

Supersymmetric Backgrounds in string theory

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(ABSTRACT)

In the first part of this dissertation, we investigate a way to find the complex structure moduli, for a given background of type IIB string theory in the presence of flux in special cases. We introduce a way to compute the complex structure and axion dilaton moduli explicitly. In the second part, we discuss $(0, 2)$ supersymmetric versions of some recent exotic $\mathcal{N} = (2, 2)$ supersymmetric gauged linear sigma models, describing intersections of Grassmannians. In the next part, we consider mirror symmetry for certain gauge theories with gauge groups F_4 , E_6 , and E_7 . In the last part of this dissertation, we study whether certain branched-double-cover constructions in Landau-Ginzburg models can be extended to higher covers.

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(GENERAL AUDIENCE ABSTRACT)

This dissertation concerns string theory, a proposal for unification of general relativity and quantum field theory. In string theory, the building block of all the particles are strings, such that different vibrations of them generate particles. String theory predicts that spacetime is 10-dimensional. In string theorist's intuition, the extra six-dimensional internal space is so small that we haven't detected it yet. The physics that string theory predicts we should observe, is governed by the shape of this six-dimensional space called a 'compactification manifold.' In particular, the possible ways in which this geometry can be deformed give rise to light degrees of freedom in the associated observable physical theory. In the first part of this dissertation, we determine these degrees of freedom, called moduli, for a large class of solutions of the so-called type IIB string theory. In the second part, we focus on constructing such spaces explicitly. We also show that there can be different equivalent ways of constructing the same internal space. The third part of the dissertation concerns mirror symmetry. Two compactification manifolds are called mirror to each other, when they both give the same four-dimensional effective theory. In this part, we describe the mirror of two-dimensional gauge theories with F_4 , E_6 , and E_7 gauge group, using the Gu-Sharpe proposal.

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Chapter 1

Introduction

1.1 Standard Model

There are four fundamental forces in nature, i.e. the gravitational, electromagnetic, weak and strong nuclear forces. Three of the fundamental forces (the electromagnetic, weak, and strong interactions) are described by the Standard Model of particle physics. It classifies all the known particles and even miraculously predicted the top quark (1995), tau neutrino (2000) and the Higgs boson (2012), the existence of the latter of which was confirmed by the LHC at CERN. Generally speaking, the Standard Model is a four-dimensional gauge theory with $SU(3) \times SU(2) \times U(1)$ gauge symmetry. It describes the electromagnetic and weak interactions in a combined framework called the electroweak interaction, and is described by an $SU(2) \times U(1)$ gauge group, describing the leptons, i.e electrons, muons, taus, and their neutrinos. On the other hand, quarks transform in a multiplet of the $SU(3)$ color group.

Although the Standard Model is beautiful, it suffers from many problems:

- (1) Gravity is not described within the Standard Model. Attempts to include gravity lead to a non-renormalizable theory.
- (2) The Standard Model is not a unified theory of the strong and electroweak interactions. One should introduce a bigger gauge group G e.g $SU(5)$ or $SO(10)$, containing $SU(3) \times SU(2) \times U(1)$ as a subgroup to which the full symmetry is broken at low energies.
- (3) The Standard Model does not contain feasible dark matter or dark energy that possesses all of the properties predicted by observational cosmology.
- (4) The prediction of the Standard Model for the cosmological constant differs from the measured value by 120 orders of magnitude!
- (5) There are around twenty independent parameters in the standard Model, that describe the fermions masses, gauge couplings, the mixing angles and the Higgs mass and its self coupling. These parameters should be measured and put into the theory. This makes the

Standard Model seem arbitrary.

(6) The Standard Model has massless neutrinos and it should be modified to reflect the nonzero masses measured in neutrino oscillation experiments.

For a theorist, the Standard Model is a playground of quantum field theory which exhibits a wide range of physical phenomena. Many have tried to extend it, in attempts to solve some of the problems listed above. One possible extension is to introduce supersymmetry. This means that for any boson there is a related fermion and vice versa. The Standard Model is not symmetric under such a transformation, so all the fields in a supersymmetric Standard Model will require a new field, known as a superpartner. The recipe is to add a new boson for each fermion and a new fermion for each boson. The most used supersymmetric extension of the Standard Model is called the *Minimal Supersymmetric Standard Model* (MSSM) which possesses $\mathcal{N} = 1$ supersymmetry. The superpartners of gauge fields are called gauginos, the superpartners of quarks are called squarks, the superpartners of leptons are called sleptons, and also the superpartner of the Higgs is called the Higgsino. If SUSY exists at high energy, it has to be broken at some energy above the observed scales and so superpartners of observed particles should be heavier than observed particles at current energy scales.

One might ask why it is desirable to introduce such a weird symmetry. This is because SUSY has several nice features:

(1) Introducing SUSY causes all the three gauge couplings, miraculously, to meet exactly at one point at an energy scale around 10^{16} GeV, the grand unification scale.

(2) If the SUSY breaking scale is at the TeV scale, then SUSY can also be the answer to Hierarchy problem for Higgs mass. The question is why the Higgs boson mass 125 GeV is so much lighter than Planck mass, $M_P \approx 10^{19}$ GeV, or grand unification scale. Quantum contributions to the Higgs mass are very large and would make it huge. In the Standard Model, an incredible fine-tuning cancellation is required to make it work. Supersymmetry explains how the Higgs mass can be protected from quantum corrections. In SUSY, for any fermion there is a related boson. So for example, the fermionic top quark loop correction to the Higgs mass would be canceled by the bosonic squark loop correction.

(3) The third problem that can be solved using SUSY is dark matter. The lightest superparticle of the MSSM is stable and massive. Then, if it is neither strongly nor electromagnetically interacting, it can be a candidate for dark matter.

Although the MSSM seems to solve a lot of problems, it has about one hundred parameters and one would ask about the origin of those parameters [1].

1.2 String theory

String theory was created by people trying to explain the spectrum of hadrons and their interactions in the late 1960s. The starting point of string theory is to replace a point particle with a finite length string. Later, this theory was rejected due to the existence of a massless spin two particle in it. In 1974, Scherk and Schwartz realized that this spin two particle can be regarded as the graviton, which turned string theory to a quantum theory of gravity. They also showed that these spin two particles in the low energy limit interact according to the laws of general relativity. String theory is built on the axiom that the building block of all the particles are strings and different vibrations of these strings produce different particles. It is also conjectured that the scale of string theory is between the scale of grand unification of the Standard Model ($\approx 10^{16}$ GeV/c) and the Planck scale ($\approx 10^{19}$ GeV), which is much higher than the energy scale of the current high energy physics experiments.

Bosonic string theory only contains bosonic strings. The main problem with it is that in addition to lacking fermions, it contains states with negative mass squared. The more realistic string theory is superstring theory. Requiring the theory be anomaly-free under conformal transformations, teaches us that strings move in a ten-dimensional space-time. String theory in this critical dimension obviously fails to reproduce the fact that we live in four dimensions. To overcome this problem, it has been suggested that a six-dimensional extra internal space is attached to each point of four-dimensional space-time, so tiny and compact that it has not been directly observed yet. This is called a compactification of string theory. Physicists explore possible six-dimensional manifolds that in the low-energy limit give us observed four-dimensional physics.

There are five different supersymmetric string theories in ten dimensions, known as the type I theory, the two heterotic theories with $E_8 \times E_8$ and $SO(32)$ gauge groups, and the type IIA and IIB theories. Before the 1990s, it was believed that the five superstring theories are distinct. String theorists thought that one of these theories, the heterotic $E_8 \times E_8$ theory, was the actual theory of everything. It is now known that the five superstring theories are different limits of a more fundamental theory, called M-theory, which is a theory in 11-dimensions. Five different types of the superstring theories are related by dualities. If two theories are related by a duality transformation, each observable of the first theory can be mapped to an observable of the second one which yields equivalent predictions.

In this context, maybe the most famous duality is T-duality. For example, in type IIA and IIB, if we take one of the nine spatial dimensions to be a circle, then type IIA on a circle of radius R is dual to type IIB on a circle of radius l_s^2/R , where l_s is the fundamental length of the string. A string traveling in the circular direction will have quantized momentum around the circle. In addition to traveling around the circle, a string may also wrap around it. The number of times the string winds around the circle is called the winding number. T-duality exchanges the momentum and the winding number of the dual theories. T-duality is actually the starting point of mirror symmetry which will be discussed more in detail

later. If the manifold M is mirror to W , then compactifying type IIA on M is dual to compactifying type IIB on W . The other duality in string theory is the S-duality. If two string theories are related by S-duality, then one theory at strong coupling is the same as the other theory at weak coupling. The interesting point about S-duality is that, a theory at strong coupling can not be understood by means of perturbation theory, but a theory at weak coupling can. Superstring theories related by S-duality are: type I superstring theory with heterotic $SO(32)$ superstring theory, and type IIB theory with itself. Also, in type IIA string theory at strong coupling, the dilaton field plays the role of the radius of an eleventh dimension of the M theory.

1.3 Introduction to $\mathcal{N} = (2, 2)$ gauge theories in 2D

$\mathcal{N} = (2, 2)$ supersymmetric gauge theories can be used to construct nonlinear sigma models (NLSMs) and Landau-Ginzburg (LG) models. A gauged linear sigma model (GLSM) can flow to either a NLSM or a LG model at low energies. Nontrivial low energy limits are superconformal field theories and can be used for the compactification of string theory.

1.3.1 $\mathcal{N} = (2, 2)$ Superfield Formalism

$\mathcal{N} = (2, 2)$ supersymmetric theories in two dimensions have two left-moving and two right-moving supersymmetry parameters. In this section we will focus on $\mathcal{N} = (2, 2)$ gauged linear sigma models in two dimensions. These theories can be driven from dimensional reduction of $\mathcal{N} = 1$ gauge theories in four dimensions.

The most efficient way to describe these theories, such that the supersymmetry is manifest, is to use the concept of *superspace*. Superspace is the space such that in addition to bosonic coordinates x^0 and x^1 , it contains four fermionic coordinates,

$$\theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-.$$

Mathematically they are called complex Grassmann numbers, and are related to each other by complex conjugation, $(\theta^\pm)^\star = \bar{\theta}^\pm$. Grassmann numbers have the property that

$$\theta^\alpha \theta^\beta = -\theta^\beta \theta^\alpha.$$

So, $\mathcal{N} = (2, 2)$ *superspace* is the space with coordinates $x^0, x^1, \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-$. Functions defined on the superspace, are called *superfields*. They can be expanded in monomials in θ^\pm and $\bar{\theta}^\pm$:

$$\mathcal{F}(x^0, x^1, \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-) = f_0(x^0, x^1) + \theta^+ f_+(x^0, x^1) + \theta^- f_-(x^0, x^1) + \dots$$

Obviously, because of the Grassmann property of some of the variables, the equation above has sixteen terms. $\mathcal{N} = (2, 2)$ supersymmetry have four generators, and they can be stated in superspace by

$$\begin{aligned} \mathcal{Q}_\pm &= \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_\pm, \\ \bar{\mathcal{Q}}_\pm &= -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm\partial_\pm, \end{aligned} \tag{1.1}$$

where ∂_\pm are derivatives with respect to $x^\pm := x^0 \pm x^1$. These generators satisfy the anti-commutation relations

$$\{\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm\} = -2i\partial_\pm,$$

with all the other anticommutation relations vanishing. It is also useful to introduce further collection of operators in superspace

$$\begin{aligned} \mathcal{D}_\pm &= \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\partial_\pm, \\ \bar{\mathcal{D}}_\pm &= -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm, \end{aligned}$$

which anticommute with \mathcal{Q}_\pm and $\bar{\mathcal{Q}}_\pm$, i.e., $\{\mathcal{D}_\pm, \mathcal{Q}_\pm\} = 0$, etc. We will use these operators to classify superfields and define different types of superfields.

The simplest type of superfields are chiral superfields Φ , that satisfy

$$\bar{\mathcal{D}}_\pm\Phi = 0.$$

They encode data about matter fields and they can be expanded:

$$\begin{aligned} \Phi(x, \theta) &= \phi - i\theta^+\bar{\theta}^+\partial_+\phi - i\theta^-\bar{\theta}^-\partial_-\phi - \theta^+\theta^-\bar{\theta}^-\bar{\theta}^+\partial_+\partial_-\phi \\ &\quad + \sqrt{2}\theta^+\psi_+ + \sqrt{2}\theta^-\psi_- - i\sqrt{2}\theta^+\theta^-\bar{\theta}^-\partial_-\psi_+ - i\sqrt{2}\theta^-\theta^+\bar{\theta}^+\partial_+\psi_- + \theta^+\theta^-F. \end{aligned}$$

In the expression above, ϕ and F are bosonic fields and ψ_\pm are fermionic superpartner. The complex conjugate of Φ is an antichiral multiplet $\bar{\Phi}$ with a similar expansion.

To encode the gauge fields and their superpartners, the gauginos, we need another type of superfield, known as a vector superfields. Vector superfields are Lie-algebra-valued functions V on superspace obeying $V = \bar{V}$. A $U(1)$ gauge transformation, for a vector superfield V and a chiral superfield Φ_i of charge Q_i are the following

$$\begin{aligned} V &\rightarrow V + i(A - \bar{A}), \\ \Phi_i &\rightarrow \exp(-iQ_iA)\Phi_i. \end{aligned} \tag{1.2}$$

where A is a chiral superfield.

If one chooses a suitable gauge, which is called Wess-Zumino gauge, then one can expand the vector superfield for a $U(1)$ GLSM in the following form

$$V = \theta^- \bar{\theta}^- (v_0 - v_1) + \theta^+ \bar{\theta}^+ (v_0 + v_1) - \theta^- \bar{\theta}^+ \sigma - \theta^+ \bar{\theta}^- \bar{\sigma} \\ + i\theta^- \theta^+ (\bar{\theta}^- \bar{\lambda}_- + \bar{\theta}^+ \bar{\lambda}_+) + i\bar{\theta}^+ \bar{\theta}^- (\theta^- \lambda_- + \theta^+ \lambda_+) + \theta^- \theta^+ \bar{\theta}^+ \bar{\theta}^- D. \quad (1.3)$$

In Wess-Zumino gauge, we can still choose $A = -a(x)$, where $a(x)$ is a real function. Then, gauge transformations will be the familiar transformations for the gauge field and matter fields

$$v \rightarrow v - da, \\ \phi_i \rightarrow \exp(iQ_i a) \phi_i.$$

One novelty in two dimensions, relative to four, is that in addition to chiral superfields we have also twisted chiral superfields, $\bar{\mathcal{D}}_+ \Sigma = \mathcal{D}_- \Sigma = 0$. An important twisted chiral superfield is defined by

$$\Sigma := \frac{1}{\sqrt{2}} \bar{\mathcal{D}}_+ \mathcal{D}_- V. \quad (1.4)$$

It includes information about the field strength of gauge fields. It is Lie algebra valued generally and in the abelian case can be written as

$$\Sigma = \sigma - i\sqrt{2}\theta^+ \lambda_+ - i\sqrt{2}\theta^- \lambda_- + \sqrt{2}\theta^+ \theta^- (D - iv_{01}) \\ - i\theta^- \theta^- (\partial_0 - \partial_1) \sigma - i\theta^+ \theta^+ (\partial_0 + \partial_1) \sigma + \sqrt{2}\bar{\theta}^- \theta^+ \theta^- (\partial_0 - \partial_1) \bar{\lambda}_+ \\ + \sqrt{2}\theta^+ \bar{\theta}^- \theta^+ (\partial_0 + \partial_1) \lambda_- - \theta^+ \bar{\theta}^- \theta^- \bar{\theta}^+ (\partial_0^2 - \partial_1^2) \sigma,$$

where $v_{01} = \partial_0 v_1 - \partial_1 v_0$ is the field strength of the gauge field.

1.3.2 The Lagrangian

The Lagrangian of an $\mathcal{N} = (2, 2)$ supersymmetric gauge theory can be divided into separate parts

$$L = L_{kin} + L_{gauge} + L_W + L_{\bar{W}}, \quad (1.5)$$

where the four terms are the kinetic energy of chiral superfields, the superpotential interaction, the kinetic energy of gauge fields, and the Fayet-Iliopoulos term and theta angle, and will be stated in the following.

The kinetic energy of chiral superfields is

$$L_{kin} = \int d^2 x d^4 \theta \sum_i \Phi_i e^{2Q_i V} \Phi_i,$$

and can be expanded to

$$\begin{aligned}
L_{kin} = \sum_i \int d^2x & \left(-D_\mu \bar{\phi}_i D^\mu \phi_i + i\bar{\psi}_{-,i}(D_0 + D_1)\psi_{-,i} + i\bar{\psi}_{+,i}(D_0 - D_1)\psi_{+,i} + |F_i|^2 \right. \\
& - 2\bar{\sigma}\sigma Q_i^2 \bar{\phi}_i \phi_i - \sqrt{2}Q_i (\bar{\sigma}\bar{\psi}_{+,i}\psi_{-,i} + \sigma\bar{\psi}_{-,i}\psi_{+,i}) + DQ_i \bar{\phi}_i \phi_i \\
& \left. - i\sqrt{2}Q_i \bar{\phi}_i (\psi_{-,i}\lambda_+ - \psi_{+,i}\lambda_-) - i\sqrt{2}Q_i \phi_i (\bar{\lambda}_-\bar{\psi}_{+,i} - \bar{\lambda}_+\bar{\psi}_{-,i}) \right),
\end{aligned}$$

where D_μ are the covariant derivatives

$$D_\mu := \partial_\mu + iv_\mu.$$

The gauge kinetic term is

$$L_{gauge} = -\frac{1}{4e^2} \int d^2x d^4\theta \bar{\Sigma}\Sigma,$$

where e is the gauge coupling constant. In components this is

$$\begin{aligned}
L_{gauge} = \frac{1}{e^2} \int d^2x & \left(-\partial_\mu \sigma \partial^\mu \bar{\sigma} + \frac{1}{2}v_{01}^2 + \frac{1}{2}D^2 \right. \\
& \left. + i\bar{\lambda}_+(\partial_0 - \partial_1)\lambda_+ + i\bar{\lambda}_-(\partial_0 + \partial_1)\lambda_- \right).
\end{aligned}$$

The other part of the Lagrangian involving chiral superfields is constructed from a gauge-invariant holomorphic function W of the chiral superfields, known as the superpotential, and they are usually a polynomial of chiral superfields. The superpotential part of the Lagrangian is

$$L_W = - \int d^2x d\theta^+ d\theta^- W(\Phi_i)|_{\theta^+ = \theta^- = 0} - h.c. \quad ,$$

and can be expanded to

$$L_W = - \int d^2x \left(F_i \frac{\partial W}{\partial \phi_i} + \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \psi_{-,i} \psi_{+,j} \right) - h.c. \quad .$$

Similarly, we can introduce a holomorphic twisted superpotential $\tilde{W}(\Sigma)$ and write:

$$L_{\tilde{W}} = \int d^2x d\theta^+ d\bar{\theta}^- \tilde{W}(\Sigma)|_{\theta^- = \bar{\theta}^+ = 0} + h.c. \quad ,$$

and it can be expanded to

$$L_{\tilde{W}} = \int d^2x \left(\tilde{W}'(\sigma)(D - iv_{01}) + \tilde{W}''(\sigma)\bar{\lambda}_+\lambda_- \right) + h.c. \quad .$$

There is an special case where $\tilde{W} = -t\Sigma$ and $t = r - i\theta$ is called the Fayet-Iliopoulos parameter. In this case we get the famous Fayet-Iliopoulos term:

$$L_{D,\theta} = \int d^2x (-rD + \theta v_{01}).$$

1.3.3 $\mathcal{N} = (2, 2)$ supersymmetry transformation

The gauge theory explained in the previous section is an example of an $\mathcal{N} = (2, 2)$ supersymmetric theory. The supersymmetry variation is given by

$$\delta = \epsilon_+ \mathcal{Q}_- - \epsilon_- \mathcal{Q}_+ - \bar{\epsilon}_+ \bar{\mathcal{Q}}_- + \bar{\epsilon}_- \bar{\mathcal{Q}}_+,$$

where \mathcal{Q}_\pm and $\bar{\mathcal{Q}}_\pm$ are the differential operators given in equation (1.1).

In the gauge transformation (1.2), for a general chiral superfield A , the Wess-Zumino gauge is not preserved, which means V would not take the form (1.3). In order to find the supersymmetry transformation of field A should be chosen wisely so after the gauge transformation of V , the result takes the form (1.3). Then, the desired form for A is

$$A = i\theta^+(\bar{\epsilon}_+\bar{\sigma} + \bar{\epsilon}_-(v_0 + v_1)) - i\theta^-(\bar{\epsilon}_-\sigma + \bar{\epsilon}_+(v_0 - v_1)) + \theta^+\theta^-(\bar{\epsilon}_-\bar{\lambda}_+ - \bar{\epsilon}_+\bar{\lambda}_-) + \dots,$$

where \dots are the derivative terms needed to make A chiral. Using this choice for A , the supersymmetry transformations of component fields of vector multiplet will be:

$$\begin{aligned} \delta v_\pm &= i\bar{\epsilon}_\pm \lambda_\pm + i\epsilon_\pm \bar{\lambda}_\pm, \\ \delta \sigma &= -i\bar{\epsilon}_+ \lambda_- - i\epsilon_- \bar{\lambda}_+, \\ \delta D &= -\bar{\epsilon}_+ \partial_- \lambda_+ - \bar{\epsilon}_- \partial_+ \lambda_- + \epsilon_+ \partial_- \bar{\lambda}_+ + \epsilon_- \partial_+ \bar{\lambda}_-, \\ \delta \lambda_+ &= i\epsilon_+(D + iF_{01}) + 2\epsilon_- \partial_+ \bar{\sigma}, \\ \delta \lambda_- &= i\epsilon_-(D - iF_{01}) + 2\epsilon_+ \partial_- \sigma. \end{aligned}$$

For charged chiral multiplets, they are:

$$\begin{aligned} \delta \phi &= \epsilon_+ \psi_- - \epsilon_- \psi_+, \\ \delta \psi_+ &= i\bar{\epsilon}_-(D_0 + D_1)\phi + \epsilon_+ F - \bar{\epsilon}_+ \bar{\sigma} \phi, \\ \delta \psi_- &= -i\bar{\epsilon}_+(D_0 - D_1)\phi + \epsilon_- F + \bar{\epsilon}_- \sigma \phi, \\ \delta F &= -i\bar{\epsilon}_+(D_0 - D_1)\psi_+ - i\bar{\epsilon}_-(D_0 + D_1)\psi_- \\ &\quad + \bar{\epsilon}_+ \bar{\sigma} \psi_- + \bar{\epsilon}_- \sigma \psi_+ + i(\bar{\epsilon}_- \bar{\lambda}_+ - \bar{\epsilon}_+ \bar{\lambda}_-)\phi. \end{aligned}$$

Since the system has the supersymmetry, we have four conserved charges, where are called Q_\pm and \bar{Q}_\pm . In addition to supersymmetry the system can have a vector R-symmetry and axial R-symmetry defined by

$$\begin{aligned} e^{i\alpha F_V} : \mathcal{F}(x^\mu, \theta^\pm, \bar{\theta}^\pm) &\mapsto e^{i\alpha q_V} : \mathcal{F}(x^\mu, e^{-i\alpha} \theta^\pm, e^{i\alpha} \bar{\theta}^\pm), \\ e^{i\beta F_A} : \mathcal{F}(x^\mu, \theta^\pm, \bar{\theta}^\pm) &\mapsto e^{i\beta q_A} : \mathcal{F}(x^\mu, e^{\mp i\beta} \theta^\pm, e^{\pm i\beta} \bar{\theta}^\pm). \end{aligned}$$

Also, in any Poincaré invariant quantum field theory, we will also have a Hamiltonian, momentum and angular momentum

$$H, P, M,$$

which are Noether charges for the time translation $\partial/\partial x^0$, spatial rotation $\partial/\partial x^1$ and Lorentz rotation $x^0\partial/\partial x^1 + x^1\partial/\partial x^0$ in two dimensions. If these symmetries are not lost in the quantum theory (which means there is not anomaly for these symmetries), we get the following (anti)commutation relations for these generators, which is called the $\mathcal{N} = (2, 2)$ supersymmetry algebra:

$$\begin{aligned}
Q_+^2 = Q_-^2 = \bar{Q}_+^2 = \bar{Q}_-^2 &= 0, \\
\{Q_\pm, \bar{Q}_\pm\} &= H \pm P, \\
\{\bar{Q}_+, \bar{Q}_-\} &= Z, & \{Q_+, Q_-\} &= Z^*, \\
\{Q_-, \bar{Q}_+\} &= \tilde{Z}, & \{Q_+, \bar{Q}_-\} &= \tilde{Z}^*, \\
[iM, Q_\pm] &= \mp Q_\pm, & [iM, \bar{Q}_\pm] &= \mp \bar{Q}_\pm, \\
[iF_V, Q_\pm] &= -iQ_\pm, & [iF_V, \bar{Q}_\pm] &= i\bar{Q}_\pm, \\
[iF_A, Q_\pm] &= \mp iQ_\pm, & [iF_A, \bar{Q}_\pm] &= \pm i\bar{Q}_\pm.
\end{aligned} \tag{1.6}$$

In the above, Z and \tilde{Z} are called central charges and commute with all operators in the theory. Z must be zero if F_V is conserved and \tilde{Z} is zero if F_A is conserved.

1.3.4 Mirror Symmetry

In the $\mathcal{N} = (2, 2)$ algebra (1.6), there is a special \mathbb{Z}_2 symmetry, which leaves the algebra invariant and it is

$$\begin{aligned}
Q_- &\longleftrightarrow \bar{Q}_-, \\
F_V &\longleftrightarrow F_A, \\
Z &\longleftrightarrow \tilde{Z},
\end{aligned} \tag{1.7}$$

without changing other operators. Suppose there are two different $\mathcal{N} = (2, 2)$ supersymmetric theories such that their supersymmetry algebra are related by above mirror map. Then we call these two theories mirror to each other. Thus, in a pair of mirror symmetric theories, a chiral multiplet of one theory is mapped to a twisted chiral multiplet of the other theory and vice versa. Also, if the axial R-symmetry in one theory is unbroken, then we can deduce that the vector R-symmetry of the other theory is unbroken and vice versa.

1.3.5 Nonlinear sigma models/Landau-Ginzburg correspondence

Gauged linear sigma models can RG flow to nonlinear sigma models and (hybrid) Landau-Ginzburg models (Appendix A) at low energies. As an example, consider a $U(1)$ gauge theory described in section 1.5. The fields D and F_i in the Lagrangian are auxiliary fields

and can be integrated out. By solving the equations of motion we get

$$D = -e^2 \left(\sum_i Q_i |\phi_i|^2 - r \right),$$

$$F_i = \frac{\partial W}{\partial \phi_i}.$$

The potential energy for the dynamical scalar fields ϕ_i , σ is

$$U(\phi_i, \sigma) = \frac{1}{2e^2} D^2 + \sum_i |F_i|^2 + 2|\sigma|^2 \sum_i Q_i^2 |\phi_i|^2. \quad (1.8)$$

Let us focus on a $U(1)$ gauge theory with five chiral superfields Φ_i of charge 1 and one superfield P of charge -5 , with gauge-invariant superpotential

$$W = PG(\Phi_1, \dots, \Phi_5),$$

where G is a homogeneous degree five polynomial. We also assume G is *transverse*, which means the system of equations

$$\frac{\partial G}{\partial \phi_1} = \dots = \frac{\partial G}{\partial \phi_5} = 0,$$

only have one solution, which is $\phi_i = 0$ trivially. This property ensures that in a case such that $[(\phi_1, \dots, \phi_5)] \in \mathbb{CP}^4$, then $G(\phi_i) = 0$ defines a smooth hypersurface in \mathbb{CP}^4 . It will be clear why we demand this property shortly.

For this GLSM, the potential is

$$U(\phi_i, p) = \frac{1}{2e^2} D^2 + |G|^2 + |p|^2 \sum_{i=1}^5 \left| \frac{\partial G}{\partial \phi_i} \right|^2 + 2|\sigma|^2 \left(\sum_{i=1}^5 |\phi_i|^2 + 5^2 |p|^2 \right),$$

with

$$D = -e^2 \left(\sum_{i=1}^5 |\phi_i|^2 - 5|p|^2 - r \right).$$

Now, let's see what is the low energy physics of this GLSM. By low energy physics, we mean: we take a vacuum, which minimize the above potential and then we integrate out all the massive fields of the theory. We will see that the low energy physics depends on the FI parameter r .

First, we consider the case where $r \gg 0$ (by $r \gg 0$ we actually mean r is much more bigger than the scale of the theory). Because the potential is a sum of positive definite terms, the minimum of the potential is zero and occurs when each term is zero separately.

In the case at hand, requiring $D = 0$ implies that not all the Φ_i can vanish. That being so, vanishing of $|p|^2 \sum_i |\partial_i G|^2$, implies that $p = 0$ (by assuming transversality of G). Therefore, the vanishing of the D term implies

$$|\phi_1|^2 + \cdots + |\phi_5| = r.$$

The space of solutions to the equation above, modulo $U(1)$ gauge transformations, is \mathbb{P}^4 . So, not all the ϕ_i 's can vanish and, demanding last term in the potential to be zero, implies $\sigma = 0$. Finally, $G(\phi_i) = 0$ implies that the space of classical vacua is isomorphic to the hypersurface $\mathbb{P}^4[5]$. It is well known that a smooth hypersurface of degree five in \mathbb{P}^4 is a Calabi-Yau manifold known as the quintic. When we expand the theory around any point of the vacua, gauge fields v_μ , σ , and p get masses proportional to r . Also the modes of ϕ_i which are perpendicular to \mathbb{P}^4 get masses and the only massless fields are the modes of ϕ_i which are tangent to \mathbb{P}^4 . After integrating out massive fields we get a non-linear sigma model (Appendix A) with \mathbb{P}^4 as its target space.

Now we analyze the case where $r \ll 0$. Still the minimum of the potential happens at zero, so all the positive definite terms should vanish separately. Vanishing of the D term requires that p can not vanish. Because, p is nonzero vanishing of

$$|p|^2 \sum_{i=1}^5 \left| \frac{\partial G}{\partial \phi_i} \right|^2$$

requires $\partial G / \partial \phi_i = 0$ for all ϕ_i . Assuming G to be transverse, results all ϕ_i to vanish. Also from the last term in the potential, because $p \neq 0$, then $\sigma = 0$. So in this case $|p| = \sqrt{-r/5}$. Using a suitable gauge transformation, we can fix the argument of the complex number p to be zero. Then, we get a unique vacuum in this case, which is

$$p = \sqrt{-r/5}, \quad \phi_1 = \cdots = \phi_5 = \sigma = 0.$$

Because the charge of p is not minimal, this choice of vacuum does not break the $U(1)$ gauge completely, instead it breaks it to the group \mathbb{Z}_5 . In the case, after integrating out massive modes, only the superfields Φ_i survive and we get a Landau-Ginzburg \mathbb{Z}_5 orbifold with the following superpotential.

$$W_{\text{eff}} = \sqrt{\frac{-r}{5}} G(\phi_i).$$

In both cases $r \gg 0$ and $r \ll 0$, one should also consider quantum corrections to the nonlinear sigma model and Landau-Ginzburg orbifold above.

1.4 Introduction to ten-dimensional supergravity

In this section we will mainly follow Washington Taylor's notes [2], as well as the prestigious textbooks in string theory by Green, Schwarz, Witten [3], Polchinski [4], and Becker, Becker, Schwarz [5].

When we look at the dynamics of the massless spectrum of worldsheet string theory, at first order in the α' expansion, they obey a 10-dimensional supergravity action. In this section we will give an introduction supergravity in 10 dimensions and in particular we will describe type IIB supergravity.

1.4.1 Supersymmetry in higher dimensions

In previous sections we reviewed different types of supersymmetry in two dimensions. Similarly, we can define supersymmetric theories in higher dimensions. As you have seen before, under supersymmetry transformations, fermions transform to bosons and vice versa. Specifically, we are interested in theories for which supersymmetry can be considered as an extension of the Poincaré symmetry group. At the level of algebra, the Lie algebra of the Poincaré group can be extended to the super Lie algebra of a supersymmetric theory. We demand that the fermionic generators of supersymmetry Q_α, \bar{Q}_α , satisfy the anti-commutation relations

$$\{Q_\alpha, \bar{Q}_\beta\} = 2P_M \Gamma_{\alpha\beta}^M,$$

where α and β are spinor indices. The anti-commutation relations above can be generalized to theories with multiple supersymmetries and central charges:

$$\{Q_\alpha^A, \bar{Q}_\beta^B\} = 2\delta^{AB} P_M \Gamma_{\alpha\beta}^M + Z^{AB} \delta_{\alpha\beta}.$$

In the above $A = 1, \dots, \mathcal{N}$, where \mathcal{N} is the number of supersymmetries and Z^{AB} is the central charge of the theory which commutes with all the other generators. The reader can find an explicit representation of gamma matrices in various dimensions in Polchinski's book [4].

We can parameterize supersymmetry transformations by a spinor parameter ϵ^A . For infinitesimal supersymmetry transformation we can write:

$$\delta\phi \sim \bar{\epsilon}_A \psi^A, \quad \delta\psi^A \sim \Gamma^M \epsilon^A \partial_M \phi.$$

Supersymmetry in a flat space-time is generally a global symmetry of the theory. However, in supergravity, a theory of gravity with supersymmetry, the supersymmetry is a local symmetry, similar to what happens to translation symmetry in a flat space-time which becomes symmetry under local diffeomorphisms in a generally covariant theory of gravity. For a theory on a curved space-time the supersymmetry parameter becomes a function of space-time. A supergravity theory contains \mathcal{N} massless spin 3/2 gravitinos $\psi_{\mu\alpha}^A$ that are partners of the graviton.

1.4.2 Supergravity in 11 dimensions

It turns out that for dimensions $D > 11$, the Clifford algebra associated with gamma matrices have dimension 64 or greater. This produces a massless particle, related to graviton by

supersymmetry, which has spin greater than two. There is no interesting theory of this kind. So, it seems the highest dimension which we can have a supersymmetric theory of gravity is 11.

In 11 dimensions there is a unique supersymmetric theory of gravity, in contrast to dimensions. This theory has $\mathcal{N} = 1$ supersymmetry with 32 supercharges Q_α due to 32 dimensional spinor representation of $SO(1, 10)$. The field content of the theory is the following:

- g_{MN} the graviton, quantum of fluctuations in the 11 dimensional space-time metric which has 44 degrees of freedom, which comes from the fact that the metric is symmetric and traceless,
- C_{MNO} a 3-form which has 84 degrees of freedom,
- $\psi_{M\alpha}$ the gravitino, the superpartner of the bosonic fields, with 128 degrees of freedom.

The low-energy action for the bosonic fields is given by:

$$S_B = \frac{1}{2\kappa_{11}^2} \left[\int \sqrt{g}(R - \frac{1}{2}|F|^2) - \frac{1}{6} \int C \wedge F \wedge F \right], \quad (1.9)$$

where κ_{11} is the coupling constant, 11-dimensional analogue of Newton constant and F is the field strength given by

$$F = dC. \quad (1.10)$$

The quantum theory of 11-dimensional supergravity is called ‘‘M-theory.’’

1.4.3 Type IIA supergravity

The type IIA supergravity theory is in 10 dimensions, and it has two sets of supersymmetry generators Q^1 and Q^2 in **16**-dimensional spinor representations, with opposite chirality. The theory has bosonic fields

$$g_{MN}, B_{MN}, \phi.$$

The field $B_{\mu\nu}$ is an anti-symmetric two-form field, and the field ϕ is a scalar field and is called dilaton. Type IIA theory has additional bosonic fields described by a 1-form and a 3-form

$$C_M^{(1)}, C_{MN}^{(3)}.$$

These fields are associated with the part of the massless spectrum arising from the Ramond-Ramond sector of the worldsheet theory. Due to this fact, they are called RR fields.

Bosonic fields in type IIA string theory have 128 degrees of freedom where it is consistent with the number of degrees of freedom for two Majorana-Weyl gravitinos (**56+56'**) and a

pair of Majorana-Weyl spinors ($\mathbf{8}+\mathbf{8}'$). Because the supersymmetry generators have opposite chirality, we have one spinor and one gravitino of each chirality and the theory is non-chiral.

The type IIA theory can be obtained from dimensional reduction of the 11-dimensional supergravity, M-theory, over a circle S^1 . For example, the dilaton can be obtained from the component of the metric tensor in 11D both in the compact direction. The complete correspondence between 11D and 10 degree of freedom for all the bosonic fields is the following:

$$\text{IIA :} \quad \begin{array}{cccccc} \text{11D SUGRA} & g_{MN}^{(11)} & g_{M\ 11}^{(11)} & g_{11\ 11}^{(11)} & C_{MN\lambda}^{(11)} & C_{MN\ 11}^{(11)} \\ \downarrow (S^1) & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbb{R}^{1,9} & g_{MN} & A_M & \phi & C_{MNO} & B_{MN} \end{array} \quad (1.11)$$

1.4.4 Type IIB supergravity

Similar to type IIA supergravity, type IIB supergravity has two supersymmetry generators. The difference here is that they have the same chirality, $Q^1, Q^2 \in \mathbf{16}$. Similarly, the type IIB theory contains g_{MN}, B_{MN}, ϕ as $\mathcal{N} = 1$ multiplet of fields as well Ramond-Ramond axion C^0 , two form C^2 , and self-dual 4-form field C^4 , whose massless states transform in the chiral $\mathbf{70}$ representation of $SO(8)$. Type IIB supergravity has a global $SL(2, \mathbb{R})$ symmetry, under which the two form fields B and $C^{(2)}$ rotate into one another as a doublet. In this theory, there are two gravitinos ψ_M^A with the same chirality ($2 \times \mathbf{56}$), and two spinors λ^A of identical chirality ($2 \times \mathbf{8}'$). In type IIB supergravity the fermions satisfy the following constraint:

$$\Gamma_{11}\psi_M^A = \psi_M^A, \quad \Gamma_{11}\lambda^A = -\lambda^A \quad A = 1, 2. \quad (1.12)$$

In this case they are doublet and we will suppress the corresponding index A .

Generally, it is not possible to write a Lagrangian such that its equation of motion gives the self-duality of the $C^{(4)}$ and we should demand this constraint separately. In this dissertation, we will use the democratic description for type IIB Lagrangian [6]. In this formulation our pseudo-action has the extended field content:

$$\text{IIB :} \quad \left\{ g_{MN}, B_{MN}, \phi, C^{(0)}, C_{MN}^{(2)}, C_{M\dots P}^{(4)}, C_{M\dots R}^{(6)}, C_{M\dots T}^{(8)}, \psi_\mu, \lambda \right\}.$$

To enforce democracy among RR potentials, self-duality is imposed on the field strength of all RR potentials ($C^{(0)}, C^{(2)}, \dots, C^{(8)}$) given by

$$F := dC - H \wedge C, \quad H = dB,$$

where C is the formal sum

$$C = C^{(0)} + C^{(2)} + \dots + C^{(8)},$$

and F is the formal sum of all odd fluxes with the self-duality constraint:

$$F_n = (-1)^{[n/2]} * F_{10-n}.$$

Their Bianchi identities are:

$$dH = 0 \quad , \quad dF^{(10)} - H \wedge F^{(10)} = 0.$$

Type IIB supergravity has the following action:

$$\begin{aligned} S_{\text{Pseudo}} = & -\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left\{ e^{-2\phi} \left[R(\omega(e)) - 4(\partial\phi)^2 + \frac{1}{2}H \cdot H + \right. \right. \\ & - 2\partial^M \phi \chi_M^{(1)} + H \cdot \chi^{(3)} + 2\bar{\psi}_M \Gamma^{MNO} \nabla_N \psi_O - 2\bar{\lambda} \Gamma^M \nabla_M \lambda + 4\bar{\lambda} \Gamma^{MN} \nabla_M \psi_N \left. \right] \\ & + \sum_{n=1,3,5,7,9} \frac{1}{4} F_n \cdot F_n + \frac{1}{2} F_n \cdot \Psi^{(n)} \left. \right\} + \text{quartic fermionic terms.} \end{aligned}$$

We have the following relation for the coupling of the above action κ_{10} :

$$\frac{1}{2\kappa_{10}} = \frac{g^2}{2\kappa^2} = \frac{2\pi}{(2\pi l_s)^8}$$

where κ^2 is the physical gravitational coupling, g is the string coupling constant and $l_s = \sqrt{\alpha'}$ is the string length scale. Also, in our convention, the inner product of forms is defined by

$$F_n \cdot F_n = \frac{1}{n!} F_{M_1 \dots M_n} F^{M_1 \dots M_n}.$$

The spin connection in the covariant derivative ∇_M is given by its zehnbein part: $\omega_M^{AB} = \omega_M^{AB}(e)$. The bosonic fields couple to the fermions via the bilinears $\chi^{(1,3)}$ and Ψ_n , which are

$$\begin{aligned} \chi_M^{(1)} &= -2\bar{\psi}_N \Gamma^N \psi_M - 2\bar{\lambda} \Gamma^N \Gamma_M \psi_N, \\ \chi_{MNO}^{(3)} &= \frac{1}{2} \bar{\psi}_A \Gamma^{[A} \Gamma_{MNO} \Gamma^{B]} \mathcal{P} \psi_B + \bar{\lambda} \Gamma_{MNO}{}^B \mathcal{P} \psi_B - \frac{1}{2} \bar{\lambda} \mathcal{P} \Gamma_{MNO} \lambda, \\ \Psi_{M_1 \dots M_n}^{(n)} &= \frac{1}{2} e^{-\phi} \bar{\psi}_A \Gamma^{[A} \Gamma_{M_1 \dots M_n} \Gamma^{B]} \mathcal{P}_n \psi_B + \frac{1}{2} e^{-\phi} \bar{\lambda} \Gamma_{M_1 \dots M_n} \Gamma^B \mathcal{P}_n \psi_B + \\ & - \frac{1}{4} e^{-\phi} \bar{\lambda} \Gamma_{[M_1 \dots M_{n-1}} \mathcal{P}_n \Gamma_{M_n]} \lambda, \end{aligned}$$

where

$$\psi_M = \begin{pmatrix} \psi_M^1 \\ \psi_M^2 \end{pmatrix}, \quad \lambda = \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix},$$

$$\begin{cases} \mathcal{P} = -\sigma^3, \\ \mathcal{P}_n = \sigma^1, & \text{for } (n+1)/2 \text{ even,} \\ \mathcal{P}_n = i\sigma^2, & \text{for } (n+1)/2 \text{ odd.} \end{cases}$$

It can be shown that the fermions satisfy

$$\Psi^{(n)} = (-)^{\text{Int}[n/2]+1} \star \Psi^{(10-n)}, \quad (1.13)$$

using properties of Γ -matrices.

1.5 Content of this dissertation

1.5.1 Moduli of Type IIB string theory

In the first part of our work, we consider type IIB string theory, which is a ten-dimensional supergravity. Using supersymmetry as in [7], we can find the constraints on its vacua. This constrains the internal space (a six dimensional manifold) and fluxes to get a low-energy four-dimensional effective theory with $\mathcal{N} = 1$ supersymmetry. One of the important features of our four-dimensional effective theory is the number of moduli, which correspond to massless fields. In the absence of fluxes, the internal space should be a Calabi-Yau manifold. Briefly, Calabi-Yau manifolds are Kähler manifolds with a nowhere-vanishing covariantly constant spinor field. There has been a great deal of work on moduli of Calabi-Yau manifolds and other features of compactifying on Calabi-Yau manifolds such as Yukawa couplings. In the presence of fluxes the internal space can be more exotic, and less is known about deriving the structure of the four-dimensional effective field theory. In [8] we propose a way to compute moduli of the effective theory corresponding to complex structure deformations of the internal space, given a solution for the background and flux. We followed the work [9] on heterotic strings and applied it to the type IIB case. The main idea is to find the moduli by finding constraints on the fluctuations of the equations of motion. Then, the next step is to state those constraints in terms of Dolbeault cohomology. In our work we show that this technique can work well in special cases of type IIB vacua. We also give an example to see how we can use these techniques explicitly in computing moduli.

1.5.2 (0,2) versions of exotic (2,2) GLSMs

GLSMs can flow in the infrared limit to non-linear sigma models, with six-dimensional target spaces. This is known as a geometric phase of a GLSM. There has been a great deal of work on GLSMs with Abelian gauge groups, $U(1)^n$. The geometric phases of these GLSMs are typically hypersurfaces or complete intersections inside toric varieties. Reference [10] studied $\mathcal{N} = (2, 2)$ GLSMs with $U(n)$ and $Sp(n)$ gauge groups, whose geometric phases are the intersections of Grassmannians. They constructed multiple GLSMs realizing such geometries, which are dual to one another. We studied (0,2) analogue of these theories in [11], and we will describe that work in this part of the dissertation. We also describe some examples of anomaly-free bundles on intersections of Grassmannians.

1.5.3 Mirror Symmetry of Exceptional groups

The third part of the dissertation is about mirror symmetry. There has been many papers on mirrors of Abelian GLSMs (see for example [12, 13] and references therein). Recently, there

was a proposal for an extension of such mirror constructions to non-Abelian theories [14]. In our work [15], which we describe in this section of our dissertation, we applied the proposal to make predictions for two-dimensional supersymmetric gauge theories with exceptional gauge groups F_4 , E_6 and E_7 . We compute the mirror Landau-Ginzburg models and predict excluded Coulomb loci and Coulomb branch relations, analogues of quantum cohomology.

1.5.4 Multicovers from Landau-Ginzburg models

In chapter 5, we attempt to generalize a construction of branched double covers to higher covers. The work [16] described GLSMs and Landau-Ginzburg models which flow in the IR to branched double covers, utilizing novel physical phenomena. In section 5.4 we describe some unpublished results attempting to extend double covers to higher covers. We do succeed in describing some eight-fold covers, but we also argue that our methods cannot yield other covers beyond that.

Chapter 2

Moduli of supersymmetric backgrounds for type IIB string theory

The contents of this chapter were adapted with minor modifications, with permission from the Journal of High Energy Physics, from our publication [8].

2.1 Introduction

There have been several papers on the moduli of compactifications of heterotic string theory, in which moduli can be written as sums of kernels of maps between Dolbeault cohomology groups [9, 17–20]. The cases studied include complex $\partial\bar{\partial}$ -manifolds which include Kähler manifolds as special cases. The kernel of the maps live inside $H^1(TX)$, $H^1(TX^\vee)$ and $H^1(\text{End}_0(V))$, which are massless degrees of freedom for Calabi-Yau compactification, so generically we have fewer massless fields in comparison to Calabi-Yau compactification. Practically, writing the moduli space in terms of kernels of maps between cohomology groups, especially in the Calabi-Yau case, makes them computable explicitly using standard techniques in algebraic geometry. Indeed, the complex structure moduli can be computed as the allowed deformations of the defining polynomials of the manifold.

In our work, we tried to see how many of the techniques developed to find moduli of heterotic string theory, can be applied to compactifications of type IIB string theory. We will see that in three special cases of compactification of type IIB string theory in the presence of background fluxes, the techniques used in heterotic string theory can be utilized. So, we will see moduli as kernels of maps between Dolbeault cohomology groups. After a general analysis, we will give an explicit example and see what simplification and complication arise in comparison to the heterotic case.

We should mention that there have been other sophisticated techniques in moduli identification of type IIB string theory that we will not present here, but we direct the reader to some examples of these works relevant to our work [21–54].

We start with a review of finding moduli in heterotic string theory in section 2.2. Then in section 2.2 we state the constraints on supersymmetric backgrounds of type IIB string theory in the presence of fluxes, following the works [7, 55]. The rest of the chapter describes our work [8] on finding moduli in certain classes of examples of type IIB string theory with fluxes.

2.2 Review of the Heterotic Case

Let's start with the simplest part of finding moduli in heterotic string explained extensively in [17, 19, 20]. One of the constraints arising when compactifying a heterotic string to a four-dimensional $\mathcal{N} = 1$ theory is that the Yang-Mills field strength $F = dA + A \wedge A$ must satisfy:

$$F_{ab} = F_{\bar{a}\bar{b}}, \quad (2.1)$$

$$g^{a\bar{b}} F_{a\bar{b}} = 0. \quad (2.2)$$

These constraints are higher dimensional version of F- and D- flatness conditions in a four dimensional theory. The first constraint (2.1) tells that the gauge bundle should be holomorphic, and from the second constraint (2.2) slope stability of the gauge bundle for a Calabi-Yau manifold can be deduced [56].

To find the part of the moduli for a Calabi-Yau threefold compactification, we find the constraints on the fluctuations of the (2.1) such that it still satisfies (2.1). Such fluctuations, correspond to the massless degree of freedom of the four dimensional effective theory. Ideally, we should fluctuate all the equations, but for simplicity, we only focus on the fluctuations of (2.1).

To do so, we write (2.1) in the following form

$$\Pi_{\bar{a}}^{(+i)} \Pi_{\bar{b}}^{(+j)} F_{ij} = 0,$$

where Π^+ is the projection operator to anti-holomorphic coordinate

$$\Pi_i^{(+j)} = \frac{1}{2}(1 + J)_i^j.$$

Then fluctuations of (2.1) will be related to fluctuations of the gauge field A and complex structure J . The following constraint is obtained [17, 19],

$$\delta J_{[\bar{a}}^d F_{\bar{b}]d}^{(0)} + i D_{[\bar{a}}^{(0)} \delta A_{\bar{b}]} = 0. \quad (2.3)$$

The (0) superscript means that $F^{(0)}$ is the initial field strength and $D^{(0)}$ is the covariant derivative with respect to the initial gauge field, before the fluctuations. It is well known that the fluctuations of the complex structure such that $J + \delta J$ is a complex structure, should be in $H^1(TX)$. The above equation gives more constraint on the fluctuations of the complex structure. Consider the map

$$f : H^1(TX) \rightarrow H^2(\text{End}_0(V)),$$

$$\delta J \mapsto \delta J_{[\bar{a} \bar{b}]d}^{(0)},$$

then equation (2.3) tells that for the allowed fluctuations of the complex structure, the image of the map f should be exact, which is the class zero in $H^2(\text{End}_0(V))$. So the allowed fluctuations of the complex structure should be in $\text{Ker}(f)$. So, we get more constraints on the initial complex structure moduli $H^1(TX)$, for a complex manifold.

Also, we have the well known constraint by Atiyah on the fluctuations of a holomorphic gauge connection to remain a holomorphic gauge connection on a holomorphic bundle, which is

$$\delta A \in H^1(\text{End}_0(V)).$$

So the deformations of A and J are

$$H^1(\text{End}_0(V)) \oplus \text{Ker} \left(H^1(TX) \rightarrow H^2(\text{End}_0(V)) \right).$$

The above space can be written as $H^1(\mathcal{Q})$ where \mathcal{Q} is defined by the following short exact sequence between bundles

$$0 \rightarrow \text{End}_0(V) \rightarrow \mathcal{Q} \rightarrow TX \rightarrow 0.$$

To see why the constraint (2.3) is related to F-flatness directly, we consider the Gukov-Vafa-Witten superpotential [57] which is a part of the heterotic superpotential

$$W_{\text{GVW}} = \int_X H \wedge \Omega. \tag{2.4}$$

In the above Ω is the holomorphic three-form. Locally the field strength H , can be written as

$$H = dB - \frac{3\alpha'}{\sqrt{2}}(\omega^{3YM} - \omega^{3L}),$$

where ω^{3YM} and ω^{3L} are Yang-Mills and Lorentz Chern-Simon three-forms.

The matter fields of the effective four-dimensional theory can be obtained from the fluctuations of the gauge degrees of freedom. So, the superpotential depends upon the matter

fields C_i through the term including ω^{3YM} . One of the conditions for a four dimensional supersymmetric Minkowski vacua is the following:

$$\frac{\partial W}{\partial C_i} = -\frac{3\alpha'}{\sqrt{2}} \int_X \Omega \wedge \frac{\partial \omega^{3YM}}{\partial C_i} = 0.$$

Fluctuating this constraint with respect to the complex structure and the gauge field, the following is obtained [17]

$$\delta \left(\frac{\partial W}{\partial C_i} \right) = \int_X \epsilon^{\bar{a}\bar{b}\bar{c}} \epsilon^{abc} \Omega_{abc} 2\bar{\omega}_{\bar{e}}^i \text{tr}(T_x T_y) \left(J_{[\bar{a}}^d F_{\bar{b}]d}^{(0)} + iD_{[\bar{a}}^{(0)} \delta A_{\bar{b}]} \right),$$

where T 's are gauge generators and $\bar{\omega}^i$ is the one form related to the matter field C_i . We see that under fluctuations, the condition on the supersymmetric Minkowski vacua derived from the Gukov-Vafa-Witten superpotential is consistent with the F-flatness condition we find in equation (2.3).

The general analysis of the F-flat moduli space of heterotic string compactification in terms of the maps between Dolbeault cohomology, has been completed in [9]. The procedure is similar to what we showed here, but as expected there are more maps involved in describing the moduli space. Finally, we should mention that other approaches to investigating the moduli of heterotic string compactifications exist in the literature. Two of the approaches with most relevance to that presented here can be found in these references [58, 59].

2.2.1 Computing in an example

The above general considerations are useful in gaining an understanding of the nature of the moduli of a heterotic Calabi-Yau compactification. However, to compute in more explicit detail, we must specialize our analysis to a given example. In particular, we must specify a Calabi-Yau threefold and a holomorphic, slope poly-stable bundle over it.

As a simple example, consider the following Calabi-Yau manifold, defined as a complete intersection in a product of projective space (or ‘‘CICY’’), and $SU(2)$ bundle, defined as an extension of two line bundles [17].

$$X = \left[\begin{array}{c|c} \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^2 & 3 \end{array} \right]^{3,75}, \quad 0 \rightarrow \mathcal{L} \rightarrow V \rightarrow \mathcal{L}^\vee \rightarrow 0. \quad (2.5)$$

Here the line bundle \mathcal{L} is taken to be $\mathcal{L} = \mathcal{O}_X(-2, -1, 2)$ and V is indeed poly-stable in appropriate regions of Kähler moduli space [17].

One could study the F-flat moduli space of this theory by pursuing the above approach of perturbing around a good choice of complex structure and bundle moduli. However, this

would only give us a limited view into the full moduli space of the system, restricted to the neighborhood of that starting choice. In addition, guessing a suitable initial point to perturb about can be difficult in many cases.

Instead, we can use the structure of bundles, such as that in (2.5), in order to obtain a more global view of the F-flat moduli space. The non-trivial extensions we are considering here are controlled by the extension group $\text{Ext}^1(\mathcal{L}^\vee, \mathcal{L}) = H^1(X, \mathcal{L}^2)$. This cohomology group actually vanishes for a generic choice of the complex structure of X . Thus generically, no such holomorphic $SU(2)$ bundle exists. However, for sub-loci of complex structure moduli space, the cohomology $H^1(X, \mathcal{L}^2)$ can jump in dimension to a non-zero value. On such loci, one can define a non-trivial holomorphic $SU(2)$ bundle of the type desired. One may then posit that if we consider a complex structure perturbation which takes the system off of this “jumping locus” that, because the above $SU(2)$ bundle can no longer remain holomorphic, the holomorphic restriction we studied at the start of this section would make such a degree of freedom massive. This is indeed the case as was shown in [17, 19]. Thus, in order to study the F-flat complex structure moduli space in such an example, we simply need to ascertain the loci where the cohomology $H^1(X, \mathcal{L}^2)$ jumps.

The jumping locus of a line bundle cohomology over a Calabi-Yau threefold can readily be obtained by making use of the Koszul sequence. For this codimension one example we have the following short exact sequence.

$$0 \rightarrow \mathcal{N}^\vee \otimes \mathcal{L}^2 \rightarrow \mathcal{L}_{\mathcal{A}}^2 \rightarrow \mathcal{L}_X^2 \rightarrow 0. \quad (2.6)$$

Here \mathcal{A} denotes the ambient space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ and \mathcal{N} the normal bundle, $\mathcal{O}_{\mathcal{A}}(2, 2, 3)$ in this case. The short exact sequence (2.6) has an associated long exact sequence in cohomology. Using the fact that, for the \mathcal{L} and \mathcal{N} given above, $H^1(\mathcal{A}, \mathcal{L}^2) = H^3(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}^2) = 0$ we can write the following.

$$0 \rightarrow H^1(X, \mathcal{L}^2) \rightarrow H^2(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}^2) \xrightarrow{P} H^2(\mathcal{A}, \mathcal{L}^2) \rightarrow H^2(X, \mathcal{L}^2) \rightarrow 0. \quad (2.7)$$

Here P is the map defined by the defining relation of the Calabi-Yau threefold. Using the theorem of Bott-Borel-Weil [60], we can describe $H^2(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}^2) = H^2(\mathcal{A}, \mathcal{O}(-6, -4, 1))$ as the space of linear combinations of monomials of degree $[-4, -2, 1]$ in the homogeneous coordinates of the ambient space $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$. Likewise, the space $H^2(\mathcal{A}, \mathcal{L}^2) = H^2(\mathcal{A}, \mathcal{O}(-4, -2, 4))$ can be described as the space of linear combinations of monomials of degree $[-2, 0, 4]$. With these explicit descriptions of the source and target spaces of (2.7), together with the explicit form of the map P , it is then easy to find a description of $H^1(X, \mathcal{L}^2)$ by taking the kernel of the mapping.

We describe the general element of the source by

$$b_i S^i \in H^2(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}^2), \quad (2.8)$$

where the S^i are a basis of monomials of the right degree and the b_i are coefficients. We then multiply this general element of the source by the defining equation,

$$P = c_a M^a, \quad (2.9)$$

where the c_a are coefficients (actually a redundant description of the complex structure moduli space) and the $\{M^a\}$ is a basis of degree $[2, 2, 3]$ monomials. We then set to zero any term in the resulting expression which is not of the degree $[-2, 0, 4]$ corresponding to the target space and this gives the image of the map in (2.7). Setting to zero the coefficient of each monomial in this image, then gives us the conditions on the b_i and the c_a for a given set of source coefficients to give rise to an element of the kernel, that is $H^1(X, \mathcal{L}^2)$, for a given complex structure of the base Calabi-Yau manifold. The resulting equations take the following bilinear form.

$$\Lambda_I^{ia} b_i c_a = 0. \quad (2.10)$$

Here the index I runs over the dimension of the target of the map and the Λ 's are simply constants. The equations (2.10) contain all of the information about what elements of $H^2(\mathcal{A}, \mathcal{N}^\vee \otimes \mathcal{L}^2)$ give rise to elements of $H^1(X, \mathcal{L}^2)$, and thus possible extension classes for the bundle V , for all possible values of the complex structure.

The set of equations (2.10), describes a reducible algebraic variety in the combined space of source coefficients and complex structure, as depicted in Figure 2.1. We can regard the

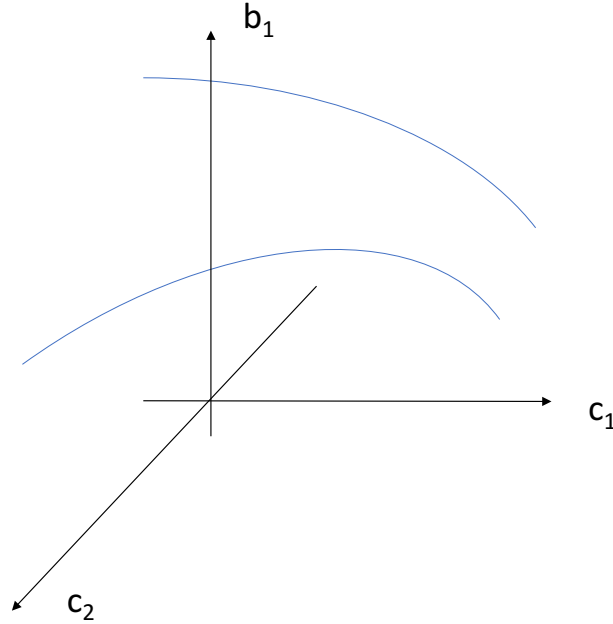


Figure 2.1: A depiction of the reducible variety given by the system (2.10). This variety lives in the combined space of complex structure, c_a , and source coefficients, b_i , of the map P in (2.7).

source space as the space of potential elements of the kernel, with the actual elements of $H^1(X, \mathcal{L}^2)$ being picked out by the solutions to these equations for a given complex structure. This algebraic variety can be broken up into its irreducible components by performing a

primary decomposition on the ideal whose generators are given by (2.10). This gives us one set of equations for each irreducible piece of the variety. By then performing an algebraic elimination of the b 's on each irreducible variety, we can find a set of loci purely in complex structure moduli space, as parameterized by the c 's. This process of primary decomposition and elimination, when applied to the toy example depicted in Figure 2.1, is depicted in Figure 2.2. These are then the loci in complex structure moduli space that the system can

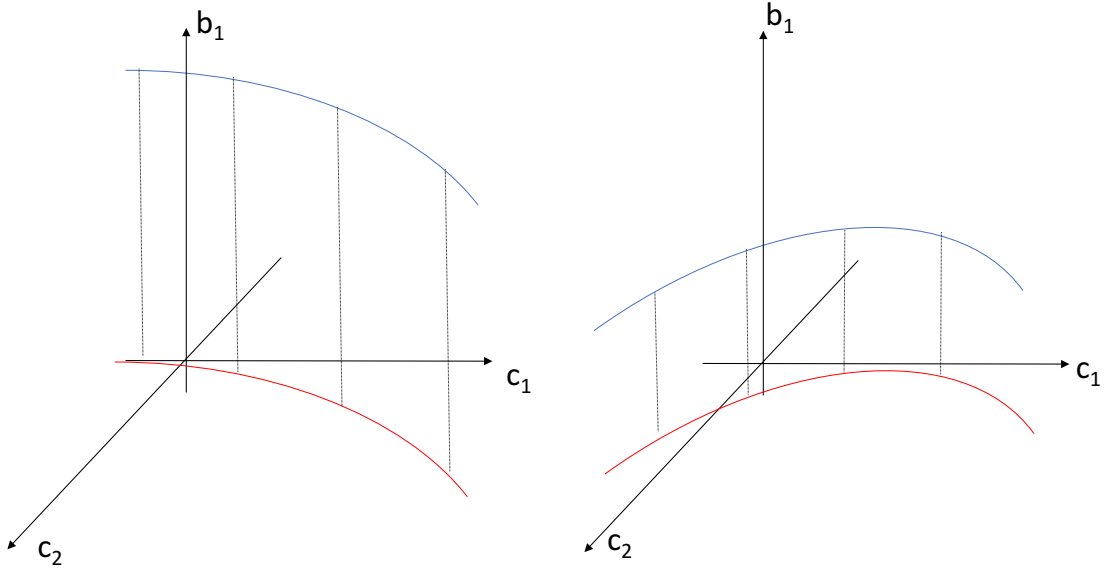


Figure 2.2: A depiction of the process of primary decomposition and then elimination as applied to the variety pictured in Figure 2.1. The red lines are the final loci obtained in complex structure moduli space.

be stabilized to, by the effects described in the previous subsection. The system will be stabilized to a particular locus if an extension class lying in the associated set of possible b 's, corresponding to points on that irreducible variety, is chosen. By applying the methodology described here, one can map out the whole moduli space associated with a given vector bundle, including all branches which are present due to different jumping phenomena in cohomology. For each irreducible variety in complex structure moduli space which is obtained in this manner, we must finally check that the Calabi-Yau threefold under consideration remains smooth for a generic complex structure on that locus.

In many examples, the structure of loci in complex structure moduli space to which the system can be stabilized turns out to be rather rich. For example, a simple case is given in [20] in which 25 non-trivial loci are found. Of these, all but one correspond to a singular Calabi-Yau initially, although it is demonstrated that some of the rest can be smoothed out by an appropriate geometrical transition.

One thing which is important to note is that, in the above, it is vital to begin by choosing a type of bundle construction with which to work. The details of which values the complex

structure gets stabilized to will depend upon the structure of the bundles in question, which is not uniquely determined by topological invariants such as Chern classes.

The case of moduli stabilization due to (or more precisely moduli identification in the presence of) a holomorphic vector bundle over a Calabi-Yau threefold is just one example of how such analyses have been applied in a heterotic setting. More generally, for example, the same type of reasoning has been used to determine the F-flat moduli space of the Strominger system [61], for cases where the compactification manifold obeys the $\partial\bar{\partial}$ -lemma [9]. The question we will try to address in the following sections is how much of this technology can be taken across to a type IIB setting? In particular, we will be interested in what simplifications we observe relative to the heterotic case and what additional complications arise in implementing such an approach to moduli identification. For simplicity, we will confine ourselves in what follows to the closed string sector.

2.3 $\mathcal{N} = 1$ Minkowski vacua of Type IIB string theory

The analysis we will follow in this subsection is a review of the work found in [7, 55].

In the type IIB string, as the name suggests, we have $N = 2$ supersymmetry in ten dimensions. For phenomenological reasons and to find a suitable four-dimensional effective field theory, we should break supersymmetry by compactifying ten-dimensional space-time on a suitable six-dimensional internal space, to get $N = 1$ in four dimensions. We are interested in vacua whose four-dimensional space admits maximal space-time symmetry, especially Minkowski space-time.

The massless bosonic fields of type IIB superstring theory are the dilaton ϕ , the metric tensor ds^2 and the two-form B in the NS-NS sector. The massless RR sector contains the axion C_0 , the two-form potential C_2 , and the four-form field C_4 with self-dual five-form field strength. The fermionic superpartners are two Majorana-Weyl gravitinos ψ_M^A , $A = 1, 2$ of the same chirality and two Majorana-Weyl dilatinos λ^A with opposite chirality to the gravitinos.

The field strength of NS flux is defined:

$$H = dB.$$

For the RR field strength, we will use the democratic formulation of [6], which considers all RR potentials $(C_0, C_2, \dots, C_{10})$ by imposing self-duality constraint on their field strength given by:

$$F^{(10)} = dC - H \wedge C,$$

where $F^{(10)}$ is the formal sum of all odd fluxes with the self-duality constraint:

$$F_n^{(10)} = (-1)^{[n/2]} * F_{10-n}^{(10)}.$$

Their Bianchi identities are:

$$dH = 0 \quad , \quad dF^{(10)} - H \wedge F^{(10)} = 0.$$

Type IIB superstring theory has $\mathcal{N} = 2$ supersymmetry with two Majorana-Weyl supersymmetry parameters ϵ^A of the same chirality as the gravitinos. A supersymmetric state is a zero energy solution to the equation of motion. In classical terms, this means that $\delta_\epsilon(\text{fields}) = 0$ gives the conditions for the vacua of the theory. Because the expectation value of a fermionic field is zero, the only conditions that we need to satisfy to get supersymmetric vacua are:

$$\begin{aligned} \delta\psi_M &= \nabla_M \epsilon + \frac{1}{4} \mathcal{H}_M \mathcal{P} \epsilon + \frac{1}{16} e^\phi \sum_n \mathcal{F}_n^{(10)} \Gamma_M \mathcal{P}_n \epsilon = 0, \\ \delta\lambda &= (\not{\partial}\phi + \frac{1}{2} \mathcal{H} \mathcal{P}) \epsilon + \frac{1}{8} e^\phi \sum_n (-1)^n (5-n) \mathcal{F}_n^{(10)} \mathcal{P}_n \epsilon = 0, \end{aligned}$$

where

$$\psi_M = \begin{pmatrix} \psi_M^1 \\ \psi_M^2 \end{pmatrix}, \quad \lambda = \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix},$$

$$\begin{cases} \mathcal{P} = -\sigma^3, \\ \mathcal{P}_n = \sigma^1, & \text{for } (n+1)/2 \text{ even,} \\ \mathcal{P}_n = i\sigma^2, & \text{for } (n+1)/2 \text{ odd.} \end{cases}$$

Also for any n-form

$$\mathcal{F} = \frac{1}{n!} F_{i_1 \dots i_n} \Gamma^{i_1 \dots i_n}.$$

We want to study flux backgrounds that preserve maximal four-dimensional symmetry. Therefore we require that:

$$F_n^{(10)} = F_n + \text{Vol}_4 \wedge \tilde{F}_{n-4}.$$

The self duality relation translates to

$$\tilde{F}_{n-4} = (-1)^{[n/2]} * F_{10-n}.$$

Also the most general ten-dimensional metric consistent with this is

$$ds^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n, \quad \mu = 0, 1, 2, 3, \quad m = 1, \dots, 6,$$

where A is a function of the internal coordinates called wrap factor, $\eta_{\mu\nu}$ is the Minkowski metric, and g_{mn} is any six dimensional metric.

The last step in breaking $\mathcal{N} = 2$ supersymmetry in ten dimensions to $\mathcal{N} = 1$ supersymmetry in four dimensions, assuming $SU(3)$ structure on the internal manifold, is to break the supersymmetry parameters in the following way:

$$\begin{aligned}\epsilon^1 &= \xi_+ \otimes (a(y)\eta_+(y)) + \xi_- \otimes (\bar{a}(y)\eta_-(y)), \\ \epsilon^2 &= \xi_+ \otimes (b(y)\eta_+(y)) + \xi_- \otimes (\bar{b}(y)\eta_-(y)),\end{aligned}$$

where ξ_+ is a constant spinor in four dimensions and $\eta_+(y)$ is nowhere vanishing spinor of the internal manifold. A minus-sign subscript indicates the complex conjugate of these spinors. For SUSY breaking we assume that there is a nowhere-vanishing spinor of the internal manifold. Having it, the internal space should possess an $SU(3)$ structure. We can encode all the $SU(3)$ structure data in a (1,1)-form J , which is called the fundamental or hermitian form, and a nowhere-vanishing (3,0)-form Ω . These are defined explicitly as:

$$\begin{aligned}J_{mn} &= -i\eta^\dagger \gamma_{mn} \gamma \eta, \\ \Omega_{mnp} &= -i\eta^\dagger \gamma_{mnp} (1 + \gamma) \eta,\end{aligned}$$

where γ is the six dimensional chirality operator. We also require them to satisfy the following constraints:

$$J \wedge \Omega = 0, \quad i\Omega \wedge \bar{\Omega} = \frac{(2J)^3}{3!}.$$

Having J and Ω , we can decompose forms with respect to them. We can decompose forms with respect to J and Ω , as we list below:

$$\begin{aligned}F_2 &= \frac{1}{3}F_2^{(1)}J + Re(F_2^{(3)} \lrcorner \bar{\Omega}) + F_2^{(8)}, \\ F_3 &= -\frac{3}{2}Im(F_3^{(1)}\bar{\Omega}) + F_3^{(3)} \wedge J + F_3^{(6)}, \\ F_4 &= \frac{1}{6}F_4^{(1)}J \wedge J + Re(F_4^{(3)}) \wedge \bar{\Omega} + F_4^{(8)}, \\ F_5 &= F_5^{(3)} \wedge J \wedge J, \\ F_6 &= \frac{1}{6}F_6^{(1)}J \wedge J \wedge J,\end{aligned}$$

where the different representations are given by

$$\begin{aligned}F_2^{(1)} &= \frac{1}{2}F_{mn}J^{mn}, & F_2^{(3)}{}_k &= \frac{1}{8}F^{ijk}\Omega_{ijk}, \\ F_3^{(1)} &= -\frac{i}{36}F_3^{ijk}\Omega_{ijk}, & F_3^{(3)}{}_i &= \frac{1}{4}F_{3imn}J^{mn}, & F_3^{(6)}{}_{ij} &= F_3^{kl}{}_{(i}\Omega_{j)kl}, \\ F_4^{(1)} &= \frac{1}{8}F^{mnpq}J_{mn}J_{pq}, & F_4^{(3)}{}_k &= \frac{1}{16}F_i^{mnpq}J_{mn}J_{pq}, \\ F_5^{(3)}{}_i &= \frac{1}{16}F_i^{mnpq}J_{mn}J_{pq}, \\ F_6^{(1)} &= \frac{1}{48}F^{mnpqrs}J_{mn}J_{pq}J_{rs}.\end{aligned}$$

It will be useful to decompose dJ and $d\Omega$ in the form:

$$\begin{aligned} dJ &= -\frac{3}{2}Im(W_1\bar{\Omega}) + W_4 \wedge J + W_3, \\ d\Omega &= W_1 J \wedge J + W_2 \wedge J + \bar{W}_5 \wedge \Omega. \end{aligned}$$

The W_i are called torsion classes and they classify the differential type of any $SU(3)$ structure. Note that in the special case that all $W_i = 0$, the internal space is a Calabi-Yau manifold. Inserting these together into the supersymmetry conditions we get:

$$\begin{aligned} [\alpha\cancel{\partial}A - \frac{i}{4}e^\phi\mathcal{F}]\eta_+ &= 0, \\ \alpha D_m\eta_+ + [\partial_m\alpha - \frac{1}{4}\beta\mathcal{H}_m - \frac{i}{8}e^\phi\mathcal{F}\gamma_m]\eta_+ &= 0, \\ \alpha\cancel{\partial}\eta_+ + [\alpha\cancel{\partial}(2A - \phi + \log\alpha) + \frac{1}{4}\beta\mathcal{H}]\eta_+ &= 0, \end{aligned}$$

where $\alpha = a + ib$ and $\beta = a - ib$ and $F = \alpha F_1 - \beta F_3 + \alpha F_5$. Also, there is a second set of equations that can be found by mapping $\alpha \leftrightarrow \beta$ and $F \rightarrow -\beta F_1 + \alpha F_3 - \beta F_5$. These equations can be split into components in the following way:

$$\begin{aligned} (Q_m + R_m)\eta_+ + (Q_{mn} + R_{mn})\gamma^n\eta_- &= 0, \\ S\eta_- + (S_m + A_m)\gamma^m\eta_+ &= 0, \\ T\eta_- + T_m\gamma^m\eta_+ &= 0, \end{aligned}$$

where:

$$\begin{aligned} A_{\bar{i}} &= \alpha\partial_{\bar{i}}A, \\ S &= \frac{3}{2}i\beta e^\phi F_3^{(1)}, \\ S_{\bar{i}} &= \frac{i}{4}e^\phi(\alpha F_1^{(\bar{3})} - 2i\beta F_3^{(\bar{3})} - 2\alpha F_5^{(\bar{3})})_{\bar{i}}, \\ Q_{\bar{i}} &= -i\partial_{\bar{i}}\alpha - \frac{i}{2}[\alpha(W_5 - W_4) + i\beta H^{(3)}]_{\bar{i}}, \\ Q_{\bar{i}j} &= \frac{1}{4}[(\alpha W_1 + 3i\beta H^{(1)})g_{\bar{i}j} + i\alpha W_{2\bar{i}j}], \\ Q_{ij} &= -\frac{1}{8}[\Omega_{ijk}(\alpha W_4 - i\beta H^{(3)})^k + \frac{i}{2}(\alpha W_3 - i\beta H^{(6)})_i{}^{kl}\Omega_{jkl}], \\ R_{\bar{i}} &= \frac{1}{4}e^\phi(\alpha F_1^{(\bar{3})} - 2i\beta F_3^{(\bar{3})} - 2\alpha F_5^{(\bar{3})})_{\bar{i}}, \\ R_{ij} &= -\frac{i}{16}e^\phi(\alpha F_1^{(\bar{3})k}\Omega_{ijk} - \beta F_{3ij}^{(6)} + 2\alpha F_5^{(\bar{3})k}\Omega_{ijk}), \\ R_{\bar{i}j} &= 0, \\ T &= \frac{3}{4}(\alpha i W_1 + \beta H^{(1)}), \\ T_{\bar{i}} &= \alpha\partial_{\bar{i}}(2A - \phi - \log\alpha) + \frac{1}{2}[\alpha(W_4 + W_5) - i\beta H^{(3)}]_{\bar{i}}. \end{aligned}$$

Then solving these equations will result in $N = 1$ Minkowski vacua. The SUSY vacua for generic $a(y), b(y)$ are defined by:

$$\begin{aligned}
\alpha W_3 - i\beta H^{(6)} - \beta e^\phi F^{(6)}|_{(1,2)} &= 0, \\
\beta W_3 - i\alpha H^{(6)} + \alpha e^\phi F^{(6)}|_{(1,2)} &= 0, \\
F_1^{(\bar{3})} &= 0, \\
e^\phi F_5^{(\bar{3})} &= \frac{4i(\alpha^2 + \beta^2)}{\beta(3\alpha^2 + \beta^2)} \bar{\partial}\beta, \\
\bar{\partial}\phi &= \frac{16\alpha^2\beta}{(3\alpha^2 + \beta^2)(\alpha^2 - \beta^2)} \bar{\partial}\beta, \\
\bar{W}_5 &= \frac{2(\alpha^2 + 3\beta^2)}{\beta(\alpha^2 - \beta^2)} \bar{\partial}\beta, \\
e^\phi F_3^{(\bar{3})} &= \frac{-8\alpha}{3\alpha^2 + \beta^2} \bar{\partial}\beta, \\
H_3^{(\bar{3})} &= \frac{-8i\alpha(\alpha^2 + \beta^2)}{(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)} \bar{\partial}\beta, \\
W_4 &= \frac{4(\alpha^2 + \beta^2)^2}{\beta(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)} \bar{\partial}\beta, \\
\bar{\partial}A &= -\frac{2(\alpha - \beta^2)}{\beta(3\alpha^2 + \beta^2)} \bar{\partial}\beta, \tag{2.11}
\end{aligned}$$

where $\alpha = a + ib, \beta = a - ib$.

We are going to determine the constraints on the moduli space in the presence of background flux. Our goal is to encode the constraints in Dolbeault cohomology so that the moduli will be extremely computable. For arbitrary a and b we will see that the moduli cannot be written in terms of Dolbeault cohomology. In general terms, the reason for this is that for general a and b , the quantities $H_3(= dB)$ and $F_3(= dC_2 - C_0 dB)$ are not exact, and we will need them to be exact to write moduli in terms of Dolbeault cohomology.

From equation (2.11), there are three special cases in which one of $F_3^{(6)}$, W_3 or $H_3^{(6)}$ are zero, where $ab = 0$, $a = \pm ib$ or $a = \pm b$. We will analyze each of these three cases extensively. We will see in these cases we can write part of the moduli in terms of kernel of maps between Dolbeault cohomology groups. We also will work through an example.

2.4 Procedure for finding moduli of a given background

In the rest of this chapter, we present our work [8]. We will compute massless fields and moduli in the effective four-dimensional theory by examining the ten-dimensional IIB compactification with flux.

To illustrate the procedure, consider the Mexican hat potential. We know that if we expand the theory around a given vacuum one of the fields gets a mass, and we get one massless field, the Goldstone boson. An intuitive way to see why we get one massless field is the following: given a vacuum, there is only one direction, such that we can move and stay on the vacua. We will use the similar method for finding moduli of type IIB string theory. In the last section we described the constraints on supersymmetric vacua of type IIB string theory in the presence of fluxes. Given a solution to the equations (2.11), we constrain the

fluctuations of these equations by requiring that fluctuations still obey equations (2.11), and then we try to state these constraints in terms of Dolbeault cohomology in special cases. Our method initially was motivated by the analysis of [9], for heterotic string theory with torsion. We follow exactly the same path in our work for type IIB string theory, and we see that the similar analysis can be done in special cases.

2.5 Type A, $a = 0$ or $b = 0$

In this case, which is usually called type A vacua, the conditions for vacua, equations 2.11, reduce to:

$$\begin{aligned} W_3 &= \pm * H_3^{(6)}, \\ W_5 &= 2W_4 = \pm 2iH_3^{(3)} = 2\partial\phi, \\ \bar{\partial}A &= \bar{\partial}a = 0, \end{aligned} \tag{2.12}$$

and all the other quantities are zero. Now we are going to simplify equation (2.12). In the simplification we use the fact (provided in appendix) that for a primitive (2,1)-form $*A_{(2,1)} = iA_{(2,1)}$. Then, without loss of generality, just considering upper signs in above equations we get:

$$\begin{aligned} (dJ - J \wedge W_4)_{(2,1)} &= i(H_3 - J \wedge H_3^{(3)}), & W_4 &= iH_3^{(3)}, \\ \Rightarrow (dJ)_{(2,1)} &= i(H_3)_{(2,1)}. \end{aligned}$$

Complex conjugate of above equation will result

$$(dJ)_{(1,2)} = -i(H_3)_{(1,2)}.$$

Combining these two and writing in more compact form gives

$$H_3 = i(\bar{\partial} - \partial)J. \tag{2.13}$$

This equation is exactly the same condition for supersymmetric vacua in heterotic string theory which is found by Strominger [61]. The moduli of such vacua have been studied comprehensively in [9].

2.6 Type B, $a = \pm ib$

In this kind of vacua, type B, the internal space should be a conformal Calabi-Yau manifold. The constraint on the fluxes, equations (2.11), reduces to:

$$e^\phi F_3^{(6)} = \mp * H_3^{(6)}, \tag{2.14a}$$

$$e^\phi F_5^{(3)} = -\frac{2}{3}iW_5 = -iW_4 = 2i\partial A = 4i\partial \log a, \tag{2.14b}$$

$$\partial\phi = 0. \tag{2.14c}$$

Manipulating equations (2.14a), with work similar to what we did before, we get

$$e^\phi(F_3 - J \wedge H_3^{(3)})_{(2,1)} = -i(H_3 - J \wedge H_3^{(3)})_{(2,1)}.$$

In addition, $H_3^{(3)} = F_3^{(3)} = 0$ in this case, which implies

$$e^\phi(F_3)_{(1,2)} = i(H_3)_{(1,2)}.$$

Combining this result with its complex conjugate we get

$$e^\phi F = i(H_{(1,2)} - H_{(2,1)}). \quad (2.15)$$

To compute moduli, we should require that the vacuum fluctuations also satisfy these equations.

Fluctuating $H_{(0,3)} = 0$ gives

$$\begin{aligned} \Pi^{(+i)}\Pi^{(+j)}\Pi^{(+k)}H_{ijk} &= 0, \\ \Rightarrow (\delta H)_{\bar{a}\bar{b}\bar{c}} + \frac{i}{2}\delta J_{\bar{a}}^d(H)_{\bar{b}\bar{c}} + \frac{i}{2}\delta J_{\bar{b}}^d(H)_{\bar{a}\bar{c}} + \frac{i}{2}\delta J_{\bar{c}}^d(H)_{\bar{a}\bar{b}} &= 0, \\ \Rightarrow (\delta H)_{\bar{a}\bar{b}\bar{c}} &= -\frac{3i}{2}\delta J_{[\bar{a}}^d(H)_{\bar{b}\bar{c}]d}, \end{aligned} \quad (2.16)$$

where $\Pi_i^{(\pm)j} = \frac{1}{2}(1 \pm iJ)_i^j$ is the projection operator into antiholomorphic and holomorphic indices. With this projection operator, we can define the following map:

$$\begin{aligned} F : H^1(TX) &\rightarrow H^3(\mathcal{O}_X), \\ \delta J_{\bar{a}}^d &\mapsto -\frac{3i}{2}\delta J_{[\bar{a}}^d(H)_{\bar{b}\bar{c}]d}. \end{aligned} \quad (2.17)$$

First, note that this is a well-defined map between cohomology groups, as $\bar{\partial}\delta J = 0$ and $\bar{\partial}H = 0$. (It is well known that δJ defines an element of $H^1(TX)$.) In particular, the image of an exact term is exact. Furthermore, for the allowed variations of complex structure, the image is exact ($\delta H = d\delta B$), hence the valid complex structure variations are defined by the kernel of the map above.

Now let's find condition on the fluctuation of $e^\phi F = i(H_{(2,1)} - H_{(1,2)})$. For this reason, we state it in the following form

$$\begin{aligned} (F - ie^{-\phi}H)_{(1,2)} &= 0, \\ \Rightarrow \Pi_i^{(+l)}\Pi_j^{(+m)}\Pi_k^{(-n)}(F - ie^{-\phi}H)_{lmn} &= 0. \end{aligned}$$

For $i = \bar{a}, j = \bar{b}, k = c$ we get:

$$\begin{aligned} \left(\frac{i}{2}\right)\delta J_{\bar{a}}^l(F - ie^{-\phi}H)_{\bar{b}c} + \left(\frac{i}{2}\right)\delta J_{\bar{b}}^m(F - ie^{-\phi}H)_{\bar{a}mc} + \\ \left(\frac{i}{2}\right)\delta J_c^{\bar{n}}(F - ie^{-\phi}H)_{\bar{a}\bar{b}\bar{n}} + \delta(F - ie^{-\phi}H)_{\bar{a}\bar{b}c} &= 0, \\ \Rightarrow \left(\frac{i}{2}\right)\delta J_{\bar{a}}^d(F - ie^{-\phi}H)_{\bar{b}cd} - \left(\frac{i}{2}\right)\delta J_{\bar{b}}^d(F - ie^{-\phi}H)_{\bar{a}cd} + \delta(F - ie^{-\phi}H)_{\bar{a}\bar{b}c} &= 0. \end{aligned}$$

Then we have:

$$\delta\left(F - ie^{-\phi}H\right)_{\bar{a}\bar{b}c} + i\delta J_{[\bar{a}}^d\left(F - ie^{-\phi}H\right)_{\bar{b}]cd} = 0. \quad (2.18)$$

F, H can be written as

$$\begin{aligned} H &= dB + H^0, \\ F &= dC_2 - C_0H + F^0, \end{aligned}$$

where H^0, F^0 are integer valued flux, then $\delta H^0 = 0, \delta F^0 = 0$ and because $F_1 = dC_0 = 0$ then C_0 is constant. Using them we get:

$$\delta\tau H_{\bar{a}\bar{b}c} - 2e^{-\phi}\delta J_{[\bar{b}}^d H_{\bar{c}]ad} = d(\delta C_2 - \tau\delta B)_{\bar{a}\bar{b}c}, \quad (2.19)$$

where $\tau = C_0 + ie^{-\phi}$. The right hand side is d -exact, to make it $\bar{\partial}$ -exact (the reason for this, is to work with Dolbeault cohomology which is more computable here) we assume that our background is a $\partial\bar{\partial}$ -manifold.

Definition: A complex manifold is called a $\partial\bar{\partial}$ -manifold if it has the property that for A , a d -closed (p,q) -form, the following statements are equivalent,

$$A = \bar{\partial}X \Leftrightarrow A = \partial Y \Leftrightarrow A = dZ \Leftrightarrow A = \partial\bar{\partial}W \Leftrightarrow A = \partial P + \bar{\partial}Q,$$

for some X, Y, Z, W, P and Q .

All the Kähler manifolds satisfy this property, but not all $\partial\bar{\partial}$ -manifolds are Kähler. Now, let's define the form

$$A = \partial_c(\delta C_2 - \tau\delta B)_{\bar{a}\bar{b}}d\bar{z}^{\bar{a}} \wedge d\bar{z}^{\bar{b}} \wedge dz^c.$$

Clearly, A is ∂ -exact so $\partial A = 0$. Also $\bar{\partial}A = 0$:

$$\begin{aligned} \bar{\partial}A &= \bar{\partial}_{\bar{c}}\partial_c(\delta C_2 - \tau\delta B)_{\bar{a}\bar{b}}d\bar{z}^{\bar{a}} \wedge d\bar{z}^{\bar{b}} \wedge d\bar{z}^{\bar{c}} \wedge dz^c, \\ &= \partial_c\left(\bar{\partial}_{\bar{c}}(\delta C_2 - \tau\delta B)_{\bar{a}\bar{b}}d\bar{z}^{\bar{a}} \wedge d\bar{z}^{\bar{b}} \wedge d\bar{z}^{\bar{c}}\right) \wedge dz^c. \end{aligned}$$

We will show that the term in the parentheses is zero. To see that, let's fluctuate $(F - ie^{-\phi}H)_{(0,3)} = 0$, then we get

$$0 = \delta(F - ie^{-\phi}H)_{\bar{a}\bar{b}c} + \frac{3i}{2}\delta J_{[\bar{a}}^d(F - ie^{-\phi}H)_{\bar{b}]cd} = \delta(F - ie^{-\phi}H)_{\bar{a}\bar{b}c}.$$

So $\bar{\partial}_{[\bar{a}}(\delta C_2 - \tau\delta B)_{\bar{b}c]} = 0$ and this completes the proof of $dA = 0$. So using $\partial\bar{\partial}$ -lemma there exists a form such that $A = \bar{\partial}\Lambda_{(1,1)}$ and we get

$$\delta\tau H_{\bar{a}\bar{b}c} - 2e^{-\phi}\delta J_{[\bar{a}}^d(H)_{\bar{b}]cd} = 2\bar{\partial}_{[\bar{a}}(\delta C_2 - \tau\delta B + \frac{1}{2}\Lambda)_{\bar{b}c]}. \quad (2.20)$$

Now consider the map

$$\begin{aligned} H^0(\mathcal{O}_X) \oplus H^1(TX) &\rightarrow H^2(TX^\vee), \\ (\delta\tau, \delta J) &\mapsto \frac{1}{2}(\delta\tau H_{\bar{a}\bar{b}c} - 2e^{-\phi}\delta J_{[\bar{a}}^d H_{\bar{b}]cd})d\bar{z}^{\bar{a}} \wedge d\bar{z}^{\bar{b}} \wedge dz^c. \end{aligned} \quad (2.21)$$

We will show next that the image of this map is closed, hence it is in $H^2(TX^\vee)$.

$$\begin{aligned}
& e^\phi F = i(H_{(1,2)} - H_{(2,1)}), \quad dF = 0, \quad d\phi = 0, \\
\Rightarrow & d(H_{(1,2)} - H_{(2,1)}) = 0, \quad dH = (H_{(1,2)} + H_{(2,1)}) = 0, \\
\Rightarrow & dH_{(2,1)} = 0, \\
\Rightarrow & \bar{\partial}H_{(2,1)} = 0, \quad \text{and similarly we will get } \bar{\partial}H_{(1,2)} = 0.
\end{aligned}$$

Then

$$\begin{aligned}
& \bar{\partial}((\delta\tau H_{\bar{a}\bar{b}\bar{c}} - 2e^{-\phi}\delta J_{[\bar{a}}^d H_{\bar{b}]cd})d\bar{z}^{\bar{a}} \wedge d\bar{z}^{\bar{b}} \wedge dz^c) = \\
& \bar{\partial}_{\bar{d}}((\delta\tau H_{\bar{a}\bar{b}\bar{c}} - 2e^{-\phi}\delta J_{[\bar{a}}^d H_{\bar{b}]cd})d\bar{z}^{\bar{d}} \wedge d\bar{z}^{\bar{a}} \wedge d\bar{z}^{\bar{b}} \wedge dz^c) = \\
& ((\delta\tau \bar{\partial}_{[\bar{d}} H_{\bar{a}\bar{b}]c} - 2e^{-\phi}\bar{\partial}_{[\bar{d}} \delta J_{\bar{a}}^d H_{\bar{b}]cd})d\bar{z}^{\bar{d}} \wedge d\bar{z}^{\bar{a}} \wedge d\bar{z}^{\bar{b}} \wedge dz^c) = \\
& ((\delta\tau \bar{\partial}_{[\bar{d}} H_{\bar{a}\bar{b}]c} - 2e^{-\phi}\delta J_{\bar{a}}^d \bar{\partial}_{[\bar{d}} H_{\bar{b}]cd})d\bar{z}^{\bar{d}} \wedge d\bar{z}^{\bar{a}} \wedge d\bar{z}^{\bar{b}} \wedge dz^c) = 0.
\end{aligned}$$

This reasoning shows that the image of the map is $\bar{\partial}$ -closed. Similarly to the previous case, the image of an exact term is exact, and so this is a map between cohomology classes.

For allowed fluctuations, the image is exact and then $(\delta\tau, \delta J)$ satisfying the equation is in the kernel of the map. This restricts moduli to the kernel, a subset of $H^0(\mathcal{O}_X) \oplus H^1(TX)$. To summarize, we have found that the allowed moduli of the system must be in the following two kernels.

$$\ker \left(H^1(TX) \xrightarrow{H_{(1,2)}} H^3(X) \right), \quad (2.22)$$

$$\ker \left(H^0(X) \oplus H^1(TX) \xrightarrow{H_{(1,2)}, H_{(2,1)}} H^2(TX^\vee) \right). \quad (2.23)$$

In this type of vacua, the internal space should be a conformal Calabi-Yau. If we take a Calabi-Yau manifold, then because of Serre duality $H^2(TX^\vee) \cong H^1(TX)$ and if $\dim H^1(TX) = n$ then our map is imposing n constraints on the number of moduli. Also $H^0(X) \cong \mathbb{C}$, then we get a total of $n + 1$ constraints on the $n + 1$ -dimensional space $H^0(X) \oplus H^1(TX)$, which reminds us of the well known statement that in terms of fluxes we can stabilize all the complex structure and axion-dilaton moduli.

2.6.1 An Explicit Example

We will begin with a simple example defined as a quotient of the quintic Calabi-Yau threefold, described by the following configuration matrix,

$$M_X = [\mathbb{P}^4 \mid 5], \quad (2.24)$$

by a freely acting $\mathbb{Z}_5 \times \mathbb{Z}_5$ symmetry. The configuration matrix (2.24) indicates that X is defined as the zero locus of a degree five polynomial inside \mathbb{P}^4 . We will denote the homogeneous coordinates on \mathbb{P}^4 as x_i where $i = 0, \dots, 4$. The freely acting $\mathbb{Z}_5 \times \mathbb{Z}_5$ symmetry by

which we will quotient X has generators acting as

$$\begin{aligned} g_1 : x_i &\rightarrow \omega^i x_i, \\ g_2 : x_i &\rightarrow x_{i+1}, \end{aligned} \tag{2.25}$$

where ω is a fifth root of unity and we define $x_5 = x_0$. The quotient manifold $X/\mathbb{Z}_5 \times \mathbb{Z}_5$ is a smooth Calabi-Yau threefold, for sufficiently generic choices of complex structure, and has $h^{1,1} = 1$ and $h^{2,1} = 5$ [62–65].

Using the sequences mentioned at the start of this section, one can compute that the complex structure moduli are encoded by the following description of the first tangent bundle valued cohomology group

$$H^1(TX) = \frac{\text{Coker} [\mathbb{C} \rightarrow [5]]}{\text{Coker} [\mathbb{C} \rightarrow [1]^{\oplus 5}]}. \tag{2.26}$$

Here the map used in defining the quotient is given by dP (the derivative of the defining relations), that in the numerator is given by P (the defining relations themselves) and that in the denominator is given by the homogeneous coordinates of the ambient space. The symbols $[n]$ where n is an integer denote the spaces of polynomials of degree n . More precisely, they are those such polynomials that are invariant under the $\mathbb{Z}_5 \times \mathbb{Z}_5$ action. Similarly, we have that

$$H^2(TX^\vee) = \text{Ker} \left[\text{Ker} [[-5] \rightarrow \mathbb{C}] \xrightarrow{dP} \text{Ker} [[-1]^{\oplus 5} \rightarrow \mathbb{C}] \right], \tag{2.27}$$

where the map in the first kernel is given by the defining relation and that in the second is given by the homogeneous coordinates. The symbols $[n]$ where n is a negative integer here denote spaces of rational functions of a given degree. More precisely, $[-|n|]$ denotes the space of rational functions constructed as a sum of terms, each of which is a rational monomial of the given degree. As in the $[|n|]$ case, only those functions that are invariant under the group action are included.

Finally, we will require the following description of this tangent bundle valued cohomology

$$H^1(\wedge^2 TX^\vee) = \text{Ker} \left[\text{Ker} \{[-10] \rightarrow [-5]\} \rightarrow \text{Ker} \left[\text{Ker} \{[-6] \rightarrow [-1]\}^{\oplus 5} \rightarrow \text{Ker} \{[-5] \rightarrow \mathbb{C}\} \right] \right].$$

Once more the maps in this expression are described by P , dP and the homogeneous coordinates, with which map is to be used being determined by which has the appropriate by degree.

With these descriptions of the relevant cohomologies in hand, let us proceed to compute the first kernel, given in (2.22). First, for simplicity in this initial example, we will choose our defining relation to be the Fermat quintic,

$$P = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5. \tag{2.28}$$

We take a general element of $H^1(TX)$, as described by (2.26)

$$\begin{aligned} & c_1 x_0 x_1 x_2 x_3 x_4 + c_2 (x_3 x_4^2 x_2^2 + x_0^2 x_1 x_2^2 + x_1^2 x_3^2 x_2 + x_0 x_1^2 x_4^2 + x_0^2 x_3^2 x_4) \\ & + c_3 (x_0^2 x_3 x_1^2 + x_2^2 x_4 x_1^2 + x_3^2 x_4^2 x_1 + x_0 x_2^2 x_3^2 + x_0^2 x_2 x_4^2) + c_4 (x_2 x_3 x_0^3 + x_1 x_3^3 x_0 + x_2^3 x_4 x_0 \\ & + x_1 x_2 x_4^3 + x_1^3 x_3 x_4) + c_5 (x_1 x_4 x_0^3 + x_3 x_4^3 x_0 + x_1^3 x_2 x_0 + x_1 x_2^3 x_3 + x_2 x_3^3 x_4). \end{aligned} \quad (2.29)$$

Note that this description of $H^1(TX)$ is in terms of degree five polynomials. As such this formulation of the complex structure is extremely easy to interpret. The elements of this space which lie in both of the kernels (2.22) and (2.23) are the fluctuations of the complex structure moduli allowed by the equations of motion. Small multiples of these polynomials can then be added to the initial defining relation (2.28) to see which family of Calabi-Yau hypersurfaces is left unstabilized by the given choice of fluxes.

To perform the mapping in (2.22) we will need a choice of flux $H_{(1,2)}$. This should be an element of $H^2(TX^\vee)$ and thus we describe it as in (2.27). In fact, we should be cautious as the flux we choose should be primitive according to the supergravity.

Consider a $(2, 1)$ field strength in any given cohomology class. The question we wish to know the answer to is, is there a field strength in the same cohomology class which is primitive? That is, if we have \mathcal{H} such that $[\mathcal{H}] \in H^{2,1}(X)$ does there exist \mathcal{H}' satisfying $[\mathcal{H}'] = [\mathcal{H}]$ such that $\mathcal{H}' \wedge J = 0$?

In the case at hand, that where X is a Calabi-Yau threefold, $\mathcal{H} \wedge J$ is an element of $H^{3,2}(X)$. Since $h^{3,2}(X) = 0$ for such a manifold we know that $\mathcal{H} \wedge J = \bar{\partial}\Lambda$ for some four-form Λ . In fact, we know a little more than this, thanks to some very well-known results.

The Hard Lefschetz theorem states that the map

$$L^k : H^{d-k}(X) \rightarrow H^{d+k}(X) \quad (2.30)$$

is an isomorphism. Here d is the complex dimension of X and L is the map on cohomology induced by the operation of performing a wedge product with the Kähler form. Taking the case where $k = 2$ we see that $L^2 : H^1(X) \rightarrow H^5(X)$ is an isomorphism. That is, any element of $H^5(X)$ can be written as $J \wedge J \wedge \alpha$ for some α . This implies that the same is true for $H^{3,2}(X)$ and thus, in the notation of the proceeding paragraph, $\bar{\partial}\Lambda = J \wedge J \wedge \bar{\partial}\gamma$ for some function γ (using the fact that $h^1(X) = 0$).

Using this information we can now easily see the desired result. By definition, $\mathcal{H}' = \mathcal{H} + \bar{\partial}\beta$ for some two-form β . Then $J \wedge \mathcal{H}' = J \wedge \mathcal{H} + J \wedge \bar{\partial}\beta = J \wedge J \wedge \bar{\partial}\gamma + J \wedge \bar{\partial}\beta$ and we see that an appropriate choice of β (namely $\beta = -J \wedge \gamma$) renders \mathcal{H}' primitive as desired.

Thus, whichever class in $H^2(TX^\vee)$ we choose, a suitable choice of primitive flux will exist within that class. An exactly analogous argument can be made for $H^1(\wedge^2 TX^\vee)$. The kernel computations we are performing here only depend upon the class of the map elements being used, and as such we do not need to know the exact form of the primitive representative to proceed.

In the case at hand, the most general possible map appearing in (2.22), as described by (2.27) and depending upon $h^2(TX^\vee) = 5$ parameters m_α , is given explicitly as follows

$$\begin{aligned}
& \frac{m_1}{x_0 x_1 x_2 x_3 x_4} + m_2 \left(\frac{1}{x_1^2 x_2 x_3^2} + \frac{1}{x_0^2 x_3^2 x_4} + \frac{1}{x_0 x_1^2 x_4^2} + \frac{1}{x_2^2 x_3 x_4^2} + \frac{1}{x_0^2 x_1 x_2^2} \right) \\
& + m_3 \left(\frac{1}{x_0 x_2^2 x_3^2} + \frac{1}{x_1^2 x_2^2 x_4} + \frac{1}{x_0^2 x_2 x_4^2} + \frac{1}{x_1 x_3^2 x_4^2} + \frac{1}{x_0^2 x_1^2 x_3} \right) \\
& + m_4 \left(\frac{1}{x_0 x_1 x_3^3} + \frac{1}{x_0 x_2^3 x_4} + \frac{1}{x_1^3 x_3 x_4} + \frac{1}{x_1 x_2 x_4^3} + \frac{1}{x_0^3 x_2 x_3} \right) \\
& + m_5 \left(\frac{1}{x_1 x_2^3 x_3} + \frac{1}{x_0^3 x_1 x_4} + \frac{1}{x_2 x_3^3 x_4} + \frac{1}{x_0 x_3 x_4^3} + \frac{1}{x_0 x_1^3 x_2} \right).
\end{aligned} \tag{2.31}$$

We multiply such a map by the general source element given in (2.29) and trim the result to only include constants - the relevant description of the target space in (2.22), $H^3(X)$. We find that, for a fluctuation of the form (2.29) to appear in the kernel (2.22) the following constraint on the coefficients c_i must hold

$$c_1 m_1 + 5c_2 m_2 + 5c_3 m_3 + 5c_4 m_4 + 5c_5 m_5 = 0. \tag{2.32}$$

So for example, if we choose the map corresponding to $m_1 = 5, m_2 = 3, m_3 = 4, m_4 = 10$ and $m_5 = 6$, then the most general fluctuation of the defining relation of the quotiented quintic which is allowed by the first constraint (2.22) is as follows

$$\begin{aligned}
& c_2 (x_3 x_4^2 x_2^2 + x_0^2 x_1 x_2^2 + x_1^2 x_3^2 x_2 - 3x_0 x_1 x_3 x_4 x_2 + x_0 x_1^2 x_4^2 + x_0^2 x_3^2 x_4) \\
& + c_3 (x_0^2 x_3 x_1^2 + x_2^2 x_4 x_1^2 + x_3^2 x_4^2 x_1 - 4x_0 x_2 x_3 x_4 x_1 + x_0 x_2^2 x_3^2 + x_0^2 x_2 x_4^2) \\
& + c_4 (x_2 x_3 x_0^3 + x_1 x_3^3 x_0 + x_2^3 x_4 x_0 - 10x_1 x_2 x_3 x_4 x_0 + x_1 x_2 x_4^3 + x_1^3 x_3 x_4) \\
& + c_5 (x_1 x_4 x_0^3 + x_3 x_4^3 x_0 + x_1^3 x_2 x_0 - 6x_1 x_2 x_3 x_4 x_0 + x_1 x_2^3 x_3 + x_2 x_3^3 x_4).
\end{aligned} \tag{2.33}$$

We see that we get one constraint on the general five parameter possible complex structure fluctuation as should be the case.

As we have seen, the first map is easily implemented, and the constraint on moduli it corresponds to can be mapped out explicitly. We now move on to consider the second kernel condition (2.23). Here we will see a complication in comparison to the heterotic case.

In the case of the second kernel condition the source space is the direct sum of (2.29), the complex structure fluctuations (which must also be constrained by the first condition), and the constants (which is the relevant description of $H^0(X)$). The target space is described by an expression of the form (2.31). Finally, the map is described by an element of $H^2(TX^\vee)$ as in (2.31) (which maps the $H^0(X)$ piece of the source to the target) together with an element of $H^1(\wedge^2 TX^\vee)$ (which maps the $H^1(TX)$ piece of the source to the target). The relevant description of this last cohomology group, depending upon $h^1(\wedge^2 TX^\vee) = 5$ parameters n_α ,

is given explicitly as follows

$$\begin{aligned}
& n_1 \left(\frac{1}{x_0^3 x_2^3 x_3^2 x_4^2} + \frac{1}{x_0^2 x_1^3 x_3^3 x_4^2} + \frac{1}{x_0^2 x_1^2 x_2^3 x_4^3} + \frac{1}{x_1^3 x_2^2 x_3^2 x_4^3} + \frac{1}{x_0^3 x_1^2 x_2^2 x_3^3} \right) \\
& + n_2 \left(\frac{1}{x_0^3 x_1^3 x_2^2 x_4^2} + \frac{1}{x_1^2 x_2^3 x_3^3 x_4^2} + \frac{1}{x_0^3 x_1^2 x_2^3 x_4^3} + \frac{1}{x_0^2 x_2^2 x_3^3 x_4^3} + \frac{1}{x_0^2 x_1^3 x_2^3 x_3^2} \right) \\
& + n_3 \left(\frac{1}{x_0^2 x_1 x_2^3 x_3^3 x_4} + \frac{1}{x_0 x_1^3 x_2^3 x_3 x_4^2} + \frac{1}{x_0^3 x_1 x_2^2 x_3 x_4^3} + \frac{1}{x_0 x_1^2 x_2 x_3^3 x_4^3} + \frac{1}{x_0^3 x_1^3 x_2 x_3^2 x_4} \right) \\
& + n_4 \left(\frac{1}{x_0 x_1^3 x_2^2 x_3^3 x_4} + \frac{1}{x_0^3 x_1 x_2 x_3^3 x_4^2} + \frac{1}{x_0^2 x_1^3 x_2 x_3 x_4^3} + \frac{1}{x_0 x_1 x_2^3 x_3^2 x_4^3} + \frac{1}{x_0^3 x_1^2 x_2^3 x_3 x_4} \right) \\
& + n_5 \frac{1}{x_0^2 x_1^2 x_2^2 x_3^2 x_4^2}.
\end{aligned} \tag{2.34}$$

The complication here arises in that the two components of this map cannot be chosen independently from the map, already specified in (2.22). The component living in $H^2(TX^\vee)$ should be proportional to the map already chosen and so this is relatively easy to determine. The component living in $H^1(\wedge^2 TX^\vee)$ is more problematic. In terms of differential forms, this map component should be the complex conjugate of the one appearing in (2.22). The problem is that these cohomologies are being described here in algebro-geometric terms and this process of complex conjugation is not transparent in such a formulation. Thus, it is rather difficult to know which component of $H^1(\wedge^2 TX^\vee)$ should be selected. Such a complication does not arise in the heterotic Atiyah class setting where there is a single map composed of a single component and no complex conjugation is required.

Note that this obstruction can be overcome in cases where the metric of the complex structure moduli space [66] is known for the Calabi-Yau in question in an appropriate form (see for example [67]). In such an instance one can combine this knowledge with the natural pairing

$$H^1(\wedge^2 TX^\vee) \times H^2(TX^\vee) \rightarrow \mathbb{C}, \tag{2.35}$$

in order to isolate the correct conjugate pairing. The point is that this pairing and the metric are essentially the same quantity up to an overall scale, and (2.35) can be computed explicitly for the polynomial descriptions of the cohomologies being utilized in this section. If the complex structure moduli space metric is given in bases for the barred and unbarred indices that are known to be conjugate, then the problem becomes soluble by performing a basis change on the cohomological spaces to match the pairing (2.35) with that metric. Such an involved computation is beyond the scope of this paper, and indeed it is dissatisfying that one needs to compute a Kähler potential in order to learn about flat directions of a superpotential in this approach. Nevertheless, in a case where one wished to have full control of the low energy theory, the Kähler potential would be required anyway and the above obstruction would be naturally overcome.

In short, one can easily determine the constraint from either (2.22) or (2.23) where the dilaton is taken to be fixed, but a complete analysis combining both constraints would require this

more subtle information. Let us give an example of the type of constraint on pure complex structure fluctuations that can arise from (2.23) in order to illustrate the more complex moduli stabilization results that can arise in this setting.

Let us choose as an example of the class of the (2, 1) field strength in (2.23) the element of (2.34) where $n_3 = n_4 = n_5 = 0$ and $n_1 = n_2 = 1$. Then, performing the map from $H^1(TX)$ to $H^2(TX^\vee)$ following a methodology analogous to that described above, we find the following for the allowed fluctuations of the defining relation (that is the complex structure)

$$P_{\text{fluctuation}} = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + \delta x_0 x_1 x_2 x_3 x_4. \quad (2.36)$$

Here δ is the fluctuation parameter. We see that this map does not give a generic result for the number of unconstrained moduli. The target space $H^2(TX^\vee)$ is of the same dimension as $H^1(TX)$ and thus we might naively expect all of the complex structure to be stabilized. With the above direct computation for this choice of map, however, we can see that this is not the case.

2.6.2 Relationship to Gukov-Vafa-Witten superpotentials

For type B the superpotential is $W = \int (F + ie^{-\phi}H) \wedge \Omega$. We are going to derive an equation for the fluctuation of the vacua from the following superpotential.

$$W = \int (F - ie^{-\phi}H) \wedge \Omega. \quad (2.37)$$

Consider the fluctuations

$$\begin{aligned} \Omega(x, y) &= \Omega^{BG}(y) + \delta\Omega(x, y), \\ \delta\Omega_{\bar{a}bc} &= -\frac{i}{2}\delta J_{\bar{a}}^d \Omega_{dbc}, \quad \delta J = z^I(x)v^I(y), \quad \text{where: } v^I \in H^1(TX), \\ \delta\Omega_{\bar{a}bc} &= -\frac{i}{2}z^I(x)(v^I(y))_{\bar{a}}^d \Omega_{dbc}. \end{aligned}$$

Then we get:

$$\begin{aligned} \frac{\partial W}{\partial z^I(x)} &= -\frac{i}{2} \int (F + ie^{-\phi}H) \wedge (v^I \star \Omega), \\ \text{where: } (v^I \star \Omega)_{(2,1)} &:= \Omega(v^I, \dots). \end{aligned}$$

Then

$$\begin{aligned} \delta \left(\frac{\partial W}{\partial z^I(x)} \right) &= -\frac{i}{2} \int \delta(F - ie^{-\phi}H) \wedge (v^I \star \Omega) + (F - ie^{-\phi}H) \wedge \delta(v^I \star \Omega), \\ &= -\frac{i}{2} \int \epsilon^{abc} \epsilon^{\bar{a}\bar{b}\bar{c}} [(F - ie^{-\phi}H)_{\bar{a}\bar{b}\bar{c}} (v^I)_{\bar{a}}^d \Omega_{dbc} - (F - ie^{-\phi}H)_{\bar{a}\bar{b}\bar{c}} \delta(v^I \star \Omega)_{\bar{a}\bar{b}\bar{c}}]. \end{aligned}$$

Note that because $(v^I \star \Omega)_{(1,2)} = 0$, we can use the same calculation to constrain fluctuations and as $\delta(v^I \star \Omega)_{\bar{a}\bar{b}c} + i\delta J_{[\bar{a}}^d(v^I \star \Omega)_{\bar{b}]cd} = 0$. Then:

$$\delta(v^I \star \Omega)_{\bar{a}\bar{b}c} = -i\delta J_{[\bar{a}}^d(v^I)_{\bar{b}]}^{d'} \Omega_{d'cd} = -i\delta J_{[\bar{a}}^d(v^I)_{\bar{b}]}^{d'} \Omega_{dd'c}.$$

For the background $(F - ie^{-\phi}H)_{\bar{a}\bar{b}c} = 0$, we have

$$\begin{aligned} \delta\left(\frac{\partial W}{\partial z^I(x)}\right) &= -\frac{i}{2} \int \epsilon^{abc} \epsilon^{\bar{a}\bar{b}\bar{c}} e^{-\phi} [\delta((F - ie^{-\phi}H)_{\bar{a}\bar{b}\bar{c}}(v^I)_{\bar{a}}^d \Omega_{dbc} \\ &\quad + i((F - ie^{-\phi}H)_{\bar{a}\bar{b}\bar{c}} \delta J_{[\bar{a}}^d(v^I)_{\bar{b}]}^{d'} \Omega_{dd'c})]. \end{aligned}$$

Now using $\Omega_{abc} = f\epsilon_{abc}$ and the properties $\epsilon_{ijk}\epsilon^{imn} = \delta_j^m \delta_k^n - \delta_j^n \delta_k^m$ and $\epsilon_{ijm}\epsilon^{ijn} = 2\delta_m^n$, we get:

$$\begin{aligned} \delta\left(\frac{\partial W}{\partial z^I(x)}\right) &= -\frac{i}{2} \int f e^{-\phi} \epsilon^{\bar{a}\bar{b}\bar{c}} \left[2(v^I)_{\bar{a}}^a \delta(F - ie^{-\phi}H)_{\bar{a}\bar{b}\bar{c}} \right. \\ &\quad \left. + i(F - ie^{-\phi}H)_{\bar{a}\bar{b}\bar{c}} \delta J_{\bar{a}}^d(v^I)_{\bar{b}}^{d'} (\delta_d^a \delta_{d'}^b - \delta_{d'}^a \delta_d^b) \right], \\ \delta\left(\frac{\partial W}{\partial z^I(x)}\right) &= -\frac{i}{2} \int f e^{-\phi} \epsilon^{\bar{a}\bar{b}\bar{c}} [2(v^I)_{\bar{a}}^a \delta((F - ie^{-\phi}H)_{\bar{a}\bar{b}\bar{c}} + 2i((F - ie^{-\phi}H)_{\bar{a}\bar{b}\bar{c}} \delta J_{\bar{a}}^a(v^I)_{\bar{b}}^b)], \\ \delta\left(\frac{\partial W}{\partial z^I(x)}\right) &= -\frac{i}{2} \int f e^{-\phi} \epsilon^{\bar{a}\bar{b}\bar{c}} (2)(v^I)_{\bar{a}}^a [\delta((F - ie^{-\phi}H)_{\bar{a}\bar{b}\bar{c}} + i\delta J_{\bar{b}}^d((F - ie^{-\phi}H)_{\bar{c}]ad})] = 0. \end{aligned}$$

Note that if $\dim H^1(TX) = n$ then $\delta(\partial W/\partial z^I(x)) = 0$, impose n constraints on the fluctuations, which is the same number that we found from the $\bar{a}\bar{b}\bar{c}$ components of the background variation. Next, write

$$W = \int (F + ie^{-\phi}H) \wedge \Omega = \int [dC_2 + (C_0 + ie^{-\phi}H)] \wedge \Omega = \int [dC_2 + \tau H] \wedge \Omega.$$

Now expanding $\tau(x, y) = \Sigma A^I(y) \tau^I(x)$, where $A^I(y)$ is a basis of functions for the compact internal manifold, we get:

$$\frac{\partial W}{\partial \tau^I(x)} = \int A(y) (H \wedge \Omega) = 0.$$

Because H is a $(2, 1) + (1, 2)$ form and Ω is a $(3, 0)$ form, we find $W = 0$ in this background.

Let's compute the fluctuation of $\partial W/\partial \tau^I$ to see if it agrees with the equations that we

found above for the fluctuation of the background.

$$\begin{aligned}
\delta \left(\frac{\partial W}{\partial \tau^I} \right) &= \int A^I(y) (\delta H \wedge \Omega + H \wedge \delta \Omega), \\
\delta \left(\frac{\partial W}{\partial \tau^I} \right) &= \int \epsilon^{abc} \epsilon^{\bar{a}\bar{b}\bar{c}} A^I(y) ((\delta H \wedge \Omega)_{abc\bar{a}\bar{b}\bar{c}} + (H \wedge \delta \Omega)_{abc\bar{a}\bar{b}\bar{c}}), \\
\delta \left(\frac{\partial W}{\partial \tau^I} \right) &= \int \epsilon^{abc} \epsilon^{\bar{a}\bar{b}\bar{c}} A^I(y) (-\delta H_{\bar{a}\bar{b}\bar{c}} \Omega_{abc} + 9 H_{\bar{a}\bar{b}\bar{c}} \delta \Omega_{abc}), \\
\delta \left(\frac{\partial W}{\partial \tau^I} \right) &= \int A^I(y) \epsilon^{\bar{a}\bar{b}\bar{c}} \epsilon^{\bar{a}\bar{b}\bar{c}} (-\delta H_{\bar{a}\bar{b}\bar{c}} \Omega_{abc} + 9 H_{\bar{a}\bar{b}\bar{c}} (-\frac{i}{2} \delta J_{\bar{a}}^d \Omega_{bcd})), \\
\delta \left(\frac{\partial W}{\partial \tau^I} \right) &= \int f A^I(y) \epsilon^{\bar{a}\bar{b}\bar{c}} (-6 \delta H_{\bar{a}\bar{b}\bar{c}} - \frac{9i}{2} H_{\bar{a}\bar{b}\bar{c}} \delta J_{\bar{a}}^d (2\delta_d^a)), \\
\delta \left(\frac{\partial W}{\partial \tau^I} \right) &= \int f A^I(y) \epsilon^{\bar{a}\bar{b}\bar{c}} (-6) (\delta H_{\bar{a}\bar{b}\bar{c}} + \frac{3i}{2} \delta J_{\bar{a}}^d H_{\bar{a}\bar{b}\bar{c}}) = 0,
\end{aligned}$$

where in the last line we used $\Omega_{abc} = f\epsilon_{abc}$. For any background, there is one dilaton modulus, in agreement with the one constraint that we found from the $\bar{a}\bar{b}\bar{c}$ components of the variation.

2.7 Type C, $a = \pm b$

In type C, we have the following non zero terms:

$$\begin{aligned}
W_3 &= \pm e^\phi * F_3^{(6)}, \\
\pm e^\phi F_3^{(3)} &= -2iW_5 = 2i\partial A = 4i\partial \log a = i\partial\phi.
\end{aligned}$$

From the first line we get:

$$(dJ - J \wedge W_4)_{(2,1)} = ie^\phi (F_3 - J \wedge F_3^{(3)})_{(2,1)}.$$

Now replacing $W_4 = 0$ and $e^\phi F_3^{(3)} = i\partial\phi$ we will get:

$$(dJ)_{(2,1)} = (ie^\phi F_3 + J \wedge \partial\phi)_{(2,1)},$$

which can be written :

$$(F_3)_{(2,1)} = -i(d(e^{-\phi} J))_{(2,1)}.$$

Combining this with its complex conjugate we get:

$$F_3 = i(\bar{\partial} - \partial)(e^{-\phi} J). \tag{2.38}$$

This equation turns into the Strominger system by the transformations $F \rightarrow H$ and $e^{-\phi} J \rightarrow J$. Then its analysis is of the same form as for type A.

2.8 F-theory like vacua

In F-theory-like vacua, which involve an imaginary self-dual 3-form flux, and additionally a nonconstant holomorphic axion-dilaton $\tau = \tau(z) = C_0 + ie^{-\phi}$, we have the following equations:

$$e^\phi F_1^{(3)} = 2e^\phi F_5^{(3)} = -iW_5 = -iW_4 = -i\partial\phi. \quad (2.39)$$

Using the first and last terms in the above equations, we get:

$$\begin{aligned} e^\phi F_a &= -i\partial_a\phi, \\ e^\phi F_{\bar{a}} &= i\partial_{\bar{a}}\phi, \end{aligned}$$

which can be written in a more compact way:

$$-i(F_{(1,0)} - F_{(0,1)}) = d(e^{-\phi}),$$

or

$$-i(\Pi^{(-)}F - \Pi^{(+)}F) = d(e^{-\phi}).$$

Taking the variation of this we get

$$-i\left(-\frac{i}{2}\delta J_a^b\right)F_b + i\delta F_{\bar{a}} = -\bar{\partial}_{\bar{a}}(\delta\phi e^{-\phi}).$$

Because $\delta F = d\delta C_0$ we get:

$$\frac{1}{2}\delta J_a^b F_b = \bar{\partial}_{\bar{a}}(\delta\phi e^{-\phi} + i\delta C_0).$$

Now consider the map

$$\begin{aligned} H^1(TX) &\rightarrow H^1(X), \\ \delta J_a^b &\mapsto \frac{1}{2}\delta J_a^b F_b. \end{aligned} \quad (2.40)$$

Now we need to show that the map is actually well defined. For that, we need to show that the image is closed. First of all, $dF_1 = 0$ which implies $d(F_{(1,0)} + F_{(0,1)}) = 0$. Also, because $-i(F_{(1,0)} - F_{(0,1)}) = d(e^{-\phi})$ then $d(F_{(1,0)} - F_{(0,1)}) = 0$, where combining them gives $dF_{(1,0)} = 0$ which results $\bar{\partial}F_{(1,0)} = 0$. This and the fact that $\bar{\partial}\delta J = 0$ proves our claim. It is clear that due to the map (2.40) only those elements of $H^1(TX)$ in the kernel are acceptable.

2.9 General case variation

Here we show that in the general case, we can not write the complex structure moduli in terms of Dolbeault cohomology. The reason for this, is that we cannot write the part of the variation which is not coupled to the complex structure as an exact piece.

For the intermediate solutions we have

$$\begin{aligned}\alpha W_3 - i\beta H^{(6)} - \beta e^\phi F^{(6)}|_{(1,2)} &= 0, \\ \beta W_3 - i\alpha H^{(6)} + \alpha e^\phi F^{(6)}|_{(1,2)} &= 0.\end{aligned}$$

Generally for the variation of a (1, 2)-form we have:

$$\Sigma|_{(1,2)} = 0 \Rightarrow (\delta\Sigma)_{a\bar{b}\bar{c}} + i\delta J_{[\bar{b}}{}^d \Sigma_{\bar{c}a]d} = 0.$$

Using this we have:

$$(\delta(\alpha W_3 - i\beta H^{(6)} - \beta e^\phi F^{(6)}))_{a\bar{b}\bar{c}} + i\delta J_{[\bar{b}}{}^d (\alpha W_3 - i\beta H^{(6)} - \beta e^\phi F^{(6)})_{\bar{c}a]d} = 0.$$

Now let's simplify the first term which is:

$$\delta(\alpha W_3 - i\beta H^{(6)} - \beta e^\phi F^{(6)}).$$

To this end, recall that

$$\begin{aligned}W_3 &= dJ - W_4 \wedge J, \\ H^{(6)} &= H - H^{(3)} \wedge J, \\ F^{(6)} &= F - F^{(3)} \wedge J,\end{aligned}$$

and

$$H = dB, \quad F = d(C_2 - C_0 B),$$

where $W_4, H^{(3)}, F^{(3)}$ can be written in terms of α and β :

$$\begin{aligned}W_4 &= \frac{4(\alpha^2 + \beta^2)^2}{\beta(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)} \bar{\partial}\beta, \\ H^{(3)} &= \frac{-8i\alpha(\alpha^2 + \beta^2)}{(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)} \bar{\partial}\beta, \\ F^{(3)} &= \frac{8\alpha}{3\alpha^2 + \beta^2} \bar{\partial}\beta.\end{aligned}$$

Now

$$\begin{aligned}\delta(\alpha W_3 - i\beta H^{(6)} - \beta e^\phi F^{(6)}) &= \\ &= \delta\alpha W_3 + \alpha \left(d\delta J - \delta \left(\frac{4(\alpha^2 + \beta^2)^2}{\beta(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)} \bar{\partial}\beta \right) \wedge J - W_4 \wedge \delta J \right) - i\delta\beta H^{(6)} \\ &\quad - i\beta \left(d(\delta B) - \delta \left(\frac{-8i\alpha(\alpha^2 + \beta^2)}{(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)} \bar{\partial}\beta \right) \wedge J - H^{(3)} \wedge \delta J \right) - \delta\beta e^\phi F^{(6)} \\ &\quad - \beta \left(\delta\phi e^\phi F + e^\phi d(\delta(C_2 - C_0 B)) + \delta \left(\frac{8\alpha}{3\alpha^2 + \beta^2} \bar{\partial}\beta \right) \wedge J - e^\phi F^{(3)} \wedge \delta J \right).\end{aligned}$$

Inserting $W_3 = i\frac{\beta}{\alpha}H^{(6)} + \frac{\beta}{\alpha}e^\phi F^{(6)}$ we get

$$\begin{aligned}
& iH^{(6)}\left(\frac{\delta\alpha}{\alpha}\beta - \delta\beta\right) + e^\phi F^{(6)}\left(\frac{\delta\alpha}{\alpha}\beta - \delta\beta\right) + \\
& \alpha d\delta J - i\beta d(\delta B) - \beta\delta\phi e^\phi F - \beta e^\phi d\left(\delta(C_2 - C_0B)\right) + \\
& \left(-\frac{\alpha}{\beta}\frac{4(\alpha^2 + \beta^2)^2}{\beta(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)} + \frac{8\alpha\beta(\alpha^2 + \beta^2)}{(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)} - \frac{8\alpha\beta}{3\alpha^2 + \beta^2}\right)\bar{\partial}\delta\beta \wedge J + \\
& \left(-\frac{\alpha}{\beta}\frac{4(\alpha^2 + \beta^2)^2}{\beta(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)} + \frac{8\alpha\beta(\alpha^2 + \beta^2)}{(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)} - \frac{8\alpha\beta}{3\alpha^2 + \beta^2}\right)\bar{\partial}\beta \wedge \delta J + \\
& \left[-\alpha\delta\left(\frac{4(\alpha^2 + \beta^2)^2}{\beta(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)}\right) + \beta\delta\left(\frac{8\alpha(\alpha^2 + \beta^2)}{(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)}\right) - \beta\delta\left(\frac{8\alpha}{3\alpha^2 + \beta^2}\right)\right]\bar{\partial}\beta \wedge J.
\end{aligned}$$

Simplifying yields:

$$\begin{aligned}
& iH^{(6)}\left(\frac{\delta\alpha}{\alpha}\beta - \delta\beta\right) + e^\phi F^{(6)}\left(\frac{\delta\alpha}{\alpha}\beta - \delta\beta\right) + \alpha d\delta J - i\beta d(\delta B) - \beta\delta\phi e^\phi F \\
& - \beta e^\phi d\left(\delta(C_2 - C_0B)\right) - \frac{4\alpha(\alpha^2 + 3\beta^2)}{\beta(\beta^2 + 3\alpha^2)}\bar{\partial}\delta\beta \wedge J - \frac{4\alpha(\alpha^2 + 3\beta^2)}{\beta(\beta^2 + 3\alpha^2)}\bar{\partial}\beta \wedge \delta J \\
& + \left[-\alpha\delta\left(\frac{4(\alpha^2 + \beta^2)}{\beta(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)}\right) + \beta\delta\left(\frac{8\alpha(\alpha^2 + \beta^2)}{(\alpha^2 - \beta^2)(3\alpha^2 + \beta^2)}\right) - \beta\delta\left(\frac{8\alpha}{3\alpha^2 + \beta^2}\right)\right]\bar{\partial}\beta \wedge J.
\end{aligned}$$

Now let's look at the coefficient of δJ . Using the equality $\beta(\beta^2 + 3\alpha^2)\bar{\partial}\alpha + \alpha(\alpha^2 + 3\beta^2)\bar{\partial}\beta = 0$, we get:

$$d(\alpha\delta J) + 3\bar{\partial}\alpha \wedge \delta J.$$

Because we can consider $d(\alpha\delta J)$ as the exact part of mapping in cohomology language, we will ignore it. So the remaining term $3\bar{\partial}\alpha \wedge \delta J$ is not closed because δJ is not. This concludes that we cannot write the variation of general case in terms of cohomological language similar to what we did in A, B, C cases.

2.10 Conclusion

In this chapter, we tried to see how many techniques developed in heterotic string theory for finding moduli in the general case can be applied to type IIB string theory.

We stated how one can get the constraint on the supersymmetric vacua of type IIB string theory for $SU(3)$ structure manifolds in the presence of fluxes. We saw that in three special cases of type IIB cases, type A, type B, and type C of vacua these methods can be directly applied. We saw that in an explicit example, that some simplifications and difficulties arises in type IIB in comparison to heterotic case. We saw that representing the cohomology groups in term of polynomials is easier and doable in type IIB case. On the other hand in type

IIB string, the maps are related by complex conjugate of the fields. When we represent the cohomology groups in terms of polynomials, it is difficult to see what is the representation of the complex conjugate of an element. In fact, we need to know the metric on the complex structure moduli space to find such a complex conjugate element, which is only known exactly for a few examples.

Finally, in an effort to see if one can apply the techniques of heterotic string to the general vacua of type IIB string, we saw that we could not write the moduli in terms of Dolbeault cohomology groups.

Chapter 3

(0,2) versions of exotic (2,2) GLSMs

The contents of this chapter were adapted with minor modifications, with permission from the International Journal of Modern Physics A, from our publication [11].

We start with a review of $\mathcal{N} = (0, 2)$ gauge theories. Then we show how one can write a $\mathcal{N} = (2, 2)$ gauge theory in $(0, 2)$ language. Then, we proceed by presenting our work. In [10], a $(2, 2)$ GLSM was given for the Calabi-Yau constructed as the self-intersection of the Grassmannian $G(2, 5)$, as well as several dual descriptions of that GLSM. In this section, we will describe the deformations of that GLSM to $(0, 2)$ in its various duality frames, and compare the results. We also introduce the way to construct anomaly free bundles over the intersection of Grassmannians.

3.1 Introduction to $\mathcal{N} = (0, 2)$ gauge theories in 2D

$\mathcal{N} = (0, 2)$ supersymmetric gauge theories are more general than $\mathcal{N} = (2, 2)$ theories, which are a special case. The name comes from the fact that there are two right-moving supersymmetries and no left-moving supersymmetry. They can be used in compactification of heterotic string theory, which makes them phenomenologically more interesting. All the constructions are similar to $\mathcal{N} = (2, 2)$ case. Finally, we will show how one can write a $\mathcal{N} = (2, 2)$ GLSM in $\mathcal{N} = (0, 2)$ language.

3.1.1 $\mathcal{N} = (0, 2)$ Superfield Formalism

Here we follow the notation of [68], (see also the reviews [69, 70]). Similarly to the $(2, 2)$ case, we should define a relevant superspace for stating $\mathcal{N} = (0, 2)$ supersymmetric theories. The $(0, 2)$ superspace is x^0, x^1, θ^+ , and $\bar{\theta}^+$, where θ^+ , and $\bar{\theta}^+$ are Grassmann variables and are related to each other by complex conjugation. The $(0, 2)$ superfields are functions on

this superspace, $\mathcal{F}(x^0, x^1, \theta^+, \bar{\theta}^+)$. The supersymmetry generators are

$$\begin{aligned}\mathcal{Q}_+ &= \frac{\partial}{\partial\theta^+} + i\bar{\theta}^+ \left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1} \right), \\ \bar{\mathcal{Q}}_+ &= -\frac{\partial}{\partial\bar{\theta}^+} - i\theta^+ \left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1} \right).\end{aligned}$$

The anticommutation relation for these operators is

$$\{\mathcal{Q}_+, \bar{\mathcal{Q}}_+\} = -2i \left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1} \right),$$

with all the other anticommutation relations vanishing. To classify superfields, we define the pair

$$\begin{aligned}D_+ &= \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+ \left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1} \right), \\ \bar{D}_+ &= -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+ \left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1} \right),\end{aligned}$$

such that they anticommute with \mathcal{Q}_+ and $\bar{\mathcal{Q}}_+$, i.e., $\{D_+, \mathcal{Q}_+\} = 0$, etc.

We want to introduce the gauge multiplet in this superspace, which encodes data about gauge fields and their superpartners. One can choose a gauge, similar to Wess-Zumino gauge in (2, 2) case, and introduce the following superfield

$$V = v_0 - v_1 - 2i\theta^+\bar{\lambda}_- - 2i\bar{\theta}^+\lambda_- + 2\theta^+\bar{\theta}^+D. \quad (3.1)$$

Also to proceed, we introduce the following gauge covariant derivatives in (0, 2) superspace:

$$\begin{aligned}\mathcal{D}_0 + \mathcal{D}_1 &= \partial_0 + \partial_1 + i(v_0 + v_1), & \mathcal{D}_0 - \mathcal{D}_1 &= \partial_0 - \partial_1 + iV, \\ \mathcal{D}_+ &= \frac{\partial}{\partial\theta^+} - i\bar{\theta}^+(\mathcal{D}_0 + \mathcal{D}_1), & \bar{\mathcal{D}}_+ &= -\frac{\partial}{\partial\bar{\theta}^+} + i\theta^+(\mathcal{D}_0 + \mathcal{D}_1).\end{aligned}$$

The gauge invariant field strength is $\Upsilon = [\bar{\mathcal{D}}_+, \mathcal{D}_0 - \mathcal{D}_1]$ and the Lagrangian for the dynamics of the gauge multiplets is

$$L_{\text{Gauge}} = \frac{1}{8e^2} \int d^2x d\theta^+ d\bar{\theta}^+ \text{Tr} \bar{\Upsilon} \Upsilon.$$

For a $U(1)$ gauge theory, this can be expanded to

$$L_{\text{Gauge}} = \frac{1}{e^2} \int d^2x \left(\frac{1}{2} F_{01}^2 + i\bar{\lambda}_-(\partial_0 + \partial_1)\lambda_- + \frac{1}{2} D^2 \right).$$

In (0, 2) GLSM's there are two kinds of matter fields. One is called chiral superfield Φ . It is in a representation of the gauge group, which satisfies

$$\bar{D}_+\Phi = 0,$$

and can be expanded in components as

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ - i\theta^+\bar{\theta}^+(D_0 + D_1)\phi.$$

If Φ_i is a superfield of charge Q_i with respect to a $U(1)$ gauge symmetry, its kinetic terms are

$$\begin{aligned} L_{\text{Chiral}} &= -\frac{i}{2} \sum_i \int d^2x d^2\theta \bar{\Phi}_i(\mathcal{D}_0 - \mathcal{D}_1)\Phi_i, \\ &= \sum_i \int d^2x \left(-|D_\mu\phi_i|^2 + \bar{\psi}_{i,+}i(D_0 - D_1)\psi_{i,+} - iQ_i\sqrt{2}\bar{\phi}_i\lambda_-\psi_{i,+} + iQ_i\sqrt{2}\bar{\psi}_{i,+}\bar{\lambda}_-\phi_i + Q_i D\bar{\phi}_i\phi_i \right). \end{aligned}$$

The other type of matter multiplet is an anticommuting, negative chirality spinor superfield Γ , in some representation of the gauge group. It is called a Fermi multiplet, obeying

$$\bar{D}_+\Gamma = \sqrt{2}E,$$

where E is a chiral superfield. The expansion of the Fermi multiplet is

$$\Gamma = \gamma_- - \sqrt{2}\theta^+G - i\theta^+\bar{\theta}^+(D_0 + D_1)\gamma_- - \sqrt{2}\bar{\theta}^+E,$$

where $E = E(\Phi_i)$ is a holomorphic function of chiral superfields, and it has the following θ expansion

$$E(\Phi_i) = E(\phi_i) + \sqrt{2}\theta^+\frac{\partial E}{\partial\phi_i}\psi_{+,i} - i\theta^+\bar{\theta}^+(D_0 + D_1)E(\phi_i).$$

The kinetic terms for Fermi multiplets are

$$L_{\text{Fermi}} = -\frac{1}{2} \sum_a \int d^2x d^2\theta \bar{\Gamma}_a \Gamma_a,$$

which can be expanded in components as

$$L_{\text{Fermi}} = \sum_{i,a} \int d^2x \left(i\bar{\gamma}_{-,a}(D_0 + D_1)\gamma_{-,a} + |G_a|^2 - |E_a(\phi_i)|^2 - \frac{\partial E_a}{\partial\phi_i}\bar{\gamma}_{-,a}\psi_{+,i} - \frac{\partial\bar{E}_a}{\partial\bar{\phi}_i}\bar{\psi}_{+,i}\gamma_{-,a} \right).$$

There are analogues of (2, 2) Fayet-Iliopoulos terms, which here are

$$\begin{aligned} L_{D,\theta} &= \frac{t}{4} \int d^2x \theta^+ \Upsilon|_{\bar{\theta}^+=0} + c.c. , \\ &= \frac{it}{2} \int d^2x (D - iF_{01}) + c.c. . \end{aligned}$$

Let $J^a(\Phi_i)$ be some chiral superfields such that $E_a J^a = 0$. Then

$$\bar{\mathcal{D}}_+(\Gamma_{-,a} J^a) = 0.$$

Using above property, we can introduce a term similar to the superpotential term in the (2, 2) case by

$$L_{\mathcal{W}} = -\frac{1}{\sqrt{2}} \int d^2x d\theta^+ \Gamma_{-,a} J^a + c.c.$$

It can be expanded in components as follows:

$$L_{\mathcal{W}} = -\sum_{i,a} \int d^2x \left(G_a J^a(\phi_i) + \gamma_{-,a} \psi_{+,i} \frac{\partial J^a}{\partial \phi_i} \right) + c.c.$$

Summing up all parts together, we get the following Lagrangian

$$L = L_{\text{Gauge}} + L_{\text{Chiral}} + L_{\text{Fermi}} + L_{D,\theta} + L_{\mathcal{W}}.$$

The potential of this theory after eliminating auxiliary fields D and G_a is

$$U(\phi_i) = \frac{1}{2} e^2 \left(\sum_i Q_i |\phi_i|^2 - r \right)^2 + \sum_a |E_a|^2 + |J_a|^2.$$

3.1.2 Writing a (2,2) GLSM in (0,2) language

In this section, we will show how to write a (2, 2) GLSM in (0, 2) language. Some of the information in a (2, 2) vector multiplet $V(x, \theta^\pm, \bar{\theta}^\pm)$ can be encoded in (0, 2) gauge multiplet described in equation (3.1). The remaining fields, namely σ , λ_+ and $\bar{\lambda}_+$ are encoded in a (0, 2) chiral multiplet Σ' .

Consider a (2, 2) chiral multiplet Φ . We can expand this multiplet in a (0, 2) chiral and Fermi multiplet as follows:

$$\begin{aligned} \Phi' &= \Phi|_{\theta^- = \bar{\theta}^- = 0}, \\ \Gamma_- &= \frac{1}{\sqrt{2}} \mathcal{D}_- \Phi|_{\theta^- = \bar{\theta}^- = 0}. \end{aligned}$$

Then

$$\bar{\mathcal{D}}_+ \Gamma_- = \frac{1}{\sqrt{2}} \{ \bar{\mathcal{D}}_+, \mathcal{D}_- \} \Phi|_{\theta^- = \bar{\theta}^- = 0} = \sqrt{2} (\sqrt{2} i Q \Sigma' \Phi') = \sqrt{2} E,$$

which means corresponding E to the Γ_- is $\sqrt{2} i Q \Sigma' \Phi'$.

3.1.3 Example: The quintic and its transformation

Consider the quintic example of section 1.3.5 in $(0, 2)$ language. This theory contains five chiral superfields Φ'_i of charge 1 and five corresponding Fermi multiplets $\Gamma_{-,i}$. It also has one chiral superfield P' of charge -5 and one corresponding Fermi multiplet Γ_0 . These Fermi superfields have E 's

$$\begin{aligned} E_i &= i\sqrt{2}\Sigma'(\Phi'_i), \\ E_0 &= i\sqrt{2}\Sigma'(-5P'). \end{aligned}$$

To complete the specification of the model, we need to pick additional functions J^i and J^0 which are given by

$$\begin{aligned} J^i &= \frac{\partial W}{\partial \Phi'_i} = P' \frac{\partial G}{\partial \Phi'_i}, \\ J^0 &= \frac{\partial W}{\partial P'} = G(\Phi'). \end{aligned}$$

We can trivially see that

$$E_0 J^0 + \sum_{i=1}^5 E_i J^i = 0. \tag{3.2}$$

One can deform this $(2, 2)$ to $(0, 2)$ theory by simply deforming E or J as long as the equation 3.2 is satisfied. One possible way to do that, is for example to deform J^i to

$$\begin{aligned} J^i &= P' \left(\frac{\partial G}{\partial \Phi'_i} + H^i(\Phi') \right), \\ J^0 &= G(\Phi'), \end{aligned}$$

such that

$$\Phi'_i H^i = 0.$$

3.2 $(0, 2)$ deformations of $G(2, N) \cap G(2, N)$

We present three different $(2, 2)$ GLSMs such that they share the geometric phase, the Calabi-Yau constructed as the self-intersection of the Grassmannian $G(2, 5)$. We show that these theories can be deformed in a same way to $(0, 2)$ GLSMs. We also show how one can construct anomaly free bundles on the geometry.

3.2.1 First description

The first (2,2) GLSM for $G(2, N) \cap G(2, N)$, presented in [10][section 4.1], was as a

$$\frac{U(1) \times SU(2) \times SU(2)}{\mathbb{Z}_2 \times \mathbb{Z}_2}$$

gauge theory with matter

- N chiral multiplets ϕ_a^i in the $(\mathbf{2}, \mathbf{1})_1$ representation,
- N chiral multiplets $\tilde{\phi}_{a'}^j$ in the $(\mathbf{1}, \mathbf{2})_1$ representation,
- $(1/2)N(N-1)$ chiral multiplets $p_{ij} = -p_{ji}$ in the $(\mathbf{1}, \mathbf{1})_{-2}$ representation,

with superpotential

$$W = \sum_{i < j} p_{ij} \left(f^{ij}(B) - \tilde{B}^{ij} \right),$$

where

$$B^{ij} = \epsilon^{ab} \phi_a^i \phi_b^j, \quad \tilde{B}^{ij} = \epsilon^{a'b'} \tilde{\phi}_{a'}^i \tilde{\phi}_{b'}^j$$

are the baryons in each $SU(2)$ factor, and $f^{ij}(x)$ define a linear isomorphism on the homogeneous coordinates of $\mathbb{P}^{(1/2)N(N-1)-1}$, defining the deformation of one of the copies of the Plücker embedding. Put another way,

$$f^{ij}(B) = f_{k\ell}^{ij} B^{k\ell}$$

for a constant invertible matrix $f_{k\ell}^{ij}$. Each \mathbb{Z}_2 factor in the gauge group linked the center of one of the two $SU(2)$'s with a \mathbb{Z}_2 subgroup of $U(1)$, and it is straightforward to check that the matter is invariant.

In this section, we shall describe (0,2) deformations of this theory.

The tangent bundle defined implicitly by this GLSM in its $r \gg 0$ phase is given¹ by the cohomology of the sequence

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O}(\mathbf{3}, \mathbf{1})_0 \oplus \mathcal{O}(\mathbf{1}, \mathbf{3})_0 \xrightarrow{E} \mathcal{O}(\mathbf{2}, \mathbf{1})_1^{\oplus N} \oplus \mathcal{O}(\mathbf{1}, \mathbf{2})_1^{\oplus N} \xrightarrow{J} \mathcal{O}(\mathbf{1}, \mathbf{1})_2^{\oplus (1/2)N(N-1)} \longrightarrow 0.$$

In our notation, $\mathcal{O}(\mathbf{n}, \mathbf{m})_p$ is the bundle defined by representation \mathbf{n} of the first $SU(2)$, \mathbf{m} of the second $SU(2)$, and charge p of the $U(1)$ factor. The leftmost factor, $\mathcal{O} \oplus \mathcal{O}(\mathbf{3}, \mathbf{1})_0 \oplus \mathcal{O}(\mathbf{1}, \mathbf{3})_0$, is defined by the gauginos in the theory. The middle factor is defined by the chiral

¹See *e.g.* [71] for a discussion of physical realizations of tangent bundles of PAX and PAXY models, which form the prototype for this observation.

multiplets $\phi_a^i, \tilde{\phi}_{a'}^i$. The rightmost factor is defined by the p_{ij} . As a consistency check, note that the rank of the resulting bundle is given by

$$2N + 2N - 7 - (1/2)N(N - 1)$$

coinciding with the expected dimension given in [10][section 3.2.1].

The $r \ll 0$ phase is realized nonperturbatively in the form of [72], so as mentioned in the introduction, we shall not try to write down a purely mathematical description of the tangent bundle.

Next, we consider $(0, 2)$ deformations. The E -terms are

$$\begin{aligned} E_{p_{i_1 i_2}} &= -\sigma \left(\tilde{N}_{i_1}^j p_{j i_2} - \tilde{N}_{i_2}^j p_{j i_1} \right), \\ E_a^i &= \sigma_a^b \phi_b^i + N_j^i \sigma \phi_a^j, \\ \tilde{E}_{a'}^i &= \tilde{\sigma}_{a'}^{b'} \tilde{\phi}_{b'}^i + \tilde{N}_j^i \sigma \tilde{\phi}_{a'}^j, \end{aligned}$$

where N_j^i, \tilde{N}_j^i are related by the constraint

$$N_{[j_1}^k f_{j_2]k}^{i_1 i_2} = \tilde{N}_k^{[i_1} f_{j_1 j_2]k}^{i_2]}, \quad (3.3)$$

with J terms

$$\begin{aligned} J_{p_{i_1 i_2}} &= f_{j_1 j_2}^{i_1 i_2} B^{j_1 j_2} - \tilde{B}^{i_1 i_2}, \\ J_{\phi_a^k} &= p_{i_1 i_2} f_{j_1 j_2}^{i_1 i_2} \frac{\partial B^{j_1 j_2}}{\partial \phi_a^k}, \\ J_{\tilde{\phi}_{a'}^k} &= -p_{i_1 i_2} \frac{\partial \tilde{B}^{i_1 i_2}}{\partial \tilde{\phi}_{a'}^k}, \end{aligned}$$

where σ_b^a is traceless. It is straightforward to check that $E \cdot J = 0$, as required by supersymmetry.

The $(2, 2)$ locus is given by taking

$$N_j^i = \delta_j^i = \tilde{N}_j^i,$$

which is easily checked to satisfy condition (3.3).

As a demonstration that other solutions to constraint (3.3) exist, the reader can verify that in the case

$$f_{j_1 j_2}^{i_1 i_2} = \frac{1}{2} (\delta_{j_1}^{i_1} \delta_{j_2}^{i_2} - \delta_{j_2}^{i_1} \delta_{j_1}^{i_2}),$$

constraint (3.3) is satisfied for

$$N_j^i = \alpha \delta_1^i \delta_j^2 = \tilde{N}_j^i,$$

where α is a constant.

3.2.2 Double dual description

Next, we turn to the ‘double dual’ of this GLSM described in [10][section 4.2], obtained by dualizing both of the $SU(2)$ factors in the GLSM for $G(2, N) \cap G(2, N)$ using the duality described in [73]. The result is a

$$\frac{U(1) \times Sp(N-3) \times Sp(N-3)}{\mathbb{Z}_2 \times \mathbb{Z}_2}$$

gauge theory with

- N fields φ_i^a in the $(\mathbf{N}-\mathbf{3}, \mathbf{1})_{-1}$ representation,
- N fields $\tilde{\varphi}_i^{a'}$ in the $(\mathbf{1}, \mathbf{N}-\mathbf{3})_{-1}$ representation,
- $(1/2)N(N-1)$ fields $b^{ij} = -b^{ji}$ in the $(\mathbf{1}, \mathbf{1})_2$ representation,
- $(1/2)N(N-1)$ fields $\tilde{b}^{ij} = -\tilde{b}^{ji}$ in the $(\mathbf{1}, \mathbf{1})_2$ representation,
- $(1/2)N(N-1)$ fields $p_{ij} = -p_{ji}$ in the $(\mathbf{1}, \mathbf{1})_{-2}$ representation,

with the superpotential

$$\begin{aligned} W &= \sum_{i < j} p_{ij} \left(f^{ij}(b) - \tilde{b}^{ij} \right) + \varphi_i^a \varphi_j^b J_{ab} b^{ij} + \tilde{\varphi}_i^{a'} \tilde{\varphi}_j^{b'} J_{a'b'} \tilde{b}^{ij}, \\ &= \left(A(p)_{ij} + \varphi_i^a \varphi_j^b J_{ab} \right) b^{ij} + \left(C(p)_{ij} + \tilde{\varphi}_i^{a'} \tilde{\varphi}_j^{b'} J_{a'b'} \right) \tilde{b}^{ij}, \end{aligned}$$

where J is the antisymmetric symplectic form, and $A(p)$, $C(p)$ are matrices that can be derived from the first line of the expression for the superpotential.

The $r \gg 0$ phase realizes geometry nonperturbatively in the sense of [72], so as described in the introduction, we shall not try to write down a purely mathematical description of the tangent bundle. The $r \ll 0$ phase, on the other hand, can be described perturbatively.

In the $r \ll 0$ phase, D terms imply that not all of the φ_i^a , $\tilde{\varphi}_i^{a'}$, and p_{ij} can vanish. The tangent bundle is built physically² as the cohomology of the complex

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O}((\text{adj}, \mathbf{1})_0) \oplus \mathcal{O}(\mathbf{1}, \text{adj})_0 \xrightarrow{E} A \xrightarrow{J} \mathcal{O}((\mathbf{1}, \mathbf{1})_{-2})^{\oplus (2)(1/2)N(N-1)} \longrightarrow 0,$$

where

$$A = \mathcal{O}((\mathbf{N}-\mathbf{3}, \mathbf{1})_{-1})^{\oplus N} \oplus \mathcal{O}((\mathbf{1}, \mathbf{N}-\mathbf{3})_{-1})^{\oplus N} \oplus \mathcal{O}((\mathbf{1}, \mathbf{1})_{-2})^{\oplus (1/2)N(N-1)}.$$

²See *e.g.* [71] for a discussion of physical realizations of tangent bundles of PAX and PAXY models, which form the prototype for this observation.

The leftmost terms are from the gauginos, the middle terms (A) are from φ_i^a , $\tilde{\varphi}_i^{a'}$, and p_{ij} , and the rightmost terms are from b^{ij} , \tilde{b}^{ij} . As a consistency check, note that for $N = 5$ the rank of this bundle is

$$(2)(5) + (2)(5) + 10 - 7 - 20 = 3,$$

as expected for a threefold.

We can describe (0,2) deformations of this theory as follows. We take

$$\begin{aligned} E_{p_{ij}} &= -\sigma \left(\tilde{N}_i^\ell p_{\ell j} - \tilde{N}_j^\ell p_{\ell i} \right), \\ E_{b^{ij}} &= \sigma \left(N_k^i b^{kj} - N_k^j b^{ki} \right), \\ E_{\tilde{b}^{ij}} &= \sigma \left(\tilde{N}_k^i \tilde{b}^{kj} - \tilde{N}_k^j \tilde{b}^{ki} \right), \\ E_{\varphi_i^a} &= \sigma_b^a \varphi_i^b - \sigma N_i^j \varphi_j^a, \\ E_{\tilde{\varphi}_i^{a'}} &= \tilde{\sigma}_{b'}^{a'} \tilde{\varphi}_i^{b'} - \sigma \tilde{N}_i^j \tilde{\varphi}_j^{a'}, \end{aligned}$$

where N_j^i , \tilde{N}_j^i are related by the constraint (3.3), namely

$$N_{[j_1}^k f_{j_2]k}^{i_1 i_2} = \tilde{N}_k^{[i_1} f_{j_1 j_2]k}^{i_2]},$$

and for J 's:

$$\begin{aligned} J_{p_{ij}} &= f_{i'j'}^{ij} b^{i'j'} - \tilde{b}^{ij}, \\ J_{b^{ij}} &= p_{i_1 i_2} f_{ij}^{i_1 i_2} + \varphi_i^a \varphi_j^b J_{ab}, \\ J_{\tilde{b}^{ij}} &= -p_{ij} + \tilde{\varphi}_i^a \tilde{\varphi}_j^b J_{ab}, \\ J_{\varphi_i^a} &= 2\varphi_k^b J_{ab} b^{ik}, \\ J_{\tilde{\varphi}_i^{a'}} &= 2\tilde{\varphi}_k^{b'} J_{a'b'} \tilde{b}^{ik}. \end{aligned}$$

It can be shown that for the deformations above, $E \cdot J = 0$, using the relation

$$\sigma_c^a J_{ab} = \sigma_b^a J_{ac}, \quad (3.4)$$

following from properties of the Sp Lie algebra.

On the (2,2) locus,

$$N_j^i = \tilde{N}_j^i,$$

just as in the original description.

3.2.3 Single dual description

In this section we dualize only one of the $SU(2)$ gauge factors of the first model discussed to $Sp(N-3)$, giving gauge group

$$\frac{U(1) \times Sp(N-3) \times SU(2)}{\mathbb{Z}_2 \times \mathbb{Z}_2}$$

with

- N fields φ_i^a in the $(\mathbf{N} - \mathbf{3}, \mathbf{1})_{-1}$ representation,
- $(1/2)N(N - 1)$ fields $b^{ij} = -b^{ji}$ in the $(\mathbf{1}, \mathbf{1})_2$ representation,
- N fields $\tilde{\phi}_{a'}^j$ in the $(\mathbf{1}, \mathbf{2})_1$ representation,
- $(1/2)N(N - 1)$ fields $p_{ij} = -p_{ji}$ in the $(\mathbf{1}, \mathbf{1})_{-2}$ representation,

with superpotential

$$W = \sum p_{ij}(f^{ij}(b) - \tilde{B}^{ij}) + \varphi_i^a \varphi_j^b J_{ab} b^{ij}$$

where $\tilde{B}^{ij} = \epsilon^{a'b'} \tilde{\phi}_{a'}^i \tilde{\phi}_{b'}^j$ and p_{ij} have charge -2 , b^{ij} have charge 2 , φ_i^a have charge -1 and $\tilde{\phi}_{a'}^i$ have charge 1 under the $U(1)$ factor in the gauge group.

In this duality frame, both the $r \gg 0$ and $r \ll 0$ phases have geometry determined in part by nonperturbative effects as in [72], so as mentioned in the introduction, at this time we cannot provide a simple monad description of either.

In $(0, 2)$ language, E deformations are

$$\begin{aligned} E_{p_{ij}} &= -\sigma \left(\tilde{N}_i^k p_{kj} - \tilde{N}_j^k p_{ki} \right), \\ E_{b^{ij}} &= \sigma \left(N_k^i b^{kj} - N_k^j b^{ki} \right), \\ E_{\varphi_i^a} &= \sigma_b^a \varphi_i^b - \sigma N_i^j \varphi_j^a, \\ E_{\tilde{\phi}_{a'}^i} &= \tilde{\sigma}_{a'}^{b'} \tilde{\phi}_{b'}^i + \sigma \tilde{N}_j^i \tilde{\phi}_{a'}^j, \end{aligned}$$

where N_j^i, \tilde{N}_j^i are related by the same constraint (3.3) as in the last two duality frames, namely

$$N_{[j_1}^k f_{j_2]k}^{i_1 i_2} = \tilde{N}_k^{[i_1} f_{j_1 j_2]k}^{i_2]},$$

and J deformations are

$$\begin{aligned} J_{p_{ij}} &= f^{ij}(b) - \tilde{B}^{ij}, \\ J_{b^{ij}} &= p_{mn} f_{ij}^{mn} + \varphi_i^a \varphi_j^b J_{ab}, \\ J_{\varphi_i^a} &= 2\varphi_j^b J_{ab} b^{ij}, \\ J_{\tilde{\phi}_{a'}^i} &= -2p_{ij} \epsilon^{a'b'} \tilde{\phi}_{b'}^j. \end{aligned}$$

It is straightforward to show that $EJ = 0$, using the symplectic property (3.4) of σ_b^a and the tracelessness of $\tilde{\sigma}_{b'}^{a'}$.

Just as in the last two duality frames, on the $(2, 2)$ locus,

$$N_j^i = \delta_j^i = \tilde{N}_j^i.$$

rank	A	B
6	$(\mathbf{2}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{2})_{-1} \oplus (\mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{1})_6^{\oplus 2}$	$(\mathbf{1}, \mathbf{1})_{10}$
7	$(\mathbf{2}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{2})_{-1} \oplus (\mathbf{1}, \mathbf{1})_2^{\oplus 2} \oplus (\mathbf{1}, \mathbf{1})_4^{\oplus 2}$	$(\mathbf{1}, \mathbf{1})_8$
7	$(\mathbf{2}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{2})_{-3} \oplus (\mathbf{1}, \mathbf{1})_4^{\oplus 2} \oplus (\mathbf{1}, \mathbf{1})_6^{\oplus 2}$	$(\mathbf{1}, \mathbf{1})_{12}$
7	$(\mathbf{2}, \mathbf{1})_{-3} \oplus (\mathbf{1}, \mathbf{2})_7 \oplus (\mathbf{1}, \mathbf{1})_4 \oplus (\mathbf{1}, \mathbf{1})_6 \oplus (\mathbf{1}, \mathbf{1})_{-2}^{\oplus 2}$	$(\mathbf{1}, \mathbf{1})_{14}$

Table 3.1: A few anomaly-free bundles on $G(2, 5) \cap G(2, 5)$.

3.2.4 Comparison of deformations

It is tempting to identify the N in any one duality frame with the N in any other, and similarly the \tilde{N} in any one duality frame with the \tilde{N} in any other. As a cautionary note, however, we observe that this is potentially too simplistic. For example, $(0,2)$ deformations of a Grassmannian $G(k, n)$ are parametrized by [74, 75] an $n \times n$ matrix B_j^i , and one might guess that the transpose would give the corresponding $(0,2)$ deformations of the dual Grassmannian $G(n-k, n)$. However, as observed in [74], merely taking the transpose of B_j^i does not generate the quantum sheaf cohomology ring of the deformed dual Grassmannian, hence the correct parameter match is more complicated than merely taking the transpose. In the present case, it is possible that the correct parameter match is more complicated than merely identifying all instances of N_j^i and \tilde{N}_j^i in different duality frames.

3.3 Examples of anomaly-free bundles on $G(2, 5) \cap G(2, 5)$

In table 3.1, we summarize several anomaly-free bundles built as kernels, in the form

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(A) \longrightarrow \mathcal{O}(B) \longrightarrow 0,$$

where A and B are representations of the gauge group.

Each of these bundles is represented by a $(0,2)$ GLSM with gauge group

$$\frac{U(1) \times SU(2) \times SU(2)}{\mathbb{Z}_2 \times \mathbb{Z}_2}$$

with matter

- 5 chiral superfields ϕ_a^i in the $(\mathbf{2}, \mathbf{1})_1$ representation,
- 5 chiral superfields $\tilde{\phi}_{a'}^i$ in the $(\mathbf{1}, \mathbf{2})_1$ representation,
- 10 Fermi superfields $\Lambda_{ij} = -\Lambda_{ji}$ in the $(\mathbf{1}, \mathbf{1})_{-2}$ representation,

- Fermi superfields Λ^α in representation A ,
- chiral superfields P_β in the dual of representation B ,

with $(0,2)$ superpotential

$$\Lambda_{ij} \left(f^{ij}(B) - \tilde{B}^{ij} \right) + P_\beta J_\alpha^\beta \Lambda^\alpha.$$

It is straightforward to check that each representation in table 3.1 is invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$, where each \mathbb{Z}_2 factor relates the center of one $SU(2)$ to a subgroup of $U(1)$.

In addition, there is also an anomaly-free bundle defined similarly by the data

rank	A	B
4	$(\mathbf{2}, \mathbf{2})_4 \oplus (\mathbf{1}, \mathbf{1})_2^{\oplus 3} \oplus (\mathbf{1}, \mathbf{1})_{-2}$	$(\mathbf{2}, \mathbf{1})_5 \oplus (\mathbf{1}, \mathbf{2})_5$

in the GLSM with gauge group

$$U(1) \times SU(2) \times SU(2)$$

and the same matter and superpotential as above. (Here, representation A is not invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$.)

In each case, Green-Schwarz anomaly cancellation requires, schematically,

$$\sum_{R_{\text{left}}} \text{tr} (T^a T^b) = \sum_{R_{\text{right}}} \text{tr} (T^a T^b).$$

Anomaly cancellation for the $U(1)$ factor can be understood in the standard form, as a sum of squares of charges. For the non-Abelian factors, anomaly cancellation in each factor can be written more explicitly as [71][section 3.1]

$$\sum_{R_{\text{left}}} (\dim R_{\text{left}}) \text{Cas}_2 (R_{\text{left}}) + (\dim \text{adj}) \text{Cas}_2 (\text{adj}) = \sum_{R_{\text{right}}} (\dim R_{\text{right}}) \text{Cas}_2 (R_{\text{right}}),$$

where we have explicitly incorporated the left-moving gauginos into the expression above. As $SU(2)$ generators are traceless, there are $SU(2) - SU(2)$ and $SU(2)' - SU(2)'$ anomalies, but no $U(1) - SU(2)$ or $SU(2) - SU(2)'$ anomalies to check. In checking such anomalies, it is handy to note that for an n -dimensional representation of $SU(2)$, Cas_2 is given by [76] [equ'n (7.27)]

$$\frac{1}{2} (n^2 - 1)$$

(using the fact that $\lambda_1 = n - 1$ in that reference's conventions).

Chapter 4

Mirror Symmetry of GLSM's with exceptional gauge groups

We start this chapter by reviewing Hori-Vafa's proposal for $U(1)$ GLSMs, and then we review Gu-Sharpe's proposal for GLSMs with non-Abelian gauge groups. In the rest, we describe our work, in which we have applied this proposal to GLSMs with F_4 , E_6 and E_7

4.1 Hori-Vafa Mirror symmetry for $U(1)$ GLSMs

In this section we will describe Hori-Vafa's proposal [13] for mirrors to $U(1)$ GLSMs without superpotential. We will mainly follow [77]. For more information on mirror symmetry we direct reader to the following literature [78–86]. Consider the following Lagrangian density

$$\mathcal{L}_0 = \int d^4\theta \left(-\frac{1}{2e^2} \bar{\Sigma} \Sigma + e^{2QV+B} - \frac{1}{2} (Y + \bar{Y}) B \right) + \frac{1}{2} \left(\int d^2\tilde{\theta} (-t\Sigma) + c.c. \right),$$

where, as we have described in section 1.3, V is a vector superfield and $\Sigma = \bar{D}_+ D_- V$ is a twisted chiral field. In addition, we have a real superfield B and a periodic twisted chiral field Y of period $2i\pi$.

In \mathcal{L}_0 superfields Y and B are auxiliary. We will first integrate over Y , and we will find a Lagrangian. Next, in \mathcal{L}_0 we will integrate over B and we will get a second Lagrangian. Because these two Lagrangians have been derived from \mathcal{L}_0 , they should be equivalent physically.

Now, let's integrate over Y . It will give the following equation of motion

$$\bar{D}_+ D_- B = D_+ \bar{D}_- B = 0.$$

The solution to the above equation is

$$B = \Psi \bar{\Psi},$$

where Ψ is a chiral superfield. If we take this back into \mathcal{L}_0 we get

$$\mathcal{L}_I = \int d^4\theta \left(-\frac{1}{2e^2} \bar{\Sigma} \Sigma + \bar{\Phi} e^{2QV} \Phi \right) + \frac{1}{2} \left(\int d^2\tilde{\theta} (-t\Sigma) + c.c. \right),$$

where $\Phi \equiv e^\Psi$. This is exactly the Lagrangian of a $U(1)$ GLSM containing one matter field of charge Q .

Now instead, integrate B out of \mathcal{L}_0 . The equation of motion for it is

$$B = -2QV + \log((Y + \bar{Y})/2).$$

Replacing B in \mathcal{L}_0 by $-2QV + \log((Y + \bar{Y})/2)$, we get

$$\mathcal{L}_{II'} = \int d^4\theta \left(-\frac{1}{2e^2} \bar{\Sigma} \Sigma + QV(Y + \bar{Y}) - \frac{1}{2}(Y + \bar{Y}) \log(Y + \bar{Y}) \right) + \frac{1}{2} \left(\int d^2\tilde{\theta} (-t\Sigma) + c.c. \right),$$

where we have used the fact that $\int d^4\theta (Y + \bar{Y}) = 0$, since Y is a twisted chiral superfield $\bar{D}_+ Y = D_- Y = 0$. There is an additional simplification in the above Lagrangian. We can write

$$\int d^4\theta QVY = -\frac{1}{4} \int d\theta^+ d\bar{\theta}^- (\bar{D}_+ D_- V) QY = \frac{1}{2} \int d^2\tilde{\theta} Q\Sigma Y.$$

So $\mathcal{L}_{II'}$ becomes

$$\mathcal{L}_{II''} = \int d^4\theta \left(-\frac{1}{2e^2} \bar{\Sigma} \Sigma - \frac{1}{2}(Y + \bar{Y}) \log(Y + \bar{Y}) \right) + \frac{1}{2} \left(\int d^2\tilde{\theta} (QY - t)\Sigma + c.c. \right).$$

This analysis is right at the classical level. Taking into account the instanton effect to $\mathcal{L}_{II''}$, coming from quantum corrections, we will get the following Lagrangian

$$\mathcal{L}_{II} = \int d^4\theta \left(-\frac{1}{2e^2} \bar{\Sigma} \Sigma - \frac{1}{2}(Y + \bar{Y}) \log(Y + \bar{Y}) \right) + \frac{1}{2} \left(\int d^2\tilde{\theta} ((QY - t)\Sigma + e^{-Y}) + c.c. \right).$$

As you see this Lagrangian has the twisted superpotential

$$\tilde{W} = (QY - t)\Sigma + e^{-Y}.$$

As an example, suppose we have a $U(1)$ GLSM with the following matter content: five chiral superfields Φ_i of charge 1 and one chiral superfield P with charge -5 . Assume the

GLSM has no superpotential. The geometric phase of this GLSM is a five-dimensional non-compact Calabi-Yau manifold, especially, the total space of $\mathcal{O}_{\mathbb{P}^4}(-5)$, a line bundle over \mathbb{P}^4 . The mirror of this theory has the effective superpotential

$$\tilde{W} = \left(\sum_{i=1}^5 Y_i - 5Y_P - t \right) \Sigma + \sum_{i=1}^5 e^{-Y_i} + e^{-Y_P}.$$

Now, if we integrate out Σ from this superpotential, the mirror Landau-Ginzburg description becomes

$$\tilde{W} = \sum_{i=1}^5 e^{-Y_i} + e^{-Y_P},$$

such that

$$\sum_{i=1}^5 Y_i - 5Y_P - t = 0.$$

Now, if we define $X_i = e^{-Y_i/5}$, the superpotential with the constraint becomes:

$$\tilde{W} = \sum_{i=1}^5 X_i^5 + e^{t/5} X_1 \cdots X_5.$$

Because the Y_i are periodic variables such that $Y_i \sim Y_i + 2\pi i$, then consequently $X_i \sim e^{2\pi i/5} X_i$. So \tilde{W} is Landau-Ginzburg \mathbb{Z}_5 -orbifold where as it explained in section 1.3.5 it is related to the quintic $\mathbb{P}^4[5]$. It shows that how $\mathcal{O}_{\mathbb{P}^4}(-5)$ is the mirror of $\mathbb{P}^4[5]$.

4.2 Brief review of the nonabelian mirror proposal

The non-Abelian mirror proposal of [14] is a generalization of the Abelian duality described in [13] (see also [87]). It takes the following form. For an A-twisted two-dimensional (2,2) supersymmetric gauge theory with connected gauge group G , the mirror is a B-twisted Landau-Ginzburg orbifold, defined by (twisted) chiral multiplets

- Y_i , corresponding to the N matter fields of the original gauge theory,
- $X_{\tilde{\mu}}$, corresponding to nonzero roots $\tilde{\mu}$ of the Lie algebra \mathfrak{g} of G , of dimension n ,
- $\sigma_a = \overline{D}_+ D_- V_a$, as many as the rank r of G , corresponding to a choice of the Cartan subalgebra of \mathfrak{g} , the Lie algebra of G .

with superpotential

$$\begin{aligned}
W = & \sum_{a=1}^r \sigma_a \left(\sum_{i=1}^N \rho_i^a Y_i - \sum_{\tilde{\mu}=1}^{n-r} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} - t_a \right) \\
& + \sum_{i=1}^N \exp(-Y_i) + \sum_{\tilde{\mu}=1}^{n-r} X_{\tilde{\mu}} - \sum_i \tilde{m}_i Y_i.
\end{aligned} \tag{4.1}$$

In the expression above, the ρ_i^a are components of weight vectors for the matter representations appearing in the original gauge theory, and $\alpha_{\tilde{\mu}}^a$ are components of nonzero roots (here viewed as defining a sublattice of the weight lattice). (Also, sometimes one uses $Z = -\ln X$ for simplicity.) The t_a are constants, corresponding to Fayet-Iliopoulos parameters of the original gauge theory, and the \tilde{m}_i are twisted masses in the original gauge theory. One then orbifolds by the Weyl group, which acts naturally on all the fields above, and leaves the superpotential invariant. The expression above was written for A-twisted gauge theories without a superpotential, but can be generalized to mirrors of gauge theories with superpotentials by assigning suitable R-charges and changing the fundamental fields accordingly, as explained in [14].

In the analysis of this theory, it was argued that some loci are dynamically excluded – specifically, loci where any $X_{\tilde{\mu}}$ vanishes. These loci turn out to reproduce excluded loci on Coulomb branches of the original gauge theories. Furthermore, critical loci of the superpotential above obey relations which correspond to relations in the OPE ring of the original A-twisted gauge theory. For gauge theories with $U(1)$ factors in G , one has continuous Fayet-Iliopoulos parameters, so one can speak of weak coupling limits, and those OPE relations are known as quantum cohomology relations. In cases in which G has no $U(1)$ factors, so that there are no continuous Fayet-Iliopoulos parameters, there is no weak coupling limit, and so referring to such relations as ‘quantum cohomology’ relations is somewhat misleading. In such cases, we refer to the relations as defining the Coulomb ring or Coulomb branch ring.

The work [14] checked the predictions of this proposal for excluded loci and Coulomb branch and quantum cohomology relations against known results for two-dimensional gauge theories in *e.g.* [72, 73, 88, 89], and gave general arguments for why correlation functions in this B-twisted theory should match correlation functions in corresponding A-twisted gauge theories, such as in *e.g.* [90–92]. It also studied mirrors to pure gauge theories, to test and refine predictions for IR behavior described in [93]. In this paper, we will apply this mirror construction to make predictions for two-dimensional (2,2) supersymmetric gauge theories with exceptional gauge groups. To make all of these comparisons, the paper [14] utilized the following operator mirror map:

$$\exp(-Y_i) = -\tilde{m}_i + \sum_{a=1}^r \sigma_a \rho_i^a, \tag{4.2}$$

$$X_{\tilde{\mu}} = \sum_{a=1}^r \sigma_a \alpha_{\tilde{\mu}}^a, \tag{4.3}$$

which we shall also use in this chapter. In this chapter we will consider the mirror Landau-Ginzburg theory for GLSM's with F_4 , E_6 and E_7 gauge groups.

4.3 F_4

In this section we will consider the mirror Landau-Ginzburg orbifold of an F_4 gauge theory with matter fields in the **26** fundamental representation, and then compute Coulomb branch relations. We also consider the pure gauge theory without matter fields.

4.3.1 Mirror Landau-Ginzburg orbifold

The mirror Landau-Ginzburg model has fields

- $Y_{i,\beta}$, $i \in \{1, \dots, n\}$, $\beta \in \{1, \dots, 26\}$, corresponding to the matter fields in n copies of the fundamental **26** dimensional representation of F_4 ,
- X_m , $m \in \{1, \dots, 48\}$, corresponding to the roots of F_4 ,
- σ_a , $a \in \{1, 2, 3, 4\}$.

We associate the roots, α_m^a , to X_m fields and the weights, $\rho_{i,\beta}^a$, of the fundamental **26** representation to $Y_{i,\beta}$.

For F_4 and all the later examples we will discuss in this paper, we would like instead to work with a basis for the roots and weights that corresponds to an integer charge lattice, so that the θ -angle periodicities take a more nearly standard form. In particular, the superpotential is invariant under such basis changes, since its terms are tensor contractions such as

$$\sum_a \sigma_a \rho_i^a Y^i.$$

We can pick any basis we like, so long as we consistently change coordinates in the tensors above. In particular, the superpotential (for this B-twisted Landau-Ginzburg model) does not depend explicitly upon *e.g.* the Cartan matrix, so the metric on the Lie algebra is not directly relevant in the presentation above. Thus, we have the flexibility to pick a basis such that the weights have integer coordinates, which yields standard θ -angle periodicities.

To be specific, we will write the weights and roots in terms of a basis of fundamental weights. Recall the fundamental weights are defined as follows. First, let $\{\alpha_\mu\}$ be a basis of simple roots, normalized so that the Cartan matrix $C_{\mu\nu}$ is given as

$$C_{\mu\nu} = 2 \frac{\alpha_\mu \cdot \alpha_\nu}{\alpha_\nu^2}. \quad (4.4)$$

The fundamental weights $\{\omega_\mu\}$ are then defined by the property that [94][section 13.1]

$$2\frac{\alpha_\mu \cdot \omega_\nu}{\alpha_\mu^2} = \delta_{\mu\nu}. \quad (4.5)$$

Furthermore, the fundamental weights form an integer basis for the weight lattice – every element of the weight lattice is a linear combination of fundamental weights with integer coefficients [94][section 13.1]. This is perfect for our purposes, as this basis yields standard θ -angle periodicities, and we will use this basis for all computations in this and later sections. To compute root and weight vectors as linear combinations of the fundamental weights, as displayed in the tables in this and later sections, we used the Mathematica package LieART [95].

The long roots and associated fields are listed in table 4.1. The short roots and associated fields are listed in table 4.2. The weights of the **26** and associated fields are listed in table 4.3.

Field	Positive root	Field	Negative root
X_1	$(1, 0, 0, 0)$	X_{25}	$(-1, 0, 0, 0)$
X_2	$(-1, 1, 0, 0)$	X_{26}	$(1, -1, 0, 0)$
X_3	$(0, -1, 2, 0)$	X_{27}	$(0, 1, -2, 0)$
X_4	$(0, 1, -2, 2)$	X_{28}	$(0, -1, 2, -2)$
X_5	$(1, -1, 0, 2)$	X_{29}	$(-1, 1, 0, -2)$
X_6	$(-1, 0, 0, 2)$	X_{30}	$(1, 0, 0, -2)$
X_7	$(0, 1, 0, -2)$	X_{31}	$(0, -1, 0, 2)$
X_8	$(1, -1, 2, -2)$	X_{32}	$(-1, 1, -2, 2)$
X_9	$(-1, 0, 2, -2)$	X_{33}	$(1, 0, -2, 2)$
X_{10}	$(1, 1, -2, 0)$	X_{34}	$(-1, -1, 2, 0)$
X_{11}	$(-1, 2, -2, 0)$	X_{35}	$(1, -2, 2, 0)$
X_{12}	$(2, -1, 0, 0)$	X_{36}	$(-2, 1, 0, 0)$

Table 4.1: Long roots of F_4 and associated fields.

Now, plugging the information above into the mirror superpotential with twisted masses

$$\begin{aligned}
W = & \sum_{a=1}^4 \sigma_a \left(\sum_{i=1}^n \sum_{\beta=1}^{26} \rho_{i,\beta}^a Y_{i,\beta} + \sum_{m=1}^{48} \alpha_m^a Z_m \right) \\
& - \sum_{i=1}^n \tilde{m}_i \sum_{\beta=1}^{26} Y_{i,\beta} + \sum_{i=1}^n \sum_{\beta=1}^{26} \exp(-Y_{i,\beta}) + \sum_{m=1}^{48} X_m, \quad (4.6)
\end{aligned}$$

where $X_m = \exp(-Z_m)$ and X_m are the fundamental fields, we get

$$W = \sum_{a=1}^4 \sigma_a \mathcal{C}^a - \sum_{i=1}^n \tilde{m}_i \sum_{\beta=1}^{26} Y_{i,\beta} + \sum_{i=1}^n \sum_{\beta=1}^{26} \exp(-Y_{i,\beta}) + \sum_{m=1}^{48} X_m, \quad (4.7)$$

Field	Positive root	Field	Negative root
X_{13}	$(0, 0, 0, 1)$	X_{37}	$(0, 0, 0, -1)$
X_{14}	$(0, 0, 1, -1)$	X_{38}	$(0, 0, -1, 1)$
X_{15}	$(0, 1, -1, 0)$	X_{39}	$(0, -1, 1, 0)$
X_{16}	$(1, -1, 1, 0)$	X_{40}	$(-1, 1, -1, 0)$
X_{17}	$(-1, 0, 1, 0)$	X_{41}	$(1, 0, -1, 0)$
X_{18}	$(1, 0, -1, 1)$	X_{42}	$(-1, 0, 1, -1)$
X_{19}	$(-1, 1, -1, 1)$	X_{43}	$(1, -1, 1, -1)$
X_{20}	$(1, 0, 0, -1)$	X_{44}	$(-1, 0, 0, 1)$
X_{21}	$(-1, 1, 0, -1)$	X_{45}	$(1, -1, 0, 1)$
X_{22}	$(0, -1, 1, 1)$	X_{46}	$(0, 1, -1, -1)$
X_{23}	$(0, -1, 2, -1)$	X_{47}	$(0, 1, -2, 1)$
X_{24}	$(0, 0, -1, 2)$	X_{48}	$(0, 0, 1, -2)$

Table 4.2: Short roots of F_4 and associated fields.

where the \mathcal{C}^a are given as follows:

$$\begin{aligned} \mathcal{C}^1 = \sum_{i=1}^n & (Y_{i,4} - Y_{i,5} + Y_{i,6} - Y_{i,7} + Y_{i,8} - Y_{i,9} + Y_{i,19} - Y_{i,20} + Y_{i,21} - Y_{i,16} + Y_{i,17} \\ & - Y_{i,18}) \\ & + Z_1 - Z_2 + Z_5 - Z_6 + Z_{16} - Z_{17} + Z_8 + Z_{18} - Z_9 - Z_{19} + Z_{20} - Z_{21} \\ & + Z_{10} - Z_{11} + 2Z_{12} - Z_{25} + Z_{26} - Z_{29} + Z_{30} - Z_{40} + Z_{41} - Z_{32} - Z_{42} \\ & + Z_{33} + Z_{43} - Z_{44} + Z_{45} - Z_{34} + Z_{35} - 2Z_{36}, \end{aligned}$$

$$\begin{aligned} \mathcal{C}^2 = \sum_{i=1}^n & (Y_{i,3} - Y_{i,4} + Y_{i,7} + Y_{i,9} - Y_{i,10} - Y_{i,11} - Y_{i,19} - Y_{i,21} + Y_{i,22} + Y_{i,23} - Y_{i,15} \\ & + Y_{i,16}) \\ & + Z_2 - Z_3 + Z_4 + Z_{15} - Z_5 + Z_7 - Z_{16} - Z_8 + Z_{19} + Z_{21} - Z_{22} + Z_{10} \\ & + 2Z_{11} - Z_{23} - Z_{12} - Z_{26} + Z_{27} - Z_{28} - Z_{39} + Z_{29} - Z_{31} + Z_{40} + Z_{32} \\ & - Z_{43} - Z_{45} + Z_{46} - Z_{34} - 2Z_{35} + Z_{47} + Z_{36}, \end{aligned}$$

$$\begin{aligned} \mathcal{C}^3 = \sum_{i=1}^n & (Y_{i,2} - Y_{i,3} + Y_{i,4} + Y_{i,5} - Y_{i,6} - Y_{i,7} + Y_{i,10} + 2Y_{i,11} - Y_{i,12} + Y_{i,19} - Y_{i,22} \\ & - 2Y_{i,23} + Y_{i,24} - Y_{i,14} + Y_{i,15} - Y_{i,16} - Y_{i,17} + Y_{i,18}) \\ & + 2Z_3 + Z_{14} - 2Z_4 - Z_{15} + Z_{16} + Z_{17} + 2Z_8 - Z_{18} + 2Z_9 - Z_{19} + Z_{22} \\ & - 2Z_{10} - 2Z_{11} + 2Z_{23} - Z_{24} - 2Z_{27} - Z_{38} + 2Z_{28} + Z_{39} - Z_{40} - Z_{41} \\ & - 2Z_{32} + Z_{42} - 2Z_{33} + Z_{43} - Z_{46} + 2Z_{34} + 2Z_{35} - 2Z_{47} + Z_{48}, \end{aligned}$$

Field	Weight	Field	Weight
$Y_{i,1}$	(0, 0, 0, 1)	$Y_{i,13}$	(0, 0, 0, -1)
$Y_{i,2}$	(0, 0, 1, -1)	$Y_{i,14}$	(0, 0, -1, 1)
$Y_{i,3}$	(0, 1, -1, 0)	$Y_{i,15}$	(0, -1, 1, 0)
$Y_{i,4}$	(1, -1, 1, 0)	$Y_{i,16}$	(-1, 1, -1, 0)
$Y_{i,5}$	(-1, 0, 1, 0)	$Y_{i,17}$	(1, 0, -1, 0)
$Y_{i,6}$	(1, 0, -1, 1)	$Y_{i,18}$	(-1, 0, 1, -1)
$Y_{i,7}$	(-1, 1, -1, 1)	$Y_{i,19}$	(1, -1, 1, -1)
$Y_{i,8}$	(1, 0, 0, -1)	$Y_{i,20}$	(-1, 0, 0, 1)
$Y_{i,9}$	(-1, 1, 0, -1)	$Y_{i,21}$	(1, -1, 0, 1)
$Y_{i,10}$	(0, -1, 1, 1)	$Y_{i,22}$	(0, 1, -1, -1)
$Y_{i,11}$	(0, -1, 2, -1)	$Y_{i,23}$	(0, 1, -2, 1)
$Y_{i,12}$	(0, 0, -1, 2)	$Y_{i,24}$	(0, 0, 1, -2)
$Y_{i,25}$	(0, 0, 0, 0)	$Y_{i,26}$	(0, 0, 0, 0)

Table 4.3: Weights of **26** of F_4 and associated fields.

$$\begin{aligned}
\mathcal{C}^4 = \sum_{i=1}^n & (Y_{i,1} - Y_{i,2} + Y_{i,6} + Y_{i,7} - Y_{i,8} - Y_{i,9} + Y_{i,10} - Y_{i,11} + 2Y_{i,12} - Y_{i,19} + Y_{i,20} \\
& + Y_{i,21} - Y_{i,22} + Y_{i,23} - 2Y_{i,24} - Y_{i,13} + Y_{i,14} - Y_{i,18}) \\
& + Z_{13} - Z_{14} + 2Z_4 + 2Z_5 + 2Z_6 - 2Z_7 - 2Z_8 + Z_{18} - 2Z_9 + Z_{19} - Z_{20} \\
& - Z_{21} + Z_{22} - Z_{23} + 2Z_{24} - Z_{37} + Z_{38} - 2Z_{28} - 2Z_{29} - 2Z_{30} + 2Z_{31} \\
& + 2Z_{32} - Z_{42} + 2Z_{33} - Z_{43} + Z_{44} + Z_{45} - Z_{46} + Z_{47} - 2Z_{48}.
\end{aligned}$$

Integrating out σ_a fields, we get four constraints $\mathcal{C}^a = 0$. So we are free to eliminate four fundamental fields. Our choice here will be $Y_{n,1}$, $Y_{n,2}$, $Y_{n,3}$ and $Y_{n,4}$.

$$\begin{aligned}
-Y_{n,1} = \sum_{i=1}^{n-1} & Y_{i,1} + \sum_{i=1}^n (Y_{i,5} + Y_{i,7} - Y_{i,8} + Y_{i,10} + Y_{i,12} - Y_{i,19} + Y_{i,20} - Y_{i,22} - Y_{i,24} - Y_{i,13} \\
& - Y_{i,17}) \\
& + Z_2 + Z_3 + Z_{13} + Z_4 + Z_5 + 2Z_6 - Z_7 + Z_{17} - Z_8 + Z_{19} - Z_{20} + Z_{22} - Z_{10} \\
& + Z_{24} - Z_{12} - Z_{26} - Z_{27} - Z_{37} - Z_{28} - Z_{29} - 2Z_{30} + Z_{31} - Z_{41} + Z_{32} - Z_{43} \\
& + Z_{44} - Z_{46} + Z_{34} - Z_{48} + Z_{36},
\end{aligned}$$

$$\begin{aligned}
-Y_{n,2} = & \sum_{i=1}^{n-1} Y_{i,2} + \sum_{i=1}^n (Y_{i,5} - Y_{i,6} + Y_{i,9} + Y_{i,11} - Y_{i,12} - Y_{i,21} - Y_{i,23} + Y_{i,24} - Y_{i,14} - Y_{i,17} \\
& + Y_{i,18}) \\
& + Z_2 + Z_3 + Z_{14} - Z_4 - Z_5 + Z_7 + Z_{17} + Z_8 - Z_{18} + 2Z_9 + Z_{21} - Z_{10} + Z_{23} \\
& - Z_{24} - Z_{12} - Z_{26} - Z_{27} - Z_{38} + Z_{28} + Z_{29} - Z_{31} - Z_{41} - Z_{32} + Z_{42} - 2Z_{33} \\
& - Z_{45} + Z_{34} - Z_{47} + Z_{48} + Z_{36},
\end{aligned}$$

$$\begin{aligned}
-Y_{n,3} = & \sum_{i=1}^{n-1} + \sum_{i=1}^n (-Y_{i,5} + Y_{i,6} + Y_{i,8} - Y_{i,10} - Y_{i,11} - Y_{i,20} + Y_{i,22} + Y_{i,23} - Y_{i,15} + Y_{i,17} \\
& - Y_{i,18}) \\
& + Z_1 - Z_3 + Z_4 + Z_{15} - Z_6 + Z_7 - Z_{17} + Z_{18} - Z_9 + Z_{20} - Z_{22} + 2Z_{10} + Z_{11} \\
& - Z_{23} + Z_{12} - Z_{25} + Z_{27} - Z_{28} - Z_{39} + Z_{30} - Z_{31} + Z_{41} - Z_{42} + Z_{33} - Z_{44} \\
& + Z_{46} - 2Z_{34} - Z_{35} + Z_{47} - Z_{36},
\end{aligned}$$

$$\begin{aligned}
-Y_{n,4} = & \sum_{i=1}^{n-1} Y_{i,4} + \sum_{i=1}^n (-Y_{i,5} + Y_{i,6} - Y_{i,7} + Y_{i,8} - Y_{i,9} + Y_{i,19} - Y_{i,20} + Y_{i,21} - Y_{i,16} + Y_{i,17} \\
& - Y_{i,18}) \\
& + Z_1 - Z_2 + Z_5 - Z_6 + Z_{16} - Z_{17} + Z_8 + Z_{18} - Z_9 - Z_{19} + Z_{20} - Z_{21} + Z_{10} \\
& - Z_{11} + 2Z_{12} - Z_{25} + Z_{26} - Z_{29} + Z_{30} - Z_{40} + Z_{41} - Z_{32} - Z_{42} + Z_{33} + Z_{43} \\
& - Z_{44} + Z_{45} - Z_{34} + Z_{35} - 2Z_{36}.
\end{aligned}$$

For convenience, we define:

$$\begin{aligned}
\Pi_1 & \equiv \exp(-Y_{n,1}), \\
& = \prod_{i=1}^n \exp(Y_{i,5} + Y_{i,7} - Y_{i,8} + Y_{i,10} + Y_{i,12} - Y_{i,19} + Y_{i,20} - Y_{i,22} - Y_{i,24} - Y_{i,13} - Y_{i,17}) \\
& \quad \cdot \prod_{i=1}^{n-1} \exp(Y_{i,1}) \cdot \frac{X_7 X_8 X_{20} X_{10} X_{12} X_{26} X_{27} X_{37} X_{28} X_{29} X_{30}^2 X_{41} X_{43} X_{46} X_{48}}{X_2 X_3 X_{13} X_4 X_5 X_6^2 X_{17} X_{19} X_{22} X_{24} X_{31} X_{32} X_{44} X_{34} X_{36}}, \tag{4.8}
\end{aligned}$$

$$\begin{aligned}
\Pi_2 & \equiv \exp(-Y_{n,2}), \\
& = \prod_{i=1}^n \exp(Y_{i,5} - Y_{i,6} + Y_{i,9} + Y_{i,11} - Y_{i,12} - Y_{i,21} - Y_{i,23} + Y_{i,24} - Y_{i,14} - Y_{i,17} + Y_{i,18}) \\
& \quad \cdot \prod_{i=1}^{n-1} \exp(Y_{i,2}) \cdot \frac{X_4 X_5 X_{18} X_{10} X_{24} X_{12} X_{26} X_{27} X_{38} X_{31} X_{41} X_{32} X_{33}^2 X_{45} X_{47}}{X_2 X_3 X_{14} X_7 X_{17} X_8 X_9^2 X_{21} X_{23} X_{28} X_{29} X_{42} X_{34} X_{48} X_{36}}, \tag{4.9}
\end{aligned}$$

$$\begin{aligned}
\Pi_3 &\equiv \exp(-Y_{n,3}), \\
&= \prod_{i=1}^n \exp(-Y_{i,5} + Y_{i,6} + Y_{i,8} - Y_{i,10} - Y_{i,11} - Y_{i,20} + Y_{i,22} + Y_{i,23} - Y_{i,15} + Y_{i,17} - Y_{i,18}) \\
&\quad \cdot \prod_{i=1}^{n-1} \exp(Y_{i,3}) \cdot \frac{X_3 X_6 X_{17} X_9 X_{22} X_{23} X_{25} X_{28} X_{39} X_{31} X_{42} X_{44} X_{34}^2 X_{35} X_{36}}{X_1 X_4 X_{15} X_7 X_{18} X_{20} X_{10}^2 X_{11} X_{12} X_{27} X_{30} X_{41} X_{33} X_{46} X_{47}}, \tag{4.10}
\end{aligned}$$

$$\begin{aligned}
\Pi_4 &\equiv \exp(-Y_{n,4}), \\
&= \prod_{i=1}^n \exp(-Y_{i,5} + Y_{i,6} - Y_{i,7} + Y_{i,8} - Y_{i,9} + Y_{i,19} - Y_{i,20} + Y_{i,21} - Y_{i,16} + Y_{i,17} - Y_{i,18}) \\
&\quad \cdot \prod_{i=1}^{n-1} \exp(Y_{i,4}) \cdot \frac{X_2 X_6 X_{17} X_9 X_{19} X_{21} X_{11} X_{25} X_{29} X_{40} X_{32} X_{42} X_{44} X_{34} X_{36}^2}{X_1 X_5 X_{16} X_8 X_{18} X_{20} X_{10} X_{12}^2 X_{26} X_{30} X_{41} X_{33} X_{43} X_{45} X_{35}}. \tag{4.11}
\end{aligned}$$

Integrating out the sigma fields and eliminating the fields, $Y_{n,1}$, $Y_{n,2}$, $Y_{n,3}$ and $Y_{n,4}$, the superpotential reduces to

$$\begin{aligned}
W &= \sum_{i=1}^{n-1} \sum_{b=1}^{26} (\exp(-Y_{i,b}) - \tilde{m}_i Y_{i,b}) \\
&\quad + (\Pi_1 + \tilde{m}_n \ln \Pi_1) + (\Pi_2 + \tilde{m}_n \ln \Pi_2) + (\Pi_3 + \tilde{m}_n \ln \Pi_3) + (\Pi_4 + \tilde{m}_n \ln \Pi_4) \\
&\quad + \sum_{a=5}^{26} (\exp(-Y_{n,a}) - \tilde{m}_n Y_{n,a}) + \sum_{m=1}^{48} X_m.
\end{aligned}$$

The superpotential is only well defined when the X_m fields in the denominator of Π_a 's are non-zero.

The critical locus is given as follows:

For $i < n$:

$$\frac{\partial W}{\partial Y_{i,1}} : \exp(-Y_{i,1}) = \Pi_1 + \tilde{m}_n - \tilde{m}_i, \tag{4.12}$$

$$\frac{\partial W}{\partial Y_{i,2}} : \exp(-Y_{i,2}) = \Pi_2 + \tilde{m}_n - \tilde{m}_i, \tag{4.13}$$

$$\frac{\partial W}{\partial Y_{i,3}} : \exp(-Y_{i,3}) = \Pi_3 + \tilde{m}_n - \tilde{m}_i, \tag{4.14}$$

$$\frac{\partial W}{\partial Y_{i,4}} : \exp(-Y_{i,4}) = \Pi_4 + \tilde{m}_n - \tilde{m}_i, \tag{4.15}$$

For $i \leq n$:

$$\frac{\partial W}{\partial Y_{i,5}} : \exp(-Y_{i,5}) = \Pi_1 + \Pi_2 - \Pi_3 - \Pi_4 - \tilde{m}_i, \quad (4.16)$$

$$\frac{\partial W}{\partial Y_{i,6}} : \exp(-Y_{i,6}) = -\Pi_2 + \Pi_3 + \Pi_4 + \tilde{m}_n - \tilde{m}_i, \quad (4.17)$$

$$\frac{\partial W}{\partial Y_{i,7}} : \exp(-Y_{i,7}) = \Pi_1 - \Pi_4 - \tilde{m}_i, \quad (4.18)$$

$$\frac{\partial W}{\partial Y_{i,8}} : \exp(-Y_{i,8}) = -\Pi_1 + \Pi_3 + \Pi_4 + \tilde{m}_n - \tilde{m}_i, \quad (4.19)$$

$$\frac{\partial W}{\partial Y_{i,9}} : \exp(-Y_{i,9}) = \Pi_2 - \Pi_4 - \tilde{m}_i, \quad (4.20)$$

$$\frac{\partial W}{\partial Y_{i,10}} : \exp(-Y_{i,10}) = \Pi_1 - \Pi_3 - \tilde{m}_i, \quad (4.21)$$

$$\frac{\partial W}{\partial Y_{i,11}} : \exp(-Y_{i,11}) = \Pi_2 - \Pi_3 - \tilde{m}_i, \quad (4.22)$$

$$\frac{\partial W}{\partial Y_{i,12}} : \exp(-Y_{i,12}) = \Pi_1 - \Pi_2 - \tilde{m}_i, \quad (4.23)$$

$$\frac{\partial W}{\partial Y_{i,13}} : \exp(-Y_{i,13}) = -\Pi_1 - \tilde{m}_n - \tilde{m}_i, \quad (4.24)$$

$$\frac{\partial W}{\partial Y_{i,14}} : \exp(-Y_{i,14}) = -\Pi_2 - \tilde{m}_n - \tilde{m}_i, \quad (4.25)$$

$$\frac{\partial W}{\partial Y_{i,15}} : \exp(-Y_{i,15}) = -\Pi_3 - \tilde{m}_n - \tilde{m}_i, \quad (4.26)$$

$$\frac{\partial W}{\partial Y_{i,16}} : \exp(-Y_{i,16}) = -\Pi_4 - \tilde{m}_n - \tilde{m}_i, \quad (4.27)$$

$$\frac{\partial W}{\partial Y_{i,17}} : \exp(-Y_{i,17}) = -\Pi_1 - \Pi_2 + \Pi_3 + \Pi_4 - \tilde{m}_i, \quad (4.28)$$

$$\frac{\partial W}{\partial Y_{i,18}} : \exp(-Y_{i,18}) = \Pi_2 - \Pi_3 - \Pi_4 - \tilde{m}_n - \tilde{m}_i, \quad (4.29)$$

$$\frac{\partial W}{\partial Y_{i,19}} : \exp(-Y_{i,19}) = -\Pi_1 + \Pi_4 - \tilde{m}_i, \quad (4.30)$$

$$\frac{\partial W}{\partial Y_{i,20}} : \exp(-Y_{i,20}) = \Pi_1 - \Pi_3 - \Pi_4 - \tilde{m}_n - \tilde{m}_i, \quad (4.31)$$

$$\frac{\partial W}{\partial Y_{i,21}} : \exp(-Y_{i,21}) = -\Pi_2 + \Pi_4 - \tilde{m}_i, \quad (4.32)$$

$$\frac{\partial W}{\partial Y_{i,22}} : \exp(-Y_{i,22}) = -\Pi_1 + \Pi_3 - \tilde{m}_i, \quad (4.33)$$

$$\frac{\partial W}{\partial Y_{i,23}} : \exp(-Y_{i,23}) = -\Pi_2 + \Pi_3 - \tilde{m}_i, \quad (4.34)$$

$$\frac{\partial W}{\partial Y_{i,24}} : \exp(-Y_{i,24}) = -\Pi_1 + \Pi_2 - \tilde{m}_i, \quad (4.35)$$

$$\frac{\partial W}{\partial Y_{i,25}} : \exp(-Y_{i,25}) = -\tilde{m}_i, \quad (4.36)$$

$$\frac{\partial W}{\partial Y_{i,26}} : \exp(-Y_{i,26}) = -\tilde{m}_i. \quad (4.37)$$

In the same way, $\partial W/\partial X_m$ gives:

$$X_1 = \Pi_3 + \Pi_4, \quad X_{25} = -\Pi_3 - \Pi_4, \quad (4.38)$$

$$X_2 = \Pi_1 + \Pi_2 - \Pi_4, \quad X_{26} = -\Pi_1 - \Pi_2 + \Pi_4, \quad (4.39)$$

$$X_3 = \Pi_1 + \Pi_2 - \Pi_3, \quad X_{27} = -\Pi_1 - \Pi_2 + \Pi_3, \quad (4.40)$$

$$X_4 = \Pi_1 + \Pi_3, \quad X_{28} = -\Pi_1 - \Pi_3, \quad (4.41)$$

$$X_5 = \Pi_1 - \Pi_2 + \Pi_4, \quad X_{29} = -\Pi_1 + \Pi_2 - \Pi_4, \quad (4.42)$$

$$X_6 = 2\Pi_1 - \Pi_4, \quad X_{30} = -2\Pi_1 + \Pi_4, \quad (4.43)$$

$$X_7 = -\Pi_1 + \Pi_2 + \Pi_3, \quad X_{31} = \Pi_1 - \Pi_2 - \Pi_3, \quad (4.44)$$

$$X_8 = -\Pi_1 + \Pi_2 + \Pi_4, \quad X_{32} = \Pi_1 - \Pi_2 - \Pi_4, \quad (4.45)$$

$$X_9 = 2\Pi_2 - \Pi_3 - \Pi_4, \quad X_{33} = -2\Pi_2 + \Pi_3 + \Pi_4, \quad (4.46)$$

$$X_{10} = -\Pi_1 - \Pi_2 + 2\Pi_3 + \Pi_4, \quad X_{34} = \Pi_1 + \Pi_2 - 2\Pi_3 - \Pi_4, \quad (4.47)$$

$$X_{11} = \Pi_3 - \Pi_4, \quad X_{35} = -\Pi_3 + \Pi_4, \quad (4.48)$$

$$X_{12} = -\Pi_1 - \Pi_2 + \Pi_3 + 2\Pi_4, \quad X_{36} = \Pi_1 + \Pi_2 - \Pi_3 - 2\Pi_4, \quad (4.49)$$

$$X_{13} = \Pi_1, \quad X_{37} = -\Pi_1, \quad (4.50)$$

$$X_{14} = \Pi_2, \quad X_{38} = -\Pi_2, \quad (4.51)$$

$$X_{15} = \Pi_3, \quad X_{39} = -\Pi_3, \quad (4.52)$$

$$X_{16} = \Pi_4, \quad X_{40} = -\Pi_4, \quad (4.53)$$

$$X_{17} = \Pi_1 + \Pi_2 - \Pi_3 - \Pi_4, \quad X_{41} = -\Pi_1 - \Pi_2 + \Pi_3 + \Pi_4, \quad (4.54)$$

$$X_{18} = -\Pi_2 + \Pi_3 + \Pi_4, \quad X_{42} = \Pi_2 - \Pi_3 - \Pi_4, \quad (4.55)$$

$$X_{19} = \Pi_1 - \Pi_4, \quad X_{43} = -\Pi_1 + \Pi_4, \quad (4.56)$$

$$X_{20} = -\Pi_1 + \Pi_3 + \Pi_4, \quad X_{44} = \Pi_1 - \Pi_3 - \Pi_4, \quad (4.57)$$

$$X_{21} = \Pi_2 - \Pi_4, \quad X_{45} = -\Pi_2 + \Pi_4, \quad (4.58)$$

$$X_{22} = \Pi_1 - \Pi_3, \quad X_{46} = -\Pi_1 + \Pi_3, \quad (4.59)$$

$$X_{23} = \Pi_2 - \Pi_3, \quad X_{47} = -\Pi_2 + \Pi_3, \quad (4.60)$$

$$X_{24} = \Pi_1 - \Pi_2, \quad X_{48} = -\Pi_1 + \Pi_2. \quad (4.61)$$

Now, plug these constraints back into (4.8)-(4.11) to get:

$$\begin{aligned} \Pi_1 = & \prod_{i=1}^{n-1} (\Pi_1 + \tilde{m}_n - \tilde{m}_i)^{-1} \prod_{i=1}^n (\Pi_1 + \Pi_2 - \Pi_3 - \Pi_4 - \tilde{m}_i)^{-1} (\Pi_1 - \Pi_4 - \tilde{m}_i)^{-1} \\ & \cdot (-\Pi_1 + \Pi_3 + \Pi_4 + \tilde{m}_n - \tilde{m}_i) (\Pi_1 - \Pi_3 - \tilde{m}_i)^{-1} (\Pi_1 - \Pi_2 - \tilde{m}_i)^{-1} \\ & \cdot (-\Pi_1 + \Pi_2 - \tilde{m}_i) (\Pi_1 - \Pi_3 - \Pi_4 - \tilde{m}_n - \tilde{m}_i)^{-1} (-\Pi_1 + \Pi_3 - \tilde{m}_i) \\ & \cdot (-\Pi_1 + \Pi_2 - \tilde{m}_i) (-\Pi_1 - \tilde{m}_n - \tilde{m}_i) (-\Pi_1 - \Pi_2 + \Pi_3 + \Pi_4 - \tilde{m}_i), \end{aligned}$$

$$\begin{aligned} \Pi_2 = & \prod_{i=1}^{n-1} (\Pi_2 + \tilde{m}_n - \tilde{m}_i)^{-1} \prod_{i=1}^n (\Pi_1 + \Pi_2 - \Pi_3 - \Pi_4 - \tilde{m}_i)^{-1} (-\Pi_2 + \Pi_3 + \Pi_4 + \tilde{m}_n - \tilde{m}_i) \\ & \cdot (\Pi_2 - \Pi_4 - \tilde{m}_i)^{-1} (\Pi_2 - \Pi_3 - \tilde{m}_i)^{-1} (\Pi_1 - \Pi_2 - \tilde{m}_i) \\ & \cdot (-\Pi_2 + \Pi_4 - \tilde{m}_i) (-\Pi_2 + \Pi_3 - \tilde{m}_i) (-\Pi_1 + \Pi_2 - \tilde{m}_i)^{-1} \\ & \cdot (-\Pi_2 - \tilde{m}_n - \tilde{m}_i) (-\Pi_1 - \Pi_2 + \Pi_3 + \Pi_4 - \tilde{m}_i) (\Pi_2 - \Pi_3 - \Pi_4 - \tilde{m}_n - \tilde{m}_i)^{-1}, \end{aligned}$$

$$\begin{aligned} \Pi_3 = & \prod_{i=1}^{n-1} (\Pi_3 + \tilde{m}_n - \tilde{m}_i)^{-1} \prod_{i=1}^n (\Pi_1 + \Pi_2 - \Pi_3 - \Pi_4 - \tilde{m}_i) (-\Pi_2 + \Pi_3 + \Pi_4 + \tilde{m}_n - \tilde{m}_i)^{-1} \\ & \cdot (-\Pi_1 + \Pi_3 + \Pi_4 + \tilde{m}_n - \tilde{m}_i)^{-1} (\Pi_1 - \Pi_3 - \tilde{m}_i) (\Pi_2 - \Pi_3 - \tilde{m}_i) \\ & \cdot (\Pi_1 - \Pi_3 - \Pi_4 - \tilde{m}_n - \tilde{m}_i) (-\Pi_1 + \Pi_3 - \tilde{m}_i)^{-1} (-\Pi_2 + \Pi_3 - \tilde{m}_i)^{-1} \\ & \cdot (-\Pi_3 - \tilde{m}_n - \tilde{m}_i) (-\Pi_1 - \Pi_2 + \Pi_3 + \Pi_4 - \tilde{m}_i)^{-1} (\Pi_2 - \Pi_3 - \Pi_4 - \tilde{m}_n - \tilde{m}_i), \end{aligned}$$

$$\begin{aligned} \Pi_4 = & \prod_{i=1}^{n-1} (\Pi_4 + \tilde{m}_n - \tilde{m}_i)^{-1} \prod_{i=1}^n (\Pi_1 + \Pi_2 - \Pi_3 - \Pi_4 - \tilde{m}_i) (-\Pi_2 + \Pi_3 + \Pi_4 + \tilde{m}_n - \tilde{m}_i)^{-1} \\ & \cdot (\Pi_1 - \Pi_4 - \tilde{m}_i) (-\Pi_1 + \Pi_3 + \Pi_4 + \tilde{m}_n - \tilde{m}_i)^{-1} (\Pi_2 - \Pi_4 - \tilde{m}_i) \\ & \cdot (-\Pi_1 + \Pi_2 - \tilde{m}_i)^{-1} (\Pi_1 - \Pi_3 - \Pi_4 - \tilde{m}_n - \tilde{m}_i) (-\Pi_2 + \Pi_4 - \tilde{m}_i)^{-1} \\ & \cdot (-\Pi_4 - \tilde{m}_n - \tilde{m}_i) (-\Pi_1 - \Pi_2 + \Pi_3 + \Pi_4 - \tilde{m}_i)^{-1} (\Pi_2 - \Pi_3 - \Pi_4 - \tilde{m}_n - \tilde{m}_i). \end{aligned}$$

The mirror maps are given by,

$$\exp(-Y_{i,\beta}) \mapsto -\tilde{m}_i + \sum_{a=1}^4 \sigma_a \rho_{i,\beta}^a, \quad X_m \mapsto \sum_{a=1}^4 \sigma_a \alpha_m^a.$$

on the critical locus, one finds

$$\begin{aligned} \Pi_1 = \exp(-Y_{n,1}) &= \sigma_4 - \tilde{m}_n, & \Pi_2 = \exp(-Y_{n,2}) &= \sigma_3 - \sigma_4 - \tilde{m}_n, \\ \Pi_3 = \exp(-Y_{n,3}) &= \sigma_2 - \sigma_3 - \tilde{m}_n, & \Pi_4 = \exp(-Y_{n,4}) &= \sigma_1 - \sigma_2 + \sigma_3 - \tilde{m}_n. \end{aligned}$$

Plugging them back in, one obtains the Coulomb branch (quantum cohomology) ring relations for F_4 :

$$\begin{aligned} & \prod_{i=1}^n (-\sigma_1 + \sigma_3 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 - \tilde{m}_i)(-\sigma_2 + \sigma_3 + \sigma_4 - \tilde{m}_i) \\ & \quad \cdot (-\sigma_3 + 2\sigma_4 - \tilde{m}_i)(-\sigma_1 + \sigma_4 - \tilde{m}_i)(\sigma_4 - \tilde{m}_i) \\ & = \prod_{i=1}^n (\sigma_1 - \sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i)(\sigma_2 - \sigma_3 - \sigma_4 - \tilde{m}_i) \\ & \quad \cdot (\sigma_3 - 2\sigma_4 - \tilde{m}_i)(-\sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_3 - \tilde{m}_i), \end{aligned} \tag{4.62}$$

$$\begin{aligned} & \prod_{i=1}^n (-\sigma_1 + \sigma_3 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \sigma_4 - \tilde{m}_i)(-\sigma_2 + 2\sigma_3 - \sigma_4 - \tilde{m}_i) \\ & \quad \cdot (\sigma_3 - 2\sigma_4 - \tilde{m}_i)(-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i)(\sigma_3 - \sigma_4 - \tilde{m}_i) \\ & = \prod_{i=1}^n (\sigma_1 - \sigma_3 + \sigma_4 - \tilde{m}_i)(-\sigma_3 + 2\sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_2 + \sigma_4 - \tilde{m}_i) \\ & \quad \cdot (\sigma_2 - 2\sigma_3 + \sigma_4 - \tilde{m}_i)(-\sigma_3 + \sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_3 - \tilde{m}_i), \end{aligned} \tag{4.63}$$

$$\begin{aligned} & \prod_{i=1}^n (\sigma_1 - \sigma_3 + \sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_4 - \tilde{m}_i)(\sigma_2 - \sigma_3 - \sigma_4 - \tilde{m}_i) \\ & \quad \cdot (\sigma_2 - 2\sigma_3 + \sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_3 - \tilde{m}_i)(\sigma_2 - \sigma_3 - \tilde{m}_i) \\ & = \prod_{i=1}^n (-\sigma_1 + \sigma_3 - \tilde{m}_i)(-\sigma_2 + \sigma_3 + \sigma_4 - \tilde{m}_i)(-\sigma_2 + 2\sigma_3 - \sigma_4 - \tilde{m}_i) \\ & \quad \cdot (-\sigma_1 + \sigma_4 - \tilde{m}_i)(-\sigma_2 + \sigma_3 - \tilde{m}_i)(-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i), \end{aligned} \tag{4.64}$$

$$\begin{aligned}
& \prod_{i=1}^n (\sigma_1 - \sigma_3 + \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i) \\
& \quad \cdot (\sigma_1 - \sigma_2 + \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_3 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_3 - \tilde{m}_i) \\
& = \prod_{i=1}^n (-\sigma_1 + \sigma_3 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_4 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_1 + \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 - \tilde{m}_i) (-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i). \tag{4.65}
\end{aligned}$$

Next, let us describe the excluded locus on the Coulomb branch. As discussed previously and in [14], part of the excluded locus is defined by the condition $X_m \neq 0$ for all m . This gives

$$\begin{aligned}
& \sigma_1(2\sigma_1 - \sigma_2)(-\sigma_1 + \sigma_2)(\sigma_1 + \sigma_2 - 2\sigma_3)(-\sigma_1 + 2\sigma_2 - 2\sigma_3)(\sigma_2 - \sigma_3)(-\sigma_1 + \sigma_3) \\
& \quad \cdot (\sigma_1 - \sigma_2 + \sigma_3)(-\sigma_2 + 2\sigma_3)(\sigma_2 - 2\sigma_4)(-\sigma_1 + 2\sigma_3 - 2\sigma_4)(\sigma_1 - \sigma_2 + 2\sigma_3 - 2\sigma_4) \\
& \quad \cdot (\sigma_1 - \sigma_4)(-\sigma_1 + \sigma_2 - \sigma_4)(\sigma_3 - \sigma_4)(-\sigma_2 + \sigma_3 - \sigma_4)\sigma_4(\sigma_1 - \sigma_3 + \sigma_4)(-\sigma_1 + 2\sigma_4) \\
& \quad \cdot (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4)(-\sigma_2 + \sigma_3 + \sigma_4)(\sigma_1 - \sigma_2 + 2\sigma_4)(\sigma_2 - 2\sigma_3 + 2\sigma_4)(-\sigma_3 + 2\sigma_4) \neq 0. \tag{4.66}
\end{aligned}$$

The second part of the excluded locus is determined by the condition that $\exp(-Y) \neq 0$. From the mirror map

$$\exp(-Y_{i,\beta}) = -\tilde{m}_i + \sum_{a=1}^4 \sigma_a \rho_{i,\beta}^a,$$

the excluded locus constraint becomes

$$-\tilde{m}_i + \sum_{a=1}^4 \sigma_a \rho_{i,\beta}^a \neq 0$$

which is encoded in the expression below:

$$\begin{aligned}
& \prod_{i=1}^n (\sigma_1 - \sigma_3 - \tilde{m}_i) (\sigma_2 - \sigma_3 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 - \tilde{m}_i) (-\sigma_1 + \sigma_3 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \tilde{m}_i) \\
& \quad \cdot (\sigma_1 - \sigma_2 + \sigma_3 - \tilde{m}_i) (\sigma_3 - 2\sigma_4 - \tilde{m}_i) (-\sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_4 - \tilde{m}_i) \\
& \quad \cdot (\sigma_2 - \sigma_3 - \sigma_4 - \tilde{m}_i) (\sigma_3 - \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i) \\
& \cdot (-\sigma_2 + 2\sigma_3 - \sigma_4 - \tilde{m}_i) (\sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_4 - \tilde{m}_i) (-\sigma_3 + 2\sigma_4 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_3 + \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_3 + \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 - \tilde{m}_i) (-\sigma_2 + \sigma_3 + \sigma_4 - \tilde{m}_i) \\
& \quad \cdot (\sigma_2 - 2\sigma_3 + \sigma_4 - \tilde{m}_i) \neq 0. \tag{4.67}
\end{aligned}$$

4.3.2 Transformation under the Weyl group of F_4

The Weyl group of F_4 has $1152 = 2^7 \cdot 3^2$ elements¹ [97][table 2.2], so explicitly listing orbits of vacua, is not feasible. (Similarly [97][table 2.2], the order of the Weyl group of E_6 is $2^7 \cdot 3^4 \cdot 5$, the order of the Weyl group of E_7 is $2^{10} \cdot 3^4 \cdot 5 \cdot 7$, so we will not be tracking orbits of vacua under the Weyl group in those cases either.) In this section, we will instead merely check that the critical locus equations transform into one another under Weyl reflections, a nontrivial check of our computations.

As reviewed earlier, the Weyl transformation acts on vectors, roots and weights:

$$S_\alpha(v^a) = v^a - 2 \frac{(\alpha, v)}{(\alpha, \alpha)} \alpha^a. \quad (4.68)$$

The Euclidean inner product takes the following metric matrix in this coordinate

$$[g_{ab}] = \begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 3 & 3/2 \\ 1 & 2 & 3/2 & 1 \end{pmatrix}. \quad (4.69)$$

The σ_a transform as co-vectors under the same Weyl transformation. F_4 has four simple roots, which can be taken to be

$$\begin{aligned} A &= (2, -1, 0, 0), & B &= (-1, 2, -2, 0), \\ C &= (0, -1, 2, -1), & D &= (0, 0, -1, 2), \end{aligned}$$

so the Weyl group of F_4 has four distinguished elements, S_A, \dots, S_D , whose actions are given by

$$\begin{aligned} S_A(v^1, v^2, v^3, v^4) &= (-v^1, v^1 + v^2, v^3, v^4), \\ S_A(\sigma_1, \sigma_2, \sigma_3, \sigma_4) &= (\sigma_2 - \sigma_1, \sigma_2, \sigma_3, \sigma_4), \end{aligned} \quad (4.70)$$

$$\begin{aligned} S_B(v^1, v^2, v^3, v^4) &= (v^1 + v^2, -v^2, 2v^2 + v^3, v^4), \\ S_B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) &= (\sigma_1, \sigma_1 - \sigma_2 + 2\sigma_3, \sigma_3, \sigma_4), \end{aligned} \quad (4.71)$$

$$\begin{aligned} S_C(v^1, v^2, v^3, v^4) &= (v^1, v^2 + v^3, -v^3, v^3 + v^4), \\ S_C(\sigma_1, \sigma_2, \sigma_3, \sigma_4) &= (\sigma_1, \sigma_2, \sigma_2 - \sigma_3 + \sigma_4, \sigma_4), \end{aligned} \quad (4.72)$$

$$\begin{aligned} S_D(v^1, v^2, v^3, v^4) &= (v^1, v^2, v^3 + v^4, -v^4), \\ S_D(\sigma_1, \sigma_2, \sigma_3, \sigma_4) &= (\sigma_1, \sigma_2, \sigma_3, \sigma_3 - \sigma_4). \end{aligned} \quad (4.73)$$

¹For the curious, information on the representation theory of the Weyl group of F_4 can be found in [96].

The superpotential is, by construction, invariant under Weyl reflections, hence to be consistent, the critical locus equations should transform into one another under these reflections. We check this below. For example, under the action of S_D , equation (4.62)

$$\begin{aligned}
& \prod_{i=1}^n (-\sigma_1 + \sigma_3 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 - \tilde{m}_i)(-\sigma_2 + \sigma_3 + \sigma_4 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_3 + 2\sigma_4 - \tilde{m}_i)(-\sigma_1 + \sigma_4 - \tilde{m}_i)(\sigma_4 - \tilde{m}_i) \\
& = \prod_{i=1}^n (\sigma_1 - \sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i)(\sigma_2 - \sigma_3 - \sigma_4 - \tilde{m}_i) \\
& \quad \cdot (\sigma_3 - 2\sigma_4 - \tilde{m}_i)(-\sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_3 - \tilde{m}_i),
\end{aligned}$$

transforms into

$$\begin{aligned}
& \prod_{i=1}^n (-\sigma_1 + \sigma_3 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \sigma_4 - \tilde{m}_i)(-\sigma_2 + 2\sigma_3 - \sigma_4 - \tilde{m}_i) \\
& \quad \cdot (\sigma_3 - 2\sigma_4 - \tilde{m}_i)(-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i)(\sigma_3 - \sigma_4 - \tilde{m}_i) \\
& = \prod_{i=1}^n (\sigma_1 - \sigma_3 + \sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_2 + \sigma_4 - \tilde{m}_i)(\sigma_2 - 2\sigma_3 + \sigma_4 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_3 + 2\sigma_4 - \tilde{m}_i)(-\sigma_3 + \sigma_4 - \tilde{m}_i)(\sigma_1 - \sigma_3 - \tilde{m}_i),
\end{aligned}$$

which is equation (4.63).

Table 4.4 schematically describes how other critical locus equations transform under these Weyl reflections.

The fact that the critical locus equations are closed under Weyl reflections associated with a set of simple roots provides a nontrivial consistency check on our results.

4.3.3 Pure gauge theory

In this section, we will consider the mirror to the pure supersymmetric F_4 gauge theory. The mirror superpotential is

$$\begin{aligned}
W = \sigma_1 \left(& Z_1 - Z_2 + Z_5 - Z_6 + Z_{16} - Z_{17} + Z_8 + Z_{18} - Z_9 - Z_{19} + Z_{20} - Z_{21} + Z_{10} \right. \\
& - Z_{11} + 2Z_{12} - Z_{25} + Z_{26} - Z_{29} + Z_{30} - Z_{40} + Z_{41} - Z_{32} - Z_{42} + Z_{33} + Z_{43} \\
& \left. - Z_{44} + Z_{45} - Z_{34} + Z_{35} - 2Z_{36} \right)
\end{aligned}$$

	Initial equ'n	Final equ'n
<i>A</i>	(4.62)	(4.62) × (4.65)
<i>A</i>	(4.63)	(4.63) × (4.65)
<i>A</i>	(4.64)	(4.64) × (4.65)
<i>A</i>	(4.65)	(4.65)
<i>B</i>	(4.62)	(4.62)
<i>B</i>	(4.63)	(4.63)
<i>B</i>	(4.64)	(4.65)
<i>B</i>	(4.65)	(4.64)
<i>C</i>	(4.62)	(4.62)
<i>C</i>	(4.63)	(4.64) × (4.65)
<i>C</i>	(4.64)	(4.63) × (4.65)
<i>C</i>	(4.65)	(4.65)
<i>D</i>	(4.62)	(4.63)
<i>D</i>	(4.63)	(4.62)
<i>D</i>	(4.64)	(4.64)
<i>D</i>	(4.65)	(4.65)

Table 4.4: Transformation of critical locus equations under four Weyl reflections.

$$\begin{aligned}
& + \sigma_2 \left(Z_2 - Z_3 + Z_4 + Z_{15} - Z_5 + Z_7 - Z_{16} - Z_8 + Z_{19} + Z_{21} - Z_{22} + Z_{10} + 2Z_{11} \right. \\
& \quad - Z_{23} - Z_{12} - Z_{26} + Z_{27} - Z_{28} - Z_{39} + Z_{29} - Z_{31} + Z_{40} + Z_{32} - Z_{43} - Z_{45} \\
& \quad \left. + Z_{46} - Z_{34} - 2Z_{35} + Z_{47} + Z_{36} \right) \\
& + \sigma_3 \left(2Z_3 + Z_{14} - 2Z_4 - Z_{15} + Z_{16} + Z_{17} + 2Z_8 - Z_{18} + 2Z_9 - Z_{19} + Z_{22} - 2Z_{10} \right. \\
& \quad - 2Z_{11} + 2Z_{23} - Z_{24} - 2Z_{27} - Z_{38} + 2Z_{28} + Z_{39} - Z_{40} - Z_{41} - 2Z_{32} + Z_{42} \\
& \quad \left. - 2Z_{33} + Z_{43} - Z_{46} + 2Z_{34} + 2Z_{35} - 2Z_{47} + Z_{48} \right) \\
& + \sigma_4 \left(Z_{13} - Z_{14} + 2Z_4 + 2Z_5 + 2Z_6 - 2Z_7 - 2Z_8 + Z_{18} - 2Z_9 + Z_{19} - Z_{20} - Z_{21} \right. \\
& \quad + Z_{22} - Z_{23} + 2Z_{24} - Z_{37} + Z_{38} - 2Z_{28} - 2Z_{29} - 2Z_{30} + 2Z_{31} + 2Z_{32} - Z_{42} \\
& \quad \left. + 2Z_{33} - Z_{43} + Z_{44} + Z_{45} - Z_{46} + Z_{47} - 2Z_{48} \right) + \sum_{m=1}^{48} X_m. \tag{4.74}
\end{aligned}$$

Now, let us consider the critical locus of the superpotential above. For each root μ , the fields X_μ and $X_{-\mu}$ appear paired with opposite signs coupling to each σ . Therefore, one implication of the derivatives

$$\frac{\partial W}{\partial X_\mu} = 0$$

is that, on the critical locus,

$$X_\mu = -X_{-\mu}. \quad (4.75)$$

(Furthermore, on the critical locus, each X_μ is determined by σ 's.) Next, each derivative

$$\frac{\partial W}{\partial \sigma_a},$$

is a product of ratios of the form

$$\frac{X_\mu}{X_{-\mu}} = -1.$$

It is straightforward to check in the superpotential above that each σ_a is multiplied by an even number of such ratios (*i.e.* the number of Z 's is a multiple of four). For example, the sum of the absolute values of the coefficients of the Z 's multiplying σ_1 and σ_2 is $32 = 4 \cdot 8$, and the sum of the absolute values of the coefficients of the Z 's multiplying σ_3 and σ_4 is $44 = 4 \cdot 11$. Thus, the constraint implied by the σ 's is automatically satisfied.

As a result, following the same analysis in [14], we see in this case, that the critical locus is nonempty, and in fact is determined by four σ 's. In other words, at the level of these topological field theory computations, we have evidence that the pure supersymmetric F_4 gauge theory in two dimensions flows in the IR to a theory of four free twisted chiral superfields.

4.4 E_6

In this section we will consider the mirror Landau-Ginzburg orbifold superpotential of E_6 gauge theory when matter fields are in **27** fundamental representation of it and then we compute the quantum cohomology ring of it. Also we will consider the pure theory without matter field.

4.4.1 Mirror Landau-Ginzburg orbifold

The mirror Landau-Ginzburg model has fields

- $Y_{i,\beta}$, $i \in \{1, \dots, n\}$, $\beta \in \{1, \dots, 27\}$, corresponding to the matter fields in n copies of the **27** representation **27**,

- X_m , $m \in \{1, \dots, 72\}$, corresponding to the roots of E_6 ,
- σ_a , $a \in \{1, 2, 3, 4, 5, 6\}$.

We associate the roots, α_m^a , to X_m fields and the weights, $\rho_{i,\beta}^a$, of fundamental **27** representation of E_6 to $Y_{i,\beta}$.

The roots of E_6 and associated fields are listed in tables 4.5, 4.6. The weights associated to the **27** of E_6 and their associated fields are listed in table 4.7.

Field	Positive root	Field	Negative root
X_1	(0, 0, 0, 0, 0, 1)	X_{37}	(0, 0, 0, 0, 0, -1)
X_2	(0, 0, 1, 0, 0, -1)	X_{38}	(0, 0, -1, 0, 0, 1)
X_3	(0, 1, -1, 1, 0, 0)	X_{39}	(0, -1, 1, -1, 0, 0)
X_4	(0, 1, 0, -1, 1, 0)	X_{40}	(0, -1, 0, 1, -1, 0)
X_5	(1, -1, 0, 1, 0, 0)	X_{41}	(-1, 1, 0, -1, 0, 0)
X_6	(-1, 0, 0, 1, 0, 0)	X_{42}	(1, 0, 0, -1, 0, 0)
X_7	(0, 1, 0, 0, -1, 0)	X_{43}	(0, -1, 0, 0, 1, 0)
X_8	(1, -1, 1, -1, 1, 0)	X_{44}	(-1, 1, -1, 1, -1, 0)
X_9	(-1, 0, 1, -1, 1, 0)	X_{45}	(1, 0, -1, 1, -1, 0)
X_{10}	(1, -1, 1, 0, -1, 0)	X_{46}	(-1, 1, -1, 0, 1, 0)
X_{11}	(1, 0, -1, 0, 1, 1)	X_{47}	(-1, 0, 1, 0, -1, -1)
X_{12}	(-1, 0, 1, 0, -1, 0)	X_{48}	(1, 0, -1, 0, 1, 0)
X_{13}	(-1, 1, -1, 0, 1, 1)	X_{49}	(1, -1, 1, 0, -1, -1)
X_{14}	(1, 0, -1, 1, -1, 1)	X_{50}	(-1, 0, 1, -1, 1, -1)
X_{15}	(1, 0, 0, 0, 1, -1)	X_{51}	(-1, 0, 0, 0, -1, 1)
X_{16}	(-1, 1, -1, 1, -1, 1)	X_{52}	(1, -1, 1, -1, 1, -1)
X_{17}	(-1, 1, 0, 0, 1, -1)	X_{53}	(1, -1, 0, 0, -1, 1)
X_{18}	(0, -1, 0, 0, 1, 1)	X_{54}	(0, 1, 0, 0, -1, -1)

Table 4.5: First set of roots of E_6 and associated fields.

The weights in the tables in this section are written as linear combinations of the fundamental weights, computed with LieART [95], as discussed earlier, so as to get conventional θ -angle periodicities.

Field	Positive root	Field	Negative root
X_{19}	$(1, 0, 0, -1, 0, 1)$	X_{55}	$(-1, 0, 0, 1, 0, -1)$
X_{20}	$(1, 0, 0, 1, -1, -1)$	X_{56}	$(-1, 0, 0, -1, 1, 1)$
X_{21}	$(-1, 1, 0, -1, 0, 1)$	X_{57}	$(1, -1, 0, 1, 0, -1)$
X_{22}	$(-1, 1, 0, 1, -1, -1)$	X_{58}	$(1, -1, 0, -1, 1, 1)$
X_{23}	$(0, -1, 0, 1, -1, 1)$	X_{59}	$(0, 1, 0, -1, 1, -1)$
X_{24}	$(0, -1, 1, 0, 1, -1)$	X_{60}	$(0, 1, -1, 0, -1, 1)$
X_{25}	$(1, 0, 1, -1, 0, -1)$	X_{61}	$(-1, 0, -1, 1, 0, 1)$
X_{26}	$(-1, 1, 1, -1, 0, -1)$	X_{62}	$(1, -1, -1, 1, 0, 1)$
X_{27}	$(0, -1, 1, -1, 0, 1)$	X_{63}	$(0, 1, -1, 1, 0, -1)$
X_{28}	$(0, -1, 1, 1, -1, -1)$	X_{64}	$(0, 1, -1, -1, 1, 1)$
X_{29}	$(0, 0, -1, 1, 1, 0)$	X_{65}	$(0, 0, 1, -1, -1, 0)$
X_{30}	$(1, 1, -1, 0, 0, 0)$	X_{66}	$(-1, -1, 1, 0, 0, 0)$
X_{31}	$(-1, 2, -1, 0, 0, 0)$	X_{67}	$(1, -2, 1, 0, 0, 0)$
X_{32}	$(0, -1, 2, -1, 0, -1)$	X_{68}	$(0, 1, -2, 1, 0, 1)$
X_{33}	$(0, 0, -1, 0, 0, 2)$	X_{69}	$(0, 0, 1, 0, 0, -2)$
X_{34}	$(0, 0, -1, 2, -1, 0)$	X_{70}	$(0, 0, 1, -2, 1, 0)$
X_{35}	$(0, 0, 0, -1, 2, 0)$	X_{71}	$(0, 0, 0, 1, -2, 0)$
X_{36}	$(2, -1, 0, 0, 0, 0)$	X_{72}	$(-2, 1, 0, 0, 0, 0)$

Table 4.6: Second set of roots of E_6 and associated fields.

Field	Weight	Field	Weight	Field	Weight
$Y_{i,1}$	$(1, 0, 0, 0, 0, 0)$	$Y_{i,2}$	$(-1, 1, 0, 0, 0, 0)$	$Y_{i,3}$	$(0, -1, 1, 0, 0, 0)$
$Y_{i,4}$	$(0, 0, -1, 1, 0, 1)$	$Y_{i,5}$	$(0, 0, 0, -1, 1, 1)$	$Y_{i,6}$	$(0, 0, 0, 1, 0, -1)$
$Y_{i,7}$	$(0, 0, 0, 0, -1, 1)$	$Y_{i,8}$	$(0, 0, 1, -1, 1, -1)$	$Y_{i,9}$	$(0, 0, 1, 0, -1, -1)$
$Y_{i,10}$	$(0, 1, -1, 0, 1, 0)$	$Y_{i,11}$	$(0, 1, -1, 1, -1, 0)$	$Y_{i,12}$	$(1, -1, 0, 0, 1, 0)$
$Y_{i,13}$	$(-1, 0, 0, 0, 1, 0)$	$Y_{i,14}$	$(0, 1, 0, -1, 0, 0)$	$Y_{i,15}$	$(1, -1, 0, 1, -1, 0)$
$Y_{i,16}$	$(-1, 0, 0, 1, -1, 0)$	$Y_{i,17}$	$(1, -1, 1, -1, 0, 0)$	$Y_{i,18}$	$(-1, 0, 1, -1, 0, 0)$
$Y_{i,19}$	$(1, 0, -1, 0, 0, 1)$	$Y_{i,20}$	$(-1, 1, -1, 0, 0, 1)$	$Y_{i,21}$	$(1, 0, 0, 0, 0, -1)$
$Y_{i,22}$	$(-1, 1, 0, 0, 0, -1)$	$Y_{i,23}$	$(0, -1, 0, 0, 0, 1)$	$Y_{i,24}$	$(0, -1, 1, 0, 0, -1)$
$Y_{i,25}$	$(0, 0, -1, 1, 0, 0)$	$Y_{i,26}$	$(0, 0, 0, -1, 1, 0)$	$Y_{i,27}$	$(0, 0, 0, 0, -1, 0)$

Table 4.7: Weights of $\mathbf{27}$ of E_6 and associated fields.

4.4.2 Superpotential

In this section, we describe the superpotential of the mirror Landau-Ginzburg orbifold. It is given by

$$W = \sum_{a=1}^6 \sigma_a \left(\sum_{i=1}^n \sum_{\beta=1}^{27} \rho_{i,\beta}^a Y_{i,\beta} + \sum_{m=1}^{72} \alpha_m^a Z_m \right) - \sum_{i=1}^n \tilde{m}_i \sum_{b=1}^{27} Y_{i,b} + \sum_{i=1}^n \sum_{\beta=1}^{27} \exp(-Y_{i,\beta}) + \sum_{m=1}^{72} X_m.$$

where $X_m = \exp(-Z_m)$ and X_m are the fundamental fields. Using the results of the previous section we get

$$W = \sum_{a=1}^6 \sigma_a \mathcal{C}^a - \sum_{i=1}^n \tilde{m}_i \sum_{\beta=1}^{27} Y_{i,\beta} + \sum_{i=1}^n \sum_{\beta=1}^{27} \exp(-Y_{i,\beta}) + \sum_{m=1}^{72} X_m. \quad (4.76)$$

where \mathcal{C}^a are given as follows:

$$\begin{aligned} \mathcal{C}^1 = & \sum_{i=1}^n (Y_{i,1} - Y_{i,2} + Y_{i,12} - Y_{i,13} + Y_{i,15} - Y_{i,16} + Y_{i,17} - Y_{i,18} + Y_{i,19} - Y_{i,20} + Y_{i,21} - Y_{i,22}) \\ & + Z_5 - Z_6 + Z_8 - Z_9 + Z_{10} + Z_{11} - Z_{12} - Z_{13} + Z_{14} + Z_{15} - Z_{16} - Z_{17} + Z_{19} + Z_{20} \\ & - Z_{21} - Z_{22} + Z_{25} - Z_{26} + Z_{30} - Z_{31} + 2Z_{36} - Z_{41} + Z_{42} - Z_{44} + Z_{45} - Z_{46} - Z_{47} \\ & + Z_{48} + Z_{49} - Z_{50} - Z_{51} + Z_{52} + Z_{53} - Z_{55} - Z_{56} + Z_{57} + Z_{58} - Z_{61} + Z_{62} - Z_{66} \\ & + Z_{67} - 2Z_{72}, \end{aligned}$$

$$\begin{aligned} \mathcal{C}^2 = & \sum_{i=1}^n (Y_{i,2} - Y_{i,3} + Y_{i,10} + Y_{i,11} - Y_{i,12} + Y_{i,14} - Y_{i,15} - Y_{i,17} + Y_{i,20} + Y_{i,22} - Y_{i,23} - Y_{i,24}) \\ & + Z_3 + Z_4 - Z_5 + Z_7 - Z_8 - Z_{10} + Z_{13} + Z_{16} + Z_{17} - Z_{18} + Z_{21} + Z_{22} - Z_{23} - Z_{24} \\ & + Z_{26} - Z_{27} - Z_{28} + Z_{30} + 2Z_{31} - Z_{32} - Z_{36} - Z_{39} - Z_{40} + Z_{41} - Z_{43} + Z_{44} + Z_{46} \\ & - Z_{49} - Z_{52} - Z_{53} + Z_{54} - Z_{57} - Z_{58} + Z_{59} + Z_{60} - Z_{62} + Z_{63} + Z_{64} - Z_{66} - 2Z_{67} \\ & + Z_{68} + Z_{72}, \end{aligned}$$

$$\begin{aligned} \mathcal{C}^3 = & \sum_{i=1}^n (Y_{i,3} - Y_{i,4} + Y_{i,8} + Y_{i,9} - Y_{i,10} - Y_{i,11} + Y_{i,17} + Y_{i,18} - Y_{i,19} - Y_{i,20} + Y_{i,24} - Y_{i,25}) \\ & + Z_2 - Z_3 + Z_8 + Z_9 + Z_{10} - Z_{11} + Z_{12} - Z_{13} - Z_{14} - Z_{16} + Z_{24} + Z_{25} + Z_{26} + Z_{27} \\ & + Z_{28} - Z_{29} - Z_{30} - Z_{31} + 2Z_{32} - Z_{33} - Z_{34} - Z_{38} + Z_{39} - Z_{44} - Z_{45} - Z_{46} + Z_{47} \\ & - Z_{48} + Z_{49} + Z_{50} + Z_{52} - Z_{60} - Z_{61} - Z_{62} - Z_{63} - Z_{64} + Z_{65} + Z_{66} + Z_{67} - 2Z_{68} \\ & + Z_{69} + Z_{70}, \end{aligned}$$

$$\begin{aligned} \mathcal{C}^4 = & \sum_{i=1}^n (Y_{i,4} - Y_{i,5} + Y_{i,6} - Y_{i,8} + Y_{i,11} - Y_{i,14} + Y_{i,15} + Y_{i,16} - Y_{i,17} - Y_{i,18} + Y_{i,25} - Y_{i,26}) \\ & + Z_3 - Z_4 + Z_5 + Z_6 - Z_8 - Z_9 + Z_{14} + Z_{16} - Z_{19} + Z_{20} - Z_{21} + Z_{22} + Z_{23} - Z_{25} \\ & - Z_{26} - Z_{27} + Z_{28} + Z_{29} - Z_{32} + 2Z_{34} - Z_{35} - Z_{39} + Z_{40} - Z_{41} - Z_{42} + Z_{44} + Z_{45} \\ & - Z_{50} - Z_{52} + Z_{55} - Z_{56} + Z_{57} - Z_{58} - Z_{59} + Z_{61} + Z_{62} + Z_{63} - Z_{64} - Z_{65} + Z_{68} \\ & - 2Z_{70} + Z_{71}, \end{aligned}$$

$$\begin{aligned}
\mathcal{C}^5 = & \sum_{i=1}^n (Y_{i,5} - Y_{i,7} + Y_{i,8} - Y_{i,9} + Y_{i,10} - Y_{i,11} + Y_{i,12} + Y_{i,13} - Y_{i,15} - Y_{i,16} + Y_{i,26} - Y_{i,27}) \\
& + Z_4 - Z_7 + Z_8 + Z_9 - Z_{10} + Z_{11} - Z_{12} + Z_{13} - Z_{14} + Z_{15} - Z_{16} + Z_{17} + Z_{18} - Z_{20} \\
& - Z_{22} - Z_{23} + Z_{24} - Z_{28} + Z_{29} - Z_{34} + 2Z_{35} - Z_{40} + Z_{43} - Z_{44} - Z_{45} + Z_{46} - Z_{47} \\
& + Z_{48} - Z_{49} + Z_{50} - Z_{51} + Z_{52} - Z_{53} - Z_{54} + Z_{56} + Z_{58} + Z_{59} - Z_{60} + Z_{64} - Z_{65} \\
& + Z_{70} - 2Z_{71},
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}^6 = & \sum_{i=1}^n (Y_{i,4} + Y_{i,5} - Y_{i,6} + Y_{i,7} - Y_{i,8} - Y_{i,9} + Y_{i,19} + Y_{i,20} - Y_{i,21} - Y_{i,22} + Y_{i,23} - Y_{i,24}) \\
& + Z_1 - Z_2 + Z_{11} + Z_{13} + Z_{14} - Z_{15} + Z_{16} - Z_{17} + Z_{18} + Z_{19} - Z_{20} + Z_{21} - Z_{22} \\
& + Z_{23} - Z_{24} - Z_{25} - Z_{26} + Z_{27} - Z_{28} - Z_{32} + 2Z_{33} - Z_{37} + Z_{38} - Z_{47} - Z_{49} - Z_{50} \\
& + Z_{51} - Z_{52} + Z_{53} - Z_{54} - Z_{55} + Z_{56} - Z_{57} + Z_{58} - Z_{59} + Z_{60} + Z_{61} + Z_{62} - Z_{63} \\
& + Z_{64} + Z_{68} - 2Z_{69}.
\end{aligned}$$

4.4.3 Coulomb ring relations

Integrating out the σ_a fields, we obtain six constraints $\mathcal{C}^a = 0$. Exponentiating these constraints, we obtain a series of equations from which the Coulomb ring relations will be derived. For reasons of notational sanity, we will also slightly simplify these expressions, as follows. To make predictions of the A model, we will evaluate the ring relations on the critical locus, where

$$\frac{X_m}{X_{m+63}} = -1.$$

It is straightforward to see that each of the constraints \mathcal{C}^a contains 22 differences of corresponding Z 's, so that the exponential of the constraints contains 22 factors of the form X_m/X_{m+63} – an even number of factors of -1 , which will cancel out. Therefore, since on the critical locus those factors will cancel out, we will omit them, and solely relate the exponentiated constraints in terms of Y s.

The exponentiated constraints are as follows.

$$\begin{aligned}
\prod_{i=1}^n \exp(Y_{i,1} - Y_{i,2} + Y_{i,12} - Y_{i,13} + Y_{i,15} - Y_{i,16} + Y_{i,17} - Y_{i,18} + Y_{i,19} - Y_{i,20} + Y_{i,21} - Y_{i,22}) \\
= 1, \quad (4.77)
\end{aligned}$$

$$\begin{aligned}
\prod_{i=1}^n \exp(Y_{i,2} - Y_{i,3} + Y_{i,10} + Y_{i,11} - Y_{i,12} + Y_{i,14} - Y_{i,15} - Y_{i,17} + Y_{i,20} + Y_{i,22} - Y_{i,23} - Y_{i,24}) \\
= 1, \quad (4.78)
\end{aligned}$$

$$\prod_{i=1}^n \exp(Y_{i,3} - Y_{i,4} + Y_{i,8} + Y_{i,9} - Y_{i,10} - Y_{i,11} + Y_{i,17} + Y_{i,18} - Y_{i,19} - Y_{i,20} + Y_{i,24} - Y_{i,25}) = 1, \quad (4.79)$$

$$\prod_{i=1}^n \exp(Y_{i,4} - Y_{i,5} + Y_{i,6} - Y_{i,8} + Y_{i,11} - Y_{i,14} + Y_{i,15} + Y_{i,16} - Y_{i,17} - Y_{i,18} + Y_{i,25} - Y_{i,26}) = 1, \quad (4.80)$$

$$\prod_{i=1}^n \exp(Y_{i,5} - Y_{i,7} + Y_{i,8} - Y_{i,9} + Y_{i,10} - Y_{i,11} + Y_{i,12} + Y_{i,13} - Y_{i,15} - Y_{i,16} + Y_{i,26} - Y_{i,27}) = 1, \quad (4.81)$$

$$\prod_{i=1}^n \exp(Y_{i,4} + Y_{i,5} - Y_{i,6} + Y_{i,7} - Y_{i,8} - Y_{i,9} + Y_{i,19} + Y_{i,20} - Y_{i,21} - Y_{i,22} + Y_{i,23} - Y_{i,24}) = 1. \quad (4.82)$$

The mirror maps are given by

$$\exp(-Y_{i,\beta}) \mapsto -\tilde{m}_i + \sum_{a=1}^6 \sigma_a \rho_{i,\beta}^a, \quad X_m \mapsto \sum_{a=1}^6 \sigma_a \alpha_m^a.$$

Applying the operator mirror maps, the Coulomb ring relations become

$$\begin{aligned} & \prod_{i=1}^n (-\sigma_1 + \sigma_2 - \tilde{m}_i) (-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\ & \quad \cdot (-\sigma_1 + \sigma_5 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_6 - \tilde{m}_i) \\ & = \prod_{i=1}^n (\sigma_1 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\ & \quad \cdot (\sigma_1 - \sigma_2 + \sigma_5 - \tilde{m}_i) (\sigma_1 - \sigma_6 - \tilde{m}_i) (\sigma_1 - \sigma_3 + \sigma_6 - \tilde{m}_i), \quad (4.83) \end{aligned}$$

$$\begin{aligned} & \prod_{i=1}^n (-\sigma_2 + \sigma_3 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\ & \quad \cdot (\sigma_1 - \sigma_2 + \sigma_5 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \sigma_6 - \tilde{m}_i) (-\sigma_2 + \sigma_6 - \tilde{m}_i) \\ & = \prod_{i=1}^n (-\sigma_1 + \sigma_2 - \tilde{m}_i) (\sigma_2 - \sigma_4 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\ & \quad \cdot (\sigma_2 - \sigma_3 + \sigma_5 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_6 - \tilde{m}_i), \quad (4.84) \end{aligned}$$

$$\begin{aligned}
& \prod_{i=1}^n (-\sigma_3 + \sigma_4 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_5 - \tilde{m}_i) \\
& \quad \cdot (\sigma_1 - \sigma_3 + \sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_6 - \tilde{m}_i) (-\sigma_3 + \sigma_4 + \sigma_6 - \tilde{m}_i) \\
& = \prod_{i=1}^n (-\sigma_2 + \sigma_3 - \tilde{m}_i) (-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_2 + \sigma_3 - \sigma_6 - \tilde{m}_i) (\sigma_3 - \sigma_5 - \sigma_6 - \tilde{m}_i) (\sigma_3 - \sigma_4 + \sigma_5 - \sigma_6 - \tilde{m}_i), \quad (4.85)
\end{aligned}$$

$$\begin{aligned}
& \prod_{i=1}^n (\sigma_2 - \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_4 + \sigma_5 - \tilde{m}_i) (\sigma_3 - \sigma_4 + \sigma_5 - \sigma_6 - \tilde{m}_i) (-\sigma_4 + \sigma_5 + \sigma_6 - \tilde{m}_i) \\
& = \prod_{i=1}^n (-\sigma_3 + \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_4 - \sigma_5 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\
& \quad \cdot (\sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 - \tilde{m}_i) (\sigma_4 - \sigma_6 - \tilde{m}_i) (-\sigma_3 + \sigma_4 + \sigma_6 - \tilde{m}_i), \quad (4.86)
\end{aligned}$$

$$\begin{aligned}
& \prod_{i=1}^n (-\sigma_5 - \tilde{m}_i) (-\sigma_1 + \sigma_4 - \sigma_5 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\
& \quad \cdot (\sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 - \tilde{m}_i) (\sigma_3 - \sigma_5 - \sigma_6 - \tilde{m}_i) (-\sigma_5 + \sigma_6 - \tilde{m}_i) \\
& = \prod_{i=1}^n (-\sigma_1 + \sigma_5 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_5 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_5 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_4 + \sigma_5 - \tilde{m}_i) (\sigma_3 - \sigma_4 + \sigma_5 - \sigma_6 - \tilde{m}_i) (-\sigma_4 + \sigma_5 + \sigma_6 - \tilde{m}_i), \quad (4.87)
\end{aligned}$$

$$\begin{aligned}
& \prod_{i=1}^n (\sigma_1 - \sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_6 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \sigma_6 - \tilde{m}_i) \\
& \quad \cdot (\sigma_4 - \sigma_6 - \tilde{m}_i) (\sigma_3 - \sigma_5 - \sigma_6 - \tilde{m}_i) (\sigma_3 - \sigma_4 + \sigma_5 - \sigma_6 - \tilde{m}_i) \\
& = \prod_{i=1}^n (-\sigma_2 + \sigma_6 - \tilde{m}_i) (\sigma_1 - \sigma_3 + \sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_6 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_3 + \sigma_4 + \sigma_6 - \tilde{m}_i) (-\sigma_5 + \sigma_6 - \tilde{m}_i) (-\sigma_4 + \sigma_5 + \sigma_6 - \tilde{m}_i). \quad (4.88)
\end{aligned}$$

These are the Coulomb ring relations, defining the analogue of the quantum cohomology ring, for E_6 .

Part of the excluded locus is defined by the vanishing locus of the X_m , and is given by

$$\begin{aligned}
& (2\sigma_1 - \sigma_2)(\sigma_1 + \sigma_2 - \sigma_3)(-\sigma_1 + 2\sigma_2 - \sigma_3)(\sigma_4 - \sigma_1)(\sigma_1 - \sigma_2 + \sigma_4)(\sigma_2 - \sigma_3 + \sigma_4) \\
& \quad \cdot (\sigma_2 - \sigma_5)(-\sigma_1 + \sigma_3 - \sigma_5)(\sigma_1 - \sigma_2 + \sigma_3 - \sigma_5)(-\sigma_3 + 2\sigma_4 - \sigma_5)(\sigma_2 - \sigma_4 + \sigma_5) \\
& \quad \cdot (-\sigma_1 + \sigma_3 - \sigma_4 + \sigma_5)(\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 + \sigma_5)(-\sigma_3 + \sigma_4 + \sigma_5)(2\sigma_5 - \sigma_4)(\sigma_3 - \sigma_6) \\
& \cdot (\sigma_1 + \sigma_3 - \sigma_4 - \sigma_6)(-\sigma_1 + \sigma_2 + \sigma_3 - \sigma_4 - \sigma_6)(-\sigma_2 + 2\sigma_3 - \sigma_4 - \sigma_6)(\sigma_1 + \sigma_4 - \sigma_5 - \sigma_6) \\
& \cdot (-\sigma_1 + \sigma_2 + \sigma_4 - \sigma_5 - \sigma_6)(-\sigma_2 + \sigma_3 + \sigma_4 - \sigma_5 - \sigma_6)(\sigma_1 + \sigma_5 - \sigma_6)(-\sigma_1 + \sigma_2 + \sigma_5 - \sigma_6) \\
& \quad \cdot (-\sigma_2 + \sigma_3 + \sigma_5 - \sigma_6)\sigma_6(\sigma_1 - \sigma_4 + \sigma_6)(-\sigma_1 + \sigma_2 - \sigma_4 + \sigma_6)(-\sigma_2 + \sigma_3 - \sigma_4 + \sigma_6) \\
& \quad \cdot (-\sigma_2 + \sigma_4 - \sigma_5 + \sigma_6)(\sigma_1 - \sigma_3 + \sigma_4 - \sigma_5 + \sigma_6)(-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 + \sigma_6) \\
& \quad \cdot (-\sigma_2 + \sigma_5 + \sigma_6)(\sigma_1 - \sigma_3 + \sigma_5 + \sigma_6)(-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_5 + \sigma_6)(2\sigma_6 - \sigma_3) \neq 0. \quad (4.89)
\end{aligned}$$

Similarly,

$$\exp(-Y_{i,\beta}) = -\tilde{m}_i + \sum_{a=1}^6 \sigma_a \rho_{i,\beta}^a$$

on the critical locus, so

$$-\tilde{m}_i + \sum_{a=1}^6 \sigma_a \rho_{i,\beta}^a \neq 0$$

which determines the remainder of the excluded locus:

$$\begin{aligned}
& (\sigma_1 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \tilde{m}_i)(-\sigma_2 + \sigma_3 - \tilde{m}_i)(\sigma_2 - \sigma_4 - \tilde{m}_i)(-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i) \\
& \quad \cdot (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i)(-\sigma_3 + \sigma_4 - \tilde{m}_i)(-\sigma_5 - \tilde{m}_i)(-\sigma_1 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\
& \cdot (\sigma_1 - \sigma_2 + \sigma_4 - \sigma_5 - \tilde{m}_i)(\sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 - \tilde{m}_i)(-\sigma_1 + \sigma_5 - \tilde{m}_i)(\sigma_1 - \sigma_2 + \sigma_5 - \tilde{m}_i) \\
& \quad \cdot (\sigma_2 - \sigma_3 + \sigma_5 - \tilde{m}_i)(-\sigma_4 + \sigma_5 - \tilde{m}_i)(\sigma_1 - \sigma_6 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \sigma_6 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_2 + \sigma_3 - \sigma_6 - \tilde{m}_i)(\sigma_4 - \sigma_6 - \tilde{m}_i)(\sigma_3 - \sigma_5 - \sigma_6 - \tilde{m}_i)(\sigma_3 - \sigma_4 + \sigma_5 - \sigma_6 - \tilde{m}_i) \\
& \cdot (-\sigma_2 + \sigma_6 - \tilde{m}_i)(\sigma_1 - \sigma_3 + \sigma_6 - \tilde{m}_i)(-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_6 - \tilde{m}_i)(-\sigma_3 + \sigma_4 + \sigma_6 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_5 + \sigma_6 - \tilde{m}_i)(-\sigma_4 + \sigma_5 + \sigma_6 - \tilde{m}_i) \neq 0. \quad (4.90)
\end{aligned}$$

4.4.4 Pure gauge theory

In this part we will consider the mirror to the pure supersymmetric E_6 gauge theory. The mirror Landau-Ginzburg superpotential is

$$\begin{aligned}
W = & \sigma_1 \left(Z_5 - Z_6 + Z_8 - Z_9 + Z_{10} + Z_{11} - Z_{12} - Z_{13} + Z_{14} + Z_{15} - Z_{16} - Z_{17} + Z_{19} + Z_{20} \right. \\
& - Z_{21} - Z_{22} + Z_{25} - Z_{26} + Z_{30} - Z_{31} + 2Z_{36} - Z_{41} + Z_{42} - Z_{44} + Z_{45} - Z_{46} - Z_{47} \\
& + Z_{48} + Z_{49} - Z_{50} - Z_{51} + Z_{52} + Z_{53} - Z_{55} - Z_{56} + Z_{57} + Z_{58} - Z_{61} + Z_{62} - Z_{66} \\
& \left. + Z_{67} - 2Z_{72} \right) \\
& + \sigma_2 \left(Z_3 + Z_4 - Z_5 + Z_7 - Z_8 - Z_{10} + Z_{13} + Z_{16} + Z_{17} - Z_{18} + Z_{21} + Z_{22} - Z_{23} - Z_{24} \right. \\
& + Z_{26} - Z_{27} - Z_{28} + Z_{30} + 2Z_{31} - Z_{32} - Z_{36} - Z_{39} - Z_{40} + Z_{41} - Z_{43} + Z_{44} + Z_{46} \\
& - Z_{49} - Z_{52} - Z_{53} + Z_{54} - Z_{57} - Z_{58} + Z_{59} + Z_{60} - Z_{62} + Z_{63} + Z_{64} - Z_{66} - 2Z_{67} \\
& \left. + Z_{68} + Z_{72} \right) \\
& + \sigma_3 \left(Z_2 - Z_3 + Z_8 + Z_9 + Z_{10} - Z_{11} + Z_{12} - Z_{13} - Z_{14} - Z_{16} + Z_{24} + Z_{25} + Z_{26} + Z_{27} \right. \\
& + Z_{28} - Z_{29} - Z_{30} - Z_{31} + 2Z_{32} - Z_{33} - Z_{34} - Z_{38} + Z_{39} - Z_{44} - Z_{45} - Z_{46} + Z_{47} \\
& - Z_{48} + Z_{49} + Z_{50} + Z_{52} - Z_{60} - Z_{61} - Z_{62} - Z_{63} - Z_{64} + Z_{65} + Z_{66} + Z_{67} - 2Z_{68} \\
& \left. + Z_{69} + Z_{70} \right) \\
& + \sigma_4 \left(Z_3 - Z_4 + Z_5 + Z_6 - Z_8 - Z_9 + Z_{14} + Z_{16} - Z_{19} + Z_{20} - Z_{21} + Z_{22} + Z_{23} - Z_{25} \right. \\
& - Z_{26} - Z_{27} + Z_{28} + Z_{29} - Z_{32} + 2Z_{34} - Z_{35} - Z_{39} + Z_{40} - Z_{41} - Z_{42} + Z_{44} + Z_{45} \\
& - Z_{50} - Z_{52} + Z_{55} - Z_{56} + Z_{57} - Z_{58} - Z_{59} + Z_{61} + Z_{62} + Z_{63} - Z_{64} - Z_{65} + Z_{68} \\
& \left. - 2Z_{70} + Z_{71} \right) \\
& + \sigma_5 \left(Z_4 - Z_7 + Z_8 + Z_9 - Z_{10} + Z_{11} - Z_{12} + Z_{13} - Z_{14} + Z_{15} - Z_{16} + Z_{17} + Z_{18} - Z_{20} \right. \\
& - Z_{22} - Z_{23} + Z_{24} - Z_{28} + Z_{29} - Z_{34} + 2Z_{35} - Z_{40} + Z_{43} - Z_{44} - Z_{45} + Z_{46} - Z_{47} \\
& + Z_{48} - Z_{49} + Z_{50} - Z_{51} + Z_{52} - Z_{53} - Z_{54} + Z_{56} + Z_{58} + Z_{59} - Z_{60} + Z_{64} - Z_{65} \\
& \left. + Z_{70} - 2Z_{71} \right) \\
& + \sigma_6 \left(Z_1 - Z_2 + Z_{11} + Z_{13} + Z_{14} - Z_{15} + Z_{16} - Z_{17} + Z_{18} + Z_{19} - Z_{20} + Z_{21} - Z_{22} + Z_{23} \right. \\
& - Z_{24} - Z_{25} - Z_{26} + Z_{27} - Z_{28} - Z_{32} + 2Z_{33} - Z_{37} + Z_{38} - Z_{47} - Z_{49} - Z_{50} + Z_{51} \\
& - Z_{52} + Z_{53} - Z_{54} - Z_{55} + Z_{56} - Z_{57} + Z_{58} - Z_{59} + Z_{60} + Z_{61} + Z_{62} - Z_{63} + Z_{64} \\
& \left. + Z_{68} - 2Z_{69} \right)
\end{aligned}$$

$$+ \sum_{m=1}^{72} X_m, \tag{4.91}$$

where as usual $Z = -\ln X$.

We can analyze this mirror in the same way as previous pure gauge theory mirrors. As discussed previously, for each root μ , the fields X_μ and $X_{-\mu}$ appear paired with opposite signs coupling to each σ . Therefore, one implication of the derivatives

$$\frac{\partial W}{\partial X_\mu} = 0$$

is that, on the critical locus,

$$X_\mu = -X_{-\mu}. \tag{4.92}$$

(Furthermore, on the critical locus, each X_μ is determined by σ s.) Next, each derivative

$$\frac{\partial W}{\partial \sigma_a}$$

is a product of ratios of the form

$$\frac{X_\mu}{X_{-\mu}} = -1.$$

It is straightforward to check that, just as in the previous examples, in the superpotential above each σ multiplies a number of Z s that is divisible by four, *i.e.* an even number of ratios $X_\mu/X_{-\mu}$. Specifically, the sum of the absolute values of the coefficients of the Z 's multiplying each σ is $44 = 4 \cdot 11$. Thus, the constraint implied by integrating out the σ 's is automatically satisfied.

As a result, following the same analysis as earlier and [14], the critical locus is nonempty, and is determined by the six σ s. Thus, at the level of these topological field theory computations, we have evidence that the pure supersymmetric E_6 gauge theory in two dimensions flows in the IR to a theory of six free twisted chiral superfields.

4.5 E_7

In this section we will consider the mirror Landau-Ginzburg orbifold to an E_7 gauge theory with matter fields in the **56** fundamental representation. As before, we will compute Coulomb branch (quantum cohomology) ring relations and excluded loci. We will also study the pure E_7 gauge theory without matter.

4.5.1 Mirror Landau-Ginzburg orbifold

The mirror Landau-Ginzburg model has superfields

- $Y_{i,\beta}$, $i \in \{1, \dots, n\}$, $\beta \in \{1, \dots, 56\}$, corresponding to the matter fields in n copies of the fundamental **56** representation of E_7 ,
- X_m , $m \in \{1, \dots, 126\}$, corresponding to the nonzero roots of E_7 ,
- σ_a , $a \in \{1, \dots, 7\}$.

We associate the roots, α_m^a , to X_m fields and the weights, $\rho_{i,\beta}^a$ to the $Y_{i,\beta}$.

The nonzero roots of E_7 are listed in tables 4.8, 4.9, and 4.10. The weights of the **56** of E_7 are listed in table 4.11. All weights are given as linear combinations of fundamental weights, as discussed earlier, and computed with LieART [95], so as to have conventional θ -angle periodicities.

4.5.2 Superpotential

Plugging into the general expression for the mirror superpotential, we find for this case that the mirror superpotential is given by

$$W = \sum_{a=1}^7 \sigma_a \left(\sum_{i=1}^n \sum_{\beta=1}^{56} \rho_{i,\beta}^a Y_{i,\beta} + \sum_{m=1}^{126} \alpha_m^a Z_m \right) - \sum_{i=1}^n \tilde{m}_i \sum_{\beta=1}^{56} Y_{i,\beta} + \sum_{i=1}^n \sum_{\beta=1}^{56} \exp(-Y_{i,\beta}) + \sum_{m=1}^{126} X_m,$$

where $X_m = \exp(-Z_m)$ and X_m are the fundamental fields, we get:

$$W = \sum_{a=1}^7 \sigma_a \mathcal{C}^a - \sum_{i=1}^n \tilde{m}_i \sum_{\beta=1}^{56} Y_{i,\beta} + \sum_{i=1}^n \sum_{\beta=1}^{56} \exp(-Y_{i,\beta}) + \sum_{m=1}^{126} X_m, \quad (4.93)$$

where \mathcal{C}^a are given as follows:

$$\begin{aligned} \mathcal{C}^1 = & \sum_{i=1}^n (Y_{i,7} - Y_{i,8} + Y_{i,9} - Y_{i,10} + Y_{i,11} - Y_{i,12} + Y_{i,13} - Y_{i,14} + Y_{i,16} - Y_{i,17} + Y_{i,19} - Y_{i,20} \\ & - Y_{i,35} + Y_{i,36} - Y_{i,37} + Y_{i,38} - Y_{i,39} + Y_{i,40} - Y_{i,41} + Y_{i,42} - Y_{i,44} + Y_{i,45} \\ & - Y_{i,47} + Y_{i,48}) \\ & + Z_1 - Z_2 + Z_{14} - Z_{15} + Z_{18} - Z_{19} + Z_{21} + Z_{22} - Z_{23} - Z_{24} + Z_{26} + Z_{27} - Z_{28} \\ & - Z_{29} + Z_{30} + Z_{31} + Z_{32} - Z_{33} - Z_{34} - Z_{35} + Z_{37} + Z_{38} - Z_{39} - Z_{40} + Z_{43} + Z_{44} \\ & - Z_{45} - Z_{46} + Z_{50} - Z_{51} + Z_{56} - Z_{57} + 2Z_{63} - Z_{64} + Z_{65} - Z_{77} + Z_{78} - Z_{81} \\ & + Z_{82} - Z_{84} - Z_{85} + Z_{86} + Z_{87} - Z_{89} - Z_{90} + Z_{91} + Z_{92} - Z_{93} - Z_{94} - Z_{95} + Z_{96} \\ & + Z_{97} + Z_{98} - Z_{100} - Z_{101} + Z_{102} + Z_{103} - Z_{106} - Z_{107} + Z_{108} + Z_{109} - Z_{113} \\ & + Z_{114} - Z_{119} + Z_{120} - 2Z_{126}, \end{aligned}$$

Field	Positive root	Field	Negative root
X_1	(1, 0, 0, 0, 0, 0, 0)	X_{64}	(-1, 0, 0, 0, 0, 0, 0)
X_2	(-1, 1, 0, 0, 0, 0, 0)	X_{65}	(1, -1, 0, 0, 0, 0, 0)
X_3	(0, -1, 1, 0, 0, 0, 0)	X_{66}	(0, 1, -1, 0, 0, 0, 0)
X_4	(0, 0, -1, 1, 0, 0, 1)	X_{67}	(0, 0, 1, -1, 0, 0, -1)
X_5	(0, 0, 0, -1, 1, 0, 1)	X_{68}	(0, 0, 0, 1, -1, 0, -1)
X_6	(0, 0, 0, 1, 0, 0, -1)	X_{69}	(0, 0, 0, -1, 0, 0, 1)
X_7	(0, 0, 0, 0, -1, 1, 1)	X_{70}	(0, 0, 0, 0, 1, -1, -1)
X_8	(0, 0, 1, -1, 1, 0, -1)	X_{71}	(0, 0, -1, 1, -1, 0, 1)
X_9	(0, 0, 0, 0, 0, -1, 1)	X_{72}	(0, 0, 0, 0, 0, 1, -1)
X_{10}	(0, 0, 1, 0, -1, 1, -1)	X_{73}	(0, 0, -1, 0, 1, -1, 1)
X_{11}	(0, 1, -1, 0, 1, 0, 0)	X_{74}	(0, -1, 1, 0, -1, 0, 0)
X_{12}	(0, 0, 1, 0, 0, -1, -1)	X_{75}	(0, 0, -1, 0, 0, 1, 1)
X_{13}	(0, 1, -1, 1, -1, 1, 0)	X_{76}	(0, -1, 1, -1, 1, -1, 0)
X_{14}	(1, -1, 0, 0, 1, 0, 0)	X_{77}	(-1, 1, 0, 0, -1, 0, 0)
X_{15}	(-1, 0, 0, 0, 1, 0, 0)	X_{78}	(1, 0, 0, 0, -1, 0, 0)
X_{16}	(0, 1, -1, 1, 0, -1, 0)	X_{79}	(0, -1, 1, -1, 0, 1, 0)
X_{17}	(0, 1, 0, -1, 0, 1, 0)	X_{80}	(0, -1, 0, 1, 0, -1, 0)
X_{18}	(1, -1, 0, 1, -1, 1, 0)	X_{81}	(-1, 1, 0, -1, 1, -1, 0)
X_{19}	(-1, 0, 0, 1, -1, 1, 0)	X_{82}	(1, 0, 0, -1, 1, -1, 0)
X_{20}	(0, 1, 0, -1, 1, -1, 0)	X_{83}	(0, -1, 0, 1, -1, 1, 0)
X_{21}	(1, -1, 0, 1, 0, -1, 0)	X_{84}	(-1, 1, 0, -1, 0, 1, 0)
X_{22}	(1, -1, 1, -1, 0, 1, 0)	X_{85}	(-1, 1, -1, 1, 0, -1, 0)
X_{23}	(-1, 0, 0, 1, 0, -1, 0)	X_{86}	(1, 0, 0, -1, 0, 1, 0)
X_{24}	(-1, 0, 1, -1, 0, 1, 0)	X_{87}	(1, 0, -1, 1, 0, -1, 0)
X_{25}	(0, 1, 0, 0, -1, 0, 0)	X_{88}	(0, -1, 0, 0, 1, 0, 0)
X_{26}	(1, -1, 1, -1, 1, -1, 0)	X_{89}	(-1, 1, -1, 1, -1, 1, 0)

Table 4.8: First set of roots of E_7 and associated fields.

$$\begin{aligned}
\mathcal{C}^2 = & \sum_{i=1}^n (Y_{i,5} + Y_{i,6} - Y_{i,7} - Y_{i,9} + Y_{i,12} + Y_{i,14} - Y_{i,15} + Y_{i,17} - Y_{i,18} + Y_{i,20} - Y_{i,21} - Y_{i,23} \\
& - Y_{i,33} - Y_{i,34} + Y_{i,35} + Y_{i,37} - Y_{i,40} - Y_{i,42} + Y_{i,43} - Y_{i,45} + Y_{i,46} - Y_{i,48} + Y_{i,49} \\
& + Y_{i,51}) \\
& + Z_2 - Z_3 + Z_{11} + Z_{13} - Z_{14} + Z_{16} + Z_{17} - Z_{18} + Z_{20} - Z_{21} - Z_{22} + Z_{25} - Z_{26} \\
& + Z_{29} - Z_{30} + Z_{34} + Z_{35} - Z_{36} + Z_{39} + Z_{40} - Z_{41} - Z_{42} + Z_{45} + Z_{46} - Z_{47} - Z_{48} \\
& + Z_{51} - Z_{52} - Z_{53} + Z_{56} + 2Z_{57} - Z_{58} - Z_{63} - Z_{65} + Z_{66} - Z_{74} - Z_{76} + Z_{77} \\
& - Z_{79} - Z_{80} + Z_{81} - Z_{83} + Z_{84} + Z_{85} - Z_{88} + Z_{89} - Z_{92} + Z_{93} - Z_{97} - Z_{98} + Z_{99} \\
& - Z_{102} - Z_{103} + Z_{104} + Z_{105} - Z_{108} - Z_{109} + Z_{110} + Z_{111} - Z_{114} + Z_{115} + Z_{116} \\
& - Z_{119} - 2Z_{120} + Z_{121} + Z_{126},
\end{aligned}$$

Field	Positive root	Field	Negative root
X_{27}	$(1, 0, -1, 0, 0, 1, 1)$	X_{90}	$(-1, 0, 1, 0, 0, -1, -1)$
X_{28}	$(-1, 0, 1, -1, 1, -1, 0)$	X_{91}	$(1, 0, -1, 1, -1, 1, 0)$
X_{29}	$(-1, 1, -1, 0, 0, 1, 1)$	X_{92}	$(1, -1, 1, 0, 0, -1, -1)$
X_{30}	$(1, -1, 1, 0, -1, 0, 0)$	X_{93}	$(-1, 1, -1, 0, 1, 0, 0)$
X_{31}	$(1, 0, -1, 0, 1, -1, 1)$	X_{94}	$(-1, 0, 1, 0, -1, 1, -1)$
X_{32}	$(1, 0, 0, 0, 0, 1, -1)$	X_{95}	$(-1, 0, 0, 0, 0, -1, 1)$
X_{33}	$(-1, 0, 1, 0, -1, 0, 0)$	X_{96}	$(1, 0, -1, 0, 1, 0, 0)$
X_{34}	$(-1, 1, -1, 0, 1, -1, 1)$	X_{97}	$(1, -1, 1, 0, -1, 1, -1)$
X_{35}	$(-1, 1, 0, 0, 0, 1, -1)$	X_{98}	$(1, -1, 0, 0, 0, -1, 1)$
X_{36}	$(0, -1, 0, 0, 0, 1, 1)$	X_{99}	$(0, 1, 0, 0, 0, -1, -1)$
X_{37}	$(1, 0, -1, 1, -1, 0, 1)$	X_{100}	$(-1, 0, 1, -1, 1, 0, -1)$
X_{38}	$(1, 0, 0, 0, 1, -1, -1)$	X_{101}	$(-1, 0, 0, 0, -1, 1, 1)$
X_{39}	$(-1, 1, -1, 1, -1, 0, 1)$	X_{102}	$(1, -1, 1, -1, 1, 0, -1)$
X_{40}	$(-1, 1, 0, 0, 1, -1, -1)$	X_{103}	$(1, -1, 0, 0, -1, 1, 1)$
X_{41}	$(0, -1, 0, 0, 1, -1, 1)$	X_{104}	$(0, 1, 0, 0, -1, 1, -1)$
X_{42}	$(0, -1, 1, 0, 0, 1, -1)$	X_{105}	$(0, 1, -1, 0, 0, -1, 1)$
X_{43}	$(1, 0, 0, -1, 0, 0, 1)$	X_{106}	$(-1, 0, 0, 1, 0, 0, -1)$
X_{44}	$(1, 0, 0, 1, -1, 0, -1)$	X_{107}	$(-1, 0, 0, -1, 1, 0, 1)$
X_{45}	$(-1, 1, 0, -1, 0, 0, 1)$	X_{108}	$(1, -1, 0, 1, 0, 0, -1)$
X_{46}	$(-1, 1, 0, 1, -1, 0, -1)$	X_{109}	$(1, -1, 0, -1, 1, 0, 1)$
X_{47}	$(0, -1, 0, 1, -1, 0, 1)$	X_{110}	$(0, 1, 0, -1, 1, 0, -1)$
X_{48}	$(0, -1, 1, 0, 1, -1, -1)$	X_{111}	$(0, 1, -1, 0, -1, 1, 1)$
X_{49}	$(0, 0, -1, 1, 0, 1, 0)$	X_{112}	$(0, 0, 1, -1, 0, -1, 0)$
X_{50}	$(1, 0, 1, -1, 0, 0, -1)$	X_{113}	$(-1, 0, -1, 1, 0, 0, 1)$
X_{51}	$(-1, 1, 1, -1, 0, 0, -1)$	X_{114}	$(1, -1, -1, 1, 0, 0, 1)$
X_{52}	$(0, -1, 1, -1, 0, 0, 1)$	X_{115}	$(0, 1, -1, 1, 0, 0, -1)$

Table 4.9: Second set of roots of E_7 and associated fields.

$$\begin{aligned}
\mathcal{C}^3 = & \sum_{i=1}^n (Y_{i,4} - Y_{i,5} + Y_{i,9} + Y_{i,10} - Y_{i,11} - Y_{i,12} + Y_{i,18} + Y_{i,21} - Y_{i,22} + Y_{i,23} - Y_{i,24} - Y_{i,26} \\
& - Y_{i,32} + Y_{i,33} - Y_{i,37} - Y_{i,38} + Y_{i,39} + Y_{i,40} - Y_{i,46} - Y_{i,49} + Y_{i,50} - Y_{i,51} + Y_{i,52} \\
& + Y_{i,54}) \\
& + Z_3 - Z_4 + Z_8 + Z_{10} - Z_{11} + Z_{12} - Z_{13} - Z_{16} + Z_{22} + Z_{24} + Z_{26} - Z_{27} + Z_{28} \\
& - Z_{29} + Z_{30} - Z_{31} + Z_{33} - Z_{34} - Z_{37} - Z_{39} + Z_{42} + Z_{48} - Z_{49} + Z_{50} + Z_{51} \\
& + Z_{52} + Z_{53} - Z_{54} - Z_{56} - Z_{57} + 2Z_{58} - Z_{59} - Z_{60} - Z_{66} + Z_{67} - Z_{71} - Z_{73} \\
& + Z_{74} - Z_{75} + Z_{76} + Z_{79} - Z_{85} - Z_{87} - Z_{89} + Z_{90} - Z_{91} + Z_{92} - Z_{93} + Z_{94} \\
& - Z_{96} + Z_{97} + Z_{100} + Z_{102} - Z_{105} - Z_{111} + Z_{112} - Z_{113} - Z_{114} - Z_{115} - Z_{116} \\
& + Z_{117} + Z_{119} + Z_{120} - 2Z_{121} + Z_{122} + Z_{123},
\end{aligned}$$

Field	Positive root	Field	Negative root
X_{53}	$(0, -1, 1, 1, -1, 0, -1)$	X_{116}	$(0, 1, -1, -1, 1, 0, 1)$
X_{54}	$(0, 0, -1, 1, 1, -1, 0)$	X_{117}	$(0, 0, 1, -1, -1, 1, 0)$
X_{55}	$(0, 0, 0, -1, 1, 1, 0)$	X_{118}	$(0, 0, 0, 1, -1, -1, 0)$
X_{56}	$(1, 1, -1, 0, 0, 0, 0)$	X_{119}	$(-1, -1, 1, 0, 0, 0, 0)$
X_{57}	$(-1, 2, -1, 0, 0, 0, 0)$	X_{120}	$(1, -2, 1, 0, 0, 0, 0)$
X_{58}	$(0, -1, 2, -1, 0, 0, -1)$	X_{121}	$(0, 1, -2, 1, 0, 0, 1)$
X_{59}	$(0, 0, -1, 0, 0, 0, 2)$	X_{122}	$(0, 0, 1, 0, 0, 0, -2)$
X_{60}	$(0, 0, -1, 2, -1, 0, 0)$	X_{123}	$(0, 0, 1, -2, 1, 0, 0)$
X_{61}	$(0, 0, 0, -1, 2, -1, 0)$	X_{124}	$(0, 0, 0, 1, -2, 1, 0)$
X_{62}	$(0, 0, 0, 0, -1, 2, 0)$	X_{125}	$(0, 0, 0, 0, 1, -2, 0)$
X_{63}	$(2, -1, 0, 0, 0, 0, 0)$	X_{126}	$(-2, 1, 0, 0, 0, 0, 0)$

Table 4.10: Third set of roots of E_7 and associated fields.

$$\begin{aligned}
\mathcal{C}^4 = & \sum_{i=1}^n (Y_{i,3} - Y_{i,4} + Y_{i,11} + Y_{i,12} - Y_{i,13} - Y_{i,14} + Y_{i,15} - Y_{i,18} + Y_{i,24} + Y_{i,26} - Y_{i,27} + Y_{i,28} \\
& - Y_{i,31} + Y_{i,32} - Y_{i,39} - Y_{i,40} + Y_{i,41} + Y_{i,42} - Y_{i,43} + Y_{i,46} - Y_{i,52} - Y_{i,54} + Y_{i,55} \\
& - Y_{i,56}) \\
& + Z_4 - Z_5 + Z_6 - Z_8 + Z_{13} + Z_{16} - Z_{17} + Z_{18} + Z_{19} - Z_{20} + Z_{21} - Z_{22} + Z_{23} \\
& - Z_{24} - Z_{26} - Z_{28} + Z_{37} + Z_{39} - Z_{43} + Z_{44} - Z_{45} + Z_{46} + Z_{47} + Z_{49} - Z_{50} \\
& - Z_{51} - Z_{52} + Z_{53} + Z_{54} - Z_{55} - Z_{58} + 2Z_{60} - Z_{61} - Z_{67} + Z_{68} - Z_{69} + Z_{71} \\
& - Z_{76} - Z_{79} + Z_{80} - Z_{81} - Z_{82} + Z_{83} - Z_{84} + Z_{85} - Z_{86} + Z_{87} + Z_{89} + Z_{91} \\
& - Z_{100} - Z_{102} + Z_{106} - Z_{107} + Z_{108} - Z_{109} - Z_{110} - Z_{112} + Z_{113} + Z_{114} + Z_{115} \\
& - Z_{116} - Z_{117} + Z_{118} + Z_{121} - 2Z_{123} + Z_{124},
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}^5 = & \sum_{i=1}^n (Y_{i,2} - Y_{i,3} + Y_{i,13} + Y_{i,14} - Y_{i,16} - Y_{i,17} + Y_{i,18} - Y_{i,21} + Y_{i,22} - Y_{i,24} + Y_{i,25} - Y_{i,28} \\
& - Y_{i,30} + Y_{i,31} - Y_{i,41} - Y_{i,42} + Y_{i,44} + Y_{i,45} - Y_{i,46} + Y_{i,49} - Y_{i,50} + Y_{i,52} - Y_{i,53} \\
& + Y_{i,56}) \\
& + Z_5 - Z_7 + Z_8 - Z_{10} + Z_{11} - Z_{13} + Z_{14} + Z_{15} - Z_{18} - Z_{19} + Z_{20} - Z_{25} + Z_{26} \\
& + Z_{28} - Z_{30} + Z_{31} - Z_{33} + Z_{34} - Z_{37} + Z_{38} - Z_{39} + Z_{40} + Z_{41} - Z_{44} - Z_{46} - Z_{47} \\
& + Z_{48} - Z_{53} + Z_{54} + Z_{55} - Z_{60} + 2Z_{61} - Z_{62} - Z_{68} + Z_{70} - Z_{71} + Z_{73} - Z_{74} \\
& + Z_{76} - Z_{77} - Z_{78} + Z_{81} + Z_{82} - Z_{83} + Z_{88} - Z_{89} - Z_{91} + Z_{93} - Z_{94} + Z_{96} - Z_{97} \\
& + Z_{100} - Z_{101} + Z_{102} - Z_{103} - Z_{104} + Z_{107} + Z_{109} + Z_{110} - Z_{111} + Z_{116} - Z_{117} \\
& - Z_{118} + Z_{123} - 2Z_{124} + Z_{125},
\end{aligned}$$

Field	Weight	Field	Weight
$Y_{i,1}$	$(0, 0, 0, 0, 0, 1, 0)$	$Y_{i,29}$	$(0, 0, 0, 0, 0, -1, 0)$
$Y_{i,2}$	$(0, 0, 0, 0, 1, -1, 0)$	$Y_{i,30}$	$(0, 0, 0, 0, -1, 1, 0)$
$Y_{i,3}$	$(0, 0, 0, 1, -1, 0, 0)$	$Y_{i,31}$	$(0, 0, 0, -1, 1, 0, 0)$
$Y_{i,4}$	$(0, 0, 1, -1, 0, 0, 0)$	$Y_{i,32}$	$(0, 0, -1, 1, 0, 0, 0)$
$Y_{i,5}$	$(0, 1, -1, 0, 0, 0, 1)$	$Y_{i,33}$	$(0, -1, 1, 0, 0, 0, -1)$
$Y_{i,6}$	$(0, 1, 0, 0, 0, 0, -1)$	$Y_{i,34}$	$(0, -1, 0, 0, 0, 0, 1)$
$Y_{i,7}$	$(1, -1, 0, 0, 0, 0, 1)$	$Y_{i,35}$	$(-1, 1, 0, 0, 0, 0, -1)$
$Y_{i,8}$	$(-1, 0, 0, 0, 0, 0, 1)$	$Y_{i,36}$	$(1, 0, 0, 0, 0, 0, -1)$
$Y_{i,9}$	$(1, -1, 1, 0, 0, 0, -1)$	$Y_{i,37}$	$(-1, 1, -1, 0, 0, 0, 1)$
$Y_{i,10}$	$(-1, 0, 1, 0, 0, 0, -1)$	$Y_{i,38}$	$(1, 0, -1, 0, 0, 0, 1)$
$Y_{i,11}$	$(1, 0, -1, 1, 0, 0, 0)$	$Y_{i,39}$	$(-1, 0, 1, -1, 0, 0, 0)$
$Y_{i,12}$	$(-1, 1, -1, 1, 0, 0, 0)$	$Y_{i,40}$	$(1, -1, 1, -1, 0, 0, 0)$
$Y_{i,13}$	$(1, 0, 0, -1, 1, 0, 0)$	$Y_{i,41}$	$(-1, 0, 0, 1, -1, 0, 0)$
$Y_{i,14}$	$(-1, 1, 0, -1, 1, 0, 0)$	$Y_{i,42}$	$(1, -1, 0, 1, -1, 0, 0)$
$Y_{i,15}$	$(0, -1, 0, 1, 0, 0, 0)$	$Y_{i,43}$	$(0, 1, 0, -1, 0, 0, 0)$
$Y_{i,16}$	$(1, 0, 0, 0, -1, 1, 0)$	$Y_{i,44}$	$(-1, 0, 0, 0, 1, -1, 0)$
$Y_{i,17}$	$(-1, 1, 0, 0, -1, 1, 0)$	$Y_{i,45}$	$(1, -1, 0, 0, 1, -1, 0)$
$Y_{i,18}$	$(0, -1, 1, -1, 1, 0, 0)$	$Y_{i,46}$	$(0, 1, -1, 1, -1, 0, 0)$
$Y_{i,19}$	$(1, 0, 0, 0, 0, -1, 0)$	$Y_{i,47}$	$(-1, 0, 0, 0, 0, 1, 0)$
$Y_{i,20}$	$(-1, 1, 0, 0, 0, -1, 0)$	$Y_{i,48}$	$(1, -1, 0, 0, 0, 1, 0)$
$Y_{i,21}$	$(0, -1, 1, 0, -1, 1, 0)$	$Y_{i,49}$	$(0, 1, -1, 0, 1, -1, 0)$
$Y_{i,22}$	$(0, 0, -1, 0, 1, 0, 1)$	$Y_{i,50}$	$(0, 0, 1, 0, -1, 0, -1)$
$Y_{i,23}$	$(0, -1, 1, 0, 0, -1, 0)$	$Y_{i,51}$	$(0, 1, -1, 0, 0, 1, 0)$
$Y_{i,24}$	$(0, 0, -1, 1, -1, 1, 1)$	$Y_{i,52}$	$(0, 0, 1, -1, 1, -1, -1)$
$Y_{i,25}$	$(0, 0, 0, 0, 1, 0, -1)$	$Y_{i,53}$	$(0, 0, 0, 0, -1, 0, 1)$
$Y_{i,26}$	$(0, 0, -1, 1, 0, -1, 1)$	$Y_{i,54}$	$(0, 0, 1, -1, 0, 1, -1)$
$Y_{i,27}$	$(0, 0, 0, -1, 0, 1, 1)$	$Y_{i,55}$	$(0, 0, 0, 1, 0, -1, -1)$
$Y_{i,28}$	$(0, 0, 0, 1, -1, 1, -1)$	$Y_{i,56}$	$(0, 0, 0, -1, 1, -1, 1)$

Table 4.11: Weights of **56** of E_7 and associated fields.

$$\begin{aligned}
C^6 = & \sum_{i=1}^n (Y_{i,1} - Y_{i,2} + Y_{i,16} + Y_{i,17} - Y_{i,19} - Y_{i,20} + Y_{i,21} - Y_{i,23} + Y_{i,24} - Y_{i,26} + Y_{i,27} + Y_{i,28} \\
& - Y_{i,29} + Y_{i,30} - Y_{i,44} - Y_{i,45} + Y_{i,47} + Y_{i,48} - Y_{i,49} + Y_{i,51} - Y_{i,52} + Y_{i,54} - Y_{i,55} \\
& - Y_{i,56}) \\
& + Z_7 - Z_9 + Z_{10} - Z_{12} + Z_{13} - Z_{16} + Z_{17} + Z_{18} + Z_{19} - Z_{20} - Z_{21} + Z_{22} - Z_{23} \\
& + Z_{24} - Z_{26} + Z_{27} - Z_{28} + Z_{29} - Z_{31} + Z_{32} - Z_{34} + Z_{35} + Z_{36} - Z_{38} - Z_{40} - Z_{41} \\
& + Z_{42} - Z_{48} + Z_{49} - Z_{54} + Z_{55} - Z_{61} + 2Z_{62} - Z_{70} + Z_{72} - Z_{73} + Z_{75} - Z_{76} \\
& + Z_{79} - Z_{80} - Z_{81} - Z_{82} + Z_{83} + Z_{84} - Z_{85} + Z_{86} - Z_{87} + Z_{89} - Z_{90} + Z_{91} - Z_{92} \\
& + Z_{94} - Z_{95} + Z_{97} - Z_{98} - Z_{99} + Z_{101} + Z_{103} + Z_{104} - Z_{105} + Z_{111} - Z_{112} + Z_{117} \\
& - Z_{118} + Z_{124} - 2Z_{125},
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}^7 = & \sum_{i=1}^n (Y_{i,2} - Y_{i,3} + Y_{i,13} + Y_{i,14} - Y_{i,16} - Y_{i,17} + Y_{i,18} - Y_{i,21} + Y_{i,22} - Y_{i,24} + Y_{i,25} - Y_{i,28} \\
& - Y_{i,30} + Y_{i,31} - Y_{i,41} - Y_{i,42} + Y_{i,44} + Y_{i,45} - Y_{i,46} + Y_{i,49} - Y_{i,50} + Y_{i,52} - Y_{i,53} \\
& + Y_{i,56}) \\
& + Z_5 - Z_7 + Z_8 - Z_{10} + Z_{11} - Z_{13} + Z_{14} + Z_{15} - Z_{18} - Z_{19} + Z_{20} - Z_{25} + Z_{26} \\
& + Z_{28} - Z_{30} + Z_{31} - Z_{33} + Z_{34} - Z_{37} + Z_{38} - Z_{39} + Z_{40} + Z_{41} - Z_{44} - Z_{46} - Z_{47} \\
& + Z_{48} - Z_{53} + Z_{54} + Z_{55} - Z_{60} + 2Z_{61} - Z_{62} - Z_{68} + Z_{70} - Z_{71} + Z_{73} - Z_{74} \\
& + Z_{76} - Z_{77} - Z_{78} + Z_{81} + Z_{82} - Z_{83} + Z_{88} - Z_{89} - Z_{91} + Z_{93} - Z_{94} + Z_{96} - Z_{97} \\
& + Z_{100} - Z_{101} + Z_{102} - Z_{103} - Z_{104} + Z_{107} + Z_{109} + Z_{110} - Z_{111} + Z_{116} - Z_{117} \\
& - Z_{118} + Z_{123} - 2Z_{124} + Z_{125}.
\end{aligned}$$

4.5.3 Coulomb ring relations

Integrating out the σ_a fields, we obtain seven constraints $\mathcal{C}^a = 0$. Exponentiating these constraints, we obtain a series of equations from which the Coulomb ring relations will be derived. For reasons of notational sanity, we will also slightly simplify these expressions, as follows. To make predictions of the A model, we will evaluate the ring relations on the critical locus, where

$$\frac{X_m}{X_{m+63}} = -1.$$

It is straightforward to see that each of the constraints \mathcal{C}^a contains 34 differences of corresponding Z 's, so that the exponential of the constraints contains 34 factors of the form X_m/X_{m+63} – an even number of factors of -1 , which will cancel out. Therefore, since on the critical locus those factors will cancel out, we will omit them, and solely relate the exponentiated constraints in terms of Y s.

The exponentiated constraints are as follows.

$$\begin{aligned}
\prod_{i=1}^n \exp (Y_{i,7} - Y_{i,8} + Y_{i,9} - Y_{i,10} + Y_{i,11} - Y_{i,12} + Y_{i,13} - Y_{i,14} + Y_{i,16} - Y_{i,17} + Y_{i,19} - Y_{i,20} \\
- Y_{i,35} + Y_{i,36} - Y_{i,37} + Y_{i,38} - Y_{i,39} + Y_{i,40} - Y_{i,41} + Y_{i,42} - Y_{i,44} + Y_{i,45} - Y_{i,47} + Y_{i,48}) \\
= 1, \quad (4.94)
\end{aligned}$$

$$\begin{aligned}
\prod_{i=1}^n \exp (Y_{i,5} + Y_{i,6} - Y_{i,7} - Y_{i,9} + Y_{i,12} + Y_{i,14} - Y_{i,15} + Y_{i,17} - Y_{i,18} + Y_{i,20} - Y_{i,21} - Y_{i,23} \\
- Y_{i,33} - Y_{i,34} + Y_{i,35} + Y_{i,37} - Y_{i,40} - Y_{i,42} + Y_{i,43} - Y_{i,45} + Y_{i,46} - Y_{i,48} + Y_{i,49} + Y_{i,51}) \\
= 1, \quad (4.95)
\end{aligned}$$

$$\prod_{i=1}^n \exp \left(Y_{i,4} - Y_{i,5} + Y_{i,9} + Y_{i,10} - Y_{i,11} - Y_{i,12} + Y_{i,18} + Y_{i,21} - Y_{i,22} + Y_{i,23} - Y_{i,24} - Y_{i,26} \right. \\ \left. - Y_{i,32} + Y_{i,33} - Y_{i,37} - Y_{i,38} + Y_{i,39} + Y_{i,40} - Y_{i,46} - Y_{i,49} + Y_{i,50} - Y_{i,51} + Y_{i,52} + Y_{i,54} \right) \\ = 1, \quad (4.96)$$

$$\prod_{i=1}^n \exp \left(Y_{i,3} - Y_{i,4} + Y_{i,11} + Y_{i,12} - Y_{i,13} - Y_{i,14} + Y_{i,15} - Y_{i,18} + Y_{i,24} + Y_{i,26} - Y_{i,27} + Y_{i,28} \right. \\ \left. - Y_{i,31} + Y_{i,32} - Y_{i,39} - Y_{i,40} + Y_{i,41} + Y_{i,42} - Y_{i,43} + Y_{i,46} - Y_{i,52} - Y_{i,54} + Y_{i,55} - Y_{i,56} \right) \\ = 1, \quad (4.97)$$

$$\prod_{i=1}^n \exp \left(Y_{i,2} - Y_{i,3} + Y_{i,13} + Y_{i,14} - Y_{i,16} - Y_{i,17} + Y_{i,18} - Y_{i,21} + Y_{i,22} - Y_{i,24} + Y_{i,25} - Y_{i,28} \right. \\ \left. - Y_{i,30} + Y_{i,31} - Y_{i,41} - Y_{i,42} + Y_{i,44} + Y_{i,45} - Y_{i,46} + Y_{i,49} - Y_{i,50} + Y_{i,52} - Y_{i,53} + Y_{i,56} \right) \\ = 1, \quad (4.98)$$

$$\prod_{i=1}^n \exp \left(Y_{i,1} - Y_{i,2} + Y_{i,16} + Y_{i,17} - Y_{i,19} - Y_{i,20} + Y_{i,21} - Y_{i,23} + Y_{i,24} - Y_{i,26} + Y_{i,27} + Y_{i,28} \right. \\ \left. - Y_{i,29} + Y_{i,30} - Y_{i,44} - Y_{i,45} + Y_{i,47} + Y_{i,48} - Y_{i,49} + Y_{i,51} - Y_{i,52} + Y_{i,54} - Y_{i,55} - Y_{i,56} \right) \\ = 1, \quad (4.99)$$

$$\prod_{i=1}^n \exp \left(Y_{i,2} - Y_{i,3} + Y_{i,13} + Y_{i,14} - Y_{i,16} - Y_{i,17} + Y_{i,18} - Y_{i,21} + Y_{i,22} - Y_{i,24} + Y_{i,25} - Y_{i,28} \right. \\ \left. - Y_{i,30} + Y_{i,31} - Y_{i,41} - Y_{i,42} + Y_{i,44} + Y_{i,45} - Y_{i,46} + Y_{i,49} - Y_{i,50} + Y_{i,52} - Y_{i,53} + Y_{i,56} \right) \\ = 1. \quad (4.100)$$

The mirror map is given by,

$$\exp(-Y_{i,\beta}) \mapsto -\tilde{m}_i + \sum_{a=1}^7 \sigma_a \rho_{i,\beta}^a, \quad X_m \mapsto \sum_{a=1}^7 \sigma_a \alpha_m^a.$$

After applying the mirror map, the constraints become

$$\begin{aligned}
& \prod_{i=1}^n (-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\
& \cdot (-\sigma_1 + \sigma_2 - \sigma_4 + \sigma_5 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_5 - \sigma_6 - \tilde{m}_i) \\
& \cdot (-\sigma_1 + \sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_5 + \sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_7 - \tilde{m}_i) \\
& \cdot (-\sigma_1 + \sigma_3 - \sigma_7 - \tilde{m}_i) (-\sigma_1 + \sigma_7 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_7 - \tilde{m}_i) \\
& = \prod_{i=1}^n (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_3 + \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\
& \cdot (\sigma_1 - \sigma_4 + \sigma_5 - \tilde{m}_i) (\sigma_1 - \sigma_6 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_5 - \sigma_6 - \tilde{m}_i) \\
& \cdot (\sigma_1 - \sigma_2 + \sigma_6 - \tilde{m}_i) (\sigma_1 - \sigma_5 + \sigma_6 - \tilde{m}_i) (\sigma_1 - \sigma_7 - \tilde{m}_i) \\
& \cdot (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_7 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_7 - \tilde{m}_i) (\sigma_1 - \sigma_3 + \sigma_7 - \tilde{m}_i), \quad (4.101)
\end{aligned}$$

$$\begin{aligned}
& \prod_{i=1}^n (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i) (-\sigma_2 + \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\
& \cdot (-\sigma_2 + \sigma_3 - \sigma_4 + \sigma_5 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \sigma_6 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_5 - \sigma_6 - \tilde{m}_i) \\
& \cdot (\sigma_1 - \sigma_2 + \sigma_6 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \sigma_5 + \sigma_6 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \sigma_7 - \tilde{m}_i) \\
& \cdot (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_7 - \tilde{m}_i) (-\sigma_2 + \sigma_7 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_7 - \tilde{m}_i) \\
& = \prod_{i=1}^n (\sigma_2 - \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\
& \cdot (-\sigma_1 + \sigma_2 - \sigma_4 + \sigma_5 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_6 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_5 - \sigma_6 - \tilde{m}_i) \\
& \cdot (\sigma_2 - \sigma_3 + \sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_5 + \sigma_6 - \tilde{m}_i) (\sigma_2 - \sigma_7 - \tilde{m}_i) \\
& \cdot (-\sigma_1 + \sigma_2 - \sigma_7 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_7 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_7 - \tilde{m}_i), \quad (4.102)
\end{aligned}$$

$$\begin{aligned}
& \prod_{i=1}^n (-\sigma_3 + \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_3 + \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 - \tilde{m}_i) \\
& \cdot (\sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_5 - \sigma_6 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_6 - \tilde{m}_i) \\
& \cdot (\sigma_1 - \sigma_3 + \sigma_7 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_7 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_7 - \tilde{m}_i) \\
& \cdot (-\sigma_3 + \sigma_5 + \sigma_7 - \tilde{m}_i) (-\sigma_3 + \sigma_4 - \sigma_6 + \sigma_7 - \tilde{m}_i) (-\sigma_3 + \sigma_4 - \sigma_5 + \sigma_6 + \sigma_7 - \tilde{m}_i) \\
& = \prod_{i=1}^n (\sigma_3 - \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i) \\
& \cdot (-\sigma_2 + \sigma_3 - \sigma_4 + \sigma_5 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \sigma_6 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \sigma_5 + \sigma_6 - \tilde{m}_i) \\
& \cdot (-\sigma_1 + \sigma_3 - \sigma_7 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \sigma_7 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_7 - \tilde{m}_i) \\
& \cdot (\sigma_3 - \sigma_5 - \sigma_7 - \tilde{m}_i) (\sigma_3 - \sigma_4 + \sigma_5 - \sigma_6 - \sigma_7 - \tilde{m}_i) \\
& \cdot (\sigma_3 - \sigma_4 + \sigma_6 - \sigma_7 - \tilde{m}_i), \quad (4.103)
\end{aligned}$$

$$\begin{aligned}
& \prod_{i=1}^n (\sigma_2 - \sigma_4 - \tilde{m}_i) (\sigma_3 - \sigma_4 - \tilde{m}_i) (-\sigma_1 + \sigma_3 - \sigma_4 - \tilde{m}_i) \\
& \cdot (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 - \tilde{m}_i) (-\sigma_4 + \sigma_5 - \tilde{m}_i) (\sigma_1 - \sigma_4 + \sigma_5 - \tilde{m}_i) \\
& \cdot (-\sigma_1 + \sigma_2 - \sigma_4 + \sigma_5 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \sigma_4 + \sigma_5 - \tilde{m}_i) (\sigma_3 - \sigma_4 + \sigma_5 - \sigma_6 - \sigma_7 - \tilde{m}_i) \\
& \cdot (\sigma_3 - \sigma_4 + \sigma_6 - \sigma_7 - \tilde{m}_i) (-\sigma_4 + \sigma_5 - \sigma_6 + \sigma_7 - \tilde{m}_i) (-\sigma_4 + \sigma_6 + \sigma_7 - \tilde{m}_i) \\
& = \prod_{i=1}^n (-\sigma_2 + \sigma_4 - \tilde{m}_i) (-\sigma_3 + \sigma_4 - \tilde{m}_i) (\sigma_1 - \sigma_3 + \sigma_4 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 - \tilde{m}_i) (\sigma_4 - \sigma_5 - \tilde{m}_i) (-\sigma_1 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\
& \quad \cdot (\sigma_1 - \sigma_2 + \sigma_4 - \sigma_5 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 - \tilde{m}_i) (\sigma_4 - \sigma_6 - \sigma_7 - \tilde{m}_i) \\
& \quad \cdot (\sigma_4 - \sigma_5 + \sigma_6 - \sigma_7 - \tilde{m}_i) (-\sigma_3 + \sigma_4 - \sigma_6 + \sigma_7 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_3 + \sigma_4 - \sigma_5 + \sigma_6 + \sigma_7 - \tilde{m}_i), \tag{4.104}
\end{aligned}$$

$$\begin{aligned}
& \prod_{i=1}^n (\sigma_4 - \sigma_5 - \tilde{m}_i) (-\sigma_1 + \sigma_4 - \sigma_5 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_4 - \sigma_5 - \tilde{m}_i) \\
& \quad \cdot (\sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 - \tilde{m}_i) (-\sigma_5 + \sigma_6 - \tilde{m}_i) (\sigma_1 - \sigma_5 + \sigma_6 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_1 + \sigma_2 - \sigma_5 + \sigma_6 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \sigma_5 + \sigma_6 - \tilde{m}_i) (\sigma_3 - \sigma_5 - \sigma_7 - \tilde{m}_i) \\
& \quad \cdot (\sigma_4 - \sigma_5 + \sigma_6 - \sigma_7 - \tilde{m}_i) (-\sigma_5 + \sigma_7 - \tilde{m}_i) (-\sigma_3 + \sigma_4 - \sigma_5 + \sigma_6 + \sigma_7 - \tilde{m}_i) \\
& = \prod_{i=1}^n (-\sigma_4 + \sigma_5 - \tilde{m}_i) (\sigma_1 - \sigma_4 + \sigma_5 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_4 + \sigma_5 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_2 + \sigma_3 - \sigma_4 + \sigma_5 - \tilde{m}_i) (\sigma_5 - \sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_5 - \sigma_6 - \tilde{m}_i) \\
& \quad \cdot (\sigma_1 - \sigma_2 + \sigma_5 - \sigma_6 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_5 - \sigma_6 - \tilde{m}_i) (\sigma_5 - \sigma_7 - \tilde{m}_i) \\
& \quad \cdot (\sigma_3 - \sigma_4 + \sigma_5 - \sigma_6 - \sigma_7 - \tilde{m}_i) (-\sigma_3 + \sigma_5 + \sigma_7 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_4 + \sigma_5 - \sigma_6 + \sigma_7 - \tilde{m}_i), \tag{4.105}
\end{aligned}$$

$$\begin{aligned}
& \prod_{i=1}^n (-\sigma_6 - \tilde{m}_i) (\sigma_1 - \sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_6 - \tilde{m}_i) \\
& \cdot (-\sigma_2 + \sigma_3 - \sigma_6 - \tilde{m}_i) (\sigma_5 - \sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_5 - \sigma_6 - \tilde{m}_i) \\
& \cdot (\sigma_1 - \sigma_2 + \sigma_5 - \sigma_6 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_5 - \sigma_6 - \tilde{m}_i) (\sigma_4 - \sigma_6 - \sigma_7 - \tilde{m}_i) \\
& \cdot (\sigma_3 - \sigma_4 + \sigma_5 - \sigma_6 - \sigma_7 - \tilde{m}_i) (-\sigma_3 + \sigma_4 - \sigma_6 + \sigma_7 - \tilde{m}_i) (-\sigma_4 + \sigma_5 - \sigma_6 + \sigma_7 - \tilde{m}_i) \\
& = \prod_{i=1}^n (\sigma_6 - \tilde{m}_i) (-\sigma_1 + \sigma_6 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_6 - \tilde{m}_i) \\
& \quad \cdot (\sigma_2 - \sigma_3 + \sigma_6 - \tilde{m}_i) (-\sigma_5 + \sigma_6 - \tilde{m}_i) (\sigma_1 - \sigma_5 + \sigma_6 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_1 + \sigma_2 - \sigma_5 + \sigma_6 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \sigma_5 + \sigma_6 - \tilde{m}_i) (\sigma_3 - \sigma_4 + \sigma_6 - \sigma_7 - \tilde{m}_i) \\
& \quad \cdot (\sigma_4 - \sigma_5 + \sigma_6 - \sigma_7 - \tilde{m}_i) (-\sigma_4 + \sigma_6 + \sigma_7 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_3 + \sigma_4 - \sigma_5 + \sigma_6 + \sigma_7 - \tilde{m}_i), \tag{4.106}
\end{aligned}$$

$$\begin{aligned}
& \prod_{i=1}^n (\sigma_1 - \sigma_7 - \tilde{m}_i) (\sigma_2 - \sigma_7 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_7 - \tilde{m}_i) \\
& \cdot (-\sigma_1 + \sigma_3 - \sigma_7 - \tilde{m}_i) (-\sigma_2 + \sigma_3 - \sigma_7 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_7 - \tilde{m}_i) \\
& \cdot (\sigma_3 - \sigma_5 - \sigma_7 - \tilde{m}_i) (\sigma_5 - \sigma_7 - \tilde{m}_i) (\sigma_4 - \sigma_6 - \sigma_7 - \tilde{m}_i) \\
& \cdot (\sigma_3 - \sigma_4 + \sigma_5 - \sigma_6 - \sigma_7 - \tilde{m}_i) (\sigma_3 - \sigma_4 + \sigma_6 - \sigma_7 - \tilde{m}_i) (\sigma_4 - \sigma_5 + \sigma_6 - \sigma_7 - \tilde{m}_i) \\
& = \prod_{i=1}^n (-\sigma_1 + \sigma_7 - \tilde{m}_i) (-\sigma_2 + \sigma_7 - \tilde{m}_i) (\sigma_1 - \sigma_2 + \sigma_7 - \tilde{m}_i) \\
& \quad \cdot (\sigma_1 - \sigma_3 + \sigma_7 - \tilde{m}_i) (\sigma_2 - \sigma_3 + \sigma_7 - \tilde{m}_i) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_7 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_5 + \sigma_7 - \tilde{m}_i) (-\sigma_3 + \sigma_5 + \sigma_7 - \tilde{m}_i) (-\sigma_3 + \sigma_4 - \sigma_6 + \sigma_7 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_4 + \sigma_5 - \sigma_6 + \sigma_7 - \tilde{m}_i) (-\sigma_4 + \sigma_6 + \sigma_7 - \tilde{m}_i) \\
& \quad \cdot (-\sigma_3 + \sigma_4 - \sigma_5 + \sigma_6 + \sigma_7 - \tilde{m}_i). \tag{4.107}
\end{aligned}$$

These equations define the relations in the Coulomb ring for this gauge theory.

Part of the excluded locus is defined by the condition that the $X_m \neq 0$. This part of the

excluded locus is encoded by

$$\begin{aligned}
& \sigma_1 (2\sigma_1 - \sigma_2) (\sigma_2 - \sigma_1) (\sigma_1 + \sigma_2 - \sigma_3) (-\sigma_1 + 2\sigma_2 - \sigma_3) (\sigma_3 - \sigma_2) (\sigma_2 - \sigma_5) (-\sigma_1 + \sigma_3 - \sigma_5) \\
& \cdot (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_5) (-\sigma_3 + 2\sigma_4 - \sigma_5) (\sigma_5 - \sigma_1) (\sigma_1 - \sigma_2 + \sigma_5) (\sigma_2 - \sigma_3 + \sigma_5) \\
& \cdot (-\sigma_1 + \sigma_4 - \sigma_6) (\sigma_1 - \sigma_2 + \sigma_4 - \sigma_6) (\sigma_2 - \sigma_3 + \sigma_4 - \sigma_6) (\sigma_2 - \sigma_4 + \sigma_5 - \sigma_6) \\
& \cdot (-\sigma_1 + \sigma_3 - \sigma_4 + \sigma_5 - \sigma_6) (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 + \sigma_5 - \sigma_6) (-\sigma_3 + \sigma_4 + \sigma_5 - \sigma_6) \\
& \cdot (-\sigma_4 + 2\sigma_5 - \sigma_6) (\sigma_2 - \sigma_4 + \sigma_6) (-\sigma_1 + \sigma_3 - \sigma_4 + \sigma_6) (\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 + \sigma_6) \\
& \cdot (-\sigma_3 + \sigma_4 + \sigma_6) (-\sigma_1 + \sigma_4 - \sigma_5 + \sigma_6) (\sigma_1 - \sigma_2 + \sigma_4 - \sigma_5 + \sigma_6) (\sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 + \sigma_6) \\
& \cdot (-\sigma_4 + \sigma_5 + \sigma_6) (2\sigma_6 - \sigma_5) (\sigma_1 + \sigma_3 - \sigma_4 - \sigma_7) (-\sigma_1 + \sigma_2 + \sigma_3 - \sigma_4 - \sigma_7) \\
& \cdot (-\sigma_2 + 2\sigma_3 - \sigma_4 - \sigma_7) (\sigma_4 - \sigma_7) (\sigma_1 + \sigma_4 - \sigma_5 - \sigma_7) (-\sigma_1 + \sigma_2 + \sigma_4 - \sigma_5 - \sigma_7) \\
& \cdot (-\sigma_2 + \sigma_3 + \sigma_4 - \sigma_5 - \sigma_7) (\sigma_3 - \sigma_4 + \sigma_5 - \sigma_7) (\sigma_3 - \sigma_6 - \sigma_7) (\sigma_1 + \sigma_5 - \sigma_6 - \sigma_7) \\
& \cdot (-\sigma_1 + \sigma_2 + \sigma_5 - \sigma_6 - \sigma_7) (-\sigma_2 + \sigma_3 + \sigma_5 - \sigma_6 - \sigma_7) (\sigma_1 + \sigma_6 - \sigma_7) (-\sigma_1 + \sigma_2 + \sigma_6 - \sigma_7) \\
& \cdot (-\sigma_2 + \sigma_3 + \sigma_6 - \sigma_7) (\sigma_3 - \sigma_5 + \sigma_6 - \sigma_7) (\sigma_1 - \sigma_4 + \sigma_7) (-\sigma_1 + \sigma_2 - \sigma_4 + \sigma_7) \\
& \cdot (-\sigma_2 + \sigma_3 - \sigma_4 + \sigma_7) (-\sigma_3 + \sigma_4 + \sigma_7) (-\sigma_2 + \sigma_4 - \sigma_5 + \sigma_7) (\sigma_1 - \sigma_3 + \sigma_4 - \sigma_5 + \sigma_7) \\
& \cdot (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_4 - \sigma_5 + \sigma_7) (-\sigma_4 + \sigma_5 + \sigma_7) (\sigma_7 - \sigma_6) (-\sigma_2 + \sigma_5 - \sigma_6 + \sigma_7) \\
& \cdot (\sigma_1 - \sigma_3 + \sigma_5 - \sigma_6 + \sigma_7) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_5 - \sigma_6 + \sigma_7) (-\sigma_2 + \sigma_6 + \sigma_7) \\
& \cdot (\sigma_1 - \sigma_3 + \sigma_6 + \sigma_7) (-\sigma_1 + \sigma_2 - \sigma_3 + \sigma_6 + \sigma_7) (-\sigma_5 + \sigma_6 + \sigma_7) (2\sigma_7 - \sigma_3) \neq 0.
\end{aligned}$$

The other part of the excluded locus is determined by the fact that $\exp(-Y) \neq 0$. Since on the critical locus,

$$\exp(-Y_{i,\beta}) = -\tilde{m}_i + \sum_{a=1}^7 \sigma_a \rho_{i,\beta}^a,$$

so

$$-\tilde{m}_i + \sum_{a=1}^7 \sigma_a \rho_{i,\beta}^a \neq 0,$$

which is given more explicitly as

$$\begin{aligned}
& \prod_{i=1}^n (-\tilde{m}_i + \sigma_2 - \sigma_4) (-\tilde{m}_i + \sigma_3 - \sigma_4) (-\tilde{m}_i - \sigma_1 + \sigma_3 - \sigma_4) (-\tilde{m}_i + \sigma_1 - \sigma_2 + \sigma_3 - \sigma_4) \\
& \cdot (-\tilde{m}_i - \sigma_2 + \sigma_4) (-\tilde{m}_i - \sigma_3 + \sigma_4) (-\tilde{m}_i + \sigma_1 - \sigma_3 + \sigma_4) (-\tilde{m}_i - \sigma_1 + \sigma_2 - \sigma_3 + \sigma_4) \\
& \cdot (-\tilde{m}_i - \sigma_1 + \sigma_4 - \sigma_5) (-\tilde{m}_i + \sigma_1 - \sigma_2 + \sigma_4 - \sigma_5) (-\tilde{m}_i + \sigma_2 - \sigma_3 + \sigma_4 - \sigma_5) \\
& \cdot (-\tilde{m}_i + \sigma_4 - \sigma_5) (-\tilde{m}_i - \sigma_4 + \sigma_5) (-\tilde{m}_i + \sigma_1 - \sigma_4 + \sigma_5) (-\tilde{m}_i - \sigma_1 + \sigma_2 - \sigma_4 + \sigma_5) \\
& \cdot (-\tilde{m}_i - \sigma_2 + \sigma_3 - \sigma_4 + \sigma_5) (-\tilde{m}_i - \sigma_6) (-\tilde{m}_i + \sigma_1 - \sigma_6) (-\tilde{m}_i - \sigma_1 + \sigma_2 - \sigma_6) \\
& \cdot (-\tilde{m}_i - \sigma_2 + \sigma_3 - \sigma_6) (-\tilde{m}_i - \sigma_1 + \sigma_5 - \sigma_6) (-\tilde{m}_i + \sigma_1 - \sigma_2 + \sigma_5 - \sigma_6) \\
& \cdot (-\tilde{m}_i + \sigma_5 - \sigma_6) (-\tilde{m}_i + \sigma_2 - \sigma_3 + \sigma_5 - \sigma_6) (\sigma_6 - \tilde{m}_i) (-\tilde{m}_i - \sigma_1 + \sigma_6) \\
& \cdot (-\tilde{m}_i + \sigma_1 - \sigma_2 + \sigma_6) (-\tilde{m}_i + \sigma_2 - \sigma_3 + \sigma_6) (-\tilde{m}_i - \sigma_5 + \sigma_6) (-\tilde{m}_i + \sigma_1 - \sigma_5 + \sigma_6) \\
& \cdot (-\tilde{m}_i - \sigma_1 + \sigma_2 - \sigma_5 + \sigma_6) (-\tilde{m}_i - \sigma_2 + \sigma_3 - \sigma_5 + \sigma_6) (-\tilde{m}_i + \sigma_1 - \sigma_7) (-\tilde{m}_i + \sigma_2 - \sigma_7) \\
& \cdot (-\tilde{m}_i - \sigma_1 + \sigma_2 - \sigma_7) (-\tilde{m}_i - \sigma_1 + \sigma_3 - \sigma_7) (-\tilde{m}_i - \sigma_2 + \sigma_3 - \sigma_7) \\
& \cdot (-\tilde{m}_i + \sigma_1 - \sigma_2 + \sigma_3 - \sigma_7) (-\tilde{m}_i + \sigma_3 - \sigma_5 - \sigma_7) (-\tilde{m}_i + \sigma_5 - \sigma_7) (-\tilde{m}_i + \sigma_4 - \sigma_6 - \sigma_7) \\
& \cdot (-\tilde{m}_i + \sigma_3 - \sigma_4 + \sigma_5 - \sigma_6 - \sigma_7) (-\tilde{m}_i + \sigma_3 - \sigma_4 + \sigma_6 - \sigma_7) (-\tilde{m}_i + \sigma_4 - \sigma_5 + \sigma_6 - \sigma_7) \\
& \cdot (-\tilde{m}_i - \sigma_1 + \sigma_7) (-\tilde{m}_i - \sigma_2 + \sigma_7) (-\tilde{m}_i + \sigma_1 - \sigma_2 + \sigma_7) (-\tilde{m}_i + \sigma_1 - \sigma_3 + \sigma_7) \\
& \cdot (-\tilde{m}_i + \sigma_2 - \sigma_3 + \sigma_7) (-\tilde{m}_i - \sigma_1 + \sigma_2 - \sigma_3 + \sigma_7) (-\tilde{m}_i - \sigma_5 + \sigma_7) (-\tilde{m}_i - \sigma_3 + \sigma_5 + \sigma_7) \\
& \cdot (-\tilde{m}_i - \sigma_3 + \sigma_4 - \sigma_6 + \sigma_7) (-\tilde{m}_i - \sigma_4 + \sigma_5 - \sigma_6 + \sigma_7) \\
& \cdot (-\tilde{m}_i - \sigma_4 + \sigma_6 + \sigma_7) (-\tilde{m}_i - \sigma_3 + \sigma_4 - \sigma_5 + \sigma_6 + \sigma_7) \neq 0.
\end{aligned}$$

4.5.4 Pure gauge theory

In this part we will consider the mirror to the pure E_7 gauge theory. The mirror superpotential is

$$\begin{aligned}
W = & \sigma_1 \left(Z_1 - Z_2 + Z_{14} - Z_{15} + Z_{18} - Z_{19} + Z_{21} + Z_{22} - Z_{23} - Z_{24} + Z_{26} + Z_{27} - Z_{28} - Z_{29} \right. \\
& + Z_{30} + Z_{31} + Z_{32} - Z_{33} - Z_{34} - Z_{35} + Z_{37} + Z_{38} - Z_{39} - Z_{40} + Z_{43} + Z_{44} - Z_{45} \\
& - Z_{46} + Z_{50} - Z_{51} + Z_{56} - Z_{57} + 2Z_{63} - Z_{64} + Z_{65} - Z_{77} + Z_{78} - Z_{81} + Z_{82} - Z_{84} \\
& - Z_{85} + Z_{86} + Z_{87} - Z_{89} - Z_{90} + Z_{91} + Z_{92} - Z_{93} - Z_{94} - Z_{95} + Z_{96} + Z_{97} + Z_{98} \\
& - Z_{100} - Z_{101} + Z_{102} + Z_{103} - Z_{106} - Z_{107} + Z_{108} + Z_{109} - Z_{113} + Z_{114} - Z_{119} \\
& \left. + Z_{120} - 2Z_{126} \right)
\end{aligned}$$

$$\begin{aligned}
& + \sigma_2 \left(Z_2 - Z_3 + Z_{11} + Z_{13} - Z_{14} + Z_{16} + Z_{17} - Z_{18} + Z_{20} - Z_{21} - Z_{22} + Z_{25} - Z_{26} + Z_{29} \right. \\
& \quad - Z_{30} + Z_{34} + Z_{35} - Z_{36} + Z_{39} + Z_{40} - Z_{41} - Z_{42} + Z_{45} + Z_{46} - Z_{47} - Z_{48} + Z_{51} \\
& \quad - Z_{52} - Z_{53} + Z_{56} + 2Z_{57} - Z_{58} - Z_{63} - Z_{65} + Z_{66} - Z_{74} - Z_{76} + Z_{77} - Z_{79} - Z_{80} \\
& \quad + Z_{81} - Z_{83} + Z_{84} + Z_{85} - Z_{88} + Z_{89} - Z_{92} + Z_{93} - Z_{97} - Z_{98} + Z_{99} - Z_{102} - Z_{103} \\
& \quad + Z_{104} + Z_{105} - Z_{108} - Z_{109} + Z_{110} + Z_{111} - Z_{114} + Z_{115} + Z_{116} - Z_{119} - 2Z_{120} \\
& \quad \left. + Z_{121} + Z_{126} \right)
\end{aligned}$$

$$\begin{aligned}
& + \sigma_3 \left(Z_3 - Z_4 + Z_8 + Z_{10} - Z_{11} + Z_{12} - Z_{13} - Z_{16} + Z_{22} + Z_{24} + Z_{26} - Z_{27} + Z_{28} - Z_{29} \right. \\
& \quad + Z_{30} - Z_{31} + Z_{33} - Z_{34} - Z_{37} - Z_{39} + Z_{42} + Z_{48} - Z_{49} + Z_{50} + Z_{51} + Z_{52} + Z_{53} \\
& \quad - Z_{54} - Z_{56} - Z_{57} + 2Z_{58} - Z_{59} - Z_{60} - Z_{66} + Z_{67} - Z_{71} - Z_{73} + Z_{74} - Z_{75} + Z_{76} \\
& \quad + Z_{79} - Z_{85} - Z_{87} - Z_{89} + Z_{90} - Z_{91} + Z_{92} - Z_{93} + Z_{94} - Z_{96} + Z_{97} + Z_{100} + Z_{102} \\
& \quad - Z_{105} - Z_{111} + Z_{112} - Z_{113} - Z_{114} - Z_{115} - Z_{116} + Z_{117} + Z_{119} + Z_{120} - 2Z_{121} \\
& \quad \left. + Z_{122} + Z_{123} \right)
\end{aligned}$$

$$\begin{aligned}
& + \sigma_4 \left(Z_4 - Z_5 + Z_6 - Z_8 + Z_{13} + Z_{16} - Z_{17} + Z_{18} + Z_{19} - Z_{20} + Z_{21} - Z_{22} + Z_{23} - Z_{24} \right. \\
& \quad - Z_{26} - Z_{28} + Z_{37} + Z_{39} - Z_{43} + Z_{44} - Z_{45} + Z_{46} + Z_{47} + Z_{49} - Z_{50} - Z_{51} - Z_{52} \\
& \quad + Z_{53} + Z_{54} - Z_{55} - Z_{58} + 2Z_{60} - Z_{61} - Z_{67} + Z_{68} - Z_{69} + Z_{71} - Z_{76} - Z_{79} + Z_{80} \\
& \quad - Z_{81} - Z_{82} + Z_{83} - Z_{84} + Z_{85} - Z_{86} + Z_{87} + Z_{89} + Z_{91} - Z_{100} - Z_{102} + Z_{106} - Z_{107} \\
& \quad + Z_{108} - Z_{109} - Z_{110} - Z_{112} + Z_{113} + Z_{114} + Z_{115} - Z_{116} - Z_{117} + Z_{118} + Z_{121} \\
& \quad \left. - 2Z_{123} + Z_{124} \right)
\end{aligned}$$

$$\begin{aligned}
& + \sigma_5 \left(Z_5 - Z_7 + Z_8 - Z_{10} + Z_{11} - Z_{13} + Z_{14} + Z_{15} - Z_{18} - Z_{19} + Z_{20} - Z_{25} + Z_{26} + Z_{28} \right. \\
& \quad - Z_{30} + Z_{31} - Z_{33} + Z_{34} - Z_{37} + Z_{38} - Z_{39} + Z_{40} + Z_{41} - Z_{44} - Z_{46} - Z_{47} + Z_{48} \\
& \quad - Z_{53} + Z_{54} + Z_{55} - Z_{60} + 2Z_{61} - Z_{62} - Z_{68} + Z_{70} - Z_{71} + Z_{73} - Z_{74} + Z_{76} - Z_{77} \\
& \quad - Z_{78} + Z_{81} + Z_{82} - Z_{83} + Z_{88} - Z_{89} - Z_{91} + Z_{93} - Z_{94} + Z_{96} - Z_{97} + Z_{100} - Z_{101} \\
& \quad + Z_{102} - Z_{103} - Z_{104} + Z_{107} + Z_{109} + Z_{110} - Z_{111} + Z_{116} - Z_{117} - Z_{118} + Z_{123} \\
& \quad \left. - 2Z_{124} + Z_{125} \right)
\end{aligned}$$

$$\begin{aligned}
& + \sigma_6 \left(Z_7 - Z_9 + Z_{10} - Z_{12} + Z_{13} - Z_{16} + Z_{17} + Z_{18} + Z_{19} - Z_{20} - Z_{21} + Z_{22} - Z_{23} + Z_{24} \right. \\
& \quad - Z_{26} + Z_{27} - Z_{28} + Z_{29} - Z_{31} + Z_{32} - Z_{34} + Z_{35} + Z_{36} - Z_{38} - Z_{40} - Z_{41} + Z_{42} \\
& \quad - Z_{48} + Z_{49} - Z_{54} + Z_{55} - Z_{61} + 2Z_{62} - Z_{70} + Z_{72} - Z_{73} + Z_{75} - Z_{76} + Z_{79} - Z_{80} \\
& \quad - Z_{81} - Z_{82} + Z_{83} + Z_{84} - Z_{85} + Z_{86} - Z_{87} + Z_{89} - Z_{90} + Z_{91} - Z_{92} + Z_{94} - Z_{95} \\
& \quad + Z_{97} - Z_{98} - Z_{99} + Z_{101} + Z_{103} + Z_{104} - Z_{105} + Z_{111} - Z_{112} + Z_{117} - Z_{118} + Z_{124} \\
& \quad \left. - 2Z_{125} \right)
\end{aligned}$$

$$\begin{aligned}
& + \sigma_7 \left(Z_5 - Z_7 + Z_8 - Z_{10} + Z_{11} - Z_{13} + Z_{14} + Z_{15} - Z_{18} - Z_{19} + Z_{20} - Z_{25} + Z_{26} + Z_{28} \right. \\
& \quad - Z_{30} + Z_{31} - Z_{33} + Z_{34} - Z_{37} + Z_{38} - Z_{39} + Z_{40} + Z_{41} - Z_{44} - Z_{46} - Z_{47} + Z_{48} \\
& \quad - Z_{53} + Z_{54} + Z_{55} - Z_{60} + 2Z_{61} - Z_{62} - Z_{68} + Z_{70} - Z_{71} + Z_{73} - Z_{74} + Z_{76} - Z_{77} \\
& \quad - Z_{78} + Z_{81} + Z_{82} - Z_{83} + Z_{88} - Z_{89} - Z_{91} + Z_{93} - Z_{94} + Z_{96} - Z_{97} + Z_{100} - Z_{101} \\
& \quad + Z_{102} - Z_{103} - Z_{104} + Z_{107} + Z_{109} + Z_{110} - Z_{111} + Z_{116} - Z_{117} - Z_{118} + Z_{123} \\
& \quad \left. - 2Z_{124} + Z_{125} \right)
\end{aligned}$$

$$+ \sum_{m=1}^{126} X_m. \tag{4.108}$$

Now, we can proceed as in previous sections. For the reasons discussed there, since each σ is multiplied by both Z_μ and $Z_{-\mu}$ with opposite signs, the critical locus equations

$$\frac{\partial W}{\partial X_\mu} = 0$$

imply that on the critical locus,

$$X_\mu = -X_{-\mu}. \tag{4.109}$$

(Furthermore, on the critical locus, each X_μ is determined by σ .) In addition, each derivative

$$\frac{\partial W}{\partial \sigma_a}$$

is a product of ratios of the form

$$\frac{X_\mu}{X_{-\mu}} = -1.$$

It is straightforward to check in the superpotential above that each σ_a is multiplied by an even number of such ratios (*i.e.* the number of Z 's is a multiple of four). Specifically, the sum of the absolute values of the Z 's multiplying each σ is $68 = 4 \cdot 17$. Thus, the constraint implied by the σ 's is automatically satisfied.

As a result, following the same analysis in [14], we see in this case, that the critical locus is nonempty, and in fact is determined by the seven σ s. In other words, at the level of these topological field theory computations, we have evidence that the pure supersymmetric E_7 gauge theory in two dimensions flows in the IR to a theory of seven free twisted chiral superfields.

Chapter 5

Landau-Ginzburg phase as multiple cover of \mathbb{P}^n

In this chapter, we will explore generalizations of the work done in [16], realizing branched double covers of projective spaces in GLSMs. We describe here some (unpublished) attempts to understand higher-rank covers. Our efforts were unsuccessful, and we discuss the issues in this chapter.

In sections 5.1 and 5.2, we will review examples discussed in [16], in which GLSMs which describe complete intersections of quadrics at $r \gg 0$ describe branched double covers at $r \ll 0$. Section 5.1 includes a novel verification that Witten indices match between the GLSM phases, a nontrivial computation since there are discrete Coulomb vacua in one phase. In sections 5.3 and 5.4, we review some pertinent mathematics. In section 5.4, we will describe our (unpublished) attempts to generalize to k -fold covers, and how the GLSM results are incompatible with such an interpretation, because of a contradiction with the Hurwitz formula.

5.1 First example: Double cover of \mathbb{P}^1

Following [16], consider a $U(1)$ GLSM with the following matter content:

- Φ_i , $i \in \{1, \dots, 2g + 2\}$, corresponding to the matter fields with charge 1
- P_1 and P_2 corresponding to the matter fields with charge -2 ,

and the superpotential

$$W = P_1 G_1(\Phi) + P_2 G_2(\Phi) = \sum_{i,j=1}^{2g+2} \Phi_i A_{ij}(P) \Phi_j,$$

where $G_1(\Phi)$ and $G_2(\Phi)$ are two quadrics and $A_{ij}(P)$ is a symmetric $(2g+2) \times (2g+2)$ matrix with entries linear function of P_1 and P_2 .

For $r \gg 0$ we get the complete intersection of two quadrics defined by G_1 and G_2 in \mathbb{P}^{2g+1} , $\mathbb{P}^{2g+1}[2, 2]$.

For $r \ll 0$, the ϕ_i are mostly massive, away from the degree $2g+2$ locus $\det A(p) = 0$. Away from that locus, the only massless fields are the p_1 and p_2 . Because the charge of p_1 and p_2 is not minimal (their charge is -2), this phase is claimed to be a double cover of \mathbb{P}^1 over the branch locus defined by

$$B = \{[p_1, p_2] \in \mathbb{P}^1 \mid \det A(p) = 0\}.$$

This space is a compact genus- g surface. As a consistency check, let compare Witten indices. For $r \gg 0$ the Witten index is the Euler characteristic of $\mathbb{P}^{2g+1}[2, 2]$ which is equal to 4 (this factor comes from the fact that a general plane in \mathbb{P}^{2g+1} intersects the complete intersection in 4 points) times the coefficient of J^{2g-1} in

$$\frac{(1+J)^{2g+2}}{(1+2J)^2},$$

where J is the Kähler structure of \mathbb{P}^{2g+1} . This coefficient is zero, so the Witten index of this phase is zero.

For $r \ll 0$ the Witten index is the Euler characteristic of the genus g surface plus the number of discrete Coulomb vacua which exist in this phase. It is well known that the Euler characteristic of a compact genus g surface is $\chi(g\text{-torus}) = 2 - 2g$. The number of discrete Coulomb vacua is equal to number of solutions of the following equation

$$(+1\sigma)^{2g+2}(-2\sigma)^{-2}(-2\sigma)^{-2} = q$$

which is $2g - 2$. So the Witten index in this phase is $(2 - 2g) + (2g - 2) = 0$. So as you see the Witten index of both phases matches.

5.2 Second example: Double cover of \mathbb{P}^2

Following [16], consider a $U(1)$ GLSM with the following matter content:

- 6 chiral superfields Φ_i of charge 1,
- 3 chiral superfields P_a of charge -2 ,

and the superpotential

$$W = \sum_{a=1}^3 P_a G_a(\Phi) = \sum_{i,j=1}^6 \Phi_i A_{ij}(P) \Phi_j,$$

where the $G_a(\Phi)$ are quadric polynomials and $A_{ij}(P_a)$ is a symmetric 6×6 matrix with entries linear in the P_a 's.

For large FI parameter $r \ll 0$ the analysis is straightforward and the GLSM describes the complete intersection $\mathbb{P}^5[2, 2, 2]$, which is a K3 surface. The Witten index of this phase is the Euler characteristic of K3 surface which is $\chi(\text{K3}) = 24$.

Now let's analyze the other phase at $r \ll 0$. From the D term, the p_a 's are not all zero, hence the superpotential defines a mass matrix for the ϕ over the space of p_a 's, \mathbb{P}^2 . Because at generic points the p_a 's are non-minimally charged and the only massless fields, physics sees a branched double cover of \mathbb{P}^2 , branched over the degree six locus defined by

$$B = \{[p_1, p_2, p_3] \in \mathbb{P}^2 \mid \det A(p) = 0\}.$$

Such a branched double cover is another K3 surface, and so we see that both phases in this model correspond to K3 surfaces. (Note that in this phase, because summation of all the charges is zero, there are no discrete Coulomb vacua in this phase.)

5.3 Hurwitz formula

Let \tilde{X} and X be two complex projective varieties of the same dimension. Consider the projection map

$$\pi : \tilde{X} \rightarrow X,$$

of degree k . This means that $\pi^{-1}(x)$ generically has k points of \tilde{X} . Points for which the number of elements of $\pi^{-1}(x)$ is less than k define the branch locus, denoted by B . Then we say that \tilde{X} is the k -fold cover of X , branched over B . We name $Z = \pi^{-1}(B)$ the ramification locus of the map. Putting all the information together, the induced map

$$\bar{\pi} : \tilde{X} \setminus Z \rightarrow X \setminus B,$$

is a k -sheeted covering space. So

$$\chi(\tilde{X} - Z) = k \times \chi(X - B).$$

By simplifying this we get a version of the Hurwitz formula

$$\chi(\tilde{X}) = k\chi(X) + \chi(Z) - k\chi(B),$$

which relates to the Euler characteristic of the ramified covering \tilde{X} to that of the original space X .

As an application of Hurwitz formula we will derive the Euler characteristic of K3. more explicitly we specify our K3 by

$$\text{K3} = \{[x_0, x_1, x_2, x_3] \in \mathbb{P}^3 \mid x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}.$$

Now we define the map

$$\begin{aligned} \pi_1 : \quad \text{K3} &\rightarrow \mathbb{P}^2 \\ [x_0, x_1, x_2, x_3] &\mapsto [x_0, x_1, x_2]. \end{aligned}$$

In this map, for any $[x_0, x_1, x_2] \in \mathbb{P}^2$ such that $x_0^4 + x_1^4 + x_2^4 \neq 0$ there are four points in K3 which project to this point. So, K3 is a quadruple cover of \mathbb{P}^2 over the branch locus B_1 defined by

$$B_1 = \{[x_0, x_1, x_2] \in \mathbb{P}^2 \mid x_0^4 + x_1^4 + x_2^4 = 0\}.$$

The ramification locus of this map is

$$Z_1 = \pi^{-1}(B_1) = \{[x_0, x_1, x_2, 0] \in \text{K3} \mid x_0^4 + x_1^4 + x_2^4 = 0\}.$$

So $Z_1 \simeq B_1$. Now, applying the Hurwitz formula to this map, we get

$$\chi(\text{K3}) = 4\chi(\mathbb{P}^2) + \chi(Z_1) - 4\chi(B_1).$$

We already know that $\chi(\mathbb{P}^n) = n + 1$, so it only remains to compute $\chi(B_1) = \chi(Z_1)$. To do that we apply the Hurwitz formula one more time. We define the following map

$$\begin{aligned} \pi_2 : \quad B_1 &\rightarrow \mathbb{P}^1 \\ [x_0, x_1, x_2] &\mapsto [x_0, x_1]. \end{aligned}$$

Similarly by this map, B_1 is a quadruple cover of \mathbb{P}^1 over the branch locus B_2 defined by

$$B_2 = \{[x_0, x_1] \in \mathbb{P}^1 \mid x_0^4 + x_1^4 = 0\},$$

with the ramification locus

$$Z_2 = \pi^{-1}(B_2) = \{[x_0, x_1, 0] \in B_1 \mid x_0^4 + x_1^4 = 0\}.$$

So $Z_2 \simeq B_2$. Reusing Hurwitz formula we get

$$\chi(B_1) = 4\chi(\mathbb{P}^1) + \chi(Z_2) - 4\chi(B_2).$$

The space B_2 is defined by a degree four polynomial in \mathbb{P}^1 , which is obviously four points in \mathbb{P}^1 . So $\chi(B_2) = \chi(Z_2) = 4$. Inserting this into above formula we get

$$\chi(B_1) = 4 \times 2 + 4 - 4 \times 4 = -4.$$

Now because $\chi(B_1) = \chi(Z_1) = -4$ we get

$$\chi(\text{K3}) = 4 \times 3 + (-4) - 4 \times (-4) = 24. \tag{5.1}$$

5.4 Feasibility of Extending to k -cover of \mathbb{P}^2 in a GLSM

In this section we want to see if we can generalize the double cover idea to a k -fold of \mathbb{P}^2 .

Consider a $U(1)$ GLSM with

- 3 chiral superfields P_a of charge $-k$,
- M chiral superfields Φ^i of charge $+a$,
- M chiral superfields $\tilde{\Phi}^j$ of charge $+b$,

and the superpotential

$$W = \sum_{a=1}^3 P_a G^a(\Phi) = \sum_{i,j,a} A_{ij}^a P_a \Phi^i \tilde{\Phi}^j,$$

where G^a are quasi homogeneous polynomials of weight k . For the superpotential to be gauge invariant, we should have $a + b = k$. Also for simplicity of the analysis, we want the geometric phase to be Calabi-Yau so $-3k + Ma + Mb = 0$. From these two equations we get $M = 3$. So $i, j \in \{1, 2, 3\}$. For $r \gg 0$, we get the geometry $\mathbb{P}_{a,a,a,b,b,b}[k, k, k]$. So we get a compact two-dimensional Calabi-Yau manifold which is a K3. The Euler characteristic of K3, as it is shown in section 5.3, is $\chi(\text{K3}) = 24$.

For $r \ll 0$, by comparison to the double cover case, we expect to get a k -fold cover of \mathbb{P}^2 , branched over the locus

$$B = \{[p_1, p_2, p_3] \in \mathbb{P}^2 \mid \det A(p) = 0\} = \mathbb{P}^2[3] \simeq \text{Torus}.$$

We will denote the ramification locus of this covering by Z . Now, the Witten indices of the phases should match. If this is a correct description of the $r \ll 0$ phase, then

$$24 = \chi(\text{K3}) = k\chi(\mathbb{P}^2) + \chi(Z) - k\chi(B). \quad (5.2)$$

If we assume $Z \simeq B$ then $\chi(Z) = \chi(B) = \chi(\text{Torus}) = 0$. From the above we have

$$24 = k \times 3 \quad \Rightarrow \quad k = 8.$$

So in this model, the $r \ll 0$ phase can be octuple cover of \mathbb{P}^2 branched over a torus.

As a special case of the above model, let's assume charges of the Φ fields are equal to the charge of the $\tilde{\Phi}$ fields, i.e. $a = b = n$. So we have the following matter content

- 3 chiral superfields P_a of charge $-2n$,
- 6 chiral superfields Φ^i of charge n

and the superpotential

$$W = \sum_{i,j=1}^6 \sum_{a=1}^3 A_{ij}^a P_a \Phi^i \Phi^j.$$

Following a similar analysis we get the same equality in equation (5.2) but with the difference that the branch locus is given by

$$B = \{[p_1, p_2, p_3] \in \mathbb{P}^2 \mid \det A(p) = 0\} = \mathbb{P}^2[6].$$

In this case $\chi(B) = \chi(Z) = -18$. Inserting this back into equation (5.2) with $k = 2n$ we get $n = 1$, which is exactly the example we introduced in section 5.2.

To conclude, we seem able to consistently describe a branched eight-fold cover of \mathbb{P}^2 , but not other branched covers, with models of this form.

Chapter 6

Conclusions

This doctoral work can be broken up into three major studies on different aspects of string theory, a proposal for a unified theory of nature that has intrigued physicists for over 40 years. Each of these studies has led to interesting, original findings regarding the compactification problem, two-dimensional model building, and mathematical application of string theory. Findings that, hopefully have added at least a small amount to the body of knowledge on string theory.

Our first study focused on compactification problem in string theory. Following the work [9], we found the constraints on the complex structure and axion-dilaton moduli, in type IIB string theory in special cases. In particular, we solved these constraints in terms of Dolbeault cohomology. We give an example to show how we can use this method explicitly. We also verified a consistency check with Gukov-Vafa-Witten superpotentials.

In the second part was about $(0, 2)$ deformations of $(2, 2)$ GLSMs. We took $(2, 2)$ GLSMs from [10]. There are three different GLSMs in [10] such that in the infrared limit flow to a NLSM with intersection of Grassmannians as its target space. We studied $(0, 2)$ analogue of these theories in [11], and we will describe that work in this part of the dissertation. We also describe some examples of anomaly-free bundles on the intersections of Grassmannians.

In the third of our work, we used the proposal for an extension of the mirror constructions to non-Abelian theories [14]. In our work [15], which we described in chapter 4 of our dissertation, we applied the proposal to make predictions for two-dimensional supersymmetric gauge theories with exceptional gauge groups F_4 , E_6 and E_7 . We computed the mirror Landau-Ginzburg models and predicted excluded Coulomb loci and Coulomb branch relations, analogues of quantum cohomology.

Appendices

Appendix A

Non-Linear Sigma Models and Landau-Ginzburg Models

Suppose we have chiral superfields Φ^1, \dots, Φ^n and a general real function $K(\Phi^i, \bar{\Phi}^{\bar{i}})$. We assume that the matrix:

$$g_{i\bar{j}} = \frac{\partial^2}{\partial \phi^i \partial \bar{\phi}^{\bar{j}}} K(\phi^i, \bar{\phi}^{\bar{i}})$$

is positive definite, where ϕ is the lowest component of Φ . Then one can consider this matrix as determining the metric on $\mathbb{C} = \{z^1, \dots, z^n\}$

$$ds^2 = g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}}$$

which defines the Levi-Civita connection $\Gamma_{jk}^i = g^{i\bar{j}} \partial_j g_{k\bar{j}}$ on the tangent bundle $T\mathbb{C}^n$.

Now we define the Lagrangian

$$L_{\text{kin}} = \int d^2x d^4\theta K(\Phi^i, \bar{\Phi}^{\bar{i}}).$$

In terms of component fields ϕ, ψ_{\pm}, F of Φ , it can be expanded as

$$\begin{aligned} L_{\text{kin}} = \int d^2x \left(-g_{i\bar{j}} \partial^\mu \phi^i \partial_\mu \bar{\phi}^{\bar{j}} + i g_{i\bar{j}} \bar{\psi}_-^{\bar{j}} (D_0 + D_1) \psi_-^i + i g_{i\bar{j}} \bar{\psi}_+^{\bar{j}} (D_0 - D_1) \psi_+^i \right. \\ \left. + R_{i\bar{j}k\bar{l}} \bar{\psi}_+^i \bar{\psi}_-^{\bar{j}} \psi_-^k \bar{\psi}_+^{\bar{l}} + g_{i\bar{j}} (F^i - \Gamma_{jk}^i \psi_+^j \psi_-^k) (\bar{F}^{\bar{j}} - \Gamma_{\bar{k}\bar{l}}^{\bar{j}} \bar{\psi}_-^{\bar{k}} \bar{\psi}_+^{\bar{l}}) \right) \end{aligned}$$

up to total derivative in x^μ . In the above D_μ is defined by

$$D_\mu \psi_\pm^i := \partial_\mu \psi_\pm^i + \partial_\mu \phi^j \Gamma_{jk}^i \psi_{\pm}^k.$$

The kinetic term are non-linear under the assumption that $g_{i\bar{j}}$ is positive definite. The action is covariant under holomorphic changes of z^1, \dots, z^n , except for the last term, which

can be eliminated by equation of motion. Also, the action is invariant under the Kähler transformation

$$K(\Phi^i, \bar{\Phi}^{\bar{i}}) \rightarrow K(\Phi^i, \bar{\Phi}^{\bar{i}}) + f(\Phi^i) + \bar{f}(\bar{\Phi}^{\bar{i}}),$$

where $f(\Phi^i)$ is a holomorphic function. This can be understood by the fact that $\int d^4\theta f(\Phi^i)$ is a total derivative if $f(\Phi^i)$ is a holomorphic function. Thus we can apply this construction for each coordinate patch of a Kähler manifold M and then glue the patches together by using the invariance of the action under coordinate transformations and Kähler transformations. So we define an action for a map of the worldsheet to a Kähler manifold by:

$$\begin{aligned} \phi &\in \Gamma(\Sigma, M), \\ \psi_{\pm} &\in \Gamma(\Sigma, \phi^*TM^{(1,0)} \otimes S_{\pm}), \\ \bar{\psi}_{\pm} &\in \Gamma(\Sigma, \phi^*TM^{(0,1)} \otimes S_{\pm}), \end{aligned}$$

where S_{\pm} are spin bundles on the worldsheet Σ . This system is called a supersymmetric non-linear sigma model on a Kähler manifold M with metric g . Note that this formalism is not global and the supersymmetry should be checked patch by patch.

If M is not compact, we can define a non-trivial holomorphic function $W(\phi)$ on M , which is called the superpotential. This allows us to add another term to L_{kin} by

$$L_W = \frac{1}{2} \left(\int d^2\theta W(\Phi) + c.c. \right).$$

In terms of the component fields it can be expanded as

$$L_W = \frac{1}{2} \int d^2x \left(F^i \partial_i W - \partial_i \partial_j \psi_+^i \psi_-^j + \bar{F}^{\bar{i}} \partial_{\bar{i}} \bar{W} - \partial_{\bar{i}} \partial_{\bar{j}} \bar{W} \bar{\psi}_-^{\bar{i}} \bar{\psi}_+^{\bar{j}} \right).$$

By construction it has $\mathcal{N} = (2, 2)$ supersymmetric theory and is called a Landau-Ginzburg model.

Appendix B

Proof of $*\omega_{(2,1)} = i\omega_{(2,1)}$ for a primitive form

The Hodge star operator is given in complex coordinates as:

$$* : \Omega_{p,q} \longmapsto \Omega_{D-q,D-p} ,$$

$$*\omega_{p,q} = \sqrt{|g|}(i)^D \frac{(-1)^{\frac{1}{2}D(D-1)+Dp}}{p!q!(D-p)!(D-q)!} \omega_{i_1 \dots i_p \bar{i}_1 \dots \bar{i}_q} \epsilon^{i_1 \dots i_p}_{\bar{j}_1 \dots \bar{j}_p} \epsilon^{\bar{i}_1 \dots \bar{i}_q}_{j_{q+1} \dots j_D} dz^{j_{q+1}} \dots dz^{j_D} d\bar{z}^{\bar{j}_{p+1}} \bar{z}^{\bar{j}_D} .$$

Using the above, the Hodge star of the (2,1)-form $\omega_{(2,1)}$ is:

$$*\omega_{(2,1)} = \sqrt{|g|} \frac{i}{4} \omega_{ij\bar{k}} \epsilon^{ij}_{\bar{a}} \epsilon^{\bar{k}}_{bc} dz^b \wedge dz^c \wedge d\bar{z}^{\bar{a}} ,$$

or equivalently

$$*\omega_{(2,1)} = \sqrt{|g|} \frac{i}{4} \omega_{ij\bar{k}} g^{i\bar{i}} g^{j\bar{j}} g^{a\bar{k}} \epsilon_{i\bar{j}\bar{a}} \epsilon_{abc} dz^b \wedge dz^c \wedge d\bar{z}^{\bar{a}} .$$

In components, the above is:

$$(*\omega)_{\bar{a}bc} = i \frac{\sqrt{|g|}}{2} \omega_{ij\bar{k}} g^{i\bar{i}} g^{j\bar{j}} g^{a\bar{k}} \epsilon_{i\bar{j}\bar{a}} \epsilon_{abc} ,$$

which can be simplified in the following way:

$$\begin{aligned} \implies g^{\bar{a}x} (*\omega)_{\bar{a}bc} &= i \frac{\sqrt{|g|}}{2} \omega_{ij\bar{k}} g^{i\bar{i}} g^{j\bar{j}} g^{\bar{a}x} \epsilon_{i\bar{j}\bar{a}} \epsilon_{abc} g^{a\bar{k}} , \\ \implies g^{\bar{a}x} (*\omega)_{\bar{a}bc} &= \frac{i}{2} \omega_{ij\bar{k}} \epsilon^{ijx} \epsilon_{abc} g^{a\bar{k}} , \end{aligned}$$

$$\begin{aligned}
\implies g^{\bar{a}x}(*\omega)_{\bar{a}bc} &= \frac{i}{2}\omega_{ij\bar{k}} \begin{vmatrix} \delta_a^i & \delta_b^i & \delta_c^i \\ \delta_a^j & \delta_b^j & \delta_c^j \\ \delta_a^x & \delta_b^x & \delta_c^x \end{vmatrix} g^{a\bar{k}}, \\
&= \frac{i}{2}\omega_{ij\bar{k}} [\delta_a^i (\delta_b^j \delta_c^x - \delta_c^j \delta_b^x) - \delta_b^i (\delta_a^j \delta_c^x - \delta_c^j \delta_a^x) + \delta_c^i (\delta_a^j \delta_b^x - \delta_b^j \delta_a^x)], \\
&= \frac{i}{2}g^{a\bar{k}} (2\omega_{ab\bar{k}}\delta_c^x + 2\omega_{bc\bar{k}}\delta_a^x + 2\omega_{ca\bar{k}}\delta_b^x).
\end{aligned}$$

So, we get:

$$(*\omega)_{\bar{a}bc} = ig^{a\bar{k}}(\omega_{ab\bar{k}}g_{\bar{a}c} + \omega_{ca\bar{k}}g_{\bar{a}b} + \omega_{bc\bar{k}}g_{\bar{a}a}). \quad (\text{B.1})$$

Because ω is primitive, $J \wedge \omega = 0$, we have:

$$g^{a\bar{k}}(\omega_{ab\bar{k}}g_{\bar{a}c} + \omega_{ca\bar{k}}g_{\bar{a}b}) = 0,$$

which simplifies (B.1) to

$$(*\omega)_{\bar{a}bc} = ig^{a\bar{k}}(\omega_{bc\bar{k}}g_{\bar{a}a}) = i\omega_{\bar{a}bc}. \quad (\text{B.2})$$

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