# THE PROBLEM OF CLASSIFYING MEMBERS OF A POPULATION ON A CONTINUOUS SCALE

by

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# I. DISCRIMINATION AND THE QUASI-RANK MULTIPLE CORRELATION COEFFICIENT

## 1.1 Basis and History of Discriminant Analysis

When individuals forming a sample can be classified into two or more groups, it is of interest to study how the classification of a given individual might be based on a set of measurements. Which measurable characteristics of an individual are relevant for this purpose is largely a matter for judgement of a specialist in the field of application.

An early example from the field of plant taxonomy is provided by Fisher (1936). Wishing to classify a given specimen of iris as Iris setosa or Iris versicolor, he utilizes measurements of sepal length, sepal width, petal length, and petal width.

A second example is given by Rao (1948) of a problem in anthropological classification in which an Indian individual is to be classified as belonging to one of three castes (Brahmin, Artisan, Korwa) on the basis of measurements upon four of his physical characteristics.

More recently, Anderson (1958) describes an example from the field of education - the admissions problem. Prospective students applying for admission into college are in one of two groups - those who have potentialities for successful completion of the work, and those who have not. Classification is based on the results of a battery of tests.

The several measurements can be combined in various ways to provide a score for the individual. Under certain standard assumptions it happens that a linear combination of the measurements is most useful in discriminating between individuals from distinct groups. In fact, one usually assumes that the vector of measurements for individuals selected at random from a given group has a multivariate normal distribution with covariance matrix $\Sigma$, $\Sigma$ being the same for each group.

Suppose there are just two groups, $\Pi_1$ and $\Pi_2$. Assume as suggested above that the vector-valued measurement $\underline{x}_i$ for an individual selected at random from $\Pi_i$ has a multivariate normal distribution with mean $\underline{\mu}_i$ and covariance matrix $\Sigma$, $i = i,2$. A randomly chosen individual with measurement vector $\underline{x}$ may then be from $\Pi_1$ or from $\Pi_2$. Let $R_1$ be the set of values of $\underline{x}$ for which the individual is classified as belonging to $\Pi_1$. Considering the problem from the standpoint of statistical decision functions (an approach first used by Wald (1944)), Anderson (1958) shows that the best region $R_1$ is of the form:

1.1.1 $$R_1 = \{ \underline{x} : \underline{x}'\Sigma^{-1} (\underline{\mu}_1 - \underline{\mu}_2) \geq k \}$$

where k depends on a-priori probabilities of $\Pi_1$ or $\Pi_2$ and also on the relative costs of misclassification. We note that the "discriminant function," $\underline{x}' \, \Sigma^{-1}(\underline{\mu}_1 - \underline{\mu}_2)$, is a linear function of the measurement components.

To form this discriminant function, one must know $\underline{\mu}_1$, $\underline{\mu}_2$, and $\Sigma$, a circumstance which can be assumed in the presence of a large amount of relevant data.

If this prior information is not available, one could for calibration purposes employ random samples $\underline{x}_{11}$, $\underline{x}_{12}$, ..., $\underline{x}_{1n_1}$ and $\underline{x}_{21}$, $\underline{x}_{22}$, ..., $\underline{x}_{2n_2}$ from $\Pi_1$, $\Pi_2$ respectively. With:

1.1.2 $$\bar{\underline{x}}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \underline{x}_{ij} \qquad i = 1,2$$

1.1.3 $$(n_1 + n_2 - 2)\, S = \sum_{i=1}^{2} \sum_{j=1}^{n_i} (\underline{x}_{ij} - \bar{\underline{x}}_i)(\underline{x}_{ij} - \bar{\underline{x}}_i)' \quad ,$$

one might use as criterion of classification (as does Wald) the statistic W:

1.1.4 $$W = \underline{x}' S^{-1} (\bar{\underline{x}}_1 - \bar{\underline{x}}_2) \quad .$$

Wald (1944) gives the large sample distribution of W and also investigates its exact distribution. His results for the exact distribution are neither simple nor in a form suitable for applicational use.

Generalizations can proceed in the direction of:

i) allowing more than two groups, or

ii) relaxing even further the assumptions concerning the underlying probability distribution.

Further discussion and references to the work of others may be found in excellent summaries by Anderson (1958) and Isaacson (1954).

## 1.2 Some Difficulties in the Standard Discriminant Analysis

Recall from above that unless large calibration samples are available one is faced with the necessity of developing better approximations to the distribution of W than are now available. There are other difficulties.

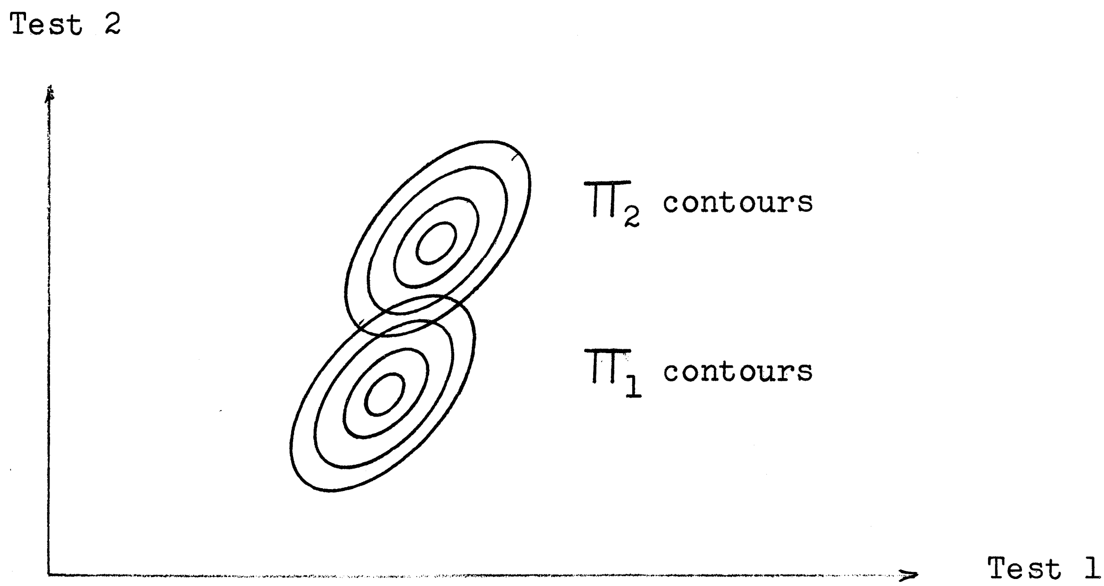
One is presumed to have a-priori probabilities of $\Pi_i$. Except in special cases, this information is at best only approximately known.

In the case of animal populations, one has the natural dichotomy of male and female. Students of plant taxonomy and anthropology proceed from the hypothesis of distinct and recognizable species, firmly established as biological responses to specialized conditions maintained over long periods of time. But one is perhaps less able to defend the hypothesis of distinct groups of human beings with respect to the possibility of achieving a given educational outcome.

Calibration samples are assumed to be drawn at random from $\Pi_1$ and $\Pi_2$. To appreciate the force of this assumption, consider the case in which just two measurements (or tests) are employed. Typical probability contours for $\Pi_1$ and $\Pi_2$ are represented in Figure I.

Figure I

Test Response Probabilities with Two Populations

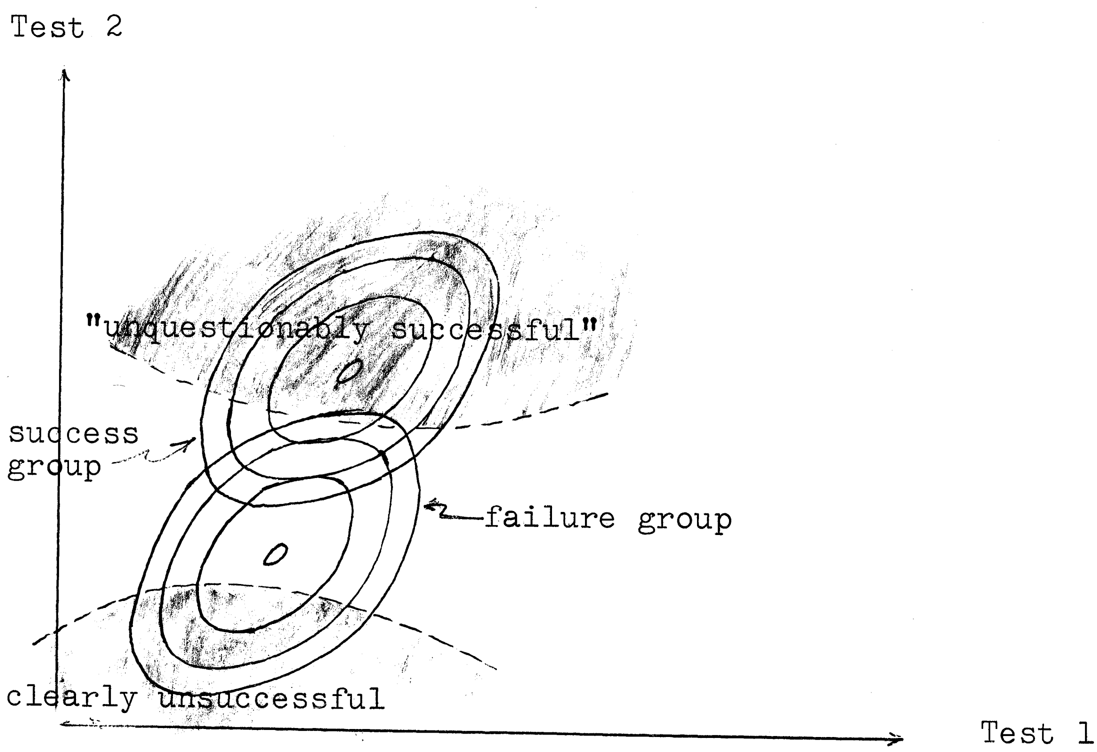



In the iris example one can readily conceive how the necessary random samples might be selected, utilizing well-identified pure plantings of iris.

With the admissions problem the situation is not so clear. It has been suggested that for calibration purposes one might use two groups - those students who are "unquestionably successful" and those "clearly unsuccessful", leaving out of consideration any students for whom a clear decision cannot be made one way or the other. An immediate consequence is that assumptions of normality and random selection of the measurements are no longer valid. Figure II is included to illustrate these remarks. If for example the failure group contours are horizontal sections of a normal surface, the probability distribution of the "clearly unsuccessful" group would hardly be normal.

Figure II

Nonnormality of the Probability Surface for Extreme Groups



The purpose of this thesis is to develop a working model for a class of problems, including the admissions problem, for which the assumptions can be more adequately justified.

## 1.3 Description and Assumptions of the Statistical Model

A battery of tests $T_1, \ldots, T_p$ is administered to each of n individuals: $\Omega_1, \ldots, \Omega_n$. The resulting (observable) scores for individual $\Omega_i$ will be denoted $x_{1i}, x_{2i}, \ldots, x_{pi}$. In addition, $\Omega_i$ has non-observable score $x_{0i}$ (on a criterion test $T_0$) which is reflected in a rank for $\Omega_i$ which can be observed. The n individuals are labeled so that this rank for $\Omega_i$ is i, and we then have the following array of data:

| TEST | INDIVIDUAL | | | |
|---|---|---|---|---|
| | $\Omega_1$ | $\Omega_2$ | $\cdots$ | $\Omega_n$ |
| $T_0$ | $(x_{01})$ | $(x_{02})$ | $\cdots$ | $(x_{0n})$ |
| $T_1$ | $x_{11}$ | $x_{12}$ | $\cdots$ | $x_{1n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $T_p$ | $x_{p1}$ | $x_{p2}$ | $\cdots$ | $x_{pn}$ |

in which $x_{01} < x_{02} < \cdots < x_{0n}$ .

To give a rather general setting for subsequent choice of a mathematical model, we note that for any individual we may observe a vector of measurements $\underline{x}$ and what we shall call an "indicator" d. The indicator may take a variety of forms; for example, one may put d = 0 if the individual lacks a characteristic (e.g., is in a failure group), d = 1 otherwise. Using the language of the admissions problem, we may further refine the classification by letting d take one of three values according as an individual is unsuccessful, unresolved, or successful. The most complete subdivision is by ranks, and it is to this situation that the work of this thesis is directed. Thus the indicator is itself a random variable correlated in some manner with the elements in vector $\underline{x}$.

A "discriminant function" which might well be used in the admissions problem is that linear combination of part scores which produces a maximum simple correlation with ranks. This maximum simple correlation, which we denote R, is acqually the multiple correlation of ranks with part scores. In the sequel we shall study the distribution of R, the "Quasi-rank" multiple correlation coefficient.

In particular, we find in Chapter 2 the null distribution of $R^2$, and that the $h^{\underline{th}}$ moment of $R^2$ for general p is expressible as the $h^{\underline{th}}$ moment of $R^2$ in the case p = 1 multiplied by a function of sample size and h alone. Chapter

3 is devoted, consequently, to the distribution of R in the case $p = 1$. We find there, for this case of $p = 1$, the first four raw moments of R. We then complete the discussion begun in Chapter 2, giving the first two raw moments of $R^2$ for general p.

At this stage, we are able to construct tests of independence based on the statistic $R^2$. Fitting a Pearson system density to the known moments of $R^2$ we approximate in Chapter 5 the power of such tests. In addition, we develop the asymptotic relative efficiency of $R^2$ compared with the standard multiple correlation coefficient, and illustrate our findings in a demonstration study.

The development in Chapter 3 required a knowledge of the first four moments of the statistic X, $X = \sum_1^n (i - \frac{n+1}{2}) w_i$, wherein the $w_i$ are the standard normal order statistics from a random sample of size n. These are found in Chapter 4.

As a linear combination of quasi-ranges, it was felt that the statistic X itself is of sufficient interest to warrant inclusion in a final chapter the joint moment generating function of $X^2$ and $S^2 = \sum_1^n (w_i - \bar{w})^2$ .

# II. SOME FIRST RESULTS ON THE DISTRIBUTION OF $R^2$

## 2.1 Invariance Properties of $R^2$

We first define some terms which are used in the sequel.

Let V be a p-square positive definite symmetric matrix and $\underline{\mu}' = (\mu_1, \mu_2, \ldots, \mu_p)$ be a vector of constants. When we require that a p-component vector $\underline{x} : \underline{x}' = (x_1, x_2, \ldots, x_p)$ have the p-variate normal distribution with mean $\underline{\mu}$ and dispersion matrix V; that is, that $\underline{x}$ have the density:

2.1.1 $$(2\pi)^{-\frac{p}{2}} |V|^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}(\underline{x} - \underline{\mu})' V^{-1} (\underline{x} - \underline{\mu})\right]$$

over the domain $-\infty < x_i < \infty \quad (i = 1,2,\ldots,n)$, we shall write for brevity:

2.1.2 $$\underline{x} \frown N_p [\ \underline{\mu} : V\ ] \ .$$

Let V be a p-square positive definite symmetric matrix and C be a p-square matrix with (i,j)-element $c_{ij} (= c_{ji})$ . When we require that matrix C have the Wishart distribution with $\nu$ degrees of freedom and dispersion matrix V; that is, that C have the density:

2.1.3 $$\frac{|V|^{-\frac{\nu}{2}} |C|^{\frac{\nu-p-1}{2}} \exp\left[\ -\tfrac{1}{2} \operatorname{tr} (V^{-1} C)\ \right]}{2^{\nu \frac{p}{2}} \pi^{\frac{1}{4}p(p-1)} \prod_{j=1}^{p} \Gamma\left[\frac{\nu+1-j}{2}\right]}$$

over the domain of all $c_{ij}$ for which C is positive definite, we shall write for brevity:

2.1.4 $$C \frown W_p\ [V : \nu] \quad .$$

It will be assumed that random vectors:

$$(x_{0i},\ x_{1i},\ \ldots,\ x_{pi}) \qquad i = 1,2,\ldots,n$$

have a (p + 1)-variate normal distribution.

Let $d_i$ be any "standardized measure of rank". That is, suppose that $d_i$ is a constant or "indicator" associated with rank i such that:

2.1.5 $$\sum_{1}^{n} d_i = 0 \qquad \text{and}$$

$$\sum_{1}^{n} d_i^2 = 1$$

Define:

2.1.7 $$\underline{x}_i' = (x_{1i},\ x_{2i},\ \ldots,\ x_{pi}) \qquad i = 1,2,\ldots,n$$

2.1.8 $$\bar{x}_t = \frac{1}{n} \sum_{i=1}^{n} x_{ti}$$

2.1.9 $$\underline{\bar{x}}' = (\bar{x}_1,\ \bar{x}_2,\ \ldots,\ \bar{x}_p)$$

The square of the multiple correlation of the $d_i$ with "part scores" $\underline{x}_i$ is then:

2.1.10 $$R^2 = \sum_{1}^{n} d_i(\underline{x}_i - \underline{\bar{x}})' \left[ \sum_{1}^{n} (\underline{x}_i - \underline{\bar{x}})(\underline{x}_i - \underline{\bar{x}})' \right]^{-1} \sum_{1}^{n} d_i(\underline{x}_i - \underline{\bar{x}})$$

It may seem that for the most complete generality one should assume:

2.1.11 $$\begin{bmatrix} x_{0i} \\ \cdots \\ \underline{x}_i \end{bmatrix} \frown N_{p+1} \left\{ \underline{\mu} : \begin{bmatrix} \sigma_{11} & \sigma'_{(1)} \\ \sigma_{(1)} & \Sigma_{22} \end{bmatrix} \right\}.$$

That $R^2$ does not depend on $\underline{\mu}$ is clear; so we may as well take $\underline{\mu} = \underline{0}$ for the discussion of $R^2$ defined by equation (2.1.10).

It is also easily seen that $R^2$ is invariant under transformations of the type:

$$\begin{bmatrix} y_{0i} \\ \cdots \\ \underline{y}_i \end{bmatrix} = \begin{bmatrix} a & \underline{0}' \\ \underline{0} & D \end{bmatrix} \begin{bmatrix} x_{0i} \\ \cdots \\ \underline{x}_i \end{bmatrix}$$

with $a \neq 0$ and D nonsingular.

Taking $a = \sigma_{11}^{-\frac{1}{2}}$ and D such that

2.1.13 $$D\Sigma_{22} D' = I_p$$

then:

2.1.14 $$\mathrm{Cov} \begin{bmatrix} y_{0i} \\ \cdots \\ \underline{y}_i \end{bmatrix} = \begin{bmatrix} 1 & \underline{\rho}' \\ \underline{\rho} & I_p \end{bmatrix}$$

where:

2.1.15 $$\underline{\rho} = \sigma_{11}^{-\frac{1}{2}} D \, \sigma_{(1)} .$$

Thus, we may actually without loss in generality take:

2.1.16 $$\begin{bmatrix} x_{0i} \\ \cdots \\ \underline{x}_i \end{bmatrix} \frown N_{p+1}\left\{ \underline{0} : \begin{bmatrix} 1 & \underline{\rho} \\ \underline{\rho} & I_p \end{bmatrix} \right\}$$

for the discussion of the distribution of $R^2$ defined in equation (2.1.10).

Now the square of the population multiple correlation coefficient between $x_{0i}$ and $\underline{x}_{(i)}$ is, under the assumption of equation (2.1.11), given by:

2.1.17 $$R_0^2 = \frac{\sigma_{(1)}' \Sigma_{22}^{-1} \sigma_{(1)}}{\sigma_{11}} .$$

But, from equation (2.1.13) we have:

2.1.18 $$\Sigma_{22}^{-1} = D'D .$$

Thus, $R_0^2$ may be written as:

$$R_0^2 = \frac{\sigma_{(1)}' D'D \sigma_{(1)}}{\sigma_{11}} = \left[\sigma_{11}^{-\frac{1}{2}} D \sigma_{(1)}\right]' \left[\sigma_{11}^{-\frac{1}{2}} D \sigma_{(1)}\right] .$$

In view of the equation (2.1.15) we have:

2.1.19 $$R_0^2 = \underline{\rho}' \underline{\rho} .$$

## 2.2 Lemmas

We list a series of lemmas which will prove useful in the development of the distribution of $R^2$. Since they are little more than special cases of well-known theorems, their proofs will be but briefly indicated.

Lemma 1.

If:

(i) $\underline{u}_i \frown N_p(\underline{0} : I)$ $\qquad j = 1,2,\ldots,N$

(ii) $\underline{u}_i$ is independent of $\underline{u}_j$ for $i \neq j$

(iii) $A = (a_{ij})$ is a symmetric idempotent matrix of rank $t$ $(t \geq p)$

then:

2.2.1
$$(\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_N)\, A\, (\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_N)'$$
$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}\, \underline{u}_i\, \underline{u}_j' \frown W_p\,(I : t) \quad .$$

Proof:

This is a special case of Corollary 7.4.1, p. 165, Anderson (1958).

Lemma 2.

If:

(i) $\underline{x}_i \frown N_p\left\{\underline{\mu}^{(i)} : M\right\}$ $\qquad i = 1,2,\ldots,n$

(ii) $\underline{x}_i$ is independent of $\underline{x}_j$ for $i \neq j$

(iii) $$\begin{bmatrix} \alpha a_1 & \bigg| & \\ \alpha a_2 & \bigg| & \\ \vdots & \bigg| & K_1 \\ \alpha a_n & \bigg| & \end{bmatrix}$$ is orthogonal,

$$\alpha^{-2} = \sum_1^n a_i^2$$

$$(K_1)_{ij} = k_{ij}$$

(iv) $$\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \end{bmatrix} = \begin{bmatrix} a_1 & \bigg| & \\ a_2 & \bigg| & \\ \vdots & \bigg| & K_1 \\ a_n & \bigg| & \end{bmatrix} \begin{bmatrix} \underline{\tilde{x}} \\ \underline{u}_1 \\ \vdots \\ \underline{u}_\nu \end{bmatrix} \qquad \nu = n - 1$$

then:

2.2.2a $$\underline{\tilde{x}} = \alpha^2 \sum_1^n a_i \, \underline{x}_i \frown N_p \quad \alpha^2 \left\{ \sum_1^n a_i \underline{\mu}^{(i)} \; : \; \alpha^2 M \right\}$$

2.2.2b $$\underline{u}_i \frown N_p \left\{ \sum_{m=1}^n k_{mi} \, \underline{\mu}^{(m)} \; : \; M \right\} \qquad i = 1,2,\ldots,\nu$$

2.2.2c $\underline{u}_i$ is independent of $\underline{u}_j$ for $i \neq j$

2.2.2d $\underline{\tilde{x}}$ is independent of $\underline{u}_i$ , $\qquad i = 1,2,\ldots,\nu$ .

2.2.2e $$\sum_1^n (\underline{x}_i - a_i \, \underline{\tilde{x}}) \, (\underline{x}_i - a_i \, \underline{\tilde{x}})' = \sum_1^\nu \underline{u}_i \, \underline{u}_i'$$

Proof:

From condition (iii):

2.2.3 $$\begin{bmatrix} a_1 & \Big| & \\ a_2 & \Big| & K_1 \\ \vdots & \Big| & \\ a_n & \Big| & \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{\alpha^2 a_1 \ \alpha^2 a_2 \ \cdots \ \alpha^2 a_n}{K_1'} \end{bmatrix} .$$

Hence condition (iv) can be written in the form:

2.2.4 $$\begin{bmatrix} \tilde{\underline{x}} \\ \underline{u}_1 \\ \vdots \\ \underline{u}_\nu \end{bmatrix} = \begin{bmatrix} \dfrac{\alpha^2 a_1 \ \alpha^2 a_2 \ \cdots \ \alpha^2 a_n}{K_1'} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \end{bmatrix} .$$

Thus (2.2.2a) and (2.2.2b) clearly follow from condition (i).

Next, identify:

2.2.5 $$\tilde{\underline{x}} = \underline{u}_n \ , \quad (\alpha^2 a_1, \alpha^2 a_2, \ldots, \alpha^2 a_n) = (k_{1n}, k_{2n}, \ldots, k_{nn})$$

Then (2.2.4) may be written in the form:

2.2.6 $$\begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_n \end{bmatrix} = \begin{bmatrix} k_{11} & k_{21} & \cdots & k_{n1} \\ k_{12} & k_{22} & \cdots & k_{n2} \\ \vdots & \vdots & & \vdots \\ k_{1n} & k_{2n} & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \end{bmatrix} .$$

It is now simple to verify that component $\alpha$ of $\underline{u}_s$ is independent of component $\beta$ of $\underline{u}_t$ for $s \neq t$ and $\begin{matrix}\alpha \\ \beta\end{matrix} = 1,2,\ldots,p$. Thus (2.2.2c) and (2.2.2d) are true.

Finally, (2.2.2e) is an algebraic fact which follows easily on writing

$$\text{2.2.7} \qquad \begin{bmatrix} \underline{x}_1 - a_1 \underline{\tilde{x}} \\ \underline{x}_2 - a_2 \underline{\tilde{x}} \\ \vdots \\ \underline{x}_n - a_n \underline{\tilde{x}} \end{bmatrix}' = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_\nu \end{bmatrix}' K_1'$$

and noting that $K_1'K_1 = I_\nu$ .

Lemma 3.

Let $\underline{x}_i' = (x_{1i}, x_{2i}, \ldots, x_{pi})$ $\qquad i = 1,2,\ldots,n$ .

Suppose that:

(i) $\qquad \begin{bmatrix} x_{0i} \\ \cdots \\ \underline{x}_i \end{bmatrix} \frown N_{p+1}\left\{\begin{bmatrix} 0 \\ \cdots \\ \underline{0} \end{bmatrix} : \begin{bmatrix} 1 & \underline{\rho}' \\ \underline{\rho} & I_p \end{bmatrix}\right\}$ $i = 1,2,\ldots,n$

and that

(ii) $\qquad x_{01} < x_{02} < \ldots < x_{0n}$ .

Then:

2.2.8a $\qquad \underline{x}_i \mid x_{0i} \frown N_p [\underline{\rho}\, x_{0i} : I_p - \underline{\rho}\, \underline{\rho}'] \qquad i = 1,2,\ldots,n$

and:

2.2.8b $\qquad f(\underline{x}_i, \underline{x}_j | x_{0i}, x_{0j}) = f(\underline{x}_i | x_{0i})\, f(\underline{x}_j | x_{0j}) \qquad i \neq j$

i.e., $\underline{x}_i$ and $\underline{x}_j$ are independent conditional on $(x_{0i}, x_{0j})$, and the distribution of $\underline{x}_i$ conditional on $(x_{0i}, x_{0j})$ is independent of $x_{0j}$.

Proof:

The strategy is to find the joint density of $\underline{x}_r | x_{0r}$ and $\underline{x}_s | x_{0s}$ for $r < s$.

To do this, consider the joint density of all the $x_{ij}$:

2.2.9 $$\frac{n!}{(\sqrt{2\pi})^{(p+1)n}\, \tau^n} \exp \left\{-\tfrac{1}{2} \sum_1^n (x_{0i}^2 + Q_i)\right\}$$

where:

2.2.10 $$\tau = |I_p - \underline{\rho}\, \underline{\rho}'|^{\frac{1}{2}} = (1 - \underline{\rho}'\underline{\rho})^{\frac{1}{2}}$$

and

2.2.11 $$Q_i = (\underline{x}_i - \underline{\rho}\, x_{0i})' [I_p - \underline{\rho}\, \underline{\rho}']^{-1} (\underline{x}_i - \underline{\rho}\, x_{0i})$$

Integrate out variates $(x_{0i} : \underline{x}_i')$ for all i except for $i = r$ and $i = s$.

Dividing this result by the joint density of $x_{0r}$, $x_{0s}$, we obtain:

2.2.12 $$\frac{1}{(\sqrt{2\pi})^p \tau} \exp\{-\tfrac{1}{2} Q_r\} \cdot \frac{1}{(\sqrt{2\pi})^p \tau} \exp\{-\tfrac{1}{2} Q_s\} \quad .$$

Thus, conclusions (2.2.8a) and (2.2.8b) are both valid.

## 2.3 Distribution of $R^2$: Some First Results Under the Alternative Hypothesis.

Lemma 3 provides a point of departure in developing the distribution of $R^2$.

Thus, we discuss the distribution of $R^2$ given by Formula (2.1.10), where

$\underline{x}_i | x_{0i} \sim N_p\{\underline{\rho}\, x_{0i} : I_p - \underline{\rho}\, \underline{\rho}'\}$

$\underline{x}_i | x_{0i}$ is independent of $\underline{x}_j | x_{0j}$ for $i \neq j$

$d_i$ is such that $\sum_1^n d_i = 0$ and $\sum_1^n d_i^2 = 1$.

To simplify the form of $R^2$, we first employ a Helmert transformation.

with:

2.3.1
$$K = \begin{bmatrix} 1 & \frac{1}{\sqrt{1.2}} & \frac{1}{\sqrt{2.3}} & \frac{1}{\sqrt{3.4}} & \cdots & \frac{1}{\sqrt{(n-1)n}} \\ 1 & \frac{-1}{\sqrt{1.2}} & \frac{1}{\sqrt{2.3}} & \frac{1}{\sqrt{3.4}} & \cdots & \frac{1}{\sqrt{(n-1)n}} \\ 1 & 0 & \frac{-2}{\sqrt{2.3}} & \frac{1}{\sqrt{3.4}} & \cdots & \frac{1}{\sqrt{(n-1)n}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & \frac{-(n-1)}{\sqrt{(n-1)n}} \end{bmatrix}$$

let

2.3.2
$$[\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n]' = K[\bar{\underline{x}}, \underline{u}_1, \ldots, \underline{u}_\nu]', \ \nu = n-1 \ .$$

Also, let:

2.3.3
$$[x_{01}, x_{02}, \ldots, x_{0n}]' = K[\bar{x}_0, u_{01}, \ldots, u_{0\nu}]'$$

and:

2.3.4
$$[d_1, d_2, \ldots, d_n]' = K[\bar{d}, e_1, \ldots, e_\nu]' \ .$$

Then

2.3.5
$$\sum_1^n d_i(\underline{x}_i - \bar{\underline{x}}) = \sum_1^\nu e_i \underline{u}_i$$

2.3.6
$$\sum_1^n (\underline{x}_i - \bar{\underline{x}})(\underline{x}_i - \bar{\underline{x}})' = \sum_1^\nu \underline{u}_i \underline{u}_i'$$

and:

2.3.7 $$R^2 = [\sum_1^{\nu} e_i\underline{u}_i'] [\sum_1^{\nu} \underline{u}_i \; \underline{u}_i']^{-1} [\sum_1^{\nu} e_i\underline{u}_i] \quad .$$

By Lemma 2,

2.3.8 $$\underline{u}_i | u_{0i} \frown N_p\{\underline{\xi}_i \;\; : \;\; I_p - \underline{\rho}\;\underline{\rho}'\} \quad i = 1,2,\ldots,n$$

and:

$\underline{u}_i | u_{0i}$ is independent of $\underline{u}_j | u_{0j}$ for $i \neq j$

where:

2.3.9 $$\underline{\xi}_i = \underline{\rho} \sum_{m=1}^{n} k_{mi}\, x_{0m} = \underline{\rho}\; u_{0i} \quad .$$

Next, let $\Omega$ be orthogonal with first row:

$$\frac{1}{\tau}\,(\rho_1, \rho_2, \ldots, \rho_p), \text{ where } \tau^2 = \sum_1^{p} \rho_i^{\,2} \quad .$$

Then:

$$\underline{w}_i = \Omega\, \underline{u}_i \qquad\qquad i = 1,2,\ldots,\nu$$

implies that

2.3.10 $$\underline{w}_i | u_{0i} \frown N_p \left\{ \begin{bmatrix} \tau u_{0i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \; : \; \begin{bmatrix} 1-\tau^2 & \underline{0}' \\ 0 & \\ \vdots & I_{p-1} \\ 0 & \end{bmatrix} \right\}$$

2.3.11 $$R^2 = \tilde{\underline{w}}' \left[\sum_1^{\nu} \underline{w}_i \, \underline{w}_i'\right]^{-1} \tilde{\underline{w}}$$

where:

2.3.12 $$\tilde{\underline{w}} = \sum_1^{\nu} e_i \, \underline{w}_i \quad .$$

Let:

2.3.13 $$M = \begin{bmatrix} 1-\tau^2 & \underline{0}' \\ \underline{0} & I_{p-1} \end{bmatrix} .$$

Employ transformation:

2.3.14 $$\underline{y}_i = M^{-\frac{1}{2}} \underline{w}_i \qquad i = 1,2,\ldots,\nu \quad .$$

Then:

2.3.15 $$\underline{y}_i \mid u_{0i} \frown N_p \left\{ \begin{bmatrix} \delta \, u_{0i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} : I_p \right\}$$

where $\delta = \dfrac{\tau}{\sqrt{1-\tau^2}}$ .

In terms of the population multiple correlation coefficient $R_0$,

$$\delta = \frac{R_0}{\sqrt{1-R_0^2}} \quad .$$

Also,

2.3.16 $$R^2 = \underline{\tilde{y}}'[\sum_1^{\nu} \underline{y}_i \, \underline{y}_i']^{-1} \underline{\tilde{y}}$$

where:

2.3.17 $$\underline{\tilde{y}} = \sum_1^{\nu} e_i \, \underline{y}_i \quad .$$

$R^2$ may be written in the alternative form:

2.3.18 $$R^2 = \underline{\tilde{y}}'[W + \underline{\tilde{y}} \, \underline{\tilde{y}}']^{-1} \, \underline{\tilde{y}}$$

where:

2.3.19 $$W = \sum_1^{\nu} (\underline{y}_i - e_i \, \underline{\tilde{y}}) \, (\underline{y}_i - e_i \, \underline{\tilde{y}})' \quad .$$

Finally, introduce $\underline{z}_i$ $i = 1,2,\ldots,\nu-1$ by an orthogonal transformation:

2.3.20 $$\begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_\nu \end{bmatrix} = \left[\begin{array}{c|c} e_1 & \\ e_2 & \\ \vdots & K_1 \\ e_\nu & \end{array}\right] \begin{bmatrix} \underline{\tilde{y}} \\ \underline{z}_1 \\ \vdots \\ \underline{z}_{\nu-1} \end{bmatrix} \quad .$$

By Lemma 2,

2.3.21 $$\underline{\tilde{y}} \sim N_p \left\{ \begin{bmatrix} \delta \tilde{u}_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} : I_p \right\}, \quad \tilde{u}_0 = \sum_1^{\nu} e_i \, u_{0i} \; .$$

2.3.22 $$\underline{z}_i \sim N_p \left\{ \begin{bmatrix} \delta \sum\limits_{m=1}^{\nu} k_{mi} u_{0m} \\ 0 \\ \vdots \\ 0 \end{bmatrix} : I_p \right\} \quad i = 1,2,\ldots,\nu-1$$

$\underline{\tilde{y}}$ and $\underline{z}_i$ are mutually independent, $i = 1,2,\ldots,\nu-1$

and:

2.3.23 $$W = \sum_1^{\nu-1} \underline{z}_i \, \underline{z}_i' \; .$$

Thus,

2.3.24 $$R^2 = \underline{\tilde{y}}' [W + H]^{-1} \, \underline{\tilde{y}}$$

where:

2.3.25 $$H = \underline{\tilde{y}} \, \underline{\tilde{y}}' \; .$$

We note that W and H are independent, conditional on the $x_{0i}$.

Now:

2.3.26
$$1 - R^2 = | 1 - R^2 |$$
$$= | 1 - \tilde{\underline{y}}' [W + H]^{-1} \tilde{\underline{y}} | \quad .$$

But:

$$\begin{vmatrix} 1 & \tilde{\underline{y}}' \\ \tilde{\underline{y}} & W + H \end{vmatrix} = | W + H | (1 - R^2)$$
$$= | W + H - H | \quad .$$

Hence:

2.3.27
$$1 - R^2 = \frac{| W |}{| W + H |} \quad .$$

Partitioning W and H:

$$W = \begin{array}{cc} & \begin{array}{cc} 1 & p-1 \end{array} \\ \begin{array}{c} 1 \\ p-1 \end{array} & \begin{bmatrix} w_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \end{array}$$

$$H = \begin{array}{cc} & \begin{array}{cc} 1 & p-1 \end{array} \\ \begin{array}{c} 1 \\ p-1 \end{array} & \begin{bmatrix} h_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \end{array}$$

we may write

2.3.28 $\quad 1 - R^2 =$

$$\frac{w_{11} \mid W_{22} - w_{11}^{-1} W_{21}W_{12} \mid}{(w_{11}+h_{11}) \mid W_{22}+H_{22}+(w_{11}+h_{11})^{-1}(W_{21}+H_{21})(W_{12}+H_{12}) \mid}$$

$$= \frac{w_{11}}{w_{11} + h_{11}} \frac{|G|}{|L|} \quad .$$

We shall now see with the aid of Lemma 1 that G and L have Wishart distributions which are independent of $w_{11}$ and $h_{11}$, and that in fact $|G| / |L|$ has a Beta distribution.

Let:

2.3.29 $$\underline{\tilde{y}} = \begin{bmatrix} \tilde{y}_1 \\ \cdots \\ \underline{\tilde{y}}_{(2)} \end{bmatrix}$$

and

2.3.30 $$\underline{z}_i = \begin{bmatrix} z_{1i} \\ \cdots \\ \underline{z}_{(2)i} \end{bmatrix} \quad i = 1,2,\ldots,\nu-1$$

so that

2.3.31 $$h_{11} = \tilde{y}_1^2$$

2.3.32 $$w_{11} = \sum_1^{\nu-1} z_{1i}^2 \quad .$$

In view of statements (2.3.21) and (2.3.22) we see that $h_{11}$ and $w_{11}$ are independent noncentral Chi-square variates with noncentrality parameters we denote by $\lambda_1$ and $\lambda_2$ respectively.

Now $\lambda_1$ and $\lambda_2$ can be related to the $x_{0i}$.

Rewriting K, definition (2.3.1),

2.3.33 $$K = \left[\begin{array}{c|c} 1 & \\ 1 & \\ \vdots & P \\ 1 & \end{array}\right]$$

we have:

2.3.34 $$P'P = I_\nu$$

2.3.35 $$P\,P' = I_n - \frac{1}{n}\,J_n$$

where $J_n$ denotes the n-square matrix with each element unity.

Using notions:

2.3.36
$$\begin{aligned}(e_1,e_2,\ldots,e_\nu)' &= \underline{e}\\ (d_1,d_2,\ldots,d_n)' &= \underline{d}\\ (\mu_{01},u_{02},\ldots,\mu_{0\nu})' &= \underline{u}_0\\ (x_{01},x_{02},\ldots,x_{0n})' &= \underline{x}_0\end{aligned}$$

and returning to equations (2.3.3) and (2.3.4), we see that:

2.3.37 $$\underline{e}' = \underline{d}'P$$

2.3.38 $$\underline{u}_0' = \underline{x}_0'P \quad .$$

Thus:

2.3.39 $$\tilde{u}_0 = \underline{e}'\underline{u}_0 = \underline{d}'PP'\underline{x}_0$$

$$= \sum_1^n d_i x_{0i} \quad ,$$

and

2.3.40 $$\lambda_1 = \tfrac{1}{2}\delta^2 \left[\sum_1^n d_i x_{0i}\right]^2 \quad .$$

Also, from (2.3.22),

2.3.41 $$\lambda_2 = \tfrac{1}{2}\delta^2 \sum_{i=1}^{\nu-1} \left[\sum_{m=1}^{\nu} k_{mi}\, u_{0m}\right]^2$$

$$= \tfrac{1}{2}\,\delta^2\, \underline{u}_0' K_1\, K_1' \underline{u}_0 \quad .$$

In view of the orthogonality of transformation (2.3.20),

2.3.42 $$K_1\, K_1' = I_\nu - \underline{e}\,\underline{e}'$$

$$= I_\nu - P'\underline{d}\,\underline{d}'P \quad .$$

Thus,

$$2.3.43 \qquad \underline{u}_0' K_1 K_1' \underline{u}_0 = \underline{x}_0' P[I_\nu - P'\underline{d}\ \underline{d}'P]\ P'\underline{x}_0$$

$$= \sum_1^n (x_{0i} - \bar{x}_0)^2 - \left[\sum_1^n d_i x_{0i}\right]^2$$

and:

$$2.3.44 \qquad \lambda_2 = \tfrac{1}{2}\,\delta^2 \left\{ \sum_1^n (x_{0i} - \bar{x}_0)^2 - \left[\sum_1^n d_i\ x_{0i}\right]^2 \right\} .$$

In summary, conditional on the $x_{0i}$; $h_{11}, w_{11}$ have independent noncentral Chi-square distributions with degrees of freedom 1, $\nu$-1 and noncentrality parameters $\lambda_1$, $\lambda_2$ respectively. That is, conditionally:

$$2.3.45 \qquad h_{11} \frown \chi_1'^2\ (\lambda_1)$$

$$2.3.46 \qquad w_{11} \frown \chi'^2_{\nu-1}\ (\lambda_2) .$$

In deriving the distribution of G, it is convenient to discuss first of all the distribution of G conditional on $z_{11}, z_{12}, \ldots z_{1:\nu-1}$ .

Using notations:

$$2.3.47 \qquad J_{\nu-1} = (1)$$

$$2.3.48 \qquad D_{\nu-1} = \text{diag}\ (z_{11}, z_{12}, \ldots, z_{1:\nu-1})$$

G may be written:

2.3.49 $$G = \sum_{s=1}^{\nu-1} \sum_{t=1}^{\nu-1} a_{st} \underline{z}_{(2)s} \underline{z}'_{(2)t}$$

where $a_{st} = (A)_{st}$

and

2.3.50 $$A = I_{\nu-1} - \frac{1}{w_{11}} D_{\nu-1} J_{\nu-1} D_{\nu-1} \quad .$$

Noting that $J_{\nu-1} D^2_{\nu-1} J_{\nu-1} = w_{11} J_{\nu-1}$, we see that A is idempotent. Also:

2.3.51 $$\text{tr } (A) = n-3$$

Thus, by Lemma 1,

2.3.52 $$G \sim W_{p-1}(I : n-3) \quad .$$

Since the conditional distribution of G given by (2.3.52) does not depend upon $z_{11}$, $z_{12}$, ..., $z_{1:\nu-1}$, it is actually the unconditional distribution of G.

In the same way, consider the distribution of L conditional on $(z_{11}, z_{12}, \ldots, z_{1:\nu-1}, \tilde{y}_1)$.

For ease in discussing this distribution, identify:

2.3.53 $$\begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \underline{\tilde{y}}_{(2)} \end{bmatrix} = \begin{bmatrix} z_{1\nu} \\ \cdots \\ \underline{z}_{(2)\nu} \end{bmatrix} \quad .$$

Then,

2.3.54 $\quad \underline{z}_{(2)i} \sim N_{p-1}\,\{\underline{0} : I\} \qquad i = 1,2,\ldots,\nu$

and $\underline{z}_{(2)i}$ is independent of $\underline{z}_{(2)j}$ for $i \neq j$.

With

2.3.55 $\quad D_\nu = \text{diag}\,(z_{11}, z_{12}, \ldots, z_{1\nu})$

L may be written:

2.3.56 $\quad L = \sum_{1}^{\nu} \sum_{1}^{\nu} b_{st}\, \underline{z}_{(2)s}\, \underline{z}'_{(2)t}$

where

$$b_{st} = (B)_{st}$$

and

2.3.57 $\quad B = I_\nu - \dfrac{1}{w_{11} + h_{11}} D_\nu J_\nu D_\nu \quad .$

Noting that $J_\nu D_\nu^2 J_\nu = (w_{11} + h_{11}) J_\nu$, we see that B is idempotent. Also:

2.3.58 $\quad \text{tr}\,(B) = n-2$ .

Thus, by Lemma 1,

2.3.59 $\quad L \sim W_{p-1}\,(I : n-2)$ ,

and this is the unconditional distribution of L.

Also, both G and L are independent of $w_{11}$ and of $h_{11}$, and thus from (2.3.28):

$$\text{2.3.60} \qquad \mathcal{E}\left\{[1-R^2]^h\right\} = \mathcal{E}\left\{\left[\frac{w_{11}}{w_{11}+h_{11}}\right]^h\right\} \mathcal{E}\left\{\frac{|G|^h}{|L|^h}\right\} ,$$

conditional on the $x_{0i}$ .

It is our next goal to show that $\frac{|G|}{|L|}$ has a Beta distribution.

To this end, consider the expression

$$\text{2.3.61} \qquad \frac{|\tilde{W}|}{|\tilde{W}+\tilde{H}|}$$

with

$$\text{2.3.62} \qquad \tilde{W} = \sum_{1}^{\nu-1} \underline{z}_i\, \underline{z}_i' \quad , \quad \tilde{H} = \underline{\tilde{y}}\, \underline{\tilde{y}}'$$

as in the allied expression (2.2.27) .

But this time we assume:

2.3.63 $\quad \underline{z}_i \frown N_p\ (\underline{0} : I) \qquad\qquad i = 1,2,\ldots,\nu-1$

$\underline{z}_i$ independent of $\underline{z}_j \quad i \neq j$

2.3.64 $\quad \underline{\tilde{y}} \frown N_p\ (\underline{0} : I)$

$\underline{\tilde{y}}$ independent of $\underline{z}_i \qquad\qquad i = 1,2,\ldots,\nu-1$

Carrying through the same partitioning and discussion of expression (2.3.60) as we did for expression (2.3.27), one obtains:

2.3.65 $$\frac{|\tilde{W}|}{|\tilde{W} + \tilde{H}|} = \frac{\tilde{w}_{11}}{\tilde{w}_{11} + \tilde{h}_{11}} \quad \frac{|G|}{|L|}$$

in which G and L have Wishart distributions identical with those previously obtained, independent of $\tilde{w}_{11}$ and $\tilde{h}_{11}$.

But now, $\tilde{w}_{11}$ and $\tilde{h}_{11}$ have independent <u>central</u> Chi-square distributions:

2.3.66 $$\tilde{w}_{11} \frown \chi^2_{n-2}$$

2.3.67 $$\tilde{h}_{11} \frown \chi^2_{1} \;.$$

Thus,

2.3.68 $$\frac{|G|}{|L|} = \frac{|\tilde{W}|}{|\tilde{W} + \tilde{H}|} \left[\frac{\tilde{w}_{11}}{\tilde{w}_{11} + \tilde{h}_{11}}\right]^{-1}$$

Now, under assumptions (2.3.62) and (2.3.63),

2.3.69 $\quad \tilde{W} \frown W_p\ (I : \nu-1)$

2.3.70 $\quad \tilde{H} \frown W_p\ (I : 1) \quad .$

Since $\tilde{W}$ and $\tilde{H}$ are independent,

2.3.71 $$\frac{|\tilde{W}|}{|\tilde{W} + \tilde{H}|} \frown U_{p:1:\nu-1}$$

and thus* $\dfrac{|\tilde{W}|}{|\tilde{W} + \tilde{H}|}$ has the density:

2.3.72 $$\frac{1}{B[\frac{\nu-p}{2}, \frac{p}{2}]} u^{\frac{\nu-p}{2}-1} (1-u)^{\frac{p}{2}-1} \qquad 0 < u < 1 \quad .$$

Also, $\dfrac{\tilde{w}_{11}}{\tilde{w}_{11} + \tilde{h}_{11}}$ is independent of $\dfrac{|G|}{|L|}$

and has the density:

2.3.73 $$\frac{1}{B[\frac{\nu-1}{2}, \frac{1}{2}]} u^{\frac{\nu-1}{2}-1} (1-u)^{\frac{1}{2}-1} \qquad 0 < u < 1 .$$

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*For a discussion of the U-statistic, see Anderson, T.W., (1958), pp. 191-202.

Thus, from equation (2.3.65),

$$\mathcal{E}\left\{\frac{|G|^h}{|L|^h}\right\} = \frac{\mathcal{E}\left\{\dfrac{|\tilde{W}|}{|\tilde{W}+\tilde{H}|}\right\}^h}{\mathcal{E}\left\{\dfrac{\tilde{w}_{11}}{\tilde{w}_{11}+\tilde{h}_{11}}\right\}^h} \tag{2.3.74}$$

If p = 1, then

$$\mathcal{E}\left\{\frac{|G|^h}{|L|^h}\right\} = 1 \qquad h = 1,2,\ldots \tag{2.3.75}$$

and hence

$$\Pr\left\{\frac{|G|}{|L|} = x\right\} \begin{array}{ll} = 1, & x = 1 \\ = 0, & x \neq 1 \end{array} . \tag{2.3.76}$$

For $p > 1$,

$$\mathcal{E}\left\{\frac{|G|^h}{|L|^h}\right\} = \frac{\left[\frac{\nu-p}{2} + h - 1\right]^{(h)}}{\left[\frac{\nu-1}{2} + h - 1\right]^{(h)}} \tag{2.3.77}$$

and hence $\dfrac{|G|}{|L|}$ has the density:

$$\frac{1}{B\left[\frac{\nu-p}{2}, \frac{p-1}{2}\right]} u^{\frac{\nu-p}{2}-1} (1-u)^{\frac{p-1}{2}-1} \qquad 0 < u < 1. \tag{2.3.78}$$

Moment equation (2.3.60) may now be written:

$$\mathcal{E}\left\{[1 - R^2]^h\right\} = \mathcal{E}\left\{\left[\frac{w_{11}}{w_{11} + h_{11}}\right]^h\right\} \text{ if } p = 1 \tag{2.3.79}$$

or:

$$\mathcal{E}\left\{[1-R^2]^h\right\} = \mathcal{E}\left\{\left[\frac{w_{11}}{w_{11} + h_{11}}\right]^h\right\} \frac{\left[\frac{\nu-p}{2} + h - 1\right]^{(h)}}{\left[\frac{\nu-1}{2} + h - 1\right]^{(h)}} \tag{2.3.80}$$

if $p > 1$.

In view of equations (2.3.79) and (2.3.80) we have complete knowledge of the moments of $1-R^2$ subject only to a discussion of the moments of $\dfrac{w_{11}}{w_{11} + h_{11}}$. As a by-product of a discussion in Chapter III of the distribution of R in the case $p = 1$, we shall obtain the first two unconditional moments of $\dfrac{w_{11}}{w_{11} + h_{11}}$.

## 2.4 Distribution of $R^2$ Under the Null Hypothesis.

Recall from Formula (2.1.19) that $R_0^2 = \underline{\rho}'\underline{\rho}$. According to the null hypothesis of no correlation in the population,

$R_0^2 = 0$. Thus:

2.4.1 $$\tau^2 = \underline{\rho}'\underline{\rho} = 0 \quad ,$$

and consequently:

2.4.2 $$\delta = \frac{\tau}{\sqrt{1-\tau^2}} = 0 \quad .$$

Returning to statements (2.3.21) and (2.3.22) we now see that $R_0^2 = 0$ implies conditions (2.3.63) and (2.3.64). Thus Formula (2.3.65) follows from the null hypothesis, and the density of $1-R^2$ is the Beta density recorded in (2.3.72).

With alternative hypothesis $R_0^2 > 0$, the critical region for a size $\alpha$ test of the null hypothesis is $R^2 > \lambda$, where $\lambda$ is found from:

2.4.3 $$\Pr[R^2 > \lambda \mid R_0 = 0] = \Pr[1-R^2 < 1-\lambda \mid R_0 = 0] = \alpha \ .$$

Thus, $1-\lambda$ is the lower $100\alpha$ % point of the Beta distribution with density:

2.4.4 $$\frac{1}{B[\frac{\nu-p}{2}, \frac{p}{2}]} u^{\frac{\nu-p}{2} - 1} (1-u)^{\frac{p}{2} - 1} \qquad 0 < u < 1 \quad .$$

The lower .5%, 1%, 2.5%, 5%, 10%, 25%, and 50% points of this Beta distribution are recorded in Pearson (1958), pp 142-155, for:

$$p = 1\ (1)\ 10,\ 12,\ 15,\ 20,\ 24,\ 30,\ 40,\ 60,\ 120$$

$$\nu - p = 1\ (1)\ 30,\ 40,\ 60,\ 120,\ \infty\ .$$

# III. DISTRIBUTION OF R IN THE CASE p=1

A major finding in Chapter 2 is expressed in Formula (2.3.80), wherein the $h^{\underline{th}}$ moment of $1-R^2$ for general p, conditional on the set $\{x_{0i}\}_1^n$, is expressed as a multiple of the $h^{\underline{th}}$ moment of $1-R^2$, conditional on the set $\{x_{0i}\}_1^n$, in the case p=1. The multiplier depends only on n and h; i.e., the multiplier is not a function of the $x_{0i}$.

In the present chapter we develop formulae for the first four unconditional moments of R in the case p=1. We can then write down the first two unconditional moments of $1-R^2$ when p=1, and thus, through Formula (2.3.80), the first two unconditional moments of $1-R^2$ for general p.

## 3.1 Conditional Density and Moments of R when p=1

Specializing Lemma 3 of Chapter 2 to the case p=1:

3.1.1 $$x_i | x_{0i} \frown N(\rho x_{0i} : 1-\rho^2) \qquad i = 1,2,\ldots,n$$

and $x_i | x_{0i}$ is independent of $x_j | x_{0j}$ for $i \neq j$ .

Also:

3.1.2 $$R = \frac{\sum_1^n d_i \; x_i}{\sqrt{\sum_1^n (x_i - \bar{x})^2}}$$

In a first transformation of R, let:

3.1.3 $$x_i = \rho x_{0i} + \sqrt{1-\rho^2}\; u_i \qquad i = 1,2,\ldots,n\ .$$

Thus,

3.1.4 $$R = \frac{\sum_1^n d_i(\delta x_{0i} + u_i)}{\sqrt{\sum_1^n [\delta(x_{0i}-\bar{x}_0) + (u_i-\bar{u})]^2}}$$

where

3.1.5 $$\delta = \frac{\rho}{\sqrt{1-\rho^2}}$$

and

3.1.6 $$u_i \frown N(0,1) \qquad i = 1,2,\ldots,n$$

and $u_i$ is independent of $u_j$ for $i \neq j$.

Let

3.1.7 $$\begin{bmatrix} u_1 + \delta(x_{01} - \bar{x}_0) \\ u_2 + \delta(x_{02} - \bar{x}_0) \\ \vdots \\ u_n + \delta(x_{0n} - \bar{x}_0) \end{bmatrix} = \left[\begin{array}{c|c} 1 & \\ 1 & \\ \vdots & P \\ 1 & \end{array}\right] \begin{bmatrix} \bar{u} \\ v_1 \\ \vdots \\ v_\nu \end{bmatrix} \qquad \nu = n-1$$

where

3.1.8
$$P = \begin{bmatrix} \frac{1}{\sqrt{1.2}} & \frac{1}{\sqrt{2.3}} & \frac{1}{\sqrt{3.4}} & \cdots & \frac{1}{\sqrt{(n-1)n}} \\ \frac{-1}{\sqrt{1.2}} & \frac{1}{\sqrt{2.3}} & \frac{1}{\sqrt{3.4}} & \cdots & \frac{1}{\sqrt{(n-1)n}} \\ 0 & \frac{-2}{\sqrt{2.3}} & \frac{1}{\sqrt{3.4}} & \cdots & \frac{1}{\sqrt{(n-1)n}} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-(n-1)}{\sqrt{(n-1)\,n}} \end{bmatrix}$$

Also, let:

3.1.9
$$\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = \left[\begin{array}{c|c} 1 & \\ 1 & \\ \vdots & P \\ 1 & \end{array}\right] \begin{bmatrix} 0 \\ e_1 \\ \vdots \\ e_\nu \end{bmatrix} .$$

The first entry in the vector on the right is zero in view of the condition $\sum_1^n d_i = 0$.

Then:

3.1.10 $$R = \frac{\sum_1^\nu e_i v_i}{\sqrt{\sum_1^\nu v_i^2}}$$

where:

3.1.11 $$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_\nu \end{bmatrix} \frown N_\nu \left\{ \delta P' \begin{bmatrix} x_{01} - \bar{x}_0 \\ x_{02} - \bar{x}_0 \\ \vdots \\ x_{0n} - \bar{x}_0 \end{bmatrix} : I_\nu \right\} .$$

Introducing the notation:

3.1.12 $$\tilde{v} = \sum_1^\nu e_i v_i \quad ,$$

$\sum_1^\nu v_i^2$ may be expressed:

3.1.13 $$\sum_1^\nu v_i^2 = \sum_1^\nu (v_i - e_i\tilde{v})^2 + \tilde{v}^2 \quad .$$

In a final transformation with

$$\left[ \begin{array}{c|c} e_1 & \\ e_2 & \\ \vdots & H \\ e_\nu & \end{array} \right]$$ orthogonal, let:

$$3.1.14 \qquad \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_\nu \end{bmatrix} = \left[ \begin{array}{c|c} e_1 & \\ e_2 & \\ \vdots & H \\ e_\nu & \end{array} \right] \begin{bmatrix} \tilde{v} \\ w_1 \\ \vdots \\ w_{\nu-1} \end{bmatrix} .$$

Then:

$$3.1.15 \qquad (\tilde{v},\ w_1,\ \ldots,\ w_{\nu-1})' \frown N_\nu \ \{\underline{\xi} : I_\nu\}$$

where:

$$3.1.16 \qquad \underline{\xi} = \delta \begin{bmatrix} e_1 \ e_2 \ \cdots \ e_\nu \\ \hline \\ H' \\ \\ \end{bmatrix} P' \begin{bmatrix} x_{01} - \bar{x}_0 \\ x_{02} - \bar{x}_0 \\ \vdots \\ x_{0n} - \bar{x}_0 \end{bmatrix}$$

Thus, $\tilde{v}$ and the $w_i$ are mutually independent,

$$2.1.17 \qquad \tilde{v} \frown N(\alpha_1, 1)$$

where

$$3.1.18 \qquad \alpha_1 = \delta \sum_1^n d_i \ x_{0i}$$

and:

$$3.1.19 \qquad \sum_1^\nu (v_i - e_i \tilde{v})^2 = \sum_1^{\nu-1} w_i^{\,2} .$$

Noting that:

3.1.20 $$HH' = I_{\nu} - P'(d_1,d_2,\ldots,d_n)'(d_1,d_2,\ldots,d_n)P \quad ,$$

it follows that $\sum_{1}^{\nu-1} w_i^2$ has a noncentral Chi-square distribution with noncentrality parameter $\lambda_2$:

3.1.21 $$\lambda_2 = \tfrac{1}{2}\delta^2 \begin{bmatrix} x_{01} - \bar{x}_0 \\ x_{02} - \bar{x}_0 \\ \vdots \\ x_{0n} \quad \bar{x}_0 \end{bmatrix}' PHH'P' \begin{bmatrix} x_{01} - \bar{x}_0 \\ x_{02} - \bar{x}_0 \\ \vdots \\ x_{0n} - \bar{x}_0 \end{bmatrix}$$

$$= \tfrac{1}{2}\delta^2 \left( \sum_{1}^{n}(x_{0i}-\bar{x}_0)^2 - [\sum_{1}^{n} d_i x_{0i}]^2 \right) \quad .$$

The joint density of $\tilde{v}$ and $\sum_{1}^{\nu-1} w_i^2$ is:

3.1.22 $$\frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\alpha_1)^2} \sum_{j=0}^{\infty} \frac{e^{-\lambda_2}\lambda_2^j \, y^{\frac{1}{2}(n+2j)-2} \, e^{-\frac{1}{2}y}}{j! \, 2^{\frac{1}{2}(n+2j)} \Gamma\left[\frac{n+2j-2}{2}\right]}$$

over the domain: $-\infty < x < \infty \, , \; 0 < y \quad .$

Letting $\lambda_1 = \frac{1}{2}\alpha_1{}^2$ and using standard techniques, the density of R conditional on the $x_{0i}$ is found to be:

3.1.23 $$\frac{1}{\sqrt{\pi}} e^{-(\lambda_1 + \lambda_2)} \cdot \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda_1{}^{\frac{1}{2}i} \lambda_2^j \Gamma\left[\frac{n+i+2j-1}{2}\right] (2r)^i (1-r^2)^{j + \frac{n-4}{2}}}{i!j!\ \Gamma\left[\frac{n+2j-2}{2}\right]}$$

over the domain: $-1 < r < 1$ .

The moments of R conditional on the $x_{0i}$ are easily derived using density (3.1.23) and the formula:

3.1.24 $$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda_1^i \lambda_2^j}{i!j!}\, i^{(a)} j^{(b)} f(i+j) =$$

$$\lambda_1^a \lambda_2^b \sum_{k=0}^{\infty} \frac{(\lambda_1+\lambda_2)^k}{k!} \quad f(k+a+b)$$

where a and b are nonnegative integers and f is an arbitrary function.

If we define $A_h\,(i,j)$ by:

$$3.1.25 \qquad A_h(i,j) = \int_{-1}^{1} r^{i+h}(1-r^2)^{j+\frac{n-4}{2}}\, dr \quad (h,i,j=0,1,2,\ldots)$$

it is readily seen on using density (3.1.23) that the expected value of $R^h$ conditional on $(x_{01}, x_{02}, \ldots, x_{0n})$ is given by:

$$3.1.26 \qquad \mathcal{E}\left[R^h \mid \{x_{0i}\}_1^n\right] =$$

$$\frac{1}{\sqrt{\pi}}\, e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(4\lambda_1)^{\frac{1}{2}i}\lambda_2^j}{i!\,j!} \; \frac{\Gamma\left[j+\frac{n+i-1}{2}\right]}{\Gamma\left[j+\frac{n-2}{2}\right]}\; A_h(i,j) \;.$$

Since $A_h(i,j)$ is an integral over a domain symmetric in $r=0$, it is clear that:

$$3.1.27 \qquad A_h(i,j) = \begin{cases} 0 & , \quad h+i \equiv 1 \pmod 2 \\ B\left[\frac{h+i+1}{2}\,,\; j+\frac{n-2}{2}\right], & \quad h+i \equiv 0 \pmod 2 \end{cases}$$

so that the odd and even conditional moments of R are best considered separately.

Case 1: $h = 2m + 1 \qquad m = 0,1,2,\ldots$

Since exponent $i+2m+1$ must be even for $A_{2m+1}$ to be nonzero, we replace i by $2\alpha+1$ in (3.1.25) and obtain:

3.1.28 $$A_{2m+1}(2\alpha+1,j) = B\left[\alpha + m + \tfrac{3}{2},\ j + \tfrac{n-2}{2}\right] \qquad (\alpha,j,m = 0,1,2,\ldots).$$

Thus:

3.1.29 $$\mathcal{E}\left[R^{2m+1} \mid \{x_{0i}\}_1^n\right] = \lambda_1^{\frac{1}{2}} e^{-(\lambda_1+\lambda_2)} \sum_{\alpha=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda_1^{\alpha}\lambda_2^{j}}{\alpha!\,j!} \frac{\Gamma\left[\alpha + \frac{3}{2} + m\right]}{\Gamma\left[\alpha + \frac{3}{2}\right]} \frac{\Gamma\left[\alpha + j + \frac{n}{2}\right]}{\Gamma\left[\alpha + j + m + \frac{n+1}{2}\right]}$$

$m = 0,1,2,\ldots$

Case 2: $h = 2m \qquad m = 0,1,2,\ldots$

Since exponent $i+2m$ must be even for $A_{2m}$ to be nonzero, we replace i by $2\alpha$ in (3.1.25) and obtain:

3.1.30 $$A_{2m} = B\left[\alpha + m + \tfrac{1}{2},\ j + \tfrac{n-2}{2}\right] \quad (\alpha,j,m = 0,1,2,\ldots).$$

Thus:

3.1.31 $\mathcal{E}\left[R^{2m} | \{x_{0i}\}_1^n\right] =$

$$e^{-(\lambda_1+\lambda_2)} \sum_{\alpha=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda_1^{\alpha}\lambda_2^{j}}{\alpha!j!} \frac{\Gamma\left[\alpha+\frac{1}{2}+m\right]}{\Gamma\left[\alpha+\frac{1}{2}\right]} \frac{\Gamma\left[\alpha+j+\frac{n-1}{2}\right]}{\Gamma\left[\alpha+j+\frac{n-1}{2}+m\right]}$$

$m = 0,1,2,\ldots$

For selected values of m, the double series in equations (3.1.29) and (3.1.31) can be expressed in simple series form by applications of summation formula (3.1.24). Thus:

3.1.32 $$\mathcal{E}\left[R | \{x_{0i}\}_1^n\right] = \lambda_1^{\frac{1}{2}} e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{\infty} \frac{(\lambda_1+\lambda_2)^k}{k!} \frac{\Gamma\left[k+\frac{n}{2}\right]}{\Gamma\left[k+\frac{n+1}{2}\right]}$$

3.1.33 $\mathcal{E}\left[R^2 | \{x_{0i}\}_1^n\right] =$

$$e^{-(\lambda_1+\lambda_2)}\left[\lambda_1 \sum_{k=0}^{\infty} \frac{(\lambda_1+\lambda_2)^k}{k!(k+\frac{n+1}{2})} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(\lambda_1+\lambda_2)^k}{k!(k+\frac{n-1}{2})}\right]$$

3.1.34 $\mathcal{E}\left[R^3 | \{x_{0i}\}_1^n\right] =$

$$\lambda_1^{\frac{3}{2}} e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{\infty} \frac{(\lambda_1+\lambda_2)^k}{k!} \frac{\Gamma\left[k+\frac{n+2}{2}\right]}{\Gamma\left[k+\frac{n+5}{2}\right]}$$

$$+ \frac{3}{2}\lambda_1^{\frac{1}{2}} e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{\infty} \frac{(\lambda_1+\lambda_2)^k}{k!} \frac{\Gamma\left[k+\frac{n}{2}\right]}{\Gamma\left[k+\frac{n+3}{2}\right]}$$

3.1.35 $$\mathcal{E}\left[R^4 \mid \{x_{0i}\}_1^n\right] =$$

$$\lambda_1^2 e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{\infty} \frac{(\lambda_1+\lambda_2)^k}{k!\left[k+\frac{n+5}{2}\right]^{(2)}}$$

$$+ 3\lambda_1 e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{\infty} \frac{(\lambda_1+\lambda_2)^k}{k!\left[k+\frac{n+3}{2}\right]^{(2)}}$$

$$+ \frac{3}{4} e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{\infty} \frac{(\lambda_1+\lambda_2)^k}{k!\left[k+\frac{n+1}{2}\right]^{(2)}} .$$

These are the first four moments about zero of R conditional on the set of unknowns $x_{0i}$. We now obtain the unconditional moments of R.

## 3.2 Unconditional Moments of R

Using notations

3.2.1 $$S^2 = \sum_1^n (x_{0i} - \bar{x}_0)^2$$

3.2.2 $$\eta^2 = \left[\sum_1^n d_i \, x_{0i}\right]^2$$

the parameters $\lambda_1$ and $\lambda_2$ of Section 1 become:

3.2.3 $$\lambda_1 = \tfrac{1}{2}\,\delta^2\,\eta^2$$

3.2.4 $$\lambda_2 = \tfrac{1}{2}\,\delta^2\left[S^2 - \eta^2\right] .$$

To simplify the exposition to follow, we state:

Theorem 3.2.1

If $x_{0i}$ $(i = 1,2,\ldots,n)$ are the standard normal order statistics from a sample of size n, then:

3.2.5 $$\frac{\eta^2}{S^2} \text{ is independent of } S^2 .$$

Proof:

Since $\bar{x}_0$ and $S^2$ are symmetric in the observations $x_{0i}$, the joint distribution of $\bar{x}_0$ and $S^2$ is the same whether the $x_{0i}$ are ordered or not. For the unordered sample, it is well known that $(\bar{x}_0, S^2)$ is sufficient for $(\mu, \sigma^2)$ and that the density of $(\bar{x}_0, S^2)$ is complete. Therefore $(\bar{x}_0, S^2)$ is statistically independent of any statistic which is independent of scale and location.

Now $\frac{\eta^2}{S^2}$ is clearly independent of scale and $S^2$ is independent of location. Further, since $\sum_1^n d_i = 0$, $\eta^2$ is also independent of location.

Thus, $\frac{\eta^2}{S^2}$ is independent of $S^2$ as stated.

Let:

3.2.6 $$\Omega = \frac{n(n^2 - 1)}{12}$$

3.2.7 $$d_i = \Omega^{-\frac{1}{2}} \left(i - \frac{n+1}{2}\right) \qquad i = 1,2,\ldots,n \ ,$$

With this choice of the $d_i$, conditions $\sum_1^n d_i = 0$ and $\sum_1^n d_i^2 = 1$ are clearly satisfied.

Further, let:

3.2.8 $$X = \sum_1^n \left(i - \frac{n+1}{2}\right) x_{0i} \quad .$$

Then:

3.2.9 $$\eta^2 = \frac{12}{n(n^2-1)} \left[\sum_1^n \left(i - \frac{n+1}{2}\right) x_{0i}\right]^2 = \Omega^{-1} X^2 \quad .$$

The first four moments of X are developed in Chapter 4.

Now:

3.2.10 $$\mathcal{E}\,[\eta^2] = \mathcal{E}\left[\frac{\eta^2}{S^2} S^2\right]$$

$$= \mathcal{E}\left[\frac{\eta^2}{S^2}\right] \mathcal{E}\left[S^2\right] = (n-1)\,\mathcal{E}\left[\frac{\eta^2}{S^2}\right]$$

3.2.11 $$\mathcal{E}\left[\eta^4\right] = \mathcal{E}\left[\frac{\eta^4}{S^4} \quad S^4\right]$$

$$= \mathcal{E}\left[\frac{\eta^4}{S^4}\right] \mathcal{E}\left[S^4\right] = (n-1)(n+1)\mathcal{E}\left[\frac{\eta^4}{S^4}\right] .$$

Thus:

3.2.12 $$\mathcal{E}\left[\frac{\eta^2}{S^2}\right] = \frac{1}{n-1}\Omega^{-1}\,\mathcal{E}\left[X^2\right]$$

and:

3.2.13 $$\mathcal{E}\left[\frac{\eta^4}{S^4}\right] = \frac{1}{(n-1)(n+1)}\Omega^{-2}\,\mathcal{E}\left[X^4\right] .$$

For obtaining the unconditional moments of R, it will be necessary to find expected values of quantities of the form $S^{2k}\exp\left[-\frac{1}{2}\delta^2 S^2\right]$ when $S^2$ has a central Chi-square distribution with n-1 degrees of freedom. These expected values are readily obtained by integration:

3.2.14
$$\mathcal{E}\left[S^{2k}\ e^{-\frac{1}{2}\ \delta^2\ S^2}\right] =$$

$$\int_0^{\infty} \frac{1}{\Gamma\left[\frac{n-1}{2}\right]2^{\frac{1}{2}(n-1)}}\ x^{\frac{n+2k-1}{2}-1}\ e^{-\frac{1}{2}(1+\delta^2)x}\ dx =$$

$$2^k\ (1+\delta^2)^{-\frac{2k+n-1}{2}}\ \frac{\Gamma\left[\frac{n-1}{2}+k\right]}{\Gamma\left[\frac{n-1}{2}\right]} .$$

Using formulae (3.2.12) and (3.2.14) and the independence of $\frac{\eta^2}{S^2}$ and $S^2$ we then have:

3.2.15
$$\mathcal{E}\left[S^{2k}\ e^{-\frac{\delta}{2}\ S^2}\ \eta^2\right] =$$

$$\mathcal{E}\left[S^{2k+2}\ e^{-\frac{\delta^2}{2}\ S^2}\ \frac{\eta^2}{S^2}\right] =$$

$$\mathcal{E}\left[S^{2k+2}\ e^{-\frac{\delta^2}{2}\ S^2}\right]\mathcal{E}\left[\frac{\eta^2}{S^2}\right] =$$

$$\Omega^{-1}\ E[X^2]\ \frac{\Gamma\left[\frac{n+1}{2}+k\right]}{\Gamma\left[\frac{n+1}{2}\right]}\ 2^k(1+\delta^2)^{-\frac{2k+n+1}{2}} .$$

Similarly:

3.2.16 $$\mathcal{E}\left[S^{2k}\, e^{-\frac{\delta^2}{2} S^2}\, \eta^4\right] =$$

$$\Omega^{-2}\, \mathcal{E}[X^4]\, \frac{\Gamma\left[\frac{n+3}{2} + k\right]}{\Gamma\left[\frac{n+3}{2}\right]}\quad 2^k(1+\delta^2)^{-\frac{2k+n+3}{2}}$$

3.2.17 $$\mathcal{E}\left[S^{2k+1}\, e^{-\frac{\delta^2}{2} S^2}\right] = \frac{\Gamma\left[\frac{n}{2} + k\right]}{\Gamma\left[\frac{n-1}{2}\right]}\, 2^{k+\frac{1}{2}}(1+\delta^2)^{-\frac{2k+n}{2}},$$

3.2.18 $$\mathcal{E}\left[S^{2k}\, e^{-\frac{\delta^2}{2} S^2}\, \eta\right] =$$

$$\Omega^{-\frac{1}{2}}\, \mathcal{E}[X]\, \frac{\Gamma\left[\frac{n}{2} + k\right]}{\Gamma\left[\frac{n}{2}\right]}\quad 2^k(1+\delta^2)^{-\frac{2k+n}{2}}$$

and:

3.2.19 $$\mathcal{E}\left[S^{2k}\, e^{-\frac{\delta^2}{2} S^2}\, \eta^3\right]$$

$$\Omega^{-\frac{3}{2}}\, \mathcal{E}[X^3]\, \frac{\Gamma\left[\frac{n+2}{2} + k\right]}{\Gamma\left[\frac{n+2}{2}\right]}\quad 2^k(1+\delta^2)^{-\frac{2k+n+2}{2}}$$

By using the relations:

3.2.20 $$\lambda_1 = \tfrac{1}{2}\delta^2\eta^2 \quad ; \quad \lambda_1 + \lambda_2 = \tfrac{1}{2}\delta^2 S^2,$$

Formula (3.1.31) can be written in the form:

3.2.21 $$\mathcal{E}\left[R \,\middle|\, \{x_{0i}\}_1^n\right] =$$

$$\sum_{k=0}^{\infty} \left[\frac{\delta^2}{2}\right]^{k+\frac{1}{2}} S^{2k} e^{-\frac{1}{2}\delta^2 S^2} \eta \frac{1}{k!} \frac{\Gamma\left[k+\frac{n}{2}\right]}{\Gamma\left[k+\frac{n+1}{2}\right]} .$$

The unconditional expectation, $\mathcal{E}[R]$, is then obtained immediately on applying Formula (3.2.18). We obtain:

3.2.22 $$\mathcal{E}[R] =$$

$$\left[\frac{\delta^2}{2}\right]^{\frac{1}{2}} (1+\delta^2)^{-\frac{n}{2}} \frac{\mathcal{E}[X]}{\Omega^{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{\Gamma^2\left[k+\frac{n}{2}\right]}{\Gamma\left[\frac{n}{2}\right]\Gamma\left[k+\frac{n+1}{2}\right]} \frac{\rho^{2k}}{k!} .$$

Similarly,

3.2.23 $$\mathcal{E}[R^2] =$$

$$\frac{\delta^2}{2} (1+\delta^2)^{-\frac{n+1}{2}} \frac{\mathcal{E}[X^2]}{\Omega} \sum_{k=0}^{\infty} \begin{bmatrix} \frac{n+1}{2} + k - 1 \\ k \end{bmatrix} \frac{\rho^{2k}}{k+\frac{n+1}{2}}$$

$$+\tfrac{1}{2} (1+\delta^2)^{-\frac{n-1}{2}} \sum_{k=0}^{\infty} \begin{bmatrix} \frac{n+1}{2} + k - 2 \\ k \end{bmatrix} \frac{\rho^{2k}}{k+\frac{n-1}{2}} ,$$

3.2.24 $\mathcal{E}[R^3] =$

$$\left[\frac{\delta^2}{2}\right]^{\frac{3}{2}} (1+\delta^2)^{-\frac{n+2}{2}} \frac{\mathcal{E}[X^3]}{\Omega^{1.5}} \sum_{k=0}^{\infty} \frac{\Gamma^2\left[k+\frac{n+2}{2}\right]}{\Gamma\left[\frac{n+2}{2}\right]\Gamma\left[k+\frac{n+5}{2}\right]} \frac{\rho^{2k}}{k!}$$

$$+ \frac{3}{2}\left[\frac{\delta^2}{2}\right]^{\frac{1}{2}} (1+\delta^2)^{-\frac{n}{2}} \frac{\mathcal{E}[X]}{\Omega^{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{\Gamma^2\left[K+\frac{n}{2}\right]}{\Gamma\left[\frac{n}{2}\right]\Gamma\left[k+\frac{n+3}{2}\right]} \frac{\rho^{2k}}{k!}$$

and:

3.2.25 $\mathcal{E}[R^4] =$

$$\left[\frac{\delta^2}{2}\right]^{2} (1+\delta^2)^{-\frac{n+3}{2}} \frac{\mathcal{E}[X^4]}{\Omega^{2}} \sum_{k=0}^{\infty} \begin{bmatrix} \frac{n+1}{2}+k \\ k \end{bmatrix} \frac{\rho^{2k}}{\left[k+\frac{n+5}{2}\right]^{(2)}}$$

$$+3\,\frac{\delta^2}{2} (1+\delta^2)^{-\frac{n+1}{2}} \frac{\mathcal{E}[X^2]}{\Omega} \sum_{k=0}^{\infty} \begin{bmatrix} \frac{n+1}{2}+k-1 \\ k \end{bmatrix} \frac{\rho^{2k}}{\left[k+\frac{n+3}{2}\right]^{(2)}}$$

$$+ \frac{3}{4} (1+\delta^2)^{-\frac{n-1}{2}} \sum_{k=0}^{\infty} \begin{bmatrix} \frac{n+1}{2}+k-2 \\ k \end{bmatrix} \frac{\rho^{2k}}{\left[k+\frac{n+1}{2}\right]^{(2)}}.$$

Having available four moments of R, it is now possible to fit a Pearson system curve to the distribution of R for selected values of n and $\rho$, and thus calculate the approximate power of a test of the null hypothesis, $R_0 = 0$.

It may be noted that these moments are expressible in terms of the generalized hypergeometric function attributed to Gauss.

Using the notation:

$(a)_n = a(a+1) \ldots (a+n-1) \qquad n = 1,2,\ldots$

$(a)_0 = 1$

the hypergeometric function of Gauss is the series:

$$F[a,b;c;z] = \sum_{k=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{n!} \quad .$$

It is easy to verify that:

$$3.2.26 \qquad \sum_{k=0}^{\infty} \frac{\Gamma^2\left[k+\frac{n}{2}\right]}{\Gamma\left[k+\frac{n+\alpha}{2}\right]} \frac{\rho^{2k}}{k!} =$$

$$\frac{\Gamma^2\left[\frac{n}{2}\right]}{\Gamma\left[\frac{n+\alpha}{2}\right]} F\left[\frac{n}{2}, \frac{n}{2}; \frac{n+\alpha}{2}; \rho^2\right] \qquad \alpha = 1,2,3$$

3.2.27 $$\sum_{k=0}^{\infty} \begin{bmatrix} \frac{n+1}{2} + k - \beta \\ k \end{bmatrix} \frac{\rho^{2k}}{k + \frac{n+3}{2} - \beta} =$$

$$\frac{1}{\frac{n+3}{2} - \beta} \; F \left[\frac{n+3}{2} - \beta, \frac{n+3}{2} - \beta; \frac{n+5}{2} - \beta; \rho^2\right] \quad \beta = 1,2$$

and

3.2.28 $$\sum_{k=0}^{\infty} \begin{bmatrix} \frac{n+1}{2} + k - \beta \\ k \end{bmatrix} \frac{\rho^{2k}}{\left[k + \frac{n+5}{2} - \beta\right]^{(2)}} =$$

$$\frac{\Gamma\left[\frac{n+3}{2} - \beta\right]}{\Gamma\left[\frac{n+7}{2} - \beta\right]} \; F \left[\frac{n+3}{2} - \beta, \frac{n+3}{2} - \beta; \frac{n+7}{2} - \beta; \rho^2\right] \quad \beta = 0,1,2 \; .$$

Then:

3.2.29 $$\mathcal{E}[R] = \left[\frac{\delta^2}{2}\right]^{\frac{1}{2}} (1+\delta^2)^{-\frac{n}{2}} \frac{\mathcal{E}[X]}{\Omega^{\frac{1}{2}}} \frac{\Gamma\left[\frac{n}{2}\right]}{\Gamma\left[\frac{n+1}{2}\right]} F\left[\frac{n}{2}, \frac{n}{2}; \frac{n+1}{2}; \rho^2\right]$$

3.2.30 $$\mathcal{E}[R^2] = \frac{\delta^2}{2} (1+\delta^2)^{-\frac{n+1}{2}} \frac{\mathcal{E}[X^2]}{\Omega} \frac{\Gamma\left[\frac{n+1}{2}\right]}{\Gamma\left[\frac{n+3}{2}\right]} F\left[\frac{n+1}{2}, \frac{n+1}{2}; \frac{n+3}{2}; \rho^2\right]$$

$$+\frac{1}{2} (1+\delta^2)^{-\frac{n-1}{2}} \frac{\Gamma\left[\frac{n-1}{2}\right]}{\Gamma\left[\frac{n+1}{2}\right]} \; F \left[\frac{n-1}{2}, \frac{n-1}{2}; \frac{n+1}{2}; \rho^2\right]$$

3.2.31 $\mathcal{E}[R^3]$ =

$$\left[\frac{\delta^2}{2}\right]^{\frac{3}{2}} (1+\delta^2)^{-\frac{n+2}{2}} \frac{\mathcal{E}[X^3]}{\Omega^{1.5}} \frac{\Gamma\left[\frac{n+2}{2}\right]}{\Gamma\left[\frac{n+5}{2}\right]} F\left[\frac{n+2}{2}, \frac{n+2}{2}, \frac{n+5}{2}; \rho^2\right]$$

$$+ \frac{3}{2} \left(\frac{\delta^2}{2}\right)^{\frac{1}{2}} (1+\delta^2)^{-\frac{n}{2}} \frac{\mathcal{E}[X]}{\Omega^{\frac{1}{2}}} \frac{\Gamma\left[\frac{n}{2}\right]}{\Gamma\left[\frac{n+3}{2}\right]} F\left[\frac{n}{2}, \frac{n}{2}; \frac{n+3}{2}; \rho^2\right]$$

3.2.32 $\mathcal{E}[R^4]$ =

$$\left[\frac{\delta^2}{2}\right]^{2} (1+\delta^2)^{-\frac{n+3}{2}} \frac{\mathcal{E}[X^4]}{\Omega^{2}} \frac{\Gamma\left[\frac{n+3}{2}\right]}{\Gamma\left[\frac{n+7}{2}\right]} F\left[\frac{n+3}{2}, \frac{n+3}{2}; \frac{n+7}{2}; \rho^2\right]$$

$$+3 \frac{\delta^2}{2} (1+\delta^2)^{-\frac{n+1}{2}} \frac{\mathcal{E}[X^2]}{\Omega} \frac{\Gamma\left[\frac{n+1}{2}\right]}{\Gamma\left[\frac{n+5}{2}\right]} F\left[\frac{n+1}{2}, \frac{n+1}{2}; \frac{n+5}{2}; \rho^2\right]$$

$$+ \frac{3}{4} (1+\delta^2)^{-\frac{n-1}{2}} \frac{\Gamma\left[\frac{n-1}{2}\right]}{\Gamma\left[\frac{n+3}{2}\right]} F\left[\frac{n-1}{2}, \frac{n-1}{2}; \frac{n+3}{2}; \rho^2\right] .$$

However, this generalized hypergeometric function is not tabulated in the literature as extensively as we require. We have thus found it convenient to program the original expressions, formulae (3.2.22) through (3.2.25), for numerical evaluation on the electronic computer (IBM 1620).

# IV. MOMENTS OF $X = \sum_{1}^{n} (i - \frac{n+1}{2}) w_i$

A major result in Chapter 3 was the set of expressions - Formulae (3.2.22), (3.2.23), (3.2.24), and (3.2.25) - for the unconditional moments $\mathcal{E}[R^h : h=1,2,3,4]$ in the case $p=1$. Contained in these expressions are moments $\mathcal{E}[X^h : h=1,2,3,4]$ which have not as yet been evaluated. We shall show in this chapter that $\mathcal{E}[X^h]$ is a polynomial of degree $2h$ in $n$, $(h=1,2,3,4)$, and actually find the coefficients.

## 4.1 First Raw Moment of $X = \sum_{1}^{n} (i - \frac{n+1}{2}) w_i$

Now:

4.1.1
$$\mathcal{E}[X] = \sum_{1}^{n} (i - \frac{n+1}{2})\, \mathcal{E}[w_i]$$

where $w_i$ is the $i^{\underline{th}}$ standard normal order statistic from a random sample of size $n$. Hence:

4.1.2
$$\mathcal{E}[X] = \int_{-\infty}^{\infty} \sum_{1}^{n} (i-1-\frac{n-1}{2}) \frac{n!}{(i-1)!(n-i)!} \Phi^{i-1}(1-\Phi)^{n-i} x\phi \, dx$$

where abbreviations $\phi$ and $\Phi$ are used rather than the more cumbersome $\phi(x)$ and $\Phi(x)$ for the density and c.d.f. respectively of the standard normal variate. When multiple integrals are considered, we shall use subscripts as in $\phi_x$, $\phi_y$, $\Phi_z$, etc., instead of $\phi(x)$, $\phi(y)$, $\Phi(z)$, etc.

But:

$$4.1.3 \quad \sum_{2}^{n} \frac{n!}{(i-2)!(n-i)!} \Phi^{i-1} (1-\Phi)^{n-i} = n^{(2)}\Phi$$

and:

$$4.1.4 \quad \sum_{1}^{n} \frac{n!}{(i-1)!(n-i)!} \Phi^{i-1} (1-\Phi)^{n-i} = n \quad .$$

Hence:

$$4.1.5 \quad \mathcal{E}[X] = n^{(2)} \int_{-\infty}^{\infty} x\phi\Phi dx - \tfrac{1}{2} n^{(2)} \int_{-\infty}^{\infty} x\phi dx$$

$$= \frac{1}{2\sqrt{\pi}} n^{(2)} \quad .$$

## 4.2 Second Raw Moment of $X = \sum_{1}^{n} (i - \frac{n+1}{2})\, w_i$

Now:

4.2.1
$$\mathcal{E}[X^2] = \sum_{1}^{n} (i - \frac{n+1}{2})^2\, \mathcal{E}[w_i^2]$$

$$+2 \sum_{i<j} (i - \frac{n+1}{2})\, (j - \frac{n+1}{2})\, \mathcal{E}[w_i w_j] \quad .$$

Considering first the term on the right in Formula (4.2.1) which is a sum of single integrals, we have:

4.2.2
$$\sum_{1}^{n} (i - \frac{n+1}{2})^2\, \mathcal{E}[w_i{}^2] =$$

$$\int_{-\infty}^{\infty} \sum_{3}^{n} \frac{n!}{(i-3)!\,(n-i)!}\, \Phi^{i-1}\, (1-\Phi)^{n-i}\, x^2 \phi\, dx$$

$$-(n-2) \int_{-\infty}^{\infty} \sum_{2}^{n} \frac{n!}{(i-2)!\,(n-i)!}\, \Phi^{i-1} (1-\Phi)^{n-i} x^2 \phi\, dx$$

$$+ \frac{1}{4}(n-1)^2 \int_{-\infty}^{\infty} \sum_{1}^{n} \frac{n!}{(i-1)!\,(n-i)!}\, \Phi^{i-1} (1-\Phi)^{n-i} x^2 \phi\, dx$$

or:

4.2.3 $$\sum_{1}^{n} (i - \tfrac{n+1}{2})^2 \,\mathcal{E}[w_i{}^2] =$$

$$n^{(3)} \int_{-\infty}^{\infty} x^2 \phi \Phi^2 dx - n^{(3)} \int_{-\infty}^{\infty} x^2 \phi \Phi dx + \tfrac{1}{4}(n-1)^2 n \int_{-\infty}^{\infty} x^2 \phi dx.$$

The following three integrals are easily evaluated using integration by parts with $dV = x\phi dx$. We obtain:

4.2.4 $$\int_{-\infty}^{\infty} x^2 \ \phi dx = 1$$

4.2.5 $$\int_{-\infty}^{\infty} x^2 \ \phi \Phi dx = \tfrac{1}{2}$$

4.2.6 $$\int_{-\infty}^{\infty} x^2 \ \phi \Phi^2 \ dx = \frac{1}{3} + \frac{1}{2\pi\sqrt{3}}$$

Hence:

4.2.7 $$\sum_{1}^{n} (i - \tfrac{n+1}{2})^2 \,\mathcal{E}[w_i{}^2] =$$

$$(\frac{1}{12} + \frac{1}{2\pi\sqrt{3}})n^{(3)} + \frac{1}{4}\ n^{(2)} \qquad .$$

The second term on the right in Formula (4.2.1) is a sum of double integrals which reduces easily to the form:

4.2.8 $$\sum_{i<j} \left(i - \frac{n+1}{2}\right)\left(j - \frac{n+1}{2}\right) \mathcal{E}[w_i w_j] =$$

$$n^{(4)} \int_{-\infty}^{\infty} \int_{-\infty}^{y} x\phi_x \Phi_x \cdot y\phi_y \Phi_y \, dxdy$$

$$-\tfrac{1}{2}\, n^{(4)} \int_{-\infty}^{\infty} \int_{-\infty}^{y} x\phi_x (\Phi_x + \Phi_y) \cdot y\phi_y \, dxdy$$

$$+ n^{(3)} \int_{-\infty}^{\infty} \int_{-\infty}^{y} x\phi_x (\Phi_x - \Phi_y) \cdot y\phi_y \, dxdy$$

But:

4.2.9 $$\int_{-\infty}^{\infty} \int_{-\infty}^{y} x\phi_x \Phi_x \cdot y\phi_y \Phi_y \, dxdy =$$

$$-\int_{-\infty}^{\infty} y\phi_y^2 \Phi_y^2 \, dy + \int_{-\infty}^{\infty} \{2_y\} y \, \phi_y \, \Phi_y \, dy \quad ,$$

where

4.2.10 $$\{2_y\} = \int_{-\infty}^{y} \phi_x^2 \, dx \quad .$$

Also,

4.2.11 $$\int_{-\infty}^{\infty}\int_{-\infty}^{y} x\phi_x\Phi_x \cdot y\ \phi_y\ dxdy = \int_{-\infty}^{\infty} y\phi_y^2\ \Phi_y\ dy$$

4.2.12 $$\int_{-\infty}^{\infty}\int_{-\infty}^{y} x\phi_x \cdot y\phi_y\Phi_y\ dxdy = -\int_{-\infty}^{\infty} y\ \phi_y^2\ \Phi_y\ dy \quad .$$

But:

4.2.13 $$\int_{-\infty}^{\infty} y\phi_y^2\ \Phi_y\ dy = \frac{1}{4\pi\sqrt{3}}$$

4.2.14 $$\int_{-\infty}^{\infty} y\phi_y^2\ \Phi_y^2\ dy = \frac{1}{4\pi\sqrt{3}}$$

4.2.15 $$\int_{-\infty}^{\infty} \{2_y\}\ y\ \phi_y\ \Phi_y\ dy = \frac{1}{4\pi\sqrt{3}} + \frac{1}{8\pi}$$

Hence:

$$4.2.16 \qquad \sum_{i<j} (i - \tfrac{n+1}{2})(j - \tfrac{n+1}{2})\,\mathcal{E}[w_i w_j] =$$

$$n^{(4)}\left[-\frac{1}{4\pi\sqrt{3}} + \frac{1}{4\pi\sqrt{3}} + \frac{1}{8\pi}\right]$$

$$-\tfrac{1}{2}\, n^{(4)}\left[\frac{1}{4\pi\sqrt{3}} - \frac{1}{4\pi\sqrt{3}}\right]$$

$$+\, n^{(3)}\left[\frac{1}{4\pi\sqrt{3}} + \frac{1}{4\pi\sqrt{3}}\right]$$

$$= \frac{1}{8\pi}\, n^{(4)} + \frac{1}{2\pi\sqrt{3}}\, n^{(3)} \qquad .$$

Using results (4.2.7) and (4.2.16) in Formula (4.2.1), we obtain:

$$4.2.17 \qquad \mathcal{E}[X^2] = \frac{1}{4\pi}\, n^{(4)} + \left(\frac{1}{12} + \frac{3}{2\pi\sqrt{3}}\right) n^{(3)} + \frac{1}{4}\, n^{(2)} \qquad .$$

4.3 Third Raw Moment of $X = \sum_{1}^{n} (i - \frac{n+1}{2})\ w_i$

Now:

4.3.1 $$\mathcal{E}[X^3] = \sum_{1}^{n} (i - \alpha)^3\ \mathcal{E}[w_i{}^3]$$

$$+ 3 \sum_{i<j} \left[(i-\alpha)(j-\alpha)^2\mathcal{E}[w_i w_j{}^2]+(i-\alpha)^2(j-\alpha)\mathcal{E}[w_i^2 w_j]\right]$$

$$+6 \sum_{i<j<k} (i - \alpha)(j - \alpha)(k - \alpha)\ \mathcal{E}\ [w_i w_j w_k] \quad ,$$

where:

4.3.2 $$\alpha = \frac{n+1}{2} \quad .$$

Introducing the normal order statistic densities appropriate to each term on the right in Formula (4.3.1) and recognizing the resulting sums as expansions of a binomial, trinomial, and quadrinomial, respectively, we have:

4.3.3 $$\mathcal{E}[X^3] = \int_{-\infty}^{\infty} A_1 x^3 \phi_x dx + \int_{-\infty}^{\infty}\int_{-\infty}^{y} A_2 x\phi_x \cdot y^2\phi_y dxdy$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{y} A_3 x^2\phi_x \cdot y\phi_y dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{z}\int_{-\infty}^{y} A_4 x\phi_x y\phi_y z\phi_z dxdydz$$

where:

$$4.3.4 \qquad A_1 = n^{(4)}\Phi_x{}^3+(6-3\alpha)n^{(3)}\Phi_x^2+(7-9\alpha+3\alpha^2)n^{(2)}\Phi_x+(1-\alpha)^3 n$$

$$4.3.5 \qquad A_2 = 3n^{(5)}\,[\Phi_x\, R^2 + \Phi_x^3 + 2\Phi_x^2\, R]$$

$$+\ 3n^{(4)}\,[(8-3\alpha)\,\Phi_x^2 + (1-\alpha)\,R^2 + (9-4\alpha)\,\Phi_x R]$$

$$+\ 3n^{(3)}\,[(2-\alpha)(7-3\alpha)\,\Phi_x + (1-\alpha)(5-2\alpha)\,R]$$

$$+\ 3n^{(2)}\,(1-\alpha)(2-\alpha)^2$$

$$4.3.6 \qquad A_3 = 3n^{(5)}\,[\Phi_x^2\, R + \Phi_x^3]$$

$$+\ 3n^{(4)}\,[(7-3\alpha)\,\Phi_x^2 + (3-2\alpha)\,\Phi_x R]$$

$$+\ 3n^{(3)}\,[(2-\alpha)(5-3\alpha)\,\Phi_x + (1-\alpha)^2\,R]$$

$$+\ 3n^{(2)}\,(2-\alpha)(1-\alpha)^2$$

and:

$$4.3.7 \qquad A_4 = 6n^{(6)}\,[\Phi_x\, RS + \Phi_x\, R^2 + 2\Phi_x^2\, R + \Phi_x^2\, S + \Phi_x^3]$$

$$+\ 6n^{(5)}\,[(10-4\alpha)\,\Phi_x R+(9-3\alpha)\Phi_x^2+(4-2\alpha)\Phi_x S+(1-\alpha)(RS+R^2)]$$

$$+\ 6n^{(4)}\,[(6-3\alpha)(3-\alpha)\,\Phi_x + (1-\alpha)(6-2\alpha)R + (1-\alpha)(2-\alpha)S]$$

$$+\ 6n^{(3)}\,(1-\alpha)(2-\alpha)(3-\alpha)$$

where:

4.3.8 $$R = \Phi_y - \Phi_x$$

and:

4.3.9 $$S = \Phi_z - \Phi_y \quad .$$

After simplifying the $A_i$, integrating by parts to reduce all multiple integrals to single integrals, and evaluating the single integrals, we obtain:

4.3.10 $$\mathcal{E}[X^3] =$$

$$\left[\left(\frac{1}{4\pi\sqrt{2\pi}} - \frac{15}{16\sqrt{\pi}} + \frac{15}{2} A\right) n^{(4)} + \frac{15}{16\sqrt{\pi}} n^{(3)} + \frac{5}{16\sqrt{\pi}} n^{(2)}\right]$$

$$+\left[3\left(\frac{1}{48\sqrt{\pi}} + \frac{1}{8\pi\sqrt{3\pi}}\right) n^{(5)} + 3\left(\frac{3}{8\pi\sqrt{2\pi}} + \frac{11}{32\sqrt{\pi}} - \frac{5}{4} A\right) n^{(4)}\right.$$

$$\left. + \frac{21}{32\sqrt{\pi}} n^{(3)} + \frac{3}{32\sqrt{\pi}} n^{(2)}\right]$$

$$+\left[3\left(\frac{1}{48\sqrt{\pi}} + \frac{1}{8\pi\sqrt{3\pi}}\right) n^{(5)} + 3\left(\frac{3}{8\pi\sqrt{2\pi}} + \frac{11}{32\sqrt{\pi}} - \frac{5}{4} A\right) n^{(4)}\right.$$

$$\left. + \frac{21}{32\sqrt{\pi}} n^{(3)} + \frac{3}{32\sqrt{\pi}} n^{(2)}\right]$$

$$+\left[\frac{1}{8\pi\sqrt{\pi}} n^{(6)} + \frac{3}{2\pi\sqrt{3\pi}} n^{(5)} + \frac{3}{2\pi\sqrt{2\pi}} n^{(4)}\right]$$

where:

4.3.11 $$A = \int_{-\infty}^{\infty} \phi^2 \, \Phi^2 \, dx \quad ,$$

and terms arising from each of the four integrals on the right side of Formula (4.3.3) are indicated separately within square brackets.

Thus:

4.3.12 $$\mathcal{E}[X^3] = \frac{1}{8\pi\sqrt{\pi}} n^{(6)} + \left(\frac{1}{8\sqrt{\pi}} + \frac{9}{4\pi\sqrt{3\pi}}\right) n^{(5)} + \left(\frac{9}{8\sqrt{\pi}} + \frac{4}{\pi\sqrt{2\pi}}\right) n^{(4)} + \frac{9}{4\sqrt{\pi}} n^{(3)} + \frac{1}{2\sqrt{\pi}} n^{(2)} .$$

4.4 <u>Fourth Raw Moment of</u> $X = \sum_{1}^{n} \left(i - \frac{n+1}{2}\right) w_i$

The approach used in producing $\mathcal{E}[X^h]$ for $h = 1,2,3$ would, if employed in this section, lay on our shoulders some formidible problems in bookkeeping. Instead, we shall show first that $\mathcal{E}[X^4]$ is a polynomial of degree eight in n and find the coefficient of $n^8$. Then we shall demonstrate the method used in finding each of the other coefficients and report the final result.

Now:

4.4.1 $$\mathcal{E}[X^4] = T_1 + T_2 + T_3 + T_4 + T_5 \quad ,$$

where:

4.4.2 $$T_1 = \sum_{1}^{n} (i - \alpha)^4 \, \mathcal{E}[w_i^{\,4}]$$

4.4.3 $$T_2 = 6 \sum_{i<j} (i - \alpha)^2 \, (j - \alpha)^2 \, \mathcal{E}[w_i^{\,2} \, w_j^{\,2}]$$

4.4.4 $$T_3 = 4 \sum_{i<j} \left[ (i-\alpha)(j-\alpha)^3 \mathcal{E}[w_i w_j^{\,3}] + (1-\alpha)^3 (j-\alpha) \mathcal{E}[w_i^3 w_j] \right]$$

4.4.5 $$T_4 = 12 \sum_{i<j<k} \Big[ (i-\alpha)^2 (j-\alpha)(k-\alpha) \mathcal{E}[w_i^{\,2} w_j w_k] +$$

$$(i-\alpha)(j-\alpha)^2 (k-\alpha) \mathcal{E}[w_i w_j^{\,2} w_k] + (i-\alpha)(j-\alpha)(k-\alpha)^2 \mathcal{E}[w_i w_j w_k^{\,2}] \Big]$$

4.4.6 $$T_5 = 24 \sum_{i<j<k<\ell} [(i-\alpha)(j-\alpha)(k-\alpha)(\ell-\alpha) \; \mathcal{E} \, [w_i w_j w_k w_\ell] \; .$$

Considering the normal order statistic densities appropriate to each term and the effects of summation, we conclude:

$T_1$ is of degree five in n;

$T_2$ and $T_3$ are of degree six in n;

$T_4$ is of degree seven in n; and

$T_5$ is of degree eight in n.

Consequently, $\mathcal{E}[X^4]$ is a polynomial of degree eight in n, so we may write:

4.4.7 $$\mathcal{E}[X^4] = \sum_{1}^{8} a_i n^{(i)} \quad .$$

Our strategy from this point shall be, first, to find $a_8$. Then, we shall find coefficients $a_i$, $i = 1,2,\ldots,7$.

To discover $a_8$ we need consider only the term $T_5$, since $T_5$ is the only term of degree eight. We express:

4.4.8 $$\begin{aligned}(i-\alpha)(j-\alpha)(k-\alpha)(\ell-\alpha) \;=\; & ijk\ell \\ & - \alpha(ijk + ij\ell + ik\ell + jk\ell) \\ & + \alpha^2 (ij + ik + i\ell + jk + j\ell + k\ell) \\ & - \alpha^3 (i + j + k + \ell) + \alpha^4 \quad .\end{aligned}$$

Now:

4.4.9 $$\alpha^4 \sum_{i<j<k<\ell} \mathcal{E}[w_i w_j w_k w_\ell] = 0 \quad .$$

Additional investigation shows that the coefficient of $n^8$ in:

4.4.10 $$\begin{aligned}\sum_{i<j<k<\ell} \Big[ & -\alpha(ijk + ij\ell + ik\ell + jk\ell) \\ & + \alpha^2(ij + ik + i\ell + jk + j\ell + k\ell) \\ & - \alpha^3(i + j + k + \ell)\Big] \, \mathcal{E}\,[w_i w_j w_k w_\ell]\end{aligned}$$

is zero.

Thus, $a_8$ is the coefficient of $n^8$ in the summation:

4.4.11 $$24 \sum_{k<j<k<\ell} ijk\ell \, \mathcal{E}[w_i w_j w_k w_\ell] \quad .$$

Now:

4.4.12 $$\begin{aligned} ijk\ell = {} & (i-1)^{(4)}+2(i-1)(j-i-1)^{(2)}(k-j-1) \\ & + (i-1)(j-i-1)(k-j-1)^{(2)}+(i-1)(j-i-1)^{(3)} \\ & + (i-1)(j-i-1)^{(2)}(\ell-k-1)+(i-1)^{(2)}(k-j-1)^{(2)} \\ & + (i-1)^{(2)}(k-j-1)(\ell-k-1)+3(i-1)^{(2)}(j-i-1)^{(2)} \\ & + 2(i-1)^{(2)}(j-i-1)(\ell-k-1)+4(i-1)^{(2)}(j-i-1)(k-j-1) \\ & + (i-1)^{(3)}(\ell-k-1)+2(i-1)^{(3)}(k-j-1) \\ & + 3(i-1)^{(3)}(j-i-1)+(i-1)(j-i-1)(k-j-1)(\ell-k-1) \\ & + \text{terms with fewer than four factors;} \end{aligned}$$

and:

4.4.13 $$\sum_{i<j<k<\ell} ijk\ell \, \mathcal{E}[w_i w_j w_k w_\ell] =$$

$$n^{(8)} \int_{-\infty}^{\infty} \int_{-\infty}^{w} \int_{-\infty}^{z} \int_{-\infty}^{y} Q(x,y,z,w) x\phi_x y\phi_y z\phi_z w\phi_w \, dxdydzdw$$

+ terms of degree less than eight in n, where:

4.4.14 $$Q(x,y,z,w) = \Phi^4 + 2\Phi R^2 S + \Phi RS^2 + \Phi R^3 + \Phi R^2 T + \Phi^2 S^2 + \Phi^2 ST + 3\Phi^2 R^2 + 2\Phi^2 RT + 4\Phi^2 RS + \Phi^3 T + 2\Phi^3 S + 3\Phi^3 R + \Phi RST .$$

The terms on the right in Formula (4.4.14) have been written in the order given to correspond respectively with the terms on the right in Formula (4.4.12), and:

4.4.15 $$\Phi = \Phi_x$$

4.4.16 $$R = \Phi_y - \Phi_x$$

4.4.17 $$S = \Phi_z - \Phi_y$$

4.4.18 $$T = \Phi_w - \Phi_z .$$

Hence:

4.4.19 $$Q(x,y,z,w) = \Phi_x \Phi_y \Phi_z \Phi_w ,$$

and:

4.4.20 $$a_8 = 24 A ,$$

where:

4.4.21 $$A = \int_{-\infty}^{\infty} \int_{-\infty}^{z} \int_{-z}^{\infty} \int_{-\infty}^{y} x\phi_x \Phi_x \cdot w\phi_w \Phi_w \cdot y\phi_y \Phi_y \cdot z\phi_z \Phi_z \, dx\,dw\,dy\,dz .$$

We now interpose a short list of definite and indefinite integrals useful in reducing integral A.

4.4.22 $$\int u\phi\Phi du = -\phi\Phi + \{2\}, \quad \{2\} = \int \phi^2 \, du$$

4.4.23 $$\int u\phi^2\Phi^2 du = -\tfrac{1}{2}\,\phi^2\Phi^2 + \int \phi^3\Phi \, du$$

4.4.24 $$\int \{2\}u\phi\Phi du = -\{2\}\phi\Phi + \tfrac{1}{2}\{2\}^2 + \int \phi^3\Phi \, du$$

4.4.25 $$\int \phi^3\Phi du = \tfrac{1}{2}\,\phi^2\Phi^2 + \int u\phi^2\Phi^2 \, du$$

Definite integrals:

4.4.26 $$\int_{-\infty}^{\infty} u\phi^3\Phi^3 \, du = \int_{-\infty}^{\infty} \phi^4\Phi^2 \, du$$

4.4.27 $$\int_{-\infty}^{\infty} u\phi^4\,\Phi^4 \, du = \int_{-\infty}^{\infty} \phi^5\,\Phi^3 \, du$$

4.4.28 $$\int_{-\infty}^{\infty} \{2\}z\phi^3\Phi^3 du = \frac{1}{3}\int_{-\infty}^{\infty} \phi^5\Phi^3 du + \int_{-\infty}^{\infty} \{2\}\phi^4\Phi^2 du$$

4.4.29 $$3\int_{-\infty}^{\infty} \{2\}^2\, u\phi^2\Phi^2 du - \int_{-\infty}^{\infty} \{2\}^3\, u\phi\Phi du =$$

$$3\int_{-\infty}^{\infty} \{2\}\,\phi^4\,\Phi^2 \, du - \frac{1}{64\,\pi^2}$$

4.4.30 $$\int_{-\infty}^{\infty} \{2\}^2\, u\phi\Phi du - 2\int_{-\infty}^{\infty} \{2\}\, u\phi^2\Phi^2 du =$$

$$- \int_{-\infty}^{\infty} \phi^4\Phi^2 du + \frac{1}{24\,\pi\,\sqrt{\pi}} .$$

Using indefinite integral (2.4.22), we have:

4.4.31 $$A =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{z} [-\phi_y\Phi_y+\{2_y\}]y\phi_y\Phi_y[\frac{1}{2\sqrt{\pi}}+\phi_z\Phi_z-\{2_z\}]z\phi_z\Phi_z dydz .$$

Indefinite integrals (4.4.23),(4.4.24), and (4.4.25) are now employed to simplify Formula (4.4.31) to the form:

4.4.32 $$A = \frac{1}{4\sqrt{\pi}}\int_{-\infty}^{\infty} z\phi^3\Phi^3 dz + \frac{1}{2}\int_{-\infty}^{\infty} z\phi^4\Phi^4 dz$$

$$- \frac{3}{2}\int_{-\infty}^{\infty} \{2\}z\phi^3\Phi^3\, dz$$

$$+ \frac{1}{2}\left[3\int_{-\infty}^{\infty} \{2\}^2 z\phi^2\Phi^2 dz - \int_{-\infty}^{\infty} \{2\}^3\, z\phi\Phi dz\right]$$

$$+ \frac{1}{4\sqrt{\pi}}\left[\int_{-\infty}^{\infty} \{2\}^2 z\phi\Phi dz - 2\int_{-\infty}^{\infty} \{2\}z\phi^2\Phi^2\, dz\right] .$$

But the definite integrals on the right side of Formula (4.4.32) are those which are listed in like order in Formulae (4.4.26) through (4.4.30), and we therefore obtain:

4.4.33 $$A = \frac{1}{384\,\pi^2} \quad .$$

Hence:

4.4.34 $$a_8 = \frac{1}{16\,\pi^2} \quad .$$

We have yet to evaluate the remaining coefficients: $a_1, a_2, \ldots, a_7$.

Returning to Formula (4.4.7), let $n = 1$. We obtain:

4.4.35 $$a_1 = \mathcal{E}[\sum_1^1 (i-1)\, w_i]^4 = 0 \quad .$$

To find $a_2, a_3, \ldots, a_7$ the same approach is used: evaluation of $\mathcal{E}[\sum_1^n (i - \frac{n+1}{2}) w_i]^4$ for particular values of $n$. As a second (and less trivial) illustration, we now find $a_2$.

Let $n = 2$ in Formula (4.4.7) to obtain:

4.4.36 $$2a_2 = \mathcal{E}[\sum_1^2 (i - \tfrac{3}{2})\, w_i\,]^4 \quad .$$

That is,

4.4.37 $$16\, a_2 = \mathcal{E}[w_2^4 | n=2] + 3\mathcal{E}\,[w_1^2\, w_2^2 | n=2] - 4\mathcal{E}\,[w_1\, w_2^3 | n=2] \quad .$$

But,

4.4.38 $$\mathcal{E}[w_2^4 | n=2] = 2 \int_{-\infty}^{\infty} x^4 \phi \Phi dx = 3 \; .$$

Also,

4.4.39 $$\mathcal{E}[w_1^2 w_2^2 | n=2] = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{y} x^2 \phi_x \cdot y^2 \phi_y \; dxdy = 1,$$

and:

4.4.40 $$\mathcal{E}[w_1 w_3^3 | n=2] = 0 \quad .$$

Thus,

4.4.41 $$a_2 = \frac{3}{8} \quad .$$

The remaining $a_i$ are found in like manner, and are listed below:

4.4.42 $$a_3 = \frac{13}{8} + \frac{27}{2\pi\sqrt{3}}$$

4.4.43 $$a_4 = \frac{21}{16} + \frac{27}{\pi\sqrt{3}} + \frac{7}{\pi}$$

4.4.44 $$a_5 = \frac{5}{16} + \frac{9}{\pi\sqrt{3}} + \frac{15}{2\pi} + \frac{125}{4\pi^2\sqrt{5}}$$

4.4.45 $$a_6 = \frac{1}{48} + \frac{3}{4\pi\sqrt{3}} + \frac{15}{8\pi} + \frac{9}{4\pi^2} + \frac{8}{\pi^2\sqrt{2}}$$

4.4.46 $$a_7 = \frac{1}{8\pi} + \frac{9}{4\pi^2\sqrt{3}} \quad .$$

We summarize findings of this chapter in three tables.

TABLE I

Decimal Values for Coefficients $a_i$ in

$$\mathcal{E}[\sum_{1}^{n} (i - \frac{n+1}{2}) w_i]^4 = \sum_{2}^{8} a_i n^{(i)} .$$

| $i$ | $\mathcal{E}[\sum_{\nu=1}^{i} (\nu - \frac{i+1}{2}) w_\nu]^4$ | $a_i$ |
|---|---|---|
| 2 | .75000 00000 | .37500 00000 |
| 3 | 26.88588 01785 | 4.10598 00298 |
| 4 | 307.10662 30236 | 8.50262 92628 |
| 5 | 1966.55240 4018 | 5.76981 74225 |
| 6 | 8839.95514 86684 | 1.55662 84254 |
| 7 | 31269.46197 4647 | .17140 88133 |
| 8 | | .00633 2573981 |

TABLE II

Values for $b_i$ and $c_i$ in $\mathcal{E}[\sum_1^n (i - \frac{n+1}{2})w_i] = b_1 n + b_2 n^2$ and $\mathcal{E}[\sum_1^n (i - \frac{n+1}{2})w_i]^2 = c_1 n + c_2 n^2 + c_3 n^3 + c_4 n^4$ .

| $i$ | $b_i$ | $c_i$ |
|---|---|---|
| 1 | -.28209 47918 | -.00946 92671 |
| 2 | .28209 47918 | .04835 88438 |
| 3 | | -.11846 70482 |
| 4 | | .07957 747155 |

TABLE III

Values for $d_i$ and $e_i$ in $\mathcal{E}[\sum_1^n (i - \frac{n+1}{2})w_i]^3 = \sum_1^6 d_i n^i$ and

$\mathcal{E}[\sum_1^n (i - \frac{n+1}{2})w_i]^4 = \sum_1^8 e_i n^i$ .

| $i$ | $d_i$ | $e_i$ |
|---|---|---|
| 1 | -.00148 16030 | -.00043 57134 |
| 2 | .00324 945134 | .00022 93585 |
| 3 | -.00393 801022 | .00096 98062 |
| 4 | .01263 341272 | -.00241 33013 |
| 5 | -.03291 16411 | .00508 83662 |
| 6 | .02244 839026 | -.00386 78320 |
| 7 | | -.00590 325817 |
| 8 | | .00633 2573981 |

# V. THE APPROXIMATE POWER OF TESTS BASED ON $R^2$

Having four exact moments of R in the case $p = 1$ it is now possible to approximate the power of tests based on $R^2$. To do this we employ the Pearson system density which agrees in these moments with the true distribution of R. Approximate power values for a two-sided test of the null hypothesis $R_0 = 0$ are shown in Section (5.3) for $n = 10$ and $n = 20$.

In Section (5.4) we obtain twenty random observations from a six-variate normal distribution of known structure to demonstrate a test of the null hypothesis $R_0 = 0$. We derive and calculate the sample discriminant function to which we referred in Chapter I.

We find that the asymptotic relative efficiency of the squared quasi-rank correlation coefficient compared with the squared standard multiple correlation coefficient is independent of the number p of predictor variables.

In a final section we discuss an alternative method of obtaining an approximation to the distribution of $R^2$ and thus an approximation to the power of tests based on $R^2$. The agreement in the case $n = 20$ with the results in Section (5.3) is excellent indeed, and provides evidence that either method will produce good approximations to the exact power.

## 5.1 <u>The Pearson System: Four-Moment Solution</u>

We are fortunate in having a treatise by W. P. Elderton (1938) which provides all formulae needed to select the appropriate type of Pearson curve and to calculate the distribution constants. The notation used below is that of Elderton.

With:

5.1.1 $$\beta_1 = \mu_3^2 \, \mu_2^{-3}$$

5.1.2 $$\beta_2 = \mu_4 \, \mu_2^{-2}$$

one calculates the "criterion" k:

5.1.3 $$k = \beta_1(\beta_2+3)^2 \cdot 4(4\beta_2-3\beta_1)(2\beta_2-3\beta_1-6)$$

to discover which type of Pearson curve to use. With the distribution of R, k will in general be negative, which demands a Type I curve.

Pearson Type I curves are solutions of the differential equation:

5.1.4 $$\frac{1}{y}\, y' = \frac{m_1}{a_1 + x} - \frac{m_2}{a_2 - x} \quad , -a_1 < x < a_2$$

in which:

5.1.5 $$a_1 > 0, \; a_2 > 0 \quad .$$

Elderton considers for mathematical convenience the subclass of solutions of equation (5.1.4) for which the mode is at $x = o$. The curve so obtained will correspond with the given distribution in variance and in shape factors $\beta_1$ and $\beta_2$. A simple translation will produce a curve having in addition the same mean as the given distribution.

We calculate in turn:

5.1.6 $$\begin{aligned} r &= m_1 + 1 + m_2 + 1 \\ &= 6(\beta_2 - \beta_1 - 1)\,(3\beta_1 - 2\beta_2 + 6)^{-1} \end{aligned}$$

5.1.7 $$\begin{aligned} \varepsilon &= (m_1 + 1)(m_2 + 1) \\ &= r^2 \left[4 + \frac{\beta_1(r+2)^2}{4(r+1)}\right]^{-1} . \end{aligned}$$

and

5.1.8 $$b^2 = \frac{\sigma^2 r^2 (r+1)}{\varepsilon}$$

At this point we can calculate $m_1 + 1$ and $m_2 + 1$ as the roots of quadratic equation:

5.1.9 $$M^2 - rM + \varepsilon = 0 \quad .$$

Now $m_1 + 1$ is the larger of these roots. To see this, we record the additional formulae:

5.1.10 $$\mu = \frac{b}{r(r-2)}\,(m_2 - m_1), \text{ where } b = a_1 + a_2$$

5.1.11 $$\mu_3 = 2b^2\ (m_1+1)(m_2+1)(r-2)[r^2(r+1)(r+2)]^{-1}\ \mu \quad .$$

Since $m_1$, $m_2$, and r-2 are positive while $\mu_3$ is negative in our calculations, Formula (5.1.11) implies that $\mu$ is negative, and thus $m_2$ must be smaller than $m_1$ (according to Formula (5.1.10)). Since $m_1 + 1$ is the <u>larger</u> of the two roots of Equation (5.1.9), we obtain:

5.1.12 $$m_1 = -1 + \tfrac{1}{2}r + \tfrac{1}{2}\sqrt{r^2 - 4\varepsilon}$$

5.1.13 $$m_2 = -1 + \tfrac{1}{2}r - \tfrac{1}{2}\sqrt{r^2 - 4\varepsilon} \quad .$$

Further,

5.1.14 $$a_1 = \frac{m_1\ b}{r - 2}$$

5.1.15 $$a_2 = \frac{m_2\ b}{r - 2}$$

This completes the preliminary work of finding the Pearson curve which corresponds with the given distribution in $\sigma^2$, $\beta_1$, and $\beta_2$.

Now $\mathcal{E}[R]$ is the mean of the given distribution, while $\mu$ is the mean of the fitted curve with mode at $x = o$.

5.1.16 $$S = \mathcal{E}[R] - \mu \quad .$$

S is positive in our calculations, and the density we are seeking has the form:

5.1.17 $$\text{Const.}\ \left(1 + \frac{x - s}{a_1}\right)^{m_1} \left(1 - \frac{x - s}{a_2}\right)^{m_2}$$

over the domain $-a_1 + s < x < a_2 + s$ .

Now for values of $\rho^2$ close to unity the appropriate Pearson curve is no longer of Type I. A method is needed to provide a supplementary approximation.

## 5.2 The Pearson System: Two-Moment Solution

To obtain a supplementary approximation for large values of $\rho^2$ one might use that curve of the Pearson system which is indicated by two moments of $R^2$. Thus we are led to consider the Beta distribution, and adjust its parameters by the first two moments of $R^2$.

It is worthy of mention that the distribution of $R^2$ under the null hypothesis is a Beta distribution. Furthermore, as we shall see, a curve of approximate power derived from the four-moment solution above appears to agree very well with the corresponding power curve derived from the two-moment solution for values of $\rho^2$ for which both solutions are available. Thus, the two-moment approximation provides a smooth extrapolation for larger values of $\rho^2$.

We now adjust the parameters $m_1$ and $m_2$ in the density

5.2.1 $$\frac{1}{B[m_1+1,\ m_2+1]}\ x^{m_1} (1 - x)^{m_2} \qquad 0 < x < 1$$

so that the equations

$$5.2.2 \qquad \frac{m_1 + 1}{m_1 + m_2 + 2} = \mathcal{E}[R^2]$$

and

$$5.2.3 \qquad \frac{(m_1 + 1)(m_2 + 1)}{(m_1 + m_2 + 2)(m_1 + m_2 + 3)} = \mathcal{E}[R^4]$$

are satisfied. Solving for $m_1$ and $m_2$, we obtain:

$$5.2.4 \qquad m_1 = \left[2(\mathcal{E}[R^2])^2 - \mathcal{E}[R^4] - \mathcal{E}[R^2]\mathcal{E}[R^4]\right] \sigma^{-2}$$

and

$$5.2.5 \qquad m_2 = \left[\mathcal{E}[R^2] - 2\mathcal{E}[R^4] + \mathcal{E}[R^2]\mathcal{E}[R^4]\right] \sigma^{-2}$$

where

$$5.2.6 \qquad \sigma^2 = \mathcal{E}[R^4] - (\mathcal{E}[R^2])^2 \quad .$$

## 5.3 Approximate Power of Tests Based on $R^2$

A size $\beta$ test of the null hypothesis $R_0 = 0$ against alternatives $R_0^2 \neq 0$ will have a critical region of the type

$$5.3.1 \qquad R^2 > \lambda_\alpha$$

where $\lambda_\alpha$ is determined from the equation

$$5.3.2 \qquad \frac{1}{B[\frac{1}{2}, \frac{n-2}{2}]} \int_{\lambda_\alpha}^{1} t^{\frac{1}{2}-1} (1-t)^{\frac{n-2}{2} - 1} dt = \alpha \quad .$$

Solutions for $\lambda_\alpha$ can be read directly from Table 13, Pearson (1958), for $\alpha$ = .001, .005, .01, .02, .05, and .1; and n - 2 = 1 (1) 20 (5) 50 (10) 100.

Knowing $\lambda_\alpha$, the four-moment solution power is easily found from the equation

$$5.3.3 \qquad \beta = \text{const}_1 \int_{-a_1 + s}^{\lambda_\alpha} \left(1 + \frac{x-s}{a_1}\right)^{m_1} \left(1 - \frac{x-s}{a_2}\right)^{m_2} dx$$

$$= \text{const}_2 \int_0^{\frac{\lambda_\alpha + a_1 - s}{a_1 + a_2}} u^{m_1} (1-u)^{m_2} du \quad ,$$

reading the value of the final integral from the chart of Table 17, Pearson (1958).

Similarly, for the two-moment solution one derives the power from

$$5.3.4 \qquad \beta = \text{const}_3 \int_0^{\lambda_\alpha} u^{m_1} (1 - u)^{m_2} du \quad .$$

where $\lambda_\alpha$ is found as before from Equation (5.3.2), and the value of the integral in Equation (5.3.4) is read from the chart of Table 17, Pearson (1958).

Values of the approximate power so derived are shown in Table IV for sample sizes $n = 10$ and $n = 20$ taking $\alpha = .05$.

TABLE IV

Approximate Power of the Two-Sided Size .05 Test of $R_0 = 0$

for n = 10 and n = 20

| $\rho^2$ | Power, n = 10<br>4-Moment Solution | <br>2-Moment Solution | Power, n = 20<br>4- Moment Solution | <br>2-Moment Solution |
|---|---|---|---|---|
| .00 | .05 | * | .05 | * |
| .05 | .08 | * | .15 | * |
| .10 | .14 | * | .25 | * |
| .15 | .17 | * | .37 | .37 |
| .20 | .22 | * | .50 | .48 |
| .25 | .28 | * | .59 | .60 |
| .30 | .36 | .35 | .70 | .70 |
| .35 | .41 | .41 | .79 | .80 |
| .40 | .49 | .48 | * | .87 |
| .45 | .55 | .55 | * | .93 |
| .50 | .63 | .62 | * | .96 |
| .55 | * | .67 | * | .99 |
| .60 | * | .73 | | |
| .65 | * | .82 | | |
| .70 | * | .87 | | |
| .75 | * | .93 | | |
| .80 | * | .97 | | |
| .85 | * | .99 | | |

* Out of the range of the Pearson Chart.

## 5.4 Demonstration Study: Test of $R_0^2 = 0$; The Discriminant Function.

In this section we illustrate the computations required to test the null hypothesis $R_0 = 0$ against alternatives $R_0 \neq 0$. In addition we derive and calculate the vector of coefficients $\hat{\underline{\beta}}$ in the linear combination $\hat{\underline{\beta}}'\underline{x}$, $\underline{x} = (x_1, x_2, \ldots, x_p)'$, which would be used to rank subsequently chosen individuals in order of merit. Thus, $\hat{\underline{\beta}}'\underline{x}$ is the discriminant function to which we referred in Chapter 1 as an alternative to the classical discriminant function of Fisher.

Given a random sample of size n from a (p + 1)-variate normal distribution, we rank the vectors $\underline{x}_i = (x_{1i}, x_{2i}, \ldots, x_{pi})$ in order of size of the $x_{0i}$: $x_{01} < x_{02} < \ldots < x_{0n}$. We form in turn

5.4.1 $$C_{01} = \sum_{1}^{n} di\ (\underline{x}_i - \bar{\underline{x}})'$$

5.4.2 $$C_{11} = \sum_{1}^{n} (\underline{x}_i - \bar{\underline{x}})(\underline{x}_i - \bar{\underline{x}})'$$

and

5.4.3 $$R^2 = C_{01}\ C_{11}^{-1}\ C_{10} \quad .$$

To determine the coefficients $\hat{\underline{\beta}}$ in the linear combination $\hat{\underline{\beta}}'\underline{x}$ such that the $\hat{\underline{\beta}}'\underline{x}_i$ have maximum simple correlation with the $d_i$, we form the simple correlation coefficient r:

$$5.4.4 \qquad r = \frac{\underline{\beta}' \; C_{10}}{\sqrt{\underline{\beta}' \; C_{11} \; \underline{\beta}}} \quad .$$

Since r is independent of scale, we take for convenience

$$5.4.5 \qquad \underline{\beta}' \; C_{11} \; \underline{\beta} = 1$$

and maximize unconditionally the expression

$$5.4.6 \qquad Q = \underline{\beta}' \; C_{10} - \tfrac{1}{2}\lambda\underline{\beta}' \; C_{11} \; \underline{\beta}$$

over all possible choices of $\underline{\beta}$.

Differentiating with respect to $\underline{\beta}$, we have

$$5.4.7 \qquad \frac{\partial Q}{\partial \underline{\beta}} = C_{10} - \lambda \; C_{11} \; \underline{\beta} \quad .$$

Denoting the critical value of $\underline{\beta}$ by $\hat{\underline{\beta}}$, we have on setting $\frac{\partial Q}{\partial \underline{\beta}} = \underline{0}$ that:

$$5.4.8 \qquad C_{10} - \lambda \; C_{11} \; \hat{\underline{\beta}} = \underline{0} \quad .$$

Multiplying both sides of Equation (5.4.8) by $\hat{\underline{\beta}}'$, we have by virtue of Equation (5.4.5) that:

$$5.4.9 \qquad \lambda = \hat{\underline{\beta}}' \; C_{10} = R$$

where R is the maximized simple correlation coefficient. Also,

5.4.10 $$\hat{\underline{\beta}} = \frac{1}{R} C_{11}^{-1} C_{10} \quad .$$

Again, since the scale of $\underline{\beta}$ can be chosen for convenience, we take

5.4.11 $$\hat{\underline{\beta}} = C_{11}^{-1} C_{10} \quad .$$

For a numerical illustration we obtain a random sample of size $n = 20$ of normal 6-component vectors $(x_{0i}, \underline{x}_i') = (x_{0i}, x_{1i}, \ldots, x_{5i})$ by expressing:

5.4.12 $$\begin{aligned} x_{0i} &= u_{1i} + u_{2i} + u_{5i} - 2u_{6i} \\ x_{1i} &= u_{1i} + u_{2i} \\ x_{2i} &= u_{1i} - 2u_{2i} \\ x_{3i} &= u_{3i} \\ x_{4i} &= u_{3i} + u_{4i} \\ x_{5i} &= u_{1i} + u_{4i} + u_{5i} \end{aligned}$$

in which $u_{mi}$ is independent of $u_{\nu j}$ for $(m, i) \neq (\nu, j)$ and each $u_{mi}$ has the standard normal distribution.

The squared population multiple correlation coefficient is then by virtue of Formula (2.1.17),

5.4.13 $$R_0^2 = \frac{3}{7} \quad .$$

We find on using Formula (2.3.80) together with Formula (3.2.23) that in this case:

5.4.14 $$\mathcal{E}[R^2] = .5366 \quad .$$

The critical region of the two-sided size $\alpha = .05$ test of the hypothesis $R_0 = 0$ is determined as outlined in Section (2.4), and consists of all $R^2$ such that:

5.4.15 $$R^2 > \lambda = .514$$

Since the asymptotic relative efficiency of $R^2$ is shown (in Section (5.5)) to be independent of p, we obtain an indication of the power of this test in the case $p = 5$ from Table IV which was developed for the case $p = 1$. The estimate so obtained is 0.90.

The random sample was generated by using Table A-2, Dixon and Massey (1957), of random normal numbers with $\mu = 0$ and $\sigma^2 = 1$.

We obtain:

5.4.16 $$R^2 = C_{01}\, C_{11}^{-1}\, C_{10}$$

in which

5.4.17 $C_{01} =$

$$(4.4519667,\ -2.9923112,\ .501151877,\ .31016855,\ 3.6394635)$$

and

5.4.18 $C_{11} =$

$$\begin{bmatrix} 42.829555 & -44.573736 & -.678252 & -7.368117 & 10.678235 \\ -44.573736 & 125.163006 & 22.868677 & 23.759471 & 6.051730 \\ -.678252 & 22.868677 & 26.460397 & 26.060996 & -1.107908 \\ -7.368117 & 23.759471 & 26.060996 & 36.849465 & 8.151393 \\ 10.678235 & 6.051730 & -1.107908 & 8.151393 & 34.573935 \end{bmatrix}$$

Calculation provides the sample value of $R^2$:

5.4.19 $R^2 = .727$

Since $R^2$ is in the example beyond the critical value $\lambda = .514$ we would correctly reject the null hypothesis: $R_0^2 = 0$.

The coefficient vector for the sample discriminant function turns out to be:

5.4.20 $\hat{\underline{\beta}}' = (.017916,\ -.032856,\ .168185,\ -.11635\ ,\ .138306)$

and the sample discriminant function is

5.4.21 $\hat{\underline{\beta}}'\underline{x} = .017916x_1 - .032856x_2 + .168185x_3 - .116356x_4$

$+ .138306x_5$ .

We now find the simple correlation coefficients of the $\hat{\underline{\beta}}'\underline{x}_i$ with the $x_{1i}$, $x_{2i}$, $x_{3i}$, $x_{4i}$, and $x_{5i}$ individually. These are listed in Table V below as $r_1$, $r_2$, $r_3$, $r_4$, and $r_5$ respectively.

TABLE V

Sample Correlations of the set $\{\hat{\underline{\beta}}'\underline{x}_i\}_1^n$ with the sets $\{x_{ki}\}_{i=1}^n$, $k = 1, 2, \ldots, 5$.

| $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ |
|---|---|---|---|---|
| .7952 | -.3164 | .1155 | .0597 | .7234 |

Since in the probability structure of the observations $x_3$ and $x_4$ are uncorrelated with $x_0$, it was to be expected that variates $x_3$ and $x_4$ would have the smallest sample correlations with the discriminant function.

In Table V, $r_1$ and $r_5$ are both large. As is the custom in the multivariate theory, our attention would be drawn to variables $x_{1i}$ and $x_{5i}$ if it is desired to reduce the dimensions of the predictor variable. This is in accordance with the structure of $(x_{0i}, x_{1i}, \ldots, x_{5i})$ wherein $x_{1i}$ and $x_{5i}$ have comparatively large population coefficients with $x_{0i}$.

Finally, by using the sample discriminant function, we can rank the 20 individuals in the given sample and compare these estimated ranks with their actual ranks based on $x_{0i}$. The following comparative ranks were obtained:

| Rank on $x_{0i}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Estimated Rank | 1 | 2 | 4 | 6 | 5 | 3 | 8 | 9 | 13 | 7 |

| Rank on $x_{0i}$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| Estimated Rank | 10 | 11 | 14 | 12 | 16 | 17 | 18 | 19 | 15 | 20 |

In no case is the discrepancy in rank more than four.

## 5.5 The Asymptotic Relative Efficiency of $R^2$

To avoid some confusion we shall in this section employ the notations:

5.5.1 $R_Q$ = the quasi-rank correlation coefficient;

5.5.2 $R_P$ = the standard multiple correlation coefficient.

Now $R_Q^2$ and $R_P^2$ have identical distributions when $R_0 = 0$. To see that this is true, recall from Chapter 2 that when $R_0 = 0$, $u = 1 - R_Q^2$ has a Pearson Type I distribution. In particular, u has the Beta distribution of the first kind given in Formula (2.3.72).

Further, $\frac{1-u}{u}$ has the Beta distribution of the second kind with density

5.5.3 $$\frac{1}{B[\frac{p}{2}, \frac{\nu-p}{2}]} \frac{x^{\frac{1}{2}p-1}}{(1+x)^{\frac{1}{2}\nu}} \qquad 0 < x < \infty$$

so that $$\frac{\nu-p}{p} \frac{1-u}{u} = \left[\frac{\nu-p}{p}\right] \frac{R_Q^2}{1-R_Q^2}$$

has the density

5.5.4 $$\frac{\Gamma[\frac{1}{2}\nu]}{\Gamma[\frac{1}{2}(\nu-p)]\Gamma[\frac{1}{2}p]} \left[\frac{p}{\nu-p}\right]^{\frac{1}{2}p} \frac{x^{\frac{1}{2}p-1}}{(1+\frac{p}{\nu-p}x)^{\frac{1}{2}\nu}} \qquad 0 < x < \infty$$

Thus, when $R_0 = 0$, $\frac{\nu-p}{p} \frac{R_Q{}^2}{1-R_Q{}^2}$ has Fisher's F distribution with p and $\nu$-p degrees of freedom. But this is precisely the distribution of $\frac{\nu-p}{p} \frac{R_P{}^2}{1-R_P{}^2}$ (Anderson (1958), p. 90).

Since the null distributions of $R_Q^2$ and $R_P^2$ are identical, the asymptotic relative efficiency of $R_Q^2$ vis-à-vis $R_P^2$ is:

5.5.5 $$\text{A.R.E. } [R_Q^2 \text{ vs } R_P^2] = \lim_{n\to\infty} \left\{ \frac{\frac{\partial}{\partial R_0{}^2} \mathcal{E}\left[R_Q^2 : R_0^2\right]_{R_0^2 = 0}}{\frac{\partial}{\partial R_0{}^2} \mathcal{E}\left[R_P^2 : R_0^2\right]_{R_0^2 = 0}} \right\}^2 .$$

Now Anderson (1958) presents the moments of $R_P^2$ in Equation (39), page 96. In particular,

5.5.6 $$\mathcal{E}[R_P^2] = \frac{(1-R_0{}^2)^{\frac{1}{2}\nu}}{\Gamma[\frac{1}{2}\nu]} \sum_{i=0}^{\infty} \frac{R_0^{2i}\, \Gamma^2[\frac{1}{2}\nu+i]\ \Gamma\ [\frac{1}{2}p+i+1]}{i!\,\Gamma[\frac{1}{2}p+i]\ \ \Gamma\ \ [\frac{1}{2}\nu+i+1]} .$$

Differentiating with respect to $R_0^2$, and evaluating this derivative at $R_0^2 = 0$, we obtain:

5.5.7 $$\frac{\partial}{\partial R_0^2} \mathcal{E}[R_P^2] \bigg|_{R_0^2 = 0} = -\tfrac{1}{2}p + \frac{\nu}{\nu+2} (\tfrac{1}{2}p + 1) .$$

Thus, $\lim_{n\to\infty} \frac{\partial}{\partial R_0^2} \mathcal{E}[R_P^2]\Big|_{R_0^2 = 0}$ is free of p and in fact:

5.5.8 $$\lim_{n\to\infty} \frac{\partial}{\partial R_0^2} \mathcal{E}[R_P^2]\bigg|_{R_0^2 = 0} = 1 .$$

Equation (3.2.23) above expresses the expected value of $R_Q^2$. Writing this expected value in terms of $R_0^2$, we have:

5.5.9 $\mathcal{E}[R_Q^2] =$

$$\frac{\mathcal{E}[X^2]}{2\Omega}(1 - R_0^2)^{\frac{1}{2}\nu} \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2}\nu + k \\ k \end{bmatrix} \frac{R_0^{2\,(k+1)}}{k + \frac{1}{2}\nu + 1} +$$

$$\frac{1}{2}(1 - R_0^2)^{\frac{1}{2}\nu} \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2}\nu + k - 1 \\ k \end{bmatrix} \frac{R_0^{2k}}{k + \frac{1}{2}\nu} \quad .$$

Differentiating with respect to ${R_0}^2$, and evaluating this derivative at ${R_0}^2$, we obtain:

5.5.10 $$\frac{\partial}{\partial R_0^2} \mathcal{E}[R_Q^2]\Bigg|_{R_0^2 = 0} = \frac{\mathcal{E}[X^2]}{\Omega(\nu+2)} - \frac{1}{\nu+2} \quad .$$

As we shall see (Chapter 4, Equation (4.2.19)),

5.5.11 $$\mathcal{E}[X^2] = \frac{1}{4\pi} n^{(4)} + \left(\frac{1}{12} + \frac{3}{2\pi\sqrt{3}}\right) n^{(3)} + \frac{1}{4} n^{(2)} \quad .$$

Hence,

5.5.12 $$\lim_{n\to\infty} \left[\frac{\mathcal{E}[X^2]}{\Omega(\nu+2)} - \frac{1}{\nu+2}\right] = \frac{3}{\pi} \quad .$$

Using the limits recorded in equations (5.5.8) and (5.5.12) in Formula (5.5.5), we have:

5.5.13 $$\text{A.R.E. } [R_Q^2 \text{ vs. } R_P^2] = \left(\frac{3}{\pi}\right)^2 = \frac{9}{\pi^2} \quad .$$

It is interesting that this result is a constant independent of the number of measured variates p.

## 5.6 An Alternative Approximation to the Distribution of $R^2$ When $p = 1$.

Though the density of R found in Chapter III did not seem to be usable for finding probabilities, we were able to generate four moments of R. In sections 1, 2, and 3 of this chapter we used these moments to find an approximate density for $R^2$ which does allow calculation of an approximate power of tests based on $R^2$.

Using a rather different approach, we find in this section an alternative approximate density of $R^2$. It will be clear from the derivation that approximate power so obtained will be increasingly accurate with larger sample sizes. Additional evidence of the asymptotic accuracy of this method is shown by comparing approximate moments of $R^2$ with exact ones for several values of n.

Good agreement of the approximate power obtained in this section with the corresponding results in Section (5.3) will be taken as evidence that either method is indeed satisfactory.

Consider once more the distribution of

$$R^2 = \frac{\left(\sum_1^n d_i y_i\right)^2}{\sum_1^n (y_i - \bar{y})^2} \tag{5.6.1}$$

under the conditions:

i) Vectors $(x_i, y_i)$ are randomly selected from the bivariate normal population with correlation parameter $\rho$, $i = 1, 2, \ldots, n$;

ii) Subscripts i in the vectors $(x_i, y_i)$ are reassigned so that $x_i$ is the $i^{\underline{th}}$ smallest of the x's;

iii) $d_i = \frac{1}{\sqrt{\Omega}}(i - \frac{n+1}{2})$ $\qquad i = 1, 2, \ldots, n$ .

In the joint conditional distribution of the y's given the set of x's, the $y_i$ are independent. $y_i$ depends only on the value of $x_i$ and in fact:

5.6.2 $\qquad y_i | x_i \frown N(\rho x_i : 1-\rho^2)$ .

In the marginal distribution of the set of x's, $x_i$ is the $i^{\underline{th}}$ smallest standard normal order statistic. We use the notations:

5.6.3 $\qquad \mathcal{E}[x_i] = \xi_i \qquad\qquad i = 1, 2, \ldots, n$

C is the n-square matrix with (i, j)-element $c_{ij}$:

5.6.4 $\qquad \mathcal{E}[(x_i-\xi_i)(x_j-\xi_j)] = (C)_{ij} = c_{ij} \qquad {i \atop j} = 1, 2, \ldots, n$

5.6.5 $\qquad \underline{\xi}' = (\xi_1, \xi_2, \ldots, \xi_n)$

5.6.6 $\qquad \underline{y}' = (y_1, y_2, \ldots, y_n)$

It is readily shown that

5.6.7 $$\mathcal{E}[\underline{y}] = \rho\,\underline{\xi}$$

and

5.6.8 $$\mathcal{E}(\underline{y}-\rho\underline{\xi})(\underline{y}-\rho\underline{\xi})'] = (1-\rho^2)I + \rho^2 C \quad .$$

Further,

5.6.9 $$\mu_3\,(y_i) = \rho^3\mu_3(x_i)$$

and

5.6.10 $$\frac{\mu_4\,(y_i)}{\mu_2^2\,(y_i)} = 3 + \frac{\rho^4\,[\,\mu_4\,(x_i) - 3\mu_2^2\,(x_i)]}{[1 - \rho^2 + \rho^2\,c_{ii}]^2}$$

$$i = 1, 2, \ldots, n \quad .$$

Since $\mu_3\,(x_i)$ is of order $O(n^{-2})$ and $\mu_4\,(x_i) - 3\mu_2^2\,(x_i)$ is of order $O\,(n^{-3})$ unless i is near to unity or to n (David and Johnson (1954), it appears that one might consider for approximation purposes that

5.6.11 $$\underline{y} \frown N_n[\rho\underline{\xi}\ :\ (1 - \rho^2)\,I + \rho^2\,C]$$

unless $\rho^2$ is very close to unity.

However, the elements of C are not simple functions of n. To have any success in developing the distribution of $R^2$ it seems that we must seek to represent $c_{ij}$ by $\tilde{c}_{ij}$ :

5.6.12 $$\tilde{c}_{ij} = \alpha_0 + \alpha_1 e_i + \alpha_2 e_j + \alpha_3 e_i e_j$$

where $e_i = i - \frac{n+1}{2}$ $\qquad i = 1, 2, \ldots, n$

and the $\alpha$'s are constants to be determined by minimizing the sum of squares:

5.6.13 $$\sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} - \tilde{c}_{ij})^2 \quad .$$

We find:

5.6.14 $$\hat{\alpha}_0 = \frac{1}{n}$$

5.6.15 $$\hat{\alpha}_1 = \hat{\alpha}_2 = 0$$

5.6.16 $$\hat{\alpha}_3 = \Omega^{-2}\{\mathcal{E}[X^2] - \frac{1}{\pi}\binom{n}{2}^2\}$$

where

5.6.17 $$X = (i - \tfrac{n+1}{2})x_i \qquad i = 1, 2, \ldots, n.$$

In Chapter IV we found that

5.6.18 $$\mathcal{E}[X^2] = \left[\frac{6}{\pi} - \frac{6}{n-2}\right]\binom{n}{4} + \left[\frac{3\sqrt{3}}{\pi} + 2\right]\binom{n}{3} \quad .$$

Hence

5.6.19 $$\hat{\alpha}_3 = \Omega^{-1}\gamma$$

where

5.6.20 $$\gamma = 1 + \frac{6}{\pi}(\sqrt{3} - 2) + \frac{6}{\pi}(5 - 3\sqrt{3})(n + 1)^{-1}$$

$$= .4882547385 - .3746235346\ (n + 1)^{-1} \quad .$$

With

5.6.21 $$d_i = \Omega^{-\frac{1}{2}} e_i \qquad i = 1, 2, \ldots, n$$

and D the n-square matrix with (i,j)-element

5.6.22 $$(D)_{ij} = d_i d_j \qquad \begin{matrix} i \\ j \end{matrix} = 1, 2, \ldots, n$$

we introduce the matrix $\tilde{C}$ :

5.6.23 $$\tilde{C} = \frac{1}{n} J + \gamma D$$

and study the distribution of

5.6.24 $$R^2 = \frac{\underline{y}' \ D \ \underline{y}}{\underline{y}' \ [I - \frac{1}{n} J] \ \underline{y}}$$

for

5.6.25 $$\underline{y} \frown N_n \ [\rho\underline{\xi} : (1-\rho^2)I + \rho^2 \tilde{C}] \quad .$$

Now

5.6.26 $$R^2 = \frac{Q_1}{Q_1 + a\ Q_2}$$

with

5.6.27 $$a = (1 - \rho^2)\ (1 - \rho^2 + \gamma\rho^2)^{-1}$$

5.6.28 $$Q_1 = (1 - \rho^2 + \gamma\rho^2)^{-1}\ \underline{y}'\ D\ \underline{y}$$

5.6.29 $$Q_2 = (1 - \rho^2)^{-1}\ \underline{y}'\ (I - \tfrac{1}{n}\ J - D)\underline{y} \quad .$$

$Q_1$ and $Q_2$ are independent noncentral Chi-square variates with degrees of freedom 1 and n-2 and noncentrality parameters

5.6.30 $$\lambda_1 = \frac{1}{2\Omega}\ \frac{\rho^2}{1 - \rho^2 + \gamma\rho^2}\ \left(\sum_1^n j\xi_j\right)^2$$

5.6.31 $$\lambda_2 = \frac{1}{2}\ \frac{\rho^2}{1-\rho^2}\left[\sum_1^n \xi_j^2 - \frac{1}{\Omega}\left(\sum_1^n j\xi_j\right)^2\right]$$

respectively.

The density of $R^2$ is easily shown to be:

5.6.32 $$e^{-(\lambda_1+\lambda_2)}$$

$$\sum_{i=0}^{\infty}\ \sum_{j=0}^{\infty}\ \frac{\lambda_1{}^i\lambda_2{}^j}{i!j!}\ \frac{a^{i+\frac{1}{2}}}{B[i+\frac{1}{2},\ j+\frac{1}{2}(n-2)]}\ \frac{u^{i-\frac{1}{2}}(i-u)^{j+\frac{1}{2}n-2}}{(au+1-u)^{i+j+\frac{1}{2}(n-1)}} \quad .$$

It seems infeasible to integrate this density for probabilities except when a=1. Setting a=1 is equivalent to taking $\hat{\alpha}_3 = 0$. Since by Formula (5.6.19) $\hat{\alpha}_3$ is of order $n^{-3}$, the probabilities should not be greatly affected by taking $\hat{\alpha}_3 = 0$ except possibly for values of n and $\rho$ such that $n^3(1-\rho^2)^2$ is very small (by virtue of Formula (5.6.10)).

For an example, we take $n = 20$. The critical region of the size $\alpha = .05$ two-sided test of $R_0^2 = 0$ is

5.6.33 $$R^2 > .197$$

and its power is

5.6.34 $$e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda_1^{\,i}\lambda_2^{\,j}}{i!j!} \frac{1}{B[i+\frac{1}{2}, j+9]} \int_{.197}^{1} u^{i-\frac{1}{2}}(1-u)^{j+8}du \; .$$

Since $\lambda_2$ is small in this example, we need consider only terms for $j = 0, 1, 2, 3$. We obtain the results given in the following table.

TABLE V

Approximate Power of the Size $\alpha=.05$

Two-Sided Test of $R_0^2=0$ for $n=20$

| $\rho^2$ | $\lambda_1$ | $\lambda_2$ | Power |
|---|---|---|---|
| .1 | .959 982 | .022 139 | .257 |
| .2 | 2.159 960 | .049 813 | .501 |
| .3 | 3.702 789 | .085 393 | .726 |
| .4 | 5.759 893 | .132 834 | .891 |
| .5 | 8.639 840 | .199 251 | .974 |

A comparison of these values with the corresponding values for $n = 20$ in Table IV shows the agreement to be very close for all $\rho^2$.

As additional evidence of the adequacy of the approximations in this section, we show in the following table exact moments $\mathcal{E}[(R^2)^h]$ for h = 1,2 and the corresponding approximate moments found from the formula:

$$\mathcal{E}[(R^2)^h] = e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda_1^{\,i}\,\lambda_2^{\,j}}{i!j!} \frac{1}{B[i+\frac{1}{2},\ j+9]} \int_0^1 u^{i+h-\frac{1}{2}}(1-u)^{j+8}du$$

TABLE VI

Approximate and True Values of $E[R^2]$ and $\mathcal{E}[R^4]$ : n = 3

| $\rho$ | $\mathcal{E}[R^2]$ | | | $\mathcal{E}[R^4]$ | | |
|---|---|---|---|---|---|---|
| | Approx. | True | % Error | Approx. | True | % Error |
| .1 | .5018 | .5021 | -.06 | .3768 | .3771 | -.08 |
| .2 | .5074 | .5084 | -.20 | .3824 | .3834 | -.26 |
| .3 | .5183 | .5192 | -.37 | .3923 | .3941 | -.43 |
| .4 | .5326 | .5350 | -.45 | .4078 | .4098 | -.49 |
| .5 | .5552 | .5566 | -.25 | .4307 | .4312 | -.12 |
| .6 | .5885 | .5854 | .53 | .4649 | .4584 | 1.42 |
| .7 | .6385 | .6237 | 2.37 | .5173 | .4965 | 4.19 |
| .8 | .7172 | .6759 | 6.11 | .6027 | .5460 | 10.38 |
| .9 | .8440 | .7524 | 12.17 | .7530 | .6171 | 22.02 |

TABLE VII

Approximate and True Values of $\mathcal{E}[R^2]$
and $\mathcal{E}[R^4]$ : n = 10

| $\bar{\rho}$ | $\mathcal{E}[R^2]$ | | | $\mathcal{E}[R^4]$ | | |
|---|---|---|---|---|---|---|
| | Approx. | True | % Error | Approx. | True | % Error |
| .1 | .1174 | .1177 | -.25 | .0332 | .0334 | -.42 |
| .2 | .1367 | .1377 | -.71 | .0424 | .0429 | -1.23 |
| .3 | .1699 | .1714 | -.89 | .0590 | .0600 | -1.77 |
| .4 | .2185 | .2195 | -.46 | .0852 | .0866 | -1.62 |
| .5 | .2851 | .2831 | .70 | .1249 | .1257 | -.57 |
| .6 | .3727 | .3636 | 2.50 | .1850 | .1820 | 1.61 |
| .7 | .4852 | .4631 | 4.75 | .2768 | .2633 | 5.12 |
| .8 | .6260 | .5849 | 7.03 | .4196 | .3819 | 9.88 |
| .9 | .7954 | .7338 | 8.39 | .6424 | .5603 | 14.66 |

TABLE VIII

Approximate and True Values of $\mathcal{E}[R^2]$
and $\mathcal{E}[R^4]$ : n = 20

| $\rho$ | $\mathcal{E}[R^2]$ | | | $\mathcal{E}[R^4]$ | | |
|---|---|---|---|---|---|---|
| | Approx. | True | % Error | Approx. | True | % Error |
| .1 | .0604 | .0606 | -.30 | .0096 | .0096 | -.55 |
| .2 | .0841 | .0846 | -.69 | .0164 | .0166 | -1.43 |
| .3 | .1242 | .1250 | -.66 | .0295 | .0301 | -1.94 |
| .4 | .1818 | .1821 | -.14 | .0522 | .0532 | -1.81 |
| .5 | .2585 | .2566 | .75 | .0898 | .0906 | -.88 |
| .6 | .3560 | .3493 | 1.90 | .1505 | .1491 | .94 |
| .7 | .4761 | .4616 | 3.14 | .2470 | .2386 | 3.54 |
| .8 | .6201 | .5949 | 4.23 | .3976 | .3732 | 6.51 |
| .9 | .7880 | .7515 | 4.86 | .6255 | .5742 | 8.94 |

## VI. Joint Moment Generating Function of $S^2 = \sum_1^n (x_i - \bar{x})^2$ and $\Omega\eta^2 = [\sum_1^n (i - \frac{n+1}{2})x_i]^2$, $x_i$ = the $i^{th}$ Standard Normal Order Statistic in n.

In a preliminary effort to investigate the distribution of R and develop the moments $\mathcal{E}[R^h]$ in the case $p = 1$, we employed the moment generating function of this chapter. Though an alternative and less cumbersome method of handling the moment problem has been given in Chapter 3, this chapter has been included for the additional insight it may provide regarding the joint distribution of the important statistic $S^2$ and the linear combination of quasi-ranges $\Omega^{\frac{1}{2}}\eta = \sum_1^n (i - \frac{n+1}{2})x_i$. No additional light is thrown on the distribution of R however.

### 6.1 Expression of $\phi(\theta, \phi) = \mathcal{E}\left[e^{\theta s^2 + \phi\Omega\eta^2}\right]$ as the Principal Quadrant Volume Bounded by an (n-1)-Dimensional Normal Surface.

Using the joint density $f(x_1, x_2, \ldots, x_n)$ of the ordered $x_i$:

$$f(x_1, x_2, \ldots, x_n) =$$

$$\frac{n!}{(2\pi)^{\frac{1}{2}n}} \exp\left[-\tfrac{1}{2}\sum_1^n x_i^2\right], \; -\infty < x_1 < x_2 < \ldots < x_n < \infty,$$

we have from the definition that:

$$6.1.2 \qquad \Phi(\theta,\phi) = \int\cdots\int \frac{n!}{(2\pi)^{\frac{1}{2}n}} \exp[-\tfrac{1}{2}Q(x_1,x_2,\ldots,x_n)]dx_1dx_2\ldots dx_n$$

$$-\infty < x_1 < x_2 < \ldots < x_n < \infty$$

where

$$6.1.3 \qquad Q(x_1, x_2, \ldots, x_n) = \sum_1^n x_i^2 - 2\theta S^2 - 2\phi\Omega\eta^2 \quad .$$

To benefit from the convenience of matrix notation we introduce the n-square matrices $J_n$ and E:

$$6.1.4 \qquad (J_n)_{ij} = 1 \qquad \begin{matrix} i \\ j \end{matrix} = 1, 2, \ldots, n$$

$$6.1.5 \qquad (E)_{ij} = \nu_{ij} = (2i - \overline{n+1})(2j - \overline{n+1}) \quad \begin{matrix} i \\ j \end{matrix} = 1,2,\ldots,n \quad .$$

Also, let:

$$6.1.6 \qquad x = 1 - 2\theta$$

$$6.1.7 \qquad \alpha = \frac{2\theta}{n}$$

$$6.1.8 \qquad \beta = \frac{\phi}{2}$$

$$6.1.9 \qquad \underline{x}' = (x_1, x_2, \ldots, x_n)$$

Then,

6.1.10 $$\phi(\theta,\phi) = \int\cdots\int \frac{n!}{(2\pi)^{\frac{1}{2}n}} \exp[-\tfrac{1}{2}\underline{x}'C\underline{x}]\, dx_1 dx_2 \cdots dx_n$$

$$-\infty < x_1 < x_2 < \ldots < x_n < \infty$$

where

6.1.11 $$C = xI_n + \alpha J_n - \beta E \quad .$$

To obtain a simpler domain of integration we employ the nonsingular transformation

6.1.12 $$\underline{x} = B\,\underline{u}$$

with

6.1.13 $$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \quad \text{(lower triangular).}$$

and

6.1.14 $$\underline{u}' = (u_1, u_2, \ldots, u_n) \quad .$$

Thus

6.1.15 $$\Phi(\theta,\phi) = \underset{\substack{-\infty < u_1 < \infty \\ 0 < u_2, \ldots, u_n < \infty}}{\int\cdots\int} \frac{n!}{(2\pi)^{\frac{1}{2}n}} \exp[-\tfrac{1}{2}\underline{u}'\Sigma^{-1}\underline{u}]du_1 du_2 \cdots du_n$$

where

6.1.16 $$\Sigma^{-1} = B' \; C \; B \;.$$

Now

6.1.17 $$(\Sigma^{-1})_{pq} = \tfrac{1}{2}x[n + 1 - p + n + 1 - q + |p - q|]$$
$$+ \alpha\,(n - p + 1)(n - q + 1)$$
$$- \beta\,(n - p + 1)(n - q + 1)(p - 1)(q - 1)$$
$$\begin{matrix} p \\ q \end{matrix} = 1,2,\ldots,n \quad .$$

Carrying out the integration over $u_1$, we obtain

6.1.18 $$\Phi(\theta,\phi) =$$
$$\underset{0 < x_2, \ldots, x_n < \infty}{\int\cdots\int} \frac{n!\,|\Sigma|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(n-1)}\,|\Sigma^{(11)}|^{\frac{1}{2}}} \;.\; \exp[-\tfrac{1}{2}\underline{u}'_{(2)}(\Sigma^{(11)})^{-1}\underline{u}_{(2)}]\; du_2, \ldots du_n \;.$$

where $\Sigma^{(11)}$ is the cofactor of $(\Sigma)_{11}$ in $\Sigma$ and $\underline{u}'_{(2)} = (u_2, u_3, \ldots, u_n)$ .

Noting from Formula (6.1.16) that

6.1.19 $$|\Sigma|^{\frac{1}{2}} = |C|^{-\frac{1}{2}}$$

we may now write

6.1.20 $$\Phi(\theta, \phi) = n!\,|C|^{-\frac{1}{2}}\, V$$

where V is the volume in the principal quadrant bounded by the (n-1)-dimensional normal surface

6.1.21 $$p\,(u_2, u_3, \ldots, u_n) = \frac{|\Sigma^{(11)}|^{-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(n-1)}} \exp[-\tfrac{1}{2}\, \underline{u}'_{(2)} (\Sigma^{(11)})^{-1}\, \underline{u}_{(2)}] ,$$

In preparation for finding a simplified form for $\Sigma^{(11)}$ in Section (6.4) we include the next two sections in which we evaluate $|C|$ and express $C^{-1}$ in terms of the latent roots and linearly independent eigenvectors of C.

## 6.2 Evaluation of $|C|$.

$|C|$ is the product of the latent roots of C. A simple corollary of Theorem 28.5, p. 73, Browne (1958) will aid us in obtaining these latent roots by inspection.

Corollary 6.1

Let C be an n-square real symmetric matrix. If the rank of $C - h I_n$ is $\nu$, then h is a latent root of C of multiplicity $n-\nu$.

We recall Equation (6.1.11):

6.2.1 $$C = xI_n + \alpha J_n - \beta E \quad .$$

Now

6.2.2 $$C - xI_n = \alpha J_n - \beta E$$

is clearly of rank 2. Thus by Corollary (6.1) C has latent root x of multiplicity n - 2.

Further, each row sum of C is 1, and hence 1 is a latent root of C.

We denote the final latent root of C as $\lambda_n$. There exists an orthogonal matrix P such that

6.2.3 $$P'CP = \mathrm{diag}(x, x, \ldots, x, 1, \lambda_n)$$

Hence

6.2.4 $$\mathrm{tr}(C) = \mathrm{tr}(P'CP) = (n-2)x + 1 + \lambda_n \quad .$$

But

6.2.5 $$\mathrm{tr}(C) = nx + n\alpha - 4\Omega\beta \quad .$$

Equating these expressions for tr(C), we obtain

6.2.6 $$\lambda_n = x - 4\Omega\beta \quad .$$

Finally,

6.2. $$|C| = x^{n-2}(x - 4\Omega\beta) \quad .$$

## 6.3 Linearly Independent Eigenvectors and the Inverse of C.

Recalling that

6.3.1 $$C = xI_n + \alpha J_n - \beta E \ ,$$

that $x + n\alpha = 1$, and that the row sums of matrix E are each zero, we have:

6.3.2 $$C\, n^{-\frac{1}{2}}(1,1,\ldots,1)' = n^{-\frac{1}{2}}(1,1,\ldots,1)' \ .$$

That is, a unit eigenvector associated with latent root $\lambda=1$ is $\underline{\xi}_1$:

6.3.3 $$\underline{\xi}_1 = (\xi_{1(1)}, \xi_{1(2)}, \ldots, \xi_{1(n)})' = n^{-\frac{1}{2}}(1,1,\ldots,1)'.$$

We define

6.3.4 $$e_i = i - \frac{n+1}{2} \qquad\qquad i = 1,2,\ldots n.$$

Now $\sum_1^n e_i = 0$, and consequently

6.3.5 $$J_n(e_1, e_2, \ldots, e_n)' = (0,0,\ldots,0)' \ .$$

Also, $\sum_1^n e_i^2 = \Omega$, so that

6.3.6 $E(e_1, e_2, \ldots, e_n)' =$

$4(e_1, e_2, \ldots, e_n)'(e_1, e_2, \ldots, e_n)(e_1, e_2, \ldots, e_n)'$

$= 4\Omega(e_1, e_2, \ldots, e_n)'$ .

Thus:

6.3.7 $C\Omega^{-\frac{1}{2}}(e_1, e_2, \ldots, e_n)' =$

$(x - 4\Omega\beta)\Omega^{-\frac{1}{2}}(e_1, e_2, \ldots, e_n)'$ .

That is, a unit eigenvector associated with latent root $\lambda = x - 4\Omega\beta$ is $\underline{\xi}_2$:

6.38 $\underline{\xi}_2 = (\xi_{2(1)}, \xi_{2(2)}, \ldots, \xi_{2(n)})' = \Omega^{-\frac{1}{2}}(e_1, e_2, \ldots, e_n)'$.

Furthermore,

6.3.9 $\underline{\xi}_1'\underline{\xi}_2 = 0$ .

That is, $\underline{\xi}_1$ and $\underline{\xi}_2$ are mutually orthogonal unit eigenvectors of the matrix C associated with latent roots $\lambda=1$ and $\lambda= x - 4\Omega\beta$, respectively.

From the discussion on p. 90, Browne (1958), we are assured that there exist n-2 unit eigenvectors of C, $(\underline{\xi}_3, \ldots, \underline{\xi}_n)$, associated with the latent root x and forming, with $\underline{\xi}_1$ and $\underline{\xi}_2$, a mutually orthogonal set. The matrix P whose columns are the eigenvectors $\underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_n$ is orthogonal and satisfies:

6.3.10 $\quad P'C\,P = \text{diag}\,(1, x - 4\Omega\beta, x, x, \ldots, x)$

Thus we may now write Formula (6.1.16) in the form:

6.3.11 $\quad \Sigma = B^{-1}P\ \text{diag}\,(1, \frac{1}{x-4\Omega\beta}, \frac{1}{x}, \ldots, \frac{1}{x})\ P'(B^{-1})'$ ,

where B is as defined in Equation (6.1.13).

## 6.4 $\Sigma^{(11)}$ Written as an Explicit Function of $\Theta$ and $\phi$

As a final step in simplifying Formula (6.1.20) for $\Phi\ (\Theta,\phi)$ to a form in which $\Theta$ and $\phi$ appear explicitly, we express $\Sigma^{(11)}$ (and thus the volume V) in terms of $\Theta$ and $\phi$.

We recall (Formula (6.3.11)) that

6.4.1 $\quad \Sigma = B^{-1}P\ \text{diag}(1, \frac{1}{x-4\Omega\beta}, \frac{1}{x}, \ldots, \frac{1}{x})\ P'(B^{-1})'$

where

6.4.2 $$B^{-1} = \begin{bmatrix} 1 & & & & \text{zeros} \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ \text{zeros} & & & 1 & \\ & & & -1 & 1 \end{bmatrix} .$$

Using the notation

6.4.3 $$F = (f_{ij}) = P'(B^{-1})'$$

we have on defining

6.4.4 $$f_{io} = 0 \qquad i = 1, 2, \ldots, n$$

that

6.4.5 $$f_{pq} = \xi_{p(q)} - \xi_{p(q-1)} \qquad \begin{matrix} p \\ q \end{matrix} = 1, 2, \ldots, n$$

where $\xi_{p(q)}$ is the $q^{\underline{th}}$ element of the $p^{th}$ eigenvector of C.

Thus

6.4.6 $$\Sigma = F' \operatorname{diag}\left(1, \frac{1}{x-4\Omega\beta}, \frac{1}{x}, \ldots \frac{1}{x}\right) F$$

and

6.4.7 $$(\Sigma)_{pq} = \left(1 - \frac{1}{x}\right) f_{1p}f_{1q} + \frac{4\Omega\beta}{x(x-4\Omega\beta)} \quad f_{2p}f_{2q}$$
$$+ \frac{1}{x} \sum_{i=1}^{n} f_{ip}f_{iq} \quad .$$

Now

6.4.8 $$f_{1q} = \begin{cases} n^{-\frac{1}{2}}, & q = 1 \\ 0 \quad , & q > 1 \end{cases}$$

6.4.9 $$f_{2q} \quad \begin{matrix} -\frac{1}{2}(n-1)\Omega^{-\frac{1}{2}} & , \quad q = 1 \\ \Omega^{-\frac{1}{2}} & , \quad q > 1 \end{matrix}$$

$$6.4.10 \qquad \sum_{i=1}^{n} f_{ip} f_{iq} = \begin{cases} 1, & p = q = 1 \\ 2, & p = q > 1 \\ -1, & |p - q| = 1 \\ 0, & |p - q| > 1 \end{cases} .$$

Thus

$$6.4.11 \qquad f_{1p} f_{1q} = \begin{cases} \frac{1}{n}, & p = q = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$6.4.12 \qquad f_{2p} f_{2q} = \begin{cases} [\frac{1}{2}(n-1)]^2 \Omega^{-1} & p = q = 1 \\ -\frac{1}{2}(n-1)\Omega^{-1} , & p=1,\ q>1 \text{ or } p>1,\ q=1 \\ \Omega^{-1} , & p > 1 \text{ and } q > 1 \end{cases}$$

and

$$6.4.13 \qquad \Sigma = \begin{bmatrix} \frac{1}{n}(1 - \frac{1}{x}) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} +$$

$$\frac{2\phi}{x(x-2\Omega\phi)} \begin{bmatrix} [\frac{1}{2}(n-1)]^2 & -\frac{1}{2}(n-1) & \cdots & -\frac{1}{2}(n-1) \\ -\frac{1}{2}(n-1) & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ -\frac{1}{2}(n-1) & 1 & \cdots & 1 \end{bmatrix} +$$

$$\frac{1}{x} \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix} .$$

The cofactor of $(\Sigma)_{11}$ in the matrix $\Sigma$ is thus:

6.4.14 $$\Sigma^{(11)} = \frac{1}{x} \begin{bmatrix} d & e & \tau & \cdots & \tau & \tau \\ e & d & e & \cdots & \tau & \tau \\ \tau & e & d & \cdots & \tau & \tau \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \tau & \tau & \tau & \cdots & d & e \\ \tau & \tau & \tau & \cdots & e & d \end{bmatrix}$$

where

6.4.15 $$\begin{aligned} d &= \tau + 2 \\ e &= \tau - 1 \\ \tau &= 2\phi[1 - 2\theta - 2\Omega\phi]^{-1} \end{aligned}$$

## 6.5 Differentiation of $\Psi(\theta, \phi)$.

We have at this point found the joint moment generating function of $S^2 = \sum_1^n (x_i - \bar{x})^2$ and $\Omega\eta^2 = [\sum_1^n (i - \frac{n+1}{2}) x_i]^2$ in the form:

6.5.1 $$\phi(\theta, \phi) = \mathcal{E}[e^{\theta s^2 + \phi\Omega\eta^2}]$$

$$= \frac{n!}{(1-2\theta)^{\frac{1}{2}(n-2)}(1-2\theta-2\Omega\phi)^{\frac{1}{2}}} \cdot V$$

where

6.5.2 $$V = \frac{|\Sigma^{(11)}|^{-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(n-1)}} .$$

$$\int_0^\infty \cdots \int \exp[-\tfrac{1}{2}\, \underline{u}'_{(2)} \, (\Sigma^{(11)})^{-1} \, \underline{u}_{(2)}] \, du_2 \ldots du_n ,$$

$\underline{u}'_{(2)} = (u'_2, u_3, \ldots, u_n)$, and $\Sigma^{(11)}$ is given as an explicit function of $\theta$ and $\phi$ in formulae (6.4.14) and (6.4.15).

However, we actually deal in Chapter 3 with moments of quantities $\lambda_1 + \lambda_2 = \frac{1}{2}\delta^2 S^2$ and $\lambda_1 = \frac{1}{2}\delta^2\eta^2$. For this reason, it seems somewhat more appropriate to discuss the differentiation of the joint moment generating function of $\lambda_1 + \lambda_2$ and $\lambda_1$.

Accordingly, we shall differentiate the function

$$\text{6.5.3} \qquad \Psi(\theta,\phi) = \mathcal{E}[e^{\frac{1}{2}\delta^2\theta S^2 + \frac{1}{2}\delta^2\phi\eta^2}]$$

$$= \Phi(\tfrac{1}{2}\delta^2\theta, \frac{1}{\Omega}\cdot\tfrac{1}{2}\delta^2\phi) \quad .$$

Thus

$$\text{6.5.4} \qquad \Psi(\theta,\phi) = \frac{n!}{(1-\delta^2\theta)^{\frac{1}{2}(n-2)}(1-\delta^2\theta-\delta^2\phi)^{\frac{1}{2}}} \cdot V \cdot$$

V is still the form given in Equation (6.5.2), except that now:

$$\text{6.5.5} \qquad \Sigma^{(11)} = \frac{1}{1-\delta^2\theta} \begin{bmatrix} d & e & \tau & \cdots & \tau & \tau \\ e & d & e & \cdots & \tau & \tau \\ \tau & e & d & \cdots & \tau & \tau \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \tau & \tau & \tau & \cdots & d & e \\ \tau & \tau & \tau & \cdots & e & d \end{bmatrix}$$

and

$$\text{6.5.6} \qquad \begin{aligned} d &= \tau + 2 \\ e &= \tau - 1 \\ \tau &= \frac{1}{\Omega}\,\delta^2\phi\,[1-\delta^2\theta - \delta^2\phi]^{-1} \end{aligned} \quad .$$

The discussion is somewhat simplified if we notice that the volume V is unaltered if $\Sigma^{(11)}$ is replaced by any matrix proportional to $\Sigma^{(11)}$.

We choose to replace $\Sigma^{(11)}$ by the (n-1)-square matrix P:

6.5.7 $$P = (1-\delta^2\theta)\ \Sigma^{(11)}$$

$$= \begin{bmatrix} d & e & \tau & \cdots & \tau & \tau \\ e & d & e & \cdots & \tau & \tau \\ \tau & e & d & \cdots & \tau & \tau \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \tau & \tau & \tau & \cdots & d & e \\ \tau & \tau & \tau & \cdots & e & d \end{bmatrix} .$$

Using the formula

6.5.8 $$1 + \Omega\tau = (1 - \delta^2\theta)(1-\delta^2\theta - \delta^2\phi)^{-1}$$

we then have:

6.5.9 $$\Psi(\theta,\phi) = \frac{n!(1+\Omega\tau)^{\frac{1}{2}}}{(1-\delta^2\theta)^{\frac{1}{2}(n-1)}} \quad \frac{|P|^{-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(n-1)}} .$$

$$\int_0^\infty \cdots \int \exp[-\tfrac{1}{2}\ \underline{u}'_{(2)}\ P^{-1}\ \underline{u}_{(2)}]\ du_2 \cdots du_n$$

wherein

6.5.10 $$\underline{u}'_{(2)} = (u_2, u_3, \ldots, u_n) \ .$$

Two lemmas are now given to facilitate a proof of Theorem (6.1).

Lemma 6.1

If $A_m$ is the m-square matrix

6.5.11
$$A_m = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} = P\Big|_{\tau=0}$$

then

6.5.12
$$|A_m| = m + 1 \qquad\qquad m = 1,2,3,\ldots$$

Proof:

It is clear that $|A_1| = 2$. The proof for general m is easily obtained by induction.

Lemma 6.2

6.5.13
$$A_m^{-1} = \frac{1}{m+1}\begin{bmatrix} m & m-1 & m-2 & \cdots & 1.2 & 1 \\ & 2(m-1) & 2(m-2) & \cdots & 2.2 & 2 \\ & & 3(m-2) & \cdots & 3.2 & 3 \\ & & \vdots & & \vdots & \vdots \\ & & & \cdots & (m-2).2 & m-2 \\ (\text{symmetric}) & & & \cdots & (m-1).2 & m-1 \\ & & & \cdots & (m-1).1 & m \end{bmatrix}$$

Proof:

We merely verify that $A_m A_m^{-1} = I_m$ .

Theorem 6.1

6.5.14 $$|P| = n + n\Omega\tau, \quad \tau = \frac{1}{\Omega}\,\delta^2\phi[1 - \delta^2\theta - \delta^2\phi]^{-1} .$$

Let M be the matrix obtained on subtracting the final row of P from each of its other rows. We obtain:

6.5.15 $$|P| = |M| = \left|\begin{array}{ccccc|cc} & & & & & 1 & -2 \\ & & & & & 1 & -2 \\ & & A_{n-3} & & & \vdots & \vdots \\ & & & & & 1 & -2 \\ & & & & & 0 & -2 \\ \hline 0 & 0 & \cdots & 0 & -1 & 3 & -3 \\ \tau & \tau & \cdots & \tau & \tau & \tau-1 & \tau+2 \end{array}\right| .$$

It is now obvious that $|P|$ is linear in $\tau$:

6.5.16 $$|P| = \alpha + \beta\tau \quad .$$

On putting $\tau = 0$ and using Lemma (6.1) with $m = n-1$ we have immediately that $\alpha = n$.

We obtain $\beta$ as $\frac{d}{d\tau}\,|P|$. Following this differentiation, we take advantage of the partitioning indicated in Formula (6.5.15) and obtain $\beta = n\Omega$. This completes the proof of Formula (6.5.16).

We may now write:

6.5.17 $$\Psi(\theta,\phi) = k_1(\theta) \int_{0<u_2,\ldots,u_n<\infty}\!\!\cdots\!\int e^{U}\, du_2 \cdots du_n$$

where

6.5.18 $$U = -\tfrac{1}{2}\, \underline{u}'_{(2)}\, P^{-1}\, \underline{u}_{(2)}$$

and

6.5.19 $$k_1(\theta) = \frac{n!}{\sqrt{n}\,(2\pi)^{\frac{1}{2}(n-1)}}\,(1-\delta^2\theta)^{-\frac{1}{2}(n-1)} \quad .$$

By inspection of the conditional moments of R given in formulae (3.1.32) through (3.1.35) it is clear that the derivatives we require are the following:

6.5.20 $$\left.\frac{\partial^k \Psi}{\partial \theta^k}\right|_{\substack{\theta = -1 \\ \phi = 0}} = \mathcal{E}[(\lambda_1+\lambda_2)^k\, e^{-(\lambda_1+\lambda_2)}]$$

6.5.21 $$\left.\frac{\partial^{k+1}}{\partial\theta^k\,\partial\phi}\right|_{\substack{\theta=-1\\ \phi=0}} = \mathcal{E}[\lambda_1(\lambda_1+\lambda_2)^k\, e^{-(\lambda_1+\lambda_2)}]$$

and

6.5.22 $$\left.\frac{\partial^{k+2}}{\partial\theta^k\,\partial\phi^2}\right|_{\substack{\theta=-1\\ \phi=0}} = \mathcal{E}[\lambda_1^2(\lambda_1+\lambda_2)^k\, e^{-(\lambda_1+\lambda_2)}] \;.$$

In preparation for finding the first of these it is convenient to use the following two lemmas.

Lemma 6.3

The sum of the elements in the $j^{\underline{th}}$ column of $A_m^{-1}$ is $\frac{1}{2}j(m-j+1)$. That is,

6.5.23 $$\sum_{i=1}^{m}(A_m^{-1})_{ij} = \tfrac{1}{2}j(m-j+1) \qquad j = 1, 2, \ldots, m \;.$$

Proof:

Since $A_m^{-1}$ is symmetric,

$$\sum_{i=1}^{m}(A_m^{-1})_{ij} = \frac{1}{m+1}[(m-j+1)\sum_{1}^{j} i + j\sum_{1}^{m-j} i]$$

$$= \tfrac{1}{2}j\,(m-j+1) \quad j = 1, 2, \ldots, m \;.$$

Lemma 6.4

Let

6.5.24 $$\underline{\xi}' = [1(n-1), 2(n-2), \ldots, (n-1).1] \ .$$

Then

6.5.25 $$\left(\frac{\partial}{\partial\tau} P^{-1}\right)\Big|_{\tau=0} = -\frac{1}{4}\,\underline{\xi}\,\underline{\xi}' \ .$$

Proof:

Differentiating both sides of the identity $P\,P^{-1} = I_{n-1}$ with respect to $\tau$, we obtain:

6.5.26 $$\left(\frac{\partial}{\partial\tau} P\right) P^{-1} + P\left(\frac{\partial}{\partial\tau} P^{-1}\right) = (0) \ .$$

Thus,

6.5.27 $$\frac{\partial}{\partial\tau} P^{-1} = -P^{-1}\left(\frac{\partial}{\partial\tau} P\right) P^{-1}$$

$$= -\,P^{-1}\,\underline{j}\,\underline{j}'\,P^{-1}$$

where

6.5.28 $\underline{j}' = (1,1,\ldots,1)$ is an $(n-1)$-component vector.

Now

6.5.29 $$P^{-1}\Big|_{\tau=0} = A_{n-1}^{-1}$$

and hence

6.5.30 $$\left(\frac{\partial}{\partial\tau} P^{-1}\right)\Big|_{\tau=0} = -\,(A_{n-1}^{-1}\,\underline{j})\,(A_{n-1}^{-1}\,\underline{j})' \quad .$$

But

6.5.31 $$A_{n-1}^{-1}\,\underline{j} = \tfrac{1}{2}\underline{\xi} \qquad \text{[Lemma (6.3)]}$$

and hence Formula (6.5.25) is demonstrated.

We are now ready to state

Theorem 6.2

6.5.32 $$\frac{\partial^k \Psi(\theta,\,\phi)}{\partial\theta^k}\Bigg|_{\substack{\theta=-1\\ \phi=0}} = \frac{\rho^{2k}\;k!}{(1+\delta^2)^{\frac{1}{2}(n-1)}}\binom{\frac{n-3}{2}+k}{k}$$

$$k = 0,\,1,\,2,\,\ldots$$

Proof:

Noting that

6.5.33 $$\frac{\partial^k\tau}{\partial\theta^k}\Bigg|_{\substack{\theta=-1\\ \phi=0}} = 0 \qquad k = 0,1,2,\ldots$$

and

6.5.34
$$\left.\frac{\partial^k (1 - \delta^2 \theta)^{-\frac{1}{2}(n-1)}}{\partial \theta^k}\right|_{\substack{\theta = -1 \\ \phi = 0}} =$$

$$\frac{\rho^{2k}\, k!}{(1+\delta^2)^{\frac{1}{2}(n-1)}} \begin{pmatrix} \frac{n-3}{2} + k \\ k \end{pmatrix} \qquad k = 0, 1, 2, \ldots$$

we obtain from Formula (6.5.17) the derivative given in Formula (6.5.32).

To simplify the development of other derivatives we present three lemmas.

Lemma 6.5

Suppose that $\{w_i\}_1^n$ is the set of standard normal order statistics from a random sample of size n.

Let

i) $\underline{u}' = (u_1, u_2, \ldots, u_n)$

ii) $\underline{u}'_{(2)} = (u_2, u_3, \ldots, u_n)$

iii) $\underline{w}' = (w_1, w_2, \ldots, w_n)$

iv) $X = \sum_1^n (i - \frac{n+1}{2})\, w_i$

Then

6.5.35 $$\mathcal{E}[X^\nu] = \frac{n!}{n^{\frac{1}{2}}\ 2^\nu\ (2\pi)^{\frac{1}{2}(n-1)}} \cdot$$

$$\int_0^\infty \cdots \int \Big[\sum_1^{n-1} j(n-j)u_{j+1}\Big]^\nu \cdot$$

$$e^{-\frac{1}{2}\underline{u}'_{(2)} A_{n-1}^{-1} \underline{u}_{(2)}}$$

$$du_2 \cdots du_n \quad \nu = 0,1,2,\ldots$$

where $A_{n-1}$ is the matrix defined in Equation (6.5.11).

Proof:

We transform the integral

6.5.36 $$\mathcal{E}[X^\nu] = \frac{n!}{(2\pi)^{\frac{1}{2}n}} \int_{-\infty<w_1<\cdots<w_n<\infty} \cdots \int X^\nu e^{-\frac{1}{2}\underline{w}'\underline{w}} dw_1 \cdots dw_n$$

by means of:

6.5.37 $$\underline{w} = B\ \underline{u}$$

with

6.5.38 $$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \cdot$$

Thus,

6.5.39 $$\mathcal{E}[X^\nu] = \frac{n!}{(2\pi)^{\frac{1}{2}n}}\,\frac{}{2^\nu} \cdot$$

$$\int_0^\infty \cdots \int \int_{-\infty}^{\infty} \left[\sum_1^{n-1} j(n-j)u_{j+1}\right]^\nu e^{-\frac{1}{2}\underline{u}' G^{-1}\underline{u}}\, du_1\, d\underline{u}_{(2)}$$

where

6.5.40 $$G = \left[\begin{array}{c|cccc} 1 & -1 & 0 & \cdots & 0 \\ \hline -1 & & & & \\ 0 & & & & \\ \vdots & & A_{n-1} & & \\ 0 & & & & \end{array}\right] .$$

Integrating with respect to $u_1$, we obtain:

6.5.41 $$\mathcal{E}[X^\nu] = \frac{n!}{(2\pi)^{\frac{1}{2}(n-1)}2^\nu} \quad \frac{|G|^{\frac{1}{2}}}{|A_{n-1}|^{\frac{1}{2}}} \cdot$$

$$\int_0^\infty \cdots \int \left[\sum_1^{n-1} j(n-j)u_{j+1}\right]^\nu$$

$$e^{-\frac{1}{2}\underline{u}'_{(2)} A_{n-1}^{-1} \underline{u}_{(2)}}\, du_2 \cdots du_n \quad .$$

The conclusion [Formula (6.5.35)] follows on noting that $|G| = 1$ and $|A_{n-1}|^{\frac{1}{2}} = n^{\frac{1}{2}}$ .

Lemma 6.6

Let

i) $$U = -\tfrac{1}{2}\, \underline{u}_{(2)}'\, P^{-1}\, \underline{u}_{(2)}$$

ii) $$\underline{j}' = (1, 1, \ldots, 1) \qquad [(n-1)\text{-components}].$$

Then

6.5.42 $$\left.\frac{\partial U}{\partial \tau}\right|_{\tau=0} = \frac{1}{8} \left[ \sum_{1}^{n-1} j(n-j)u_{j+1} \right]^2$$

and

6.5.43 $$\left.\frac{\partial^2 U}{\partial \tau^2}\right|_{\tau=0} = -\frac{\Omega}{4} \left[ \sum_{1}^{n-1} j(n-j)u_{j+1} \right]^2 \quad .$$

Proof:

Formula (6.5.42) follows immediately from Lemma (6.4).

Now

6.5.44 $$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial}{\partial \tau} \left[\tfrac{1}{2}\, \underline{u}_{(2)}'\, P^{-1}\, \underline{j}\, \underline{j}'\, P^{-1}\, \underline{u}_{(2)}\right]$$

$$= \underline{u}_{(2)}'\, P^{-1}\, \underline{j}(\underline{j}'\, P^{-1} \underline{j})\, \underline{j}'\, P^{-1}\, \underline{u}_{(2)} \quad .$$

But

6.5.45 $$\left.(\underline{j}\, P^{-1}\, \underline{j}')\right|_{\tau=0} = \Omega$$

and

6.5.46 $$\left.\frac{\partial^2 U}{\partial \tau^2}\right|_{\tau=0} = \Omega\,\underline{u}_{(2)}'\left.\left(\frac{\partial}{\partial\tau} P^{-1}\right)\right|_{\tau=0} \underline{u}_{(2)}$$

$$= -\frac{\Omega}{4}\,(\underline{u}_{(2)}'\,\underline{\xi})^2 \quad \text{[Lemma (6.4)]}$$

$$= -\frac{\Omega}{4}\left[\sum_{1}^{n-1} j(n-j)u_{j+1}\right]^2 .$$

Lemma 6.7

Let

i) $$k_1(\theta) = \frac{n!}{\sqrt{n}\,(2\pi)^{\frac{1}{2}(n-1)}}\,(1-\delta^2\theta)^{-\frac{1}{2}(n-1)}$$

ii) $$k_2(\theta) = \frac{n!\,\delta^2}{\sqrt{n}\,\Omega\,(2\pi)^{\frac{1}{2}(n-1)}}\,(1-\delta^2\theta)^{-\frac{1}{2}(n+1)}$$

iii) $$k_3(\theta) = \frac{n!\,\delta^4}{\sqrt{n}\,\Omega^2\,(2\pi)^{\frac{1}{2}(n-1)}}\,(1-\delta^2\theta)^{-\frac{1}{2}(n+3)} .$$

Then

6.5.47 $$\frac{\partial\Psi}{\partial\phi} = k_2(\theta)f_1(\tau)$$

6.5.48 $$\frac{\partial^2\Psi}{\partial\phi^2} = k_3(\theta)f_2(\tau)$$

where

$$6.5.49 \qquad f_1(\tau) = (1+\Omega\tau)^2 \int_0^\infty \cdots \int e^U \frac{\partial U}{\partial \tau} du_2 \cdots du_n$$

$$6.5.50 \qquad f_2(\tau) = 2\Omega(1+\Omega\tau)^3 \int_0^\infty \cdots \int e^U \frac{\partial U}{\partial \tau} du_2 \cdots du_n$$

$$+ (1+\Omega\tau)^4 \int_0^\infty \cdots \int e^U [(\frac{\partial U}{\partial \tau})^2 + \frac{\partial^2 U}{\partial \tau^2}] du_2 \cdots du_n \quad .$$

Making use of the derivative

$$6.5.51 \qquad \frac{\partial \tau}{\partial \phi} = \frac{\delta^2}{\Omega} \frac{(1+\Omega\tau)^2}{1 - \delta^2\theta}$$

and the chain rule

$$6.5.52 \qquad \frac{\partial \Psi(\theta,\phi)}{\partial \phi} = \frac{\partial \Psi(\theta,\phi)}{\partial \tau} \cdot \frac{\partial \tau}{\partial \phi}$$

we differentiate Formula (6.5.17) and obtain:

$$6.5.53 \qquad \frac{\partial \Psi(\theta,\phi)}{\partial \phi} = k_2(\theta)(1+\Omega\tau)^2 \int_0^\infty \cdots \int e^U \frac{\partial U}{\partial \tau} du_2 \cdots du_n$$

$$= k_2(\theta)\ f_1(\tau) \quad .$$

Likewise,

$$6.5.54 \qquad \frac{\partial^2 \Psi(\theta,\phi)}{\partial \phi^2} = \frac{\partial}{\partial \tau} [\frac{\partial \Psi(\theta,\phi)}{\partial \phi}] \cdot \frac{\partial \tau}{\partial \phi}$$

$$= k_3(\theta)\ f_2(\tau) \quad .$$

Theorem 6.2

If $\Psi(\theta,\phi)$ is the moment generating function given in Formula (6.5.17), then:

6.5.55 $$\left.\frac{\partial^{k+1}\Psi(\theta,\phi)}{\partial\theta^k\,\partial\phi}\right|_{\substack{\theta=-1\\ \phi=0}} = \frac{\delta^2}{2\Omega}\,\frac{\rho^{2k}k!}{(1+\delta^2)^{\frac{1}{2}(n+1)}}\binom{\frac{n-1}{2}+k}{k}\mathcal{E}[X^2]$$

and

6.5.56 $$\left.\frac{\partial^{k+2}\Psi(\theta,\phi)}{\partial\theta^k\,\partial\phi^2}\right|_{\substack{\theta=-1\\ \phi=0}} = \frac{\delta^4}{4\Omega^2}\,\frac{\rho^{2k}k!}{(1+\delta^2)^{\frac{1}{2}(n+3)}}\binom{\frac{n+1}{2}+k}{k}\mathcal{E}[X^4]$$

$k = 0, 1, 2, \ldots$

Proof:

Noting that

6.5.57 $$\left.\frac{\partial^k\tau}{\partial 0^k}\right|_{\substack{\theta=-1\\ \phi=0}} = 0 \qquad k = 0, 1, 2, \ldots$$

and that

6.5.58 $$\left.\frac{\partial^k \, k_2(\theta)}{\partial\theta^k}\right|_{\theta=-1} = \frac{n! \; \delta^2\rho^{2k} \; k!}{\sqrt{n}\,\Omega\,(2\pi)^{\frac{1}{2}(n-1)}(1+\delta^2)^{\frac{1}{2}(n+1)}} \binom{\frac{n-1}{2}+k}{k}$$

$k = 0,1,2,\ldots$

we obtain

6.5.59 $$\left.\frac{\partial^{k+1}\Psi(\theta,\phi)}{\partial\theta^k \; \partial\phi}\right|_{\substack{\theta=-1\\ \phi=0}} = \frac{n! \; \delta^2\rho^{2k} \; k! \; f_1(0)}{\sqrt{n}\,\Omega\,(2\pi)^{\frac{1}{2}(n-1)}(1+\delta^2)^{\frac{1}{2}(n+1)}} \binom{\frac{n-1}{2}+k}{k}$$

$k = 0,1,2,\ldots$

But, by Lemma (6.6),

6.5.60 $$f_1(0) = \frac{1}{8}\int_0^{\infty}\!\!\cdots\!\int \Big[\sum_{1}^{n-1} j(n-j)u_{j+1}\Big]^2 \, e^{-\frac{1}{2}\underline{u}'(2)\, A_{n-1}^{-1}\underline{u}(2)}\, du_2\ldots du_n \quad .$$

and thus, by (Lemma 6.5),

6.5.61 $$f_1(0) = \frac{n^{\frac{1}{2}}\,(2\pi)^{\frac{1}{2}(n-1)}}{2n!}\;\mathcal{E}[X^2] \quad .$$

Replacing $f_1(0)$ in Formula (6.5.59) by the expression in Formula (6.5.61), we obtain Formula (6.5.55).

Similarly, we note that

$$6.5.62 \qquad \left.\frac{\partial^k k_3(\theta)}{\partial\theta^k}\right|_{\theta=-1} = \frac{n!\,\delta^4\rho^{2k}\,k!}{\sqrt{n}\,\Omega^2\,(2\pi)^{\frac{1}{2}(n-1)}(1+\delta^2)^{\frac{1}{2}(n+3)}}\binom{\frac{n+1}{2}+k}{k}$$

$$k = 0, 1, 2, \ldots$$

and obtain

$$6.5.63 \qquad \left.\frac{\partial^{k+2}\Psi(\theta,\phi)}{\partial\theta^k\,\partial\phi}\right|_{\substack{\theta=-1\\ \phi=0}} = \frac{n!\,\delta^4\rho^{2k}k!\,f_2(0)}{\sqrt{n}\,\Omega^2\,(2\pi)^{\frac{1}{2}(n-1)}(1+\delta^2)^{\frac{1}{2}(n+3)}}\binom{\frac{n+1}{2}+k}{k}$$

$$k = 0, 1, 2, \ldots$$

But $f_2(0)$ is, by use of Lemmas (6.5) and (6.6),

$$6.5.64 \qquad f_2(0) = \frac{n^{\frac{1}{2}}\,(2\pi)^{\frac{1}{2}(n-1)}}{4\,n!}\,\mathcal{E}[X^4] \qquad .$$

Replacing $f_2(0)$ in Formula (6.5.63) we obtain Formula (6.5.56).

We remark that formulae (6.5.34), (6.5.55), and (6.5.56) could be used to pass immediately from Formula (3.1.33) and Formula (3.1.35) to Formula (3.2.23) and Formula (3.2.25) respectively.

## SUGGESTIONS FOR FURTHER RESEARCH

A generalization of the model in this thesis is obtained by relaxing the requirement that individuals in the calibration sample be strictly ranked on the criterion of interest. Thus, it may be that these individuals can be assigned to groups so that individuals in distinct groups are clearly different, but individuals in the same group are indistinguishable on the criterion of interest. This generalization is the subject of the Ph.D. thesis of Mr. Roger Flora now being directed at Virginia Polytechnic Institute by Dr. J. G. Saw.

If the individuals in the calibration sample are imperfectly ranked there will be some disturbance in the power of tests based on the quasi-rank multiple correlation coefficient and in the ranking of subsequently chosen individuals. No work has as yet been done to assess the magnitude of this disturbance for various errors in ranking of the calibration sample. It is clear that if there is serious difficulty in obtaining a complete ranking of the calibration sample, the generalization of Mr. Flora would be useful.

It would be of considerable interest to have probabilities of errors in ranking subsequently selected individuals on the basis of the discriminant function. Thus, we should like to know the probability that no predicted rank differs by more than k from the true rank, or the probability that the rank correlation of predicted ranks with true ranks is less than $\gamma$, etc.

Again, some work might be done in assessing the practical value of the method outlined in this thesis. Thus, it could well be, at least for large sample sizes, that one would lose very little in power by replacing ranks by normal scores and using a standard regression analysis of the data.

Finally, the statistic $\eta/S$ introduced in Chapter III promises to be useful as a measure of normality sensitive to skewness. Much of the groundwork having been laid in this thesis, it is the author's intention to investigate this problem at a later date.

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## ABSTRACT

Having available a vector of measurements for each individual in a random sample from a multivariate population, we assume in addition that these individuals can be ranked on some criterion of interest. As an example of this situation, we may have measured certain physiological characteristics (blood pressure, amounts of certain chemical substances in the blood, etc.) in a random sample of schizophrenics. After a series of treatments (perhaps shock treatments, doses of a tranquillizer, etc.) these individuals might be ranked on the basis of favorable response to treatment. We shall in general be interested in predicting which individuals in a new group would respond most favorably. Thus, in the example, we should wish to know which individuals would most likely benefit from the series of treatments.

Some difficulties in applying the classical discriminant function analysis to problems of this type are noted.

We have chosen to use the multiple correlation coefficient of ranks with measured variates as a statistic in testing whether ranks are associated with measurements. We give to this coefficient the name "quasi-rank multiple correlation coefficient", and proceed to find its first four exact moments under the assumption that the underlying probability distribution is multivariate normal.

Two methods are used to approximate the power of tests based on the quasi-rank multiple correlation coefficient in the case of just one measured variate. The agreement for a sample size of twenty is quite good.

The asymptotic relative efficiency of the squared quasi-rank coefficient vis-à-vis the squared standard multiple correlation coefficient is $9/\pi^2$, a result which does not depend on the number of measured variates.

If the null hypothesis that ranks are not associated with measurements is rejected, it is appropriate to use the measurements in some way to predict the ranks. The quasi-rank multiple correlation coefficient is, however, the maximized simple correlation of ranks with linear combinations of the measured variates. The maximizing linear combination of measured variates is taken as a discriminant function, and its values for subsequently chosen individuals is used to rank these individuals in order of merit.

A demonstration study is included in which we employ a random sample of size twenty from a six-variate normal distribution of known structure (for which the population multiple correlation coefficient is .655). The null hypothesis of no association of ranks with measurements is rejected in a two-sided size .05 test. The discriminant function is obtained and is used to "predict" the true ranks of the twenty individuals in the sample. The predicted ranks represent the true ranks rather well, with no predicted rank more than four places from the true rank. For other populations in which the population multiple correlation coefficient is greater than .655 we should expect to obtain even better sets of predicted ranks.

In developing the moments of the quasi-rank multiple correlation coefficient it was necessary to obtain exact moments of a certain linear combination of quasi-ranges in a random sample from a normal population. Since this quasi-range statistic may be useful in other investigations, we include also its moment generating function and some derivatives of this moment generating function.