APPLICATIONS OF A GENERALIZED INTEGRAL TRANSFORM TO VIBRATIONS OF CONTINUOUS MEDIA

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LIST OF SYMBOLS

A, A_o	Cross-sectional area
$A_r, B_r, C_s, D_s, E_i, F_i$	Coefficients in series expansions
A_n, a_n	Coefficients in linear differential operator
a, b	Plate dimensions
B_i	Linear differential operators
D	Flexural rigidity for isotropic plate
D_x, D_{xy}, D_y, D_1	Flexural rigidities for orthotropic plate
E	Modulus of elasticity
F_n, F_{ij}	Generalized forces
f	Forcing function, arbitrary function
G_{jk}	Coefficients in series expansion
H	$D_1 + 4D_{xy}$
h	Plate thickness
I	Moment of inertia
I_0, I_1	Modified Bessel functions
J_0, J_1	Bessel functions
$[k]$	Stiffness matrix
L	Linear differential operator
ℓ	Length of rod or beam
$M(P)$	Mass distribution
M_0	Uniform mass distribution
M, M_i	Concentrated mass
$M(t)$	Time-dependent bending moment
M^n	Bending moment normal to boundary

M^{ns}	Twisting moment
$[m]$	Mass matrix
m	Mass per-unit-length
m_r	Eigenvalues
N	Generalized force
n	Normal coordinate
$P, P(t)$	General spatial point, axial force on rod
p	Natural frequency, order of differential operator
Q	Shear force
$q(P,t), q$	Forcing function, order of differential operator
R	Region of integration, radius of curvature, plate corner reaction, and mass ratio
r, θ	Polar coordinates
S	Boundary of region R
s	Tangential coordinate
T	Transpose of matrix
t	Time
$\{u\}$	Eigenvector
$[u]$	Matrix of eigenvectors
$u(t)$	Longitudinal displacement of rod
V, V^n	Kirchhoff shear
$W(P,t)$	Transverse deflection
x, y	Rectangular coordinates
α, β	Integers
α_n	Coefficient in beam eigenfunction
α_{ij}, β_{ij}	Coefficients in Levy eigenfunction
β_n	Eigenvalues for beam

Γ_{ij}, γ_{ij}	Coefficients in plate eigenfunctions
δ	Dirac delta function
δ_{ij}	Kronecker delta
$\eta(t)$	Generalized time function
λ_i, λ_{ij}	Eigenvalue
ν	Poisson's ratio
ρ, ρ_0	Density per-unit-area
τ	Dummy time variable of integration
ϕ_i, Φ_i	Eigenfunctions
ψ_s	Eigenfunction
ω	Natural frequency

I. INTRODUCTION

In the development of the theory of vibrations of continuous elastic bodies a great deal of effort has been made to formulate and solve boundary value problems and obtain exact mathematical solutions. The most powerful method of obtaining solutions has been by the method of modal analysis. The technique of modal analysis consists of solving the related free vibration problem to obtain infinite sets of eigenfunctions and associated eigenvalues. The dynamic response is then obtained in the form of an infinite series as the product of the eigenfunctions and a corresponding set of generalized time functions. A large class of problems has been solved in a straightforward manner using this approach. In some cases, however, although the boundary value problem can be rigorously formulated, mathematical difficulty is encountered in solving the differential equations governing the eigenfunctions. In other cases, although the differential equation can be solved to yield the eigenfunctions, difficulty is encountered in solving for the generalized time functions required for the dynamic response.

The class of problems for which closed form solutions exist for the eigenfunctions is usually characterized by relatively simple distributions of stiffness and mass. For the most part closed form solutions exist for the eigenfunctions only for cases with uniform distributions because they lead to differential equations with constant coefficients. Some mathematical solutions for the eigenfunctions also exist for special non-uniform distributions leading to differential equations with variable coefficients that possess solutions expressible in terms of power series.

The difficulty encountered in obtaining exact solutions to problems with non-uniform mass and stiffness distributions has led to the development of a variety of schemes for finding approximate solutions for the eigenvalue problem. Generally these methods are characterized by replacing the continuous system by a finite-degree-of-freedom system. By various techniques the eigenvalue problem for the continuous system can be converted into a matrix eigenvalue problem analogous to the eigenvalue problem for a discrete system.

The dynamic response of a continuous elastic system may be initiated or sustained by three classes of excitations: (1) initial distributions of displacements and velocities throughout the system, (2) forcing functions applied to the system, or (3) motions introduced by the time dependence of the supports of the system. With the use of modal analysis, the determination of the generalized time functions associated with the first two types of excitations is relatively straightforward. For many problems with time-dependent boundary conditions, modal analysis can also be used to obtain the generalized time functions provided a transformation of the dependent variable is made. This transformation converts the problem with non-homogeneous boundary conditions into another problem with homogeneous boundary conditions.

Another way of solving problems with time-dependent boundary conditions is to use an integral transform, e.g., the finite sine or cosine, which will remove the dependence upon spatial variables by integration and yields the integral transform of the generalized time function. The inversion of the transformed problem then leads to the desired solution of the original problem. In the past, the method of integral transforms has been limited to only a few selected types of transforms chosen to satisfy simple boundary conditions; in each case the kernel of these transforms has coincided with the eigenfunctions of the system.

In general if the eigenfunctions for a particular eigenvalue problem are selected as the kernel, it is possible to introduce the concept of a generalized integral transform and to state its corresponding inversion series. The fundamental objective of this dissertation is to demonstrate that the concept of a generalized integral transform gives a unified and systematic approach to solving three general classes of problems in the vibrations of continuous media.

A method for determining the eigenfunctions and eigenvalues and dynamic response for a continuous media with non-uniform mass and stiffness distributions will be developed. The integral transform of a general partial differential equation will be performed for a self-adjoint eigenvalue problem. The general theory will be established, and then applications will be given for a number of selected boundary value problems. For convenience, the general case with both non-uniform stiffness and mass distributions will be treated first, and then the special case with only a non-uniform mass distribution will be considered.

Finally the dynamic response of continuous media with time-dependent boundary conditions will be considered. The integral transform for the governing partial differential equations will be performed to obtain the general time functions. One dimensional continuous media will be treated as well as isotropic and orthotropic flat plates with time-dependent boundary conditions.

II. LITERATURE REVIEW

This chapter will present a chronological account of the significant previous investigations which have been made in this subject. For convenience, the review of literature is subdivided into four sections corresponding to the main chapters of this dissertation.

Finite Integral Transforms

In the past the method of integral transforms has been applied with advantage to the solution of a number of boundary value and initial value problems in mathematical physics. In a problem for which one of the independent variables is, say x , the use of an integral transform of the type

$$\bar{f}(\xi) = \int_{\alpha}^{\beta} f(x) K(\xi, x)\, dx$$

will reduce a partial differential equation in n independent variables to one in $n-1$ independent variables. The quantity $\bar{f}(\xi)$ is the integral transform of $f(x)$ corresponding to the kernel $K(\xi, x)$. In several instances repeated application of the integral transform will reduce the partial differential equation to an ordinary differential equation which then can be solved by elementary methods for the transformed variable. For certain kernels, inversion theorems are available which may be used to obtain the desired solution.

For an interval $\alpha = 0$ to $\beta = \infty$, the kernel $K(\xi, x) = e^{-\xi x}$ defines the well-known Laplace transform. In this case the inversion is performed by means of a contour integral in the complex plane. Numerous engineering applications of this transform have been made for both ordinary and partial differential equations [1].* Frequent use of

*Numbers in brackets [ ] denote references in the bibliography.

the Laplace transform has been made in the study of transient phenomena and wave propagation problems. Experience has shown that difficulties are frequently encountered in evaluating the inversion integral. Quite often these difficulties lead to a numerical evaluation of definite integrals in the final solution.

Vibrations of semi-infinite and infinite bodies have been studied by means of the Fourier transform. In these problems the kernel is usually a sine or cosine function. Solutions to a number of vibrations problems for strings, beams, and plates were given by Sneddon [2]. Important applications of the Fourier transform have also been made in other areas such as heat conduction. For the Fourier transform the inversion is performed by an integration in the real domain.

Numerous other transforms have been employed for an infinite range of the independent variable. The kernel chosen depends upon the partial differential equation and the coordinate system used. For problems in cylindrical coordinates, for example, frequent use has been made of the Hankel transform whose kernel involves a Bessel function.

In cases where the range of χ is finite, integral transforms have been utilized to a lesser degree. This fact is somewhat surprising since they possess the advantage of performing the inversion by means of an infinite series. The finite transforms involving the trigonometric functions were first suggested, according to Scott [1], by Doetsch [3] in 1935. The extension to finite transforms involving Bessel functions was attributed to Sneddon [4] in 1946. In his text [2], Sneddon applied the finite Fourier sine transform to vibrations of a beam with simple supports and a rectangular membrane and the finite Hankel transform to the symmetrical vibrations of a circular plate. Later, the Legendre transform was introduced by Tranter [5] and the Jacobi transform by Scott [6]. In 1954 Eringen [7] considered in a general manner the finite transform associated with the second order Sturm-Liouville system. With mathematical rigor he established the

Sturm-Liouville transform for a general kernel and obtained a solution for a heat conduction problem for a region bounded by two conical surfaces. A generalization of finite transforms for solving boundary value problems in rectangular, cylindrical, and spherical coordinates for the wave equation and the heat conduction equation was presented by Kaplan and Sonnemann [8]. In addition a brief general treatment of generalized integral transforms for second order ordinary differential equations was given by Churchill in his text [9].

The application of more general finite integral transforms to the partial differential equations governing the vibrations of elastic solids apparently did not receive attention until the first part of this decade. In 1960, Solecki [10] considered a generalized finite transform to investigate the harmonic vibrations of a triangular plate on an elastic foundation subjected to in-plane forces and edge moments. As the kernel in the integral transform, he used the eigenfunctions for the corresponding boundary value problem with homogeneous boundary conditions. In a later paper [11] he investigated the harmonic vibrations of an orthotropic plate on an elastic foundation supporting an arbitrary mass distribution. Using the same technique, he obtained a general expression for the transformed variable for non-homogeneous boundary conditions. In this pair of papers, Solecki called the generalized transform the eigentransform and the method of obtaining solutions the eigentransform method.

In 1966, Cinelli [12] applied the finite transform technique to the vibration of the Euler beam. He considered in detail the transient response of a viscously damped cantilever beam subjected to external loading and time-dependent boundary conditions. In an appendix he listed the appropriate transforms and eigenfunctions for several combinations of beam boundary conditions. Later in 1966, Pilkey [13], in a discussion to a paper, extended Cinelli's work to the vibration of a viscoelastic Timoshenko beam with time-dependent boundary conditions. In 1967 [14] Pilkey treated the case of an elastic

Timoshenko beam and showed how to obtain the general solution for time-dependent boundary conditions. In other recent papers, Sharp [15, 16] investigated the symmetrical and nonsymmetrical vibration problem for the annular membrane, and Cobble and Fang [17] examined the damped vibrations of an elastically supported cantilever beam.

Vibration of Non-Uniform Continuous Media

The formulation of the eigenvalue problem associated with the non-uniform continuum is considered in detail in the Appendix. However, since the problem concerns a differential equation with variable coefficients, only a few problems exist for which exact solutions have been found. Most of the exact solutions result from problems having cross-sectional variation such that the solution leads to a differential equation for which a series solution has been tabulated, e.g., Bessel's equation. Kirchhoff [18] in 1879 treated the vibration of wedge-shaped and cone-shaped beams in this way. Considerable work on the vibration of variable cross-sectional bars has been done since then. A long list of references on these problems was presented in a recent paper by Wang [19].

The more difficult problem of plates of variable thickness has received a more limited treatment. Bounds for the first eigenvalue of a rectangular plate were given by Appl and Byers in [20]. Circular plate problems with linearly varying thickness were solved by Conway, Becker and Dubil [21]; in this case the differential equation was a form of Bessel's equation and exact solutions were obtained.

Vibration of Continuous Media with Non-Uniform Mass Distribution

The vibration of one dimensional structural elements including supported masses has been treated extensively in the literature by a variety of methods. The classical technique is to isolate a concentrated mass as a free body and write its equations of motion. These equations, together with pertinent continuity conditions at the mass location, are sufficient to formulate the eigenvalue problem. This approach has been illustrated in several books, including Prescott's [22]. A number of papers in the literature have demonstrated this technique, especially for the flexural vibrations of beams, [23, 24, 25, 26, 27, 28, 29]. In recent years, integral transform techniques have also been used to analyze the one dimensional problem. Symbolic functions, such as the Dirac function, have been used to represent concentrated masses as part of the structural element's mass distribution. With Laplace transforms the string [30] and Euler beam [31, 32, 33] have been effectively treated in this manner. Amba-Rao [34, 35] has also demonstrated that the finite Fourier sine transform can be employed to solve the Euler beam with concentrated masses.

In the solutions for the vibrations of plates with concentrated masses, similar techniques have been applied. In an article by Das and Navaratna [36] the harmonic vibrations of a rectangular plate with a concentrated mass, spring and dashpot were examined by isolating the single-degree-of-freedom system. This system and the plate were subjected to a harmonic forcing function and the resulting displacements were matched. Stokey and Zorowski [37] treated the rectangular plate by an energy approach using Lagrange's equations. Wah [38] investigated vibrations of rectangular plates with Levy supports by expanding Dirac representations of the concentrated masses into infinite series of the eigenfunctions for the uniform plate. Amba-Rao [39] analyzed vibrations of the simply supported rectangular plate with a concentrated mass using a finite double Fourier sine transform. Solecki [11] considered the orthotropic plate with concentrated masses by the eigentransform. In each of the last four papers, the frequency equation derived was an infinite series which required a laborious trial and error solution. Because of the amount of work involved, only limited numerical results were obtained. Circular plates bearing a concentrated mass have been treated in two papers by Roberson [40,41].

The vibration of shells with attached masses has become of interest in applications to submarines and space vehicles. In a company report [42] Chen considered the vibrations of a cylindrical panel in a method similar to Wah [38]. Geers and others [43, 44, 45] have investigated circular cylinders with concentrated masses by the energy approach. Lee [46] made an analysis of the vibration of a shallow spherical shell with a concentrated mass by expanding the mass in terms of the eigenfunctions.

Vibration of Continuous Media with Time-Dependent Boundary Conditions

The earliest investigation of the vibrations of structures with time-dependent boundary conditions was conducted by Nothmann [47]. This investigation treated beams with time-dependent boundary conditions using the Laplace transform. However, to avoid the difficulty encountered by Nothmann in performing the inversion of the Laplace transform, a new approach was published by Mindlin and Goodman [48]. The Mindlin-Goodman technique consists of separating the solution into two parts of the form

$$W(x,t) = S(x,t) + \sum_{i=1}^{4} f_i(t) g_i(x)$$

where the second part is a product of unknown spatial functions $g_i(x)$ and the specified boundary functions $f_i(t)$. The $g_i(x)$ functions are chosen so that the boundary conditions on $S(x,t)$ become homogeneous.

The method of Mindlin and Goodman was extended to the general elastic body by Berry and Naghdi [49]. Ojalvo [50] prepared a similar paper for a general boundary value problem.

In his dissertation Falgout [51] applied the Mindlin-Goodman technique to orthotropic flat plates with time-dependent boundary conditions. In his solution, the boundary conditions were restricted to a uniform variation along each plate edge. Six cases of

Levy type supports were considered and the eigenfunctions and g_i functions were tabulated. For four of the six cases, the g_i functions corresponding to certain types of time-dependent boundary conditions were not given. In his book Meirovitch [52] also applied the method of Mindlin-Goodman for the case of general one-dimensional structural elements.

The finite integral transform has been applied in some special cases of problems with time-dependent boundary conditions. As mentioned earlier, Cinelli [12] has considered the Euler beam, Pilkey [13, 14] the Timoshenko beam; and Sharp [15, 16] the annular membrane.

III. THE GENERALIZED INTEGRAL TRANSFORM IN VIBRATIONS OF CONTINUOUS MEDIA

In this chapter the definition and some fundamental properties of the eigentransform will be presented. Then the specific classes of vibration problems that will be considered in Chapters IV, V, and VI will be described.

The Eigentransform

In this dissertation a one dimensional continuous media is specified by a single space variable, say X ; a two dimensional media will require two spatial coordinates to specify a general point. When either a one or two dimensional region R is intended the space variable(s) will be denoted by P .

For a given boundary value problem where there exist eigenfunctions $\Phi_r(P)$ which are orthogonal with respect to a weighting function $r(P)$, the eigentransform of a function $f(P)$ is given by the definition, [11],

$$\bar{f}_r = T\left\{f(P)\right\} = \int_R r(P) f(P)\, \Phi_r(P)\, dR. \tag{3.01}$$

The inverse transform is determined by means of the inversion series

$$f(P) = T^{-1}\left\{\bar{f}_r\right\} = \sum_r \frac{\bar{f}_r(P)}{\| \Phi_r \|} \Phi_r(P) \tag{3.02}$$

where the norm of the eigenfunction is

$$\| \Phi_r \| = \int_R r(P)\, \Phi_r^2(P)\, dR. \tag{3.03}$$

For convenience in the remainder of this dissertation the eigenfunctions will be normalized so that

$$\| \Phi_r \| = 1, \tag{3.04}$$

and (3.02) will be correspondingly simplified.

One other result due to Solecki [11] will be introduced at this point. In the derivations that follow in Chapter IV, it will be necessary to calculate the eigentransform of the product of two functions, say $f(P)\,a(P)$. Starting from the definition (3.01), it follows that

$$T\left\{f(P)a(P)\right\} = \int_R r(P)\,f(P)\,a(P)\,\Phi_r(P)\,dR. \tag{3.05}$$

Since the eigenfunctions $\Phi_s(P)$ form a complete set (See Appendix A) then the function $a(P)\,\Phi_r(P)$ can be expanded in a series of the eigenfunctions to yield

$$a(P)\,\Phi_r(P) = \sum_s b_{rs}\,\Phi_s(P)\,. \tag{3.06}$$

Next (3.06) is multiplied by $r(P)\,\Phi_r(P)$ and integrated over the region R . Since the eigenfunctions are orthogonal and are normalized according to (3.04) then

$$\int_R r(P)\,\Phi_r(P)\,\Phi_s(P)\,dR = \delta_{rs}\,, \tag{3.07}$$

where δ_{rs} is the Kronecker delta. Thus the coefficients b_{rs} are found to be given by

$$b_{rs} = \int_R r(P)\,a(P)\,\Phi_r(P)\,\Phi_s(P)\,dR\,. \tag{3.08}$$

Then substitution of (3.06) in (3.05) results in

$$T\left\{f(P)a(P)\right\} = \sum_s b_{rs}\int_R r(P)\,f(P)\,\Phi_s(P)\,dR$$

and with the definition (3.01) the transform of the product of two functions reduces to

$$T\left\{f(P)a(P)\right\} = \sum_s b_{rs}\,\bar{f}_s(P). \tag{3.09}$$

Applications to the Vibrations of Continuous Media

The eigentransform will be applied to the class of vibrations problems governed by partial differential equations of the type

$$L\left[W(P,t)\right] + M(P)\,\frac{\partial^2 W(P,t)}{\partial t^2} = F(P,t) \tag{3.10}$$

where L is a linear homogeneous differential operator of the type

$$L = A_0(P) + A_1(P)\frac{\partial}{\partial x} + A_2(P)\frac{\partial}{\partial y} + A_3(P)\frac{\partial^2}{\partial x \partial y} + \cdots$$

and $A_0(P), \ldots$ are known functions. Here $M(P)$ is associated with the mass distribution of the continuum. The external forcing function $F(P,t)$ is assumed expressible as a product of a function of the space variables and a function of time, t The boundary conditions, in general, may either be homogeneous or specified as a function of time and position on the boundary. The equation of motion (3.10) assumes the motion to be undamped.

The approach to be followed consists of transforming the partial differential equation (3.10) into a set of ordinary differential equations in the transformed dependent variable $\overline{W}_r(t)$. These equations are then solved and the result is substituted into an inversion series of the form of (3.02). The success of the method depends upon performing the integral transform of each term in (3.10). The technique followed will make use of the definition of a self-adjoint eigenvalue problem and the transformation formula for the product of two functions derived above. Three classes of problems are considered:

(1) Vibration of non-uniform continuous media: The cross section of the continuous media is a function of position and a non-homogeneous material is permitted. Thus $A_0(P), \ldots$ and $M(P)$ are variables. The

boundary conditions will be homogeneous.

(2) Vibration of continuous media with an arbitrary mass distribution: The cross section is uniform but a variation in $M(P)$ is considered. The boundary conditions will be homogeneous.

(3) Vibrations with time-dependent boundary conditions: The cross section is uniform and the mass distribution is also uniform. The medium is excited by time-dependent boundary motions.

These problems may be treated together in a general manner; however, for clarity they will be treated separately in Chapters IV, V, and VI.

IV. VIBRATION OF NON-UNIFORM CONTINUOUS MEDIA

In this chapter the vibrations of non-uniform continuous media will be considered using the eigentransform to establish the basic equations. The theory will be developed first and then as an application, the eigenvalues and eigenfunctions will be obtained for a particular problem.

General Development of Theory

Application of the Eigentransform

A number of problems of vibrations of continuous media with variable cross sections are governed by a partial differential equation of the type

$$L\left[w(P,t)\right] + M(P)\frac{\partial^2 w(P,t)}{\partial t^2} = q(P,t). \tag{4.01}$$

Here L is a linear differential operator with variable coefficients consisting of derivatives of the order through $2p$, where p is an integer. The boundary conditions are homogeneous and of the form

$$B_i\left[w(P,t)\right] = 0 \qquad i = 1, 2, \ldots p \tag{4.02}$$

where B_i are linear differential operators of order $2p-1$

The associated problem of a uniform continuous media is governed by a similar partial differential equation but with constant coefficients and a similar set of boundary conditions. The solution of the associated problem consists of a set of eigenvalues λ_r and their corresponding eigenfunctions $\phi_r(P)$. The eigenfunctions are orthogonal with respect to a constant weighting function and can be normalized so that

$$\int_R \phi_r(P)\,\phi_s(P)\,dR = \delta_{rs}, \tag{4.03}$$

where the weighting function is taken as one for convenience and R is the region under consideration.

The transformed differential equation is obtained by taking the eigentransform of (4.01) with $\phi_r(P)$ as the kernel:

$$\int_R L[w(P,t)]\phi_r(P)dR + \frac{\partial^2}{\partial t^2}\int_R w(P,t)M(P)\phi_r(P)dR = \bar{q}_r(t)\,. \tag{4.04}$$

To evaluate the transform of L it is assumed that the partial differential equation (4.01) and boundary conditions (4.02) together pose a self-adjoint eigenvalue problem (see Appendix). Then, since $\phi_r(P)$ of the associated system satisfies the boundary conditions on $w(P,t)$, equation (A5) of the appendix gives by the definition of self-adjointness,

$$\int_R L[w(P,t)]\phi_r(P)dR = \int_R w(P,t)L[\phi_r(P)]dR\,. \tag{4.05}$$

Now $L[\phi_r(P)]$ is expanded into a series of the eigenfunctions, ϕ_s. From the orthogonality condition there results,

$$L[\phi_r(P)] = \sum_{s=1}^{\infty} k_{rs}\,\phi_s(P) \tag{4.06}$$

where

$$k_{rs} = \int_R L[\phi_r(P)]\,\phi_s(P)\,dR. \tag{4.07}$$

By equation (A11) it can be seen that the coefficients k_{rs} are symmetric. Substitution of (4.06) into (4.05) gives the desired transformation of the first term in (4.04):

$$\int_R L[w(P,t)]\phi_r(P)dR = \sum_{s=1}^{\infty} k_{rs}\int_R w(P,t)\phi_s(P)dR = \sum_{s=1}^{\infty} k_{rs}\,\overline{w}_s(t)\,. \tag{4.08}$$

To transform the remaining term in (4.04) use is made of the transformation for the product of two functions, equation (3.09). This transformation gives

$$\int_R w(P,t)M(P)\phi_r(P)\,dR = \sum_{s=1}^{\infty} m_{rs}\overline{w}_s(t)$$

where

$$m_{rs} = \int_R M(P)\phi_r(P)\phi_s(P)\,dR. \tag{4.09}$$

Now, using (4.08) and (4.09), the final form of the transformed equation (4.04) is

$$\sum_{s=1}^{\infty} k_{rs}\overline{w}_s(t) + \sum_{s=1}^{\infty} m_{rs}\frac{d^2\overline{w}_s}{dt^2} = \overline{q}_r(t). \tag{4.10}$$

Equation (4.10) represents an infinite set of coupled differential equations of the same form as those obtained in the analysis of the vibration of discrete masses. Because of this analogy, it is possible to perform a transformation of coordinates leading to a set of uncoupled equations that can easily be solved.

Eigenvalue Problem

Before an analysis of the dynamic response is made, the free vibration problem must be considered. For convenience, equations (4.10) can be written in matrix form as

$$[m]\{\ddot{\overline{w}}\} + [k]\{\overline{w}\} = \{0\} \tag{4.11}$$

where

$[m]$ is a square symmetric matrix with elements m_{rs} ,

$[k]$ is a square symmetric matrix with elements k_{rs} ,

braces indicate column matrices, and

dots denote differentiations with respect to time.

For free vibrations a solution of the form

$$\overline{w}_s = u_s e^{i\omega t}$$

can be assumed. Substitution of this in (4.11) yields a set of linear, homogeneous algebraic equations:

$$\Big[[k] - \omega^2[m]\Big]\{u\} = 0 . \tag{4.12}$$

By Cramer's rule, for a non-trivial solution, the determinant of the coefficients must vanish;

hence

$$\Big| [k] - \omega^2[m] \Big| = 0 . \tag{4.13}$$

Equation (4.13) is the frequency equation for a non-uniform continuous media. Since the matrices involved are of infinite size, only approximate values of the frequencies can be obtained from (4.13).

If equations (4.12) are approximated by, say an $n \times n$ system, then the resulting set of equations is sometimes denoted in the literature as Galerkin's equations. This set of equations is mathematically associated with the problem of minimization of a function of several variables [53]. In vibration theory, this set of equations has been derived by means of the Rayleigh-Ritz method and also by Galerkin's method, [52]. It has been shown that if the system is *positive definite* the eigenvalues obtained from (4.13) will provide upper bounds for the true eigenvalues. It has been further established that as n is increased these eigenvalues approach the true values from above. In vibration theory, this result is customarily reasoned from the fact that the larger mathematical representation reduces the constraints imposed on the system and consequently the approximate values converge from above.

If each of the eigenvalues ω_r^2 are substituted back into (4.12) the corresponding eigenvectors are obtained. Each eigenvector is a column matrix denoted as

$$\{u^{(r)}\} .$$

Each column matrix is determined within only a multiplicative constant since (4.12) is homogeneous. If it is required that the eigenvectors satisfy the relation

$$\left\{u^{(r)}\right\}^T [m] \left\{u^{(r)}\right\} = 1 , \tag{4.14}$$

then the elements are determined uniquely. The eigenvectors are orthogonal with respect to the mass matrix $[m]$,

thus

$$\left\{u^{(r)}\right\}^T [m] \left\{u^{(s)}\right\} = 0. \tag{4.15}$$

The last two equations can be combined into a single equation:

$$\left\{u^{(r)}\right\}^T [m] \left\{u^{(s)}\right\} = \delta_{rs} . \tag{4.16}$$

Dynamic Response

With the eigenvalues and eigenvectors determined, equations (4.10) can be solved. Written in matrix form these are

$$[m]\left\{\ddot{\bar{w}}\right\} + [k]\left\{\bar{w}\right\} = \left\{\gamma\right\} . \tag{4.17}$$

These equations may be uncoupled by forming a square transformation matrix of the eigenvectors determined above. The matrix $[u]$ is introduced which is defined as

$$[u] = \left[\left\{u^{(1)}\right\} \left\{u^{(2)}\right\} \left\{u^{(3)}\right\} \ldots \right] . \tag{4.18}$$

Next, a new time function $\eta(t)$ is introduced and is related to $\bar{w}$ by the transformation equation

$$\left\{\bar{w}\right\} = [u]\left\{\eta(t)\right\} . \tag{4.19}$$

The last equation substituted into (4.17) gives

$$[m][u]\{\ddot{\eta}\} + [k][u]\{\eta\} = \{q\} ,$$

which when premultiplied by $[u]^T$ yields

$$[u]^T[m][u]\{\ddot{\eta}\} + [u]^T[k][u]\{\eta\} = [u]^T\{q\} .$$

From the orthogonality conditions of (4.16), the following equality results:

$$[u]^T[m][u] = [I], \tag{4.20}$$

and from normal mode theory (see [52]) it follows that

$$[u]^T[k][u] = [\omega^2] . \tag{4.21}$$

For convenience, let

$$\{N\} = [u]^T\{q\} \tag{4.22}$$

then the transformed equation becomes, using (4.20) and (4.21),

$$\{\ddot{\eta}\} + [\omega^2]\{\eta\} = \{N\} . \tag{4.23}$$

The last equation represents a set of uncoupled ordinary differential equations of the type

$$\ddot{\eta}_r + \omega_r^2 \eta_r = N_r , \qquad r = 1, 2, \ldots .$$

These equations have the solution

$$\eta_r(t) = \eta_r(0) \cos \omega_r t + \frac{\dot{\eta}_r(0)}{\omega_r} \sin \omega_r t + \frac{1}{\omega_r} \int_0^t N_r(\tau) \sin \omega_r (t-\tau) d\tau, \tag{4.24}$$

where $\eta_r(0)$ and $\dot{\eta}_r(0)$ can be determined from the initial conditions on $w(P, t)$. From (4.19) these are found by premultiplying by the inverse of $[u]$ so that

$$\{\eta(0)\} = [u]^{-1}\{\overline{w}(0)\} \tag{4.25a}$$

and $$\left\{\dot{\eta}(0)\right\} = [u]^{-1}\left\{\dot{\bar{w}}(0)\right\} . \tag{4.25b}$$

To proceed to the final solution (4.19) is written as

$$\bar{w}_r(t) = \sum_{s=1}^{\infty} u_r^{(s)} \eta_s(t)$$

and hence,

$$\bar{w}_r(t) = \sum_{s=1}^{\infty} u_r^{(s)}\left[\eta_s(0)\cos\omega_s t + \frac{\dot{\eta}_s(0)}{\omega_s}\sin\omega_s t + \frac{1}{\omega_s}\int_0^t N_s(\tau)\sin\omega_s(t-\tau)d\tau\right]. \tag{4.26}$$

The inversion series (3.02) yields the final solution:

$$w(P,t) =$$

$$\sum_{s=1}^{\infty}\sum_{r=1}^{\infty} u_r^{(s)}\phi_r(P)\left[\eta_s(0)\cos\omega_s t + \frac{\dot{\eta}_s(0)}{\omega_s}\sin\omega_s t + \frac{1}{\omega_s}\int_0^t N_s(\tau)\sin\omega_s(t-\tau)d\tau\right]. \tag{4.27}$$

This solution may be interpreted as expanding the true but unknown mode shapes in terms of a series of known mode shapes from the eigenvalue problem for the uniform continuum. Hence from (4.27) the mode shapes $\psi_s(P)$ of the non-uniform continuum may be written as

$$\psi_s(P) = \sum_{r=1}^{\infty} u_r^{(s)}\phi_r(P) \tag{4.28}$$

and then (4.27) becomes

$$w(P,t) =$$

$$\sum_{s=1}^{\infty}\psi_s(P)\left[\eta_s(0)\cos\omega_s t + \frac{\dot{\eta}_s(0)}{\omega_s}\sin\omega_s t + \frac{1}{\omega_s}\int_0^t N_s(\tau)\sin\omega_s(t-\tau)d\tau\right]. \tag{4.29}$$

Application to the Vibration of a Non-Uniform Rod

The procedure for solving for the frequencies and mode shapes for a problem with a non-uniform cross section consists of using the eigenfunctions from the associated uniform problem to calculate the elements of the stiffness and mass matrices, (4.07) and (4.09). The eigenvalues and eigenvectors are then computed to the desired degree of accuracy by taking successively larger arrays for the eigenvalue problem (4.12). With the eigenvectors from this result, the mode shapes are found by summation using (4.28).

To illustrate this procedure the example of the longitudinal vibrations of a tapered rod is considered. Although the problem illustrated here is an elementary one, it possesses an exact solution for comparative purposes. The rod under consideration is shown in Figure 1.

The governing equation for the axial displacement $u(x,t)$ is

$$-\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right]+\rho A(x)\frac{\partial^2 u(x,t)}{\partial t^2}=0 \tag{4.30}$$

where ρ is the density and A is the cross-sectional area. In this case

$$A(x)=A_0\left(1-\frac{x}{\ell}\right) \tag{4.31}$$

where A_0 is the area at $x=0$.

Hence, in this case, by comparison with (4.01), the result is

$$L=-\frac{\partial}{\partial x}\left[EA(x)\frac{\partial}{\partial x}\right], \qquad M(x)=\rho A(x). \tag{4.32}$$

The boundary conditions are

$$u(0,t)=0$$

$$EA(x)\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=\ell}=0. \tag{4.33}$$

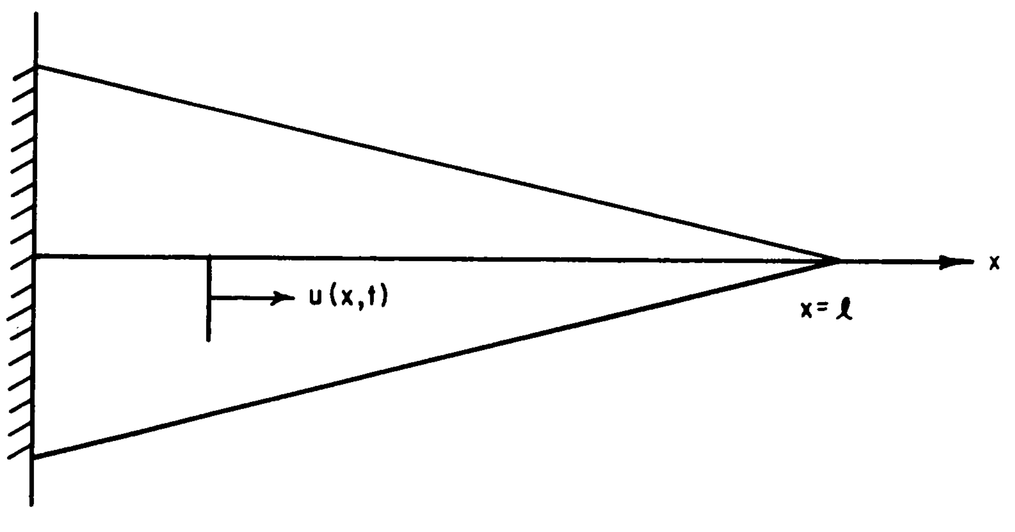


Figure 1 - Longitudinal Vibration of a Tapered Rod

This eigenvalue problem can be solved in closed form to yield the normalized eigenfunctions

$$\Phi_i(x) = \sqrt{\frac{2}{\rho A_0 \ell}} \quad \frac{J_0\left[\lambda_i\left(1-\frac{x}{\ell}\right)\right]}{J_1(\lambda_i \ell)} \tag{4.34}$$

where λ_i are the roots of

$$J_0(\lambda_i \ell) = 0 \tag{4.35}$$

and the frequencies are

$$\omega_i = \lambda_i \sqrt{\frac{E}{\rho}} \; . \tag{4.36}$$

For the associated uniform rod the normalized eigenfunctions are

$$\phi_r(x) = \sqrt{\frac{2}{\ell}} \, \sin \beta_r x \tag{4.37}$$

and

$$\beta_r = \frac{(2r-1)\pi}{2\ell} \, . \tag{4.38}$$

The stiffness coefficients are thus calculated from (4.07) as

$$k_{rs} = \int_0^\ell -\frac{\partial}{\partial x}\left[EA(x)\frac{\partial \phi_r}{\partial x}\right]\phi_s(x)\,dx \tag{4.39}$$

and the mass coefficients are determined by (4.09) as

$$m_{rs} = \rho \int_0^\ell A(x)\,\phi_r(x)\,\phi_s(x)\,dx. \tag{4.40}$$

Substitution of (4.37) and integration leads to

$$k_{rr} = \frac{EA_0}{4\ell^2}\left[3 + (-1)^{2r-1} + \frac{(2r-1)^2\pi^2}{2}\right]$$

$$k_{rs} = \frac{EA_0(2r-1)(2s-1)}{4\ell^2}\left[\frac{1-(-1)^{s-r}}{(s-r)^2} + \frac{1-(-1)^{s+r-1}}{(s+r-1)^2}\right] \qquad r \neq s \tag{4.41}$$

and

$$m_{rr} = \frac{\rho A_o}{2}\left[1 - \frac{2}{\pi^2}\,\frac{1-(-1)^{2r-1}}{(2r-1)^2}\right]$$

$$m_{rs} = \frac{\rho A_o}{\pi^2}\left[\frac{1-(-1)^{s-r}}{(s-r)^2} - \frac{1-(-1)^{s+r-1}}{(s+r-1)^2}\right] \qquad r \neq s. \qquad (4.42)$$

Equations (4.41) and (4.42) are then used to calculate the elements in the frequency determinant, equation (4.13). This determinant has the form,

$$\begin{vmatrix} k_{11}-\omega^2 m_{11} & k_{12}-\omega^2 m_{12} & \cdots \\ k_{21}-\omega^2 m_{21} & k_{22}-\omega^2 m_{22} & \\ \vdots & & \ddots \end{vmatrix} = 0. \qquad (4.43)$$

As a first approximation to the fundamental frequency, $\omega_1^{(1)}$, a 1 X 1 determinant is taken. This gives

$$\omega_1^{(1)} = \sqrt{\frac{k_{11}}{m_{11}}}\,.$$

Now k_{11} and m_{11} from (4.41) and (4.42) give the approximate frequency, that is,

$$\omega_1^{(1)} = \frac{2.41460}{\ell}\sqrt{\frac{E}{\rho}}\,. \qquad (4.44)$$

This value is a good approximation to the first mode; it is about 0.5 percent higher than the actual value. To ascertain the rate of convergence and the higher modes, the eigenvalue problem was solved on the IBM 1130 for successively larger arrays. The eigenvalues from these calculations are shown for the first four modes in Table I. For comparison the exact values from equation (4.35) are also given in Table I.

The results of Table I demonstrate that the convergence is monotonic from above, and a 12 X 12 matrix is required for the first mode frequency to agree to six significant figures. The second, third, and fourth modes then agree to five figures. This degree of

TABLE I

Comparison of Eigenvalues of a Tapered Rod

Matrix Size	Eigenvalues			
	Mode 1 $\lambda_1 \ell$	Mode 2 $\lambda_2 \ell$	Mode 3 $\lambda_3 \ell$	Mode 4 $\lambda_4 \ell$
1 x 1	2.41460			
2 x 2	2.40619	5.52974		
3 x 3	2.40526	5.52167	8.66310	
4 x 4	2.40502	5.52066	8.65536	11.80070
5 x 5	2.40493	5.52036	8.65435	11.79316
6 x 6	2.40489	5.52024	8.65404	11.79217
7 x 7	2.40487	5.52018	8.65391	11.79186
8 x 8	2.40485	5.52014	8.65385	11.79173
9 x 9	2.40484	5.52012	8.65381	11.79166
10 x 10	2.40484	5.52011	8.65379	11.79162
11 x 11	2.40484	5.52010	8.65377	11.79160
12 x 12	2.40483	5.52010	8.65376	11.79158
Exact Values Eq(4.35)	2.40483	5.52008	8.65372	11.79153

accuracy is, however, stringent for engineering applications and a smaller matrix would give acceptable answers. The eigenvectors from the 8×8 solution were then used with equations (4.37) and (4.28) to calculate the mode shapes for comparison with the exact mode shapes, equation (4.34). The approximate results, the sum of eight terms, agree with (4.34) to three significant figures. These mode shapes are plotted in Figures 2 and 3. The circled points are the approximate values and for the accuracy of the plot they coincide with the exact values.

This problem has demonstrated the application of the theory developed in the previous section. For this case the convergence to acceptable values for the frequencies was reasonably rapid and with the aid of a digital computer higher approximations were found without difficulty.

This example concludes the treatment of the vibration of non-uniform continuous media. In the next chapter the vibration of continuous media with a non-uniform mass distribution will be considered.

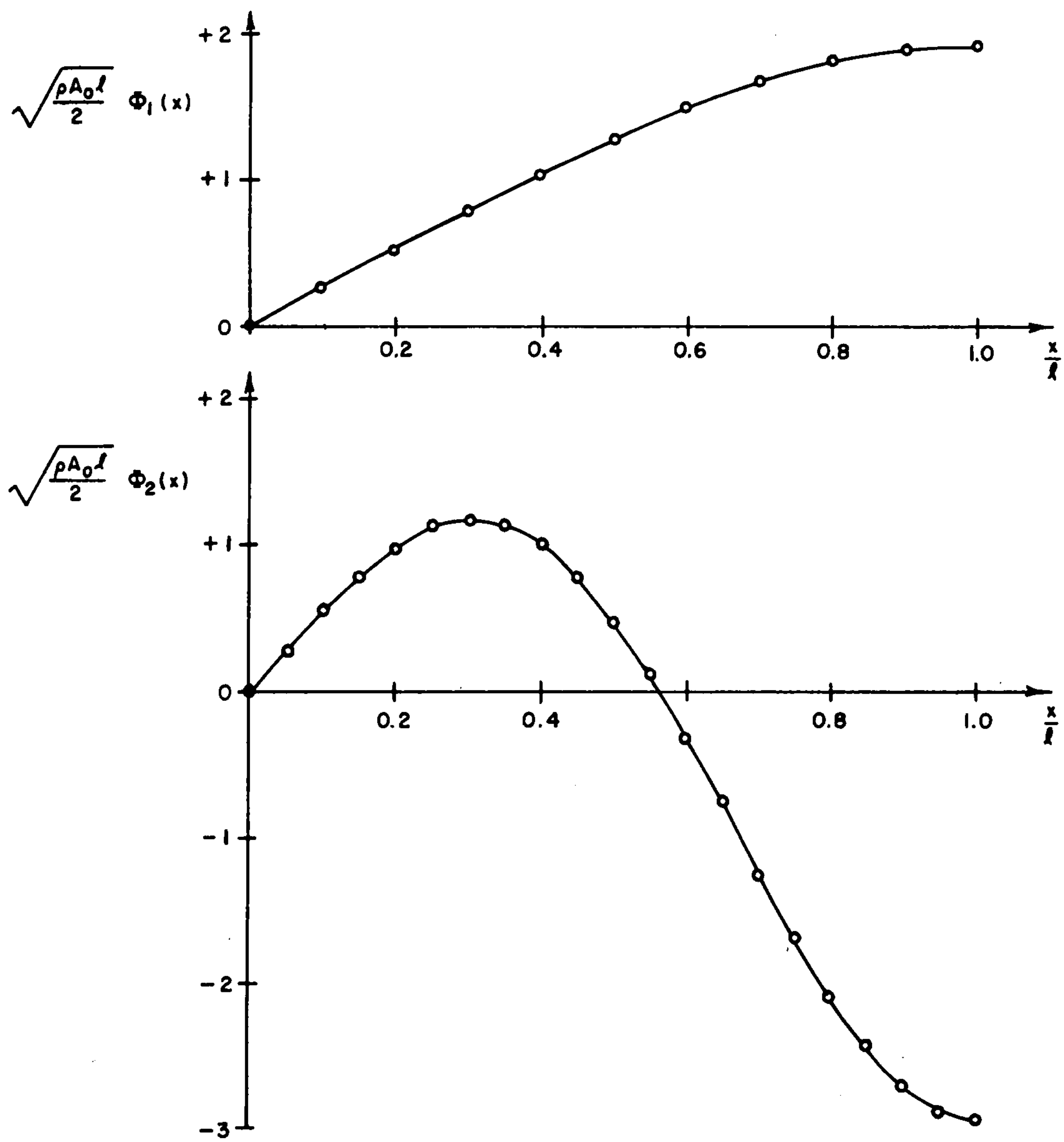


Figure 2 - Mode Shapes for a Tapered Rod, $\Phi_1(x)$, $\Phi_2(x)$

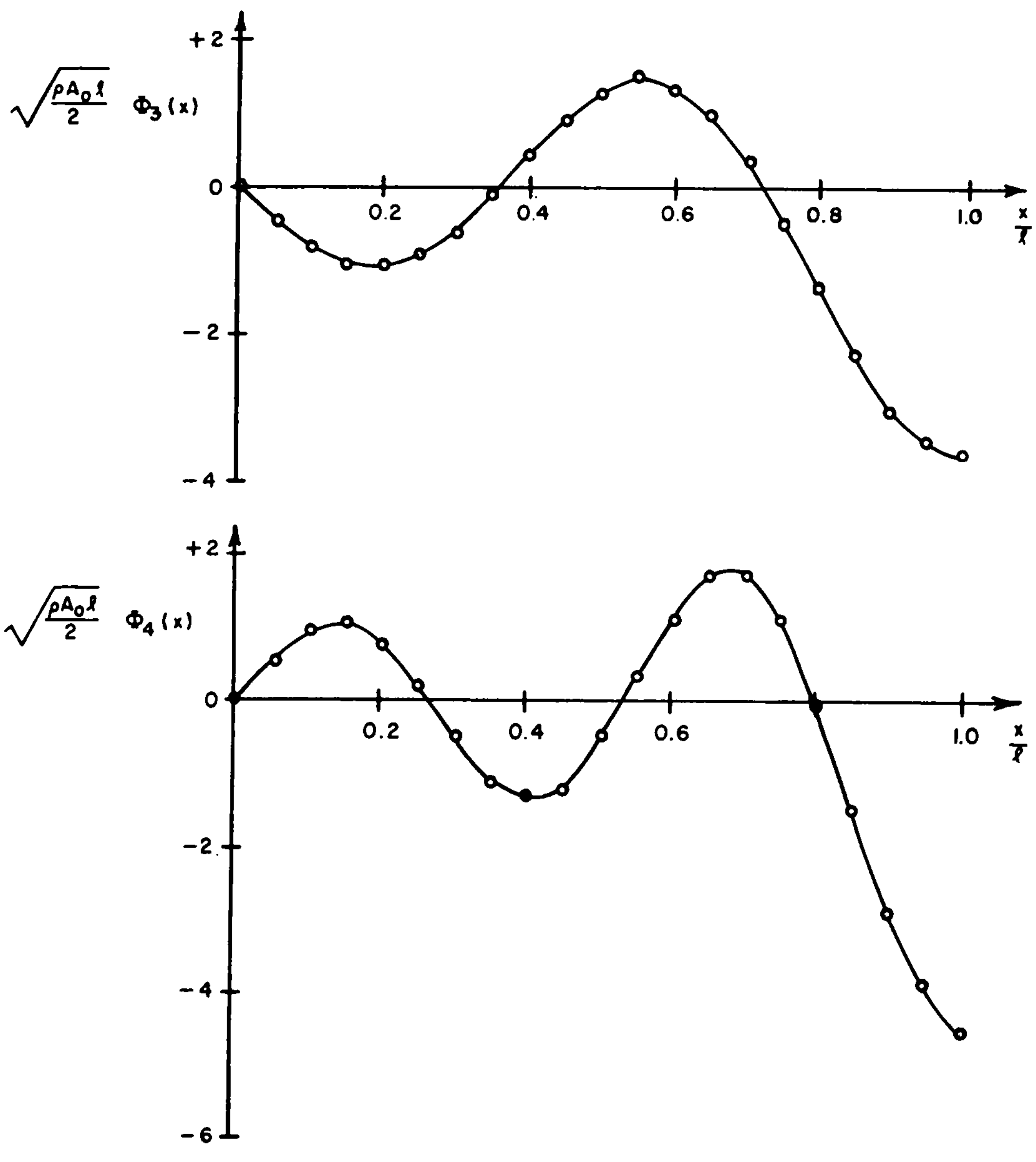


Figure 3 - Mode Shapes for a Tapered Rod, $\Phi_3(x)$, $\Phi_4(x)$

V. VIBRATION OF CONTINUOUS MEDIA WITH ARBITRARY MASS DISTRIBUTION

In this chapter the eigentransform is used to obtain solutions for the vibration of structural elements with uniform stiffness but an arbitrary mass distribution. The general problem is considered first and then several applications to beams and plates with concentrated masses are presented.

General Development of Theory

Application of Eigentransform

Consider a one or two dimensional continuous media with an arbitrary mass distribution. The density of the material may be variable and the structural element may be loaded by an arbitrary mass distribution. The cross section of the element is uniform. The behavior of such a system is characterized by a partial differential equation of order $2p$,

$$L[w(P,t)] + M(P)\frac{\partial^2 w(P,t)}{\partial t^2} = q(P,t) \tag{5.01}$$

and boundary conditions of order $2p-1$,

$$B_i[w(P,t)] = 0 . \tag{5.02}$$

The problem is assumed to be self-adjoint.

Corresponding to the problem posed above is an associated problem with uniform mass distribution. The associated problem is characterized by a set of eigenvalues λ_r and a set of eigenfunctions $\phi_r(P)$ which satisfy the differential equation

$$L[\phi_r(P)] = \lambda_r \phi_r(P) \tag{5.03}$$

and the boundary conditions (5.02). The operator L is the same for both problems due to assumption of a uniform cross section. The eigenfunctions are orthogonal and are normalized so that

$$\int_R \phi_r(P)\phi_s(P)\,dR = \delta_{rs}. \tag{5.04}$$

Using ϕ_r as the kernel, the eigentransform of (5.01) is performed. This gives

$$\int_R L[W(P,t)]\,\phi_r(P)\,dR + \frac{\partial^2}{\partial t^2}\int_R M(P)\,W(P,t)\,\phi_r(P)\,dR = \bar{q}_r(t). \tag{5.05}$$

Since $w(P,t)$ and $\phi_r(P)$ satisfy the same boundary conditions, the definition of a self-adjoint eigenvalue problem can be used to write

$$\int_R L[W(P,t)]\,\phi_r(P)\,dR = \int_R W(P,t)\,L[\phi_r(P)]\,dR$$

and then (5.03) can be substituted to give

$$\int_R L[W(P,t)]\phi_r(P)\,dR = \lambda_r \int_R W(P,t)\phi_r(P)\,dR = \lambda_r \bar{W}_r(t). \tag{5.06}$$

The transform of the mass term may be performed using the transformation for the product of two functions, (3.09);

thus
$$\int_R M(P)W(P)\,\phi_r(P)\,dR = \sum_{s=1}^{\infty} m_{rs}\bar{W}_s(t), \tag{5.07}$$

where
$$m_{rs} = \int_R M(P)\,\phi_r(P)\,\phi_s(P)\,dR. \tag{5.08}$$

Substitution of (5.06) and (5.07) into (5.05) yields the transformed equation,

$$\lambda_r \bar{W}_r(t) + \sum_{s=1}^{\infty} m_{rs}\,\ddot{\bar{W}}_s(t) = \bar{q}_r(t), \tag{5.09}$$

which can be written in matrix form as

$$[m]\{\ddot{\bar{w}}\} + [\lambda]\{\bar{w}\} = \{\bar{q}\}\ . \tag{5.10}$$

Thus, the problem with an arbitrary mass distribution is converted into an infinite set of coupled differential equations in a manner similar to the problem of variable cross section. In the present problem, however, only the set of mass coefficients must be calculated; the stiffness matrix is a diagonal matrix of the eigenvalues known from the associated uniform problem.

Eigenvalue Problem and Dynamic Response

With the procedure of Chapter IV, (5.10) can be uncoupled by means of a transformation matrix of the eigenvectors for the matrix eigenvalue problem. Proceeding in the same way, the final solution is obtained as

$$w(P,t) = \sum_{s=1}^{\infty} \psi_s(P)\,\eta_s(t) \tag{5.11}$$

where $\eta_s(t)$ are the solutions to the uncoupled set of ordinary differential equations

$$\ddot{\eta}_s + \omega_s^2 \eta_s = N_s(t) \tag{5.12}$$

and $N_s(t)$ are the generalized forces found from

$$\{N\} = [u]\{\gamma\} \, . \tag{5.13}$$

As before, the eigenfunctions are found from

$$\psi_s(P) = \sum_{r=1}^{\infty} u_r^{(s)} \phi_r(P) . \tag{5.14}$$

It is again seen that the solution amounts to expanding the true but unknown mode shapes into an infinite series of known mode shapes. For small deviations from the uniform continuum it is reasonable to expect the uniform solutions to be an approximate representation. Equation (5.14) gives a mathematical procedure for finding this representation when the deviation is not small. The above solution, of course, is inexact to the extent of approximating the eigenvalues and eigenvectors of the infinite matrices. The practicality of these approximations will be examined by specific applications to a number of problems in the following sections.

Applications to the Euler Beam

Fundamental Frequency for Beams with Concentrated Masses

In this section an approximate expression is derived for the fundamental frequency of an Euler beam supporting an arbitrary number of concentrated masses located arbitrarily along the beam. The beam considered can have any combination of the customary beam boundary conditions, and the transverse deflection w is governed by

$$EI \frac{\partial^4 w}{\partial x^4} + M(x) \frac{\partial^2 w}{\partial t^2} = 0$$

where E is the modulus of elasticity, I is the moment of inertia of the cross-sectional area, and $M(x)$ is the mass distribution per-unit-length. For a uniform beam loaded by a total of N point masses, M_i located at X_i , the mass distribution has the form

$$M(x) = m + \sum_{i=1}^{N} M_i \, \delta(X - X_i) \tag{5.15}$$

where $\delta(X - X_i)$ is the Dirac delta function, and m is the mass per-unit-length of the uniform beam.

The associated uniform beam together with the same boundary conditions has a set of eigenfunctions $\phi_r(x)$ and eigenvalues λ_r . The eigenvalues are related to the natural frequencies ω_r by

$$\lambda_r = m \omega_r^2 .$$

The general frequency equation for the beam with non-uniform mass is the determinant

$$\begin{vmatrix} m\omega_1^2 - m_{11}\omega^2 & -m_{12}\omega^2 & \cdots \\ -m_{12}\omega^2 & m\omega_2^2 - m_{22}\omega^2 & \\ \vdots & & \ddots \end{vmatrix} = 0. \tag{5.16}$$

As the first approximation a determinant of one element gives

$$\omega = \omega_1 \sqrt{\frac{m}{m_{11}}} \, . \tag{5.17}$$

The elements of the mass matrix are found by substituting (5.15) into (5.08) which gives, after integration,

$$m_{rs} = m\,\delta_{rs} + \sum_{i=1}^{N} M_i\, \phi_r(x_i)\, \phi_s(x_i) . \tag{5.18}$$

From (5.18) m_{11} is found and is substituted into (5.17) to give

$$\omega = \frac{\omega_1}{\sqrt{1 + \sum_{i=1}^{N} \frac{M_i}{m} \phi_1^2(x_i)}} \, . \tag{5.19}$$

Equation (5.19) gives the fundamental frequency ω for a beam loaded with N arbitrary point masses M_i located at points x_i . The fundamental frequency for the associated uniform beam is ω_1 . When used with a table of eigenfunctions, such as Young and Felgar [54], the formula (5.19) gives a convenient way of estimating the fundamental frequency.

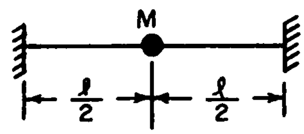
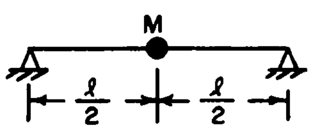
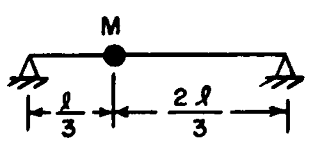
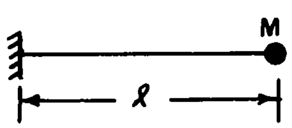
To determine the accuracy of this result, a number of comparisons with known exact solutions were made. Comparisons were made with existing solutions for the simply supported and clamped-clamped beam with a single mass M and a cantilever with a tip mass. These comparisons are given in Table II, where λ is defined by

$$\omega = \frac{\lambda}{\ell^2} \sqrt{\frac{EI}{m}} \, . \tag{5.20}$$

TABLE II

Comparison of Approximate Fundamental Frequencies for

Beams with Concentrated Masses

Problem	Reference No.	Mass Ratio $\frac{M}{m\ell}$	Exact Eigenvalue λ	Approximate Eigenvalue λ(Equation 5.19)	Percent Error
	[26]	1	3.440	3.453	0.38
		10	2.074	2.090	0.77
		100	1.176	1.186	0.85
	[26]	1	2.3832	2.3871	0.16
		10	1.4627	1.4676	0.33
		100	0.8314	0.8344	0.36
	[29]	1	2.4867	2.4984	0.43
		10	1.5514	1.5708	1.25
		100	0.8837	0.8962	1.42
	[23]	1	1.2479	1.2540	0.49
		6	0.8328	0.8386	0.70

From these results it can be seen that generally (5.19) gave very good results. In each case, of course, the approximate frequency was higher than the exact value. For small mass ratios the agreement was excellent, but it became progressively poorer as the mass ratio increased. This is logical since (5.19) is based upon representing the first mode of the beam with additional mass in terms of the first mode of the uniform beam. As the mass ratio becomes larger additional modes of the uniform beam will be required for an accurate representation of the true mode shape.

In the next section the effect of including higher approximations will be considered and correlations of several frequencies and modes shapes with an exact solution will be made.

Frequencies and Mode Shapes for the Simply Supported Beam with Central Mass

In the last section it was seen that a good estimate of the fundamental frequency could be found by using a one element approximation of the frequency determinant. In this section the effect of higher approximations will be investigated. The problem of the simply supported beam with a central mass was chosen since an exact solution for the frequencies and mode shapes can be easily obtained, see [26] or [32].

The beam under consideration is shown in Figure 4. It is governed by

$$EI \frac{\partial^4 W}{\partial x^4} + \left[m + M\,\delta(x - \tfrac{\ell}{2}) \right] \frac{\partial^2 W}{\partial t^2} = 0$$

where m denotes the beam mass per-unit-length and $\delta(x - \frac{\ell}{2})$ is the Dirac delta function representing the concentrated mass M at mid-span.

The associated problem of the uniform beam has normalized eigenfunctions given by

$$\phi_r(x) = \sqrt{\frac{2}{\ell}}\, \sin \frac{r\pi x}{\ell} \tag{5.21}$$

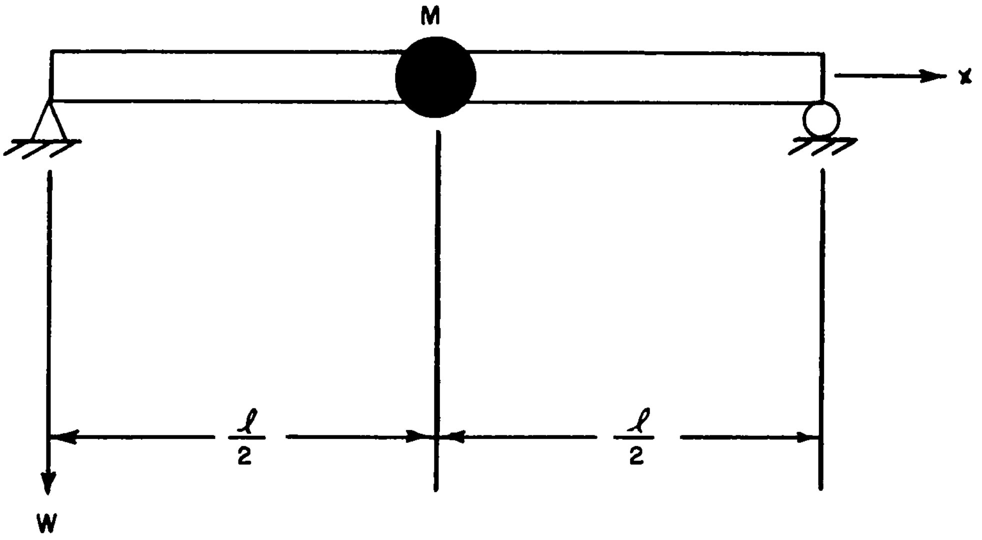


Figure 4 - Simply Supported Beam with Central Mass

and eigenvalues

$$\lambda_r = m\omega_r^2 \tag{5.22}$$

where

$$\omega_r = \frac{r^2\pi^2}{\ell^2}\sqrt{\frac{EI}{m}} \; . \tag{5.23}$$

Substitution in (5.08) gives,

$$m_{rs} = \int_0^\ell \left[m + M\delta(x - \tfrac{\ell}{2})\right] \phi_r \phi_s \, dx$$

which yields

$$m_{rs} = m\,\delta_{rs} + \frac{2M}{\ell} \sin\frac{r\pi}{2} \sin\frac{s\pi}{2} \; . \tag{5.24}$$

The matrix eigenvalue problem in this case has the form

$$\Big[\big[\lambda\big] - \omega^2 \big[m\big] \Big] \{u\} = 0 \tag{5.25}$$

where the frequency equation is the determinant of the square matrix. The determinant has the form

$$\begin{vmatrix} m\omega_1^2 - m_{11}\omega^2 & 0 & -m_{13}\omega^2 & \cdots \\ 0 & m\omega_2^2 - m_{22}\omega^2 & 0 & \\ -m_{13}\omega^2 & 0 & m\omega_3^2 - m_{33}\omega^2 & \\ \vdots & & & \ddots \end{vmatrix} = 0 \, , \tag{5.26}$$

In this problem the mass is located at the midpoint of the span, and it will affect only the symmetric modes of vibration. It is located at a nodal point for the antisymmetric modes and since rotatory inertia is neglected the antisymmetric modes will be identical to those of the uniform beam. These results would appear automatically from (5.25); however, recognizing this fact reduces the size of the eigenvalue problem to be solved. For this reason it is expedient to delete the antisymmetric modes from (5.25) by striking out the even rows and columns prior to performing the numerical computations. The solution for the eigenvalue problem then gives the eigenvalues and their corresponding eigenvectors for the symmetric modes. Using the eigenvectors the mode shapes for the beam with the mass $\psi_s(x)$ are found as a summation of the modes for the uniform beam.

The exact solution for the symmetric modes of this problem is given by the frequency equation [26]

$$\omega_s = \frac{\lambda_s^2}{\ell^2}\sqrt{\frac{EI}{m}} \tag{5.27}$$

where λ_s are the roots of

$$2 - \frac{M}{m\ell}\,\frac{\lambda_s}{2}\left(\tan\frac{\lambda_s}{2} - \tanh\frac{\lambda_s}{2}\right) = 0 \tag{5.28}$$

and the mode shapes are [32]

for $0 < x < \frac{\ell}{2}$,

$$\Phi_s(x) = \frac{M}{m\ell}\,\frac{\lambda_s}{4}\left[\frac{\sinh\lambda_s\frac{x}{\ell}}{\cosh\frac{\lambda_s}{2}} - \frac{\sin\lambda_s\frac{x}{\ell}}{\cos\frac{\lambda_s}{2}}\right]$$

for $\frac{\ell}{2} < x < \ell$,

$$\Phi_s(x) = \frac{M}{m\ell}\left\{\frac{\lambda_s}{4}\left[\frac{\sinh\lambda_s\frac{x}{\ell}}{\cosh\frac{\lambda_s}{2}} - \frac{\sin\lambda_s\frac{x}{\ell}}{\cos\frac{\lambda_s}{2}}\right] - 2\left[\sinh\lambda_s\left(\frac{x}{\ell}-\frac{1}{2}\right) - \sin\lambda_s\left(\frac{x}{\ell}-\frac{1}{2}\right)\right]\right\} \tag{5.29}$$

The mode shapes of (5.29) are normalized in this particular case so that

$$\Phi_s\left(\frac{\ell}{2}\right) = 1 .$$

To compare the present method to the exact solution, (5.28) and (5.29), numerical computations were performed on the IBM 1130 computer. For the comparison, the ratio of supported mass M to beam mass $m\ell$ was taken as 2, and the frequencies and mode shapes for the first four symmetric modes were calculated from (5.28) and (5.29). Next the eigenvalue problem was solved repeatedly, increasing the matrix size starting from a 2 X 2 matrix until agreement with the exact first mode frequency was obtained. The successive approximations for the first four eigenvalues and the exact values are shown in Table III.

The results of Table III demonstrate that the approximate frequencies converge to the exact values monotonically from above as expected. For engineering accuracy it can be seen that using the rule that the matrix size be two times the number of required modes gives very good accuracy in this case. Selecting an 8 X 8 matrix, for example, gives an error of only 0.25 percent in the fourth symmetric frequency.

With the eigenvalues known, the mode shapes were calculated by summation using equations (5.21) and (5.14) and summing eight terms. These results are shown as points plotted on the exact shapes in Figures 5 and 6 where $0 \leq \frac{x}{\ell} \leq \frac{1}{2}$ since the shapes are symmetric. The solid line denotes the exact solution, and the circled points denote the values obtained by summation.

The agreement, as expected, is excellent for the first mode with the values identical to four significant figures. This agreement decreases slightly with each successive mode; the typical error for the fourth symmetrical mode is about 2 percent. Hence, for the mode shapes, the rule of adding a number of terms equal to two times the number of required modes gives answers that are within engineering accuracy, although not quite as accurate as the eigenvalues.

In this section the effect of higher approximations has been investigated for a beam with relatively simple and symmetric mode shapes. To illustrate the generality of the method, a problem with more complex boundary conditions is considered in the next section.

TABLE III

Convergence of Eigenvalues for Simply Supported Beam with Central Mass

Matrix Size	Eigenvalues for Mass Ratio, $\frac{M}{m\ell} = 2$			
	Mode 1 λ_1	Mode 3 λ_3	Mode 5 λ_5	Mode 7 λ_7
2 x 2	4.39641	66.4695		
3 x 3	4.39417	65.5679	208.230	
4 x 4	4.39359	65.3429	205.467	430.125
5 x 5	4.39337	65.2613	204.540	425.246
6 x 6	4.39327	65.2248	204.135	423.327
7 x 7	4.39322	65.2061	203.928	422.382
8 x 8	4.39319	65.1955	203.812	421.856
9 x 9	4.39317	65.1891	203.742	421.540
10 x 10	4.39316	65.1851	203.697	421.338
11 x 11	4.39314	65.1823	203.666	421.203
12 x 12	4.39313	65.1804	203.646	421.109
Exact Values Eq(5.28)	4.39313	65.1740	203.575	420.793

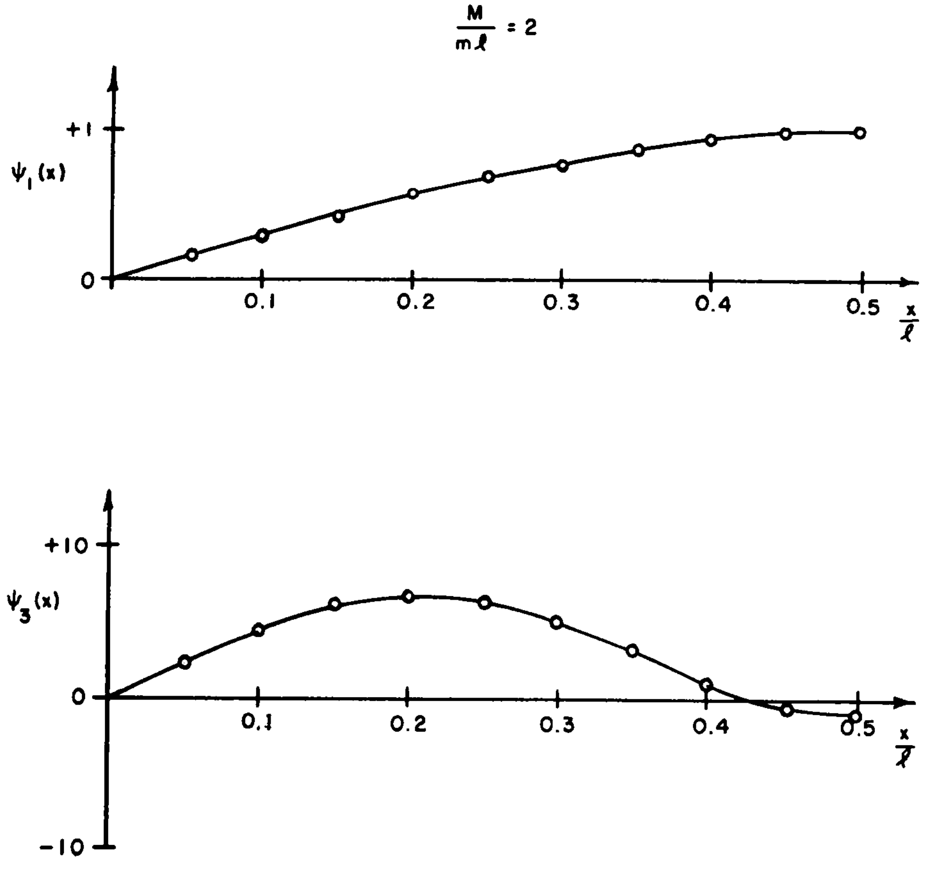


Figure 5 - Mode Shapes for Simply Supported Beam with Central Mass, $\psi_1(x)$, $\psi_3(x)$

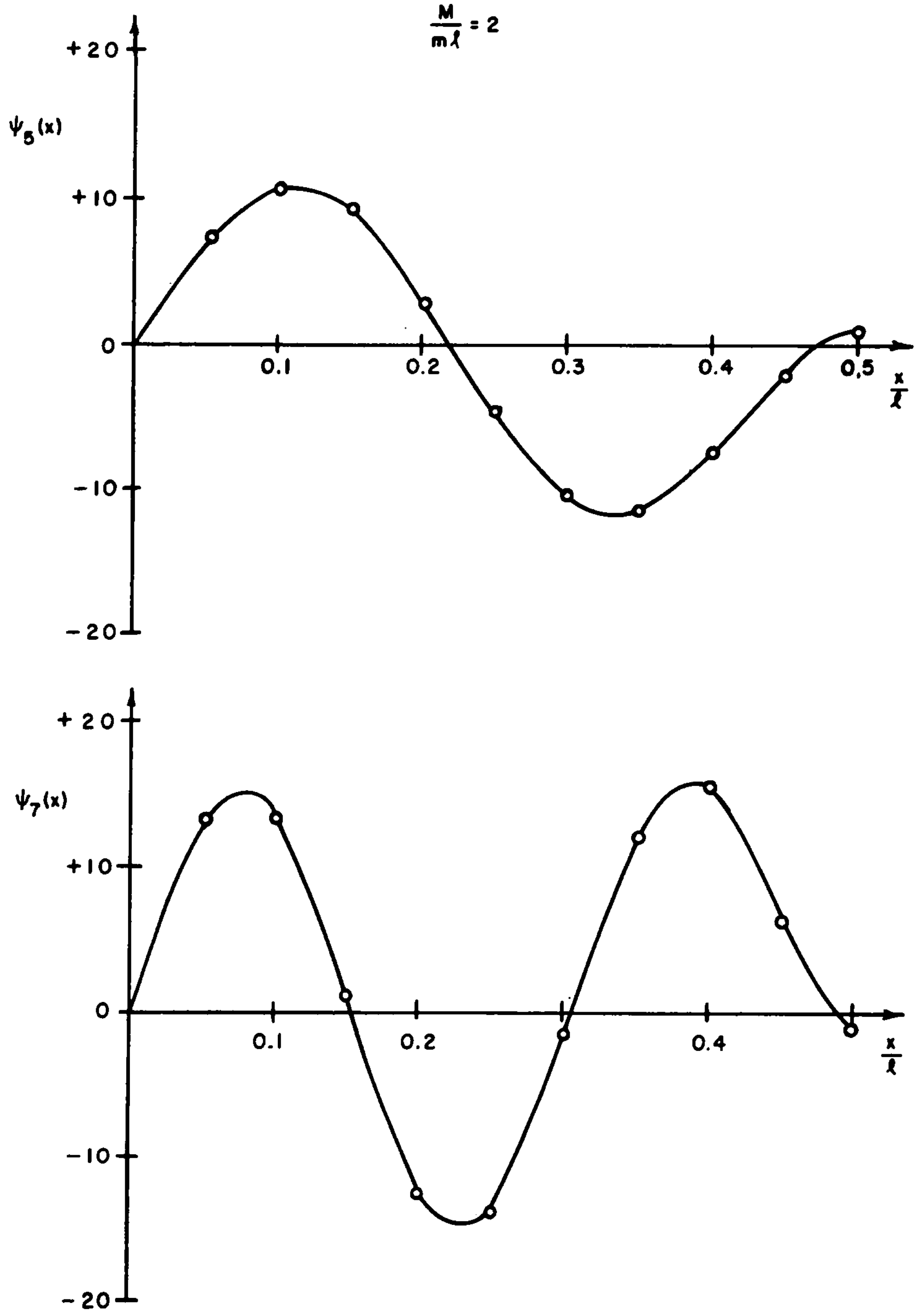


Figure 6 - Mode Shapes for Simply Supported Beam with Central Mass, $\psi_5(x)$, $\psi_7(x)$

Frequencies and Mode Shapes for a Clamped-Supported Beam with Central Mass

To illustrate the generality of the method, the problem of a clamped-supported beam is considered (Figure 7). The solution of this particular problem and other beam problems is greatly facilitated by the use of tables, such as those of Young and Felgar [54]. In this case the associated problem has as its normalized eigenfunctions,

$$\phi_n(x) = \frac{1}{\sqrt{\ell}}\left[\left(\cosh\beta_n\frac{x}{\ell} - \cos\beta_n\frac{x}{\ell}\right) - \alpha_n\left(\sinh\beta_n\frac{x}{\ell} - \sin\beta_n\frac{x}{\ell}\right)\right] \tag{5.30}$$

where β_n are the roots of

$$\tan\beta_n\ell = \tanh\beta_n\ell \tag{5.31}$$

and

$$\alpha_n = \frac{\cos\beta_n\ell + \cosh\beta_n\ell}{\sin\beta_n\ell + \sinh\beta_n\ell} . \tag{5.32}$$

The natural frequencies are given by

$$\omega_n = \frac{\beta_n^2}{\ell^2}\sqrt{\frac{EI}{m}} . \tag{5.33}$$

Both α_n and β_n are tabulated in [54].

The mass distribution in this case is

$$M(x) = m + M\delta\left(x - \frac{\ell}{2}\right) .$$

Substitution of the mass distribution into (5.08) and integration gives the elements of the mass matrix, i.e.,

$$m_{rs} = m\,\delta_{rs} + M\,\phi_r\left(\frac{\ell}{2}\right)\phi_s\left(\frac{\ell}{2}\right) . \tag{5.34}$$

With the eigenvalues β_n and the mass matrix (5.34) known, the problem reduces to solving the eigenvalue problem to the degree of accuracy required. In the present case an 8×8 array was selected, and computations were performed for various ratios of $\frac{M}{m\ell}$. The first four frequencies are given in Table IV and are plotted in Figure 8. The corresponding mode shapes are shown in Figures 9 and 10.

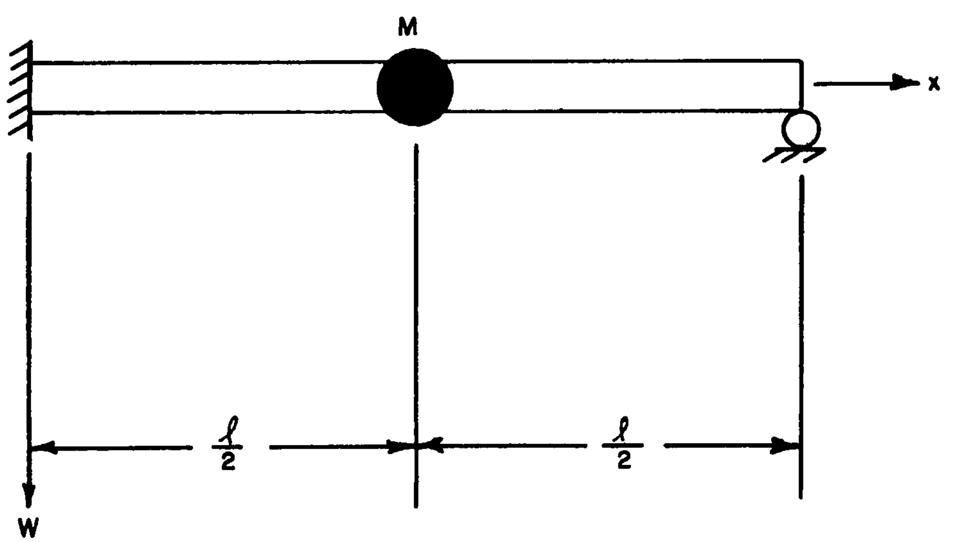


Figure 7 - Clamped-Supported Beam with Central Mass

TABLE IV

Eigenvalues for Clamped-Supported Beam

with Central Mass

Mass Ratio R	Eigenvalues Frequency, $\omega_n = \frac{\lambda_n^2}{\ell^2}\sqrt{\frac{EI}{m}}$ Mode 1 λ_1^2	Mode 2 λ_2^2	Mode 3 λ_3^2	Mode 4 λ_4^2
0	15.42	49.96	104.3	178.3
0.01	15.26	49.89	103.4	178.0
0.02	15.11	49.81	102.6	177.8
0.05	14.67	49.59	100.5	177.2
0.1	14.02	49.29	97.65	176.4
0.2	12.93	48.80	93.78	175.3
0.5	10.73	47.94	88.21	173.8
1	8.699	47.29	84.85	172.9
2	6.696	46.79	82.63	172.4
5	4.489	46.40	81.07	171.9
10	3.242	46.24	80.50	171.8

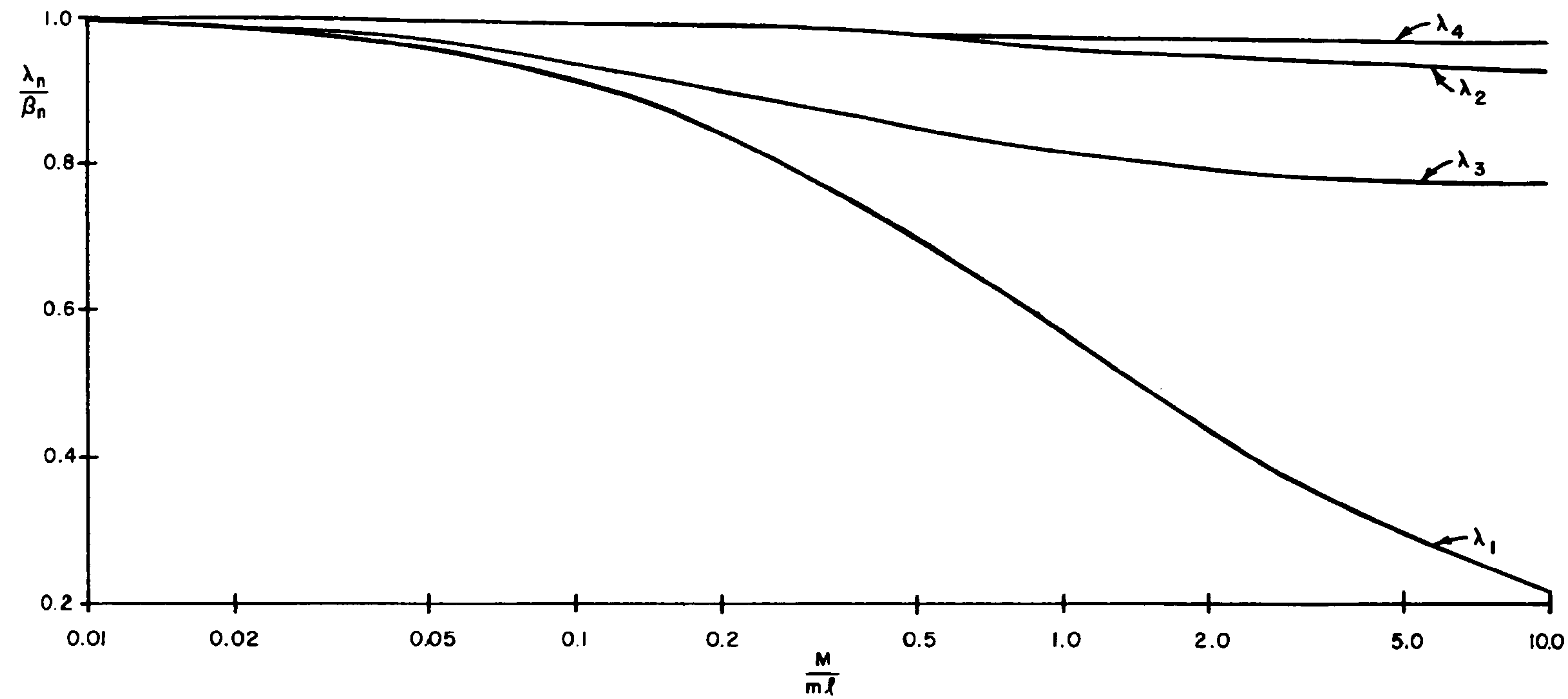


Figure 8 - Ratio of Eigenvalues against Mass Ratio for Clamped-Supported Beam with Central Mass

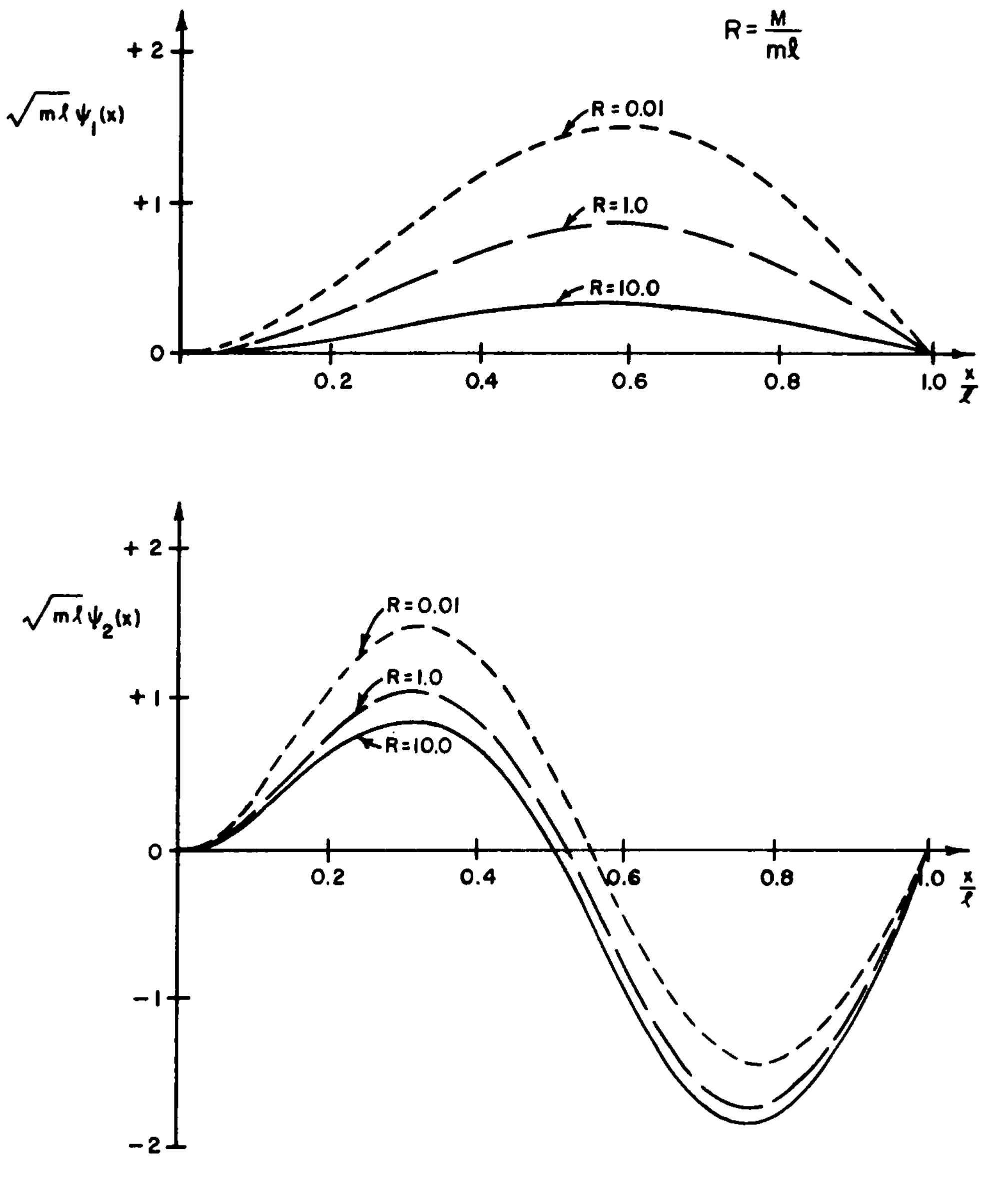


Figure 9 - Mode Shapes for Clamped-Supported Beam with Central Mass, $\psi_1(x)$, $\psi_2(x)$

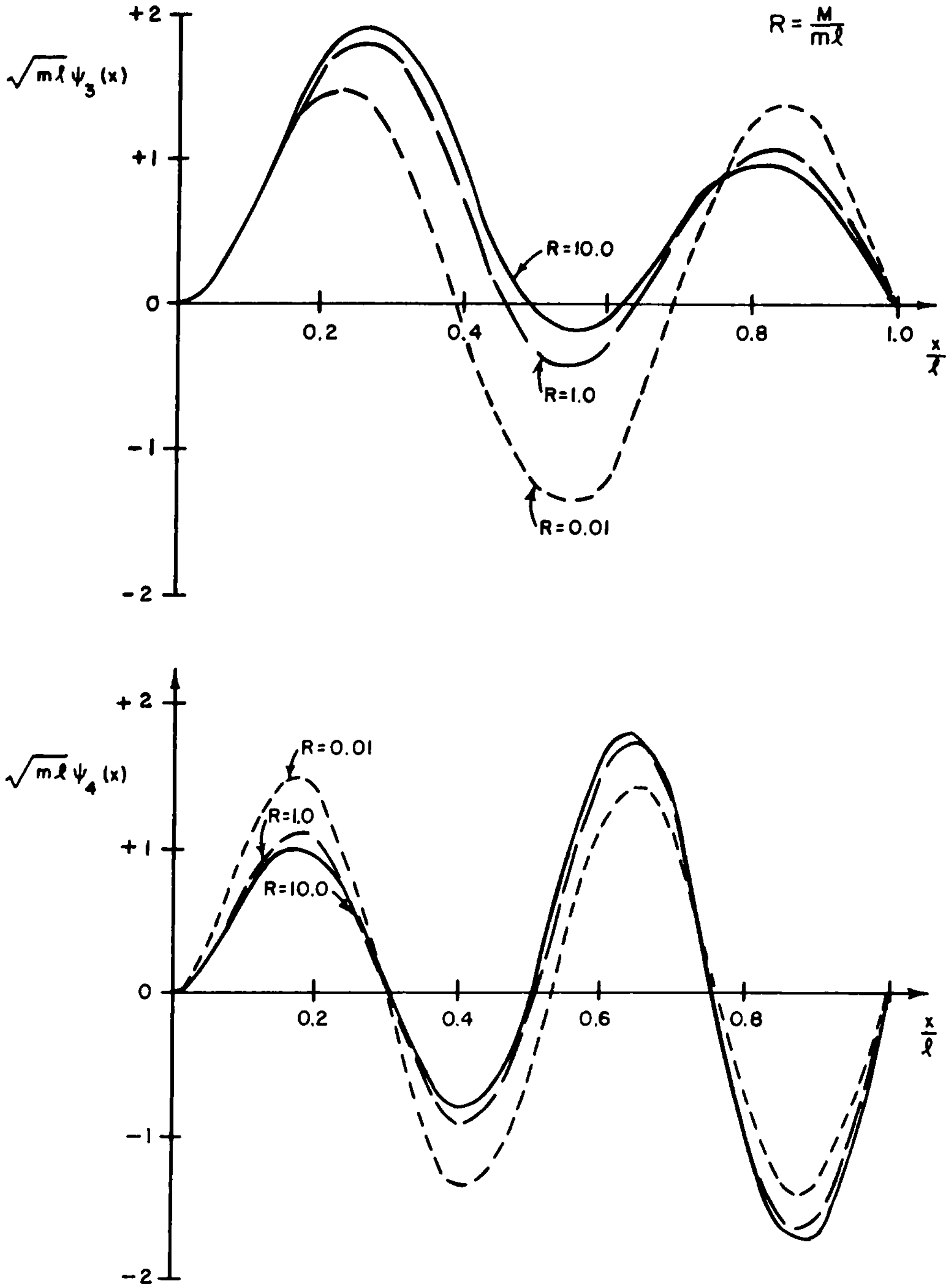


Figure 10 - Mode Shapes for Clamped-Supported Beam with Central Mass, $\psi_3(x)$, $\psi_4(x)$

The accuracy of the present solution corresponds to that of the simply supported beam. This was verified by using as a check, for $\frac{M}{m\ell} = 2$, a 12 X 12 eigenvalue problem. This solution differed from the results of the 8 X 8 problem by less than 0.01 percent in the first mode, and 0.1 percent in the fourth mode and hence the 8 X 8 solution given in Table IV was found to be acceptable.

Figure 8 shows that increasing the supported mass has an appreciable effect on the fundamental frequency but significantly less effect on the higher modes. In this particular problem the second mode is altered only slightly since the nodal point is very close to the mass as can be seen in Figure 9.

This problem completes the applications that will be given for beams. In the next section applications will be made to the vibrations of rectangular plates with concentrated masses.

Applications to Isotropic Plates

Fundamental Frequencies for Plates with Concentrated Masses

In this section an approximate formula for the fundamental frequency of isotropic plates with concentrated masses will be derived. This is analogous to that developed previously for the Euler beam. An isotropic plate loaded by a total of N concentrated masses M_i located at points P_i is considered. The governing equation on the transverse displacement $w(P,t)$ for free vibrations and general boundary conditions is

$$D \nabla^4 w(P,t) + \rho(P) \frac{\partial^2 w}{\partial t^2}(P,t) = 0 \tag{5.35}$$

where D is the flexural rigidity and ρ is the mass distribution per-unit-area. For the problem under consideration the mass distribution takes the form

$$\rho(P) = \rho_o + \sum_{i=1}^{N} M_i \, \delta(P-P_i) \tag{5.36}$$

where ρ_o is the mass per-unit-area of the associated uniform plate and $\delta(P-P_i)$ is the Dirac delta function.

The associated problem of the uniform plate with the same boundary conditions leads to a set of eigenvalues λ_{ij} and a set of normalized eigenfunctions $\phi_{ij}(P)$. The eigenvalues are related to the natural frequencies by

$$\lambda_{ij} = \rho_o \, \omega_{ij}^2 \, . \tag{5.37}$$

As the first approximation to the general frequency equation (5.16) a determinant of one element gives in this case

$$\omega = \omega_{11} \sqrt{\frac{\rho_o}{m_{11}}} \, . \tag{5.38}$$

To find m_{11} (5.36) is substituted into (5.08) which gives

$$m_{11} = \int_R \left[\rho_o + \sum_{i=1}^{N} M_i \, \delta(P-P_i) \right] \phi_{11}^2(P) \, dR$$

hence

$$m_{11} = \rho_o + \sum_{i=1}^{N} M_i \, \phi_{11}^2(P) \tag{5.39}$$

after integration. Substitution of (5.39) into (5.38) gives the desired result

$$\omega = \frac{\omega_{11}}{\sqrt{1 + \sum_{i=1}^{N} \frac{M_i}{\rho_o} \phi_{11}^2(P)}} \tag{5.40}$$

which is exactly analogous to the equation developed for beams (5.19).

The simplest example of applying (5.40) is for the problem of a simply supported rectangular plate (Figure 11) with a central mass, M. For this case the normalized

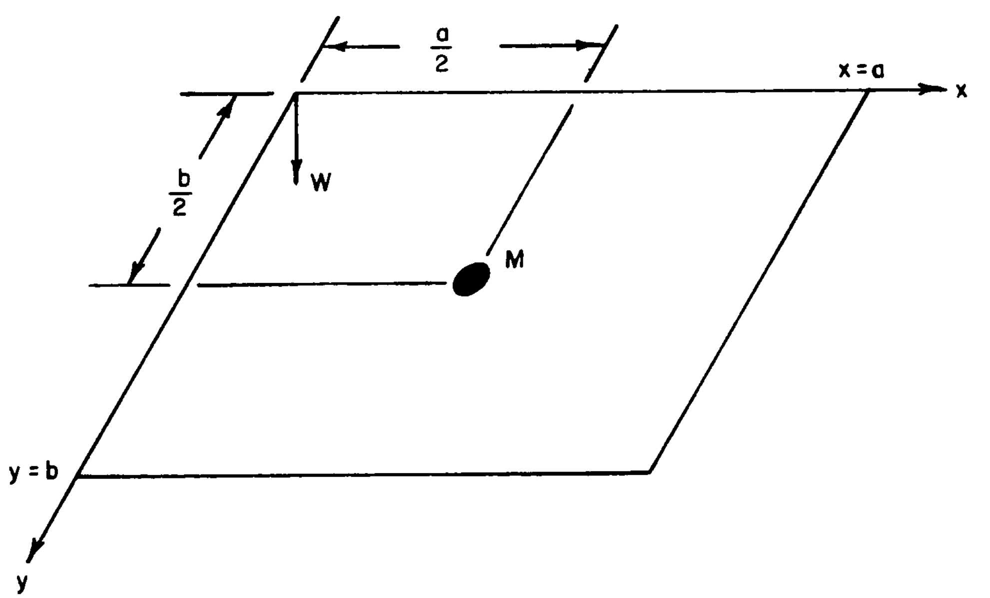


Figure 11 - Rectangular Plate with Central Mass

eigenfunctions are

$$\phi_{ij}(x,y) = \frac{2}{\sqrt{ab}} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} \tag{5.41}$$

and the natural frequencies are given by

$$\omega_{ij} = \left(\frac{i^2}{a^2} + \frac{j^2}{b^2}\right) \pi^2 \sqrt{\frac{D}{\rho_o}} \quad . \tag{5.42}$$

Substituting (5.41) into (5.40) yields

$$\omega = \frac{\omega_{11}}{\sqrt{1 + 4 \frac{M}{\rho_o a b}}} \quad . \tag{5.43}$$

This result, again showing the analogy between beams and plates, appears in the paper [38] by Wah. A comparison of this result with more extensive solutions of the general eigenvalue problem shows that for $\frac{M}{\rho_o a b} = 1$, (5.43) gives a result about 3.5 percent too high.

As a second example a clamped circular plate with a central mass is considered. The plate-mass system is shown in Figure 12. For symmetric modes the normalized eigenfunctions are

$$\phi_i(r) = \frac{1}{\sqrt{2\pi}\, a} \left[\frac{J_o(\lambda_i r)}{J_o(\lambda_i a)} - \frac{I_o(\lambda_i r)}{I_o(\lambda_i a)} \right] \tag{5.44}$$

where the eigenvalues λ_i are the roots of

$$J_o(\lambda_i a) I_1(\lambda_i a) + J_1(\lambda_i a) I_o(\lambda_i a) = 0 \tag{5.45}$$

and the natural frequencies are

$$\omega_i = \lambda_i^2 \sqrt{\frac{D}{\rho_o}} \quad . \tag{5.46}$$

Substitution of (5.44) evaluated at $r = 0$ in (5.40) gives

$$\omega = \frac{\omega_{11}}{\sqrt{1 + \frac{M}{\pi a^2 \rho_o} \frac{1}{2 J_o^2(\lambda_i a)}}} \quad . \tag{5.47}$$

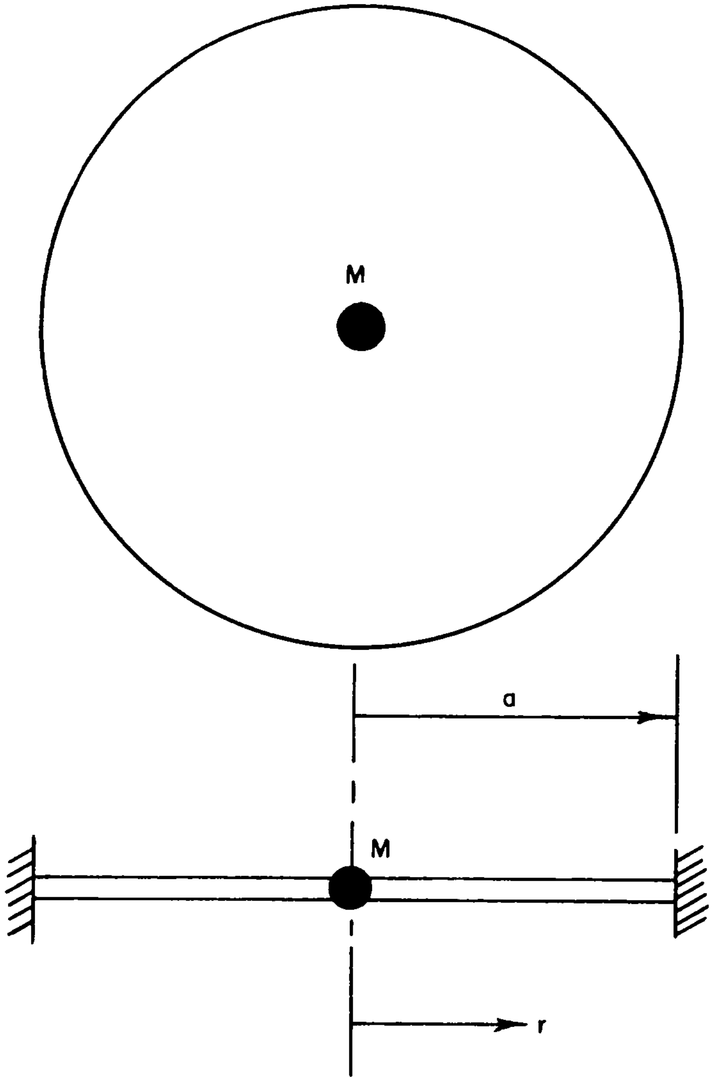


Figure 12 - Clamped Circular Plate with Central Mass

The first root of (5.45) is $\lambda_1 a = 3.19593$ and (5.47) reduces to

$$\omega = \frac{10.21}{a^2 \sqrt{1 + 4.91 \frac{M}{\pi a^2 \rho_o}}} \sqrt{\frac{D}{\rho_o}} \ . \tag{5.48}$$

In this section an approximate formula for the estimation of fundamental frequencies has been derived and applications to two particular problems have been made. In the next section the effect of higher approximations will be considered by solving the general eigenvalue problem.

Simply Supported Plate with Central Mass

In this section the general eigenvalue problem will be solved to demonstrate the rate of convergence and to obtain values for some higher eigenvalues. The plate under consideration was shown in Figure 11, and the eigenfunctions ϕ_{ij} were given in (5.41) and the natural frequencies ω_{ij} are found from (5.42). In the present problem for i or j even the central mass lies on a nodal line and consequently it has no effect on these modes. Thus these modes need not be considered, and only the symmetrical modes will be treated.

To calculate the coefficients of the mass matrix, it is convenient to convert the two dimensional arrays for λ_{ij} and ϕ_{ij} into one dimensional arrays. Thus the eigenvalues and eigenfunctions are arranged in ascending order and are denoted by λ_α and ϕ_α , respectively. Using as the mass distribution

$$\rho = \rho_o + M\,\delta(x - \tfrac{a}{2})\,\delta(y - \tfrac{b}{2}) \tag{5.49}$$

and substituting in (5.08),

$$m_{\alpha\beta} = \rho_o\,\delta_{\alpha\beta} + M\,\phi_\alpha(\tfrac{a}{2}, \tfrac{b}{2})\,\phi_\beta(\tfrac{a}{2}, \tfrac{b}{2}) \ . \tag{5.50}$$

After using the last equation to calculate the mass matrix, the general eigenvalue problem was solved on the IBM 1130. To examine the rate of convergence numerical calculations were performed for a square plate with the ratio of the supported mass to the plate mass, $\frac{M}{\rho_0 a b} = 1$. The results of these calculations are shown in Table V for the first four modes, where the natural frequencies are

$$\omega_\alpha = \frac{\lambda_\alpha \pi^2}{a^2} \sqrt{\frac{D}{\rho_0}} \quad . \tag{5.51}$$

From an examination of these results it can be seen that the rate of convergence is not as rapid as was experienced for beams with masses. This is undoubtably due to the more complex nature of the mode shapes for plates and is, of course, dependent upon the mass ratio. For smaller mass ratios convergence is more rapid. The results, however, are sufficiently accurate for engineering calculations and if greater accuracy is required then larger eigenvalue problems may be solved leading to more terms in the expansions for the mode shapes.

The variation of the first four frequencies with the mass ratio is shown in Figure 13. These frequencies have been non-dimensionalized with respect to the corresponding frequencies without the supported mass, equation (5.42). The corresponding mode shapes are shown in Figures 14 through 17. These were calculated by summation, (5.14), using nine terms.

From these figures it can be seen that the third mode is independent of the mass ratio. This is a consequence of choosing a square plate. The square plate without a mass has repeated frequencies $\omega_{13} = \omega_{31}$, etc. so that two distinct eigenfunctions ϕ_{13} and ϕ_{31} correspond to the same frequency. For repeated eigenvalues any linear combination of the corresponding natural modes is also a natural mode, [55]. The mode shown in Figure 16 represents one of these possible modes, the difference of the ϕ_{13} and ϕ_{31} modes, $\phi_{13} - \phi_{31}$. Its nodal lines are the diagonals of the square, and consequently the mass lies at a nodal point and exerts no influence on the response.

TABLE V

Convergence of Eigenvalues for Simply Supported Square Plate with Central Mass

Matrix Size	Eigenvalues for Mass Ratio $\frac{M}{\rho_0 ab} = 1$			
	Mode I λ_{I}	Mode II λ_{II}	Mode III λ_{III}	Mode IV λ_{IV}
1 x 1	0.8944			
2 x 2	0.8830	7.5497	10.0	
3 x 3	0.8720	6.3616	10.0	
4 x 4	0.8686	6.1766	10.0	16.2745
5 x 5	0.8670	6.0962	10.0	16.0643
6 x 6	0.8654	6.0173	10.0	15.8311
7 x 7	0.8645	5.9730	10.0	15.7145
8 x 8	0.8636	5.9291	10.0	15.5934
9 x 9	0.8632	5.9093	10.0	15.5432
10 x 10	0.8627	5.8897	10.0	15.4924
11 x 11	0.8623	5.8702	10.0	15.4410
12 x 12	0.8620	5.8558	10.0	15.4035

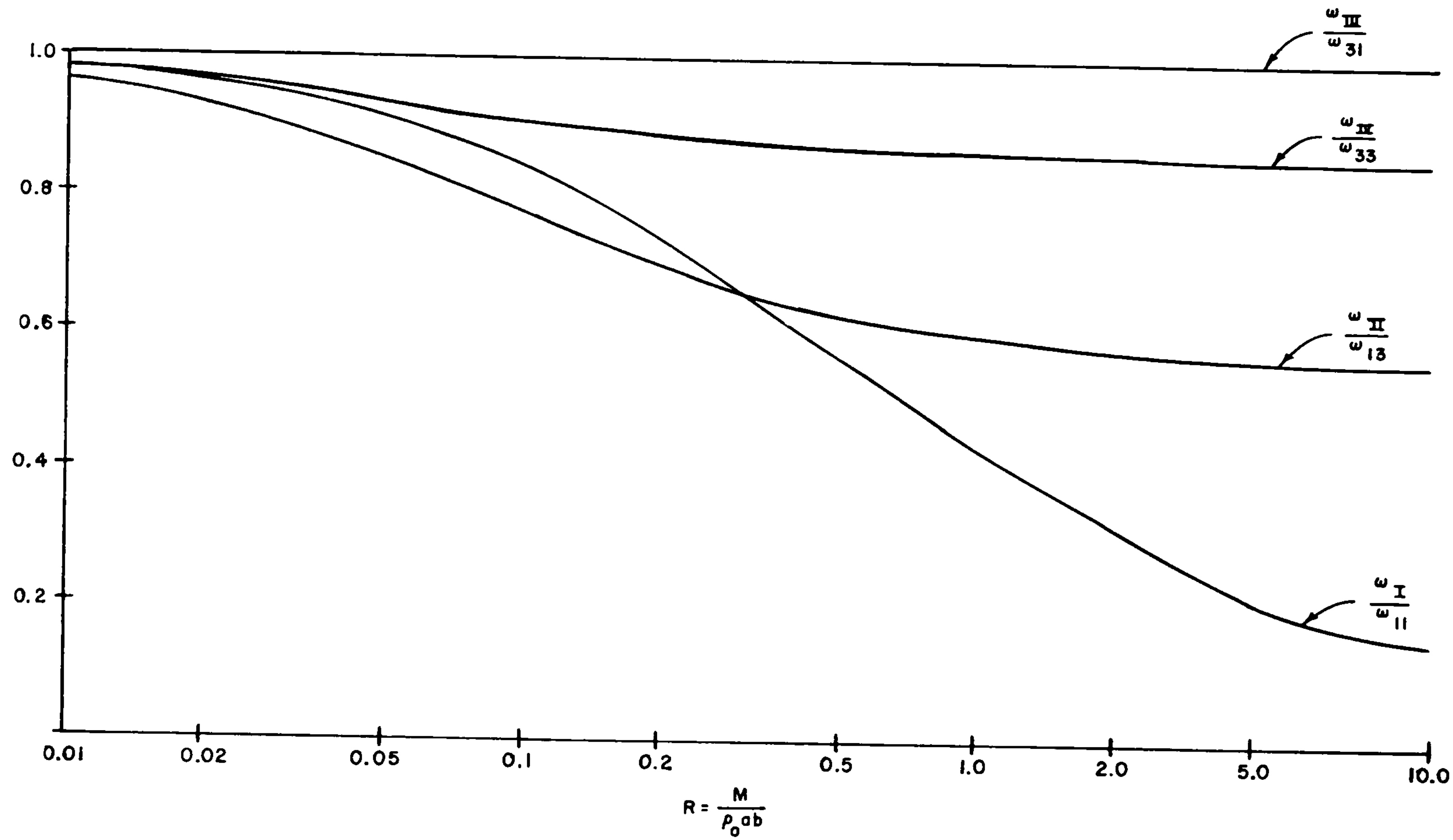


Figure 13 - Ratio of Frequencies against Mass Ratio for Simply Supported Plate with Central Mass

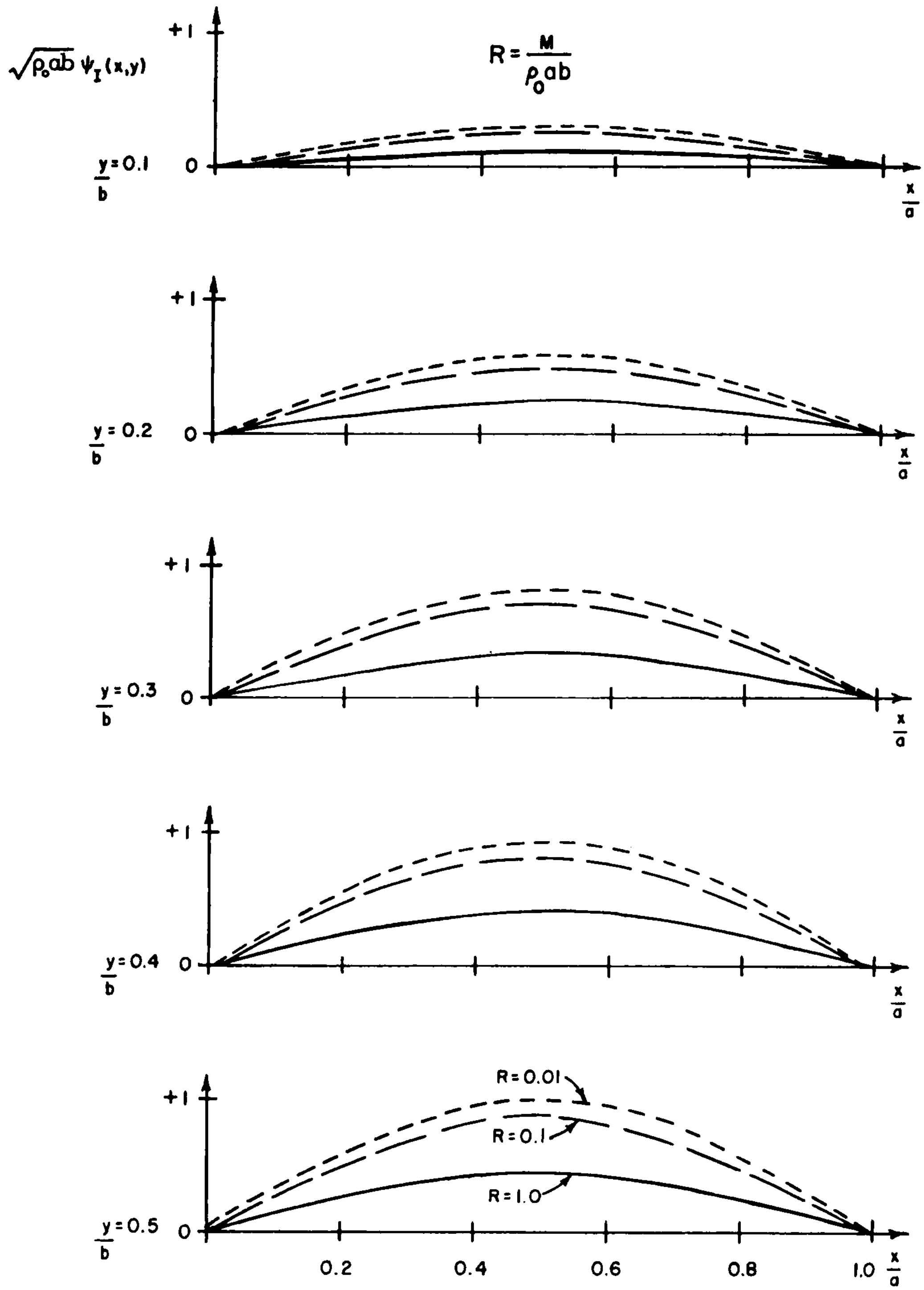


Figure 14 - Mode Shape for Simply Supported Plate with Central Mass, $\psi_I(x,y)$

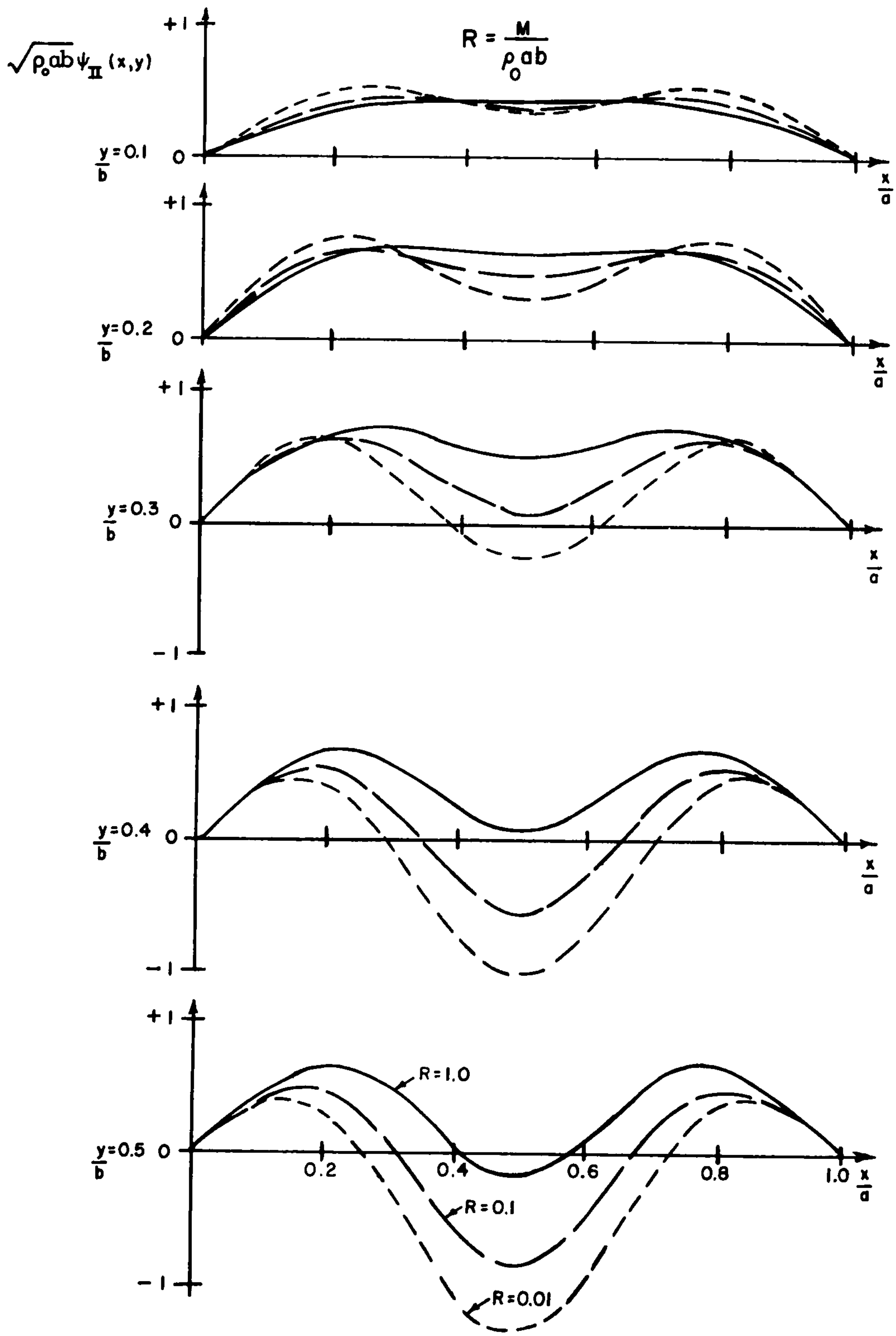


Figure 15 - Mode Shape for Simply Supported Plate with Central Mass, $\psi_{II}(x, y)$

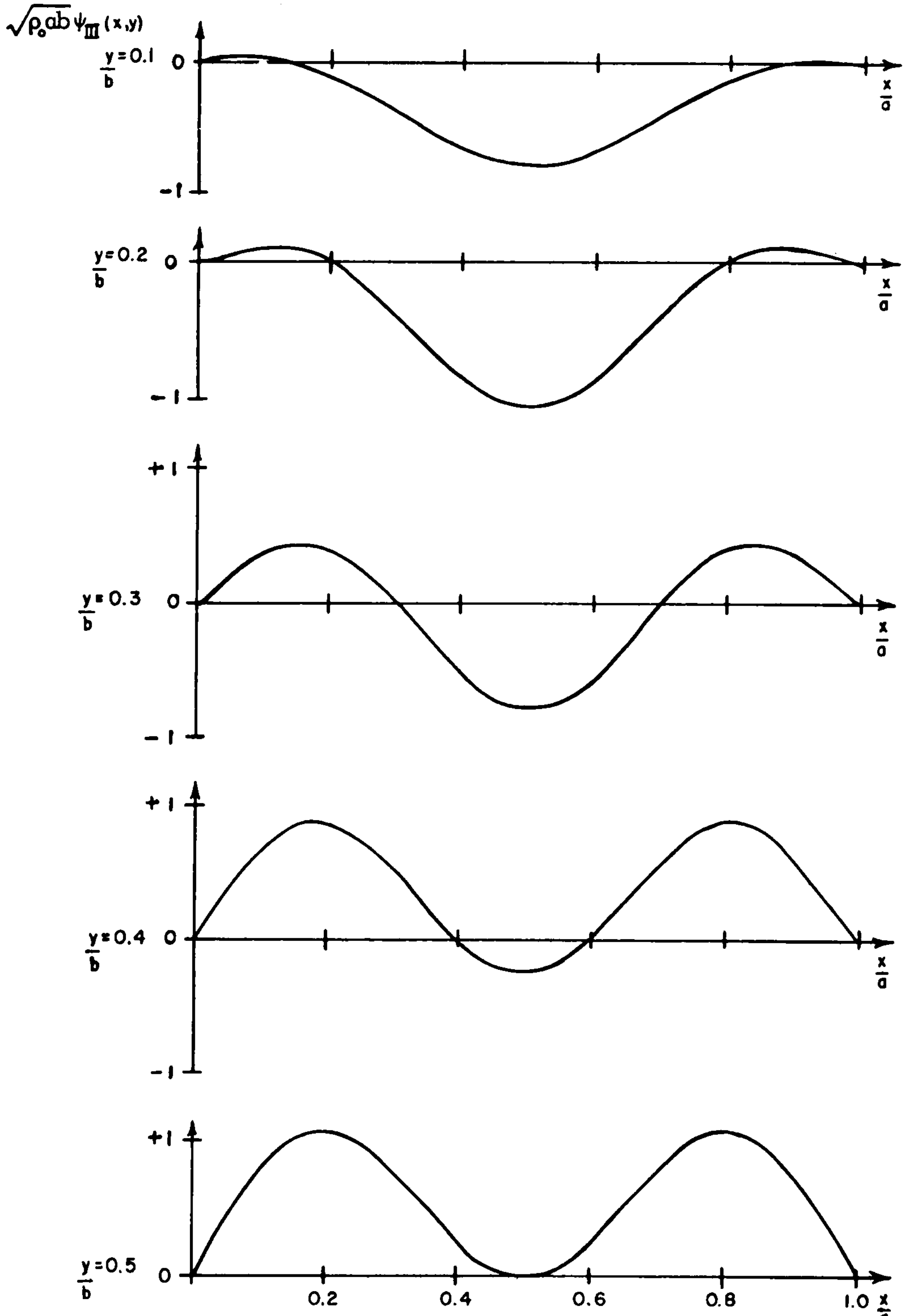


Figure 16 - Mode Shape for Simply Supported Plate with Central Mass, $\psi_{III}(x, y)$

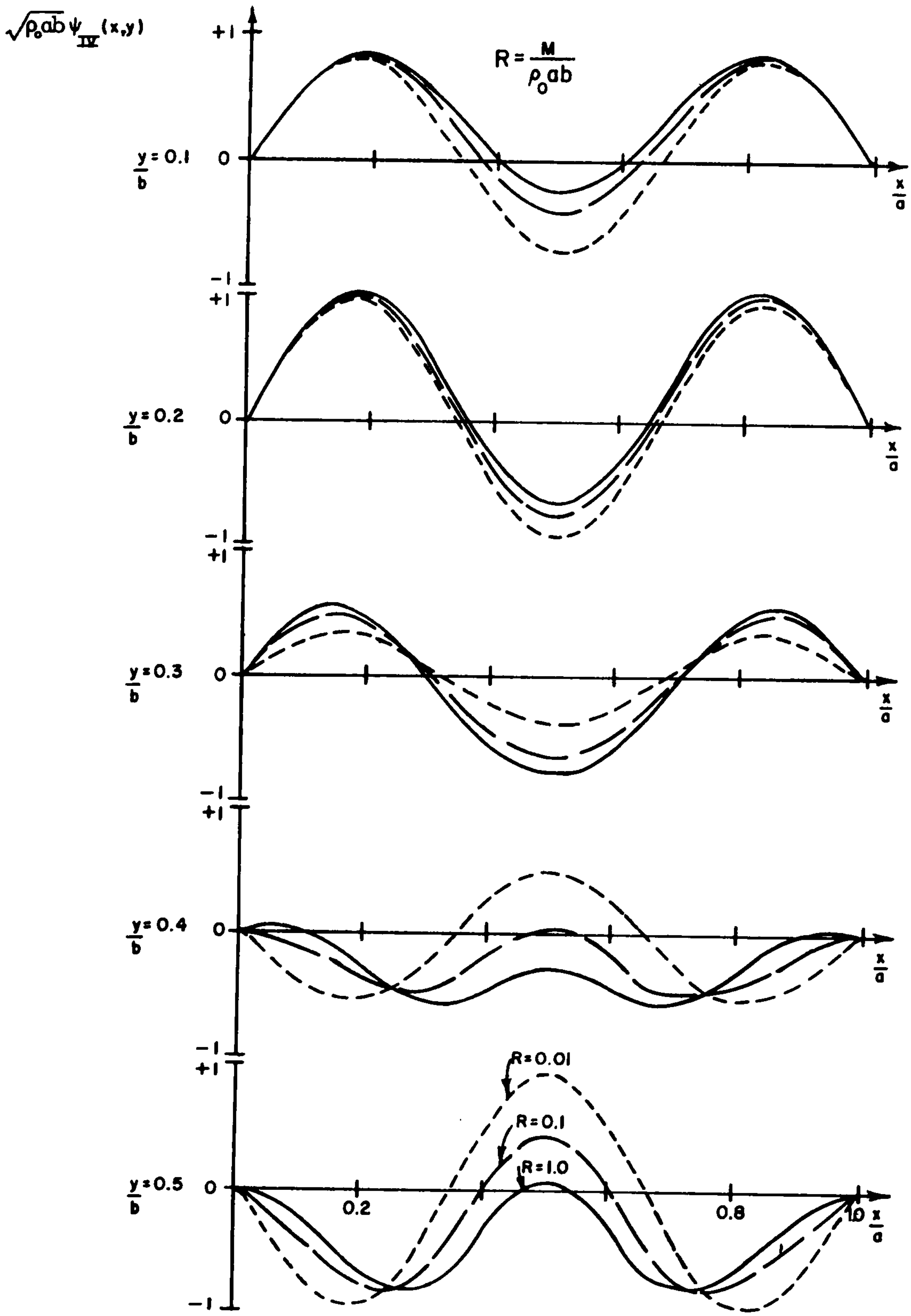


Figure 17 - Mode Shape for Simply Supported Plate with Central Mass, $\psi_{IV}(x,y)$

In this section frequencies and mode shapes for the simply supported plate with a central mass have been derived. These converge to acceptable accuracy with a reasonable size eigenvalue problem and the numerical calculations were performed in a straightforward way on the IBM 1130 computer.

The next section will consider a problem with more complex eigenfunctions to illustrate the generality of the present method.

Clamped-Supported Plate with Central Mass

This section will consider the problem of a plate supporting a central mass having two opposite edges simply supported and two opposite edges clamped. The associated eigenvalue problem without the mass is a special case of the more general Levy solutions. For plate problems with two opposite edges simply supported and any combination of boundary conditions on the other two edges the frequency equations and eigenfunctions have been listed in references [38, 51, and 56].

The plate with the central mass is shown in Figure 18. For antisymmetrical modes the mass will lie on a nodal line and the mass will not affect the response. For symmetry about the X axis the frequency equation [38], is

$$\omega_{ij} = \frac{\alpha_{ij}^2 + \beta_{ij}^2}{2b^2}\sqrt{\frac{D}{\rho_0}} \tag{5.52}$$

where

$$\alpha_{ij}^2 = \beta_{ij}^2 + 2\left(\frac{i\pi b}{a}\right)^2 \tag{5.53}$$

and α_{ij} and β_{ij} satisfy

$$\beta_{ij}\tan\frac{\beta_{ij}}{2} + \alpha_{ij}\tanh\frac{\alpha_{ij}}{2} = 0. \tag{5.54}$$

The normalized eigenfunctions for modes symmetrical about the X axis are

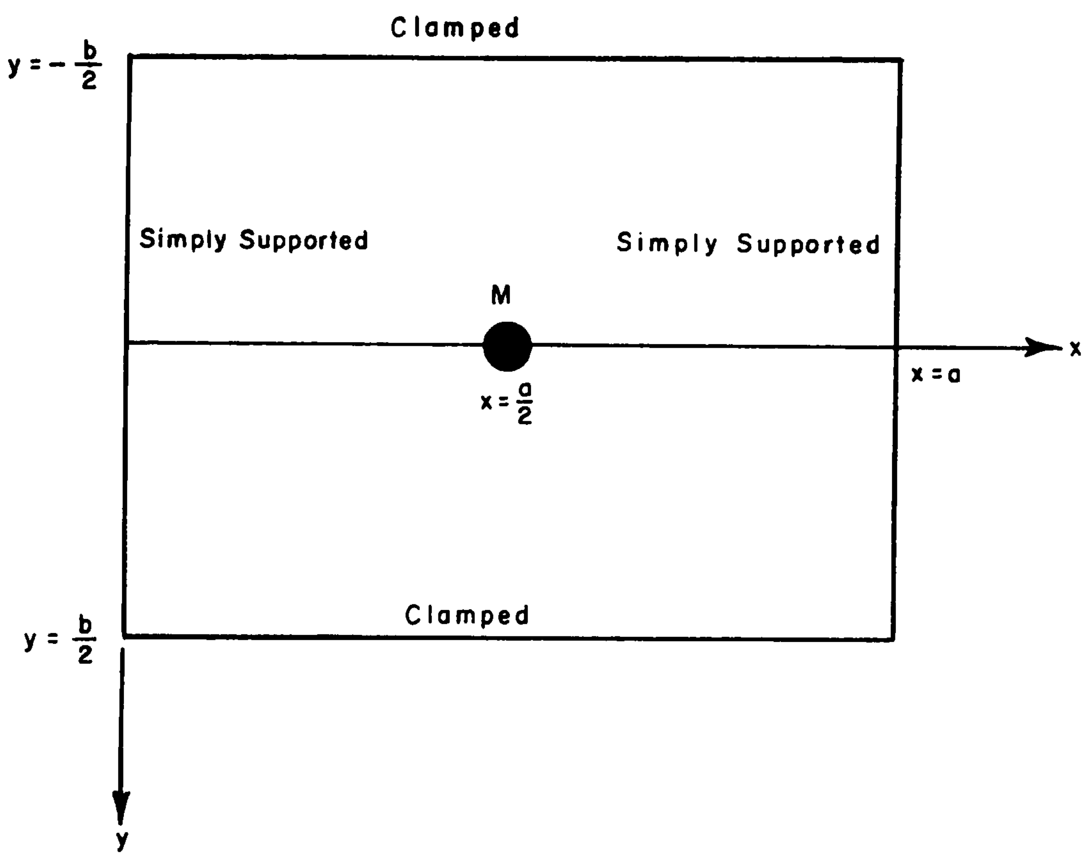


Figure 18 - Simply Supported-Clamped Plate with Central Mass

$$\phi_{ij}(x,y) = \frac{2}{\sqrt{ab}} \frac{1}{m_{ij}} \left(\cosh \frac{\alpha_{ij}}{b} y - \gamma_{ij} \cos \frac{\beta_{ij}}{b} y \right) \sin \frac{i\pi x}{a} \tag{5.55}$$

where

$$m_{ij} = \left[\left(1 + \frac{\sinh \alpha_{ij}}{\alpha_{ij}} \right) + \gamma_{ij}^2 \left(1 + \frac{\sin \beta_{ij}}{\beta_{ij}} \right) \right]^{\frac{1}{2}} \tag{5.56}$$

and

$$\gamma_{ij} = \frac{\cosh \frac{\alpha_{ij}}{2}}{\cos \frac{\beta_{ij}}{2}} .$$

A square plate was selected for numerical calculations and the first 15 roots of (5.54) were found for symmetrical vibrations. These values were then arranged, in ascending order, and the corresponding values of the eigenfunctions for (5.50) were calculated using (5.55) and (5.56). Then using the eigenvalues and the mass matrix found from (5.50) as input, the general eigenvalue problem was solved for successively larger arrays.

The results of these computations showed that the rate of convergence was comparable to that for the simply supported plate. For example, an increase in the matrix size from 9 X 9 to 12 X 12 produces a change of 0.16 percent in the first mode frequency, and 0.60 percent in the fourth mode.

The variation of the first four natural frequencies with the mass ratio is given in Table VI. A non-dimensional plot of these results is shown in Figure 19. The corresponding mode shapes are shown in Figures 20 through 23.

This section concludes the treatment of continuous media with non-uniform mass distributions. In the next chapter the vibration of continuous media with time-dependent boundary conditions will be considered.

TABLE VI

Eigenvalues for Simply Supported-Clamped Plate

with Central Mass

Mass Ratio $R = \frac{M}{\rho_0 ab}$	Eigenvalues Mode I λ_{I}	Mode II λ_{II}	Mode III λ_{III}	Mode IV λ_{IV}
0	28.95	102.2	129.1	199.8
0.01	28.27	99.85	126.8	195.9
0.02	27.63	97.53	125.1	192.7
0.05	25.90	91.46	122.1	186.4
0.1	23.55	84.66	120.1	181.6
0.2	20.22	77.80	118.8	177.9
0.5	15.02	71.37	117.9	175.1
1	11.34	68.66	117.6	174.1
2	8.305	67.19	117.4	173.6
5	5.370	66.27	117.3	173.3
10	3.826	65.95	117.3	173.2

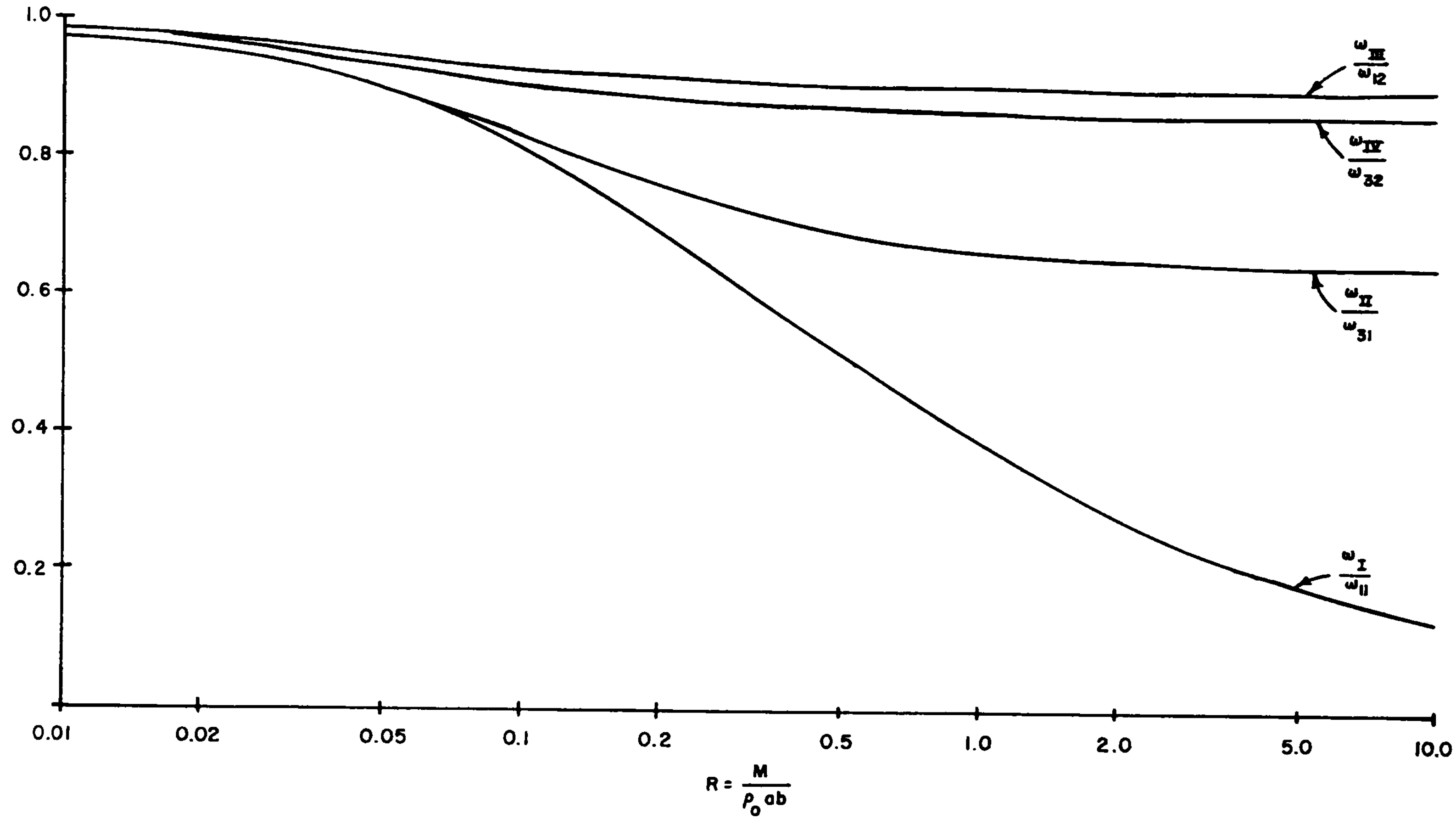


Figure 19 - Ratio of Frequency against Mass Ratio for Square Simply Supported

Clamped Plate with Central Mass

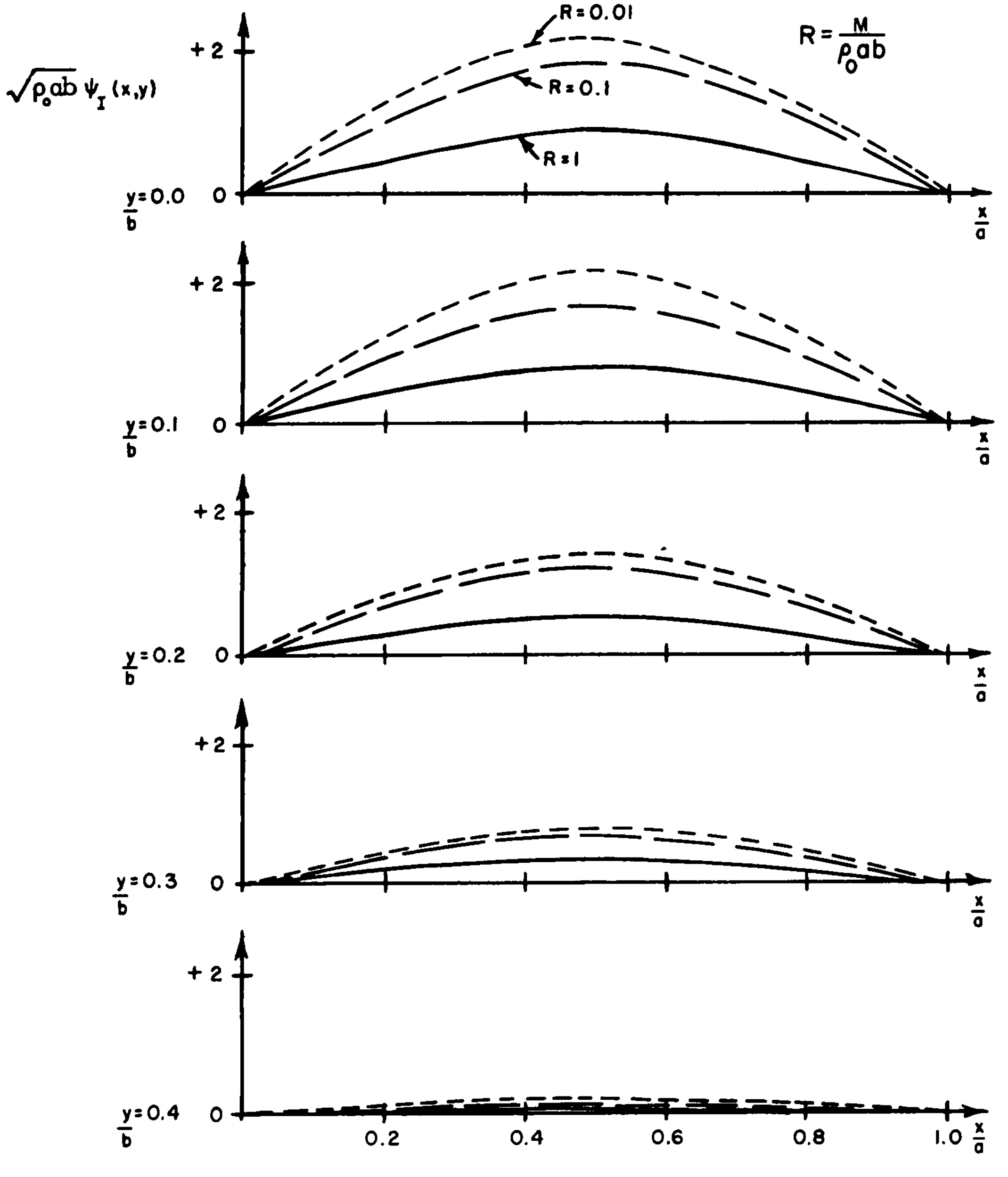


Figure 20 - Mode Shape for Square Simply Supported Clamped Plate with Central Mass, $\psi_I(x,y)$

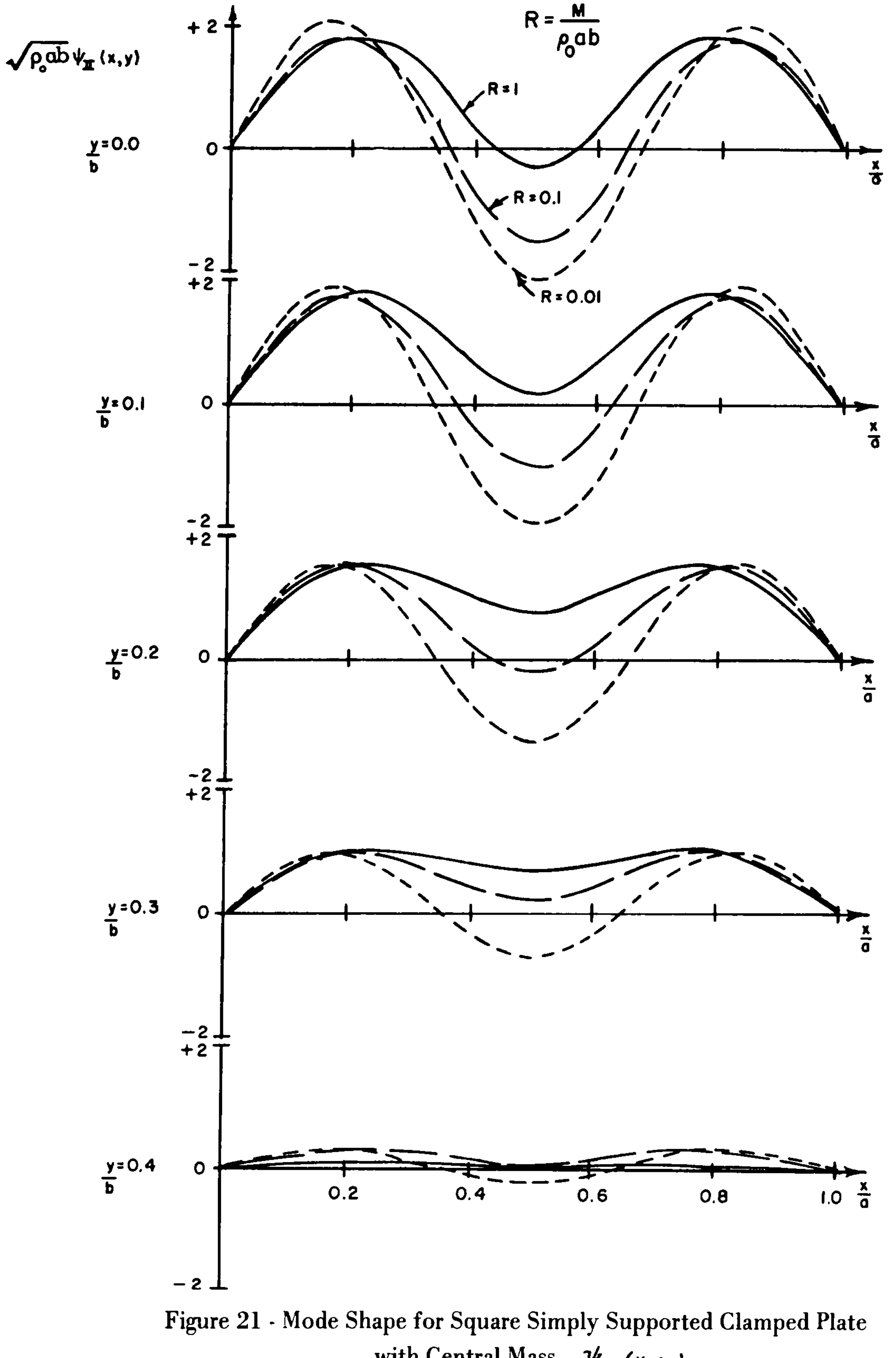


Figure 21 - Mode Shape for Square Simply Supported Clamped Plate with Central Mass, $\psi_{II}(x,y)$

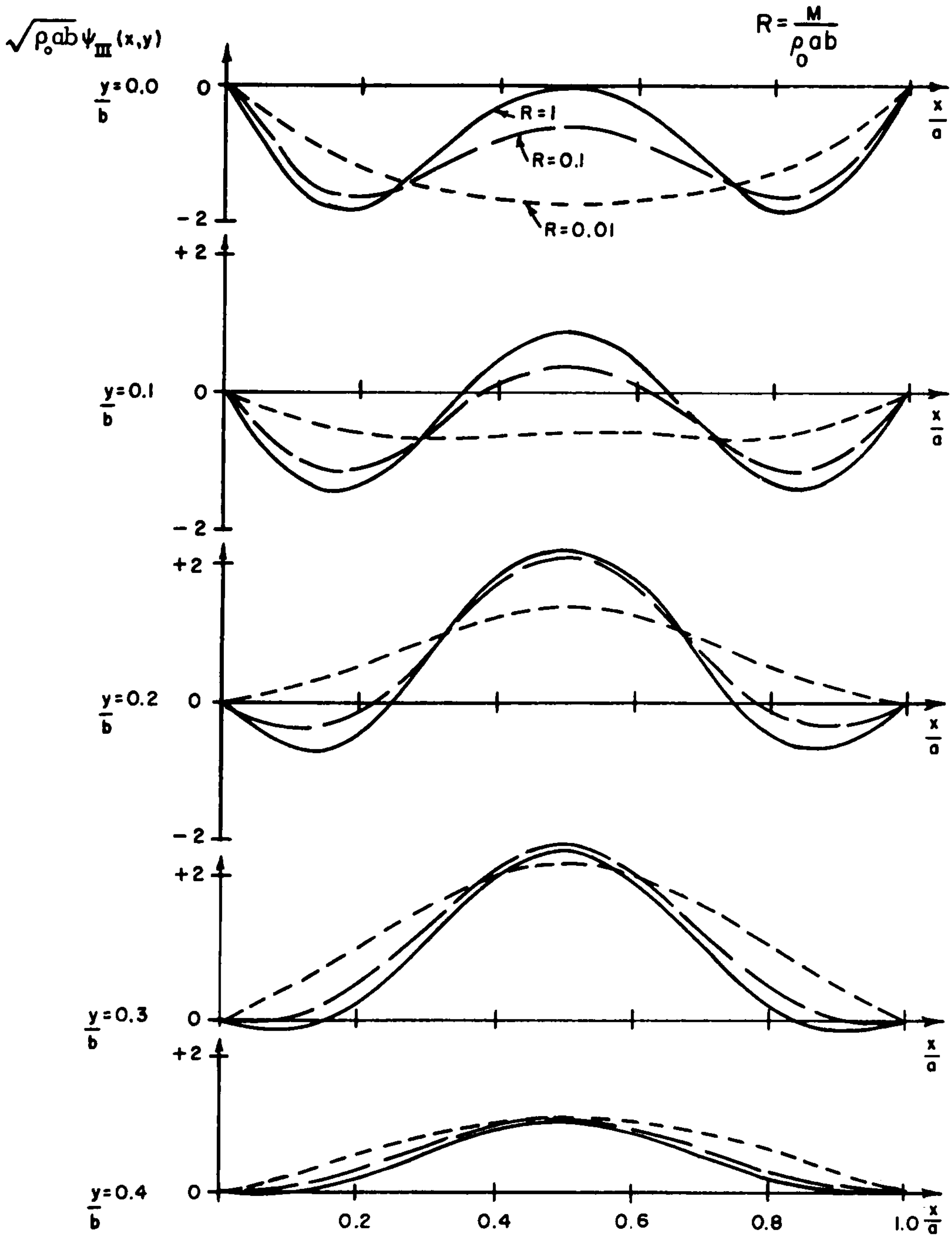


Figure 22 - Mode Shape for Square Simply Supported Clamped Plate with Central Mass, $\psi_{III}(x, y)$

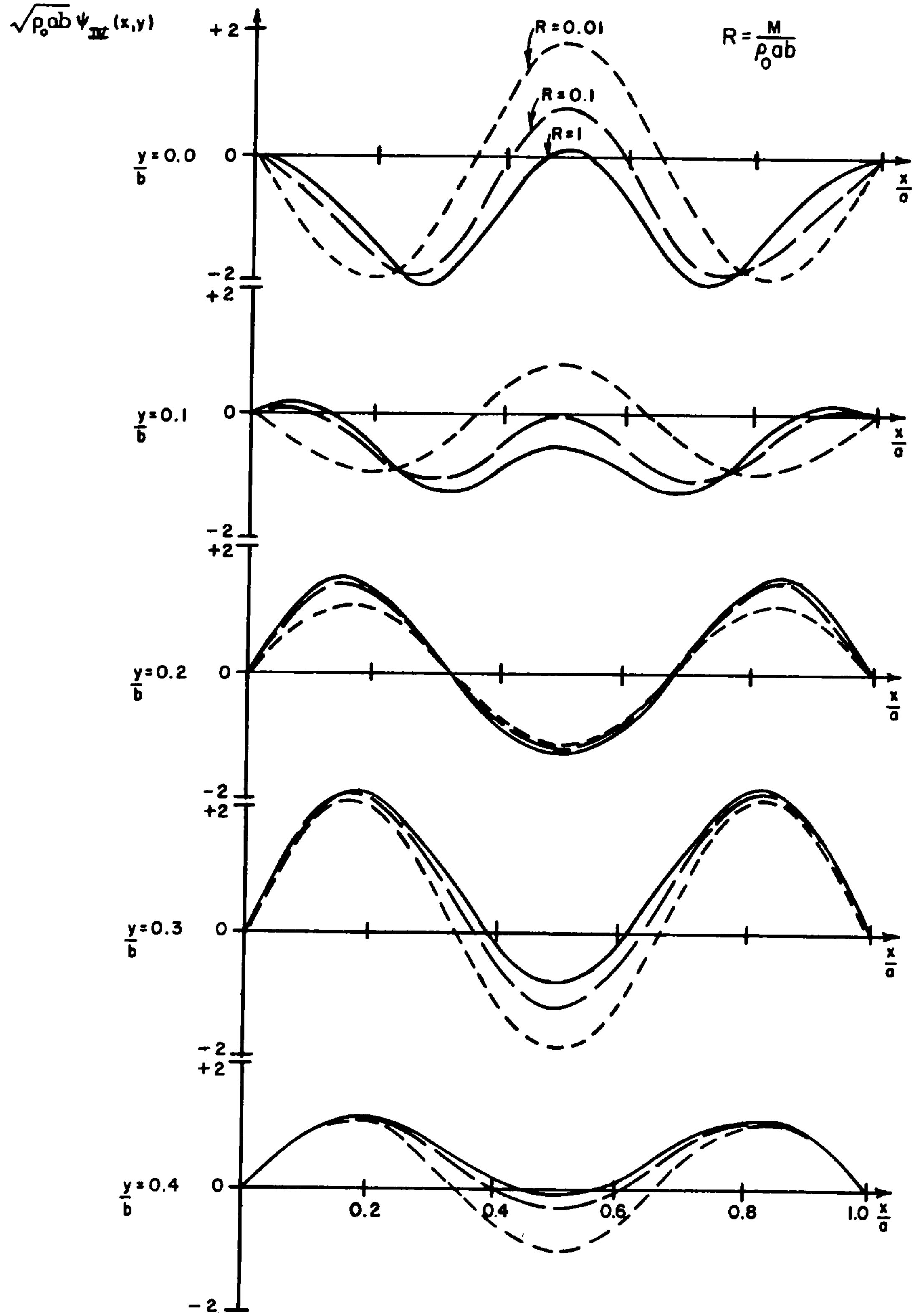


Figure 23 - Mode Shape for Square Simply Supported Clamped Plate with Central Mass, $\psi_{IV}(x,y)$

VI. VIBRATIONS OF CONTINUOUS MEDIA WITH TIME-DEPENDENT BOUNDARY CONDITIONS

This chapter will consider the application of the eigentransform technique to the solution of problems involving time-dependent boundary conditions. Two classes of problems are considered: (1) one dimensional continuous media such as the rod, string or Euler beam, and (2) isotropic and orthotropic flat plates.

One Dimensional Media with Time-Dependent Boundary Conditions

The vibration problems of the uniform rod, string, and Euler beam are governed by partial differential equations which are of even order in the derivatives with respect to the spatial variable. This is true, for example, for the Euler beam supported on an elastic foundation with constant in-plane forces. It is convenient then to consider this class of problems together by suitably generalizing a differential operation in even spatial derivatives. The solutions given by Cinelli [12] will then be given as a special case of the present solution. The problem will be treated by giving first a general development of the theory which is followed by applications to the vibrations of a rod and a beam.

General Development

The partial differential equation is taken in the form

$$L[W(x,t)] + M_o \frac{\partial^2 W(x,t)}{\partial t^2} = q(x,t) \tag{6.01}$$

where

L is a linear differential operator in even derivatives of order $2p$ having the form

$$L = a_o + a_2 \frac{\partial^2}{\partial x^2} + a_4 \frac{\partial^4}{\partial x^4} + \cdots + a_{2p} \frac{\partial^{2p}}{\partial x^{2p}} = \sum_{n=0,2}^{2p} a_n \frac{\partial^n}{\partial x^n}. \tag{6.02}$$

M_o is constant, and

$g(x,t)$ is the forcing function.

In taking the eigentransform, the integral

$$\int_0^{\ell} L[w(x,t)]\,\phi_r(x)\,dx \tag{6.03}$$

must be evaluated where $\phi_r(x)$ are the eigenfunctions corresponding to (6.01) with homogeneous boundary conditions. The eigenfunctions thus satisfy

$$L[\phi_r(x)] = \lambda_r\,\phi_r(x) \tag{6.04}$$

where $\lambda_r = \omega_r^2 M_o$; ω_r are the natural frequencies of the system. In this case the eigenfunctions are orthogonal with respect to a weighting function of 1. Substitution of (6.02) into (6.03) gives

$$\int_0^{\ell} \sum_{n=0,2}^{2p} a_n \frac{\partial^n w}{\partial x^n}\,\phi_r(x)\,dx = \sum_{n=0,2}^{2p} a_n \int_0^{\ell} \frac{\partial^n w}{\partial x^n}\,\phi_r(x)\,dx$$

which after repeated integrations by parts can be expressed as

$$\int_0^{\ell} \sum_{n=0,2}^{2p} a_n \frac{\partial^n w}{\partial x^n}\,\phi_r(x)\,dx = \sum_{n=2,4}^{2p} a_n F_n(t) + \int_0^{\ell} \sum_{n=0,2}^{2p} a_n \frac{d^n \phi_r}{dx^n}\,w(x,t)\,dx \tag{6.05}$$

where

$$F_n(t) = \Big[\overset{(0)}{\phi_r}(x)\,\overset{(n-1)}{w}(x,t) - \overset{(1)}{\phi_r}(x)\,\overset{(n-2)}{w}(x,t) + \cdots$$

$$\cdots + \overset{(n-2)}{\phi_r}(x)\,\overset{(1)}{w}(x,t) - \overset{(n-1)}{\phi_r}(x)\,\overset{(0)}{w}(x,t) \Big]_0^{\ell} \tag{6.06}$$

in which numbers in parenthesis denote differentiations with respect to x. The last term on the right hand side of (6.05) can then be simplified using (6.02) and (6.04) to yield the result

$$\int_0^{\ell} L[w(x,t)]\,\phi_r(x)\,dx = \sum_{n=2,4}^{2p} a_n F_n(t) + \lambda_r \int_0^{\ell} w(x,t)\,\phi_r(x)\,dx\,;$$

hence

$$\int_0^{\ell} L[w(x,t)]\phi_r(x)dx = \sum_{n=2,4}^{2p} a_n F_n(t) + \lambda_r \overline{w}_r(t). \tag{6.07}$$

Thus by means of the eigentransform, the original equation is reduced to the ordinary differential equation

$$\sum_{n=2,4}^{2p} a_n F_n(t) + \lambda_r \overline{w}_r(t) + M_o \frac{d^2\overline{w}_r}{dt^2} = \overline{q}_r(t)$$

or

$$\frac{d^2\overline{w}_r}{dt^2} + \omega_r^2 \overline{w}_r = \frac{1}{M_o}\left[\overline{q}_r(t) - \sum_{n=2,4}^{2p} a_n F_n(t)\right]. \tag{6.08}$$

Let

$$N_r(t) = \frac{1}{M_o}\left[\overline{q}_r(t) - \sum_{n=2,4}^{2p} a_n F_n(t)\right] \tag{6.09}$$

then (6.08) reduces to

$$\frac{d^2\overline{w}_r}{dt^2} + \omega_r^2 \overline{w}_r = N_r(t)$$

which has the general solution

$$\overline{w}_r(t) = \overline{w}_r(0)\cos\omega_r t + \frac{\dot{\overline{w}}_r(0)}{\omega_r}\sin\omega_r t + \frac{1}{\omega_r}\int_0^t N_r(\tau)\sin\omega_r(t-\tau)\,d\tau \tag{6.10}$$

where

$$\overline{w}_r(0) = \int_0^{\ell} w(x,0)\,\phi_r(x)\,dx \tag{6.11}$$

and

$$\dot{\overline{w}}_r(0) = \int_0^{\ell} \frac{\partial w}{\partial t}(x,0)\,\phi_r(x)dx\,.$$

The final solution to the problem can be written as

$$w(x,t) = \sum_{r=1}^{\infty} \overline{w}_r(t)\phi_r(x) \tag{6.12}$$

where

$$\int_0^{\ell} \phi_r(x)\,\phi_s(x)\,dx = \delta_{rs}.$$

Applications

1. Longitudinal Vibration of a Uniform Rod

As the first example the longitudinal vibration of a uniform rod clamped at the end $x = 0$ with a time-dependent force, $P(t)$, acting at $x = \ell$ is considered. The classical method of treating this problem is to compute the kinetic and potential energies in terms of a generalized time coordinate and apply Lagrange's equations of motion. It has also been solved by the Laplace transform [57] or the Mindlin-Goodman technique [52].

For the rod shown in Figure 24 the governing equation on the longitudinal displacement $u(x,t)$ is

$$EA\,\frac{\partial^2 u}{\partial x^2} - m\,\frac{\partial^2 u}{\partial t^2} = 0 \tag{6.13}$$

where m is the mass per-unit-length. The boundary conditions require

$$u(0,t) = 0$$

$$EA\,\frac{\partial u}{\partial x}(\ell,t) = P(t). \tag{6.14}$$

In this case

$$L = EA\,\frac{\partial^2 u}{\partial x^2} \qquad M_0 = -m \qquad q(x,t) = 0$$

and hence from (6.02),

$$a_0 = 0$$

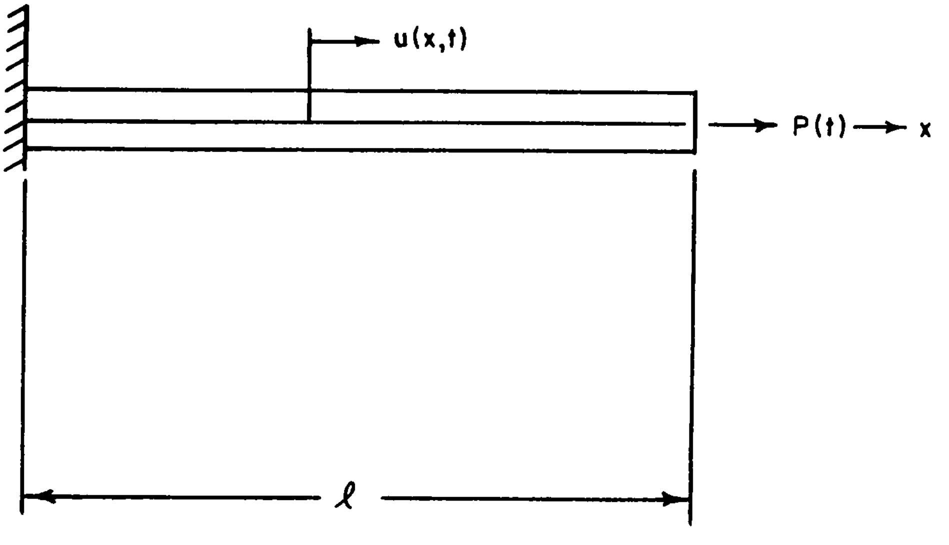


Figure 24 - Longitudinal Vibration of a Rod with Axial Force

and

$$a_2 = EA.$$

With these results the generalized forcing function is found from (6.09) to be

$$N_r(t) = \frac{EA}{m} F_2(t) \tag{6.15}$$

where from (6.06)

$$F_2(t) = \left[\phi_r(x) \frac{\partial u}{\partial x}(x,t) - \frac{d\phi_r}{dx}(x)\, u(x,t) \right]_0^{\ell} . \tag{6.16}$$

The eigenvalue problem associated with the free vibration of the rod consists of solving

$$\phi_r'' = -\tilde{\lambda}_r^2 \phi_r$$

subject to the homogeneous boundary conditions

$$\phi_r(0) = 0$$
$$\phi_r'(\ell) = 0$$

where $\tilde{\lambda}_r^2 = \dfrac{m\omega^2}{EA}$, for convenience. This problem has as its solution the eigenvalues

$$\tilde{\lambda}_r = (2r-1)\frac{\pi}{2\ell} \tag{6.17}$$

and normalized eigenfunctions

$$\phi_r(x) = \sqrt{\frac{2}{\ell}} \sin \tilde{\lambda}_r x \ . \tag{6.18}$$

Using the eigenfunctions (6.18) and the specified boundary conditions, (6.14), $F_2(t)$ is found from (6.16) to be given by

$$F_2(t) = \frac{2}{\ell} (-1)^{r-1} \frac{P(t)}{EA} \tag{6.19}$$

and hence from (6.15)

$$N_r(t) = \frac{2}{\ell} (-1)^{r-1} \frac{P(t)}{m} . \tag{6.20}$$

For homogeneous initial conditions (6.10) then gives

$$\bar{u}_r(t) = \frac{1}{m} \; \frac{2}{l} \; \frac{(-1)^{r-1}}{\omega_r} \int_0^t P(\tau) \sin \omega_r (t-\tau)\, d\tau \tag{6.21}$$

and with the inversion series, the final result is obtained:

$$u(x,t) = \frac{2}{ml} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{\omega_r} \sin (2r-1) \frac{\pi}{2} \frac{x}{l} \int_0^t P(\tau) \sin \omega_r (t-\tau)\, d\tau . \tag{6.22}$$

2. Transverse Vibration of a Beam

As the second illustration, a clamped-supported beam with the end displacement specified as a function of time will be considered. This problem is solved by Mindlin and Goodman in their paper [48]. Following their conventions, the coordinates are shown in Figure 25.

The governing equation is

$$\frac{\partial^4 W}{\partial x^4} + \frac{m}{EI} \frac{\partial^2 W}{\partial t^2} = 0 \tag{6.23}$$

and the boundary conditions are

$$\begin{aligned} &W(0,t) = f(t) \qquad && W(l,t) = 0 \\ &EI\, W''(0,t) = 0 \qquad && W'(l,t) = 0 \end{aligned} \tag{6.24}$$

where E is the modulus of elasticity,

I is the moment of inertia, and

m is the mass per-unit-length. The beam is assumed to start from rest so all of the initial conditions are zero. In this case

$$L = \frac{\partial^4}{\partial x^4} \qquad M_0 = \frac{m}{EI}$$

and hence

$$a_0 = a_2 = 0 \qquad a_4 = 1.$$

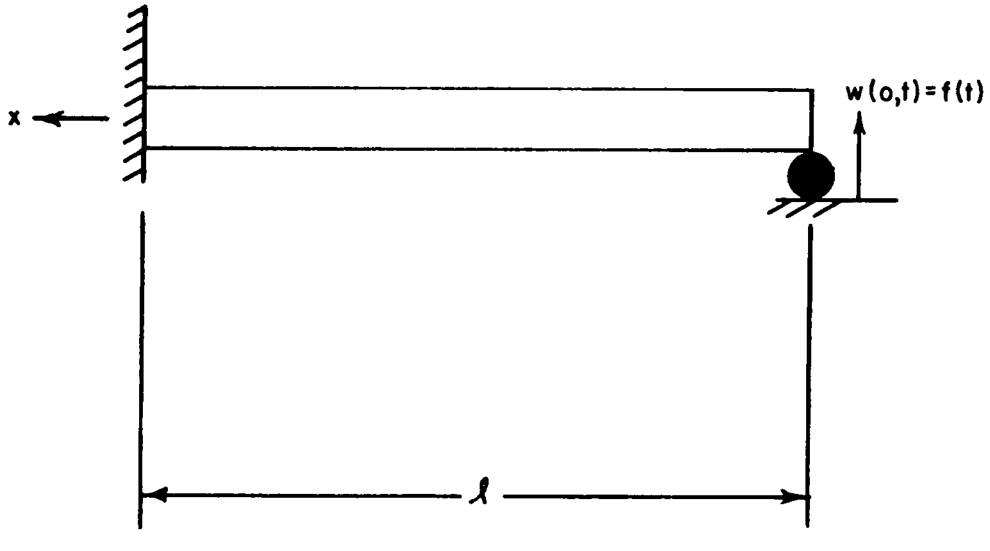


Figure 25 - Clamped-Supported Beam with Prescribed Support Motion

Referring to (6.09),

$$N_r(t) = -\frac{EI}{m} F_4(t) \tag{6.25}$$

where from (6.06),

$$F_4(t) = \Big[\phi_r(x) W'''(x,t) - \phi_r'(x) W''(x,t) + \phi_r''(x) W'(x,t) - \phi_r'''(x) W(x,t)\Big]_0^{\ell} . \tag{6.26}$$

The associated eigenvalue problem consists of solving

$$\phi_r^{IV}(x) = \tilde{\lambda}_r^4 \phi_r(x) \tag{6.27}$$

subject to the homogeneous boundary conditions

$$\begin{array}{ll} \phi_r(0) = 0 & \phi_r(\ell) = 0 \\ \phi_r''(0) = 0 & \phi_r'(\ell) = 0 \end{array} \tag{6.28}$$

where, for convenience, $\tilde{\lambda}_r^4 = \frac{\omega_r^2 m}{EI}$.

The solution to this problem yields the normalized eigenfunctions

$$\phi_r(x) = \sqrt{\frac{2}{\ell}} \; \frac{\sinh m_r \sin m_r \frac{x}{\ell} - \sin m_r \sinh \frac{m_r x}{\ell}}{(\sinh^2 m_r - \sin^2 m_r)^{1/2}} \tag{6.29}$$

where m_r are the roots of

$$\tan m_r = \tanh m_r \tag{6.30}$$

and $m_r = \tilde{\lambda}_r \ell$.

From the boundary conditions given in (6.24) and (6.28), $F_4(t)$ is found from (6.26) to reduce to

$$F_4(t) = -\sqrt{\frac{2}{\ell}} \left(\frac{m_r}{\ell}\right)^3 \frac{\sinh m_r + \sin m_r}{(\sinh^2 m_r - \sin^2 m_r)^{1/2}} f(t) . \tag{6.31}$$

Substitution of this in (6.25) then gives $N_r(t)$, and this result in (6.10) gives

$$\bar{w}_r(t) = \frac{EI}{m\,\omega_r} \sqrt{\frac{2}{\ell}} \left(\frac{m_r}{\ell}\right)^3 \frac{\sinh m_r + \sin m_r}{(\sinh^2 m_r - \sin^2 m_r)^{1/2}} \int_0^t f(\tau) \sin \omega_r (t-\tau)\, d\tau . \tag{6.32}$$

Finally, from (6.12) the desired solution is

$$w(x,t) = \sum_{r=1}^{\infty} \frac{2\omega_r W_r(x)}{\sinh m_r - \sin m_r} \int_0^t f(\tau) \sin \omega_r (t-\tau) d\tau, \tag{6.33}$$

where

$$W_r(x) = \sinh m_r \sin \frac{m_r x}{\ell} - \sin m_r \sinh \frac{m_r x}{\ell}, \tag{6.34}$$

and the natural frequencies are given by

$$\omega_r = \frac{m_r^2}{\ell^2} \sqrt{\frac{EI}{m}} \quad . \tag{6.35}$$

The expansion given in (6.33) represents the solution within the region $0 < x < \ell$. The displacement at $x = \ell$ is the specified displacement.

Isotropic Plates with Time-Dependent Boundary Conditions

General Development

An isotropic flat plate of arbitrary boundary of uniform thickness and mass per-unit-area ρ_o will be considered. The plate is subjected to a transverse load $q(P,t)$, per-unit-area which can be expressed as a product of space and time functions. The governing equation on the transverse displacement w for general forced vibrations is

$$D \nabla^4 w(P,t) + \rho_o \frac{\partial^2 w(P,t)}{\partial t^2} = q(P,t) \tag{6.36}$$

where $q(P,t)$ is the forcing function per-unit-plate-area. For generality, the problem will be expressed in curvilinear coordinates, normal and tangential to the boundary of the plate. The conventions are shown in Figure 26 where the radius of curvature of the boundary is R .

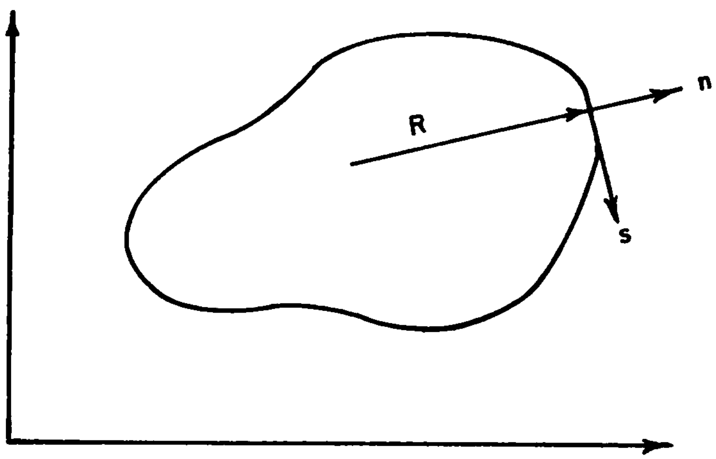


Figure 26 - Coordinates for Arbitrary Isotropic Plate

The stress resultants in these coordinates can be expressed [52] as

$$M^{n} = -D\nabla^{2}W + (1-\nu)D\left(\frac{1}{R}\frac{\partial W}{\partial n} + \frac{\partial^{2}W}{\partial s^{2}}\right) \tag{6.37a}$$

$$M^{ns} = (1-\nu)D\left(\frac{\partial^{2}W}{\partial n\,\partial s} - \frac{1}{R}\frac{\partial W}{\partial s}\right) \tag{6.37b}$$

$$Q^{n} = -D\,\frac{\partial}{\partial n}(\nabla^{2}W) \tag{6.37c}$$

where

$$\nabla^{2} = \frac{\partial^{2}}{\partial n^{2}} + \frac{1}{R}\frac{\partial}{\partial n} + \frac{\partial^{2}}{\partial s^{2}} \tag{6.37d}$$

and the Kirchhoff shear is

$$V^{n} = Q^{n} - \frac{\partial M^{ns}}{\partial s}. \tag{6.37e}$$

The positive conventions for these resultants are shown in Figure 27.

The method of separation of variables leads to the eigenvalue problem associated with

$$\nabla^{4}\phi = \lambda^{4}\phi \tag{6.38}$$

and homogeneous boundary conditions. This problem leads to the eigenvalues λ_{ij} and eigenfunctions ϕ_{ij} . The eigenvalues λ_{ij} are related to the natural frequencies ω_{ij} by

$$\omega_{ij} = \lambda_{ij}^{2}\sqrt{\frac{D}{\rho_{o}}}\;. \tag{6.39}$$

The eigenfunctions are orthogonal with respect to a weighting function of 1 and are normalized so that

$$\int_{A} \phi_{ij}(P)\,\phi_{mn}(P)\,dA = \delta_{im}\,\delta_{jn} \tag{6.40}$$

where A is the plane area of the plate.

In taking the transform of (6.36) the crucial step is to transform the first term. For a non-isotropic plate the integration must be performed by parts. In this case an

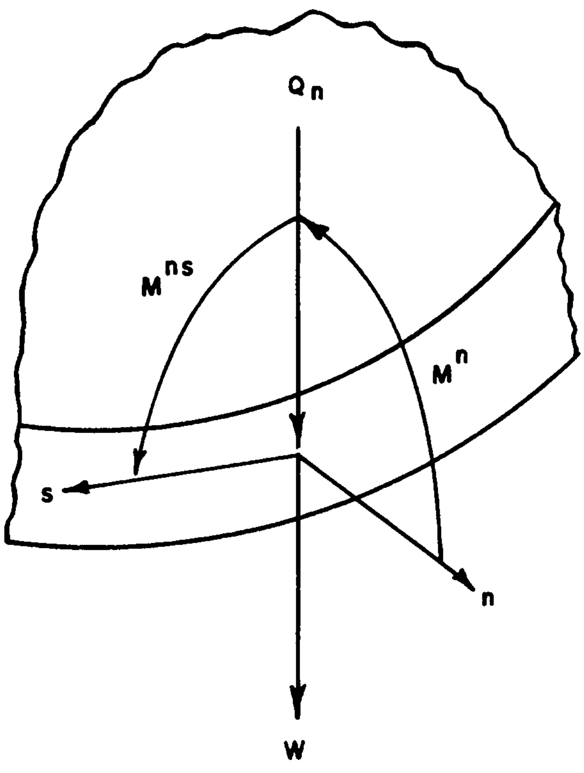


Figure 27 - Positive Stress Resultants for
Arbitrary Isotropic Plate

integral theorem of vector analysis can be used. The eigentransform of this term is defined by

$$\int_A \nabla^4 W(P,t)\,\phi_{ij}(P)\,dA \;.$$

Using some identities of vector analysis, the integrand may be rewritten so that

$$\int_A \nabla^4 W(P,t)\,\phi_{ij}(P)dA = \int_A W(P,t)\,\nabla^4\phi_{ij}(P)dA$$

$$+\int_A\left\{\vec{\nabla}\cdot\left[\phi_{ij}\,\vec{\nabla}(\nabla^2 W) - (\nabla^2 W)\,\vec{\nabla}\phi_{ij} - W\vec{\nabla}(\nabla^2\phi_{ij}) + \nabla^2\phi_{ij}(\vec{\nabla}W)\right]\right\}dA\,. \tag{6.41}$$

The last integral on the right-hand side can then be transformed into a line integral around the boundary by using the divergence theorem. Hence, (6.41) reduces to

$$\int_A \nabla^4 W(P,t)\phi_{ij}(P)dA = \int_A W(P,t)\nabla^4\phi_{ij}(P)\,dA$$

$$+\oint\left[\phi_{ij}\,\frac{\partial(\nabla^2 W)}{\partial n} - W\frac{\partial}{\partial n}(\nabla^2\phi_{ij}) + \nabla^2\phi_{ij}\,\frac{\partial W}{\partial n} - \nabla^2 W\,\frac{\partial\phi_{ij}}{\partial n}\right]ds. \tag{6.42}$$

As a consequence of (6.38), the first term on the right-hand side of the last equation reduces to

$$\int_A W(P,t)\nabla^4\phi_{ij}(P)dA = \lambda_{ij}^4\int_A W(P,t)\phi_{ij}(P)dA = \lambda_{ij}^4\,\overline{W}_{ij}(t). \tag{6.43}$$

Equation (6.37) can be rewritten in the form

$$\nabla^2 W = -\frac{M^n}{D} + (1-\nu)\left(\frac{1}{R}\frac{\partial W}{\partial n} + \frac{\partial^2 W}{\partial s^2}\right)$$

and

$$\frac{\partial(\nabla^2 W)}{\partial n} = -\frac{V^n}{D} - (1-\nu)\frac{\partial}{\partial s}\left(\frac{\partial^2 W}{\partial n\,\partial s} - \frac{1}{R}\frac{\partial W}{\partial s}\right),$$

and can be used to rewrite the last term of (6.42). After some rearrangement, these simplifications yield

$$\int_A \nabla^4 w\, \phi_{ij}\, dA = \lambda_{ij}^4\, \overline{w}_{ij}(t)$$
$$+ \frac{1}{D}\oint \left(w V_{ij}^n - \phi_{ij} V^n + \frac{\partial \phi_{ij}}{\partial n} M^n - \frac{\partial w}{\partial n} M_{ij}^n \right) ds$$
$$+ (1-\nu)\oint \left[\frac{\partial w}{\partial n}\frac{\partial^2 \phi_{ij}}{\partial s^2} - \phi_{ij}\left(\frac{\partial^2 w}{\partial n \partial s} - \frac{1}{R}\frac{\partial w}{\partial s}\right) - \frac{\partial \phi_{ij}}{\partial n}\frac{\partial^2 w}{\partial s^2} - w\frac{\partial}{\partial s}\left(\frac{\partial^2 \phi_{ij}}{\partial n \partial s} - \frac{1}{R}\frac{\partial \phi_{ij}}{\partial s}\right)\right] ds \tag{6.44}$$

where the Kirchhoff shear V_{ij}^n and moment M_{ij}^n associated with the homogeneous boundary conditions have been introduced. If the last integral is integrated by parts it can be evaluated at the end points of the path of integration, say $0 \leq s \leq \ell$. Introducing the twisting moment M_{ij}^{ns} associated with the homogeneous boundary conditions, the transform for $\nabla^4 w$ finally reduces to

$$\int_A \nabla^4 w\, \phi_{ij}\, dA = \lambda_{ij}^4\, \overline{w}_{ij} + \frac{1}{D}\oint \left(w V_{ij}^n - \phi_{ij} V^n + \frac{\partial \phi_{ij}}{\partial n} M^n - \frac{\partial w}{\partial n} M_{ij}^n \right) ds$$
$$+ \frac{1}{D}\left[w M_{ij}^{ns} - \phi_{ij} M^{ns}\right]_0^{\ell} + (1-\nu)\left[\frac{\partial \phi_{ij}}{\partial s}\frac{\partial w}{\partial n} - \frac{\partial w}{\partial s}\frac{\partial \phi_{ij}}{\partial n}\right]_0^{\ell}. \tag{6.45}$$

Then for convenience let

$$F_{ij}(t) = \frac{1}{D}\oint \left(w V_{ij}^n - \phi_{ij} V^n + \frac{\partial \phi_{ij}}{\partial n} M^n - \frac{\partial w}{\partial n} M_{ij}^n \right) ds$$
$$+ \frac{1}{D}\left[w M_{ij}^{ns} - \phi_{ij} M^{ns}\right]_0^{\ell} + (1-\nu)\left[\frac{\partial \phi_{ij}}{\partial s}\frac{\partial w}{\partial n} - \frac{\partial w}{\partial s}\frac{\partial \phi_{ij}}{\partial n}\right]_0^{\ell} \tag{6.46}$$

and hence,

$$\int_A \nabla^4 w(P,t)\, \phi_{ij}(P)\, dA = \lambda_{ij}^4\, \overline{w}_{ij}(t) + F_{ij}(t). \tag{6.47}$$

In the evaluation of the terms in (6.46) for a particular plate a distinction must be made between plates with corners and those without corners. For plates with corners, to find $F_{ij}(t)$, the line integration must be broken up into segments and the $\frac{\partial w}{\partial n}$ evaluated as the corners are approached from the interior. For plates with no corners, the terms evaluated at $s=0$ and $s=\ell$ will cancel identically as the integration is performed around a closed path.

Using (6.47), the transform of the governing plate equation (6.36) becomes

$$D\lambda_{ij}^4 \bar{w}_{ij}(t) + DF_{ij}(t) + \rho_o \frac{d^2\bar{w}_{ij}}{dt^2} = \bar{q}_{ij}(t)$$

or

$$\frac{d^2\bar{w}_{ij}}{dt^2} + \omega_{ij}^2 \bar{w}_{ij} = N_{ij}(t) \tag{6.48}$$

where

$$N_{ij}(t) = \frac{1}{\rho_o}\bar{q}_{ij}(t) - \frac{D}{\rho_o}F_{ij}(t)\,. \tag{6.49}$$

Then the solution for the transformed variable can be written as

$$\bar{w}_{ij}(t) = \bar{w}_{ij}(0)\cos\omega_{ij}t + \frac{\dot{\bar{w}}_{ij}(0)}{\omega_{ij}}\sin\omega_{ij}t + \frac{1}{\omega_{ij}}\int_0^t N_{ij}(\tau)\sin\omega_{ij}(t-\tau)d\tau \tag{6.50}$$

where

$$\bar{w}_{ij}(0) = \int_A w(P,0)\,\phi_{ij}(P)dA$$

and

$$\dot{\bar{w}}_{ij}(0) = \int \frac{\partial w(P,0)}{\partial t}\phi_{ij}(P)dA\,. \tag{6.51}$$

The final solution for the plate response can be found by substituting (6.50) into the inversion series

$$w(P,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \bar{w}_{ij}(t)\,\phi_{ij}(P)\,. \tag{6.52}$$

Applications to Rectangular Plates

As the first application of the procedure described above a rectangular plate will be considered. The plate under consideration is shown in Figure 28.

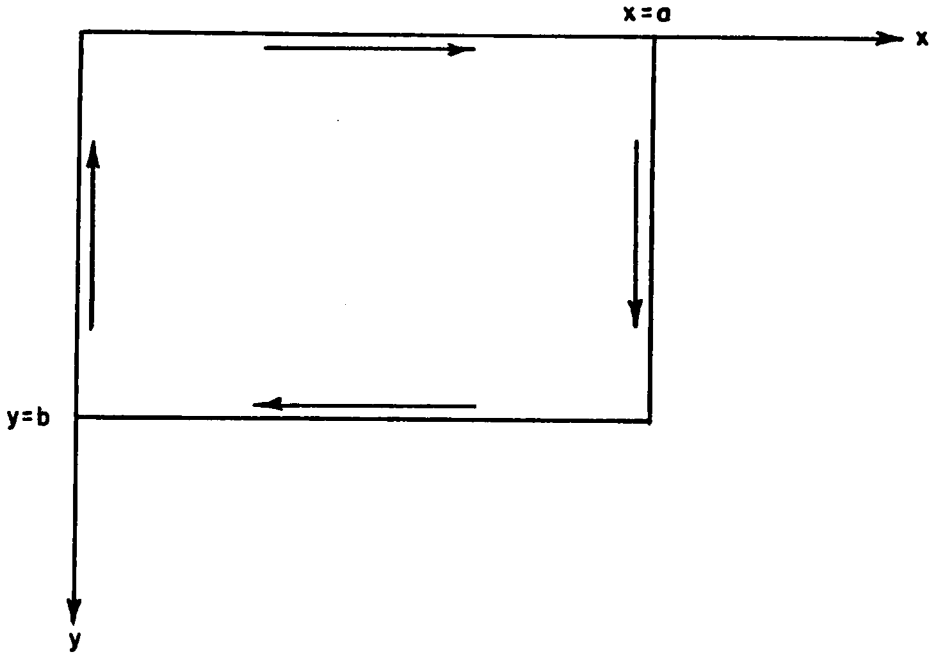


Figure 28 - Path of Integration for Rectangular Plate

The possible time-dependent boundary conditions on an edge, such as $y = 0$ include any consistent pair of

$$
\begin{array}{ll}
w(x,0,t) & M^y(x,0,t) \\
\dfrac{\partial w}{\partial y}(x,0,t) & V^y(x,0,t) .
\end{array}
$$

The solution for the rectangular plate can be expressed in terms of these edge values by evaluating (6.46) for the given plate. This involves evaluating line integrals along each edge and evaluating the normal derivatives at each corner from two directions. As a result of these operations, the last term in (6.46) vanishes but contributions are received from the term containing the twisting moments. Thus,

$$
\begin{aligned}
F_{ij}(t) = \frac{1}{D}\int_0^a \Big\{ & \Big[w V_{ij}^y - \phi_{ij} V^y - \frac{\partial \phi_{ij}}{\partial y} M^y + \frac{\partial w}{\partial y} M_{ij}^y \Big]_{y=0} \\
& + \Big[w V_{ij}^y - \phi_{ij} V^y + \frac{\partial \phi_{ij}}{\partial y} M^y - \frac{\partial w}{\partial y} M_{ij}^y \Big]_{y=b} \Big\} dx \\
+ \frac{1}{D}\int_0^b \Big\{ & \Big[w V_{ij}^x - \phi_{ij} V^x - \frac{\partial \phi_{ij}}{\partial x} M^x + \frac{\partial w}{\partial x} M_{ij}^x \Big]_{x=0} \\
& + \Big[w V_{ij}^x - \phi_{ij} V^x + \frac{\partial \phi_{ij}}{\partial x} M^x - \frac{\partial w}{\partial x} M_{ij}^x \Big]_{x=a} \Big\} dy \\
& + \frac{1}{D}\Big[w R_{ij} - \phi_{ij} R \Big]_{0,0}^{a,b} ,
\end{aligned}
\tag{6.53}
$$

where the corner reactions have been introduced by the definition

$$
R = 2(1-\nu) D \frac{\partial^2 w}{\partial x \partial y} .
$$

If the customary sign convention for the Kirchhoff shear is used the result given in (6.53) can be written more compactly. Thus let

$$V^x = -D\left[\frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial x \partial y^2}\right]$$
$$V^y = -D\left[\frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial y \partial x^2}\right] \tag{6.54}$$

then (6.53) can be written as

$$F_{ij}(t) = \frac{1}{D}\int_0^a \left\{\left[W V_{ij}^y - \phi_{ij} V^y + \frac{\partial \phi_{ij}}{\partial y} M^y - \frac{\partial W}{\partial y} M_{ij}^y\right]_{y=0}^{y=b}\right\} dx$$
$$+ \frac{1}{D}\int_0^b \left\{\left[W V_{ij}^x - \phi_{ij} V^x + \frac{\partial \phi_{ij}}{\partial x} M^x - \frac{\partial W}{\partial x} M_{ij}^x\right]_{x=0}^{x=a}\right\} dy + \frac{1}{D}\left[W R_{ij} - \phi_{ij} R\right]_{0,0}^{a,b}. \tag{6.55}$$

Numerous applications of the last result can be made. One of the most important includes the determination of the plate response due to known boundary motions. Another is the determination of frequencies and mode shapes for other boundary conditions by appropriately selecting the boundary inputs.

1. Response of Plates with Prescribed Input Motions

Equation (6.55) can be used to calculate the plate response due to a variety of input motions. These may include time-dependent displacements, slopes, moments, and shears. Any spatial distribution of these may be specified subject to the evaluation of the integrals which appear. Any type of plate boundary condition may be treated assuming that the corresponding eigenvalue problem can be solved.

To illustrate the procedure consider the problem of a rectangular plate with two edges simply supported and the two opposite edges clamped. The two simply supported edges are subjected, for simplicity, to a uniform time-dependent moment $M(t)$. The plate under consideration is shown in Figure 29, where the coordinate system has been located to take advantage of symmetry.

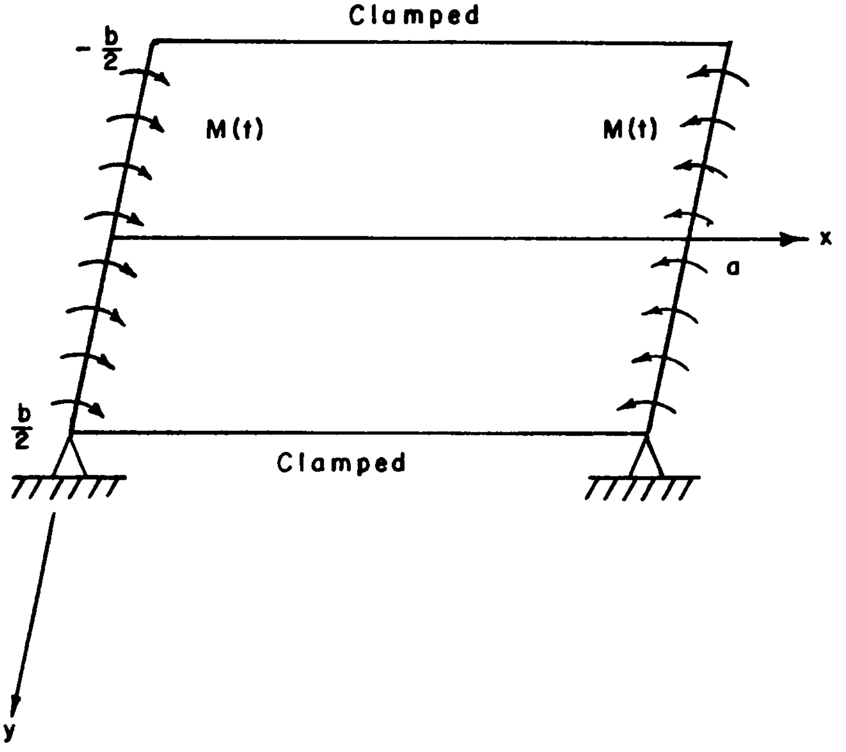


Figure 29 - Clamped-Supported Plate with Edge Moments

The normalized eigenfunctions for this problem were given in Chapter V, but they are repeated here, for convenience,

$$\phi_{ij}(x,y) = \frac{2}{\sqrt{ab}} \frac{1}{m_{ij}} \left(\cosh \alpha_{ij} \frac{y}{b} - \gamma_{ij} \cos \beta_{ij} \frac{y}{b} \right) \sin \frac{i\pi x}{a} \tag{6.56a}$$

where

$$m_{ij} = \left[\left(1 - \frac{\sinh \alpha_{ij}}{\alpha_{ij}}\right) + \gamma_{ij} \left(1 + \frac{\sin \beta_{ij}}{\beta_{ij}}\right) \right]^{1/2}, \tag{6.56b}$$

and

$$\gamma_{ij} = \frac{\cosh \frac{\alpha_{ij}}{2}}{\cos \frac{\beta_{ij}}{2}} . \tag{6.56c}$$

The natural frequencies are given by

$$\omega_{ij} = \frac{\alpha_{ij}^2 + \beta_{ij}^2}{2b^2} \sqrt{\frac{D}{\rho_o}} \tag{6.57}$$

where α_{ij} and β_{ij} are related by

$$\alpha_{ij}^2 - \beta_{ij}^2 = 2\left(\frac{i\pi b}{a}\right)^2 \tag{6.58}$$

and satisfy

$$\beta_{ij} \tan \frac{\beta_{ij}}{2} - \alpha_{ij} \tanh \frac{\alpha_{ij}}{2} = 0 . \tag{6.59}$$

For the present boundary conditions $F_{ij}(t)$ reduces to

$$F_{ij}(t) = \int_{-\frac{b}{2}}^{\frac{b}{2}} \left\{ \left[-\frac{\partial \phi_{ij}}{\partial x} M^x \right]_{x=0} + \left[\frac{\partial \phi_{ij}}{\partial x} M^x \right]_{x=a} \right\} dy$$

and after substitution of (6.56a) and integration yields

$$F_{ij}(t) = -\frac{8\pi}{D} \sqrt{\frac{b}{a^3}} \frac{i}{m_{ij}} \left(\frac{\sinh \frac{\alpha_{ij}}{2}}{\alpha_{ij}} - \gamma_{ij} \frac{\sin \frac{\beta_{ij}}{2}}{\beta_{ij}} \right) M(t). \quad (i=1,3,5,\ldots) \tag{6.60}$$

Next substitution in (6.49) and then in (6.50) gives

$$\overline{W}_{ij}(t) = \frac{8\pi}{\rho_o}\sqrt{\frac{b}{a^3}}\,\frac{i}{m_{ij}}\left(\frac{\sinh\frac{\alpha_{ij}}{2}}{\alpha_{ij}} - \gamma_{ij}\frac{\sin\frac{\beta_{ij}}{2}}{\beta_{ij}}\right)\frac{1}{\omega_{ij}}\int_0^t M(\tau)\sin\omega_{ij}(t-\tau)\,d\tau\,, \tag{6.61}$$

where, for convenience, all initial conditions are assumed to be zero. When substituted in the inversion series, this result gives the final solution

$$W(x,y,t) = \frac{16}{\rho_o a^2}\sum_{i=1,3,}^{\infty}\sum_{j=1,2}^{\infty}\frac{i\Gamma_{ij}}{\omega_{ij}}\left(\cosh\alpha_{ij}\frac{y}{b} - \gamma_{ij}\cos\beta_{ij}\frac{y}{b}\right)\sin\frac{i\pi x}{a}\int_0^t M(\tau)\sin\omega_{ij}(t-\tau)\,d\tau, \tag{6.62}$$

where

$$\Gamma_{ij} = \frac{\dfrac{\sinh\frac{\alpha_{ij}}{2}}{\alpha_{ij}} - \gamma_{ij}\dfrac{\sin\frac{\beta_{ij}}{2}}{\beta_{ij}}}{1 + \dfrac{\sinh\alpha_{ij}}{\alpha_{ij}} + \gamma_{ij}\left(1 + \dfrac{\sin\beta_{ij}}{\beta_{ij}}\right)}\,. \tag{6.63}$$

2. Determination of Mode Shapes and Frequencies for Combinations of Simple and Clamped Supports

Closed form solutions for frequencies and mode shapes of rectangular plates are tractable for the Levy type of supports. For combinations of boundary conditions other than with two opposite edges simply supported, closed form solutions are not yet known. A variety of approximate methods have been used to treat these problems. For plates with clamped and simply supported edges the frequencies and mode shapes may be found [57] by applying harmonically varying edge moments of unknown magnitude to simply supported edges and then adjusting these moments to satisfy the required clamped boundary conditions. The frequency equations which result are infinite series but can be solved with a computer.

Equation (6.55) can be used to solve problems of this type. Consider the rectangular plate of Figure 30 which is simply supported on all edges. The plate is subjected to time-dependent edge moments of unknown frequency p. These have the form

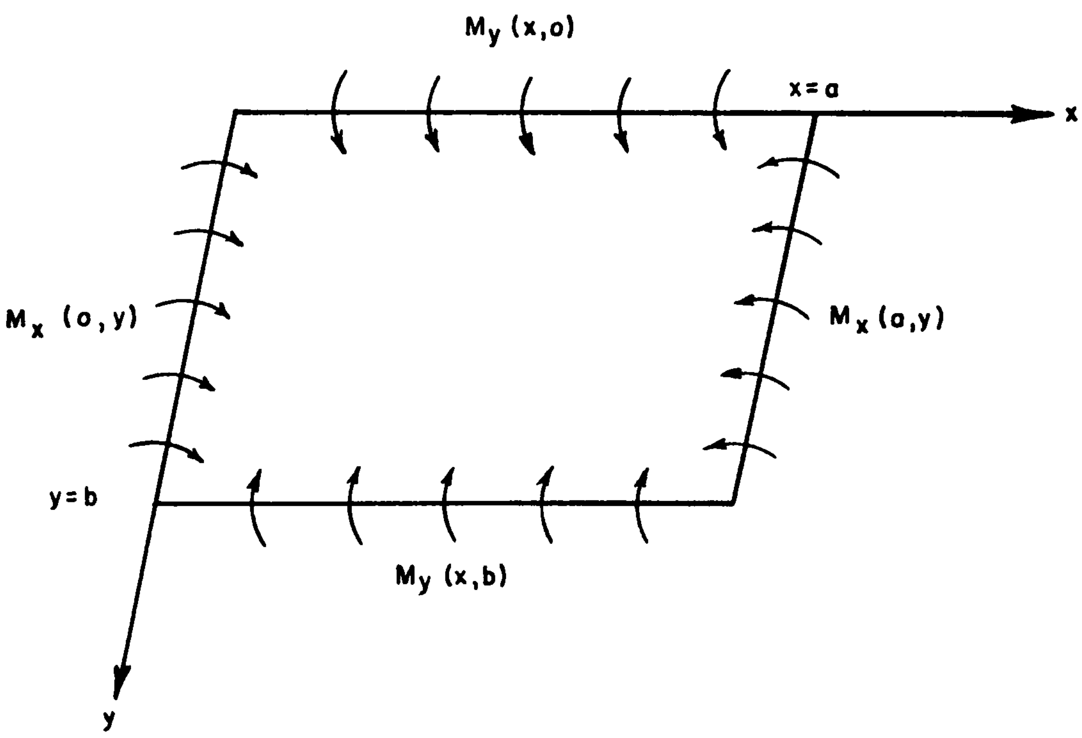


Figure 30 - Simply Supported Plate with Edge Moments

$$M^x(0,y,t) = M_x(0,y)\sin pt$$
$$M^x(a,y,t) = M_x(a,y)\sin pt$$
$$M^y(x,0,t) = M_y(x,0)\sin pt$$
$$M^y(x,b,t) = M_y(x,b)\sin pt$$

so that (6.55) reduces to

$$F_{ij}(t) = \sin pt \int_0^a \left[M_y(x,y)\frac{\partial \phi_{ij}}{\partial y}\right]_0^b dx + \sin pt \int_0^b \left[M_x(x,y)\frac{\partial \phi_{ij}}{\partial x}\right]_0^a dy. \tag{6.64}$$

For the simply supported plate the normalized mode shapes are

$$\phi_{ij} = \frac{2}{\sqrt{ab}} \sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b} \tag{6.65}$$

and the natural frequencies are

$$\omega_{ij} = \pi^2\left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2\right]\sqrt{\frac{D}{\rho_0}} \quad . \tag{6.66}$$

If equation (6.65) is substituted in (6.64) then

$$F_{ij}(t) = \frac{2}{\sqrt{ab}} \frac{j\pi}{b} \sin pt \int_0^a \left[(-1)^j M_y(x,b) - M_y(x,0)\right] \sin \frac{i\pi x}{a}\, dx$$

$$+ \frac{2}{\sqrt{ab}} \frac{i\pi}{a} \sin pt \int_0^b \left[(-1)^i M_x(a,y) - M_x(0,y)\right] \sin \frac{j\pi y}{b}\, dy. \tag{6.67}$$

The unknown edge moments are now expanded into series of the form

$$M_y(x,b) = \sum_r A_r \sin \frac{r\pi x}{a}$$
$$M_y(x,0) = \sum_r B_r \sin \frac{r\pi x}{a}$$
$$M_x(a,y) = \sum_s C_s \sin \frac{s\pi y}{b}$$
$$M_x(0,y) = \sum_s D_s \sin \frac{s\pi y}{b} \quad . \tag{6.68}$$

If these expansions are substituted into (6.67) and use is made of the orthogonality conditions then

$$F_{ij}(t) = \frac{\pi}{\sqrt{ab}}\left\{ j\,\frac{a}{b}\left[(-1)^j A_i - B_i\right] + i\,\frac{b}{a}\left[(-1)^i C_j - D_j\right]\right\} \sin pt. \tag{6.69}$$

The final solution for the steady-state plate response is found from (6.49), (6.50) and (6.52) to be

$$w(x,y,t) = \frac{2\pi}{\rho_0 ab} \sum_{i,j} \frac{j\frac{a}{b}\left[(-1)^{j+1}A_i + B_i\right] + i\,\frac{b}{a}\left[(-1)^{i+1}C_j - D_j\right]}{\omega_{ij}^2 - p^2} \cdot$$

$$\cdot \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \sin pt \,. \tag{6.70}$$

This last equation may be used to obtain the frequencies and mode shapes for a plate with one edge clamped, or two or more adjacent edges clamped. The latter cases are, of course, not included in the Levy solution. The equation is derived in Nowacki's book [57] using the finite sine transform which is, of course, a special case of the present method.

For the case of one edge clamped, a particularly simple solution is obtained. Let the edge $x = 0$ be required to have the clamped boundary condition; then $A_i = B_i = C_j = 0$, and (6.70) reduces to

$$w(x,y,t) = -\frac{2\pi}{\rho_0 a^2} \sum_{i,j} \frac{i\,D_j}{\omega_{ij}^2 - p^2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \sin pt \,. \tag{6.71}$$

Imposing the boundary condition that

$$\frac{\partial w}{\partial x}(0,y,t) = 0$$

gives the frequency equation

$$\sum_i \frac{i^2}{\omega_{ij}^2 - p^2} = 0 . \qquad j = 1, 2, \ldots \tag{6.72}$$

This equation was solved on the IBM 1130 computer for the roots shown in Table VII, where

$$p_{kj} = \lambda_{kj} \frac{\pi^2}{a^2}\sqrt{\frac{D}{\rho_0}} \,. \tag{6.73}$$

TABLE VII

Eigenvalues for Plate with One Edge Clamped

Plate Dimensions $\frac{a}{b}$	Eigenvalues λ_{11}	λ_{12}	λ_{21}	λ_{22}
0.5	1.762	2.402	5.299	5.963
1	2.402	5.241	5.963	8.748
2	5.241	17.124	8.748	20.459

With the plate frequencies known, the corresponding mode shapes can be calculated from (6.71). Corresponding to p_{kj},

$$\Phi_{kj} = \sum_i \frac{i\,D_j}{\omega_{ij}^2 - p_{kj}^2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \tag{6.74}$$

so that the mode shapes have the form

$$\Phi_{kj}(x,y) = D_j X_{kj}(x) \sin\frac{j\pi y}{b} \tag{6.75}$$

where

$$X_{kj}(x) = \sum_i \frac{i \sin\frac{i\pi x}{a}}{\omega_{ij}^2 - p_{kj}^2} . \tag{6.76}$$

As the second application, consider the rectangular plate with two adjacent edges clamped, say the edges $x = 0$ and $y = 0$. In this case $A_i = C_j = 0$ and (6.70) reduces to

$$w(x,y,t) = \frac{2\pi}{\rho_0 b^2} \sum_{i,j} \frac{j B_i - i\left(\frac{b}{a}\right)^2 D_j}{\omega_{ij}^2 - p^2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \sin pt . \tag{6.77}$$

Imposing the boundary conditions

$$\frac{\partial w}{\partial x}(0,y,t) = 0 \qquad\qquad \frac{\partial w}{\partial y}(x,0,t) = 0$$

gives the following two equations

$$j \sum_i \frac{i\,B_i}{\omega_{ij}^2 - p^2} + \left(\frac{b}{a}\right)^2 \sum_i \frac{i^2}{\omega_{ij}^2 - p^2} D_j = 0 \qquad j = 1,2,\dots \tag{6.78}$$

$$\sum_j \frac{j^2}{\omega_{ij}^2 - p^2} B_i + \left(\frac{b}{a}\right)^2 \sum_j \frac{ij}{\omega_{ij}^2 - p^2} D_j = 0 . \qquad i = 1,2,\dots \tag{6.79}$$

Let

$$E_i = \sum_j \frac{j^2}{\omega_{ij}^2 - p^2}$$

and

$$F_j = \sum_i \frac{i^2}{\omega_{ij}^2 - p^2}$$

then by eliminating B_i a single set of simultaneous equations is obtained for D_j. These are

$$D_j - \sum_k G_{jk} D_k = 0 \qquad j = 1, 2, \cdots \tag{6.80}$$

where

$$G_{jk} = \frac{jk}{F_j} \sum_i \frac{i^2}{(\omega_{ij}^2 - p^2)(\omega_{ik}^2 - p^2) E_i} \; . \tag{6.81}$$

Equation (6.80) is an infinite set of homogeneous simultaneous equations. For a non-trivial solution the determinant of the coefficients must vanish. The fundamental frequency of a rectangular plate was obtained from (6.80) for two plate geometries. A 3 X 3 determinant was used together with 100 terms in each of the series. These results are given in Table VIII

where

$$p_{11} = \lambda_{11} \frac{\pi^2}{a^2} \sqrt{\frac{D}{\rho_0}} \; . \tag{6.82}$$

It should be noted that even with a computer this computation was laborious. Since equation (6.81) represents a series for which each term is also a series, the results of Table VIII involved calculating 10,000 terms for each assumed value of the desired root p. The infinite series expansion for each mode are found in a manner similar to that of the previous problem.

These examples conclude the treatment of rectangular plates with time-dependent boundary conditions. In the next section applications will be made to circular plates.

TABLE VIII

Eigenvalues for Plate with

Two Adjacent Edges Clamped

Plate Dimensions $\frac{a}{b}$	Eigenvalues λ_{11}
1	2.74
2	7.23

Applications to Circular Plates

The application of (6.46) to circular plates is particularly simple, since there are no corners and the last two terms vanish. Thus

$$F_{ij}(t) = \frac{1}{D}\oint \left(w V_{ij}^{n} - \phi_{ij} V^{n} + \frac{\partial \phi_{ij}}{\partial n} M^{n} - \frac{\partial w}{\partial n} M_{ij}^{n} \right) ds. \qquad (6.83)$$

For generality this last result will be evaluated for an annular plate. Consider the plate of Figure 31. The line integral can be evaluated by the sum

$$\oint = \oint_{DA} + \int_{AB} + \oint_{BC} + \int_{CD}$$

but, since the integrand of (6.83) is continuous within the plate, the line integrals along AB and CD cancel. Then

$$F_{ij}(t) = \frac{1}{D}\int_0^{2\pi} \left(w V_{ij}^{r} - \phi_{ij} V^{r} + \frac{\partial \phi_{ij}}{\partial r} M^{r} - \frac{\partial w}{\partial r} M_{ij}^{r} \right)_{r=a} a\,d\theta$$

$$- \frac{1}{D}\int_0^{2\pi} \left(w V_{ij}^{r} - \phi_{ij} V^{r} - \frac{\partial \phi_{ij}}{\partial r} M^{r} - \frac{\partial w}{\partial r} M_{ij}^{r} \right)_{r=b} b\,d\theta.$$

If the Kirchhoff shear is now given a consistent definition throughout the plate, i.e.,

$$V^{r} = -D\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \right) - \frac{(1-\nu)D}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial^2 w}{\partial r\,\partial \theta} - \frac{1}{r}\frac{\partial w}{\partial \theta} \right) \qquad (6.84)$$

then the last equation can be written as

$$F_{ij}(t) = \frac{1}{D}\int_0^{2\pi} \left[r\left(w V_{ij}^{r} - \phi_{ij} V^{r} + \frac{\partial \phi_{ij}}{\partial r} M^{r} - \frac{\partial w}{\partial r} M_{ij}^{r} \right) \right]_{r=b}^{r=a} d\theta. \qquad (6.85)$$

Equation (6.85) can be used to solve a variety of circular plate problems with time-dependent boundary conditions. These include determining plate response due to known input motions and finding frequencies and mode shapes for other plate boundary conditions.

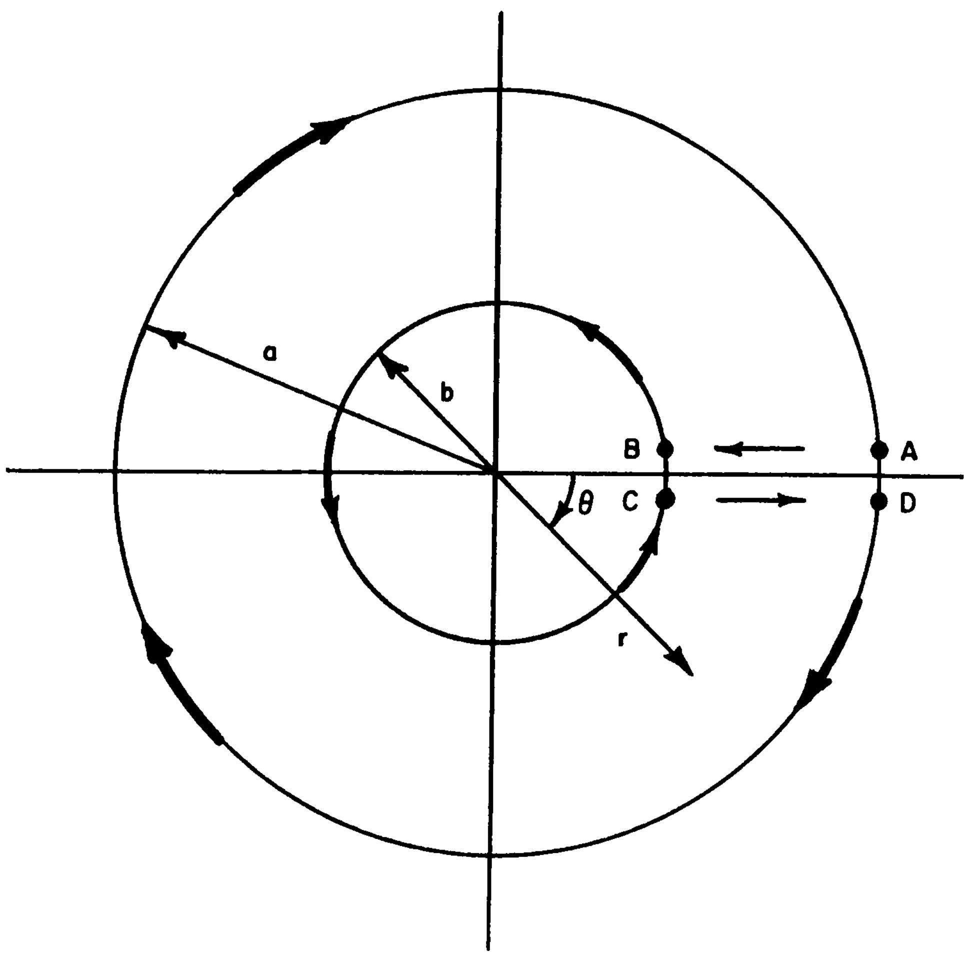


Figure 31 - Path of Integration for Circular Plate

As an example the determination of the response of a clamped circular plate with a time-dependent edge displacement is considered. For simplicity a uniform edge displacement is prescribed. The plate under consideration is the same as shown in Figure 12.

For the given boundary conditions the normalized eigenfunctions are

$$\phi_i(r) = \frac{1}{\sqrt{2\pi}\, a}\left[\frac{J_o(\lambda_i r)}{J_o(\lambda_i a)} - \frac{I_o(\lambda_i r)}{I_o(\lambda_i a)}\right] \tag{6.86}$$

where J_o is the Bessel function of order zero and I_o is a modified Bessel function of order zero. The eigenvalues λ_i are the roots of

$$J_o(\lambda_i a)\, I_1(\lambda_i a) + J_1(\lambda_i a)\, I_o(\lambda_i a) = 0, \tag{6.87}$$

and the natural frequencies in this case are given by,

$$\omega_i = \lambda_i^2 \sqrt{\frac{D}{\rho_o}}, \tag{6.88}$$

where by the assumption of the uniform boundary motion only symmetric modes are produced. This is indicated by the mode shape's independence of θ .

For a solid circular plate (6.85) reduces to

$$F_{ij}(t) = \frac{a}{D}\int_0^{2\pi} \left(w V_{ij}^r - \phi_{ij} V^r + \frac{\partial \phi_{ij}}{\partial r} M^r - \frac{\partial w}{\partial r} M_{ij}^r \right)_{r=a} d\theta . \tag{6.89}$$

For the clamped plate with a uniform edge displacement this reduces further to simply

$$F_i(t) = \frac{a}{D}\int_0^{2\pi} w(a,t) V_i^r(a)\, d\theta = \frac{2\pi a}{D} w(a,t)\, V_i^r(a) .$$

Substituting for the Kirchhoff shear using (6.84) and (6.86) gives

$$F_i(t) = -\frac{2\sqrt{2\pi}\,\lambda_i^3\, J_1(\lambda_i a)}{J_o(\lambda_i a)}\, w(a,t) . \tag{6.90}$$

Next substituting in (6.49) and (6.50) the transformed displacements is given by

$$\bar{w}_i(t) = 2\sqrt{2\pi}\,\frac{J_1(\lambda_i a)}{J_0(\lambda_i a)}\,\frac{\omega_i}{\lambda_i}\int_0^t W(a,\tau)\sin\omega_i(t-\tau)\,d\tau \tag{6.91}$$

where all initial conditions are taken as zero. The plate response is then obtained from the inversion series as

$$w(r,t) = \frac{2}{a}\sum_i^{\infty}\frac{\omega_i}{\lambda_i}\,\frac{J_1(\lambda_i a)}{J_0(\lambda_i a)}\left[\frac{J_0(\lambda_i r)}{J_0(\lambda_i a)} - \frac{I_0(\lambda_i r)}{I_0(\lambda_i a)}\right]\int_0^t W(a,\tau)\sin\omega_i(t-\tau)\,d\tau . \tag{6.92}$$

Equation (6.85) can also be used to determine natural frequencies and mode shapes for annular plates with various edge conditions. The procedure is similar to the one followed in the previous section for rectangular plates. It will not be pursued further here since for circular plates the eigenvalue problem can be solved in closed form for all combinations of edge conditions.

This section concludes the development and applications which will be given for isotropic plates. The next section of this chapter will be devoted to orthotropic plates.

Rectangular Orthotropic Plates with Time-Dependent Boundary Conditions

General Development

A rectangular orthotropic plate of uniform thickness having a mass per-unit-area ρ_o will be considered. The plate is subjected to time-dependent boundary conditions and is loaded by a transverse load per-unit-area $q(x,y,t)$. The governing equation on the transverse deflection W [56] is

$$D_x\frac{\partial^4 W}{\partial x^4} + 2H\frac{\partial^4 W}{\partial x^2\partial y^2} + D_y\frac{\partial^4 W}{\partial y^4} + \rho_o\frac{\partial^2 W}{\partial t^2} = q(x,y,t) \tag{6.93}$$

and the stress resultants are related to the deflection w by

$$M_x = -\left(D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2} \right)$$

$$M_y = -\left(D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x^2} \right)$$

$$M_{xy} = -M_{yx} = 2 D_{xy} \frac{\partial^2 w}{\partial x \partial y}$$

$$Q_x = -\left(D_x \frac{\partial^3 w}{\partial x^3} + H \frac{\partial^3 w}{\partial x \partial y^2} \right)$$

$$Q_y = -\left(D_y \frac{\partial^3 w}{\partial y^3} + H \frac{\partial^3 w}{\partial y \partial x^2} \right)$$

$$V_x = Q_x - \frac{\partial M_{xy}}{\partial y} = -\left[D_x \frac{\partial^3 w}{\partial x^3} + (D_1 + 4D_{xy}) \frac{\partial^3 w}{\partial x \partial y^2} \right]$$

$$V_y = Q_y + \frac{\partial M_{yx}}{\partial x} = -\left[D_y \frac{\partial^3 w}{\partial y^3} + (D_1 + 4D_{xy}) \frac{\partial^3 w}{\partial y \partial x^2} \right] \tag{6.94}$$

where $H = D_1 + 4 D_{xy}$. The stress resultants are shown acting in their positive directions on an element in Figure 32. The plate considered is shown in Figure 33.

The eigenvalue problem associated with (6.93) consists of solving

$$D_x \frac{\partial^4 \phi}{\partial x^4} + 2H \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 \phi}{\partial y^4} = \lambda^4 \phi \tag{6.95}$$

subject to homogeneous boundary conditions. This problem leads to a set of eigenvalues λ_{ij} and eigenfunctions ϕ_{ij}. The eigenfunctions ϕ_{ij} are orthogonal (see [56]) and they can be normalized so that

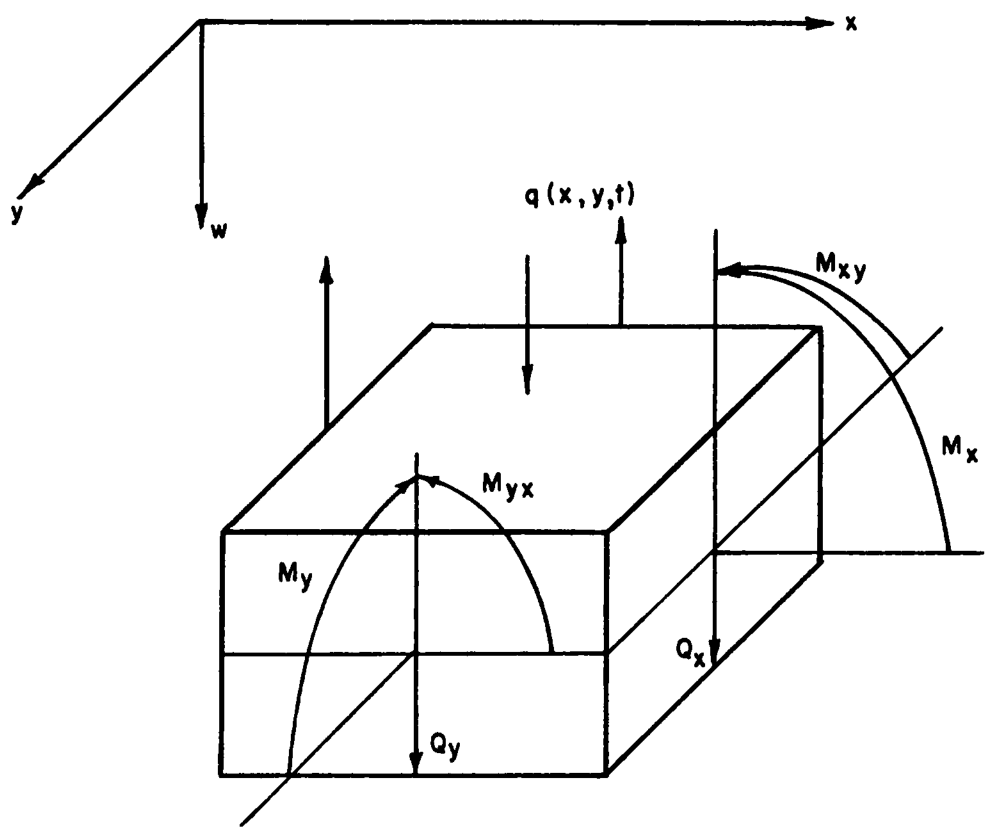


Figure 32 - Stress Resultants for Orthotropic Plate

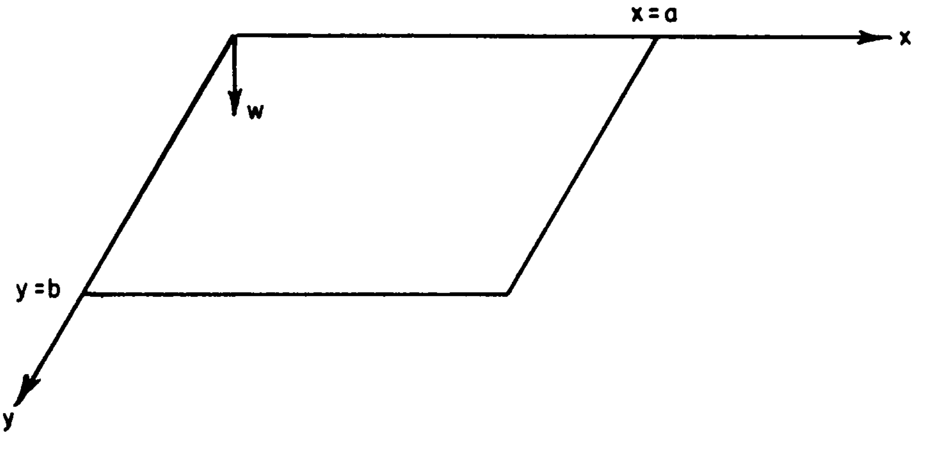


Figure 33 - Rectangular Orthotropic Plate

$$\int_0^a\int_0^b \phi_{ij}\,\phi_{mn}\,dx\,dy = \delta_{im}\,\delta_{jn}\,. \qquad (6.96)$$

The transform of the governing equation is accomplished by multiplying (6.93) by the eigenfunction ϕ_{ij} and integrating over the plate area. This gives

$$\int_0^a\int_0^b \left(D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} \right) \phi_{ij}\,dx\,dy + \rho_0 \frac{d^2 \bar{w}_{ij}}{dt^2} = \bar{q}_{ij}(t) \qquad (6.97)$$

where the definition of the transform has been used for the last two terms. In this case the transform of the differential operator in the space variables is accomplished by integration by parts. After the initial integrations are performed the term is given by

$$\int_0^a\int_0^b \left(D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} \right) \phi_{ij}\,dx\,dy =$$

$$\int_0^a\int_0^b \left(D_x \frac{\partial^4 \phi_{ij}}{\partial x^4} + 2H \frac{\partial^4 \phi_{ij}}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 \phi_{ij}}{\partial y^4} \right) w\,dx\,dy$$

$$+ \int_0^b \left\{ D_x \left[\phi_{ij} \frac{\partial^3 w}{\partial x^3} - \frac{\partial \phi_{ij}}{\partial x} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi_{ij}}{\partial x^2} \frac{\partial w}{\partial x} - \frac{\partial^3 \phi_{ij}}{\partial x^3} w \right]_0^a \right.$$

$$\left. + 2H \left[\phi_{ij} \frac{\partial^3 w}{\partial x \partial y^2} - \frac{\partial \phi_{ij}}{\partial x} \frac{\partial^2 w}{\partial y^2} \right]_0^a \right\} dy$$

$$+ \int_0^a \left\{ D_y \left[\phi_{ij} \frac{\partial^3 w}{\partial y^3} - \frac{\partial \phi_{ij}}{\partial y} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 \phi_{ij}}{\partial y^2} \frac{\partial w}{\partial y} - \frac{\partial^3 \phi_{ij}}{\partial y^3} w \right]_0^b \right.$$

$$\left. + 2H \left[\frac{\partial w}{\partial y} \frac{\partial^2 \phi_{ij}}{\partial x^2} - w \frac{\partial^3 \phi_{ij}}{\partial y \partial x^2} \right]_0^b \right\} dx\,. \qquad (6.98)$$

This result can be considerably simplified by using (6.95) and introducing the moment and shear stress resultants. After these substitutions, there results

$$\int_0^a\int_0^b \left(D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4}\right)\phi_{ij}\,dx\,dy = \lambda_{ij}^4\,\bar{w}_{ij}$$

$$+ \int_0^a \left[M_y \frac{\partial \phi_{ij}}{\partial y} - M_{ij}^y \frac{\partial w}{\partial y} + V_{ij}^y w - V_y \phi_{ij}\right]_0^b dx$$

$$+ \int_0^b \left[M_x \frac{\partial \phi_{ij}}{\partial x} - M_{ij}^x \frac{\partial w}{\partial x} + V_{ij}^x w - V_x \phi_{ij}\right]_0^a dy$$

$$+\int_0^b \left[D_1\, \phi_{ij} \frac{\partial^3 w}{\partial x \partial y^2} + (D_1 + 4D_{xy})\, w \frac{\partial^3 \phi_{ij}}{\partial x \partial y^2} - (D_1 + 4D_{xy}) \frac{\partial^2 w}{\partial y^2} \frac{\partial \phi_{ij}}{\partial x} - D_1 \frac{\partial w}{\partial x} \frac{\partial^2 \phi_{ij}}{\partial y^2}\right]_0^a dy$$

$$+\int_0^a \left[-D_1 w \frac{\partial^3 \phi_{ij}}{\partial y \partial x^2} - (D_1 + 4D_{xy})\, \phi_{ij} \frac{\partial^3 w}{\partial y \partial x^2} + (D_1 + 4D_{xy}) \frac{\partial w}{\partial y} \frac{\partial^2 \phi_{ij}}{\partial x^2} + D_1 \frac{\partial \phi_{ij}}{\partial y} \frac{\partial^2 w}{\partial x^2}\right]_0^b dx. \tag{6.99}$$

If the last two integrals on the right-hand side are integrated by parts several cancellations occur and equation (6.99) simplifies considerably. The final result can then be written as

$$\int_0^a\int_0^b \left(D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4}\right)\phi_{ij}\,dx\,dy = \lambda_{ij}^4\,\bar{w}_{ij} + F_{ij}(t) \tag{6.100}$$

where

$$F_{ij}(t) = \int_0^a \left[M_y \frac{\partial \phi_{ij}}{\partial y} - M_{ij}^y \frac{\partial w}{\partial y} + V_{ij}^y w - V_y \phi_{ij}\right]_0^b dx$$

$$+ \int_0^b \left[M_x \frac{\partial \phi_{ij}}{\partial x} - M_{ij}^x \frac{\partial w}{\partial x} + V_{ij}^x w - V_x \phi_{ij}\right]_0^a dy + \left[w R_{ij} - \phi_{ij} R\right]_{0,0}^{a,b} \tag{6.101}$$

and the corner reactions R have been introduced as

$$R = 4 D_{xy} \frac{\partial^2 w}{\partial x \partial y} . \tag{6.102}$$

Using the result given in (6.100) the transform of the governing equation becomes from (6.97),

$$\lambda_{ij}^4 \bar{W}_{ij} + F_{ij}(t) + \rho_0 \frac{d^2 \bar{W}_{ij}}{dt^2} = \bar{q}_{ij}(t)$$

or

$$\frac{d^2 \bar{W}_{ij}}{dt^2} + \omega_{ij}^2 \bar{W}_{ij} = N_{ij}(t) \tag{6.103}$$

where

$$\omega_{ij} = \frac{\lambda_{ij}^2}{\sqrt{\rho_0}} \tag{6.104}$$

and

$$N_{ij} = \frac{\bar{q}_{ij} - F_{ij}}{\rho_0} . \tag{6.105}$$

Then the solution for the transformed variable can be written as

$$\bar{W}_{ij}(t) = \bar{W}_{ij}(0) \cos \omega_{ij} t + \frac{\dot{\bar{W}}_{ij}(0)}{\omega_{ij}} \sin \omega_{ij} t + \frac{1}{\omega_{ij}} \int_0^t N_{ij}(\tau) \sin \omega_{ij}(t-\tau) d\tau \tag{6.106}$$

where

$$\bar{W}_{ij}(0) = \int_0^a \int_0^b w(x,y,0)\, \phi_{ij}(x,y)\, dx\, dy \tag{6.107}$$

and

$$\dot{\bar{W}}_{ij}(0) = \int_0^a \int_0^b \frac{\partial w}{\partial t}(x,y,0)\, \phi_{ij}(x,y)\, dx\, dy .$$

The final solution for the plate response may be found by substituting into the inversion series

$$w(x,y,t) = \sum_i \sum_j \bar{w}_{ij}(t)\, \phi_{ij}(x,y) . \tag{6.108}$$

Applications

As a special case of the orthotropic plate with time-dependent boundary conditions, the problem of a simply supported plate with the edge displacements specified as a function of time and position is considered. The simple supports are assumed for simplicity in this example; actually, any combination of supports may be used provided the corresponding eigenvalue problem can be solved.

For simple support conditions the normalized eigenfunctions are found to be

$$\phi_{ij}(x,y) = \frac{2}{\sqrt{ab}} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \tag{6.109}$$

and the natural frequencies are

$$\omega_{ij} = \pi^2 \left[\frac{\frac{i^4}{a^4} D_x + \frac{2 i^2 j^2}{a^2 b^2} H + \frac{j^4}{b^4} D_y}{\rho_0} \right]^{\frac{1}{2}} . \tag{6.110}$$

Using (6.109) the function $F_{ij}(t)$ can be evaluated. This involves evaluating the Kirchhoff shears on each edge and the corner reactions. On the edges of the plate, the displacements are specified as

$$\begin{aligned} w(x,0,t) &= W_1(x,t) \\ w(a,y,t) &= W_2(y,t) \\ w(x,b,t) &= W_3(x,t) \\ w(0,y,t) &= W_4(y,t) . \end{aligned} \tag{6.111}$$

At the corners, the plate displacements must be continuous and are given by

$$
\begin{aligned}
W_1(0,t) &= W_4(0,t) = f_1(t) \\
W_1(a,t) &= W_2(0,t) = f_2(t) \\
W_2(b,t) &= W_3(a,t) = f_3(t) \\
W_3(0,t) &= W_4(b,t) = f_4(t).
\end{aligned}
\tag{6.112}
$$

Using (6.109), (6.111), and (6.112), equation (6.101) reduces to

$$
\begin{aligned}
F_{ij}(t) &= \\
&\frac{2\pi^3}{\sqrt{ab}} \frac{j}{b}\left[D_y\left(\frac{j}{b}\right)^2 + (D_1 + 4D_{xy})\left(\frac{i}{a}\right)^2\right] \int_0^a \left[(-1)^j W_3(x,t) - W_1(x,t)\right] \sin\frac{i\pi x}{a}\, dx \\
&+ \frac{2\pi^3}{\sqrt{ab}} \frac{i}{a} \, D_x\left(\frac{i}{a}\right)^2 + (D_1 + 4D_{xy})\left(\frac{j}{b}\right)^2\Big]\int_0^b \left[(-1)^i W_2(y,t) - W_4(y,t)\right] \sin\frac{j\pi y}{b}\, dy \\
&+ \frac{8\pi^2}{\sqrt{ab}} \frac{ij}{ab} D_{xy}\left[f_1(t) + (-1)^{i+1} f_2(t) + (-1)^i(-1)^j f_3(t) + (-1)^{j+1} f_4(t)\right].
\end{aligned}
\tag{6.113}
$$

When the edge displacements are uniform, say $f(t)$, the last equation takes an especially simple form. In that case,

$$
F_{ij}(t) = -\frac{8}{\pi^2}\rho_0 \sqrt{ab}\, \frac{\omega_{ij}^2}{ij} f(t) \qquad i,j = 1,3,5,\dots \, . \tag{6.114}
$$

If there is no transverse load, then from (6.105)

$$
N_{ij}(t) = \frac{8}{\pi^2}\sqrt{ab}\, \frac{\omega_{ij}^2}{ij} f(t). \tag{6.115}
$$

For the case where the plate starts from rest, the time function is given by

$$\overline{W}_{ij}(t) = \frac{8}{\pi^2}\sqrt{ab}\,\frac{\omega_{ij}}{ij}\int_0^t f(\tau)\sin\omega_{ij}(t-\tau)\,d\tau \tag{6.116}$$

and the final solution, from the inversion series (6.108) is

$$W(x,y,t) = \frac{16}{\pi^2}\sum_{i=1,3}^{\infty}\sum_{j=1,3}^{\infty}\frac{\omega_{ij}}{ij}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\int_0^t f(\tau)\sin\omega_{ij}(t-\tau)\,d\tau .$$

The problem considered above used simple boundary conditions and thus yielded a simple result. The general solution is not, however, restricted to any particular combination of boundary conditions such as the Levy conditions. In numerous cases the Levy conditions may be conveniently used in practical applications. Since the mode shapes and frequency equations for the Levy supports have been tabulated, then for these problems it is particularly convenient to treat time-dependent boundary conditions by substituting into the equations derived above.

VII. DISCUSSION

In this dissertation a finite generalized integral transform has been applied to three general classes of problems in the vibrations of continuous media. The integral transform uses as its kernel the eigenfunction of an associated eigenvalue problem, and it is denoted as the eigentransform. The eigentransform was applied first to the vibrations of continuous media with both non-uniform stiffness and non-uniform mass distributions. Then it was applied to continuous media with a uniform stiffness but a non-uniform mass distribution. Finally, the eigentransform was used to treat problems with time-dependent boundary conditions.

In Chapter IV a general method was presented for treating vibrations of continuous media with non-uniform stiffness and mass distributions. The eigenvalue problem considered was assumed to be self-adjoint. The eigentransform was applied to the governing partial differential equation and subsequently, the transformed displacement was found to satisfy an infinite set of coupled ordinary differential equations similar to those encountered in the vibrations of discrete masses. For free vibrations this set of equations led to a matrix eigenvalue problem from which approximate eigenvalues and eigenvectors were obtained. For forced vibrations, the differential equations were uncoupled with a transformation matrix of the eigenvectors and solved for the generalized time function. The inversion series for the transform was then used to obtain the solution for the dynamic response.

The frequencies and mode shapes for the longitudinal vibrations of a tapered rod were determined as an example. Using successively larger matrices the frequencies and mode shapes were determined and compared to the known exact solution. These results demonstrated that convergence to the exact frequencies was monotonic from above.

Agreement with the exact solution was obtained without difficulty.

The reduction of the eigenvalue problem for non-uniform continuous media to that of a matrix eigenvalue problem has been accomplished previously by other methods. These have included the Rayleigh-Ritz and the Galerkin methods. An application of the eigentransform arrives at the same result from a different approach.

In Chapter IV the eigentransform was used to develop a general procedure for treating continuous media with a non-uniform mass distribution. These results, similar to those of Chapter IV, reduced the non-uniform mass distribution problem to a matrix eigenvalue problem for determination of the eigenvalues. The mode shapes were determined by summation using the eigenvectors and mode shapes for the uniform continuous media.

A variety of problems for beams and plates with concentrated masses was solved as examples. A secondary result of these examples was the derivation of approximate formulas for the fundamental frequencies of beams and plates with concentrated masses. Comparison with known exact solutions showed that the formula derived for a beam estimated the frequency with an error usually less than 1 percent. The formula for plates was less accurate; it gave results with an error of about 3 percent.

The general matrix eigenvalue problem was solved for the first four modes for a number of beam and plate problems with a single central point mass. Convergence was reasonably rapid for both beams and plate, although the convergence was usually more rapid for beams. However, for both beams and plates as a rough rule a ($2n \times 2n$) matrix eigenvalue problem gave n frequencies and n mode shapes with acceptable engineering accuracy. A value of $n = 8$ was used for the beam problems and $n = 9$ for the plate problems to find four modes.

In the past treatments of plates with attached masses, the frequency equations were determined as infinite series. The present method using the matrix eigenvalue problem approach has definite computational advantages.

Vibrations of continuous media with time-dependent boundary conditions were treated in Chapter VI. One dimensional media were considered first and then isotropic and orthotropic plates.

One dimensional continuous media with time-dependent boundary conditions were treated in a general way by specifying a differential operator of even order in the spatial derivatives. Examples of applications to rods and beams were presented.

The vibration of isotropic plates with time-dependent boundary conditions was treated in a general way using normal and tangential coordinates. The eigentransform of the governing equation was performed using an identity and theorem of vector analysis. The time-dependent boundary conditions were allowed to have an arbitrary variation along the boundary. Detailed applications were then made to rectangular and circular plates.

For the rectangular plate an example of determining the response history was given, and the determination of frequencies and mode shapes for combinations of simply supported and clamped edges were considered. The frequency equations for these problems involved infinite series and infinite determinants which were found to be laborious to solve for more than one clamped edge.

The general equation for treating annular circular plates with time-dependent conditions was derived. This result was applied to the determination of the symmetric response of a clamped plate with a uniform time-dependent edge displacement.

Orthotropic rectangular plates with time-dependent boundary conditions were also considered in Chapter VI. The eigentransform of the orthotropic plate equation was performed by integration by parts. Again the boundary conditions were permitted to vary arbitrarily around the boundary. The response of a simply supported plate with an arbitrary edge displacement was determined as an illustration.

VIII. CONCLUSIONS

The present investigation of applications of a generalized finite integral transform, the eigentransform, led to the following conclusions.

1. The eigentransform when applied to the governing partial differential equation for non-uniform continuous media leads to a matrix eigenvalue problem of a form which had previously been derived by minimization methods, such as Rayleigh-Ritz. For the example considered, the convergence to the eigenvalues and mode shapes was rapid and was conveniently accomplished with a digital computer.
2. The eigentransform when applied to problems with non-uniform mass distribution leads also to a matrix eigenvalue problem. For plates with masses this represents an improvement over previous methods of calculating frequencies and mode shapes. With a digital computer the eigenvalues and mode shapes for plates with masses and Levy type boundary conditions can be obtained in a straightforward manner.
3. Vibration problems with time-dependent boundary conditions may be solved in a direct way using the eigentransform. In previous treatments of vibrations of plates with time-dependent boundary conditions it has been necessary to perform part of the analysis by trial and error. In addition these analyses have been restricted to uniform edge motions. The eigentransform solves the problem in a direct manner and removes the restriction of uniform edge motions. Non-uniform edge motions may be considered for both isotropic and orthotropic plates.
4. The eigentransform is a logical generalization of other finite integral transforms, such as the finite sine or cosine transform. In the past the advantageous use of

these transforms was restricted to only certain simple boundary conditions. The concept of a generalized finite integral transform extends the advantages of integral transforms to a much broader class of boundary value problems.

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APPENDIX A

General Formulation of the Eigenvalue Problem in Vibration of Continuous Media

The eigenvalue problems encountered in the vibration of structural elements such as rods, beams, and plates have been formulated in a general way by a number of writers. Discussions of the general problem appear in the books by Courant and Hilbert [55], Crandall [58], and Meirovitch [52]. An important feature of these discussions is that criteria are established for the existence and uniqueness of the eigenvalues and the orthogonality of the eigenfunctions. For convenience, some of these results will be summarized in this appendix. The presentation follows that of Meirovitch.

A large class of eigenvalue problems (see Table IX) are governed by a partial differential equation of the type

$$L[\phi(x,y)] = \lambda M[\phi(x,y)] \tag{A1}$$

where λ is a parameter and L and M are linear, homogeneous differential operators of orders $2p$ and $2q$, respectively. For example, L has the form

$$L = A_1(x,y) + A_2(x,y)\frac{\partial}{\partial x} + A_3(x,y)\frac{\partial}{\partial y} + A_4(x,y)\frac{\partial^2}{\partial x\,\partial y} + \cdots, \tag{A2}$$

where the coefficients A_1, A_2, $\cdots$ are known functions of the spatial variables x and y. The operator M has a form similar to L and is of order $2q$ such that $p > q$. Equation (A1) must be satisfied at every point within a one or two dimensional region R. Associated with the partial differential equation there are p boundary conditions that $w(x,y)$ must satisfy at every point on the boundary S of the region R. The boundary conditions may be written in the general form

$$B_i[\phi] = \lambda C_i[\phi] \tag{A3}$$

where B_i and C_i are linear differential operators of order $2p-1$ involving derivatives normal to the boundary and along the boundary. Equation (A3) gives a total of

TABLE IX

Typical Operators for Eigenvalue Problems

Continuous System	Spatial Operator L	Mass Distribution M	Region
String	$\frac{\partial}{\partial x}\left(T\frac{\partial}{\partial x}\right)$	$\rho\,(x)$	$0 < x < \ell$
Rod	$\frac{\partial}{\partial x}\left(EA\frac{\partial}{\partial x}\right)$	$\rho\,A(x)$	$0 < x < \ell$
Beam	$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2}{\partial x^2}\right)$	$\rho\,A(x)$	$0 < x < \ell$
Membrane	$T\,\nabla^2$	ρ	Arbitrary Plane
Uniform Plate	$D\,\nabla^4$	ρ	Arbitrary Plane

p boundary conditions that must be satisfied at every point on the boundary.

The eigenvalue problem consists of seeking the values of the parameter for which there exist non-trivial solutions satisfying the partial differential equation (A1) and boundary conditions (A3). Such parameters λ are called eigenvalues and the corresponding functions of $\phi(x,y)$ are called eigenfunctions.

The problem as posed above is called a general eigenvalue problem. For the case when M is not a differential operator and is only a function of the spatial variables, i.e., $M = M(x,y)$, the problem is called a special eigenvalue problem.

The remainder of the discussion will be restricted to the class of problems involving homogeneous boundary conditions so that at every point on S the boundary conditions take the form

$$B_i\,[\phi] = 0. \tag{A4}$$

The eigenvalue problem posed by (A1) and (A4) is defined to be self-adjoint if, for any two function u and v satisfying the boundary conditions (A4), the following conditions are met:

$$\int_R u\,L[v]\,dR = \int_R v\,L[u]\,dR \tag{A5}$$

$$\int_R u\,M[v]\,dR = \int_R v\,M[u]\,dR. \tag{A6}$$

The conditions (A5) and (A6) can be verified for a one dimensional region by integration by parts and for a two dimensional region either by integration by parts or by the integral theorems of vector analysis.

If for any function u ,

$$\int_R u\,L[u]\,dR \geq 0 \tag{A7}$$

the operator L is said to be *positive*. The operator L is said to be *positive-definite* if the integral is zero only when u is identically zero. Similar statements hold for the M operator. If both L and M are positive definite the eigenvalue problem is said to be positive definite. When L is only positive and M is positive definite the problem is *semi-definite*. In many cases the operator $M = M(x,y)$ can be recognized as the distributed mass. In these cases it is obvious that M is positive definite.

The solution of the eigenvalue problem, (A1) and (A4), consists of an infinite set of eigenvalues ($\lambda_1, \lambda_2, \ldots$) and a corresponding sequence of eigenfunctions ($\phi_1, \phi_2, \ldots$). Since the problem is homogeneous the amplitudes of the eigenfunctions ϕ_r ($r = 1, 2, \ldots$) are arbitrary, and only the shapes can be determined uniquely. If the system is positive definite all eigenvalues λ_r are positive. In the case of a semi-definite system in which L is only positive, $\lambda_r = 0$ is also an eigenvalue.

Let λ_r and λ_s be two distinct eigenvalues and ϕ_r and ϕ_s be the corresponding eigenfunctions of a self-adjoint eigenvalue problem. Using (A1) gives

$$L[\phi_r] = \lambda_r M[\phi_r] \tag{A8}$$

and

$$L[\phi_s] = \lambda_s M[\phi_s] \tag{A9}$$

Multiplying (A8) by ϕ_s and (A9) by ϕ_r, subtracting, and integrating over the region R gives

$$\int_R (\phi_s L[\phi_r] - \phi_r L[\phi_s])\, dR = \int_R (\lambda_r \phi_s M[\phi_r] - \lambda_s \phi_r M[\phi_s])\, dR. \tag{A10}$$

Now since ϕ_r and ϕ_s are solutions of a self-adjoint eigenvalue problem, they satisfy (A5) and (A6). Thus,

$$\int_R \phi_s L[\phi_r]\, dR = \int_R \phi_r L[\phi_s]\, dR, \tag{A11}$$

and

$$\int_R \phi_s M[\phi_r]\, dR = \int_R \phi_r M[\phi_s]\, dR \tag{A12}$$

which may be substituted in (A10) to give

$$(\lambda_r - \lambda_s)\int_R \phi_r M[\phi_s]\,dR = 0 .$$

Then since $\lambda_r \neq \lambda_s$

$$\int_R \phi_r M[\phi_s]\,dR = 0. \tag{A13}$$

Equation (A13) is known as the generalized orthogonality condition. From (A9) it follows that

$$\int_R \phi_r L[\phi_s]\,dR = 0, \qquad \lambda_r \neq \lambda_s . \tag{A14}$$

The eigenfunction ϕ_r are said to be normalized with respect to M if the integral is defined as

$$\int_R \phi_r M[\phi_r]\,dR = 1 \tag{A15}$$

which determines the otherwise arbitrary amplitudes of the eigenfunction ϕ_r. Then (A13) and (A15) may be written by the single equation

$$\int_R M[\phi_r]\phi_s\,dR = \delta_{rs} \tag{A16}$$

where δ_{rs} is the Kronecker delta.

In the special cases (shown for example in Table IX) where M is not a differential operator, the orthogonality condition reduces to

$$\int_R M\,\phi_r\,\phi_s\,dR = \delta_{rs} . \tag{A17}$$

The family of eigenfunctions satisfying (A16) has been shown by Courant and Hilbert [55] to constitute a complete set. This means that any function f satisfying the homogeneous boundary conditions and for which $L[f]$ is continuous may be represented by an absolutely and uniformly convergent series in the eigenfunctions in the form

$$f = \sum_r c_r \phi_r \tag{A18}$$

where the coefficients c_r are given by

$$c_r = \int_R f M[\phi_r] \, dR, \qquad r = 1, 2, \ldots \quad . \tag{A19}$$

This representation is known as the expansion theorem.

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APPLICATIONS OF A GENERALIZED INTEGRAL TRANSFORM
TO VIBRATIONS OF CONTINUOUS MEDIA

by

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ABSTRACT

A finite generalized integral transform was applied to three general classes of problems in the vibrations of continuous media. For its kernel the eigenfunction of an associated eigenvalue problem was used and the result was denoted the eigentransform. The eigentransform was applied to (1) continuous media with both non-uniform stiffness and mass distributions; (2) continuous media with uniform stiffness but non-uniform mass distribution; and (3) to problems with time-dependent boundary conditions.

A general method was presented for treating vibrations of continuous media with non-uniform stiffness and mass distributions. The eigentransform was applied to the governing partial differential equation and subsequently the transformed displacement was found to satisfy an infinite set of coupled ordinary differential equations similar to those encountered in the vibrations of discrete masses. These equations led to a matrix eigenvalue problem from which approximate eigenvalues and eigenvectors were obtained. The differential equations were uncoupled using a transformation matrix of the eigenvectors and then were solved for the generalized time function. Finally, the inversion series for the transform was used to obtain the solution for the dynamic response. To illustrate the method, the first four frequencies and mode shapes were determined for the longitudinal vibration of a tapered rod.

The eigentransform was used to develop a general procedure for treating continuous media with uniform stiffness but non-uniform mass distribution. These results, similar to

those for the general non-uniform problem, reduced this problem to a matrix eigenvalue problem. The mode shapes were determined by summation using the eigenvectors and mode shapes for the uniform continuous media. Several problems for beams and plates with concentrated masses were solved as examples. This approach demonstrated definite computational advantages for plates over past treatments where frequency equations were determined as infinite series.

Vibrations of continuous media with time-dependent boundary conditions were then treated using the eigentransform. One dimensional media were considered first and next isotropic and orthotropic plates. One-dimensional continuous media were treated in a general way by specifying a differential operator of even order in the spatial derivatives. Applications to rods and beams were presented. The vibration of isotropic plates for an arbitrary shape was treated by expressing the equations in normal and tangential coordinates. The eigentransform of the governing equation was performed using an identity and theorem of vector analysis. The time-dependent boundary conditions were allowed to have an arbitrary variation along the boundary. Detailed applications were then made to rectangular and circular isotropic plates. The eigentransform of the orthotropic plate equation was performed by integration by parts. Again the boundary conditions were permitted to vary arbitrarily around the boundary. The response of a simply supported plate with an arbitrary edge displacement was determined as an illustration.

This investigation demonstrated that the eigentransform is a logical generalization of other finite integral transforms. The concept of a generalized finite integral transform extends the advantages of integral transforms to a much broader class of boundary value problems.