THE FINE TOPOLOGY AND OTHER TOPOLOGIES ON C(X,Y)

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DEDICATION

I dedicate this work to a loving wife, for without her it would never have been a reality, and to our lovely children and

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I would like to express my gratitude to Prof. R. A. McCoy in his suggestion of a topic, and for his help in pursuing simplified ways of approaching different problems.

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TABLE OF CONTENTS

INTRODUCTION

"The Fine Topology" on C(X,Y) where (Y,d) is a metric space is referred to, in an exercise in [14], as the topology generated by basic open neighborhoods of the form $B(f,\epsilon) = \{g: d(f(x),g(x)) < \epsilon(x)\}$ where ϵ is a positive continuous real valued function. So in the fine topology, a function g is close to f if g(x) is continuously close to f(x); whereas in the uniform topology, g(x) must be uniformly close to f(x), that is, within a constant distance of f(x). So the fine topology is an obvious refinement of the uniform topology.

This topology has not been extensively studied before, and it is the purpose of this paper to see how the fine topology fits in with the lattice of other well studied topologies on C(X,Y), and to study some properties of this topology in itself. Furthermore, other results on these well studied topologies will be examined and compared with the fine topology.

Most of the termilology used is described as used. Some exceptions are the following. The symbols Q and R refer respectively to the rational and real numbers. The symbols $[0,\Omega]$ and $[0,\omega]$ refer to the set of ordinals starting at zero up to and including respectively the first uncountable ordinal, Ω, and the first infinite ordinal, ω, with the order topology. Analagous definitions can be made for the symbols $[0,\Omega)$ and $[0,\omega)$. A space X is said to have non-trivial path if there are distinct points x_1 and x_2 in X, and a continuous function $\phi: [0,1] \longrightarrow X$ such that $\phi(0) = x_1$ and $\phi(1) = x_2$. A space X is said to be hemicompact if there is a sequence $\{K_j\}$ of compact sets in X such that if K is any compact set in X, then $K \subseteq K_n$ for some n.

CHAPTER I

A LATTICE OF TOPOLOGIES ON C(X,Y)

Let X and Y be topological spaces, and let C(X,Y) be the set of all continuous functions from X to Y. In this chapter we will look at the different topologies on C(X,Y), and see how they fit together to form a lattice.

1. Set-open topologies

If τ is a topology on C(X,Y), then $C_\tau(X,Y)$ will stand for the set C(X,Y) with topology τ. If τ_1 and τ_2 are two topologies on C(X,Y), then we will write $C_{\tau_1}(X,Y) \leq C_{\tau_2}(X,Y)$, $C_{\tau_1}(X,Y) \not\leq C_{\tau_2}(X,Y)$, $C_{\tau_1}(X,Y) < C_{\tau_2}(X,Y)$, $C_{\tau_1}(X,Y) = C_{\tau_2}(X,Y)$, and $C_{\tau_1}(X,Y) \neq C_{\tau_2}(X,Y)$ if $\tau_1 \subseteq \tau_2$, $\tau_1 \not\subseteq \tau_2$, $\tau_1 \subsetneq \tau_2$, $\tau_1 = \tau_2$, and $\tau_1 \neq \tau_2$ respectively. Furthermore, we will write C(X) and $C_\tau(X)$ for C(X,Y) and $C_\tau(X,Y)$ respectively when Y = R.

Let S be a family of subsets of X, and define (E,V) = {f ε C(X,Y): f(E) $\subseteq$ V}. The {(E,V): E ε S and V open in Y} forms a subbase for a topology on C(X,Y) called a set-open topology. If S = {{x}: x ε X}, then the topology generated is called the point-open topology and is indicated by π. If S = {K: K compact in X}, then the topology is called the compact-open topology and is denoted by κ. An obvious result is that $C_\pi(X,Y) \leq C_\kappa(X,Y)$.

2. Topologies generated by uniformities

If Y is Tychonoff with compatible uniformity μ, then we can define uniformities on C(X,Y) in the following ways:

a) Define a base for a uniformity π_μ by $\{E_{D,F}: D \in \mu \text{ and } F \text{ finite}\}$, where $E_{D,F} = \{(f,g): (f(x),g(x)) \in D \text{ and } x \in F\}$.

b) Define a base for a uniformity κ_μ by $\{E_{D,K}: D \in \mu \text{ and } K \text{ compact in } X\}$, where $E_{D,K} = \{(f,g): (f(x),g(x)) \in D \text{ and } x \in K\}$.

c) Define a base for a uniformity u_μ (just call it u) by $\{E_D: D \in \mu\}$, where $E_D = \{(f,g): (f(x),g(x)) \in D \text{ and } x \in X\}$.

The uniform spaces given in a), b), and c) are denoted $C_{\pi_\mu}(X,Y)$, $C_{\kappa_\mu}(X,Y)$, and $C_u(X,Y)$ respectively. The uniformities are called the uniformity of pointwise convergence, the uniformity of compact convergence (or uniform convergence on compacta), and the uniformity of uniform convergence respectively, and are denoted by π_μ, κ_μ, and u respectively. The topologies generated are called respectively the topology of pointwise convergence, the topology of compact convergence (or uniform convergence on compacta), and the topology of uniform convergence (or the uniform topology). We also let the symbols $C_{\pi_\mu}(X,Y)$, $C_{\kappa_\mu}(X,Y)$, and $C_u(X,Y)$ represent the generated topological spaces.

The fact that $C_\pi(X,Y) = C_{\pi_\mu}(X,Y)$ and $C_\kappa(X,Y) = C_{\kappa_\mu}(X,Y)$ can be seen in [18]. Therefore, $C_{\pi_\mu}(X,Y)$ and $C_{\kappa_\mu}(X,Y)$ are independent of our choice of uniformity on Y. However, this is not true for the uniform topology.

If $E_{D,K}[f]$ is a basic open neighborhood of f in $C_{\kappa_\mu}(X,Y)$, then $E_D[f] \subseteq E_{D,K}[f]$ and since $E_D[f]$ is a basic open neighborhood of f in $C_u(X,Y)$, $E_{D,K}[f]$ is open in $C_u(X,Y)$. Consequently $C_{\kappa_\mu}(X,Y) \leq C_u(X,Y)$, and we have formed the composite lattice structure:

$$C_\pi(X,Y) \leq C_\kappa(X,Y) \leq C_u(X,Y)$$

for all uniformities μ on Y.

If (Y,d) is a metric space with bounded metric, define $\bar{d}$ on $C(X,Y)$ by $\bar{d}(f,g) = \sup\{d(f(x),g(x)): x \in X\}$. The metric space so obtained is denoted by $C_d(X,Y)$ and the topological space generated by this metric is likewise denoted by $C_d(X,Y)$

Now if μ is a uniformity generated by a bounded metric d, then $C_u(X,Y) = C_d(X,Y)$.

Let (Y,d) be a metric space (d is not necessarily bounded). Put $d' = \min\{d,1\}$, and μ and μ' be the uniformities generated by d and d' respectively. Then $\mu=\mu'$, and furthermore

(*) $$C_u(X,Y) = C_{u'}(X,Y) = C_{d'}(X,Y).$$

So we need not be too particular about how we call them, and will in the future write $C_u(X,Y)$ or $C_d(X,Y)$, and will for the purpose of proving results, assume that d is bounded.

Since we use the symbol $C_u(X,Y)$ as both a uniform space and a topological space, the term "$C_u(X,Y)$ is metrizable" will mean as a topological space, not as a uniform space. So by (*), if (Y,d) is a metric space, and if μ is the uniformity generated by d, then $C_u(X,Y)$ is metrizable.

3. <u>Open cover topology</u>

Let $\Gamma(Y) = \{\mathcal{V}: \mathcal{V}$ is an open cover of $Y\}$, and define $\mathcal{V}(f) = \{g \in C(X,Y):$ for all $x \in X$ there is a $V \in \mathcal{V}$ such that $f(x),g(x) \in V\}$ where $\mathcal{V} \in \Gamma(Y)$. Let γ be the topology generated by the subbase $\{\mathcal{V}(f): f \in C(X,Y)$ and $\mathcal{V} \in \Gamma(Y)\}$.

It is shown in [11], that; for all topological spaces Y, $C_\kappa(X,Y) \leq C_\gamma(X,Y)$; and for all metric spaces (Y,d), $C_d(X,Y) \leq C_\gamma(X,Y)$.

So if Y is a topological space, we have

$$C_\pi(X,Y) \leq C_\kappa(X,Y) \leq C_\gamma(X,Y)$$

and for (Y,d) a metric space, we have

$$C_\pi(X,Y) \leq C_\kappa(X,Y) \leq C_d(X,Y) \leq C_\gamma(X,Y).$$

4. Graph and m-topologies

If $f \in Y^X$, the graph of f, denoted by G(f), is defined by $G(f) = \{(x,f(x)): x \in X\}$. For $E \subseteq X \times Y$, let $N(E) = \{f \in C(X,Y): G(f) \subseteq E\}$. The graph and m-topologies on C(X,Y) are therefore defined as follows:

a) The graph topology, g, is generated by the basis $\{N(U):\ U \text{ is open in } X \times Y\}$.

b) The m-topology, m, is generated by the basis $\{N(C):\ C \text{ is a cozero set in } X \times Y\}$.

These two topologies have been widely studied, and some results about them can be found in [6], [15], and [16].

It can easily be seen that $C_m(X,Y) \leq C_g(X,Y)$.

Theorem 1.4.1: $C_\gamma(X,Y) \leq C_g(X,Y)$.

Proof: Let $g \in \mathcal{V}(f)$. We need only show there is an open set N(U) in $C_g(X,Y)$ such that $g \in N(U) \subseteq \mathcal{V}(f)$.

Let $x \in X$. Since $g \in \mathcal{V}(f)$, there is an open set $V_x \in \mathcal{V}$ such that $f(x), g(x) \in V_x$. By the continuity of f and g, there is an open neighborhood W_x of x such that if $y \in W_x$, then $f(y), g(y) \in V_x$. Put $U_x = W_x \times V_x$, and $U = \cup\{U_x : x \in X\}$. We will be through if we can show $g \in N(U) \subseteq \mathcal{V}(f)$.

For each $x \in X$, $(x,g(x)) \in W_x \times V_x = U_x \subseteq U$. So $G(g) \subseteq U$ and $g \in N(U)$.

Let $h \in N(U)$. By definition $G(h) \subseteq U$. Let $y \in X$. Then $(y,h(y)) \in U = \cup\{U_x : x \in X\}$. So $(y,h(y)) \in U_x = W_x \times V_x$ for some x, and $h(y) \in V_x$ for some x. But by the definition of W_x, $f(y) \in V_x$, and so $h(y), f(y) \in V_x \in \mathcal{V}$. Consequently, $h \in \mathcal{V}(f)$ and $N(U) \subseteq \mathcal{V}(f)$.

Q.E.D.

<u>Theorem 1.4.2</u>: If Y is Tychonoff, then $C_\kappa(X,Y) \leq C_m(X,Y)$.

<u>Proof</u>: Let (K,U) be a subbasic open set in $C_\kappa(X,Y)$. We wish to show that it is open in $C_m(X,Y)$. To do this, let $f \in (K,U)$. We have to find a cozero set $C \subseteq X \times Y$ such that $f \in N(C) \subseteq (K,U)$.

Now $f(K)$ is compact and $f(K) \subseteq U$. Therefore, since Y is Tychonoff, there is a continuous function $g:Y \longrightarrow [0,1]$ such that $g(f(K)) = \{0\}$ and $g(U^c) = \{1\}$. Define the continuous functions $h:Y \longrightarrow [0,1]$ by $h = 1 - g$, and $\phi:X \longrightarrow [0,1]$ by $\phi = g \circ f$. Now $\bar{Z} = \phi^{-1}(0)$ is a zero set in X, and $Z' = h^{-1}(0)$ is a zero set in Y. Therefore $Z_1 = \bar{Z} \times Y$ and $Z_2 = X \times Z'$ are zero sets in $X \times Y$. So put $C = Z^c$ where $Z = Z_1 \cap Z_2$ is a zero set in $X \times Y$. We need only show $f \in N(C) \subseteq (K,U)$.

Assume $x \in X$. We would like to conclude that $(x,f(x)) \in C$. If not, $(x,f(x)) \in Z = Z_1 \cap Z_2$. That is, $(x,f(x)) \in Z_1 = \bar{Z} \times Y$ and $(x,f(x)) \in Z_2 = X \times Z'$. So $x \in \bar{Z}$ and $f(x) \in Z'$. But if $x \in \bar{Z}$, then $\phi(x) = 0$, and if $f(x) \in Z'$, then $h(f(x)) = 0$, that is $\phi(x) = g(f(x)) = 1$; which is a contradiction. Therefore $(x,f(x)) \in C$, and $G(f) \subseteq C$. So $f \in N(C)$.

Now $\phi(x) = 0$ on K and $\phi(x) = 1$ off $f^{-1}(U)$, since if $x \notin f^{-1}(U)$, then $f(x) \notin U$, that is $f(x) \in U^c$. But $g(U^c) = 1$, so $\phi(x) = g(f(x)) = 1$. Therefore, $K \subseteq \bar{Z} \subseteq f^{-1}(U)$. Also $h(x) = 0$ on U^c, so $U^c \subseteq Z'$ and $Z'^c \subseteq U$.

So now if $l \in N(C)$ and $x \in K$, then $(x,l(x)) \in C$ where $C = (Z_1 \cap Z_2)^c = Z_1^c \cup Z_2^c$. Therefore $(x,l(x)) \in Z_1^c$ or $(x,l(x)) \in Z_2^c$. If $(x,l(x)) \in Z_1^c = (\bar{Z} \times Y)^c$, then $x \notin \bar{Z}$. But $x \in K \subseteq \bar{Z}$, which is a contradiction. Therefore $(x,l(x)) \in Z_2^c = (X \times Z')^c$ and $l(x) \in Z'^c$. But $Z'^c \subseteq U$, so $l(x) \in U$ and $l \in (K,U)$. Consequently $N(C) \subseteq (K,U)$.

Q.E.D.

Because of this Theorem, and other previous results, we have the following lattice structures:

a) If X, Y are topological spaces, then

$$C_\pi(X,Y) \le C_\kappa(X,Y) \begin{array}{c} \swarrow\!\!\!\!\!\!\le \;\; C_\gamma(X,Y) \;\; \le \\ \\ \quad\; C_m(X,Y) \;\; \le \end{array} C_g(X,Y).$$

b) If X is topological space and Y is Tychonoff, then

$$C_\pi(X,Y) \le C_\kappa(X,Y) \begin{array}{c} \le \; C_u(X,Y) \quad\;\; \\ \le \; C_\gamma(X,Y) \; \le \\ \le \; C_m(X,Y) \; \le \end{array} C_g(X,Y).$$

5. The Fine Topology

If (Y,d) is a metric space, then $C_d(X,Y)$ is generated by basic open sets of the form $S(f,\epsilon)$ where

$$S(f,\epsilon) = \{g \in C(X,Y): \; d(f(x),g(x)) < \epsilon \text{ for all } x \in X\}$$

where $\epsilon \in R^+$ and R^+ signifies all positive real numbers.

The most obvious generalization of this topology is to define a topology f_d on $C(X,Y)$ which is generated by the basic open sets of the form $B(f,\epsilon)$ where

$$B(f,\epsilon) = \{g \varepsilon C(X,Y): \ d(f(x),g(x)) < \epsilon(x) \text{ for all } x \varepsilon X\} \text{ and}$$
$\epsilon \varepsilon C(X,R^+)$.

Note: a) In $S(f,\epsilon)$, ϵ is a positive real number, while in $B(f,\epsilon)$, ϵ is a continuous positive real valued function on X.

b) The topology, f_d, is called the fine topology generated by (the metric) d, or if d is understood it is called the fine topology and denoted f.

c) Also $C_{f_d}(X,Y) = C_{f_{d'}}(X,Y)$ where $d' = \min\{d,1\}$. So for the purpose of proving results, we can assume without loss of generality that d is bounded.

Obviously $C_d(X,Y) \leq C_{f_d}(X,Y)$, and $C_{f_d}(X,Y)$ fits very nicely into the existing lattice with the following result.

<u>Theorem 1.5.1</u>: If (Y,d) is a metric space, then $C_{f_d}(X,Y) \leq C_m(X,Y)$.

<u>Proof</u>: Let $B(f,\epsilon)$ be a basic open neighborhood of f in $C_{f_d}(X,Y)$. We need to show that $B(f,\epsilon)$ is open in $C_m(X,Y)$. To do this, it will suffice to find a cozero set $C \subseteq X \times Y$ such that $f \varepsilon N(C) \subseteq B(f,\epsilon)$.

So define $F: X \times Y \longrightarrow R$ by $F(x,y) = \max\{\epsilon(x) - d(y,f(x)),0\}$. Clearly F is continuous on $X \times Y$. So put $Z = F^{-1}(0)$ and $C = Z^c$.

$$\text{Now } F(x,f(x)) = \max\{\epsilon(x) - d(f(x),f(x)),0\}$$
$$= \epsilon(x) > 0,$$

so $(x,f(x)) \varepsilon C$ and $G(f) \subseteq C$. Therefore $f \varepsilon N(C)$.

If $h \varepsilon N(C)$, then $G(h) \subseteq C$ and $F(x,h(x)) > 0$ for all x X. That is, $\epsilon(x) - d(h(x),f(x)) > 0$ for all $x \varepsilon X$. Consequently,

$d(h(x),f(x)) < \epsilon(x)$ for all $x \in X$, and $h \in B(f,\epsilon)$. Therefore, $N(C) \subseteq B(f,\epsilon)$.

Q.E.D.

Therefore, we have the following lattice structure:

$$C_\pi(X,Y) \leq C_\kappa(X,Y) \leq C_d(X,Y) \begin{matrix} \lesssim & C_\gamma(X,Y) & \lesssim \\ \lesssim & C_{f_d}(X,Y) \leq C_m(X,Y) & \lesssim \end{matrix} C_g(X,Y)$$

for X a topological space, and (Y,d) a metric space.

We can also define a uniformity f_u on C(X,Y), call it the fine uniformity, by choosing $\mathcal{B} = \{u(\epsilon): \epsilon \in C(X,R^+)\}$ where

$$u(\epsilon) = \{(f,g): d(f(x),g(x)) < \epsilon(x) \text{ for all } x \in X\}.$$

That $\mathcal{B}$ is a base for a uniformity f_u can be seen by the following:

a) For all $u(\epsilon) \in \mathcal{B}$, $\Delta = \{(f,f): f \in C(X,Y)\} \subseteq u(\epsilon)$.

b) If $u(\epsilon)$, $u(\delta) \in \mathcal{B}$, then if $\eta = \min\{\epsilon,\delta\}$ then $u(\eta) \in \mathcal{B}$ and $u(\eta) \subseteq u(\epsilon) \cap u(\delta)$.

c) If $u(\epsilon) \in \mathcal{B}$, then $u(\delta) \circ u(\delta) \subseteq u(\epsilon)$ for $\delta = \frac{1}{2}\epsilon$ and $u(\delta) \in \mathcal{B}$.

d) If $u(\epsilon) \in \mathcal{B}$, then $u(\epsilon)^{-1} = u(\epsilon) \in \mathcal{B}$.

We shall use the symbol $C_{f_u}(X,Y)$ to denote the set C(X,Y) with uniformity f_u. This symbol shall also be used for the generated topological space.

<u>Theorem 1.5.2</u>: $C_{f_u}(X,Y) = C_{f_d}(X,Y)$.

<u>Proof</u>: The $\{B(f,\epsilon): f \in C(X,Y) \text{ and } \epsilon \in C(X,R^+)\}$ forms a basis for $C_{f_d}(X,Y)$, and $\{u(\epsilon)[f]: f \in C(X,Y) \text{ and } \epsilon \in C(X,R^+)\}$ forms a basis for $C_{f_u}(X,Y)$ as a topological space.

However, $u(\epsilon)[f] = \{g: (g,f) \varepsilon u(\epsilon)\}$

$= \{g: d(f(x),g(x)) < \epsilon(x) \text{ for all } x \varepsilon X\}$

$= B(f,\epsilon)$.

Q.E.D.

We have now completed our discussion on the topologies of C(X,Y). In chapter 2 we will discuss situations where these inequalities are strict, and characterize some conditions for which equality holds. In chapters 3, 4, and 5 we will answer questions of separation, countability, and completeness, especially for $C_f(X,Y)$.

CHAPTER II

EQUALITIES AND NONEQUALITIES BETWEEN TOPOLOGIES OF C(X,Y)

1. Characterization Theorems

The first five characterization theorems will be indicated here without proofs; as these results are already known, and their location in the literature is so indicated.

Theorem 2.1.1 [8]: If X is a Tychonoff space, and (Y,d) is a metric space, then $C_d(X,Y) = C_\kappa(X,Y)$ if and only if X is compact.

It should be noted that this theorem does not imply that X need be compact for $C_\kappa(X,Y)$ to be metrizable. Hemicompactness of X would be sufficient to force $C_\kappa(X,Y)$ to be metrizable.

Theorem 2.1.2 [9] & [11]: If X is a Tychonoff space, and (Y,d) is a metric space with non-trivial path, then $C_d(X,Y) = C_\gamma(X,Y)$ if and only if X is pseudocompact.

The following two theorems are characterizations of when $C_u(X,Y) = C_g(X,Y)$. The difference between the two is that they assume completely different restrictions on Y.

Theorem 2.1.3 [6]: If X is a topological space, and (Y,d) is a metric space with a non-isolated point, then $C_d(X,Y) = C_g(X,Y)$ if and only if X is countably compact.

Theorem 2.1.4 [16]: If X is a topological space, and Y is a first countable non-discrete locally compact topological group, then

$C_u(X,Y) = C_g(X,Y)$ if and only if X is countably compact.

As in Theorem 2.1.4, Theorem 2.1.5 requires that Y be a locally compact topological group.

Theorem 2.1.5 [16]: If X is a topological space, and Y is a non-discrete locally compact topological group, then $C_u(X,Y) = C_m(X,Y)$ if and only if X is pseudocompact.

Finally we come to a characterization theorem for the fine topology, which is not found in the literature.

Theorem 2.1.6: If X is a topological space, and (Y,d) is a metric space with non-isolated point, then X is pseudocompact if and only if $C_d(X,Y) = C_{f_d}(X,Y)$.

Proof: ($\longrightarrow$) We already know that $C_d(X,Y) \leq C_{f_d}(X,Y)$, so we need only show $C_{f_d}(X,Y) \leq C_d(X,Y)$ if X is pseudocompact.

Thus assume X is pseudocompact, and $B(f,\epsilon)$ is a basic open set in $C_{f_d}(X,Y)$. Since X is pseudocompact, ϵ is bounded away from zero, and there is a $\delta > 0$ such that $\delta < \epsilon(x)$ for all $x \in X$. So $f \in S(f,\delta) \subseteq B(f,\epsilon)$, and $B(f,\epsilon)$ is open in $C_d(X,Y)$. Therefore, $C_{f_d}(X,Y) = C_d(X,Y)$.

($\longleftarrow$) Assume X is not pseudocompact. Then there is a C-embedded copy of N in X, call it $\bar{X} = \{x_1,x_2,x_3,\ldots\}$, and a positive continuous real valued function ϵ defined on X such that $\epsilon(x_i) = 1/i$.

Since Y has a non-isolated point y_0, $S(y_0,1/n)$ contains points other than y_0 for all positive integers n. For each positive integer n, pick $y_n \in S(y_0,1/n)$ such that $y_n \neq y_0$, and let $\bar{y}_j$ be the constant function $\bar{y}_j(x) = y_j$ for all $x \in X$ and $j = 0,1,2,\ldots$. Now $\bar{y}_0 \in B(\bar{y}_0,\epsilon)$ and if

$B(\bar{y}_0,\epsilon)$ were open in $C_d(X,Y)$, it would have to contain an open set of the form $S(\bar{y}_0,1/n)$ for some positive integer n.

We wish to show that this is not true, and have to show that $S(\bar{y}_0,1/n) \not\subseteq B(\bar{y}_0,\epsilon)$ for any n. So let n be a fixed but arbitrary positive integer. Now $\bar{y}_n \in S(\bar{y}_0,1/n)$, and $d(\bar{y}_0(x),\bar{y}_n(x)) = d(y_0,y_n) > 0$ for all $x \in X$. So pick k such that $1/k < d(y_0,y_n)$. Therefore,

$$d(\bar{y}_0(x_k),\bar{y}_n(x_k)) = d(y_0,y_n) > 1/k = \epsilon(x_k), \text{ and } \bar{y}_n \notin B(\bar{y}_0,\epsilon).$$

Consequently, $S(\bar{y}_0,1/n) \not\subseteq B(\bar{y}_0,\epsilon)$ for any n, and so $B(\bar{y}_0,\epsilon)$ is not open in $C_d(X,Y)$. Therefore, $C_{f_d}(X,Y) \neq C_d(X,Y)$.

Q.E.D.

Theorem 2.1.6 tells us that, if X is pseudocompact, then $C_{f_d}(X,Y)$ is metrizable. With a little stronger hypotheses on both X and Y, we will actually see that it is also necessary that X be pseudocompact for $C_{f_d}(X,Y)$ to be metrizable.

Theorem 2.1.7: If X is Tychonoff, and (Y,d) is a metric space with non-trivial path, then the following are equivalent:

i) X is pseudocompact.

ii) $C_{f_d}(X,Y) = C_d(X,Y)$.

iii) $C_{f_d}(X,Y)$ is metrizable.

iv) $C_{f_d}(X,Y)$ is first countable.

Proof: i) $\longrightarrow$ ii) This is by Theorem 2.1.6.

ii) $\longrightarrow$ iii) This is immediate.

iii) $\longrightarrow$ iv) This is immediate.

iv) $\longrightarrow$ i) Assume X is not pseudocompact. Let $\bar{X} = \{x_1,x_2,x_3,\ldots\}$ be a C-embedded copy of N in X, and let

$\phi\colon I \xrightarrow{\text{onto}} P \subseteq Y$ be an arc in Y with $y_0 = \phi(0)$. Put $\bar{y}_0$ as the constant function $\bar{y}_0(x) = y_0$ for all $x \in X$. Let $\mathcal{B} = \{B(\bar{y}_0, \epsilon_n)\}_{n=1}^{\infty}$ be any countable collection of basic open neighborhoods of $\bar{y}_0$. In order to show that $C_{f_d}(X,Y)$ is not first countable, we need only find a positive continuous real valued function ϵ such that $B(\bar{y}_0, \epsilon_n) \not\subseteq B(\bar{y}_0, \epsilon)$ for any n. To do this, we will a) find functions $f_n \in C(X,Y)$ such that $f_n \neq \bar{y}_0$ and $f_n \in B(\bar{y}_0, \epsilon_n)$, and b) define ϵ such that $f_n \notin B(\bar{y}_0, \epsilon)$ for all n.

a) Let n be a fixed but arbitrary positive integer, and put $\bar{\epsilon}_n = \min\{\epsilon_n, 1\}$. Pick W to be a neighborhood of x_n such that $\bar{\epsilon}_n(W) \subseteq (\bar{\epsilon}_n(x_n)/2, 3\bar{\epsilon}_n(x_n)/2)$. Let $g(y) = d(y_0, y)$ and put $V = g^{-1}([0, \bar{\epsilon}_n(x_n)/2))$. Let $[0,\alpha)$ be a neighborhood of 0 in I such that $\phi([0,\alpha)) \subseteq V \cap P$. Now pick a continuous function h_n on X (by complete regularity of X) such that $h_n\colon X \longrightarrow [0,\alpha/2]$ with $h_n(x_n) = \alpha/2$ and $h_n(W^c) = 0$, and let $f_n = \phi \circ h_n$. We want to show that $f_n \neq \bar{y}_0$, and $f_n \in B(\bar{y}_0, \epsilon_n)$.

Now $h_n(x_n) = \alpha/2 \neq 0$, so $f_n(x_n) = \phi(h_n(x_n)) = \phi(\alpha/2) \neq y_0$. Therefore, $f_n \neq \bar{y}_0$.

If $x \in X$, then $x \in W$ or $x \in W^c$. So if $x \in W$, then $\bar{\epsilon}_n(x) > \bar{\epsilon}_n(x_n)/2$. Now $h_n(x) \in [0,\alpha)$, so $f_n(x) = \phi(h_n(x)) \in \phi([0,\alpha)) \subseteq V \cap P \subseteq V$, and $d(\bar{y}_0(x), f_n(x)) = d(y_0, f_n(x)) = g(f_n(x)) < \bar{\epsilon}_n(x_n)/2 < \bar{\epsilon}_n(x)$. And if $x \in W^c$, then $h_n(x) = 0$, and $f_n(x) = \phi(h_n(x)) = \phi(0) = y_0$. So $d(\bar{y}_0(x), f_n(x)) = d(y_0, y_0) = 0 < \bar{\epsilon}_n(x)$. Therefore, in all cases $d(\bar{y}_0(x), f_n(x)) < \bar{\epsilon}_n(x) \leq \epsilon_n(x)$, and $f_n \in B(\bar{y}_0, \epsilon_n)$.

b) Now for every n, $f_n(x_n) \neq y_0$, so for each n, put $m_n = d(y_0, f_n(x_n)) > 0$, and define $\bar{\epsilon}$ on $\bar{X}$ by $\bar{\epsilon}(x_n) = m_n/2$. Since $\bar{X}$ is C-embedded in X, extend $\bar{\epsilon}$ to ϵ on X. But

$$d(\bar{y}_0(x_n), f_n(x_n)) = d(y_0, f_n(x_n)) = m_n > m_n/2 = \epsilon(x_n),$$

and therefore $f_n \notin B(\bar{y}_0, \epsilon)$. Consequently $B(\bar{y}_0, \epsilon_n) \not\subseteq B(\bar{y}_0, \epsilon)$ for any n, and $C_{f_d}(X,Y)$ is not first countable.

Q.E.D.

A similar statement can be made with respect to Theorem 2.1.2, but requires considerably more restrictions on both X and Y. For this purpose we make the following definitions. Space Y will be called an extensor of space X if every continuous function from a closed subspace of X into Y has a continuous extension from X into Y. Space Y will be called a local extensor of X if Y has no isolated points and for every $y \in Y$ and open neighborhood V of y in Y, there exists an open W in Y such that $y \in W \subseteq V$ and W is an extensor of X.

It should be noted in the next theorem, that for metric spaces, compactness and pseudocompactness are equivalent.

<u>Theorem 2.1.8</u> [11]: Let X and (Y,d) be metric spaces such that Y is a local extensor of X, and Y contains a non-trivial path in every non-empty open set. Then the following are equivalent:

i) $C_\gamma(X,Y) = C_d(X,Y)$

ii) $C_\gamma(X,Y)$ is metrizable.

iii) $C_\gamma(X,Y)$ is first countable.

iv) X is compact.

Because of Theorems 2.1.2 and 2.1.6, we can state the following result:

If X is a Tychonoff space, and (Y,d) is a metric space with non-trivial path, then the following are equivalent:

i) X is pseudocompact.

ii) $C_\gamma(X,Y) = C_d(X,Y)$.

iii) $C_{f_d}(X,Y) = C_d(X,Y)$.

This suggests some kind of relationship between $C_\gamma(X,Y)$ and $C_{f_d}(X,Y)$. In fact it will be shown in the next section that $C_\gamma(X,Y) \leq C_{f_d}(X,Y)$ for X binormal.

2. Another Equality Theorem

X is said to be binormal if X is normal and countably paracompact.

Themrem 2.2.1 [4]: X is binormal if and only if for all lsc (lower semicontinuous) l and usc (upper semicontinuous) u such that $u,l:X \longrightarrow R$ with $u < l$ there is an $f \in C(X)$ such that $u < f < l$.

Lemma 2.2.2: Let (Y,d) be a metric space. Let $f \in C(X,Y)$, W open in X, and V open in Y. Define $u^f_{W,V}:X \longrightarrow R$ by

$$u^f_{W,V}(x) = \begin{cases} d(f(x),V^c), & \text{if } x \in W. \\ 0, & \text{otherwise} \end{cases}$$

Then $u^f_{W,V}$ is lsc.

Proof: If x_0 is a fixed but arbitrary element in W^c, then $u^f_{W,V}(x_0) = 0$. Assume $\alpha < u^f_{W,V}(x_0) = 0$, then for every neighborhood U of x_0, $x \in U$ implies that $u^f_{W,V}(x) \geq 0 > \alpha$. So $u^f_{W,V}$ is lsc on W^c. On W, $u^f_{W,V}(x) = d(f(x),V^c)$ is continuous, and therefore, $u^f_{W,V}$ is lsc on W. Consequently,

$u^f_{W,V}$ is lsc on X.

Q.E.D.

Theorem 2.2.3: If X is binormal, and (Y,d) is a metric space, then $C_g(X,Y) = C_{f_d}(X,Y)$.

Proof:

($\geq$) This follows from the fact that by Theorem 1.5.1 $C_{f_d}(X,Y) \leq C_m(X,Y)$, and the obvious fact that $C_m(X,Y) \leq C_g(X,Y)$.

($\leq$) Let N(U) be a basic open neighborhood of f in $C_g(X,Y)$ where U is open in $X \times Y$. To show N(U) is open in $C_{f_d}(X,Y)$, we need only show there is a positive function $\epsilon \in C(X)$ such that $B(f,\epsilon) \subseteq N(U)$.

Now $G(f) \subseteq U$, and U is open in $X \times Y$. So for each $x \in X$, there exist open sets W_x and V_x in X and Y respectively such that $(x,f(x)) \in W_x \times V_x \subseteq U$. So define a cover $\mathfrak{W}$ of X by $\mathfrak{W} = \{W_x: x \in X\}$, and define $u(x) = \sup \{d(f(x),V_y^c): (x,f(x)) \in W_y \times V_y \text{ and } W_y \in \mathfrak{W}\}$. This defines a function $u:X \longrightarrow R$, and in fact, it is easy to see that $u = \sup \{u^f_{W_x,V_x} : x \in X\}$. But by Lemma 2.2.2 each $u^f_{W_x,V_x}$ is lsc, and so u is lsc. Now u is obviously positive, so by Theorem 2.2.1 there is a function $\epsilon \in C(X)$ such that $0 < \epsilon < u$. We need now only show that $B(f,\epsilon) \subseteq N(U)$.

Let $h \in B(f,\epsilon)$ and $x \in X$. So $d(f(x),h(x)) < \epsilon(x) < u(x) = \sup \{d(f(x),V_y^c): (x,f(x)) \in W_y \times V_y \text{ and } W_y \in \mathfrak{W}\}$. Therefore, there exists a $W_y \in \mathfrak{W}$ such that $d(f(x),h(x)) < \epsilon(x) \leq d(f(x),V_y^c)$, and $(x,h(x)) \in W_y \times V_y \subseteq U$. Therefore, for all $x \in X$ $(x,h(x)) \in U$. So $G(h) \subseteq U$, and $h \in N(U)$.

Q.E.D.

<u>Corollary 2.2.4</u>: If X is binormal, and (Y,d) is a metric space, then $C_\gamma(X,Y) \leq C_{f_d}(X,Y)$.

<u>Proof</u>: Now $C_\gamma(X,Y) \leq C_g(X,Y)$ by Theorem 1.4.1

$= C_{f_d}(X,Y)$ by Theorem 2.2.3.

Q.E.D.

It should be noted that, binormality of X is a sufficient condition in Theorem 2.2.3 and Corollary 2.2.4, but it is certainly not a necessary condition. If X is countably compact but not normal (for example, the Tychonoff Corkscrew Topology of [17] p. 109), then

$$C_\gamma(X,Y) = C_{f_d}(X,Y) = C_g(X,Y)$$

<u>Theorem 2.2.5</u> [16]: If X is a topological space, then $C_f(X) = C_m(X)$.

3. <u>Nonequalities</u>

a) $C_\kappa(R) \not\leq C_\pi(R)$ is obvious.

b) $C_d(R) \not\leq C_\kappa(R)$ by Theorem 2.1.1.

c) $C_\gamma(R) \not\leq C_d(R)$ by Theorem 2.1.2.

d) $C_g(X) \not\leq C_\gamma(X)$ by Theorems 2.1.3 and 2.1.2 where $X = [0,\Omega] \times [0,\omega] \setminus \{(\Omega\ \omega)\}$. X is pseudocompact, but not countably compact.

e) $C_g(X) \not\leq C_m(X)$ by Theorems 2.1.3 and 2.1.5 where X is pseudocompact, but not countably compact.

f) $C_{f_d}(R) \not\leq C_d(R)$ by Theorem 2.1.6.

g) $C_{f_d}(R,Q) \not\leq C_\gamma(R,Q)$ since as shown in [9] $C_d(R,Q) = C_\gamma(R,Q)$, but $C_d(R,Q) < C_{f_d}(R,Q)$ by Theorem 2.1.6.

The above list would be complete if we could find an X and Y such that $C_m(X,Y) \not\leq C_{f_d}(X,Y)$. If such an X and Y can be found, then it has been ascertained that X can not be binormal, and Y can neither be R nor any non-discrete locally compact topological group which is pseudocompact.

In Theorem 2.1.2 it was stated that Y had to contain a non-trivial path. That this condition is necessary, can be seen in example g) above. However, in Theorem 2.1.7 this condition is stated without counterexample to prove its necessity.

CHAPTER III

SEPARATION PROPERTIES

In this chapter, as before, $\bar{y}$ will represent the constant function in $C(X,Y)$ defined by $\bar{y}(x) = y$ for all $x \in X$, where $y \in Y$.

Lemma 3.1: If y_1, y_2 are distinct points of Y such that all neighborhoods of y_1 contain y_2, then all neighborhoods of $\bar{y}_1$ in $C_g(X,Y)$ contain $\bar{y}_2$.

Proof: Let $N(U)$ be a basic open neighborhood of $\bar{y}_1$. We need only show that $\bar{y}_2 \in N(U)$.

If $x \in X$, then $(x,y_1) = (x,\bar{y}_1(x)) \in U$. Now there are open sets U_1 in X, and V_1 in Y such that $(x,y_1) \in U_1 \times V_1 \subseteq U$. So $y_1 \in V_1$, and therefore by hypothesis $y_2 \in V_1$. Thus $(x,\bar{y}_2(x)) = (x,y_2) \in U_1 \times V_1 \subseteq U$. Consequently, $G(\bar{y}_2) \subseteq U$, and $\bar{y}_2 \in N(U)$.

Q.E.D.

Theorem 3.2: If Y is not T_0, then $C_g(X,Y)$ is not T_0.

Proof: If Y is not T_0, there are $y_1 \neq y_2$ in Y such that every neighborhood of y_1 contains y_2, and every neighborhood of y_2 contains y_1. But by Lemma 3.1 every neighborhood of $\bar{y}_1$ in $C_g(X,Y)$ contains $\bar{y}_2$ and every neighborhood of $\bar{y}_2$ in $C_g(X,Y)$ contains $\bar{y}_1$. Therefore, $C_g(X,Y)$ is not T_0.

Q.E.D.

Theorem 3.3: If Y is not T_1, then $C_g(X,Y)$ is not T_1.

Proof: If Y is not T_1, there are $y_1 \neq y_2$ in Y such that every neighborhood of y_1 contains y_2, or every neighborhood of y_2 contains y_1. Without loss of generality, assume that every neighborhood of y_1 contains y_2. Thus by Lemma 3.1, every neighborhood of $\bar{y}_1$ in $C_g(X,Y)$ contains $\bar{y}_2$. Therefore, $C_g(X,Y)$ is not T_1.

Q.E.D.

Theorem 3.4: If $\phi: Y \longrightarrow C_\tau(X,Y)$ is defined by $\phi(y) = \bar{y}$, then ϕ is an embedding for $\tau = \pi$, κ, and u.

Proof: ($\tau = \kappa$) To show continuity, let (K,U) be a subbasic open neighborhood of $\bar{y}_0 = \phi(y_0)$ in $C_\kappa(X,Y)$. We will be through, if we show that $\phi(U) \subseteq (K,U) \cap \phi(Y)$. But if $y \in U$, then $\bar{y}(K) = \{y\} \subseteq U$, and $\phi(y) = \bar{y} \in (K,U)$.

To show that ϕ is open, we need show that $\phi(U) = (\{x_0\},U) \cap \phi(Y)$ for U open in Y and some $x_0 \in X$. Actually, we pick x_0 to be arbitrary in X. But, $y \in U$ if and only if $\bar{y}(x_0) = y \in U$; that is, if and only if $\phi(y) = \bar{y} \in (\{x_0\},U)$. Consequently, ϕ is an embedding from Y to $C_\kappa(X,Y)$.

($\tau = \pi$) The fact that we have $\phi(U) = (\{x_0\},U) \cap \phi(Y)$ for each fixed x_0 in X, and every open set U in Y, is sufficient to guarantee that ϕ is an embedding from Y to $C_\pi(X,Y)$.

(τ = u) Let $\mathcal{D}$ be a base for the uniformity μ on Y of open, symmetric subsets of Y × Y. Now $\{D[y]: y \in Y, D \in \mathcal{D}\}$ forms a basis for Y as a topological space, and $\{E_D[\bar{y}] \cap \phi(Y): y \in Y, D \in \mathcal{D}\}$ forms a basis for $\phi(Y)$ in $C_u(X,Y)$ as a topological space. So we need only show that $\phi(D[y_0]) = E_D[\bar{y}_0] \cap \phi(Y)$ where y_0 is a fixed but arbitrary element in Y.

But $y \in D[y_0]$ if and only if $(y,y_0) \in D$, and this is true if and only if $(\bar{y}(x),\bar{y}_0(x)) = (y,y_0) \in D$ for all $x \in X$; that is if and only if $\bar{y} \in E_D[\bar{y}_0] \cap \phi(Y)$. Consequently, ϕ is an embedding from Y to $C_u(X,Y)$.

Q.E.D.

Theorem 3.5: The following are equivalent:

i) Y is T_1 (respectively T_0).

ii) $C_\pi(X,Y)$ is T_1 (respectively T_0).

iii) $C_\kappa(X,Y)$ is T_1 (respectively T_0).

iv) $C_\gamma(X,Y)$ is T_1 (respectively T_0).

v) $C_g(X,Y)$ is T_1 (respectively T_0).

Proof: If Y is T_0, T_1 respectively, then Y^X (with the Tychonoff topology) is T_0, T_1 respectively. Since $C_\pi(X,Y)$ is a subspace of Y^X, $C_\pi(X,Y)$ is respectively T_0, and T_1. Since

$$C_\pi(X,Y) \leq C_\kappa(X,Y) \leq C_\gamma(X,Y) \leq C_g(X,Y),$$

then $C_\kappa(X,Y)$, $C_\gamma(X,Y)$, and $C_g(X,Y)$ are respectively T_0, and T_1.

If $C_\pi(X,Y)$ or $C_\kappa(X,Y)$ or $C_\gamma(X,Y)$ is T_0, T_1 respectively, then since $C_\pi(X,Y) \leq C_\kappa(X,Y) \leq C_\gamma(X,Y) \leq C_g(X,Y)$, $C_g(X,Y)$ is T_0, T_1 respectively. Therefore, Y is T_0, T_1 respectively by Theorem 3.2 and 3.3.

Q.E.D.

Theorem 3.6: If Y is T_2, then $C_\gamma(X,Y)$ and $C_g(X,Y)$ are T_2.

Proof: If Y is T_2, then Y^X (with the Tychonoff topology) is T_2. Since $C_\pi(X,Y)$ is a subspace of Y^X, $C_\pi(X,Y)$ is T_2 also. Since

$$C_\pi(X,Y) \leq C_\gamma(X,Y) \leq C_g(X,Y),$$

$C_\gamma(X,Y)$ and $C_g(X,Y)$ are T_2 also.

Q.E.D.

Theorem 3.7: The following are equivalent:

i) Y is T_2 (respectively T_3, $T_{3\frac{1}{2}}$).

ii) $C_\pi(X,Y)$ is T_2 (respectively T_3, $T_{3\frac{1}{2}}$).

iii) $C_\kappa(X,Y)$ is T_2 (respectively T_3, $T_{3\frac{1}{2}}$).

Proof: Assume Y is T_2, then as in the proof of Theorem 3.6 $C_\pi(X,Y)$ is T_2, and since $C_\pi(X,Y) \leq C_\kappa(X,Y)$, $C_\kappa(X,Y)$ is T_2 also.

If Y is T_3, then as done in [7] p. 151, it can be shown that $C_\kappa(X,Y)$ is T_3. An analogous proof shows that $C_\pi(X,Y)$ is T_3.

If Y is $T_{3\frac{1}{2}}$, then Y is uniformizable. But as was emphasized in Chapter I, this makes $C_\pi(X,Y)$ and $C_\kappa(X,Y)$ uniformizable. Consequently, $C_\pi(X,Y)$ and $C_\kappa(X,Y)$ are $T_{3\frac{1}{2}}$.

Assume $C_\pi(X,Y)$ or $C_\kappa(X,Y)$ is T_2, T_3, $T_{3\frac{1}{2}}$ respectively. Then by Theorem 3.4 $\phi: Y \longrightarrow C_\tau(X,Y)$ where $\tau = \pi$, κ is an embedding, and therefore, Y can be treated as a subspace. Consequently Y is T_2, T_3, $T_{3\frac{1}{2}}$ respectively.

Q.E.D.

Theorem 3.8: If Y is Tychonoff, then $C_u(X,Y)$ is Tychonoff and $C_m(X,Y)$ is at least T_2.

Proof: If Y is Tychonoff, then Y is uniformizable, and consequently $C_u(X,Y)$ also. Therefore, $C_u(X,Y)$ is Tychonoff.

If Y is Tychonoff, $C_\kappa(X,Y) \leq C_m(X,Y)$ by Theorem 1.4.2. But by

Theorem 3.7 $C_\kappa(X,Y)$ is at least T_2, so $C_m(X,Y)$ is at least T_2 also.

Q.E.D.

<u>Theorem 3.9</u>: If (Y,d) is a metric space, then $C_{f_d}(X,Y)$ is Tychonoff.

<u>Proof</u>: By theorem 1.5.2 $C_{f_d}(X,Y)$ is uniformizable. Consequently $C_{f_d}(X,Y)$ is Tychonoff.

Q.E.D.

CHAPTER IV

COUNTABILITY PROPERTIES

In metric spaces the concept of Lindelöf, second countable, and separable coincide. Here, we will list some necessary and sufficient conditions for $C_\tau(X,Y)$ to be second countable, separable, and/or Lindelöf for all the topologies τ we are considering in this paper.

A topological space X has a metrizable compression if its topology contains some metrizable topology on X. Such a space is also called submetrizable. A similar definition can be given for a space to have a separable metrizable compression.

Theorem 4.1 [13]: If X is a Tychonoff space, then $C_\pi(X)$ is separable if and only if X has a separable metrizable compression.

A space X is a Urysohn space if C(X) separates points of X.

Theorem 4.2 [12]: If X is a Urysohn space, then $C_\kappa(X)$ is separable if and only if X has a separable metrizable compression.

Lemma 4.3 [12]: If X is locally compact, then:

i) If X and Y are second countable, then $C_\kappa(X,Y)$ is second countable.

ii) If $C_\kappa(X)$ is second countable, then X is second countable.

Theorem 4.4: If X is locally compact, and Y is a topological space containing an arc, then $C_\kappa(X,Y)$ is second countable if and only if X and Y are second countable.

<u>Proof</u>: ($\longleftarrow$) This is Lemma 4.3(i).

($\longrightarrow$) Let P be an arc in Y with end points p_0, p_1, and put $\bar{P} = P \smallsetminus \{p_0, p_1\}$. Then $C_\kappa(X)$ is second countable, since

$$C_\kappa(X) \simeq C_\kappa(X,\bar{P}) \subseteq C_\kappa(X,Y).$$

Therefore, by Lemma 4.3(ii), X is second countable. Also, by Theorem 3.4, $Y \simeq \phi(Y) \subseteq C_\kappa(X,Y)$, so Y is second countable.

Q.E.D.

<u>Lemma 4.5</u> [3]: Let X be Tychonoff, then X is compact and metric if and only if $C_d(X)$ is second countable.

<u>Theorem 4.6</u>: Let X be Tychonoff, and (Y,d) a metric space with non-trivial path. Then the following are equivalent:

i) X is compact metric, and Y is second countable (equivalently separable, Lindelöf).

ii) $C_d(X,Y)$ is second countable (equivalently separable, Lindelöf).

iii) $C_{f_d}(X,Y)$ is second countable.

iv) $C_{f_d}(X,Y)$ is separable.

v) $C_{f_d}(X,Y)$ is Lindelöf.

vi) $C_\gamma(X,Y)$ is second countable.

vii) $C_\gamma(X,Y)$ is separable.

viii) $C_\gamma(X,Y)$ is Lindelöf.

ix) $C_m(X,Y)$ is second countable.

x) $C_m(X,Y)$ is separable.

xi) $C_m(X,Y)$ is Lindelöf.

xii) $C_g(X,Y)$ is second countable.

xiii) $C_g(X,Y)$ is separable.

xiv) $C_g(X,Y)$ is Lindelöf.

Proof: ($i \longrightarrow j$ where j = ii, iii,...,xiv) Since X is compact, then by Theorems 2.1.1 and 2.1.3, $C_\kappa(X,Y) = C_d(X,Y) = C_g(X,Y)$. So $C_\kappa(X,Y) = C_\tau(X,Y)$ for $\tau = d, f_d, \gamma, m$, or g by the lattice structure on page 10. Now since both X and Y are second countable, then by Lemma 4.3, $C_\kappa(X,Y) = C_\tau(X,Y)$ are second countable. But since $C_d(X,Y)$ is metric, $C_\tau(X,Y) = C_d(X,Y)$ are also separable and Lindelöf.

(xii $\longrightarrow$ xiii,xiv), (ix $\longrightarrow$ x,xi), (vi $\longrightarrow$ vii,viii), and (iii $\longrightarrow$ iv,v) This is obvious, since second countable always implies both separable and Lindelöf.

($j \longrightarrow$ ii, for j = iv, v, vii, viii, x, xi, xiii, xiv) This is immediate, since for any two topologies $\tau_1 \supseteq \tau_2$ on a set Z, if (Z,τ_1) is separable (respectively Lindelöf), then (Z,τ_2) is separable (respectively Lindelöf).

(ii $\longrightarrow$ i) Let P be an arc in Y with end points p_0, p_1, and put $\bar{P} = P \setminus \{p_0, p_1\}$. Then $C_d(X)$ is second countable, since

$$C_d(X) \cong C_d(X,\bar{P}) \subseteq C_d(X,Y).$$

Consequently, by Lemma 4.5, X is compact and metric. But by Theorem 3.4, $Y \cong \phi(Y) \subseteq C_d(X,Y)$, so Y is second countable.

Q.E.D.

CHAPTER V

COMPLETENESS

In this chapter, we want to discuss some completeness properties for function spaces. The terms we will be using are "complete" both as a uniform space and as a metric space, "Cech-complete", "pseudo-complete", and "Baire".

If (Y,d) is a metric space, then a sequence $\{x_n\}$ is Cauchy; if for every $\epsilon > 0$ there is an N such that, if $n,m \geq N$ then $d(x_n,x_m) < \epsilon$. A metric space (Y,d) is complete if every cauchy sequence converges in Y. A topological space is completely metrizable if there can be found a complete metric on Y which generates the same topology.

If (Y,μ) is a uniform space, then a net $\{x_\lambda\}_{\lambda \in \Lambda}$ is Cauchy; if for every $D \in \mu$ there is a $\lambda_0 \in \Lambda$ such that, if $\lambda_1,\lambda_2 \geq \lambda_0$ then $(x_{\lambda_1},x_{\lambda_2}) \in D$. A uniform space (Y,μ) is complete if every cauchy net converges in Y. If (Y,d) is a metric space, and μ is a uniformity generated by d, then (Y,μ) is complete as a uniform space if and only if (Y,d) is complete as a metric space.

A Tychonoff space Y is Cech-complete if Y is G_δ in βY. In [5], it is shown that closed and G_δ subspaces of Cech-complete spaces are Cech-complete. In metrizable space, Cech-complete is equivalent with completely metrizable.

A pseudo-base for Y is a collection of non-empty open subsets of Y such that every non-empty open subset of Y contains a member of this collection. A space is quasi-regular if every non-empty open set contains the closure of some non-empty open set. A space is pseudo-complete provided that it is a quasi-regular space having a sequence $\{\mathcal{P}_n\}$ of pseudo-bases such that if $P_n \in \mathcal{P}_n$ and $cl(P_{n+1}) \subseteq P_n$ for each n, then $\bigcap_{n=1}^{\infty} P_n \neq \emptyset$. Closed subsets of pseudo-complete spaces are not necessarily pseudo-complete. As is indicated in [1], a metrizable space, in fact a Moore space, is pseudo-complete if and only if it contains a dense completely metrizable subspace. Cech-complete implies pseudo-complete.

A space Y is said to be Baire if for any countable collection $\{A_n\}$ of closed sets in Y such that $int(A_n) = \emptyset$, the union has empty interior. It is a well known result that, Y is Baire if and only if every open set in Y is second category (in itself). It can be easily seen from this, that open subspaces of Baire spaces are Baire, but closed subspaces are not necessarily Baire. Also, it should be noted that, pseudo-complete implies Baire.

<u>Theorem 5.1</u>: If ϕ is the embedding from Y to $C_\tau(X,Y)$ where $\tau = \pi$, κ, or u in Theorem 3.4 and Y is T_2, then $\phi(Y)$ is closed in $C_\tau(X,Y)$.

<u>Proof</u>: Because of the lattice structure on C(X,Y), we need only show that $\phi(Y)$ is closed in $C_\pi(X,Y)$. If $f \notin \phi(Y)$, then there are $x_1 \neq x_2$ such that $f(x_1) \neq f(x_2)$. Since Y is T_2, there are disjoint open sets V_1 and V_2 in Y respectively containing $f(x_1)$ and $f(x_2)$. But $(x_1,V_1) \cap (x_2,V_2)$ is an open neighborhood of f in $C_\pi(X,Y)$ containing no constant functions, and so $(\phi(Y))^c$ is open.

Q.E.D.

<u>Theorem 5.2</u>: If $C_{\tau_\mu}(X,Y)$ is a complete uniform space where $\tau = \pi$, κ, or u, then (Y,μ) is a complete uniform space.

Proof: It is easy to see that $\{y_\lambda\}$ is a cauchy net in Y if and only if $\{\bar{y}_\lambda\}$ is a cauchy net in $\phi(Y)$. But $\phi(Y)$ is closed in the complete space $C_{\tau_\mu}(X,Y)$, so $\{\bar{y}_\lambda\}$ converges to a point $\bar{y}$ in $\phi(Y)$. Consequently, $\{y_\lambda\}$ converges to the point y in Y, and Y is complete.

Q.E.D.

Theorem 5.3: (Y,μ) is a complete uniform space if and only if $C_u(X,Y)$ is a complete uniform space.

Proof: This follows immediately from Theorem 5.2 and [18] on page 281.

Q.E.D.

A topological space X will be called a k_R-space if continuity on compact subsets implies continuity.

Theorem 5.4: If X is a k_R-space, then $C_{\kappa_\mu}(X,Y)$ is a complete uniform space if and only if (Y,μ) is a complete uniform space.

Proof: This follows immediately from Theorem 5.2 and [18] pages 285-286.

Q.E.D.

Because of Theorem 5.2, we might wonder if the condition in Theorem 5.4, that X be a k_R-space, could be weakened. The next theorem indicates that it, in fact, can not be weakened.

Theorem 5.5 [2]: If X is Tychonoff, then X is a k_R-space if and only if $C_\kappa(X)$ is complete.

Theorem 5.6: (Y,d) is complete metric if and only if $C_d(X,Y)$ is complete metric.

Proof: ($\longrightarrow$) If (Y,d) is complete metric, let $\{f_n\}$ be a cauchy sequence in $C_d(X,Y)$. So for all $x \in X$, $\{f_n(x)\}$ is cauchy in Y. But (Y,d) is complete, so there is an f(x) such that $f_n(x) \longrightarrow f(x)$. It can easily be seen that $f \in C(X,Y)$ and $f_n \longrightarrow f$ in $C_d(X,Y)$.

($\longleftarrow$) Now $\phi(Y)$ is closed in $C_d(X,Y)$. So if $C_d(X,Y)$ is complete metric, then so too is $\phi(Y)$. But $d(y_1,y_2) = \bar{d}(\bar{y}_1,\bar{y}_2)$, and Y is complete.

Q.E.D.

Theorem 5.7: For Y Tychonoff, if $C_u(X,Y)$ is Cech-complete then Y is Cech-complete.

Proof: Now $Y \simeq \phi(Y)$, and $\phi(Y)$ is closed in $C_u(X,Y)$. But closed subspaces of Cech-complete spaces are Cech-complete. Therefore $\phi(Y)$ and Y are Cech-complete.

Q.E.D.

Theorem 5.8: If (Y,d) is complete metric, then $C_{f_d}(X,Y)$ is a Baire space.

Proof: Let $\{F_n\}$ be a sequence of closed meager sets in $C_{f_d}(X,Y)$, and $B(f_0,\epsilon_0)$ be an arbitrary basic open set in $C_{f_d}(X,Y)$. We want to show that there is a function $f \in B(f_0,\epsilon_0)$ such that $f \notin F_n$ for any n; that is, that $B(f_0,\epsilon_0) \not\subseteq \cup F_n$.

Pick $f_1 \in B(f_0,\epsilon_0/3)$ such that $f_1 \notin F_1$, and choose $\epsilon_1 \leq \min\{1,\epsilon_0/3\}$ such that $B(f_1,\epsilon_1) \cap F_1 = \emptyset$. It is easily seen that $B(f_1,\epsilon_1) \subseteq B(f_0,\epsilon_0)$. Pick $f_2 \in B(f_1,\epsilon_1/3)$ such that $f_2 \notin F_2$, and choose $\epsilon_2 \leq \epsilon_1/3$ such that $B(f_2,\epsilon_2) \cap F_2 = \emptyset$. Again, it is easily seen that $B(f_2,\epsilon_2) \subseteq B(f_1,\epsilon_1)$. Assume f_j and ϵ_j have been chosen such that $f_j \notin F_j$, $f_j \in B(f_{j-1},\epsilon_{j-1})$, $\epsilon_j \leq \epsilon_{j-1}/3$, $B(f_j,\epsilon_j) \cap F_j = \emptyset$, and $B(f_j,\epsilon_j) \subseteq B(f_{j-1},\epsilon_{j-1})$ for all

$j = 1,2,\ldots,i$. Pick $f_{i+1} \in B(f_i,\epsilon_i/3)$ such that $f_{i+1} \notin F_{i+1}$, and choose $\epsilon_{i+1} \le \epsilon_i/3$ such that $B(f_{i+1},\epsilon_{i+1}) \cap F_{i+1} = \emptyset$. Again $B(f_{i+1},\epsilon_{i+1}) \subseteq B(f_i,\epsilon_i)$, and it is easily seen that if $g \in B(f_{j+1},\epsilon_{j+1})$, then $d(g(x),f_j(x)) < 2\epsilon_j(x)/3$ for each $j = 1,2,\ldots,i$.

So $\{f_n\}$ has been defined inductively; and for each N and $x \in X$, $d(f_n(x),f_m(x)) < 2/3^N$ for all $n,m \ge N$. Consequently, $\{f_n(x)\}$ is cauchy in Y. Since Y is complete, there is an $f(x)$ such that $f_n(x) \longrightarrow f(x)$. It is easily shown that $f \in C(X,Y)$, and we need only show that $f \in \bigcap_{i=1}^{\infty} B(f_i,\epsilon_i)$.

If $x \in X$, we want to show that $d(f(x),f_i(x)) < \epsilon_i(x)$. So let i be fixed but arbitrary. Since $f(x) = \lim_{n\to\infty} f_n(x)$, there is an N such that $d(f(x),f_n(x)) < \epsilon_{i+1}(x)$ for all $n \ge N$. If $n \ge N, i+1$, then $f_n \in B(f_{i+1},\epsilon_{i+1})$, and $d(f_n(x),f_i(x)) < 2\epsilon_i(x)/3$. Therefore,

$$\begin{aligned} d(f(x),f_i(x)) &\le d(f(x),f_n(x)) + d(f_n(x),f_i(x)) \\ &< \epsilon_{i+1}(x) + 2\epsilon_i(x)/3 \\ &< \epsilon_i(x)/3 + 2\epsilon_i(x)/3 \\ &= \epsilon_i(x). \end{aligned}$$

Therefore, $f \in B(f_i,\epsilon_i)$ for all i, and $f \notin \bigcup_{n=1}^{\infty} F_n$.

Q.E.D.

However, $C_{f_d}(X,Y)$ (or $C_d(X,Y)$) Baire does not even imply that Y is pseucocomplete, much less complete metric. To see this, pick X to be a one point set and (Y,d) to be Baire but not pseudo-complete. Here $Y \simeq C_{f_d}(X,Y) = C_d(X,Y)$. It is also my suspicion that we can find a metric space (Y,d) which is not Baire but such that $C_{f_d}(X,Y)$ (or $C_d(X,Y)$) is Baire.

A metric space (Y,d) is convex if for each $x_0 \neq x_1$ there is an $x_{\frac{1}{2}}$ such that $d(x_0,x_{\frac{1}{2}}) = d(x_{\frac{1}{2}},x_1) = \frac{1}{2}d(x_0,x_1)$. So let $x_0 \neq x_1$ be two fixed points in the convex metric space (Y,d). We will define inductively a one-to-one correspondence between the diatic rationals D in $[0,1]$ and $Z \subseteq Y$ such that for each t, t' in D with $t \leq t'$, we have $d(x_t,x_{t'}) = (t' - t)d(x_0,x_1)$.

(k = 0) Put $\bar{D}_0 = \{0,1\}$, and $\bar{Z}_0 = \{x_0,x_1\}$. There is an $x_{\frac{1}{2}}$ such that $d(x_0,x_{\frac{1}{2}}) = d(x_{\frac{1}{2}},x_1) = \frac{1}{2}d(x_0,x_1)$.

(k = 1) Put $\bar{D}_1 = \{\frac{1}{2}\}$, $\bar{Z}_1 = \{x_{\frac{1}{2}}\}$, $D_1 = \bar{D}_0 \cup \bar{D}_1$, and $Z_1 = \bar{Z}_0 \cup \bar{Z}_1$. For each t_1, $t_2 \in D_1$, $t_1 \leq t_2$, $d(x_{t_1},x_{t_2}) = (t_2 - t_1)d(x_0,x_1)$. Now there are $x_{\frac{1}{4}}$, $x_{\frac{3}{4}}$ such that $d(x_0,x_{\frac{1}{4}}) = d(x_{\frac{1}{4}},x_{\frac{1}{2}}) = \frac{1}{2}d(x_0,x_{\frac{1}{2}}) = \frac{1}{4}d(x_0,x_1)$, and $d(x_{\frac{1}{2}},x_{\frac{3}{4}}) = d(x_{\frac{3}{4}},x_1) = \frac{1}{2}d(x_{\frac{1}{2}},x_1) = \frac{1}{4}d(x_0,x_1)$. Thus

$$\begin{aligned} d(x_0,x_{\frac{3}{4}}) &\leq d(x_0,x_{\frac{1}{2}}) + d(x_{\frac{1}{2}},x_{\frac{3}{4}}) \\ &= \tfrac{1}{2}d(x_0,x_1) + \tfrac{1}{4}d(x_0,x_1) \\ &= \tfrac{3}{4}d(x_0,x_1), \text{ and} \\ d(x_0,x_1) &\leq d(x_0,x_{\frac{3}{4}}) + d(x_{\frac{3}{4}},x_1) \\ &= d(x_0,x_{\frac{3}{4}}) + \tfrac{1}{4}d(x_0,x_1). \end{aligned}$$

So $d(x_0,x_{\frac{3}{4}}) \geq \frac{3}{4}d(x_0,x_1)$, and $d(x_0,x_{\frac{3}{4}}) = \frac{3}{4}d(x_0,x_1)$. It can similarly be shown that $d(x_{\frac{1}{4}},x_1) = \frac{3}{4}d(x_0,x_1)$.

(k = 2) So put $\bar{D}_2 = \{\frac{1}{4},\frac{3}{4}\}$, $\bar{Z}_2 = \{x_{\frac{1}{4}},x_{\frac{3}{4}}\}$, $D_2 = D_1 \cup \bar{D}_2$, and $Z_2 = Z_1 \cup \bar{Z}_2$. Now for t_1, $t_2 \in D_2$ with $t_1 \leq t_2$, $d(x_{t_1},x_{t_2}) = (t_2 - t_1)d(x_0,x_1)$.

(Assume true for k = n) That is, if $\bar{D}_n = \{a/2^n\colon\ 0 < a < 2^n$, and a odd$\}$, $D_n = \bar{D}_n \cup D_{n-1} = \bigcup_{j=0}^{n} \bar{D}_j$, $Z_n = \{x_t\colon\ t \in D_n\}$, then for t_1, $t_2 \in D_n$ with $t_1 \leq t_2$, $d(x_{t_1},x_{t_2}) = (t_2 - t_1)d(x_0,x_1)$.

(Show for k = n+1) Put $\bar{D}_{n+1} = \{a/2^{n+1}\colon\ 0 < a < 2^{n+1}$, a odd$\}$. To define $\bar{Z}_{n+1}$, let t be a fixed but arbitrary element in $\bar{D}_{n+1}$, put $t_1 = t - 1/2^{n+1}$ and $t_2 = t + 1/2^{n+1}$. The symbols indicated here will be referred to as (*). Now there is an x_t such that $d(x_{t_1},x_t) = d(x_t,x_{t_2}) = \frac{1}{2}d(x_{t_1},x_{t_2}) = (1/2^{n+1})d(x_0,x_1)$. So for t, t_1, and t_2 in (*), $d(x_{t_1},x_t) = (t - t_1)d(x_0,x_1)$ and $d(x_t,x_{t_2}) = (t_2 - t)d(x_0,x_1)$. Therefore, put $\bar{Z}_{n+1} = \{x_t: \ t \in \bar{D}_{n+1}\}$, and it will be shown in Lemmas 5.9 and 5.10 that the formula holds for k = n+1.

Lemma 5.9: i) If $t \in \bar{D}_{n+1}$ and $t' < t$ with $t' \in D_n$, then $d(x_{t'},x_t) = (t - t')d(x_0,x_1)$.

ii) If $t \in \bar{D}_{n+1}$ and $t < t''$ with $t'' \in D_n$, then $d(x_t,x_{t''}) = (t'' - t)d(x_0,x_1)$.

Proof: i) Let t, t_1, and t_2 be as in (*). Then $t' \leq t_1 < t < t_2$, and

$$\begin{aligned} d(x_{t'},x_t) &\leq d(x_{t'},x_{t_1}) + d(x_{t_1},x_t) \\ &= (t_1 - t')d(x_0,x_1) + (t - t_1)d(x_0,x_1) \\ &= (t - t')d(x_0,x_1). \end{aligned}$$

Also $d(x_{t'},x_{t_2}) \leq d(x_{t'},x_t) + d(x_t,x_{t_2})$, so $(t_2 - t')d(x_0,x_1) \leq d(x_{t'},x_t) + (t_2 - t)d(x_0,x_1)$. Therefore, $d(x_{t'},x_t) \geq (t_2 - t')d(x_0,x_1) - (t_2 - t)d(x_0,x_1) = (t - t')d(x_0,x_1)$, and $d(x_{t'},x_t) = (t - t')d(x_0,x_1)$.

ii) Let t, t_1, and t_2 be as in (*). Then $t_1 < t < t_2 \leq t''$, and

$$\begin{aligned} d(x_t,x_{t''}) &\leq d(x_t,x_{t_2}) + d(x_{t_2},x_{t''}) \\ &= (t_2 - t)d(x_0,x_1) + (t'' - t_2)d(x_0,x_1) \\ &= (t'' - t)d(x_0,x_1). \end{aligned}$$

Also $d(x_{t_1},x_{t''}) \leq d(x_{t_1},x_t) + d(x_t,x_{t''})$, so $(t'' - t_1)d(x_0,x_1) \leq (t - t_1)d(x_0,x_1) + d(x_t,x_{t''})$. Therefore,

$d(x_t, x_{t''}) \geq (t'' - t_1)d(x_0,x_1) - (t - t_1)d(x_0,x_1) = (t'' - t)d(x_0,x_1)$, and $d(x_t, x_{t''}) = (t'' - t)d(x_0,x_1)$.

Q.E.D.

Lemma 5.10: If t', and t'' are in $\bar{D}_{n+1}$, and $t' \leq t''$, then $d(x_{t'}, x_{t''}) = (t'' - t')d(x_0,x_1)$.

Proof: We need only show the formula works for the case $t' < t''$. Pick $\bar{t}$, $t^* \in D_n$ such that $t' < \bar{t} < t'' < t^*$, then by Lemma 5.9

$$\begin{aligned} d(x_{t'}, x_{t''}) &\leq d(x_{t'}, x_{\bar{t}}) + d(x_{\bar{t}}, x_{t''}) \\ &= (\bar{t} - t')d(x_0,x_1) + (t'' - \bar{t})d(x_0,x_1) \\ &= (t'' - t')d(x_0,x_1). \end{aligned}$$

Now $d(x_{t'}, x_{t^*}) \leq d(x_{t'}, x_{t''}) + d(x_{t''}, x_{t^*})$, so by Lemma 5.9 $(t^* - t')d(x_0,x_1) \leq d(x_{t'}, x_{t''}) + (t^* - t'')d(x_0,x_1)$. Therefore, $d(x_{t'}, x_{t''}) \geq (t^* - t')d(x_0,x_1) - (t^* - t'')d(x_0,x_1) = (t'' - t')d(x_0,x_1)$, and $d(x_{t'}, x_{t''}) = (t'' - t')d(x_0,x_1)$.

Q.E.D.

So the diatic rationals $D = \bigcup_{n=0}^{\infty} D_n$, and we define $Z = \bigcup_{n=0}^{\infty} Z_n$.

Theorem 5.11: If Y is convex, metric and $x_0 \neq x_1$ in Y, then if D is the diatic rationals in [0,1], there is a subset Z of Y and a one-to-one function $\phi: D \xrightarrow{\text{onto}} Z$ such that $\phi(t) = x_t$, and for $t' \leq t''$, $d(x_{t'}, x_{t''}) = (t'' - t')d(x_0,x_1)$. In particular, if t is a diatic rational such that $0 \leq t \leq 1$, then $d(x_0,x_t) = td(x_0,x_1)$ and $d(x_t,x_1) = (1 - t)d(x_0,x_1)$.

Proof: The proof is done by induction. We define the sets D_n and Z_n inductively as was done in pages 34-35. Then Z is defined as was done above, and the function ϕ is defined inductively on the sets $\bar{D}_n$ by

$\phi(t) = x_t$ as was done on pages 34-35. The equality was shown to hold for $k = 0,1,2$ on page 34. Assuming the equality holds for $k = n$, it is shown to also hold for the case $k = n+1$ in Lemmas 5.9 and 5.10. Therefore, by induction it holds on all of D.

Q.E.D.

<u>Theorem 5.12</u>: If (Y,d) is a complete, convex metric space, and $x_0 \neq x_1 \in Y$, then there is a one-to-one function $\phi:[0,1] \xrightarrow{\text{onto}} Z \subseteq Y$ such that if $r' \leq r''$, then $d(x_{r'},x_{r''}) = (r'' - r')d(x_0,x_1)$.

<u>Proof</u>: Let r be a fixed but arbitrary element in $[0,1]$. Then there is a sequence $\{d_i\} \subseteq D$ such that $d_i \longrightarrow r$. Now since $\{x_{d_i}\}$ is cauchy and since Y is complete, there is an $x_r \in Y$ such that $x_{d_i} \longrightarrow x_r$.

Put $Z = \{x_r: \ r \in [0,1]\}$, and $\phi(r) = x_r$. It is only necessary to show the formula works for $r' < r''$. So let $\{d_i'\}$ and $\{d_i''\}$ be two sequences such that $d_i' \longrightarrow r'$ and $d_i'' \longrightarrow r''$. But $x_{d_i'} \longrightarrow x_{r'}$ and $x_{d_i''} \longrightarrow x_{r''}$, and there is an N such that for all $i \geq N$ $d_i'' > d_i'$. So

$$\begin{aligned} d(x_{r'},x_{r''}) &= \lim_{i\to\infty} d(x_{d_i'},x_{d_i''}) \\ &= \lim_{\substack{i\to\infty \\ i\geq N}} d(x_{d_i'},x_{d_i''}) \\ &= \lim_{\substack{i\to\infty \\ i\geq N}} (d_i'' - d_i')d(x_0,x_1) \\ &= (r'' - r')d(x_0,x_1). \end{aligned}$$

Q.E.D.

The following two lemmas will relate the concept of closure in $C_{f_d}(X,Y)$ and $C_d(X,Y)$. If $F \subseteq C(X,Y)$, then closure in $C_{f_d}(X,Y)$ will be written $cl_f(F)$, and closure in $C_d(X,Y)$ will be written $cl_u(F)$. Also keep in mind that $B(g,\epsilon) = \{f: d(f(x),g(x)) < \epsilon(x) \text{ for all } x \in X\}$ where $\epsilon \in C(X,R^+)$ and R^+ signifies all positive real numbers, while $S(g,\epsilon) = \{f: d(f(x),g(x)) < \epsilon \text{ for all } x \in X\}$ where $\epsilon \in R^+$.

Lemma 5.13: If $h \in cl_u(B(g,\epsilon))$, then $d(g(x),h(x)) \leq \epsilon(x)$ for all $x \in X$.

Proof: Assume the hypothesis is true and the conclusion is false. So there is an $x_0 \in X$ such that $d(g(x_0),h(x_0) > \epsilon(x_0)$. Put $1 = d(g(x_0),h(x_0)) - \epsilon(x_0)$, and $k \in S(h,1)$. Then $d(h(x),k(x)) < 1$ for all $x \in X$, and in particular, $d(h(x_0),k(x_0)) < 1$. Now

$$d(g(x_0),h(x_0)) \leq d(g(x_0),k(x_0)) + d(k(x_0),h(x_0))$$
$$< d(g(x_0),k(x_0)) + 1.$$ Therefore

$d(g(x_0),h(x_0)) < d(g(x_0),k(x_0)) + d(g(x_0),h(x_0)) - \epsilon(x_0)$, and $d(g(x_0),k(x_0)) > \epsilon(x_0)$. Therefore $k \in (B(g,\epsilon)^c$, and $S(h,1) \subsetneq (B(g,\epsilon))^c$, which is a contradiction.

Q.E.D.

A metric space Y is said to have the unique convergence property if for each $r \in [0,1]$, $x \neq y$; $d(z_n,x) \longrightarrow d(z_0,x) = rd(x,y)$, and $d(z_n,y) \longrightarrow d(z_0,y) = (1 - r)d(x,y)$ implies $z_n \longrightarrow z_0$. In a complete, convex metric space Y with the unique convergence property, there is for each $r \in [0,1]$ a unique z_0 such that $d(z_0,x) = rd(x,y)$ and $d(z_0,y) = (1 - r)d(x,y)$ for each pair of distinct points $x, y \in Y$.

Lemma 5.14: If Y is a complete, convex metric space with the unique convergence property, then $cl_u(B(f,\epsilon)) = cl_f(B(f,\epsilon))$.

Proof: ($\supseteq$) Now $cl_u(B(f,\epsilon))$ is a closed set in $C_d(X,Y)$ containing $B(f,\epsilon)$, so $cl_u(B(f,\epsilon))$ is a closed set in $C_{f_d}(X,Y)$. Since $cl_f(B(f,\epsilon))$ is the smallest closed set in $C_{f_d}(X,Y)$ containing $B(f,\epsilon)$, $cl_f(B(f,\epsilon)) \subseteq cl_u(B(f,\epsilon))$.

($\subseteq$) let $g \in cl_u(B(f,\epsilon))$, and $B(g,\delta)$ a basic neighborhood of g in $C_{f_d}(X,Y)$. We need only find $k \in b(g,\delta) \cap B(f,\epsilon)$. Put

$$p(x) = \min \{ \frac{d(f(x),g(x))}{\epsilon(x)} , \frac{\delta(x)}{d(f(x),g(x)) + 1} \},$$

and define $k(x)$ to be that unique z such that $d(g(x),z) = p(x)d(f(x),g(x))$ and $d(f(x),z) = (1 - p(x))d(f(x),g(x))$.

If $f(x) = g(x)$, then $d(g(x),z) = 0 < \delta(x)$. If $f(x) \neq g(x)$, then

$$\begin{aligned} d(g(x),z) &= p(x)d(f(x),g(x)) \\ &\leq \frac{\delta(x)}{d(f(x),g(x)) + 1} d(f(x),g(x)) \\ &< \delta(x). \end{aligned}$$

Therefore $d(g(x),k(x)) < \delta(x)$.

If $f(x) = g(x)$, then $d(f(x),z) = 0 < \epsilon(x)$. If $f(x) \neq g(x)$, then

$$\begin{aligned} d(f(x),z) &= (1 - p(x))d(f(x),g(x)) \\ &< d(f(x),g(x)) \\ &\leq \epsilon(x) \text{ by Lemma 5.13.} \end{aligned}$$

Therefore $d(f(x),k(x)) < \epsilon(x)$.

Consequently, we only need to show that $k \in C(X,Y)$. So let $\epsilon > 0$, and let x_0 be a fixed but arbitrary element in X. Since Y has the unique convergence property, there is a $\delta > 0$ such that if $|d(z,g(x_0)) - d(z_0,g(x_0))| < \delta$, and $|d(z,f(x_0)) - d(z_0,f(x_0))| < \delta$; then $d(z,z_0) < \epsilon$. Since $p(x)d(f(x),g(x))$ and $(1 - p(x))d(f(x),g(x))$ are continuous, there is an open neighborhood U_1 of x_0 such that if $x \in U_1$, then $|p(x)d(f(x),g(x)) - p(x_0)d(f(x_0),g(x_0))| < \delta/2$, and

$|(1 - p(x))d(f(x),g(x)) - (1 - p(x_0))d(f(x_0),g(x_0))| < \delta/2$. That is, $|d(g(x),k(x)) - d(g(x_0),k(x_0))| < \delta/2$, and $|d(f(x),k(x)) - d(f(x_0),k(x_0))| < \delta/2$. Since g(x) and f(x) are continuous, there is a U_2 such that if $x \in U_2$, then $d(g(x),g(x_0)) < \delta/2$ and $d(f(x),f(x_0)) < \delta/2$. So if $x \in U = U_1 \cap U_2$, it can be shown by manipulating the inequalities, that $|d(f(x_0),k(x)) - d(f(x_0),k(x_0))| < \delta$, and $|d(g(x_0),k(x)) - d(g(x_0),k(x_0))| < \delta$; and therefore $d(k(x),k(x_0)) < \epsilon$. Consequently, k is continuous at x_0.

Q.E.D.

Theorem 5.15: If Y is a convex metric space with the unique convergence property, then (Y,d) complete implies $C_{f_d}(X,Y)$ is pseudo-complete.

Proof: Since $C_{f_d}(X,Y)$ is Tychonoff, $C_{f_d}(X,Y)$ is certainly quasiregular. So define $\mathcal{P}_n = \{B(f,\epsilon): \epsilon \leq 1/2^n\}$, and assume $\{B(f_n,\epsilon_n)\}$ is such that $B(f_n,\epsilon_n) \in \mathcal{P}_n$ and $cl_f(B(f_{n+1},\epsilon_{n+1}) \subseteq B(f_n,\epsilon_n)$. We need only show that $\bigcap_{n=1}^{\infty} B(f_n,\epsilon_n) \neq \emptyset$. Now $\{f_n\}$ is a cauchy sequence in $C_d(X,Y)$. But $C_d(X,Y)$ is complete, so there is an $f \in C(X,Y)$ such that $f_n \longrightarrow f$ in $C_d(X,Y)$ However, for each n, $f_m \in B(f_{n+1},\epsilon_{n+1})$ for all m > n, and since $f_n \longrightarrow f$ in $C_d(X,Y)$, we have $f \in cl_u(B(f_{n+1},\epsilon_{n+1})) = cl_f(B(f_{n+1},\epsilon_{n+1}))$ by Lemma 5.14 $\subseteq B(f_n,\epsilon_n)$.

Therefore $f \in \bigcap_{n=1}^{\infty} B(f_n,\epsilon_n)$, and $C_{f_d}(X,Y)$ is pseudo-complete.

Q.E.D.

Corollary 5.16: If $Y = R^n$ with the usual metric d, then $C_{f_d}(X,Y)$ is pseudo-complete.

It is also my suspicion that the unique convergence property occurs in Banach spaces, and if this is true, the result of Corollary 5.16 will also hold for Banach spaces Y.

Now $C_{f_d}(X,Y)$ ($C_d(X,Y)$) pseudo-complete does not imply that Y is complete metric. To see this, pick X to be a single point set, and (Y,d) to be pseudo-complete but not complete. Also, Y in Theorems 5.4, 5.6, 5.8, and 5.15 can not be weakened to pseudo-complete. In [10] there is defined a pseudo-complete separable metric space Y such that $C_\pi(I,Y)$, $C_\kappa(I,Y)$, and $C_d(I,Y) = C_{f_d}(I,Y)$ where $I = [0,1]$ are first category, and therefore obviously not Baire.

BIBLIOGRAPHY

1. Aartz, J. M., and D. J. Lutzer, Pseudo-completeness and the product of Baire spaces, Pacific J. Math., 48 (1973), 1-10.

2. Beckenstein, E., L. Narici, and C. Suffel, Topological algebras, Notas de Matematica (60), North-Holland Publishing Company, New York, 1977.

3. Bourbaki, N., General topology, Addison-Wesley, Reading, Mass., 1966.

4. Dowker, C. H., On countably paracompact spaces, Canadian J. Math., 3 (1951), 219-224.

5. Engelking, R., Outline of general topology, North-Holland Publishing Company, New York, 1968.

6. Hansard, J. D., Function space topologies, Pacific J. Math., 35 (1970), No. 2, 381-388.

7. Hu, Sze-Tsen, Elements of general topology, Holden-Day, Inc., San Francisco, 1964.

8. Jackson, J. R., Comparison of topologies on function spaces, Proc. Amer. Math. Soc., 3 (1952), 156-158.

9. McCoy, R. A., Characterization of pseudocompactness by the topology of uniform convergence on function spaces, (to appear in J. Australian Math. Soc.).

10. McCoy, R. A., Function spaces with intervals as domain spaces, Fund. Math., 90 (1976), 189-198.

11. McCoy, R. A., The open-cover topology of function spaces, (to appear in Fund. Math.).

12. McCoy, R. A., Second countable and separable function spaces, (to appear in Amer. Math. Monthly).

13. McCoy, R. A., Submetrizable spaces and almost σ-compact function spaces, (to appear in Proc. Amer. Math. Soc.).

14. Munkries, J. R., Topology, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975.

15. Naimpally, S. A., Graph topology for function spaces, Trans. Amer. Math. Soc., 123 (1966), 267-272.

16. Noble, N., Products with closed projections, Trans. Amer. Math. Soc., 140 (1969), 381-391.

17. Steen, L. A. and J. A. Seebach, Jr., Counterexamples in topology, Holt, Rinehart and Winston, Inc., New York, 1970.

18. Willard, S., General topology, Addison-Wesley, Reading, Mass., 1970.

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THE FINE TOPOLOGY AND OTHER

TOPOLOGIES ON C(X,Y)

by

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(ABSTRACT)

The fine topology on C(X,Y) is defined as the topology generated by basic open sets $B(f,\epsilon) = \{g: d(f(x),g(x)) < \epsilon(x) \text{ and } x \in X\}$ where $f \in C(X,Y)$, d is a metric on Y, and ϵ is a positive continuous real valued function on X. This topological space is denoted by $C_{f_d}(X,Y)$ (or $C_f(X,Y)$ if the metric d is understood), and the topology is an obvious refinement to the topology for the uniform space $C_d(X,Y)$.

The fine topology is shown to fit in with the lattice of topologies on C(X,Y) which include the point-open and compact-open topologies, the open cover topology, and both the graph and m-topologies. The m-topology is the topology generated by basic open sets N(C) where C is a cozero set in $X \times Y$.

A characterization is made as to when $C_{f_d}(X,Y) = C_d(X,Y)$, and similar characterizations are noted between other topologies in the lattice. Separation properties are discussed for the topologies in the lattice, and it is shown, in particular, that $C_f(X,Y)$ is Tychonoff. Also, a characterization is made as to when $C_f(X,Y)$ is second countable (separable, Lindelöf), and some conditions are introduced which force $C_f(X,Y)$ to be pseudo-complete.