The Space of Left Orders on a Group

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(ABSTRACT)

The study of orderable groups is a topic that is all too often overlooked as a topic in algebra. The subject of orderable groups is a field of study which is directly associated with algebraic group theory, algebraic topology, and set theory. This paper will act as a guide into the world of orderable groups. It begins by enlightening the reader to the fundamental axioms of orderable groups, as well as, noting various important groups on which orders are often considered. We will then consider more interesting groups, on which the placement of orders is considered less often.

After considering the orderings placed on various groups, we wish to consider in further detail the topologies of the sets of these orders. In particular, it is important to consider whether the set of orders placed on a particular group is finite or uncountable. We prove the latter by creating a homeomorphism from the group to the Cantor set, a set which is known for its uncountability.

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Chapter 1

Introduction

The study of algebra heavily focuses on the characterization and study of the behavior of groups. One such way to characterize groups is to place an ordering on the group. However, there are degrees of ordering a group. That is, we can either totally order a group or partially order it. To begin we explore what it means to place an order on a group.

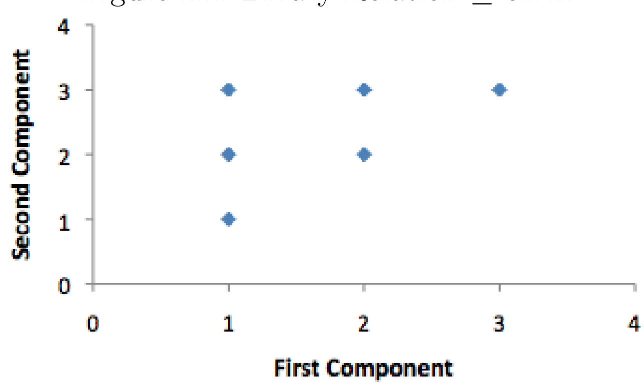
Recall that a binary relation R on a set G is defined as a collection of ordered pairs of elements of G. This type of relation is commonly used in mathematics to compare terms. Such comparisons include equality and congruency of terms. Graph theory uses binary relations to determine adjacency of vertices.

Example 1.0.1. *Let $H = \{1, 2, 3\}$. Suppose we wish to place the relation $x \leq y$ on this set. The result of this relation is*

$$\{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\},$$

where we can also represent this relation graphically, as shown below.

Figure 1.1: Binary Relation $\leq$ on H



We say that such a binary operation is antisymmetric if for all $x, y \in G$ such that $R(x, y) = R(y, x)$, then $x = y$. We also note that a binary relation is said to be total if for all elements $x, y \in G$, either x is related to y and/or y is related to x.

Definition 1.0.2. *A total order on a set G is a relation, $\leq$, that is transitive, antisymmetric, and total. Thus, we note that for all $x, y, z \in G$:*
i. $x \leq y$ *or* $y \leq x$
ii. *if* $x \leq y$, $y \leq z$, *then* $x \leq z$
iii. *if* $x \leq y$ *and* $y \leq x$, *then* $x = y$.

A slightly less restricted order is called a partially ordered set or poset.

Definition 1.0.3. *A partial order on a set G is a relation, $\leq$, that is reflexive, antisymmetric, and transitive. Let $x, y, z \in G$,*
i. $x \leq x$
ii. *if* $x \leq y$ *and* $y \leq x$, *then* $x = y$
iii. *if* $x \leq y$ *and* $y \leq z$, *then* $x \leq z$.

Thus, we note that a total order is simply a partial order under which all elements must be related. That is, for any two elements of a set, either $x \leq y$ or $y \leq x$. Note that in a partial order, there need not exist a relationship between two arbitrary elements of a set.

An example of a partial order that is not a total order is the study of the human population under genealogy. We can consider those persons biologically related to one another as orderable. That is, we may say that a son "$<$" father, however, there is no relationship between two people who are not biologically related.

1.1 Left, Right, and Biorderable Groups

Previously, we have shown what qualifies as an order on a set. That is, we know what it means to be given two arbitrary elements and order the pair. We must now consider what it means for an entire group to be orderable.

Definition 1.1.1. *A group G is orderable if there exists a total order relation, $\leq$, on G such that for $g_1, g_2 \in G$, $g_1 \leq g_2$ implies $xg_1y \leq xg_2y$ for all $x, y \in G$. This is also called a biorderable group.*

Examples of biorderable groups include $\mathbb{Z}$ and $\mathbb{Q}$ under the typical ordering. One group, considered in greater detail later in the paper, is the Heisenberg group, which is a biorderable matrix group.

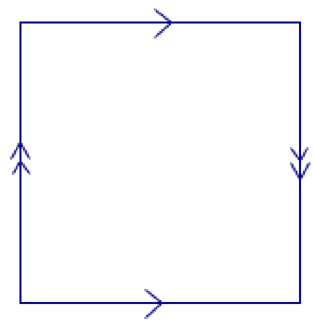
We note that a group need not be biorderable to have a relation on it. We will also consider left orderable and right orderable groups. In fact the focus of this paper will be on the set of left orders of a group.

Definition 1.1.2. *We say that a group is left orderable if there exists a total order relation,* $\leq$*, on* G *such that for* $g_1, g_2 \in G$*,* $g_1 \leq g_2$ *implies* $xg_1 \leq xg_2$ *for all* $x \in G$*.*

We define a right orderable group similarly. Let us consider an example of a group that is left orderable but not biorderable. One such group is the Klein Bottle group, which we will now discuss in more detail.

Example 1.1.3. *To define the Klein bottle group we will describe its construction. We begin with a square* $[0, 1] \times [0, 1]$*.*

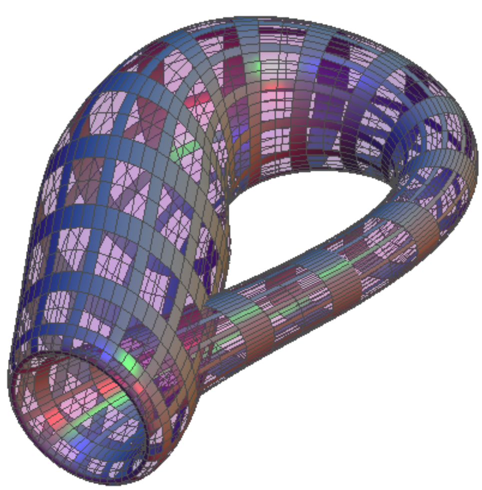
Figure 1.2: Fundamental Square



We glue the sides of the square together using glueing maps, in such a way as to preserve the direction of the side. Define $\phi_1 : [0, 1] \times \{0\} \to [0, 1] \times \{1\}$ *by* $\phi_1(x, 0) = (x, 1)$*. This mapping glues the bottom edge to the top. We glue the left edge* $\{1\} \times [0, 1]$ *to the right edge* $\{0\} \times [0, 1]$ *using the glueing map* $\phi_2(x) : \{1\} \times [0, 1] \to \{0\} \times [0, 1]$ *defined by* $\phi_2(1, y) = (0, 1 - y)$*. [1]*

When we are done attaching the sides we have the following image. [2]

Figure 1.3: The Klein Bottle



Then note that we can also define the Klein bottle group as $\langle x, y : x^{-1}yx = y^{-1}\rangle$. *These relations come from the directional edges that generate the fundamental square and direct the connection of the surface.*

We will use the aforementioned relations to prove that the Klein bottle group is not biorderable. Note that for a group G *to be biorderable we must have that for all* $x, y \in G$, $1 < y$ *if and only if* $1 < x^{-1}yx$ *(where this is the same as saying that the orders are bivariant under multiplication from the group* G*). Suppose* y *is an element of the Klein group such that* $1 < y$. *Then* $y^{-1} < 1$. *Now we consider* $xyx^{-1} = y^{-1} < 1$. *Thus, we conclude that the Klein group is not biorderable.*

Commonly the set of left orders and right orders are denoted $\mathrm{LO}(G)$ and $\mathrm{RO}(G)$, respectively. We note that there is a one-to-one correspondence between the set of right and left orders on G when G is a group. Let $<_r$ be a right order on G. That is, for all $x, y \in G$ $1 <_r x^{-1}y \iff 1 <_l yx^{-1}$, for some left order $<_l$ on G. [13]

We now discuss what makes a group a right/left orderable group biorderable. That is, we wish to discover what characteristics of a group affect its orderability. When a group has a fixed order on it, we say that it is an ordered group.

Proposition 1.1.4. *If a left orderable group* G *is abelian then every left ordering is a bi-ordering.*

Proof. Assume G be a left orderable group. Fix $< \in LO(G)$ and fix $x, g \in G$. Then we know that $1 < x$ if and only if $g < gx$. This is because left orders are invariant under left multiplication from G. Thus, for $h \in G$, $hg < hgx = hxg$. But g and h we arbitrary elements. Thus, we conclude from Definition 1.1.1 that $<$ is a bi-ordering. □

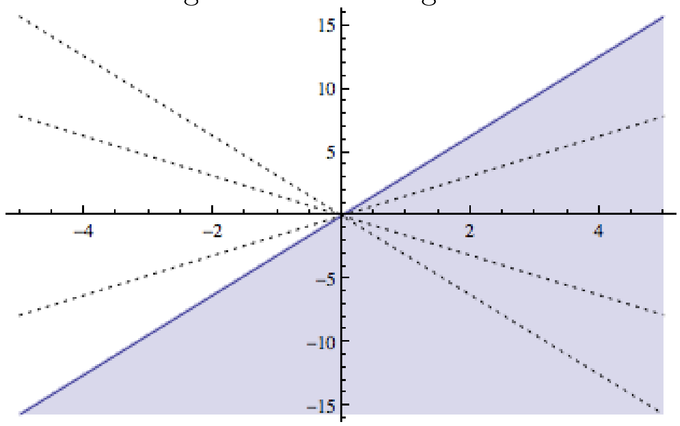
Later in our paper the size of $\mathrm{LO}(G)$ will become a topic of great scrutiny. For the present we will consider $\mathrm{LO}(\mathbb{Z}^2)$, the space of all left orders on $\mathbb{Z}^2$. Previously, we have considered individual ways to order a group; however, now we wish to consider a group and consider multiple ways to order its elements.

Example 1.1.5. *We wish to determine possible orderings on* $\mathbb{Z}^2$. *Consider lines through the origin of irrational slope,* α. *That is, consider* $y = \alpha x$. *Then we say that* $(x, y) >_\alpha 0$ *if and only if* $y - \alpha x > 0$. *Specifically, we consider* $y = \pi x$, *the solid line in the graph below.*

We consider points not on the line. For example, consider the point $(2, \pi)$. *Then* $\pi - \pi \cdot 2 = -\pi <_\pi 0$, *so we conclude that* $(2, \pi) < 0$. *But then we note that all points* (x, y) *to the right of our line, in the shaded portion of the graph, are such that* $(x, y) <_\pi 0$.

We see that every line with irrational slope determines a distinct ordering of $\mathbb{Z}^2$. *In particular, the dotted lines in the graph above have rational slope* α_i. *Thus, we can note that each of these lines through the origin determines an ordering* $<_{\alpha_i}$ *on* $\mathbb{Z}^2$, *where we can order the points* (x, y) *to the left of the line as* $(x, y) <_{\alpha_i} 0$. *But since we know there are*

Figure 1.4: Orderings of $\mathbb{Z}^2$



uncountably many irrational numbers, we intuitively expect that there are uncountably many ways of ordering the elements of $\mathbb{Z}^2$.

Determining the number of orderings on $\mathbb{Z}^n$ will warrant greater discussion in chapter three.

Definition 1.1.6. *Let G be a group that is left orderable. Then we call the positive cone of G,*

$$\{x \in G : x > 1\}.$$

We note that the positive cone of G has the following properties:
i. $g, h \in P \implies gh \in P$.
ii. $g \in P \implies g^{-1} \notin P$.
iii. $1 \neq g \in G$, *then either g or g^{-1} is in P. [12]*

Note that we can define a left order on G for any subset P satisfying conditions (i), (ii), and (iii) stated above. [12] That is, every subset P described above defines an ordering $\leq$ on G.

Proposition 1.1.7. *Left orderable groups are torsion-free.*

Proof. Let G be a group. We want to show that if G is left orderable, then G is torsion-free. Fix $g \in G$. Since G is left orderable we may assume without loss of generality that $1 < g$ for $<\in LO(G)$. We know that the left orders on G are invariant under left multiplication in G. So $g < g^2$. This implies that $g^2 > 1$ by transitivity, since $<$ is a total order. But then this implies that $g^n > 1$ for all $n > 0$. We conclude that $g^n \neq 1$ for any $n \in \mathbb{N}$. Therefore, we may conclude that all left orderable groups are torsion-free. Note that the case for $g < 1$ follows similarly. □

The proposition above proves to be extremely useful in our characterization of left orderable groups. It follows directly from the proposition that left orderable groups have no elements of finite order. That is, finite groups are not left orderable.

Definition 1.1.8. *We say that a group is locally indicable if and only if every finitely generated subgroup, not equal to 1, has an infinite cyclic quotient.[11]*

The above definition is equivalent to noting that every finite subgroup of a locally indicable group G can be mapped homomorphically onto $\mathbb{Z}$.

Remark 1.1.9. *The positive cone of G has the following properties:*
i. $PP \subseteq P$
ii. $P \cap P^{-1} = \{1\}$, *where we define* $P^{-1} = \{x^{-1}; x \in P\}$
iii. $P \cup P^{-1} = G$.

Note that if G is biorderable $P^g = g^{-1}Pg \subseteq P$. A partial order Q on G is said to be an extension of a partial order P on G if $P \subseteq Q$. Every partial order on G can be extended to a maximal partial order using Zorn's Lemma.[13]

We now wish to explore other ways of extending orderings on a group G. To do this we must review ways in which we may extend a group. Recall that if we are given two modules A, C, there exists a module B containing an isomorphic copy of A such that C is isomorphic to the quotient group B/A. [3] Before we can discuss these types of extensions in more detail, we must study what it means for a mapping to be order isomorphic.

1.2 Mappings of Orderings

Another important aspect of considering orderable groups, is the preservation of an ordering through mappings. That is, we wish to consider what happens when we map orderable groups.

Definition 1.2.1. *Let G_1 and G_2 be ordered groups, and let $\theta : G_1 \to G_2$ be an isomorphism. Then we say that θ is an order isomorphism if $x \le y$ implies that $\theta(x) \le \theta(y)$ for all $x, y \in G$.[13]*

The only order isomorphism from $\mathbb{Z}$ to itself is the identity map. Suppose we wish to consider $\mathbb{R}$ as the domain for our function. Note that $f(x) = 2x$ is an order isomorphism for $f : \mathbb{R} \to \mathbb{R}$. This observation is based on the fact that

$$2x = f(x) \le f(y) = 2y \iff x \le y.$$

A specific example of an order isomorphism is an order automorphism. An order automorphism is a mapping from an ordered group to itself preserving the ordering on a group. We wish to use our knowledge of order isomorphisms to discuss extensions of an ordered group.

Definition 1.2.2. *For multiplicative groups A, B, C, the sequence $1 \to A \xrightarrow{\phi} B \xrightarrow{\phi} C \to 1$ is called a short exact sequence if and only if ψ is injective, ϕ is surjective, and im$(\psi) = \ker(\phi)$.*

Such a group B as in the above definition is said to be an extension of C by A.[3] We wish to use short exact sequences to extend left orderable groups.

Proposition 1.2.3. *Let*

$$1 \to K \xrightarrow{\psi} G \xleftarrow{\phi} H \to 1$$

be a short exact sequence. If K and H are left orderable, then G is left orderable.

Let P_K denote the positive cone of K and let P_H denote the positive cone of H. Note that proving the above proposition is equivalent to showing that the subset of G,

$$P_G = \psi(P_K) \cup \phi^{-1}(P_H),$$

is the positive cone of G. Recall that showing such a subset of G exists is equivalent to proving that G is left orderable. We will not go into the details of this proof as it is relatively straightforward, but will instead advise the reader that the proof may be completed using the properties of the positive cone of K and H.

Thus, we have shown one way to extend a left ordering on a group. It should be noted that if in the above proposition K and H are biorderable, then G is biorderable if and only if $g^{-1}\ \psi(P_K)g \subseteq \psi(P_K)$ for all $g \in G$.

1.3 Specific Orderings

Now that we have defined what an ordering on a group is, let us consider some specific ways of ordering groups. That is, although we have defined what it means for $<$ to be a left order, we have yet to consider any interesting ways of ordering the elements of a group.

1.3.1 Lexicographic Ordering

One of the most commonly used ways of ordering a group is the lexicographic ordering. This ordering is placed on the Cartesian product of two ordered sets and is defined below.

Definition 1.3.1. *Let G and H be two partially ordered sets. Then the lexicographic order on $G \times H$ is given by the relation $(g_1, h_1) \leq (g_2, h_2)$ if and only if $g_1 < g_2$ or $g_1 = g_2$ and $h_1 \leq h_2$, for $g_1, g_2 \in G, h_1, h_2 \in H$.*

Note that we can extend the lexicographic ordering to larger sets. Suppose we have an n-tuple of sets, $\{G_1, G_2, .., G_n\}$, with respective orderings $<_i$ on each G_i. Let $G = G_1 \times G_2 \times ... \times G_n$. Then we can define the lexicographic ordering, $<$, on the Cartesian product as follows: given $(g_1, g_2, ..., g_n), (h_1, h_2, ..., h_n) \in G$,

$$(g_1, g_2, .., g_n) < (h_1, h_2, ..., h_n)$$

if and only if $g_k <_k h_k$ for some $k \leq n$ and $g_j = h_j$ for $j \in [1, k-1]$.

In our studies we will most commonly apply the lexicographic ordering to matrix groups. Thus, we wish to consider explicitly what one such ordering looks like.

Example 1.3.2. *Let* $G = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \text{where } a, b, c \in \mathbb{Z} \right\}$.

Note that this group is commonly called the Heisenberg group and we will later discuss this group in more detail. We wish to order this group of matrices under the lexicographic ordering. Then note that this means that

$$\begin{bmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{bmatrix} < \begin{bmatrix} 1 & a_2 & c_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix}$$

if (1) $a_1 < a_2$, *(2)* $a_1 = a_2$ *and* $b_1 < b_2$, *or (3)* $a_1 = a_2, b_1 = b_2$ *and* $c_1 < c_2$. *Otherwise the matrices are equal.*

Consider matrices $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$, *and* $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \in G$.

Thus, we note that under the order specified we may conclude that

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} < \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} < \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Although it relatively straightforward to lexicographically order vectors and matrices of dimension 3, it proves more difficult to lexicographically order matrices of larger dimension. For this reason we created a code for ordering square upper triangular matrices of dimension n (see Appendix 1). We will put our code to work for two 5×5 matrices, which we wish to order lexicographically.

Example 1.3.3. *Let*

$$A = \begin{bmatrix} 1 & 9 & -4 & 5 & 3 \\ 0 & 1 & 8 & 2 & 1 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 9 & -4 & 2 & 6 \\ 0 & 1 & 9 & 2 & -5 \\ 0 & 0 & 1 & -17 & -3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We apply our program to order our matrices.

```
>>A=[1 9 -4 5 3; 0 1 8 2 1; 0 0 1 4 7; 0 0 0 1 6; 0 0 0 0 1];
>>B=[1 9 -4 2 6; 0 1 9 2 -5; 0 0 1 -17 -3; 0 0 0 1 -5; 0 0 0 0 1];
>>LM(A,B);
Matrix 1 <= Matrix 2
```

Thus, we conclude that $A \leq B$.

The program allows us to easily apply the lexicographic ordering to matrix groups of dimension n, n large, which may have previously been tedious to order.

Note that after studying the lexicographic ordering, we can provide an alternate proof sketch for Proposition 1.2.3. Note that given left orderable groups K and H with short exact sequence $1 \to K \xrightarrow{\psi} G \xleftarrow{\phi} H \to 1$, we can lexicographically order G. View $H = G/K$. Then for $g_1, g_2 \in G$, we note that $g_2 < g_1$ if and only if $g_2^{-1}g_1K > K$ or $g_2^{-1}g_1 \in K$ and $g_2^{-1}g_1 > 1$.[14]

1.3.2 Archimedean Ordering

Another well-known and commonly used ordering is the Archimedean ordering.

Definition 1.3.4. *We say that an ordering* $<$ *on a group* G *is Archimedean if whenever* $1 < x < y$*, then there exists some* $n \in \mathbb{N}$ *such that* $y < x^n$.

Note that the set of integers, the rational numbers, and the real numbers are all examples of additive groups which exhibit Archimedean orderings.

Theorem 1.3.5. *(Hölder's Theorem) Let* $\leq$ *be an ordering on a biorderable group* G*. Then* $\leq$ *is an Archimedean order if and only if* G *is order isomorphic to a subgroup of the additive group of real numbers under the natural order.*

Note that a detailed proof of this theorem is provide both in Mura and Rhemtulla's book Orderable Groups (Theorem 7.2.1)[13], as well as, in Constance Wilmarth's thesis "Orderable Groups and Topology," in which the entirety of chapter two is devoted to an exhaustive proof of this theorem.[17] The direct consequence of this theorem is the following corollary.

Corollary 1.3.6. *(Conrad) If G is left orderable and Archimedean, then G is biorderable.*

Note that this corollary is proven by the aforementioned theorem in conjunction with the fact that G is biorderable if and only if $P \triangleleft G$. Recall that $P \triangleleft G$ is equivalent to noting that $gPg^{-1} \subseteq G$ for all $g \in G$. We have stated this in earlier remarks. Before we can discuss any more properties of Archimedean orderings we must consider what it means for a subgroup of G to be convex.

1.4 Convex Subgroups

We now wish to consider a specific type of left ordered group. The reason for this is we wish to consider which characterizations of groups guarantee an ordering on a group.

Definition 1.4.1. *A subset H of G is said to be convex under a left ordering $<$, if $h \in H$ whenever $h_1 \leq h \leq h_2$ for some $h_1, h_2 \in H$.*

Perhaps, the most common convex subsets are intervals, antichain, and the empty set. Note that we define an anitchain as a subset of a poset where no two elements are comparable. Note that a convex subgroup is any subgroup H of G which is also a convex subset of G. We will consider some specific examples of convex subgroups.

Proposition 1.4.2. *$\mathbb{R}$ has no proper nontrivial convex subgroups.*

Proof. Let $C \subset \mathbb{R}$ such that C is convex and $C \neq 1$. Fix $0 \neq c \in \mathbb{R}$. Let $x \in \mathbb{R}$ such that $x \geq 0$. WLOG assume $c > 0$. Then $0 \leq x < nc$ for some $n \in \mathbb{N}$. But since C is convex and $0, nc \in \mathbb{R}$, we conclude that $x \in C$. Consequently, $\mathbb{R} \subseteq C$. Thus, we conclude that $C = \mathbb{R}$. But C was an arbitrary convex subgroup. Thus, we conclude that $\mathbb{R}$ does not have any nontrivial proper convex subgroup. □

Example 1.4.3. *Let $G = \mathbb{R}^2$ be a group under the lexicographic ordering. That is, for $(a, b), (c, d) \in \mathbb{R}^2$, $(a, b) \leq (c, d)$ if and only if $a < c$ or $a = c$ and $b \leq d$. We wish to determine some of the convex subgroups of G.*

Begin by noting that the trivial subgroup is always a convex subgroup of G.

Let $H_1 = \{(0, b) : b \in \mathbb{R}\}$. Suppose $(0, b_1) \leq (a, b) \leq (0, b_2)$ for $b_1, b_2, a, b \in \mathbb{R}$. Then we know that the above inequality holds if and only if $0 \leq a \leq 0$. Thus, we conclude that $a = 0$. But then $(a, b) = (0, b)$, where $b \in \mathbb{R}$. Thus, $(0, b) \in H_1$. Also, we note that if $(0, b) \in H_1$ then $(0, b-1) \leq (0, b) \leq (0, b+1)$, where $b, b-1, b+1 \in \mathbb{R}$. Thus, we may conclude that H_1 is a convex subgroup.

Let $H_2 = \{(a, b) : a, b \in \mathbb{R}, a \geq 0\}$. Suppose $(a_1, b_1) \leq (a_2, b_2) \leq (a_3, b_3)$ for some $a_i, b_i \in \mathbb{R}$. But this implies that $a_1 \leq a_2 \leq a_3$. Thus, $a_2 \geq 0$ since $a_1 \geq 0$. We conclude that

$(a_2, b_2) \in H_2$. Now suppose that $(a, b) \in H_2$. Then $(a, b-1) \leq (a, b) \leq (a+1, b+1)$, where $(a, b-1), (a+1, b+1) \in H_2$. Thus, we can conclude that H_2 is a convex subset. However, H_2 is not a subgroup.

We wish to show H_1 is the only nontrivial convex subgroup of G. Recall that the convex subgroups form a chain. Thus, if H_1 is not the only nontrivial convex subgroup, then either H_1 contains a convex subgroup, or H_1 is contained in a convex subgroup. But we can map H_1 surjectively onto $\mathbb{R}$ using the group homomorphism $\theta(0, b) = b$. Note that this homomorphism has trivial kernel. Thus, we conclude that $H_1 \cong \mathbb{R}$. However, $\mathbb{R}$ has no nontrivial convex subgroup. Thus, it must be that H_1 is contained in a convex subgroup.

Thus, $H_1 \subseteq C \subset \mathbb{R}$ for C convex. But then we consider G/H_1, which is isomorphic to $\mathbb{R}$. But $\mathbb{R}$ has no nontrivial convex subgroup. Thus, by subgroup correspondence theorem we cannot find C such that $H_1 \subset C \subset \mathbb{R}^2$. Thus, we conclude that H_1 is the only proper nontrivial convex subgroup of $\mathbb{R}^2$.

This example helps us begin to characterize which types of groups are convex.

Proposition 1.4.4. *An Archimedean left ordered group does not have any proper convex subgroup.*

The above proposition is given without proof. A detailed proof can be found in Manuela Ioana Haias's thesis, "On the Value of Exponential and Differential Ordered Fields" (Lemma 1.2.1)[6] We now consider other characterizations of convex subgroups.

Definition 1.4.5. *Let G be a group. Then we say that G is solvable if and only if we can write G as $1 = G_0 \triangleleft G_1 \triangleleft \ldots. \triangleleft G_n = G$, where G_{i+1}/G_i is abelian.*

Solvability gives us a very specific way of characterizing groups. Consequently, we get some very interesting results based on this property. Another group for which we have interesting results is the commutator subgroup.

Definition 1.4.6. *Let x and y be elements of a group G. Then we use the notation $[x, y]$ to denote the commutator, $x^{-1}y^{-1}xy$.*

It is a simple observation that if G is a abelian group then the commutator of all elements is the identity element. That is, the commutator is the identity element, if x and y commute.

Definition 1.4.7. *We define the commutator subgroup, or derived subgroup, of a group G as the subgroup generated by all the commutators of the group. It is commonly denoted G'.*

The commutator subgroup is the smallest normal subgroup H such that G/H is abelian. [3]

Remark 1.4.8. *We note that G/N is abelian if and only if N contains the commutator subgroup.*

Theorem 1.4.9. *If G is an ordered group that does not have normal convex subgroups under the given order, then G' does not have normal convex subgroups under the induced order either.*

Rhemtulla and Mura provide a detailed proof of this proposition (Theorem 2.6.8) in their book Orderable Groups.[13]

Corollary 1.4.10. *Let G be a solvable ordered group. Then either G is Archimedean or G has a proper non-trivial normal convex subgroup.*

Proof. We will provide an exposition of Rhemtulla and Mura's proof (Corollary 2.6.9). Let G be a solvable ordered group. We want to show that G is either Archimedean or it has a proper non-trivial normal convex subgroup. Suppose for contradiction that G is neither and has minimal length. Then $\langle 1 \rangle \neq G'$. By Theorem 1.4.9, we may conclude that G' does not have any convex subgroups under the given order. Also, by the way we chose G, we know that G' is Archimedean.

Let C be a convex subgroup of G. Since we have assumed that C is not normal, we conclude that either C contains G' or $G' \cap C = 1$. If we have the former case then $C \triangleleft G$, since it contains the commutator. But this contradicts our assumption that C is not normal in G.

Thus, we assume the latter case. That is, $C \cap G' = 1$. Then we have that

$$C \cong \frac{C}{C \cap G'} \cong \frac{G'C}{G'} \leq \frac{G}{G'}.$$

Thus, we $C \leq \frac{G}{G'}$ and C is abelian. But then gCg^{-1} is a convex subgroup. Thus, since the convex subgroups of G form a chain. Therefore, either $C \leq gCg^{-1}$ or $gCg^{-1} \leq C$. WLOG assume that $gCg^{-1} \lneq C$. But this implies that

$$g \cdot \frac{CG'}{G'} \cdot g^{-1} \lneq \frac{CG'}{G'}.$$

But $\frac{CG'}{G'} = \frac{G}{G'}$ by the above, which is abelian. Thus, $g \cdot \frac{CG'}{G'} \cdot g^{-1} = \frac{CG'}{G'}$, which contradicts our assumption. Thus, we have shown that if G is a solvable group, then either G is Archimedean or G has a proper non-trivial normal convex subgroup. □

The previous corollary will prove to be extremely helpful in determining the numbers of orders of a solvable group in chapter three.

Definition 1.4.11. *We say a convex subgroup is absolutely convex if it is convex for any total order. If a convex subgroup is convex for only a specific total order, we say that it is a relatively convex subgroup.*

When we defined a convex subgroup originally it was in terms of a specific order. Now we wish to extend our understanding of convexity to all possible orders on a group G. In order to improve our intuition as to what makes an order relatively convex rather than absolutely convex, we will consider these groups relations to each other.

Theorem 1.4.12. *(Kopytov, Mamaev) Let H denote the intersection of all relatively convex subgroups of a bi-orderable group G. Then H is an absolutely convex subgroup of G.*

The proof of this theorem is found in Mamaev and Kopytov's paper "The Absolute Convexity of Certain Subgroups of an Ordered Group." [8] The proof is based on Proposition 1.4.4, and has been omitted due to its length not its complexity.

Remark 1.4.13. *Let G be a group, and let H_1 and H_2 be convex subgroups of G. We claim that $H_1 \cap H_2$ is a convex subgroup of G.*

We know that $H_1 \cap H_2 \leq G$. We must only show that $H_1 \cap H_2$ is convex. To observe this we fix $x, y \in H_1 \cap H_2$. Suppose that $x \leq z \leq y$. But $x, y \in H_1$ and H_1 is convex. Thus, $z \in H_1$. Also, $x, y \in H_2$ and H_2 is convex, so $z \in H_2$. But then $z \in H_1$ and H_2, so $z \in H_1 \cap H_2$. Therefore, we conclude that $H_1 \cap H_2$ is a convex subgroup of G.

Remark 1.4.14. *The convex subgroups of a group G form a chain. That is, if H_1 and H_2 are convex subgroups of a left ordered group G, then either H_1 is contained in H_2 or H_2 is contained in H_1.*

We now wish to consider a broader range of subgroup. To do this we define the convex hull of a subgroup H of G.

Definition 1.4.15. *Let G be a left ordered group and let H be a subset of G. Then we defined the convex hull of H as*

$$hull_G(H) = \{g \in G : h_1 \leq g \leq h_2 \text{ for some } h_1, h_2 \in H\}.$$

Let H be a subgroup of G. We observe that H is convex if $H = \text{hull}_G(H)$. If H is convex, then for all $h_1, h_2 \in H$, $h_1 \leq h \leq h_2$ implies $h \in H$. Thus, $\text{hull}_G(H) \subseteq H$. But we define H as $H = \{h \in H \subseteq G : h_1 \leq h \leq h_2 \text{ for some } h_1, h_2 \in H\}$. Therefore, $H \subseteq \text{hull}_G(H)$.

Lemma 1.4.16. *(Chiswell and Kropholler) If G is a left ordered group and H is a normal subgroup of G, then $hull_G(H)$ is a subgroup.*

Proof. The identity element will always be in $\text{hull}_G(H)$. Now we must show that $\text{hull}_G(H)$ is closed under multiplication and multiplicative inverses. Let $g_1, g_2 \in \text{hull}_G(H)$. Then by definition, there exist $h_1, h_2, h_3, h_4 \in H$ such that $h_1 \leq g_1 \leq h_2$ and $h_3 \leq g_2 \leq h_4$.

We wish to show that $g_1^{-1} \in \text{hull}_G(H)$. Then note that since $g_1 \leq h_2$ and left orders are invariant under left multiplication in G, we conclude

$$\begin{aligned} (g_1^{-1}h_2^{-1})g_1 &\leq (g_1^{-1}h_2^{-1})h_2 \\ &= g_1^{-1} \\ &= (g_1^{-1}h_1^{-1})h_1 \\ &\leq (g_1^{-1}h_1^{-1})g_1 \end{aligned}$$

But then note that since H is a normal subgroup of G, we know that $(g_1^{-1}h_2^{-1})g_1, (g_1^{-1}h_1^{-1})g_1 \in H$ and $(g_1^{-1}h_2^{-1})g_1 \leq g_1^{-1} \leq (g_1^{-1}h_1^{-1})g_1$. Thus, $g_1^{-1} \in \text{hull}_G(H)$ and we may conclude that $\text{hull}_G(H)$ is closed under multiplicative inverses.

Finally, we must show that $\text{hull}_G(H)$ is closed under multiplication. That is, $g_2g_1 \in \text{hull}_G(H)$. Then we note that since $h_3 \leq g_2$ and left orderings are invariant under left multiplication in G, we conclude

$$\begin{aligned} (g_2h_1g_2^{-1})h_3 &\leq (g_2h_1g_2^{-1})g_2 \\ &= g_2h_1 \\ &\leq g_2g_1 \\ &\leq g_2h_2 \\ &= (g_2h_2g_2^{-1})g_2 \\ &\leq (g_2h_2g_2^{-1})h_4 \end{aligned}$$

Since H is normal in G, we know that $(g_2h_1g_2^{-1})h_3, (g_2h_2g_2^{-1})h_4 \in H$. Thus, since $(g_2h_1g_2^{-1})h_3 \leq g_2g_1 \leq (g_2h_2g_2^{-1})h_4$, we may conclude that $g_2g_1 \in \text{hull}_G(H)$.

But then we have shown that $\text{hull}_G(H)$ contains the identity element, and is closed under multiplication and multiplicative inverses. Thus, $\text{hull}_G(H)$ is a subgroup of G. □

Definition 1.4.17. *Let G be a left ordered group, and let $x, y \in G$. We say that x is infinitely larger than y, $x \gg y$, if either $x > y^k$ for all $k \in \mathbb{Z}$ or $x^{-1} > y^k$ for all $k \in \mathbb{Z}$.*

Notice that if $H \triangleleft G$, and $x \in \text{hull}_G(H)$, then if $x \ll y$, $y \notin \text{hull}_G(H)$. This is because for y to be an element of $\text{hull}_G(H)$, there must exist some element z in $\text{hull}_G(H)$ such that $y < z$. But since $x \ll y$, this must not be the case.

Lemma 1.4.18. *Let x and y be elements of a groups G. If $[x, y]$ commutes with both x and y, then $[y^n, x^m] = [x, y]^{-mn}$ for all integers m and n.*

Proof. We will provide an exposition of the proof provided by Daniel Gorenstein in his book Finite Groups.[4] Suppose that $[x, y] = z$. Then we have

$$\begin{aligned} & x^{-1}y^{-1}xy &= z \\ \Longrightarrow \quad & y^{-1}xy &= xz. \end{aligned}$$

But x and z commute by assumption so note that

$$y^{-1}x^m y = (y^{-1}xy)^m = (xz)^m = x^m z^m.$$

We now conjugate by y and note that we have

$$y^{-2}x^m y^2 = y^{-1}x^m z^m y = (y^{-1}x^m y)z^m = (x^m z^m)z^m = x^m z^{2m},$$

since y and z commute. But then we note that if we conjugate n times we will get

$$\begin{array}{rrcl} & y^{-n}x^m y^n & = & x^m z^{nm} \\ \Longrightarrow & x^{-m}y^{-n}x^m y^n & = & z^{nm} \\ \Longrightarrow & [x^m, y^n] & = & z^{mn} \end{array}$$

However, we know that

$$[x, y]^{-1} = (x^{-1}y^{-1}xy)^{-1} = y^{-1}x^{-1}yx = [y, x].$$

If $[x, y] = z$ commutes with both x and y, then $[x, y]^{-1} = [x^{-1}, y] = [x.y^{-1}]$ by a similar calculation.[4] Therefore, we conclude that

$$[y^m, x^n] = z^{-mn} = [x, y]^{-mn}.$$

□

We will use Lemma 1.4.18 in the proof of the next theorem.

Theorem 1.4.19. *(Ault) Let x, y, z be non-identity elements of a left ordered group H, with $[x, y] = z^k$ for some nonzero $k \in \mathbb{Z}$ and $[x, z] = [y, z] = 1$. Then either $z \ll x$ or $z \ll y$.*

Proof. We will provide an exposition of the proof found in Dave Witte's article "Arithmetic Groups of Higher $\mathbb{Q}$-Rank Cannot Act on 1-Manifolds."[18] He concisely sums up Ault's original proof of the theorem. Note that $x^j, y^j, z^{jk} \geq 1$, where $j = \pm 1$. This follows from the definition of orderable groups. We will assume that $x, y, z^k \geq 1$, where to do this we may also have to interchange x and y. To simplify the proof we replace z with z^k (thus, assuming $k = 1$). If the proof holds for $k = 1$, it holds for larger k as well.

We wish to show that either $z \ll x$ or $z \ll y$. If $z \ll x$ then we are done, so assume that $z \not\ll x$. Then $x < z^i$ for some positive integer i. We replace wish to replace z^i with z. To do this we let $[x, y^i] = z^i$. Then replacing y and z with y^i and z^i we may assume $x < z$.

Then we note that $zx^{-1}, y, zx > 1$. Therefore, any power of the aforementioned elements is positive, as is their product. We also know that since $[x, z] = [y, z] = 1$, z commutes with both x and y. Thus, for all positive integers r,

$$\begin{aligned} 1 &< (zx^{-1})^{3r}y^3(zx)^{3r} \\ &= z^{3r}x^{-3r}y^3z^{3r}x^{3r} \\ &= x^{-3r}y^3x^{3r}z^{3r}z^{3r} \\ &= y^3y^{-3}x^{-3r}y^3x^{3r}z^{3r}z^{3r} \\ &= y^3[y^3, x^{3r}]z^{6r} \\ &= y^3z^{-9r}z^{6r} \\ &= (yz^{-r})^3 \end{aligned}$$

We conclude that $1 < (yz^{-r})^3$. But this implies that $1 < yz^{-r}$ and consequently $z^r < y$ for all integers r. That is, $c \ll b$, which is what we wanted to show. □

The result of this theorem is that the convex hull of a group G is not the entire group. This result proves invaluable in some of our proofs in chapter four.

Theorem 1.4.20. *Let G be a left orderable group. Let H be a normal convex subgroup of G. Then G/H is left orderable.*

Proof. Let G be a left orderable group, and let $<$ be a specific left order on G. Let H be the convex hull of G. Then note that we wish to show if $Hx \neq Hy$, then $Hx < Hy$ if and only if $x < y$, for $x, y \in G$. That is, $Hx < Hy$ implies $y^{-1}x < 1$.

Suppose $Hx = Hx_1$ and $Hy = Hy_1$. Thus, $Hx_1 < Hy_1$. We wish to show that $x_1 < y_1$, that is, $1 \leq x_1^{-1}y_1$. We can note that since $Hx = Hx_1$, $x_1x^{-1} \in H$, so $x_1 = h_1x$ for some $h_1 \in H$. Similarly, we can note that since $Hy = Hy_1$, $y_1 = h_2y$ for some $h_2 \in H$.

Note that since we have assumed $Hx \neq Hy$, $y^{-1}x \notin H$. Recall H is the convex hull. Thus, since $y^{-1}x < 1$, we wish to show that $y^{-1}x < h$ for all $h \in H$. Suppose not. That is, for some $h_1 \in H$, $h_1 \leq y^{-1}x < h$. But by the convexity of H, we can conclude that $y^{-1}x \in H$, contradicting our assumption that $y^{-1}x \notin H$. Thus, no such h_1 exists and we conclude that $y^{-1}x < h$ for all $h \in H$.

But note that since $y \in G$, and H is a normal subgroup of G, that $k = y^{-1}h_1^{-1}h_2y \in H$, since certainly $h_1^{-1}h_2 \in H$.

Thus, we can conclude that

$$\begin{aligned} & y^{-1}x < k \\ \Longrightarrow\ & y^{-1}x < y^{-1}h_1^{-1}h_2y \\ \Longrightarrow\ & x < h_1^{-1}h_2y \\ \Longrightarrow\ & h_1x < h_2y \\ \Longrightarrow\ & x_1 < y_1, \end{aligned}$$

which is exactly what we wanted to show.

Thus, we conclude that G/H is left orderable. □

We now have a basic understanding of not only what makes a group orderable, but of the fundamental structure of the subgroups of an ordered group. This chapter has also shown various methods for extending an ordering on a group. We now consider how many ways there are to order a group. The remainder of this paper is devoted to building the machinery to answer this question.

Chapter 2

The Cantor Set

Knowing that a set is left orderable is not as useful as being able to describe the space of left orders on a set. Perhaps, the most informative aspect of the space of left orders is its cardinality. That is, it is very useful if we discern the number of elements in the space of left orders of a group. For the sake of this paper we will concern ourselves primarily with groups whose space of left orders has an uncountable number of elements. This task is not always easy, and this chapter is spent building the machinery to make such a proof possible.

2.1 Homeomorphisms

One of the most frequently used methods of determining the cardinality of a set is the creation of a bijection between the sets. However, the space of left orders is in actuality a topological space.

Definition 2.1.1. *Let G be a set with a collection of subsets $\{G_i\}$, collectively known as open sets. Then we say that G is a topological space if it satisfies the following conditions:*
(i) The union of any collection of open sets is an open set.
(ii) The intersection of any pair of open sets is an open set.
(iii) The empty set and G are both open.

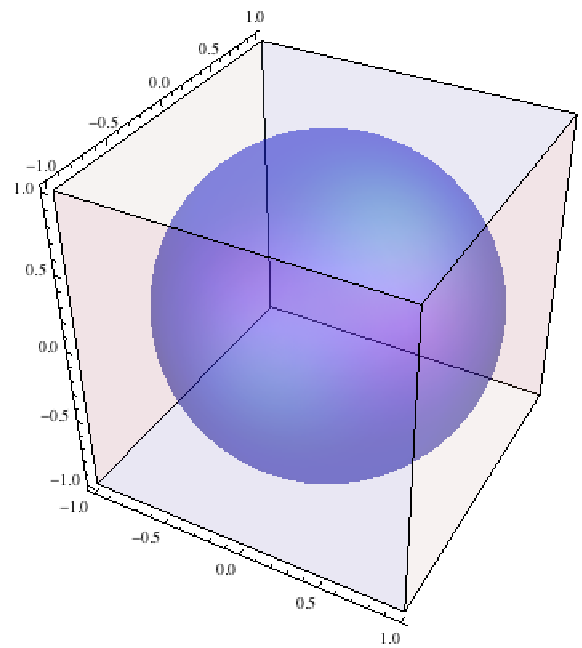
We now wish to adapt a method for determining the cardinality of a topological space. To do this we must consider how to create a bijection between two topological spaces.

Definition 2.1.2. *Let X and Y be topological spaces and let $f : X \to Y$. Then we say that f is a homeomorphism if it satisfies the following properties:*
(i) f is one-to-one and onto
(ii) for any subset S of X, p is a limit point of S if and only if $f(p)$ is a limit point of $f(S)$.

Equivalently we may note that a mapping between two topological spaces is a homeomorphism if and only if f is a continuous function with a continuous inverse function. Essentially, f is the analogue to a conformal map in analysis [7].

Now let us consider some examples of homeomorphisms. We will use an example commonly found in topology textbooks, the deformation of a cube into a sphere using a homeomorphism [10]. We will specifically use the mapping given by John Lee in his textbook Introduction to Topological Manifolds.[10]

Figure 2.1: Cube Deformed to Sphere



Example 2.1.3. *Let* $X = \{x \in \mathbb{R}^3 : \|x\| = 1\}$, *where* $\|x\|$ *denotes the norm of* x *in* $\mathbb{R}^3$. *That is, let* X *be the unit sphere in* $\mathbb{R}^3$. *Let* $Y = \{(x, y, z) \in \mathbb{R}^3 : max(|x|, |y|, |z|) = 1\}$. *Thus,* Y *is a cube centered at the origin with side length 2.*

Define $f : Y \to X$ *by*

$$f(x, y, z) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}.$$

We can immediately note that this function is continuous, since $\sqrt{x^2 + y^2 + z^2} \neq 0$ *for any* $(x, y, z) \in Y$.

Now to show that f *is a homeomorphism we will consider its inverse mapping,* $f^{-1} : X \to Y$. *Note that we argue that*

$$g(x, y, z) = \frac{(x, y, z)}{max(|x|, |y|, |z|)}$$

is the inverse of f.

We notice that the function given by g is continuous as max$(|x|,|y|,|z|) \neq 0$, since $\sqrt{x^2+y^2+z^2} = 1$ for all $(x,y,z) \in X$. We must ask ourselves if f^{-1} and g are equal. To determine such we fix $(x,y,z) \in X$ and discern if $f(g(x,y,z)) = (x,y,z)$. Assume WLOG max$(|x|,|y|,|z|) = |x|$. Observe

$$\begin{aligned} f(g(x,y,z)) &= f\left(\frac{(x,y,z)}{\max(|x|,|y|,|z|)}\right) = f\left(\frac{1}{|x|}(x,y,z)\right) \\ &= \frac{\left(\frac{x}{|x|},\frac{y}{|x|},\frac{z}{|x|}\right)}{\sqrt{\left(\frac{x}{|x|}\right)^2+\left(\frac{y}{|x|}\right)^2+\left(\frac{z}{|x|}\right)^2}} = \frac{\left(\frac{x}{|x|},\frac{y}{|x|},\frac{z}{|x|}\right)}{\sqrt{\frac{x^2+y^2+z^2}{|x|^2}}} \\ &= \frac{\left(\frac{x}{|x|},\frac{y}{|x|},\frac{z}{|x|}\right)}{\sqrt{\frac{1}{|x|^2}}} = \frac{\left(\frac{x}{|x|},\frac{y}{|x|},\frac{z}{|x|}\right)}{\frac{1}{|x|}} = (x,y,z) \end{aligned}$$

Thus, we conclude that $g = f^{-1}$. Therefore, we have found a continuous bijective mapping for f. We conclude that f is a homeomorphism.

Given two topological spaces we can show that the spaces share the same cardinality if a homeomorphism exists between the two spaces. Therefore, since proving the cardinality of a space of left orders is uncountable is often challenging, we will instead show that the space of left orders is homeomorphic to the Cantor set. Before we can do this we must discern the cardinality of the Cantor set. This requires a basic knowledge of the Cantor set.

2.2 Cantor Ternary Set

The Cantor set is often used in topology to provide an example of a perfect set that is nowhere dense. Recall that a perfect set is a closed set in which every point is a limit point of the set. Describing a set as nowhere dense is equivalent to noting that the closure of the set has no interior points. We will go into greater detail as to the properties of the set after discussing its construction.

The German mathematician Georg Cantor introduced the Cantor middle-third or ternary set in 1883. [5] The set is constructed by taking the interval $I = [0,1]$ and continually removing the open middle third. That is, we begin by removing $(1/3, 2/3)$ from $[0,1]$, letting our first interval be $E_1 = [0,1/3] \cup [2/3,1]$. We remove the open middle third of each of the remaining segments. Then we have $E_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$. We note that the Cantor set is

$$\mathcal{C} = \bigcap_{i=1}^{\infty} E_i,$$

where we define $E_i = \frac{E_{i-1}}{3} \cup \left(\frac{2}{3} + \frac{E_{i-1}}{3}\right)$. That is,$E_{i+1} = E_i$ minus the open middle third.

Below we demonstrate pictorially the first four steps in constructing the Cantor set.

Figure 2.2: Cantor Set Construction

We now wish to discuss the measure of the Cantor set. Note that in measure theory, a measure on a set is way to assign each relevant subset a number that may be intuitively indicative of the size of the subset.

Proposition 2.2.1. *The Cantor set has measure zero.*

The idea behind this proof is that in constructing E_1, we removed from $[0,1]$ $1(=2^0)$ subinterval of length 3^{-1}. In constructing E_2 we remove from E_1 $2(=2^1)$ subintervals of length 3^{-2}. Thus, in constructing E_n we remove from E_{n-1} 2^{n-1} intervals of length 3^{-n}. So the total length of the subintervals removed is

$$\sum_{n=1}^{\infty} 2^{n-1}3^{-n} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{2}\cdot\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{1-2/3} = 1.$$

Therefore, the complement in $[0,1]$ of the Cantor set has length 1. We conclude that the Cantor set has measure 0.

Note that by construction the Cantor set is compact. We note that the Cantor set, $\mathcal{C}$, is the intersection of closed subsets of the unit interval. We also note that $\mathcal{C} \subset [0,1]$. Therefore, we have a set that is closed and bounded, and consequently by the Heine-Borel Theorem, compact.

Recall that a perfect set is one that contains its limit points. We wish to show that the Cantor set is perfect.

Proposition 2.2.2. *The Cantor set is perfect.*

Proof. Let $\epsilon > 0$ be given. Fix $x \in \mathcal{C}$, the Cantor set. Thus, we note that $x \in E_i$ for all $i \in \mathbb{N}$. Let I_i denote that interval of E_i to which x belongs in E_i. Note that we can find N large enough such that $I_N \subset B_\epsilon(x)$, the ball of radius ϵ centered at x. Let x_1 and x_2 be the endpoints of I_N, and WLOG assume $x_1 \leq x_2$. Since the endpoints are never removed when constructing the Cantor set, $x_1, x_2 \in \mathcal{C}$. Therefore, we can conclude that x is not isolated. But x was an arbitrary point, so $\mathcal{C}$ is perfect. □

The fact that the Cantor set has no isolated points is an important property that we will put to good use in our later work.

Now we wish to show that despite that fact that $\mathcal{C}$ has no isolated points, it is a totally disconnected set.

Proposition 2.2.3. *The Cantor set is totally disconnected.*

Proof. Fix two distinct points $x, y \in \mathcal{C}$. Note that this implies that $x, y \in E_i$ for all $i \in \mathbb{N}$. But note that since x and y are distinct points, we can find N large enough such that $\frac{1}{3^N} < |x - y|$. Therefore, in E_N, x and y belong to two different intervals. But this implies that there exists at least one interval between the interval containing x and the interval containing y that does not belong to the E_N. But if this interval is removed from E_N, then it is certainly removed from $\mathcal{C}$. Then for any point z in this interval, $z \notin C$, but z is between x and y. Thus, we have shown that $\mathcal{C}$ is totally disconnected. □

Thus, we note that the Cantor set has no non-trivial connected subsets.

Another important property of the Cantor set is the fact that it is metrizable. We say that a metric space X is metrizable if there exists a metric ρ on the set X that induces the topology of X. Thus, the Cantor set is a perfect, totally disconnected, compact, metrizable set.

In their book, Topology, Hocking and Young explore these characterizations.[7] We wish to show that if a set has the same characterization as the Cantor set, namely, that it is compact, totally disconnected, compact and metrizable, then it is homeomorphic to the Cantor set. This result is not immediately obvious.

We will provide a concise summary of the set of results leading to this conclusion, as presented in Hocking and Young's book. Begin by noting that if $\mathcal{U}$ is a covering of a metric space X composed of open sets, and $n \in \mathbb{Z}$, then there exists a refinement $\mathcal{V}$ of $\mathcal{U}$, composed of open sets of diameter strictly less than $1/n$. In fact, if X is a compact metric space, then $\mathcal{V}$ is finite.

Theorem 2.2.4. *Let X be a compact totally disconnected metric space. Then it has a sequence $\mathcal{U}_1, \mathcal{U}_2, ...$ of finite coverings with each $\mathcal{U}_n$ being a collection of disjoint sets of diameter strictly less than $1/n$ that are both open and closed, where $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ for each value of n.*

The idea behind the proof is that if X_1 is a subset of X and if U is an open set containing X_1, then there is an open and closed set V lying in U containing X_1. We use this property and induction to infer the conclusion.

Next we take the sequence of coverings of X, $\mathcal{U}_1, \mathcal{U}_2, ...$, described above and construct an inverse limit sequence. That is for each index n, let $\mathcal{U}_n^*$ be the space whose points are the open sets of $\mathcal{U}_n$ (which has the discrete topology). We wish to define a map from $f_n : \mathcal{U}_n^* \to \mathcal{U}_{n-1}^*$, for $n > 1$. Note that the elements of $\mathcal{U}_{n-1}$ are disjoint. Fix $U_{n,i} \in \mathcal{U}_n$. There exists $U_{n-1,j} \in \mathcal{U}_{n-1}$, which contains $U_{n,i}$. Let $f_n(U_{n,i}) = U_{n-1,j}$. Thus, $\{\mathcal{U}_n, f_n\}$ is an inverse limit sequence of compact sets.

Theorem 2.2.5. *Let X be a compact, totally disconnected metric space. Then X is homeomorphic to the inverse limit space of an inverse limit sequence of finite, discrete spaces.*

The idea behind this proof is we can construct the inverse limit sequence of finite, discrete spaces as described above. Then we denote the inverse limit space of the sequence $\{\mathcal{U}_n, f_n\}$ as $\mathcal{U}_\infty$. By construction we know that $\mathcal{U}_\infty$ is nonempty. We can then define a map from $h : \mathcal{U}_\infty \to X$. Proving that h is a homeomorphism proves Theorem 2.2.5.

Theorem 2.2.6. *If U is an open set in a totally disconnected perfect topological space, and n is an integer, then U is the union of n disjoint nonempty open sets.*

Proof. We will provide an exposition of the proof present in Hocking and Young's book. Note that this result is true for $n = 1$. Suppose that $n = k > 1$. Then we write $U = U_1 \cup ... \cup U_k$, where the U_i are open, disjoint and nonempty. Since the space is totally disconnected we can immediately note that U_k is not connected. Then $U_k = U_{k,1} \cup U_{k,2}$, where $U_{k,1}$ and $U_{k,2}$ are open in U_k and disjoint. Thus, we can note that $U = U_1 \cup ... \cup U_{k-1} \cup U_{k,1} \cup U_{k,2}$ gives the result. □

We now have the machinery to begin proving results that are directly applicable to our work. That is, we wish to show that two sets with the same properties, namely, both sets being perfect, compact, totally disconnected and metrizable are homeomorphic.

Theorem 2.2.7. *Any two totally disconnected, perfect, compact metric spaces are homeomorphic.*

The proof of this theorem is quintessential in discerning the cardinality on the space of left orders on numerous groups in chapter three. The idea behind the proof is to express each space in terms of its open coverings. Then we can express each open covering as the union of open sets, $\{\mathcal{U}_i\}$ and $\{\mathcal{V}_j\}$, respectively. If both spaces have the same number of open sets, we are done as we can create a homeomorphism between them. If not, we can decompose one of the open sets as done in the proof of the previous theorem. This may have to be done repeatedly. Once the sets have the same number of elements, we define a continuous mapping from $\{\mathcal{U}_n^*\} \to \{\mathcal{V}_n^*\}$ by induction and define another map in terms of projection. The result is that using the previous theorems, we can show that U_∞ can be mapped onto V_∞ via a homeomorphism.

Corollary 2.2.8. *Any compact totally disconnected perfect metric space is homeomorphic to the Cantor set.*

This follows directly from the previous theorem and our characterization of the Cantor set. Note that this result is the basis for many of the proofs in the next chapter, where we discern the space of left orders on various groups. It will allows us to compare the cardinality of these topologies to a space whose cardinality is well known.

2.3 Proof of Uncountability

In the previous section we derived a method for proving that if a set is totally disconnected, perfect, compact and metrizable, then it is homeomorphic to the Cantor set. The reason that this is so valuable, is that the Cantor set is one of the most famous examples of an uncountable set.

Definition 2.3.1. *A countably infinite set has an injective function mapping it to the natural numbers.*

We say that a set is uncountable if it has too many elements to be countable. Thus, a set is uncountable if it is neither countably infinite nor finite. Rather than proving directly that the Cantor set is uncountable, we will begin by considering the the real numbers on the interval $[0, 1]$.

Cantor developed a method, now known as Cantor's diagonalization argument, to show that $S = \{0, 1\}^{\mathbb{N}}$, the set of possible sequences of 0s and 1s, is uncountable.

Theorem 2.3.2. $S = \{0, 1\}^{\mathbb{N}}$*, the set of possible sequences of 0s and 1s, is uncountable.*

Proof. For contradiction, suppose that $S = \{0, 1\}^{\mathbb{N}}$ is countable. Write $S = \{S_1, S_2, ...\}$, where

$$S_1 = \{s_{11}, s_{12}, s_{13}, ...\}$$
$$S_2 = \{s_{21}, s_{22}, s_{23}, ...\}$$
$$\vdots$$
$$S_i = \{s_{i1}, s_{i2}, s_{i3}, ...\},$$

where each s_{ij} is either 0 or 1.

Next we then define $\tau \in S$ by $\tau = \{\tau_1, \tau_2, ..., \tau_m\}$, where

$$\tau_1 = \begin{cases} 0 & \text{if } s_{11} = 1 \\ 1 & \text{if } s_{11} = 0, \end{cases} \quad \tau_2 = \begin{cases} 0 & \text{if } s_{22} = 1 \\ 1 & \text{if } s_{22} = 0, \end{cases} \quad \dots \quad \tau_n = \begin{cases} 0 & \text{if } s_{nn} = 1 \\ 1 & \text{if } s_{nn} = 0. \end{cases}$$

That is, τ_i depends on the diagonal element s_{ii}, in that if $s_{ii} = 1$ the $\tau_i = 0$.

But then note that $\tau \in S$, but $\tau \neq S_1$, since $\tau_1 \neq s_{11}$, and $\tau \neq S_2$, since $\tau_2 \neq s_{22}$. Thus, $\tau \neq S_n$ for any $n \in \mathbb{N}$, since $\tau_n \neq s_{nn}$. Thus, there does not exist a bijection between S and the natural numbers. Therefore, we conclude that S is uncountable. □

Note that any number in the set $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ can be expressed as a binary decimal. Thus, it follows from the previous theorem that $(0, 1)$, as a subset of $\mathbb{R}$, is uncountable. However, note that $(0, 1) \subset \mathbb{R}$. Therefore, since $(0, 1)$ is uncountable $\mathbb{R}$ is uncountable.

We now wish to show that the Cantor set is uncountable. To do this we use the Cantor function to map the Cantor set to the closed interval $[0, 1]$.

Definition 2.3.3. *The Cantor function, or the Devil's staircase, is a continuous function* $f : [0, 1] \to [0, 1]$*, which is defined as follows:*

1) Express $x \in [0, 1]$ *in base 3.*
2) If x *contains a 1, replace every digit after the first 1 by 0.*
3) Replace all 2s with 1s.
4) Interpret the result as the binary number $f(x)$*.*

Thus, we see that every point in the Cantor set can be written as an infinite binary sequence. But we have shown previously that S is uncountable. Thus, we conclude that the Cantor set is uncountable. This proof can also be done by writing each element of the Cantor set as a ternary expression consisting of only 0s and 2s. The result of this is that we can use an adaption of Cantor's diagonalization argument shown above in the proof that S is uncountable.

Chapter 3

Orderable Groups

Now that we have familiarized ourselves with what an orderable group is, as well as, the most important characteristics of such a group, we proceed forward by considering specific important groups on which we wish to consider the the space of left orders.

In the previous chapter we discussed the importance of cardinality in characterizing a group. This proves to be particularly interesting in studying the space of left orders on a group. This chapter considers well-known and studied groups, and proceeds to construct the space of left orders for said groups. We then seek to characterize the space of left orders, choosing to focus particularly on the cardinality of the space of left orders. To do this we must derive some results on the topology of the space of left orders.

3.1 Topology on LO(G)

In order to indicate the topology on orders, we will use the notation $U_{x,y} \subset \mathrm{LO}(G)$ to represent the set of all left orderings such that $1 < x^{-1}y$. The notation used in discussing the topology on the space of left orders is adopted from Adam Sikora's "Topology on the Spaces of Orderings of Groups." The theorems are his, and all proofs are expositions of work done in his article.[15]

We now wish to consider how we can place a topology on the space of left orders on G. Note that if G is a left orderable group, then for all $<_i \in \mathrm{LO}(G)$, and for all distinct $x, y \in G$, we can order these elements under $<_i$. Now we note that the set of all orderings $<_i$ such that $x <_i y$ forms an open set.

Definition 3.1.1. *Let $U_{x,y}$ denote the set of left orderings $<$ on G for which $x < y$.*

By construction $U_{x,y} \subset \mathrm{LO}(G)$. Any open set in $\mathrm{LO}(G)$ is composed of some union of sets of the form $U_{x_1,y_1} \cap U_{x_2,y_2} \cap ... \cap U_{x_n,y_n}$. That is, we notice that sets of the form

$U_{x_1,y_1} \cap U_{x_2,y_2} \cap ... \cap U_{x_n,y_n}$ generate the topology on LO(G).

Definition 3.1.2. *Define* $V_{x_1^{-1}y_1, x_2^{-1}y_2, ..., x_n^{-1}y_n} = U_{x_1,y_1} \cap U_{x_2,y_2} \cap ... \cap U_{x_n,y_n}$. *Let* $V = \{V_{x_i^{-1}y_i}\}$. We may write $V(G)_{x_1^{-1}y_1,...,x_n^{-1}y_n}$ when it is unclear as to which group's base we are referring.[12]

Although the above definition gives us that the topology on the space of left orders of a group is composed of open sets, we have yet to discuss a way of describing the distance between points in the space of left orders. Thus, we will redefine the metric where we discuss distance between orders in terms on the number of sets on which the orders agree.

Definition 3.1.3. *We define a filtration as an indexed set G_i of sub-objects of a given algebraic structure with index i, for some index set I. We note that a filtration is a totally ordered set subject to the condition $i \leq j$ in I, then $G_i \subseteq G_j$. We note that these sets are normally indexed by the natural numbers.*

Note that a topology associated with a filtration on a group G makes G into a topological group. We will only consider countable groups G, and whenever we discuss filtrations, the reader should assume they are composed of finite subsets of G. This allows us to prove that our topological space is metrizable. The topology associated with a filtration G_n on a group G is Hausdorff if and only if $\bigcap G_n = \{1\}$.

We now wish to devise a method of determining the distance between elements in our set LO(G). Since the elements of our set are orderings on a group, our metric will not be Euclidean. Instead we will use the same method as Sikora to compare two orders in our set.

Definition 3.1.4. *(Sikora) Let $G_0 \subset G_1 \subset ... \subset G_n = G$ be a complete filtration. That is, suppose $\bigcup_{i \in I} G_i = G$. Then for $<_1, <_2 \in LO(G)$ let*
(i) $\rho(<_1, <_2) = \frac{1}{2^r}$, $r = \max\{s : <_1 = <_2$ on $G_s\}$.
(ii) $\rho(<_1, <_2) = 0$ if $r = \infty$. That is, $\rho(<_1, <_2) = 0$ if the orders of $<_1$ and $<_2$ never coincide.

Thus, we have now given a more concrete way of defining the distance between two orders. We wish to show that ρ is a distance function that turns G into an ultrametric space. To do this we must first define what it means for a group to be an ultrametric space.

Definition 3.1.5. *We define an ultrametric space as a set of points G with an associated distance function $d : G \times G \to \mathbb{R}$ such that for all $x, y, z \in G$,*
(i) $d(x, y) \geq 0$
(ii) $d(x, y) = 0 \iff x = y$
(iii) $d(x, y) = d(y, x)$
(iv) $d(x, z) \leq max\{d(x, y), d(y, z)\}$

We wish to show that ρ is in fact an ultrametric on LO(G). Note that this result directly follows from Sikora's paper, however, as no proof is given we have provided one below.

Proposition 3.1.6. *(Sikora) The function ρ is an ultrametric on $LO(G)$.*

Proof. Fix $<_1, <_2, <_3 \in \mathrm{LO}(G)$. Note that $\rho(<_1, <_2) = \frac{1}{2^r}$ for some $r \geq 0, r \in \mathbb{Z}$. Thus, for all r $\rho(<_1, <_2) \geq 0$. Also, note that if $\rho(<_1, <_2) = \frac{1}{2^r}$, this implies that the largest set on which $<_1$ and $<_2$ coincide is G_r. But then we note that this is the same as the largest set on which $<_2$ and $<_1$ coincide. That is, $\rho(<_1, <_2) = \frac{1}{2^r} = \rho(<_2, <_1)$.

We wish to show that $\rho(<_1, <_2) = 0 \iff <_1 = <_2$. Suppose that $\rho(<_1, <_2) = 0$. Then $\rho(<_1, <_2) = \frac{1}{2^r} = 0$. This implies that $r = \infty$. Thus, $<_1$ and $<_2$ coincide for all G_r. Thus, we conclude that $<_1 = <_2$. Now we suppose that $<_1 = <_2$. That is, $<_1$ coincides with $<_2$ on G_r for all r. Then the largest set on which $<_1$ and $<_2$ coincides is G_∞. But then we have the ∞ is the largest value of r for which $<_1$ and $<_2$ coincide on G_r. Thus, $\rho(<_1, <_2) = \frac{1}{2^\infty} = 0$.

Finally, we must show that $\rho(x, z) \leq \max\{\rho(<_1, <_2), \rho(<_2, <_3)\}$. Suppose that the largest set on which $<_1$ and $<_2$ coincide is G_r, and the largest set on which $<_2$ and $<_3$ coincide is G_s, for some $r, s \geq 0, r, s \in \mathbb{Z}$. WLOG we may assume that $r \leq s$. Thus, by definition $G_r \subseteq G_s$. But then we can note that $<_1$ and $<_3$ agree on G_r. That is, $\rho(<_1, <_3) = \frac{1}{2^r} = \max\left\{\frac{1}{2^r}, \frac{1}{2^s}\right\} = \max\{\rho(<_1, <_2), \rho(<_2, <_3)\}$.

Thus, we have shown that ρ is an ultrametric. □

Proposition 3.1.7. *(Sikora) The topology $LO(G)$ induced by this metric is the same as the topology $U_{x,y}$, which is the set of all orders for which $x < y$.*

Proof. Let $<_0 \in LO(G)$. We want to show that $B_{\frac{1}{2^r}}(<_0) = \left\{<_i \in LO(G) : \rho(<_0, <_i) < \frac{1}{2^r}\right\}$ is an open ball in $U_{x_1,y_1} \cap ... \cap U_{x_n,y_n}$. Recall that $<_1 \in B_{\frac{1}{2^r}}(<_0)$ if and only if $\rho(<_0, <_1) < \frac{1}{2^r}$, which is true if and only if and only if $<_0$ and $<_1$ coincide on at least G_{r+1}. Then we consider all pairs $(x_i, y_i) \in G$ such that $1 <_0 x_i^{-1} y_i$, $i \leq n$.

Therefore, $B_{\frac{1}{2^r}}(<_0) = \cap U_{x_i,y_i}$. Recall that $U_{x,y}$ denotes the set of left orders such that $1 < x^{-1}y$. That is, $B_{\frac{1}{2^r}}(<_0)$ is equal to the intersection of all left orders for which $1 < x^{-1}y$. But then we have that $\cap U_{x_i,y_i} \subset B_{\frac{1}{2^{r-1}}}(<_0)$.

We now wish to show that any set of the form $U_{x_1,y_1} \cap ... \cap U_{x_n,y_n}$ is open with respect to the metric ρ. To do this we must show that there exists some natural number r such that $B_{\frac{1}{2^r}}(<_0) \subset U_{x_1,y_1} \cap ... \cap U_{x_n,y_n}$. Let G_r be an arbitrary set in the filtration. Then there exist pairs $(x_i, y_i) \in G$ such that $x_1, y_1, ..., x_k, y_k \in G_r$. But then we have that $B_{\frac{1}{2^r}}(<_0) \subset U_{x_1,y_1} \cap ... \cap U_{x_k,y_k}$. □

We know that the elements of V are open with respect to the topology on $\mathrm{LO}(G)$. We also know that every open set in $\mathrm{LO}(G)$ can be written as a union of elements V. Therefore, we conclude that V is a base for $\mathrm{LO}(G)$; that is, V generates the topology in $\mathrm{LO}(G)$.

Now that we have means of discussing the topology on $\mathrm{LO}(G)$. That is, now we have a good understanding of the metric ρ on $\mathrm{LO}(G)$. Now we can begin characterizing $\mathrm{LO}(G)$.

Note that we wish to show that LO(G) has the same properties as the Cantor set, namely, that LO(G) is compact, totally disconnected, perfect and metrizable. The last of the aforementioned characteristics was shown in the above proof. That is, we have shown that LO(G) is metrizable. Next we show that the space of left orders is totally disconnected. Thus, we must show that if two points are distinct, then they are contained in disjoint open sets which cover the space. [15]

Theorem 3.1.8. *(Sikora) LO(G) is totally disconnected metric space.*

Proof. Note for any two left orders $<_1, <_2 \in LO(G)$, there exists $x, y \in G$ such that $<_1 \in U_{x,y}$ and $<_2 \in U_{y,x}$. That is, $1 <_1 x^{-1}y$ and $1 <_2 y^{-1}x$. Note that for all $<_i \in LO(G)$ we have that either $1 <_i x^{-1}y$ or $1 <_i y^{-1}x$. That is, $<_i \in U_{x,y}$ or $<_i \in U_{y,x}$. But then we note that $LO(G) = U_{x,y} \cup U_{y,x}$. Also, we know that since x and y are distinct that $U_{x,y} \cap U_{y,x} = \emptyset$, since it is impossible that $1 <_i x^{-1}y$ and $1 <_i y^{-1}x$. But recall that $<_1$ and $<_2$ are distinct orders in disjoint sets. We have consequently shown that LO(G) is the union of two disjoint nonemtpy open subsets. Thus, we conclude that LO(G) is totally disconnected. □

Definition 3.1.9. *Let X be topological space. We say X is a Hausdorff space if any two distinct points of X can be separated by neighborhoods. That is, if x and y are distinct points in X such that there exists a neighborhood U_x of x and a neighborhood U_y of y such that $U_x \cap U_y = \emptyset$.*

Note that almost all spaces encountered in analysis are Hausdorff. Examples of Hausdorff spaces are all metric spaces. Consequently, allowing us to discuss LO(G) as a Hausdorff space will allow us to readily apply more topological results in characterizing LO(G).

Remark 3.1.10. *Let X be a topological space and let d be a metric on X. We wish to show that (X, d) is Hausdorff.*

To demonstrate the veracity of the above remark, fix $x, y \in X$ distinct. Note that since x and y are distinct it follows that $d(x, y) \neq 0$. Let $\delta = \frac{d(x,y)}{2}$. Then $B_\delta(x)$ and $B_\delta(y)$ are open sets in X. We must show that $B_\delta(x) \cap B_\delta(y) = \emptyset$. Fix $z \in X$. Suppose that $z \in B_\delta(x)$ and $z \in B_\delta(y)$. Then $d(z, x) < \delta$ and $d(z, y) < \delta$. But then we have that

$$\begin{aligned} d(z,x) + d(z,y) &< \delta + \delta \\ &= \frac{d(x,y)}{2} + \frac{d(x,y)}{2} \\ &= d(x,y). \end{aligned}$$

But we know that by the triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$. Thus, we have a contradiction, and we note that there exists no $z \in X$ such that $z \in B_\delta(x)$ and $z \in B_\delta(y)$. Thus, $B_\delta(x) \cap B_\delta(y) = \emptyset$. But then we can conclude that the metric topology given to X by d is Hausdorff. It follows directly from the above results that LO(G) is Hausdorff.

Before we show that LO(G) is compact, we will discuss what it means for a metric space to be compact. In metric spaces countable compactness, the fact that every countable open cover has a finite subcover, is the same as compactness. We also note that every sequentially compact space is countably compact. That is, a metric space is compact if every sequence has a convergent subsequence. We introduce some new terminology to do this.

Definition 3.1.11. *Let $<^{\infty}$ be a left order, which we define as follows $1 <^{\infty} x^{-1}y$ for $x, y \in G$ if and only if $1 <^{n} x^{-1}y$ for almost all n.*

Theorem 3.1.12. *(Sikora) LO(G) is compact.*

Proof. We must now show that LO(G) is compact. Consider a complete filtration of G. That is, consider $G_0 \subset G_1 \subset G_2 \subset ... \subset G$, where $G = \bigcup G_i$, and the G_i are finite.

We will show LO(G) is sequentially compact. To do this we must show that every sequence in LO(G) has a convergent subsequence. Let $<_1, <_2, ...$ be an arbitrary infinite sequence in LO(G). We wish to construct a convergent subsequence. Begin by considering G_1 in our filtration. Note that there are only finitely many ways to totally order the elements of G_1. Then there is an infinite subsequence $<_{i_1^1}, <_{i_2^1}, ...$ of $<_1, <_2, ..,$ where the elements of our subsequence induce the same total order on G_1.

Now from our first subsequence we choose an infinite subsequence of orders, $<_{i_1^2}, <_{i_2^2}, ...,$ that agree on G_2. Continue this process of constructing subsequences. That is, we get $<_{i_1^n}, <_{i_2^n}$ is a subsequence of the previous subsequence $<_{i_1^{n-1}}, <_{i_2^{n-1}}$ that agrees on G_n. Now consider $<^1, <^2, ...,$ where $<^k = <_{i_k^k}$ for $k = 1, 2, 3....$

We wish to show that the sequence $<^1, <^2, ...$ converges to the left order $<^{\infty}$. Consider the set G_r. Fix $x, y \in G_r$. Then note that either $1 <^i x^{-1}y$ for $i > r$ or $1 <^i y^{-1}x$ for $i > r$. Thus, $<^{\infty}$ is a total order. It should be clear that $<^{\infty}$ is a left order. Then consider $\rho(<^n, <^{\infty}) \leq \frac{1}{2^n}$. Thus, we note that for n large, $<^1, <^2, ...$ converges to $<^{\infty}$. Thus, $<^1, <^2, ...$ is a convergent subsequence of $<_1, <_2, ...$ We conclude that LO(G) is sequentially compact, which implies that LO(G) is compact. □

But note that we have shown that LO(G) is compact and completely disconnected. Also, in the case that G is countable we have shown that LO(G) is metrizable. We wish to show now that the cardinality of LO(G) is either finite or uncountable. If $G = 1$, then there exists exactly one left order.[12] Note if $G \neq 1$ and G is finite LO(G) $= \emptyset$ by Proposition 1.1.7.

One of the most accepted methods of proof in determining the cardinality of a topological set is using a homeomorphism to map the space to another topological space with a well-known cardinality. Thus, we will consider homeomorphisms on the space of left orders. Before we do this we discuss the mappings of ordered sets in greater detail.

Let us define a G-action on LO(G) such that for $g \in G$ and $< \in LO(G)$, we say $1 <_g x^{-1}y \iff 1 < x^{-g}y^g$ for all $x, y \in G$ [12]. Recall that we have previously defined $x^g =$

gxg^{-1}. That is, any left orderable group G comes with an action by homeomorphisms defined above.

Definition 3.1.13. *Let $H \triangleleft G$. We say that H is G-orderable if it admits an order $<$ that is invariant under conjugation. That is, if $1 <_g x^{-1}y \iff 1 < x^{-g}y^g$ for all $x, y \in G$.*

Remark 3.1.14. *Note that $<_{gh} = (<_g)_h$ for all $g, h \in G$.*

Fix $x, y \in G, < \in LO(G)$ and $g, h \in G$. Then note that

$$\begin{aligned} x <_{gh} y &\iff x^{gh} < y^{gh} \\ &\iff (gh)x(gh)^{-1} < (gh)y(gh)^{-1} \\ &\iff ghxh^{-1}g^{-1} < ghyh^{-1}g^{-1} \\ &\iff g(hxh^{-1})g^{-1} < g(hyh^{-1})g^{-1} \\ &\iff (hxh^{-1}) <_g (hyh^{-1} \\ &\iff x\ (<_g)_h\ y, \end{aligned}$$

which is exactly what we wanted to show.

Let $H \leq G$. Then any left ordering on G can be restricted to a left ordering on H.[12]

Proposition 3.1.15. *Let $\rho : LO(G) \to LO(H)$ be defined by $\rho(<) = <_H$, the order $<$ restricted to H, for $< \in LO(G)$. Then ρ is a well-defined continuous map.*

Proof. We wish to show that ρ is a continuous and well-defined. To see that ρ is well-defined fix $<_1, <_2 \in LO(G)$ such that $<_1 = <_2$. That is, for all $g_1, g_2 \in G$, $g_1 <_1 g_2 \iff g_1 <_2 g_2$. Then $\rho(<_1)$ and $\rho(<_2)$ produce orders restricted to $H \leq G$. Thus, since $<_1$ and $<_2$ agree on the larger group, they will certainly agree on the smaller group as well. Thus, ρ is well-defined.

We now wish to show that ρ is continuous. To do this we will consider the inverse map. That is, we will consider ρ^{-1} which takes mappings restricted to H to LO(G). We will consider this by considering where we map the base of H. That is $\rho^{-1}(V(H)_h) = V(G)_h$ for all $h \in H$. [12] But then the inverse image of any open set is open. Thus, we conclude that ρ is continuous. □

Recall that we have previously shown that given a normal convex subgroup N, then we can define an ordering on G/N. We now wish to extend an ordering on a subgroup H of G to G, where H has the special property that it is a convex subgroup of itself. That is, $h_1 \prec x \prec h_2$ for $h_1, h_2 \in H$ if and only if $x \in H$. We will extend an ordering $< \in LO(H)$ to G using the lexicographic ordering. That is, we define $\omega_{\prec}(<) = <^G$ by $g <^G h$ for $g, h \in G$ if and only if $g^{-1}h \in H$ and $h < g$ or $g^{-1}h \notin H$ and $g \prec h$.

Recall that we have previously defined $V(G)$. We now note that $V(G)_g = \{< \in LO(G) : 1 < g\}$.

Proposition 3.1.16. *$\omega_{\prec} : LO(G) \to LO(H)$ is continuous.*

Proof. Let g be a non-identity element of G. We will show that the map is continuous by defining its inverse. Let

$$\omega_{\prec}^{-1}\left(V(G)_g\right) = \begin{cases} \emptyset, & \text{if } g \notin H \text{ and } g \prec 1 \\ LO(H), & \text{if } g \notin H \text{ and } 1 \prec g \\ V(H)_g, & \text{if } g \in H. \end{cases}$$

Thus, we conclude that $\omega_{\prec}$ is continuous. □

It requires a straightforward computation to show that Also $\rho\omega_{\prec}$ is the identity map on $LO(H)$ and $(\prec_H)^G = \prec$. Suppose we add an additional stipulation to our subgroup H. That is, suppose that $H \triangleleft G$. We use this normality to define an order $\overline{\prec}$ on G/H by $H \overline{\prec} gH$ if and only if $1 \prec g$ for $g \in G \setminus H$. Also, for $< \in LO(G/H)$ we can define another left order $\psi_{\prec}(<) = <_l$ by $g <_l h$ if $g^{-1}h \in H$, $1 \prec g^{-1}h$ for $g, h \in G$, and $g <_l h$ if $g^{-1}h \notin H$ and $gH < hH$. Then $\overline{\prec}_l = \prec$.

Proposition 3.1.17. *$\psi_{\prec} : LO(G/H) \to LO(G)$ is continuous.*

Proof. Let g be a non-identity element of G. We will show that $\psi_{\prec}$ is continuous by producing its inverse function. Then note

$$\psi_{\prec}^{-1}\left(V(G)_g\right) = \begin{cases} \emptyset, & \text{if } g \in H \text{ and } g \prec 1 \\ LO(G/H), & \text{if } g \in H \text{ and } 1 \prec g \\ V(G/H)_{gH}, & \text{if } g \notin H. \end{cases}$$

Thus, we conclude that $\psi_{\prec}$ is continuous. □

Note that $\psi_{\prec}$ is injective. This result can be proven directly, and is left to the reader as an exercise. Note that if G is a finitely generated nilpotent group with left order $\prec$, which is isolated in $LO(G)$. If G is not abelian, then there exists a nontrivial proper normal convex subgroup H in G such that either H or G/H is noncyclic. If H is noncyclic, by induction we conclude that $LO(H)$ has no isolated points. Thus, we apply Proposition 3.1.16. If G/H is noncylic, then we can apply Proposition 3.1.17.

Note that the previously mentioned results in conjunction with an understanding of the topology on Hausdorff spaces can be used to prove an extremely useful result.

Theorem 3.1.18. *The space of left orders of a group G is either finite or uncountable. [12]*

This theorem is given without proof, however, a detailed proof is proved in [12] (Theorem 1.3). A direct result of this proof is that no group has a countably infinite number of left orders.

3.2 Groups with Finitely Many Orders

We begin our exploration by considering groups whose space of left orders has a finite number of elements.

Proposition 3.2.1. *If $LO(G)$ is finite, then $|LO(G)|$ is even .*

Proof. Recall that if $<$ is a left order on G, then $<^{-1}$ is also a left order on G. That is, for every order $<$ such that $x < y$, there exists $<^{-1}$ such that $y <^{-1} x$ for $x, y \in G$. Thus, we note that the number of orders must be even. □

Proposition 3.2.1 gives us some intuition as to the cardinality of $\text{LO}(G)$ when $\text{LO}(G)$ is finite. We wish to come up with a sharper estimate. However, to give any such estimate we must know more about our group G.

Theorem 3.2.2. *(Kopytov) If G is a solvable group with finitely many orders, then the number of these orders is either 2 or a multiple of 4.*

Proof. We will provide an exposition of Kopytov's proof as is found in Rhemtulla's book, Orderable Groups [13]. Begin by supposing the G has more than two orders. Then we know by Corollary 1.4.10 that the number of normal convex subgroups for each order is finite and greater than zero. Fix $<\in LO(G)$ and let H be the minimal convex subgroup of $<$ in G.

Then since H is convex, we know that G/H is left orderable. Thus, suppose that $2k_1$ is the number of left orders on G/H, where can assume such a number is even by Proposition 3.2.1. Let $2k_2$ be the number of orders, $<_{g_i}$ invariant under conjugation of H, such that $<_{g_i}$ does not have convex subgroup normal in G. Then the number of orders with H as its minimal normal subgroup is $(2k_1)(2k_2) = 4k_1k_2$. But then we see that the total number of orders of G is a multiple of 4. □

Now we will consider the number of left orders on a specific example, namely, the integers. It is appropriate the first group we consider is $\mathbb{Z}$. This group under multiplication provides a basic example of a group whose space of left orders is relatively well understood, and whose proof is easy to follow.

Theorem 3.2.3. *There are only two distinct left orders on $\mathbb{Z}$.*

Proof. Let $<_i\in LO(G)$. Note that we have two possible ways of comparing the distinct elements 0 and 1. Namely, for $<_i\in LO(G)$, $0 <_i 1$ or $1 <_i 0$. We must show that these orders extend to orderings on all of the elements of $\mathbb{Z}$.

Let us begin by considering the former case. Fix $<_1\in LO(G)$ such that $0 <_1 1$. Now let us begin by considering $n \in \mathbb{Z}^+$. Note that $0 <_1 n$ for all $n \in \mathbb{Z}^+$. The analogous condition for $0 <_1 1$ is $-1 <_1 0$. Then note $-n <_1 0$ for $n \in \mathbb{Z}^+$. But this analogous to saying $n <_1 0$ for $n \in \mathbb{Z}^-$. Thus, we have extended $<_1$ to an ordering on $\mathbb{Z}$.

Now consider $<_2 \in LO(G)$ such that $1 <_2 0$. Then we note that $n <_2 0$ and $0 <_2 -n$ for $n \in \mathbb{Z}^+$. Thus, we can extend $<_2$ to an ordering on $\mathbb{Z}$. Therefore, we conclude that there are two distinct orders on $\mathbb{Z}$. □

3.3 The Space of Left Orders on $\mathbb{Z}^n$

Our next task is to discern a topology on the space of left orders of $\mathbb{Z}^n$. The work in the following section is based on Adam Sikora's paper "Topology on the Spaces of Orderings of Groups." The theorems credited to Sikora come from his article and all proofs are expositions of his work. [15]

Note that we have previously shown in the first section of this chapter that LO(G) is totally disconnected and compact. We still need to show that LO(G) is perfect. That is, we must show that LO(G) contains its limit points. Note that this is the same as condition (ii) of the corollary statement below.

Corollary 3.3.1. *(Sikora) LO(G) is homeomorphic to the Cantor set if and only if*
i. $LO(G) \neq \emptyset$
ii. any sequence $x_1, y_1, .., x_n, y_n$ of elements of G, the set $U_{x_1,y_1} \cap U_{x_2,y_2} \cap ... \cap U_{x_n,y_n}$ is either empty or infinite.

Proof. We have previously shown that LO(G) is compact, totally disconnected and metrizable. Thus, by Corollary 2.2.8 we know that if LO(G) is nonempty and perfect, then it is homeomorphic to the Cantor set. But condition (ii) says that LO(G) is perfect. □

Now we have the machinery to show that LO(G) is homeomorphic to the Cantor set. That is, we know that for countable G, LO(G) is metrizable, compact, and totally disconnected.

We wish to use these criteria to show that LO($\mathbb{Z}^n$) for $n > 1$ is homeomorphic to the Cantor set. For the construction of such a proof it is necessary for us to use topological concepts that have remained previously undiscussed in our paper. Thus, we will begin by looking at hyperplanes of $\mathbb{R}^n$.

Definition 3.3.2. *Consider the vector space $\mathbb{R}^n$. Suppose that $\boldsymbol{n}$ and $\boldsymbol{x}$ are vectors in $\mathbb{R}^n$ such that $\boldsymbol{n} \neq 0$. The set of all vectors $\boldsymbol{y} \in \mathbb{R}^n$ such that*

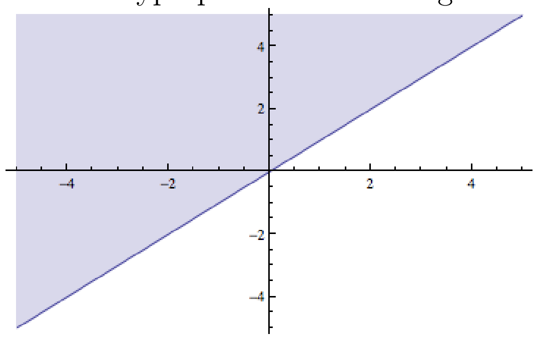
$$\boldsymbol{n} \cdot (\boldsymbol{y} - \boldsymbol{x}) = 0,$$

where $\cdot$ is the dot product in n-dimensional space, is called the hyperplane through point $\boldsymbol{x}$. We call $\boldsymbol{n}$ the normal vector for the hyperplane.[16]

In $\mathbb{R}^k$ a hyperplane is a subset of dimension $k-1$. Thus, in $\mathbb{R}^2$ a hyperplane is a line and in $\mathbb{R}^3$ a hyperplane is a plane. We now consider some examples in two and three-dimensional space.

Example 3.3.3. *We wish to consider the hyperplane through the origin in $\mathbb{R}^2$. Then we note that we have a line through the origin. We are considering the set of vectors $\boldsymbol{y} \in \mathbb{R}^2$ such that $\boldsymbol{n} \cdot \boldsymbol{y} = 0$. That is, $(n_1, n_2) \cdot (y_1, y_2) = n_1y_1 + n_2y_2 = 0$. If we fix $(n_1, n_2) = (1, -1)$ then the set of points that make up our hyperplane is of the form (y_1, y_1). That is, we get the line $y = x$.*

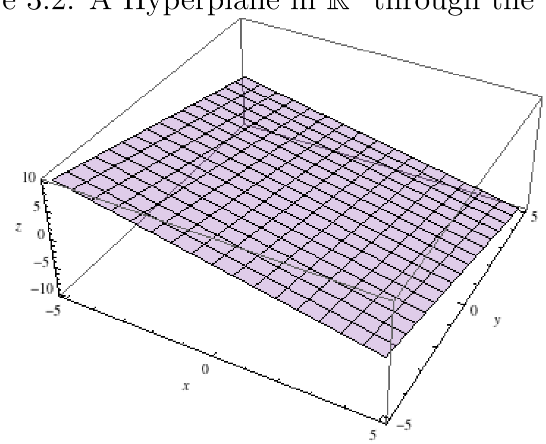
Figure 3.1: A Hyperplane in $\mathbb{R}^2$ through the Origin



Then note that since $y = x$, we conclude that $y - x = 0$. The points to the left of the line, shaded in the above graph, are such that $y - x > 0$ and all points to the right of the line are such that $y - x < 0$.

Suppose we wish to construct a hyperplane through the origin in $\mathbb{R}^3$. Then we have a plane through the origin. We are considering the set of vectors $\boldsymbol{y} \in \mathbb{R}^2$ such that $\boldsymbol{n} \cdot \boldsymbol{y} = 0$. That is, $(n_1, n_2, n_3) \cdot (y_1, y_2, y_3) = n_1y_1 + n_2y_2 + n_3y_3 = 0$. Let $\boldsymbol{n} = (n_1, n_2, n_3) = (1, 1, 1)$. We get the plane $x + y + z = 0$.

Figure 3.2: A Hyperplane in $\mathbb{R}^3$ through the Origin



Note that for points above the plane, $x+y+z > 0$ and for points below the plane, $x+y+z < 0$.

Consider $P^+ \subset \mathbb{R}^n$ such that $P^+ = \{x \in \mathbb{R}^n : x > 0\}$. Then note that P^+ is a convex set. Similarly, $P^- = \{x \in \mathbb{R}^n : x < 0\}$ is also a convex set. Note that $P^+ \cap P^- = \emptyset$. We want to show that P^+ and P^- are separated by a hyperplane. That is, $\mathbb{R}^n \setminus (P^+ \cup P^-) = H$ is a hyperplane.

Theorem 3.3.4. *(Geometric Version of Hahn-Banach Theorem) Let K be a nonempty open convex set in a real topological space X, such that the operations of addition and scalar multiplication are continuous operations in X. Let L be a linear variety that does not intersect K. That is, let L be a subset of X such that $L = x_0 + L_0$ for some fixed vector and L_0 is a subspace of X, that does not intersect K. Then there exists a closed hyperplane M that contains L and is such that K lies strictly on one side of M.*

We state this without proof, however, a detailed proof of this theorem can be found in Lay and Taylor's book Functional Analysis (Theorem 2,4, page 127).[9] The direct consequence of the above theorem is the following fact.

Remark 3.3.5. *If K_1 and K_2 are nonempty, nonintersecting convex sets and if K_1 is open, there exists a closed hyperplane M such that K_1 is one of the two closed half spaces determined by M and K_2 is in the other. If K_2 is also open, M can be chosen so that K_1 and K_2 are strictly on opposite sides of M.[9]*

We will give an idea of how to prove the above fact. Let K_1 and K_2 be as described above. That is, K_1 and K_2 are convex and disjoint. Then $K = K_1 - K_2 = \{k = k_1 - k_2 : k_1 \in K_1, k_2 \in K_2\}$ is a non-empty convex set. Also, since K_1 is open $K = K_1 - K_2$ is open. Finally, note that $K_1 \cap K_2 = \emptyset$ implies that there does not exist $k_1 \in K_1$ and $k_2 \in K_2$ such that $k_1 - k_2 = 0$. Thus, $0 \notin K = K_1 - K_2$. So there exists a closed hyperplane through 0 and not intersecting K_2. It follows that there exists a hyperplane separating $K_1 - K_2$ from 0. We translate this hyperplane to M.

We now have all the tools necessary to show that LO(G) is homeomorphic to the Cantor set.

Theorem 3.3.6. *(Sikora) For $n > 1$, LO$(\mathbb{Z}^n)$ is homeomorphic to the Cantor set.*

Proof. Suppose not. We will provide a detailed exposition of Sikora's proof. That is, for contradiction, suppose there exists some N such that $N > 1$ and LO$(\mathbb{Z}^N)$ is not homeomorphic to the Cantor set.

We begin with case where N is finite. Then by Corollary 3.3.1 we note that since LO$(\mathbb{Z}^N) \neq \emptyset$, for $x_1, y_1, ..., x_k, y_k \in \mathbb{Z}^N \times \mathbb{Z}^N$, $U_{x_1,y_1} \cap U_{x_2,y_2} \cap ... \cap U_{x_k,y_k}$ is neither empty nor infinite. We conclude that there exists a finite set of pairs $(x_1, y_1), ..., (x_k, y_k) \in \mathbb{Z}^N \times \mathbb{Z}^N$ such that the number of orderings on $\mathbb{Z}^N$ such that $z_i = y_i - x_i > 0$ for $i = 1, 2, ..., k$ is positive and finite. We can assume there is only one such order $<$, where $z_i > 0, i = 1, 2, .., s$, by adding pairs $(x_i, y_i) \in \mathbb{Z}^N \times \mathbb{Z}^N$ $k < i \leq s$, if necessary, to guarantee uniqueness.

We can assume that $y_j - x_j$ is not a rational multiple of $y_i - x_i$ for $i \neq j$. This is because if it were, we would have $py_i - px_i = qy_j - qx_j$ for $p, q \in \mathbb{Z}$, and consequently,

$$\begin{array}{rcccll} & 1 & < & (py_i)^{-1}(px_i) & = & (qy_j)^{-1}(qx_j) \\ \Longrightarrow & 1 & < & p^{-1}y_i^{-1}px_i & = & q^{-1}y_j^{-1}qx_j \\ \Longrightarrow & 1 & < & p^{-1}py_i^{-1}x_i & = & q^{-1}qy_j^{-1}x_j, \qquad \text{since } p, q \in \mathbb{Z} \text{ are scalars} \\ \Longrightarrow & 1 & < & y_i^{-1}x_i & = & y_j^{-1}x_j. \end{array}$$

We wish to extend our ordering $<$ on the subset of $\mathbb{Z}^N$ to an ordering on $\mathbb{Q}^N$. We can do this by defining $v_1 < v_2$ for $v_1, v_2 \in \mathbb{Q}^N$ if and only if $kv_1 < kv_2$ for all $k \in \mathbb{Z}$ such that $kv_1, kv_2 \in \mathbb{Z}^N$. We can do this since for $kv_1 = k\left(\frac{p_1}{q_1}, ..., \frac{p_N}{q_N}\right) \in \mathbb{Z}^N, p_1, q_i \in \mathbb{Z}$ when $k = \text{lcm}(q_1, q_2, .., q_N)$.

Define $H \subset \mathbb{Q}^N \otimes \mathbb{R}$, where we note that $\mathbb{Q}^N \otimes \mathbb{R} = \mathbb{R}^N$, such that for all $x \in H$ every neighborhood of x contains both positive and negative elements. That is, for $\epsilon > 0$, $B_\epsilon(x)$ contains positive and negative elements of $\mathbb{R}^N$. We should note for the sake of clarity that we are considering these neighborhoods with respect to the Euclidean topology on $\mathbb{R}^N$. We also clarify that a vector v is considered positive if and only if $v > 0$. That is, if $0 < v \in \mathbb{R}^N$, then every coordinate of v is positive. Note that H is a closed subspace of $\mathbb{R}^N$, since by construction it contains its limit points.

We now wish to consider $\mathbb{R}^N \setminus H$, which we know to have two connected components. Note that these two components are $H^+ = \{x \in \mathbb{R}^N : x > 0\}$ and $H^- = \{x \in \mathbb{R}^N : x < 0\}$. Consider (x_i, y_i) such that $1 < x_i^{-1}y_i$. Then $y_i - x_i > 0$, so $y_i - x_i \in H^+$ or $y_i - x_i \in H$.

Consider $H^+ \setminus H$ and $H^- \setminus H$. We wish to show that these sets are open and convex. The former characterization comes from the fact that H is a closed set and consequently its complement is open. The convexity of the sets must be proven. We know that H^+ and H^- are convex.

We will now show that $H^+ \setminus H$. Fix $x \in H^+ \setminus H$. Then $B_\epsilon(x) > 0$. Every neighborhood around $B_\epsilon(x)$ contains only positive elements. It should be clear that we can choose $y_1, y_2 \in B_\epsilon(x)$ such that $y_1 \leq x \leq y_2$ under the lexicographic ordering. Now suppose that y is an element such that $x_1 \leq y \leq x_2$ for $x_1, x_2 \in H^+ \setminus H$. Then x_1, x_2 are positive elements that contain no negative elements in any neighborhood. Thus, y is a positive element such that no neighborhood contains negative elements. Therefore, $y \in H^+ \setminus H$. But then we have shown that $H^+ \setminus H$ is convex. The argument for $H^- \setminus H$ is similar.

By the previous theorem and remark we know that since $H^+ \setminus H$ and $H^- \setminus H$ are open, convex and disjoint we can conclude that there is a hyperplane H' separating them. Notice that $H' \subseteq H$ by construction, and so H is a hyperplane. But then either H' and H have codimension one, that is $\dim(H) - \dim(H') = 1$, or $H = \mathbb{R}^N$. But we know that $H \neq \mathbb{R}^N$ by construction. Thus, we conclude that $H = H'$.

Let I denote the set of indices such that $y_i - x_i \in H$. Consider $H \cap \mathbb{Z}^N$. Note that $H \cap \mathbb{Z}^N$ is composed of $x \in \mathbb{Z}^N$ such that in a neighborhood of x there are both positive and negative points. Then we note that $<$ is the only order for which $1 < x_i^{-1}y_i$ for $i \in I$. Otherwise, this would contradict the uniqueness of $<$.

We note that since the number of left orders that satisfy $z_i = y_i - x_i > 0$ on $H \cap \mathbb{Z}^N$ is finite, (namely one, $<$), we have failed to satisfy condition (ii) of Corollary 3.3.1. Note that $H \cap \mathbb{Z}^N = \mathbb{Z}^k$ for $k < N$. Since for $N > k > 1$, $LO(\mathbb{Z}^k)$ is homeomorphic to the Cantor set and satisfies Corollary 3.3.1(ii), we can conclude that either $H \cap \mathbb{Z}^N = 0$ or $H \cap \mathbb{Z}^N = \mathbb{Z}$ by our assumption on the minimality of N. Note that if $H \cap \mathbb{Z}^N = 0$, then we have that $I = \emptyset$. This implies that $y_i - x_i \notin H$ for any i. That is, $y_i - x_i \in H^+$ for all values of i. Then by perturbing H slightly we can create other hyperplanes $H_j \subseteq \mathbb{R}^N$, since we have a great deal of freedom in choosing H to satisfy the condition that every element of H_j contains both positive and negative elements. Note that for each H_j we can define an ordering $<_i$ on $\mathbb{Q}^N$ such that $1 <_i x_i^{-1}y_i$ for $i = 1, ..., k$.

Thus, $H \cap \mathbb{Z}^N = \mathbb{Z}$. But recall that none of the vectors $y_i - x_i$ is a rational multiple of another. Suppose $y_k - x_k \in H$. Then since H is a hyperplane through the origin it satisfies the equation $n \cdot (y_k - x_k) = 0$ for some normal vector n. But the only way that another element of $\mathbb{Z}^N$ satisfies the above equation is if it is rational multiple of $y_k - x_k$. Thus, $y_k - x_k$ is the only vector in H. But then once again we can see with slight perturbations to our hyperplane H, we can create infinitely many hyperplanes H_i such that $y_k - x_k$ lies in the same component of $\mathbb{R}^N \setminus H_i$ as the other vectors $y_i - x_i$. Thus, each of these hyperplanes induces an ordering $<_i$ such that $1 <_i x_i^{-1}y_i$ for $i = 1, .., k$. But then we have a contradiction, so we can conclude that $\mathrm{LO}(\mathbb{Z}^n)$ is homeomorphic to the Cantor set for all $n > 1$.

Now we consider the case where N is infinite. Suppose $<$ is an isolated point in $\mathrm{LO}(\mathbb{Z}^N)$. Then there exist finitely many $z_i = y_i - x_i$ in G such that $<$ is the only order satisfying $z_i > 0$ for all i. Then there exist m finite such that z_i is in $\mathbb{Z}^m$ for all i. We restrict $<$ to an order on $\mathbb{Z}^m$, $<_{\mathbb{Z}^m}$. Then $<_{\mathbb{Z}^m}$ is the only order on $\mathbb{Z}^m$ satisfying $z_i > 0$ for all i, and consequently is an isolated point in $\mathrm{LO}(\mathbb{Z}^m)$. □

Corollary 3.3.7. *(Sikora) The set of left orders on $\mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to the Cantor set.*

Proof. This statement follows directly from the previous theorem. □

3.4 The Space of Left Orders on $\mathbb{Q}^* \times \mathbb{Q}^*$

Recall in the previous section we showed that $\mathrm{LO}(\mathbb{Z}^n)$ is homeomorphic to the Cantor set for $n > 1$. We now wish to consider $\mathbb{Q}^* \times \mathbb{Q}^*$, that is, the direct product of positive rational numbers with itself.

Proposition 3.4.1. $\mathbb{Q}^* \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}$.

Proof. Every element of x of $\mathbb{Q}^*$ has a unique prime factorization $x = p_1^{n_1}p_2^{n_2}...p_m^{n_m}$ for p_i prime, $n_i \in \mathbb{Z}$. Let $\theta : \mathbb{Q}^* \to \bigoplus_{n=1}^{\infty} \mathbb{Z}$ be defined by $\theta(p_1^{n_1}p_2^{n_2}...p_m^{n_m}) = (n_1, n_2, .., n_m, 0, 0, 0, ...)$. We wish to show that θ is an isomorphism.

We must first show that θ preserves the group operation. Fix $x_1, x_2 \in \mathbb{Q}^*$. Then we can write $x_1 = p_1^{n_1}p_2^{n_2}...p_m^{n_m}, x_2 = p_1^{s_1}p_2^{s_2}...p_m^{s_m}$, for p_i prime and $s_i, n_i \in \mathbb{Z}$. We can assume that both elements of $\mathbb{Q}^*$ contain all values of $p_i, i \in \{1, 2, .., m\}$ in their factorization since we have allowed that n_i and s_i may be 0.

Note that $\mathbb{Q}^*$ is a group under multiplication and $\bigoplus_{n=1}^{\infty} \mathbb{Z}$ is a group under addition. Thus, we wish to show that $\theta(x_1x_2) = \theta(x_1) + \theta(x_2)$. Then note that

$$\begin{aligned}
\theta(x_1) + \theta(x_2) &= \theta(p_1^{n_1}p_2^{n_2}...p_m^{n_m}) + \theta(p_1^{s_1}p_2^{s_2}...p_m^{s_m}) \\
&= (n_1, n_2, ..., n_m, 0, 0, ...) + (s_1, s_2, .., s_m, 0, 0, ...) \\
&= (n_1 + s_1, n_2 + s_2, .., n_m + s_m, 0, 0...) \\
&= \theta(p_1^{n_1+s_1}p_2^{n_2+s_2}...p_m^{n_m+s_m}) \\
&= \theta((p_1^{n_1}p_2^{n_2}...p_m^{n_m})(p_1^{s_1}p_2^{s_2}...p_m^{s_m})).
\end{aligned}$$

Thus, θ preserves the group operation and we can conclude that θ is a group homomorphism.

We now wish to show that θ is one-to-one. Fix $x_1, x_2 \in \mathbb{Q}^*$ and suppose that $\theta(x_1) = \theta(x_2)$. We wish to show that $x_1 = x_2$. We will assume that x_1 and x_2 have the prime factorizations designated in the above proof of closure under the mapping θ. Note

$$\begin{aligned}
& & \theta(x_1) &= \theta(x_2) \\
&\implies & \theta(p_1^{n_1}p_2^{n_2}...p_m^{n_m}) &= \theta(p_1^{s_1}p_2^{s_2}...p_m^{s_m}) \\
&\implies & (n_1, n_2, ..., n_m, 0, 0, ...) &= (s_1, s_2, ..., s_m, 0, 0, ...) \\
&\implies & n_i &= s_i \text{ for all } i
\end{aligned}$$

But then by the uniqueness of prime factorization, we can conclude that $x_1 = x_2$. Thus, θ is one-to-one.

Finally, we wish to show that θ is onto. Fix $x = (a_1, a_2, .., a_m, ...) \in \bigoplus_{n=1}^{\infty} \mathbb{Z}$. That is, define x as an infinite integer vector with coordinates $a_i \in \mathbb{Z}$, where a_i denotes the ith coordinate. Then note that $p_1^{a_1}p_2^{a_2}....p_m^{a_m}... \in \mathbb{Q}^*$. We can find such primes, since there exist an infinite number of primes in $\mathbb{Z}$. Thus, $\theta(p_1^{a_1}p_2^{a_2}....p_m^{a_m}..) = (a_1, a_2, .., a_m, ...)$, so we conclude that θ is onto. □

Now we wish to consider the direct product $\mathbb{Z}^\infty \oplus \mathbb{Z}^\infty$.

Proposition 3.4.2. *There exists a bijective mapping from* $\bigoplus_{n=1}^{\infty} \mathbb{Z}$ *to* $\mathbb{Z}^\infty \oplus \mathbb{Z}^\infty$.

Proof. Let $\phi : \bigoplus_{n=1}^{\infty} \mathbb{Z} \to \mathbb{Z}^\infty \oplus \mathbb{Z}^\infty$ be defined by $\phi(a_1, a_2, a_3, ...) = ((a_1, a_3, a_5, ..), (a_2, a_4, a_6, ...))$. We wish to show that θ is a group homomorphism. Fix $(a_1, a_2, a_3, ...), (b_1, b_2, b_3, ..) \in \bigoplus_{n=1}^{\infty} \mathbb{Z}$.

Note that

$$\begin{aligned}
\theta((a_1,a_2,a_3,...)+(b_1,b_2,b_3,..)) &= \theta((a_1+b_1,a_2+b_2,a_3+b_3,...)) \\
&= ((a_1+b_1,a_3+b_3,...),(a_2+b_2,a_4+b_4,...)) \\
&= ((a_1,a_3,a_5,..),(a_2,a_4,a_6,..))+((b_1,b_3,b_5,...),(b_2,b_4,b_6,...)) \\
&= \theta(a_1,a_2,a_3,a_4,..)+\theta(b_1,b_2,b_3,b_4,..)
\end{aligned}$$

So θ preserves addition. Thus, θ is a group homomorphism.

We wish to show that θ is one-to-one. Fix $(a_1,a_2,a_3,...),(b_1,b_2,b_3,..) \in \bigoplus_{n=1}^{\infty}\mathbb{Z}$ such that $\theta((a_1,a_2,a_3,...)) = \theta((b_1,b_2,b_3,..))$. Then note that

$$\begin{aligned}
\theta((a_1,a_2,a_3,...)) &= \theta((b_1,b_2,b_3,..)) \\
\Longrightarrow \quad ((a_1,a_3,a_5,...),(a_2,a_4,..)) &= ((b_1,b_3,b_5,...),(b_2,b_4,...))
\end{aligned}$$

But then $(a_1,a_3,a_5,...) = (b_1,b_3,b_5,..)$ and $(a_2,a_4,a_6,...) = (b_2,b_4,b_6,...)$. Thus, we can conclude that $a_i = b_i$ for all i. Thus, the inputs are the same, which is what we wanted to show. Therefore, θ is one-to-one.

Finally, we must show that θ is onto. Fix $((a_1,a_3,a_5,....),(a_2,a_4,a_6,...)) \in \mathbb{Z}^\infty \oplus \mathbb{Z}^\infty$. Then $\theta((a_1,a_2,a_3,a_4,...)) = ((a_1,a_3,a_5,....),(a_2,a_4,a_6,...))$ for $(a_1,a_2,a_3,a_4,...) \in \bigoplus_{n=1}^{\infty}\mathbb{Z}$. Thus, we can conclude that θ is onto. Thus, there exists a bijective mapping from $\bigoplus_{n=1}^{\infty}\mathbb{Z}$ to $\mathbb{Z}^\infty \oplus \mathbb{Z}^\infty$. □

Theorem 3.4.3. *The set of left orders on $\mathbb{Q}^* \times \mathbb{Q}^*$ is isomorphic to the Cantor set.*

Proof. Note that by Proposition 3.4.1 $\mathbb{Q}^* \cong \bigoplus_{n=1}^{\infty}\mathbb{Z}$. Then we apply Proposition 3.4.2 and note that we may conclude that $\mathbb{Q}^* \times \mathbb{Q}^* \cong \mathbb{Z}^\infty \oplus \mathbb{Z}^\infty \cong \bigoplus_{n=1}^{\infty}\mathbb{Z}$. Recall from Theorem 3.3.6 that $\mathrm{LO}(\bigoplus_{n=1}^{\infty}\mathbb{Z})$ is homeomorphic to the Cantor set. Thus, we conclude that $\mathrm{LO}(\mathbb{Q}^* \times \mathbb{Q}^*)$ is homeomorphic to the Cantor set. □

The results proven in this chapter have given us a foundation to begin considering the space of left orders of more interesting groups. In the next chapter we use use the results on cardinality proven in this chapter to infer the cardinality of various matrix groups, including the Heisenberg group.

Chapter 4

Matrix Groups whose Space of Left Orders is Homeomorphic to the Cantor Set

Perhaps, even more valuable then the results of this section, is the method by which these results are obtained. As seen previously, it is often challenging to create a method of proof by which the set of left orders on a group is shown to be uncountable. Most mathematicians do this by proving that no order is isolated. That is, they shown that the set of left orders on a group is isomorphic to the Cantor set.

We wish to construct a method of showing that the set of orders on an matrix groups, such as the Heisenberg group, is uncountable. The method by which we plan on showing such is via constructing an isomorphism from our set of matrices to a group on which we have previously shown the topology of left orders to be homeomorphic to the Cantor Set.

The idea behind this construction is to create a method, which may prove useful as a means of proof for numerous groups, with special consideration on groups of matrices. This would prove not only beneficial for our own results in this paper, but for others to use this method of proof in the future as a simplified means of determining if the set of left orders on a group is uncountable. Although this is our endgame we must first set up the tools with which we may construct this proof.

Many of our proofs will be based on extending orders on groups to a larger group. For example, given a group G with normal subgroup H, we wish to be able to extend an ordering on H to an ordering on the group G.

We also wish to consider a group G with normal subgroup H, where we can place on ordering on the quotient group G/H. Then given an ordering on G/H and a separate ordering on H, we wish to use the lexicographic ordering to place an ordering on G.

Theorem 4.0.4. *Let* $H \triangleleft G$*. If*
(i) $<_r \in LO(G/H)$*, converges to an order* $<$ *on* G/H*, and*
(ii) $\ll_r \in LO(H)$*, converges to an order* $\ll$ *on* H*,*
then $(<_r, \ll_r)$ *converges to the order* $(<, \ll)$*.*

Proof. Given that the set of orders on G/H, $<_r$, converges to $<$, we know that there exists $N_1 \in \mathbb{Z}^+$ such that for all $r > N_1$, $H <_r Hg$ for all $g \in G$. We also now that since the set of orders $\ll_r$ on H converges to $\ll$, there exists $N_2 \in \mathbb{Z}^+$ such that for all $r > N_2$, $1 \ll_r h$ for all $h \in H$. Let $N = \max\{N_1, N_2\}$. Then for all $r > N$, $H <_r Hg$ if $g \notin H$, and $1 \ll_r g$ if $g \in H$. But then we have shown that $(<_r, \ll_r)$ converges to $(<, \ll)$. □

4.1 Heisenberg Group

We now turn our focus to a specific matrix group, namely the Heisenberg group, which is often used in mathematics due to its unique structure.

Definition 4.1.1. *The Heisenberg group is a group of upper triangular* 3×3 *matrices of the form*

$$H_3(D) = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},$$

where x, y, z *are elements of an abelian ring* D*.*

The most commonly used domains over which the Heisenberg group is considered are the abelian rings $\mathbb{Z}$ and $\mathbb{R}$.

The Heisenberg group is a nonabelian, infinte nilpotent group.

Proposition 4.1.2. *All elements of* $H_3(D)$ *are invertible.*

Proof. The Heisenberg group consists upper triangular matrices with ones on the diagonal. We know that for a matrix to be invertible, it must have nonzero determinant. We also know that the determinant of an upper triangular matrix is the product of the elements on its diagonal. Since $\det(h) = 1$ for all $h \in H_3(D)$, we may conclude that all matrices in the Heisenberg group are invertible. □

We wish to derive a normal subgroup of $G = H_3(D)$. Perhaps, the most obvious choice is the center of our group, $Z(G)$. Since the center of G is always a normal subgroup of G we need not test normality. We must only ascertain the form of the elements in the center of G.

Proposition 4.1.3. *The center of* $G = H_3(D)$ *is*

$$H = \left\{ \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, w \in D \right\}$$

where D *is an abelian ring.*

Proof. We wish to show that $Z(G) = H$, where H is the group indicated above. We will do this through double set inclusion.

Begin by fixing $h \in H$. Then $h = \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for some $w \in D$. Note that

$$\begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x & w+z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x & z+w \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

since by definition D is an abelian ring.

Now fix $z \in Z(G)$. Then note that $zg = gz$ for all $g \in G$. That is, $z = gzg^{-1}$ for all $g \in G$, since we know all elements of $G = H_3(D)$ are invertible. Then let $z \in Z(G)$ and $g \in G$,

$$\begin{aligned} z &= gzg^{-1} \\ &= \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -x & -z+xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -x & -z+xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a & -ya+c+bx \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

But since we know $-ya+c+bx = c$, it must be that $-ya+bx = 0$. Then for this expression to be true for arbitrary values of y and x it must be that $a = c = 0$. But then $z \in H$.

Thus, we have shown through double set inclusion that $H = Z(G)$. □

Thus, $H \triangleleft G$. However, we can say nothing about the set of orders on G/H, so we wish to show that G/H is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. We will use this relationship later on to discuss the space of left orders on G, as we know from chapter three that the set of orders on $\mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to the Cantor set.

Proposition 4.1.4. *The quotient group $G/H \cong \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. Let $\theta : G \to \mathbb{Z} \oplus \mathbb{Z}$ be defined by $\theta\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\right) = (a, c)$. We must show that this is a group homomorphism. Then fix $\begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{bmatrix} \in G$. Note that

$$\begin{aligned} \theta\left(\begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{bmatrix}\right) &= \theta\left(\begin{bmatrix} 1 & a_1 + a_2 & b_2 + a_1 c_2 + b_1 \\ 0 & 1 & c_1 + c_2 \\ 0 & 0 & 1 \end{bmatrix}\right) \\ &= (a_1 + a_2, c_1 + c_2) \\ &= \theta\left(\begin{bmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}\right) + \theta\left(\begin{bmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{bmatrix}\right) \end{aligned}$$

Thus, θ is a group homomorphism.

Fix $(a, c) \in \mathbb{Z} \oplus \mathbb{Z}$. Then note that $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in G$ and $\theta\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\right) = (a, c)$. Thus, we may conclude that θ is onto.

Finally, let us consider $\ker(\theta)$. Note that

$$\begin{aligned} \ker(\theta) &= \left\{\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in H : \theta\left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\right) = 0\right\} \\ &= \left\{\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in H : (a, c) = (0, 0)\right\} \\ &= \left\{\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = H \end{aligned}$$

Then if we apply the Fundamental Homomorphism Theorem, we can note that $G/H \cong \mathbb{Z} \oplus \mathbb{Z}$. □

Proposition 4.1.5. *$H \cong \mathbb{Z}$.*

Proof. Let $\psi : H \to \mathbb{Z}$ be defined by $\psi\left(\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = b$.

Fix $\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & b_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H$. Then note that

$$\begin{aligned} \psi\left(\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & b_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) &= \psi\left(\begin{bmatrix} 1 & 0 & b_1+b_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\ &= b_1 + b_2 \\ &= \psi\left(\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) + \psi\left(\begin{bmatrix} 1 & 0 & b_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \end{aligned}$$

Thus, ψ is a group homomorphism.

We wish to show that ψ is onto. Fix $b \in \mathbb{Z}$. Then note that $\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H$ and $\psi\left(\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = b$. Thus, ψ is onto.

Now we wish to consider $\ker(\psi)$. Then note that

$$\begin{aligned} \ker(\psi) &= \left\{\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H : \psi\left(\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0\right\} \\ &= \left\{\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H : b = 0\right\} \\ &= \left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = 1 \end{aligned}$$

Then we may conclude from Fundamental Homomorphism Theorem that

$$H \cong \mathbb{Z}.$$

□

Thus, we have shown that

$$G \cong \frac{G/H}{H} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z}}.$$

We now wish to consider the convex hull of $Z(G) = H$. Recall that the since $H \triangleleft G$, we know by Lemma 1.4.16 that $C = \text{hull}_G(H)$ is a subgroup of G.

Note that we may conclude that $C \neq G$ by Theorem 1.4.15. We also observe that $Z(G) = H \subseteq C$. We know that $\frac{G}{Z(G)} = \frac{G}{H}$ is abelian, since we have shown that $G/H \cong \mathbb{Z} \oplus \mathbb{Z}$. Therefore, we conclude that $H \leq C \leq G$, where $C/H \triangleleft G/H$ by subgroup correspondence theorem. Thus, we may also use the subgroup correspondence theorem to conclude that $C \triangleleft G$.

However, recall that if a subgroup of G is normal and convex in G, then its quotient with G is left orderable by Theorem 1.4.16. Thus, we are now ready to show that the set of left orders on the Heisenberg group is isomorphic to the Cantor set.

Theorem 4.1.6. *The set of orders on G is isomorphic to the Cantor set.*

Proof. Recall from chapter three that the set of left orders on $\mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to the Cantor set. Thus, we must only prove that the set of left orders on G is isomorphic to the set of left orders on $\mathbb{Z} \oplus \mathbb{Z}$. We must consider two cases. That is, we must consider the case where $C = H$ and the case where this is not true.

We begin be supposing that we have the former case. That is, suppose that $C = H$. Then $G/H = G/C$. Recall that $G/H \cong Z \oplus Z$. Thus, we have previously shown by Theorem 4.0.4 that given a set of orders on the quotient group G/C, we can extends the ordering to G. But we know that the set of orders on the group $\mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to the Cantor set. Thus, the set of orders on G/C, which we extend to a set of orders on G, must also be isomorphic to the Cantor set. Thus, we may conclude that the set of orders on the Heisenberg group is isomorphic to the Cantor set.

We now suppose that $C \neq H$. Then since we know that $G/Z \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H \subset C \subset G$, that $G/C \cong \mathbb{Z}$. Then we can also conclude that $C/H \cong \mathbb{Z}$. Recall that we have shown that $H \cong \mathbb{Z}$. But then we know that $C \cong \mathbb{Z} \oplus \mathbb{Z}$ by the second isomorphism theorem. Thus, we can conclude that the set of orders on C are isomorphic to the Cantor set. By Theorem 4.0.4 we can extend the ordering on C to an ordering on G. Thus, we conclude that the space of left orders on G is isomorphic to the Cantor set. □

4.2 Future Work on Matrix Groups of Dimension Two

We will begin our task of determining the topology of the left order on matrix groups of dimension two. All of our matrix groups are upper triangular matrices.

4.2.1 Left Order Topology Isomorphic to $\mathbb{Q}^* \times \mathbb{Q}^*$

We will begin by considering

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, c \in \mathbb{Q}^*, b \in \mathbb{Q} \right\} \text{ and } H = \left\{ \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} : d \in \mathbb{Q} \right\}.$$

Proposition 4.2.1. $H \triangleleft G$

Proof. We wish to show that H is a normal subgroup of G. To prove that $H \triangleleft G$ we must show that $ghg^{-1} \in H$ for all $g \in G, h \in H$. Fix $h = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \in H$ and $g = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in G$.

We must begin by finding g^{-1}. Then note that $\det\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = ac$. Thus,

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^{-1} = \frac{1}{ac}\begin{bmatrix} c & -b \\ 0 & a \end{bmatrix} = \begin{bmatrix} \frac{c}{ac} & -\frac{b}{ac} \\ 0 & \frac{a}{ac} \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} ghg^{-1} &= \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{bmatrix} \\ &= \begin{bmatrix} a & ad+b \\ 0 & c \end{bmatrix}\begin{bmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\frac{b}{c} + \frac{ad+b}{c} \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

where $-\frac{b}{c} + \frac{ad+b}{c} \in \mathbb{Q}$. But then $ghg^{-1} \in H$. But g was an arbitrary element of G and h was an arbitrary element of H. We conclude $ghg^{-1} \in H$ for all $g \in G, h \in H$. So $H \triangleleft G$. □

Thus, we have found a matrix group with a normal subgroup such that $H \triangleleft G$.

Proposition 4.2.2. $H \cong \mathbb{Q}$

Proof. We want to show that $H \cong \mathbb{Q}$. Define $\theta : H \to \mathbb{Q}$ by $\theta\left(\begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}\right) = q$. Now we must show that θ is a group homomorphism. Fix $\begin{bmatrix} 1 & q_1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & q_2 \\ 0 & 1 \end{bmatrix} \in H$. Note that

$$\theta\left(\begin{bmatrix} 1 & q_1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & q_2 \\ 0 & 1 \end{bmatrix}\right) = \theta\left(\begin{bmatrix} 2 & q_1+q_2 \\ 0 & 2 \end{bmatrix}\right) = q_1+q_2 = \theta\left(\begin{bmatrix} 1 & q_1 \\ 0 & 1 \end{bmatrix}\right) + \theta\left(\begin{bmatrix} 1 & q_2 \\ 0 & 1 \end{bmatrix}\right).$$

Thus, θ preserves addition and is a group homomorphism.

We now wish to show that $\ker(\theta) = 1$. Note that

$$\begin{aligned}
\ker(\theta) &= \{x \in H : \theta(x) = 0\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H : \theta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = 0 \right\} \\
&= \left\{ \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}, q \in \mathbb{Q} : \theta\left(\begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \right) = 0 \right\} \\
&= \left\{ \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}, q \in \mathbb{Q} : q = 0 \right\} \\
&= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} = 1
\end{aligned}$$

Thus, we have shown that $\ker(\theta) = 1$.

Finally, we must show that θ is onto. Fix $q \in \mathbb{Q}$ and note that $\begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \in H$. Note that $\theta\left(\begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \right) = q$. Thus, θ is onto.

But note that by the Fundamental Homomorphism Theorem we may conclude that $H \cong \mathbb{Q}$. □

Proposition 4.2.3. $G/H \cong \mathbb{Q}^* \times \mathbb{Q}^*$

Proof. We want to show that $G/H \cong \mathbb{Q}^* \times \mathbb{Q}^*$. Define $\psi : G \to \mathbb{Q}^* \times \mathbb{Q}^*$ by $\psi\left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \right) = (a, b), a, b \in \mathbb{Q}^*, x \in \mathbb{Q}$. We wish to show that ψ is a group homomorphism. Fix $\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix}, \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \in G$. Note that

$$\begin{aligned}
\psi\left(\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \right) &= \psi\left(\begin{bmatrix} a_1a_2 & a_1x_2 + b_2x_1 \\ 0 & b_1b_2 \end{bmatrix} \right) \\
&= (a_1a_2, b_1b_2) \\
&= (a_1, b_1)(a_2, b_2) \\
&= \psi\left(\begin{bmatrix} a_1 & x_1 \\ 0 & b_1 \end{bmatrix} \right) \psi\left(\begin{bmatrix} a_2 & x_2 \\ 0 & b_2 \end{bmatrix} \right).
\end{aligned}$$

Thus, ψ preserves addition and is a group homomorphism.

We also want to show that $\ker(\psi) = H$. Then note that

$$\begin{aligned}
\ker(\psi) &= \{x \in G : \psi(x) = 1\} \\
&= \left\{ \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in G : \psi\left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \right) = (1,1) \right\} \\
&= \left\{ \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in G : (a,b) = (1,1) \right\} \\
&= \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} x \in \mathbb{Q} \right\} = H
\end{aligned}$$

So $\ker(\psi) = H$.

Now we must show that ψ is onto. Fix $(q_1, q_2) \in \mathbb{Q}^* \times \mathbb{Q}^*$. Then note that $\begin{bmatrix} q_1 & x \\ 0 & q_2 \end{bmatrix} \in G, x \in \mathbb{Q}$. Also, note that $\psi\left(\begin{bmatrix} q_1 & x \\ 0 & q_2 \end{bmatrix} \right) = (q_1, q_2)$. Thus, ψ is onto.

Then we can note by the Fundamental Homomorphism Theorem that $G/H \cong \mathbb{Q}^* \times \mathbb{Q}^*$. □

We now wish to consider the convex hull of H. We note that since $H \triangleleft G$, we know by Lemma 1.4.16, that $C = \text{hull}_G(H)$ is a subgroup of G. However, G is not a nilpotent group like the Heisenberg group. Therefore, we can no longer conclude that $C \neq G$. Further study will create theory needed to prove the following result.

Theorem 4.2.4. *The set of left orders on* $G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, c \in \mathbb{Q}^*, b \in \mathbb{Q} \right\}$ *is isomorphic to the Cantor set.*

While the above result is intuitively true, we have yet to create the machinery needed to provide proof of such. We have included Theorem 3.4.3 to aid our intuition in ascertaining the veracity of the above fact.

4.2.2 Left Order Topology Isomorphic to $\mathbb{Z} \times \mathbb{Z}$

Next we consider the matrix group

$$G = \left\{ \begin{bmatrix} p^n & x \\ 0 & p^m \end{bmatrix} : p \text{ prime}, m, n \in \mathbb{Z}, x \in \mathbb{Z}\left[\frac{1}{p}\right] \right\} \text{ and } H = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{Z}\left[\frac{1}{p}\right] \right\}.$$

Proposition 4.2.5. $H \triangleleft G$

Proof. We wish to show that H is a normal subgroup of G. To prove that $H \triangleleft G$ we must show that $ghg^{-1} \in H$ for all $g \in G, h \in H$. Fix $h = \begin{bmatrix} 1 & x_2 \\ 0 & 1 \end{bmatrix} \in H$ and $g = \begin{bmatrix} p^n & x_1 \\ 0 & p^m \end{bmatrix} \in G$ for some $x_1, x_2 \in \mathbb{Z}[1/p], m, n \in \mathbb{Z}$, p prime.

We must begin by finding g^{-1}. Then note that $\det\left(\begin{bmatrix} p^n & x_1 \\ 0 & p^m \end{bmatrix}\right) = p^n p^m = p^{n+m}$. Then note that

$$\begin{bmatrix} p^n & x_1 \\ 0 & p^m \end{bmatrix}^{-1} = \frac{1}{p^{m+n}} \begin{bmatrix} p^m & -x_1 \\ 0 & p^n \end{bmatrix} = \begin{bmatrix} p^{-n} & -x_1\, p^{-m-n} \\ 0 & p^{-m} \end{bmatrix}.$$

Then note that

$$\begin{aligned} ghg^{-1} &= \begin{bmatrix} p^n & x_1 \\ 0 & p^m \end{bmatrix} \begin{bmatrix} 1 & x_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p^{-n} & -x_1\, p^{-m-n} \\ 0 & p^{-m} \end{bmatrix}. \\ &= \begin{bmatrix} p^n & p^n x_2 + x_1 \\ 0 & p^m \end{bmatrix} \begin{bmatrix} p^{-n} & -x_1\, p^{-m-n} \\ 0 & p^{-m} \end{bmatrix}. \\ &= \begin{bmatrix} 1 & -x_1 p^{-m} + x_2 p^n p^{-m} + p^{-m} x_1 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

where $x_1, x_2 \in \mathbb{Z}[1/p]$. Therefore, $x_1 = bp^j, x_2 = ap^k$ for $a, b, j, k \in \mathbb{Z}$. Then we note that $-x_1 p^{-m} + x_2 p^n p^{-m} + p^{-m} x_1 = -bp^j p^{-m} + ap^k p^n p^{-m} + p^{-m} bp^j = -bp^{j-m} + ap^{k+n-m} + bp^{j-m} \in \mathbb{Z}[1/p]$. But then $ghg^{-1} \in H$. But g was an arbitrary element of G and h was an arbitrary element of H. Thus, $ghg^{-1} \in H$ for all $g \in G, h \in H$. So $H \triangleleft G$. □

Thus, we have found a matrix group with a normal subgroup such that $H \triangleleft G$.

Proposition 4.2.6. $H \cong \mathbb{Z}[1/p]$

Proof. We want to show that $H \cong \mathbb{Z}[1/p]$. Define $\theta : H \to \mathbb{Z}[1/p]$ by $\theta\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) = x$. We wish to show that θ is a group homomorphism. Fix $\begin{bmatrix} 1 & x_1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & x_2 \\ 0 & 1 \end{bmatrix} \in H$. Note that

$$\theta\left(\begin{bmatrix} 1 & x_1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & x_2 \\ 0 & 1 \end{bmatrix}\right) = \theta\left(\begin{bmatrix} 2 & x_1 + x_2 \\ 0 & 2 \end{bmatrix}\right) = x_1 + x_2 = \theta\left(\begin{bmatrix} 1 & x_1 \\ 0 & 1 \end{bmatrix}\right) + \theta\left(\begin{bmatrix} 1 & x_2 \\ 0 & 1 \end{bmatrix}\right).$$

Thus, θ preserves addition and is a group homomorphism.

We now wish to show that $\ker(\theta) = 1$. Note that

$$\begin{aligned}
\ker(\theta) &= \{x \in H : \theta(x) = 0\} \\
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H : \theta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = 0 \right\} \\
&= \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, x \in \mathbb{Z}[1/p] : \theta\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right) = 0 \right\} \\
&= \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, x \in \mathbb{Z}[1/p] : x = 0 \right\} \\
&= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} = 1
\end{aligned}$$

Thus, we have shown that $\ker(\theta) = 1$.

Now we must show that θ is onto. Fix $x \in \mathbb{Z}[1/p]$ and note that $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in H$. Note that $\theta\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right) = x$. Thus, θ is onto.

But note that by the Fundamental Homomorphism Theorem we may conclude that

$$H \cong \mathbb{Z}[1/p].$$

□

Proposition 4.2.7. *$G/H \cong \mathbb{Z} \times \mathbb{Z}$*

Proof. We want to show that $G/H \cong \mathbb{Z} \times \mathbb{Z}$. Define $\psi : G \to \mathbb{Z} \times \mathbb{Z}$ by $\psi\left(\begin{bmatrix} p^m & x \\ 0 & p^n \end{bmatrix} \right) = (m, n), m, n \in \mathbb{Z}, x \in \mathbb{Z}[1/p]$. We wish to show that ψ is a group homomorphism. Fix $\begin{bmatrix} p^{m_1} & x_1 \\ 0 & p^{n_1} \end{bmatrix}, \begin{bmatrix} p^{m_2} & x_2 \\ 0 & p^{n_2} \end{bmatrix} \in G$. Note that

$$\begin{aligned}
\psi\left(\begin{bmatrix} p^{m_1} & x_1 \\ 0 & p^{n_1} \end{bmatrix} \begin{bmatrix} p^{m_2} & x_2 \\ 0 & p^{n_2} \end{bmatrix} \right) &= \psi\left(\begin{bmatrix} p^{m_1+m_2} & p^{m_1}x_2 + p^{n_2}x_1 \\ 0 & p^{n_1+n_2} \end{bmatrix} \right) \\
&= (m_1 + m_2, n_1 + n_2) \\
&= (m_1, n_1) + (m_2, n_2) \\
&= \psi\left(\begin{bmatrix} p^{m_1} & x_1 \\ 0 & p^{n_1} \end{bmatrix} \right) + \psi\left(\begin{bmatrix} p^{m_2} & x_2 \\ 0 & p^{n_2} \end{bmatrix} \right).
\end{aligned}$$

Thus, ψ preserves the group operation and is a group homomorphism.

We also want to show that $\ker(\psi) = H$. Then note that

$$\begin{aligned}
\ker(\psi) &= \{x \in G : \psi(x) = 0\} \\
&= \left\{ \begin{bmatrix} p^m & x \\ 0 & p^n \end{bmatrix} \in G : \psi\left(\begin{bmatrix} p^m & x \\ 0 & p^n \end{bmatrix} \right) = (0,0) \right\} \\
&= \left\{ \begin{bmatrix} p^m & x \\ 0 & p^n \end{bmatrix} \in G : (m,n) = (0,0) \right\} \\
&= \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} x \in \mathbb{Z}[1/p] \right\} = H
\end{aligned}$$

So $\ker(\psi) = H$.

We wish to show that ψ is onto. Fix $(m,n) \in \mathbb{Z} \times \mathbb{Z}$. Then note that $\begin{bmatrix} p^m & x \\ 0 & p^n \end{bmatrix} \in G, x \in \mathbb{Z}[1/p]$. Also, note that $\psi\left(\begin{bmatrix} p^m & x \\ 0 & p^n \end{bmatrix} \right) = (m,n)$. Thus, ψ is onto.

Then we can note by the Fundamental Homomorphism Theorem that $G/H \cong \mathbb{Z} \times \mathbb{Z}$. □

We now wish to consider the convex hull of H. We note that since $H \lhd G$, we know by Lemma 1.4.16, that $C = \text{hull}_G(H)$ is a subgroup of G. However, we can not apply Theorem 1.4.19, since G is not a nilpotent group (unlike the Heisenberg group). We would like to adapt a methodology similar to the method used above to prove that the following theorem.

Theorem 4.2.8. *The set of left orders on* $G = \left\{ \begin{bmatrix} p^m & x \\ 0 & p^n \end{bmatrix} : n, m \in \mathbb{Z}, x \in \mathbb{Z}[1/p] \right\}$ *is isomorphic to the Cantor set.*

This result is almost certainly true, however, we have yet to provide the necessary theorems to provide proof of such at this time. Future work will adapt another another method of proof similar to that used in the previous section to show Theorem 4.2.8 is in fact true.

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Appendix A

Lexicographic Ordering Program

```
function [M ] = LM(A,B)
%This program was created to lexicographically order two upper triangular
%matrices with ones on the diagonal and integer entries.  Note that this
%program may be easily modified to lexicographically order a set of
%matrices.  We assume that the two matrices input have the same dimension.
%Otherwise they would not be in the same group and thus, we would be unable
%to order them.
%INPUT: -two square upper triangular matrices with ones on the diagonal and
%        integer entries.
%OUTPUT -a message ordering the two input matrices

m=size(A);
n=factorial(m(1,1)-1);
%After generating the size of the matrix, we note that the only entries we
%need to compare are the entries above the diagonal.  Thus, the number of
%entries we are comparing will be (size(A)-1)!
\vspace{-1mm}
nums=zeros(1,n);
for i=1:1:n
    nums(i)=i;
end

M=zeros(2,n+1);
index=1;
for j=1:1:m-1
    for i=1:1:m-j
        M(1, index)=A(i,i+j);
        M(2, index)=B(i,i+j);
        index=index+1;
```

```
    end
    M(1,n+1)=1;
    M(2,n+1)=2;
end
M=sortrows(M, nums);
if (M(1,n+1)==1)
    fprintf('Matrix 1 <= Matrix 2\n')
else
    fprintf('Matrix 2 <= Matrix 1 \n')
end
end
```