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Modeling and Response Analysis of Thin-Walled Beam Structures Constructed of Advanced Composite Materials

by

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(ABSTRACT)

Thin-walled beam structures are adopted as structural members in various fields of modern technology including aeronautical/aerospacial, naval, mechanical and civil ones.

With the advent of advanced composite material systems, there is a vital need to reformulate the classical theory of thin-walled beams in a wider framework.

This dissertation is intended to incorporate several essential effects which have a considerable importance for the rational design of composite thin-walled beam structures. These effects are the transverse shear deformation, the warping constraint, the secondary warping as well as the hygrothermal and the dynamic ones.

The field equations of laminated composite thin-walled beams of either open or closed single and multicell cross-sections are derived through the application of Hamilton's variational principle. The Laplace Transform technique is used to obtain exact solutions.

In this dissertation, the aeroelastic divergence instability of aircraft wings modelled as thin-walled beams as well as the eigenfrequency problem of cantilevered composite thin-walled beams of closed cross-section are considered in the framework of a refined theory incorporating non-classical effects.

The numerical results reveal the great role played by non-classical effects as well as by the tailoring technique applied to the problems studied in this dissertation.

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Chapter I

Introduction

1.1 Motivation

The requirement of weight-saving and structural efficiency of structural members has stimulated a wide use of thin-walled beam structures in various fields of modern technology encompassing aeronautical/aerospacial, naval, mechanical and civil ones. A thin-walled beam structure may be assimilated with a prismatic or cylindrical shell whose thickness is small compared with any of the cross-sectional dimensions while the cross-sectional dimensions are considerably smaller than the length of the beam. In order to obtain solutions which could be used in practice, the 3-D problem is converted, in the case of thin-walled beams, to an equvalent 1-D one. Thus, based on some simplifying assumptions, the 1-D theory of thin-walled beams has been developed.

The classical bending theory of beams (referred to as the Bernoulli-Euler beam theory) is based on the assumption that the cross-sections, after deformation, remain plane and normal to the bent axis of the beam [1,46,51,60]. The Bernoulli-Euler theory of bending postulates a linear strain distribution across the cross-section and ignores the influence of transverse shear deformations.

As concerns torsion, within St. Venant's type of theory [1,13,34,51], it is assumed that the cross-sections of the beam maintain their original shape although they are free to warp in the axial direction. The warping displacement is postulated to be proportional to the rate of twist, which is assumed to be constant along the beam axis. However, these classical theories of bending and torsion may result in erroneous predictions, especially when the warping constraint is present. The warping constraint introduces a torsion - bending coupling, and as a result, the rate of twist can not be assumed to remain constant along the axis of the beam.

Another element which has constituted a great stimulation towards further development and refinement of the theory of thin-walled beams was the advent of new material systems, such as composites. It is well known that the new and exotic material systems provide outstanding specific strength and stiffness characteristics as well as a flexibility of tailoring the stiffness and strength properties according to prescribed design requirements. The use of composite materials can also result in advantages such as greater corrosion resistance, greater fatigue resistance, better damping and lower thermal expansion characteristics.

However, anisotropic composite material structures exhibit a complex behavior different from that of their metallic counterparts. As has become apparent, in spite of its evident importance, the theory of thin-walled composite beams is the most under-investigated area in the field of composite structures when compared with their plate and shell counterparts. This work is intended to incorporate, in a unified way, several essential effects having importance in the design and the analysis of composite thin-walled beam structures, namely :

(a) Transverse shear deformation

Composite material structures exhibit great flexibility in transverse shear, contradicting the usual assumption of an infinite rigidity in transverse shear postualated by the classical theory [36]. As a result, the transverse shear effect constitutes an important factor in the behavior of composite thin-walled beams and, hence, this effect will be taken into account in the analysis.

(b) The warping restraint effect

As is well known, torsion related nonuniform warping occurs when a section is restrained against out of plane deformation [6,10,20,56,64,72] and/or when a non-uniform distributed torque is applied along the span of the beam. When a cross-section exhibits a warping restraint, St. Venant's principle stipulates that this warping inhibition effect tends to decay away from the restrained section. However, for anisotropic composite material, the decay length can be much larger than that for the conventional isotropic beam counterpart [5]. Therefore, as was reported [14,26,37,38,52,59,66], the free warping assumption may result in erroneous predictions of the behavior of cantilevered type structures (such as, e.g., an aircraft wing). Consequently, the warping restraint effect will be incorporated into the theory.

(c) Incorporation of secondary warping

For metallic thin-walled beam structures, the warping displacement on the middle surface (which is referred to as the primary warping) is usually more predominant over the secondary warping which is assumed to vary across the thickness [20]. As a result, warping displacement is usually assumed to be constant across the thickness and the secondary warping effect was often neglected in previous analyses. However, when the thickness of the wall is not so small compared with the other dimensions of the beam and/or for composite structures which in general exhibit a weak rigidity in transverse shear, the secondary warping may constitute a dominant part of the warping displacement. In addition, for special shapes of the cross-section of beams (e.g., circular cross-section) for which the primary warping displacement vanishes, the secondary warping still exists [51]. In the present work this effect will also be included.

(d) Incorporation of the thermal and hygral fields

In spite of their evident importance, the hygrothermal and dynamic effects have been almost overlooked throughout research works on thin-walled beams considered so far (a very recent work devoted to the free vibration of thin-walled composite beams is due to Rehfield, Atigan, and Hodges [58]). In this dissertation, a structural model incorporating a series of non-classical effects, among them the hygrothermal and dynamic ones, will be developed.

Within this work, the modelling of single and multicell closed cross-section thin-walled beams will be considered. In addition, the refined theory of open cross-section beam structures will be substantiated.

In summary, it is expected that this general theory incorporating in a unified manner a series of important effects will have a significant impact on the design of structures made of thin-walled beams as, e.g., flight vehicle structures, helicopter blades, naval structures, space deployable structures, and civil constructions (as, e.g., bridges), etc.

1.2 Historical Review

A great deal of research activity has been devoted to the substantiation of the theory of thin-walled beams made of conventional isotropic materials. A comprehensive theory of thin-walled beams with open and closed cross-sections was developed in the late 30's by Vlasov [72], and more recently by Gjelsvik [20]. The underlying idea of Vlasov's theory lies in the replacement of the two-dimensional shell quantities with equivalent one-dimensional quantities that are independent of the contour coordinate. Vlasov has introduced the concept of sectorial area connected with the primary warping displacement of the cross-section. The non-uniform warping effect is taken into account through a variable rate of twist along the beam axis. Vlasov has extended his work to the static, dynamic, dynamic stability, and thermoelastic analyses of thin-walled beams.

Gjelsvik [20] extended a theory of metallic thin-walled beams to both open and closed cross-section cases. The secondary warping effect was also introduced in his monograph.

A general theory of thin-walled beams with open cross-section was developed independently by Timoshenko [64] where the warping restraint effect for an I-beam assuming non-constant rate of twist was considered. Furthermore, Timoshenko has introduced the effect of transverse shear strain, in the bending vibration problem, as an additional kinematic variable [63]. A theory relating warping displacements and rotations to the applied torque was developed for thin-walled multicell beams by Benscoter [10]. Gere [19] also has studied the torsional vibration problem of thin-walled open beams including the warping constraint effect. Theories for single and multicell thin-walled beams, particularly of conical and cylindrical shape, have been developed by Argyris and Dunne [2]. Their analysis was concentrated on the refinement of the ordinary engineering theory. The research works mentioned above were devoted to the theory of thin-walled beam structures made of conventional isotropic materials.

For thin-walled beams made of composite materials, several refinements of their theory have been accomplished in the last decade. A thin-walled beam model was developed by Mansfield and Sobey [45] for composite helicopter rotor blade applications. The authors reveal in their study the powerful effects played by the elastic tailoring in rotor blade applications. By using variational principles, Bauchau [5] has derived a thin-walled beam theory based on the assumption of the non-deformability of the cross sections. The engineering theory of bending, stipulating that the initially plane cross-sections remain plane after deformation and normal to the bent axis, is relaxed in order to obtain solutions in agreement with the real behavior of composite structures. Recently, an extension of Vlasov's theory to laminated composite open beams was performed by Bauld and Tzeng [8] in the framework of the classical lamination theory (Love-Kirchhoff assumption). Bauchau, Coffenberry, and Rehfield [6] have presented a comprehensive analysis of composite thin-walled beams with closed cross-section. They have incorporated in their theory two non-classical effects, i.e., the transverse shear deformations and the warping restraint effect which are more significant for composite material structures. Libove [35] has conducted a research for composite closed cross-section thin-walled beams with free warping end condition. In his study, it was shown that the anisotropicity of the structures can result, even without warping inhibition, in a variable rate of twist. Bank and Melehn [3,4] has developed a theory of composite thin-walled beams which accounts for shear deformations in the framework of the Timoshenko beam theory. In his work, in order to incorporate the shear deformation effect for composite structures, a modified shear coefficient was introduced. In addition, his work was expanded for the case of multicell thin-walled composite orthotropic beams.

Rehfield, Hodges, and Atilgan [55-58] have conducted a pioneering and definitive research activity concerning the structural modelling of composite rotor blades. Their thin-walled beam model includes two non-classical effects, namely transverse shear and non-uniform warping, in a unique and rational way. The inclusion of transverse shear effect in the structural model can be considered as an especially significant and creative contribution to the theory of thin-walled beam structures. Their research encompasses both static and dynamic problems of thin-walled beams. Hoddes and Nixson [23] performed FEM analysis to compare with the results from the analytical works.

Soler [62] has extended the theory of thin-walled beams with open cross-section to include the initial pretwist. The resulting theory reveals the existance of elastic coupling between various deformation modes. Krenk [29-32] has presented a static analysis of closed, thin-walled pretwisted beams with variable cross-section. Warping of cross-sections is accounted for by a modified sector coordinate. Tsuiji [66] presents the derivation of equations of motion of pretwisted thin-walled beams under axial loading. In his work the coupled torsional and axial vibrations of a cantilevered beam with open cross-section was analyzed in the framework of the linear theory. The torsional vibration of short aspect ratio rotating metallic beams was treated by Kaza and Kielb [28] where the effects of pretwist and warping restraint on the torsional vibration were considered. Rosen [61] has derived a beam theory describing the nonlinear behavior of pretwisted beams subjected to axial tension and torque. Within his analysis, the St. Venant concept of free torsion was considered. The differential equations of motion were derived for flexural and torsional motions of pretwisted beams applicable to helicopter and propeller blades [25]. Within his work, the classical beam theory was adopted. A notable work [22] was performed by Hodges and Dowell to develop a nonlinear solid beam theory with applications in the dynamic response and aeroelastic stability of helicopter rotor blades. The equations of motion are derived through Hamilton's variational principle. The increase of torsional stiffness in the presence of tensile force is discussed within the framework of pretwisted solid beam analysis [21] by Hodges. All the works mentioned above have considered the case of metallic beams.

Mansfield and Sobey [45] have performed research work concerning the modelling of a composite helicopter blade possessing an initial twist in the unloaded state. Within their work, the initial twist and changes in the initial twist were assumed to be small, so that the analysis was done in a linear framework. A finite element method was used to obtain static and dynamic responses of pretwisted composite solid beams by Bauchau and Hong [7]. The non-classical effects such as transverse shear and warping restraint were included in this work.

An aeroelastic stability analysis of a composite rotor blade was investigated by Hong and Chopra [24]. Within their analysis, the rotor blade was modelled as a single cell rectangular box beam composed of an arbitrary lay-up of composite layers. The coupling of the various stiffnesses is known to be of extreme importance in the aeroelastic stability analysis of composite lifting surfaces [see e.g., 37,38,41,42]. As is well known, by using composite material structures it was possible in the case of the Grumann X29 fighter to prevent, for the first time, the occurrence of the aeroelastic divergence instability, a chronic problem facing the swept-forward wing aircraft.

1.3 Expected Results

As has become apparent, the theory of thin-walled beams is the most under-investigated area in the field of composite structures when compared with the composite plate and shell counterparts. This dissertation is intended to incorporate several essential effects which have a considerable importance in the accurate prediction of the behavior of composite thin-walled beams. These effects are transverse shear deformation and primary and secondary warping constraint effects.

The most general field equations of laminated composite thin-walled beams with either open or closed (single and multicell) cross-sections will be derived in the framework of a refined theory incorporating these non-classical effects. Hamilton's variational principle will be adopted to derive the equations of motion and the associated boundary conditions.

According to the problem at hand, the system of governing equations can be expressed in terms of either displacement variables or stress resultants and stress couples The governing equations with associated boundary conditions will be solved exactly by using a Laplace Transform technique. This solution methodology constitutes another contribution to the state of the art.

A general anisotropic material structure is considered, allowing one to perform a tailoring analysis. Such an analysis is accomplished in chapter 3. The general governing equations derived in this dissertation are expected to be used in the study of the following structural problems:

- Free vibration
- Static aeroelastic instability (divergence)
- Dynamic aeroelastic instability (flutter)
- Static (stress and deformation) analysis
- Buckling analysis
- Thermal flutter analysis
- Dynamic response analysis

Static aeroelastic instability and free vibration problems are already treated in this dissertation.

Chapter II

Kinematics

2.1 General Considerations

Formulation of the kinematic relationships constitutes an important part of the analysis of the behavior of any structure. Moreover, adoption of physically acceptable kinematic assumptions enables one to obtain more realistic predictions of their behavior. A prismatic thin-walled beam and two associated coordinate systems are shown in Fig. 1. A right-handed orthogonal curvilinear coordinate system, (s,z,n), is associated with the middle surface lying midway through the thickness of the thin-walled beam. The contour line (or profile line) is defined as the intersection of the middle surface with a plane normal to the generators. The generator lines are parallel to the axial coordinate of the thin- walled beam. The thin-walled beam is considered as a prismatic one, and hence the wall thickness is assumed to be constant along the axial direction of the beam. However it can be a variable quantity along the contour line, i.e., $h = h(s)$. The configuration and the cross-sectional coordinate systems are shown in Fig. 1. The contour coordinate s has its origin at a conveniently chosen point s_0 (x_0, y_0), and is considered as positive in the direction depicted in Fig. 1.

2.2 *Kinematics of Thin-Walled Open Beams*

2.2.1 Basic Assumptions

A number of the kinematic assumptions traditionally used to substantiate the theory of conventional isotropic thin-walled beams of open cross-section, mainly of a geometric nature, will be adopted in the case of their composite counterparts as well. In addition, by incorporating the non-classical effects which are known to be far more prominent for composite structures than for their conventional metallic counterparts, the refined theory of thin-walled beam structures made of advanced composite materials will rely upon the following kinematic assumptions :

1) The original shape of each cross-section remains unchanged after deformation [20,39,40,55-58,72]. This fact implies that the cross-sections do not deform in their own plane. As a result, the in-plane strains are assumed to be negligibly small when compared with the axial strain. This assumption basically distinguishes the thin-walled beam theory from the two-dimensional shell theory for which the local distortions are taken into account.

2) For conventional metallic beams, owing to the great flexibility of the open cross-section [9,18,48], the membrane shear strain in the middle surface of a beam is usually assumed to be negligible [18-20,64,72]. However, for composite structures , the transverse shear effect is far more significant than for conventional metallic structures. These transverse shear strains will produce membrane shear strain in the mid-plane of thin-walled beams. Then, by using the strain transformation law [16], the membrane shear strain can be expressed in terms of the transverse shear strains. For a thin-walled open beam, the membrane shear strain is assumed to have a linear

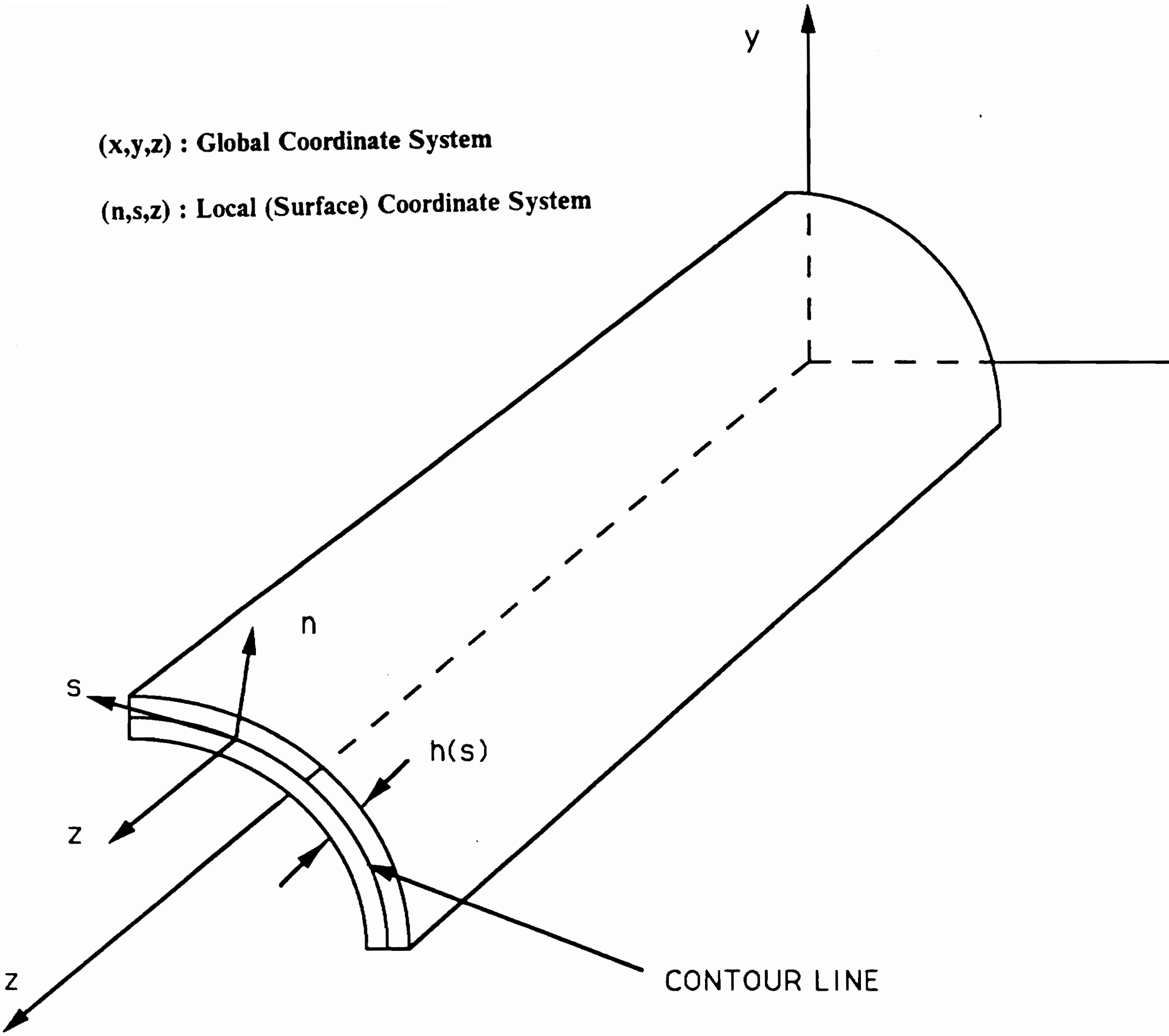


Figure 1. Configuration of a Thin-Walled Beam

variation across the beam thickness when a torsional moment is applied [13,46,50,51]. Therefore, the membrane shear strain, γ_{sz}, can be expressed as

$$\gamma_{sz}(s,z,n) = \gamma_{xz}(z)\frac{dx}{ds} + \gamma_{yz}(z)\frac{dy}{ds} + 2n\phi'(z) \tag{2.1}$$

In Eq 2.1 as well as in the following the primes denote the derivative with respect to the axial coordinate z while $\gamma_{xz}(z)$ and $\gamma_{yz}(z)$ denote the transverse shear strains in the x-z and y-z planes, respectively. The transverse shear components are assumed to be constant for each cross-section and may vary along the longitudinal direction of the beam only. The cross-section, then, is assumed to remain plane and unaffected by transverse shear as in the framework of Timoshenko beam theory. When the transverse shear strains are neglected according to Eq 2.1, the membrane shear strain of the midsurface will become zero. Such a statement is considered in many works dealing with the classical theory of thin-walled beam structures.

(3) The quantity, $\phi'(z)$, referred to as the rate of twist can not be assumed to be constant along the beam axis when warping restraint exists [10,20,56-58,72].

(4) For thin-walled beam structures, the warping displacement of the middle surface (which is referred to as the primary warping) is usually more prominent than the secondary warping which is assumed to vary across the thickness [20,51]. As a result, the secondary warping effect was often neglected in the previous works. When the wall thickness is, however, not small compared with the other dimensions of the beam and/or in the case of composite structures which exhibit a high flexibility in transverse shear, the secondary warping can not be considered negligibly small. In fact, in special cases, it may constitute a dominant part of the warping displacement. For special shapes of the cross-section beams for which the primary warping displacement becomes zero (e.g., square closed cross-section beams), the secondary warping still exists [51].

2.2.2 Kinematics of Thin-Walled Open Beams

Denote the lateral rigid body displacements of a cross-section in the x and y directions as $u_0(z)$ and $v_0(z)$, respectively, and the rotation of a cross-section about the z-axis as $\phi(z)$. Since the cross-section is assumed to be nondeformable in its own plane, the contour of the cross-section can be considered as embedded in a rigid disk lying in the x-y plane. Consequently, these kinematic variables (u_0, v_0, ϕ) are functions of the axial coordinate z only. A number of basic relationships between the displacement of a point A on the contour line and these kinematic variables (u_0, v_0, and ϕ) could be derived (see in this sense Fig. 2). This assumption is fulfilled in practice. It is well known that, e.g., in aeronautics the wing and fuselage structures are reinforced transversely by rigid bulkheads which ensure in fact this nondeformability property. The same feature is exhibited by naval constructions. Consider first several relationships concerning the geometry of thin-walled beam structures. Let us define the position vector $\vec{r}(s)$ of a point located on the mid-line of the thin-walled beam. Let us consider that the arbitrary origin for $\vec{r}$ lies in the plane of the cross-section (see Fig. 3). This gives

$$\vec{r}(s) = x(s)\vec{i} + y(s)\vec{j} \tag{2.2}$$

where $\vec{i}, \vec{j}$, and $\vec{k}$ denote the unit vectors along the x,y, and z axes, respectively. The unit tangent vector to the contour line is

$$\vec{t} = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} \tag{2.3}$$

The outward unit vector normal to the middle surface can be defined as

$$\vec{n} = \vec{t} \times \vec{k} = -\frac{dx}{ds}\vec{j} + \frac{dy}{ds}\vec{i} \tag{2.4}$$

In general, the displacement vector $\vec{\delta}(x,y,z;t)$ can be written as (see Fig. 2)

$$\vec{\delta} = u\vec{i} + v\vec{j} + w\vec{k} \tag{2.5}$$

where u,v, and w denote the displacement components of a point of the beam in the x,y, and z directions, respectively. By considering the geometric relations obtainable from Fig. 4, the lateral displacements u_ϕ and v_ϕ of the point A on the contour line , due to the rotation of the cross-section, can be expressed as

$$\begin{aligned} u_\phi &= x - R\cos(\phi + \alpha) \\ &= x - R(\cos\phi\cos\alpha - \sin\phi\sin\alpha) \\ &= x - R(\cos\phi \frac{x}{R} - \sin\phi \frac{y}{R}) \\ &= x(1 - \cos\phi) + y\sin\phi \\ &\simeq y\phi \end{aligned} \tag{2.6}$$

$$\begin{aligned} v_\phi &= -y + R\sin(\phi + \alpha) \\ &= -y + R(\sin\phi\cos\alpha + \cos\phi\sin\alpha) \\ &= -y + R(\sin\phi \frac{x}{R} + \cos\phi \frac{y}{R}) \\ &= y(-1 + \cos\phi) + x\sin\phi \\ &\simeq x\phi \end{aligned} \tag{2.7}$$

In the above deductions, the angle of rotation ϕ is assumed to be small so that $\sin\phi = \phi$ and $\cos\phi = 1$. In accordance with Fig. 2, the displacement components in the x and y directions, denoted by u and v, respectively, are expressed as

$$u(x,y,z) = u_0(z) - u_\phi(z) = u_0(z) - y\phi(z) \tag{2.8}$$

$$v(x,y,z) = v_0(z) + v_\phi(z) = v_0(z) + x\phi(z) \tag{2.9}$$

It may be shown that the displacement field fulfils the assumption of the cross-section nondeformability, i.e.,

$$\varepsilon_{xx} = \frac{\partial u(y,z)}{\partial x} = 0 \tag{2.10}$$

$$\varepsilon_{yy} = \frac{\partial v(x,z)}{\partial y} = 0 \tag{2.11}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\phi + \phi = 0 \tag{2.12}$$

The membrane shear strain, γ_{sz}, is given by Eq 2.1.:

$$\gamma_{sz}(s,z,n) = \gamma_{xz}(z)\frac{dx}{ds} + \gamma_{yz}(z)\frac{dy}{ds} + 2n\phi'(z) \tag{2.1}$$

By definition, the membrane shear strain is given by

$$\gamma_{sz} = \frac{\partial v_t}{\partial z} + \frac{\partial w}{\partial s} \tag{2.13}$$

where v_t denotes the tangential displacement. It is expressed as :

$$\begin{aligned} v_t &= \vec{\delta} \cdot \vec{t} \\ &= (u\vec{i} + v\vec{j} + w\vec{k}) \cdot (\frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j}) \\ &= u\frac{dx}{ds} + v\frac{dy}{ds} \\ &= (u_0 - y\phi)\frac{dx}{ds} + (v_0 + x\phi)\frac{dy}{ds} \\ &= u_0(z)\frac{dx}{ds} + v_0(z)\frac{dy}{ds} + r_n(s)\phi(z) \end{aligned} \tag{2.14}$$

where (see also Fig. 3)

$$r_n(s) = \vec{r} \cdot \vec{n} = (x\vec{i} + y\vec{j}) \cdot (-\frac{dx}{ds}\vec{j} + \frac{dy}{ds}\vec{i}) = x\frac{dy}{ds} - y\frac{dx}{ds} \tag{2.15}$$

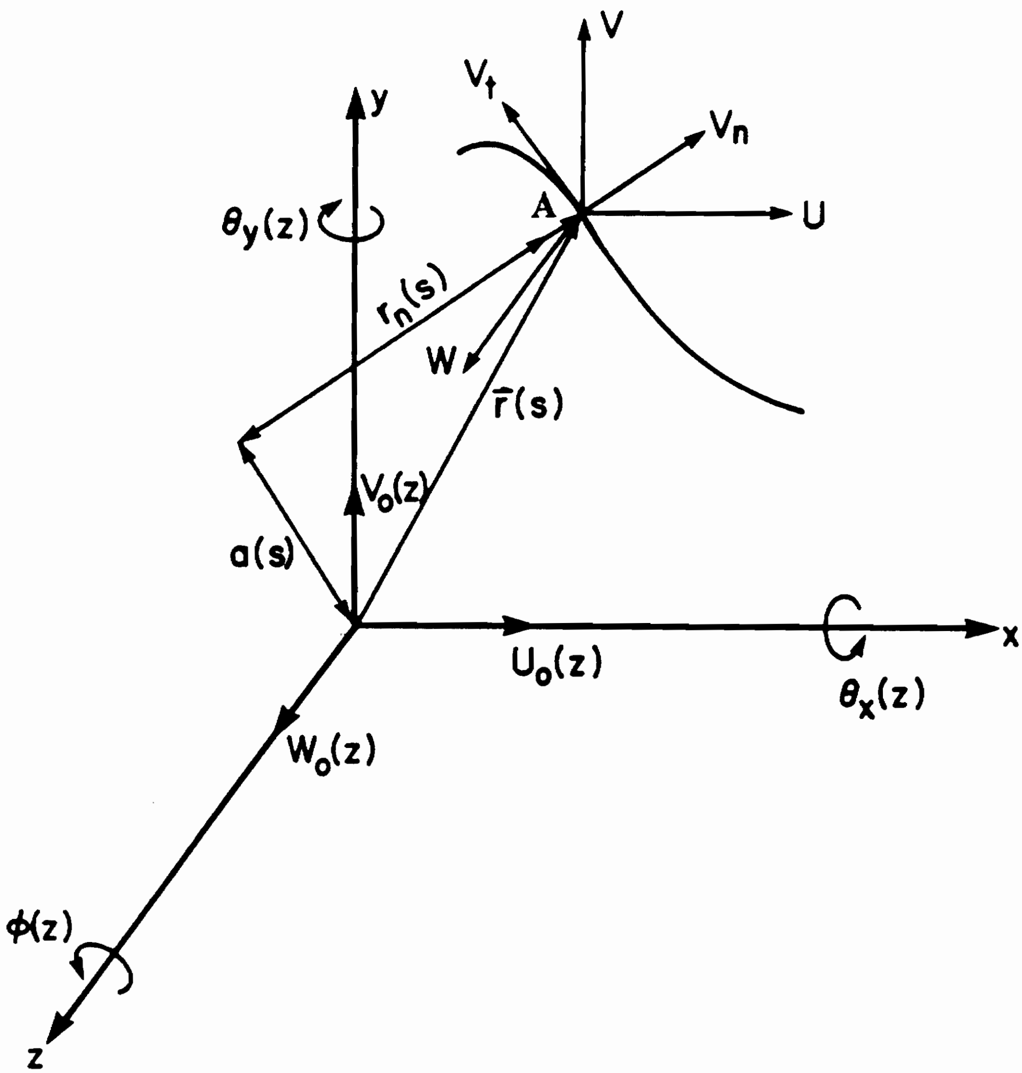
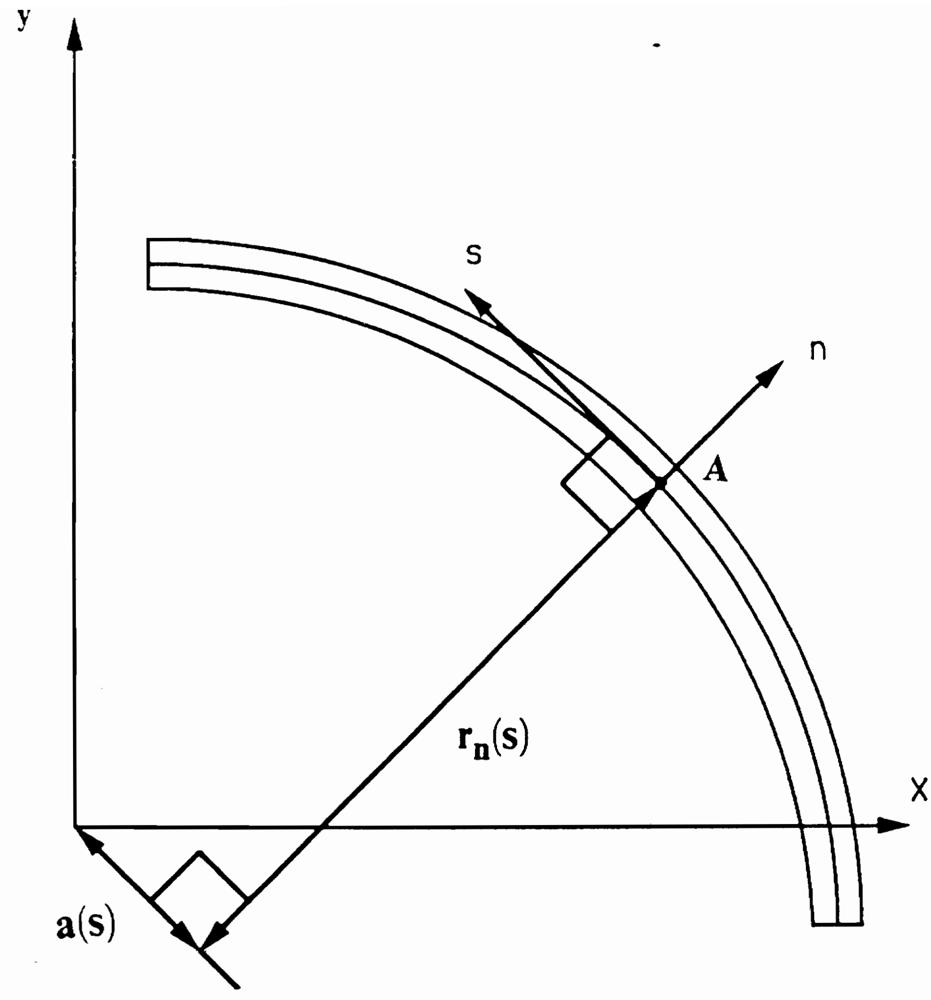


Figure 2. Displacement Field



$$r_n(s) = x\frac{dy}{ds} - y\frac{dx}{ds}$$

$$a(s) = -y\frac{dy}{ds} - x\frac{dx}{ds}$$

Figure 3. Configuration of A Cross-section

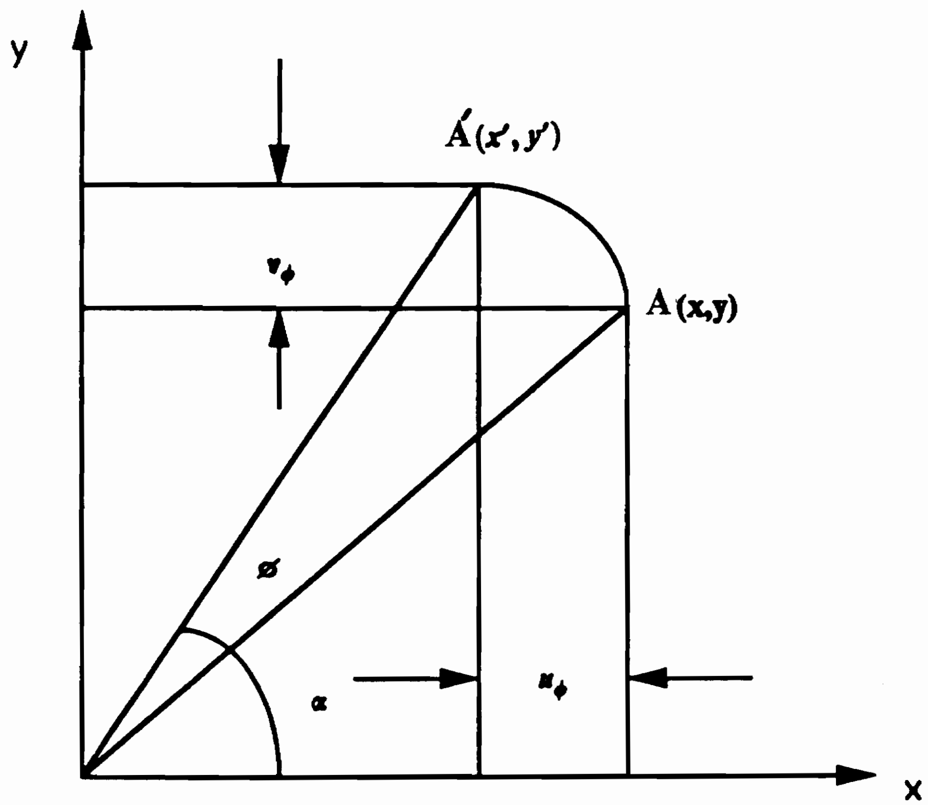


Figure 4. Displacements Due To the Rotation

Replacement of Eq's 2.14 and 2.1 in Eq 2.13 results in

$$\gamma_{xz}(z)\frac{dx}{ds} + \gamma_{yz}(z)\frac{dy}{ds} + 2n\phi'(z) = u_0'(z)\frac{dx}{ds} + v_0'(z)\frac{dy}{ds} + r_n(s)\phi'(z) + \frac{\partial w}{\partial s} \qquad (2.16)$$

Furthermore, multiplication of both sides of Eq. 2.16 by ds followed by its integration around the contour coordinate results in the expression of the axial displacement at an arbitrary point s as follows :

$$w(s,z,n) = \overline{w_0}(z) + (\gamma_{xz} - u_0')\int_0^s \frac{dx}{ds}\,ds + (\gamma_{yz} - v_0')\int_0^s \frac{dy}{ds}\,ds - \phi'(z)\int_0^s [r_n(s) - 2n]ds \quad (2.17)$$

where the origin for s is conveniently chosen. An equivalent form for Eq. 2.17 is

$$.w(s,z,n) = w_0(z) + x(s)\theta_y(z) + y(s)\theta_x(z) - \phi'(z)F_\omega(n,s) \quad (2.18)$$

where

$$w_0(z) = \overline{w_0}(z) - x_0\theta_y(z) - y_0\theta_x(z) \quad (2.19)$$

$$\theta_y(z) = \gamma_{xz}(z) - u_0'(z) \quad (2.20)$$

$$\theta_x(z) = \gamma_{yz}(z) - v_0'(z) \quad (2.21)$$

$$F_\omega(n,s) = \int_0^s [r_n(s) - 2n]ds = \Omega(s) - 2ns \quad (2.22)$$

In Eq 2.19 $\overline{w_0}(z)$ denotes the axial displacement of the origin of the s coordinate while $\theta_y(z)$ and $\theta_x(z)$ indicate the rotations about the x and y coordinates, respectively. The integral on the right-hand side of Eq 2.17 is taken along the contour line. The last terms in Eqs 2.17 and 2.18 represent the torsion related warping displacement of the beam. For $n = 0$, Eq 2.22 reduces to the expression of $\Omega(s)$, referred to as the sectorial area [72]. As shown in Fig. 5, $\Omega(s)$ represents twice the area swept by the radius vector $\vec{r}(s)$. This sectorial area is connected with the warping of the cross-section and will be referred to as the primary warping function. The term 2n in Eq 2.22 is referred to as the torsional function for thin-walled open beams.

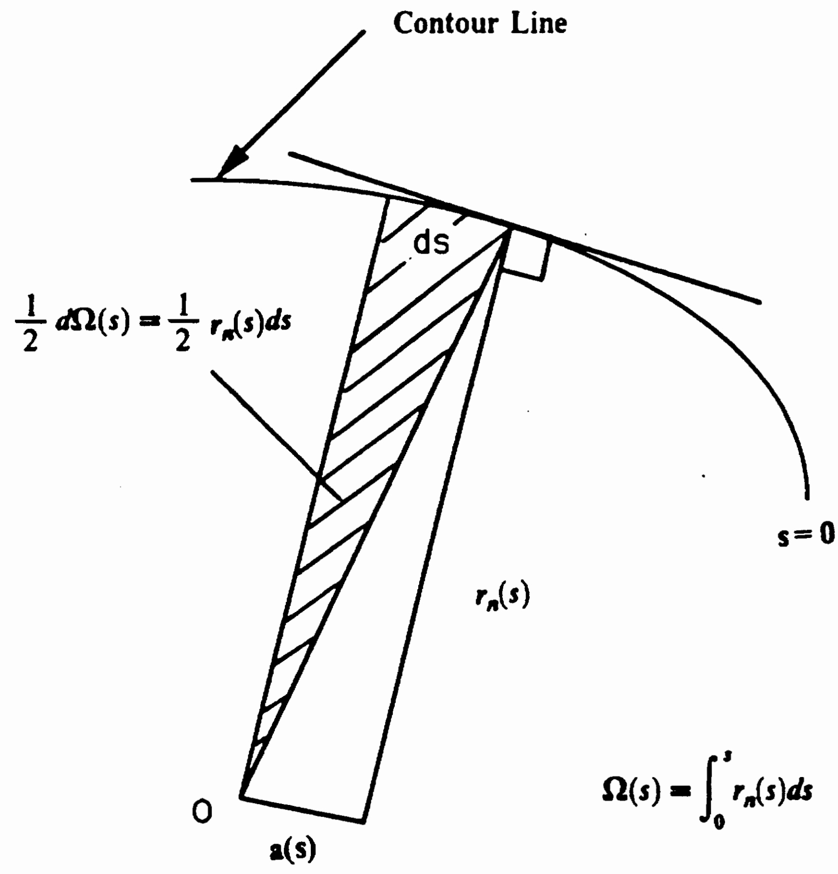


Figure 5. Sectorial Area

Consistent with Eq 2.18 the axial strain, without considering the secondary warping effect, can be expressed as

$$\varepsilon_{zz}(s,z,n) = w_0'(z) + \theta_y'(z)x(s) + \theta_x'(z)y(s) - \phi''(z)F_\omega(n,s) \tag{2.23}$$

Next, the secondary warping effect is going to be considered. According to the strain transformation law, the shear strain in the n-z plane, γ_{nz}, can be expressed in terms of transverse shear strain components as:

$$\gamma_{nz}(s,z) = \gamma_{xz}(z)\frac{dy}{ds} - \gamma_{yz}(z)\frac{dx}{ds} = [\theta_y(z) + u_0'(z)]\frac{dy}{ds} - [\theta_x(z) + v_0'(z)]\frac{dx}{ds} \tag{2.24}$$

By definition

$$\gamma_{nz} = \frac{\partial w^s}{\partial n} + \frac{\partial v_n}{\partial z} \tag{2.25}$$

where w^s denotes the secondary warping displacement while v_n denotes the displacement component in the direction of the outward normal to the middle surface as shown in Fig. 3. It is given by

$$\begin{aligned} v_n &= \vec{\delta}\cdot\vec{n} \\ &= (u\vec{i} + v\vec{j} + w\vec{k})\cdot\left(-\frac{dx}{ds}\vec{j} + \frac{dy}{ds}\vec{i}\right) \\ &= u\frac{dy}{ds} - v\frac{dx}{ds} \\ &= (u_0 - y\phi)\frac{dy}{ds} - (v_0 + x\phi)\frac{dx}{ds} \\ &= u_0(z)\frac{dy}{ds} - v_0(z)\frac{dx}{ds} + a(s)\phi(z) \end{aligned} \tag{2.26}$$

where (see Fig. 2)

$$a(s) = -\left(y\frac{dy}{ds} + x\frac{dx}{ds}\right) \tag{2.27}$$

Consideration of Eqs 2.24 and 2.26 in 2.25 yields

$$\frac{\partial w^s}{\partial n} = \theta_y\frac{dy}{ds} - \theta_x\frac{dx}{ds} - a(s)\phi' \tag{2.28}$$

Integration of Eq. (2.28) yields the warping displacement expressed as

$$w^s(s,z,n) = [\theta_y(z)\frac{dy}{ds} - \theta_x(z)\frac{dx}{ds} - a(s)\phi'(z)]n \tag{2.29}$$

or alternatively as

$$w^s(s,z,n) = [\{\gamma_{xz}(z) - u_0'(z)\}\frac{dy}{ds} - \{\gamma_{yz}(z) - v_0'(z)\}\frac{dx}{ds} - a(s)\phi'(z)]n \tag{2.30}$$

Eq. (2.30) implies that the secondary warping has a linear variation across the wall thickness. By virtue of Eqs. (2.18) and (2.30) the total axial displacement results in

$$w(s,z,n) = w_0(z) + \theta_y(z)[x(s) + n\frac{dy}{ds}] + \theta_x(z)[y(s) - n\frac{dx}{ds}] - \phi'(z)[\Omega(s) - 2ns + na(s)] \tag{2.31}$$

where $\Omega(s)$ and $na(s)$ are referred to as the primary(contour) and the secondary(thickness) warping functions, respectively. Setting n equal to zero in Eq. (2.31) results in the expression for the primary warping. Based on Eq. 2.31 the total axial strain is expressed as:

$$\begin{aligned}\varepsilon_{zz}(s,z,n) = {} & w_0'(z) + \theta_y'(z)[x(s) + n\frac{dy}{ds}] + \theta_x'(z)[y(s) - n\frac{dx}{ds}] \\ & -\phi''(z)[\Omega(s) - 2ns + na(s)]\end{aligned} \tag{2.32}$$

In an equivalent form, the axial strain can be represented as:

$$\varepsilon_{zz}(s,z,n) = \varepsilon_{zz}^{0}(s,z) + n\,\varepsilon_{zz}^{n}(s,z) \tag{2.33}$$

where

$$\varepsilon_{zz}^{0}(s,z) = w_0'(z) + \theta_y'(z)x(s) + \theta_x'(z)y(s) - \phi''(z)\Omega(s) \tag{2.34}$$

and

$$\varepsilon_{zz}^{n}(s,z) = \theta_y'\frac{dy}{ds} - \theta_x'\frac{dx}{ds} - \phi''(z)[a(s) - 2s] \tag{2.35}$$

In an alternative form Eqs 2.34 and 2.35 are expressed as:

$$\varepsilon_{zz}^{0}(s,z) = w_0'(z) + [\gamma_{xz}'(z) - u_0''(z)]x(s) + [\gamma_{yz}'(z) - v_0''(z)]y(s) - \phi''(z)\Omega(s) \tag{2.36}$$

$$\varepsilon_{zz}^{n}(s,z) = [\gamma_{xz}'(z) - u_0''(z)]\frac{dy}{ds} - [\gamma_{yz}'(z) - v_0''(z)]\frac{dx}{ds} - \phi''(z)[a(s) - 2s] \tag{2.37}$$

When the non-classical effects are discarded, i.e., when the transverse shear strain components $\gamma_{xz}(z)$ and $\gamma_{yz}(z)$, the torsion related primary warping, and the secondary warping effect are not considered in the analysis, the expression of the axial strain given by Eq 2.31 reduces to the one encountered in the classical beam theory, i.e.,

$$\varepsilon_{zz}(s,z) = w_0'(z) - u_0''(z)x(s) - v_0''(z)y(s) \tag{2.38}$$

Next, the membrane shear strain is expressed as

$$\gamma_{sz}(s,z,n) = \gamma_{xz}(z)\frac{dx}{ds} + \gamma_{yz}(z)\frac{dy}{ds} + 2n\phi'(z) \tag{2.1}$$

or in different notation as

$$\gamma_{sz}(s,z,n) = [\theta_y(z) + u_0'(z)]\frac{dx}{ds} + [\theta_x(z) + v_0'(z)]\frac{dy}{ds} + 2n\phi'(z) \tag{2.39}$$

When the transeverse shear strains, $\gamma_{xz}(z)$ and $\gamma_{yz}(z)$, are not included in the analysis, Eq 2.1 reduces to the expression for the membrane shear strain of thin-walled open beams subjected to pure torsion in the framework of the classical theory. The shear strain in the n-z plane, γ_{nz}, was also expressed by Eq 2.24:

$$\gamma_{nz}(s,z) = \gamma_{xz}(z)\frac{dy}{ds} - \gamma_{yz}(z)\frac{dx}{ds} = [\theta_y(z) + u_0'(z)]\frac{dy}{ds} - [\theta_x(z) + v_0'(z)]\frac{dx}{ds} \tag{2.24}$$

According to the assumption of rigid cross-sections, the other strain components in the plane of the cross-section are zero. Six kinematic variables (u_0, v_0,w_0,θ_y,θ_x,ϕ) which are functions of z only can wholly define the displacement field and, hence, the strain field. Within the framework of the St.Venant torsion model, the rate of twist, ϕ', is assumed to be constant along the beam axis so that the last term in Eq 2.19 becomes immaterial. It is evident that, within the framework of the classical St. Venant theory, there is no axial strain produced by twisting (see Eq 2.23).

2.3 *Kinematics of Thin-Walled Closed Beams-Single Cell*

2.3.1 Basic Assumptions

The assumption of rigid cross-sections formulated in section 2.2.1 for open cross-section beams will be retained for closed cross-section beams also. For this case, in contrast to the case of open beams, the membrane shear strain in the middle surface due to torsion can not be assumed to be zero. Such an assumption would result in a closed beam infinitely rigid in torsion, i.e., in a beam whose shear modulus goes to infinity. Let $\gamma_t(z)$ be a membrane shear strain due to torsion. For a closed thin-walled beam, the shear flow and the membrane shear strain associated with torsion is assumed to be independent of the contour coordinate s. Taking into account transverse shear and torsional strain components, the membrane shear strain of the middle surface can be expressed as [55]:

$$\gamma_{sz}(s,z) = \gamma_{xz}(z)\frac{dx}{ds} + \gamma_{yz}(z)\frac{dy}{ds} + \gamma_t(z) \tag{2.40}$$

where $\gamma_t(z)$ denotes the shear strain due to torsion. This function will be determined in the future analysis. The secondary warping effect is included in the analysis of the closed beam in the same manner as in the case of open beams.

2.3.2 Kinematics of Thin-Walled Closed Beams

By definition, the membrane shear strain in the middle surface of a closed beam can be expressed as

$$\gamma_{sz}(s,z) = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z} \tag{2.13}$$

The displacement component tangent to the contour line was expressed as

$$v_t(s,z) = u_0(z)\frac{dx}{ds} + v_0(z)\frac{dy}{ds} + r_n(s)\phi(z) \tag{2.14}$$

Inserting Eqs 2.40 and 2.14 into Eq 2.13 yields

$$\frac{\partial w}{\partial s} = [\gamma_{xz}(z) - u_0'(z)]\frac{dx}{ds} + [\gamma_{yz}(z) - v_0'(z)]\frac{dy}{ds} + \gamma_t(z) - r_n(s)\phi'(z) \tag{2.41}$$

or alternatively

$$\frac{\partial w}{\partial s} = \theta_y(z)\frac{dx}{ds} + \theta_x(z)\frac{dy}{ds} + \gamma_t(z) - r_n(s)\phi'(z) \tag{2.42}$$

For the closed beams, the axial displacement should be continuous around the perimeter of the cross-section and should satisfy the periodicity condition

$$\int_C \frac{\partial w}{\partial s}\, ds = 0 \tag{2.43}$$

Substitution of Eq 2.42 into Eq 2.43 yields

$$\theta_y(z)\int_C \frac{dx}{ds}\, ds + \theta_x(z)\int_C \frac{dy}{ds}\, ds + \int_C \gamma_t(z) ds - \phi'(z)\int_C r_n(s) ds = 0 \tag{2.44}$$

However, it can be easily verified, for closed cross-sections, that

$$\int_C \frac{dx}{ds}\, ds = \int_C \frac{dy}{ds}\, ds = 0 \qquad (2.45)$$

Eq.2.44 then gives

$$\gamma_t(z)\int_C ds = \phi'(z)\int_C r_n(s)ds \qquad (2.46)$$

Then

$$\gamma_t(z) = \phi'(z)\,\frac{2A_c}{\beta} \qquad (2.47)$$

where

$$2A_c = \int_C r_n(s)\, ds \qquad (2.48)$$

$$\beta = \int_C ds \qquad (2.49)$$

A_c denotes the cross-sectional area bounded by the mid-line of the contour while β represents the total length of the mid-line of the contour. The axial displacement of the middle surface is finally given by

$$\begin{aligned} w(s,z) = w_0(z) + [\gamma_{xz}(z) - u_0'(z)]x(s) + [\gamma_{yz}(z) - v_0'(z)]y(s) \\ -\phi'(z)\int_0^s r_n(s)ds + \frac{s}{\beta}\,\phi'(z)\int_C r_n(s)ds \end{aligned} \qquad (2.50)$$

or in different notation

$$w(s,z) = w_0(z) + \theta_y(z)x(s) + \theta_x(z)y(s) - \phi'(z)F_\omega(s) \tag{2.51}$$

where $F_\omega(s)$ is referred to as the warping function for the closed cross-section beam and has the form:

$$F_\omega(s) = \int_0^s [r_n(s) - 2\frac{A_c}{\beta}]ds = \Omega(s) - 2\frac{A_c}{\beta}s \tag{2.52}$$

where $2\frac{A_c}{\beta}$ is the torsional function for thin-walled closed beams denoted as:

$$\Psi_T = 2\frac{A_c}{\beta} \tag{2.53}$$

It is found that the warping function for closed beams is a function of s only. The axial strain component is derived from the displacement field given by Eq 2.50 or Eq 2.51. The axial strain of the middle surface is

$$\varepsilon_{zz}(s,z) = w_0'(z) + \theta_y'(z)x(s) + \theta_x'(z)y(s) - \phi''(z)F_\omega(s) \tag{2.54}$$

An alternative form of Eq. 2.54 is

$$\varepsilon_{zz}(s,z) = w_0'(z) + [\gamma_{xz}(z)' - u_0''(z)]x(s) + [\gamma_{yz}(z)' - v_0''(z)]y(s) - \phi''(z)F_\omega(s) \tag{2.55}$$

By following the procedure devised previously for the case of open cross-section beams, the secondary warping of closed beams results in the same expression as that for open beams. For the sake of completeness, the expression of the secondary warping for the open beam is repeated here:

$$w^s = [\theta_y(z)\frac{dy}{ds} - \theta_x(z)\frac{dx}{ds} - a(s)\phi'(z)]n \tag{2.29}$$

or

$$w^s = [\{\gamma_{xz}(z) - u_0'(z)\} \frac{dy}{ds} - [\{\gamma_{yz}(z) - v_0'(z)\}] \frac{dx}{ds} - a(s)\phi'(z)]n \qquad (2.30)$$

By adding this secondary warping displacement to the primary one, given by Eq 2.51, the total axial displacement for closed section beams is given by

$$w(s,z,n) = w_0(z) + \theta_y(z)[x(s) + n \frac{dy}{ds}] + \theta_x(z)[y(s) - n \frac{dx}{ds}] - \phi'(z)[F_\omega(s) + na(s)] \qquad (2.56)$$

where $F_\omega(s)$ and $na(s)$ are referred to as the primary (contour) and the secondary (thickness) warping functions, respectively. Setting n equal to zero in Eq. 2.56 results in an expression of the warping displacement which does not account for the secondary warping effect.

The axial strain can be written as

$$\begin{aligned}\varepsilon_{zz}(s,z,n) = w_0'(z) + \theta_y'(z)[x(s) + n \frac{dy}{ds}] + \theta_x'(z)[y(s) - n \frac{dx}{ds}] \\ -\phi''(z)[F_\omega(s) + na(s)]\end{aligned} \qquad (2.57)$$

or equivalently as:

$$\varepsilon_{zz}(s,z,n) = \varepsilon_{zz}^{0}(s,z) + n\, \varepsilon_{zz}^{n}(s,z) \qquad (2.58)$$

where

$$\varepsilon_{zz}^{0}(s,z) = w_0'(z) + \theta_y'(z)x(s) + \theta_x'(z)y(s) - \phi''(z)F_\omega(s) \qquad (2.59)$$

$$\varepsilon_{zz}^{n}(s,z) = \theta_y' \frac{dy}{ds} - \theta_x' \frac{dx}{ds} - \phi''(z)a(s) \qquad (2.60)$$

Other equivalent forms for Eqs 2.59 and 2.60 are

$$\varepsilon_{zz}^{0}(s,z) = w_0'(z) + [\gamma_{xz}'(z) - u_0''(z)]x(s) + [\gamma_{yz}'(z) - v_0''(z)]y(s) - \phi''(z)F_\omega(s) \qquad (2.61)$$

$$\varepsilon_{zz}^{n}(s,z) = [\gamma_{xz}'(z) - u_0''(z)] \frac{dy}{ds} - [\gamma_{yz}'(z) - v_0''(z)] \frac{dx}{ds} - \phi''(z)a(s) \qquad (2.62)$$

As in the case of open cross-section beams, the last term in Eq. 2.57 is associated with the non-uniform warping strain. When the non-classical effects are not included in the anlysis, the axial strain given by Eq 2.57 is reduced to the one associated with classical beam theory:

$$\varepsilon_{zz}(s,z) = w_0'(z) - u_0''(z)x(s) - v_0''(z)y(s) \tag{2.63}$$

The membrane shear strain was given by Eq 2.40:

$$\gamma_{sz}(s,z) = \gamma_{xz}(z)\frac{dx}{ds} + \gamma_{yz}(z)\frac{dy}{ds} + \gamma_t(z) \tag{2.40}$$

When the transverse shear strains, $\gamma_{xz}(z)$ and $\gamma_{yz}(z)$, are discarded, Eq 2.40 is reduced to the expression of the classical membrane shear strain of thin-walled closed section beams subjected to a pure torsion. The membrane shear strain is obtained by considering Eqs 2.40 and 2.47:

$$\gamma_{sz}(s,z) = \gamma_{xz}(z)\frac{dx}{ds} + \gamma_{yz}(z)\frac{dy}{ds} + \phi'\frac{2A_c}{\beta} \tag{2.64}$$

or

$$\gamma_{sz}(s,z) = [\theta_y(z) + u_0'(z)]\frac{dx}{ds} + [\theta_x(z) + v_0'(z)]\frac{dy}{ds} + \phi'(z)\Psi_T \tag{2.65}$$

As in the case of the open beam, the shear strain in the n-z plane, γ_{nz}, was also given by Eq 2.24

$$\gamma_{nz}(s,z) = \gamma_{xz}(z)\frac{dy}{ds} - \gamma_{yz}(z)\frac{dx}{ds} = [\theta_y(z) + u_0'(z)]\frac{dy}{ds} - [\theta_x(z) + v_0'(z)]\frac{dx}{ds} \tag{2.24}$$

The other strains in the plane of the cross-section are identically zero, which is consistent with the assumption of rigid cross-section beams. Six kinematic variables (u_0, v_0,w_0,θ_y,θ_x,ϕ) which are functions of z only can fully define the displacement field and, hence, the strain field within the theory of thin-walled closed section beams.

2.4 *Kinematic Equations for Both Open and Closed Beams*

Except for the torsional function, the expressions of kinematic relations are the same for both open and closed cross-section beams. Now, it is possible to express the kinematic equations in a general form which can be applied to both open and closed beam cases as follows:

Considering Eqs 2.31 and 2.56, the axial displacement can be expressed as

$$\begin{aligned} w(s,z,n) = w_0(z) + \theta_y(z)[x(s) + n\frac{dy}{ds}] + \theta_x(z)[y(s) - n\frac{dx}{ds}] \\ -\phi'(z)[F_\omega + na(s)] \end{aligned} \tag{2.66}$$

where the warping function, F_ω, is given by

$$F_\omega = \int_0^s [r_n(s) - \Psi_T]ds = \int_0^s [r_n(s) - \delta_o 2n - \delta_c 2\frac{A_c}{\beta}]\,ds \tag{2.67}$$

The torsional functions, Ψ_T, for the open and the closed beams are 2n and $2\frac{A_c}{\beta}$, respectively. By inserting Eq 2.67 into Eq 2.66, the axial displacement can be expressed as

$$\begin{aligned} w(s,z,n) = w_0(z) + \theta_y(z)[x(s) + n\frac{dy}{ds}] + \theta_x(z)[y(s) - n\frac{dx}{ds}] \\ -\phi'(z)[\Omega(s) - \delta_c 2\frac{A_c}{\beta}s + n\{a(s) - \delta_o 2s\}] \end{aligned} \tag{2.68}$$

and the axial strain is

$$\begin{aligned} \varepsilon_{zz}(s,z,n) = w_0'(z) + \theta_y'(z)[x(s) + n\frac{dy}{ds}] + \theta_x'(z)[y(s) - n\frac{dx}{ds}] \\ -\phi'(z)[\Omega(s) - \delta_c 2\frac{A_c}{\beta}s + n\{a(s) - \delta_o 2s\}] \end{aligned} \tag{2.69}$$

where δ_o and δ_c are the tracers identifying the open and the closed beams, respectively (i.e., $\delta_o = 1$ and $\delta_c = 0$ for the open beams while $\delta_o = 0$ and $\delta_c = 1$ for the closed beams). It is found that the warping function for open beams is a function of both s and n coordinates, while the warping function for closed beams is a function of s only. Eq 2.67 reveals the difference in the warping functions for the open and closed beam cases.

Next, the membrane shear strain is

$$\gamma_{sz} = \gamma_{xz}(z)\frac{dx}{ds} + \gamma_{yz}(z)\frac{dy}{ds} + \gamma_t \tag{2.40}$$

where, γ_t, is given by

$$\gamma_t = \delta_o 2n\phi'(z) + \delta_c 2\frac{A_c}{\beta}\phi'(z) \tag{2.70}$$

The shear strain in the n-z plane, γ_{nz}, is given by Eq 2.24 as

$$\gamma_{nz}(s,z) = \gamma_{xz}(z)\frac{dy}{ds} - \gamma_{yz}(z)\frac{dx}{ds} = [\theta_y(z) + u_0'(z)]\frac{dy}{ds} - [\theta_x(z) + v_0'(z)]\frac{dx}{ds} \tag{2.24}$$

2.5 Refinement of Kinematics with Respect to Warping Measure

Within the framework of the classical St.Venant torsion theory of beams, the rate of twist is assumed to be constant so that the axial strain due to torsion becomes a zero quantity. In our derivation, the warping displacement is assumed to be proportional to the rate of twist, $\phi'(z)$, which provides the warping measure. The concept of variable rate of twist provides a non-uniform warping measure enabling one to include the warping restraint effect in the analysis. Now we will introduce an additional refinement of the warping restraint effect by introducing an independent kinematic variable as a warping measure. An independent kinematic variable, $\Theta(z)$, not necessarily a rate of twist, is introduced as a warping measure in the kinematic equations [49,67]. The resulting expression of the axial displacement is given by

$$\begin{aligned} w(s,z,n) = w_0(z) + \theta_y(z)[x(s) + n\frac{dy}{ds}] + \theta_x(z)[y(s) - n\frac{dx}{ds}] \\ -\Theta(z)[\Omega(s) - \delta_c 2\frac{A_c}{\beta} + n\{a(s) - \delta_o 2s\}] \end{aligned} \tag{2.71}$$

The membrane shear strain in the middle surface of a beam was given by

$$\gamma_{sz}(s,z) = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z} \tag{2.13}$$

The displacement component tangent to the contour line was given by

$$v_t(s,z) = u_0(z)\frac{dx}{ds} + v_0(z)\frac{dy}{ds} + r_n(s)\phi(z) \tag{2.14}$$

Then, the axial strain can be written as

$$\begin{aligned} \varepsilon_{zz}(s,z,n) = w_0'(z) + \theta_y'(z)[x(s) + n\frac{dy}{ds}] + \theta_x'(z)[y(s) - n\frac{dx}{ds}] \\ -\Theta'(z)[\Omega(s) - \delta_c 2\frac{A_c}{\beta} + n\{a(s) - \delta_o 2s\}] \end{aligned} \tag{2.72}$$

The membrane shear strain is

$$\begin{aligned} \gamma_{sz} = [\theta_y(z) + u_0'(z)]\frac{dx}{ds} + [\theta_x(z) + v_0'(z)]\frac{dy}{ds} + r_n(s)\phi'(z) \\ -\Theta(z)[r_n(s) - \delta_o 2n - \delta_c 2\frac{A_c}{\beta}] \end{aligned} \tag{2.73}$$

In this derivation, the variation of the secondary warping along the contour is neglected. Similary, the shear strain in the n-z plane can be expressed as

$$\gamma_{nz}(s,z) = [\theta_y(z) + u_0'(z)]\frac{dy}{ds} - [\theta_x(z) + v_0'(z)]\frac{dx}{ds}$$
$$+a(s)\phi'(z) - [a(s) - \delta_o 2s]\Theta(z) \tag{2.74}$$
$$= \gamma_{xz}(z)\frac{dy}{ds} - \gamma_{yz}(z)\frac{dx}{ds} + a(s)\phi'(z) - [a(s) - \delta_o 2s]\Theta(z) \cdot$$

2.6 *Alternative Expressions of the Kinematic Equations*

The various kinematic equations incorporating the non-classical effects for either the open or closed cross-section beams can be expressed in a compact fashion by introducing a system of tracing quantities whose purpose is to identify the various effects. In such a way, the displacement components can be expressed as

$$u(s,z) = u_0(z) - y[\delta_{w1}\phi(z) + \delta_{w2}\phi(z) + \delta_{w3}\phi_c] \tag{2.75}$$

$$v(s,z) = v_0(z) + x[\delta_{w1}\phi(z) + \delta_{w2}\phi(z) + \delta_{w3}\phi_c] \tag{2.76}$$

$$w(s,z,n) = w_0(z) + [\delta_t\theta_y(z) - \delta_{nt}u_0'(z)][x(s) + \delta_s n\frac{dy}{ds}]$$
$$+ [\delta_t\theta_x(z) - \delta_{nt}v_0'(z)][y(s) - \delta_s n\frac{dx}{ds}] \tag{2.77}$$
$$- [\delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c][\widetilde{F}_\omega(s) + n\widetilde{a}(s)]$$

where

$$\widetilde{F}_\omega(s) = \int_0^s r_n(s)\, ds - \delta_c 2\frac{A_c}{\beta}s \tag{2.78}$$

$$\widetilde{a}(s) = \delta_s a(s) - \delta_o 2s \tag{2.79}$$

The displacement component tangent to the contour line can be expressed as

$$v_t(s,z) = u_0(z)\frac{dx}{ds} + v_0(z)\frac{dy}{ds} + r_n(s)[\delta_{w1}\phi + \delta_{w2}\phi(z) + \delta_{w3}\phi_c] \tag{2.80}$$

The displacement component in the outward normal direction to the middle surface can be expressed as

$$v_n = u_0(z)\frac{dy}{ds} - v_0(z)\frac{dx}{ds} + a(s)[\delta_{w1}\phi(z) + \delta_{w2}\phi(z) + \delta_{w3}\phi_c] \tag{2.81}$$

where, δ_s is the tracer identifying the secondary warping effect., δ_t and δ_{nt} are the tracers related to the transverse shear effect., δ_{w1}, δ_{w2}, and δ_{w3} are the tracers related to the warping measures, δ_s is 0 or 1 depending on whether the secondary warping is neglected or incorporated, respectively, δ_t is 0 or 1 and δ_{nt} is simultaneously 1 or 0 depending on whether transverse shear deformation is disregarded or included, respectively; δ_{w1}, δ_{w2}, and δ_{w3} are taken as 1 according to whether $\phi'(z)$, $\Theta(z)$ or ϕ' (constant) are considered as warping measures, respectively. Otherwise, these tracing quantities are taken as zero.

In such a general form, the axial strain component is given by

$$\begin{aligned}\varepsilon_{zz} = {} & w_0'(z) + [\delta_t\theta_y'(z) - \delta_{nt}u_0''(z)][x(s) + \delta_s n\frac{dy}{ds}] \\ & + [\delta_t\theta_x'(z) - \delta_{nt}v_0''(z)][y(s) - \delta_s n\frac{dx}{ds}] \\ & - [\delta_{w1}\phi''(z) + \delta_{w2}\Theta'(z)][\tilde{F}_\omega(s) + n\tilde{a}(s)]\end{aligned} \tag{2.82}$$

The membrane shear strain is

$$\begin{aligned}\gamma_{sz} = {} & [\delta_t\theta_y(z) - \delta_{nt}u_0'(z)][\frac{dx}{ds} + \delta_s n\frac{d^2y}{ds^2}] \\ & + [\delta_t\theta_x(z) - \delta_{nt}v_0'(z)][\frac{dy}{ds} - \delta_s n\frac{d^2x}{ds^2}] \\ & - [\delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c'][r_n(s) - \Psi_T + n\frac{d\tilde{a}}{ds}] \\ & + u_0'(z)\frac{dx}{ds} + v_0'(z)\frac{dy}{ds} + r_n(s)[\delta_{w1}\phi'(z) + \delta_{w2}\phi'(z) + \delta_{w3}\phi_c']\end{aligned} \tag{2.83}$$

This equation can be expressed in a different way as

$$\begin{aligned}\gamma_{sz} = &[\delta_t\theta_y(z) - \delta_{nt}u_0'(z)]\frac{dx}{ds} + [\delta_t\theta_x(z) - \delta_{nt}v_0'(z)]\frac{dy}{ds} \\ &- [\delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c'][r_n(s) - \Psi_T] \\ &+ u_0'(z)\frac{dx}{ds} + v_0'(z)\frac{dy}{ds} + r_n(s)[\delta_{w1}\phi'(z) + \delta_{w2}\phi'(z) + \delta_{w3}\phi_c'] \\ &+ \delta_s n[\{\delta_t\theta_y(z) - \delta_{nt}u_0'(z)\}\frac{d^2y}{ds^2} - \{\delta_t\theta_x(z) - \delta_{nt}v_0'(z)\}\frac{d^2x}{ds^2}] \\ &- n\{\delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c'\}\frac{d\tilde{a}}{ds}\end{aligned} \tag{2.84}$$

where $\frac{da}{ds}$ is

$$\frac{da}{ds} = -(1 + y\frac{d^2y}{ds^2} + x\frac{d^2x}{ds^2}) \tag{2.85}$$

However, for simplicity, the last part of the membrane shear strain produced by the secondary warping effect can be neglected. As a result, for closed section beams, the membrane shear strain is assumed to be constant across the thickness while for open section beams, since the torsional function, Ψ_T, depends on the thickness coordinate, it would vary across the thickness. Now, the membrane shear strain is expressed as:

$$\begin{aligned}\gamma_{sz} = &[\delta_t\theta_y(z) - \delta_{nt}u_0'(z)]\frac{dx}{ds} + [\delta_t\theta_x(z) - \delta_{nt}v_0'(z)]\frac{dy}{ds} \\ &- [\delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c'][r_n(s) - \Psi_T] \\ &+ u_0'(z)\frac{dx}{ds} + v_0'(z)\frac{dy}{ds} + r_n(s)[\delta_{w1}\phi'(z) + \delta_{w2}\phi'(z) + \delta_{w3}\phi_c']\end{aligned} \tag{2.86}$$

Eq 2.86 can be rewritten as

$$\begin{aligned}\gamma_{sz} = &[\delta_t\theta_y(z) - \delta_{nt}u_0'(z)]\frac{dx}{ds} + [\delta_t\theta_x(z) - \delta_{nt}v_0'(z)]\frac{dy}{ds} \\ &+ u_0'(z)\frac{dx}{ds} + v_0'(z)\frac{dy}{ds} + \delta_{w2}r_n(s)[\phi'(z) - \Theta(z)] \\ &+ \Psi_T[\delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c']\end{aligned} \tag{2.87}$$

Next, the shear strain in the n-z plane can be expressed as

$$\gamma_{nz} = \delta_s[\delta_t\theta_y(z) - \delta_{nt}u_0'(z)]\frac{dy}{ds} - \delta_s[\delta_t\theta_x(z) - \delta_{nt}v_0'(z)]\frac{dx}{ds}$$
$$- [\delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c']\tilde{a}(s) \tag{2.88}$$
$$+ u_0'(z)\frac{dy}{ds} - v_0'(z)\frac{dx}{ds} + [\delta_{w1}\phi'(z) + \delta_{w2}\phi'(z) + \delta_{w3}\phi_c']a(s)$$

In order to emphasize several of the effects included in the expressions of the strains ε_{zz}, γ_{sz}, and γ_{nz}, other expressions of them will be provided next.

• The axial strain can be expressed in the form

$$\varepsilon_{zz}(s,z,n) = \varepsilon_{zz}{}^0(s,z) + n\varepsilon_{zz}{}^n(s,z) \tag{2.89}$$

where $\varepsilon_{zz}{}^0(s,z)$ and $\varepsilon_{zz}{}^n(s,z)$ are

$$\varepsilon_{zz}{}^0(s,z) = w_0'(z) + [\delta_t\theta_y'(z) - \delta_{nt}u_0''(z)]x(s) + [\delta_t\theta_x'(z) - \delta_{nt}v_0''(z)]y(s)$$
$$- [\delta_{w1}\phi''(z) + \delta_{w2}\Theta'(z)]\tilde{F}_\omega(s) \tag{2.90}$$

$$\varepsilon_{zz}{}^n(s,z) = \delta_s[\delta_t\theta_y'(z) - \delta_{nt}u_0''(z)]\frac{dy}{ds} - \delta_s[\delta_t\theta_x'(z) - \delta_{nt}v_0''(z)]\frac{dx}{ds}$$
$$- [\delta_{w1}\phi''(z) + \delta_{w2}\Theta'(z)]\tilde{a}(s) \tag{2.91}$$

Neglecting the secondary warping effect, the second part of Eq 2.88 will vanish.

• The membrane shear strain can be considered as a combination of two different parts as (see Eq 2.87)

$$\gamma_{sz} = \gamma_{sz}{}^0 + \gamma_{sz}{}^t \tag{2.92}$$

where $\gamma_{sz}{}^0$ is expressed as

$$\gamma_{sz}{}^0 = [\delta_t\theta_y(z) - \delta_{nt}u_0'(z)]\frac{dx}{ds} + [\delta_t\theta_x(z) - \delta_{nt}v_0'(z)]\frac{dy}{ds}$$
$$+ u_0'(z)\frac{dx}{ds} + v_0'(z)\frac{dy}{ds} + \delta_{w2}r_n(s)[\phi'(z) - \Theta(z)] \tag{2.93}$$

while $\gamma_{sz}{}^t$ can be expressed as

$$\gamma_{sz}{}^{t} = \Psi_T W_M = (\delta_o 2n + \delta_c 2\frac{A_c}{\beta})[\delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c'] \tag{2.94}$$

which stands for the membrane shear strain caused by torsion. The torsional functions, Ψ_T, for the open and the closed beam cases are 2n and $2\frac{A_c}{\beta}$, respectively. The warping measure, W_M, is defined as

$$W_M = \delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c' \tag{2.95}$$

- The shear strain in the n-z plane was given by Eq 2.88 as

$$\begin{aligned}\gamma_{nz} = {} & \delta_s[\delta_t\theta_y(z) - \delta_{nt}u_0'(z)]\frac{dy}{ds} - \delta_s[\delta_t\theta_x(z) - \delta_{nt}v_0'(z)]\frac{dx}{ds} \\ & - [\delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c']\tilde{a}(s) \\ & + u_0'(z)\frac{dy}{ds} - v_0'(z)\frac{dx}{ds} + [\delta_{w1}\phi'(z) + \delta_{w2}\phi'(z) + \delta_{w3}\phi_c']a(s)\end{aligned} \tag{2.88}$$

The above equations will be used in the derivation of constitutive equations and of the governing equations.

2.7 Kinematics of Multicell Closed Section Thin-walled Beams

2.7.1 Basic Assumptions

A multicell thin-walled beam is composed of several closed cross-section cells. As is evident that a part of the kinematic assumptions used for single cell closed beams can be applied for this more intricate cases too. However, the multicell thin-walled beam structure requires an additional consideration which concerns the compatibility conditions for the rate of twist characteristics. The analysis of a multicell thin-walled beam structure subjected to a torsional moment is a statically indeterminate problem since an extra unknown, namely, the shear flow associated with each additional cell, is introduced. For a thin-walled beam with N-cells, N compatibility conditions concerning the rate of twist of each cell should supplement the single equilibrium equation of the torsional moment. N number of shear flows and the value of the rate of twist (considered to be the same for all cells) can be obtained from the N compatibility conditions plus one equilibrium equation of torsion [1,10,46,60].

Rehfield, et al [56] have accomplished a pioneering work on the formulation of kinematics for multicell composite thin-walled beam structures. To the best knowledge of the author, this work is the first one dealing with multicell thin-walled beams made of composite materials including the warping restraint effect. In this dissertation, the formulation of kinematics on the multicell composite beams will be complemented by incorporating the secondary effect.

First of all, the St. Venant torsional model of multicell thin-walled beams will be considered to obtain the variation of the shear strain along the contour line. Within the St. Venant torsional model the shear strain is assumed to be constant along the beam axis, which implies the absence of the warping restraint effect. As a result, the membrane shear stress resultant (N_{sz}), which is referred to as the shear flow, is assumed to be constant along the contour line of each cell. This membrane shear stress resultant (shear flow) is denoted by q in articles and monographs. The equilibrium of an element of the wall, considering the axial stress resultant (N_{zz}), yields

$$\frac{\partial N_{zz}}{\partial z} + \frac{\partial N_{sz}}{\partial s} = 0 \tag{2.96}$$

where $\frac{\partial N_{sz}}{\partial s}$ is assumed to vanish and, hence, N_{zz} is constant along the beam axis,i.e., N_{zz} is not a function of the axial coordinate z. Furthermore, the axial stress resultant (N_{zz}) should be zero everywhere in the body in order to satisfy the free lateral boundary condition and the force and moment equilibrium conditions at the free ends where only a torsional moment is applied. These conditions are given by

$$T_z = \int_C N_{zz} ds = 0 \tag{2.97}$$

$$M_x = \int_C y(s) N_{zz} ds = 0 \tag{2.98}$$

$$M_y = \int_C x(s) N_{zz} ds = 0 \tag{2.99}$$

2.7.2 Kinematics of Multicell Cross-Section Beams

A free torsion problem of a multicell thin-walled beam subjected to equal and opposite torques at both ends without warping restraints will be considered first. This is done in order to formulate the kinematics for the structure with warping restraint effect. The warping function obtained from the free torsion model will provide a cross-sectional distribution of the warping displacement. The rate of twist of the R-th cell of the beam having N number of cells can be determined through the following procedure.

The membrane shear strain is, by definition, given by

$$\gamma_{sz} = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z} \tag{2.13}$$

where v_t is given in Eq 2.14. Differentiation of Eq 2.14 with respect to the axial coordinate z gives

$$v_t'(s,z) = u_0'(z)\frac{dx}{ds} + v_0'(z)\frac{dy}{ds} + r_n(s)\phi'(z) \tag{2.100}$$

Substituting Eq 2.100 into Eq 2.13 and integrating along the contour of the R-th cell yields

$$\int_R (\gamma_{sz})_R ds = \int_R \frac{\partial w}{\partial s} ds + \int_R [u_0'(z) \frac{dx}{ds}]ds + \int_R [v_0'(z) \frac{dy}{ds}]ds + \phi'_R(z) \int_R r_n(s)ds \quad (2.101)$$

The axial displacement w(s,z) should be continuous along the contour line of the R-th cell. By considering Eq 2.101 where the first three terms of the right hand side become zero, the rate of twist of the R-th cell is

$$\phi'_R = \frac{1}{2} A_R \int_R (\gamma_{sz})_R ds \quad (2.102)$$

where $2A_R$, denoting twice the enclosed area of the R-th cell, is expressed as

$$2A_R = \int_R r_n(s)ds \quad (2.103)$$

The shear flow distribution in the R-th cell is illustrated in Fig. 6.

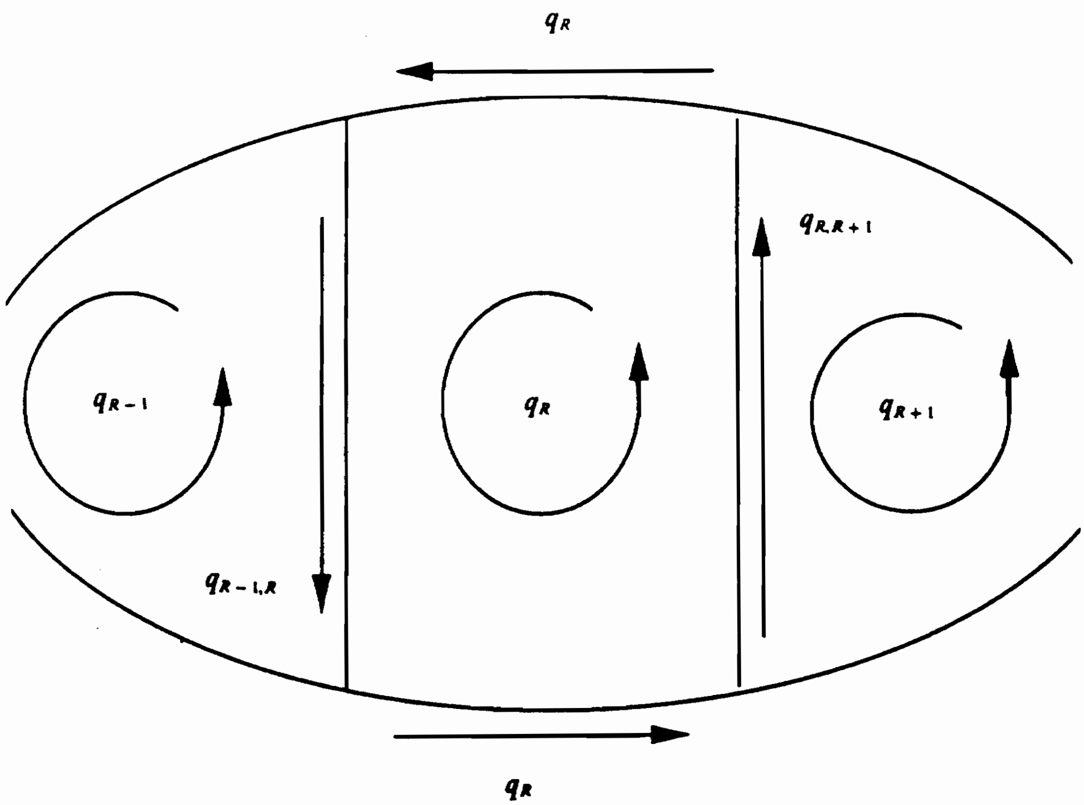


Figure 6. Shear Flow Distribution in the R-th Cell

When there is no internal pressure inside the thin-walled beams, the hoop stress, N_{ss}, is usually quite small and can be neglected. The generalized constitutive relations are given in matrix form in Eq 2.104:

$$\begin{bmatrix} N_{zz} \\ N_{sz} \end{bmatrix} = \begin{bmatrix} K_{11} K_{12} \\ K_{12} K_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{zz} \\ \gamma_{sz} \end{bmatrix} \tag{2.104}$$

The derivation of Eq 2.104 is detailed in the next chapter, i.e., Chapter 3. In Eq 2.104, K_{ij} denoting modified stiffness are

$$K_{11} = A_{22} - \frac{A_{12}^2}{A_{11}} \tag{2.105}$$

$$K_{12} = A_{26} - \frac{A_{12}A_{16}}{A_{11}} \tag{2.106}$$

$$K_{22} = A_{66} - \frac{A_{16}^2}{A_{11}} \tag{2.107}$$

As was mentioned before, for St.Venant's torsional model the axial stress resultant, N_{zz} , is zero everywhere in the body. Then, as is usually denoted in the literature,

$$N_{sz} \equiv q = (K_{22} - \frac{K_{12}^2}{K_{11}}) \tag{2.108}$$

where, q denotes the shear flow. Denote α as

$$\alpha \equiv \frac{1}{K_{22}(1 - \frac{K_{12}}{K_{11}} K_{22})} \tag{2.109}$$

Here $\frac{1}{\alpha}$ denotes the generalized torsional stiffness. Then the shear flow can be expressed as

$$q = \frac{1}{\alpha} \gamma_{sz} \quad \text{or} \quad \gamma_{sz} = \alpha q \tag{2.110}$$

Substituting Eq 2.110 into Eq 2.102 gives

$$\begin{aligned} \phi'_R &= \frac{1}{2} A_R \int_R [q\alpha] ds \\ &= \frac{1}{2} A_R [q_{R-1} \textstyle\int_{R-1,R} \alpha ds + q_R \int_R \alpha ds + q_{R+1} \int_{R,R+1} \alpha ds] \end{aligned} \tag{2.111}$$

where $\int_{R,R+1}(\cdots)ds$ denotes the curvilinear integral taken along the wall separating the R-th and (R + 1)-th cells. By considering every cell of the cross-section, Eq 2.111 can be expressed in the matrix form

$$\begin{bmatrix} \phi'_1 \\ \phi'_2 \\ \bullet \\ \bullet \\ \phi'_N \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \bullet & \bullet & S_{1N} \\ S_{21} & S_{22} & \bullet & \bullet & S_{2N} \\ \bullet & \bullet & \bullet & \bullet & S_{3N} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ S_{N1} & S_{N2} & \bullet & \bullet & S_{NN} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \bullet \\ \bullet \\ q_N \end{bmatrix} \tag{2.112}$$

where

$$S_{RR} = \frac{1}{2} A_R \int_R \alpha ds \tag{2.113}$$

$$S_{R,R+1} = \frac{1}{2} A_R \int_{R,R+1} \alpha ds \tag{2.114}$$

$$S_{R,R-1} = \frac{1}{2} A_R \int_{R-1,R} \alpha ds \tag{2.115}$$

Now, the compatibility conditions requiring that all the cells must rotate through the same angle of twist per unit length should be applied. This requires:

$$\phi'_1 = \phi'_2 = \phi'_3 = \bullet = \bullet = \phi'_N \equiv \phi' \tag{2.116}$$

Substituting Eq 2.116 into Eq 2.112 results in a constitutive relation expressed in matrix form as

$$\{\phi'\} = [S]\{q\} \tag{2.117}$$

and

$$\{q\} = [S]^{-1}\{\phi'\} \tag{2.118}$$

where {q} and {ϕ'} denote the shear flows and the rate of twist vectors, respectively; [S] denotes the compliance matrix relating the shear flows and the rate of twist. Next, the local shear flow of each wall segment of the cross-section will be obtained. Having in view Fig. 6, the local shear flow can be calculated from the equations given by

$$q_{R-1,R} = q_R - q_{R-1} \tag{2.119}$$

and

$$q_{R,R+1} = q_R - q_{R+1} \tag{2.120}$$

Then, for each wall segment, the relationship between the shear flow and the rate of twist is expressed in a vectorial form

$$\begin{bmatrix} q_1 \\ q_{1,2} \\ q_2 \\ q_{2,3} \\ \bullet \\ \bullet \\ q_{N-1,N} \\ q_N \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 - q_1 \\ q_2 \\ q_3 - q_2 \\ \bullet \\ \bullet \\ q_n - q_{N-1} \\ q_N \end{bmatrix} = \begin{bmatrix} H_1 \\ H_{21} \\ H_2 \\ H_{32} \\ \bullet \\ \bullet \\ H_{N,N-1} \\ H_N \end{bmatrix} \phi' \tag{2.121}$$

where

$$H_{R,R-1} = \frac{S_{R-1,R-1} + S_{R,R-1} - S_{RR} - S_{R,R-1}}{S_{RR}S_{R-1,R-1} - S_{R,R-1}S_{R,R-1}} \qquad R = 1,2,3,\dots\dots,N \tag{2.122}$$

and

$$\{H_R\} = [S]^{-1}\{I\} \tag{2.123}$$

where {I} stands for the identity vector. The membrane shear strain for each wall segment between nodes can be expressed in a vectorial form by considering Eqs 2.110 and 2.121:

$$\begin{bmatrix} \gamma_{sz}^{1} \\ \gamma_{sz}^{12} \\ \gamma_{sz}^{2} \\ \gamma_{sz}^{23} \\ \gamma_{sz}^{3} \\ \bullet \\ \bullet \\ \gamma_{sz}^{N,N-1} \\ \gamma_{sz}^{N} \end{bmatrix} = \begin{bmatrix} \alpha_1 H_1 \\ \alpha_{12} H_{21} \\ \alpha_2 H_2 \\ \alpha_{23} H_{23} \\ \alpha_3 H_3 \\ \bullet \\ \bullet \\ \alpha_{N,N-1} H_{N,N-1} \\ \alpha_N H_n \end{bmatrix} \phi' = \{\alpha H\}\phi' \qquad (2.124)$$

Now the relationship between the shear strain and the rate of twist is derived for the St.Venant torsional model. In Eqs 2.121 and 2.124, for the free torsion model the rate of twist, ϕ', is not a function of the beam axial coordinate z, but a constant-valued quantity. For thin-walled beams affected by the warping restraint effect, the shear flow can be expressed by introducing a variable rate of twist such as

$$\{\gamma_{sz}\}_i = \{\alpha H\}_i \phi'(z) \qquad i = 1,2,3,........,3N-3 \qquad (2.125)$$

where the rate of twist is no longer a constant quantity but a function of the axial coordinate z. Furthermore, considering the transverse shear strains, $\gamma_{xz}(z)$ and $\gamma_{yz}(z)$, the shear strain for any wall segment can be expressed in a vectorial form such as

$$\gamma_{sz}(s,z) = \gamma_{xz}(z)\frac{dx}{ds} + \gamma_{yz}(z)\frac{dy}{ds} + \{\alpha H\}\phi'(z) \qquad (2.126)$$

where the vector $\{\alpha H\}$ is given by Eq 2.124. Considering the strain-displacement relationship and Eq 2.13 yield

$$\gamma_{sz} = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z} = \frac{\partial w}{\partial s} + u_0'(z)\frac{dx}{ds} + v_0'(z)\frac{dy}{ds} + r_n(s)\phi'(z) \tag{2.127}$$

Combining Eqs 2.126 and 2.127 yields

$$\begin{aligned}\gamma_{sz} &= \gamma_{xz}(z)\frac{dx}{ds} + \gamma_{yz}(z)\frac{dy}{ds} + \{\alpha H\}\phi'(z) \\ &= \frac{\partial w}{\partial s} + u_0'(z)\frac{dx}{ds} + v_0'(z)\frac{dy}{ds} + r_n(s)\phi'(z)\end{aligned} \tag{2.128}$$

Integrating Eq 2.128 along the contour line of the cross-section results in the axial displacement of the contour line where n is set to zero:

$$\begin{aligned}w(s,z) &= w_0(z) \\ &+ \int_0^s \{[\gamma_{xz}(z) - u_0'(z)]\frac{dx}{ds} + [\gamma_{yz}(z) - v_0'(z)]\frac{dy}{ds} + [\{\alpha H(s)\} + r_n(s)]\phi'(z)\}ds\end{aligned} \tag{2.129}$$

Then

$$\begin{aligned}w(s,z) = w_0(z) &+ [\gamma_{xz}(z) - u_0'(z)]x(s) + [\gamma_{yz}(z) - v_0'(z)]y(s) \\ &+\phi'(z)\int_0^s [\{\alpha H(s)\} - r_n(s)]ds + \Pi\phi'(z)\end{aligned} \tag{2.130}$$

or

$$w(s,z) = w_0(z) + \theta_y(z)x(s) + \theta_x y(s) + \phi'(z)\int_0^s [\{\alpha H(s)\} - r_n(s)]ds + \Pi\phi'(z) \tag{2.131}$$

where Π can be determined from the continuity condition of the warping displacement between each cell. Next, the secondary warping can be included in the axial displacement in the same manner as for the case of single cell beams. The axial displacement including the secondary warping can be expressed as

$$w(s,z) = w_0(z) + [\gamma_{xz}(z) - u_0'(z)][x(s) + n\frac{dy}{ds}] + [\gamma_{yz}(z) - v_0'(z)][y(s) - n\frac{dx}{ds}]$$
$$+\phi'(z)[\int_0^s [\{\alpha H(s)\} - r_n(s)]ds + na(s)] + \Pi\phi'(z) \qquad (2.132)$$

or

$$w(s,z) = w_0(z) + \theta_y(z)x(s) + \theta_x y(s) + \phi'(z)[\int_0^s [\{\alpha H(s)\} - r_n(s)]ds + na(s)] + \Pi\phi'(z) \qquad (2.133)$$

The axial strain is

$$\varepsilon_{zz}(s,z) = w_0'(z) + \theta_y'(z)[x(s) + n\frac{dy}{ds}] + \theta_x'(z)[y(s) - n\frac{dx}{ds}]$$
$$+\phi''[\int_0^s (\{\alpha H(s)\} - r_n(s))ds + na(s) + \Pi] \qquad (2.134)$$

or

$$\varepsilon_{zz}(s,z,n) = \varepsilon_{zz}{}^0(s,z) + n\,\varepsilon_{zz}{}^n(s,z) \qquad (2.135)$$

where

$$\varepsilon_{zz}{}^0(s,z) = w_0'(z) + \theta_y'(z)x(s) + \theta_x'(z)y(s) + \phi''[\int_0^s (\{\alpha H(s)\} - r_n(s))ds + \Pi] \qquad (2.136)$$

$$\varepsilon_{zz}{}^n(s,z) = \theta_y'(z)\frac{dy}{ds} - \theta_x'(z).\frac{dx}{ds} + \phi''(z)a(s) \qquad (2.137)$$

Also, the membrane shear strain can be written as

$$\gamma_{sz}(s,z) = [\theta_y'(z) + u_0'(z)]\frac{dx}{ds} + [\theta_x'(z) + v_0'(z)]\frac{dy}{ds} + \{\alpha H(s)\}\phi'(z) \qquad (2.138)$$

or

$$\gamma_{sz}(s,z) = \gamma_{xz}(z)\frac{dx}{ds} + \gamma_{yz}(z)\frac{dy}{ds} + \{\alpha H(s)\}\phi'(z) \tag{2.139}$$

The shear strain in the n-z plane for a multicell beam is the same as that for the single cell beam case:

$$\gamma_{nz}(s,z) = \gamma_{xz}(z)\frac{dy}{ds} - \gamma_{yz}(z)\frac{dx}{ds} = [\theta_y(z) + u_0'(z)]\frac{dy}{ds} - [\theta_x(z) + v_0'(z)]\frac{dx}{ds} \tag{2.140}$$

The above expressions of strains for multicell thin-walled beams are similar to those of their single-cell counterparts. It is found that the warping function has a more complicated vectorial form in the case of multicell thin-walled beams as compared to the single cell case.

Chapter III

The Constitutive Relationships

3.1 3-D Stress-Strain Relations

When the principal material directions of an orthotropic material coincide with the geometrical coordinates of the beam, the 3-D stress-strain relations including the thermal and hygro-thermal effects [17,54,70,71], may be expressed as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} - \alpha_1 T - \beta_1 M \\ \varepsilon_{22} - \alpha_2 T - \beta_2 M \\ \varepsilon_{33} - \alpha_3 T - \beta_3 M \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} \tag{3.1}$$

Here Q_{ij}'s denote the stiffnesses in the principal material coordinate system while α_i and β_i denote the thermal and the hygro-thermal expansion coefficients, respectively. Considering the transformation matrix between the material coordinate system and the global one, the constitutive relations in the global coordinate system (s,z,n) are expressed as

$$\begin{bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{nn} \\ \sigma_{nz} \\ \sigma_{sn} \\ \sigma_{sz} \end{bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{13} & 0 & 0 & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{23} & 0 & 0 & \overline{Q}_{26} \\ \overline{Q}_{13} & \overline{Q}_{23} & \overline{Q}_{33} & 0 & 0 & \overline{Q}_{36} \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{36} & 0 & 0 & \overline{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{ss} - \alpha_s T - \beta_s M \\ \varepsilon_{zz} - \alpha_z T - \beta_z M \\ \varepsilon_{nn} - \alpha_n T - \beta_n M \\ \gamma_{zn} \\ \gamma_{sn} \\ \gamma_{sz} - \alpha_{sz} T - \beta_{sz} M \end{bmatrix} \tag{3.2}$$

where $\overline{Q}_{ij}$ (i,j = 1,6) represent the stiffnesses with respect to the global coordinate system (s,z,n). In Eqs 3.1 and 3.2, T and M denote the temperature change and the increase of moisture measured in percentage weight increase, respectively. Details concerning the transformation of stresses and strains from one coordinate system to another are presented in Appendix A.

3.2 Stress Resultants and Couples

The stress resultants and couples in terms of strain measures could be obtained through the integration of the 3-D constitutive equations across the laminate thickness [17,27,36]. As a result, these stress resultants and couples become two dimensional quantities which are functions of the contour and the axial coordinates s and z, respectively. The dependency on the thickness coordinate n is eliminated through an integration across the thickness direction.

The stress resultants are given by

$$\begin{bmatrix} N_{ss} \\ N_{zz} \\ N_{sz} \end{bmatrix} = \sum_{k=1}^{N} \int_{h(k)} \begin{bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{sz} \end{bmatrix}^{(k)} dn \tag{3.3}$$

The shear stress resultants are given by

$$\begin{bmatrix} N_{nz} \\ N_{sn} \end{bmatrix} = \sum_{k=1}^{N} \int_{h(k)} \begin{bmatrix} \sigma_{nz} \\ \sigma_{sn} \end{bmatrix}^{(k)} dn \tag{3.4}$$

The stress couples are given by

$$\begin{bmatrix} L_{zz} \\ L_{sz} \end{bmatrix} = \sum_{k=1}^{N} \int_{h(k)} \begin{bmatrix} \sigma_{zz} \\ \sigma_{sz} \end{bmatrix}^{(k)} ndn \tag{3.5}$$

In Eqs 3.3 through 3.5 it was assumed that the ratio of the thickness to the radius of curvature, $\frac{h}{\kappa}$, of the middle surface of the beams is negligibly small in the sense that $\frac{h}{\kappa} << 1$.

3.3 Constitutive Relations for Thin-Walled Open Section Beams

The stress resultants and couples can be obtained by substituting Eq 3.2 into Eqs 3.3 through 3.5. For the sake of completeness, the expressions of various strain components for the open beams derived in Chapter 2 will be repeated here:

$$\begin{aligned} \varepsilon_{zz}{}^{0}(s,z) &= w_0'(z) + [\delta_t\theta_y'(z) - \delta_{nt}u_0''(z)]x(s) + [\delta_t\theta_x'(z) - \delta_{nt}v_0''(z)]y(s) \\ &\quad - [\delta_{w1}\phi''(z) + \delta_{w2}\Theta'(z)]\widetilde{F}_{\omega}(s) \end{aligned} \tag{3.6}$$

$$\varepsilon_{zz}{}^{n}(s,z) = \delta_s[\delta_t\theta_y'(z) - \delta_{nt}u_0''(z)]\frac{dy}{ds} - \delta_s[\delta_t\theta_x'(z) - \delta_{nt}v_0''(z)]\frac{dx}{ds} - [\delta_{w1}\phi''(z) + \delta_{w2}\Theta'(z)]\tilde{a}(s) \quad (3.7)$$

$$\gamma_{sz}{}^{0} = [\delta_t\theta_y(z) - \delta_{nt}u_0'(z)]\frac{dx}{ds} + [\delta_t\theta_x(z) - \delta_{nt}v_0'(z)]\frac{dy}{ds} + u_0'(z)\frac{dx}{ds} + v_0'(z)\frac{dy}{ds} + \delta_{w2}r_n(s)[\phi'(z) - \Theta(z)] \quad (3.8)$$

$$\gamma_{sz}{}^{t} = \Psi_T W_M \quad (3.9)$$

where the torsional function, Ψ_T, for the open beams is

$$\Psi_T = 2n \quad (3.10)$$

while the warping measure, W_M, is given by

$$W_M = \delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c' \quad (3.11)$$

$$\gamma_{nz} = \delta_s[\delta_t\theta_y(z) - \delta_{nt}u_0'(z)]\frac{dy}{ds} - \delta_s[\delta_t\theta_x(z) - \delta_{nt}v_0'(z)]\frac{dy}{ds} - \delta_s[\delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c']\tilde{a}(s) + u_0'(z)\frac{dy}{ds} - v_0'(z)\frac{dx}{ds} + [\delta_{w1}\phi'(z) + \delta_{w2}\phi'(z) + \delta_{w3}\phi_c']a(s) \quad (3.12)$$

The hoop stress resultant is obtained as

$$
\begin{aligned}
N_{ss}(s,z) &= \sum_{k=1}^{N} \int_{h(k)} \sigma_{ss}^{(k)} dn \\
&= \sum_{k=1}^{N} \int_{h(k)} [\overline{Q}_{11}(\varepsilon_{ss} - \alpha_s T - \beta_s M) + \overline{Q}_{12}(\varepsilon_{zz} - \alpha_z T - \beta_z M) \\
&+ \overline{Q}_{13}(\varepsilon_{nn} - \alpha_n T - \beta_n M) + \overline{Q}_{16}(\gamma_{sz} - \alpha_{sz} T - \beta_{sz} M)]^{(k)} dn \\
&= \sum_{k=1}^{N} \int_{h(k)} [\overline{Q}_{11}(\varepsilon_{ss} - \alpha_s T - \beta_s M) + \overline{Q}_{12}(\varepsilon_{zz}{}^{0} + n\varepsilon_{zz}{}^{n} - \alpha_z T - \beta_z M) \\
&+ \overline{Q}_{13}(\varepsilon_{nn} - \alpha_n T - \beta_n M) + \overline{Q}_{16}(\gamma_{sz}{}^{0} + \gamma_{sz}{}^{t} - \alpha_{sz} T - \beta_{sz} M)]^{(k)} dn \\
&= A_{11}\varepsilon_{ss} + A_{12}\varepsilon_{zz}{}^{0} + B_{12}\varepsilon_{zz}{}^{n} + A_{16}\gamma_{sz}{}^{0} + 2B_{16}W_M - N_s{}^{T} - N_s{}^{M}
\end{aligned}
\tag{3.13}
$$

In the derivation of Eq 3.13, both Eq 2.89 and the assumption of the cross-section non-deformability (implying $\varepsilon_{nn} = 0$) have been used. In addition, in Eq 3.13

$$A_{ij} = \sum_{k=1}^{N} \int_{h(k)} \overline{Q}_{ij}{}^{(k)} dn \tag{3.14}$$

$$B_{ij} = \sum_{k=1}^{N} \int_{h(k)} \overline{Q}_{ij}{}^{(k)} n dn \tag{3.15}$$

$$N_s{}^{T}(s,z) = \sum_{k=1}^{N} \int_{h(k)} T[\overline{Q}_{11}\alpha_s + \overline{Q}_{12}\alpha_z + \overline{Q}_{13}\alpha_n + \overline{Q}_{16}\alpha_{sz}]^{(k)} dn \tag{3.16}$$

$$N_s{}^{M}(s,z) = \sum_{k=1}^{N} \int_{h(k)} M[\overline{Q}_{11}\beta_s + \overline{Q}_{12}\beta_z + \overline{Q}_{13}\beta_n + \overline{Q}_{16}\beta_{sz}]^{(k)} dn \tag{3.17}$$

Similarly, the axial stress resultant can be obtained as

$$
\begin{aligned}
N_{zz}(s,z) &= \sum_{k=1}^{N} \int_{h(k)} \sigma_{zz}^{(k)} dn \\
&= \sum_{k=1}^{N} \int_{h(k)} [\overline{Q}_{12}(\varepsilon_{ss} - \alpha_s T - \beta_s M) + \overline{Q}_{22}(\varepsilon_{zz} - \alpha_z T - \beta_z M) \\
&+ \overline{Q}_{23}(\varepsilon_{nn} - \alpha_n T - \beta_n M) + \overline{Q}_{26}(\gamma_{sz} - \alpha_{sz} T - \beta_{sz} M)]^{(k)} dn \\
&= A_{12}\varepsilon_{ss} + A_{22}\varepsilon_{zz}{}^{0} + B_{22}\varepsilon_{zz}{}^{n} + A_{26}\gamma_{sz}{}^{0} + 2B_{26}W_M - N_z{}^{T} - N_z{}^{M}
\end{aligned}
\tag{3.18}
$$

where

$$
N_z{}^{T}(s,z) = \sum_{k=1}^{N} \int_{h(k)} T\,[\overline{Q}_{12}\alpha_s + \overline{Q}_{22}\alpha_z + \overline{Q}_{23}\alpha_n + \overline{Q}_{26}\alpha_{sz}]^{(k)}\, dn \tag{3.19}
$$

$$
N_z{}^{M}(s,z) = \sum_{k=1}^{N} \int_{h(k)} M[\overline{Q}_{12}\beta_s + \overline{Q}_{22}\beta_z + \overline{Q}_{23}\beta_n + \overline{Q}_{26}\beta_{sz}]^{(k)}\, dn \tag{3.20}
$$

The membrane shear stress resultant is obtained as

$$
\begin{aligned}
N_{sz}(s,z) &= \sum_{k=1}^{N} \int_{h(k)} \sigma_{sz}^{(k)} dn \\
&= \sum_{k=1}^{N} \int_{h(k)} [\overline{Q}_{16}(\varepsilon_{ss} - \alpha_s T - \beta_s M) + \overline{Q}_{26}(\varepsilon_{zz} - \alpha_z T - \beta_z M) \\
&+ \overline{Q}_{26}(\varepsilon_{nn} - \alpha_n T - \beta_n M) + \overline{Q}_{66}(\gamma_{sz} - \alpha_{sz} T - \beta_{sz} M)]^{(k)} dn \\
&= A_{16}\varepsilon_{ss} + A_{26}\varepsilon_{zz}{}^{0} + B_{26}\varepsilon_{zz}{}^{n} + A_{66}\gamma_{sz}{}^{0} + 2B_{66}W_M - N_{sz}{}^{T} - N_{sz}{}^{M}
\end{aligned}
\tag{3.21}
$$

where

$$
N_{sz}{}^{T}(s,z) = \sum_{k=1}^{N} \int_{h(k)} T\,[\overline{Q}_{16}\alpha_s + \overline{Q}_{26}\alpha_z + \overline{Q}_{36}\alpha_n + \overline{Q}_{66}\alpha_{sz}]^{(k)}\, dn \tag{3.22}
$$

$$N_{sz}{}^{M}(s,z) = \sum_{k=1}^{N} \int_{h(k)} M[\overline{Q}_{16}\beta_s + \overline{Q}_{26}\beta_z + \overline{Q}_{36}\beta_n + \overline{Q}_{66}\beta_{sz}]^{(k)}\, dn \tag{3.23}$$

The shear stress resultants are given by

$$N_{nz}(s,z) = A_{44}\gamma_{nz} + A_{45}\gamma_{sn} \tag{3.24}$$

$$N_{sn}(s,z) = A_{45}\gamma_{nz} + A_{55}\gamma_{sn} \tag{3.25}$$

In light of the assumption of cross-section non-deformability $\gamma_{sn} = 0$. As a result

$$N_{nz} = A_{44}\gamma_{nz} \tag{3.26}$$

$$N_{sn} = A_{45}\gamma_{nz} \tag{3.27}$$

where

$$A_{ij} = \frac{5}{4} \sum_{k=1}^{N} \overline{Q}_{ij}[h_k - h_{k-1} - \frac{4}{3}(h_k{}^3 - h_{k-1}{}^3)\frac{1}{h^2}] \qquad i,j = 4,5 \tag{3.28}$$

Now the stress resultants can be expressed in matrix form

$$\begin{bmatrix} N_{ss} \\ N_{zz} \\ N_{sz} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz}{}^0 \\ \gamma_{sz}{}^0 \end{bmatrix} + \begin{bmatrix} B_{12} & B_{16} \\ B_{22} & B_{26} \\ B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{zz}{}^n \\ 2W_M \end{bmatrix} - \begin{bmatrix} N_s{}^T \\ N_z{}^T \\ N_{sz}{}^T \end{bmatrix} - \begin{bmatrix} N_s{}^M \\ N_z{}^M \\ N_{sz}{}^M \end{bmatrix} \tag{3.29}$$

When the internal pressure is absent, the hoop stress resultant, N_{ss}, is assumed to be negligibly small. From Eq 3.13

$$N_{ss} = 0 = A_{11}\varepsilon_{ss} + A_{12}\varepsilon_{zz}{}^0 + B_{12}\varepsilon_{zz}{}^n + A_{16}\gamma_{sz}{}^0 + 2B_{16}W_M - N_s{}^T - N_s{}^M \tag{3.30}$$

Solving for ε_{ss} yields

$$\varepsilon_{ss} = \frac{-A_{12}\varepsilon_{zz}{}^{0} - A_{16}\gamma_{sz}{}^{0} - 2B_{16}W_M - B_{12}\varepsilon_{zz}{}^{n} + N_s{}^T + N_s{}^M}{A_{11}} \tag{3.31}$$

Inserting Eqs 3.31 into Eqs 3.18 and 3.21 results in

$$\begin{aligned} N_{zz} &= \frac{A_{12}}{A_{11}}(-A_{12}\varepsilon_{zz}{}^{0} - A_{16}\gamma_{sz}{}^{0} - 2B_{16}W_M - B_{12}\varepsilon_{zz}{}^{n} + N_s{}^T + N_s{}^M) \\ &+ A_{22}\varepsilon_{zz}{}^{0} + B_{22}\varepsilon_{zz}{}^{n} + A_{26}\gamma_{sz}{}^{0} + 2B_{26}W_M - N_z{}^T - N_z{}^M \end{aligned} \tag{3.32}$$

$$\begin{aligned} N_{sz} &= \frac{A_{16}}{A_{11}}(-A_{12}\varepsilon_{zz}{}^{0} - A_{16}\gamma_{sz}{}^{0} - 2B_{16}W_M - B_{12}\varepsilon_{zz}{}^{n} + N_s{}^T + N_s{}^M) \\ &+ A_{26}\varepsilon_{zz}{}^{0} + B_{26}\varepsilon_{zz}{}^{n} + A_{66}\gamma_{sz}{}^{0} + 2B_{66}W_M - N_{sz}{}^T - N_{sz}{}^M \end{aligned} \tag{3.33}$$

Eqs 3.32 and 3.33 could be expressed in a more compact form as

$$\begin{bmatrix} N_{zz} \\ N_{sz} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \end{bmatrix} \begin{bmatrix} \varepsilon_{zz}{}^{0} \\ \gamma_{sz}{}^{0} \\ W_M \\ \varepsilon_{zz}{}^{n} \end{bmatrix} - \begin{bmatrix} N_1{}^T \\ N_2{}^T \end{bmatrix} - \begin{bmatrix} N_1{}^M \\ N_2{}^M \end{bmatrix} \tag{3.34}$$

where the modified stiffnesses K_{ij} are given by

$$K_{11} = A_{22} - \frac{A_{12}{}^2}{A_{11}} \tag{3.35}$$

$$K_{12} = A_{26} - \frac{A_{12}A_{16}}{A_{11}} = K_{21} \tag{3.36}$$

$$K_{13} = 2(B_{26} - \frac{A_{12}B_{16}}{A_{11}}) \tag{3.37}$$

$$K_{14} = B_{22} - \frac{A_{12}B_{12}}{A_{11}} \tag{3.38}$$

$$N_1^T = N_z^T - \frac{A_{12}N_s^T}{A_{11}} \tag{3.39}$$

$$N_1^M = N_z^M - \frac{A_{12}N_s^M}{A_{11}} \tag{3.40}$$

$$K_{22} = A_{66} - \frac{A_{16}^2}{A_{11}} \tag{3.41}$$

$$K_{23} = 2(B_{66} - \frac{A_{16}B_{16}}{A_{11}}) \tag{3.42}$$

$$K_{24} = B_{26} - \frac{A_{16}B_{12}}{A_{11}} \tag{3.43}$$

$$N_2^T = N_{sz}^T - \frac{A_{16}N_s^T}{A_{11}} \tag{3.44}$$

$$N_2^M = N_{sz}^M - \frac{A_{16}N_s^M}{A_{11}} \tag{3.45}$$

By inserting Eq 3.2 into Eq 3.5, the local stress couples with respect to the middle surface of the beam are obtained as:

$$\begin{aligned} L_{zz}(s,z) &= \sum_{k=1}^{N} \int_{h(k)} \sigma_{zz}^{(k)}\, n\, dn \\ &= \sum_{k=1}^{N} \int_{h(k)} \overline{Q}_{12}(\varepsilon_{ss} - \alpha_s T - \beta_s M) + \overline{Q}_{22}(\varepsilon_{zz} - \alpha_z T - \beta_z M) \\ &+ \overline{Q}_{23}(\varepsilon_{nn} - \alpha_n T - \beta_n M) + \overline{Q}_{26}(\gamma_{sz} - \alpha_{sz} T - \beta_{sz} M)\, n\, dn \\ &= B_{12}\varepsilon_{ss} + B_{22}\varepsilon_{zz}^0 + D_{22}\varepsilon_{zz}^n + B_{26}\gamma_{sz}^0 + 2D_{26}W_M - L_z^T - L_z^M \end{aligned} \tag{3.46}$$

$$
\begin{aligned}
L_{sz}(s,z) &= \sum_{k=1}^{N} \int_{h(k)} \sigma_{sz}{}^{(k)}\, n\, dn \\
&= \sum_{k=1}^{N} \int_{h(k)} \overline{Q}_{16}(\varepsilon_{ss} - \alpha_s T - \beta_s M) + \overline{Q}_{26}(\varepsilon_{zz} - \alpha_z T - \beta_z M) \\
&+ \overline{Q}_{36}(\varepsilon_{nn} - \alpha_n T - \beta_n M) + \overline{Q}_{66}(\gamma_{sz} - \alpha_{sz} T - \beta_{sz} M)\, n\, dn \\
&= B_{16}\varepsilon_{ss} + B_{26}\varepsilon_{zz}{}^{0} + D_{66}\varepsilon_{zz}{}^{n} + B_{66}\gamma_{sz}{}^{0} + 2D_{66}W_M - L_{sz}{}^{T} - L_{sz}{}^{M}
\end{aligned}
\tag{3.47}
$$

where

$$
D_{ij} = \sum_{k=1}^{N} \int_{h(k)} \overline{Q}_{ij}{}^{(k)} n^2\, dn \tag{3.48}
$$

$$
L_z{}^{T}(s,z) = \sum_{k=1}^{N} \int_{h(k)} T\, [\overline{Q}_{12}\alpha_s + \overline{Q}_{22}\alpha_z + \overline{Q}_{23}\alpha_n + \overline{Q}_{26}\alpha_{sz}]^{(k)}\, n\, dn \tag{3.49}
$$

$$
L_z{}^{M}(s,z) = \sum_{k=1}^{N} \int_{h(k)} M\, [\overline{Q}_{12}\beta_s + \overline{Q}_{22}\beta_z + \overline{Q}_{23}\beta_n + \overline{Q}_{26}\beta_{sz}]^{(k)}\, n\, dn \tag{3.50}
$$

$$
L_{sz}{}^{T}(s,z) = \sum_{k=1}^{N} \int_{h(k)} T\, [\overline{Q}_{16}\alpha_s + \overline{Q}_{26}\alpha_z + \overline{Q}_{36}\alpha_n + \overline{Q}_{66}\alpha_{sz}]^{(k)}\, n\, dn \tag{3.51}
$$

$$
L_{sz}{}^{M}(s,z) = \sum_{k=1}^{N} \int_{h(k)} M\, [\overline{Q}_{16}\beta_s + \overline{Q}_{26}\beta_z + \overline{Q}_{36}\beta_n + \overline{Q}_{66}\beta_{sz}]^{(k)}\, n\, dn \tag{3.52}
$$

Substituting Eqn 3.31 into Eqn's 3.46 and 3.47 yields

$$
\begin{aligned}
L_{zz} &= \frac{B_{12}}{A_{11}}(- A_{12}\varepsilon_{zz}{}^{0} - A_{16}\gamma_{sz}{}^{0} - 2B_{16}W_M - B_{12}\varepsilon_{zz}{}^{n} - N_s{}^{T} - N_s{}^{M}) \\
&+ B_{22}\varepsilon_{zz}{}^{0} + B_{26}\gamma_{sz}{}^{0} + 2D_{26}W_M + D_{22}\varepsilon_{zz}{}^{n} - L_z{}^{T} - L_z{}^{M}
\end{aligned}
\tag{3.53}
$$

$$L_{sz} = \frac{B_{16}}{A_{11}}(-A_{12}\varepsilon_{zz}{}^{0} - A_{16}\gamma_{sz}{}^{0} - 2B_{16}W_M - B_{12}\varepsilon_{zz}{}^{n} - N_s{}^{T} - N_s{}^{M})$$
$$+ B_{26}\varepsilon_{zz}{}^{0} + B_{66}\gamma_{sz}{}^{0} + 2D_{66}W_M + D_{26}\varepsilon_{zz}{}^{n} - L_{sz}{}^{T} - L_{sz}{}^{M} \quad (3.54)$$

In a more compact form, Eqs 3.53 and 3.54 may be expressed as:

$$\begin{bmatrix} L_{zz} \\ L_{sz} \end{bmatrix} = \begin{bmatrix} K_{41} & K_{42} & K_{43} & K_{44} \\ K_{51} & K_{52} & K_{53} & K_{54} \end{bmatrix} \begin{bmatrix} \varepsilon_{zz}{}^{0} \\ \gamma_{sz}{}^{0} \\ W_M \\ \varepsilon_{zz}{}^{n} \end{bmatrix} - \begin{bmatrix} N_4{}^{T} \\ N_5{}^{T} \end{bmatrix} - \begin{bmatrix} N_4{}^{M} \\ N_5{}^{M} \end{bmatrix} \quad (3.55)$$

where the modified stiffnesses K_{ij} are

$$K_{41} = B_{22} - \frac{B_{12}A_{12}}{A_{11}} = K_{14} \quad (3.56)$$

$$K_{42} = B_{26} - \frac{B_{12}A_{16}}{A_{11}} = K_{24} \quad (3.57)$$

$$K_{43} = 2(D_{26} - \frac{B_{12}B_{16}}{A_{11}}) \quad (3.58)$$

$$K_{44} = D_{22} - \frac{B_{12}{}^{2}}{A_{11}} \quad (3.59)$$

$$K_{51} = B_{26} - \frac{B_{16}A_{12}}{A_{11}} \quad (3.60)$$

$$K_{52} = B_{66} - \frac{B_{16}A_{16}}{A_{11}} \quad (3.61)$$

$$K_{53} = 2(D_{66} - \frac{B_{16}{}^{2}}{A_{11}}) \quad (3.62)$$

$$K_{54} = D_{26} - \frac{B_{12}B_{16}}{A_{11}} \tag{3.63}$$

$$N_4^{\,T} = L_z^{\,T} - \frac{B_{12}N_s^{\,T}}{A_{11}} \tag{3.64}$$

$$N_4^{\,M} = L_z^{\,M} - \frac{B_{12}N_s^{\,M}}{A_{11}} \tag{3.65}$$

$$N_5^{\,T} = L_{sz}^{\,T} - \frac{B_{16}N_s^{\,T}}{A_{11}} \tag{3.66}$$

$$N_5^{\,M} = L_{sz}^{\,M} - \frac{B_{16}N_s^{\,M}}{A_{11}} \tag{3.67}$$

3.4 Constitutive Relation for Thin-Walled Closed Cross-Section Beams

The procedure of deriving the constitutive relationships for the closed beams is similar to that for the open beams. For a thin-walled beam with closed cross-section, a difference lies in the expression of the torsional function. The membrane shear strain has the expression

$$\gamma_{sz} = \gamma_{sz}^{\,0} + \gamma_{sz}^{\,t} \tag{3.68}$$

where γ_{sz}^0 was presented in Eq 3.8. The membrane shear strain due to torsion, γ_{sz}^t, is

$$\gamma_{sz}^{\,t} = \Psi_T W_M \tag{3.69}$$

where

$$\Psi_T = 2\frac{A_c}{\beta} \tag{3.70}$$

$$W_M = \delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c' \tag{3.71}$$

As a result, for closed beams, $\gamma_{sz}{}^t$ can be expressed as

$$\gamma_{sz}{}^t = 2\frac{A_c}{\beta}\,[\delta_{w1}\phi'(z) + \delta_{w2}\Theta(z) + \delta_{w3}\phi_c'] \tag{3.72}$$

The hoop stress resultant for closed beams is given by

$$\begin{aligned} N_{ss}(s,z) &= \sum_{k=1}^{N}\int_{h(k)} \sigma_{ss}{}^{(k)}dn \\ &= \sum_{k=1}^{N}\int_{h(k)} [\overline{Q}_{11}(\varepsilon_{ss} - \alpha_s T - \beta_s M) + \overline{Q}_{12}(\varepsilon_{zz} - \alpha_z T - \beta_z M) \\ &+ \overline{Q}_{13}(\varepsilon_{nn} - \alpha_n T - \beta_n M) + \overline{Q}_{16}(\gamma_{sz} - \alpha_{sz} T - \beta_{sz} M)]^{(k)}dn \\ &= A_{11}\varepsilon_{ss} + A_{12}\varepsilon_{zz}{}^0 + B_{12}\varepsilon_{zz}{}^n + A_{16}\gamma_{sz}{}^0 + A_{16}2\frac{A_c}{\beta}W_M - N_s{}^T - N_s{}^M \end{aligned} \tag{3.73}$$

where

$$N_s{}^T(s,z) = \sum_{k=1}^{N}\int_{h(k)} T\,[\overline{Q}_{11}\alpha_s + \overline{Q}_{12}\alpha_z + \overline{Q}_{13}\alpha_n + \overline{Q}_{16}\alpha_{sz}]^{(k)}\,dn \tag{3.74}$$

$$N_s{}^M(s,z) = \sum_{k=1}^{N}\int_{h(k)} M\,[\overline{Q}_{11}\beta_s + \overline{Q}_{12}\beta_z + \overline{Q}_{13}\beta_n + \overline{Q}_{16}\beta_{sz}]^{(k)}\,dn \tag{3.75}$$

Silmilarly, the axial stress resultant is

$$
\begin{aligned}
N_{zz}(s,z) &= \sum_{k=1}^{N} \int_{h(k)} \sigma_{zz}^{(k)} dn \\
&= \sum_{k=1}^{N} \int_{h(k)} [\overline{Q}_{12}(\varepsilon_{ss} - \alpha_s T - \beta_s M) + \overline{Q}_{22}(\varepsilon_{zz} - \alpha_z T - \beta_z M) \\
&+ \overline{Q}_{23}(\varepsilon_{nn} - \alpha_n T - \beta_n M) + \overline{Q}_{26}(\gamma_{sz} - \alpha_{sz} T - \beta_{sz} M)]^{(k)} dn \\
&= A_{12}\varepsilon_{ss} + A_{22}\varepsilon_{zz}{}^{0} + B_{22}\varepsilon_{zz}{}^{n} + A_{26}\gamma_{sz}{}^{0} + A_{26} 2 \frac{A_c}{\beta} W_M - N_z{}^T - N_z{}^M
\end{aligned}
\tag{3.76}
$$

where

$$
N_z{}^T(s,z) = \sum_{k=1}^{N} \int_{h(k)} T\, [\overline{Q}_{12}\alpha_s + \overline{Q}_{22}\alpha_z + \overline{Q}_{23}\alpha_n + \overline{Q}_{26}\alpha_{sz}]^{(k)}\, dn \tag{3.77}
$$

$$
N_z{}^M(s,z) = \sum_{k=1}^{N} \int_{h(k)} M\, [\overline{Q}_{12}\beta_s + \overline{Q}_{22}\beta_z + \overline{Q}_{23}\beta_n + \overline{Q}_{26}\beta_{sz}]^{(k)}\, dn \tag{3.78}
$$

The membrane shear stress resultant is given by

$$
\begin{aligned}
N_{sz}(s,z) &= \sum_{k=1}^{N} \int_{h(k)} \sigma_{sz}^{(k)} dn \\
&= \sum_{k=1}^{N} \int_{h(k)} [\overline{Q}_{16}(\varepsilon_{ss} - \alpha_s T - \beta_s M) + \overline{Q}_{26}(\varepsilon_{zz} - \alpha_z T - \beta_z M) \\
&+ \overline{Q}_{36}(\varepsilon_{nn} - \alpha_n T - \beta_n M) + \overline{Q}_{66}(\gamma_{sz} - \alpha_{sz} T - \beta_{sz} M)]^{(k)} dn \\
&= A_{16}\varepsilon_{ss} + A_{26}\varepsilon_{zz}{}^{0} + B_{26}\varepsilon_{zz}{}^{n} + A_{66}\gamma_{sz}{}^{0} + A_{66} 2 \frac{A_c}{\beta} W_M - N_{sz}{}^T - N_{sz}{}^M
\end{aligned}
\tag{3.79}
$$

where

$$
N_{sz}{}^T(s,z) = \sum_{k=1}^{N} \int_{h(k)} T\, [\overline{Q}_{16}\alpha_s + \overline{Q}_{26}\alpha_z + \overline{Q}_{36}\alpha_n + \overline{Q}_{66}\alpha_{sz}]^{(k)}\, dn \tag{3.80}
$$

$$N_{sz}^{M}(s,z) = \sum_{k=1}^{N} \int_{h(k)} M\,[\overline{Q}_{16}\beta_s + \overline{Q}_{26}\beta_z + \overline{Q}_{36}\beta_n + \overline{Q}_{66}\beta_{sz}]^{(k)}\, dn \tag{3.81}$$

The shear stress resultants are the same for the open beam case:

$$N_{nz} = A_{44}\gamma_{nz} + A_{45}\gamma_{sn} \tag{3.82}$$

$$N_{sn} = A_{45}\gamma_{nz} + A_{55}\gamma_{sn} \tag{3.83}$$

Assuming γ_{sn} to be zero yields

$$N_{nz} = A_{44}\gamma_{nz} \tag{3.84}$$

$$N_{sn} = A_{45}\gamma_{nz} \tag{3.85}$$

where

$$A_{ij} = \frac{5}{4} \sum_{k=1}^{N} \overline{Q}_{ij}[h_k - h_{k-1} - \frac{4}{3}(h_k^{\,3} - h_{k-1}^{\,3})\frac{1}{h^2}] \qquad i,j = 4,5 \tag{3.86}$$

The stress resultants can be expressed in the matrix form

$$\begin{bmatrix} N_{ss} \\ N_{zz} \\ N_{sz} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & (A_{16}2\frac{A_c}{\beta}) \\ A_{12} & A_{22} & A_{26} & (A_{26}2\frac{A_c}{\beta}) \\ A_{16} & A_{26} & A_{66} & (A_{66}2\frac{A_c}{\beta}) \end{bmatrix} \begin{bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz}^{\,0} \\ \gamma_{sz}^{\,0} \\ W_M \end{bmatrix} + \begin{bmatrix} B_{12} \\ B_{22} \\ B_{26} \end{bmatrix} \varepsilon_{zz}^{\,n} - \begin{bmatrix} N_s^{\,T} \\ N_z^{\,T} \\ N_{sz}^{\,T} \end{bmatrix} - \begin{bmatrix} N_s^{\,M} \\ N_z^{\,M} \\ N_{sz}^{\,M} \end{bmatrix} \tag{3.87}$$

As in the open beam case, when the internal pressure is absent N_{ss} is assumed to be negligibly small. Thus

$$N_{ss} = 0 = A_{11}\varepsilon_{ss} + A_{12}\varepsilon_{zz}^{\,0} + B_{12}\varepsilon_{zz}^{\,n} + A_{16}\gamma_{sz}^{\,0} + (A_{16}2\frac{A_c}{\beta}W_M) - N_s^{\,T} - N_s^{\,M} \tag{3.88}$$

Solving for ε_{ss},

$$\varepsilon_{ss} = \frac{-A_{12}\varepsilon_{zz}{}^{0} - A_{16}\gamma_{sz}{}^{0} - (A_{16}2\frac{A_c}{\beta})W_M - B_{12}\varepsilon_{zz}{}^{n} + N_s{}^{T} + N_s{}^{M}}{A_{11}} \tag{3.89}$$

Inserting Eq 3.89 into Eqs 3.76 and 3.79 yields

$$\begin{aligned} N_{zz}(s,z) = & \frac{A_{12}}{A_{11}}(-A_{12}\varepsilon_{zz}{}^{0} - A_{16}\gamma_{sz}{}^{0} - A_{16}2\frac{A_c}{\beta}W_M - B_{12}\varepsilon_{zz}{}^{n} + N_s{}^{M} + N_s{}^{M}) \\ & + A_{22}\varepsilon_{zz}{}^{0} + B_{22}\varepsilon_{zz}{}^{n} + A_{26}\gamma_{sz}{}^{0} + A_{26}2\frac{A_c}{\beta}W_M - N_z{}^{T} - N_z{}^{M} \end{aligned} \tag{3.90}$$

The membrane stress resultant is given by

$$\begin{aligned} N_{sz}(s,z) = & \frac{A_{16}}{A_{11}}(-A_{12}\varepsilon_{zz}{}^{0} - A_{16}\gamma_{sz}{}^{0} - A_{16}2\frac{A_c}{\beta}W_M - B_{12}\varepsilon_{zz}{}^{n} + N_s{}^{T} + N_s{}^{M}) \\ & + A_{26}\varepsilon_{zz}{}^{0} + B_{26}\varepsilon_{zz}{}^{n} + A_{66}\gamma_{sz}{}^{0} + A_{66}2\frac{A_c}{\beta}W_M - N_{sz}{}^{T} - N_{sz}{}^{M} \end{aligned} \tag{3.91}$$

In a unified form, Eqs 3.90 and 3.91 may be expressed in matrix form as

$$\begin{bmatrix} N_{zz} \\ N_{sz} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \end{bmatrix} \begin{bmatrix} \varepsilon_{zz}{}^{0} \\ \gamma_{sz}{}^{0} \\ W_M \\ \varepsilon_{zz}{}^{n} \end{bmatrix} - \begin{bmatrix} N_1{}^{T} \\ N_2{}^{T} \end{bmatrix} - \begin{bmatrix} N_1{}^{M} \\ N_2{}^{M} \end{bmatrix} \tag{3.92}$$

where

$$K_{11} = A_{22} - \frac{A_{12}{}^{2}}{A_{11}} \tag{3.93}$$

$$K_{12} = A_{26} - \frac{A_{12}A_{16}}{A_{11}} = K_{21} \tag{3.94}$$

$$K_{13} = (A_{26} - \frac{A_{12}A_{16}}{A_{11}})2\frac{A_c}{\beta} \quad (3.95)$$

$$K_{14} = B_{22} - \frac{A_{12}B_{12}}{A_{11}} \quad (3.96)$$

$$N_1{}^T = N_z{}^T - \frac{A_{12}N_s{}^T}{A_{11}} \quad (3.97)$$

$$N_1{}^M = N_z{}^M - \frac{A_{12}N_s{}^M}{A_{11}} \quad (3.98)$$

$$K_{22} = A_{66} - \frac{A_{16}{}^2}{A_{11}} \quad (3.99)$$

$$K_{23} = (A_{66} - \frac{A_{16}{}^2}{A_{11}})2\frac{A_c}{\beta} \quad (3.100)$$

$$K_{24} = B_{26} - \frac{A_{16}B_{12}}{A_{11}} \quad (3.101)$$

$$N_2{}^T = N_{sz}{}^T - \frac{A_{16}N_s{}^T}{A_{11}} \quad (3.102)$$

$$N_2{}^M = N_{sz}{}^M - \frac{A_{16}N_s{}^M}{A_{11}} \quad (3.103)$$

It is found that the above expression of the stiffness quantities is similar to the one of their open cross-section beam counterparts. However, the torsion related modified stiffnesses, K_{13} and K_{23}, have different expressions for each of two cases. The local stress couples with respect to the middle surface of the beam are given by Eq 3.5:

$$L_{zz}(s,z) = \sum_{k=1}^{N} \int_{h(k)} \sigma_{zz}^{(k)} \, n \, dn$$

$$= \sum_{k=1}^{N} \int_{h(k)} [\overline{Q}_{12}(\varepsilon_{ss} - \alpha_s T - \beta_s M) + \overline{Q}_{22}(\varepsilon_{zz} - \alpha_z T - \beta_z M) \tag{3.104}$$

$$+ \overline{Q}_{23}(\varepsilon_{nn} - \alpha_n T - \beta_n M) + \overline{Q}_{26}(\gamma_{sz} - \alpha_{sz} T - \beta_{sz} M)] \, n \, dn$$

$$= B_{12}\varepsilon_{ss} + B_{22}\varepsilon_{zz}{}^{0} + D_{22}\varepsilon_{zz}{}^{n} + B_{26}\gamma_{sz}{}^{0} + B_{26} 2 \frac{A_c}{\beta} W_M - L_z{}^{T} - L_z{}^{M}$$

$$L_{sz}(s,z) = \sum_{k=1}^{N} \int_{h(k)} \sigma_{sz}^{(k)} \, n \, dn$$

$$= \sum_{k=1}^{N} \int_{h(k)} [\overline{Q}_{16}(\varepsilon_{ss} - \alpha_s T - \beta_s M) + \overline{Q}_{26}(\varepsilon_{zz} - \alpha_z T - \beta_z M) \tag{3.105}$$

$$+ \overline{Q}_{36}(\varepsilon_{nn} - \alpha_n T - \beta_n M) + \overline{Q}_{66}(\gamma_{sz} - \alpha_{sz} T - \beta_{sz} M)] \, n \, dn$$

$$= B_{16}\varepsilon_{ss} + B_{26}\varepsilon_{zz}{}^{0} + D_{66}\varepsilon_{zz}{}^{n} + B_{66}\gamma_{sz}{}^{0} + B_{66} 2 \frac{A_c}{\beta} W_M - L_{sz}{}^{T} - L_{sz}{}^{M}$$

where

$$D_{ij} = \sum_{k=1}^{N} \int_{h(k)} \overline{Q}_{ij}{}^{(k)} n^2 \, dn \tag{3.106}$$

$$L_s{}^{T}(s,z) = \sum_{k=1}^{N} \int_{h(k)} T \, [\overline{Q}_{11}\alpha_s + \overline{Q}_{12}\alpha_z + \overline{Q}_{13}\alpha_n + \overline{Q}_{16}\alpha_{sz}]^{(k)} \, n \, dn \tag{3.107}$$

$$L_s{}^{M}(s,z) = \sum_{k=1}^{N} \int_{h(k)} M \, [\overline{Q}_{11}\beta_s + \overline{Q}_{12}\beta_z + \overline{Q}_{13}\beta_n + \overline{Q}_{16}\beta_{sz}]^{(k)} \, n \, dn \tag{3.108}$$

$$L_z{}^{T}(s,z) = \sum_{k=1}^{N} \int_{h(k)} T \, [\overline{Q}_{12}\alpha_s + \overline{Q}_{22}\alpha_z + \overline{Q}_{23}\alpha_n + \overline{Q}_{26}\alpha_{sz}]^{(k)} \, n \, dn \tag{3.109}$$

$$L_z^M(s,z) = \sum_{k=1}^{N} \int_{h(k)} M\,[\overline{Q}_{12}\beta_s + \overline{Q}_{22}\beta_z + \overline{Q}_{23}\beta_n + \overline{Q}_{26}\beta_{sz}]^{(k)}\, n\, dn \tag{3.110}$$

$$L_{sz}^T(s,z) = \sum_{k=1}^{N} \int_{h(k)} T\,[\overline{Q}_{16}\alpha_s + \overline{Q}_{26}\alpha_z + \overline{Q}_{36}\alpha_n + \overline{Q}_{66}\alpha_{sz}]^{(k)}\, n\, dn \tag{3.111}$$

$$L_{sz}^M(s,z) = \sum_{k=1}^{N} \int_{h(k)} M\,[\overline{Q}_{16}\beta_s + \overline{Q}_{26}\beta_z + \overline{Q}_{36}\beta_n + \overline{Q}_{66}\beta_{sz}]^{(k)}\, n\, dn \tag{3.112}$$

Substituting Eqn 3.89 into Eqn's 3.90 and 3.91 yields

$$\begin{aligned} L_{zz} = \frac{B_{12}}{A_{11}}\Big(-A_{12}\varepsilon_{zz}^{\;0} - A_{16}\gamma_{sz}^{\;0} - A_{16}2\frac{A_c}{\beta}W_M - B_{12}\varepsilon_{zz}^{\;n} + N_s^{\;T} + N_s^{\;M}\Big) \\ + B_{22}\varepsilon_{zz}^{\;0} + B_{26}\gamma_{sz}^{\;0} + B_{26}2\frac{A_c}{\beta}W_M + D_{22}\varepsilon_{zz}^{\;n} - L_z^{\;T} - L_z^{\;M} \end{aligned} \tag{3.113}$$

$$\begin{aligned} L_{sz} = \frac{B_{16}}{A_{11}}\Big(-A_{12}\varepsilon_{zz}^{\;0} - A_{16}\gamma_{sz}^{\;0} - A_{16}2\frac{A_c}{\beta}W_M - B_{12}\varepsilon_{zz}^{\;n} + N_s^{\;T} + N_s^{\;M}\Big) \\ + B_{26}\varepsilon_{zz}^{\;0} + B_{66}\gamma_{sz}^{\;0} + B_{66}2\frac{A_c}{\beta}W_M + D_{26}\varepsilon_{zz}^{\;n} - L_{sz}^{\;T} - L_{sz}^{\;M} \end{aligned} \tag{3.114}$$

Eqs 3.113 and 3.114 can be expressed in matrix form as

$$\begin{bmatrix} L_{zz} \\ L_{sz} \end{bmatrix} = \begin{bmatrix} K_{41} & K_{42} & K_{43} & K_{44} \\ K_{51} & K_{52} & K_{53} & K_{54} \end{bmatrix} \begin{bmatrix} \varepsilon_{zz}^{\;0} \\ \gamma_{sz}^{\;0} \\ W_M \\ \varepsilon_{zz}^{\;n} \end{bmatrix} - \begin{bmatrix} N_4^{\;T} \\ N_5^{\;T} \end{bmatrix} - \begin{bmatrix} N_4^{\;M} \\ N_5^{\;M} \end{bmatrix} \tag{3.115}$$

where

$$K_{41} = B_{22} - \frac{B_{12}A_{12}}{A_{11}} = K_{14} \tag{3.116}$$

$$K_{42} = B_{26} - \frac{B_{12}A_{16}}{A_{11}} = K_{24} \tag{3.117}$$

$$K_{43} = (B_{26} - \frac{B_{12}A_{16}}{A_{11}})2\frac{A_c}{\beta} \tag{3.118}$$

$$K_{44} = D_{22} - \frac{B_{12}^{\ 2}}{A_{11}} \tag{3.119}$$

$$K_{51} = B_{26} - \frac{B_{16}A_{12}}{A_{11}} \tag{3.120}$$

$$K_{52} = B_{66} - \frac{B_{16}A_{16}}{A_{11}} \tag{3.121}$$

$$K_{53} = (B_{66} - \frac{B_{16}A_{16}}{A_{11}})2\frac{A_c}{\beta} \tag{3.122}$$

$$K_{54} = D_{26} - \frac{B_{12}B_{16}}{A_{11}} \tag{3.123}$$

$$N_4^{\ T} = L_z^{\ T} - \frac{B_{12}N_s^{\ T}}{A_{11}} \tag{3.124}$$

$$N_4^{\ M} = L_z^{\ M} - \frac{B_{12}N_s^{\ M}}{A_{11}} \tag{3.125}$$

$$N_5^{\ T} = L_{sz}^{\ T} - \frac{B_{16}N_s^{\ T}}{A_{11}} \tag{3.126}$$

$$N_5^{\ M} = L_{sz}^{\ M} - \frac{B_{16}N_s^{\ M}}{A_{11}} \tag{3.127}$$

The constitutive equations for the multicell closed beams are the same as those for single cell closed cross-section beams. Thus, Eqs 3.92 and 3.115 can be applied to multicell beam structures too.

3.5 *A Unified Form of the Constitutive Relationships*

An unified expression of the constitutive equations applicable to both open and closed cross-section beams will be displayed next in matrix form. They are given as

$$\begin{bmatrix} N_{zz} \\ N_{sz} \\ L_{zz} \\ L_{sz} \\ N_{nz} \\ N_{sn} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & 0 \\ K_{21} & K_{22} & K_{23} & K_{24} & 0 \\ K_{41} & K_{42} & K_{43} & K_{44} & 0 \\ K_{51} & K_{52} & K_{53} & K_{54} & 0 \\ 0 & 0 & 0 & 0 & A_{44} \\ 0 & 0 & 0 & 0 & A_{45} \end{bmatrix} \begin{bmatrix} \varepsilon_{zz}^{0} \\ \gamma_{sz}^{0} \\ W_{M} \\ \varepsilon_{zz}^{n} \\ \gamma_{nz} \end{bmatrix} - \begin{bmatrix} N_1^T \\ N_2^T \\ N_4^T \\ N_5^T \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} N_1^M \\ N_2^M \\ N_4^M \\ N_5^M \\ 0 \\ 0 \end{bmatrix} \tag{3.128}$$

However it should be kept in mind that K_{13}, K_{23}, K_{43}, and K_{53} are different for open and closed section beams. With this precaution in mind it is possible to use the constitutive equations as expressed by Eq 3.128 for both open and closed section beams. For the sake of comparison, we will display here the constitutive equations of thin-walled beam theory, under the form encountered in the literature [Bauchu, Rehfield et al, etc]:

$$\begin{bmatrix} N_{zz} \\ N_{sz} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{zz}^{0} \\ \gamma_{sz}^{0} \end{bmatrix} \tag{3.129}$$

Chapter IV

The Governing Equations and Their Associated Boundary Conditions

4.1 Hamilton's Variational Principle

There are a variety of ways allowing one to derive the equations of motion of thin-walled beam structures. One of the most convenient ways is the energetic approach. Herein this method is going to be used. For this purpose, in order to derive the equations of motion and their associated boundary conditions, Hamilton's variational principle [12,17,47,65] is applied. Hamilton's principle states that of all displacements $u_i = u_i(x_1, x_2, x_3, t)$ that satisfy the boundary conditions $u_i = \tilde{u}_i$ over Ω_v for $\forall t > 0$ and that fulfil also the condition $\delta u_i = 0$ at $t = t_0$ and $t = t_1$ for $\forall x_i \in \tau$, where τ denotes the 3-D volume while the symbol $\forall$ has a meaning of "all", the actual displacements (i.e., the ones fulfilling the equations of motion and the boundary conditions) minimize the functional

$$J = \int_{t_0}^{t_1} (-U + K + W_e + W_d)\, dt \qquad (4.1)$$

where U, K, W_e, and W_d denote the strain energy, kinetic energy, work done by the external loadings, and work done by the damping forces, respectively.

The condition of a minimum of the functional J may be replaced by the condition of stationarity of the energy functional J. The stationarity condition is written as

$$\delta J = \int_{t_0}^{t_1} (-\delta U + \delta K + \delta W_e + \delta W_d)\, dt = 0 \qquad (4.2)$$

where δ denotes the variation sign. When applied to static problems, Hamilton's variational principle reduces to the principle of minimum potential energy, i.e.

$$\delta J = -\delta U + \delta W_e + \delta W_d = 0 \qquad (4.3)$$

4.2 Strain Energy

4.2.1 General Considerations

The elastic strain energy stored in a 3-D body may be expressed as

$$U = \frac{1}{2} \int_{\tau} \sigma_{ij} \varepsilon_{ij}\, d\tau \qquad (4.4)$$

Having in view the assumption of non-deformability of the cross-section of thin-walled beams implying that ε_{ss}, ε_{nn}, and γ_{sn} should be zero, the strain energy stored in a laminated composite thin-walled beam (with either open or closed cross-section) can be written as

$$
\begin{aligned}
U &= \frac{1}{2}\int_{\tau}\sigma_{ij}\varepsilon_{ij}d\tau \\
&= \frac{1}{2}\int_0^L\int_A \sigma_{ij}\varepsilon_{ij}\,dA\,dz \\
&= \frac{1}{2}\int_0^L\int_C\sum_{k=1}^{N}\int_{h(k)}[\sigma_{zz}\varepsilon_{zz}+\sigma_{sz}\gamma_{sz}+\sigma_{nz}\gamma_{nz}]^{(k)}dndsdz \\
&= \frac{1}{2}\int_0^L\int_C\sum_{k=1}^{N}\int_{h(k)}[\sigma_{zz}(\varepsilon_{zz}{}^0+n\varepsilon_{zz}{}^n)+\sigma_{sz}(\gamma_{sz}{}^0+\gamma_{sz}{}^t)+\sigma_{nz}\gamma_{nz}]^{(k)}dndsdz \\
&= \frac{1}{2}\int_0^L\int_C\sum_{k=1}^{N}\int_{h(k)}[\sigma_{zz}(\varepsilon_{zz}{}^0+n\varepsilon_{zz}{}^n)+\sigma_{sz}(\gamma_{sz}{}^0+\Psi_T W_M)+\sigma_{nz}\gamma_{nz}]^{(k)}dndsdz \\
&= \frac{1}{2}\int_0^L\int_C[N_{zz}\varepsilon_{zz}{}^0+L_{zz}\varepsilon_{zz}{}^n+N_{sz}\gamma_{sz}{}^0+(\delta_o 2L_{sz}+\delta_c N_{sz}2\frac{A_c}{\beta})W_M+N_{nz}\gamma_{nz}]dsdz
\end{aligned}
\tag{4.5}
$$

The tracing quantities δ_o and δ_c are introduced to identify the warping inhibition for open and closed section beams. In the case of open beams, $\delta_o = 1$ and $\delta_c = 0$ while in the latter one $\delta_c = 1$ and $\delta_o = 0$.

The variation of the strain energy is

$$
\begin{aligned}
\delta U &= \delta[\frac{1}{2}\int_{\tau}\sigma_{ij}\varepsilon_{ij}\,d\tau] \\
&= \frac{1}{2}\int_{\tau}2\sigma_{ij}\delta\varepsilon_{ij}\,d\tau \\
&= \int_{\tau}\sigma_{ij}\delta\varepsilon_{ij}\,d\tau
\end{aligned}
\tag{4.6}
$$

Then

$$\delta U = \int_0^L \int_C \sum_{k=1}^{N} \int_{h(k)} [\sigma_{zz}\delta\varepsilon_{zz} + \sigma_{sz}\delta\gamma_{sz} + \sigma_{nz}\delta\gamma_{nz}]^{(k)} dndsdz$$
$$= \int_0^L \int_{C} \sum_{k=1}^{N} \int_{h(k)} [\sigma_{zz}(\delta\varepsilon_{zz}{}^0 + n\delta\varepsilon_{zz}{}^n) + \sigma_{sz}(\delta\gamma_{sz}{}^0 + \delta\gamma_{sz}{}^t) + \sigma_{nz}\delta\gamma_{nz}]^{(k)} dndsdz \tag{4.7}$$

Eq 4.7 becomes

$$\delta U = \int_0^L \int_C [N_{zz}\delta\varepsilon_{zz}{}^0 + L_{zz}\delta\varepsilon_{zz}{}^n + N_{sz}\delta\gamma_{sz}{}^0 + (\delta_o 2L_{sz} + \delta_c N_{sz} 2\frac{A_c}{\beta})\delta W_M + N_{nz}\delta\gamma_{nz}]dsdz \tag{4.8}$$

Inserting the strain components into Eq 4.8 yields

$$\begin{aligned}
\delta U = \int_0^L \int_C [& N_{zz}\{\delta w_0' + (\delta_t\delta\theta_y' - \delta_{nt}\delta u_0'')x + (\delta_t\delta\theta_x' - \delta_{nt}\delta v_0'')y - (\delta_{w1}\delta\phi'' + \delta_{w2}\delta\Theta')\tilde{F}_\omega\} \\
& + L_{zz}\{\delta_s(\delta_t\delta\theta_y' - \delta_{nt}\delta u_0'')\frac{dy}{ds} - \delta_s(\delta_t\delta\theta_x' - \delta_{nt}\delta v_0'')\frac{dx}{ds} - (\delta_{w1}\delta\phi'' + \delta_{w2}\delta\Theta')\tilde{a}\} \\
& + N_{sz}\{(\delta_t\delta\theta_y - \delta_{nt}\delta u_0')\frac{dx}{ds} + (\delta_t\delta\theta_x - \delta_{nt}\delta v_0')\frac{dy}{ds} + \delta u_0'\frac{dx}{ds} + \delta v_0'\frac{dy}{ds} + \delta_{w2}r_n(\delta\phi' - \delta\Theta)\} \\
& + (\delta_o 2L_{sz} + \delta_c N_{sz} 2\frac{A_c}{\beta})(\delta_{w1}\delta\phi' + \delta_{w2}\delta\Theta + \delta_{w3}\delta\phi_c') \\
& + N_{nz}\{\delta_s(\delta_t\delta\theta_y - \delta_{nt}\delta u_0')\frac{dy}{ds} - \delta_s(\delta_t\delta\theta_x - \delta_{nt}\delta v_0')\frac{dx}{ds} \\
& + \delta u_0'\frac{dy}{ds} - \delta v_0'\frac{dx}{ds} + (\delta_{w1}\delta\phi' + \delta_{w2}\delta\phi' + \delta_{w3}\delta\phi_c')a \\
& - (\delta_{w1}\delta\phi' + \delta_{w2}\delta\Theta + \delta_{w3}\delta\phi_c')\tilde{a}\}]\, ds\, dz
\end{aligned} \tag{4.9}$$

By introducing the concept of global stress resultants and couples, Eq 4.9 can be rewritten in a more compact form as:

$$\begin{aligned}
\delta U = \int_0^L [& T_A\delta w_0' + \delta_t M_y\delta\theta_y' + \delta_t M_x\delta\theta_x' + \delta_t Q_x\delta\theta_y + \delta_t Q_y\delta\theta_x - \delta_{w1}B_\omega\delta\phi'' \\
& - \delta_{w2}B_\omega\delta\Theta' + (\delta_{w1}M_p + \delta_{w2}M_t)\delta\phi' + \delta_{w2}(M_p - M_t)\delta\Theta + \delta_{w3}M_p\delta\phi_c' \\
& - \delta_{nt}M_y\delta u_0'' - \delta_{nt}M_x\delta v_0'' + Q_{xo}\delta u_0' + Q_{yo}\delta v_0']dz
\end{aligned} \tag{4.10}$$

The above equation can be expressed as

$$\delta U = \int_0^L [T_A \delta w_0' + \delta_t M_y \delta\theta_y' + \delta_t M_x \delta\theta_x' + \delta_t Q_x \delta\theta_y + \delta_t Q_y \delta\theta_x - \delta_{w1} B_\omega \delta\phi'' - \delta_{w2} B_\omega \delta\Theta' + (\delta_{w1} M_p + \delta_{w2} M_t)\delta\phi' + \delta_{w2}(M_p - M_t)\delta\Theta + \delta_{w3} M_p \delta\phi_c' - \delta_{nt} M_y \delta u_0'' - \delta_{nt} M_x \delta v_0'' + (-\delta_{nt} Q_x + \hat{Q}_x)\delta u_0' + (-\delta_{nt} Q_y + \hat{Q}_y)\delta v_0']dz \tag{4.11}$$

where the global stress resultants and couples are defined as [see Fig. 7]:

$$T_A(z) \equiv \int_C N_{zz}\, ds \tag{4.12}$$

$$M_y(z) \equiv \int_C (x N_{zz} + \delta_s L_{zz} \frac{dy}{ds}) ds \tag{4.13}$$

$$M_x(z) \equiv \int_C (y N_{zz} - \delta_s L_{zz} \frac{dx}{ds}) ds \tag{4.14}$$

$$Q_x(z) \equiv \int_C (N_{sz} \frac{dx}{ds} + \delta_s N_{nz} \frac{dy}{ds})\, ds \tag{4.15}$$

$$Q_y(z) \equiv \int_C (N_{sz} \frac{dy}{ds} - \delta_s N_{nz} \frac{dx}{ds})\, ds \tag{4.16}$$

$$B_\omega(z) \equiv \int_C [\tilde{F}_\omega(s) N_{zz} + \tilde{a}(s) L_{zz}] ds \tag{4.17}$$

$$M_p(z) \equiv \int_C [\{a(s) - \tilde{a}(s)\} N_{nz} + (\delta_o 2 L_{sz} + \delta_c N_{sz} 2 \frac{A_c}{\beta})]\, ds \tag{4.18}$$

$$M_t(z) \equiv \int_C [r_n(s)N_{sz} + a(s)N_{nz}]\ ds \tag{4.19}$$

$$\begin{aligned} Q_{xo}(z) &\equiv \int_C [-\delta_{nt}(N_{sz}\frac{dx}{ds} + \delta_s N_{nz}\frac{dy}{ds}) + (N_{nz}\frac{dy}{ds} + N_{sz}\frac{dx}{ds})]\ ds \\ &\equiv -\delta_{nt}Q_x + \hat{Q}_x \end{aligned} \tag{4.20}$$

$$\begin{aligned} Q_{yo}(z) &\equiv \int_C [-\delta_{nt}(N_{sz}\frac{dy}{ds} - \delta_s N_{nz}\frac{dx}{ds}) + (N_{sz}\frac{dy}{ds} - N_{nz}\frac{dx}{ds})]\ ds \\ &\equiv -\delta_{nt}Q_y + \hat{Q}_y \end{aligned} \tag{4.21}$$

where

$$\hat{Q}_x \equiv \int_C (N_{nz}\frac{dy}{ds} + N_{sz}\frac{dx}{ds})ds \tag{4.22}$$

$$\hat{Q}_y \equiv \int_C (N_{sz}\frac{dy}{ds} - N_{nz}\frac{dx}{ds})ds \tag{4.23}$$

The above global stress resultants and couples given by Eqs 4.12 through 4.23 are obtained through the integration of their 2-D counterparts along the contour. As a result, these quantities become functions of the beam axial coordinate z only. Integration by parts in Eq 4.11 gives

$$\begin{aligned} \delta U = [&T_A\delta w_0 + \delta_t M_y\delta\theta_y + \delta_t M_x\delta\theta_x - \delta_{w1}B_\omega\delta\phi' + (\delta_{w1}B_\omega' + \delta_{w1}M_p + \delta_{w2}M_t)\delta\phi \\ &- \delta_{w2}B_\omega\delta\Theta + \delta_{w3}M_p\delta\phi_c - \delta_{nt}M_y\delta u_0' - \delta_{nt}M_x\delta v_0' \\ &+ \{\delta_{nt}(M'_y - Q_x) + \hat{Q}_x\}\delta u_0 + \{\delta_{nt}(M'_x - Q_y) + \hat{Q}_y\}\delta v_0\Big]_{z=0}^{z=L} \\ &- \int_0^L [T'_A\delta w_0 + \delta_t(M'_y - Q_x)\delta\theta_y + \delta_t(M'_x - Q_y)\delta\theta_x \\ &+ (\delta_{w1}B_\omega'' + \delta_{w1}M_p' + \delta_{w2}M_t')\delta\phi - \delta_{w2}(B_\omega' + M_p - M_t)\delta\Theta + \delta_{w3}M_p'\delta\phi_c \\ &+ (\delta_{nt}M''_y - \delta_{nt}Q'_x + \hat{Q}_x')\delta u_0 + (\delta_{nt}M''_x - \delta_{nt}Q'_y + \hat{Q}_y')\delta v_0]dz \end{aligned} \tag{4.24}$$

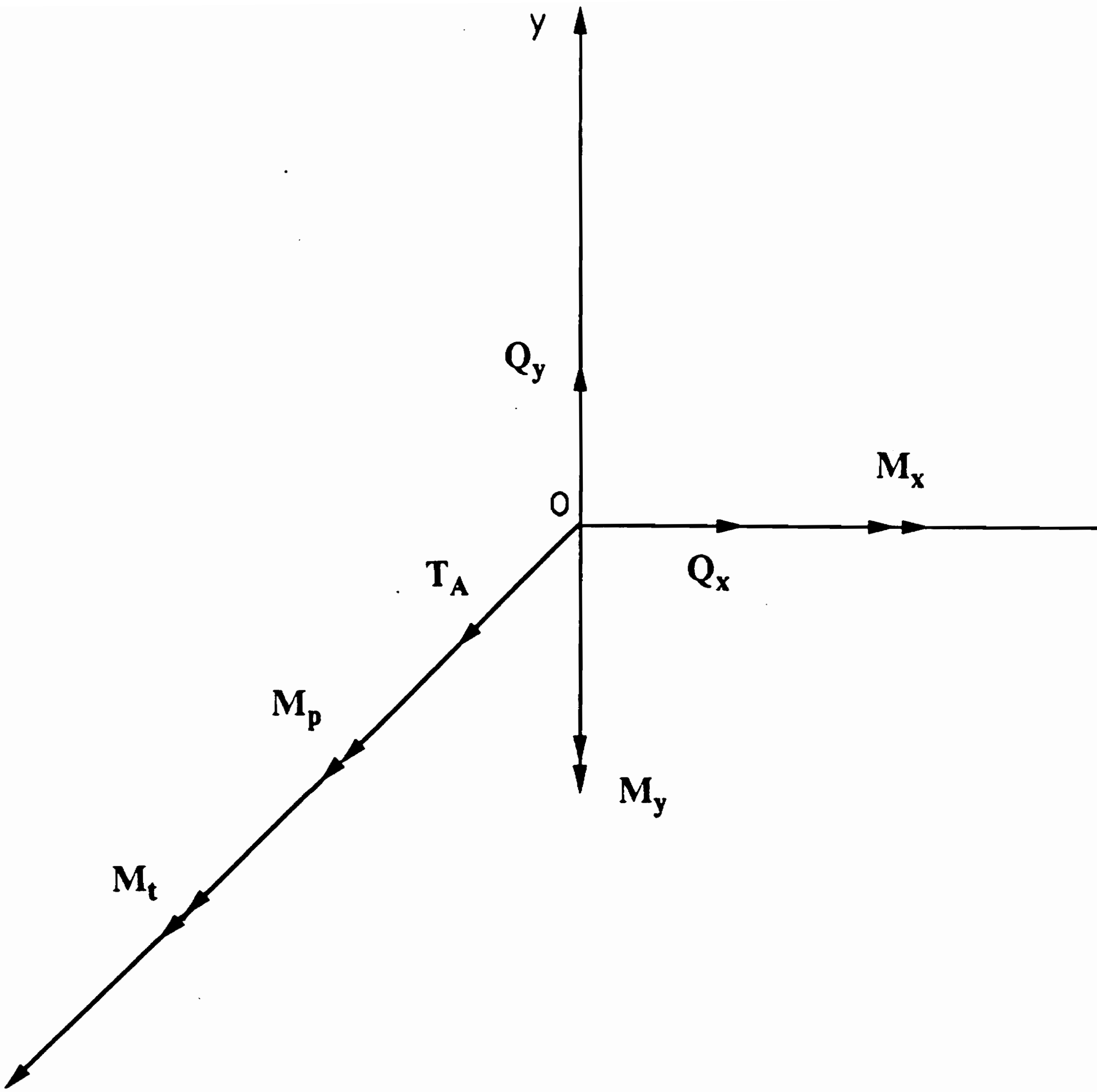


Figure 7. Gloabal Stress Resultants and Couples

4.2.2 Global Stress Resultants and Couples

Utilization of constitutive relations given by Eq 3.128 in Chapter 3 enables one to express the above global stress resultants and couples in terms of displacement variables.

The axial force T_A is given by

$$
\begin{aligned}
T_A &= \int_C N_{zz} ds \\
&= \int_C (K_{11}\varepsilon_{zz}{}^0 + K_{12}\gamma_{sz}{}^0 + K_{13}W_M + K_{14}\varepsilon_{zz}{}^n - N_1{}^T - N_1{}^M) ds \\
&= \int_C \{K_{11}[w_0' + (\delta_t\theta_y' - \delta_{nt}u_0'')x + (\delta_t\theta_x' - \delta_{nt}v_0'')y - (\delta_{w1}\phi'' + \delta_{w2}\Theta')\tilde{F}_\omega] \\
&+ K_{12}[(\delta_t\theta_y + (1-\delta_{nt})u_0')\frac{dx}{ds} + (\delta_t\theta_x + (1-\delta_{nt})v_0')\frac{dy}{ds} + \delta_{w2}r_n(\phi' - \Theta)] \\
&+ K_{13}[\delta_{w1}\phi' + \delta_{w2}\Theta + \delta_{w3}\phi_c'] \\
&+ K_{14}[\delta_s(\delta_t\theta_y' - \delta_{nt}u_0'')\frac{dy}{ds} - \delta_s(\delta_t\theta_x' - \delta_{nt}v_0'')\frac{dx}{ds} - (\delta_{w1}\phi'' + \delta_{w2}\Theta')\tilde{a}] \\
&- N_1{}^T - N_1{}^M\} ds
\end{aligned}
\tag{4.25}
$$

Collection of terms in Eq 4.25 yields the following expression for T_A:

$$
\begin{aligned}
T_A(z) &= a_{11}w_0' + a_{12}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{13}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ a_{14}[\delta_t\theta_y + (1-\delta_{nt})u_0'] + a_{15}[\delta_t\theta_x + (1-\delta_{nt})v_0'] + a_{16}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ a_{17}\phi' + \delta_{w2}a_{18}\Theta + \delta_{w3}a_{19}\phi_c' - h_1
\end{aligned}
\tag{4.26}
$$

The above equation can be expressed in a different way as:

$$
\begin{aligned}
T_A(z) &= a_{11}w_0' + a_{12}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{13}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ a_{14}\delta_t(\theta_y + u_0') + a_{15}\delta_t(\theta_x + v_0') + a_{16}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ a_{17}\phi' + \delta_{w2}a_{18}\Theta + \delta_{w3}a_{19}\phi_c' - h_1
\end{aligned}
\tag{4.27}
$$

Making use of the kinematic and constitutive relationships displayed before, the other global stress resultants and couples can be expressed in terms of displacement variables as:

$$M_y(z) = a_{21}w_0' + a_{22}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{23}(\delta_t\theta_x' - \delta_{nt}v_0'') + a_{24}\delta_t(\theta_y + u_0') + a_{25}\delta_t(\theta_x + v_0') + a_{26}(\delta_{w1}\phi'' + \delta_{w2}\Theta') + a_{27}\phi' + \delta_{w2}a_{28}\Theta + \delta_{w3}a_{29}\phi_c' - h_2 \tag{4.28}$$

$$M_x(z) = a_{31}w_0' + a_{32}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{33}(\delta_t\theta_x' - \delta_{nt}v_0'') + a_{34}\delta_t(\theta_y + u_0') + a_{35}\delta_t(\theta_x + v_0') + a_{36}(\delta_{w1}\phi'' + \delta_{w2}\Theta') + a_{37}\phi' + \delta_{w2}a_{38}\Theta + \delta_{w3}a_{39}\phi_c' - h_3 \tag{4.29}$$

$$Q_x(z) = a_{41}w_0' + a_{42}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{43}(\delta_t\theta_x' - \delta_{nt}v_0'') + a_{44}\delta_t(\theta_y + u_0') + a_{45}\delta_t(\theta_x + v_0') + a_{46}(\delta_{w1}\phi'' + \delta_{w2}\Theta') + a_{47}\phi' + \delta_{w2}a_{48}\Theta + \delta_{w3}a_{49}\phi_c' - h_4 \tag{4.30}$$

$$Q_y(z) = a_{51}w_0' + a_{52}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{53}(\delta_t\theta_x' - \delta_{nt}v_0'') + a_{54}\delta_t(\theta_y + u_0') + a_{55}\delta_t(\theta_x + v_0') + a_{56}(\delta_{w1}\phi'' + \delta_{w2}\Theta') + a_{57}\phi' + \delta_{w2}a_{58}\Theta + \delta_{w3}a_{59}\phi_c' - h_5 \tag{4.31}$$

$$B_\omega(z) = a_{61}w_0' + a_{62}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{63}(\delta_t\theta_x' - \delta_{nt}v_0'') + a_{64}\delta_t(\theta_y + u_0') + a_{65}\delta_t(\theta_x + v_0') + a_{66}(\delta_{w1}\phi'' + \delta_{w2}\Theta') + a_{67}\phi' + \delta_{w2}a_{68}\Theta + \delta_{w3}a_{69}\phi_c' - h_6 \tag{4.32}$$

$$M_p(z) = a_{71}w_0' + a_{72}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{73}(\delta_t\theta_x' - \delta_{nt}v_0'') + a_{74}\delta_t(\theta_y + u_0') + a_{75}\delta_t(\theta_x + v_0') + a_{76}(\delta_{w1}\phi'' + \delta_{w2}\Theta') + a_{77}\phi' + \delta_{w2}a_{78}\Theta + \delta_{w3}a_{79}\phi_c' - h_7 + (1 - \delta_s)(g_1u_0' - g_2v_0') \tag{4.33}$$

$$M_t(z) = a_{81}w_0' + a_{82}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{83}(\delta_t\theta_x' - \delta_{nt}v_0'') + a_{84}\delta_t(\theta_y + u_0') + a_{85}\delta_t(\theta_x + v_0') + a_{86}(\delta_{w1}\phi'' + \delta_{w2}\Theta') + a_{87}\phi' + \delta_{w2}a_{88}\Theta + \delta_{w3}a_{89}\phi_c' - h_8 + (1 - \delta_s)(g_3u_0' - g_4v_0') \tag{4.34}$$

$$\hat{Q}_x(z) = s_{41}w_0' + s_{42}(\delta_t\theta_y' - \delta_{nt}u_0'') + s_{43}(\delta_t\theta_x' - \delta_{nt}v_0'') + s_{44}\delta_t(\theta_y + u_0') + s_{45}\delta_t(\theta_x + v_0') + s_{46}(\delta_{w1}\phi'' + \delta_{w2}\Theta') + s_{47}\phi' + \delta_{w2}s_{48}\Theta + \delta_{w3}s_{49}\phi_c' - h_4 + (1 - \delta_s)(g_5u_0' - g_6v_0') \tag{4.35}$$

$$\hat{Q}_y(z) = s_{51}w_0' + s_{52}(\delta_t\theta_y' - \delta_{nt}u_0'') + s_{53}(\delta_t\theta_x' - \delta_{nt}v_0'') + s_{54}\delta_t(\theta_y + u_0') + s_{55}\delta_t(\theta_x + v_0') + s_{56}(\delta_{w1}\phi'' + \delta_{w2}\Theta') + s_{57}\phi' + \delta_{w2}s_{58}\Theta + \delta_{w3}s_{59}\phi_c' - h_5 + (1 - \delta_s)(g_7u_0' - g_8v_0') \tag{4.36}$$

In Eqs 4.27 through 4.36, a_{ij}, g_1, and g_2 denote the stiffness coefficients while h_i represent the combination of thermal and hygrothermal loading terms.

As an example, h_1 is given by

$$h_1 \equiv \int_C (N_1{}^T + N_1{}^M)ds \tag{4.37}$$

We note that when the secondary warping effect is incorporated (which implies $\delta_s = 1$) the last terms in Eqs 4.33 and 4.34 will become immaterial. Having in view the meaning of the various tracing quantities, a number of special cases of practical importance could be obtained.

4.3 Kinetic Energy Term

The expression of kinetic energy [17] of a body in motion is given by

$$K = \frac{1}{2}\int_\tau \rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} d\tau = \frac{1}{2}\int_0^L \int_C ((\frac{\partial u}{\partial t})^2 + (\frac{\partial v}{\partial t})^2 + (\frac{\partial w}{\partial t})^2)\rho h ds dz \tag{4.38}$$

where ρ denotes the mass density of the body and h denotes thickness of the wall while u,v, and w denote the displacement components in the x,y, and z directions, respectively. It is assumed that the mass density and thickness of the wall are allowed to vary along the contour line while being constant along the beam axis. This means that $\rho \equiv \rho(s)$ and $h \equiv h(s)$. The variation of the kinetic energy is

$$
\begin{aligned}
\delta K &= \frac{1}{2}\int_\tau 2\delta(\frac{\partial u_i}{\partial t})(\frac{\partial u_i}{\partial t})\rho d\tau \\
&= \int_\tau \delta(\frac{\partial u_i}{\partial t})(\frac{\partial u_i}{\partial t})\rho d\tau \\
&= \int_\tau \frac{\partial}{\partial t}(\delta u_i \frac{\partial u_i}{\partial t})\rho d\tau - \int_\tau \delta u_i \frac{\partial^2 u_i}{\partial t^2}\rho d\tau
\end{aligned}
\tag{4.39}
$$

Integration of δK with respect to time and application of the conditions:

$$\delta u_i = 0 \qquad \text{for} \;\; t = t_0 \;\; \text{and} \;\; t = t_1$$

yield

$$
\begin{aligned}
\int_{t_0}^{t_1} \delta K \, dt &= -\int_{t_0}^{t_1}\int_\tau \frac{\partial^2 u_i}{\partial t^2}\delta u_i \, \rho \, d\tau \, dt \\
&= -\int_{t_0}^{t_1}\int_0^L\int_C\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial^2 u_i}{\partial t^2}\delta u_i \, \rho \, dn \, ds \, dz \, dt \\
&= -\int_{t_0}^{t_1}\int_0^L\int_C\int_{-\frac{h}{2}}^{\frac{h}{2}} (\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 v}{\partial t^2}\delta v + \frac{\partial^2 w}{\partial t^2}\delta w)\rho \, dn \, ds \, dz \, dt
\end{aligned}
\tag{4.40}
$$

Let

$$w = w_1 + n w_2 \tag{4.41}$$

where w_1 and w_2 represent the axial displacement of the points of the contour line associated with the primary and secondary warping, respectively:

$$w_1 = w_0 + (\delta_t\theta_y - \delta_{nt}u_0')x + (\delta_t\theta_x - \delta_{nt}v_0')y - (\delta_{w1}\phi' + \delta_{w2}\Theta + \delta_{w3}\phi_c')\widetilde{F}_\omega \tag{4.42}$$

$$w_2 = \delta_s(\delta_t\theta_y - \delta_{nt}u_0')\frac{dy}{ds} - \delta_s(\delta_t\theta_x - \delta_{nt}v_0')\frac{dx}{ds} - (\delta_{w1}\phi' + \delta_{w2}\Theta + \delta_{w3}\phi_c')\tilde{a} \tag{4.43}$$

Then, Eq 4.40 can be rewritten as

$$\begin{aligned}\int_{t_0}^{t_1}\delta K\, dt = &-\int_{t_0}^{t_1}\int_0^L\int_C\int_{-\frac{h}{2}}^{\frac{h}{2}}\Big[\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 v}{\partial t^2}\delta v \\ &+ \Big(\frac{\partial^2 w_1}{\partial t^2} + n\frac{\partial^2 w_2}{\partial t^2}\Big)(\delta w_1 + n\delta w_2)\Big]\rho\, dn\, ds\, dz\, dt\end{aligned} \tag{4.44}$$

or in another form as :

$$\begin{aligned}\int_{t_0}^{t_1}\delta K\, dt = &-\int_{t_0}^{t_1}\int_0^L\int_C\int_{-\frac{h}{2}}^{\frac{h}{2}}\Big[\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 v}{\partial t^2}\delta v + \frac{\partial^2 w_1}{\partial t^2}\delta w_1 \\ &+ n\Big(\frac{\partial^2 w_1}{\partial t^2}\delta w_2 + \frac{\partial^2 w_2}{\partial t^2}\delta w_1\Big) + n^2\frac{\partial^2 w_2}{\partial t^2}\delta w_2\Big]\rho\, dn\, ds\, dz\, dt\end{aligned} \tag{4.45}$$

It is evident that the fourth term in the integrand is not going to contribute to the kinetic energy. The above equation reduces to

$$\int_{t_0}^{t_1}\delta K dt = -\int_{t_0}^{t_1}\int_0^L\int_C\int_{-\frac{h}{2}}^{\frac{h}{2}}\Big[\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 v}{\partial t^2}\delta v + \frac{\partial^2 w_1}{\partial t^2}\delta w_1 + n^2\frac{\partial^2 w_2}{\partial t^2}\delta w_2\Big]\rho dndsdzdt \tag{4.46}$$

Integration with respect to the thickness coordinate yields

$$\int_{t_0}^{t_1}\delta K\, dt = -\int_{t_0}^{t_1}\int_0^L\int_C\Big[\frac{\partial^2 u}{\partial t^2}\delta u + \frac{\partial^2 v}{\partial t^2}\delta v + \frac{\partial^2 w_1}{\partial t^2}\delta w_1 + \frac{h^3}{12}\frac{\partial^2 w_2}{\partial t^2}\delta w_2\Big]\rho dsdzdt \tag{4.47}$$

Substitution of the expression for the displacement components (see Chapter 2) into Eq 4.47 and collection of terms yield

$$\int_{t_0}^{t_1} \delta K \, dt = -\int_{t_0}^{t_1}\int_0^L [K_1\delta u_0 + K_2\delta v_0 + K_3\delta w_0 + (\delta_{w1} + \delta_{w2})K_4\delta\phi + \delta_{w3}K_{12}\delta\phi_c + K_5\delta_t\delta\theta_y + K_6\delta_{nt}\delta u_0' + K_7\delta_t\delta\theta_x + K_8\delta_{nt}\delta v_0' + K_9\delta_{w1}\delta\phi' + K_{10}\delta_{w2}\delta\Theta + K_{11}\delta_{w3}\delta\phi_c'] \, dz \, dt \tag{4.48}$$

Integration of the above equation by parts with respect to the z coordinate gives

$$\begin{aligned}\int_{t_0}^{t_1} \delta K \, dt = &-\int_{t_0}^{t_1}\int_0^L [K_1\delta u_0 + K_2\delta v_0 + K_3\delta w_0 + (\delta_{w1} + \delta_{w2})K_4\delta\phi + \delta_{w3}K_{12}\delta\phi_c \\ &+ K_5\delta_t\delta\theta_y - K_6'\delta_{nt}\delta u_0 + K_7\delta_t\delta\theta_x - K_8'\delta_{nt}\delta v_0 \\ &- K_9'\delta_{w1}\delta\phi + K_{10}\delta_{w2}\delta\Theta - K_{11}'\delta_{w3}\delta\phi_c] \, dz \, dt \\ &- \int_{t_0}^{t_1} [K_6\delta_{nt}\delta u_0\Big]_0^L + K_8\delta_{nt}\delta v_0\Big]_0^L + K_9\delta_{w1}\delta\phi\Big]_0^L + K_{11}\delta_{w3}\delta\phi_c\Big]_0^L] \, dt\end{aligned} \tag{4.49}$$

Collection of terms in Eq 4.49 yields

$$\begin{aligned}\int_{t_0}^{t_1} \delta K \, dt = &-\int_{t_0}^{t_1}\int_0^L [(K_1 - \delta_{nt}K_6')\delta u_0 + (K_2 - \delta_{nt}K_8')\delta v_0 + K_3\delta w_0 \\ &+ (\delta_{w1}K_4 - \delta_{w1}K_9' + \delta_{w2}K_4)\delta\phi \\ &+ \delta_t K_5\delta\theta_y + \delta_t K_7\delta\theta_x + \delta_{w2}K_{10}\delta\Theta + \delta_{w3}(K_{12} - K_{11}')\delta\phi_c] \, dz \, dt \\ &- \int_{t_0}^{t_1} [\delta_{nt}K_6\delta u_0\Big]_0^L + \delta_{nt}K_8\delta v_0\Big]_0^L + \delta_{w1}K_9\delta\phi\Big]_0^L + \delta_{w3}K_{11}\delta\phi_c\Big]_0^L] \, dt\end{aligned} \tag{4.50}$$

where $(\ )' \equiv \frac{\partial(\ )}{\partial z}$. The coefficients K_1 through K_{12} are given by

$$K_1 \equiv \int_C \{\ddot{u}_0 - (\delta_{w1} + \delta_{w2})y\ddot{\phi} - \delta_{w3}y\ddot{\phi}_c\}h\rho \, ds \tag{4.51}$$

$$K_2 \equiv \int_C \{\ddot{v}_0 + (\delta_{w1} + \delta_{w2})x\ddot{\phi} + \delta_{w3}x\ddot{\phi}_c\}h\rho \, ds \tag{4.52}$$

$$K_3 \equiv \int_C \{\ddot{w}_0 + x(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') + y(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') - \tilde{F}_\omega(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c')\}h\rho\, ds \tag{4.53}$$

$$K_4 \equiv \int_C \{(x^2 + y^2)\ddot{\phi} - y\ddot{u}_0 + x\ddot{v}_0\}h\rho\, ds \tag{4.54}$$

$$K_5 \equiv \int_C [xh\ddot{w}_0 + x^2h\ddot{\theta}_y + xyh\ddot{\theta}_x - xh\tilde{F}_\omega(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c') + \delta_s\frac{h^3}{12}\{\delta_s\frac{dy}{ds}\frac{dy}{ds}\ddot{\theta}_y - \delta_s\frac{dx}{ds}\frac{dy}{ds}\ddot{\theta}_x - \frac{dy}{ds}\tilde{a}(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c')\}]\rho\, ds \tag{4.55}$$

$$K_6 \equiv \int_C [-xh\ddot{w}_0 + x^2h\ddot{u}_0' + xyh\ddot{v}_0' + xh\tilde{F}_\omega(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c') + \delta_s\frac{h^3}{12}\{\delta_s\frac{dy}{ds}\frac{dy}{ds}\ddot{u}_0' - \delta_s\frac{dx}{ds}\frac{dy}{ds}\ddot{v}_0' + \frac{dy}{ds}\tilde{a}(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c')\}]\rho\, ds \tag{4.56}$$

$$K_7 \equiv \int_C [yh\ddot{w}_0 + y^2h\ddot{\theta}_x + xyh\ddot{\theta}_y - yh\tilde{F}_\omega(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c') + \delta_s\frac{h^3}{12}\{\delta_s\frac{dx}{ds}\frac{dx}{ds}\ddot{\theta}_x - \delta_s\frac{dx}{ds}\frac{dy}{ds}\ddot{\theta}_y + \frac{dx}{ds}\tilde{a}(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c')\}]\rho\, ds \tag{4.57}$$

$$K_8 \equiv \int_C [-yh\ddot{w}_0 + y^2h\ddot{v}_0' + xyh\ddot{u}_0' + yh\tilde{F}_\omega(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c') + \delta_s\frac{h^3}{12}\{\delta_s\frac{dx}{ds}\frac{dx}{ds}\ddot{v}_0' - \delta_s\frac{dx}{ds}\frac{dy}{ds}\ddot{u}_0' - \frac{dx}{ds}\tilde{a}(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c')\}]\rho\, ds \tag{4.58}$$

$$K_9 \equiv \int_C [-\tilde{F}_\omega h\ddot{w}_0 - xh\tilde{F}_\omega(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') - yh\tilde{F}_\omega(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') + \tilde{F}_\omega{}^2h\ddot{\phi}' + \frac{h^3}{12}\{\delta_s\frac{dx}{ds}\tilde{a}(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') - \delta_s\frac{dy}{ds}\tilde{a}(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') + (\tilde{a})^2\ddot{\phi}'\}]\rho\, ds \tag{4.59}$$

$$K_{10} \equiv \int_C [- \widetilde{F}_\omega h\ddot{w}_0 - x\widetilde{F}_\omega h(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') - y\widetilde{F}_\omega h(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') + \widetilde{F}_\omega{}^2 h\ddot{\Theta}$$
$$+ \frac{h^3}{12}\{\delta_s \frac{dx}{ds}\widetilde{a}(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') - \delta_s\frac{dy}{ds}\widetilde{a}(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') + (\widetilde{a})^2\ddot{\Theta}\}]\rho \, ds \qquad (4.60)$$

$$K_{11} \equiv \int_C [- \widetilde{F}_\omega h\ddot{w}_0 - xh\widetilde{F}_\omega(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') - yh\widetilde{F}_\omega(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') + \widetilde{F}_\omega{}^2 h\ddot{\phi}_c'$$
$$+ \frac{h^3}{12}\{\frac{dx}{ds}\delta_s\widetilde{a}(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') - \frac{dy}{ds}\delta_s\widetilde{a}(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') + (\widetilde{a})^2\ddot{\phi}_c'\}]\rho \, ds \qquad (4.61)$$

$$K_{12} \equiv \int_C \{(x^2 + y^2)\ddot{\phi}_c - y\ddot{u}_0 + x\ddot{v}_0\}h\rho \, ds \qquad (4.62)$$

A more compact form of the above equations is given by

$$K_1 \equiv b_1\ddot{u}_0 - (\delta_{w1} + \delta_{w2})b_2\ddot{\phi} - \delta_{w3}b_2\ddot{\phi}_c \qquad (4.63)$$

$$K_2 \equiv b_1\ddot{v}_0 + (\delta_{w1} + \delta_{w2})b_3\ddot{\phi} + \delta_{w3}b_3\ddot{\phi}_c \qquad (4.64)$$

$$K_3 \equiv b_1\ddot{w}_0 + b_3(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') + b_2(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') - b_7(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c') \qquad (4.65)$$

$$K_4 \equiv b_3\ddot{v}_0 - b_2\ddot{u}_0 + (b_4 + b_5)\ddot{\phi} \qquad (4.66)$$

$$K_5 \equiv b_3\ddot{w}_0 + (b_5 + \delta_s b_{15})\ddot{\theta}_y + (b_6 - \delta_s b_{13})\ddot{\theta}_x - (b_9 + \delta_s b_{17})(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c') \qquad (4.67)$$

$$K_6 \equiv - b_3\ddot{w}_0 + (b_5 + \delta_s b_{15})\ddot{u}_0' + (b_6 - \delta_s b_{13})\ddot{v}_0'$$
$$+ (b_9 + \delta_s b_{17})(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c') \qquad (4.68)$$

$$K_7 \equiv b_2\ddot{w}_0 + (b_4 + \delta_s b_{14})\ddot{\theta}_x + (b_6 - \delta_s b_{13})\ddot{\theta}_y$$
$$- (b_8 - \delta_s b_{16})(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c') \qquad (4.69)$$

$$K_8 \equiv -b_2\ddot{w}_0 + (b_6 - \delta_s b_{13})\ddot{u}_0' + (b_4 + \delta_s b_{14})\ddot{v}_0' + (b_8 - \delta_s b_{16})(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c') \tag{4.70}$$

$$K_9 \equiv -b_7\ddot{w}_0 - (b_9 + \delta_s b_{17})(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') - (b_8 - \delta_s b_{16})(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') + (b_{10} + b_{18})\ddot{\phi}' \tag{4.71}$$

$$K_{10} \equiv -b_7\ddot{w}_0 - (b_9 + \delta_s b_{17})(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') - (b_8 - \delta_s b_{16})(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') + (b_{10} + b_{18})\ddot{\Theta} \tag{4.72}$$

$$K_{11} \equiv -b_7\ddot{w}_0 - (b_9 + \delta_s b_{17})(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') - (b_8 - \delta_s b_{16})(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') + (b_{10} + b_{18})\ddot{\phi}_c' \tag{4.73}$$

$$K_{12} \equiv b_3\ddot{v}_0 - b_2\ddot{u}_0 + (b_4 + b_5)\ddot{\phi}_c \tag{4.74}$$

where the coefficients b_i are defined in Appendix D.

4.4 Work Done by the External Loading

4.4.1 General Considerations

The work done by the external and body forces on the beam can be expressed as [17] :

$$W_e = \int_\tau \rho H_i u_i d\tau + \int_\Omega \tilde{\sigma}_i u_i d\Omega \tag{4.75}$$

where H_i denote the components of the body force vector while $\tilde{\sigma}_i$ denote the prescribed traction vector on the external surface of the body. The total surface Ω of the beam may be considered to be $\Omega = \Omega_L U \Omega_E$ where Ω_L and Ω_E are defined by $n = \pm \frac{h}{2}$ and $z = 0$, L, respectively. Eq 4.75 can be rewritten as

$$W_e = \int_\tau \rho H_i u_i d\tau + \int_{\Omega_L} \tilde{\sigma}_i u_i d\Omega + \int_{\Omega_E} \tilde{\sigma}_i u_i d\Omega \tag{4.76}$$

where $d\tau = dn\ ds\ dz$, $d\Omega_L = ds\ dz$, $d\Omega_E = dn\ ds$.

The variation of the work done by the external and body forces can be expressed as

$$\begin{aligned} \delta W_e = & \int_0^L \int_C \int_{-\frac{h}{2}}^{\frac{h}{2}} [\rho H_x \delta u + \rho H_y \delta v + \rho H_z \delta w]\, dndsdz \\ & + \int_0^L \int_C [\tilde{\sigma}_x \delta u + \tilde{\sigma}_y \delta v + \tilde{\sigma}_z \delta w]_{n = \pm h_2} dsdz \\ & + \int_C \sum_{k=1}^{N} \int_{h(k)} [\tilde{\sigma}_x \delta u + \tilde{\sigma}_y \delta v + \tilde{\sigma}_z \delta w]_{z=0}^{z=L} dnds \end{aligned} \tag{4.77}$$

4.4.2 Work Done by the Body Forces

Substitution of the kinematic equations into Eq 4.77 yields the variation of the work done by the body forces:

$$
\begin{aligned}
\delta W_e{}^b = \int_0^L \int_C \int_{-\frac{h}{2}}^{\frac{h}{2}} & [\rho H_x \delta\{u_0 - (\delta_{w1} + \delta_{w2}) y\phi - \delta_{w3} y\phi_c\} \\
& + \rho H_y \delta\{v_0 + (\delta_{w1} + \delta_{w2}) x\phi + \delta_{w3} x\phi_c\} \\
& + \rho H_z \delta\{w_0 + (x + \delta_s n \frac{dy}{ds})(\delta_t \delta\theta_y - \delta_{nt} \delta u_0') \\
& + (y - \delta_s \frac{dx}{ds})(\delta_t \delta\theta_x - \delta_{nt} \delta v_0') \\
& - (\tilde{F}_\omega + n\tilde{a})(\delta_{w1} \delta\phi' + \delta_{w2} \delta\Theta + \delta_{w3} \delta\phi_c')\}] \, dndsdz
\end{aligned} \tag{4.78}
$$

Denote

$$
F_i(s,z) \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho H_i dn \tag{4.79}
$$

$$
E_i(s,z) \equiv \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho H_i n dn \tag{4.80}
$$

where i = x,y,z. As a result of the integration of the 3-D quantities across the thickness of the beam, F_i and E_i, denoting body resultants and couples, are 2-D quantities.

Introduction of the notation given by Eqs 4.79 and 4.80 into Eq 4.78 and collection of terms yields

$$
\begin{aligned}
\delta W_e{}^b = \int_0^L & [F_1 \delta u_0 + F_2 \delta v_0 + F_3 \delta w_0 + F_4 \delta_t \delta\theta_y + F_5 \delta_t \delta\theta_x \\
& + F_6 \delta\phi + F_7 \delta_{w2} \delta\Theta + F_8 \delta_{w3} \delta\phi_c] \, dz
\end{aligned} \tag{4.81}
$$

where

$$
F_1 \equiv \int_C [F_x + \delta_{nt}(x + \delta_s \frac{dy}{ds}) E_z'] \, ds \tag{4.82}
$$

$$F_2 \equiv \int_C [F_y + \delta_{nt}(y - \delta_s \frac{dx}{ds})E_z'] \, ds \tag{4.83}$$

$$F_3 \equiv \int_C F_z \, ds \tag{4.84}$$

$$F_4 \equiv \int_C [xF_z + \delta_s \frac{dy}{ds} E_z] \, ds \tag{4.85}$$

$$F_5 \equiv \int_C [yF_z - \delta_s \frac{dx}{ds} E_z] \, ds \tag{4.86}$$

$$F_6 \equiv \int_C [(-yF_x + xF_y)(\delta_{w1} + \delta_{w2}) + (F_z'\widetilde{F}_\omega + \widetilde{a}E_z')\delta_{w1}] \, ds \tag{4.87}$$

$$F_7 \equiv -\int_C (F_z\widetilde{F}_\omega + \widetilde{a}E_z) ds \tag{4.88}$$

$$F_8 \equiv \int_C [(-yF_x + xF_y) + (F_z'\widetilde{F}_\omega + E_z'\widetilde{a})] \, ds \tag{4.89}$$

4.4.3 Work Done by the Surface Tractions

Next the variation of the work done by the surface tractions can be expressed as

$$\delta W_e^{\,s} = \int_0^L \int_C [\tilde{\sigma}_x \delta u + \tilde{\sigma}_y \delta v + \tilde{\sigma}_z \delta w]^+ dsdz + \int_0^L \int_C [\tilde{\sigma}_x \delta u + \tilde{\sigma}_y \delta v + \tilde{\sigma}_z \delta w]^- dsdz$$
$$+ \int_C \sum_{k=1}^{N} \int_{h(k)} \left[\tilde{\sigma}_x \delta u + \tilde{\sigma}_y \delta v + \tilde{\sigma}_z \delta w\right]_{z=0}^{z=L} dnds \tag{4.90}$$

where $[\quad]^+$ and $[\quad]^-$ denote the quantities evaluated at $n = +\frac{h}{2}$ and $n = -\frac{h}{2}$, respectively. Let us denote

$$\tilde{\sigma}_x^{\,0} \equiv \tilde{\sigma}_x^{\,+} + \tilde{\sigma}_x^{\,-} \tag{4.91}$$

$$\tilde{\sigma}_y^{\,0} \equiv \tilde{\sigma}_y^{\,+} + \tilde{\sigma}_y^{\,-} \tag{4.92}$$

$$\tilde{\sigma}_z^{\,0} \equiv \tilde{\sigma}_z^{\,+} + \tilde{\sigma}_z^{\,-} \tag{4.93}$$

$$\tilde{\sigma}_z^{\,1} \equiv \tilde{\sigma}_z^{\,+} - \tilde{\sigma}_z^{\,-} \tag{4.94}$$

whereas for the axial displacement we have

$$w(s,z,n) = w_1(s,z) + nw_2(s,z) \tag{4.95}$$

By virtue of Eqs 4.91 through 4.95, Eq 4.90 can be rewritten as

$$\delta W_e^{\,s} = \int_0^L \int_C [\tilde{\sigma}_x^{\,0} \delta u + \tilde{\sigma}_y^{\,0} \delta v + \tilde{\sigma}_z^{\,0} \delta w_1 + \tilde{\sigma}_z^{\,1} \frac{h}{2} \delta w_2]\, dsdz$$
$$+ \int_C \sum_{k=1}^{N} \int_{h(k)} \left[\tilde{\sigma}_x \delta u + \tilde{\sigma}_y \delta v + \tilde{\sigma}_z \delta w_1 + \tilde{\sigma}_z n \delta w_2\right]_{z=0}^{z=L} dnds \tag{4.96}$$

Introduction of the displacement quantities into Eq 4.96 gives

$$
\begin{aligned}
\delta W_e{}^s = \int_0^L \int_C \; & \tilde{\sigma}_x{}^0[\delta u_0 - y(\delta_{w1}\delta\phi + \delta_{w2}\delta\phi + \delta_{w3}\delta\phi_c)] \\
& + \tilde{\sigma}_y{}^0[\delta v_0 + x(\delta_{w1}\delta\phi + \delta_{w2}\delta\phi + \delta_{w3}\delta\phi_c)] \\
& + \tilde{\sigma}_z{}^0[\delta w_0 + x(\delta_t\delta\theta_y - \delta_{nt}\delta u_0{}') + y(\delta_t\delta\theta_x - \delta_{nt}\delta v_0{}') \\
& - \tilde{F}_\omega(\delta_{w1}\delta\phi' + \delta_{w2}\delta\Theta + \delta_{w3}\delta\phi_c')] \\
& + \tilde{\sigma}_z{}^1 \frac{h}{2}[\delta_s \frac{dy}{ds}(\delta_t\delta\theta_y - \delta_{nt}\delta u_0{}') - \delta_s \frac{dx}{ds}(\delta_t\delta\theta_x - \delta_{nt}\delta v_0{}') \\
& - \tilde{a}(\delta_{w1}\delta\phi' + \delta_{w2}\delta\Theta + \delta_{w3}\delta\phi_c')] \; dsdz \\
& + \int_C \sum_{k=1}^{N} \int_{h(k)} \tilde{\sigma}_x[\delta u_0 - y(\delta_{w1}\delta\phi + \delta_{w2}\delta\phi + \delta_{w3}\delta\phi_c)] \\
& + \tilde{\sigma}_y[\delta v_0 + x(\delta_{w1}\delta\phi + \delta_{w2}\delta\phi + \delta_{w3}\delta\phi_c)] \\
& + \tilde{\sigma}_z[\delta w_0 + x(\delta_t\delta\theta_y - \delta_{nt}\delta u_0{}') + y(\delta_t\delta\theta_x - \delta_{nt}\delta v_0{}') \\
& - \tilde{F}_\omega(\delta_{w1}\delta\phi' + \delta_{w2}\delta\Theta + \delta_{w3}\delta\phi_c')] \\
& + \tilde{\sigma}_z[n\delta_s \frac{dy}{ds}(\delta_t\delta\theta_y - \delta_{nt}\delta u_0{}') - n\delta_s \frac{dx}{ds}(\delta_t\delta\theta_x - \delta_{nt}\delta v_0{}') \\
& - n\tilde{a}(\delta_{w1}\delta\phi' + \delta_{w2}\delta\Theta + \delta_{w3}\delta\phi_c')] \; dnds
\end{aligned}
\qquad (4.97)
$$

4.4.4 Total Work Done by the External Loading

Eqs 4.81 and 4.97 yield the expression of the variation of the work done by the external loading as:

$$
\begin{aligned}
\delta W_e = \int_0^L & [(F_1 + p_x + \delta_{nt}m_y{}')\delta u_0 + (F_2 + p_y + \delta_{nt}m_x{}')\delta v_0 + (F_3 + p_z)\delta w_0 \\
& + (F_4 + m_y)\delta_t\delta\theta_y + (F_5 + m_x)\delta_t\delta\theta_x + [(\delta_{w1} + \delta_{w2})(F_6 + m_z) + \delta_{w1}b_\omega{}']\delta\phi \\
& + (F_8 + b_\omega{}' + m_z)\delta_{w3}\delta\phi_c - (F_7 + b_\omega)\delta_{w2}\delta\Theta]dz \\
& + [\tilde{T}_A\delta w_0 + \tilde{M}_y\{\delta_t\delta\theta_y - \delta_{nt}\delta u_0{}'\} + \tilde{M}_x\{\delta_t\delta\theta_x - \delta_{nt}\delta v_0{}'\} \\
& + (\delta_{w1} + \delta_{w2})\tilde{M}_z\delta\phi + \delta_{w3}\tilde{M}_z\delta\phi_c + \tilde{Q}_x\delta u_0 + \tilde{Q}_y\delta v_0 \\
& + \tilde{B}_\omega\{\delta_{w1}\delta\phi + \delta_{w2}\delta\Theta + \delta_{w3}\delta\phi_c\}\Big]_{z=0}^{z=L}
\end{aligned}
\qquad (4.98)
$$

where

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \equiv \int_C \begin{bmatrix} \tilde{\sigma}_x{}^0 \\ \tilde{\sigma}_y{}^0 \\ \tilde{\sigma}_z{}^0 \end{bmatrix} ds \tag{4.99}$$

$$\begin{bmatrix} m_x \\ m_y \\ m_z \\ b_\omega \end{bmatrix} \equiv \int_C \begin{bmatrix} y\tilde{\sigma}_z{}^0 - \delta_s \frac{h}{2}\frac{dx}{ds}\tilde{\sigma}_z{}^1 \\ x\tilde{\sigma}_z{}^0 + \delta_s \frac{h}{2}\frac{dy}{ds}\tilde{\sigma}_z{}^1 \\ x\tilde{\sigma}_y{}^0 - y\tilde{\sigma}_x{}^0 \\ \tilde{F}_\omega\tilde{\sigma}_z{}^0 + \tilde{a}\frac{h}{2}\tilde{\sigma}_z{}^1 \end{bmatrix} ds \tag{4.100}$$

$$\tilde{Q}_x \equiv \int_C \sum_{k=1}^{N} \int_{h(k)} [\tilde{\sigma}_x] dnds \tag{4.101}$$

$$\tilde{Q}_y \equiv \int_C \sum_{k=1}^{N} \int_{h(k)} [\tilde{\sigma}_y] dnds \tag{4.102}$$

$$\tilde{T}_A \equiv \int_C \sum_{k=1}^{N} \int_{h(k)} [\tilde{\sigma}_z] dnds \tag{4.103}$$

$$\tilde{M}_y \equiv \int_C \sum_{k=1}^{N} \int_{h(k)} [x\tilde{\sigma}_z + \delta_s \frac{dy}{ds} n\tilde{\sigma}_z] dnds \tag{4.104}$$

$$\tilde{M}_x \equiv \int_C \sum_{k=1}^{N} \int_{h(k)} [y\tilde{\sigma}_z - \delta_s \frac{dx}{ds} n\tilde{\sigma}_z] dnds \tag{4.105}$$

$$\tilde{M}_z \equiv \int_C \sum_{k=1}^{N} \int_{h(k)} [-y\tilde{\sigma}_x + x\tilde{\sigma}_y] dnds \tag{4.106}$$

$$\widetilde{B}_\omega \equiv -\int_C \sum_{k=1}^{N} \int_{h(k)} [\widetilde{F}_\omega \widetilde{\sigma}_z + n\widetilde{a}\widetilde{\sigma}_z]dnds \tag{4.107}$$

where $(\sim)$ identifies a prescribed quantity at the ends of the beam (i.e., at $z=0$ and $z=L$).

4.5 *Viscous Damping Term*

The work done by the viscous damping forces can be expressed as

$$W_d = -\int_\tau F_{di} u_i d\tau \tag{4.108}$$

where F_{di} denotes the damping force per unit volume of the beam in the x_i direction where $i=x,y,z$. The variation of the work done by the viscous damping forces is

$$\delta W_d = -\int_\tau F_{di} \delta u_i d\tau = -\int_\tau C_i \dot{u}_i \delta u_i d\tau \tag{4.109}$$

where C_i denote the damping coefficients in the x_i directions. By virtue of Eq 4.41, the above equation can be expressed as

$$\begin{aligned} \delta W_d &= = -\int_0^L (C_1 \dot{u} \delta u + C_2 \dot{v} \delta v + C_3 \dot{w} \delta w) dz \\ &= -\int_0^L [C_1 \dot{u} \delta u + C_2 \dot{v} \delta v + C_3 (\dot{w}_1 + n\dot{w}_2) \delta (w_1 + nw_2)] dz \end{aligned} \tag{4.110}$$

Introduction of the displacement components in accordance with Eq 4.110 gives

$$\delta W_d = -\int_0^L \int_C \int_{-\frac{h}{2}}^{\frac{h}{2}} [C_1\{\ddot{u}_0 - (\delta_{w1} + \delta_{w2})y\dot{\phi} - \delta_{w3}y\dot{\phi}_c\}\{\delta u_0 - (\delta_{w1} + \delta_{w2})y\delta\phi - \delta_{w3}y\delta\phi_c\}$$

$$+ C_2\{\dot{v}_0 + (\delta_{w1} + \delta_{w2})x\dot{\phi} + \delta_{w3}x\dot{\phi}_c\}\{\delta v_0 + (\delta_{w1} + \delta_{w2})x\delta\phi + \delta_{w3}x\delta\phi_c\}$$

$$+ C_3\{\dot{w}_0 + x(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') + y(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - \widetilde{F}_\omega(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')\} \tag{4.111}$$

$$\{\delta w_0 + x(\delta_t\delta\theta_y - \delta_{nt}\delta u_0') + y(\delta_t\delta\theta_x - \delta_{nt}\delta v_0') - \widetilde{F}_\omega(\delta_{w1}\delta\phi' + \delta_{w2}\delta\Theta + \delta_{w3}\delta\phi_c')$$

$$+ C_3 n^2\{\delta_s\frac{dy}{ds}(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') - \delta_s\frac{dx}{ds}(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - \widetilde{a}(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')\}$$

$$\{\delta_s\frac{dy}{ds}(\delta_t\delta\theta_y - \delta_{nt}\delta u_0') - \delta_s\frac{dx}{ds}(\delta_t\delta\theta_x - \delta_{nt}\delta v_0') - \widetilde{a}(\delta_{w1}\delta\phi' + \delta_{w2}\delta\Theta + \delta_{w3}\delta\phi_c')\}]\,dndsdz$$

Integration of the above equation across the thickness yields

$$\delta W_d = -\int_0^L [D_1\delta u_0 + D_2\delta v_0 + D_3\delta w_0 + (\delta_{w1} + \delta_{w2})D_4\delta\phi + D_5\delta_t\delta\theta_y + D_6\delta_{nt}\delta u_0'$$
$$+ D_7\delta_t\delta\theta_x + D_8\delta_{nt}\delta v_0' + D_9\delta_{w1}\delta\phi' + D_{10}\delta_{w2}\delta\Theta + \delta_{w3}(D_{11}\delta\phi_c' + D_{12}\delta\phi_c)]dz \tag{4.112}$$

Integrating by parts gives

$$\delta W_d = -[\delta_{nt}D_6\delta u_0 + \delta_{nt}D_8\delta v_0 + \delta_{w1}D_9\delta\phi + \delta_{w3}D_{11}\delta\phi_c]_{z=0}^{z=L}$$
$$-\int_0^L [(D_1 - \delta_{nt}D_6')\delta u_0 + (D_2 - \delta_{nt}D_8')\delta v_0 + D_3\delta w_0 + \delta_t D_5\delta\theta_y + \delta_t D_7\delta\theta_x$$
$$+ \delta_{w2}D_{10}\delta\Theta + (\delta_{w1}D_4 + \delta_{w2}D_4 - \delta_{w1}D_9')\delta\phi + \delta_{w3}(D_{12} - D_{11}')\delta\phi_c]\,dz \tag{4.113}$$

Here the D coefficients are given by

$$D_1 \equiv \int_C C_1\{h\ddot{u}_0 - (\delta_{w1} + \delta_{w2})yh\dot{\phi} - \delta_{w3}yh\dot{\phi}_c\}\,ds \tag{4.114}$$

$$D_2 \equiv \int_C C_2\{h\dot{v}_0 + (\delta_{w1} + \delta_{w2})xh\dot{\phi} + \delta_{w3}xh\dot{\phi}_c\}\,ds \tag{4.115}$$

$$D_3 \equiv \int_C C_3[h\dot{w}_0 + xh(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') + yh(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - h\widetilde{F}_\omega(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')]\, ds \tag{4.116}$$

$$D_4 \equiv \int_C [(C_1 y^2 + C_2 x^2)h\dot{\phi} - C_1 yh\dot{u}_0 + C_2 xh\dot{v}_0]\, ds \tag{4.117}$$

$$\begin{aligned} D_5 \equiv \int_C [&C_3\{xh\dot{w}_0 + x^2h\dot{\theta}_y + xyh\dot{\theta}_x - xh\widetilde{F}_\omega(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')\} \\ &+ \frac{h^3}{12} C_3\{\delta_s \frac{dy}{ds}\frac{dy}{ds}\dot{\theta}_y - \delta_s \frac{dx}{ds}\frac{dy}{ds}\dot{\theta}_x - \widetilde{a}\frac{dy}{ds}(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')\}]\, ds \end{aligned} \tag{4.118}$$

$$\begin{aligned} D_6 \equiv \int_C [&C_3\{-xh\dot{w}_0 + x^2h\dot{u}_0' + xyh\dot{v}_0' + xh\widetilde{F}_\omega(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')\} \\ &+ \frac{h^3}{12} C_3\{\delta_s \frac{dy}{ds}\frac{dy}{ds}\dot{u}_0' - \delta_s \frac{dx}{ds}\frac{dy}{ds}\dot{v}_0' + \widetilde{a}\frac{dy}{ds}(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')\}]\, ds \end{aligned} \tag{4.119}$$

$$\begin{aligned} D_7 \equiv \int_C [&C_3\{yh\dot{w}_0 + xyh\dot{\theta}_y + y^2h\dot{\theta}_x - yh\widetilde{F}_\omega(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')\} \\ &+ \frac{h^3}{12} C_3\{\delta_s \frac{dx}{ds}\frac{dy}{ds}\dot{\theta}_y - \delta_s \frac{dx}{ds}\frac{dx}{ds}\dot{\theta}_x - \widetilde{a}\frac{dx}{ds}(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')\}]\, ds \end{aligned} \tag{4.120}$$

$$\begin{aligned} D_8 \equiv \int_C [&C_3\{-yh\dot{w}_0 + xyh\dot{u}_0' + y^2h\dot{v}_0' + yh\widetilde{F}_\omega(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')\} \\ &+ \frac{h^3}{12} C_3\{\delta_s \frac{dx}{ds}\frac{dx}{ds}\dot{v}_0' - \delta_s \frac{dx}{ds}\frac{dy}{ds}\dot{u}_0' - \widetilde{a}\frac{dx}{ds}(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')\}]\, ds \end{aligned} \tag{4.121}$$

$$\begin{aligned} D_9 \equiv \int_C [&C_3\widetilde{F}_\omega\{-yh(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - xh(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') - h\dot{w}_0 + \widetilde{F}_\omega h\dot{\phi}'\} \\ &+ \frac{h^3}{12} C_3\widetilde{a}\{\delta_s \frac{dx}{ds}(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - \delta_s \frac{dy}{ds}(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') + \widetilde{a}\dot{\phi}'\}]\, ds \end{aligned} \tag{4.122}$$

$$D_{10} \equiv \int_C [C_3\widetilde{F}_\omega\{-yh(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - xh(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') - h\dot{w}_0 + h\widetilde{F}_\omega\dot{\Theta}\}$$
$$+ \frac{h^3}{12} C_3\widetilde{a}\{\delta_s \frac{dx}{ds}(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - \delta_s \frac{dy}{ds}(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') + \widetilde{a}\dot{\Theta}\}]\, ds \quad (4.123)$$

$$D_{11} \equiv \int_C [C_3\widetilde{F}_\omega\{-yh(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - xh(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') - h\dot{w}_0 + h\widetilde{F}_\omega\dot{\phi}_c'\}$$
$$+ \frac{h^3}{12} C_3\widetilde{a}\{\delta_s \frac{dx}{ds}(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - \delta_s \frac{dy}{ds}(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') + \widetilde{a}\dot{\phi}_c'\}]\, ds \quad (4.124)$$

$$D_{12} \equiv \int_C [C_1(y^2h\dot{\phi}_c - yh\dot{u}_0) + C_2(x^2h\dot{\phi}_c + xh\dot{v}_0)]ds \quad (4.125)$$

A more compact form of the above expressions is

$$D_1 \equiv d_1\dot{u}_0 - (\delta_{w1} + \delta_{w2})d_2\dot{\phi} - \delta_{w3}d_2\dot{\phi}_c \quad (4.126)$$

$$D_2 \equiv d_1\dot{v}_0 + (\delta_{w1} + \delta_{w2})d_3\dot{\phi} + \delta_{w3}d_3\dot{\phi}_c \quad (4.127)$$

$$D_3 \equiv d_{20}\dot{w}_0 + d_{22}(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') + d_{21}(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - d_7(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c') \quad (4.128)$$

$$D_4 \equiv d_3\dot{v}_0 - d_2\dot{u}_0 + (d_4 + d_5)\dot{\phi} \quad (4.129)$$

$$D_5 \equiv d_{22}\dot{w}_0 + (d_{24} + \delta_s d_{15})\dot{\theta}_y + (d_6 - \delta_s d_{13})\dot{\theta}_x$$
$$- (d_9 + \delta_s d_{17})(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c') \quad (4.130)$$

$$D_6 \equiv -d_{22}\dot{w}_0 + (d_{24} + \delta_s d_{15})\dot{u}_0' + (d_6 - \delta_s d_{13})\dot{v}_0'$$
$$+ (d_9 + \delta_s d_{17})(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c') \quad (4.131)$$

$$D_7 \equiv d_{21}\dot{w}_0 + (d_{23} + \delta_s d_{14})\dot{\theta}_x + (d_6 - \delta_s d_{13})\dot{\theta}_y$$
$$- (d_8 - \delta_s d_{16})(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c') \quad (4.132)$$

$$D_8 \equiv -d_{21}\dot{w}_0 + (d_6 - \delta_s d_{13})\dot{u}_0' + (d_{23} + \delta_s d_{14})\dot{v}_0' + (d_8 - \delta_s d_{16})(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c') \quad (4.133)$$

$$D_9 = -d_7\dot{w}_0 - (d_9 + \delta_s d_{17})(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') - (d_8 - \delta_s d_{16})(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') + (d_{10} + d_{18})\dot{\phi}' \quad (4.134)$$

$$D_{10} = -d_7\dot{w}_0 - (d_9 + \delta_s d_{17})(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') - (d_8 - \delta_s d_{16})(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') + (d_{10} + d_{18})\dot{\Theta} \quad (4.135)$$

$$D_{11} = -d_7\dot{w}_0 - (d_9 + \delta_s d_{17})(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') - (d_8 - \delta_s d_{16})(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') + (d_{10} + d_{18})\dot{\phi}_c' \quad (4.136)$$

$$D_{12} = d_3\dot{v}_0 - d_2\dot{u}_0 + (d_4 + d_5)\dot{\phi}_c \quad (4.137)$$

where the coefficients d_i are defined in Appendix E.

4.6 The Governing Equations and the Associated Boundary Conditions

The variations of the strain energy, of the kinetic energy, and of the work done by external loadings and of damping forces have been derived in sections 4.2 through 4.5. Substitution of the corresponding expressions into Eq 4.4 and collection of the terms which correspond to the same variations of kinematic variables yields the governing equations and the associated boundary conditions. These variations of the kinematic variables are independent and arbitrary within any instant of time $t \in [t_0, t_1]$. As a result, the coefficients of these variations should vanish in order to satisfy Hamilton's principle. This yields the equations of motion and the associated boundary conditions.

4.6.1 The Governing Equations

The equations of motion are:

δu_0:

$$\delta_{nt}M_y'' - \delta_{nt}Q'_x + \hat{Q}_x' - (K_1 - \delta_{nt}K_6') - (D_1 - \delta_{nt}D_6') + F_1 + p_x + \delta_{nt}m_y' = 0 \tag{4.138}$$

δv_0:

$$\delta_{nt}M_x'' - \delta_{nt}Q'_y + \hat{Q}_y' - (K_2 - \delta_{nt}K_8') - (D_2 - \delta_{nt}D_8') + F_2 + p_y + \delta_{nt}m_x' = 0 \tag{4.139}$$

δw_0:

$$T_A' - K_3 - D_3 + F_3 + p_z = 0 \tag{4.140}$$

$\delta \phi$:

$$\begin{aligned} &\delta_{w1}(B_\omega'' + M_p') + \delta_{w2}M_t' - (\delta_{w1}K_4 - \delta_{w1}K_9' + \delta_{w2}K_4) \\ &- (\delta_{w1}D_4 - \delta_{w1}D_9' + \delta_{w2}D_4) + (\delta_{w1} + \delta_{w2})F_6 + (\delta_{w1} + \delta_{w2})m_z + \delta_{w1}b_\omega' = 0 \end{aligned} \tag{4.141}$$

$\delta_t\delta\theta_y$:

$$M'_y - Q_x - K_5 - D_5 + F_4 + m_y = 0 \tag{4.142}$$

$\delta_t\delta\theta_x$:

$$M'_x - Q_y - K_7 - D_7 + F_5 + m_x = 0 \tag{4.143}$$

$\delta_{w2}\delta\Theta$:

$$-B_{\omega}' - M_p + M_t - K_{10} - D_{10} - F_7 - b_{\omega} = 0 \tag{4.144}$$

$\delta_{w3}\delta\phi_c$:

$$M_p' - (K_{12} - K_{11}') - (D_{12} - D_{11}') + F_8 + m_z + b_{\omega}' = 0 \tag{4.145}$$

Substitution of the generalized forces and couples as well as of the dissipative forces in terms of the displacement quantities results in the governing equations. These are given by:

δu_0:

$$\begin{aligned}
&\delta_{nt}[a_{21}w_0''' + a_{22}(\delta_t\theta_y''' - \delta_{nt}u_0'''') + a_{23}(\delta_t\theta_x''' - \delta_{nt}v_0'''') + a_{24}\delta_t(\theta_y'' + u_0''') \\
&+ a_{25}\delta_t(\theta_x'' + v_0''') + a_{26}(\delta_{w1}\phi'''' + \delta_{w2}\Theta''') + a_{27}\phi''' + a_{28}\delta_{w2}\Theta'' - h_2''] \\
&- \delta_{nt}[a_{41}w_0'' + a_{42}(\delta_t\theta_y'' - \delta_{nt}u_0''') + a_{43}(\delta_t\theta_x'' - \delta_{nt}v_0''') + a_{44}\delta_t(\theta_y' + u_0'') \\
&+ a_{45}\delta_t(\theta_x' + v_0'') + a_{46}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') + a_{47}\phi'' + a_{48}\delta_{w2}\Theta' - h_4'] \\
&+ [s_{41}w_0'' + s_{42}(\delta_t\theta_y'' - \delta_{nt}u_0''') + s_{43}(\delta_t\theta_x'' - \delta_{nt}v_0''') + s_{44}\delta_t(\theta_y' + u_0'') \\
&+ s_{45}\delta_t(\theta_x' + v_0'') + s_{46}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') + s_{47}\phi'' + s_{48}\delta_{w2}\Theta' - h_4' \\
&+ (1 - \delta_s)(g_5u_0'' - g_6v_0'')] - [b_1\ddot{u}_0 - (\delta_{w1} + \delta_{w2})b_2\ddot{\phi} - \delta_{w3}b_2\ddot{\phi}_c] \\
&+ \delta_{nt}[-b_3\ddot{w}_0' + (b_5 + \delta_s b_{15})\ddot{u}_0'' + (b_6 - \delta_s b_{13})\ddot{v}_0'' + (b_9 + \delta_s b_{17})(\delta_{w1}\ddot{\phi}'' + \delta_{w2}\ddot{\Theta}')] \\
&- [d_1\dot{u}_0 - (\delta_{w1} + \delta_{w2})d_2\dot{\phi} - \delta_{w3}d_2\dot{\phi}_c] \\
&+ \delta_{nt}[-d_{22}\dot{w}_0' + (d_{24} + \delta_s d_{15})\dot{u}_0'' + (d_6 - \delta_s d_{13})\dot{v}_0'' \\
&+ (d_9 + \delta_s d_{17})(\delta_{w1}\dot{\phi}'' + \delta_{w2}\dot{\Theta}')] + F_1 + p_x + \delta_{nt}m_y' = 0
\end{aligned} \tag{4.146}$$

δv_0:

$$\begin{aligned}
&\delta_{nt}[a_{31}w_0''' + a_{32}(\delta_t\theta_y''' - \delta_{nt}u_0'''') + a_{33}(\delta_t\theta_x''' - \delta_{nt}v_0'''') + a_{34}\delta_t(\theta_y'' + u_0''') \\
&+ a_{35}\delta_t(\theta_x'' + v_0''') + a_{36}(\delta_{w1}\phi'''' + \delta_{w2}\Theta''') + a_{37}\phi''' + a_{38}\delta_{w2}\Theta'' - h_3''] \\
&- \delta_{nt}[a_{51}w_0'' + a_{52}(\delta_t\theta_y'' - \delta_{nt}u_0''') + a_{53}(\delta_t\theta_x'' - \delta_{nt}v_0''') + a_{54}\delta_t(\theta_y' + u_0'') \\
&+ a_{55}\delta_t(\theta_x' + v_0'') + a_{56}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') + a_{57}\phi'' + a_{58}\delta_{w2}\Theta' - h_5'] \\
&+[s_{51}w_0'' + s_{52}(\delta_t\theta_y'' - \delta_{nt}u_0''') + s_{53}(\delta_t\theta_x'' - \delta_{nt}v_0''') + s_{54}\delta_t(\theta_y' + u_0'') \\
&+ s_{55}\delta_t(\theta_x' + v_0'') + s_{56}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') + s_{57}\phi'' + s_{58}\delta_{w2}\Theta' - h_5' \\
&+ (1 - \delta_s)(g_7u_0'' - g_8v_0'')] - [b_1\ddot{v}_0 + (\delta_{w1} + \delta_{w2})b_3\ddot{\phi} + \delta_{w3}b_3\ddot{\phi}_c] \\
&+ \delta_{nt}[-b_2\ddot{w}_0' + (b_6 - \delta_sb_{13})\ddot{u}_0'' + (b_4 + \delta_sb_{14})\ddot{v}_0'' + (b_8 - \delta_sb_{16})(\delta_{w1}\ddot{\phi}'' + \delta_{w2}\ddot{\Theta}')] \\
&-[d_{19}\dot{v}_0 + (\delta_{w1} + \delta_{w2})d_3\dot{\phi} + \delta_{w3}d_3\dot{\phi}_c] \\
&+ \delta_{nt}[-d_{21}\dot{w}_0' + (d_6 - \delta_sd_{13})\dot{u}_0'' + (d_{23} + \delta_sd_{14})\dot{v}_0'' \\
&+ (d_8 - \delta_sd_{16})(\delta_{w1}\dot{\phi}'' + \delta_{w2}\dot{\Theta}')] + F_2 + p_y + \delta_{nt}m_x' = 0
\end{aligned} \tag{4.147}$$

δw_0:

$$\begin{aligned}
&a_{11}w_0'' + a_{12}(\delta_t\theta_y'' - \delta_{nt}u_0''') + a_{13}(\delta_t\theta_x'' - \delta_{nt}v_0''') + a_{14}\delta_t(\theta_y' + u_0'') \\
&+ a_{15}\delta_t(\theta_x' + v_0'') + a_{16}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') + a_{17}\phi'' + a_{18}\delta_{w2}\Theta' - h_1' \\
&-[b_1\ddot{w}_0 + b_3(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') + b_2(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') - b_7(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c')] \\
&-[d_{20}\dot{w}_0 + d_{21}(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') + d_{22}(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') \\
&- d_7(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')] + F_3 + p_z = 0
\end{aligned} \tag{4.148}$$

$\delta_t\delta\theta_y$:

$$\begin{aligned}
&a_{21}w_0'' + a_{22}(\delta_t\theta_y'' - \delta_{nt}u_0''') + a_{23}(\delta_t\theta_x'' - \delta_{nt}v_0''') + a_{24}\delta_t(\theta_y' + u_0'') \\
&+ a_{25}\delta_t(\theta_x' + v_0'') + a_{26}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') + a_{27}\phi'' + a_{28}\delta_{w2}\Theta' - h_2' \\
&-[a_{41}w_0' + a_{42}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{43}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ a_{44}\delta_t(\theta_y + u_0') + a_{45}\delta_t(\theta_x + v_0') + a_{46}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ a_{47}\phi' + a_{48}\delta_{w2}\Theta + a_{49}\delta_{w3}\phi_c' - h_4] \\
&-[b_3\ddot{w}_0 + (b_5 + \delta_sb_{15})\delta_t\ddot{\theta}_y + (b_6 - \delta_sb_{13})\delta_t\ddot{\theta}_x \\
&- (b_9 + \delta_sb_{17})(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c')] \\
&-[d_{22}\dot{w}_0 + (d_{24} + \delta_sd_{15})\delta_t\dot{\theta}_y + (d_6 - \delta_sd_{13})\delta_t\dot{\theta}_x \\
&- (d_9 + \delta_sd_{17})(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')] + F_4 + m_y = 0
\end{aligned} \tag{4.149}$$

$\delta_t\delta\theta_x$:

$$
\begin{aligned}
&a_{31}w_0'' + a_{32}(\delta_t\theta_y'' - \delta_{nt}u_0''') + a_{33}(\delta_t\theta_x'' - \delta_{nt}v_0''') + a_{34}\delta_t(\theta_y' + u_0'') \\
&+ a_{35}\delta_t(\theta_x' + v_0'') + a_{36}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') + a_{37}\phi'' + a_{38}\delta_{w2}\Theta' - h_3' \\
&-[a_{51}w_0' + a_{52}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{53}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ a_{54}\delta_t(\theta_y + u_0') + a_{55}\delta_t(\theta_x + v_0') + a_{56}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ a_{57}\phi' + a_{58}\delta_{w2}\Theta + a_{59}\delta_{w3}\phi_c' - h_5] \\
&-[b_2\ddot{w}_0 + (b_6 - \delta_s b_{13})\delta_t\ddot{\theta}_y + (b_4 + \delta_s b_{14})\delta_t\ddot{\theta}_x \\
&- (b_8 - \delta_s b_{16})(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c')] \\
&-[d_{21}\dot{w}_0 + (d_6 - \delta_s d_{13})\delta_t\dot{\theta}_y + (d_{23} + \delta_s d_{14})\delta_t\dot{\theta}_x \\
&- (d_8 - \delta_s d_{16})(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')] + F_5 + m_x = 0
\end{aligned}
\tag{4.150}
$$

$\delta\phi$:

$$
\begin{aligned}
&\delta_{w1}[a_{61}w_0''' + a_{62}(\delta_t\theta_y''' - \delta_{nt}u_0'''') + a_{63}(\delta_t\theta_x''' - \delta_{nt}v_0'''') \\
&+ a_{64}\delta_t(\theta_y'' + u_0''') + a_{65}\delta_t(\theta_x'' + v_0''') + a_{66}(\delta_{w1}\phi'''' + \delta_{w2}\Theta''') \\
&+ a_{67}\phi''' + a_{68}\delta_{w2}\Theta'' - h_6'' \\
&+ a_{71}w_0'' + a_{72}(\delta_t\theta_y'' - \delta_{nt}u_0''') + a_{73}(\delta_t\theta_x'' - \delta_{nt}v_0''') \\
&+ a_{74}\delta_t(\theta_y' + u_0'') + a_{75}\delta_t(\theta_x' + v_0'') + a_{76}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') \\
&+ a_{77}\phi'' + a_{78}\delta_{w2}\Theta' + (1 - \delta_s)(g_1u_0'' - g_2v_0'') - h_7'] \\
&+ \delta_{w2}[a_{81}w_0'' + a_{82}(\delta_t\theta_y'' - \delta_{nt}u_0''') + a_{83}(\delta_t\theta_x'' - \delta_{nt}v_0''') \\
&+ a_{84}\delta_t(\theta_y' + u_0'') + a_{85}\delta_t(\theta_x' + v_0'') + a_{86}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') \\
&+ a_{87}\phi'' + a_{88}\delta_{w2}\Theta' + (1 - \delta_s)(g_3u_0'' - g_4v_0'') - h_8'] \\
&- (\delta_{w1} + \delta_{w2})[b_3\ddot{v}_0 - b_2\ddot{u}_0 + (b_4 + b_5)\ddot{\phi}] \\
&+ \delta_{w1}[- b_7\ddot{w}_0' - (b_9 + \delta_s b_{17})(\delta_t\ddot{\theta}_y' - \delta_{nt}\ddot{u}_0'') - (b_8 - \delta_s b_{16})(\delta_t\ddot{\theta}_x' - \delta_{nt}\ddot{v}_0'') \\
&+ (b_{10} + b_{18})\ddot{\phi}''] - (\delta_{w1} + \delta_{w2})[(d_4 + d_5)\dot{\phi} - d_2\dot{u}_0 + d_3\dot{v}_0] \\
&+ \delta_{w1}[- d_7\dot{w}_0' - (d_8 - \delta_s d_{16})(\delta_t\dot{\theta}_x' - \delta_{nt}\dot{v}_0'') - (d_9 + \delta_s d_{17})(\delta_t\dot{\theta}_y' - \delta_{nt}\dot{u}_0'') \\
&+ (d_{10} + d_{18})\dot{\phi}''] + (\delta_{w1} + \delta_{w2})(F_6 + m_z) + \delta_{w1}b_\omega' = 0
\end{aligned}
\tag{4.151}
$$

$\delta_{w2}\delta\Theta$:

$$
\begin{aligned}
&-[a_{61}w_0'' + a_{62}(\delta_t\theta_y'' - \delta_{nt}u_0''') + a_{63}(\delta_t\theta_x'' - \delta_{nt}v_0''') + a_{64}\delta_t(\theta_y' + u_0'') \\
&+ a_{65}\delta_t(\theta_x' + v_0'') + a_{66}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') + a_{67}\phi'' + a_{68}\delta_{w2}\Theta' - h_6'] \\
&-[a_{71}w_0' + a_{72}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{73}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ a_{74}\delta_t(\theta_y + u_0') + a_{75}\delta_t(\theta_x + v_0') + a_{76}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ a_{77}\phi' + a_{78}\delta_{w2}\Theta + a_{79}\delta_{w3}\phi_c' + (1-\delta_s)(g_1u_0' - g_2v_0') - h_7] \\
&+[a_{81}w_0' + a_{82}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{83}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ a_{84}\delta_t(\theta_y + u_0') + a_{85}\delta_t(\theta_x + v_0') + a_{86}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ a_{87}\phi' + a_{88}\delta_{w2}\Theta + a_{89}\delta_{w3}\phi_c' + (1-\delta_s)(g_3u_0' - g_4v_0') - h_8] \\
&-[-b_7\ddot{w}_0 - (b_9 + \delta_s b_{17})(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') - (b_8 - \delta_s b_{16})(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') + (b_{10} + b_{18})\ddot{\Theta}] \\
&-[-d_7\dot{w}_0 - (d_8 - \delta_s d_{16})(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - (d_9 + \delta_s d_{17})(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') \\
&+ (d_{10} + d_{18})\dot{\Theta}] - F_7 - b_\omega = 0
\end{aligned} \tag{2.152}
$$

$\delta_{w3}\delta\phi_c$:

$$
\begin{aligned}
&a_{71}w_0'' + a_{72}(\delta_t\theta_y'' - \delta_{nt}u_0''') + a_{73}(\delta_t\theta_x'' - \delta_{nt}v_0''') \\
&+ a_{74}\delta_t(\theta_y' + u_0'') + a_{75}\delta_t(\theta_x' + v_0'') + a_{76}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') \\
&+ a_{77}\phi'' + a_{78}\delta_{w2}\Theta' + (1-\delta_s)(g_1u_0'' - g_2v_0'') - h_7' \\
&-[b_3\ddot{v}_0 - b_2\ddot{u}_0 + (b_4 + b_5)\ddot{\phi}_c] \\
&+[-b_7\ddot{w}_0' - (b_9 + \delta_s b_{17})(\delta_t\ddot{\theta}_y' - \delta_{nt}\ddot{u}_0'') - (b_8 - \delta_s b_{16})(\delta_t\ddot{\theta}_x' - \delta_{nt}\ddot{v}_0'') \\
&+ (b_{10} + b_{18})\ddot{\phi}_c''] \\
&+[-d_7\dot{w}_0' - (d_8 - \delta_s d_{16})(\delta_t\dot{\theta}_x' - \delta_{nt}\dot{v}_0'') - (d_9 + \delta_s d_{17})(\delta_t\dot{\theta}_y' - \delta_{nt}\dot{u}_0'') \\
&+ (d_{10} + d_{18})\dot{\phi}_c''] - [(d_4 + d_5)\dot{\phi}_c - d_2\dot{u}_0 + d_3\dot{v}_0] + F_8 + m_z + b_\omega' = 0
\end{aligned} \tag{4.153}
$$

4.6.2 The Boundary Conditions

The boundary conditions at the ends (z = 0, L) of the beam are expressed as

$$
[\delta_{nt}(M'_y - Q_x) + \hat{Q}_x + \delta_{nt}(K_6 + D_6) - \tilde{Q}_x]\delta u_0 = 0 \tag{4.154}
$$

$$
[\delta_{nt}(M'_x - Q_y) + \hat{Q}_y + \delta_{nt}(K_8 + D_8) - \tilde{Q}_y]\delta v_0 = 0 \tag{4.155}
$$

$$[T_A - \widetilde{T}_A]\delta w_0 = 0 \tag{4.156}$$

$$[M_y - \widetilde{M}_y](\delta_t \delta\theta_y - \delta_{nt}\delta u_0') = 0 \tag{4.157}$$

$$[M_x - \widetilde{M}_x](\delta_t \delta\theta_x - \delta_{nt}\delta v_0') = 0 \tag{4.158}$$

$$[B_\omega - \widetilde{B}_\omega](\delta_{w1}\delta\phi' + \delta_{w2}\delta\Theta) = 0 \tag{4.159}$$

$$[\delta_{w1}(B_\omega' + M_p + K_9 + D_9 - \widetilde{M}_z) + \delta_{w2}(M_t - \widetilde{M}_z)]\delta\phi = 0 \tag{4.160}$$

$$[M_p + K_{11} + D_{11} - \widetilde{M}_z]\delta_{w3}\delta\phi_c = 0 \tag{4.161}$$

From Eqs 4.154 through 4.161 the static boundary conditions are:

$$\delta_{nt}(M'_y - Q_x) + \hat{Q}_x + \delta_{nt}(K_6 + D_6) - \widetilde{Q}_x = 0 \tag{4.162}$$

$$\delta_{nt}(M'_x - Q_y) + \hat{Q}_y + \delta_{nt}(K_8 + D_8) - \widetilde{Q}_y = 0 \tag{4.163}$$

$$T_A - \widetilde{T}_A = 0 \tag{4.164}$$

$$M_y - \widetilde{M}_y = 0 \tag{4.165}$$

$$M_x - \widetilde{M}_x = 0 \tag{4.166}$$

$$(\delta_{w1} + \delta_{w2})(B_\omega - \widetilde{B}_\omega) = 0 \tag{4.167}$$

$$\delta_{w1}(B_\omega' + M_p + K_9 + D_9) + \delta_{w2}M_t - (\delta_{w1} + \delta_{w2})\widetilde{M}_z = 0 \tag{4.168}$$

$$\delta_{w3}(M_p + K_{11} + D_{11}) - \delta_{w3}\widetilde{M}_z = 0 \tag{4.169}$$

Next, the geometric boundary conditions can be expressed as:

$$u_0 = \tilde{u}_0 \tag{4.170}$$

$$v_0 = \tilde{v}_0 \tag{4.171}$$

$$w_0 = \tilde{w}_0 \tag{4.172}$$

$$\delta_t \theta_y - \delta_{nt} u_0{}' = \delta_t \tilde{\theta}_y - \delta_{nt} \tilde{u}_0{}' \tag{4.173}$$

$$\delta_t \theta_x - \delta_{nt} v_0{}' = \delta_t \tilde{\theta}_x - \delta_{nt} \tilde{v}_0{}'' \tag{4.174}$$

$$\delta_{w1} \phi' + \delta_{w2} \Theta = \delta_{w1} \tilde{\phi}' + \delta_{w2} \tilde{\Theta} \tag{4.175}$$

$$\phi' = \tilde{\phi}' \tag{4.176}$$

$$\phi_c = \tilde{\phi}_c \tag{4.177}$$

The static boundary conditions can be expressed in terms of the displacement quantities as:

δu_0:

$$\begin{aligned}
&\delta_{nt}[a_{21}w_0'' + a_{22}(\delta_t\theta_y'' - \delta_{nt}u_0''') + a_{23}(\delta_t\theta_x'' - \delta_{nt}v_0''') \\
&+ a_{24}\delta_t(\theta_y' + u_0'') + a_{25}\delta_t(\theta_x' + v_0'') + a_{26}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') \\
&+ a_{27}\phi'' + a_{28}\delta_{w2}\Theta' - h_2'] \\
&- \delta_{nt}[a_{41}w_0' + a_{42}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{43}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ a_{44}\delta_t(\theta_y + u_0') + a_{45}\delta_t(\theta_x + v_0') + a_{46}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ a_{47}\phi' + a_{48}\delta_{w2}\Theta + a_{49}\delta_{w3}\phi_c' - h_4] \\
&+ s_{41}w_0' + s_{42}(\delta_t\theta_y' - \delta_{nt}u_0'') + s_{43}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ s_{44}\delta_t(\theta_y + u_0') + s_{45}\delta_t(\theta_x + v_0') + s_{46}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ s_{47}\phi' + s_{48}\delta_{w2}\Theta + s_{49}\delta_{w3}\phi_c' - h_4 + (1 - \delta_s)(g_5u_0' - g_6v_0') \\
&+ \delta_{nt}[-b_3\ddot{w}_0 + (b_5 + \delta_s b_{15})\ddot{u}_0' + (b_6 - \delta_s b_{13})\ddot{v}_0' \\
&+ (b_9 + \delta_s b_{17})(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c')] \\
&+ \delta_{nt}[-d_{22}\dot{w}_0 + (d_{24} + \delta_s d_{15})\dot{u}_0' + (d_6 - \delta_s d_{13})\dot{v}_0' \\
&+ (d_9 + \delta_s d_{17})(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')] - \widetilde{Q}_x = 0
\end{aligned} \qquad (4.178)$$

δv_0:

$$\begin{aligned}
&\delta_{nt}[a_{31}w_0'' + a_{32}(\delta_t\theta_y'' - \delta_{nt}u_0''') + a_{33}(\delta_t\theta_x'' - \delta_{nt}v_0''') \\
&+ a_{34}\delta_t(\theta_y' + u_0'') + a_{35}\delta_t(\theta_x' + v_0'') + a_{36}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') \\
&+ a_{37}\phi'' + a_{38}\delta_{w2}\Theta' - h_3'] \\
&- \delta_{nt}[a_{51}w_0' + a_{32}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{53}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ a_{54}\delta_t(\theta_y + u_0') + a_{55}\delta_t(\theta_x + v_0') + a_{56}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ a_{57}\phi' + a_{58}\delta_{w2}\Theta + a_{59}\delta_{w3}\phi_c' - h_5] \\
&+ s_{51}w_0' + s_{52}(\delta_t\theta_y' - \delta_{nt}u_0'') + s_{53}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ s_{54}\delta_t(\theta_y + u_0') + s_{55}\delta_t(\theta_x + v_0') + s_{56}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ s_{57}\phi' + s_{58}\delta_{w2}\Theta + s_{59}\delta_{w3}\phi_c' - h_5 + (1 - \delta_s)(g_7u_0' - g_8v_0') \\
&+ \delta_{nt}[-b_2\ddot{w}_0 + (b_6 - \delta_s b_{13})\ddot{u}_0' + (b_4 + \delta_s b_{14})\ddot{v}_0' \\
&+ (b_8 - \delta_s b_{16})(\delta_{w1}\ddot{\phi}' + \delta_{w2}\ddot{\Theta} + \delta_{w3}\ddot{\phi}_c')] \\
&+ \delta_{nt}[-d_{21}\dot{w}_0 + (d_6 - \delta_s d_{13})\dot{u}_0' + (d_{23} + \delta_s d_{14})\dot{v}_0' \\
&+ (d_8 - \delta_s d_{16})(\delta_{w1}\dot{\phi}' + \delta_{w2}\dot{\Theta} + \delta_{w3}\dot{\phi}_c')] - \widetilde{Q}_y = 0
\end{aligned} \qquad (4.179)$$

δw_0:

$$\begin{aligned} &a_{11}w_0' + a_{12}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{13}(\delta_t\theta_x' - \delta_{nt}v_0'') \\ &+ a_{14}\delta_t(\theta_y + u_0') + a_{15}\delta_t(\theta_x + v_0') + a_{16}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\ &+ a_{17}\phi' + a_{18}\delta_{w2}\Theta + a_{19}\delta_{w3}\phi_c' - h_1 - \widetilde{T}_A = 0 \end{aligned} \tag{4.180}$$

$\delta_t\delta\theta_y$ or $-\delta_{nt}\delta u_0'$:

$$\begin{aligned} &a_{21}w_0' + a_{22}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{23}(\delta_t\theta_x' - \delta_{nt}v_0'') \\ &+ a_{24}\delta_t(\theta_y + u_0') + a_{25}\delta_t(\theta_x + v_0') + a_{26}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\ &+ a_{27}\phi' + a_{28}\delta_{w2}\Theta + a_{29}\delta_{w3}\phi_c' - h_2 - \widetilde{M}_y = 0 \end{aligned} \tag{4.181}$$

$\delta_t\delta\theta_x$ or $-\delta_{nt}\delta v_0'$:

$$\begin{aligned} &a_{31}w_0' + a_{32}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{33}(\delta_t\theta_x' - \delta_{nt}v_0'') \\ &+ a_{34}\delta_t(\theta_y + u_0') + a_{35}\delta_t(\theta_x + v_0') + a_{36}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\ &+ a_{37}\phi' + a_{38}\delta_{w2}\Theta + a_{39}\delta_{w3}\phi_c' - h_3 - \widetilde{M}_x = 0 \end{aligned} \tag{4.182}$$

$\delta_{w1}\delta\phi'$ or $\delta_{w2}\delta\Theta$:

$$\begin{aligned} &a_{61}w_0' + a_{62}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{63}(\delta_t\theta_x' - \delta_{nt}v_0'') \\ &+ a_{64}\delta_t(\theta_y + u_0') + a_{65}\delta_t(\theta_x + v_0') + a_{66}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\ &+ a_{67}\phi' + a_{68}\delta_{w2}\Theta - h_6 - \widetilde{B}_\omega = 0 \end{aligned} \tag{4.183}$$

$\delta\phi$:

$$
\begin{aligned}
&\delta_{w1}[a_{61}w_0'' + a_{62}(\delta_t\theta_y'' - \delta_{nt}u_0''') + a_{63}(\delta_t\theta_x'' - \delta_{nt}v_0''') + a_{64}\delta_t(\theta_y' + u_0'') \\
&+ a_{65}\delta_t(\theta_x' + v_0'') + a_{66}(\delta_{w1}\phi''' + \delta_{w2}\Theta'') + a_{67}\phi'' + a_{68}\delta_{w2}\Theta' - h_6' \\
&+ a_{71}w_0' + a_{72}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{73}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ a_{74}\delta_t(\theta_y + u_0') + a_{75}\delta_t(\theta_x + v_0') + a_{76}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ a_{77}\phi' + a_{78}\delta_{w2}\Theta + a_{79}\delta_{w3}\phi_c' + (1 - \delta_s)(g_1u_0' - g_2v_0') - h_7] \\
&+ \delta_{w2}[a_{81}w_0' + a_{82}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{83}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ a_{84}\delta_t(\theta_y + u_0') + a_{85}\delta_t(\theta_x + v_0') + a_{86}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ a_{87}\phi' + a_{88}\delta_{w2}\Theta + a_{89}\delta_{w3}\phi_c' + (1 - \delta_s)(g_3u_0' - g_4v_0') - h_8] \\
&+ \delta_{w1}[-b_7\ddot{w}_0 - (b_9 + \delta_sb_{17})(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') - (b_8 - \delta_sb_{16})(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') \\
&+ (b_{10} + b_{18})\ddot{\phi}'] \\
&+ \delta_{w1}[-d_7\dot{w}_0 - (d_8 - \delta_sd_{16})(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - (d_9 + \delta_sd_{17})(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') \\
&+ (d_{10} + d_{18})\dot{\phi}'] - (\delta_{w1} + \delta_{w2})\widetilde{M}_z = 0
\end{aligned}
\tag{4.184}
$$

$\delta_{w3}\delta\phi_c$:

$$
\begin{aligned}
&a_{71}w_0' + a_{72}(\delta_t\theta_y' - \delta_{nt}u_0'') + a_{73}(\delta_t\theta_x' - \delta_{nt}v_0'') \\
&+ a_{74}\delta_t(\theta_y + u_0') + a_{75}\delta_t(\theta_x + v_0') + a_{76}(\delta_{w1}\phi'' + \delta_{w2}\Theta') \\
&+ a_{77}\phi' + a_{78}\delta_{w2}\Theta + a_{79}\delta_{w3}\phi_c' + (1 - \delta_s)(g_1u_0' - g_2v_0') - h_7 \\
&- b_7\ddot{w}_0 - (b_9 + \delta_sb_{17})(\delta_t\ddot{\theta}_y - \delta_{nt}\ddot{u}_0') - (b_8 - \delta_sb_{16})(\delta_t\ddot{\theta}_x - \delta_{nt}\ddot{v}_0') \\
&+ (b_{10} + b_{18})\ddot{\phi}_c' \\
&+ \delta_{w1}[-d_7\dot{w}_0 - (d_8 - \delta_sd_{16})(\delta_t\dot{\theta}_x - \delta_{nt}\dot{v}_0') - (d_9 + \delta_sd_{17})(\delta_t\dot{\theta}_y - \delta_{nt}\dot{u}_0') \\
&+ (d_{10} + d_{18})\dot{\phi}_c'] - \widetilde{M}_z = 0
\end{aligned}
\tag{4.185}
$$

As an example, when the cross-section is clamped, the geometric boundary conditions are

$$
\begin{aligned}
&u_0 = v_0 = w_0 = \delta_t\theta_y - \delta_{nt}v_0' = \delta_t\theta_x - \delta_{nt}u_0' \\
&= (\delta_{w1} + \delta_{w2})\phi + \delta_{w3}\phi_c = \delta_{w1}\phi' + \delta_{w2}\Theta = 0
\end{aligned}
\tag{4.186}
$$

4.6.3 Compact Form of the Governing Equations and Boundary Conditions

In operational form, the governing equations and the associated boundary conditions can be expressed as

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} & L_{16} & L_{17} & L_{18} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} & L_{26} & L_{27} & L_{28} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} & L_{36} & L_{37} & L_{38} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} & L_{46} & L_{47} & L_{48} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} & L_{56} & L_{57} & L_{58} \\ L_{61} & L_{62} & L_{63} & L_{64} & L_{65} & L_{66} & L_{67} & L_{68} \\ L_{71} & L_{72} & L_{73} & L_{74} & L_{75} & L_{76} & L_{77} & L_{78} \\ L_{81} & L_{82} & L_{83} & L_{84} & L_{85} & L_{86} & L_{87} & L_{88} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ w_0 \\ \delta_t \delta\theta_y \\ \delta_t \delta\theta_x \\ \phi \\ \delta_{w2}\Theta \\ \delta_{w3}\phi_c \end{bmatrix} = - \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix} \tag{4.187}$$

Similarly in a condensed form the static boundary conditions can be expressed as:

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} & M_{17} & M_{18} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & M_{26} & M_{27} & M_{28} \\ M_{31} & M_{32} & M_{33} & M_{34} & M_{35} & M_{36} & M_{37} & M_{38} \\ M_{31} & M_{42} & M_{43} & M_{44} & M_{45} & M_{46} & M_{47} & M_{48} \\ M_{51} & M_{52} & M_{53} & M_{54} & M_{55} & M_{56} & M_{57} & M_{58} \\ M_{61} & M_{62} & M_{63} & M_{64} & M_{65} & M_{66} & M_{67} & M_{68} \\ M_{71} & M_{72} & M_{73} & M_{74} & M_{75} & M_{76} & M_{77} & M_{78} \\ M_{81} & M_{82} & M_{83} & M_{84} & M_{85} & M_{86} & M_{87} & M_{88} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ w_0 \\ (\delta_t \delta\theta_y - \delta_{nt}\delta u_0') \\ (\delta_t \delta\theta_x - \delta_{nt}\delta v_0') \\ (\delta_{w1} + \delta_{w2})\phi \\ (\delta_{w1}\phi' + \delta_{w2}\Theta) \\ \delta_{w3}\phi_c \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \end{bmatrix} \tag{4.188}$$

where L_{ij}, M_{ij}, f_i, and m_i are given in Appendices B and C. For special cases of practical importance, the governing equations and the associated boundary conditions could be specialized by discarding several of the effects previously introduced. For example, in the absence of the transverse shear effect, the tracers are:

$$\delta_t = 0 \quad \delta_{nt} = 1$$
$$\delta_s = 1 \tag{4.189}$$
$$\delta_{w1} = 1 \quad \delta_{w2} = \delta_{w3} = 0$$

In Eq 4.189, both warping effects are incorporated and the warping measure is chosen to be the rate of twist, $\phi'(z)$. The governing equations along with the boundary conditions for various cases can be derived from the most general expressions given by Eqs 4.146 through 4.185. In the most general case, the order of the governing equations is fourteen and, as a result, seven boundary conditions have to be prescribed at each end of the beam. Various forms of the governing equations and the associated boundary conditions can be obtained depending on the cross-sectional geometry, the ply lay-up, etc. As an example, for orthotropic thin-walled beams with double symmetry of their cross-section, the governing equations and the boundary conditions could be decoupled into bending, torsion, and extension parts.

Chapter V

Static Aeroelastic Instability of Swept Wing Structures

This chapter is devoted to an analysis of the aeroelastic divergence instability of swept wing structures made of anisotropic composite materials. As is well known, the plate-beam and solid-beam models have been widely used in the study of the aeroelasticity of lifting surfaces. To the best knowledge of the author, a single exception concerns the paper [53] where the classical thin-walled beam model was used to analyze the divergence instability of metallic straight wings. Here the divergence instability of a swept wing is analyzed and, in this context, the model of thin-walled composite beam structures is adopted [39].

5.1 Governing Equations and Boundary Conditions

The equations of equilibrium and the associated boundary conditions of cantilevered thin-walled closed cross-section beams can be obtained by specializing the general counterparts derived in chapter 4. Dealing with a static instability problem, the kinetic and the viscous damping terms have to be discarded.

The equations of equilibrium are :

δu_0:

$$\delta_{nt} M_y'' - \delta_{nt} Q'_x + \hat{Q}_x' = 0 \tag{5.1}$$

δv_0:

$$\delta_{nt} M_x'' - \delta_{nt} Q'_y + \hat{Q}_y' + p_y = 0 \tag{5.2}$$

δw_0:

$$T_A' = 0 \tag{5.3}$$

$\delta\phi$:

$$\delta_{w1}(B_\omega'' + M_p') + \delta_{w2} M_t' + (\delta_{w1} + \delta_{w2}) m_z = 0 \tag{5.4}$$

$\delta_t \delta\theta_y$:

$$M'_y - Q_x = 0 \tag{5.5}$$

$\delta_t \delta\theta_x$:

$$M'_x - Q_y = 0 \tag{5.6}$$

$\delta_{w2} \delta\Theta$:

$$B_\omega' + M_p - M_t = 0 \tag{5.7}$$

$\delta_{w3} \delta\phi_c$:

$$M_p' + m_z = 0 \tag{5.8}$$

In Eqs 5.2 and 5.4 (associated with the plunging and torsional displacements), the expressions of the aerodynamic lift (p_y) and torsional moment (m_z) have been expressed in conjunction with the strip-theory aerodynamics of incompressible flows [11,39,41,42]. These quantities are

$$p_y = q_n c a_0 (\phi - v_0' \tan\Lambda) \tag{5.9}$$

$$m_z = q_n c a_0 e (\phi - v_0' \tan\Lambda) \tag{5.10}$$

where q_n $(\equiv \frac{\rho}{2} U_n^2)$ denotes the dynamic pressure normal to the leading edge of the swept wing; c denotes the uniform chord of the wing; a_0 denotes the "corrected lift " curve slope coefficient; Λ stands for the angle of sweep (considered positive for swept-back wings) while e denotes the offset between the aerodynamic and reference axis (the latter one selected to coincide with the centroidal axis of the section).

The boundary conditions at the root (z = 0) are

$$\begin{aligned} u_0 = v_0 = w_0 = \delta_t\theta_y - \delta_{nt}u_0' = \delta_t\theta_x - \delta_{nt}v_0' \\ = (\delta_{w1} + \delta_{w2})\phi + \delta_{w3}\phi_c = \delta_{w1}\phi' + \delta_{w2}\Theta = 0 \end{aligned} \tag{5.11}$$

and at the tip of the beam (z = L)

δu_0:

$$\delta_{nt}(M'_y - Q_x) + \hat{Q}_x = 0 \tag{5.12}$$

δv_0:

$$\delta_{nt}(M'_x - Q_y) + \hat{Q}_y = 0 \tag{5.13}$$

δw_0:

$$T_A = 0 \tag{5.14}$$

$[\delta_t\delta\theta_y]$ or $[\delta_{nt}\delta u_0']$:

$$M_y = 0 \tag{5.15}$$

$[\delta_t\delta\theta_x]$ or $[\delta_{nt}\delta v_0']$:

$$M_x = 0 \tag{5.16}$$

$[\delta_{w1}\delta\phi']$ or $[\delta_{w2}\delta\Theta]$:

$$B_\omega = 0 \tag{5.17}$$

$\delta\phi$:

$$\delta_{w1}(B_\omega' + M_p) + \delta_{w2}M_t = 0 \tag{5.18}$$

$\delta_{w3}\delta\phi_c$:

$$M_p = 0 \tag{5.19}$$

The equations governing the aeroelastic equilibrium could be obtained by expressing the stress resultants and stress couples in terms of the displacement quantities. The same procedure is valid for the static boundary conditions. Adequate manipulation of the tracing quantities could result in various groups of governing equations incorporating or discarding the various effects. In a general form, these are given by

δu_0:

$$s_{41}w_0'' + s_{42}\theta_y'' + s_{43}\theta_x'' + s_{44}(\theta_y' + u_0'') + s_{45}(\theta_x' + v_0'') + s_{46}\phi''' + s_{47}\phi'' = 0 \quad (5.20)$$

δv_0:

$$\begin{aligned} &s_{51}w_0'' + s_{52}\theta_y'' + s_{53}\theta_x'' + s_{54}(\theta_y' + u_0'') + s_{55}(\theta_x' + v_0'') \\ &+ s_{56}\phi''' + s_{57}\phi'' + q_n c a_0(\phi - v_0'\tan\Lambda) = 0 \end{aligned} \quad (5.21)$$

δw_0:

$$a_{11}w_0'' + a_{12}\theta_y'' + a_{13}\theta_x'' + a_{14}(\theta_y' + u_0'') + a_{15}(\theta_x' + v_0'') + a_{16}\phi''' + a_{17}\phi'' = 0 \quad (5.22)$$

$\delta\phi$:

$$\begin{aligned} &a_{61}w_0''' + a_{62}\theta_y''' + a_{63}\theta_x''' + a_{64}(\theta_y'' + u_0''') + a_{65}(\theta_x'' + v_0''') + a_{66}\phi'''' + a_{67}\phi''' \\ &+ a_{71}w_0'' + a_{72}\theta_y'' + a_{73}\theta_x'' + a_{74}(\theta_y' + u_0'') + a_{75}(\theta_x' + v_0'') + a_{76}\phi''' + a_{77}\phi'' \\ &+ q_n c a_0 e(\phi - v_0'\tan\Lambda) = 0 \end{aligned} \quad (5.23)$$

$\delta\theta_y$:

$$\begin{aligned} &a_{21}w_0'' + a_{22}\theta_y'' + a_{23}\theta_x'' + a_{24}(\theta_y' + u_0'') + a_{25}(\theta_x' + v_0'') \\ &+ a_{26}\phi''' + a_{27}\phi'' - a_{41}w_0' - a_{42}\theta_y' - a_{43}\theta_x' \\ &- a_{44}(\theta_y + u_0') - a_{45}(\theta_x + v_0') - a_{46}\phi'' - a_{47}\phi' = 0 \end{aligned} \quad (5.24)$$

$\delta\theta_x$:

$$\begin{aligned} &a_{31}w_0'' + a_{32}\theta_y'' + a_{33}\theta_x'' + a_{34}(\theta_y' + u_0'') + a_{35}(\theta_x' + v_0'') \\ &+ a_{36}\phi''' + a_{37}\phi'' - a_{51}w_0' - a_{52}\theta_y' - a_{53}\theta_x' \\ &- a_{54}(\theta_y + u_0') - a_{55}(\theta_x + v_0') - a_{56}\phi'' - a_{57}\phi' = 0 \end{aligned} \quad (5.25)$$

The boundary conditions at the root ($z=0$) are

$$u_0 = v_0 = w_0 = \theta_y = \theta_x = \phi = \phi' = \Theta = 0 \quad (5.26)$$

while the ones at the tip of the beam ($z=L$) become

δu_0:

$$s_{41}w_0' + s_{42}\theta_y' + s_{43}\theta_x' + s_{44}(\theta_y + u_0') + s_{45}(\theta_x + v_0') + s_{46}\phi'' + s_{47}\phi' = 0 \quad (5.27)$$

δv_0:

$$s_{51}w_0' + s_{52}\theta_y' + s_{53}\theta_x' + s_{54}(\theta_y + u_0') + s_{55}(\theta_x + v_0') + s_{56}\phi'' + s_{57}\phi' = 0 \quad (5.28)$$

δw_0:

$$a_{11}w_0' + a_{12}\theta_y' + a_{13}\theta_x' + a_{14}(\theta_y + u_0') + a_{15}(\theta_x + v_0') + a_{16}\phi'' + a_{17}\phi' = 0 \quad (5.29)$$

$\delta\theta_y$:

$$a_{21}w_0' + a_{22}\theta_y' + a_{23}\theta_x' + a_{24}(\theta_y + u_0') + a_{25}(\theta_x + v_0') + a_{26}\phi'' + a_{27}\phi' = 0 \quad (5.30)$$

$\delta\theta_x$:

$$a_{31}w_0' + a_{32}\theta_y' + a_{33}\theta_x' + a_{34}(\theta_y + u_0') + a_{35}(\theta_x + v_0') + a_{36}\phi'' + a_{37}\phi' = 0 \quad (5.31)$$

$\delta\phi'$:

$$a_{61}w_0' + a_{62}\theta_y' + a_{63}\theta_x' + a_{64}(\theta_y + u_0') + a_{65}(\theta_x + v_0') + a_{66}\phi'' + a_{67}\phi' = 0 \quad (5.32)$$

$\delta\phi$:

$$\begin{aligned} &a_{61}w_0'' + a_{62}\theta_y'' + a_{63}\theta_x'' + a_{64}(\theta_y' + u_0'') + a_{65}(\theta_x' + v_0'') \\ &+ a_{66}\phi''' + a_{67}\phi'' + a_{71}w_0' + a_{72}\theta_y' + a_{73}\theta_x' \\ &+ a_{74}(\theta_y + u_0') + a_{75}(\theta_x + v_0') + a_{76}\phi'' + a_{77}\phi' = 0 \end{aligned} \quad (5.33)$$

5.2 Numerical Illustrations

The case of a cantilevered swept wing modelled as an angle-ply composite box beam will be considered in the numerical illustrations (see Fig. 8). The beam is constructed of a Graphite-Epoxy material. Its elastic characteristics are

$$\begin{gathered} E_1 = 30 \text{ x } 10^6 \text{ psi} \\ E_2 = E_3 = 0.75 \text{ x } 10^6 \text{ psi} \\ G_{13} = G_{23} = 0.37 \text{ x } 10^6 5\text{psi} \\ G_{12} = 0.45 \text{ x } 10^6 \text{ psi} \\ \mu_{12} = \mu_{23} = \mu_{13} = 0.25 \end{gathered} \quad (5.34)$$

Its geometric dimensions are

$$\begin{gathered} h = 0.4 \text{ inch} \\ b = 2.0 \text{ inch} \\ c = 10.0 \text{ inch} \\ L = 96.0 \text{ feet} \end{gathered} \quad (5.35)$$

The governing equations in conjunction with the boundary conditions were solved by utilizing the Laplace Transform technique, enabling one to obtain exact solutions. A MACSYMA software [43,44] in which symbolic manipulations are available, has been used in the solution procedure.

In Figs 9 and 10 the effect of tailoring on the divergence speed of a swept-forward wing structure is emphasized. The associated results have been obtained by discarding the warping inhibition and transverse shear effects. In Fig. 11, the effect played by the warping inhibition, and in Fig. 12 the effect played by the ply angle in conjunction with the aspect ratio of the wing (AR = 2L/c) on the divergence speed, are emphasized. In addition, other numerical results allowing one to put into evidence the role played by the transverse shear deformability and the secondary warping are shown in Tables 1, 2 and 3, respectively.

5.3 Discussions

Within this study a more reliable analysis of the aeroelastic divergence instability of a swept cantilevered composite wing was accomplished. The analysis based on a refined model of composite thin-walled closed cross-section beams reveals that the aeroelastic tailoring technique considered within the framework of this more realistic model provides a strong basis for an enhancement of the aeroelastic properties of forward swept wing structures. The results emerging from these graphs reveal that for each sweep angle (Λ), an optimum ply angle yielding the highest divergence instability speed could be obtained. The same figures reveal that the tailoring technique yields a relatively stronger enhancement (from the divergence instability point of view) for forward swept wings than for their straight wing counterparts (for which case the divergence instability does not constitute in general a critical case).

As concerns the warping inhibition, its effect, as revealed for this case in Fig. 11, is a beneficial one. As is seen, the sweep angle plays a great role in this enhancement. The same is valid for the ply angles. For the two considered cases of high aspect ratio wings, Fig. 12 reveals that the aeroelastic tailoring technique could also result in a strong enhancement of the aeroelastic properties of high aspect ratio wings.

As concerns transverse shear deformation (which is so characteristic of composite material structures), the character of its influence on the divergence speed is different in the case of swept-back and swept-forward wings. For a better understanding of this difference, we have to remeber the well-known result according to which, in the case of a swept wing, the effective angle of attack ϕ_{eff} [11,39,41,42] is

$$\phi_{eff} = \phi - v_0' \tan\Lambda \tag{5.36}$$

Equation 5.36 reveals the well-known fact that for swept-back wings ($\Lambda > 0$) the bending deformation tends to reduce the effective angle of attack, while in the case of swept-forward wings the opposite effect is valid. On the other hand, the transverse shear flexibility yields an increase of the bending deflection and, as a result, the wash-out and wash-in effects characterizing the swept-back and swept-forward wings, respectively, are further exacerbated. This expected result was proven by using the thin-walled beam model considered in this dissertation. The numerical findings show that, at least for the case considered here, the transverse shear flexibility could yield an increase of the divergence speed for a sweep angle $\Lambda = 45°$ of about 20% as compared to the one obtained within the case of infinite rigidity in transverse shear. However, for swept-forward wings, the decrease of the divergence speed due to the same effect is less drastic than the increase resulting in the case of swept-back wings. The numerical results concerning the transverse shear effect with respect to the sweep angle are given in Table 5.1.

As concerns the secondary warping, its effect results in an increase of the divergence speed. However, the amount of its increase in the present case is only about 6%. It is believed, however, that for other cases, a larger contribution of this effect could occur. The numerical results concerning the secondary warping effect are given in Tables 5.2 and 5.3.

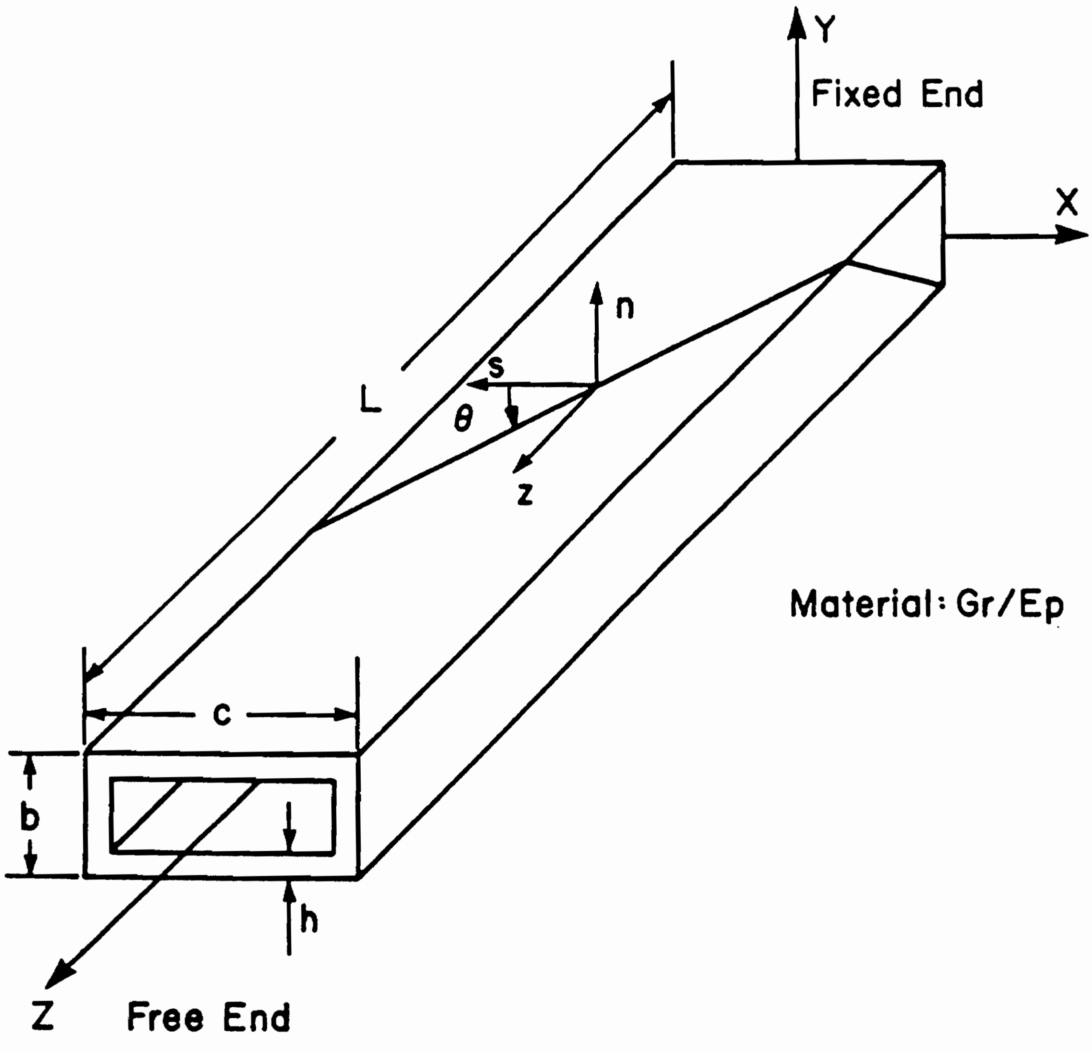


Figure 8. Geometry of the Box Beam

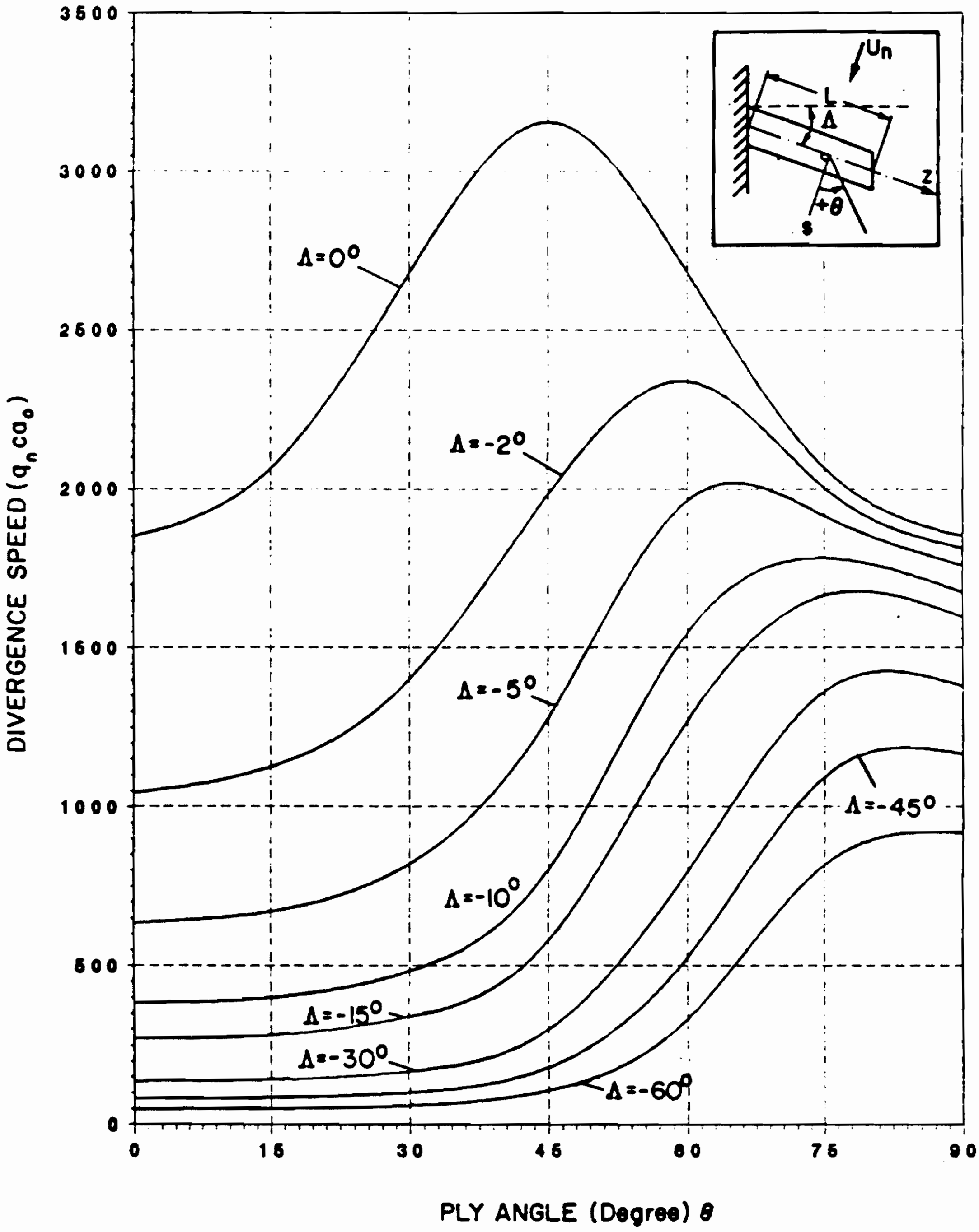


Figure 9. Variation of the Divergence Speed vs. Ply Angle for Given Values of Sweep Angle

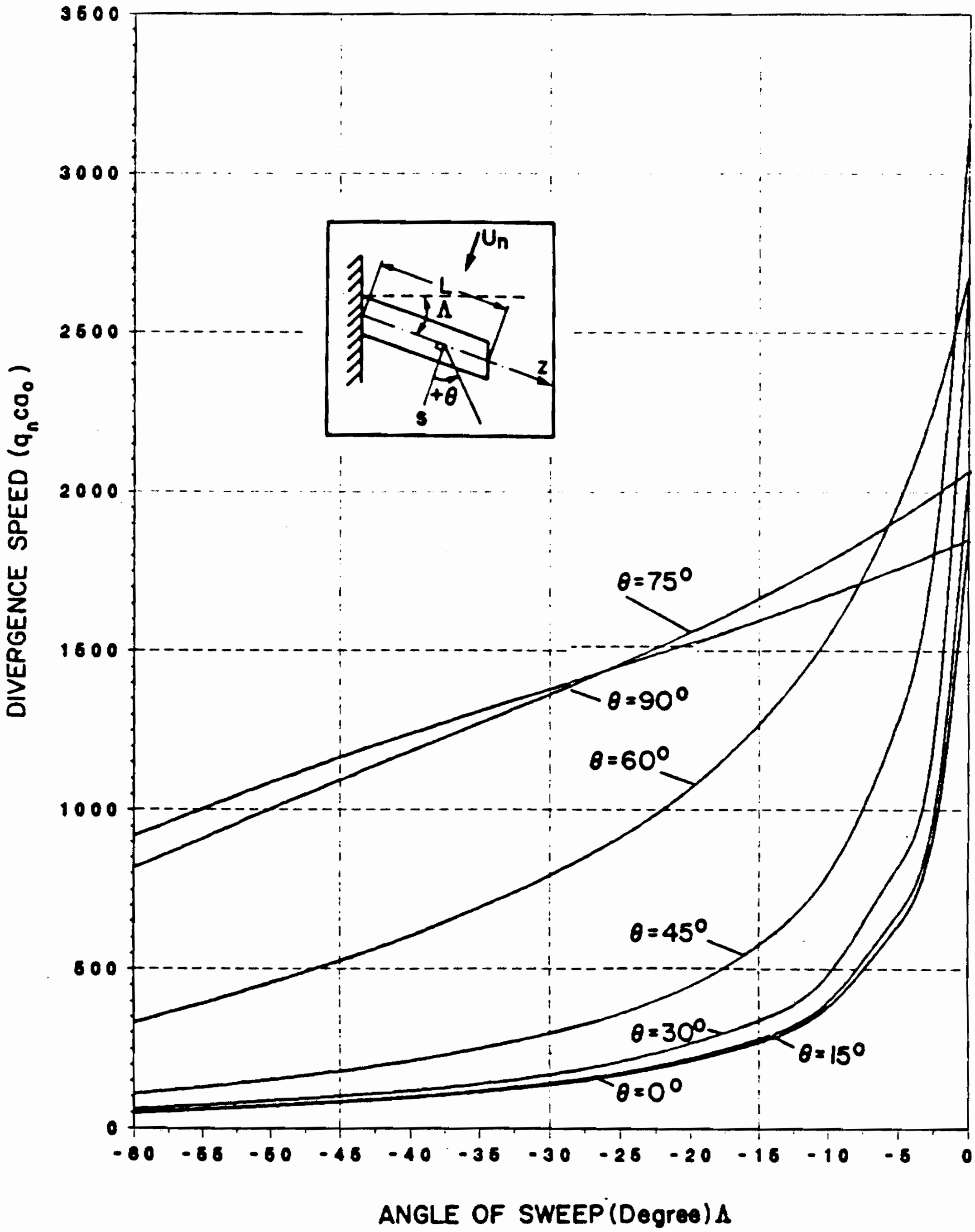


Figure 10. Variation of the Divergence Speed vs. Sweep Angle for Given Values of Ply Angle

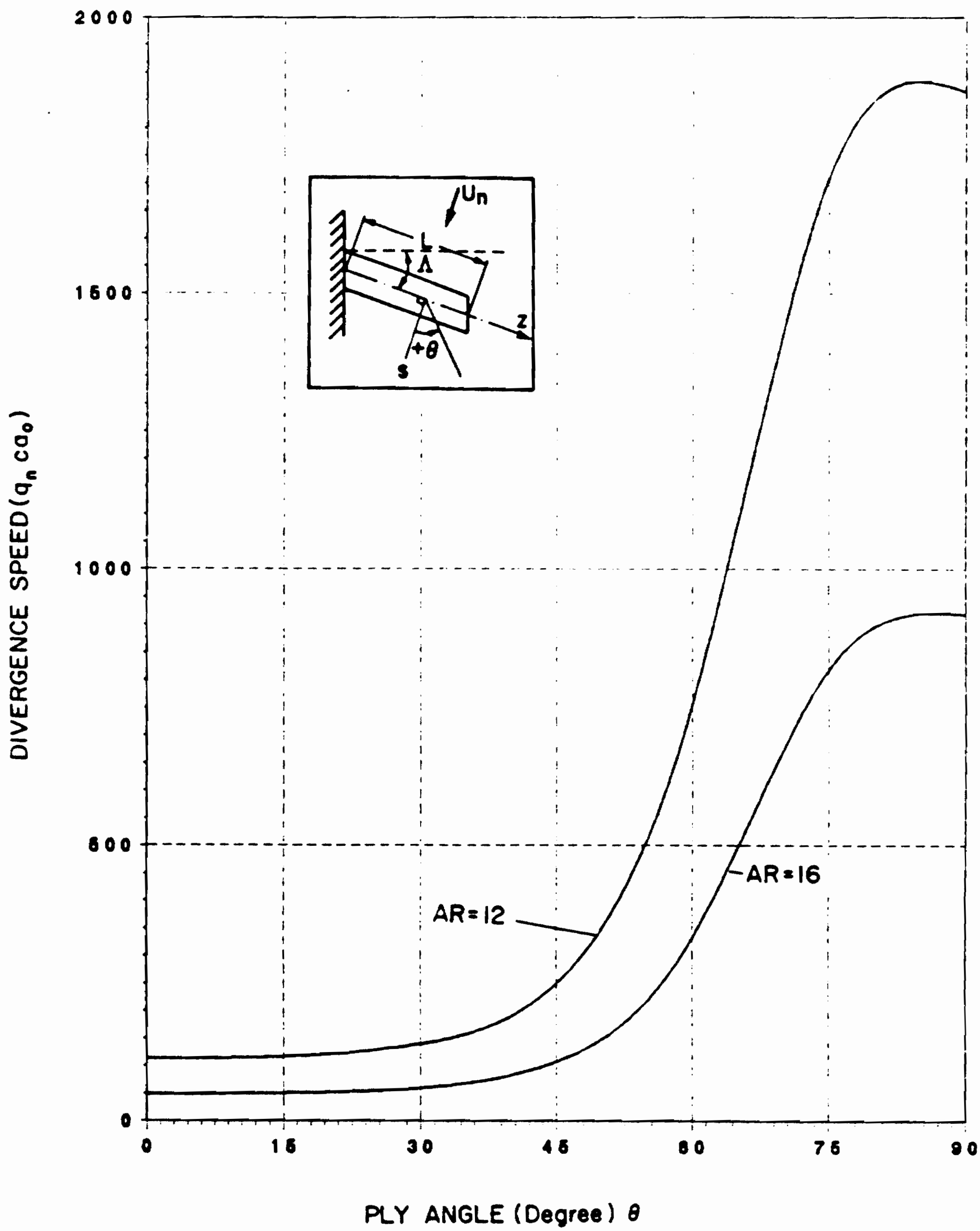


Figure 11. Variation of the Divergence Speed vs. Ply Angle for Two Values of Aspect Ratio

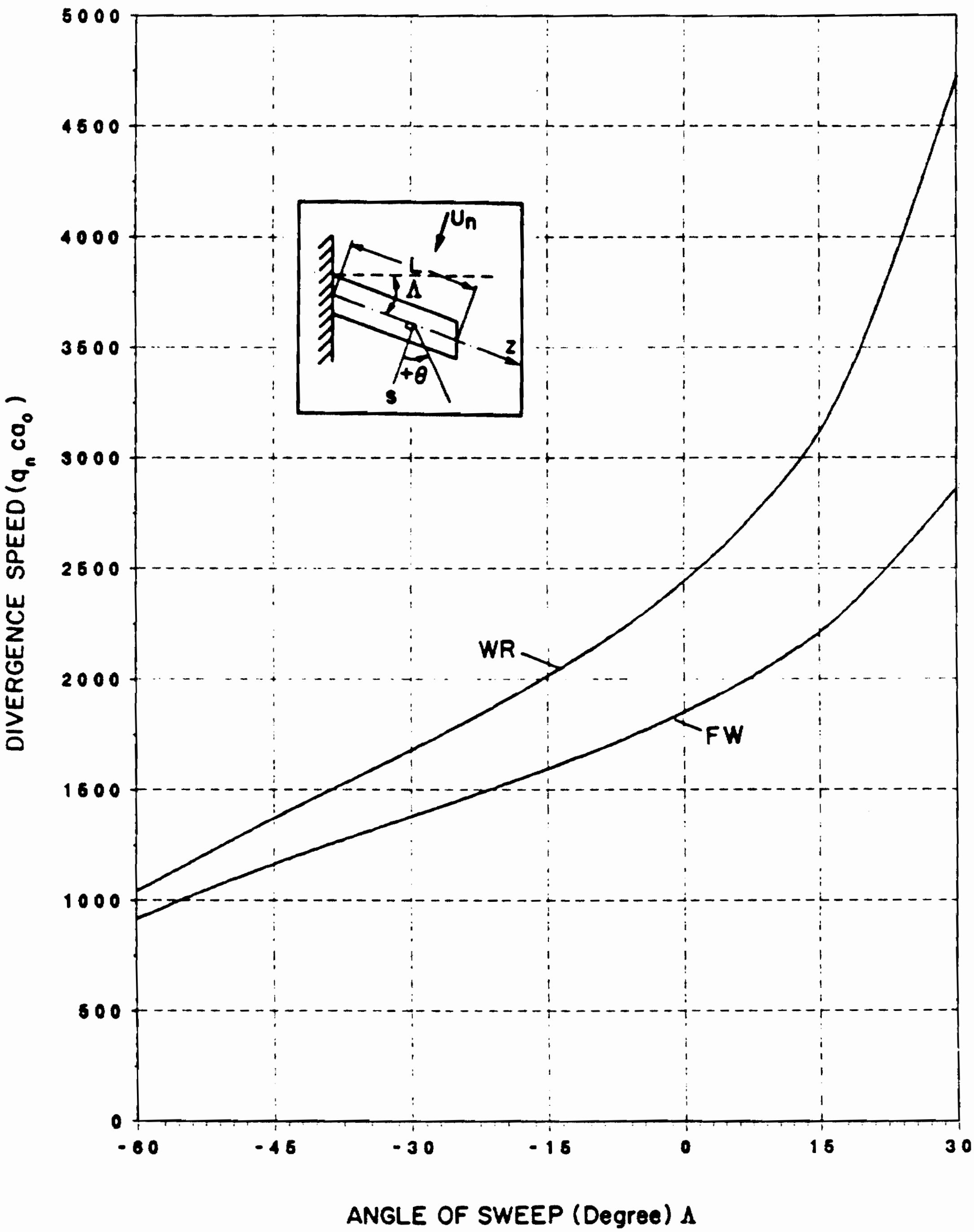


Figure 12. Variations of the Divergence Speed vs. Ply Angle for Free and Warping Restraint Models

Table 1. Variation of the Divergence Speed vs. Sweep Angle Concerning the Transverse Shear Effect (Ply Angle = 90 degree)

Sweep Angle (deg)	Divergence Speed	
	I	II
- 60	918	906 (-1.3)
- 45	1165	1152 (-1.1)
- 30	1380	1368 (-0.9)
- 15	1596	1589 (-0.4)
0	1851	1851 (0)
15	2208	2222 (0.6)
30	2863	2925 (2.2)
45	5330	6417 (20.4)
60	no divergence	no divergence

( ): % difference

Ply Angle = 90 degree

I: Without Transverse Shear Deformability

II: With Transverse Shear Deformability

Table 2. Variation of the Divergence Speed vs. Ply Angle Concerning the Secondary Warping Effect (Sweep Angle = -30 degree)

Ply Angle (deg)	Divergence Speed	
	I	II
0	314	317 (1.0)
15	324	328 (1.2)
30	379	393 (3.6)
45	651	685 (5.0)
60	1673	1711 (2.2)
75	2635	2646 (0.4)
90	2609	2617 (0.3)

( ): % difference

Sweep Angle = -30 degree

I: Without Secondary Warping Effect

II: With Secondary Warping Effect

Table 3. Variation of the Divergence Speed vs. Ply Angle Concerning the Secondary Warping Effect (Sweep Angle = -60 degree)

Ply Angle (deg)	Divergence Speed	
	I	II
0	111	113 (1.7)
15	114	116 (1.7)
30	134	139 (3.6)
45	235	249 (5.6)
60	732	755 (3.0)
75	1694	1708 (0.8)
90	1854	1865 (0.6)

( ): % difference

Sweep Angle = -60 degree

I: Without Secondary Warping Effect

II: With Secondary Warping Effect

Chapter VI

Free Vibration Analysis

This chapter is devoted to an analysis of the free vibration of anisotropic composite aircraft wing structures. The thin-walled beam structural model is adopted for this analysis. The results of the free vibration analysis are essentially needed in aeroelastic flutter studies. As is well known the accurate determination of eigenfrequencies constitutes a necessary requirement for a reliable prediction of the flutter instability of aircraft wing structures [11,15].

6.1 Governing Equations and Boundary Conditions

The governing equations and the associated boundary conditions of vibrating cantilevered thin-walled closed cross-section beams could be obtained by specializing the general governing equations and boundary conditions derived previously in chapter 4. Dealing with the undamped free vibration problem, the loading and damping terms could be discarded.

The equations of motion are:

δu_0:

$$\delta_{nt}(M_y'' - Q'_x) + \hat{Q}_x' - (K_1 - \delta_{nt}K_6') = 0 \tag{6.1}$$

δv_0:

$$\delta_{nt}(M_x'' - Q'_y) + \hat{Q}_y' - (K_2 - \delta_{nt}K_8') = 0 \tag{6.2}$$

δw_0:

$$T_A' - K_3 = 0 \tag{6.3}$$

$\delta\phi$:

$$\delta_{w1}(B_\omega'' + M_p') + \delta_{w2}M_t' - (\delta_{w1}K_4 - \delta_{w1}K_9' + \delta_{w2}K_4) = 0 \tag{6.4}$$

$\delta_t\delta\theta_y$:

$$M'_y - Q_x - K_5 = 0 \tag{6.5}$$

$\delta_t\delta\theta_x$:

$$M'_x - Q_y - K_7 = 0 \tag{6.6}$$

$\delta_{w2}\delta\Theta$:

$$B_\omega' + M_p - M_t + K_{10} = 0 \tag{6.7}$$

$\delta_{w3}\delta\phi_c$:

$$M_p' - (K_{12} - K_{11}') = 0 \tag{6.8}$$

Substitution in the above equations of the generalized forces and couples in terms of the displacement quantities results in the governing equations of the problem. Appropriate manipulation of the tracing quantities could result in various sets of approximate governing equation systems. In the general case these are given by

δu_0:

$$\begin{aligned} & s_{41}w_0'' + s_{42}\theta_y'' + s_{43}\theta_x'' + s_{44}(\theta_y' + u_0'') + s_{45}(\theta_x' + v_0'') + s_{46}\Theta'' + s_{47}\phi'' + s_{48}\Theta' \\ & - (b_1\ddot{u}_0 - b_2\ddot{\phi}) = 0 \end{aligned} \tag{6.9}$$

δv_0:

$$\begin{aligned} & s_{51}w_0'' + s_{52}\theta_y'' + s_{53}\theta_x'' + s_{54}(\theta_y' + u_0'') + s_{55}(\theta_x' + v_0'') + s_{56}\Theta'' + s_{57}\phi'' + s_{58}\Theta' \\ & - (b_1\ddot{v}_0 + b_3\ddot{\phi}) = 0 \end{aligned} \tag{6.10}$$

δw_0:

$$\begin{aligned} & a_{11}w_0'' + a_{12}\theta_y'' + a_{13}\theta_x'' + a_{14}(\theta_y' + u_0'') + a_{15}(\theta_x' + v_0'') + a_{16}\Theta'' + a_{17}\phi'' + a_{18}\Theta' \\ & - (b_1\ddot{w}_0 + b_3\ddot{\theta}_y + b_2\ddot{\theta}_x - b_7\ddot{\Theta}) = 0 \end{aligned} \tag{6.11}$$

$\delta\phi$:

$$\begin{aligned} & a_{81}w_0'' + a_{82}\theta_y'' + a_{83}\theta_x'' + a_{84}(\theta_y' + u_0'') + a_{85}(\theta_x' + v_0'') + a_{86}\Theta'' + a_{87}\phi'' + a_{88}\Theta' \\ & -\{b_3\ddot{v}_0 - b_2\ddot{u}_0 + (b_4 + b_5)\ddot{\phi}\} \\ & +\{ - b_7\ddot{w}_0' - (b_9 + b_{17})\ddot{\theta}_y' - (b_8 - b_{16})\ddot{\theta}_x' + (b_{10} + b_{18})\ddot{\phi}''\} = 0 \end{aligned} \tag{6.12}$$

$\delta\theta_y$:

$$\begin{aligned} & a_{21}w_0'' + a_{22}\theta_y'' + a_{23}\theta_x'' + a_{24}(\theta_y' + u_0'') + a_{25}(\theta_x' + v_0'') + a_{26}\Theta'' + a_{27}\phi'' + a_{28}\Theta' \\ & -\{a_{41}w_0' + a_{42}\theta_y' + a_{43}\theta_x' + a_{44}(\theta_y + u_0') + a_{45}(\theta_x + v_0') + a_{46}\Theta' + a_{47}\phi' + a_{48}\Theta\} \\ & -\{b_3\ddot{w}_0 + (b_5 + b_{15})\ddot{\theta}_y + (b_6 - b_{13})\ddot{\theta}_x - (b_9 + b_{17})\ddot{\Theta}\} = 0 \end{aligned} \tag{6.13}$$

$\delta\theta_x$:

$$\begin{aligned} & a_{31}w_0'' + a_{32}\theta_y'' + a_{33}\theta_x'' + a_{34}(\theta_y' + u_0'') + a_{35}(\theta_x' + v_0'') + a_{36}\Theta'' + a_{37}\phi'' + a_{38}\Theta' \\ & -\{a_{51}w_0' + a_{52}\theta_y' + a_{53}\theta_x' + a_{54}(\theta_y + u_0') + a_{55}(\theta_x + v_0') + a_{56}\Theta' + a_{57}\phi' + a_{58}\Theta\} \\ & -\{b_2\ddot{w}_0 + (b_6 - b_{13})\ddot{\theta}_y + (b_4 + b_{14})\ddot{\theta}_x - (b_8 - b_{16})\ddot{\Theta}\} = 0 \end{aligned} \tag{6.14}$$

$\delta\Theta$:

$$\begin{aligned} & a_{61}w_0'' + a_{62}\theta_y'' + a_{63}\theta_x'' + a_{64}(\theta_y' + u_0'') + a_{65}(\theta_x' + v_0'') + a_{66}\Theta'' + a_{67}\phi'' + a_{68}\Theta' \\ & + a_{71}w_0' + a_{72}\theta_y' + a_{73}\theta_x' + a_{74}(\theta_y + u_0') + a_{75}(\theta_x + v_0') + a_{76}\Theta' + a_{77}\phi' + a_{78}\Theta \\ & -\{a_{81}w_0' + a_{82}\theta_y' + a_{83}\theta_x' + a_{84}(\theta_y + u_0') + a_{85}(\theta_x + v_0') + a_{86}\Theta' + a_{87}\phi' + a_{88}\Theta\} \\ & - b_7\ddot{w}_0 - (b_9 + b_{17})\ddot{\theta}_y - (b_8 - b_{16})\ddot{\theta}_x + (b_{10} + b_{18})\ddot{\Theta} = 0 \end{aligned} \tag{6.15}$$

The boundary conditions at the root are:

$$u_0 = v_0 = w_0 = \theta_y - \delta_{nt} u_0' = \theta_x - \delta_{nt} v_0'$$
$$= (\delta_{w1} + \delta_{w2})\phi + \delta_{w3}\phi_c = \delta_{w1}\phi' + \delta_{w2}\Theta = 0 \qquad (6.16)$$

while at z = L (at the tip of the beam) these are:

δu_0:

$$\delta_{nt}(M'_y - Q_x) + \hat{Q}_x + \delta_{nt} K_6 = 0 \qquad (6.17)$$

δv_0:

$$\delta_{nt}(M'_x - Q_y) + \hat{Q}_y + \delta_{nt} K_8 = 0 \qquad (6.18)$$

δw_0:

$$T_A = 0 \qquad (6.19)$$

$\delta\theta_y$ or $\delta_{nt}\delta u_0'$:

$$M_y = 0 \qquad (6.20)$$

$\delta\theta_x$ or $\delta_{nt}\delta v_0'$:

$$M_x = 0 \qquad (6.21)$$

$\delta_{w1}\delta\phi'$ or $\delta_{w2}\delta\Theta$:

$$B_\omega = 0 \qquad (6.22)$$

$\delta\phi$:

$$\delta_{w1}(B_\omega' + M_p) + \delta_{w2} M_t + \delta_{w1} K_9 = 0 \qquad (6.23)$$

$\delta_{w3}\delta\phi_c$:

$$M_p + K_{11} = 0 \qquad (6.24)$$

When $\delta_t = \delta_{w2} = 1$, the boundary conditions at the root are:

$$u_0 = v_0 = w_0 = \theta_y = \theta_x = \phi = \Theta = 0 \qquad (6.25)$$

In terms of the displacement quantities, the boundary conditions at z = L are:

δu_0:

$$s_{41} w_0' + s_{42}\theta_y' + s_{43}\theta_x' + s_{44}(\theta_y + u_0') + s_{45}(\theta_x + v_0') + s_{46}\Theta' + s_{47}\phi' + s_{48}\Theta = 0 \qquad (6.26)$$

δv_0:

$$s_{51} w_0' + s_{52}\theta_y' + s_{53}\theta_x' + s_{54}(\theta_y + u_0') + s_{55}(\theta_x + v_0') + s_{56}\Theta' + s_{57}\phi' + s_{58}\Theta = 0 \qquad (6.27)$$

δw_0:

$$a_{11} w_0' + a_{12}\theta_y' + a_{13}\theta_x' + a_{14}(\theta_y + u_0') + a_{15}(\theta_x + v_0') + a_{16}\Theta' + a_{17}\phi' + a_{18}\Theta = 0 \qquad (6.28)$$

$\delta\theta_y$:

$$a_{21}w_0' + a_{22}\theta_y' + a_{23}\theta_x' + a_{24}(\theta_y + u_0') + a_{25}(\theta_x + v_0') + a_{26}\Theta' + a_{27}\phi' + a_{28}\Theta = 0 \quad (6.29)$$

$\delta\theta_x$ equation

$$a_{31}w_0' + a_{32}\theta_y' a_{33}\theta_x' + a_{34}(\theta_y + u_0') + a_{35}(\theta_x + v_0') + a_{36}\Theta' + a_{37}\phi' + a_{38}\Theta = 0 \quad (6.30)$$

$\delta\Theta$:

$$a_{61}w_0' + a_{62}\theta_y' + a_{63}\theta_x' + a_{64}(\theta_y + u_0') + a_{65}(\theta_x + v_0') + a_{66}\Theta' + a_{67}\phi' + a_{68}\Theta = 0 \quad (6.31)$$

$\delta\phi$:

$$a_{81}w_0' + a_{82}\theta_y' + a_{83}\theta_x' + a_{84}(\theta_y + u_0') + a_{85}(\theta_x + v_0') + a_{86}\Theta' + a_{87}\phi' + a_{88}\Theta = 0 \quad (6.32)$$

6.2 Numerical Illustrations

The case of a cantilevered wing modelled as an angle-ply composite box beam will be considered in the numerical illustrations (see Fig. 8). The material of the beam is Graphite-Epoxy. Its elastic characteristics are given by Eq 5.34. Its geometric dimensions are (see Fig. 8)

$$\begin{aligned} h &= 0.04 \text{ inch} \\ b &= 0.2 \text{ inch} \\ c &= 1.0 \text{ inch} \\ L &= 10.0 \text{ inch} \end{aligned} \quad (6.33)$$

The governing equations and associated boundary conditions are reduced to the eigenvalue problem and solved by using the Laplace Transform technique to obtain exact solutions. In the solution procedure, a MACSYMA software [43,44] in which symbolic manipulations are available, has been utilized.

In Fig. 13 the effect of tailoring on the eigenfrequencies of the lateral and plunging bending modes is emphasized. The associated results have been obtained by incorporating all the non-classical effects, i.e., the warping inhibition, the transverse shear deformation, and the secondary warping. In Figs 14 and 15, the effect of the transverse shear on the fundamental eigenfrequencies of the lateral and plunging vibration modes, respectively, for various ply angles is presented. In Fig. 16, the variation of the fundamental eigenfrequency associated with the extension and torsion coupled mode vs. the variation of the ply angle is depicted. The numerical results, allowing one to put into evidence the role played by the secondary warping, are displayed in Tables 4 and 5, while in Table 6 the eigenfrequencies of the axial vibration mode for two orthotropic beam cases are presented. Within the framework of Table 7, the effect played by the warping inhibition is emphasized.

6.3 Discussions

Within this study a more reliable analysis of the free vibration characteristics of a cantilevered composite wing is performed. The analysis is based on a refined model of thin-walled closed cross-section composite beams. Figs 13 through 16 reveal the effect of the tailoring technique on the free vibration characteristics of a wing structure. The results emerging from Fig. 13 reveal that higher eigenfrequencies are expected for larger ply angles. This fact is justifiable by comparing the magnitudes of the bending stiffness for various ply angles. Fig. 13 also reveals that the eigenfrequencies of the lateral bending mode are higher than those of the plunging mode. This result can be predicted by comparing the bending stiffnesses for each bending vibration mode. Due to the geometry of the cross-section of the beam considered here, the magnitude of the moment of inertia about the y-axis is larger than that about the x-axis.

In Fig. 14 and Fig. 15, the effect of the transverse shear deformability on the eigenfrequencies of two vibrational modes is presented. It is revealed that the incorportion of the transverse shear flexibility results in lower eigenfrequencies than their rigid counterpart in transverse shear. This implies that the transverse shear effect provides an extra flexibility to the structures. In Tables 8 and 9, the effects of the transverse shear on the fundamental and second frequencies are presented. Table 8 reveals that in the case of the bending vibration mode the effect of the transverse shear is more prominent for the second frequencies than for the first ones. For the plunging vibration mode, this trend was not detected (see Table 9).

In Fig. 16 the effect of tailoring on the frequencies of the coupled mode (axial and torsional vibration modes) is revealed. This figure shows the effect played by the angle variation on the fundamental eigenfrequencies of the axial and torsional vibration coupled mode.

As concerns the warping inhibition, its effect, as revealed in Table 7 for the two considered cases, results in an increase of 13 % in the eigenfrequencies of the torsional vibration mode. This fact implies that the warping restraint effect actually provides an extra torsional stiffness to the structure. This phenomenon is similar to the effect of the structural coupling considered in Fig. 16 where the effect of coupling between the axial and torsional modes results in an increase of the torsional vibration frequencies. This Table also shows that the effect of warping restraint is more severe in the higher modes than in the lower ones.

In Table 6, the variation of the eigenfrequencies in the axial vibration mode for two different ply angles is illustrated. Dramatic changes in the eigenfrequencies for the two different ply angle cases are detected. The effect of the structural tailoring is clearly revealed in this numerical result.

The effect of the secondary warping effect on the eigenfrequencies for the two bending vibration modes is given in Table 4 and Table 5. The numerical results show that the inclusion of the secondary warping effect yields an increase of bending eigenfrequencies. This fact is explainable, having in view that the secondary warping provides both an extra bending stiffness as well as an extra torsional stiffness. As was mentioned before, the importance of the secondary warping effect is dependent on the magnitude of the thickness of the considered structures. In this numerical illustration, the inclusion of the secondary warping effect results in an increase of 5 % of the eigenfrequency. It is believed, however, that for other cases, larger contributions of this effect could be detected. For example, their influence could become more prominent for relatively thick-walled beams and for special cross-section beams (e.g., square closed cross-section ones, for which case the primary warping becomes immaterial).

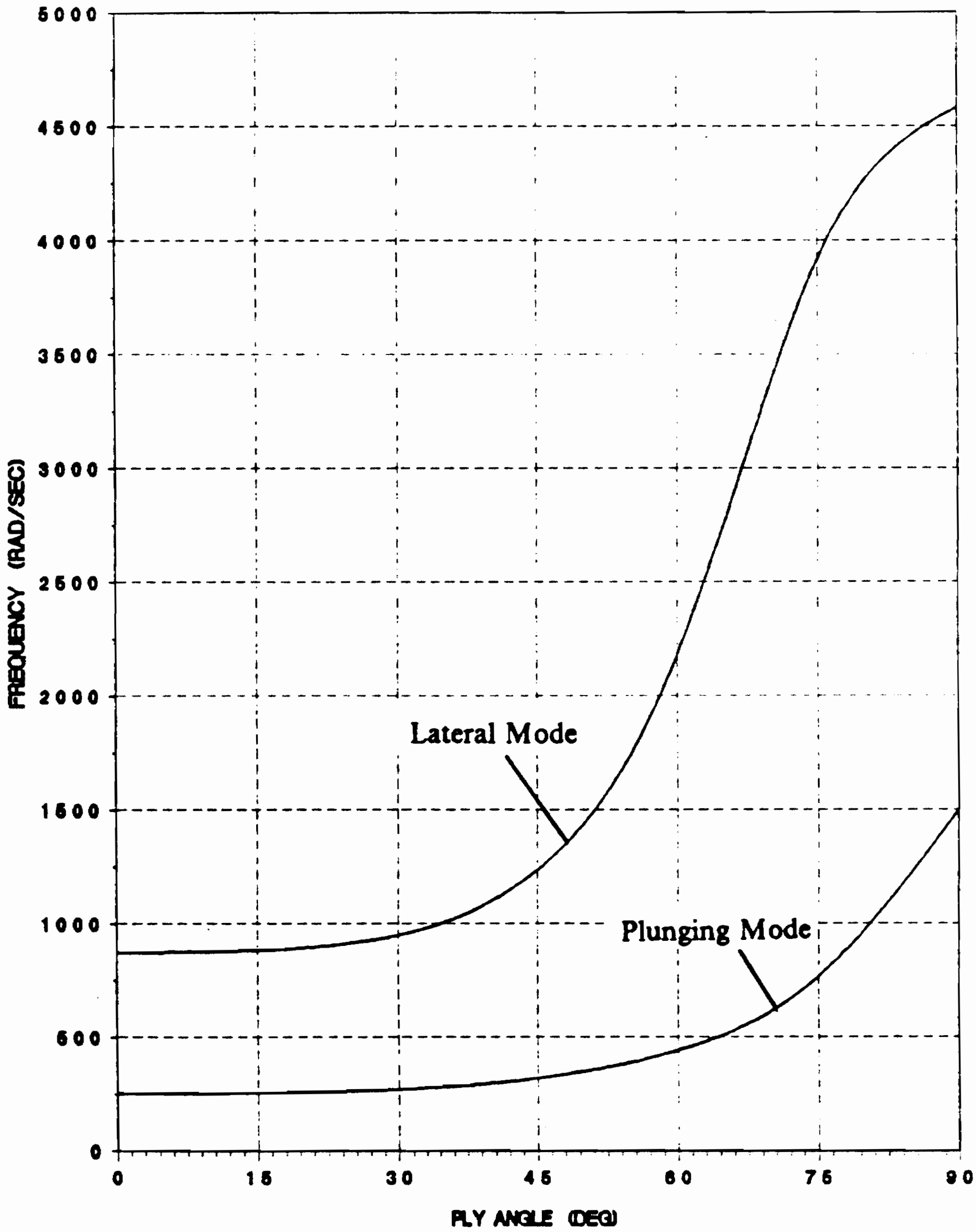


Figure 13. Variation of the Fundamental Eigenfrequencies vs. Ply Angle for Lateral and Plunging Vibration Modes

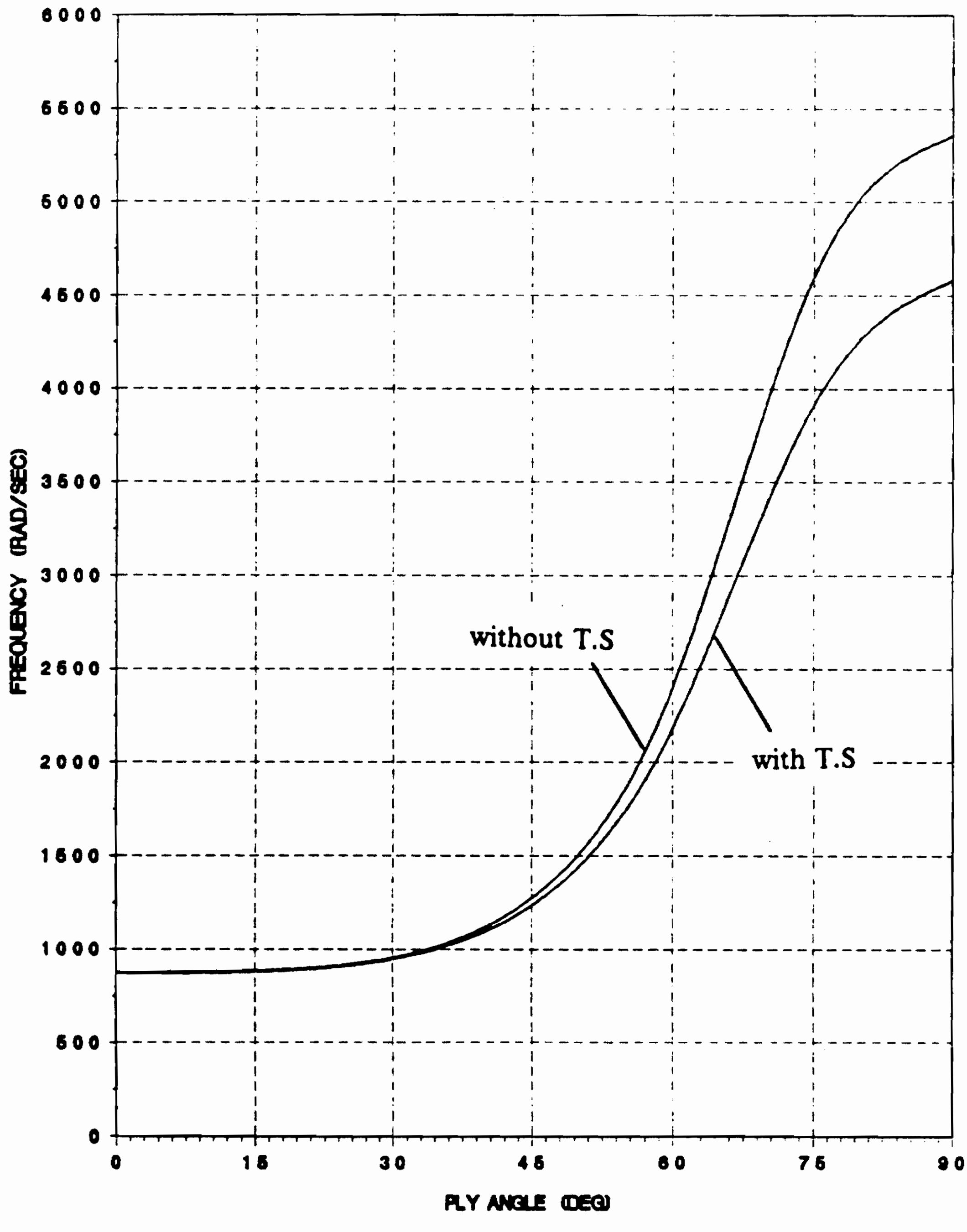


Figure 14. Effect of the Transverse Shear on the Fundamental Eigenfrequencies of the Lateral Mode

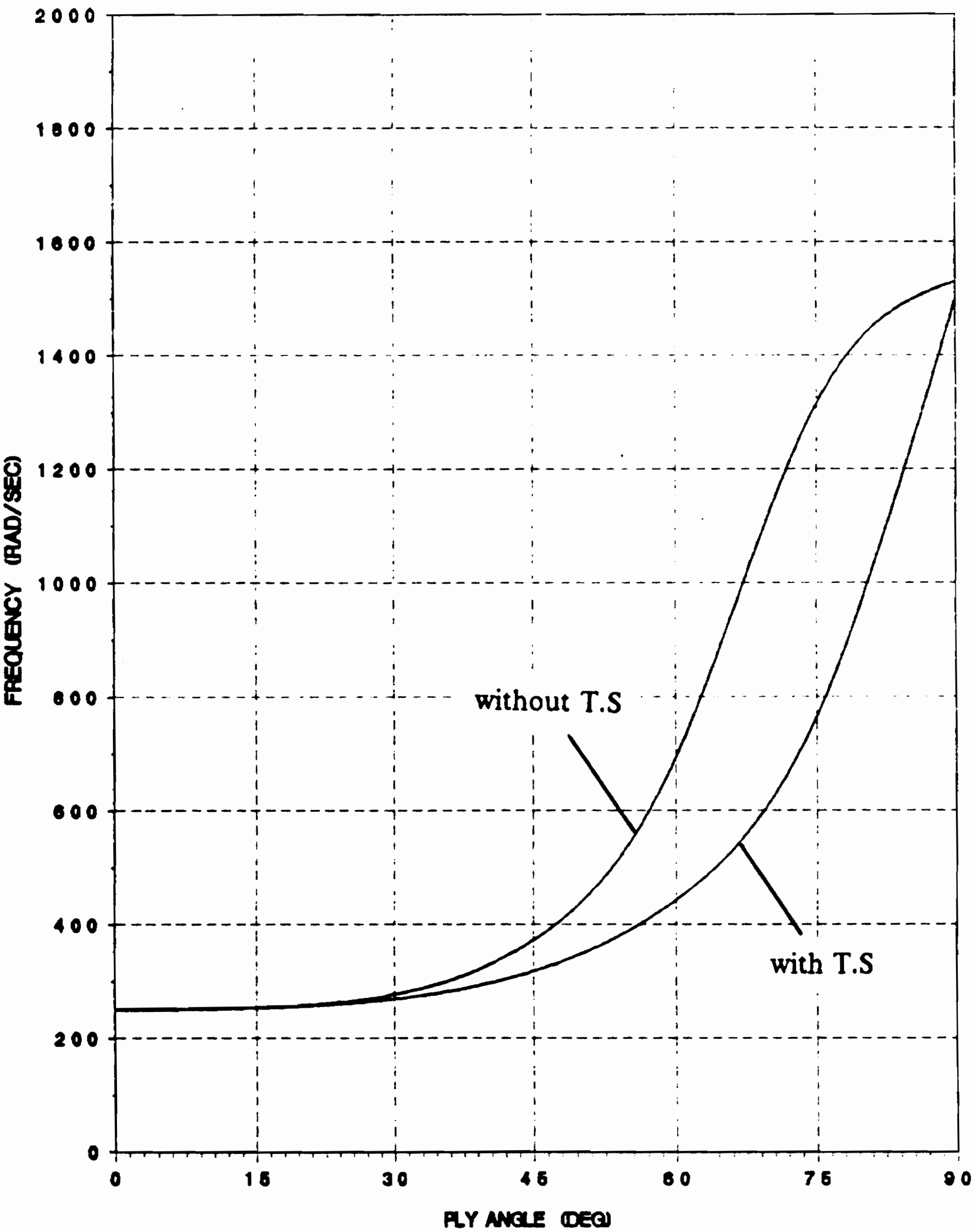


Figure 15. Effect of the Transverse Shear on the Fundamental Eigenfrequencies of the Plunging Mode

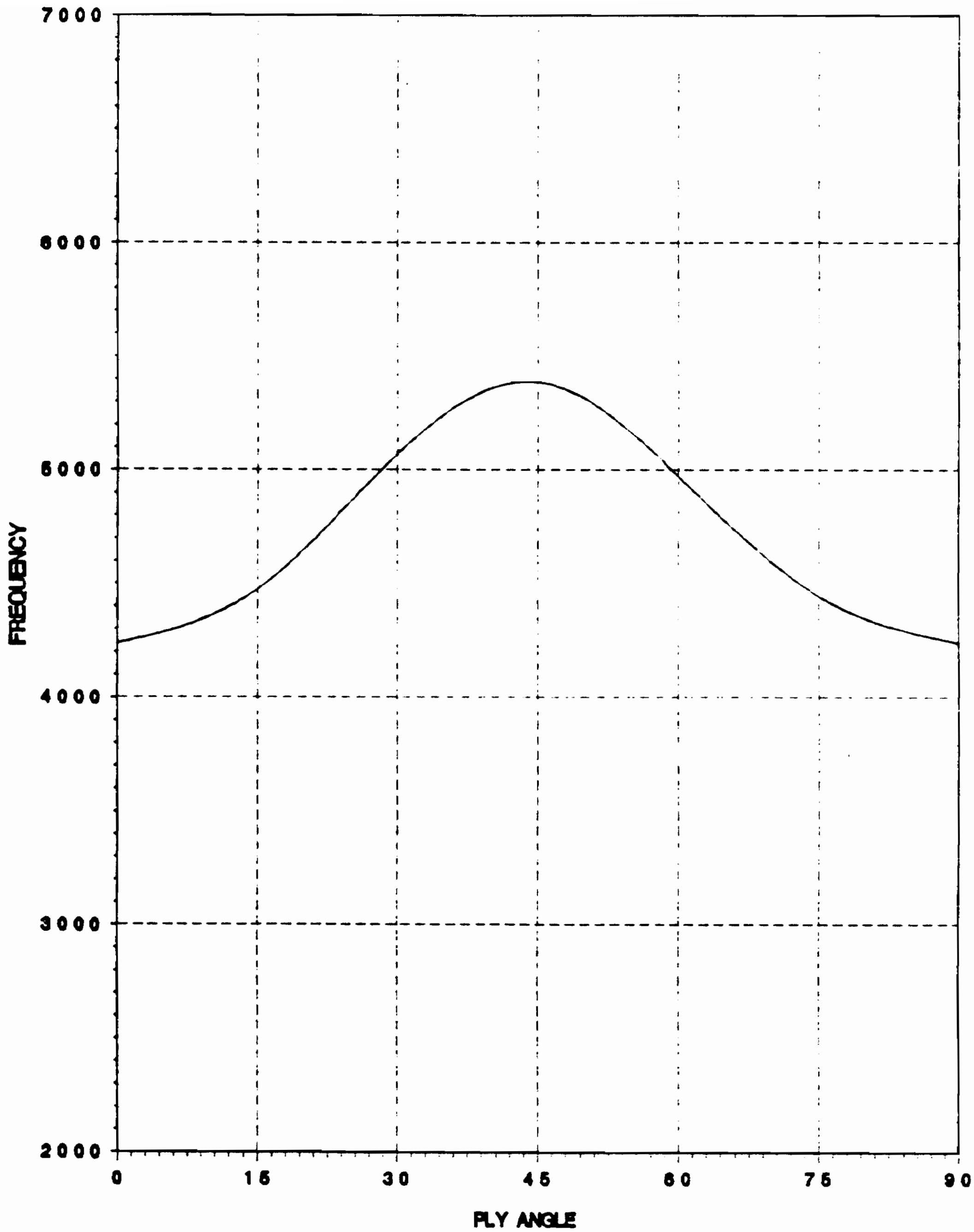


Figure 16. Variation of the Fundamental Eigenfrequencies vs. Ply Angle for the Extension-Torsion Coupled Mode

Table 4. Variation of the Eigenfrequency vs. Ply Angle by Including and Discarding the Secondary Warping Effect (Lateral Mode)

Ply Angle (deg)	Fundamental Frequency	
	I	II
0	870	871
15	880	881
30	947	949
45	1234	1236
60	2177	2182
75	3896	3916
90	4579	4583

unit : rad/sec

I : Without Secondary Warping Effect

II : With Secondary Warping Effect

Table 5. Variation of the Eigenfrequency vs. Ply Angle by Including and Discarding the Secondary Warping Effect (Plunging Mode)

Ply Angle (deg)	Fundamental		Second	
	I	II	I	II
0	248	249	1548	1558
15	251	253	1564	1577
30	264	269	1646	1676
45	306	318	1906	1985
60	425	443	2672	2786
75	749	765	4853	4934
90	1493	1502	8473	8502

unit : rad/sec

I : Without Secondary Warping Effect

II: With Secondary Warping Effect

Table 6. Eigenfrequencies in the Axial Vibration Mode for Two Ply Angles

Ply Angle (deg)	Axial Vibration	
	Fundamental	Second
0	11749	35247
90	72007 (513.0)	216021 (513.0)

( ): % difference

Table 7. Comparison of the Eigenfrequencies of the Torsional Mode Concerning the Warping Restraint Effect (Ply Angle = 90 degree)

	Torsional Vibration Mode	
	Fundamental	Second
F.W	4239	12717
W.R	4640 (8.6)	14595 (13.0)

( ) : % difference

FW : Free Warping Case

WR : Warping Restraint Case

Table 8. Effect of the Transverse Shear on the First Two Eigenfrequencies (Lateral Mode)

ply angle (deg)	fundamental		second	
	I	II	I	II
15	886	880 (-0.6)	5463	4383 (-19.8)
30	955	949 (-0.6)	5898	4657 (-21.0)
45	1276	1236 (-9.0)	7877	5490 (-40.5)
60	2399	2182 (-9.0)	14804	7554 (-49.0)
75	4598	3916 (-14.8)	28380	12333 (-56.5)

unit : rad/sec

( ) : % difference

I : Without Transverse Shear Deformability

II: With Transverse Shear Deformability

Table 9. Effect of the Transverse Shear on the First Two Eigenfrequencies (Plunging Mode)

ply angle (deg)	fundamental		second	
	I	II	I	II
0	250	249 (-0.4)	1564	1558 (-0.4)
15	254	253 (-0.4)	1585	1577 (-0.5)
30	277	269 (-2.9)	1728	1676 (-3.0)
45	373	318 (-14.7)	2335	1985 (-15.0)
60	693	443 (-36.1)	4337	2786 (-35.8)
75	1315	765 (-41.8)	8230	4934 (-40.0)
90	1530	1502 (-1.8)	9576	8502 (-11.2)

unit : rad/sec

( ) : % difference

I : Without Transverse Shear Deformability

II: With Transverse Shear Deformability

Chapter VII

Concluding Remarks and Future Developments

The widespread interest in the employment of advanced composite material systems as well as of thin-walled structures in the various fields of technology and especially in aeronautical/aerospace structures requires a reformulation of the classical theory of thin-walled beams. In spite of the fact that thin-walled beam structures are used in aircraft, space, naval and civil construction, the theory of thin-walled beams is the most under-investigated area in the field of composite thin-walled structures when compared with their composite plate and shell counterparts. This dissertation is intended to incorporate several essential effects which have a considerable importance for the rational design of composite thin-walled beam structures. These effects are the transverse shear deformation, the warping constraint, the secondary warping as well as the hygrothermal and dynamic ones.

The constitutive equations considered in this work are far more general than their counterparts contained in the specialized literature. This fact enables one to analyze and obtain additional conclusions concerning the power of the tailoring technique applied to these kinds of structures.

In this dissertation the field equations of laminated composite thin-walled beams with either open or closed (single and multicell) cross-sections were obtained in a very general and unified fashion through the application of Hamilton's principle. The governing equations in conjunction with the boundary conditions were solved by using the Laplace Transform technique, enabling one to obtain exact solutions aided by a software (MACSYMA) in which symbolic manipulations have been carried out.

The numerical results reveal the great role played on the response characteristics by the non-classical effects accompanying the cases studied here. It should be mentioned that in our numerical examples, which concern the aeroelastic divergence and free vibration problems the considered structure was modelled as a thin-walled box beam manufactured by the filament winding techniques. In this case, due to the double symmetry of the cross-section of the box beam and the antisymmetric lay-up characterizing the filament winding manufacturing technique, the structural coupling between bending and torsion was eliminated. However, in the divergence analysis of swept wings, there is a bending-torsion coupling of an aeroelastic nature.

In order to describe its possible effects on the response characteristics of various problems, the structural bending-torsion coupling will also be incorporated in forthcoming works.

In the greatest majority of works dealing with the theory of thin-walled beams, the cross-section is assumed to be rigid in its own plane. This fact results in the discarding of in-plane strain components. In the future work, the cross-sectional deformability effect allowing for the presence of distorsions in their own plane will also be included.

In another context, pretwisted thin-walled beams have important applications in aircraft and aerospace structures. Propellers, helicopter rotor blades, and turbine blades are usually pretwisted to meet certain aerodynamic requirements. For the pretwisted laminated composite thin-walled beams, the coupling of various elastic deformations results from both the inherent anisotropy of composite materials and the geometrical effect of pretwist. This effect will also be incorporated in future work.

A structural model incorporating various stiffening members such as stringers, webs, etc. should also be developed in the future. Such a model has the potential of improving and enlarging the applicability in the design of future generations of aeronautical/aerospace as well as naval structures made of composite materials.

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Appendix A

Constitutive Relationships of a Lamina Referred to Global Axes

Let (x_1', x_2', x_3') be a material coordinte sytem of the lamina, while the global coordinate system for the laminate is denoted by (s,z,n) or (x_1, x_2, x_3) (see Fig. 17). The constitutive relations for each lamina should be transformed to the global coordinate system in order to obtain the laminate constitutive relations. Considering the transformation of stresses and strains [16,17]

$$\sigma_{mn}' = a_{mi} a_{nj} \sigma_{ij} = a_{mi} \sigma_{ij} a_{nj}^{T} \tag{A.1}$$

and

$$\varepsilon_{mn}' = a_{mi} a_{nj} \varepsilon_{ij} = a_{mi} \varepsilon_{ij} a_{nj}^{T} \tag{A.2}$$

where σ and ε denote the stress and strain tensors, respectively while a_{ij} denotes the transformation matrix. T denotes the operation of transpose of the matrix (m,n,i,j = 1,2,3). As it can be inferred from Fig 17, the direction cosines a_{ij} (i,j = 1,2,3) are given by the ones in Table 10.

Table 10. Directional Cosines

	x_1(s)	x_2(z)	x_3(n)
x'_1	m	n	0
x'_2	-n	m	0
x'_3	0	0	1

The stresses and strains with respect to the material coordinate system can be obtained from Eqs A.1 and A.2. The results can be expressed in a vectorial form by considering the symmetry conditions of stress and strain tensors. The results are

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = [T] \begin{bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{nn} \\ \sigma_{nz} \\ \sigma_{sn} \\ \sigma_{sz} \end{bmatrix} \qquad (A.3)$$

and

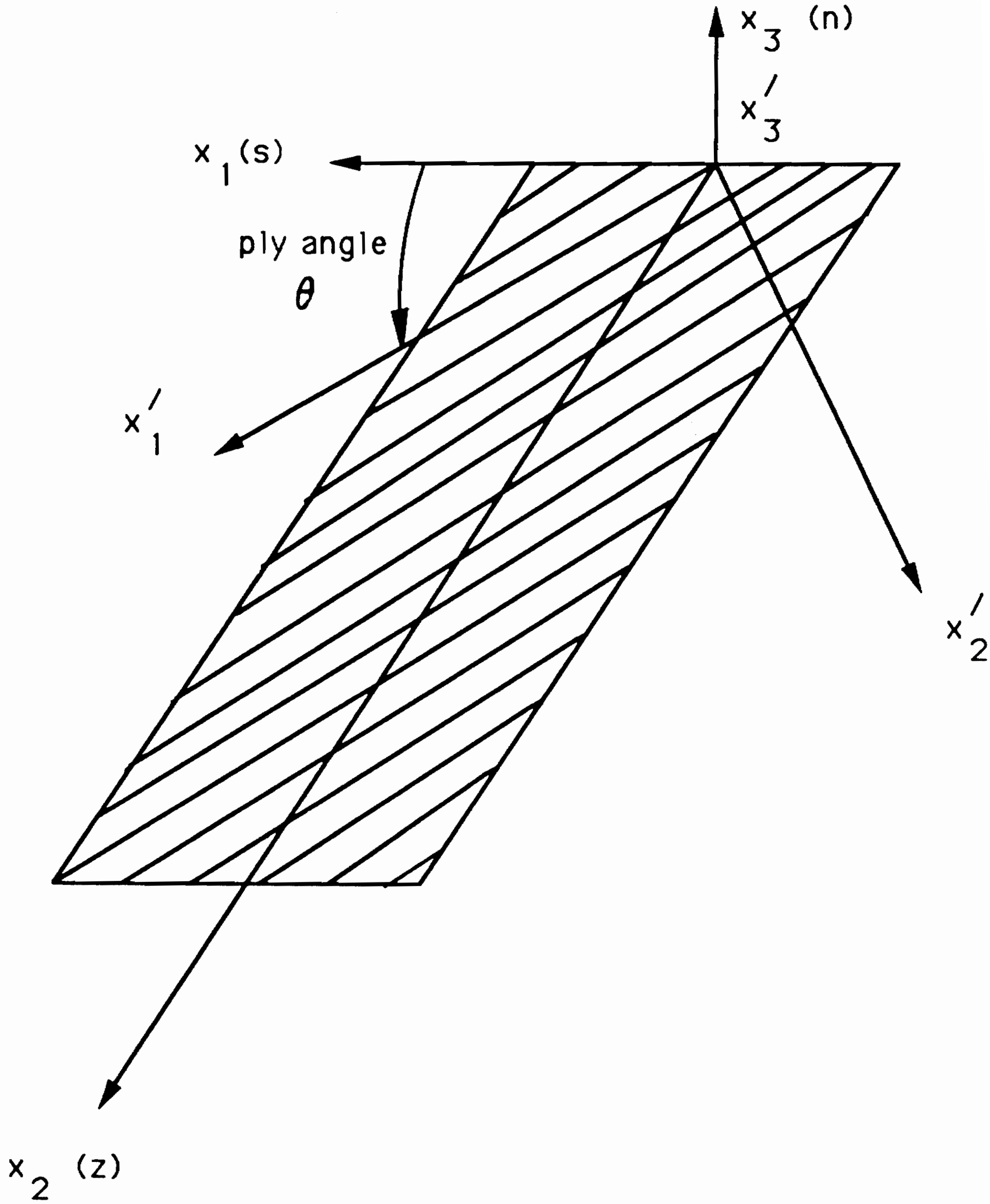

Figure 17. Material and Gloabal Coordinate Systems

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix} = [T] \begin{bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz} \\ \varepsilon_{nn} \\ \varepsilon_{nz} \\ \varepsilon_{sn} \\ \varepsilon_{sz} \end{bmatrix} \tag{A.4}$$

where the transformation matrix [T] is given by

$$[T] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & (m^2 - n^2) \end{bmatrix} \tag{A.5}$$

Inverting Eqs A.3 and A.4 yields

$$\begin{bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{nn} \\ \sigma_{nz} \\ \sigma_{sn} \\ \sigma_{sz} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} \tag{A.6}$$

and

$$\begin{bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz} \\ \varepsilon_{nn} \\ \varepsilon_{nz} \\ \varepsilon_{sn} \\ \varepsilon_{sz} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix} \tag{A.7}$$

where the inverse of [T] is given by

$$[T]^{-1} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & -2mn \\ n^2 & m^2 & 0 & 0 & 0 & 2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & n & 0 \\ 0 & 0 & 0 & -n & m & 0 \\ mn & -mn & 0 & 0 & 0 & (m^2 - n^2) \end{bmatrix} \tag{A.8}$$

Considering the generalized Hooke's law which incorporates the thermal and hygro-thermal effects, the stress strain relations with respect to the material coordinate system may be expressed as:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} - \alpha_1 T - \beta_1 M \\ \varepsilon_{22} - \alpha_2 T - \beta_2 M \\ \varepsilon_{33} - \alpha_3 T - \beta_3 M \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} \tag{A.9}$$

where Q_{ij} denote the stiffnesses in the material coordinate system while α_i and β_i identify the thermal and the hygro-thermal expansion coefficients, respectively. Q_{ij} are related with the engineering constants for an orthotropic material as follows:

$$Q_{11} = \frac{E_{11}(1 - \mu_{23}\mu_{32})}{\Delta} \tag{A.10}$$

$$Q_{22} = \frac{E_{22}(1 - \mu_{31}\mu_{13})}{\Delta} \tag{A.11}$$

$$Q_{33} = \frac{E_{33}(1 - \mu_{12}\mu_{21})}{\Delta} \tag{A.12}$$

$$Q_{12} = \frac{E_{11}(\mu_{21} + \mu_{31}\mu_{23})}{\Delta} \tag{A.13}$$

$$Q_{13} = \frac{E_{11}(\mu_{31} + \mu_{21}\mu_{32})}{\Delta} \tag{A.14}$$

$$Q_{23} = \frac{E_{22}(\mu_{32} + \mu_{12}\mu_{31})}{\Delta} \tag{A.15}$$

$$Q_{44} = G_{23} \tag{A.16}$$

$$Q_{55} = G_{13} \tag{A.17}$$

$$Q_{66} = G_{12} \tag{A.18}$$

where

$$\Delta = 1 - \mu_{12}\mu_{21} - \mu_{23}\mu_{32} - \mu_{31}\mu_{13} - 2\mu_{21}\mu_{32}\mu_{13} \tag{A.19}$$

Eqs A.3 and A.4 can be expressed in contracted form as:

$$\{\sigma\}' = [T]\{\sigma\} \tag{A.20}$$

$$\{\varepsilon\}' = [T]\{\varepsilon\} \tag{A.21}$$

Eqs A.6 and A.7 also can be expressed as

$$\{\sigma\} = [T]^{-1}\{\sigma\}' \tag{A.22}$$

$$\{\varepsilon\} = [T]^{-1}\{\varepsilon\}' \tag{A.23}$$

Eq A.9 in a contracted form is

$$\{\sigma\}' = [Q]\{\varepsilon\}' \tag{A.24}$$

where { } denotes the vector quantity while [ ] denotes the matrix. The prime (′) identifies the quanties defined in the material coordinate system, while the unprimed quantities are defined in the laminate coordinte system.

By considering Eqs A.20 through A.24, Eq A.6 becomes

$$\{\sigma\} = [T]^{-1}\{\sigma\}' = [T]^{-1}[Q][T]\{\varepsilon\} \tag{A.25}$$

Consequently, the constitutive equations for the laminate axes are

$$\{\sigma\} = \overline{[Q]}\{\varepsilon\} \tag{A.26}$$

or in an explicit form

$$\begin{bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{nn} \\ \sigma_{nz} \\ \sigma_{sn} \\ \sigma_{sz} \end{bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{13} & 0 & 0 & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{23} & 0 & 0 & \overline{Q}_{26} \\ \overline{Q}_{13} & \overline{Q}_{23} & \overline{Q}_{33} & 0 & 0 & \overline{Q}_{36} \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{36} & 0 & 0 & \overline{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{ss} - \alpha_s T - \beta_s M \\ \varepsilon_{zz} - \alpha_z T - \beta_z M \\ \varepsilon_{nn} - \alpha_n T - \beta_n M \\ \gamma_{zn} \\ \gamma_{sn} \\ \gamma_{sz} - \alpha_{sz} T - \beta_{sz} M \end{bmatrix} \tag{A.27}$$

where $\overline{Q}_{ij}$ (i,j = 1,6) represent the stiffnesses with respect to the global coordinate system, given by

$$\overline{Q}_{11} = Q_{11}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}n^4 \tag{A.28}$$

$$\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})m^2n^2 + Q_{12}(m^4 + n^4) \tag{A.29}$$

$$\overline{Q}_{13} = Q_{13}m^2 + Q_{23}n^2 \tag{A.30}$$

$$\overline{Q}_{16} = -mn^3Q_{22} + m^3nQ_{11} - mn(m^2 - n^2)(Q_{12} + 2Q_{66}) \tag{A.31}$$

$$\overline{Q}_{22} = Q_{22}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{11}n^4 \tag{A.32}$$

$$\overline{Q}_{23} = n^2Q_{13} + m^2Q_{23} \tag{A.33}$$

$$\overline{Q}_{26} = mn^3Q_{11} - m^3nQ_{22} + mn(m^2 - n^2)(Q_{12} + 2Q_{66}) \tag{A.34}$$

$$\overline{Q}_{33} = Q_{33} \tag{A.35}$$

$$\overline{Q}_{36} = (Q_{13} - Q_{23})mn \tag{A.36}$$

$$\overline{Q}_{44} = Q_{44}m^2 + Q_{55}n^2 \tag{A.37}$$

$$\overline{Q}_{45} = (Q_{55} - Q_{44})mn \tag{A.38}$$

$$\overline{Q}_{55} = Q_{44}n^2 + Q_{55}m^2 \tag{A.39}$$

$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12})m^2n^2 + Q_{66}(m^2 - n^2)^2 \tag{A.40}$$

$$\alpha_s = \alpha_1 m^2 + \alpha_2 n^2 \tag{A.41}$$

$$\alpha_z = \alpha_2 m^2 + \alpha_1 n^2 \tag{A.42}$$

$$\alpha_n = \alpha_3 \tag{A.43}$$

$$\alpha_{sz} = 2(\alpha_1 - \alpha_2)mn \tag{A.44}$$

$$\beta_s = \beta_1 m^2 + \beta_2 n^2 \tag{A.45}$$

$$\beta_z = \beta_2 m^2 + \beta_1 n^2 \tag{A.46}$$

$$\beta_n = \beta_3 \tag{A.47}$$

$$\beta_{sz} = 2(\beta_1 - \beta_2)mn \tag{A.48}$$

where $m \equiv \cos\theta$ and $n \equiv \sin\theta$ and θ denotes the angle between the two coordinate systems measured from the positive s-direction to the positive z-direction (see Fig. 17)

Appendix B

The Expressions of the Operators in the Governing Equations

$$L_{11} = \delta_{nt}[-a_{22}\frac{\partial^4}{\partial z^4} + (a_{42} - s_{42})\frac{\partial^3}{\partial z^3} + (b_5 + \delta_s b_{15})\frac{\partial^4}{\partial z^2 \partial t^2} + (d_{24} + \delta_s d_{15})\frac{\partial^3}{\partial z^2 \partial t}] + \delta_t s_{44}\frac{\partial^2}{\partial z^2} - b_1\frac{\partial^2}{\partial t^2} - d_1\frac{\partial}{\partial t} + (1 - \delta_s)g_5\frac{\partial^2}{\partial z^2} \tag{B.1}$$

$$L_{12} = \delta_{nt}[-a_{23}\frac{\partial^4}{\partial z^4} + (a_{43} - s_{43})\frac{\partial^3}{\partial z^3} + (b_6 - \delta_s b_{13})\frac{\partial^4}{\partial z^2 \partial t^2} + (d_6 - \delta_s d_{13})\frac{\partial^3}{\partial z^2 \partial t}] + \delta_t s_{45}\frac{\partial^2}{\partial z^2} - (1 - \delta_s)g_6\frac{\partial^2}{\partial z^2} \tag{B.2}$$

$$L_{13} = \delta_{nt}\{a_{21}\frac{\partial^3}{\partial z^3} - a_{41}\frac{\partial^2}{\partial z^2} - b_3\frac{\partial^3}{\partial z \partial t^2} - d_{22}x\frac{\partial^2}{\partial z \partial t}\} + s_{41}\frac{\partial^2}{\partial z^2} \tag{B.3}$$

$$L_{14} = s_{42}\frac{\partial^2}{\partial z^2} + s_{44}\frac{\partial}{\partial z} \tag{B.4}$$

$$L_{15} = s_{43}\frac{\partial^2}{\partial z^2} + s_{45}\frac{\partial}{\partial z} \tag{B.5}$$

$$L_{16} = \delta_{nt}\{\delta_{w1}a_{26}\frac{\partial^4}{\partial z^4} + a_{27}\frac{\partial^3}{\partial z^3} - \delta_{w1}a_{46}\frac{\partial^3}{\partial z^3} - a_{47}\frac{\partial^2}{\partial z^2}$$
$$+ \delta_{w1}(b_9 + \delta_s b_{17})\frac{\partial^4}{\partial z^2 \partial t^2} + \delta_{w1}(d_9 + \delta_s d_{17})\frac{\partial^3}{\partial z^2 \partial t}\} \tag{B.6}$$
$$+ \delta_{w1}s_{46}\frac{\partial^3}{\partial z^3} + s_{47}\frac{\partial^2}{\partial z^2} + (\delta_{w1} + \delta_{w2})(b_2\frac{\partial^2}{\partial t^2} + d_2\frac{\partial}{\partial t})$$

$$L_{17} = \delta_{nt}[a_{26}\frac{\partial^3}{\partial z^3} + a_{28}\frac{\partial^2}{\partial z^2} - a_{46}\frac{\partial^2}{\partial z^2} - a_{48}\frac{\partial}{\partial z} + (b_9 + \delta_s b_{17})\frac{\partial^3}{\partial z \partial t^2}$$
$$+ (d_9 + \delta_s d_{17})\frac{\partial^2}{\partial z \partial t}] + s_{46}\frac{\partial^2}{\partial z^2} + s_{48}\frac{\partial}{\partial z} \tag{B.7}$$

$$L_{18} = b_2\frac{\partial^2}{\partial t^2} + d_2\frac{\partial}{\partial t} \tag{B.8}$$

$$L_{21} = \delta_{nt}\{-a_{32}\frac{\partial^4}{\partial z^4} + (a_{52} - s_{52})\frac{\partial^3}{\partial z^3} + (b_6 - \delta_s b_{13})\frac{\partial^4}{\partial z^2 \partial t^2} + (d_6 - \delta_s d_{13})\frac{\partial^3}{\partial z^2 \partial t}\}$$
$$+ \delta_t s_{54}\frac{\partial^2}{\partial z^2} + (1 - \delta_s)g_7\frac{\partial^2}{\partial z^2} \tag{B.9}$$

$$L_{22} = \delta_{nt}\{-a_{33}\frac{\partial^4}{\partial z^4} + (a_{53} - s_{53})\frac{\partial^3}{\partial z^3} + (b_4 + \delta_s b_{14})\frac{\partial^4}{\partial z^2 \partial t^2}$$
$$+ (d_{23} + \delta_s d_{14}).\frac{\partial^3}{\partial z^2 \partial t}\} + \delta_t s_{55}\frac{\partial^2}{\partial z^2} - b_1\frac{\partial^2}{\partial t^2} - d_{19}\frac{\partial}{\partial t} - (1 - \delta_s)g_8\frac{\partial^2}{\partial z^2} \tag{B.10}$$

$$L_{23} = \delta_{nt}(a_{31}\frac{\partial^3}{\partial z^3} - a_{51}\frac{\partial^2}{\partial z^2} - b_2\frac{\partial^3}{\partial z \partial t^2} - d_{21}\frac{\partial^2}{\partial z \partial t}) + s_{51}\frac{\partial^2}{\partial z^2} \tag{B.11}$$

$$L_{24} = s_{52}\frac{\partial^2}{\partial z^2} + s_{54}\frac{\partial}{\partial z} \tag{B.12}$$

$$L_{25} = s_{53}\frac{\partial^2}{\partial z^2} + s_{55}\frac{\partial}{\partial z} \tag{B.13}$$

$$L_{26} = \delta_{nt}\{\delta_{w1} a_{36} \frac{\partial^4}{\partial z^4} + a_{37} \frac{\partial^3}{\partial z^3} - \delta_{w1} a_{56} \frac{\partial^3}{\partial z^3} - a_{57} \frac{\partial^2}{\partial z^2} + \delta_{w1}(b_8 - \delta_s b_{16}) \frac{\partial^4}{\partial z^2 \partial t^2}$$
$$+ \delta_{w1}(d_9 + \delta_s d_{17}) \frac{\partial^3}{\partial z^2 \partial t}\} + \delta_{w1} s_{56} \frac{\partial^3}{\partial z^3} + s_{57} \frac{\partial^2}{\partial z^2} - (\delta_{w1} + \delta_{w2})(b_3 \frac{\partial^2}{\partial t^2} + d_3 \frac{\partial}{\partial t}) \tag{B.14}$$

$$L_{27} = \delta_{nt}[a_{36} \frac{\partial^3}{\partial z^3} + a_{38} \frac{\partial^2}{\partial z^2} - a_{56} \frac{\partial^2}{\partial z^2} - a_{58} \frac{\partial}{\partial z} + (b_8 - \delta_s b_{16}) \frac{\partial^3}{\partial z \partial t^2}$$
$$+ (d_8 - \delta_s d_{16}) \frac{\partial^2}{\partial z \partial t}] + s_{56} \frac{\partial^2}{\partial z^2} + s_{58} \frac{\partial}{\partial z} \tag{B.15}$$

$$L_{28} = - b_3 \frac{\partial^2}{\partial t^2} - d_3 \frac{\partial}{\partial t} \tag{B.16}$$

$$L_{31} = \delta_{nt}\{ - a_{12} \frac{\partial^3}{\partial z^3} + b_3 \frac{\partial^3}{\partial z \partial t^2} + d_{22} \frac{\partial^2}{\partial z \partial t}\} + \delta_t a_{14} \frac{\partial^2}{\partial z^2} \tag{B.17}$$

$$L_{32} = \delta_{nt}\{ - a_{13} \frac{\partial^3}{\partial z^3} + b_2 \frac{\partial^3}{\partial z \partial t^2} + d_{21} \frac{\partial^2}{\partial z \partial t}\} + \delta_t a_{15} \frac{\partial^2}{\partial z^2} \tag{B.18}$$

$$L_{33} = a_{11} \frac{\partial^2}{\partial z^2} - b_1 \frac{\partial^2}{\partial t^2} - d_{20} \frac{\partial}{\partial t} \tag{B.19}$$

$$L_{34} = a_{12} \frac{\partial^2}{\partial z^2} - b_3 \frac{\partial^2}{\partial t^2} - d_{22} \frac{\partial}{\partial t} + a_{14} \frac{\partial}{\partial z} \tag{B.20}$$

$$L_{35} = a_{13} \frac{\partial^2}{\partial z^2} - b_2 \frac{\partial^2}{\partial t^2} - d_{21} \frac{\partial}{\partial t} + a_{15} \frac{\partial}{\partial z} \tag{B.21}$$

$$L_{36} = \delta_{w1}(a_{16} \frac{\partial^3}{\partial z^3} + b_7 \frac{\partial^3}{\partial z \partial t^2} + d_7 \frac{\partial^2}{\partial z \partial t}) + a_{17} \frac{\partial^2}{\partial z^2} \tag{B.22}$$

$$L_{37} = a_{16} \frac{\partial^2}{\partial z^2} + b_7 \frac{\partial^2}{\partial t^2} + d_7 \frac{\partial}{\partial t} + a_{18} \frac{\partial}{\partial z} \tag{B.23}$$

$$L_{38} = b_7 \frac{\partial^3}{\partial z \partial t^2} + d_7 \frac{\partial^2}{\partial z \partial t} \tag{B.24}$$

$$L_{41} = (a_{24} \frac{\partial^2}{\partial z^2} - a_{44} \frac{\partial}{\partial z}) \tag{B.25}$$

$$L_{42} = (a_{25} \frac{\partial^2}{\partial z^2} - a_{45} \frac{\partial}{\partial z}) \tag{B.26}$$

$$L_{43} = a_{21} \frac{\partial^2}{\partial z^2} - a_{41} \frac{\partial}{\partial z} - b_3 \frac{\partial^2}{\partial t^2} - d_{22} \frac{\partial}{\partial t} \tag{B.27}$$

$$L_{44} = a_{22} \frac{\partial^2}{\partial z^2} + (a_{24} - a_{42}) \frac{\partial}{\partial z} - a_{44} - (b_5 + \delta_s b_{15}) \frac{\partial^2}{\partial t^2} - (d_{24} + \delta_s d_{15}) \frac{\partial}{\partial t} \tag{B.28}$$

$$L_{45} = a_{23} \frac{\partial^2}{\partial z^2} + (a_{25} - a_{43}) \frac{\partial}{\partial z} - a_{45} - (b_6 - \delta_s b_{13}) \frac{\partial^2}{\partial t^2} - (d_6 - \delta_s d_{13}) \frac{\partial}{\partial t} \tag{B.29}$$

$$\begin{aligned} L_{46} = \{&a_{26} \frac{\partial^3}{\partial z^3} - a_{46} \frac{\partial^2}{\partial z^2} + (b_9 + \delta_s b_{17}) \frac{\partial^3}{\partial z \partial t^2} + (d_9 + \delta_s d_{17}) \frac{\partial^2}{\partial z \partial t} \} \\ &+ a_{27} \frac{\partial^2}{\partial z^2} - a_{47} \frac{\partial}{\partial z} \end{aligned} \tag{B.30}$$

$$L_{47} = a_{26} \frac{\partial^2}{\partial z^2} + (a_{28} - a_{46}) \frac{\partial}{\partial z} - a_{48} + (b_9 + \delta_s b_{17}) \frac{\partial^2}{\partial t^2} + (d_9 + \delta_s d_{17}) \frac{\partial}{\partial t} \tag{B.31}$$

$$L_{48} = - a_{49} \frac{\partial}{\partial z} + (b_9 + \delta_s b_{17}) \frac{\partial^3}{\partial z \partial t^2} + (d_9 + \delta_s d_{17}) \frac{\partial^2}{\partial z \partial t} \tag{B.32}$$

$$L_{51} = (a_{34} \frac{\partial^2}{\partial z^2} - a_{54} \frac{\partial}{\partial z}) \tag{B.33}$$

$$L_{52} = (a_{35} \frac{\partial^2}{\partial z^2} - a_{55} \frac{\partial}{\partial z}) \tag{B.34}$$

$$L_{53} = a_{31} \frac{\partial^2}{\partial z^2} - a_{51} \frac{\partial}{\partial z} - b_2 \frac{\partial^2}{\partial t^2} - d_{21} \frac{\partial}{\partial t} \tag{B.35}$$

$$L_{54} = a_{32} \frac{\partial^2}{\partial z^2} + (a_{34} - a_{52}) \frac{\partial}{\partial z} - a_{54} - (b_6 - \delta_s b_{13}) \frac{\partial^2}{\partial t^2} - (d_6 - \delta_s d_{13}) \frac{\partial}{\partial t} \tag{B.36}$$

$$L_{55} = a_{33}\frac{\partial^2}{\partial z^2} + (a_{35} - a_{53})\frac{\partial}{\partial z} - a_{55} - (b_4 + \delta_s b_{14})\frac{\partial^2}{\partial t^2} - (d_{23} + \delta_s d_{14}).\frac{\partial}{\partial t} \tag{B.37}$$

$$\begin{aligned} L_{56} = \{&a_{36}\frac{\partial^3}{\partial z^3} - a_{56}\frac{\partial^2}{\partial z^2} + (b_8 - \delta_s b_{16})\frac{\partial^3}{\partial z\partial t^2} \\ &+ (d_8 - \delta_s d_{16})\frac{\partial^2}{\partial z\partial t}\} + a_{37}\frac{\partial^2}{\partial z^2} - a_{57}\frac{\partial}{\partial z} \end{aligned} \tag{B.38}$$

$$L_{57} = a_{36}\frac{\partial^2}{\partial z^2} + (a_{38} - a_{56})\frac{\partial}{\partial z} - a_{58} + (b_8 - \delta_s b_{16})\frac{\partial^2}{\partial t^2} + (d_8 - \delta_s d_{16})\frac{\partial}{\partial t} \tag{B.39}$$

$$L_{58} = -a_{59}\frac{\partial}{\partial z} + (b_8 - \delta_s b_{16})\frac{\partial^3}{\partial z\partial t^2} + (d_8 - \delta_s d_{16})\frac{\partial^2}{\partial z\partial t} \tag{B.40}$$

$$\begin{aligned} L_{61} = \delta_{w1}\{&-\delta_{nt}a_{62}\frac{\partial^4}{\partial z^4} + \delta_t a_{64}\frac{\partial^3}{\partial z^3} - \delta_{nt}a_{72}\frac{\partial^3}{\partial z^3} + \delta_t a_{74}\frac{\partial^2}{\partial z^2} \\ &+ (1 - \delta_s)g_1\frac{\partial^2}{\partial z^2} + \delta_{nt}(b_9 + \delta_s b_{17})\frac{\partial^4}{\partial z^2\partial t^2} + \delta_{nt}(d_9 + \delta_s d_{17})\frac{\partial^3}{\partial z^2\partial t}\} \\ &+ \delta_{w2}\{-\delta_{nt}a_{82}\frac{\partial^3}{\partial z^3} + \delta_t a_{84}\frac{\partial^2}{\partial z^2} + (1 - \delta_s)g_3\frac{\partial^2}{\partial z^2}\} \\ &+ (\delta_{w1} + \delta_{w2})\{b_2\frac{\partial^2}{\partial t^2} + d_2\frac{\partial}{\partial t}) \end{aligned} \tag{B.41}$$

$$\begin{aligned} L_{62} = \delta_{w1}\{&-\delta_{nt}a_{63}\frac{\partial^4}{\partial z^4} + \delta_t a_{65}\frac{\partial^3}{\partial z^3} - \delta_{nt}a_{73}\frac{\partial^3}{\partial z^3} + \delta_t a_{75}\frac{\partial^2}{\partial z^2} \\ &- (1 - \delta_s)g_2\frac{\partial^2}{\partial z^2} + \delta_{nt}(b_8 - \delta_s b_{16})\frac{\partial^4}{\partial z^2\partial t^2} + \delta_{nt}(d_8 - \delta_s d_{16})\frac{\partial^3}{\partial z^2\partial t}\} \\ &+ \delta_{w2}\{-\delta_{nt}a_{83}\frac{\partial^3}{\partial z^3} + \delta_t a_{85}\frac{\partial^2}{\partial z^2} - (1 - \delta_s)g_4\frac{\partial^2}{\partial z^2}\} \\ &- (\delta_{w1} + \delta_{w2})\{b_3\frac{\partial^2}{\partial t^2} + d_3\frac{\partial}{\partial t}) \end{aligned} \tag{B.42}$$

$$L_{63} = \delta_{w1}(a_{61}\frac{\partial^3}{\partial z^3} + a_{71}\frac{\partial^2}{\partial z^2} - b_7\frac{\partial^3}{\partial z\partial t^2} - d_7\frac{\partial^2}{\partial z\partial t}) + \delta_{w2}a_{81}\frac{\partial^2}{\partial z^2} \tag{B.43}$$

$$L_{64} = \delta_{w1}\{a_{62}\frac{\partial^3}{\partial z^3} + (a_{64} + a_{72})\frac{\partial^2}{\partial z^2} + a_{74}\frac{\partial}{\partial z} - (b_9 + \delta_s b_{17})\frac{\partial^3}{\partial z \partial t^2} - (d_9 + \delta_s d_{17})\frac{\partial^2}{\partial z \partial t}\} + \delta_{w2}(a_{82}\frac{\partial^2}{\partial z^2} + a_{84}\frac{\partial}{\partial z}) \tag{B.44}$$

$$L_{65} = \delta_{w2}\{a_{63}\frac{\partial^3}{\partial z^3} + (a_{65} + a_{73})\frac{\partial^2}{\partial z^2} + a_{75}\frac{\partial}{\partial z} - (b_8 - \delta_s b_{16})\frac{\partial^3}{\partial z \partial t^2} - (d_8 - \delta_s d_{16})\frac{\partial^2}{\partial z \partial t}\} + \delta_{w2}(a_{83}\frac{\partial^2}{\partial z^2} + a_{85}\frac{\partial}{\partial z}) \tag{B.45}$$

$$L_{66} = \delta_{w1}[a_{66}\frac{\partial^4}{\partial z^4} + (a_{67} + a_{76})\frac{\partial^3}{\partial z^3} + a_{77}\frac{\partial^2}{\partial z^2} + (b_{10} + b_{18})\frac{\partial^4}{\partial z^2 \partial t^2} + (d_{10} + d_{18})\frac{\partial^3}{\partial z^2 \partial t}] + \delta_{w2}a_{87}\frac{\partial^2}{\partial z^2} - (\delta_{w1} + \delta_{w2})[(b_4 + b_5)\frac{\partial^2}{\partial t^2} + (d_4 + d_5)\frac{\partial}{\partial t}] \tag{B.46}$$

$$L_{67} = a_{86}\frac{\partial^2}{\partial z^2} + a_{88}\frac{\partial}{\partial z} \tag{B.47}$$

$$L_{68} = 0 \tag{B.48}$$

$$L_{71} = -\delta_{nt}[-a_{62}\frac{\partial^3}{\partial z^3} + (a_{82} - a_{72})\frac{\partial^2}{\partial z^2} + (b_9 + \delta_s b_{17})\frac{\partial^3}{\partial z \partial t^2} + (d_9 + \delta_s d_{17})\frac{\partial^2}{\partial z \partial t}] - \delta_t[a_{64}\frac{\partial^2}{\partial z^2} + (a_{74} - a_{84})\frac{\partial}{\partial z}] - (1 - \delta_s)(g_1 - g_3)\frac{\partial}{\partial z} \tag{B.49}$$

$$L_{72} = -\delta_{nt}[-a_{63}\frac{\partial^3}{\partial z^3} + (a_{83} - a_{73})\frac{\partial^2}{\partial z^2} + (b_8 - \delta_s b_{16})\frac{\partial^3}{\partial z \partial t^2} + (d_8 - \delta_s d_{16})\frac{\partial^2}{\partial z \partial t}] - \delta_t[a_{65}\frac{\partial^2}{\partial z^2} + (a_{75} - a_{85})\frac{\partial}{\partial z}] + (1 - \delta_s)(g_2 - g_4)\frac{\partial}{\partial z} \tag{B.50}$$

$$L_{73} = -[a_{61}\frac{\partial^2}{\partial z^2} + (a_{71} - a_{81})\frac{\partial}{\partial z} - b_7\frac{\partial^2}{\partial t^2} - d_7\frac{\partial}{\partial t}] \tag{B.51}$$

$$L_{74} = -[a_{62}\frac{\partial^2}{\partial z^2} + (a_{64} + a_{72} - a_{82})\frac{\partial}{\partial z} + a_{74} - a_{84} - (b_9 + \delta_s b_{17})\frac{\partial^2}{\partial t^2} - (d_9 + \delta_s d_{17})\frac{\partial}{\partial t}] \tag{B.52}$$

$$L_{75} = -[a_{63}\frac{\partial^2}{\partial z^2} + (a_{65} + a_{73} - a_{83})\frac{\partial}{\partial z} + a_{75} - a_{85} - (b_8 - \delta_s b_{16})\frac{\partial^2}{\partial t^2} - (d_8 - \delta_s d_{16})\frac{\partial}{\partial t}] \tag{B.53}$$

$$L_{76} = -a_{67}\frac{\partial^2}{\partial z^2} - (a_{77} + a_{87})\frac{\partial}{\partial z} \tag{B.54}$$

$$L_{77} = -[a_{66}\frac{\partial^2}{\partial z^2} + (a_{68} + a_{76} - a_{86})\frac{\partial}{\partial z} + a_{78} - a_{88} + (b_{10} + b_{18})\frac{\partial^2}{\partial t^2} + (d_{10} + d_{18})\frac{\partial}{\partial t}] \tag{B.55}$$

$$L_{78} = 0 \tag{B.56}$$

$$L_{81} = \delta_{nt}\{-a_{72}\frac{\partial^3}{\partial z^3} + (b_9 + \delta_s b_{17})\frac{\partial^4}{\partial z^2 \partial t^2} + (d_9 + \delta_s d_{17})\frac{\partial^3}{\partial z^2 \partial t}\} + \delta_t a_{74}\frac{\partial^2}{\partial z^2} + (1 - \delta_s)g_1\frac{\partial^2}{\partial z^2} + b_2\frac{\partial^2}{\partial t^2} + d_2\frac{\partial}{\partial t} \tag{B.57}$$

$$L_{82} = \delta_{nt}\{-a_{73}\frac{\partial^3}{\partial z^3} + (b_8 - \delta_s b_{16})\frac{\partial^4}{\partial z^2 \partial t^2} + (d_8 - \delta_s d_{16})\frac{\partial^3}{\partial z^2 \partial t}\} + \delta_t a_{75}\frac{\partial^2}{\partial z^2} - (1 - \delta_s)g_2\frac{\partial^2}{\partial z^2} - b_3\frac{\partial^2}{\partial t^2} - d_3\frac{\partial}{\partial t} \tag{B.58}$$

$$L_{83} = a_{71}\frac{\partial^2}{\partial z^2} - b_7\frac{\partial^3}{\partial z \partial t^2} - d_7\frac{\partial^2}{\partial z \partial t} \tag{B.59}$$

$$L_{84} = \delta_t[a_{72}\frac{\partial^2}{\partial z^2} + a_{74}\frac{\partial}{\partial z} - (b_9 + \delta_s b_{17})\frac{\partial^3}{\partial z \partial t^2} - (d_9 + \delta_s d_{17})\frac{\partial^2}{\partial z \partial t}] \tag{B.60}$$

$$L_{85} = \delta_t[a_{73} \frac{\partial^2}{\partial z^2} + a_{75} \frac{\partial}{\partial z} - (b_8 - \delta_s b_{16}) \frac{\partial^3}{\partial z \partial t^2} - (d_8 - \delta_s d_{16}) \frac{\partial^2}{\partial z \partial t}] \quad (B.61)$$

$$L_{86} = a_{77} \frac{\partial^2}{\partial z^2} \quad (B.62)$$

$$L_{87} = 0 \quad (B.63)$$

$$L_{88} = -(b_4 + b_5) \frac{\partial^2}{\partial t^2} + (b_{10} + b_{18}) \frac{\partial^4}{\partial z^2 \partial t^2} + (d_{10} + d_{18}) \frac{\partial^3}{\partial z^2 \partial t} - (d_4 + d_5) \frac{\partial}{\partial t} \quad (B.64)$$

$$f_1 = \delta_{nt}(-h_2'' + h_4' + m_y') - h_4' + F_1 + p_x \quad (B.65)$$

$$f_2 = \delta_{nt}(-h_3'' + h_5' + m_x') - h_5' + F_2 + p_y \quad (B.66)$$

$$f_3 = -h_1' + F_3 + p_z \quad (B.67)$$

$$f_4 = \delta_t(-h_2' + h_4 + F_4 + m_y) \quad (B.68)$$

$$f_5 = \delta_t(-h_3' + h_5 + F_5 + m_x) \quad (B.69)$$

$$f_6 = \delta_{w1}(-h_6'' - h_7' + b_\omega' + m_z) + \delta_{w2}(m_z - h_8') + F_6 \quad (B.70)$$

$$f_7 = -\delta_{w2}(-h_6' - h_7 + h_8 + F_7 + b_\omega) \quad (B.71)$$

$$f_8 = \delta_{w3}(-h_7' + F_8 + m_z + b_\omega') \quad (B.72)$$

Appendix C

The Expressions of the Operators in the Boundary Conditions

$$M_{11} = \delta_{nt}\{ - a_{22} \frac{\partial^3}{\partial z^3} + (a_{42} - s_{42}) \frac{\partial^2}{\partial z^2} + (b_5 + \delta_s b_{15}) \frac{\partial^3}{\partial z \partial t^2} + (d_{24} + \delta_s d_{15}) \frac{\partial^2}{\partial z \partial t} \} + \delta_t s_{44} \frac{\partial}{\partial z} + (1 - \delta_s) g_5 \frac{\partial}{\partial z} \tag{C.1}$$

$$M_{12} = \delta_{nt}\{ - a_{23} \frac{\partial^3}{\partial z^3} + (a_{43} - s_{43}) \frac{\partial^2}{\partial z^2} + (b_6 - \delta_s b_{13}) \frac{\partial^3}{\partial z \partial t^2} + (d_6 - \delta_s d_{13}) \frac{\partial^2}{\partial z \partial t} \} + \delta_t s_{45} \frac{\partial}{\partial z} - (1 - \delta_s) g_6 \frac{\partial}{\partial z} \tag{C.2}$$

$$M_{13} = \delta_{nt}\{a_{21} \frac{\partial^2}{\partial z^2} - a_{41} \frac{\partial}{\partial z} - b_3 \frac{\partial^2}{\partial t^2} - d_{22} \frac{\partial}{\partial t}\} + s_{41} \frac{\partial}{\partial z} \tag{C.3}$$

$$M_{14} = s_{42} \frac{\partial}{\partial z} + s_{44} \tag{C.4}$$

$$M_{15} = s_{43} \frac{\partial}{\partial z} + s_{45} \tag{C.5}$$

$$M_{16} = \delta_{nt}\{\delta_{w1}a_{26}\frac{\partial^3}{\partial z^3} + a_{27}\frac{\partial^2}{\partial z^2} - \delta_{w1}a_{46}\frac{\partial^2}{\partial z^2} - a_{47}\frac{\partial}{\partial z} + \delta_{w1}(b_9 + \delta_s b_{17})\frac{\partial^3}{\partial z \partial t^2} + \delta_{w1}(d_9 + \delta_s d_{17})\frac{\partial^2}{\partial z \partial t}\} + \delta_{w1}s_{46}\frac{\partial^2}{\partial z^2} + s_{47}\frac{\partial}{\partial z} \tag{C.6}$$

$$M_{17} = \delta_{nt}\{a_{26}\frac{\partial^2}{\partial z^2} + a_{28}\frac{\partial}{\partial z} - a_{46}\frac{\partial}{\partial z} - a_48 + (b_9 + \delta_s b_{17})\frac{\partial^2}{\partial t^2} + (d_9 + \delta_s d_{17})\frac{\partial}{\partial t}\} + s_{46}\frac{\partial}{\partial z} + s_{48} \tag{C.7}$$

$$M_{18} = \delta_{nt}\{-a_{49}\frac{\partial}{\partial z} + (b_9 + \delta_s b_{17})\frac{\partial^3}{\partial z \partial t^2} + (d_9 + \delta_s d_{17})\frac{\partial^2}{\partial z \partial t}\} + s_{49}\frac{\partial}{\partial z} \tag{C.8}$$

$$M_{21} = \delta_{nt}\{-a_{32}\frac{\partial^3}{\partial z^3} + (a_{52} - s_{52})\frac{\partial^2}{\partial z^2} + (b_6 - \delta_s b_{13})\frac{\partial^3}{\partial z \partial t^2} + (d_6 - \delta_s d_{13})\frac{\partial^2}{\partial z \partial t}\} + \delta_t s_{54}\frac{\partial}{\partial z} + (1 - \delta_s)g_7\frac{\partial}{\partial z} \tag{C.9}$$

$$M_{22} = \delta_{nt}\{-a_{33}\frac{\partial^3}{\partial z^3} + (a_{53} - s_{53})\frac{\partial^2}{\partial z^2} + (b_4 + \delta_s b_{14})\frac{\partial^3}{\partial z \partial t^2} + (d_{23} + \delta_s d_{14}).\frac{\partial^2}{\partial z \partial t}\} + \delta_t s_{55}\frac{\partial}{\partial z} - (1 - \delta_s)g_8\frac{\partial}{\partial z} \tag{C.10}$$

$$M_{23} = \delta_{nt}\{a_{31}\frac{\partial^2}{\partial z^2} - a_{51}\frac{\partial}{\partial z} - b_2\frac{\partial^2}{\partial t^2} - d_{21}\frac{\partial}{\partial t}\} + s_{51}\frac{\partial}{\partial z} \tag{C.11}$$

$$M_{24} = s_{52}\frac{\partial}{\partial z} + s_{54} \tag{C.12}$$

$$M_{25} = s_{53}\frac{\partial}{\partial z} + s_{55} \tag{C.13}$$

$$M_{26} = \delta_{nt}\{\delta_{w1}a_{36}\frac{\partial^3}{\partial z^3} + a_{37}\frac{\partial^2}{\partial z^2} - \delta_{w1}a_{56}\frac{\partial^2}{\partial z^2} - a_{57}\frac{\partial}{\partial z} + \delta_{w1}(b_8 - \delta_s b_{16})\frac{\partial^3}{\partial z \partial t^2} + \delta_{w1}(d_8 - \delta_s d_{16})\frac{\partial^2}{\partial z \partial t}\} + \delta_{w1}s_{56}\frac{\partial^2}{\partial z^2} + s_{57}\frac{\partial}{\partial z} \tag{C.14}$$

$$M_{27} = \delta_{nt}\{a_{36}\frac{\partial^2}{\partial z^2} + (a_{38} - a_{56})\frac{\partial}{\partial z} - a_{58} + (b_8 - \delta_s b_{16})\frac{\partial^2}{\partial t^2} + (d_8 - \delta_s d_{16})\frac{\partial}{\partial t}\} + s_{56}\frac{\partial}{\partial z} + s_{58} \quad (C.15)$$

$$M_{28} = s_{59}\frac{\partial}{\partial z} + \delta_{nt}\{-a_{59}\frac{\partial}{\partial z} + (b_8 - \delta_s b_{16})\frac{\partial^3}{\partial z \partial t^2} + (d_8 - \delta_s d_{16})\frac{\partial^2}{\partial z \partial t}\} \quad (C.16)$$

$$M_{31} = -\delta_{nt}a_{12}\frac{\partial^2}{\partial z^2} + \delta_t a_{14}\frac{\partial}{\partial z} \quad (C.17)$$

$$M_{32} = -\delta_{nt}a_{13}\frac{\partial^2}{\partial z^2} + \delta_t a_{15}\frac{\partial}{\partial z} \quad (C.18)$$

$$M_{33} = a_{11}\frac{\partial}{\partial z} \quad (C.19)$$

$$M_{34} = a_{12}\frac{\partial}{\partial z} + a_{14} \quad (C.20)$$

$$M_{35} = a_{13}\frac{\partial}{\partial z} + a_{15} \quad (C.21)$$

$$M_{36} = \delta_{w1}a_{16}\frac{\partial^2}{\partial z^2} + a_{17}\frac{\partial}{\partial z} \quad (C.22)$$

$$M_{37} = a_{16}\frac{\partial}{\partial z} + a_{18} \quad (C.23)$$

$$M_{38} = a_{19}\frac{\partial}{\partial z} \quad (C.24)$$

$$M_{31} = -\delta_{nt}a_{22}\frac{\partial^2}{\partial z^2} + \delta_t a_{24}\frac{\partial}{\partial z} \quad (C.25)$$

$$M_{42} = -\delta_{nt}a_{23}\frac{\partial^2}{\partial z^2} + \delta_t a_{25}\frac{\partial}{\partial z} \quad (C.26)$$

$$M_{43} = a_{21} \frac{\partial}{\partial z} \tag{C.27}$$

$$M_{44} = a_{22} \frac{\partial}{\partial z} + a_{24} \tag{C.28}$$

$$M_{45} = a_{23} \frac{\partial}{\partial z} + a_{25} \tag{C.29}$$

$$M_{46} = \delta_{w1} a_{26} \frac{\partial^2}{\partial z^2} + a_{27} \frac{\partial}{\partial z} \tag{C.30}$$

$$M_{47} = a_{26} \frac{\partial}{\partial z} + a_{28} \tag{C.31}$$

$$M_{48} = a_{29} \frac{\partial}{\partial z} \tag{C.32}$$

$$M_{51} = -\delta_{nt} a_{32} \frac{\partial^2}{\partial z^2} + \delta_t a_{34} \frac{\partial}{\partial z} \tag{C.33}$$

$$M_{52} = -\delta_{nt} a_{33} \frac{\partial^2}{\partial z^2} + \delta_t a_{35} \frac{\partial}{\partial z} \tag{C.34}$$

$$M_{53} = a_{31} \frac{\partial}{\partial z} \tag{C.35}$$

$$M_{54} = a_{32} \frac{\partial}{\partial z} + a_{34} \tag{C.36}$$

$$M_{55} = a_{33} \frac{\partial}{\partial z} + a_{35} \tag{C.37}$$

$$M_{56} = \delta_{w1} a_{36} \frac{\partial^2}{\partial z^2} + a_{37} \frac{\partial}{\partial z} \tag{C.38}$$

$$M_{57} = a_{36} \frac{\partial}{\partial z} + a_{38} \tag{C.39}$$

$$M_{58} = a_{39} \frac{\partial}{\partial z} \tag{C.40}$$

$$\begin{aligned} M_{61} = \{ &- \delta_{nt} a_{62} \frac{\partial^3}{\partial z^3} + \delta_t a_{64} \frac{\partial^2}{\partial z^2} - \delta_{nt} a_{72} \frac{\partial^2}{\partial z^2} + \delta_t a_{74} \frac{\partial}{\partial z} \\ &+ (1 - \delta_s) g_1 \frac{\partial}{\partial z} + \delta_{nt}(b_9 + \delta_s b_{17}) \frac{\partial^3}{\partial z \partial t^2} + \delta_{nt}(d_9 + \delta_s d_{17}) \frac{\partial^2}{\partial z \partial t} \} \\ &+ \delta_{w2}\{ - \delta_{nt} a_{82} \frac{\partial^2}{\partial z^2} + \delta_t a_{84} \frac{\partial}{\partial z} + (1 - \delta_s) g_3 \frac{\partial}{\partial z} \} \end{aligned} \tag{C.41}$$

$$\begin{aligned} M_{62} = \{ &- \delta_{nt} a_{63} \frac{\partial^3}{\partial z^3} + \delta_t a_{65} \frac{\partial^2}{\partial z^2} - \delta_{nt} a_{73} \frac{\partial^2}{\partial z^2} + \delta_t a_{75} \frac{\partial}{\partial z} \\ &- (1 - \delta_s) g_2 \frac{\partial}{\partial z} + \delta_{nt}(b_8 - \delta_s b_{16}) \frac{\partial^3}{\partial z \partial t^2} + \delta_{nt}(d_8 - \delta_s d_{16}) \frac{\partial^2}{\partial z \partial t} \} \\ &+ \delta_{w2}\{ - \delta_{nt} a_{83} \frac{\partial^2}{\partial z^2} + \delta_t a_{85} \frac{\partial}{\partial z} - (1 - \delta_s) g_4 \frac{\partial}{\partial z} \} \end{aligned} \tag{C.42}$$

$$M_{63} = \delta_{w1}(a_{61} \frac{\partial^2}{\partial z^2} + a_{71} \frac{\partial}{\partial z} - b_7 \frac{\partial^2}{\partial t^2} - d_7 \frac{\partial}{\partial t}) + \delta_{w2} a_{81} \frac{\partial}{\partial z} \tag{C.43}$$

$$\begin{aligned} M_{64} = \delta_{w1}\{ &a_{62} \frac{\partial^2}{\partial z^2} + (a_{64} + a_{72}) \frac{\partial}{\partial z} + a_{74} - (b_9 + \delta_s b_{17}) \frac{\partial^2}{\partial t^2} \\ &- (d_9 + \delta_s d_{17}) \frac{\partial}{\partial t} \} + \delta_{w2}(a_{82} \frac{\partial}{\partial z} + a_{84}) \end{aligned} \tag{C.44}$$

$$\begin{aligned} M_{65} = \delta_{w1}\{ &a_{63} \frac{\partial^2}{\partial z^2} + (a_{65} + a_{73}) \frac{\partial}{\partial z} + a_{75} - (b_8 - \delta_s b_{16}) \frac{\partial^2}{\partial t^2} \\ &- (d_8 - \delta_s d_{16}) \frac{\partial}{\partial t} \} + \delta_{w2}(a_{83} \frac{\partial}{\partial z} + a_{85}) \end{aligned} \tag{C.45}$$

$$\begin{aligned} M_{66} = \delta_{w1}\{ &a_{66} \frac{\partial^3}{\partial z^3} + (a_{67} + a_{76}) \frac{\partial^2}{\partial z^2} + a_{77} \frac{\partial}{\partial z} + (b_{10} + b_{18}) \frac{\partial^3}{\partial z \partial t^2} \\ &+ (d_{10} + d_{18}) \frac{\partial^2}{\partial z \partial t} \} + \delta_{w2} a_{87} \frac{\partial}{\partial z} \end{aligned} \tag{C.46}$$

$$M_{67} = a_{86} \frac{\partial}{\partial z} + a_{88} \tag{C.47}$$

$$M_{68} = 0 \tag{C.48}$$

$$M_{71} = -\delta_{nt}a_{62}\frac{\partial^2}{\partial z^2} + \delta_t a_{64}\frac{\partial}{\partial z} \quad \text{(C.49)}$$

$$M_{72} = -\delta_{nt}a_{63}\frac{\partial^2}{\partial z^2} + \delta_t a_{65}\frac{\partial}{\partial z} \quad \text{(C.50)}$$

$$M_{73} = a_{61}\frac{\partial}{\partial z} \quad \text{(C.51)}$$

$$M_{74} = a_{62}\frac{\partial}{\partial z} + a_{64} \quad \text{(C.52)}$$

$$M_{75} = a_{63}\frac{\partial}{\partial z} + a_{65} \quad \text{(C.53)}$$

$$M_{76} = \delta_{w1}a_{66}\frac{\partial^2}{\partial z^2} + a_{67}\frac{\partial}{\partial z} \quad \text{(C.54)}$$

$$M_{77} = a_{66}\frac{\partial}{\partial z} + a_{68} \quad \text{(C.55)}$$

$$M_{78} = 0 \quad \text{(C.56)}$$

$$\begin{aligned} M_{81} = {} & \delta_{nt}\{-a_{72}\frac{\partial^2}{\partial z^2} + (b_9 + \delta_s b_{17})\frac{\partial^3}{\partial z \partial t^2} + (d_9 + \delta_s d_{17})\frac{\partial^2}{\partial z \partial t}\} \\ & + \delta_t a_{74}\frac{\partial}{\partial z} + (1 - \delta_s)g_1\frac{\partial}{\partial z} \end{aligned} \quad \text{(C.57)}$$

$$\begin{aligned} M_{82} = {} & \delta_{nt}\{-a_{73}\frac{\partial^2}{\partial z^2} + (b_8 - \delta_s b_{16})\frac{\partial^3}{\partial z \partial t^2} + (d_8 - \delta_s d_{16})\frac{\partial^2}{\partial z \partial t}\} \\ & + \delta_t a_{75}\frac{\partial}{\partial z} - (1 - \delta_s)g_2\frac{\partial}{\partial z} \end{aligned} \quad \text{(C.58)}$$

$$M_{83} = a_{71}\frac{\partial}{\partial z} - b_7\frac{\partial^2}{\partial t^2} - d_7\frac{\partial}{\partial t} \quad \text{(C.59)}$$

$$M_{84} = a_{72}\frac{\partial}{\partial z} + a_{74} - (b_9 + \delta_s b_{17})\frac{\partial^2}{\partial t^2} - (d_9 + \delta_s d_{17})\frac{\partial}{\partial t} \quad \text{(C.60)}$$

$$M_{85} = a_{73}\frac{\partial}{\partial z} + a_{75} - (b_8 - \delta_s b_{16})\frac{\partial^2}{\partial t^2} - (d_8 - \delta_s d_{16})\frac{\partial}{\partial t} \tag{C.61}$$

$$M_{86} = a_{77}\frac{\partial}{\partial z} \tag{C.62}$$

$$M_{87} = 0 \tag{C.63}$$

$$M_{88} = a_{79}\frac{\partial}{\partial z} + (b_{10} + b_{18})\frac{\partial^3}{\partial z \partial t^2} + (d_{10} + d_{18})\frac{\partial^2}{\partial z \partial t} \tag{C.64}$$

$$m_1 = \delta_{nt}(h_2' - h_4) + h_4 + \tilde{Q}_x \tag{C.65}$$

$$m_2 = \delta_{nt}(h_3' - h_5) + h_5 + \tilde{Q}_y \tag{C.66}$$

$$m_3 = h_1 + \tilde{T}_A \tag{C.67}$$

$$m_4 = h_2 + \tilde{M}_y \tag{C.68}$$

$$m_5 = h_3 + \tilde{M}_x \tag{C.69}$$

$$m_6 = \delta_{w1}(h_6' + h_7 + \tilde{M}_z) + \delta_{w2}(h_8 + \tilde{M}_z) \tag{C.70}$$

$$m_7 = (\delta_{w1} + \delta_{w2})(h_6 + \tilde{B}_\omega) \tag{C.71}$$

$$m_8 = \delta_{w3}(h_7 + \tilde{M}_z) \tag{C.72}$$

Appendix D

The Expressions of the Mass Coefficients

$$b_1 = \int_C \rho h ds \tag{D.1}$$

$$b_2 = \int_C \rho h y ds \tag{D.2}$$

$$b_3 = \int_C \rho h x ds \tag{D.3}$$

$$b_4 = \int_C \rho h y^2 ds \tag{D.4}$$

$$b_5 = \int_C \rho h x^2 ds \tag{D.5}$$

$$b_6 = \int_C \rho h x y ds \tag{D.6}$$

$$b_7 = \int_C \rho h \widetilde{F}_\omega ds \tag{D.7}$$

$$b_8 = \int_C \rho h y \widetilde{F}_\omega ds \tag{D.8}$$

$$b_9 = \int_C \rho h x \widetilde{F}_\omega ds \tag{D.9}$$

$$b_{10} = \int_C \rho h \widetilde{F}_\omega{}^2 ds \tag{D.10}$$

$$b_{11} = \int_C \frac{dx}{ds} hds \tag{D.11}$$

$$b_{12} = \int_C \frac{dy}{ds} hds \tag{D.12}$$

$$b_{13} = \int_C \frac{dx}{ds}\frac{dy}{ds} hds \tag{D.13}$$

$$b_{14} = \int_C \frac{dx}{ds}\frac{dx}{ds} hds \tag{D.14}$$

$$b_{15} = \int_C \frac{dy}{ds}\frac{dy}{ds} hds \tag{D.15}$$

$$b_{16} = \int_C \tilde{a} \frac{dx}{ds} hds \tag{D.16}$$

$$b_{17} = \int_C \tilde{a} \frac{dy}{ds} hds \tag{D.17}$$

$$b_{18} = \int_C \rho h \tilde{a}^2 hds \tag{D.18}$$

Appendix E

The Expressions of the Coefficients in the Damping Term

$$d_1 = \int_C C_1 h ds \tag{E.1}$$

$$d_2 = \int_C C_1 y h ds \tag{E.2}$$

$$d_3 = \int_C C_2 x h ds \tag{E.3}$$

$$d_4 = \int_C C_1 y^2 h ds \tag{E.4}$$

$$d_5 = \int_C C_2 x^2 h ds \tag{E.5}$$

$$d_6 = \int_C C_3 xyhds \tag{E.6}$$

$$d_7 = \int_C C_3 \widetilde{F}_\omega hds \tag{E.7}$$

$$d_8 = \int_C y\widetilde{F}_\omega hds \tag{E.8}$$

$$d_9 = \int_C x\widetilde{F}_\omega hds \tag{E.9}$$

$$d_{10} = \int_C \widetilde{F}_\omega{}^2 h\, ds \tag{E.10}$$

$$d_{11} = \int_C C_3 \frac{h^3}{12} \frac{dx}{ds} ds \tag{E.11}$$

$$d_{12} = \int_C C_3 \frac{h^3}{12} \frac{dy}{ds} ds \tag{E.12}$$

$$d_{13} = \int_C C_3 \frac{h^3}{12} \frac{dx}{ds} \frac{dy}{ds} ds \tag{E.13}$$

$$d_{14} = \int_C C_3 \frac{h^3}{12} \frac{dx}{ds} \frac{dx}{ds} ds \tag{E.14}$$

$$d_{15} = \int_C C_3 \frac{h^3}{12} \frac{dy}{ds} \frac{dy}{ds} ds \tag{E.15}$$

$$d_{16} = \int_C C_3 \frac{h^3}{12} \tilde{a} \frac{dx}{ds} ds \tag{E.16}$$

$$d_{17} = \int_C C_3 \frac{h^3}{12} \tilde{a} \frac{dy}{ds} ds \tag{E.17}$$

$$d_{18} = \int_C C_3 \frac{h^3}{12} \tilde{a}^2 ds \tag{E.18}$$

$$d_{19} = \int_C C_2 h ds \tag{E.19}$$

$$d_{20} = \int_C C_3 h ds \tag{E.20}$$

$$d_{21} = \int_C C_3 y h ds \tag{E.21}$$

$$d_{22} = \int_C C_3 x h ds \tag{E.22}$$

$$d_{23} = \int_C C_3 y^2 h ds \tag{E.23}$$

$$d_{24} = \int_C C_3 x^2 h ds \tag{E.24}$$

Appendix F

The Expressions of the Thermal and Hygral Loading Terms

$$h_1 = \int_C (N_1{}^T + N_1{}^M) ds \tag{F.1}$$

$$h_2 = \int_C [x(N_1{}^T + N_1{}^M) + \delta_s \frac{dy}{ds} (N_4{}^T + N_4{}^M)] ds \tag{F.2}$$

$$h_3 = \int_C [y(N_1{}^T + N_1{}^M) - \delta_s \frac{dx}{ds} (N_4{}^T + N_4{}^M)] ds \tag{F.3}$$

$$h_4 = \int_C (N_2{}^T + N_2{}^M) \frac{dx}{ds} ds \tag{F.4}$$

$$h_5 = \int_C (N_2{}^T + N_2{}^M) \frac{dy}{ds} ds \tag{F.5}$$

$$h_6 = \int_C [\tilde{F}_\omega(N_1{}^T + N_1{}^M) + \tilde{a}(N_4{}^T + N_4{}^M)]ds \tag{F.6}$$

$$h_7 = \int_C [2\delta_o(N_5{}^T + N_5{}^M) + \delta_c(N_2{}^T + N_2{}^M)]\ ds \tag{F.7}$$

$$h_8 = \int_C r_n(N_2{}^T + N_2{}^M)\ ds \tag{F.8}$$

Appendix G

Several Elements of the Laplace Integral Transform

The Laplace integral transform method is used to solve exactly the governing equations along with the boundary conditions for the aeroelastic divergence and eigenfrequency problems considered as numerical examples in this dissertation. The underlying idea of the Laplace transform is to replace complex problems with simple ones by transforming the differential equations into algebraic ones. Several elements related to the Laplace transform technique [33,39-42] applied to the specific examples considered here are given.

The Laplace transform of the n-th order derivative of the generic function $\Phi = \Phi(z)$ is

$$\mathscr{L}\{\Phi^{(n)}(z)\} = s^n f(s) - s^{n-1}\Phi(0) - s^{n-2}\Phi^{(1)}(0) - s^{n-3}\Phi^{(2)}(0) \cdots\cdot \\ -s\,\Phi^{(n-2)}(0) - \Phi^{(n-1)}(0) \tag{G.1}$$

where s denotes the Laplace transformed domain while $f(s) \equiv \mathscr{L}\{\Phi(z)\}$ and

$$\Phi^{(n-1)}(0) \equiv \frac{\partial^{(n-1)}\Phi}{\partial z^{n-1}} \tag{G.2}$$

By using Eq G.2, a set of differential equations can be reduced to a system of simultaneous algebraic equations which can be solved in the usual manner. At this stage, the boundary conditions at the root (z = 0) are applied to replace the unknowns produced by the Laplace transform with the known boundary conditions at the root. In the transformed domain (s-domain), the simultaneous algebraic equations can be solved. Afterward, the solutions in the s - domain, i.e., $\tilde{\Phi}(s)$, should be inversely transformed to the actual (original) domain. To perform the inverse transform, we need an important rule known as the Heaviside expansion theorem regarding the inverse Laplace transform of partial fractioned expressions as follows:

G.1 When There is no Repeated Root of the Denominator of the Expression

In this case, the partial fraction of f(s) is expressed as

$$f(s) = \frac{N(s)}{D(s)} = \sum_{k=1}^{n} \frac{A_k}{s - r_k} \tag{G.3}$$

where N(s) and D(s) are polynomials in s and have no common factors between them, r_k and n denote the k-th root and the order of the polynomial D(s), respectively.

The inverse transform is

$$\mathscr{L}^{-1}(f) = \sum_{k=1}^{n} A_k e^{r_k z} \tag{G.4}$$

where A_k is given by either of the two expressions

$$A_k = \Pi_{r_k}(r_k) \tag{G.5}$$

or

$$A_k = \frac{N(s)}{\left[\frac{dD(s)}{ds}\right]_{s=r_k}} \tag{G.6}$$

where

$$\Pi_{r_k}(s) = \frac{(s - r_k)N(s)}{D(s)} \tag{G.7}$$

G.2 When There are Repeated Roots of the Denominator of the Expression

In this case, the partial fraction of f(s) is

$$f(s) = \frac{A_1}{(s - r_k)} + \frac{A_2}{(s - r_k)^2} + \frac{A_{m-1}}{(s - r_k)^{m-1}} + \frac{A_m}{(s - r_k)^m} + R(s) \tag{G.8}$$

where r_k is the repeated root while R(s) denotes the remainder. The inverse transform of this expression is

$$\mathscr{L}^{-1}(f) = e^{r_k z}[A_1 + A_2 \frac{z}{1!} + \cdots + A_m \frac{z^{m-1}}{(m-1)!}] + \mathscr{L}^{-1}(R) \tag{G.9}$$

where

$$A_m = \Pi_{r_k}(r_k) \tag{G.10}$$

$$A_p = \frac{1}{(m-p)!} \frac{d^{m-p}\Pi_{r_k}(s)}{ds^{m-p}} \Bigg]_{s=r_k} \qquad (p = 1,2,......,m-1) \tag{G.11}$$

here,

$$\Pi_{r_k}(s) = \frac{(s-r_k)^m N(s)}{D(s)} \tag{G.12}$$

here r_k can be a real or complex number.

Vita

The author was born in 1954 in Puyeo, Korea. He received his B.S degree in Mechanical Engineering from the Seoul National University in 1978. Upon graduation, he worked for the Agency for Defense Development in Korea from 1978 to 1984 as a research engineer. He was then admitted to the Master's program in Mechanical Engineering at the New Jersey Institute of Technology, New Jersey, in 1984. After completing his Master's degree in New Jersey, he transferred to Virginia Polytechnic Institute and State University in January, 1987. Since then, he has been working towards the degree of Doctor of Philosophy in Engineering Mechanics. He is a student member of AIAA and ASME as well as of Tau Beta Pi. The author is married to Bo K. Chun and has one daughter, Jae-Young.

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