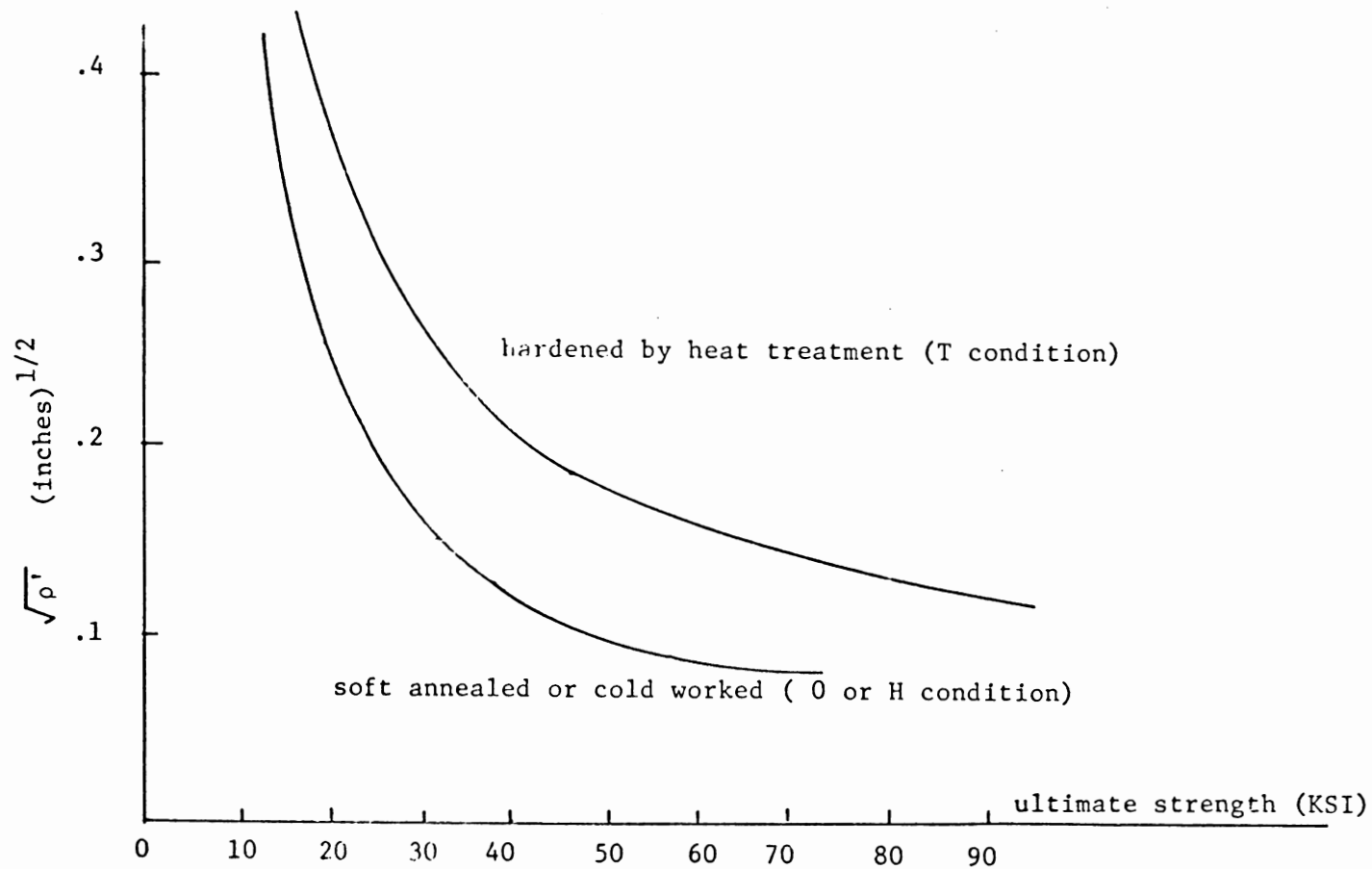
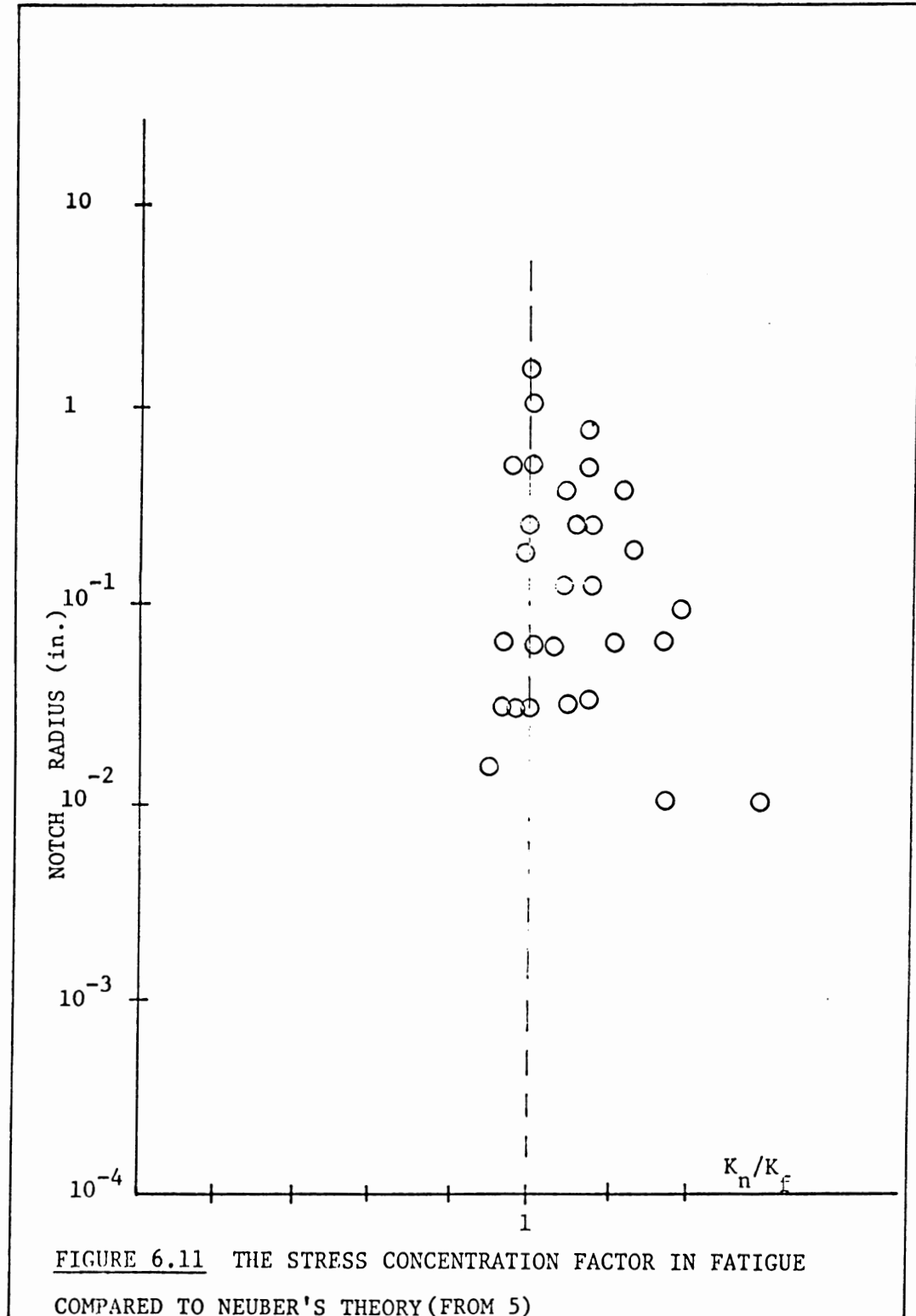


FIGURE 6.10 NEUBER CONSTANT FOR WROUGHT ALUMINUM ALLOYS (FROM 5)

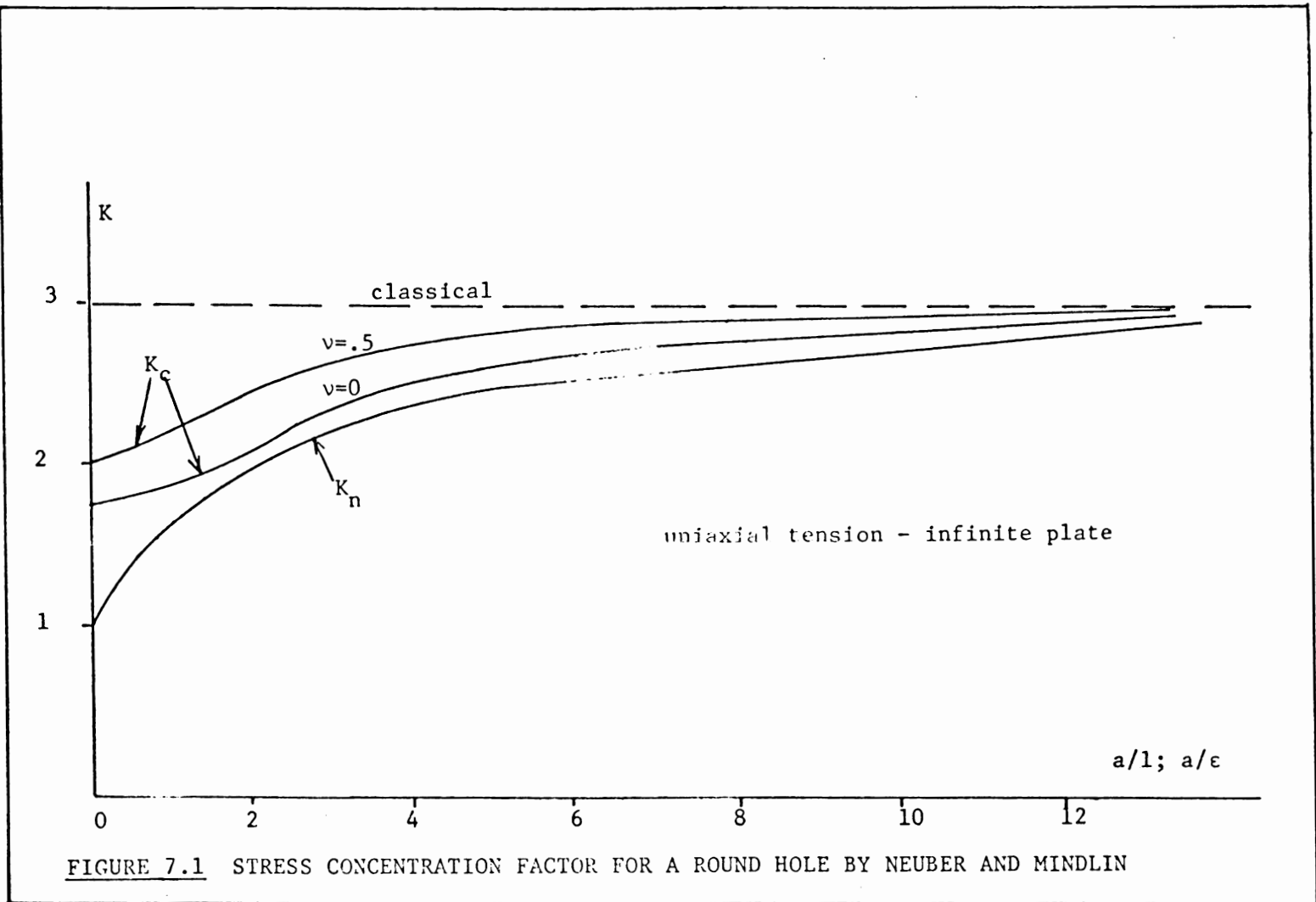


VII CORRELATION OF THEORY

We now find ourselves with experimental data and an empirical model for the behavior of notch sensitive metals on the one hand and an analytical theory which apparently gives good qualitative agreement on the other. Both the empirical and analytical approaches involve a new material constant which has units of length and both show a size effect, reduction of stress concentration factor, which depends on the relationship of the material constant to the characteristic size of the specimen (notch radius). Figure 7.1 shows the stress concentration factors K_C and K_N plotted together versus a/ℓ and a/ϵ .

The similarity of reasoning behind the two approaches becomes even more clear when we consider the manner in which the stress gradient is treated. Neuber (2) assumed that across his "element" stress was a constant and replaced the gradient in that region by an average computed from classical theory. Mindlin (6) and his contemporaries replaced the stress gradient across the element in their concept of a continuum with microstructure by an average, in a sense, and then added a couple to maintain local equilibrium. Figure 7.2 demonstrates these concepts.

Obviously, the next logical step would be to examine the analytical formulation to ascertain whether its effects are of the proper order of magnitude in real materials to match or account for the experimental observations. To begin this task, let us determine what value the analytical constant, ℓ , would have to have in order to explain the elastic portion of the size effect.



From figure 6.10 we will take a moderate value of Neuber's constant for aluminum and substitute it into equation 6.25 to determine K_n for a round hole in an infinite plate in terms of the radius, a , of the hole. Then, choosing an appropriate hole radius, we can easily determine what K_n would be. If the couple stress theory would account for the effect, then the theoretical value of K_c should be near that calculated using the empirical methods.

Following such a scheme, we select $\rho = .02$ inches, and $\sqrt{\rho'} = .10$ inches^{1/2}. From equation 6.25, with $K_E = 3$, we have

$$K_n \approx 1 + \frac{2}{1 + \frac{.10}{.14}} = 2.2$$

Now, referring to figure 7.1, let us pick off a value of a/ℓ which would give a value of K_c near 2.2. For $\nu = 1/3$, it is easy to see that a/ℓ would have to be nearly unity to yield such a value.

Thus, we are lead to the conclusion that the constant ℓ would have to be near the order of magnitude of .02 inches in aluminum to adequately explain the elasticity of the size effect analytically. Koiter (10) was lead to a similar conclusion for the order of magnitude of ℓ using data from reference 1. Following a similar procedure and using data for steel in reference 1, it can be shown that ℓ should be on the order of magnitude of .01 inches in soft steel.

We must now devise a method for testing these quantities to determine the validity of our comparison. Direct measurements of the stress concentration factor in very small stress raisers would be quite difficult in metals and photoelastic materials, which have no

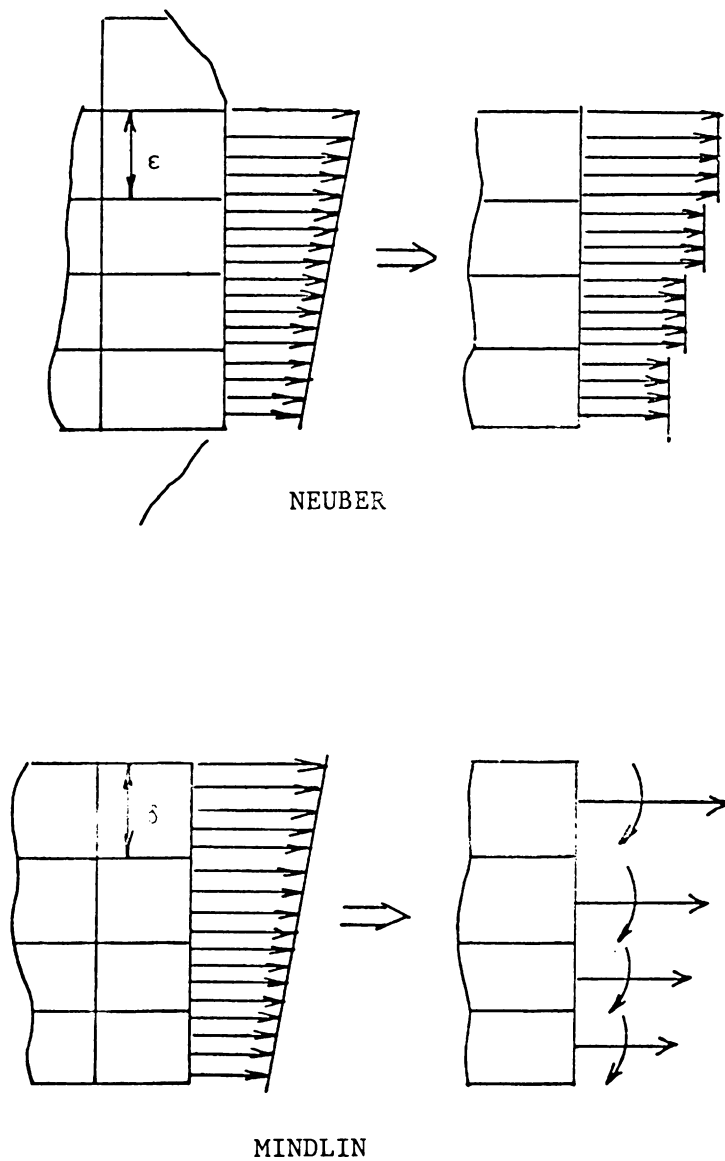


FIGURE 7.2 COMPARISON OF THE NEUBER AND MINDLIN
CONCEPTS

granular structure as such, would probably not exhibit a pronounced couple stress effect (6). Such measurements would probably not add a great deal to the knowledge of such factors applicable to this analysis. Mindlin (9), suggested determination of ℓ by experimental consideration of thickness-shear modes of vibration in plates. He points out, however, that only the high modes would exhibit a measurable effect of couple stress and such an investigation was deemed impractical as a first step in the determination of the order of magnitude of the constant.

Koiter (10) presented a three dimensional solution for the torsion of a cylindrical shaft which involved two couple stress constants (see Appendix I) and showed the effect of couple stress to increase as the diameter of the shaft was decreased. His expression for the torsional stiffness was:

$$S_t = \frac{\pi}{32} G d^2 [d^2 + 48 (1+\eta)\ell^2] \quad 7.1$$

where d = diameter of the shaft

If the torsional rigidity of a series of fine wires of the same material were divided by d^2 and then plotted versus d^2 , the resulting straight line would have an intercept which would be a measure of $(1 + \eta)\ell^2$. The need here, however, was for the separate constants and this separation could not be accomplished using this configuration only. (It was felt that all experiments in this first series should be performed on specimens of the same general configuration and fabrication history since it was not certain what effect, if any, these factors might have on the constants.)

Koiter (10) also presented the plane stress solution for the pure bending of a flat plate to a cylindrical surface. This analysis was extended by Hoffman (21) to include other pure bending problems and showed that the effect of couple stress increased as the ratio of h (plate thickness) to ℓ decreased (see chapter VIII).

The plate bending experiments indicated by Koiter and Hoffman, however, left little room for control of boundary conditions, practical instrumentation, and simple procedure. It was decided to develop a thin plate theory, in the frame-work of the couple stress analysis, similar to classical thin plate theory, in order to allow wide choice of problems for the experimental program.

VIII THIN PLATE THEORY

If the thickness, h , of a body is very small when compared to its other dimensions and if such a body has a middle surface which is initially a plane, it is called a thin plate. We shall now develop a simplified theory of elasticity to deal with the deformation of such bodies in the frame work of couple stress theory. We will make use of the basic assumptions used in classical thin plate theory (22,23,24). That is:

1. The deflection is limited to magnitudes which are small in the classical sense ($< h/2$).
2. Normals to the middle surface before deformation remain straight lines normal to the middle surface after deformation.
3. The normal stress normal to the middle surface is small when compared with the other stresses.
4. The effect of normal strain (ϵ_z) is negligible in comparison with the deflection of the middle surface (i.e. $w = w(x,y)$).
5. The middle surface is a neutral surface of the force stress.
6. The strains ϵ_{xz} and ϵ_{yz} are small compared to the other strains but are not zero (see 22).

In addition we shall assume:

7. The curvatures η_{xz} and η_{yz} are small compared to the other curvatures.

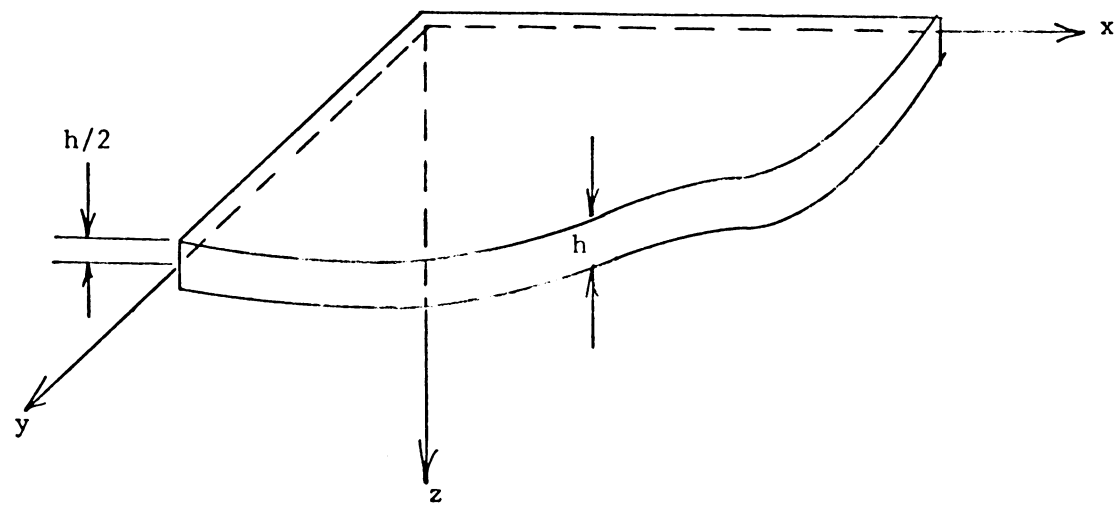


FIGURE 8.1 A THIN PLATE ORIENTED IN SPACE

8. The couple stress m_{33} is zero or negligible compared to the other couple stresses.

9. The trace, μ_{kk} , of the couple stress tensor is zero.

Assumption seven can be shown to be consistent with number six from classical theory. In assumption number eight we state that there is no local twist along the "z" axis which is also consistent with the classical concept of a thin plate. A more detailed discussion of the assumptions is given in Appendix III.

Following Wang (22), based on the first six assumptions we have:

$$u = -z \frac{\partial \omega}{\partial x} = \text{displacement in the "x" direction}$$

$$v = -z \frac{\partial \omega}{\partial y} = \text{displacement in the "y" direction}$$

$$\omega = \omega(x,y) = \text{displacement in the "z" direction} \quad 8.1$$

Thus, from the strain displacement relations, 6.11:

$$\epsilon_x = -z \frac{\partial^2 \omega}{\partial x^2}$$

$$\epsilon_y = -z \frac{\partial^2 \omega}{\partial y^2}$$

$$\epsilon_{xy} = -z \frac{\partial^2 \omega}{\partial x \partial y} \quad 8.2$$

and, from 6.10:

$$\omega_x = \frac{1}{2} \left[\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial z} \right] = \frac{\partial \omega}{\partial y}$$

$$\omega_y = \frac{1}{2} \left[\frac{\partial u}{\partial z} - \frac{\partial \omega}{\partial x} \right] = - \frac{\partial \omega}{\partial x}$$

$$\omega_z = \frac{1}{2} \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] = 0 \quad 8.3$$

Thus, the curvature-twist relations, 6.12, yield

$$\eta_x = \frac{\partial \omega_x}{\partial x} = \frac{\partial^2 \omega}{\partial x \partial y}$$

$$\eta_y = \frac{\partial \omega_y}{\partial y} = - \frac{\partial^2 \omega}{\partial x \partial y}$$

$$\begin{aligned}\eta_{xy} &= \frac{\partial \omega x}{\partial y} = \frac{\partial^2 \omega}{\partial y^2} \\ \eta_{yx} &= \frac{\partial \omega y}{\partial x} = - \frac{\partial^2 \omega}{\partial x^2}\end{aligned}\tag{8.4}$$

Upon substitution into the stress-strain relations (6.13) and the couple stress-curvature relations (6.14), making use of our assumptions we have:

$$\begin{aligned}\sigma_x^S &= \frac{-zE}{1-\nu^2} \left(\frac{\partial^2 \omega}{\partial x^2} + \nu \frac{\partial^2 \omega}{\partial y^2} \right) \\ \sigma_y^S &= \frac{-zE}{1-\nu^2} \left(\frac{\partial^2 \omega}{\partial y^2} + \nu \frac{\partial^2 \omega}{\partial x^2} \right) \\ \tau_{xy}^S &= \frac{-zE}{1+\nu} \frac{\partial^2 \omega}{\partial x \partial y}\end{aligned}\tag{8.6}$$

and:

$$\begin{aligned}m_x &= 4 G \ell^2 (1+\eta) \frac{\partial^2 \omega}{\partial x \partial y} \\ m_y &= - 4 G \ell^2 (1+\eta) \frac{\partial^2 \omega}{\partial x \partial y} \\ m_{xy} &= - 4 G \ell^2 \left(\frac{\partial^2 \omega}{\partial x^2} - \eta \frac{\partial^2 \omega}{\partial y^2} \right) \\ m_{yx} &= 4 G \ell^2 \left(\frac{\partial^2 \omega}{\partial y^2} - \eta \frac{\partial^2 \omega}{\partial x^2} \right)\end{aligned}\tag{8.6}$$

Furthermore, if the trace of the couple stress tensor is taken to be zero as mentioned in chapter VI, it can be easily shown (see Equation A1.2) that:

$$\tau_{xy}^A = 0$$

Where we have taken throughout the development that the body forces and couples are zero.

Figure 8.2 shows the orientation of these stresses and couple stresses on an element of the plate.

Now, following the procedure of classical theory, we shall define

and formulate the resultants of the stresses shown in the above figure. Thus:

$$\begin{aligned}
 M_y &= \int_h \sigma_y Z \, dZ - \int_h m_{yx} \, dZ \\
 M_x &= \int_h \sigma_x Z \, dZ + \int_h m_{xy} \, dZ \\
 M_{yx} &= \int_h \tau_{xy} Z \, dZ + \int_h m_y \, dZ \\
 M_{xy} &= \int_h \tau_{xy} Z \, dZ - \int_h m_x \, dZ
 \end{aligned} \tag{8.7}$$

and:

$$\begin{aligned}
 Q_x &= \int_h \tau_{xz} \, dZ \\
 Q_y &= \int_h \tau_{yz} \, dZ
 \end{aligned} \tag{8.8}$$

Where m_x , m_y , m_{xy} , m_{yx} are moments per unit length along the edge of an element and Q_x , Q_y are forces per unit length along an edge. Figure 8.3 shows these stress resultants acting on an element of a plate.

Substituting into 8.7 from 8.5 and 8.6 and integrating as indicated, after a bit of algebra, we have:

$$\begin{aligned}
 M_y &= \frac{-Eh^3}{12(1-\nu^2)} \left[\frac{\partial^2 \omega}{\partial y^2} + \nu \frac{\partial^2 \omega}{\partial x^2} \right] - 4G \ell^2 h \left[\frac{\partial^2 \omega}{\partial y^2} - \eta \frac{\partial^2 \omega}{\partial x^2} \right] \\
 M_x &= \frac{-Eh^3}{12(1-\nu^2)} \left[\frac{\partial^2 \omega}{\partial x^2} + \nu \frac{\partial^2 \omega}{\partial y^2} \right] - 4G \ell^2 h \left[\frac{\partial^2 \omega}{\partial x^2} - \eta \frac{\partial^2 \omega}{\partial y^2} \right] \\
 M_{xy} = m_{yx} &= - \left[\frac{Eh^3}{12(1+\nu)} + 4G \ell^2 (1+\eta) h \right] \frac{\partial^2 \omega}{\partial x \partial y}
 \end{aligned} \tag{8.9}$$

Now, by summing moments on the element shown in figure 8.3 we can obtain:

$$\begin{aligned}
 \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= 0 \\
 \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y &= 0
 \end{aligned} \tag{8.10a}$$

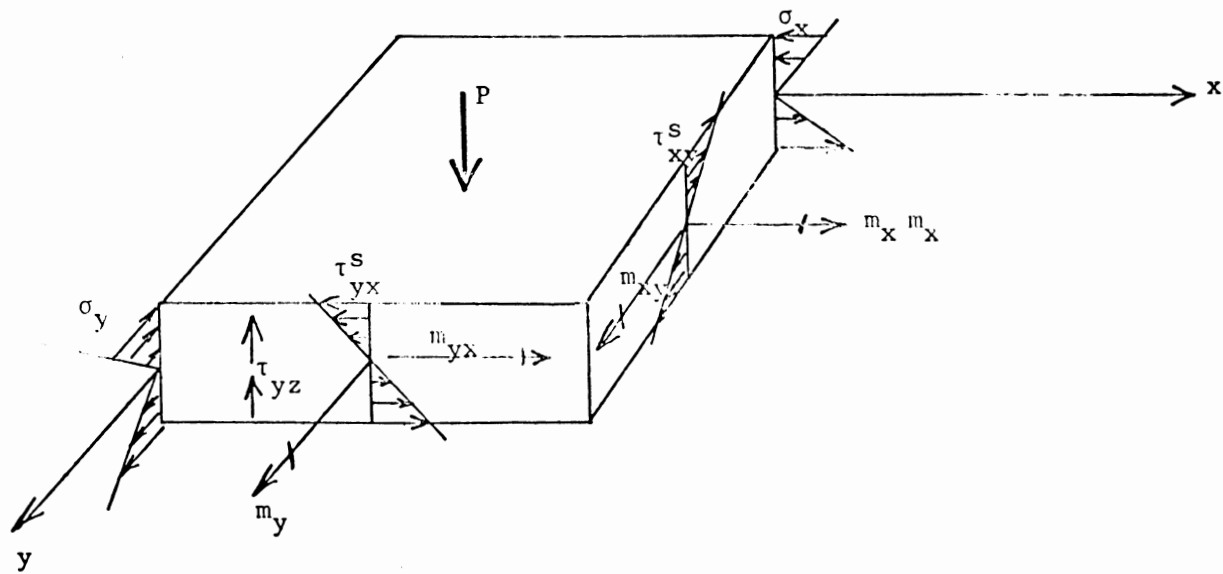


FIGURE 8.2 DISTRIBUTION OF POSITIVE STRESS ON AN ELEMENT OF A THIN PLATE

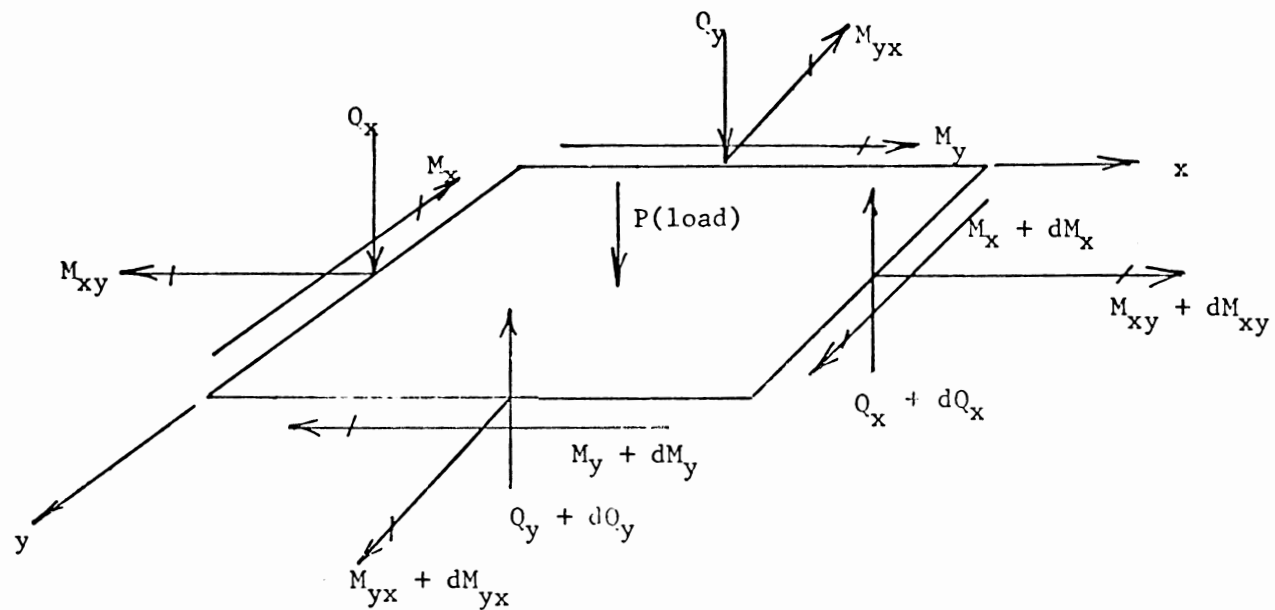


FIGURE 8.3 POSITIVE STRESS RESULTANTS ON AN ELEMENT OF A THIN PLATE

and from the equilibrium of forces in the "z" direction

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + P(x,y) = 0 \quad 8.10b$$

where $P(x,y)$ is the loading normal to the middle surface.

From 8.10a we can compute the shear resultants by substitution from 8.9. Thus, after simplifying:

$$\begin{aligned} Q_x &= -\left[\frac{Eh^3}{12(1-\nu^2)} + 4G \ell^2 h\right] \frac{\partial}{\partial x} \nabla^2 \omega \\ Q_y &= -\left[\frac{Eh^3}{12(1-\nu^2)} + 4G \ell^2 h\right] \frac{\partial}{\partial y} \nabla^2 \omega \end{aligned} \quad 8.11$$

Inserting this result into 8.10b, we find the governing equation for our theory:

$$-\left[\frac{Eh^3}{12(1-\nu^2)} + 4G \ell^2 h\right] \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \nabla^2 \omega + P(xy) = 0$$

or,

$$\left[\frac{Eh^3}{12(1-\nu^2)} + 4G \ell^2 h\right] \nabla^4 \omega = P(x,y) \quad 8.12$$

Using the theory we have formulated, we shall now determine the flexural rigidity of a thin plate bending to a circular cylindrical surface and of one in pure twist and compare the results to those of Koiter (10) and Hoffman (21) who obtained their results from a plane stress approach.

For the case of cylindrical bending, let us examine an infinite strip oriented in such a way that its two edges are parallel to the "y" axis (see figure 8.4a). If a uniform moment, $M_x = -M$, is applied along the two edges we can write, from classical theory (24):

$$\omega = \frac{x^2}{2R} \quad 8.13$$

where R is the radius of curvature of the plate. Inserting this deflection into 8.9 we have:

$$M_x = -\left[\frac{Eh^3}{12(1-\nu^2)} + 4G \ell^2 h\right] \frac{1}{R} = -M \quad 8.14$$

The flexural rigidity is defined in terms of the moment and curvature as:

$$M_x = -D \frac{1}{R}$$

Thus :

$$D = \left[\frac{Eh^3}{12(1-\nu^2)} + 4G \ell^2 h\right] \quad 8.15$$

which is identical to Koiter's results for pure bending.

In pure twist (see figure 8.4b), we can write, according to classical theory:

$$\omega = \frac{xy}{R} \quad 8.16$$

Inserting this relation into 8.9 we have:

$$M_{xy} = -\left[\frac{Eh^3}{12(1+\nu)} + 4G \ell^2 (1+\eta)h\right] \frac{1}{R}$$

But in the case of pure twist, the flexural rigidity (23) is related to the moment and curvature as:

$$M_{xy} = -D (1-\nu) \frac{1}{R}$$

Thus, the rigidity for pure twist is given by:

$$D = \frac{Eh^3}{12(1-\nu^2)} + \frac{4G\ell^2(1+\eta)h}{1-\nu} \quad 8.17$$

Which matches Hoffman's result in the case of pure twist.

There appears then to be good agreement between our plate theory and more exact solutions in the restricted problem above. In our formulation, however, we have a wide choice of problems which we can solve easily and adopt for experiments. A few such problems are included near the end of this chapter.

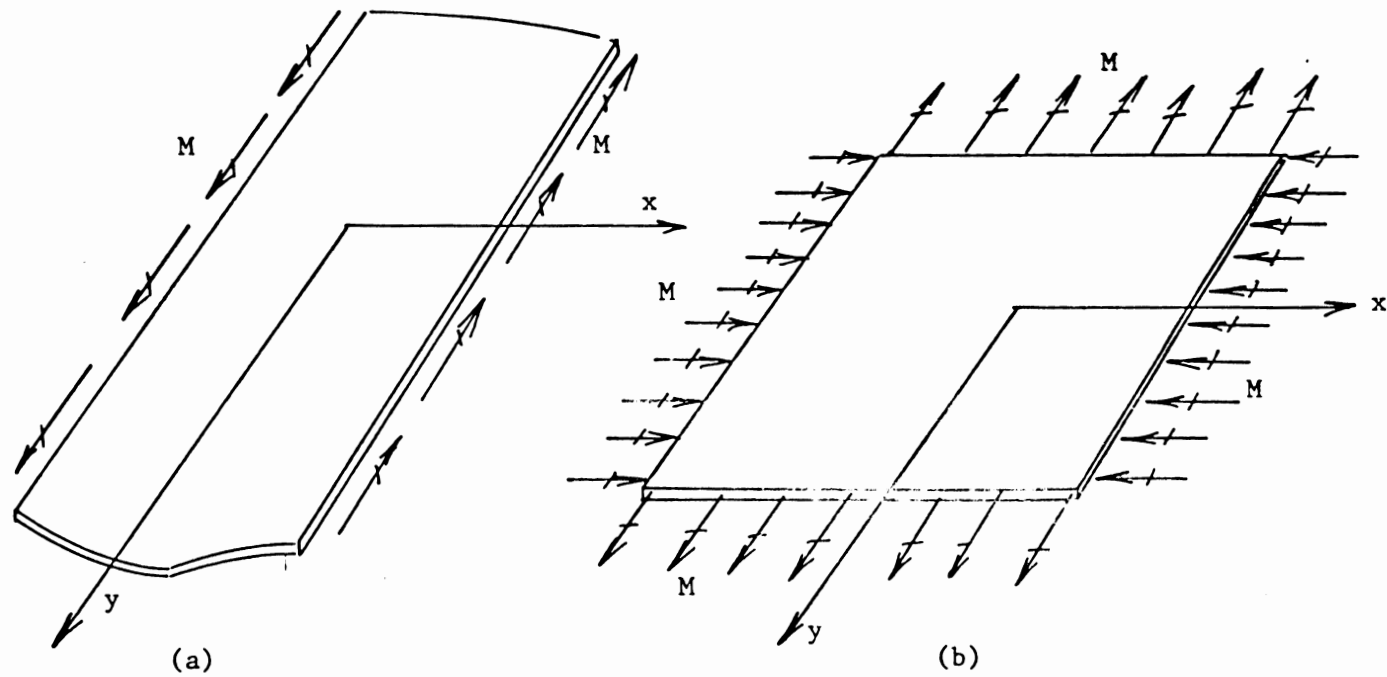


FIGURE 8.4 PURE BENDING AND PURE TWIST OF THIN PLATES

To complete our thin plate theory, we shall relax the assumption that the middle surface is a neutral surface of the force stress and allow that surface to deform in its plane as well as bend out of it. Thus, as in the classical analysis (22), we define the membrane stress resultants as:

$$\begin{aligned} N_x &= \int_h \sigma_x dz \\ N_y &= \int_h \sigma_y dz \\ N_{xy} &= N_{yx} = \int_h \tau_{xy} dz \end{aligned} \quad 8.18$$

Where N_x , N_y , and N_{xy} are forces per unit length in the plane of the middle surface.

Considering the equilibrium of forces then, we have from figure 8.5 (see 24):

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= 0 \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + P(x,y) + N_x \frac{\partial^2 \omega}{\partial x^2} + N_y \frac{\partial^2 \omega}{\partial y^2} + 2N_{xy} \frac{\partial^2 \omega}{\partial x \partial y} &= 0 \end{aligned} \quad 8.19$$

The equilibrium of moments yields a result the same as before since the additional force terms lie in the middle surface of the plate. Thus, substituting from 8.11 which still holds into 8.19 we obtain the governing equation for the present case:

$$\left[\frac{Eh^2}{12(1-\nu^2)} + 4G l^2 h \right] \nabla^4 \omega = P(x,y) + N_x \frac{\partial^2 \omega}{\partial x^2} + N_y \frac{\partial^2 \omega}{\partial y^2} + N_{xy} \frac{\partial^2 \omega}{\partial x \partial y} \quad 8.20$$

We shall restrict ourselves to cases in which N_x , N_y , and N_{xy} are known.

Thus the governing equations for thin plate theory considering

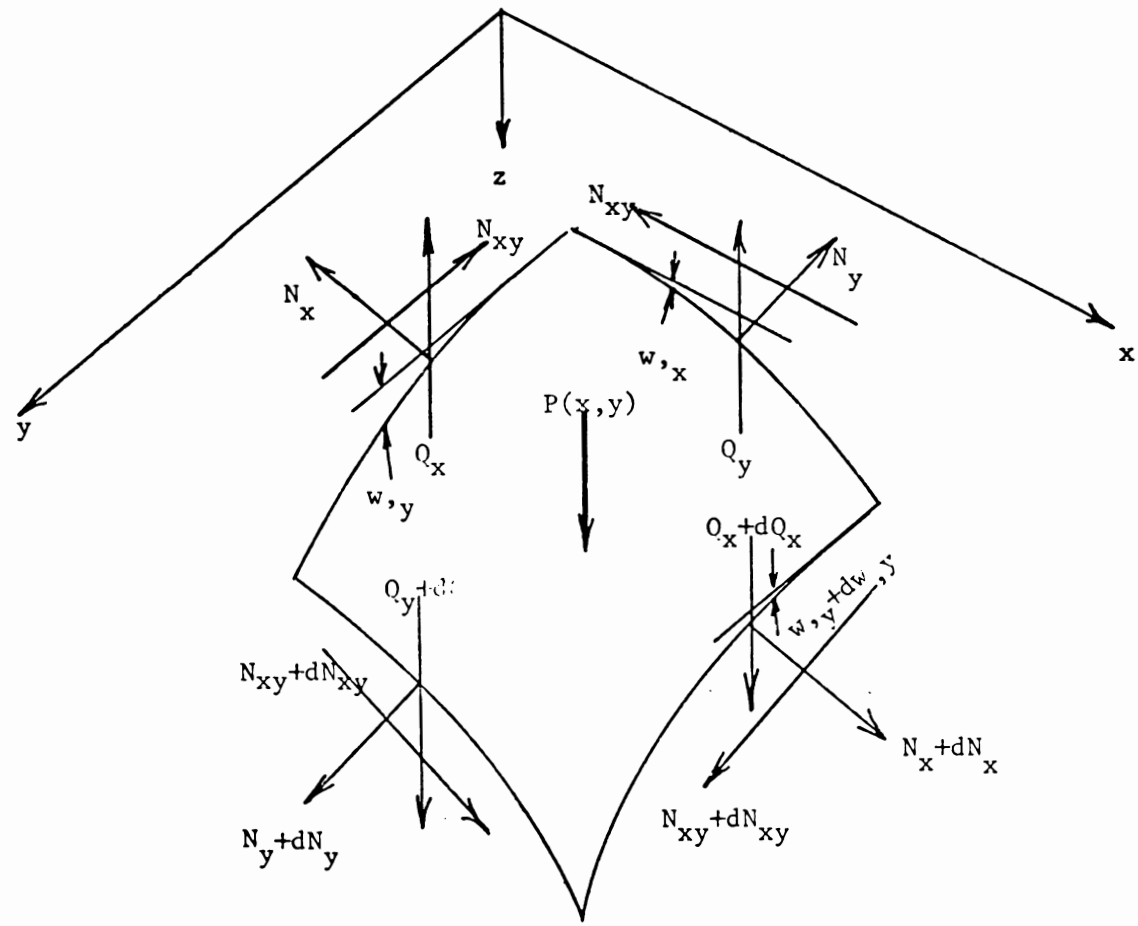


FIGURE 8.5 AN ELEMENT OF A THIN PLATE SHOWING MEMBRANE FORCES

couple stress effects are similar to those of classical theory with only the constants involved being changed. For this reason, the vast number of solutions known in classical thin plate theory can easily be modified to suit our couple stress formulation. Bending configurations would be the same with only the magnitude of deflections and slopes changed. The general effect of the couple stress terms, as we have used them, would be, as we can easily see from an examination of the governing equations, to stiffen a given plate to an extent which depends on the magnitude of the thickness, h , compared to the material constants of the formulation.

The boundary conditions which we shall apply will be based on the displacements and stress resultants as they were in the classical case. This is valid since the stress resultants include the integrated effects of the couple stresses as well as the force stresses. Thus we have (see 23), for a given condition at the edge, $x = \text{Constant} = c$, of a plate:

$$\begin{aligned}
 & \text{Simple Support} \quad \left\{ \begin{array}{l} \omega(b,y) = 0 \\ \frac{\partial^2 \omega}{\partial x^2}(b,y) = 0 \end{array} \right. \\
 & \text{Fixed Edge} \quad \left\{ \begin{array}{l} \omega(b,y) = \frac{\partial \omega}{\partial x}(b,y) = 0 \end{array} \right. \\
 & \text{Prescribed Deflection} \quad \omega(b,y) = F(y) \qquad \qquad \qquad 8.21 \\
 & \text{and Moment} \quad \left\{ \begin{array}{l} \frac{\partial^2 \omega}{\partial x^2} = - \frac{G(y)}{\left[\frac{Eh^3}{12(1-\nu^2)} + 4G \ell^2 h \right]} \end{array} \right. \\
 & \text{Free Edge} \quad \left\{ \begin{array}{l} M_x(b,y) = 0 \\ V'_x(b,y) = 0 \end{array} \right.
 \end{aligned}$$

Where:

$$V'_x = Q_x + \frac{\partial M_{xy}}{\partial y} = - \frac{Eh^3}{12(1-\nu^2)} \left[\frac{\partial^3 \omega}{\partial x^3} + (2-\nu) \frac{\partial^3 \omega}{\partial x \partial y^2} \right] - 4G\ell^2 h \left[\frac{\partial^3 \omega}{\partial x^2} + (2+\nu) \frac{\partial^3 \omega}{\partial x \partial y^2} \right]$$

is a modified Kirchhoff Shear arrived at using the same analysis that is used in the usual theory.

In order to fix the effect of the couple stress formulation in our thoughts and to arrive at certain relations which will be useful later, we shall consider a few simple examples.

Example 1: As a first example, let us consider the buckling of a simply supported rectangular plate (see figure 8.6) in the frame work of our theory. Following (24) with the inclusion of our modifications we obtain:

$$\left[\frac{Eh^3}{12(1-\nu^2)} + 4G\ell^2 h \right] \nabla^4 \omega + N \frac{\partial^2 \omega}{\partial x^2} = 0$$

Assuming a solution for the deflection as:

$$\sum_p \sum_q W_{pq} \sin \frac{p\pi x}{a} \sin \frac{q\pi y}{b}$$

which satisfies the simple support boundary conditions on the edges of the plate, and substituting into the governing equation, we find that, in order to have a non-trivial solution for the deflection, we must have

$$N = \frac{\pi^2}{b^2} \left(\frac{Eh^3}{12(1-\nu^2)} + 4G\ell^2 h \right) \left(\frac{pb}{a} + \frac{q^2 a}{pb} \right)^2$$

But, the critical value occurs, as in the classical problem at $q = \ell$, $Pb/a = \ell$. Thus, the critical buckling load would be:

$$P_{cr.} = \frac{4\pi^2}{b^2} \left[\frac{Eh^3}{12(1-\nu^2)} + 4G\ell^2 h \right]$$

which is greater by $16 G \ell^2 h \pi^2 / b^2$ than the classical prediction. The mode shapes remain the same. If ℓ is small but not negligible compared

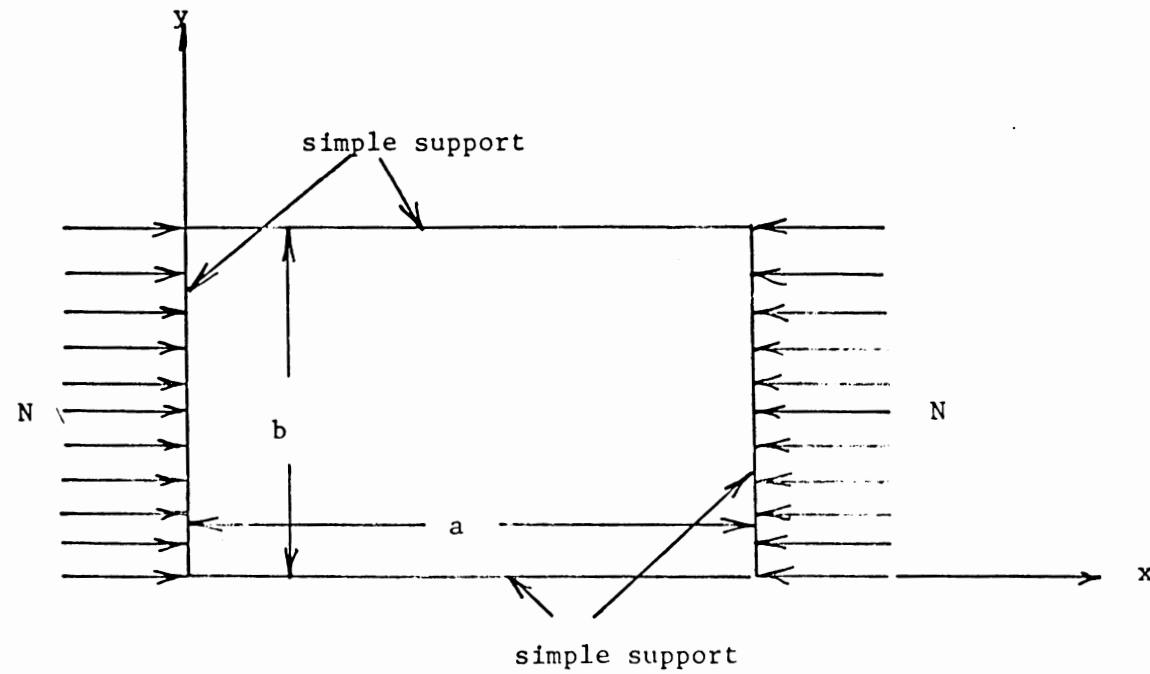


FIGURE 8.6 SIMPLY SUPPORTED PLATE WITH A BUCKLING LOADING

to h then we can see that the couple stress term would add a small stiffening effect which, conceivably, could have been overlooked in experiments with real plates. Of course, if $\ell = 0$ or is very small compared to h , classical theory would hold.

Example 2: Next, we will consider the uniaxial bending of a thin plate simply supported on one edge and under the action of a uniform moment on its opposite edge (see figure 8.7a). The boundary conditions are:

$$\omega(L, y) = 0$$

$$\omega(0, y) = 0$$

$$\frac{d^2\omega}{dx^2}(L, y) = 0$$

$$\frac{d^2\omega}{dx^2}(0, y) = \frac{M_o}{\frac{Eh^3}{12(1-\nu^2)} + 4G\ell^2h}$$

and the governing equation reduces to:

$$\frac{d^4\omega}{dx^4} = 0$$

We shall assume a polynomial solution in the form

$$\omega = A_x^3 + B_x^2 + C_x + F$$

and substitute into the boundary conditions. Doing this we find:

$$\omega = \frac{M_o}{12\left\{\frac{Eh^3}{12(1-\nu^2)} + 4G\ell^2h\right\}} \left\{ \frac{-2x^3}{L} + 6x^2 - 4Lx \right\} \quad 8.22$$

Thus we see that in this problem couple stress produces a stiffening effect on the plate as before. If we reduce the thickness of the plate to the order of magnitude of ℓ , the couple stress effect should give appreciable differences from the classical formulation which can be obtained from above by setting ℓ equal to zero. If, on the other hand, h is large relative to ℓ it is easy to see how such an effect could be

masked in experiment.

Example 3: To broaden our collection of solutions, we will now solve a similar problem with alternate boundary conditions (see figure 8.7b). This time consider a plate fixed on one edge with a constant moment on the opposite edge. The boundary conditions are:

$$\omega(o,y) = 0$$

$$\omega(L,y) = 0$$

$$\frac{d^2\omega}{dx^2}(o,y) = \frac{M_o}{\frac{Eh^3}{12(1-\nu^2)} + 4G\ell^2h}$$

$$\frac{d\omega}{dx}(L,y) = 0$$

Proceeding as before, it is easy to show that:

$$\omega = \frac{M_o}{2} \frac{\left\{ \frac{-x^3}{2L} + x^2 \frac{-Lx}{2} \right\}}{\left\{ \frac{Eh^3}{12(1-\nu^2)} + 4G\ell^2h \right\}} \quad 8.23$$

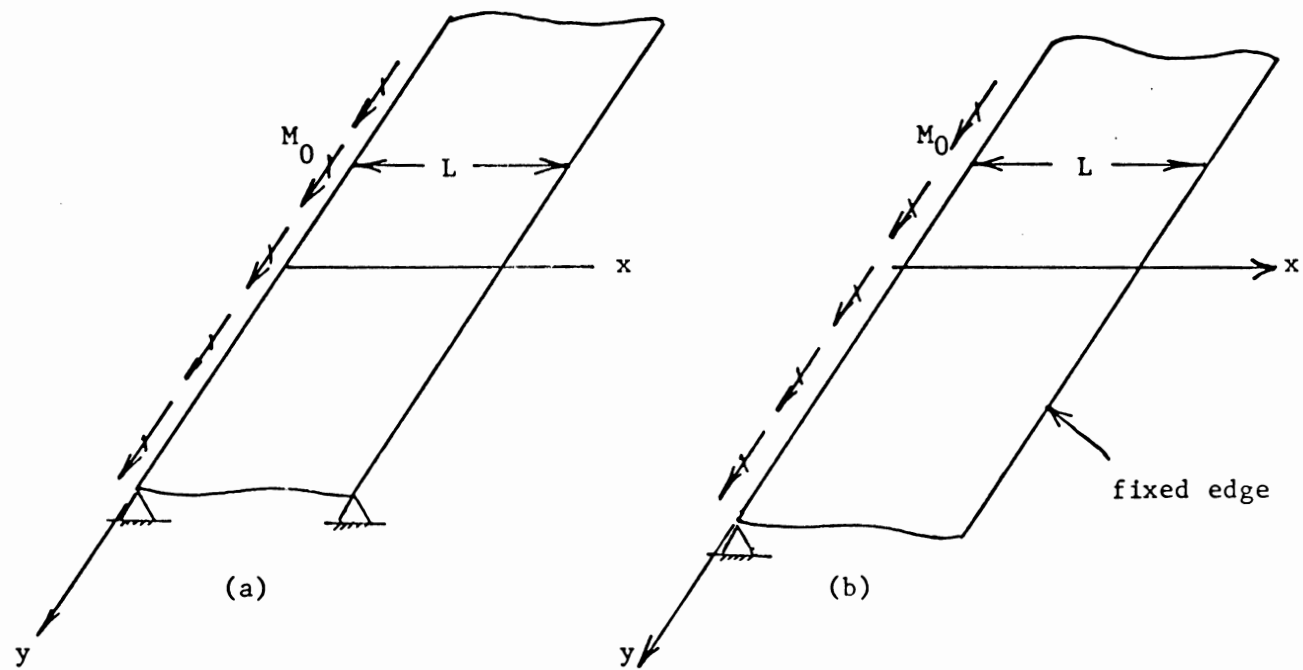


FIGURE 8.7 ORIENTATION OF PLATES FOR BENDING EXPERIMENTS

IX EXPERIMENTAL PROGRAM

Armed with the results of examples 2 and 3 from chapter VIII, we can go directly to the formulation of our experimental program.

According to our foundation in chapter VI, we know that for soft aluminum we should expect ℓ to be on the order of .02 inches if the couple stress theory is to be of use in explaining size effects. Thus, according to our example problems in plate bending, if we reduce the thickness, h , to near this order of magnitude we should be able to determine the importance of Mindlin's theories in the mechanics of materials.

Let us now define D as the flexural rigidity of a thin plate such that:

$$D = \frac{Eh^3}{12(1-\nu^2)} + 4G\ell^2h \quad 9.1$$

If we divide this quantity by Eh and multiply by $12(1-\nu^2)$ we can obtain the parameter $\tilde{D}$ such that:

$$\tilde{D} = \frac{D(12)(1-\nu^2)}{Eh} = h^2 + 24(1-\nu)\ell^2 \quad 9.2$$

If we now plot this parameter versus h^2 we should obtain a straight line whose slope is unity and whose intercept at $h^2 = 0$ is $24(1-\nu)\ell^2$. Figure 9.1 shows the classical result ($\ell=0$) and the couple stress result for $\ell = .02$ inches plotted together with a $\pm 5\%$ error envelope for the classical theory for aluminum. Examining this figure, it is easy to see that, in order for the couple stress effect to be clearly observable, we must test very thin plates and that in thicker plates the effect is completely clouded by a reasonable experimental error and could have

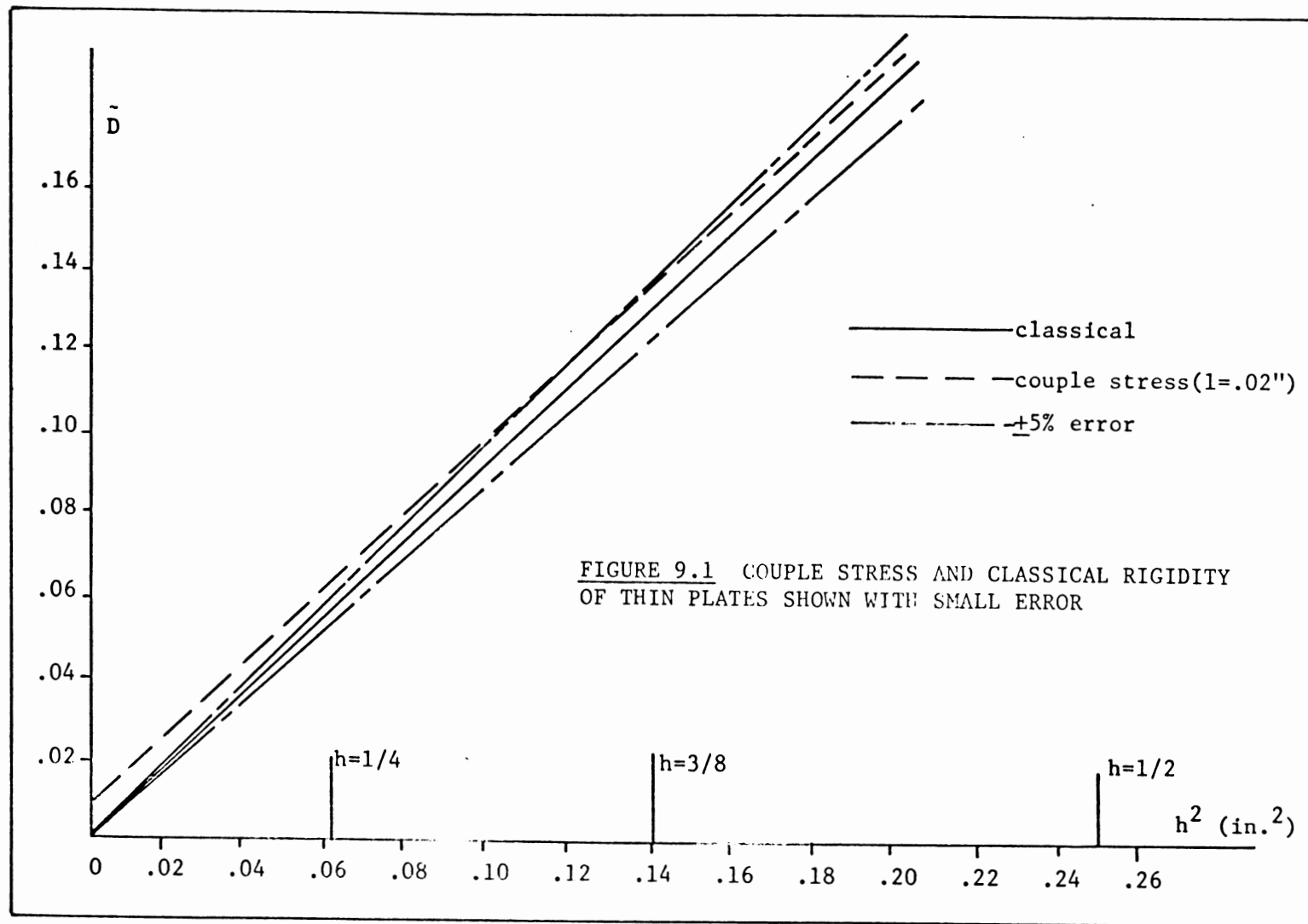


FIGURE 9.1 COUPLE STRESS AND CLASSICAL RIGIDITY OF THIN PLATES SHOWN WITH SMALL ERROR

conceivably been overlooked in previous work not oriented to its detection.

In light of these considerations, it was decided to test plates fabricated from the thinnest gage sheet material readily commercially available. The plan was to determine the parameter D/h or $\tilde{D}$ experimentally for a series of thicknesses of a given material and plot it versus h^2 in order to determine the location of the $h=0$ intercept by extrapolation. Such extrapolation, it was felt, would tend to minimize systematic error in the data.

With regard to choice of materials, it was decided to use soft, large grained wrought metals in order to maximize the tendency toward size effects. Since the large body of the literature dealing with such effects concerns steel and aluminum, it was decided to test samples of these metals. Considering the above requirements along with availability and fabrication properties, the following metals were chosen:

Aluminum sheet 1100-0, .020 inch to .050 inch thickness. This is nearly pure (99.0+%) aluminum in the soft annealed condition.

Aluminum sheet 1100-H-14, .025 inch - .050 inch. This is 99.0+% aluminum in the cold worked (half hard) condition. This was obtained for purpose of comparison with the annealed temper.

Low carbon steel, soft annealed .018 to .038 inches in thickness.

The aluminum was obtained from Reynold's Supply Company, Richmond, Virginia and the steel from Ryerson Steel, Cincinnati, Ohio.

One inch wide by nine inch long coupons were cut from representative sheets oriented in directions both parallel and perpendicular to the

direction of rolling and static tensile tests were performed to determine the elastic properties, yield stress and ultimate strength of the materials used. Foil strain gages (BLH FA-25-12S9L and FA-25-12S6L) attached with "Eastman 910" cement and monitored using an SR4 type L strain indicator and SR4 switching and Balancing unit were used to determine elastic strains. Loading was accomplished using a Tinius Olsen, mechanical test machine with the range of 0 - 1200 pounds in one pound increments. Figure 9.2 shows the configuration of strain gages on a specimen. Elastic properties and strengths are reported in Chapter X.

Sections of representative sheets were cut, mounted, polished, etched and examined metallographically to determine the order of magnitude of grain size and configuration. Steels were etched with "2% Nital" and aluminum with a hot, saturated caustic solution. Photomicrographs and discussion of structure are included in Chapter X.

The plates to be tested would have very low rigidity and deflections would have to be kept small compared to their thickness. For these reasons, special concern had to be given to fixtures and equipment for loading the plates and measuring deformations. Since it was desired to measure deflections in uniaxial bending, plates and loading techniques would have to be designed to prevent end curling or minimize its effects.

After consideration of factors of sag, magnitude of loading and deflections, it was decided to fabricate test plates with a gage width (L) of four inches and a length of 12 inches. The loading would be accomplished by application of concentrated forces to the end of tabs extending over the plate support giving the effect of a moment

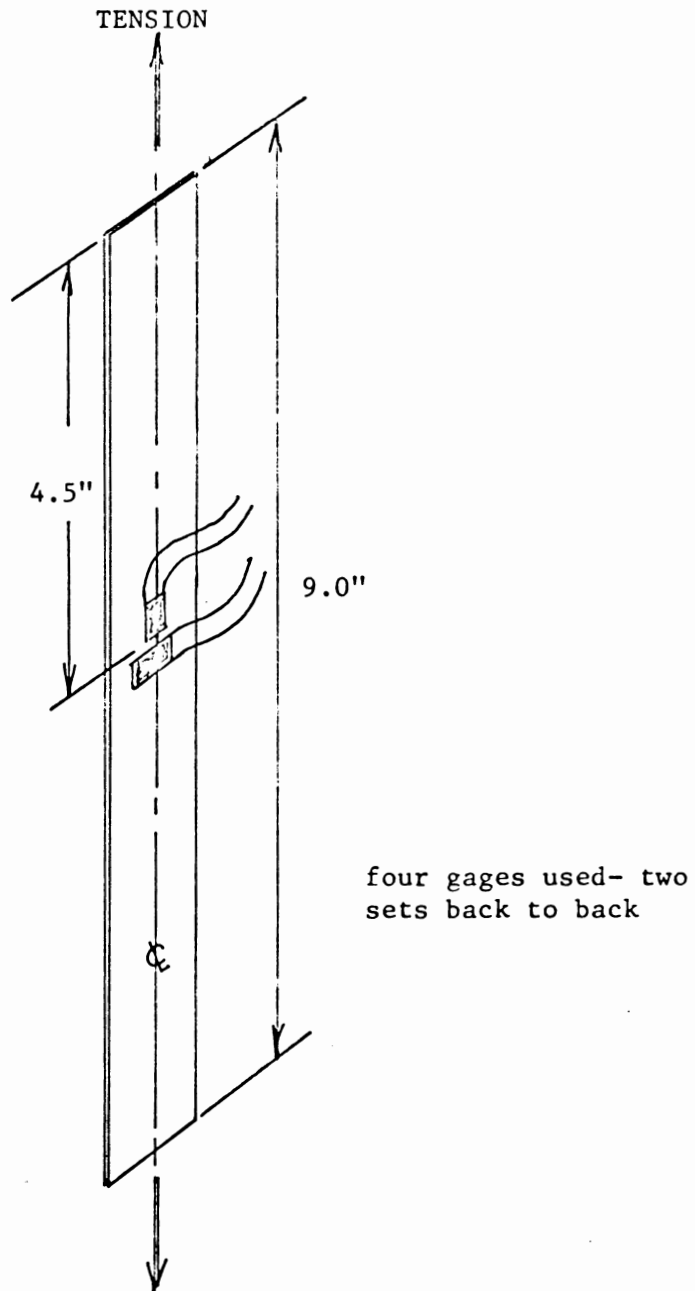


FIGURE 9.2 TENSILE TEST SPECIMEN

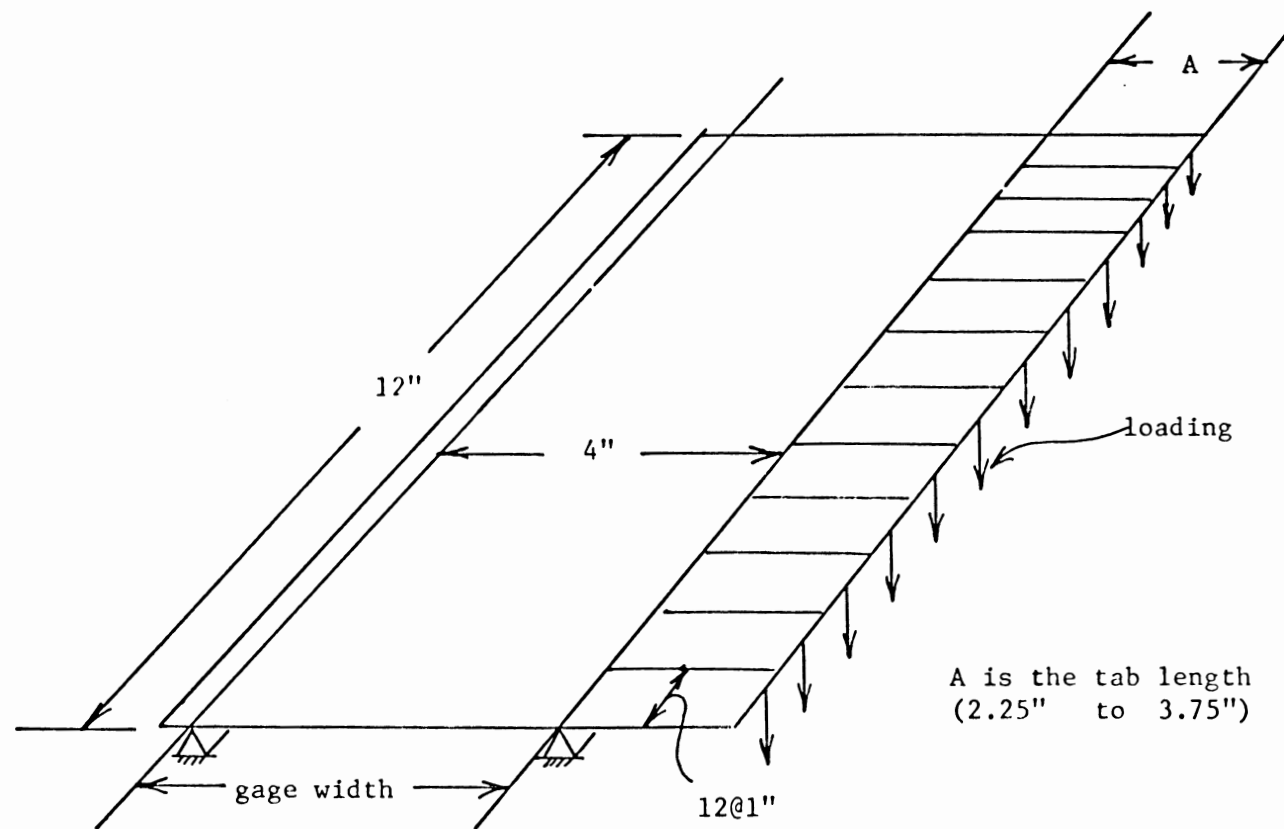
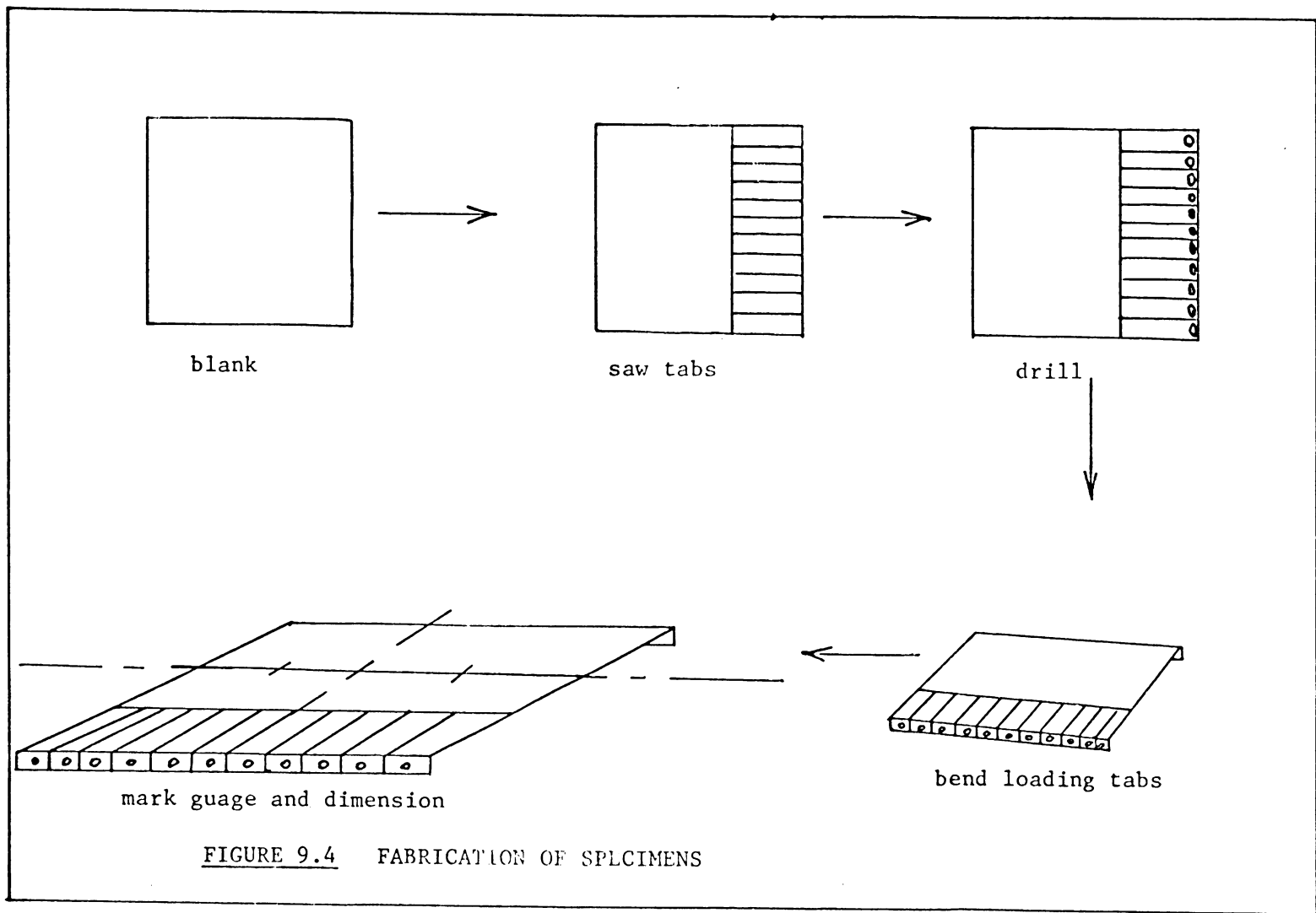


FIGURE 9.3 TYPICAL BENDING SPECIMEN

distributed along an edge. (See figure 9.3) The specimens were sheared into rectangular blanks, slots sawed to lengths determined by the magnitude of the moment desired, holes drilled in each tab to accept a loading rig, and the edges parallel to the long dimension crimped to give added resistance to end curling. (See figure 9.4) The edges parallel to the width were filed to eliminate any crimping from the shear. In final preparation, the center of each gage section was marked as accurately as possible as were the points of support, the support surfaces were inspected for flatness and polished using a fine (600 grit) emery paper, and the plate was cleaned using soap and water.

The loading fixture consisted of a 5 inch by 12 inch platform with two parallel 60° V grooves running along the long dimension 4 inches apart (see figure 9.5) The grooves were machined to hold 1/8 inch ground stainless steel rods to act as simple supports. The platform was supported on two legs which also supported a second platform on which to rest instrumentation. This fixture was fabricated from 1/2 inch, 2024-T3 aluminum plate and fastened together with 5/16" steel flat head screws.

It was decided to make use of the results of both examples 2 and 3 from the previous chapter in order to take into account the effect of real support in the experimental program. Thus two kinds of tests were run: one with a "simple support" on one edge and distributed moment on the other; one with a "clamped support" on one edge and distributed moment on the other.



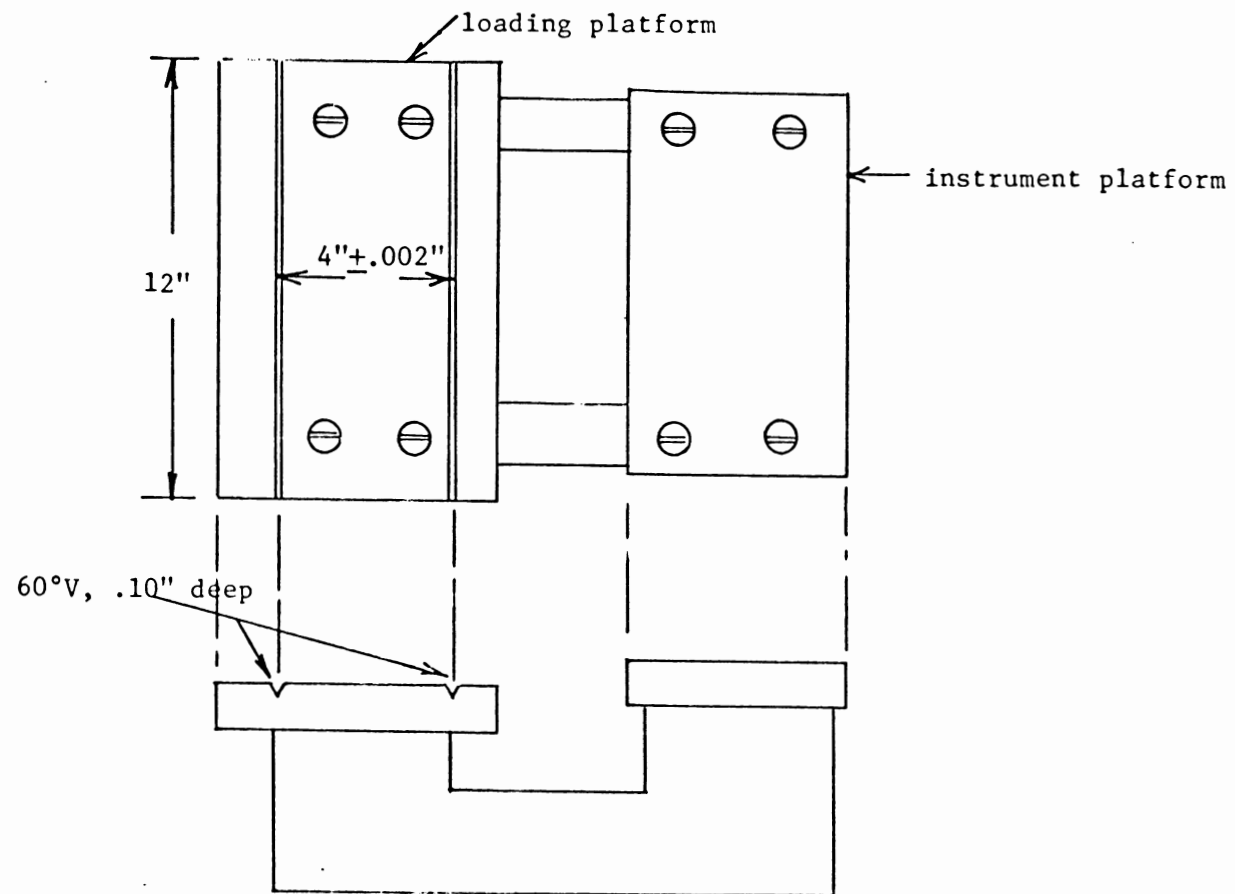


FIGURE 9.5 LOADING PLATFORM FOR BENDING OF VERY THIN PLATES

To accomplish the simple support, the 1/8 inch steel rod was laid in the groove at the back of the loading platform. The plate was positioned on top of this rod and then an upper support was placed directly over the 1/8 inch rod, on top of the plate. (See figure 9.6) This upper support consisted of two 12 inch long, 3/16 inch diam. ground rods, side by side and taped to a 1/4 inch by 3/4 inch by 12 inch steel strip. This support was free to rotate with the plate but held it firmly against the lower rod.

This upper support was necessary only at the rear of the plate (away from the distributed moment) since the action of the forces applied to the tabs on the specimen served to create good contact between specimen and support at the front. However, to test the effect of the upper support, pilot tests were run with it in position and out of position over the front supports without any effect on the experiment.

For a fixed or clamped end condition, the 1/8 inch rod was removed from the rear V groove and a steel shim, 1 inch wide, about 12 1/2 inches long and .065" thick was laid on the loading platform (See figure 9.7). The plate was then positioned on top of the shim and a second shim was placed on the plate. A 3/8 inch square steel rod, about 12 1/2 inches long, was then laid over the edge of the second shim and clamped tight with two 4 inch heavy steel C clamps.

It was necessary in applying the loading to the tabs, that each tab receive as nearly as possible the same load and that such loading be accomplished with the minimum disturbance to the experimental set-up. The method finally selected used small brass fishing snaps and

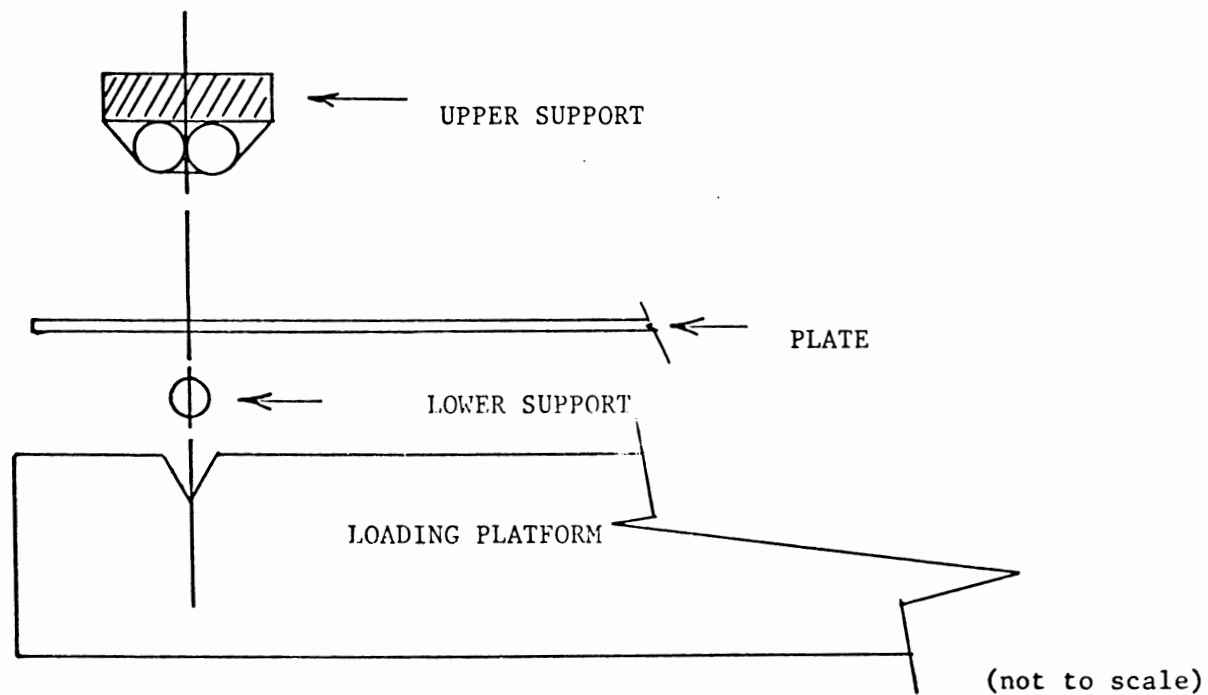
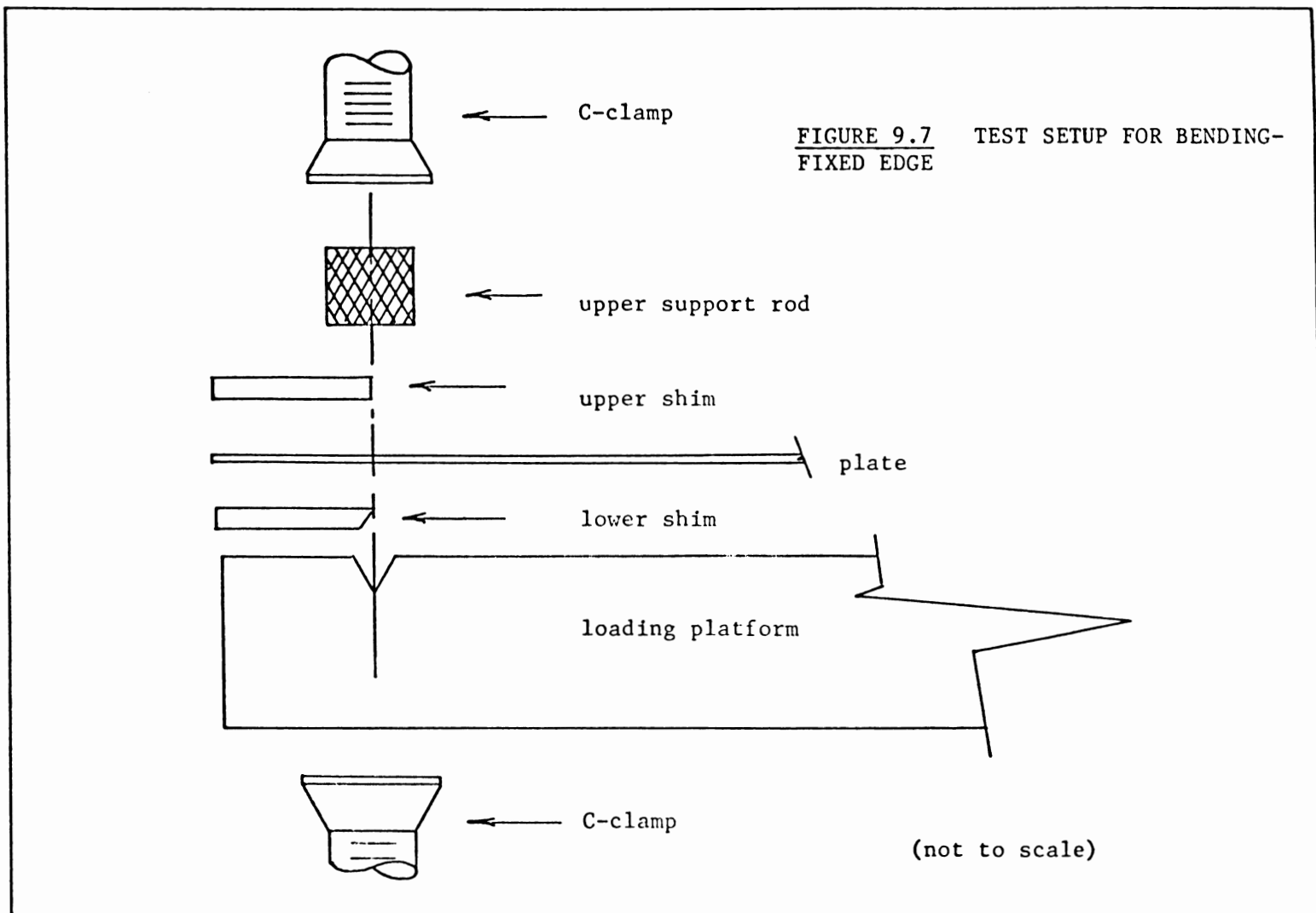


FIGURE 9.6 TEST SETUP FOR BENDING- SIMPLE SUPPORT



swivels through the holes on the plate. Number 50 cotton thread was woven from the swivel to eyes on a 1 inch x 1/4 inch x 12 inch hardwood beam. A smaller beam was similarly attached below this one and a hook was on this element to accept the weights for loading. This configuration not only gave a good distribution of load by allowing a certain amount of self adjustment, but also facilitated the loading procedure (only one weight needed) and gave a small, but appreciable, initial load from its dead weight which was useful in obtaining support contact and stability.

To monitor deformation, it was found impossible to use ordinary dial gages since deflections would be on the order of two to three thousandth of an inch with readings necessary to 1/10000ths. of an inch. At this level, such gages are rather inaccurate. In addition, dial gages require a large force, considering the stiffness of the plates concerned, to actuate and it was reasoned that they would have a detrimental effect on the data. Pilot tests bore out this line of reasoning.

Strain gages, though useful for some of the thicker plates, were not considered for the very thin ones because the strains obtained there (perhaps 10-30 μ in) would be near the limit of detection of available instrumentation.

The Linear Variable Differential Transformer (LVDT), Schaevitz 100 ML, monitored and powered by a Dyna-Mike, model 129D was chosen to make the majority of deflection measurements. It was calibrated, using a micrometer and its internal calibration circuits such that a

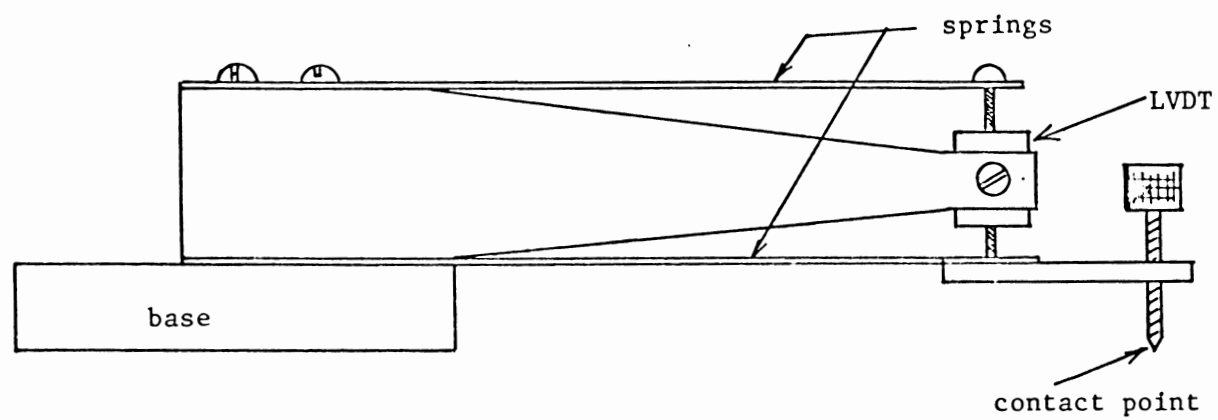


FIGURE 9.8 LVDT MOUNTING FIXTURE FOR BENDING TESTS

full scale deflection of its integral meter corresponded to a .005" deflection of the LVDT core. It was tested and found to give a linear relation between meter deflection and LVDT position such that deflections could be obtained in increments of .00013 inches.

The LVDT was mounted in a fixture and supported by two 6" by 1/2" cantilever springs of .020 aluminum (see figure 9.8) such that the force required to produce deflections on the order of our requirements was very small. The fixture was clamped to the instrument platform of the loading jig using a C clamp and extended over the plate. A screw actuated point could be adjusted to rest on any point of the top surface of the plate. Plate 9.1 shows the LVDT, Dyna-Mike and a plate set up for testing with a fixed end and distributed moment.

When performing experiments, it was found that a thirty minute warm up period for the Dyna-Mike was sufficient and that it was more stable when an electric fan was allowed to circulate air at the rear of its console.

In tests using the LVDT, the following general procedure was followed. After the warm up period for the equipment, the plate having been positioned in the fixture, the loading jig attached and the pointer on the LVDT head positioned on the center of the plate, the Dyna-Mike was zeroed according to the operating instructions for the instrument. This zero then was referred to the plate under the weight of the loading jig with good support contact. The instrument was then brought into calibration using its internal calibration circuits. Weights were applied to the loading beam and deflections

recorded for various values of load. After each load was applied and removed, calibration was rechecked.

The LVDT was then moved to a position along the centerline of the plate two inches to the left or right and the entire procedure repeated. If any differences were noted in deflections, the supports were examined, instrumentation checked, and loading rig adjusted until differences became insignificant. Final readings were then obtained. In a selected number of cases, the entire set up was dismantled, reassembled and the experiments reperformed to determine repeatability. It was found that results easily could be repeated to within the accuracy of the instrumentation.

With the deflection-load data along the center line of the plate in hand, the rigidity of the plates was calculated using the formula for a simple support, for $L = 2''$:

$$D = \frac{M_o}{12} \left[\frac{12}{W} \right] = \frac{M_o}{W} \quad 9.3$$

and for a fixed edge

$$D = \frac{M_o}{2W} \quad 9.4$$

These data are presented in Chapter X.

In order to double check the data and be certain that the procedure and fixtures were giving exactly the loading we expected, it was decided to test a few of the thicker plates using strain gages to monitor strain rather than measuring deflection directly. Foil gages (FA-25-12S9L and FA-25-12S6L) were attached to plate as shown in figure 9.9 with Eastman 910 cement. Leads were soldered to the extended tabs of the gages, supported on the instrument platform, and allowed

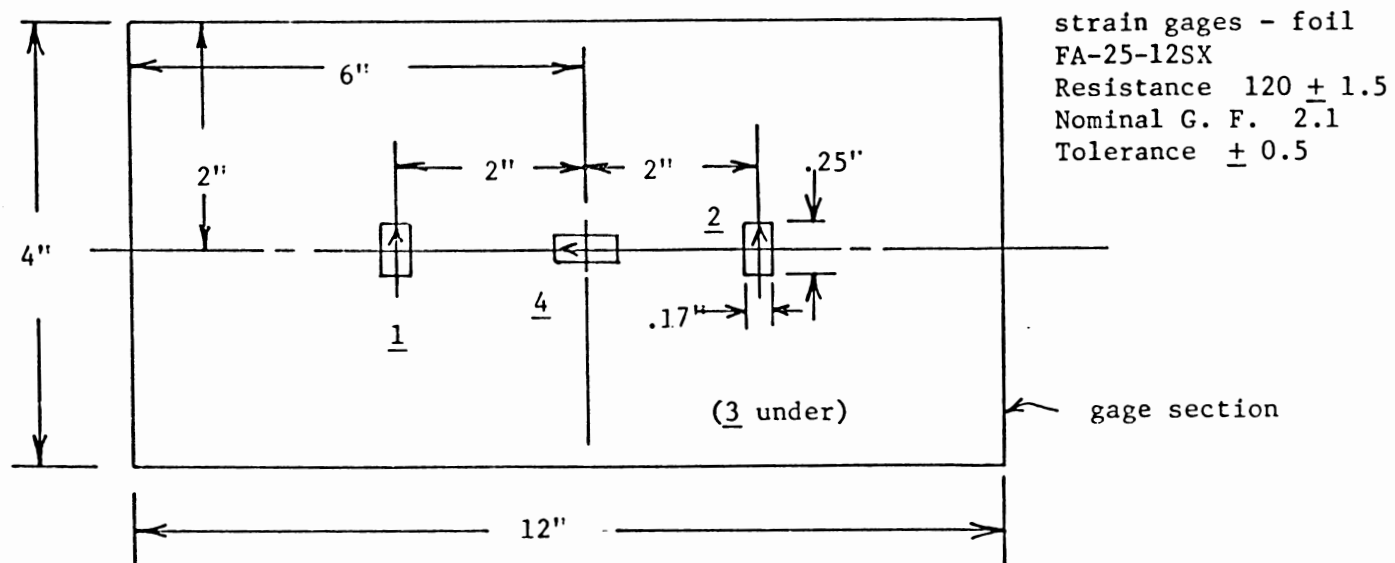


FIGURE 9.9 PLATE WITH STRAIN GAGES FOR BENDING TESTS

to cantilever over the specimens so that they were not resting on or secured to the plate. Strains were obtained using an SR4 switching and balancing unit and SR4, Type L strain indicator. Loading was accomplished and supports obtained using the same methods as previously discussed. The rigidity of the plates was calculated using the formula for simple end condition

$$D = \frac{M_o h}{4\epsilon_x}$$

and for a fixed edge

$$D = \frac{M_o h}{8\epsilon_x}$$

where ϵ_x was the strain at the center of the plate in the x direction.

Result of these tests are reported in Chapter X.



PLATE 9.1 TEST SETUP WITH LOADING FIXTURE, LVDT, AND FIXED EDGE

X. RESULTS

Static tensile tests performed on the materials used in these tests have the following average value for their tensile properties:

	E	ν	σ yield	σ ult.
Low Carbon Steel	30.7×10^6	.28	38000	44000 psi
1100-0 Aluminum	10×10^6	.33	-----	12300

Table 10.1

Samples of steel taken from directions parallel and perpendicular to the rolling direction gave essentially the same results for the elastic properties while the ultimate strength and yield strength were slightly different; the parallel specimen being about 2000 psi stronger in each case.

Aluminum specimens from the different orientations gave the same results for the elastic constant E but Poisson's ratio, ν , varied from .32 for the parallel specimen to .34 for the perpendicular orientation. It was felt that the small variation was tolerable since ν appears squared in our theory. Ultimate strengths were essentially the same.

Metallographic examination of the steel and aluminum before loading showed typical grain structure of the annealed material (25, 26, 27). Grains were large and equiax showing no deformation or apparent orientation. There was some variation in grain size from one gage to another, especially in the steel but the diameter was generally on the order of magnitude of .001 inches for steel. The structure of the

nearly pure aluminum was most difficult to obtain by etching but reasonable photomicrographs were produced. These showed a grain diameter of about the same order of magnitude for the aluminum. Plates 10.1, 10.2 and 10.3 show the micro-structure of the steel used. Plates 10.4 and 10.5 show the aluminum structure.

Figures 10.1, 10.2, 10.3 and 10.4 together with tables 10.2 and 10.3 give the results of uniaxial bending tests using the LVDT.

The data plotted in figure 10.1 is intended to show the scatter in the data for the aluminum specimens. As we would have expected the fixed end condition, when attempted in reality, gave results slightly lower than the mean while the simple support yielded results slightly higher. This trend is easily explainable since any real "clamp" allows a small but finite rotation of elements at the support while any real "simple support" must contribute some friction. These effects would be more pronounced with soft materials where resistance to rotation is smaller and friction coefficient is larger. In addition, upon examination of Table 10.2, it can be seen that specimens cut with their width transverse (perpendicular) to the rolling direction generally gave slightly higher results than those cut parallel to this direction. This phenomenon can be partially explained by the small variation observed in ν for the two directions. Other scatter can be explained by allowing for reasonable tolerance in manufacture of specimens - their thickness and the length of loading tabs, and for the error inherent in the measurements of deflection which were made to the nearest .00013 inches (usually about 5%). Since we were in search of

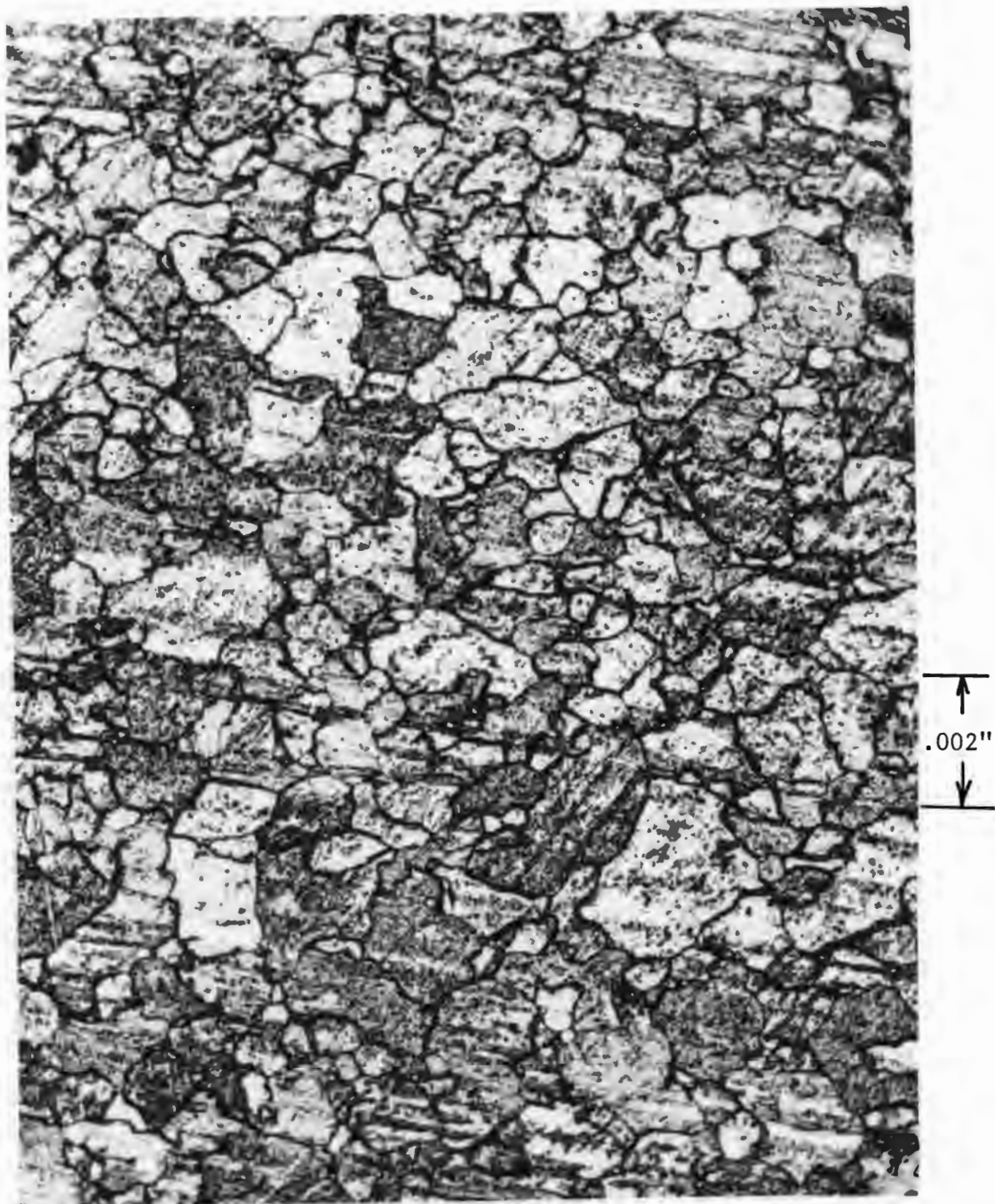


PLATE 10.1 LOW CARBON STEEL, ANNEALED TOP SURFACE
 NITAL ETCH (FROM .0365 SHEET)

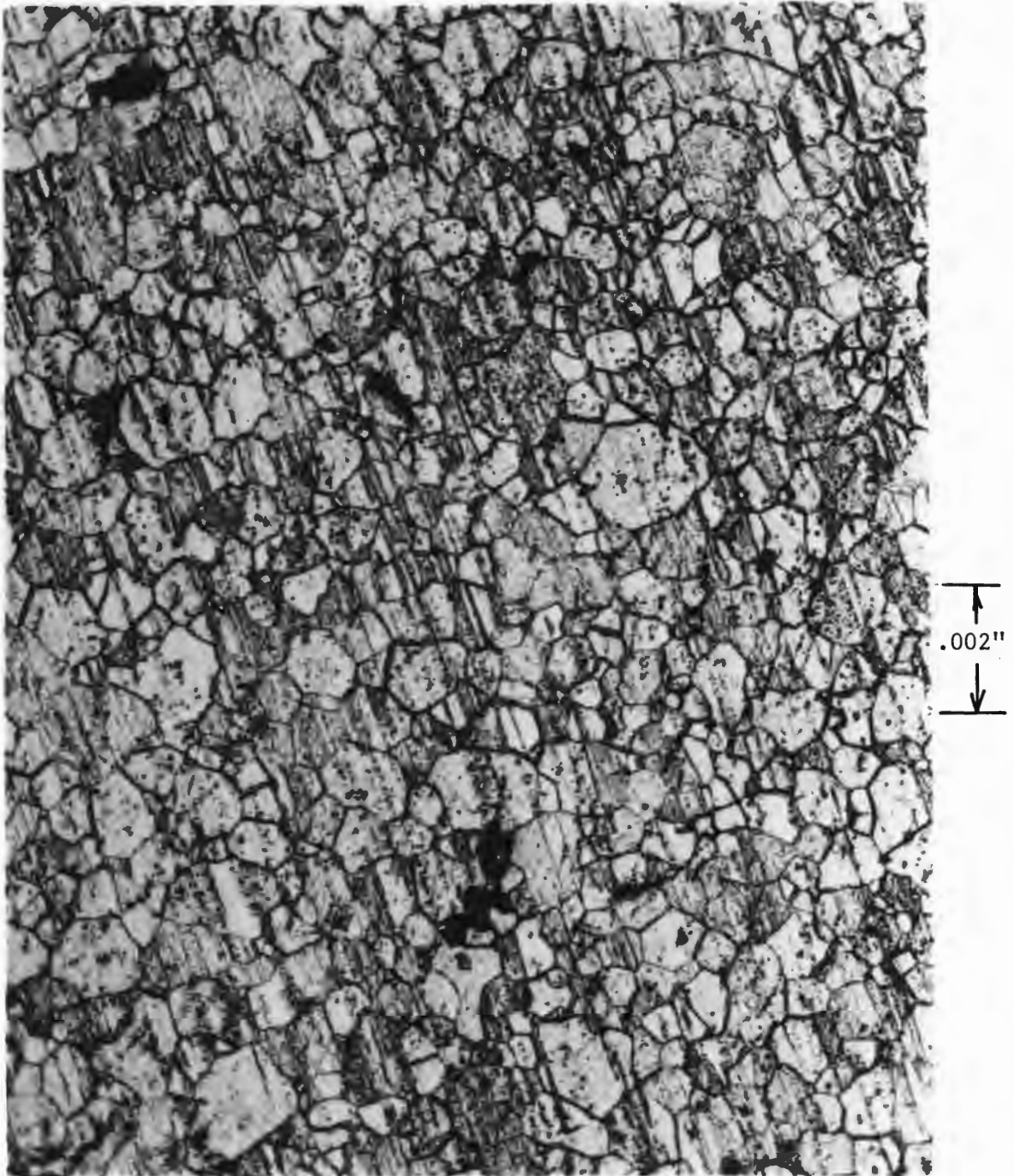


PLATE 10.2 LOW CARBON STEEL, ANNEALED TOP SURFACE
 NITAL ETCH (FROM .018 SHEET)

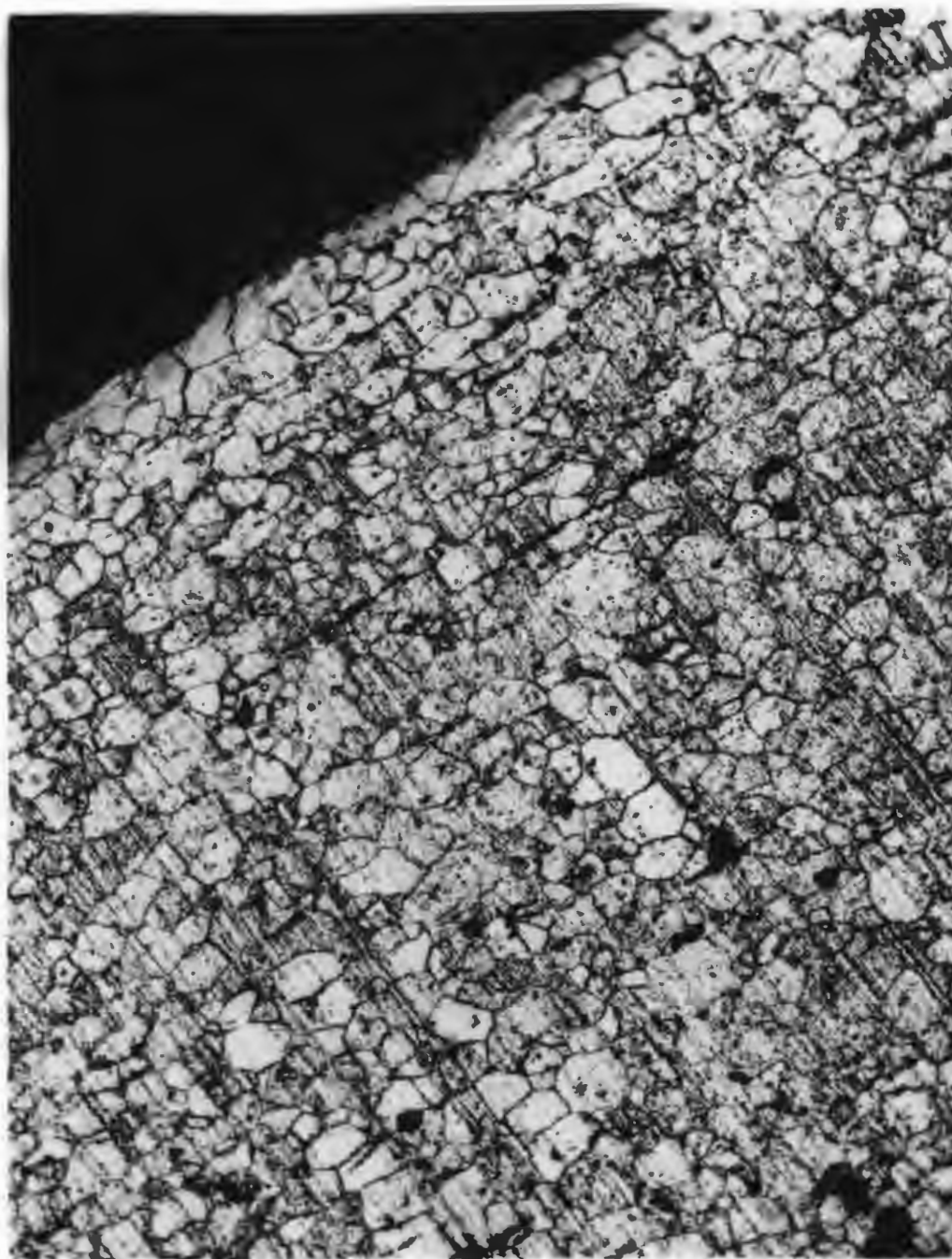


PLATE 10.3

LOW CARBON STEEL, ANNEALED
.018 SHEET NITAL ETCH

TRANSVERSE SECTION OF

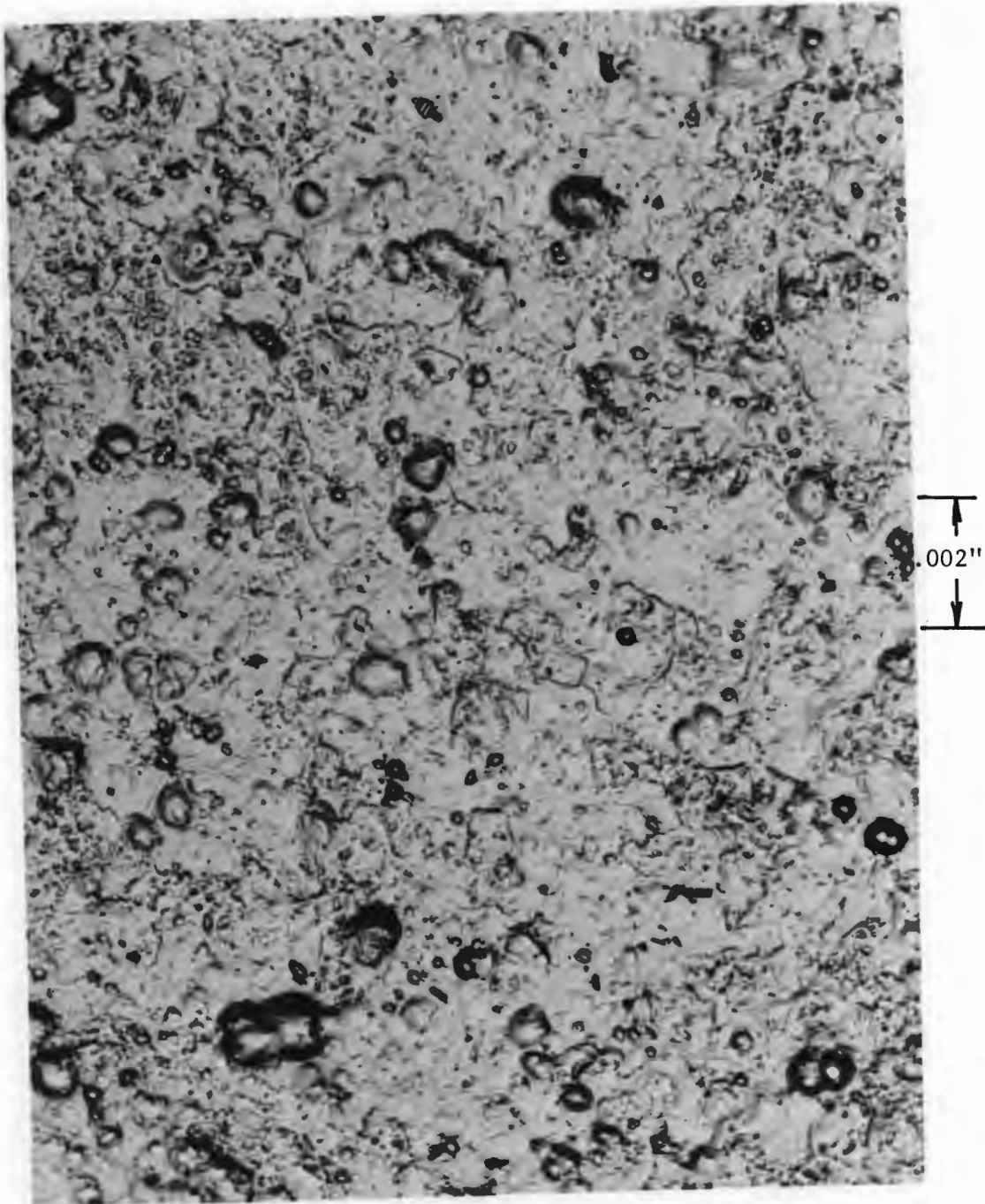


PLATE 10.4 1100 ALUMINUM, ANNEALED TOP SURFACE
HOT CAUSTIC ETCH (FROM .020 SHEET)

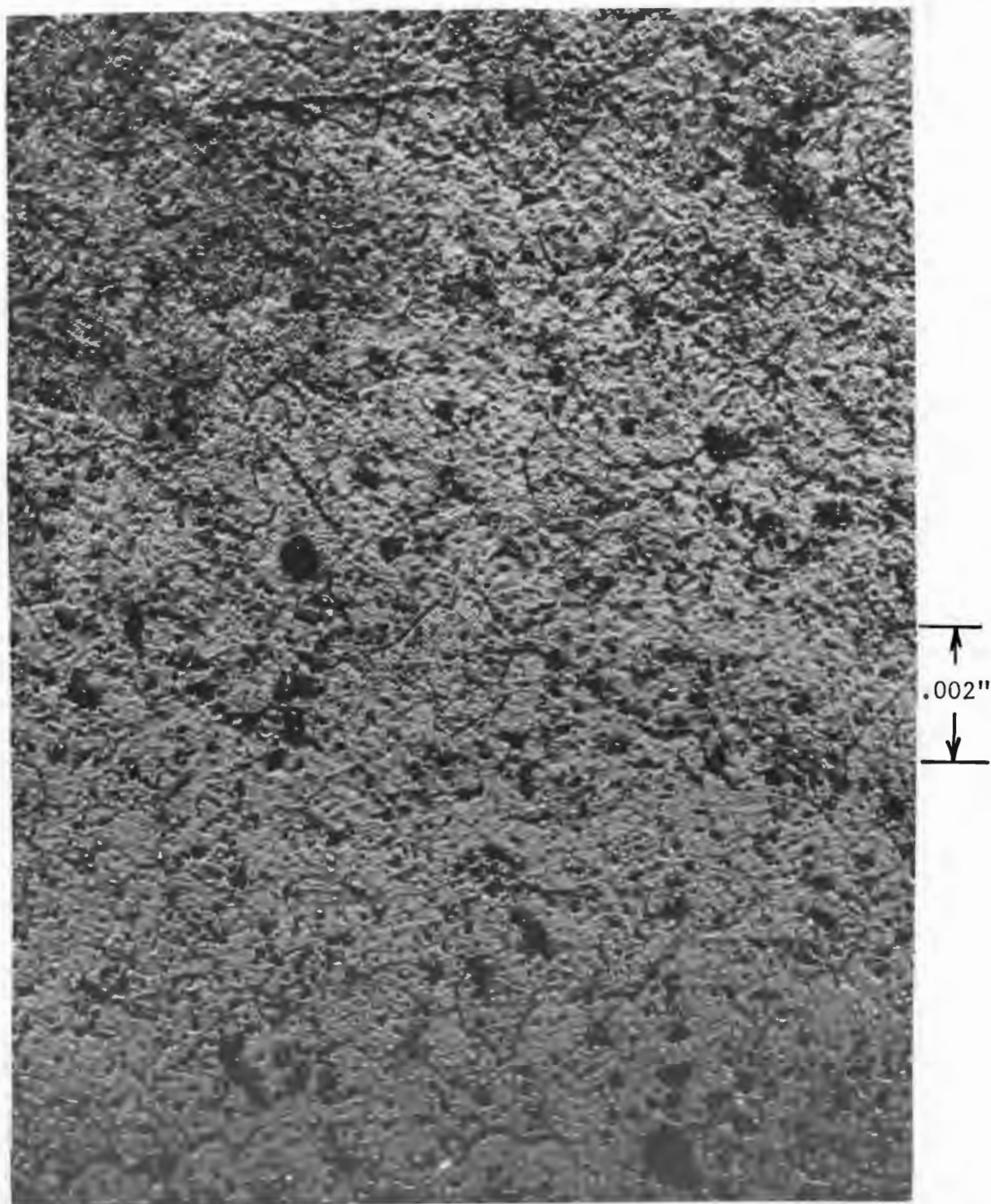


PLATE 10.5 1100 ALUMINUM, ANNEALED TOP SURFACE
HOT CAUSTIC ETCH (FROM .044 SHEET)

order of magnitude of the constant ℓ , the scatter obtained was considered quite tolerable. The data for temper H-14 (cold worked) fell along the same line as did the annealed material showing that at the range of our testing the temper condition had no effect on elastic action.

Figure 10.2 shows a similar representation of data for annealed low carbon steel. It can be seen that the scatter is somewhat smaller in this case since the material is inherently stronger and more resistant to rotation at the fixed edge and has a lower friction coefficient. Due to its rigidity, it is also more readily fabricated to low tolerance.

The plot given in Figure 10.3 shows the average value of $\bar{D}$ plotted versus h^2 for the aluminum plates and graphically illustrates the fit of the data to the line for $\ell = 0$ (classical result). Special attention should be paid to the data in relationship to the line for $\ell = .02$ inches which we should have expected to approach if the couple stress theory, as we have presented it, could have explained size effects as has been suggested (6).

Figure 10.4 shows a similar plot for steel which also fits the curve $\ell = 0$ well.

The results of the bending tests on some of the thicker plates using strain gages are given in Table 10.4. These tests verified the values of D which were reported previously to within the accuracy of the instrumentation (generally about $\pm 8\%$ for strains at the level of our tests). Most important, the comparison of gages mounted at different points along the center line of the plate and back to back

showed that the strains were quite consistent with our loading scheme. There was no significant difference between the magnitude of strains along the plates showing a good uniformity in loading and support while the strains on the top surface and bottom surface were identical showing a pure bending. The strain ϵ_y measured at the plate center was essentially zero agreeing well with the prediction of our plate theory.

Table 10.2

BENDING OF ALUMINUM PLATES USING LVDT

spec.	h(in.)	end	M_0 (in.-lb)	w	D	D/hX 10 ⁻²	h^2 X 10 ⁴
1	.020	ss	.0206 .0244 .0221	.00263 .00312 .00283	7.83	3.91	4.00
1	.020	sf	.0206 .0362 .0427	.00150 .00267 .00313	6.82	3.41	4.00
2	.0202	ss	.0206 .0244	.00238 .00275	8.75	4.35	4.10
2	.0202	sf	.0206 .0362	.00133 .00238	7.68	3.80	4.10
3	.020	ss	.0130	.00170	7.65	3.80	4.00
4	.020	ss	.0206 .0244	.00250 .00300	8.20	4.38	4.00
5	.025	ss	.0410 .0641	.00287 .00437	14.50	5.80	6.25
5	.025	sf	.0410 .0641	.00175 .00263	12.0	5.13	6.25
6	.025	ss	.0410 .0641	.00250 .00388	16.5	6.60	6.25
7	.025	ss	.0410 .0641	.00250 .00400	16.2	6.95	6.25
7	.025	sf	.0410 .0641	.00150 .00245	13.4	5.3	6.25
8	.033	ss	.0600 .0930	.00180 .00280	33.3	10.1	10.75
8	.033	sf	.1490 .2090	.00238 .00340	31.2	9.5	10.75

Table 10.2 continued

spec.	h(in.)	end	M ₀ (in.-lb)	w	D	D/h X 10 ⁻²	h ² X 10 ⁴
9	.033	ss	.0425 .0660 .1060	.00125 .00183 .00288	35.7	10.8	10.55
10	.033	ss	.0600 .0940 .1490	.00175 .00275 .00413	34.9	10.6	10.75
10	.033	sf	.1490 .2090	.00250 .00363	29.3	8.9	10.75
11	.045	sf	.1700 .2400 .3190	.00100 .00150 .00200	82.0	18.2	20.00
12	.044	sf	.1700 .2400 .3190	.0010 .00150 .00213	80.0	18.2	19.40
12	.044	ss	.1700 .2400	.00200 .00300	83.0	18.8	19.40
13	.024	ss	.0290 .0510	.00225 .00400	12.9	5.38	5.76
13	.024	sf	.0290 .0510	.00138 .00250	10.3	4.38	5.76
14	.024	ss	.0283 .0510	.00200 .00350	14.2	5.92	5.76
14	.024	sf	.0283 .0510	.00125 .00225	11.2	4.66	5.76
15	.050	sf	.1730 .2450	.00075 .00113	112	22.4	25.0
16	.050	sf	.1730 .2450	.00075 .00108	115	23.0	25.0

1100-0 plates (longitudinal) 1,4,5,6,8,9,11,12

1100-0 plates (transverse) 2,7,10,13

1100-H14 plates 14,15,16

Table 10.3

BENDING OF STEEL PLATES USING LVDT

spec.	h(in.)	end	M_0 (in.-lb)	w	D	$D/h \times 10^{-2}$	$h^2 \times 10^4$
17	.018	ss	.051 .029	.00313 .00175	16.5	9.2	3.24
17	.018	sf	.051 .029	.00163 .00100	15.1	8.4	3.24
18	.018	sf	.029 .051	.00100 .00175	14.6	8.1	3.24
18	.018	ss	.029 .051	.00175 .00325	16.2	9.0	3.24
19	.0238	ss	.051 .081 .111	.00138 .00213 .00300	37.4	15.7	5.65
19	.0238	sf	.051 .081 .111	.00075 .00113 .00163	34.7	14.5	5.65
20	.0238	ss	.051 .081 .111	.00125 .00213 .00288	39.0	16.4	5.65
20	.0238	sf	.051 .081 .111	.00075 .00113 .00150	35.5	14.9	5.65
21	.0365	ss	.2080 .2920	.00150 .00213	138	37.8	13.3
21	.0365	sf	.2080 .2920	.00075 .00113	134	36.8	13.3
22	.0365	sf	.2080 .2920	.00075 .00113	134	36.8	13.3

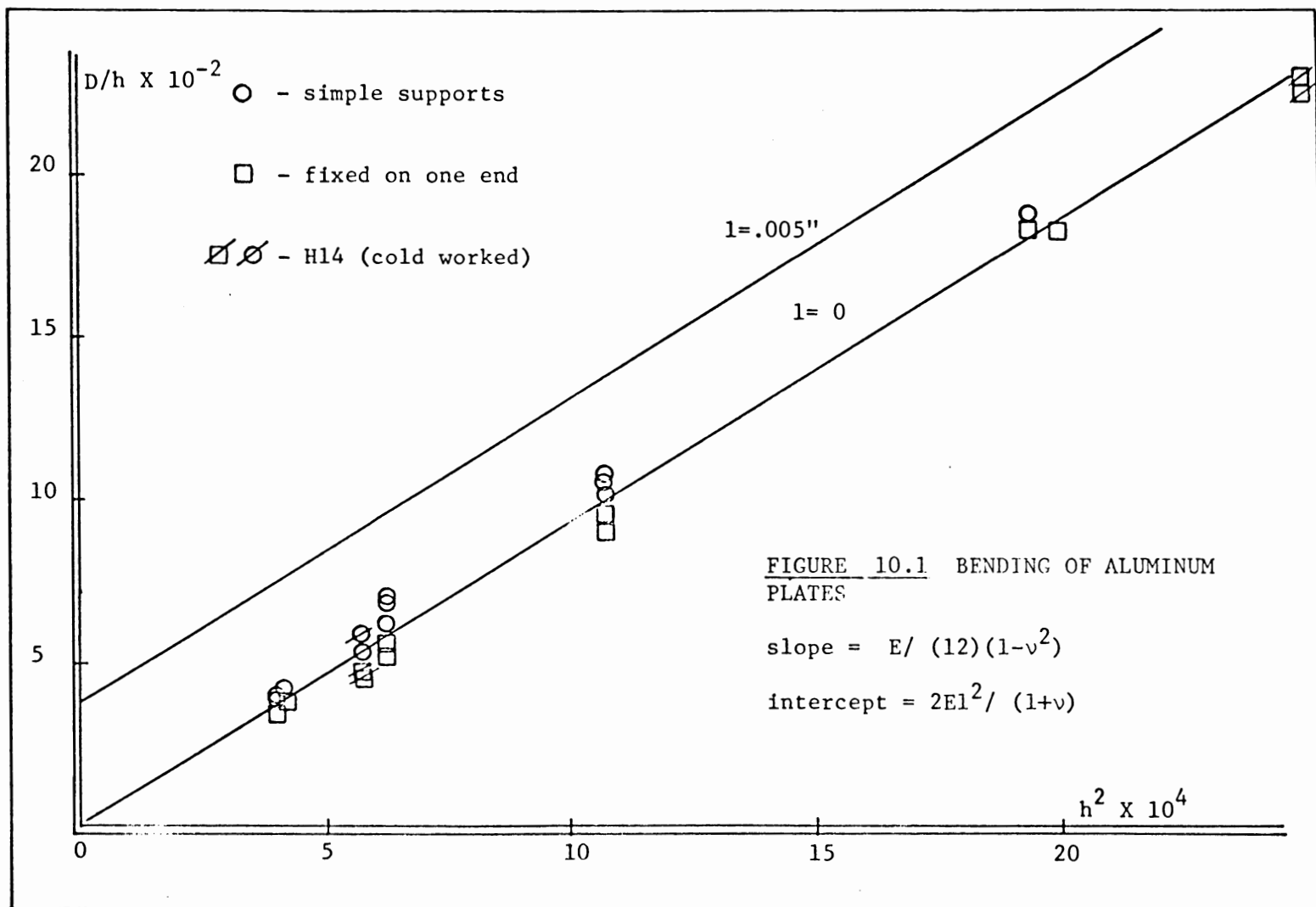
Steel plates (longitudinal) 18,19,21

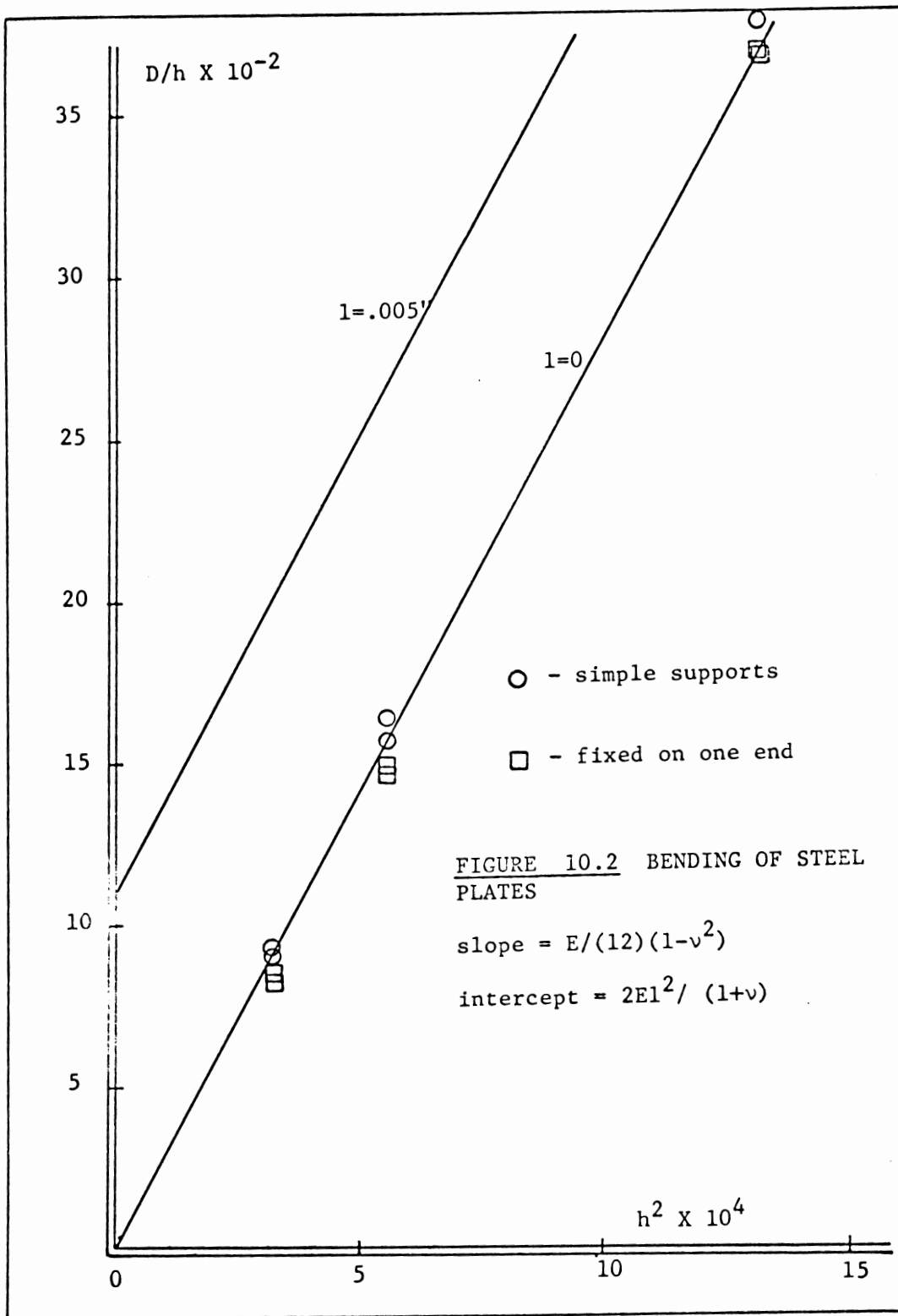
Steel plates (transverse) 17,20,22

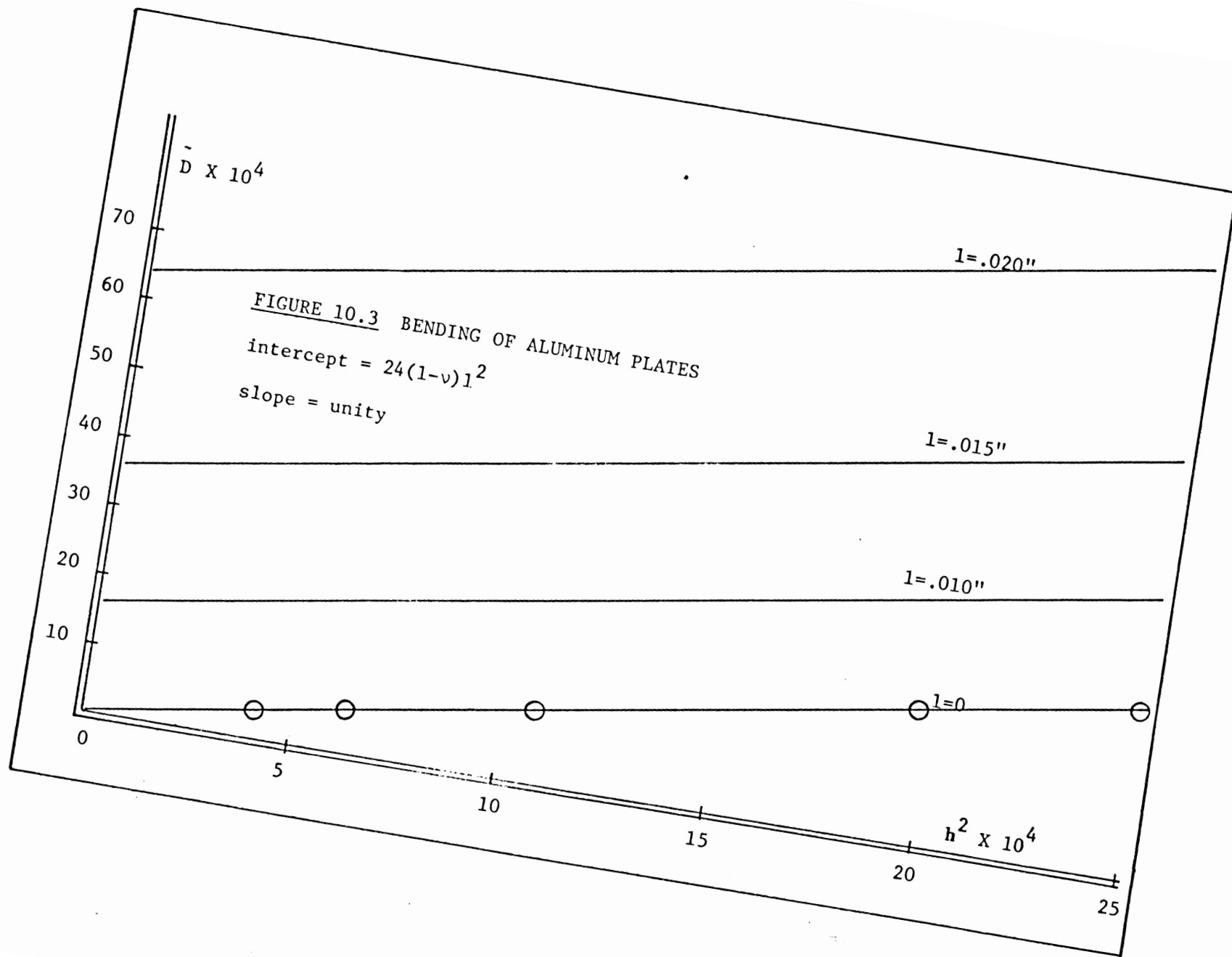
Table 10.4

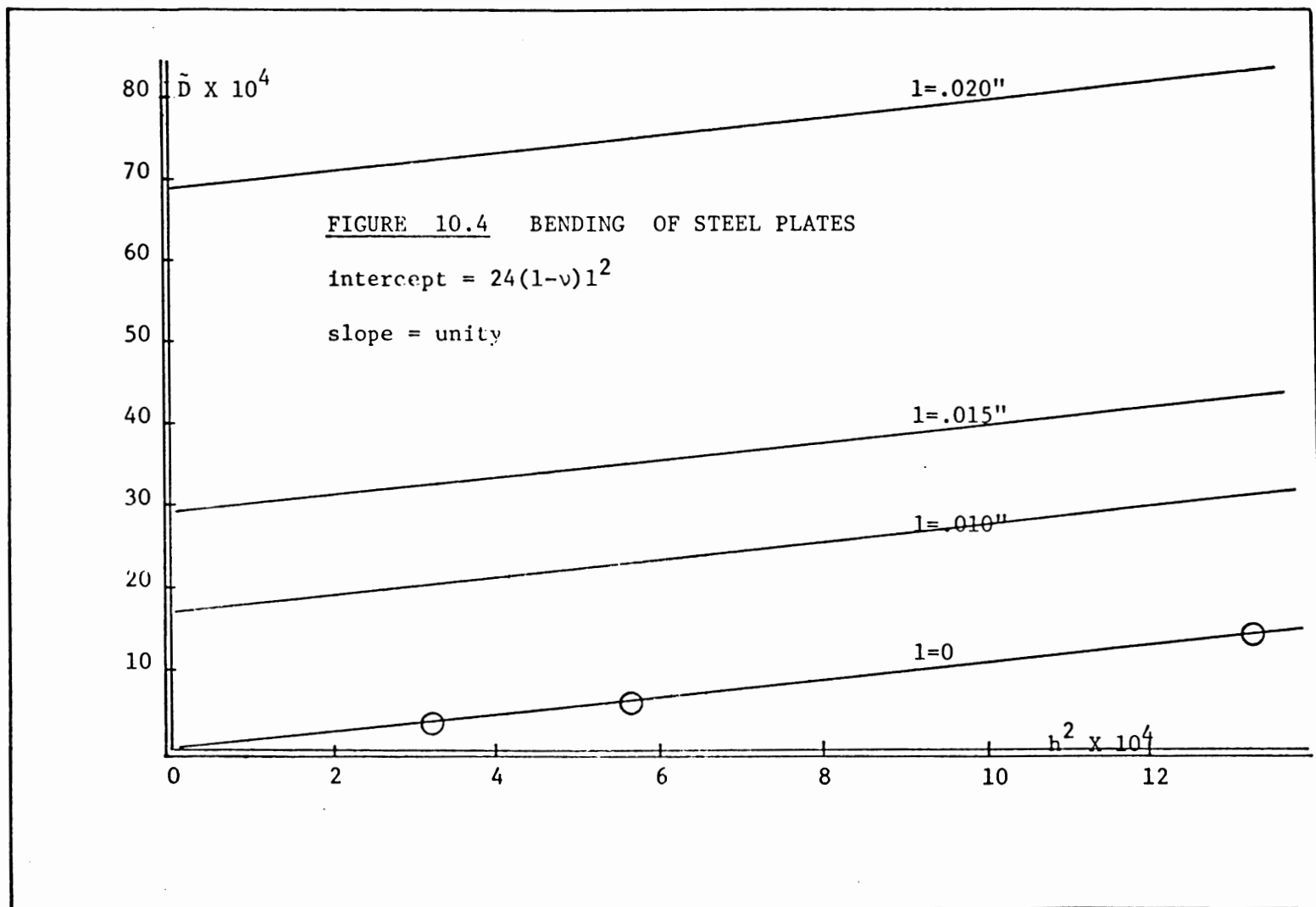
BENDING OF STEEL AND ALUMINUM PLATES WITH STRAIN GAGES

spec.	end	M_0 (in.-lb)	ϵ_y	$\epsilon_x(1)$	$\epsilon_x(2)$	$\epsilon_x(3)$	D
(micro inches/inch)							
5	ss	.102	0	45	45	-45	14.2
		.192	0	85	85	-85	
5	sf	.192	0	50	50	-50	12.0
8	ss	.151	0	38	38	-38	33.0
		.281	0	70	70	-70	
20	ss	.234	0	35	35	-35	38.0
		.265	0	45	45	-45	









XI. DISCUSSION AND CONCLUSIONS

It is obvious from an examination of Figures 10.3 and 10.4 that the order of magnitude of the constant ℓ is not near the values of .02 inches for aluminum or .01 inches for steel necessary to account for observable size effect in these metals. In fact, from an examination of Figure 10.1 and 10.2, after considering the width of the scatterband, we can consider the magnitude of the constants as somewhat less than .003 inches in aluminum and .002 inches in steel at most.

For a hole of radius .02 inches in an infinite steel plate then a/ℓ would be at least 10. An examination of Figure 6.5 indicates that this ratio would give a stress concentration factor very near the classical value and far from that we needed to explain size effects in Chapter VII. A similar argument holds for aluminum.

On the other hand, suppose we are interested in a hole of .002 inches in radius, could couple stress be of major importance? For the sake of illustration let us consider $\ell = .002$ inches for steel, the maximum possible from our results. The grain sizes of our steel specimens ranged between about .002 inches and .001 inches. For our illustration, let us take an average of about .0015 inches for grain diameter. Figure 11.1 shows a .004 inch hole, a diameter which should give significant couple stress effects for $\ell = .002$ inches, in an idealized matrix of cubic grains having dimensions of .0015 inches.

Obviously, at this level we no longer have a material which obeys the same generalizations as the material on a whole. Each grain,

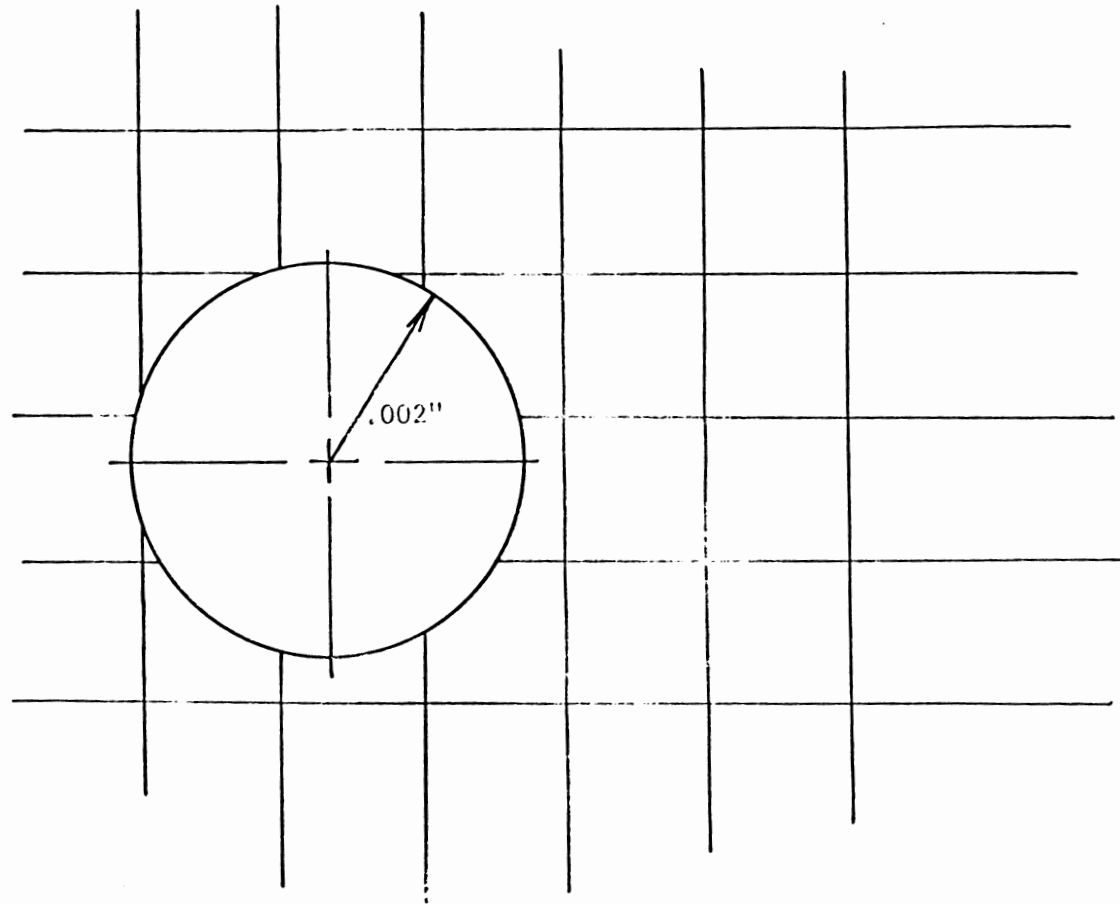


FIGURE 11.1 SMALL HOLE IN AN IDEALIZED MICROSTRUCTURE

oriented randomly in the structure, is an anisotropic continuum in its own right having basically the symmetry properties of the characteristic grain structure of the metal in single phase alloys or perhaps properties varying greatly from one grain to another in multiphase materials.

If grain size were small compared to the radius of our hole and stress gradients small compared with grain width, we might expect a theory based on a macroscopic model (such as classical theory or couple stress theory) to be valid. How can we expect such a model, however, to hold when the most significant part of a stress gradient can occur entirely across one single grain when we consider the foregoing comments? Figure 11.2 shows two cases of a round hole in an infinite plate. One (a) where a macroscopic theory would be expected to hold and the other (b) where a significant portion of the stress gradient can occur across one grain.

A comparison of our upper limit of l and the grain size of the material makes it obvious that, at best, we would have a configuration such as 11.2b if we produced a round notch in a steel plate in which we might expect a significant couple stress effect. Plate 11.1 shows two holes superimposed on the microstructure of one of our steel specimens. The larger hole has a diameter of .02 inches and could conceivably be treated using a macroscopic theory since grain size is small compared to radius. The smaller hole, diameter .002 inches, however, it seems safe to say, could hardly be expected to give stress distributions based on the macroscopic, homogeneous, isotropic theory.

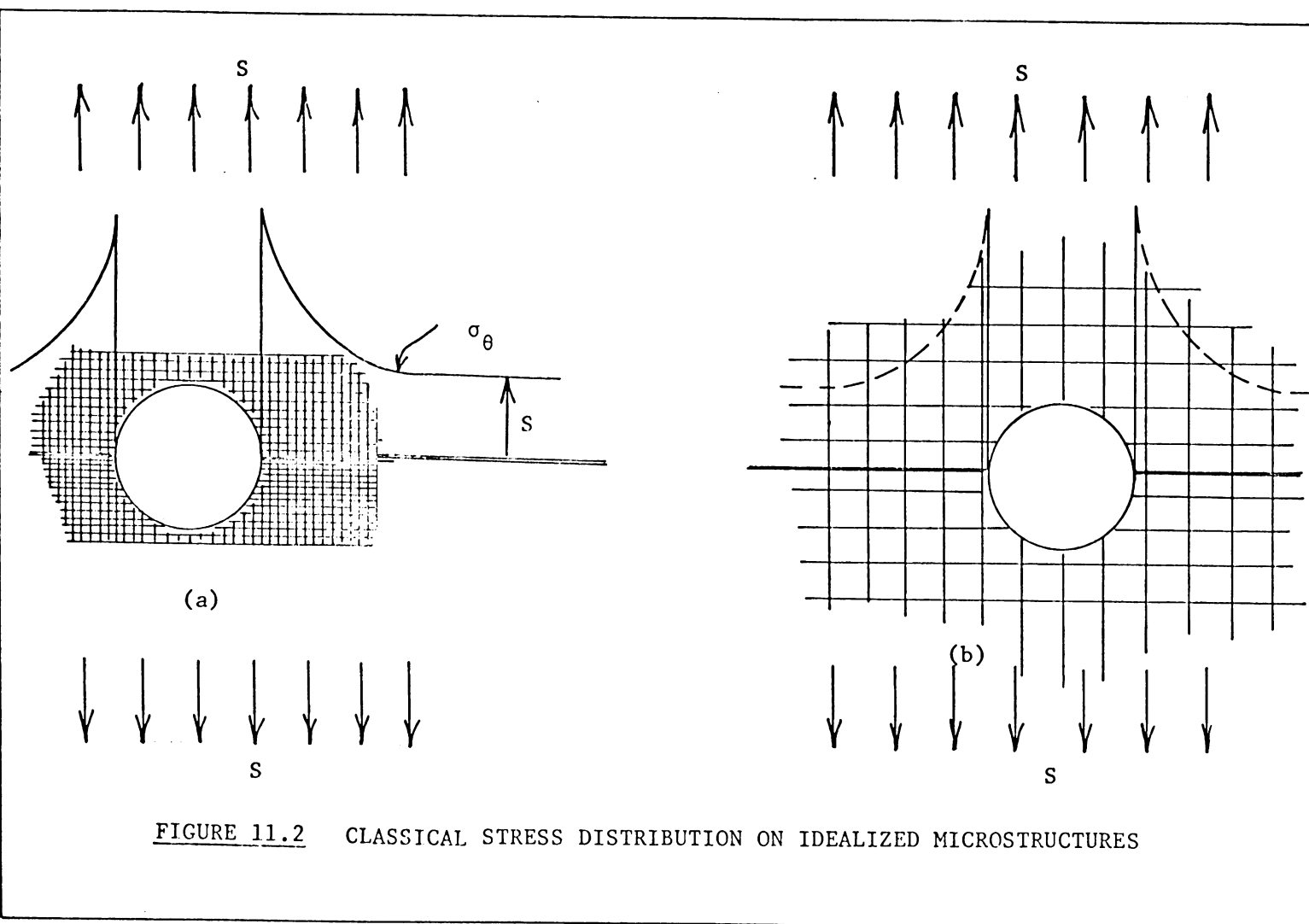


FIGURE 11.2 CLASSICAL STRESS DISTRIBUTION ON IDEALIZED MICROSTRUCTURES

If we allow the fact that ℓ can be considerably smaller than our maximum value, it is quite conceivable that a hole radius at which the ratio of a/ℓ would be expected to give significant couple stress effects could pierce a single grain. Then, perhaps we would need to consider the interaction of the surrounding material on the single grain but the governing factors certainly would involve the symmetry, orientation and mechanical behavior of that grain and not the macroscopic behavior of the material itself.

We are lead to the unmistakable conclusion that, on the one hand, the observed size effects in steel and aluminum are at least one order of magnitude too large to be explained by couple stress and on the other hand, the couple stress effect is so small, at best, that any significant effect we might have expected would occur at a level near the grain size of the material. A conservative interpretation of this lower limit would be that the couple stress effect could give, at the best, only a very approximate idea of behavior. It is felt that the order of this approximation is such that the couple stress effect is not really significant in the mechanics of metallic materials of the types tested. It does not seem reasonable to neglect the inhomogeneity and anisotropy at the microstructural level while retaining such terms.

In our argument we have made extensive use of Mindlin's results (6) for a round hole in an infinite plate since this problem involves well known easy to visualize concepts. Similar conclusions on the general applicability of the couple-stress theory in our materials



PLATE 11.1 STEEL STRUCTURE WITH LARGE (.02") AND SMALL (.002")
HOLES

can be shown for other problems as well. For example, in Dietrich's problems (15) in order to obtain the same order magnitude of effect as was reported for $\ell = .02$, we would have to move from a point $y = .05$ to a point near .005 inches below the loading for $\ell = .002$ inches. Again, this value would be dangerously near the grain size of the material at best and very close to the point of application of the load.

XII. CLOSING REMARKS

The development and effects treated in this research have dealt with the couple stress theory and effects as presented and alluded to by Mindlin (6, 9, 11) Stokes (7), Koiter (10), Wietsman (16) etc. in polycrystalline metals. In amorphous or semi-amorphous materials, made up of discrete elements such as soils, ceramics, concrete etc. couple stress could conceivably be of importance especially where energy of rotation is significant (28). In consideration of single crystals of metals (32) it is also conceivable that, near the atomic level, couple stress theories might be of value in dealing with behavior but this level is well below the order of our consideration. Other theories, based on ideas similar to our couple stress theory (13, 29, 30, 31) have not been considered.

It is evident from our analysis, that for the present at least, the Neuber method is the best that we can offer to handle size effect on an engineering level. One cannot help but wonder, based on our knowledge of microstructure and symmetry properties of metals, why a theory has not been formulated elastically to take these factors into account. On the surface at least, such a theory would certainly seem to give a more realistic model of mechanical behavior and could be formulated in such a way that the statistical nature of grain orientation could be included. This would seem attractive since we know that size effects are quite statistical in nature (see Figure 6.10 and Reference 1). In addition, a theory of this type could, it seems, be developed to include all three regions of the notch size-grain size Spectrum, i.e.

- I grain size very small compared to stress gradient (Classical Theory)
- II grain size such that a significant part of a stress gradient takes place across one grain (Intermediate)
- III grain size large compared to notch size (properties of a single grain govern)

Naturally in such a theory, some simplifying assumptions would have to be made regarding the interaction of grains and the effect of macroscopic properties on the microscopic level. Guided by previous discussions, let us postulate a theory including the effect of microstructure.

We know that as long as stress gradients are moderate compared to grain width, classical elastic theory for isotropic, homogeneous bodies is quite adequate to explain mechanical behavior. It is only the grains near stress raiser, where stress gradients are steep that would have a primary size effect. In addition, we are most interested in the stress across a particular section; that section containing the maximum stress. Let us assume that we can determine the section of maximum stress from classical theory and that classical theory holds over most of the body. However, let us assert that, around the edge of a stress concentration, there is a layer of material which obeys the symmetry of a single crystal of the metal for a single phase alloy or which has the properties of one phase of a multiphase alloy. In keeping with our previous examples, Figure 12.1 shows such

a model for a round hole in an infinite plate.

The thickness of the layer would be determined by a consideration of the average grain size of the material and its orientation in the problem would be considered statistically or at least a most favorable (lowest stress concentration factor) and least probable (highest stress concentration factor) orientation treated. For multiphase alloys, the problem would be repeated for each phase present, using size and properties characteristic of each to get a smallest and largest stress concentration factor.

In the particular case of a round hole in an infinite plate, referring to Figure 12.1, the boundary conditions would be:

$$\sigma_{\theta\theta} = S \quad (r \rightarrow \infty)$$

$$\sigma_i = 0 \quad (r = a)$$

$$\sigma_i(I) = \sigma_i(II) \quad (r = \Omega)$$

$$u_i(I) = u_i(II) \quad (r = \Omega)$$

In the region $r > \Omega$ the ordinary elastic equations would be used and for $a < r < \Omega$, a set of elastic equations based on the symmetry of the metal itself (body centered cubic, face centered cubic, etc.) would be applied. The stress concentration factors, using the coordinate system in Figure 12.1 would be calculated for $\theta = 0$.

If we allowed a/Ω to approach one, our result should reduce to the classical theory while, in the limit of a/Ω as zero, we should have two cases. First for Ω approaching infinity, we would obtain the result for a hole in a plate with symmetry properties of the single grain and second, if a approaches 0 we would have no hole.

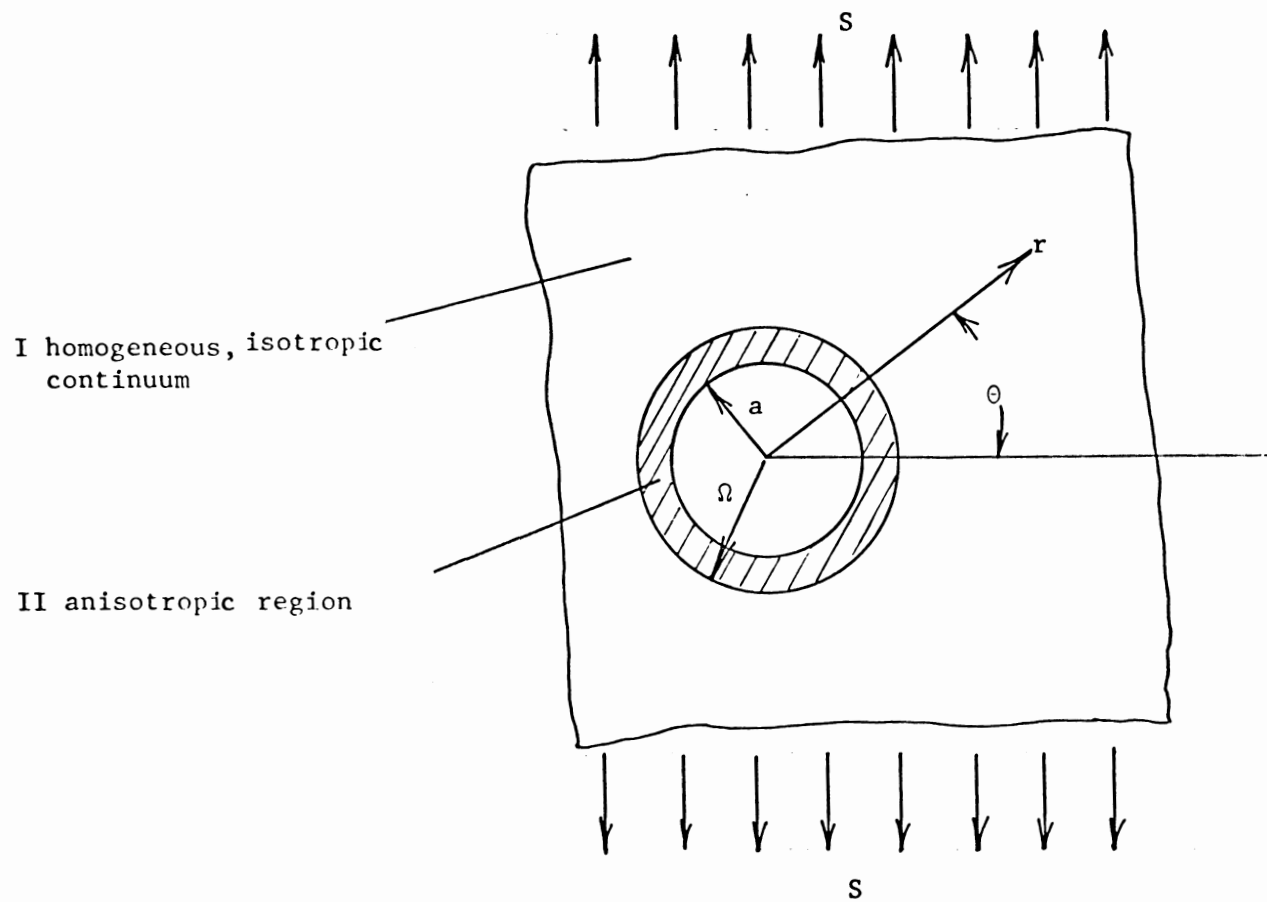


FIGURE 12.1 AN ALTERNATE MODEL TO EXPLAIN THE EFFECTS OF MICROSTRUCTURE

Besides the academic fact that this model considers the effect of microstructure, results based on such calculations might have practical engineering application. Microstructural analysis might be applied on a quantitative basis in mechanics of materials rather than in terms of generalities as it is now. Such an analysis might give important insight into the statistical nature of size effects as well as the effect of preferred orientation, heat treatment and cold work.

With regard to the couple-stress theory of Mindlin, etc., one might wish to devise a set of experiments to measure its constants to an order of magnitude lower than this investigation was intended. To do this, it still appears that plate theory offers the best choice of experiments. Plates should range near .002 inches thick, judging from our examples, to pick up the lower order of magnitude and would be only a few grains thick. Deflections would have to be kept very small (.0002 inches) and measurements would have to be made in at least .00002 inches increments. For these reasons plates would undoubtedly have to be fabricated using special techniques to insure uniformity and grain size control. Test jigs, supports and loading rigs would have to be designed for very small, flexible specimens while no mechanical or electronic (LVDT for example) measuring device which contacts the plate could be used. Optical measurements would probably be needed. In general, the procedure would be indeed difficult to obtain the next lower order of magnitude for ℓ . Still lower order would involve increasingly more elaborate procedures.

One cannot resist asking the question whether this line of research is necessary, on a practical level, for polycrystalline metals in light of our arguments in Chapter XI.

XIII. SUMMARY

The couple stress theory as presented by Mindlin, Koiter, Stokes and others is reviewed with special attention to stress concentration and size effects. That theory is compared on a qualitative and quantitative basis with Neuber's Theory of stress concentration and with observed notch sensitivity effects in aluminum and steel.

A thin plate theory is developed for application to experiments to determine the order of magnitude of the couple stress constant ℓ which is of primary importance in the couple stress theory of stress concentration. A series of experiments is described using the plate theory.

It is found that the couple stress effect in aluminum and steel is at least one order of magnitude too small to account for observed size effects as Mindlin alluded. In addition, the magnitude of ℓ is found to be such that its effects would undoubtedly be clouded or overshadowed by the inhomogeneity and anisotropy of the metals at the microstructural level.

These results are the only experimental results available, to the best knowledge of the author, at the time of completion of this research and represent the next obvious step in the development of the theory which, up to now, has proceeded on a purely mathematical plane.

A theory to account for microstructural effects of anisotropy and inhomogeneity in stress raisers is discussed.

XIV ACKNOWLEDGEMENTS

The author wishes to express appreciation to professor C. W. Smith of the Engineering Mechanics department at Virginia Polytechnic Institute for his aid in the preparation of this dissertation. His significant and material contributions in the theory and experimentation are gratefully acknowledged. The department of Engineering Mechanics and the Engineering Experiment Station of V.P.I. are given thanks for their financial support of the experimental phase of this research.

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XVII APPENDICES

Appendix I Elastic Equations with Couple Stress

In classical elasticity, we consider surface and body forces which act across some surface of a body or on the particles of a body. Here we will also consider that surface and body couples act across surfaces, internal and external, or on the particles of our continuum. The following development follows primarily from the papers of Koiter (10), Stokes (7), and Mindlin (6).

Letting σ_{ij} and μ_{ij} denote the components of the stress (σ_i) and couple stress (μ_i) tensors, already defined in chapter six, and by applying the equilibrium condition to a small tetrahedron (figure A1.1) with three sides coincident with the coordinate axes, we can find the relation between the stress and couple stress vectors and their respective vectors (14). The result of such manipulation is:

$$\begin{aligned}\sigma_i &= \sigma_{ji}n_j \\ \mu_i &= \mu_{ji}n_j\end{aligned}\tag{A1.1}$$

The stress component σ_{12} for example is a force per unit area acting on a plane perpendicular to the y_1 axis in the y_2 direction. Similarly, the couple stress μ_{12} is a couple per unit area on a plane perpendicular to the y_1 axis acting in the y_2 direction (using the right hand rule to denote a couple vector).

In our cartesian coordinate system, σ_{ij} and μ_{ij} are second order tensor quantities and transform as such.

We shall now consider the equilibrium of a region R of some body

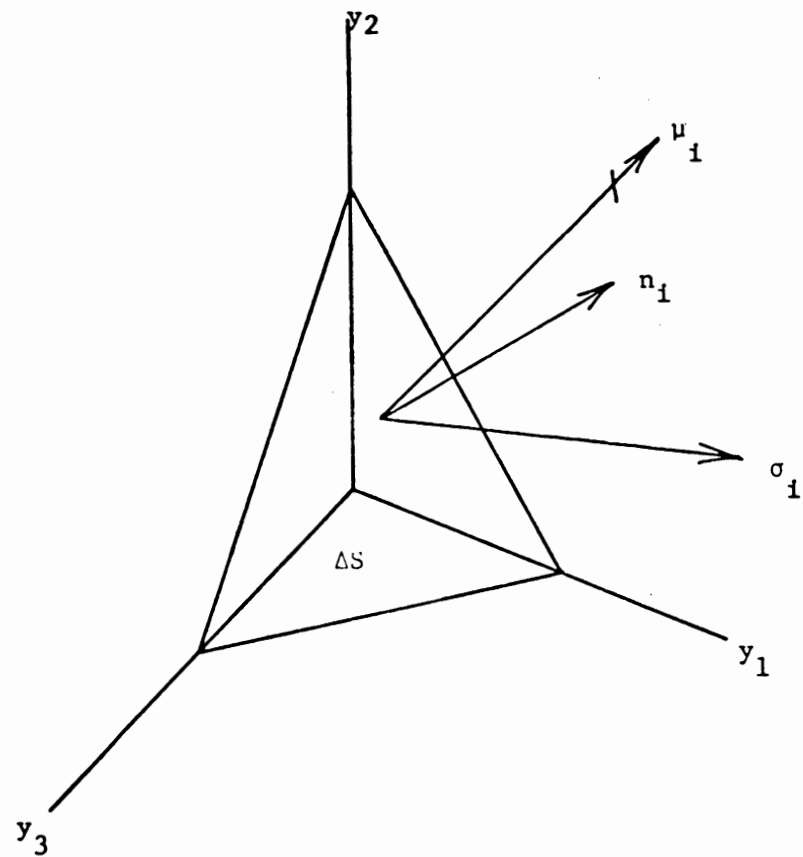


FIGURE A1.1 A FREE BODY OF A PLANE NEAR THE ORIGIN IN A CONTINUUM

bounded by the closed surface S (see figure 6.3). First the force equilibrium equation is found by setting:

$$\Sigma R_i = 0$$

where R_i are the components of all the forces, including inertia forces, acting in the region or on its surface. Thus, including inertia in the body force terms we have:

$$\int_R \rho \cdot f_i + \int_S \sigma_i dS = 0$$

Making use of equation A1.1 and the Gauss theorem, this can be written as:

$$\int_R (\rho \cdot f_i + \sigma_{ji,j}) dv = 0$$

But, since the choice of the region is arbitrary, the integral above must vanish such that:

$$\sigma_{ji,j} + \rho \cdot f_i \quad 6.4$$

The moment equilibrium equation is obtained by setting the resultant couple equal to zero for the region. Thus:

$$\begin{aligned} \int_R \rho \cdot g_i dv + \int_R \epsilon_{ijk} y_j \rho \cdot f_k dv \\ + \int_S \mu_i dS + \int_S \epsilon_{ijk} y_j \sigma_k dS = 0 \end{aligned}$$

applying A1.1 and the Gauss theorem we have, after considering the fact that the choice of the region is arbitrary:

$$\epsilon_{ijk} (y_j \sigma_k)_{,l} + \mu_{li,l} + \rho \cdot g_i + \epsilon_{ijk} y_j \rho \cdot f_k = 0$$

or, making use of the identity $y_{j,i} = \delta_{ji}$;

$$\epsilon_{ijk} \sigma_{jk} + \mu_{li,l} + \rho \cdot g_i = 0 \quad 6.5$$

In classical elasticity where μ_i is neglected and g_i is taken as zero, the above equation leads to the symmetry of the stress tensor.

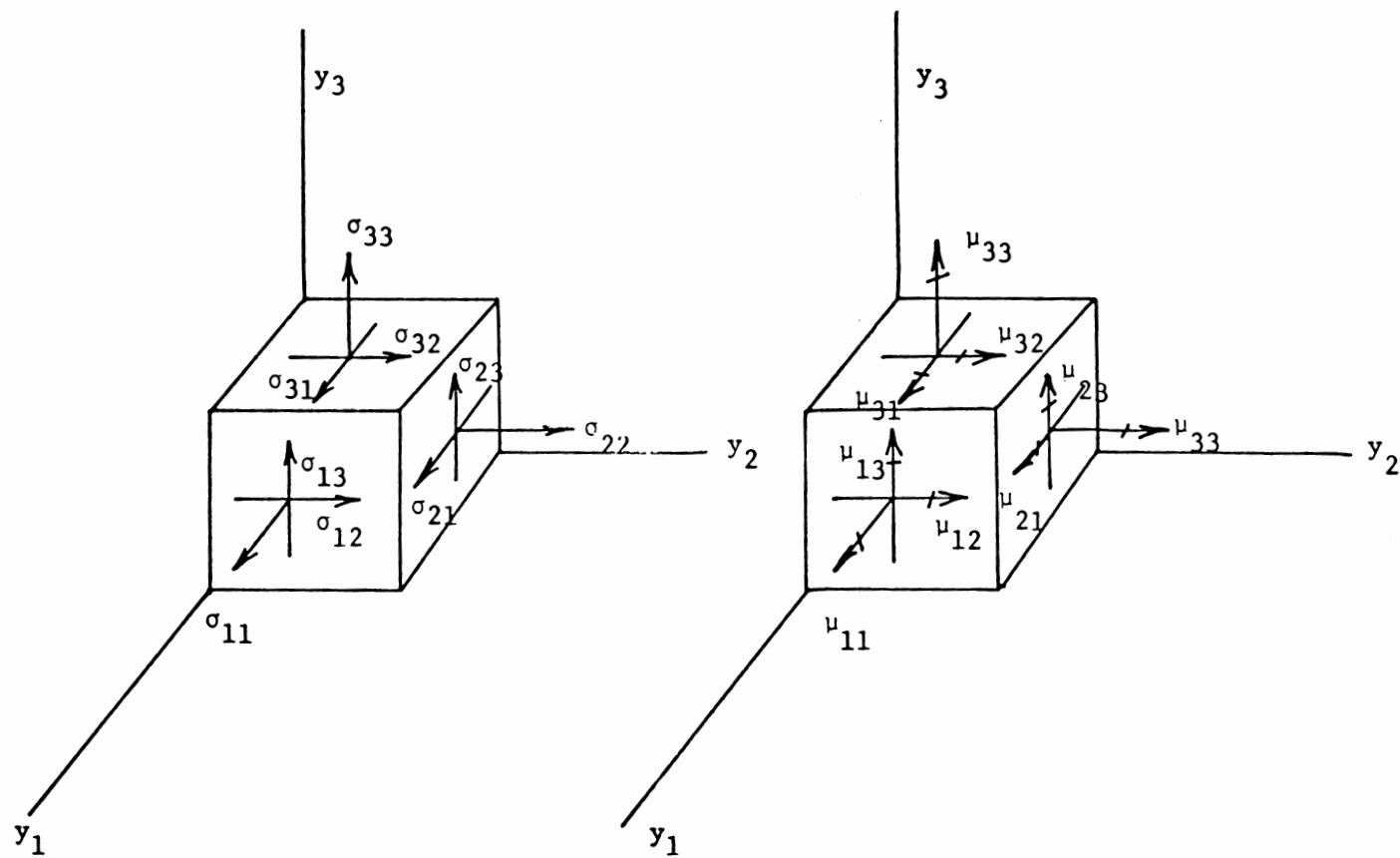


FIGURE A1.2 A POSITIVE STATE OF STRESS AND COUPLE STRESS

Obviously, this is not generally true in the present theory. We can however introduce a symmetric and anti-symmetric stress tensor as shown below:

$$\sigma_{ij}^S = \frac{1}{2} [\sigma_{ij} + \sigma_{ji}] \quad 6.6$$

$$\sigma_{ij}^A = \frac{1}{2} [\sigma_{ij} - \sigma_{ji}] \quad 6.7$$

$$\sigma_{ij} = \sigma_{ij}^S + \sigma_{ij}^A \quad 6.8$$

To further reduce the equilibrium equations, we shall operate on 6.5 to obtain an expression for the anti-symmetric stress tensor in terms of the couple stress tensor and the body moment. We can insert this into 6.4 to obtain a single set of equations (10).

$$\epsilon_{imn} [\epsilon_{ijk} \sigma_{jk} + \mu_{\ell i, \ell} + \rho_0 g_i] = 0$$

making use of the identity $\epsilon_{imn} \epsilon_{ijk} = \delta_{mj} \delta_{nk} - \delta_{mk} \delta_{nj}$ we have:

$$\sigma_{mn} - \sigma_{nm} + \epsilon_{imn} (\mu_{\ell i, \ell} + \rho_0 g_i) = 0$$

or, from 6.6-6.8:

$$\sigma_{mn}^A = -\frac{1}{2} \epsilon_{imn} (\mu_{\ell i, \ell} + \rho_0 g_i) \quad A1.2$$

substituting into 6.4

$$(\sigma_{mn}^S + \sigma_{mn}^A)_{,m} + \rho f_n = 0$$

$$\sigma_{mn,m}^S - \frac{1}{2} \epsilon_{imn} (\mu_{\ell i, \ell m} + \rho_0 g_{i,m}) + \rho f_n = 0$$

or, introducing the couple stress deviator tensor $m_{ij} = \mu_{ij} - \frac{1}{3} \mu_{kk} \delta_{ij}$ the equilibrium equation reduces to

$$\sigma_{mn,m}^S - \frac{1}{2} \epsilon_{imn} [m_{\ell i, \ell m} + (\rho_0 g_i)_{,m}] + \rho_0 f_n = 0 \quad 6.9$$

As has been done in classical continuum mechanics, the small strain

tensor and the small rotation vector are defined in terms of the displacement vector u_i in the following manner:

$$\epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad 6.11$$

$$\omega_i = \frac{1}{2} \epsilon_{ijk} u_{k,j} \quad 6.10$$

The strain tensor is, by definition, symmetric with respect to its range indices. The diagonal terms of the tensor describe the specific change in length parallel to the coordinate axes while the mixed terms are a measure of the change in angle between two lines originally parallel to two coordinate axes.

The components of the rotation vector, give the infinitesimal rotations of an element about the coordinate axes, with the direction of rotation determined by the right hand rule.

The curvature-twist tensor, η_{ij} , is defined as:

$$\eta_{ij} = \omega_{i,j} = \frac{1}{2} \epsilon_{ikl} u_{k,jl} \quad 6.12$$

This tensor is not generally symmetric. Its physical meaning can be seen if we consider the deformation of the region in figure A1.3.

At point $P(y_1, y_2, y_3)$, the small rotation about the y_3 axis is given by ω_3 while at the point $P'(y_1, y_2, y_3 + dy_3)$ that quantity is given by $\omega_3 + \frac{\partial \omega_3}{\partial y_3} dy_3$

Obviously then, the two points undergo a relative twist of

$$\omega_3 + \frac{\partial \omega_3}{\partial y_3} dy_3 - \omega_3 = \frac{\partial \omega_3}{\partial y_3} dy_3$$

The relative twist per unit length along the y_3 axis is then:

$$\frac{\partial \omega_3}{\partial y_3} dy_3 / dy_3 = \frac{\partial \omega_3}{\partial y_3} = \eta_{33}$$

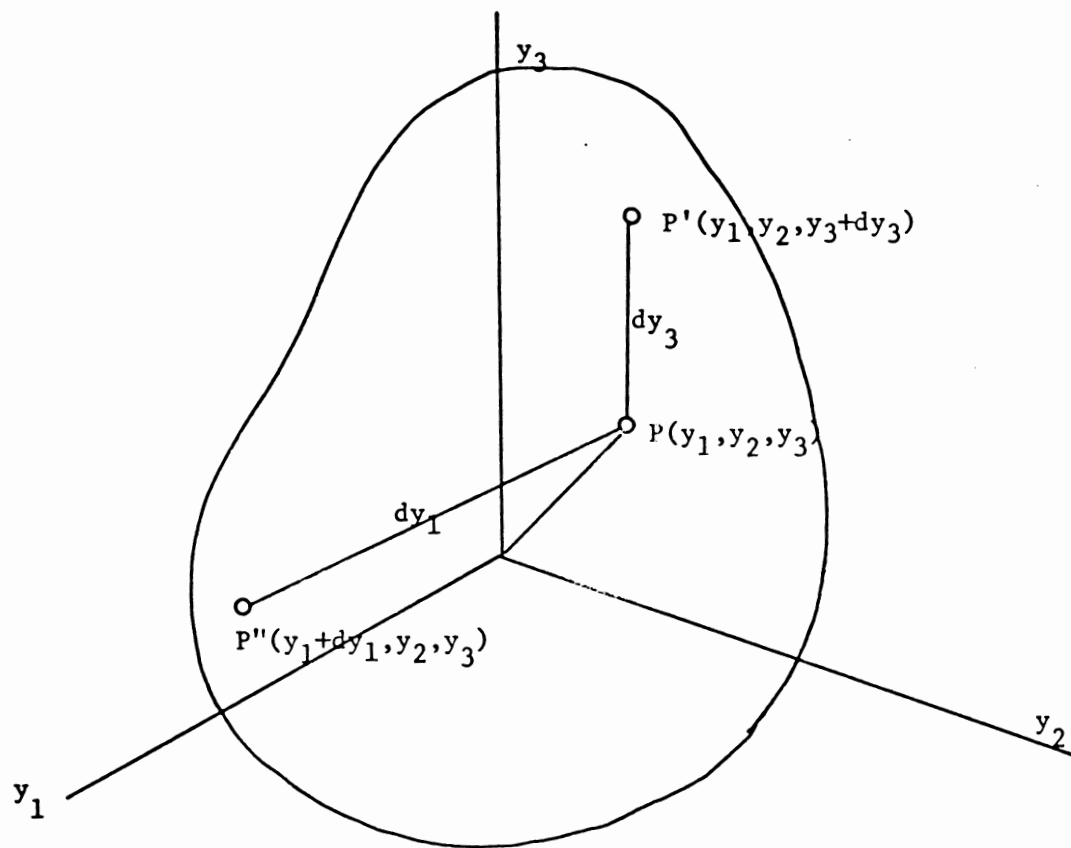


FIGURE A1.3

ADJACENT POINTS IN A CONTINUUM

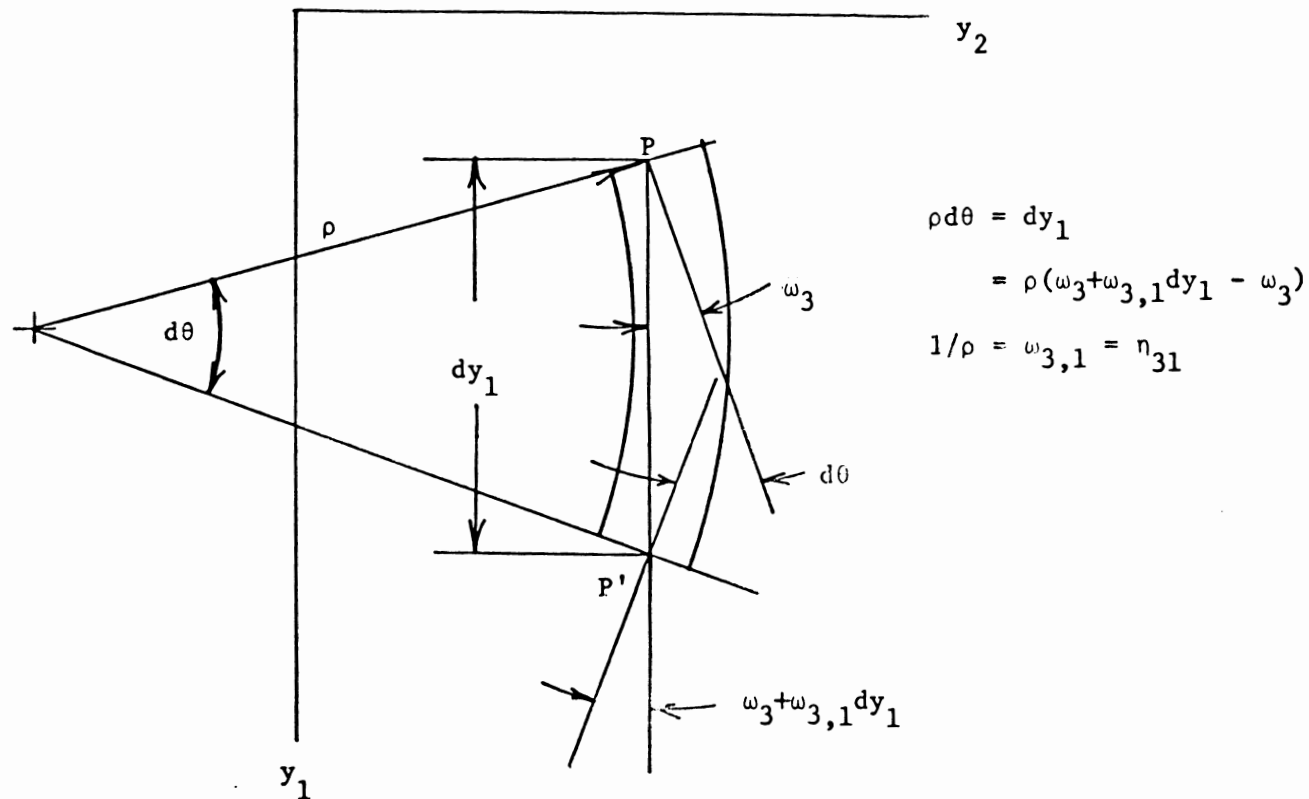


FIGURE A1.4 CURVATURE OF AN ELEMENT IN THE $y_1 - y_2$ PLANE

Similarly, η_{11} and η_{22} are the relative twists per unit length along the y_1 and y_2 axes.

Now, at $P'' (y_1 + dy_1, y_2, y_3)$ the small rotation about the y_3 axes is given by $\omega_3 + \frac{\partial \omega_3}{\partial y_1} dy_1$. If we consider a plane section of the region through the points P and P'' parallel to y_2 as shown in Figure A1.4 we can see that curvature in this plane (normal to the y_3 axes) of the element parallel to y_1 is precisely η_{31} .

Similarly, all of the diagonal terms $\eta_{\alpha\beta}$ can be shown to be curvatures in planes perpendicular to the y_α axes of elements oriented along the y_β axes.

Internal Energy Equation

If the region of Figure 6.3 is allowed to undergo deformation under the influence of the surface and body forces and moments present, in the absence of heat transfer or other influences, the increase in the internal strain energy of the region should by the first law of thermodynamics be equal to the work done by the forces and moments. Thus, if we let u be the internal energy per unit mass, we have:

$$\begin{aligned} \int_R \rho_0 u dv &= \int_S \left(\frac{\sigma_{ji} u_{ji}}{2} + \frac{\mu_{ji} \omega_{ji}}{2} \right) dS \\ &+ \int_R \rho_0 \left(\frac{f_{ji} u_{ji}}{2} + \frac{g_{ji} \omega_{ji}}{2} \right) dv \end{aligned}$$

Making use of A1.1 and the Gauss Theorem this can be written as:

$$\int_R \rho_0 u dv = \int_R \left(\frac{\sigma_{jji} u_{ji}}{2} + \frac{\mu_{jji} \omega_{ji}}{2} \right)_{,j} + \rho_0 \left(\frac{f_{ji} u_{ji}}{2} + \frac{g_{ji} \omega_{ji}}{2} \right) dv$$

Thus:

$$\int_R \rho_0 u dv = \int_R \left[(\sigma_{jji,j} + \rho f_{ji}) \frac{u_{ji}}{2} + \frac{\sigma_{jji} u_{ji,j}}{2} + (\mu_{jji,j} + \rho g_{ji}) \frac{\omega_{ji}}{2} + \mu_{jji} \frac{\omega_{ji,j}}{2} \right] dv$$

applying 6.4 and substituting for σ_{ji} from 6.6-6.8

$$\int_R \rho_0 u dv = \int_R [\sigma_{ji}^S \frac{u_{i,j}}{2} + \sigma_{ji}^A \frac{u_{i,j}}{2} + (\mu_{ji,j} + \rho g_i) \frac{\omega_i}{2} + \mu_{ji} \frac{\omega_{i,j}}{2}] dv$$

Finally, substituting for ω_i and σ_{ij}^A from A1.2 and inserting the identity

$$\sigma_{ji}^S u_{i,j} = \sigma_{ji}^S \epsilon_{ij} \quad \text{we have:}$$

$$\int_R (\rho_0 u - \frac{\sigma_{ji}^S}{2} - \mu_{ji} \frac{\omega_{i,j}}{2}) dv = 0$$

again, since R can be arbitrarily chosen

$$\rho_0 u = \frac{\sigma_{ji}^S \epsilon_{ij}}{2} + \frac{\mu_{ji} \omega_{i,j}}{2}$$

or, in terms of m_{ij} and η_{ij}

$$\begin{aligned} \rho_0 u &= \frac{\sigma_{ji}^S \epsilon_{ij}}{2} + (m_{ji} - \frac{1}{3} \mu_{kk} \delta_{ij}) \frac{\eta_{ij}}{2} \\ &= \frac{\sigma_{ji}^S \epsilon_{ij}}{2} + \frac{m_{ji} \eta_{ij}}{2} - \frac{1}{3} \mu_{kk} \delta_{ij} \frac{\eta_{ij}}{2} \end{aligned}$$

$$\text{but } \delta_{ij} \eta_{ij} = \eta_{ii} = 0 \quad \text{by definition}$$

Thus

$$\rho_0 u = \frac{\sigma_{ij}^S \epsilon_{ij}}{2} + \frac{m_{ij} \eta_{ij}}{2} \quad \text{A1.3}$$

Significantly then, we have shown that the internal energy involves only the symmetric stress tensor and the couple stress deviator tensor. The trace of μ_{ij} and the antisymmetric stress tensor do not appear.

Linear Constitutive Equations for an Isotropic Body

In terms of infinitesimal changes in the displacements and rotations, the first law of thermodynamics can be written, ignoring heat transfer (10,14):

$$\rho_0 du = \sigma_{ji}^S d\epsilon_{ij} + m_{ji} d\eta_{ij}$$

The internal energy is a function (10) of the deformation variables ϵ_{ij}

and η_{ij} . Thus we can write:

$$\rho_0 du = \rho_0 \frac{\partial u}{\partial \epsilon_{ij}} d\epsilon_{ij} + \rho_0 \frac{\partial u}{\partial \eta_{ij}} d\eta_{ij}$$

Therefore:

$$\sigma_{ji}^S = \rho_0 \frac{\partial u}{\partial \epsilon_{ij}}$$

$$m_{ji} = \rho \frac{\partial u}{\partial \eta_{ij}} \quad A1.4$$

In keeping with our assumption of a linear stress-strain, couple stress-curvature relationship we choose $\rho_0 u$ to be a function of the strain and curvature invariant in the following form following Koiter (10).

$$\rho_0 u = G \left[\frac{\nu}{1-2\nu} (\epsilon_{ii})^2 + \epsilon_{ij} \epsilon_{ij} + 2\ell^2 (\eta_{ij} \eta_{ij} + \eta_{ji} \eta_{ji}) \right] \quad A1.5$$

Each term on the right gives a linear result when inserted in A1.4 and the state of no stress coincides with that of no strain. The first invariant of the curvature tensor is zero. Inserting into A1.4 we have

$$\sigma_{ij}^S = 2G \left[\epsilon_{ij} + \frac{\nu}{1-2\nu} \epsilon_{kk} \delta_{ij} \right] \quad 6.13$$

$$m_{ji} = 4G\ell^2 (\eta_{ij} + \eta_{ji}) \quad 6.14$$

Where:

- G is the shear modulus (psi)
- ν is Poisson's Ratio (dimensionless)
- ℓ is Mindlin's constant (length)
- η is a dimensionless constant

The strain energy we will assert to be positive definite for any strain state. Thus G must be positive, ν must be greater than zero and less than 1/2, ℓ must be real and η must be between minus one and plus 1 (10).

This is verified by inspection of A1.5.

It has been established that the symmetric stress tensor leads to ordinary strains. Mindlin (6) and Stokes (7) have shown that the antisymmetric stress tensor leads to a local rigid body rotation.

Boundary Conditions

Koiter (10) showed that the boundary conditions in the couple stress theory reduce to five rather than the six we would expect by allowing the stress vector and the couple stress vector to vary at will on the boundary of our system.

Starting with the six conditions for specifying stress and couple stress we have:

$$\sigma_{ji} n_j = S_i (\zeta)$$

$$\mu_{ji} n_j = T_i (\zeta)$$

where n_i is again the unit normal to the surface and ζ is a variable on the surface.

Using 6.6-6.8, A1.2 and the definition of the couple stress deviator tensor, these relationships can be written:

$$[\sigma_{ji}^S - \frac{1}{2} \epsilon_{mji} (m_{jm, \ell} + \frac{\mu_{kk}}{3} ,_{,m} + \rho g_m)] n_j = S_i$$

$$m_{ji} n_j + \frac{\mu_{kk}}{3} \delta_{ji} n_j = T_i (\zeta)$$

Multiplying the second equation by n_i , we can solve for the trace of the couple stress tensor. Substituting this result above yields:

$$\begin{aligned} \{ \sigma_{ji}^S - \frac{1}{2} \epsilon_{kji} [m_{jk, \ell} - (m_{pq} n_p n_q)_{,k} + \rho g_k] \} n_j \\ = S_i (\zeta) - \frac{1}{2} \epsilon_{jki} (T_p n_p)_{,k} n_j \end{aligned}$$

$$m_{ji}n_j - m_{pq}n_p n_q n_i = T_i(\zeta) - T_p n_p n_i$$

or, using the notation of Koiter we have:

$$\sigma_{ji}^S + \frac{1}{2} \epsilon_{jki} (m_{lk, l} - m_{(nn)},_k + \rho g_k) n_j = \bar{S}_i(\zeta)$$

$$m_{ji}n_j - m_{(nn)}n_i = \bar{T}_i(\zeta)$$

A1.6

where

$$m_{pq}n_p n_q = m_{(nn)} \quad (n \text{ not summed})$$

$\bar{T}_i$ are the tangential components of T_i

$S_i(\zeta) - \frac{1}{2} \epsilon_{jki} (T_p n_p),_i n_j = \bar{S}_i(\zeta)$ are reduced stresses on the surface

Furthermore, at an edge between two adjacent smooth parts of the surface, if μ_{kk} is continuous, Koiter has shown

$$(m_{pq}n_p n_q)_+ - (m_{pq}n_p n_q)_- = (T_p n_p)_+ - (T_p n_p)_-$$

Thus the boundary conditions have been shown to be free from the trace of the couple stress tensor. Since the equilibrium equation 6.9 and the energy equation A1.3 are also free from this quantity, we shall choose it arbitrarily equal to zero (10) without affecting the stresses, strains or rotations.

Compatibility Equations

If we formulate an elasticity problem in terms of stresses, we arrive at the displacements through the six strain displacement equations. Thus we are faced with determination of three unknowns from six conditions.

Obviously, the strains must be under some constraint such that the displacements

thus obtained are unique. To find this constraint, we can simply eliminate displacement from (eqn. 6.11) by operating with the curl operator such that we obtain:

$$\epsilon_{ipq} \epsilon_{jmn} \epsilon_{np,mq} = 0 \quad A1.7$$

Similarly, when couple stress is specified, the curvature twist tensor gives the three components of rotation in terms of nine curvatures. We eliminate rotation from (eqn. 6.12) to obtain:

$$\epsilon_{irs} \eta_{si,r} = 0 \quad A1.8$$

Appendix II - Plane Strain [Stokes (7), Mindlin (6)]

In this section we shall assume that the displacement is a constant in the y_3 direction and that the other displacement are functions of y_1 and y_2 only (plane strain) such that:

$$\begin{aligned} u_1 &= u_1(y_1, y_2) \\ u_2 &= u_2(y_1, y_2) \\ u_3 &= \text{Const.} \end{aligned} \tag{A2.1}$$

Inserting this into (6.11) we immediately see that

$$\begin{aligned} \epsilon_{i3} &= 0 \\ \epsilon_{ij} &= \epsilon_{ij}(y_1, y_2) \end{aligned}$$

From 6.13 we have

$$\begin{aligned} \sigma_{11} &= 2G \left\{ \epsilon_{11} + \frac{\nu}{1-2\nu} (\epsilon_{11} + \epsilon_{22}) \right\} \\ \sigma_{22} &= 2G \left\{ \epsilon_{22} + \frac{\nu}{1-2\nu} (\epsilon_{11} + \epsilon_{22}) \right\} \\ \sigma_{33} &= 2G \left\{ \frac{\nu}{1-2\nu} (\epsilon_{11} + \epsilon_{22}) \right\} \end{aligned}$$

Thus:

$$\sigma_{33} = \nu (\sigma_{11} + \sigma_{22}) \tag{A2.2}$$

Upon inverting (6.13) and inserting (A2.2) we find:

$$\begin{aligned} \epsilon_{11} &= \frac{1+\nu}{E} [\sigma_{11} - \nu (\sigma_{11} + \sigma_{22})] \\ \epsilon_{22} &= \frac{1+\nu}{E} [\sigma_{22} - \nu (\sigma_{11} + \sigma_{22})] \\ \epsilon_{12} &= \frac{1}{2G} \sigma_{12}^S \end{aligned} \tag{A2.3}$$

Inserting the definition (A2.1) into (6.10) we have

$$\omega_1 = \frac{1}{2} (u_{3,2} - u_{2,3}) = 0$$

$$\omega_2 = \frac{1}{2} (u_{1,3} - u_{3,1}) = 0$$

$$\omega_3 = \frac{1}{2} (u_{2,1} - u_{1,2})$$

A2.4

From (A2.4) and (6.12) we have

$$\eta_{31} = \omega_{3,1} = \frac{1}{2} (u_{2,11} - u_{1,12})$$

$$\eta_{32} = \omega_{3,2} = \frac{1}{2} (u_{2,12} - u_{1,22})$$

$$\text{All other } \eta_{ij} = 0$$

A2.5

Now, inserting this result into (6.14) we have:

$$m_{13} = 2G\ell^2 (u_{2,11} - u_{1,12}) = 2G\ell^2 \eta_{31}$$

$$m_{23} = 2G\ell^2 (u_{2,12} - u_{1,22}) = 2G\ell^2 \eta_{32}$$

$$m_{31} = \eta m_{13}$$

$$m_{32} = \eta m_{23}$$

A2.6

Since the stresses are functions of y_1 and y_2 only, the equilibrium equations (6.9) can be written:

$$\sigma_{11,1} + \sigma_{21,2} + \frac{1}{2} (\sigma_{13,12} + \sigma_{23,22}) = 0$$

$$\sigma_{12,1} + \sigma_{22,2} - \frac{1}{2} (\sigma_{13,11} + \sigma_{23,21}) = 0$$

A2.7

The compatibility equations (A1.7, A1.8) are:

$$2 \epsilon_{12,12} = \epsilon_{22,11} + \epsilon_{11,22} \quad (a)$$

$$\eta_{31,2} - \eta_{32,1} = 0 \quad (b)$$

A2.8

Stress Functions

Let us now rewrite (A2.7) in the following form and observe the form of the equation:

$$\frac{\partial \sigma_{11}}{\partial y_1} + \frac{\partial}{\partial y_2} \left(\sigma_{21}^S + \frac{m_{13,1} + m_{23,2}}{2} \right) = 0$$

$$\frac{\partial \sigma_{22}}{\partial y_2} + \frac{\partial}{\partial y_1} \left(\sigma_{12}^S - \frac{m_{13,1} + m_{23,2}}{2} \right) = 0$$

These equations can be identically satisfied if we take two scalar functions such that:

$$\begin{aligned} \sigma_{11} &= \partial \alpha / \partial y_2 & ; & & G_{22} &= \frac{\partial \beta}{\partial y_1} \\ \sigma_{21}^S + \frac{m_{13,1} + m_{23,2}}{2} &= - \partial \alpha / \partial y_1 \\ \sigma_{12}^S - \frac{m_{13,1} + m_{23,2}}{2} &= - \partial \beta / \partial y_2 \end{aligned} \tag{A2.9}$$

where

$$\alpha = \alpha (y_1, y_2) \quad ; \quad \beta = \beta (y_1, y_2)$$

from (A2.8b) and (A2.6) we can write the following identity:

$$m_{13,2} - m_{23,1} = 0 \tag{A2.10}$$

Thus, using the same reasoning as before, (A2.10) can be satisfied if:

$$m_{13} = \partial \psi / \partial y_1 \tag{6.17}$$

$$m_{23} = \partial \psi / \partial y_2 \tag{6.18}$$

as a consequence of (6.17, 6.18) and (A2.6) we also have:

$$m_{31} = \eta \partial \psi / \partial y_1$$

$$m_{32} = \eta \partial \psi / \partial y_2 \tag{A2.11}$$

where

$$\psi = \psi (y_1, y_2)$$

Now, subtracting (A2.9d) from (A2.9b) we have

$$\frac{m_{13,1} + m_{23,2} + m_{13,1} + m_{23,2}}{2} = \frac{\partial \beta}{\partial y_2} - \frac{\partial \alpha}{\partial y_1}$$

substituting for the couple stress terms from (6.17, 6.18) and (A2.11) we can write:

$$\frac{\partial^2 \psi}{\partial y_1^2} + \frac{\partial^2 \psi}{\partial y_2^2} = \frac{\partial \beta}{\partial y_2} - \frac{\partial \alpha}{\partial y_1} \quad \text{A2.12}$$

$$\text{or} \quad \frac{\partial}{\partial y_1} \left(\frac{\partial \psi}{\partial y_1} + \alpha \right) + \frac{\partial}{\partial y_2} \left(\frac{\partial \psi}{\partial y_2} - \beta \right) = 0$$

Obviously, we can satisfy (A2.12) then if:

$$\frac{\partial \psi}{\partial y_1} + \alpha = \partial \phi / \partial y_2$$

$$\frac{\partial \psi}{\partial y_2} - \beta = - \frac{\partial \phi}{\partial y_1}$$

Therefore, (A2.9) can be rewritten as:

$$\begin{aligned} \sigma_{11} &= \frac{\partial}{\partial y_2} \left(\frac{\partial \phi}{\partial y_2} - \frac{\partial \psi}{\partial y_1} \right) = \frac{\partial^2 \phi}{\partial y_2^2} - \frac{\partial^2 \psi}{\partial y_1 \partial y_2} \\ \sigma_{22} &= \frac{\partial}{\partial y_1} \left(\frac{\partial \phi}{\partial y_1} + \frac{\partial \psi}{\partial y_2} \right) = \frac{\partial^2 \phi}{\partial y_1^2} + \frac{\partial^2 \psi}{\partial y_1 \partial y_2} \\ \sigma_{12}^S &= - \frac{1}{2} \left(\frac{\partial \alpha}{\partial y_1} + \frac{\partial \beta}{\partial y_2} \right) = - \frac{\partial^2 \phi}{\partial y_1 \partial y_2} + \frac{1}{2} \left(\frac{\partial^2 \psi}{\partial y_1^2} - \frac{\partial^2 \psi}{\partial y_2^2} \right) \end{aligned}$$

Since the stress functions ϕ and ψ are based on stresses, any problems formulated in terms of these must include the compatability equations (A2.8a) substituting from (A2.3) into (A2.8a) we have

$$\sigma_{11,22} + \sigma_{22,11} - \nu (\sigma_{11} + \sigma_{22})_{,ii} = 2\sigma_{12,12}^S$$

Inserting the relationships (6.19-6.21) into this expression we have, after some algebra

$$\nabla^4 \phi = 0 \quad \text{6.15}$$

where

$$\nabla^4 = \left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) \left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right)$$

Thus we see that ϕ (which corresponds to the Airy stress function in ordinary elasticity) is a biharmonic function.

We have already used (A2.8b) in (A2.10). We must, however, see to it that the displacements obtained by way of the symmetric force stress tensor are compatible with those obtained through the couple stresses. To do this we write (A2.5) in the following form:

$$\eta_{31} = \frac{1}{2} (u_{2,11} - u_{1,12} + u_{1,12} - u_{1,12}) = \epsilon_{21,1} - \epsilon_{11,2}$$

$$\eta_{32} = \frac{1}{2} (u_{2,12} - u_{1,22} + u_{2,21} - u_{2,21}) = \epsilon_{22,1} - \epsilon_{12,2}$$

substituting for the strains and curvatures from (A2.3) and (A2.6) we have:

$$\mu_{13} = \ell^2 \{ \sigma_{21,1}^S - 2 [\sigma_{11,2} - \nu (\sigma_{11,2} + \sigma_{22,2})] \}$$

$$\mu_{23} = \ell^2 \{ -\sigma_{12,2}^S + 2 [\sigma_{22,1} - \nu (\sigma_{11,1} + \sigma_{22,1})] \}$$

Inserting from (6.19-6.21) and (6.17,6.18) we have, after some manipulation:

$$\begin{aligned} \frac{\partial}{\partial y_1} [\psi - \ell^2 \nabla^2 \psi] &= -2(1-\nu) \ell^2 \frac{\partial}{\partial y_2} \nabla^2 \phi \\ \frac{\partial}{\partial y_2} [\psi - \ell^2 \nabla^2 \psi] &= 2(1-\nu) \ell^2 \frac{\partial}{\partial y_1} \nabla^2 \phi \end{aligned} \quad \text{A2.14}$$

We recognize (A2.14) as the Cauchy-Riemann equations from elementary complex variables. Thus, from the properties of this equation we know:

$$\nabla^2 (\psi - \ell^2 \nabla^2 \psi) = 0$$

$$\text{or} \quad \nabla^2 \psi - \ell^2 \nabla^4 \psi = 0 \quad 6.16$$

The relations (6.15) and (6.16) are the governing equations plane strain. Further the stress related functions $\psi - \ell^2 \nabla^2 \psi$ and $2(1-\nu) \ell^2 \nabla^2 \phi$ must be harmonic conjugate functions. (Satisfy one of A2.14).

Appendix III - Comments on Plate Theory Assumptions

The assumptions made in derivation of the thin plate theory were basically the same as those made in the classical theory of thin plates as presented by Wang (22) and others. The first six, at least, are standard.

Upon examination of equation 8.11, one might at first conclude that shear resultants are increased by the addition of $4G\ell^2h$ to the classical rigidity. However, upon consideration of equilibrium, we realize that the stress resultant, for a given loading would be the same with or without couple stress. Obviously then, the gradient of displacement must be reduced as the stiffness of the plate increases due to couple stress.

Since shear strains ϵ_{xz} and ϵ_{yz} depend only on the symmetric force stresses τ_{xz}^S and τ_{yz}^S and since the addition of couple stress only tends to disperse such stresses (15), we can easily conclude that shear deflections will be negligible in our couple stress theory if they are negligible in a classical plate of the same dimension.

In our formulation, we tacitly assumed that the loading on the top surface of the plate was in the form of a distributed normal force only. Thus the tangential components of couple stress on the top surface and bottom surface would be zero. Since the plates must be thin, we can generalize, by an argument similar to that used in plane stress, that the couple stress terms m_{zy} and m_{zx} must be zero or very small throughout the plate. Thus, by examination of equation

6.14 and inclusion of the fact that the rotation about the z axis is zero from previous assumptions, we can see that η_{yz} and η_{xz} must be zero or very small throughout the plate; thus leading to assumption seven.

By similar reasoning if m_{33} is zero on the top and bottom surfaces and the plate is thin, it must be negligible throughout the region (assumption eight). Taking μ_{kk} as zero is consistent with work shown in Appendix I.

ON COUPLE STRESS IN THE
MECHANICS OF MATERIALS

by

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ABSTRACT

The couple stress theory as presented by Mindlin, Koiter, Stokes and others is reviewed with particular attention being given to stress concentration and size effects. That theory is compared on a qualitative and quantitative basis with Neuber's theory for stress concentration for very small or sharp notches and with observed notch sensitivity effects in aluminum and steel.

It is established that, in order to explain observed elastic size effects in these materials, the new material constant ℓ which is characteristic of the couple stress theory must be on the order of magnitude of .02". In order to determine order of magnitude of the constant, a series of experiments was designed based on a thin plate analysis developed here.

The plate theory uses the basic assumptions of classical plate theory as well as several, as explained in the text, necessary when couple stress is included. The formulation resulted in a set of equations for plate deflection including classical and couple stress terms in such a way that when the constant ℓ was allowed to be zero, the classical equations resulted.

Experimentally, it was found that the couple stress effect in steel

and aluminum is at least one order of magnitude too small to account for observed size effects as was suggested by Mindlin. In addition, the magnitude of l was found to be such that its effect would undoubtedly be clouded by inhomogeneity and anisotropy of metals at the microstructural level.

The results represent the only known experimental data at the time of publication dealing with couple stress in metals and would seem to be the next obvious step in the development of the theory which has proceeded on a purely mathematical plane.

An alternate approach to account for the effects of microstructure in notch sensitivity is discussed.