CHAPTER 6

HEURISTIC REDUCTION OF THE NETWORK BASED ON ESTIMATED COMPLETION TIMES

Suppose that we are finding a shortest path from node s to node t in a network or digraph $G(N, A)$, where N is the set of nodes, A is the set of (directed) arcs, and where for each $(i, j) \in A$, c_{ij} represents the travel time. For the sake of simplicity in exposition, we present our discussion in this section for such a **static** shortest path problem; its extension to the time-dynamic label-constrained case follows an identical process.

Suppose that in the application of the prescribed shortest path algorithm, we are examining the forward-star $FS(k)$ of some node $k \in N$ that has been selected to be considered in the reduced network. (To initialize, this node would be node s.) Let $l(k)$ denote the *level* of node k in the current shortest path tree. Then, the revised scanning and update rule that we employ is as follows.

Denote

w_i = current shortest path labels, $\forall\, i \in N$;

$d(i, t)$ = estimate for the travel time from i to t;

b_i = parameter (≥ 1) whose choice is specified in the sequence, and

T = upper bound on an acceptable total travel time.

Then, for each $i \in FS(k)$, if $w_i' \equiv (w_k + c_{ki}) < w_i$ (6.1a)

and if $w_i' + b_i\, d(i, t) < T$, (6.1b)

then update $w_i \leftarrow w_i'$, and set DOWN $(i) = k$ and $l(i) = l(k) + 1$, and include node i in the list (NEXT) for further updating subsequent nodes.

The function $d(i, t)$ is based on the **Euclidean distance** between the locations of nodes i and t. Denoting $x^i = (x_1^i, x_2^i)$ to be the coordinates of node i in some two-dimensional Cartesian space, we have

$$d(i, t) = \frac{\left\| x^i - x^t \right\|}{v} \tag{6.2}$$

where v is some average estimated velocity of travel. Note that nodes whose labels are not revised according to equation (6.1) are not considered in further computations. Hence, this is akin to dynamically curtailing the network being considered during the process of the algorithm. Also, the role of the parameter b_i in equation (6.1b) is to reflect the fact that the travel time measure equation (6.2), which is based on the Euclidean distance, is likely to be more accurate when i is in the relative vicinity of t, and is likely to be a relatively weaker lower bound on the actual travel time value when i is further away from t. Accordingly, it might be beneficial to vary b_i from a value greater than 1 toward a value of unity as the path progresses from s to t. Below, we prescribe four such possible strategies that we will evaluate and compare experimentally.

Case (i) (Standard Base-Case) $b_i = 1 \quad \forall i \in N$.

Case (ii) Network Sectioning Technique.

Consider a linear transformation of the coordinate system from the x-space to a y-space according to the relationship

$$x = x^s + By \tag{6.3a}$$

where $B = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$, and where $\begin{pmatrix} p \\ q \end{pmatrix} = \frac{(x^t - x^s)}{\left\| x^t - x^s \right\|}$. (6.3b)

The transformation equation (3) shifts the origin to x^s and rotates the axes so that the y_1 axis is oriented along the vector $(x^t - x^s)$ and the y_2 axis is orthogonal to this axis, being rotated 90^{o} in the anti-clockwise direction with respect to it (see Figure 23).

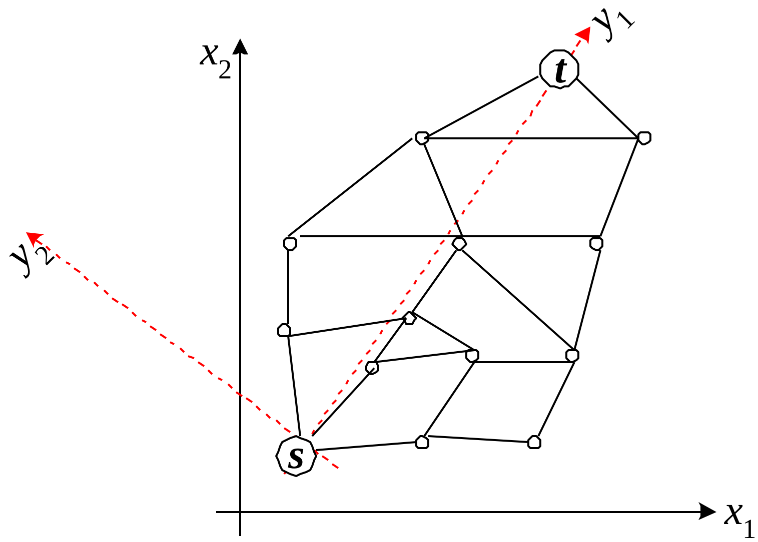


Figure 23: Transformation of Space.

Note that since B is an orthonormal matrix, we have $B^{-1} \equiv B^T$, and so, the inverse transformation to equation (3) is given by

$$y = B^T(x - x^s). \tag{6.4}$$

Accordingly, suppose that we have *a priori* computed the coordinates y^i = (y_1^i, y_2^i) $\forall\, i \in N$ by using equation (6.4). Let us now define (with respect to the terminus node t)

$$t_0 = 0,\ t_1 = \frac{y_1^t}{3},\ t_2 = \frac{2}{3} y_1^t,\ t_3 = y_1^t \tag{6.5a}$$

and let us section the nodes in N into the sets

$$S_r = \left\{ i \in N : t_{r-1} \le y_1^i < t_r \right\} \text{ for } r = 1, 2, 3 \tag{6.5b}$$

(where for r = 3, we include i in S_r if $y_1^i \equiv t_3$). Correspondingly, we define

$$b_i \equiv q_r\ \forall\, i \in S_r, \text{ where } q_r = 1 + (3 - r)q \quad \forall\, r = 1, 2, 3, \tag{6.6}$$

and where q > 0 is some (to-be-empirically determined) parameter. (For example, we can try q = 0.25.) This provides a section-wise linear progressively decreasing sequence for the b -parameters.

Case (iii) Level-Based Technique

Let $|N| = n$ be the number of nodes in the problem. In this case, we vary b_i according to its level or depth away from the starting node s by using one of the following two relationships:

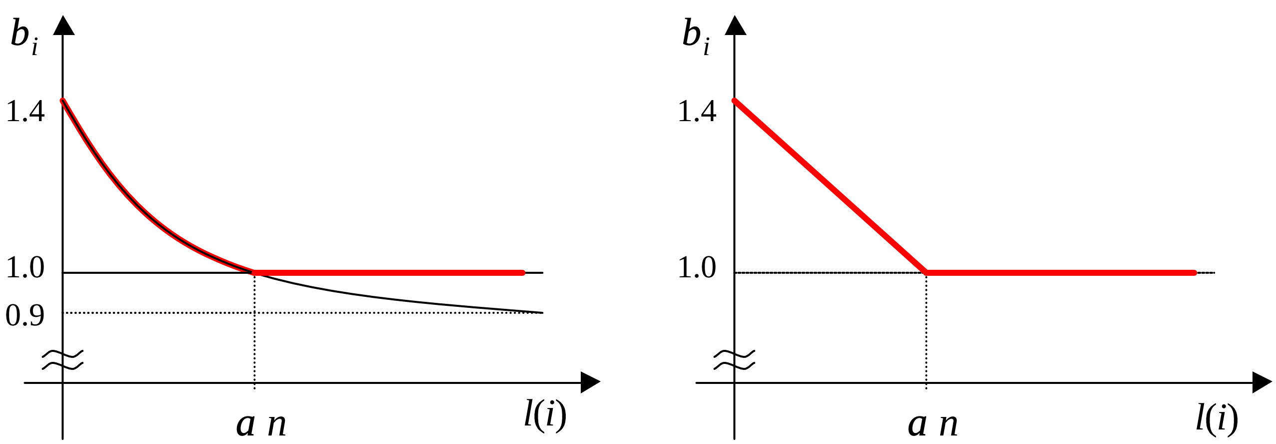
$$b_i = \max\left\{1,\ 0.9 + 0.5e^{-l\, l(i)}\right\}\ \forall\, i \in N, \text{ where } l = \frac{-\ln(0.2)}{a\, n} \tag{6.7a}$$

or

$$b_i = \max\left\{1,\ 1.4 - \left(\frac{0.4}{a\, n}\right) l(i)\right\}\ \forall\, i \in N, \tag{6.7b}$$

where $b_t \equiv 1$, and where $0 < a \le 1$.

Figure 24 depicts the schema based on which equations (6.7a) and (6.7b) are designed. The value 1.4 is selected as an approximation to $\sqrt{2}$, which represents an upper bound on the ratio of the rectilinear to the Euclidean distance measure.



(a) Depiction of Equation (6.7a) (b) Depiction of Equation (6.7b)

Figure 24: Prescription of b Based on the Level Measure.

The equation (6.7b) is computationally less expensive to implement, being linear. The breakpoint $a\, n$ used in either case is a value that could be experimented with, based on the philosophy that we would like $b_i = 1$ when i is in the relative vicinity of node t. (For example, we might try $a = 0.5$.)

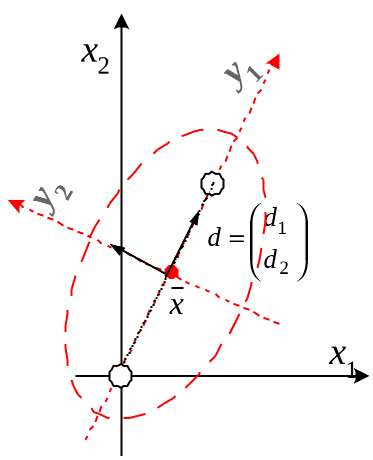
Case (iv) Ellipsoidal Region Technique

In this case, we curtail the network under construction based on three ellipsoidal regions defined using:

- the line connecting nodes s and t as the major axis for the first ellipsoidal region (E_1),
- the line connecting node s and the known freeway entrance in the corresponding zone as the major axis for the second ellipsoidal region (E_2),
- the line connecting node t and the known freeway exit in the corresponding zone as the major axis for the third ellipsoidal region (E_3),

with the centers being at the midpoints ($\bar{x}$) of these line segments, and with their major and minor axes lengths being parameters as specified below (see Figure 25).

Consider the linear transformation



$$x = \bar{x} + By \tag{6.8a}$$

where $B = \begin{bmatrix} d_1 & -d_2 \\ d_2 & d_1 \end{bmatrix}$,

so that the inverse transformation that defines the y-space coordinates $y^i \ \forall i \in N$ is given by

$$y = B^T \left[x - \bar{x}\right]. \tag{6.8b}$$

Hence, consider the ellipsoidal region in y-space given by

$$E = \left\{ y : \frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} \leq 1 \right\} \tag{6.9a}$$

where $$a = g\ y_1^t \text{, and } b = y\ a, \tag{6.9b}$$

and where $g > 1$ and $0 < y < 1$ (6.9c)

are prescribed parameters. (For example, we can take $g = 1.25$, and $y = 0.75$.)

The ellipse (6.9a) is of the form $y^{\mathrm{T}} F\, y \leq 1$ where $F = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix}$

In the x-space, this translates to

$$(x - \bar{x})^{\mathrm{T}} B\, F\, B^{\mathrm{T}} (x - \bar{x}) \leq 1.$$

Then, defining the ellipsoids E_2 and E_3 in x-space in this manner, we let

$$b_i = \begin{cases} 0 & \text{if } y^i \in E_1 \cup E_2 \cup E_3 \cup \text{Freeway} \\ \infty & \text{otherwise} \end{cases}. \qquad (6.10)$$

Note that in general, one could actually also include nodes that lie on some known paths between s and t and between these nodes and suitable freeway entrances/exits. In essence, the rule in Equation (6.10) simply performs the usual shortest path computations on a reduced network that contains only those nodes $i \in N$ that lie within the designated ellipsoidal regions E_1, E_2, E_3, or the freeway. The motivation here is that users will typically explore local vicinity routes *as well as* high speed corridors that are reasonably accessible between any origin-destination pair.

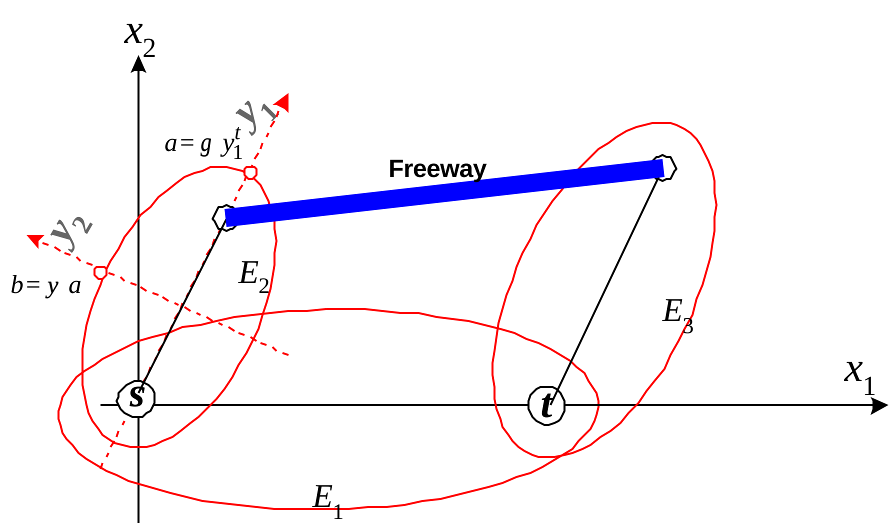


Figure 25: Ellipsoidal Regions E_1, E_2, and *E_3 to Curtail the Network.*