SYSTEM MODELING AND MODIFICATION
VIA MODAL ANALYSIS

by

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Dedication

To my beloved parents, , and my brothers and sister who have constantly encouraged me to pursue a higher education.

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NOMENCLATURE

A = acceleration at physical coordinate

a = constant used in least square error fit

b = constant used in least square error fit

[C] = physical viscous damping matrix

$[\bar{C}]$ = physical viscous damping matrix of the modified system

$[\Delta C]$ = physical viscous damping modification matrix

$[\ulcorner c \lrcorner]$ = diagonal modal viscous damping matrix

$[\bar{c}]$ = modal viscous damping matrix of the modified system

$[\Delta c]$ = modal viscous damping modification matrix

c = cosine

c = constant used in least square error fit

c_j = j^{th} modal viscous damping

c_{ik}^{j} = effective viscous damping in the j^{th} mode measured from the Y_{ik} mobility or its related functions

d_i = x component of the external force at i point

d_j = x component of the external force at j point

E = error function

e_i = y component of the external force at i point

e_j = y component of the external force at j point

F_i = amplitude of $F_i e^{i\omega t}$ for steady-state response

F_j = amplitude of $F_j e^{i\omega t}$ for steady-state response

$F_i(t)$ = arbitrary external point force applied at i

f_i = z component of the external force at i point

f_j = z component of the external force at j point

g = structural damping coefficient

g_j = j^{th} modal structural damping

g_{ik}= effective structural damping in the j^{th} mode measured from the Y_{ik} mobility or its related functions

$[I]$ = identity matrix

Im = imaginary part

$i = \sqrt{-1}$

$[K]$ = physical stiffness matrix

$[\bar{K}]$ = physical stiffness matrix of modified system

$[\Delta K]$ = physical stiffness modification matrix

$[\ulcorner k \lrcorner]$ = diagonal modal stiffness matrix

$[\bar{k}]$ = modal stiffness matrix of modified system

$[\Delta k]$ = modal stiffness modification matrix

Δk = coefficient of stiffness modification

k_j = j^{th} modal stiffness

k_{ik}^{j} = effective stiffness in the j^{th} mode measured from the Y_{ik} mobility or its related functions

L = length of an element

ℓ = number of modes

$[M]$ = physical mass matrix

$[\bar{M}]$ = physical mass matrix of modified system

$[\Delta M]$ = physical mass modification matrix

$[\ulcorner m \lrcorner]$ = diagonal modal mass matrix

$[\bar{m}]$ = diagonal modal mass matrix of modified system

$[\Delta m]$ = modal mass modification matrix

Δm = coefficient of mass modification

m_j = j^{th} modal mass

m_{ik}^{j} = effective mass in the j^{th} mode measured from the Y_{ik} mobility or its related functions

n = number of degree-of-freedom

$[P]$ = modal matrix, each column is an undamped modal vector, $\{P\}^j$

$\{P\}^j$ = undamped modal vector for the j^{th} mode

$\{q\}$ = normal coordinate

q_j = j^{th} element of $\{q\}$

R = radius of circle

r = number of data points for fitting

s = sine

u_i = x-component of displacement at i point

u_j = x-component of displacement at j point

$\{V\}^j$ = physical coordinate velocity vector owing only to response in the j^{th} mode

v_k^j = k^{th} element of $\{V\}^j$

v_i = y-component of displacement at i point

v_j = y-component of displacement at j point

w_i = z-component of displacement at i point

w_j = z-component of displacement at j point

$\{X\}$ = physical coordinate displacement vector

$\{X\}^j$ = physical coordinate displacement vector owing only to response in the j^{th} mode

x = variable used in least square error fit

x_k = k^{th} physical coordinate displacement, an element of $\{X\}$

x_k^j = kth element of $\{X\}^j$

Y_{ij} = mobility where response is measured at i with a force at j provided all other external forces are zero

y = variable used in least square error fit

Z = impedance

Z^* = complex number

ω = circular frequency, rad/sec.

ω_j = resonant circular frequency in the j^{th} mode, rad/sec.

ω_n^j = natural

λ = eigenvalue

ϕ = angle between a stiffness member and the axis

SUPERSCRIPT

$+$ = pseudo-inverse of a matrix

T = transpose of a matrix

-1 = ordinary inverse of a square, non-singular matrix

SUBSCRIPT

I = modal space I

II = modal space II

x = x direction

y = y direction

z = z direction

Chapter 1

INTRODUCTION

The rapid advancement of the modern computer technology has produced the microprocessor-based Fast Fourier Transform (FFT) Analyzers. In addition, more and more engineers are concerned about the dynamic behavior of the structures they design so that their design will have improved product reliability, higher performance, reduced noise and vibration levels, more efficient material usage, and reduced design, manufacturing, and operating costs. These demands for improved product performance have caused many designers to turn to structural dynamic testing for better understanding of the structure's behavior in order to be able to control or improve it. These relatively inexpensive microprocessor-based FFT analyzers provide them with a powerful tool for acquisition and analysis of vibration data needed for characterizing the dynamic behavior of a structure.

If sufficient data is obtained from a vibration test, one can obtain the description of the dynamic characteristics of a structure via a process called modal analysis. This process characterizes the structure by noting its natural modes of vibration. Each natural mode of vibration has associated with it certain modal properties, i.e. mass, stiffness, damping, mode shape, and frequency characteristics. These characteristics are global properties of the structure and can be measured at almost any point on the structure except, of course, at the node point whose motion is almost stationary. The total dynamic response of a structure to any arbitrary input force is the sum of the responses of each of the modes of vibration. By identifying the modal properties of each individual mode, a mathematical model can be derived that will describe the dynamic characteristics of a structure in various vibration environments.

Most techniques that reduce a physical system to a mathematical model require a large amount of test data, much computation, and an experienced and knowledgeable analyst. The microprocessor-based system has limited memory space and is slow in computation speed compared to mini- or main-framed computer. Therefore, a simple technique for experimentally determining the system parameters of a structure is needed. When solving a vibration problem, one frequently needs to know how the structure's responses change when a mass, stiffness, or damping modification is made to the structure. This can be done by actually modifying the structure and re-testing it to find new dynamic character. But this trial-and-error approach could be a very costly and time-consuming process. If a mathematical model can be derived from the vibration test, the mass, stiffness, or damping properties of the structure can be altered analytically, and the resultant dynamic characteristics of the "modified" structure determined. In this way, any number of modifications can be studied analytically. This allows the designers to evaluate many alternate solutions to the problem and to choose the most desirable or optimum solution before the structure is actually modified.

Conventionally, people used finite element modeling methods to develop the mathematical models to describe the dynamic behavior of a structure. Then modifications can be added to the model to predict the new responses. The problem with this analytical approach is that it requires large main-frame computers for large problems, is difficult to formulate, needs a great deal of memory space and computational time, and is almost impossible to check the accuracy of the analytical model unless an actual test is performed. The test is needed so that the analytical results can be compared to the experimental results. In view of this, an analytical method that can efficiently predict the dynamic characteristics of the modified structure is extremely useful.

This dissertation describes a new modeling method which uses a single degree-of-freedom model. The dynamic response of a structure, a multi-degree-of-freedom system, is represented by a summation of the response of individual modes of vibration. To represent the response analytically, the dynamic characteristics describing each mode of vibration are assumed to behave as a single degree-of-freedom system. The system parameters--effective mass, stiffness, damping, natural frequency, and mode shape--of each mode are determined from experimental data. The structure is assumed to be describable by a linear, proportionally and lightly damped, lumped parameter model. Two types of damping models, viscous and structural damping, are provided. The system parameters can be converted to global modal mass, stiffness, and damping, and then to physical mass, stiffness, and damping for both complete and truncated system. This method, therefore, deals with the general case where a structure has a relatively large number of points of interest, but only a relatively small number of modes can be practically identified.

This mathematical model, developed from experimental data, can be further used for analytically predicting the change of the structure's dynamic response due to a mass, stiffness, or damping modification on the structure before they are actually made. There are three ways that modifications can be made. They are: 1) modifications made in the physical coordinates model; 2) modifications made in both the physical and modal coordinates models; and 3) modifications made in the modal coordinates model. Because of the limitations of microprocessor-based systems, the most efficient ways have to be determined. The derivations of mass, stiffness, and damping modification matrices for a general structure are also presented.

Sometimes, a designer may want to reduce the vibration level of a particular frequency. He can do so by shifting the resonance of the structure out of that frequency range. But he will need to know the modifications required to accomplish this. This is just the opposite of the modification method. A resonant frequency is specified, and the amount of a particular modification is to be determined. The resonance specification technique will accomplish this.

Also, there is a need for synthesizing a frequency response function (output to input ratios given in the frequency domain) which has or has not been measured in a vibration test because it takes too much time and memory space to store every frequency response function. Besides, some measurement may be difficult or physically impossible to obtain. Therefore, it is very useful to be able to synthesize a frequency response function between any two test points on the structure using the system modal parameters obtained from either the experimental data of the structure or the analytical data of the modified structure. The frequency response function synthesis developed will do this.

The method for determining the effective mass, stiffness, damping, natural frequency, and animated mode shape is implemented in a computer program entitled BASIC MODAL®+, an animated modal analysis package. Also, the techniques of simulating the dynamic response of a modified structure having mass, stiffness, or damping modifications, as well as the resonance specification and frequency response function synthesis, all done in modal coordinates, are implemented in a computer program called BASIC MODIFICATION®+. Both programs are written in BASIC language and are used on a Tektronix 4052 microprocessor with a 54K memory interfacing with a Zonic 6080 FFT analyzer, available in the Noise and Vibration Laboratory of the Mechanical Engineering

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+BASIC MODAL® and BASIC MODIFICATION® are trademarks of Zonic Corporation, Milford, Ohio, U.S.A. 45150.

Department of the Virginia Polytechnic Institute and State University, together with a Tektronix 4631 hard copy machine for producing both numerical and graphical outputs from the CRT.

Chapter 2

LITERATURE REVIEW

Several approaches to deriving a mathematical model of a test structure have been developed. Kennedy and Pancu [1]* presented one of the first normal mode methods for determining system parameters from test data. Their technique is to use four vibrators to separate normal modes such that the responses of the test structure are found to be exactly in or out of phase. A polar plot representing the variation of the response vector with forcing frequency is used for determining the modal characteristics. The basic concept is that such a plot for a single mode is a circle. As the vibrator goes through a certain resonance, the tip of the total response vector describes an arc approaching that of the pure mode response. Identification of these arcs gives the natural frequency, peak, amplitude, and damping.

Ramey [2] had computed values for mass, stiffness, and damping coefficients of the dynamic equations associated with a particular input response from data obtained in vibration tests of a structure. The theory was based on the dynamic properties of multi-degree-of-freedom systems. He identified each mode separately corresponding

*Numbers in brackets refer to reference listed at the end of this dissertation.

to frequencies nearly coincident with resonance. Then the coefficients of mass, stiffness, and damping are determined by solving a system of simultaneous equations obtained from a few frequencies near each major structure resonance. The structure is assumed to be lightly damped and linear.

Vatz [3] determined only the modal masses of the system. Although the method of determining the generalized parameters of mass, stiffness, and damping was derived, the method was not applied because of limitation in the available data.

Engblom [4] developed expressions for the mobility and impedance matrices for damped lumped parameter linear dynamic systems utilizing a "normal mode" approach. He found that the impedance matrix expressions are valid only when the reduced modal matrix is square and nonsingular. Therefore, the number of accurate modes must be equal to or greater than the number of points to be measured.

Young and On [5] reviewed efforts carried out to generate mathematical models in the areas of automatic controls, aerodynamics, and mechanical structures. In the case of structural dynamic analysis, the identification is based on the assumption of orthogonality of the modes with respect to masses, stiffness, and damping coefficients. The strengths and weaknesses of various modeling techniques were discussed. The conclusion was that all the modeling techniques belong in the analyst's bag of tools and it is his task to select the technique that best matches the particular job.

Flannelly, McGarrey, and Berman [6] had treated the problem from an impedance approach. The equations of motion, modal vectors, and natural frequencies of an n degree-of-freedom, structurally damped, linear system can be determined from test data. This method has the limitation that the number of test points times their coordinate degrees-of-freedom must be equal to the number of mode measured during testing.

Klosterman [7] investigated various techniques to determine an analytical representation of the dynamic characteristics of a system from the test results. The techniques include real and complex modal representation of the system. But the system parameters such as mass, stiffness, and damping are not identified; only modal representations are determined.

Ross [8] presented methods for synthesizing the mass and stiffness matrices from experimentally derived data in a way which preserves the physical significance of the individual mass and stiffness elements. The synthesizing procedures allow for the incorporation of other mass and stiffness data, whether empirical or based on the analyst's insight. He also discussed three ways of dealing with truncated system. They are: 1) reduced the system order by limiting the number of coordinates to the number of modes identified, thus producing a square and non-singular modal matrix; 2) added arbitrary linearly independent vectors to fill out modal matrix, thus making it a non-singular matrix. These arbitrary eigenvectors should be chosen such that their eigenvalues are very large, so large that they are modes out of frequency range of interest; and 3) synthesize the flexibility matrix instead of stiffness matrix.

Berman and Flannelly [9] presented a method for identifying parameters in a linear, discrete, structurally damped model by using measured normal modes. The structure considered has a relatively large number of points of interest but a relatively small number of modes identified. The characteristics of this model is discussed. They used a pseudo-inverse to compute the mass matrix. They started with a system of linear equations in the form of $A\bar{m} = R$, where A is a matrix formed by the coefficients of the orthogonal equation with respect to the unknown mass, $\bar{m}$, is a column vector made up of elements of the mass matrix whose magnitudes are unknown; and R is a column vector made up of masses of known magnitude, such as the total mass of the structure or any known generalized modal masses. A column vector, m_A, which is an approximation to $\bar{m}$ and also a diagonal matrix, W, which is a weighting function, are needed. Each element of the weighting function is a measure of the analyst's confidence in the corresponding approximation. He found that the unknown mass is $\bar{m} = W^{-1} (AW^{-1})^{+} (R-Am_A) + Im_A$ where the plus sign indicates a pseudo-inverse. This method relies heavily on the quantitative information from tests and the qualitative information from analysis in order that a valid and useful model can be obtained.

Ibanez [10] developed a method based on an estimation of the whole modal characteristics. The assumption of diagonal mass matrix is used.

Thoren [11] used orthonormal modal vectors computed from dynamic test response data to derive mass, stiffness, and damping matrices for a discrete, proportionally damped model of a distributed elastic system. This method has the limitation of requiring the model to have the same number of degrees-of-freedom and modes identified during the test.

Richardson and Potter [12] derived a technique to identify modal parameters from the transfer function data. They applied an analytical transfer function expression through a least square estimation to measured data from linear structures. In another paper [13], they derived the mass, stiffness, and damping matrices corresponding to a lumped equivalent model of the structure, using complex mode theory.

Miramand, Billaud, Leleux, and Kernevez [14] identified the modal parameters of a structure with arbitrary viscous damping. The coefficients of the mobility-displacement function of the mathematical model are calculated by a least square fit of theoretical values to the experimental results. A quadratic functional of the form $J(Z) = \frac{1}{2} (AZ, Z) - (b,Z)$ where A is a non-negative definite matrix, b is a vector function of experimental data, Z is the vector of coefficients to be identified, is used.

Durham and Russell [15] outlined the procedure for developing the animated deformation plots from experimental frequency response function. The data are collected by a digital signal processing system using the Fast Fourier Transform technique. These plots provide a very useful tool for studying complex structures vibrating at resonant frequencies.

Structural Measurement Systems, Inc. [16] has developed a software package called Structural Dynamics Modification System, which will allow the user to investigate the effects of potential design modifications to a test structure. The modifications permitted are lumped mass, linear stiffness, and damping. The system is assumed to be linear and proportionally damped. Some analytical and one experimental examples are included to illustrate the software capabilities, but no detail information on the theories or methods are shown.

This dissertation has developed a new system modeling technique and presented the theories required in system modification. Three ways of making modifications are identified and compared to find the most efficient method.

Chapter 3

SYSTEM MODELING VIA MODAL ANALYSIS

3.1 GENERAL

Although some simpler components which approximate uniform beams and plates can be treated by an exact analysis, most practical systems are represented by a discretized model, i.e. an assemblage of discrete rigid masses connected together with linear springs and dampers. Therefore, the mathematical model is defined by a set of mass, stiffness, and damping matrices which will describe the system's dynamic behavior.

The method used is a single degree-of-freedom model. Each mode of the multi-degree-of-freedom system is considered as a single degree-of-freedom. This assumption is reasonable in the case where the system is proportionally and lightly damped or the modes are well separated. However, corrections are made for adjacent modes.

3.1.1 VISCOUS DAMPING MODEL

The equations of motion for a multi-degree-of-freedom system with viscous damping may be written

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{F(t)\} \tag{3.1}$$

These equations, with the assumption of proportional damping, can be decoupled in terms of the normal coordinate, q, by using the transformation

$$\{X\} = [P]\,\{q\} \tag{3.2}$$

If one writes Eq. 3.2 in terms of the component of $\{X\}$, x_k, one gets

$$x_k = \sum_{j=1}^{\ell} p_k^j\, q_j \tag{3.3}$$

Following Hurty and Rubinstein [17] one applies

$$\lceil m \rfloor \{\ddot{q}\} + \lceil c \rfloor \{\dot{q}\} + \lceil k \rfloor \{q\} = [P]^T \{F(t)\} \tag{3.4}$$

where

$$\left.\begin{aligned} \lceil m \rfloor &= [P]^T\,[M]\,[P] \\ \lceil c \rfloor &= [P]^T\,[C]\,[P] \\ \lceil k \rfloor &= [P]^T\,[K]\,[P] \end{aligned}\right\} \tag{3.5}$$

The equation for the j^{th} normal coordinate is

$$m_j \ddot{q}_j + c_j \dot{q}_j + k_j q_j = \sum_{i=1}^{n} p_i^j \; F_i(t) \tag{3.6}$$

For only one external forcing function applied at i, Eq. 3.6 becomes [2]

$$m_j \ddot{q}_j + c_j \dot{q}_j + k_j q_j = p_i^j \; F_i(t) \tag{3.7}$$

Equation 3.2 may be written

$$\{x\} = \sum_{j=1}^{\ell} \{p\}^j \; q_j \tag{3.8}$$

So that the response in the j^{th} mode is

$$\{x\}^j = \{p\}^j \; q_j \tag{3.9}$$

or

$$x_k^j = p_k^j \; q_j \tag{3.10}$$

and

$$q_j = \frac{x_k^j}{p_k^j} \tag{3.11}$$

Substituting Eq. 3.11 into Eq. 3.7 and dividing by p_i^j, one gets

$$\frac{m_j}{p_i^j \; p_k^j} \; \ddot{x}_k^j + \frac{c_j}{p_i^j \; p_k^j} \; \dot{x}_k^j + \frac{k_j}{p_i^j \; p_k^j} \; x_k^j = F_i(t) \tag{3.12}$$

or

$$m_{ik}^{j}\ \ddot{x}_{k}^{j} + c_{ik}\ \dot{x}_{k}^{j} + k_{ik}^{j}\ x_{k}^{j} = F_{i}(t) \tag{3.13}$$

Equation 3.13 represents the system dynamics for response at point k owing to a force at point i where $F_i(t)$ is an arbitrary, external, point forcing function. For steady-state response, $F_i(t) = F_i \sin\omega t$, with $\omega = \omega_j$, only one equation of the form of Eq. 3.13 is required to represent the system response. We call m_{ik}^j, c_{ik}^j, and k_{ik}^j in Eq. 3.13 as effective mass, damping, and stiffness.

Equating the coefficients of Eqs. 3.12 and 3.13, one gets a relation between the modal parameters and effective parameters

$$\begin{aligned} m_j &= m_{ik}^{j}\ p_i^j\ p_k^j \\ c_j &= c_{ik}^{j}\ p_i^j\ p_k^j \\ k_j &= k_{ik}^{j}\ p_i^j\ p_k^j \end{aligned} \tag{3.14}$$

The total system response at k owing to any time independent force $F_i(t)$ at i is found by superposition of the solutions of each of the set of the following equations

$$m_{ik}^{j}\ \ddot{x}_{k}^{j} + c_{ik}^{j}\dot{x}_{k}^{j} + k_{ik}^{j}x_{k}^{j} = F_{i}(t) \text{ where } j=1 \text{ to } \ell \tag{3.15}$$

and the total displacement is

$$x_k = \sum_{j=1}^{\ell} x_k^j \tag{3.16}$$

3.1.2 STRUCTURAL DAMPING MODEL

The equations of motion for a multi-degree-of-freedom system with structural damping is

$$[M]\{\ddot{X}\} + [K + iKg]\{X\} = \{F(t)\} \tag{3.17}$$

If the development for the viscous damping model is repeated for the structurally damped system, the equations in terms of the normal coordinate, q, are

$$\lceil m \rfloor \{\ddot{q}\} + \lceil k \rfloor \{q\} = [P]^T \{F(t)\} \tag{3.18}$$

where

$$\lceil m \rfloor = [P]^T [M] [P]$$
$$\lceil k \rfloor = [P]^T [K+iKg] [P] \tag{3.19}$$

The equations for the total system response at k owing to any time independent force $F_i(t)$ at i are

$$m^j_{ik}\ddot{x}^j_k + k^j_{ik}x^j_k = F_i(t) \text{ where } j=1 \text{ to } \ell \tag{3.20}$$

and the total displacement is the same as Eq. 3.16. The relationship between modal parameters and effective parameters are

$$m_j = m^j_{ik}\, p^j_i\, p^j_k$$
$$k_j = k^j_{ik}\, p^j_i\, p^j_k \tag{3.21}$$

The natural frequency for both damping models is

$$\omega_n^j = \sqrt{\frac{k_j}{m_j}} \tag{3.22}$$

3.2 SELECTING A SUITABLE TYPE OF FREQUENCY RESPONSE FUNCTION

So far, only one of the several different types of frequency response functions (FRF) has been used. There are six types of frequency response functions as can be seen in Table 1. Although these frequency response functions--Dynamic Compliance, Mobility, and Inertance, and their inverses--all describe essentially the same properties, each has its own advantages.

Conventionally, the analysis and parameter extraction are done from the dynamic compliance type, (X/F), for both viscous and structural damping models. A Nyquist plot is obtained by plotting the real part of the frequency response function against the imaginary part of the frequency response function. This is frequently used to extract the modal parameters. In this plot, the frequency is not shown explicitly, but rather appears as a parameter along the curve. Many fast Fourier Transform Systems "circle fit" this data. A closer look at the actual frequency response function shows that the data does not always lie exactly on a circle.

3.2.1 VISCOUS DAMPING MODEL

For the case of viscous damping, the effective equation at j^{th} mode is given in Eq. 3.15. The real and imaginary part of the frequency response function for steady-state response are

Table 1. Types of Frequency Response Function [18]

Response Parameter	Displacement	Velocity	Acceleration
Response / Force	Dynamic Compliance	Mobility	Inertance
Force / Response	Apparent Stiffness	Mechanical Impedence	Apparent Mass

$$\left.\begin{aligned} \mathrm{Re}\left(\frac{x_k^j}{F_i}\right) &= \frac{k_{ik}^j - m_{ik}^j\omega^2}{(k_{ik}^j - m_{ik}^j\omega^2)^2 + (c_{ik}^j\omega)^2} \\ \mathrm{Im}\left(\frac{x_k^j}{F_i}\right) &= \frac{-\,c_{ik}^j\omega^2}{(k_{ik}^j - m_{ik}^j\omega^2)^2 + (c_{ik}^j\omega)^2} \end{aligned}\right\} \quad (3.23)$$

One can plot these functions in the Nyquist plane. It is "almost" a circle, but is somewhat distorted as shown in Fig. 1. The curve starts at a point where the imaginary part is zero, but the real part is $\frac{1}{k_{ik}^j}$. Besides, the diameter of the circle is $\frac{1}{c_{ik}^j}$, which depends on the variable circular frequency. This is certainly not a circle.

The same conclusion can be drawn from a theorem in complex variable theory which states that a complex inversion maps circles and lines onto lines and circles, respectively. Specifically, if a complex number Z^* traces out a straight line not including the origin in the complex plane, then $\frac{1}{Z^*}$ traces out a circle. The converse is also true. If one calls the complex plane of Eq. 3.23 the $\frac{1}{Z^*}$ plane, then in the Z^* plane the real and imaginary part of the solution to Eq. 3.15 becomes

$$\left.\begin{aligned} \mathrm{Re}\left(\frac{F_i}{x_k^j}\right) &= k_{ik}^j - m_{ik}^j\omega^2 \\ \mathrm{Im}\left(\frac{F_i}{x_k^j}\right) &= c_{ik}^j\omega \end{aligned}\right\} \quad (3.24)$$

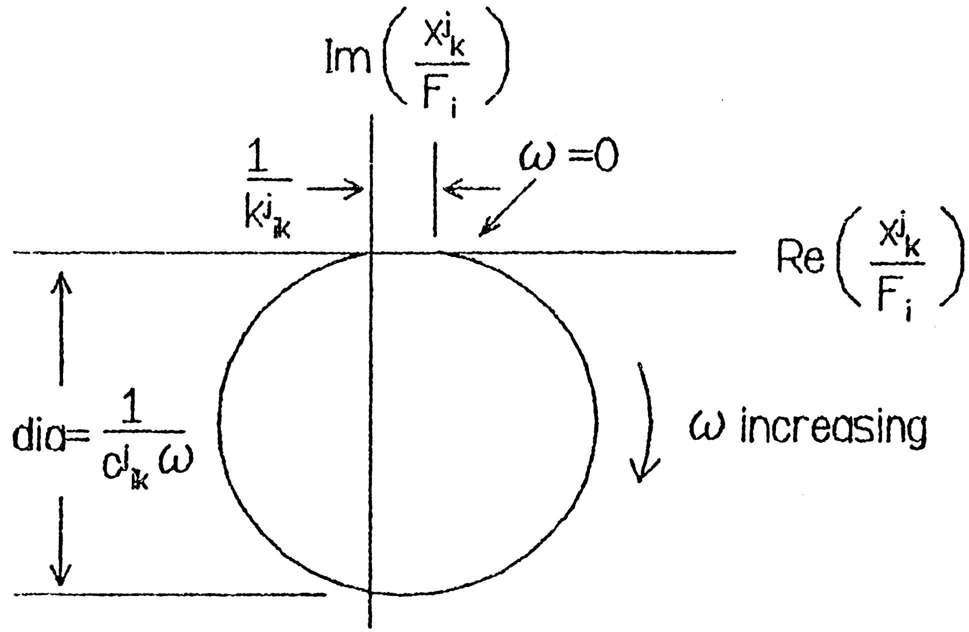


Fig. 1 Nyquist plot of x/F with viscous damping; $1/z^*$ plane.

Equation 3.24 traces out a parabola in the complex Z^* plane, as shown in Fig. 2. Therefore, according to the complex variable theory, the complex inverse will not trace out a circle. This agrees with the observations concerning the non-circular form in Fig. 1.

Next, one looks at the mobility type of frequency response function with viscous damping. The steady-state solution of the effective equation of motion at the j^{th} mode is

$$i m^j_{ik}\omega\, V^j_k + c^j_{ik}\, V^j_k + \frac{k^j_{ik}}{i\omega} V^j_k = F_i e^{i\omega t} \tag{3.25}$$

and

$$\left.\begin{aligned} \mathrm{Re}\left(\frac{V^j_k}{F_i}\right) &= \frac{c^j_{ik}}{(c^j_{ik})^2 + (m^j_{ik} \quad - \underline{k^j_{ik}})^2} \\ \mathrm{Im}\left(\frac{V^j_k}{F_i}\right) &= \frac{-(m^j_{ik}\omega - \frac{k^j_{ik}}{\omega})}{(c^j_{ik})^2 + (m^j_{ik}\omega - \frac{k^j_{ik}}{\omega})^2} \end{aligned}\right\} \tag{3.26}$$

The Nyquist plot of V/F is a circle as shown in Fig. 3. Also, the diameter of this circle is $\frac{1}{c^j_{ik}}$, which is a constant value. The real and imaginary part in Z^* plane is

$$\left.\begin{aligned} \mathrm{Re}\left(\frac{F_i}{V^j_k}\right) &= c^j_{ik} \\ \mathrm{Im}\left(\frac{F_i}{V^j_k}\right) &= m^j_{ik}\omega - \frac{k^j_{ik}}{\omega} \end{aligned}\right\} \tag{3.27}$$

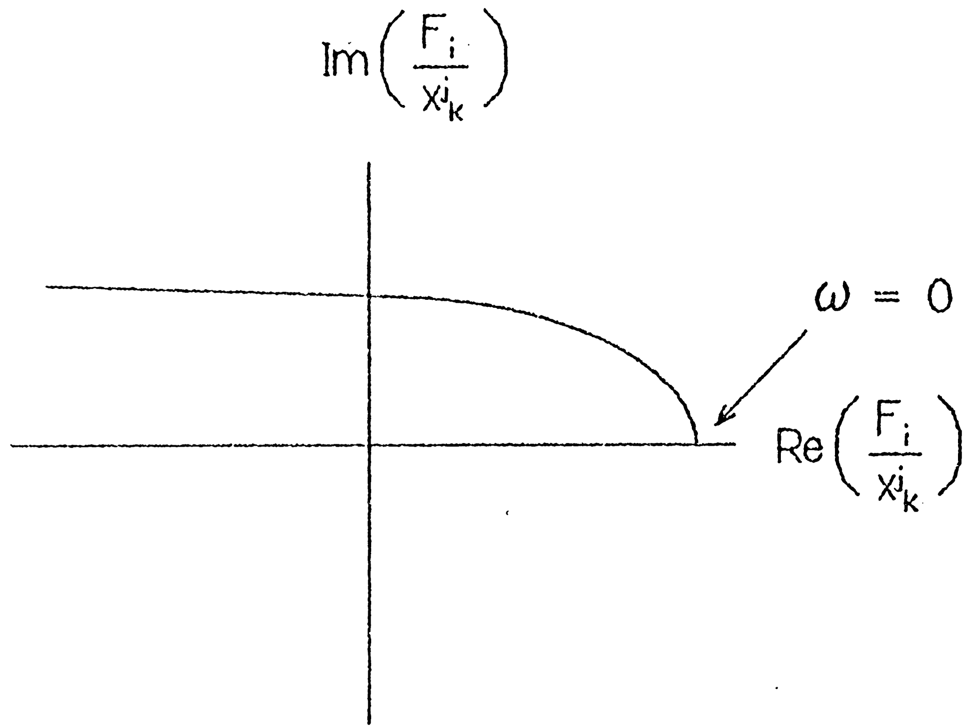


Fig. 2 Nyquist plot of F/x with viscous damping; z^* plane.

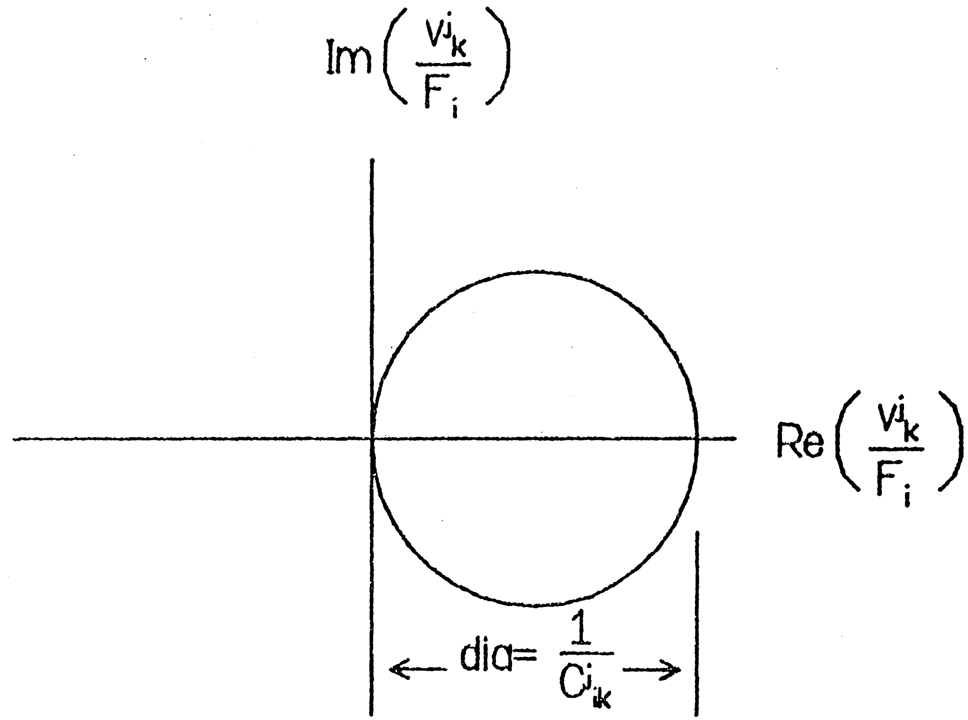


Fig. 3 Nyquist plot of v/F with viscous damping; $1/z^*$ plane.

Equation 3.27 describes a straight line in the Z^* complex plane as shown in Fig. 4. Therefore, by the complex variable theorem, a Nyquist plot in $\frac{1}{Z^*}$ plane is a perfect circle if the frequency response function is expressed in terms of velocity rather than displacement for the viscously damped case.

If similar analysis is performed using inertance, $(\frac{A}{F})$, it can be shown that the Nyquist plot in $\frac{1}{Z^*}$ plane is not a perfect circle, and the plot in Z^* plane is not a straight line, as in the case of dynamic compliance.

3.2.2 STRUCTURAL DAMPING MODEL

For the case of structural damping, the effective equation of motion at j^{th} mode is given in Eq. 3.20. The steady-state solution in the $\frac{1}{Z^*}$ complex plane is

$$\left.\begin{aligned} Re\left(\frac{x_k^j}{F_i}\right) &= \frac{k_{ik}^j - m_{ik}^j\omega^2}{(k_{ik}^j - m_{ik}^j\omega^2)^2 + (k_{ik}^j\, g_{ik}^j)^2} \\ Im\left(\frac{x_k^j}{F_i}\right) &= \frac{-k_{ik}^j\, g_{ik}^j}{(k_{ik}^j - m_{ik}^j\omega^2)^2 + (k_{ik}^j\, g_{ik}^j)^2} \end{aligned}\right\} \quad (3.28)$$

In the complex Z^* plane, the real and imaginary parts are

$$\left.\begin{aligned} Re\left(\frac{F_i}{x_k^j}\right) &= k_{ik}^j - m_{ik}^j\omega^2 \\ Im\left(\frac{F_i}{x_k^j}\right) &= k_{ik}^j\, g_{ik}^j \end{aligned}\right\} \quad (3.29)$$

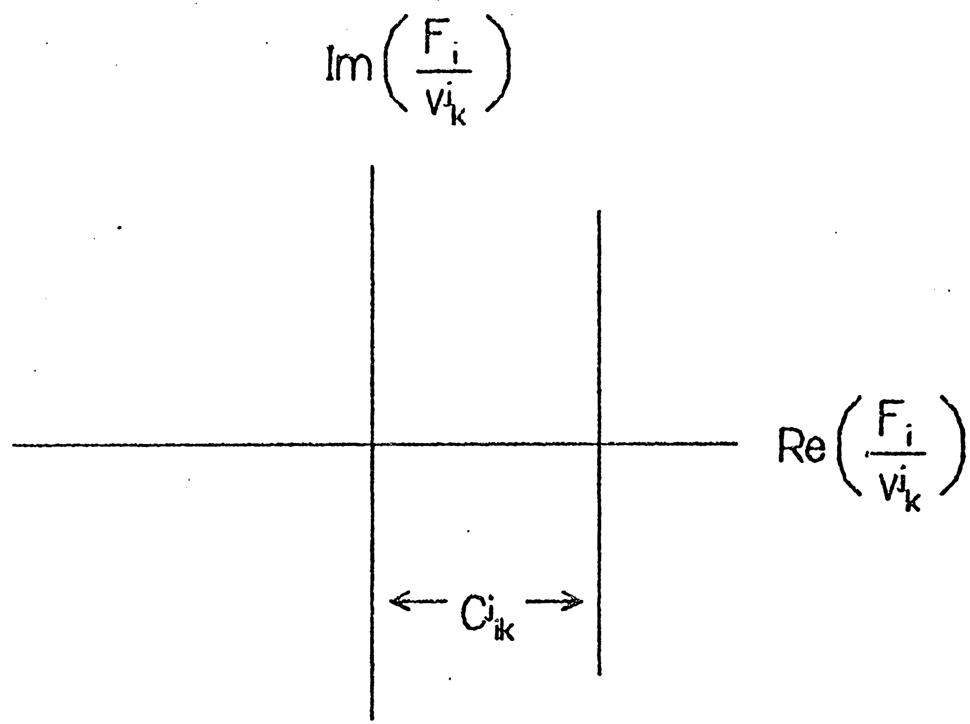


Fig. 4 Nyquist plot of F/v with viscous damping; z^* plane.

in Z^* plane is a straight line of non-infinite extent. This is shown in Fig. 5 and 6. In this case, since the straight line in the $\frac{1}{Z^*}$ complex plane is not of infinite extent in both the positive and negative directions, the circle in the Z^* complex plane does not close. However, this does not prevent valid circle fits of the data.

If similar analysis is performed for the mobility and inertance types of frequency response function with structural damping, it can be shown that the Nyquist plot in $\frac{1}{Z^*}$ complex plane is not a circle, and in Z^* complex plane, the plot is not a straight line.

Since only the Nyquist plot of mobility with viscous damping and Nyquist plot of dynamic compliance with structural damping are circles, only these two data types are used in the extraction of effective parameters. Fitting of frequency response function data in any other forms adds needless error to the measurement process. Also, the peak of these two frequency response function types occurs at the natural undamped frequency for lightly damped system.

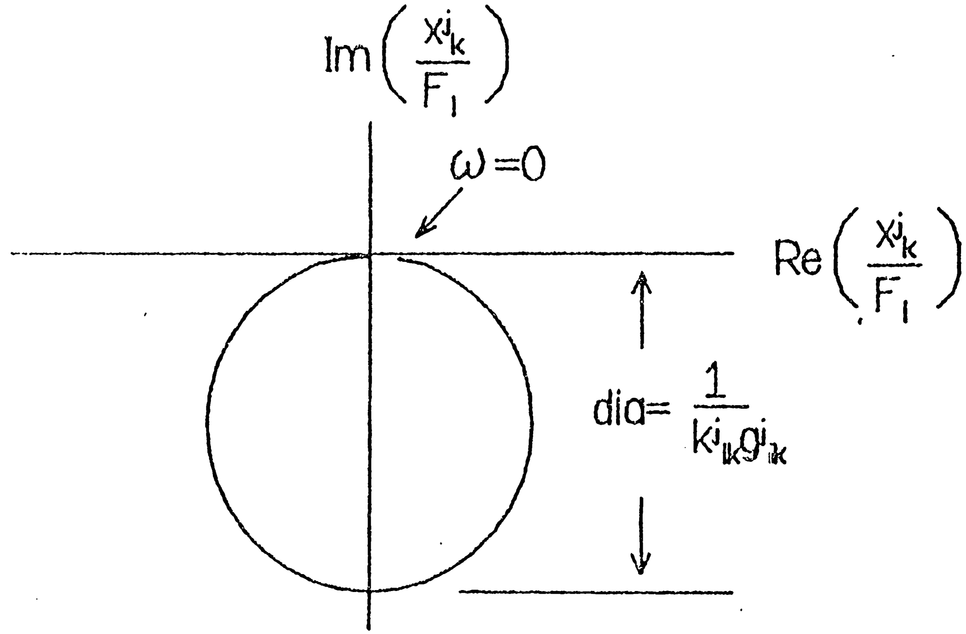


Fig. 5 Nyquist plot of X/F with structural damping; $1/z^*$ plane.

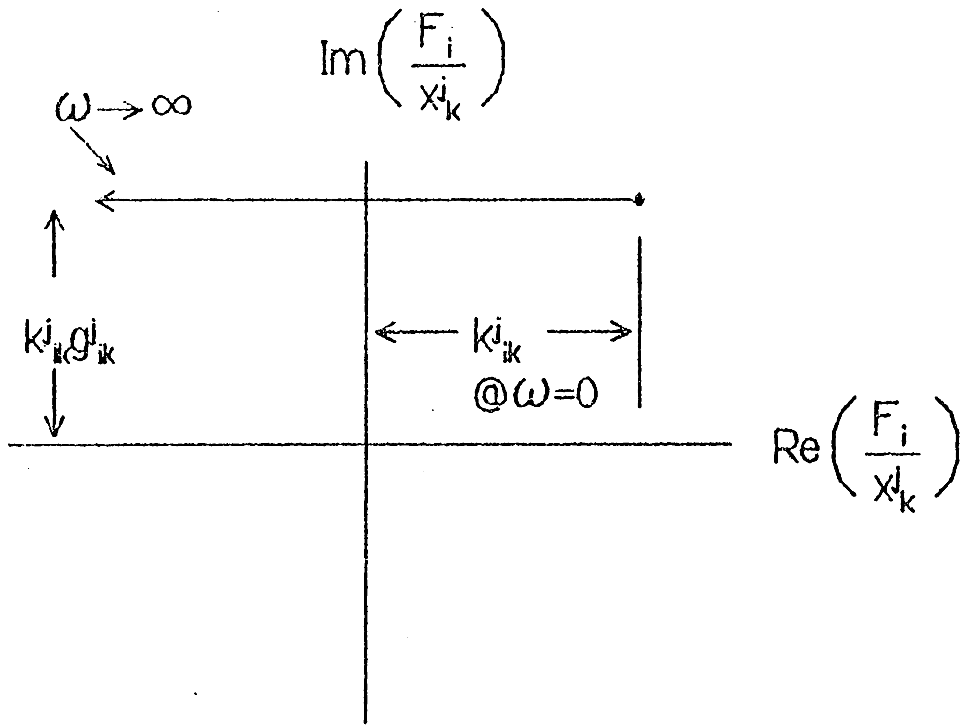


Fig. 6 Nyquist plot of F/x with structural damping; z^* plane.

3.3 ADJACENT MODE CORRECTION

The measured frequency response function is actually a multi-degree-of-freedom system. But in the case where the system is only lightly damped or the modes are well separated, that is, damping is not a dominant factor in system response, then near the resonance points one mode predominates and can be approximated in modal form. The single degree-of-freedom model will give good results. When the effect of the nearby modes dominates, the circle in the Nyquist plot will shift away from the origin. Kennedy and Pancu [1] found that improved results can be obtained by moving the circle back to origin tangency, as shown in Fig. 7 or Fig. 8. The effective parameters are therefore extracted from the data of this corrected circle.

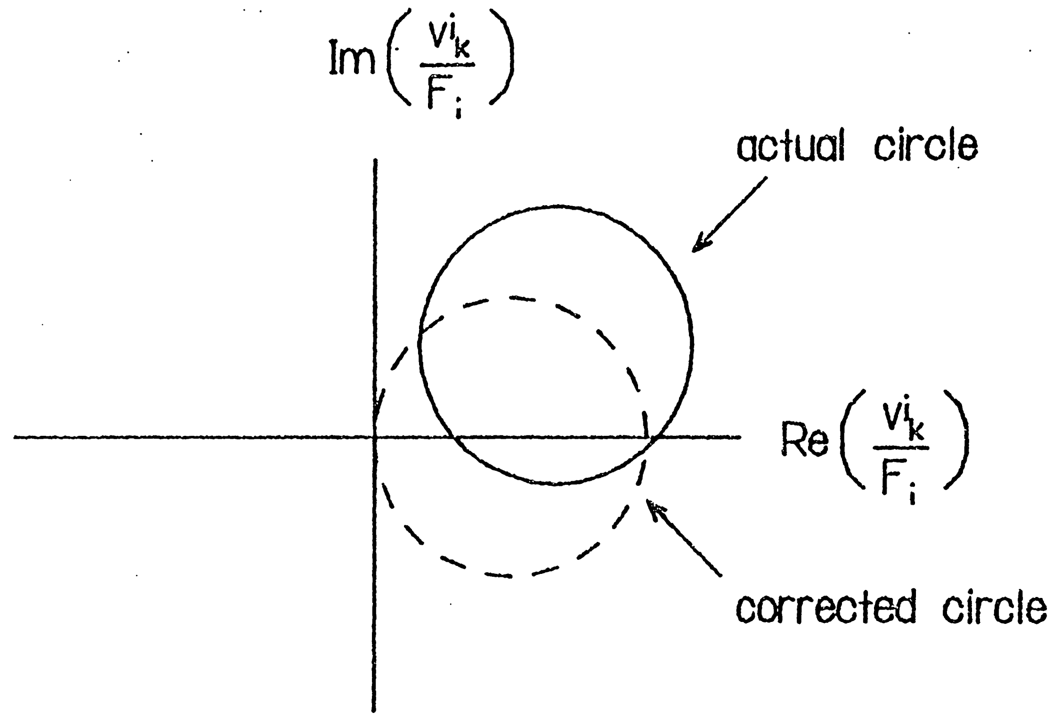


Fig. 7 Corrected circle for viscous damping model; $1/z^*$ plane.

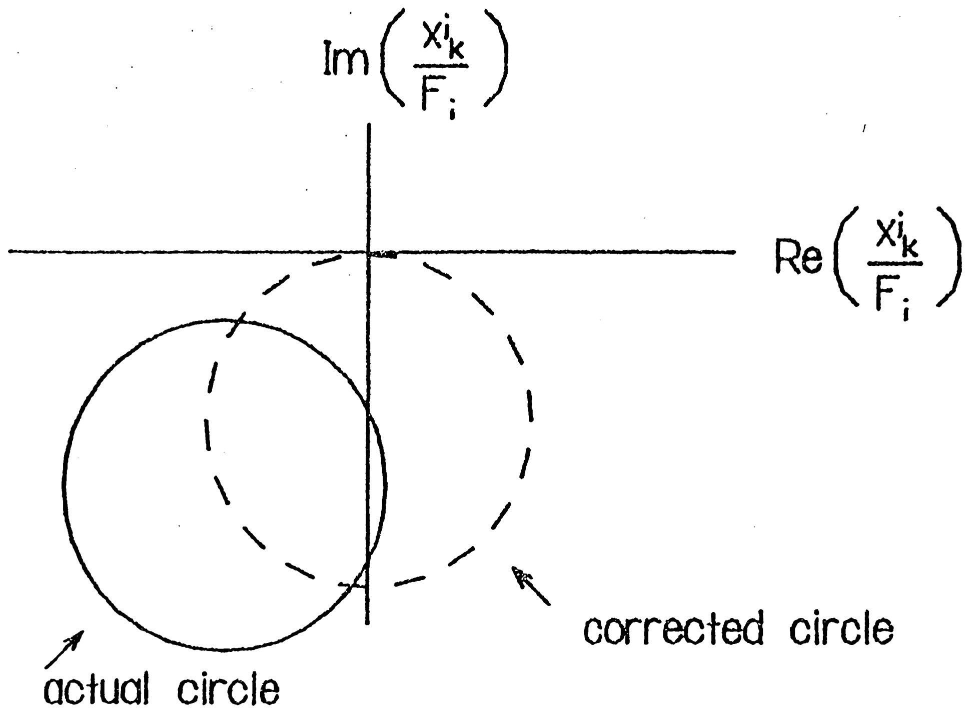


Fig. 8 Corrected circle for structural damping model; $1/z^*$ plane.

3.4 MODAL MATRIX EXTRACTION

The modal matrix or mode shape is extracted by the real part of the mobility type freuqncy response function, or the imaginary part of the dynamic compliance and inertance type frequency response function.

If the "circle fit" method is used, the measured frequency response function data is fitted to a circle using a least square error fit. The center and diameter are obtained and the modal displacement is determined from the diameter of the "circle fit" circle [19]. The procedure is repeated for all points in the structure. This provides sufficient deflection vector information to determine the deflected dynamic mode shape. Also, an animated plot can be displayed at each mode. If all the vectors are combined together, this matrix is called modal matrix.

3.5 EXTRACTION OF SYSTEM PARAMETERS

The effective parameters: effective mass, stiffness, and damping are extracted from the straight line curve in the Z^* plane through a regression analysis technique.

3.5.1 VISCOUS DAMPING MODEL

The effective equation of motion at the j^{th} mode with viscous damping for a mobility type frequency response function is given in Eq. 3.25. In the Z^* complex plane, the real and the imaginary part is given in Eq. 3.27. A polynomial regression is set up for the imaginary part.

$$\mathrm{Im}\left(\frac{F_i}{v_k^j}\right) = m_{ik}^j \omega - k_{ik}^j \omega^{-1} \tag{3.30}$$

or

$$a = b\omega - c\omega^{-1} \tag{3.31}$$

where

$$a = \mathrm{Im}\left(\frac{F_i}{v_k^j}\right), \; b = m_{ik}^j, \; c = k_{ik}^j$$

Rearranging, one gets (3.32)

$$b\omega - c\omega^{-1} - a = 0$$

Using a least square error fit, and let the error be E, one gets

$$\sum_{i=1}^{r} E^2 = \sum_{i=1}^{r} (b\omega_i - c\omega_i^{-1} - a_i)^2 \qquad (3.33)$$

Taking the derivative of Eq. 3.33 with respect to B and D respectively,

$$\left.\begin{aligned} \frac{\partial(\Sigma E^2)}{\partial b} &= 2\sum_{i=1}^{r} (b\omega_i - c\omega_i^{-1} - a_i)\omega_i \\ \frac{\partial(\Sigma E^2)}{\partial c} &= 2\sum_{i=1}^{r} (b\omega_i - c\omega_i^{-1} - a_i)(-\omega_i^{-1}) \end{aligned}\right\} (3.34)$$

Setting the above equations to zero, and writing them in matrix form, one gets

$$\begin{bmatrix} \sum_{i=1}^{r} \omega_i^2 & -r \\ -r & \sum_{i=1}^{r} \omega_i^{-2} \end{bmatrix} \begin{Bmatrix} b \\ c \end{Bmatrix} = \begin{bmatrix} \sum_{i=1}^{r} a_i\omega_i \\ -\sum_{i=1}^{r} a_i\omega_i^{-1} \end{bmatrix} \qquad (3.35)$$

The unknowns b and c can be solved by

$$\left\{\begin{array}{c} b \\ \hline c \end{array}\right\} = \left[\begin{array}{c|c} \sum_{i=1}^{r} \omega_i^2 & -r \\ \hline -r & \sum_{i=1}^{r} \omega_i^{-2} \end{array}\right]^{-1} \left[\begin{array}{c} \sum_{i=1}^{r} a_i \omega_i \\ \hline -\sum_{i=1}^{r} a_i \omega_1^{-1} \end{array}\right] \qquad (3.36)$$

where $m_{ik}^j = b$, $k_{ik}^j = c$ and c_{ik}^j can be obtained from the statistical average of the real part, $\mathrm{Re}\left(\frac{F_i}{v_k^j}\right)$. Thus, effective mass, stiffness, and damping have been determined.

3.5.2 STRUCTURAL DAMPING MODEL

The effective equation of motion at the j^{th} mode with structural damping is given in Eq. 3.20, and the real and imaginary part in the Z^* complex plane is given in Eq. 3.29. A polynomial regression is set up for the real part.

$$\mathrm{Re}\left(\frac{F_i}{x_k^j}\right) = k_{ik}^j - m_{ik}^j \omega^2 \qquad (3.37)$$

or

$$a = b\omega^2 + c \qquad (3.38)$$

where

$$a = \mathrm{Re}\left(\frac{F_i}{x_k^j}\right), \; b = m_{ik}^j, \; c = k_{ik}^j$$

Rearranging,

$$-b\omega^2 + c - a = 0 \tag{3.39}$$

Using a least square error fit gives

$$\left\{ \begin{array}{c} b \\ \hline c \end{array} \right\} = \left[\begin{array}{c|c} \sum_{i=1}^{r} \omega_i^4 & -\sum_{i=1}^{r} \omega_i^2 \\ \hline -\sum_{i=1}^{r} \omega_i^2 & r \end{array} \right]^{-1} \left[\begin{array}{c} -\sum_{i=1}^{r} \omega_i^2 a_i \\ \hline \sum_{i=1}^{r} a_i \end{array} \right] \tag{3.40}$$

where $m_{ik}^j = b$, $k_{ik}^j = c$

3.6 MODAL PARAMETERS FROM EFFECTIVE PARAMETERS

The modal mass, stiffness, and damping can be determined from the effective parameters by using Eq. 3.14 for the viscous damping model, or Eq. 3.21 for the structural damping model.

3.7 PHYSICAL SPACE MODELS FROM MODAL PARAMETERS

From the modal parameters a physical space model can be obtained. This procedure can be described in two parts: complete or truncated systems.

3.7.1 COMPLETE SYSTEMS

If the modal description is a complete one, that is, the number of degrees-of-freedom is the same as the number of modes identified, the physical mass, stiffness, and damping can be found from Eq. 3.5 by pre- and post-multiplying by the appropriate inverses to get

For a viscous damping model,

$$\left.\begin{aligned} [M] &= [P^T]^{-1} \lceil m \rfloor [P]^{-1} \\ [K] &= [P^T]^{-1} \lceil k \rfloor [P]^{-1} \\ [C] &= [P^T]^{-1} \lceil c \rfloor [P]^{-1} \end{aligned}\right\} \quad (3.41)$$

where $[M]$, $[K]$, $[C]$, $\lceil m \rfloor$, $\lceil k \rfloor$, $\lceil c \rfloor$, $[P]$ are all of order $n \times n$.

For a structural damping model,

$$\left.\begin{aligned} [M] &= [P^T]^{-1} \lceil m \rfloor [P]^{-1} \\ [K+iKg] &= [P^T]^{-1} \lceil k \rfloor [P]^{-1} \end{aligned}\right\} \quad (3.42)$$

where $[M]$, $[K+iKg]$, $\lceil m \rfloor$, $\lceil k \rfloor$, $[P]$ are all of order $n \times n$.

3.7.2 TRUNCATED SYSTEM

When one measures test data experimentally from a structure, it is usually not possible to obtain a full modal description, i.e., a complete system as described in previous sections, because the structure will have an almost-infinite number of degrees of freedom, possible coordinates, and modes of vibration. It is, therefore, necessary to make a simplified model using a greatly reduced number of measurement points and, usually, an even smaller number of modes. Usually, only a few modes can be identified in the frequency range that is of interest. When the number of measurement points exceeds the number of measured modes, one calls this a truncated modal description. The modal matrix will come out to be non-square, so the ordinary matrix inverse cannot be applied. To deal with this, one needs to use a special inverse called pseudo-inverse [19]. The properties and method to generate this inverse are given in Appendix III. The pseudo-inverse is a generalized inverse that can deal with non-square or square but singular matrix. The physical mass, stiffness, and damping can be found by using the pseudo-inverse. The relation for the development of a physical space mass matrix is given as

$$[M] = [P^T]^+ \lceil m \rfloor [P]^+ \tag{3.43}$$

The proof for Eq. 3.43 will now be given. Using condition (i) from the definition of pseudo-inverse in Appendix III and substituting it into the mass equation of Eq. 3.5. Equation 3.5 was developed originally for a complete system, but it can be shown that it is valid for both complete and truncated systems. The proof is shown in Appendix VI. It is now used for a truncated system.

$$[P]^T [P^T]^+ [P]^T [M] [P] [P]^+ [P] = \lceil m \rfloor \qquad (3.44)$$

Comparing Eq. 3.5 and 3.44, one arrives at

$$[M] = [P^T]^+ [P]^T [M] [P] [P]^+ \qquad (3.45)$$

From Eq. 3.5 we know that $[P]^T [M] [P] = \lceil m \rfloor$.

Therefore, Eq. 3.55 becomes

$$[M] = [P^T]^+ \lceil m \rfloor [P]^+ \qquad (3.46)$$

which is the same as Eq. 3.43. Thus, the proof is established.

Similarly, the physical stiffness and damping can be proved in the same way. The results are listed below.

For a viscous damping model,

$$\begin{aligned} [M] &= [P^T]^+ \lceil m \rfloor [P]^+ \\ [K] &= [P^T]^+ \lceil k \rfloor [P]^+ \\ [C] &= [P^T]^+ \lceil c \rfloor [P]^+ \end{aligned} \qquad (3.47)$$

where $[M]$, $[K]$, $[C]$ are of order $n \times n$

$\lceil m \rfloor$, $\lceil k \rfloor$ $\lceil c \rfloor$ are of order $\ell \times \ell$

$[P]$ is of order $n \times \ell$

$[P]^+$ is of order $\ell \times n$

For a structural damping model

$$\begin{aligned} [M] &= [P^T]^+ \lceil m \rfloor [P]^+ \\ [K+iKg] &= [P^T]^+ \lceil k \rfloor [P]^+ \end{aligned} \qquad (3.48)$$

where [M], [K+iKg] are of order n x n

$\lceil m \rfloor$, $\lceil k \rfloor$ are of order $\ell \times \ell$

[P] is of order $n \times \ell$

$[P]^+$ is of order $\ell \times n$

Chapter 4

SYSTEM MODIFICATION VIA MODAL ANALYSIS

4.1 GENERAL

In chapter three, a mathematical model that will describe the dynamic characteristic of a general physical structure is developed. Also, the way to uncouple or diagonalize the equations of motion is shown, that is, to transform the system from physical coordinates to modal coordinates. This transformation requires the system to have proportional damping, that is, the damping matrix must be proportional to the mass and/or stiffness matrices, or the structure exhibits light damping. The difference between complete and truncated systems were also discussed in chapter three. A complete system will yield a square and non-singular modal matrix, [P], while a truncated system will yield a rectangular modal matrix. A pseudo-inverse, therefore, has to be used in the truncated system if a physical space model is needed. Since the test data obtained experimentally from a physical structure is usually a truncated system, one needs to derive all the equations for the truncated system, which will include the special case of a complete system.

The system modification will make use of the mathematical model developed to predict the effects of physical changes on a structure's dynamic properties before they are actually made. This will eliminate the uncertainty, time-consuming and costly process of a trial-and-error prototype development.

The modifications allowed are point mass, linear spring, and linear dashpot (damper) or a combination of all three. There are three ways that modifications can be made. They are: 1) modifications made in the physical coordinates model; 2) modifications made in both the physical and modal coordinates models; and 3) modifications made in the modal coordinates model. One would like to determine the simplest direct and computationally efficient method in physical coordinates.

4.2 VISCOUS DAMPING MODEL

The equations of motion in physical coordinates (Eq. 3.1) is repeated here as

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{F(t)\} \tag{4.1}$$

The equivalent equations of motion in modal coordinates (Eq. 3.4), which will be labeled as modal space I, are repeated here from the last chapter.

$$\lceil m \rfloor_I \{\ddot{q}\}_I + \lceil c \rfloor \{\dot{q}\}_I + \lceil k \rfloor_I \{q\}_I = [P]_I^T \{F(t)\} \tag{4.2}$$

where

$$\left.\begin{aligned} \lceil m \rfloor_I &= [P]_I^T [M] [P]_I \\ \lceil c \rfloor_I &= [P]_I^T [C] [P]_I \\ \lceil k \rfloor_I &= [P]_I^T [K] [P]_I \end{aligned}\right\} \tag{4.3}$$

One wishes to make mass, stiffness, or damping modifications to the dynamical system described by the equations of motion depicted by Eq. 4.1. These modifications are represented by matrix addition to the original mass, stiffness, and damping matrices, that is, a mass modification can be represented by

$$[M] + [\Delta M]$$

where $[\Delta M]$ is a matrix of mass modification in physical space. Similarly, the damping and stiffness modification in physical space can be represented the same way. Derivation of mass, stiffness, and damping modification matrices will be discussed in later sections.

4.2.1 MODIFICATIONS MADE IN THE PHYSICAL COORDINATES MODEL

If the mass, damping, and stiffness modification matrices are added to the original system, a new set of equations of motion will be developed.

$$\Big[[M] + [\Delta M]\Big]\{\ddot{X}\} + \Big[[c] + [\Delta c]\Big]\{\dot{X}\} + \Big[[K] + [\Delta K]\Big]\{X\} = \{F(t)\} \quad (4.4)$$

or

$$[\bar{M}]\{\ddot{X}\} + [\bar{C}]\{\dot{X}\} + [\bar{K}]\{X\} = \{F(t)\} \quad (4.5)$$

where
$$\left.\begin{aligned} [\bar{M}] &= [M] + [\Delta M] \\ [\bar{C}] &= [C] + [\Delta C] \\ [\bar{K}] &= [K] + [\Delta K] \end{aligned}\right\} \quad (4.6)$$

Using the same transformation as in Eq. 3.2 and assuming the structure is proportionally damped, we can transform the equations of motion of Eq. 4.4 to modal coordinates

$$[M+\Delta M]\,[P]_I\{\ddot{q}\}_I + [C+\Delta C]\,\{\dot{q}\}_I + [K+\Delta K]\,[P]_I\{q\}_I = \{F(t)\} \quad (4.7)$$

Notice that the modal matrix used is from the original or unmodified structure. Pre-multiplying Eq. 4.7 by $[P]_I^T$ gives the following equation

$$[\bar{m}]_I \{\ddot{q}\}_I + [\bar{c}]_I \{\dot{q}\}_I + [\bar{k}]_I \{q\}_I = [P]_I^T \{F(t)\} \tag{4.8}$$

where
$$\left.\begin{aligned} [\bar{m}]_I &= [P]_I^T [M+\Delta M] [P]_I \\ [\bar{c}]_I &= [P]_I^T [C+\Delta C] [P]_I \\ [\bar{k}]_I &= [P]_I^T [K+\Delta K] [P]_I \end{aligned}\right\} \tag{4.9}$$

These modal parameters can be rewritten as

$$\left.\begin{aligned} [\bar{m}]_I &= \lceil m \rfloor_I + [\Delta m]_I \\ [\bar{c}]_I &= \lceil c \rfloor_I + [\Delta c]_I \\ [\bar{k}]_I &= \lceil k \rfloor_I + [\Delta k]_I \end{aligned}\right\} \tag{4.10}$$

where
$$\left.\begin{aligned} [\Delta m]_I &= [P]_I^T [\Delta M] [P]_I \\ [\Delta c]_I &= [P]_I^T [\Delta C] [P]_I \\ [\Delta k]_I &= [P]_I^T [\Delta K] [P]_I \end{aligned}\right\} \tag{4.11}$$

Since the modal matrix, [P], is a non-square matrix in a truncated system, the pseudo-inverse method will be used to recover the physical mass of the modified structure. From Eq. 4.9 the physical mass of the modified system can be obtained as

$$\begin{aligned} [M+\Delta M] &= [P^T]_I^+ [\bar{m}]_I [P]_I^+ \\ &= [P^T]_I^+ \lceil m \rfloor_I [P]_I^+ + [P^T]_I^+ [\Delta m] [P]_I^+ \end{aligned} \tag{4.12}$$

The first term can be identified as the physical mass recovered by pseudo-inverse from Eq. 4.3. Therefore, Eq. 4.12 becomes

$$[\bar{M}] = [M] + [P^T]_I^+ [\Delta m]_I [P]_I^+ \tag{4.13}$$

Substituting Eq. 4.11 into Eq. 4.13

$$[\bar{M}] = [M] + [P^T]_I^+ \; [P]_I^T \; [\Delta M] \, [P]_I \; [P]_I^+ \tag{4.14}$$

When modifications are made on the physical coordinate equations the added physical mass has to be pre-conditioned by the pseudo-inverse. The modification is <u>not</u> just $[\Delta M]$, but is $[P^T]_I^+ \; [P]_I^T \; [\Delta M] \, [P]_I \, [P]_I^+$. The reason for this is that pseudo-inverses are not and do not follow the rules of ordinary inverses. Thus, $[P]_I \, [P]_I^+ \neq [I]$.

In the case of a complete system which uses ordinary inverses and has the property $[P]_I \, [P]_I^{-1} = [I]$, the term $[P^T]_I^{-1} \, [P]_I^T \, [\Delta M]_I \, [P]_I \, [P]_I^{-1}$ will collapse to $[\Delta M]$. Therefore, Eq. 4.6 is valid only for making modifications for complete systems.

Similarly, the physical damping and stiffness of the modified structure for truncated systems are

$$\left.\begin{aligned} [\bar{C}] &= [C] + [P^T]_I^+ \; [P]_I^T \, [\Delta C] \, [P]_I \, [P]_I^+ \\ [\bar{K}] &= [K] + [P^T]_I^+ \; [P]_I^T \, [\Delta K] \, [P]_I \, [P]_I^+ \end{aligned}\right\} \tag{4.15}$$

In this method, modifications are added by using Eq. 4.14 and 4.15.

After modifications are made, a new set of eigenvalues and eigenvectors can be obtained by performing an eigenanalysis on

$$\left.\begin{aligned} \left| [\bar{K}]^+ \, [\bar{M}] - \lambda [I] \right| &= 0 \\ \left[[\bar{K}]^+ \, [\bar{M}] - \lambda [I] \right] \{X\} &= \{0\} \end{aligned}\right\} \tag{4.16}$$

n eigenvalues and n eigenvectors will be determined, but only ℓ of them are valid; the rest are for the non-existing part of the system which should be ignored.

At first view this way would have seemed to be the most straightforward and direct way. The modal mass, damping, and stiffness from the unmodified structure, which are obtained by the system modeling described in the last chapter, are converted to physical parameters. The modifications are added in the physical space to yield a mathematical model of the modified structure. Then the dynamic characteristics can be determined from these new physical mass, damping, and stiffness. The modified system is completely defined. But the introduction of the pseudo-inverse in dealing with a truncated system makes this approach more complicated and computationally expensive. The modification is not just a matrix addition. Instead it consists of a product of five matrices, and two of them have to be inverted by the pseudo-inverse method, which is quite complicated as shown by an algorithm described in Appendix V. The inversion is done column by column, and as the order of the matrix goes up, considerable computational time and memory storage is used. Also, the order of physical matrices are of n x n, which is quite large. Therefore, this way is not efficient.

4.2.2 MODIFICATIONS MADE IN BOTH THE PHYSICAL AND MODAL COORDINATES MODELS

Using the system modeling described in chapter three, the modal parameters of the unmodified structure are determined. The modification matrices can be converted from physical coordinates to modal coordinates by Eq. 4.11. Then these modal modification matrices can be added to the modal matrices of the unmodified structure by using Eq. 4.10. Now, the new modified physical matrices can be recovered by using

$$\left.\begin{aligned} [\bar{M}] &= [P^T]_I^+ \, [\bar{m}]_I \, [P]_I^+ \\ [\bar{C}] &= [P^T]_I^+ \, [\bar{c}]_I \, [P]_I^+ \\ [\bar{K}] &= [P^T]_I^+ \, [\bar{k}]_I \, [P]_I^+ \end{aligned}\right\} \qquad (4.17)$$

These equations can be shown easily by matrix operations that they are equivalent to Eq. 4.14 and 4.15. After these physical matrices are obtained, the same eigenanalysis as in Eq. 4.16 can be performed to obtain the new eigenvalues and eigenvectors.

By converting the modifications to modal space I and adding them to the original system in modal space, some computations can be saved comparing to the full physical space method. These resultant modal matrices are then converted to physical mass, damping, and stiffness to form the model of the modified system. The dynamic behavior can be derived from this model. This method also has the disadvantage in that it needs the pseudo-inverse to recover the physical parameters from modal space I. Besides, the order of the matrices are mostly of n x n, which could be quite large in a vibration test. Therefore, this method is not efficient enough.

4.2.3 MODIFICATIONS MADE IN THE MODAL COORDINATES MODEL

The modification matrices in modal space I, as shown in Eq. 4.9, are not diagonalized. To see this, one can look at the mass matrix from Eq. 4.9 in expanded form as

$$[\bar{m}]_I = [P]_I^T [M] [P]_I + [P]_I^T [\Delta M] [P]_I \qquad (4.18)$$

or

$$[\bar{m}]_I = [\,^\frown m_\smile\,] + [\Delta m] \qquad (4.19)$$

The first term on the right-hand side of Eq. 4.19, $[\,^\frown m_\smile\,]$, is a diagonalized mass matrix of the unmodified structure, but the second term, $[\Delta m]$, is not diagonalized because the modal matrix of an unmodified structure will only uncouple the original physical mass matrix and not the mass modification matrix. Therefore, an eigen-analysis is performed on Eq. 4.8 to find a new modal matrix, $[P]_{II}$ in modal space II, to uncouple $[\bar{m}]_I$, $[\bar{c}]_I$, and $[\bar{k}]_I$. The eigen-analysis needed for this operation is

$$\left.\begin{aligned} &\left| [\bar{k}]_I^{-1} [\bar{m}]_I - \lambda[I] \right| = 0 \\ &\left[[\bar{k}]_I^{-1} [\bar{m}]_I - \lambda[I] \right] \{X\} = \{0\} \end{aligned}\right\} \qquad (4.20)$$

The ordinary inverse can be used in Eq. 4.20 because the matrix is square, $\ell \times \ell$, and non-singular.

Equation 4.8, again with the assumption of proportional damping, can be decoupled in terms of another normal coordinate, q_{II}, by using the transformation

$$\{q\}_I = [P]_{II} \{q\}_{II} \tag{4.21}$$

The resultant equations of motion in modal space II is

$$[\bar{m}]_{II} \{\ddot{q}\}_{II} + [\bar{c}]_{II} \{\dot{q}\}_{II} + [\bar{k}]_{II} \{q\}_{II} = [P]_{II}^T [P]_I^T \{F(t)\} \tag{4.22}$$

where

$$\left.\begin{aligned} [\bar{m}]_{II} &= [P]_{II}^T [\bar{m}]_I [P]_{II} \\ [\bar{c}]_{II} &= [P]_{II}^T [\bar{c}]_I [P]_{II} \\ [\bar{k}]_{II} &= [P]_{II}^T [\bar{k}]_I [P]_{II} \end{aligned}\right\} \tag{4.23}$$

The physical matrices can be recovered from modal space II by substituting Eq. 4.9 into Eq. 4.23. Then one arrives at

$$\left.\begin{aligned} [\bar{m}]_{II} &= [P]_{II}^T [P]_I^T [M+\Delta M] [P]_I [P]_{II} \\ [\bar{c}]_{II} &= [P]_{II}^T [P]_I^T [C+\Delta C] [P]_I [P]_{II} \\ [\bar{k}]_{II} &= [P]_{II}^T [P]_I^T [K+\Delta K] [P]_I [P]_{II} \end{aligned}\right\} \tag{4.24}$$

Then the physical matrices are given by

$$\left.\begin{aligned}[\bar{m}] &= \left[[P]_{II}^{T}\ [P]_{I}^{T}\right]^{+} [\bar{m}]_{II} \left[[P]_{I}\ [P]_{II}\right]^{+} \\ [\bar{c}] &= \left[[P]_{II}^{T}\ [P]_{I}^{T}\right]^{+} [\bar{c}]_{II} \left[[P]_{I}\ [P]_{II}\right]^{+} \\ [\bar{k}] &= \left[[P]_{II}^{T}\ [P]_{I}^{T}\right]^{+} [\bar{k}]_{II} \left[[P]_{I}\ [P]_{II}\right]^{+}\end{aligned}\right\} \quad (4.25)$$

The order of modal matrix in modal space I is $n \times \ell$ in general, but the one in modal space II is $\ell \times \ell$; the product of these two is $n \times \ell$.

Full modal coordinate modification requires the computations stay in modal space. The physical space modification matrices are converted to modal space I by Eq. 4.11. They are added on to the original modal matrices as in Eq. 4.10. A modal matrix in modal space II can be obtained by performing an eigenanalysis on Eq. 4.22. Thus, modal mass, damping, and stiffness in modal space II can be determined. The modified system is now completely defined by modal mass, damping, and stiffness in modal space II and the new modified modal matrix, $\left[[P]_{I}\ [P]_{II}\right]$. This method requires only ordinary matrix inverse if the physical mass, damping, and stiffness need not be determined. Another advantage of this method is that all the matrices are in the order of $\ell \times \ell$, where ℓ is usually much smaller than n in a vibration test. This reduces the size of matrices considerably. Thus, memory, as well as computational time, can be reduced greatly. This is the most efficient way of making modifications as shown in Table 5.

4.3 STRUCTURAL DAMPING MODEL

If the development for the viscous damping model is repeated for the structurally damped system, a similar set of equations can be developed.

The equations of motion in physical coordinates for proportional damping is repeated here as

$$[M]\{\ddot{X}\} + (1+ig)\,[K]\{X\} = \{F(t)\} \tag{4.26}$$

After transformation is applied, a set of equations in terms of Modal Space I can be obtained.

$$\lceil m \rfloor_I \{\ddot{q}\}_I + \lceil k \rfloor_I \{q\}_I = \{F(t)\} \tag{4.27}$$

where
$$\left.\begin{aligned} \lceil m \rfloor_I &= [P]_I^T\ [M]\ [P]_I \\ \lceil k \rfloor_I &= [P]_I^T\ [(1+ig)[K]]\ [P]_I \end{aligned}\right\} \tag{4.28}$$

4.3.1 MODIFICATIONS MADE IN THE PHYSICAL COORDINATES MODEL

If the mass and stiffness modification matrices are added to the original system, a new set of equations of motion will be developed.

$$\Big[[M] + [\Delta M]\Big]\{\ddot{X}\} + \Big[(1+ig)\big[[K] + [\Delta K]\big]\Big]\{X\} = \{F(t)\} \tag{4.29}$$

or

$$[\bar{M}]\{\ddot{X}\} + \left[(1+ig)[\bar{K}]\right]\{X\} = \{F(t)\} \tag{4.30}$$

$$\left.\begin{aligned} \text{where } [\bar{M}] &= [M] + [\Delta M] \\ [\bar{K}] &= [K] + [\Delta K] \end{aligned}\right\} \quad (4.31)$$

Using the same transformation as in Eq. 4.27 and using the assumption of proportionally damping, the equations of motion in modal space I are

$$[\bar{m}]_I\{\ddot{q}\}_I + [\bar{k}]_I\{q\}_I = [P]_I^T \{F(t)\} \tag{4.32}$$

$$\left.\begin{aligned} \text{where } [\bar{m}]_I &= [P]_I^T [M+\Delta M] [P]_I \\ [\bar{k}]_I &= [P]_I^T \left[(1+ig)[k+\Delta k]\right]_I [P]_I \end{aligned}\right\} \quad (4.33)$$

The modal parameters can be rewritten as

$$\left.\begin{aligned} [\bar{m}]_I &= \lceil m \rfloor_I + [\Delta m]_I \\ [\bar{k}]_I &= \lceil k \rfloor_I + [\Delta k]_I \end{aligned}\right\} \quad (4.34)$$

$$\left.\begin{aligned} \text{where } [\Delta m]_I &= [P]_I^T [\Delta M] [P]_I \\ [\Delta k]_I &= [P]_I^T \left[(1+ig)[\Delta K]\right] [P]_I \end{aligned}\right\} \quad (4.35)$$

Using the pseudo-inverse, the physical mass and stiffness of the modified structure for truncated system are:

$$\begin{aligned} [\bar{M}] &= [M] + [P^T]_I^+ [P]_I^T [\ M] [P]_I [P]_I^+ \\ (1+ig)[\bar{K}] &= (1+ig)[K] + [P^T]_I^+ + [P]_I^T \left[(1+ig)[\Delta K]\right] [P]_I [P]_I^+ \end{aligned} \quad (4.36)$$

A new set of eigenvalues and eigenvectors of the modified system can be obtained by performing an eigenanalysis as in Eq. 4.16. Again, n eigenvalues and n eigenvectors are obtained, but only ℓ

of them are valid. The rest are for the non-existing part of the system which should be ignored.

4.3.2 MODIFICATIONS MADE IN BOTH THE PHYSICAL AND MODAL COORDINATES MODELS

Equation 4.35 should be used in this method to convert the modification matrices from physical coordinates to modal coordinates. Then these modification matrices can be added to the modal matrices of the unmodified system by using Eq. 4.34. The new modified physical matrices can be recovered by using

$$\left.\begin{aligned} [\bar{M}] &= [P^T]_I^+ \, [\bar{m}]_I \, [P]_I^+ \\ (1+ig)[\bar{K}] &= [P^T]_I^+ \, [\bar{k}] \, [P]_I^+ \end{aligned}\right\} \quad (4.37)$$

The same eigenanalysis as in Eq. 4.16 should be performed to get the new eigenvalues and eigenvectors.

4.3.3 MODIFICATIONS MADE IN THE MODAL COORDINATES MODEL

In order to diagonalize the modifications matrices in modal space I, another eigenanalysis has to be performed on Eq. 4.32. The eigenanalysis needed is

$$\left.\begin{aligned} &\left| [\bar{k}]_I^{-1} \, [\bar{m}]_I - \lambda[I] \right| = 0 \\ &\left[[\bar{k}]_I^{-1} \, [\bar{m}]_I - \lambda[I] \right] \{X\} = \{0\} \end{aligned}\right\} \quad (4.38)$$

The resultant equations of motion in modal space II is

$$[\bar{m}]_{II} \{\ddot{q}\}_{II} + [\bar{k}]_{II} \{q\}_{II} = [P]_I^T [P]_I^T \{F(t)\} \tag{4.39}$$

where

$$\left.\begin{aligned} [\bar{m}]_{II} &= [P]_{II}^T [\bar{m}]_I [P]_{II} \\ [\bar{k}]_{II} &= [P]_{II}^T [\bar{k}]_I [P]_{II} \end{aligned}\right\} \tag{4.40}$$

The physical matrices can be recovered from modal space II matrices by substituting Eq. 4.33 into Eq. 4.40.

$$\left.\begin{aligned} [\bar{m}]_{II} &= [P]_{II}^T [P]_I^T \Big[[M+\Delta M]\Big] [P]_I [P]_{II} \\ [\bar{k}]_{II} &= [P]_{II}^T [P]_I^T \Big[(1+ig)[K+\Delta K]\Big] [P]_I [P]_{II} \end{aligned}\right\} \tag{4.41}$$

And physical matrices are given by

$$\left.\begin{aligned} [\bar{M}] &= \Big[[P]_{II}^T [P]_I^T\Big]^+ [\bar{m}]_{II} \Big[[P]_I [P]_{II}\Big]^+ \\ (1+ig)[\bar{K}] &= \Big[[P]_{II}^T [P]_I^T\Big]^+ [\bar{k}]_{II} \Big[[P]_I [P]_{II}\Big]^+ \end{aligned}\right\} \tag{4.42}$$

The equations needed for this method are Eq. 4.35, 4.34, and 4.39. Then the modified system is completely defined by modal mass, stiffness in modal space II, and the new modified modal matrix, $\Big[[P]_I [P]_{II}\Big]$.

This method, like the one for viscous damping model, requires only ordinary inverse if the physical mass, damping, and stiffness are not needed. All the matrices involved are in the order of $\ell \times \ell$. Thus, the matrix operation is very efficient. This is the most efficient way of making modification to a system with structural damping as shown in Table 5.

4.4 DERIVATION OF MODIFICATION MATRIX

The development of the modification matrix is sometimes not easy to do, especially when the system has all three dimensions, x, y, and z. Therefore, the derivation of the modification matrix is discussed. There are three types of modification matrices: mass, stiffness, and damping modification matrices. All are in physical coordinates. The modification matrices are the same for both viscous and structural damping model.

4.4.1 MASS MODIFICATION MATRIX

The mass modification matrix is the simplest one among the three. A three dimensional physical mass modification matrix is shown in Fig. 9. It is a diagonal matrix with non-zero entries at locations where the mass modification is to be effective. The rest of the entries are zero. The mass modification matrix in Fig. 9 shows a case where a mass of magnitude Δm is added at point number two, and it is effective in all x, y, and z directions. If a mass removal is needed instead, a negative mass coefficient should be placed in the appropriate entries of the matrix.

$$
[\Delta M] = \begin{array}{c}
\begin{array}{cccccccccccccccc} 1x & 1y & 1z & 2x & 2y & 2z & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & nx & ny & nz & \end{array} \\
\left[\begin{array}{ccccccccccccccc}
0 & & & & & & & & & & & & & & \\
 & 0 & & & & & & & & & & & & & \\
 & & 0 & & & & & & & & & & & & \\
 & & & \Delta m & & & & & & & & & 0 & & \\
 & & & & \Delta m & & & & & & & & & & \\
 & & & & & \Delta m & & & & & & & & & \\
 & & & & & & \cdot & & & & & & & & \\
 & & & & & & & \cdot & & & & & & & \\
 & & & & & & & & \cdot & & & & & & \\
 & & & & & & & & & \cdot & & & & & \\
 & & & & & & & & & & \cdot & & & & \\
0 & & & & & & & & & & & \cdot & & & \\
 & & & & & & & & & & & & 0 & & \\
 & & & & & & & & & & & & & 0 & \\
 & & & & & & & & & & & & & & 0
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1x \\ 1y \\ 1z \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ nx \\ ny \\ nz \end{array}
$$

Fig. 9 A three dimensional mass modification matrix with a mass of magnitude Δm added to point #2, effective in x, y, and z directions.

4.4.2 STIFFNESS MODIFICATION MATRIX

If the system is one dimensional only, the stiffness modification matrix can be obtained by isolating the lumped mass as a free-body, summing the vertical forces. It can also be done by any number of classical or operational techniques [20].

Another way is to use a "force node" technique [21], which says: "At each point of common velocity (force node, usually a mass), equate the force generator output to the sum of impedances attached to that point multiplied by the velocity of the force node while subtracting the impedance of each element multiplied by the velocity of its extreme connection point."

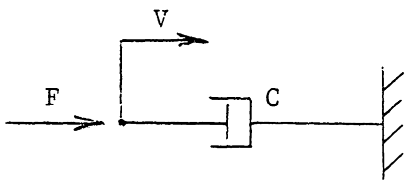
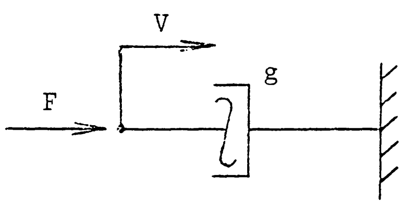
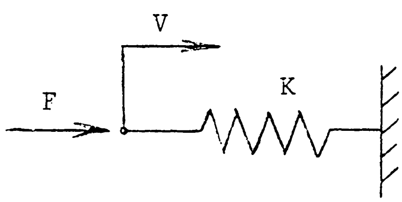
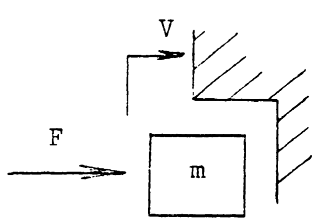
Impedance is the matrix inverse of the mobility. Mobility is defined as $Y_{ki} = \frac{V_k}{F_i}$ when Y_{ki} is the velocity response at point k with a unit dynamic force applied at point i with no dynamic force applied at all other points in the structure. To use the force node technique, one must review the impedance/mobility functions in Table 2.

For example, a stiffness, Δk, is added between mass i and j, where k_1, k_2, and k_3 are the stiffness originally in the structure, as shown in Fig. 10. Assuming steady-state motion and using the force node technique, one arrives at the two following equations:

$$\left.\begin{aligned} F_i &= (Z_{mi} + Z_{k1} + Z_{k2} + Z_{\Delta k})V_i - (Z_{k2} + Z_{\Delta k})V_j - (Z_{k1})V_{i-1} \\ F_j &= (Z_{mj} + Z_{k2} + Z_{\Delta k} + Z_{k3})V_j - (Z_{k2} + Z_{\Delta k})V_i - (Z_{k3})V_{j+1} \end{aligned}\right\} \quad (4.44)$$

Since we are interested only at obtaining the stiffness modification

Table 2. Impedance/mobility functions for simple elements [22]

Element	Impedance	Mobility
 Viscous Damper	$Z_C = C$	$Y_C = \frac{1}{C}$
 Structural Damper	$Z_S = \frac{gK}{\omega}$	$Y_S = \frac{\omega}{gK}$
 Stiffness	$Z_K = \frac{-iK}{\omega}$	$Y_K = \frac{i\omega}{K}$
 Mass	$Z_m = i\, m\omega$	$Y_m = \frac{-i}{m\omega}$

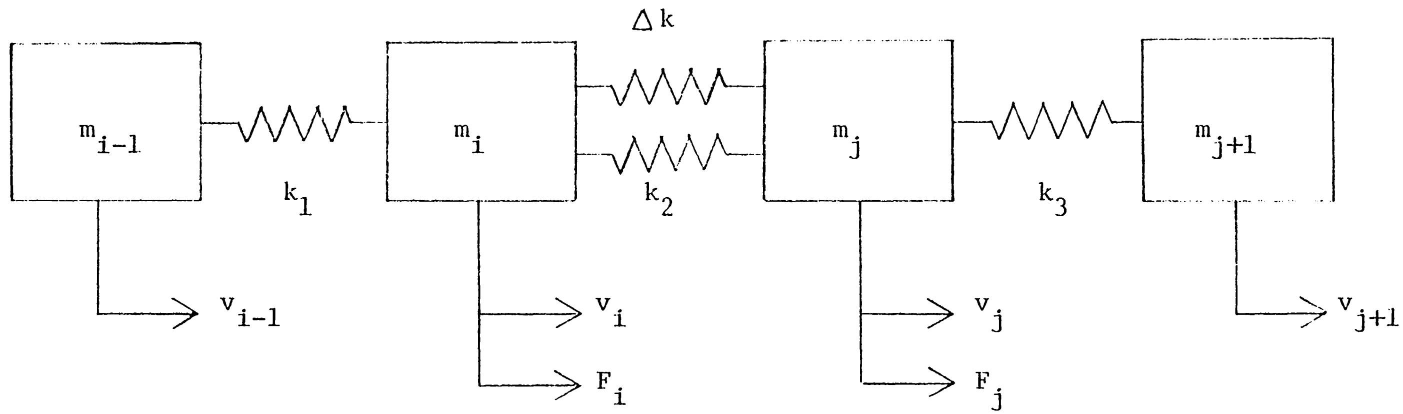


Fig. 10 Stiffness modification, k, added between two lumped masses i and j

matrix, all the terms except for the stiffness modification, Z_Δ , can be set to zero. Writing them in matrix form, one arrives at

$$\begin{bmatrix} Z_{\Delta k} & -Z_{\Delta k} \\ -Z_{\Delta k} & Z_{\Delta k} \end{bmatrix} \begin{Bmatrix} V_i \\ V_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix} \tag{4.45}$$

Converting back to mechanical compliance by using the relations

$$Z_{\Delta k} = \frac{-i\Delta k}{\omega} \tag{4.46}$$

$$i\omega V = X \tag{4.47}$$

one arrives at

$$\Delta k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} x_i \\ x_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix} \tag{4.48}$$

The stiffness modification matrix is the square and symmetric matrix with the coefficient Δk.

For a general three-dimensional system, the development is much more difficult by the above-mentioned methods. Therefore, a general method of deriving the stiffness modification matrix of three-dimensional structures is desired. Cook [23] has derived a two-dimensional stiffness matrix. The three-dimensional one presented here is just an extension of his development. Martin [24] has developed the same matrix for a space truss.

In the following, the member is assumed to be uniform, pin-connected at its ends, linearly elastic, and axially loaded. The actual displacements are assumed to be small.

Figure 11 shows a single element with an i, j node on each end, arbitrarily placed in the x-y-z space. The coefficient of stiffness modification, Δk, is in terms of the constant cross-sectional area,

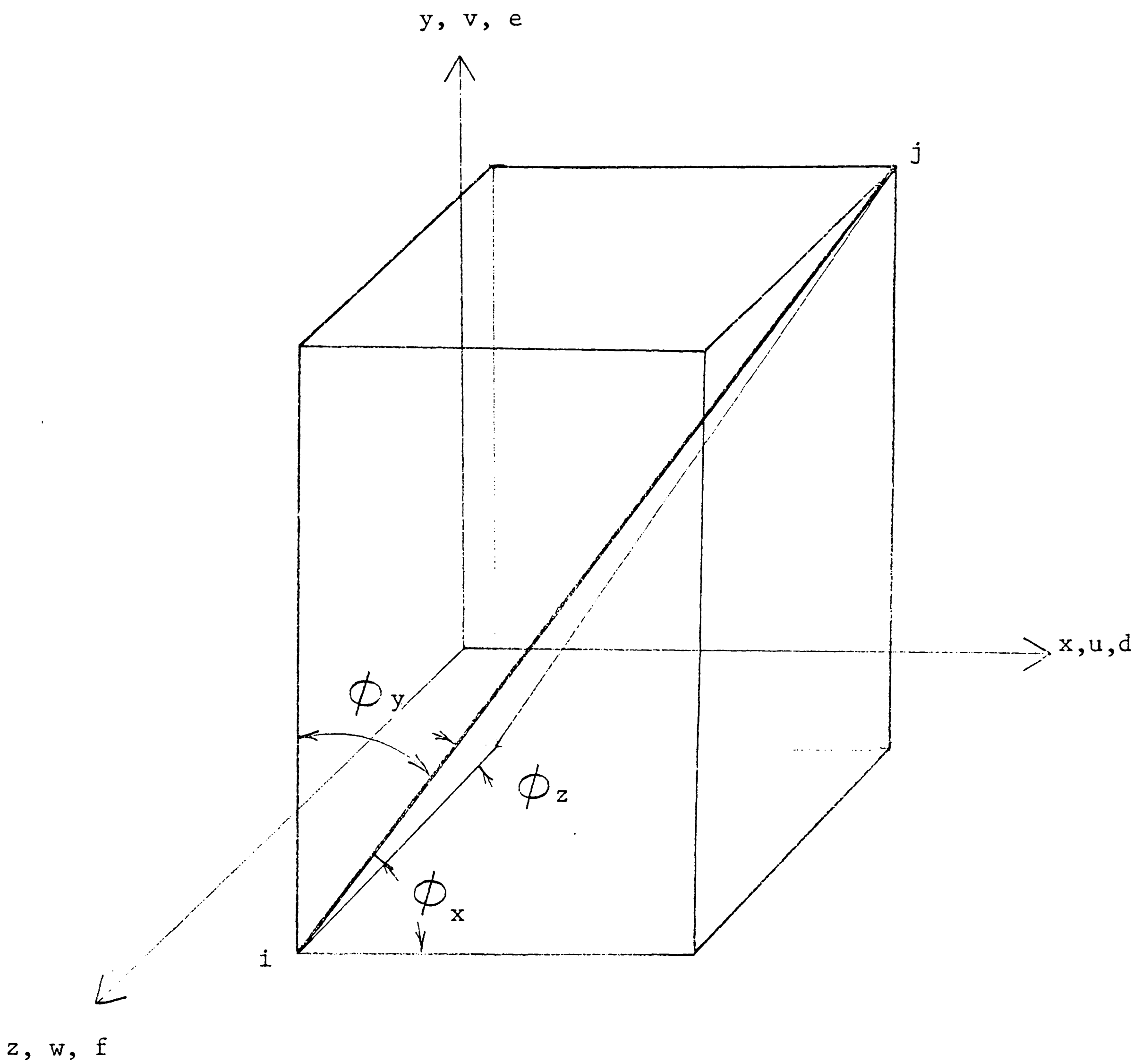


Fig. 11 Single element connecting between node i and j in x, y, z space.

the elastic modulus, and the element length, L. The sine and cosine of the angle ϕ are denoted by s and c, subscripted by x, y, z for directions, may be evaluated from the nodal coordinates. These are normally termed direction sines and cosines.

$$\left.\begin{aligned}
c_x &= \cos\phi_x = \frac{x_j - x_i}{L} \\
s_x &= \sin\phi_x = \frac{\sqrt{(y_j - y_i)^2 + (z_j - z_i)^2}}{L} \\
c_y &= \cos\phi_y = \frac{y_j - y_i}{L} \\
s_y &= \sin\phi_y = \frac{\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}}{L} \\
c_z &= \cos\phi_z = \frac{z_j - z_i}{L} \\
s_z &= \sin\phi_z = \frac{\sqrt{(x_j - x_i)^2 + (z_j - z_i)^2}}{L}
\end{aligned}\right\} \quad (4.49)$$

where $L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$

If each node is displaced a small amount in each coordinate direction, while all other nodal displacements are prohibited, the amount, $c_x u_i$, which is compressed axially will produce an axial compressive force ($\Delta k c_x u_i$). In order to maintain static equilibrium in the system, the x, y, and z components of this force must be equilibrated by the external forces d_i, e_i, f_i, d_j, e_j, and f_j. Therefore,

$$\Delta k \begin{Bmatrix} c_x^2 \\ c_x c_y \\ c_x c_z \\ -c_x^2 \\ -c_x c_y \\ -c_x c_z \end{Bmatrix} u_i = \begin{Bmatrix} d_i \\ e_i \\ f_i \\ d_j \\ e_j \\ f_j \end{Bmatrix} \qquad (4.50)$$

If similar analyses are performed with the displacements v_i, w_i, u_j, v_j, and w_j, the result is the following matrix.

$$\Delta k \begin{bmatrix} c_x^2 & c_x c_y & c_x c_z & -c_x^2 & -c_x c_y & -c_x c_z \\ c_x c_y & c_y^2 & c_y c_z & -c_x c_y & -c_y^2 & -c_y c_z \\ c_x c_z & c_y c_z & c_z^2 & -c_x c_z & -c_y c_z & -c_z^2 \\ -c_x^2 & -c_x c_y & -c_x c_z & c_x^2 & c_x c_y & c_x c_z \\ -c_x c_y & -c_y^2 & -c_y c_z & c_x c_y & c_y^2 & c_y c_z \\ -c_x c_z & -c_y c_z & -c_z^2 & c_x c_z & c_y c_z & c_z^2 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \\ u_j \\ v_j \\ w_j \end{Bmatrix} = \begin{Bmatrix} d_i \\ e_i \\ f_i \\ d_j \\ e_j \\ f_j \end{Bmatrix} \qquad (4.51)$$

The square and symmetric matrix, including the coefficient Δk, is the element stiffness modification matrix. This matrix requires only the cosine of the angle ϕ. This method is a systematic and easily automated way of building a three-dimensional stiffness modification matrix of an arbitrary element connected between two points.

If the system has only one dimension, say x, then

$L = x_j - x_i$

for $y_j - y_i = 0$

$z_j - z_i = 0$

Also, the cosine of the angle ϕ are

$c_x = 1$

$c_y = 0$

$c_z = 0$

Using the above, the stiffness modification matrix will collapse to

$$\Delta k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} d_i \\ d_j \end{Bmatrix} \qquad (4.52)$$

which is the same result obtained by the conventional classical approach or the "force node" technique.

If removal of stiffness is required, a negative coefficient of the stiffness modification should be used.

4.4.3 DAMPING MODIFICATION MATRIX

The same derivation for developing the stiffness modification matrix can be used for the damping, except that, of course, the stiffness coefficient should be replaced by the damping coefficient.

Chapter 5

RESONANCE SPECIFICATION AND FREQUENCY RESPONSE FUNCTION

5.1 RESONANCE SPECIFICATION

Sometimes, a designer or engineer may wish to reduce the vibration level at a certain frequency. This can be done by shifting the resonance peak away from that frequency. This part will determine the amount of physical mass or stiffness modifications needed to move a particular resonance peak to the desired frequency.

Resonance specification can be done in either physical or modal coordinates.

5.1.1 RESONANCE SPECIFICATION IN PHYSICAL COORDINATES

For both viscous and structural damping model, the determinant equation for computing the required modification at a specified frequency is

$$\left| [\bar{k}]^{+} [\bar{M}] - \lambda [I] \right| = 0$$

5.1.2 RESONANCE SPECIFICATION IN MODAL COORDINATES

The determinant equation for both viscous and structural damping models is

$$\left| [\bar{k}]^{-1} [\bar{m}] - \lambda [I] \right| = 0 \tag{5.2}$$

A half-interval search is used to find the amount of mass or stiffness modifications that will make the determinant reduced to zero at the specified frequency.

One has to be cautious when a three-dimensional system is used. Otherwise, the half-interval search may not converge. For example, suppose a structure has been measured only in the z direction so that the modal matrix has only z direction data. If a stiffness modification results in x and y directions, the product of the transpose of the modal matrix and the stiffness modification matrix will always be zero. Thus, the resultant stiffness modification matrix in modal space will also be zero. Then the system will remain unmodified even though many increments of the modification coefficient have been used because the modification term in the determinant equation is always zero. In this way, the search will never converge. Therefore, if only z direction data of the modal matrix is available, only modifications in the z direction can be allowed to the system.

5.2 FREQUENCY RESPONSE FUNCTION SYNTHESIS

It is practically impossible to store all the frequency response functions during a vibration test. Sometimes, the measurement may be difficult or physically impossible to obtain for certain points on the structure. Also, the frequency response function synthesis can be used to verify the accuracy of the experimentally measured frequency response function which is frequently contaminated by noise or other sources. Therefore, a capability to synthesize a frequency response function between test points on the structure which were or were not measured is needed. There are two ways that a frequency response function synthesis can be done: one in physical coordinates and the other in modal coordinates.

5.2.1 FREQUENCY RESPONSE FUNCTION SYNTHESIS IN PHYSICAL COORDINATES

If the physical mass, stiffness, and damping are determined for a system, a frequency response function can be synthesized.

For a viscous damping model, the dynamic compliance is

$$\left[\frac{X}{F}\right] = \Big[[K] - \omega^2[M] + i\omega[C]\Big]^{+} \tag{5.3}$$

For a structural damping model, the dynamic compliance is

$$\left[\frac{X}{F}\right] = \Big[(1+ig)[K] - \omega^2[M]\Big]^{+} \tag{5.4}$$

The 2n x 2n resultant matrix contains all frequency response function, and any one can be chosen. The disadvantage here is that it needs to compute the inverse of the system matrix for each data point plotted, which will require a great deal of computational time and memory space in order for one frequency response function to be synthesized!

5.2.2 FREQUENCY RESPONSE FUNCTION SYNTHESIS IN MODAL COORDINATES

For a viscous damping model with the assumption of harmonic excitation, one substitutes

$$\{x\} = \{X\}\, e^{i\omega t} \tag{5.5}$$

into Eq. 3.1, and obtain

$$\Big[[K] - \omega^2[M] + i\omega[C]\Big]\{X\} = \{F\} \tag{5.6}$$

or

$$\Big[[K] - \omega^2[M] + i\omega[C]\Big] = \left[\frac{F}{X}\right] \tag{5.7}$$

Equation 5.7 is in physical space; one can transform it to modal space I with the assumption of proportional damping.

$$[P]_I^T \left[[K] - \omega^2[M] + i\omega[c]\right] [P]_I = [P]_I^T \left[\frac{F}{X}\right] [P]_I \tag{5.8}$$

The term at the right-hand side of Eq. 5.8 is equal to a diagonal matrix

$$[P]_I^T \left[\frac{F}{X}\right] [P]_I = \left[\diagdown \frac{F}{X} \diagdown\right]_I \tag{5.9}$$

Using the pseudo-inverse on the modal matrix, $[P]_I$, for general case, one can arrive

$$\left[\frac{F}{X}\right] = [P^T]_I^+ \left[\diagdown \frac{F}{X} \diagdown\right]_I [P]_I^+ \tag{5.10}$$

The inverse of Eq. 5.10 is then

$$\left[\frac{X}{F}\right] = [P]_I \left[\diagdown \frac{X}{F} \diagdown\right]_I [P]_I^T \tag{5.11}$$

or in component form

$$\frac{x_i}{F_k} = \sum_{j=1}^{\ell} P_i^j \left(\frac{x}{F}\right)^j P_k^j \tag{5.12}$$

The $\left(\frac{x}{F}\right)^j$ term in Eq. 5.12 can be obtained from Eq. 3.13, which is the dynamic compliance in modal space I. This term contains both real and imaginary part of the frequency response function. The advantage of using Eq. 5.12 for synthesizing a frequency response function is that every term of the summation is a scalar; no matrix operation is involved. It sums up for all the modes required.

If the modal parameters can be identified for a system, any one of the frequency response function can be synthesized. Therefore, only the modal parameters need to be stored for the synthesized. In this way, very little memory space is used.

For a structural damping model, the same procedure for the viscous damping model can be followed; the resultant scalar equation is exactly the same as Eq. 5.12.

Chapter 6

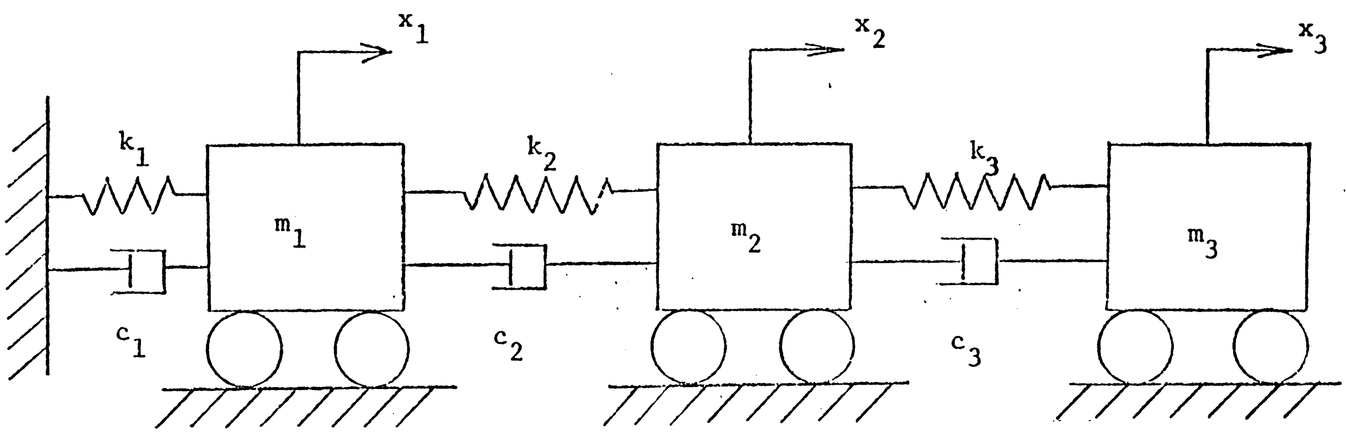
VERIFICATION OF THE SYSTEM MODELING AND MODIFICATION

A theoretical three degree-of-freedom system and two experimental systems were used to verify the theories which were developed in Chapters 3, 4, and 5.

6.1 THEORETICAL THREE DEGREE-OF-FREEDOM SYSTEM

The theoretical three degree-of-freedom system is one which has proportional and viscous damping, as shown in Fig. 12. Its physical parameters are also listed in Fig. 12. The equation of motion of this system is shown in Fig. 13 in matrix form. An eigenanalysis was performed on the system to obtain the natural frequencies and modal matrix listed in Fig. 13. Using this modal matrix and the physical mass, stiffness, and damping, the theoretical modal mass, stiffness, and damping are computed using Eq. 3.5, also listed in Fig. 13.

The system modeling is first verified by generating the real and imaginary parts of frequency response function from the physical mass, stiffness, and damping of the system. Then these data are passed to the BASIC MODAL®, a modal analysis program. The data are converted to mobility type frequency response function and are used



$m_1 = 1$ kg	$c_1 = 5 \frac{\text{N-sec}}{\text{m}}$	$k_1 = 100{,}000$ N/m
$m_2 = 2$ kg	$c_2 = 10 \frac{\text{N-sec}}{\text{m}}$	$k_2 = 200{,}000$ N/m
$m_3 = 4$ kg	$c_3 = 15 \frac{\text{N-sec}}{\text{m}}$	$k_3 = 300{,}000$ N/m

Fig. 12 Theoretical three degree-of-freedom system with viscous damping.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{bmatrix} 15 & -10 & 0 \\ -10 & 25 & -15 \\ 0 & -15 & 15 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{bmatrix} 300{,}000 & -200{,}000 & 0 \\ -200{,}000 & 500{,}000 & -300{,}000 \\ 0 & -300{,}000 & 300{,}000 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ 0 \end{Bmatrix} \sin(\omega) t$$

Natural Frequencies:

1 = 15.4908 Hz

2 = 68.0964 Hz

3 =104.6630 Hz

Modal Matrix

$$[P] = \begin{bmatrix} -.4126 & -.8147 & .8282 \\ -.5993 & -.4764 & -.5485 \\ -.6860 & .3306 & .1151 \end{bmatrix}$$

Modal Mass

$$[m] = \begin{bmatrix} 2.7709 & 0 & 0 \\ 0 & 1.5548 & 0 \\ 0 & 0 & 1.3406 \end{bmatrix}$$

Modal Stiffness

$$[k] = \begin{bmatrix} 26{,}250.0713 & 0 & 0 \\ 0 & 284{,}630.0905 & 0 \\ 0 & & 579{,}754.9297 \end{bmatrix}$$

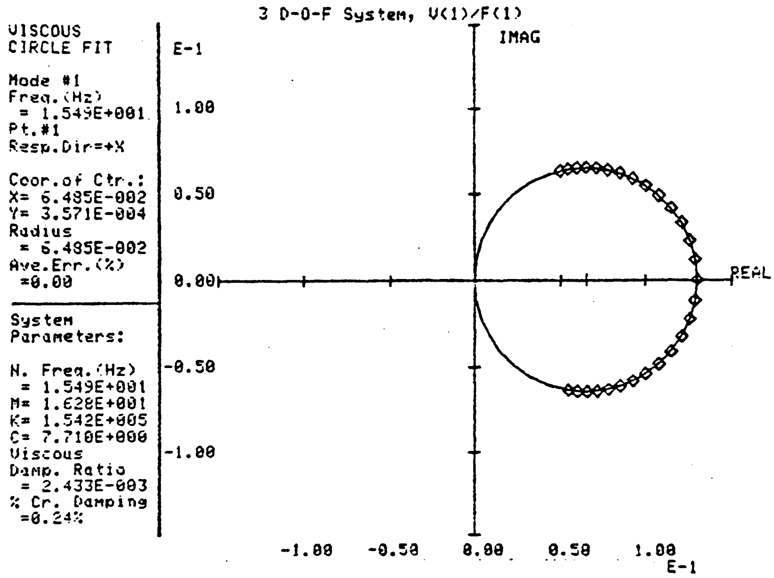
Modal Damping

$$[c] = \begin{bmatrix} 1.3125 & 0 & 0 \\ 0 & 14.2315 & 0 \\ 0 & 0 & 28.9877 \end{bmatrix}$$

Fig. 13 Equation of motion and modal parameters of the theoretical three degree-of-freedom system.

in the circle fitting because only mobility type frequency response function is a perfect circle in a Nyquist plane for a system with viscous damping. The data are then corrected for any adjacent mode effect and inverted in an inverse Nyquist plane. The effective parameters--effective mass, stiffness, and damping--are extracted using the method described in Chapter 3. Two typical circle fittings, the effective parameters, and the inverse Nyquist plots for modes #1 and 3 in $V1_{/F_1}$ and $V1_{/F_3}$ type of frequency response function are shown in Figs. 14 and 15. The straight line in the inverse Nyquist plane of Fig. 15 is inclined at an angle because the effect of the adjacent mode dominates. A summary of the results of all circle fittings are listed in Table 3. The negative sign is added to the radius of the circle when the circle is located on the negative real axis to indicate a 180° phase shift. The modal matrix is obtained by using the diameter of the circle and is listed in Table 4a. The numerical values look different than the one obtained theoretically. But it can be shown that they generate the same modal matrix when the numeric values are normalized such that the first entry of each column is unity.

Next, this modal matrix is normalized so that it is the same as the theoretical one for the purpose of comparing the modal parameters. The effective parameters obtained from BASIC MODAL® are converted to modal parameters using Eq. 3.14. The results are listed in Table 4b. The effective parameters from any one of the frequency response function converts to approximate the same global



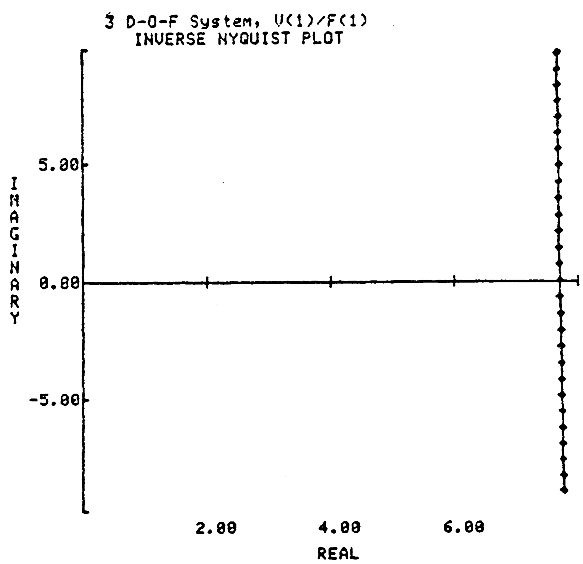
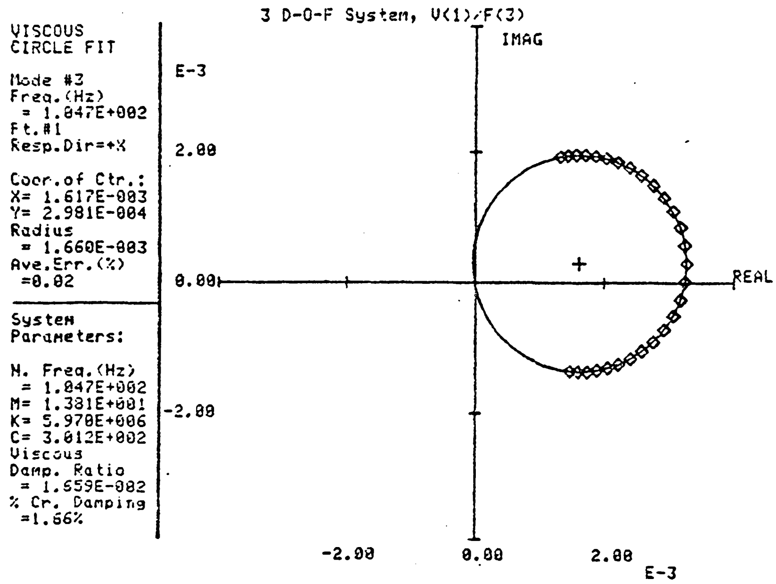


Fig. 14 Nyquist and inverse Nyquist plots of the theoretical three degree-of-freedom systems, mode #1.



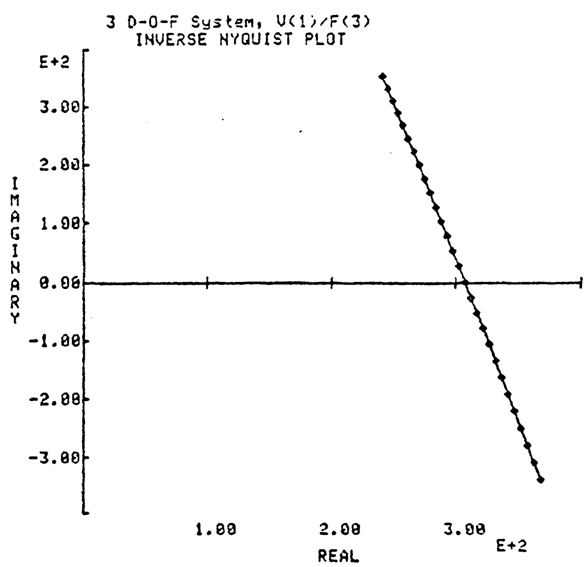


Fig. 15 Nyquist and inverse Nyquist plots of the theoretical three degree-of-freedom systems, mode #3.

Table 3 Effective parameters and radius of circle of the theoretical three degree-of-freedom obtained by BASIC MODAL®.

FRF type	Mode NO.	Nat Freq.	Effective Mass	Effective Stiffness	Effective Damping	Radius of Circle
V_1/F_1	1	15.49	16.28	1.542×10^{5}	7.710	6.485×10^{-2}
	2	68.10	2.347	4.297×10^{5}	21.46	2.330×10^{-2}
	3	104.70	1.970	8.519×10^{5}	42.43	1.178×10^{-2}
V_1/F_2	1	15.49	11.21	1.062×10^{5}	5.308	9.420×10^{-2}
	2	68.10	3.999	7.320×10^{5}	36.64	1.365×10^{-2}
	3	104.70	2.930	1.267×10^{6}	63.57	-7.865×10^{-3}
V_1/F_3	1	15.49	9.790	9.275×10^{4}	4.637	1.078×10^{-1}
	2	68.10	5.765	1.055×10^{6}	52.80	-9.469×10^{-3}
	3	104.70	13.81	5.970×10^{6}	301.2	1.660×10^{-3}

Table 4a Modal matrix used as a basis to convert effective parameters to modal parameters given in Table 4b.

$$[p] = \begin{bmatrix} .1297 & .0466 & .0236 \\ .1884 & .0273 & -.0157 \\ .3560 & -.0189 & .0033 \end{bmatrix}$$

Table 4b Modal parameters of the theoretical three degree-of-freedom system from various frequency response function, obtained by BASIC MODIFICATION®

FRF type	Mode No.	Natural Frequency (Hz)	Modal Mass	Modal Stiffness	Modal Damping
V_1/F_1	1	15.49	2.771	2.625×10^4	1.313
	2	68.10	1.555	2.846×10^5	14.23
	3	104.70	1.341	5.798×10^5	28.99
V_1/F_2	1	15.49	2.771	2.626×10^4	1.313
	2	68.10	1.552	2.841×10^5	14.22
	3	104.70	1.331	5.756×10^5	28.88
V_1/F_3	1	15.49	2.771	2.625×10^4	1.313
	2	68.10	1.553	2.842×10^5	14.22
	3	104.70	1.316	5.690×10^5	28.71

modal parameters. The discrepancies are due to the fact that limited number of digits are used to describe the effective parameters and the modal matrix. Moreover, errors are generated in the least square error fit in both the Nyquist plane and inverse Nyquist plane.

Now that the mathematical model in modal coordinates has been developed, if physical parameters are desired they can be obtained by converting the modal parameters using Eq. 3.51, which is for a complete system. The resulting physical mass, stiffness, and damping are listed in Fig. 16. They are the same number as in Fig. 13. An eigenanalysis is performed, and the natural frequencies and modal matrix are exactly the same as in Fig. 13.

In order to check the theory of truncated system, one of the three modes is dropped to make the complete system become a truncated one. The third mode will be dropped, and only the first two modes are used. The modal parameters of the truncated system are given in Fig. 17. The modal matrix and its pseudo-inverse are also listed in Fig. 17. Notice that the modal matrix is of order 3 by 2 because the third column is dropped, and its pseudo-inverse is of order 2 by 3. Using the pseudo-inverse method, the physical parameters of the truncated system are obtained by Eq. 3.57 and are shown in Fig. 18. Their values are not the same as the original physical parameters, but both sets of parameters have the same dynamic characteristics, i.e., the same eigenvalues and eigenvectors, or natural frequencies and the same modal matrix. This can be seen

Physical Mass:

1.0000	0.0000	0.0000
0.0000	2.0000	0.0000
0.0000	0.0000	4.0000

Physical Stiffness:

300000.0000	-200000.0000	0.0000
-200000.0000	500000.0000	-300000.0000
0.0000	-300000.0000	0.0000

Physical Damping:

15.0000	-10.0000	0.0000
-10.0000	25.0000	-15.0000
0.0000	-15.0000	15.0000

Fig. 16 Physical parameters obtained from modal parameters.

Modal Mass:

$$\begin{bmatrix} 2.7709 & 0.0000 \\ 0.0000 & 1.5548 \end{bmatrix}$$

Modal Stiffness:

$$\begin{bmatrix} 26250.0713 & 0.0000 \\ 0.0000 & 284630.0905 \end{bmatrix}$$

Modal Damping:

$$\begin{bmatrix} 1.3125 & 0.0000 \\ 0.0000 & 14.2315 \end{bmatrix}$$

Modal Matrix:

$$\begin{bmatrix} -0.4126 & -0.8147 \\ -0.5993 & -0.4764 \\ -0.6860 & 0.3306 \end{bmatrix}$$

Pseudo-inverse of Modal Matrix:

$$\begin{bmatrix} -0.1077 & -0.4872 & -0.9673 \\ -0.7722 & -0.2840 & 0.7126 \end{bmatrix}$$

Fig. 17 Modal parameters of the truncated system using only the first two modes.

Physical Mass:

0.9593	0.4863	-0.5669
0.4863	0.7831	0.9913
-0.5669	0.9913	3.3823

Physical Stiffness:

170037.3458	63794.0211	-153885.1938
63794.0211	29183.9764	-45223.0990
-153885.1938	-45223.0990	169081.9824

Physical Damping:

8.5019	3.1897	-7.6943
3.1897	1.4592	-2.2612
-7.6943	-2.2612	8.4541

Natural Frequencies:

1. 15.4911 Hz
2. 68.0964 Hz
3. 9.6261E+8 Hz

Modal Matrix:

-0.4126	-0.8147	0.5713
-0.5993	-0.4764	-0.7568
-0.6860	0.3306	0.3176

Fig. 18 Physical parameters of the truncated system, obtained by using the pseudo-inverse method.

by comparing the natural frequencies and modal matrix of the first two modes of the complete system in Fig. 13 with that for the truncated one shown in Fig. 18. The third natural frequencies and the third column of the modal matrix of the truncated system should be discarded because they represent the non-existing part of the system.

Now that the mathematical model has been developed, one can go on to make modifications on the system. The system will be modified in four different independent ways: 1) 1 kg. of mass is added to point number 2; 2) 50,000 N/m of stiffness is added between point number 1 and ground; 3) 200,000 N/m of stiffness is added between point number 1 and 2; 4) 1 kg. of mass is added to point number 2 and 200,000 N/m of stiffness is added between point number 1 and 2. All four modifications will be performed by three methods so that the results from each method can be compared. The three methods are: 1) modifications made in the physical coordinates model of the complete system; 2) modifications made in the physical coordinates model of the truncated system, using the pseudo-inverse method; and 3) modifications made in the modal coordinates model of the truncated system. The first method is used as the standard for comparison. Figures 19 and 20 are the results from each method. The natural frequencies of all three methods agree well. The natural frequencies of the truncated system, modified in physical and modal coordinates, have exactly the same value. The modal matrix seems to vary among the three methods, but they are quite close to each other, as can be seen in Figs. 21 and 22. The magnitude and

Modification I

Add 1 kg of mass to point #2:

1. Complete system, modification made in the physical coordinates model.

   Natural Frequencies:

   1 = 14.5652 Hz
   2 = 63.6374 Hz
   3 = 97.2559 Hz

$$[P] = \begin{bmatrix} -.4145 & -.7305 & -.9353 \\ -.6044 & -.5118 & .3433 \\ -.6804 & .4522 & -.0863 \end{bmatrix}$$

2. Truncated system, modification made in the physical coordinates model.

   Natural Frequencies:

   1 = 14.5690 Hz
   2 = 64.1085 Hz

$$[P] = \begin{bmatrix} -.4183 & -.8023 \\ -.6014 & -.4334 \\ -.6806 & .4105 \end{bmatrix}$$

3. Truncated system, modification made in the modal coordinates model.

   Natural Frequencies:

   1 = 14.5690 Hz
   2 = 64.1085 Hz

$$[P] = \begin{bmatrix} -.4198 & -.7715 \\ -.6035 & -.4168 \\ -.6830 & .3948 \end{bmatrix}$$

Modification II

Add 50,000 N/m of stiffness between point #1 and ground.

1. Complete system, modification made in the physical coordinates model.

   Natural Frequencies:

   1 = 17.4810 Hz
   2 = 71.6293 Hz
   3 = 107.9889 Hz

$$[P] = \begin{bmatrix} -.3556 & -.7600 & -.8717 \\ -.6008 & -.5603 & .4811 \\ -.7160 & .3294 & -.0936 \end{bmatrix}$$

2. Truncated system, modification made in the physical coordinates model.

   Natural Frequencies:

   1 = 17.5821 Hz
   2 = 72.0172 Hz

$$[P] = \begin{bmatrix} -.3746 & -.8172 \\ -.5847 & -.4890 \\ -.7196 & .3050 \end{bmatrix}$$

3. Truncated system, modification made in the modal coordinates model.

   Natural Frequencies:

   1 = 17.5821 Hz
   2 = 72.0172 Hz

$$[P] = \begin{bmatrix} -.3662 & -.8273 \\ -.5716 & -.4950 \\ -.7035 & .3088 \end{bmatrix}$$

Fig. 19 Natural frequencies and modal matrix of modification I and II.

Modification III

Add 200,000 N/m of stiffness between point #1 and point #2.

1. Complete system, modification made in the physical coordinates model.
   Natural Frequencies:

   1 = 16.6214 Hz
   2 = 69.3578 Hz
   3 = 135.4391 Hz

$$[P] = \begin{bmatrix} -.4692 & -.7338 & .8709 \\ -.5737 & -.5689 & -.4881 \\ -.6713 & .3713 & .0564 \end{bmatrix}$$

2. Truncated system, modification made in the physical coordinates model.
   Natural Frequencies:

   1 = 17.2823 Hz
   2 = 70.8185 Hz

$$[P] = \begin{bmatrix} -.4402 & -.8123 \\ -.6092 & -.4660 \\ -.6596 & .3507 \end{bmatrix}$$

3. Truncated system, modification made in the modal coordinates model.
   Natural Frequencies:

   1 = 17.2823 Hz
   2 = 70.8185 Hz

$$[P] = \begin{bmatrix} -.4478 & -.8044 \\ -.6196 & -.4615 \\ -.6709 & .3473 \end{bmatrix}$$

Modification IV

Add 1 kg of mass to point #2 and 200,000 N/m of stiffness between points #1 and 2.

1. Complete system, modification made in the physical coordinates model.
   Natural Frequencies:

   1 = 15.6755 Hz
   2 = 64.0385 Hz
   3 = 126.9987 Hz

$$[P] = \begin{bmatrix} -.4722 & -.6672 & .9454 \\ -.5788 & -.5639 & -.3232 \\ -.6648 & .4867 & .0431 \end{bmatrix}$$

2. Truncated system, modification made in the physical coordinates model.
   Natural Frequencies:

   1 = 16.1885 Hz
   2 = 66.9405 Hz

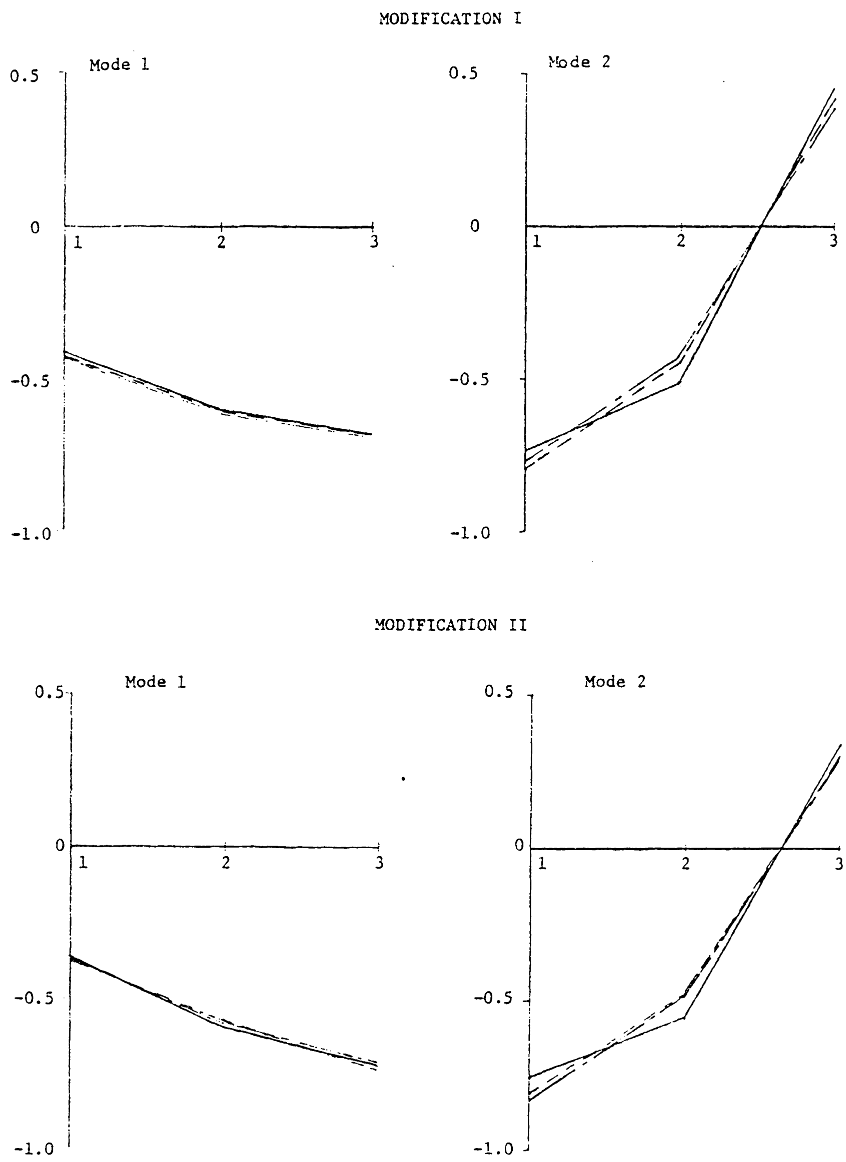
$$[P] = \begin{bmatrix} -.4465 & -.7977 \\ -.6113 & -.4211 \\ -.6535 & .4317 \end{bmatrix}$$

3. Truncated system, modification made in the modal coordinates model.
   Natural Frequencies:

   1 = 16.1685 Hz
   2 = 66.9405 Hz

$$[P] = \begin{bmatrix} -.4559 & -.7591 \\ -.6241 & -.4007 \\ -.6672 & .4108 \end{bmatrix}$$

Fig. 20 Natural frequencies and modal matrix of modification III and IV.



Keys:

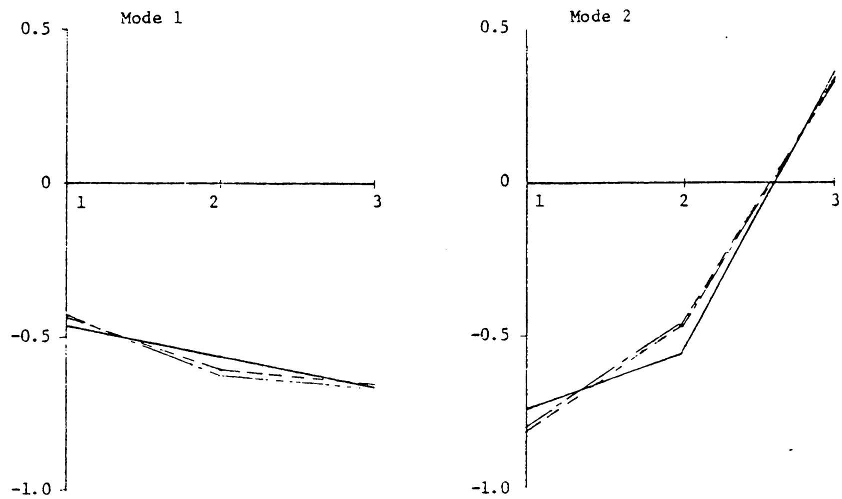
___ complete system, modification made in the physical coordinates model

--- truncated system, modification made in the physical coordinates model

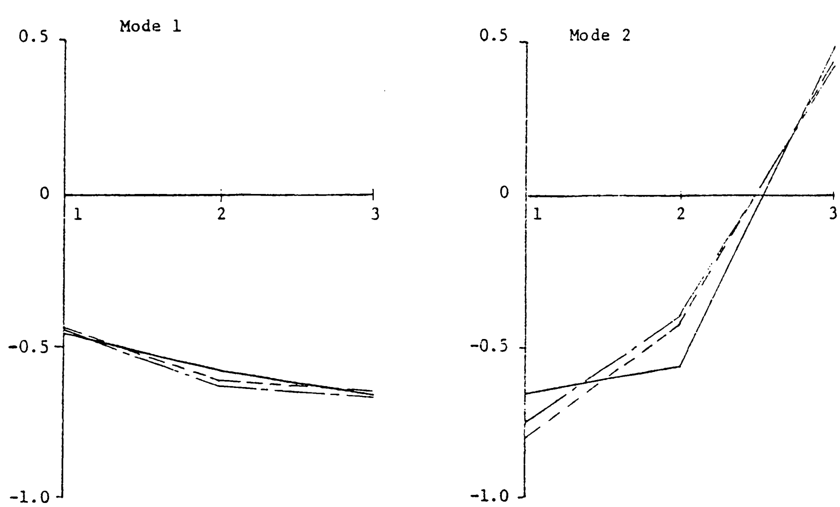
_ _ _ _ truncated system, modification made in the modal coordinates model

Fig. 21 Mode shapes of modification I and II.

MODIFICATION III



MODIFICATION IV



Keys:

___ complete system, modification made in the physical coordinates model

--- truncated system, modification made in the physical coordinates model

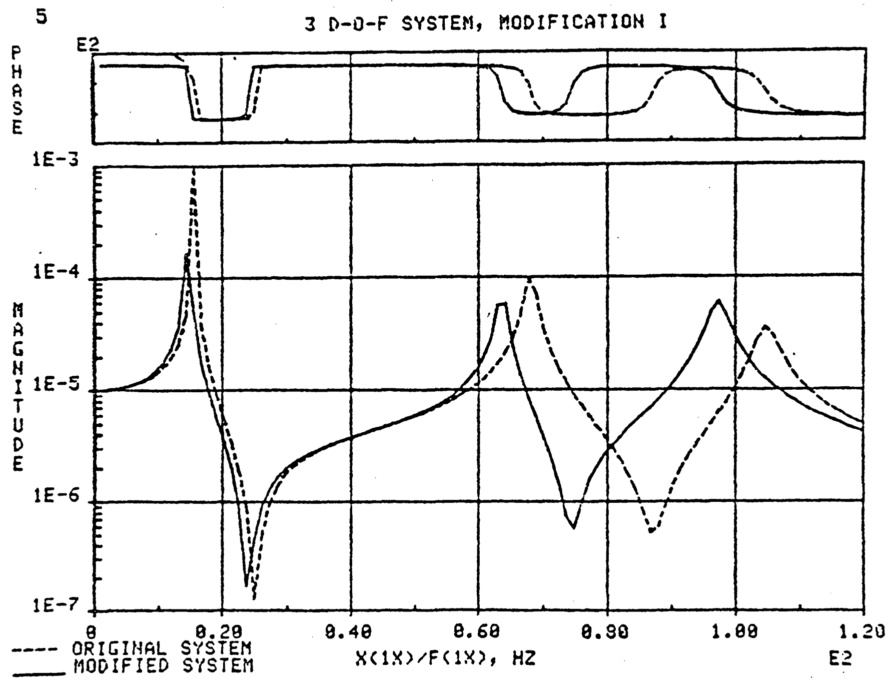
_ _ _ _ truncated system, modification made in the modal coordinates model

Fig. 22 Mode shapes of modification III and IV.

phase of frequency response function of all four modifications compared to the original system are also generated and shown in Fig. 23 and 24.

The resonance specification is just the opposite of the modification. It computes the amount of physical changes needed to move the resonance peak to a specified frequency. The original system is used, but the frequency specified is for the modified system so that the required modification can be determined. All three frequencies are used in the resonance specification routine in BASIC MODIFICATION ®. The interactive session and results of modification III are shown in Fig. 25. The results come out very close to the correct solution. By performing the resonance specification on all modes, one can find out which mode is most sensitive to the modification. When stiffness modification has to be added or removed between a point of the structure and ground, the routine will ask for the x, y, z coordinates of the ground point, with respect to the axis of the structure, so that the direction cosine can be computed.

The frequency response function synthesis is performed on the original system in both physical and modal coordinates, shown in Fig. 26. Then the frequency response function of the truncated system is also synthesized in both physical and modal coordinates, shown in Fig. 27. The frequency response function, synthesized from both methods, agree with each other quite well. Thus, all the theories described in previous chapters are verified for the theoretical three degree-of-freedom system.



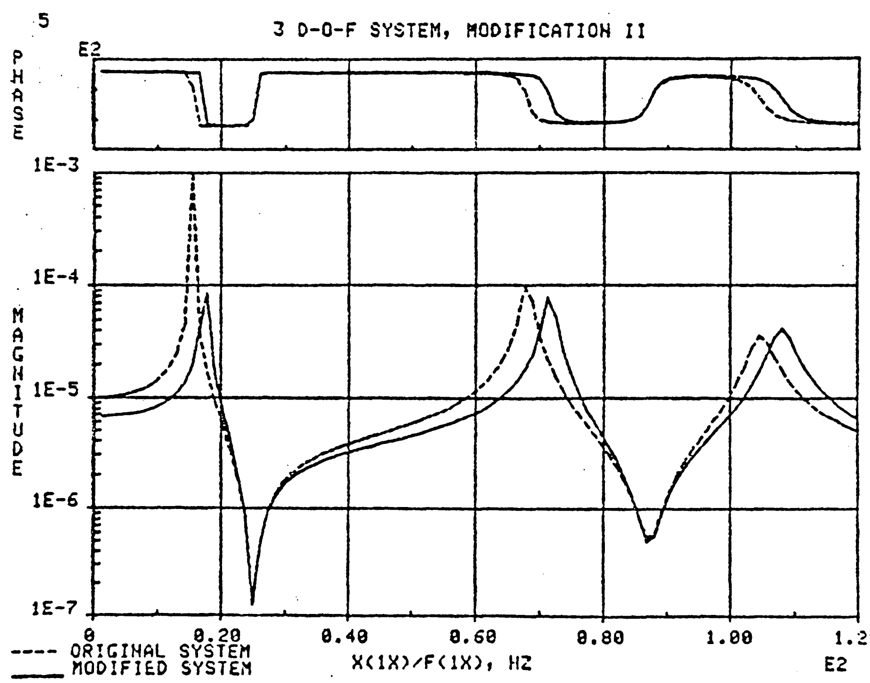
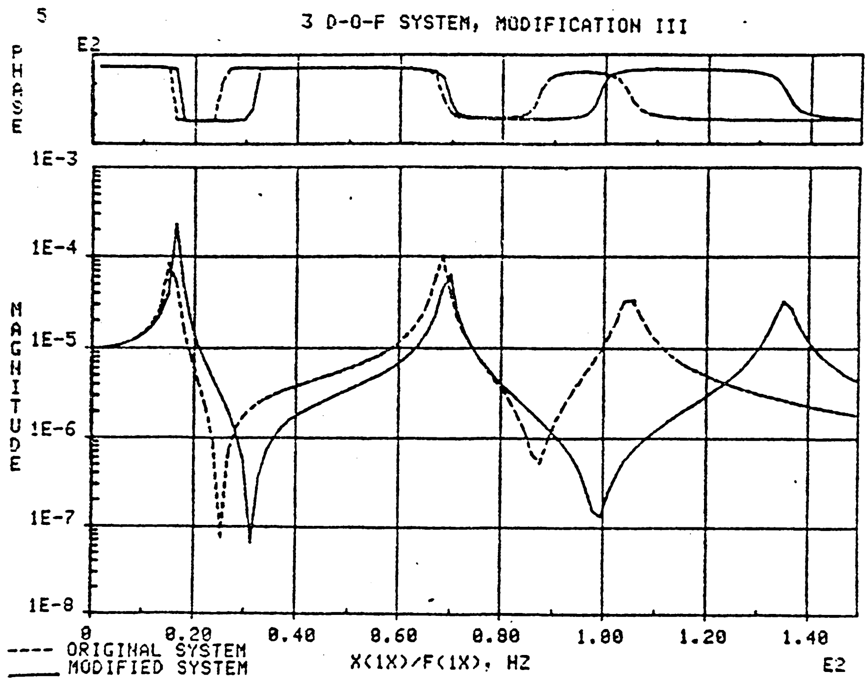


Fig. 23 Magnitude and phase of frequency response function of the original and modified system for modification I and II.



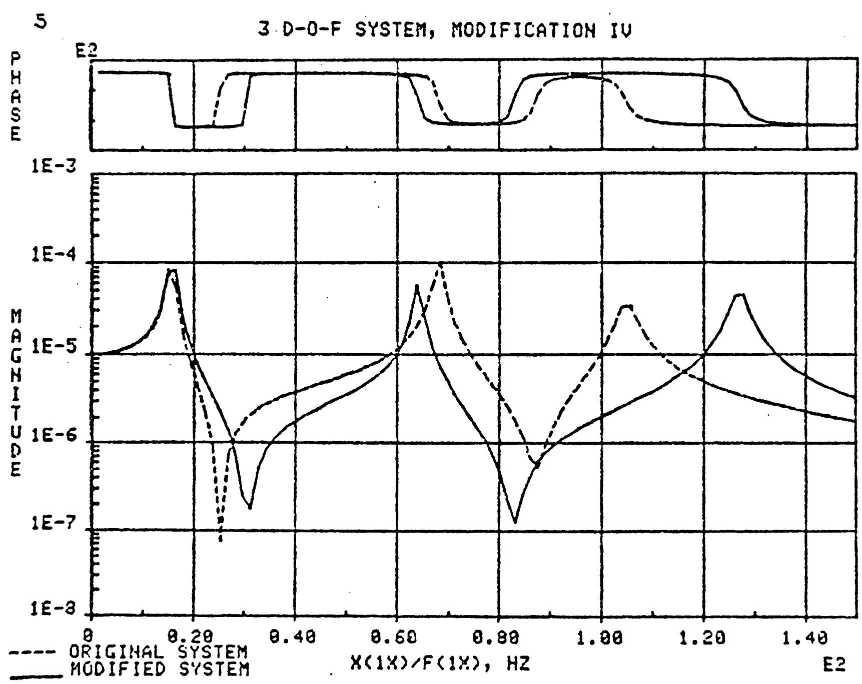
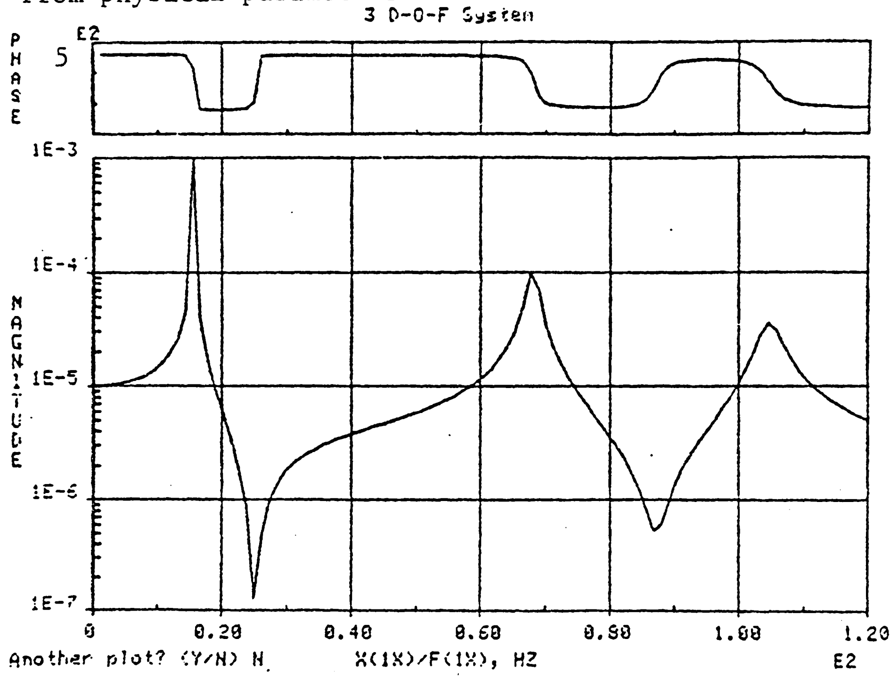


Fig. 24 Magnitude and phase of frequency response function of the original and modified system for modification III and IV.

```
*** RESONANCE SPECIFICATION:
Title: 3 D-O-F System

Enter unit desired (1=in-lb-sec, 2=ft-lb-sec, 3=m-N-sec) = 3


Enter Desired Natural Frequency (Hz) = 16.6214
Type of Modification allowed (1=Mass, 2=Stiff.)= 2
Enter first location # = 1
Enter second location # (0=ground) = 2

*** SOLUTION:
Stiffness change from location #   1 to  2 = 200012.1312 N/m

Want another Resonance to be specified? (Y/N)Y


Enter Desired Natural Frequency (Hz) = 69.3578
Type of Modification allowed (1=Mass, 2=Stiff.)= 2
Enter first location # = 1
Enter second location # (0=ground) = 2

*** SOLUTION:
Stiffness change from location #   1 to  2 = 199996.8808 N/m

Want another Resonance to be specified? (Y/N)Y


Enter Desired Natural Frequency (Hz) = 135.4391
Type of Modification allowed (1=Mass, 2=Stiff.)= 2
Enter first location # = 1
Enter second location # (0=ground) = 2

*** SOLUTION:
Stiffness change from location #   1 to  2 = 200000.0241 N/m
```

Fig. 25 Interactive session of resonance specification for modification III.

From physical parameters



From modal parameters

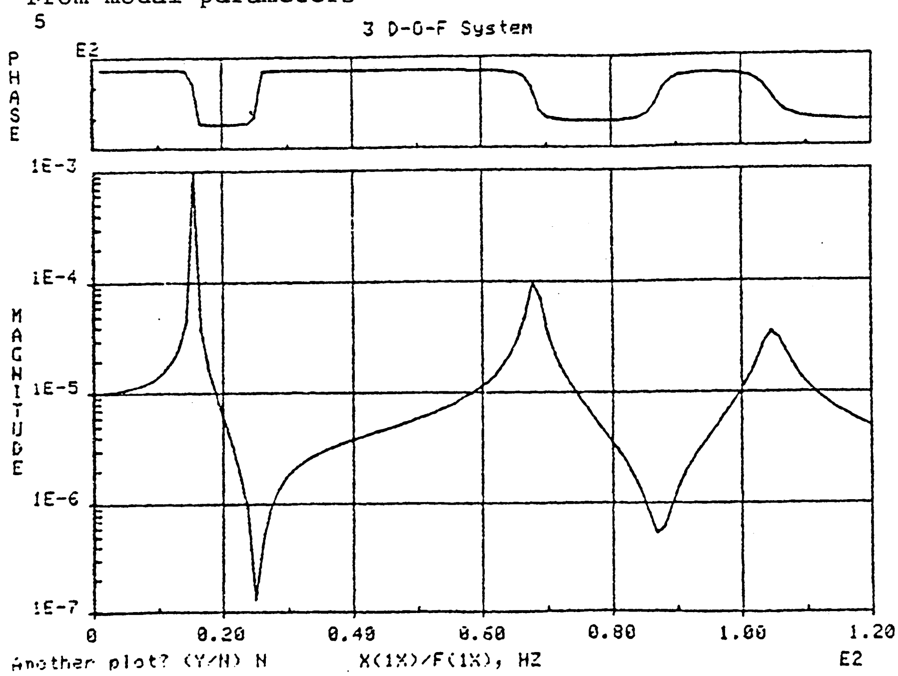
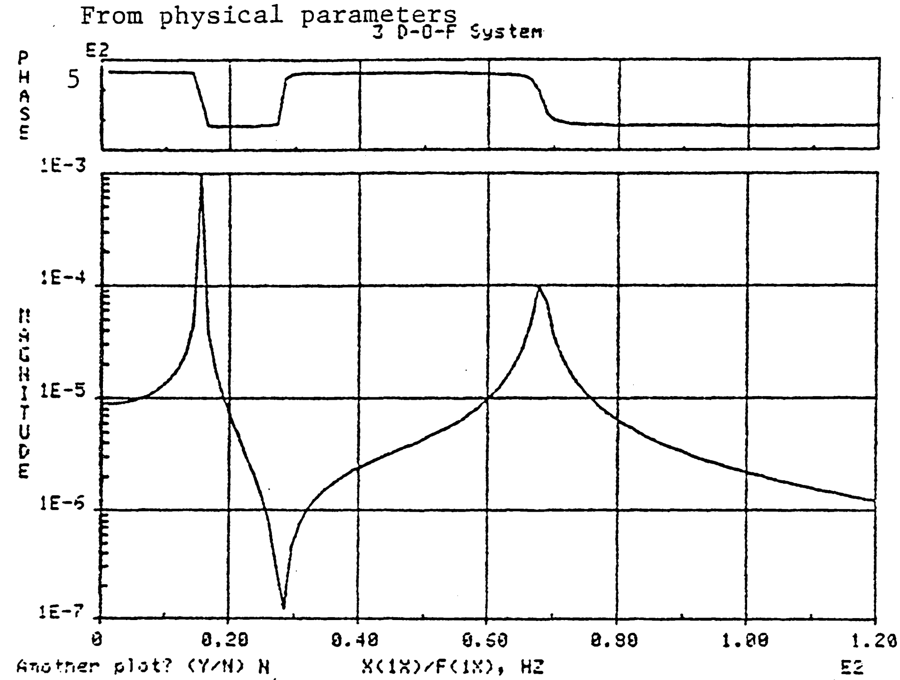


Fig. 26 Magnitude and phase of frequency response function of the truncated system.

From physical parameters



From modal parameters

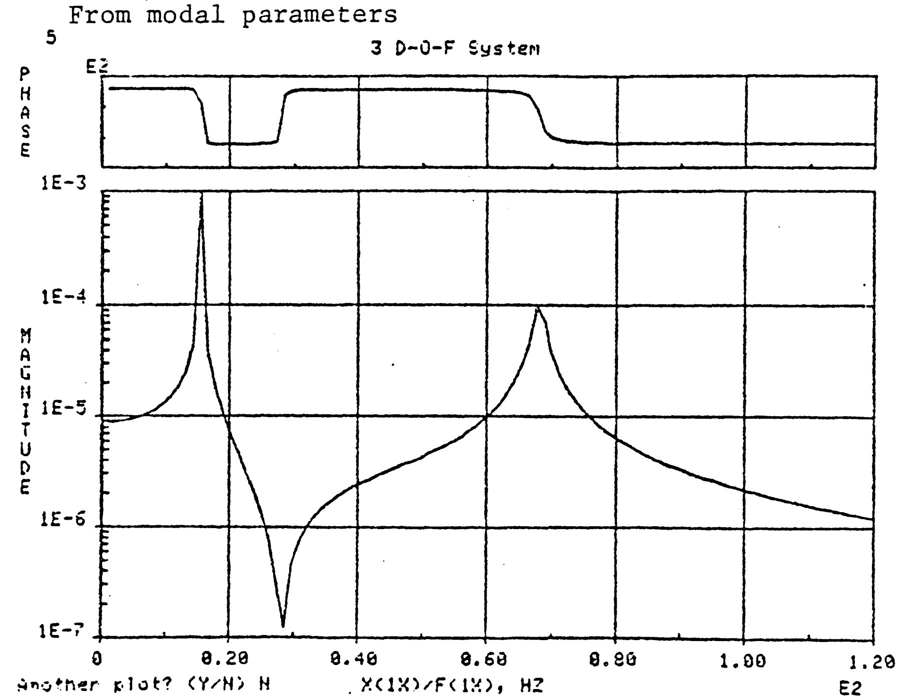


Fig. 27 Magnitude and phase of frequency response function of the truncated system.

6.2 EXPERIMENTAL SYSTEMS

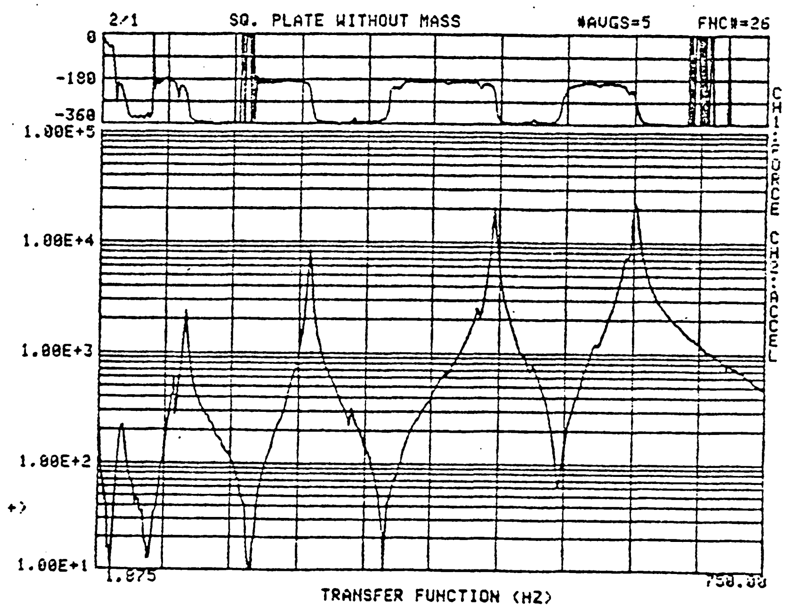
A square plate and a 0.305 meter C-clamp were used to verify the theories experimentally. The square plate has nine measurement points (data from z direction only) with a total of nine degrees-of-freedom, while the C-clamp has nine measurement points (data from x, y, z directions) with a total of twenty-seven degrees-of-freedom. The square plate is used for verifying the mass modification method, while the C-clamp is used for verifying the stiffness modification.

6.2.1 SQUARE PLATE WITH MASS MODIFICATION

The square plate is made of steel and is uniformly supported over its entire surface by a piece of foam. Before the data acquisition is done by the modal analysis program--BASIC MODAL®, the Zonic 6080 FFT analyzer has to be set up properly. The impact test is used because it is the simplest way to acquire vibration data. An impact hammer is used to generate impulse signal. After the signal is Fourier transformed, the energy density of both input and output signals are checked to see if sufficient energy is applied at all frequencies of interest. The energy is sufficient, and it shows clearly four dominated peaks in the frequency range of 0 to 750 Hz. A peak search is performed to find the resonant frequency of these four peaks, which are required as inputs to <u>BASIC MODAL</u>®.

Since only four modes are used in the analysis of a nine degree-of-freedom system, a truncated system results. Now, one can acquire the data. The frequency response function at driving point, both magnitude and phase plots, and real and imaginary plots are shown in Fig. 28. A viscous damping model is used. Circle fitting is used to construct the frequency damping table. A typical circle fit of mode #4 is shown in Fig. 29. The circle is then corrected for adjacent mode effect, and the data is converted to inverse Nyquist plane. The frequency damping table is constructed at the driving point and the effective parameters are listed in Fig. 30. After the data acquisition is completed, the deflected plots of all four modes are drawn, as shown in Fig. 31. Next, we proceed to use BASIC MODIFICATION®. The modal parameters are first converted from the effective parameters to the modal parameters also shown in Fig. 30. Using the frequency response function synthesis routine, the magnitude and phase, real and imaginary plots of the frequency response function at driving point are synthesized and shown in Fig. 32. One can compare these plots to Fig. 28, which is generated by the FFT analyzer experimentally. They agree very closely to each other. Thus, it shows that the mathematical model in modal coordinates obtained from BASIC MODAL® is corrected and can be used to synthesize the same frequency response function plots obtained experimentally.

With the mathematical model on hand, one can modify the original system analytically. A mass of 0.072 kg. is added to measure-



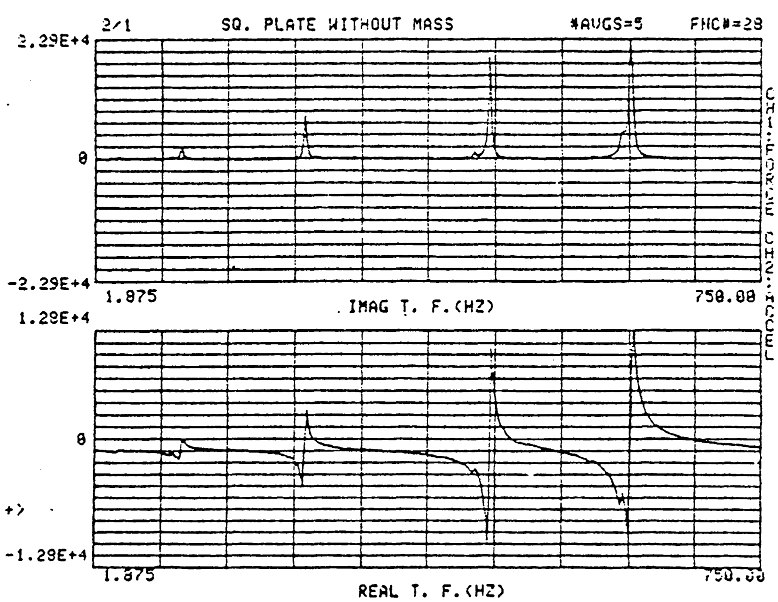


Fig. 28 Frequency response function at driving point of square plate without mass.

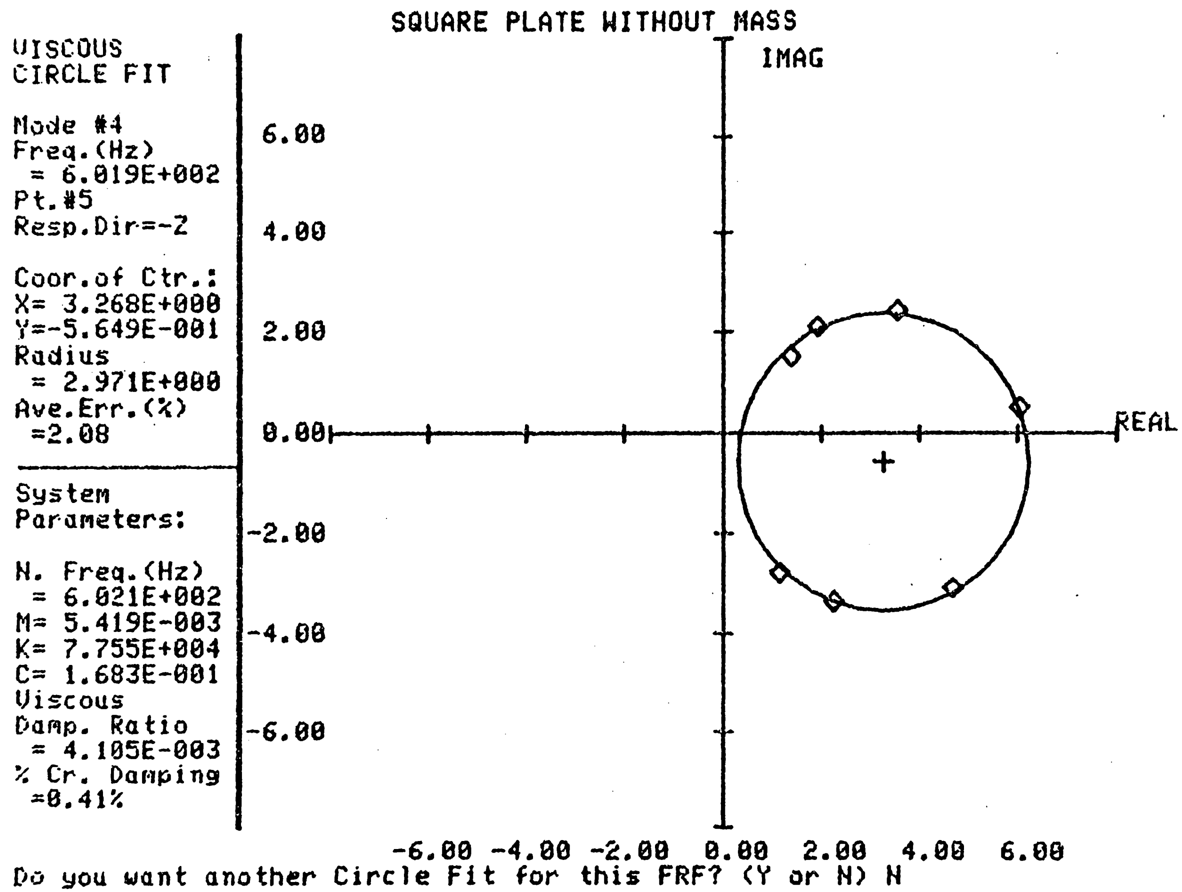


Fig. 29 Circle fitting in Nyquist plane for square plate without mass, mode #4.

VISCOUS DAMPING MODELING

COEFFICIENTS DERIVED FROM POINT#5

FREQUENCY, DAMPING, MASS AND STIFFNESS TABLE

Mode and Dir.	Peak Frequency (Hz)	Natural Frequency (Hz)	Effective Mass	Effective Stiffness	Effective Viscous Damping,C	% Critical Damping (%)
1:Z	99.38	97.97	1.366E-002	5.176E+003	2.717E-001	1.62
2:Z	238.13	237.76	9.481E-003	2.116E+004	1.842E-001	0.65
3:Z	444.39	444.74	5.960E-003	4.654E+004	1.297E-001	0.39
4:Z	601.88	602.08	5.419E-003	7.755E+004	1.683E-001	0.41

Mode and Dir.	Peak Frequency (Hz)	Natural Frequency (Hz)	Modal Mass	Modal Stiffness	Modal Damping	% Critical Damping (%)
1:Z	99.38	97.97	2.538E-003	9.618E+002	5.049E-002	1.62
2:Z	238.13	237.76	6.518E-004	1.455E+003	1.266E-002	0.65
3:Z	444.38	444.74	9.695E-004	7.571E+003	2.110E-002	0.39
4:Z	601.88	602.08	8.415E-004	1.204E+004	2.613E-002	0.41

Title: SQUARE PLATE WITHOUT MASS
Deflection data: (Read points in ascending order from left to right)

Dir:Z	Z	Z	Z	Z	Z
Mode #1					
0.46325	0.06220	0.26594	0.28524	-0.43108	0.27237
0.40749	0.39891	0.21039			
Mode #2					
-0.35858	0.30167	-0.34871	0.34542	-0.26219	0.31746
-0.39477	-0.39477	-0.24146			
Mode #3					
-0.13119	-0.42455	-0.24412	-0.42880	-0.40333	-0.37573
-0.29719	-0.29931	-0.27808			
Mode #4					
0.38719	0.22786	0.29125	-0.13209	-0.39404	-0.14460
0.43516	0.43687	0.37348			

Fig. 30 Effective and modal parameters of square plate without mass.

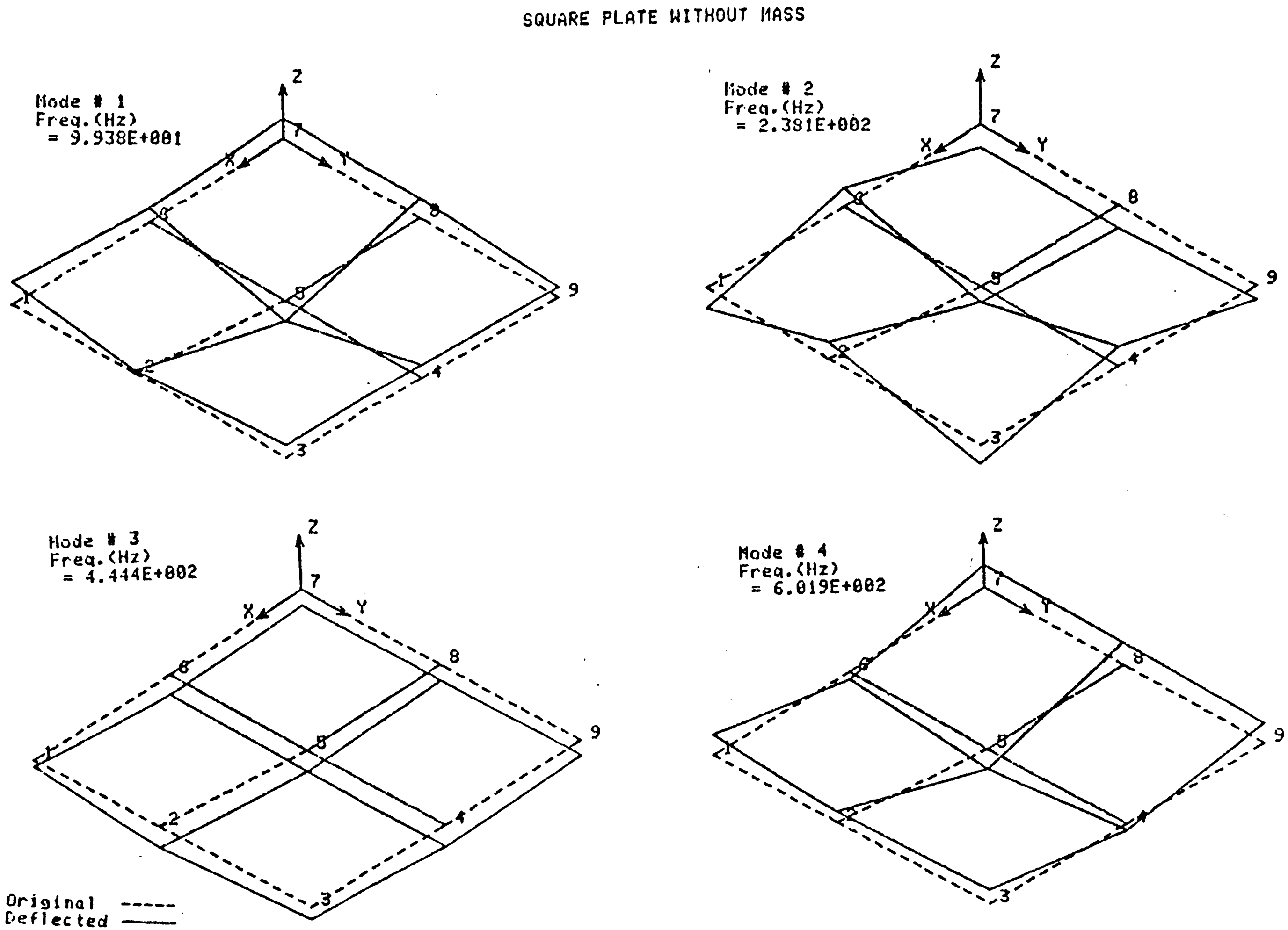


Fig. 31 Deflected plots of square plate without mass for mode #1, 2, 3, and 4.

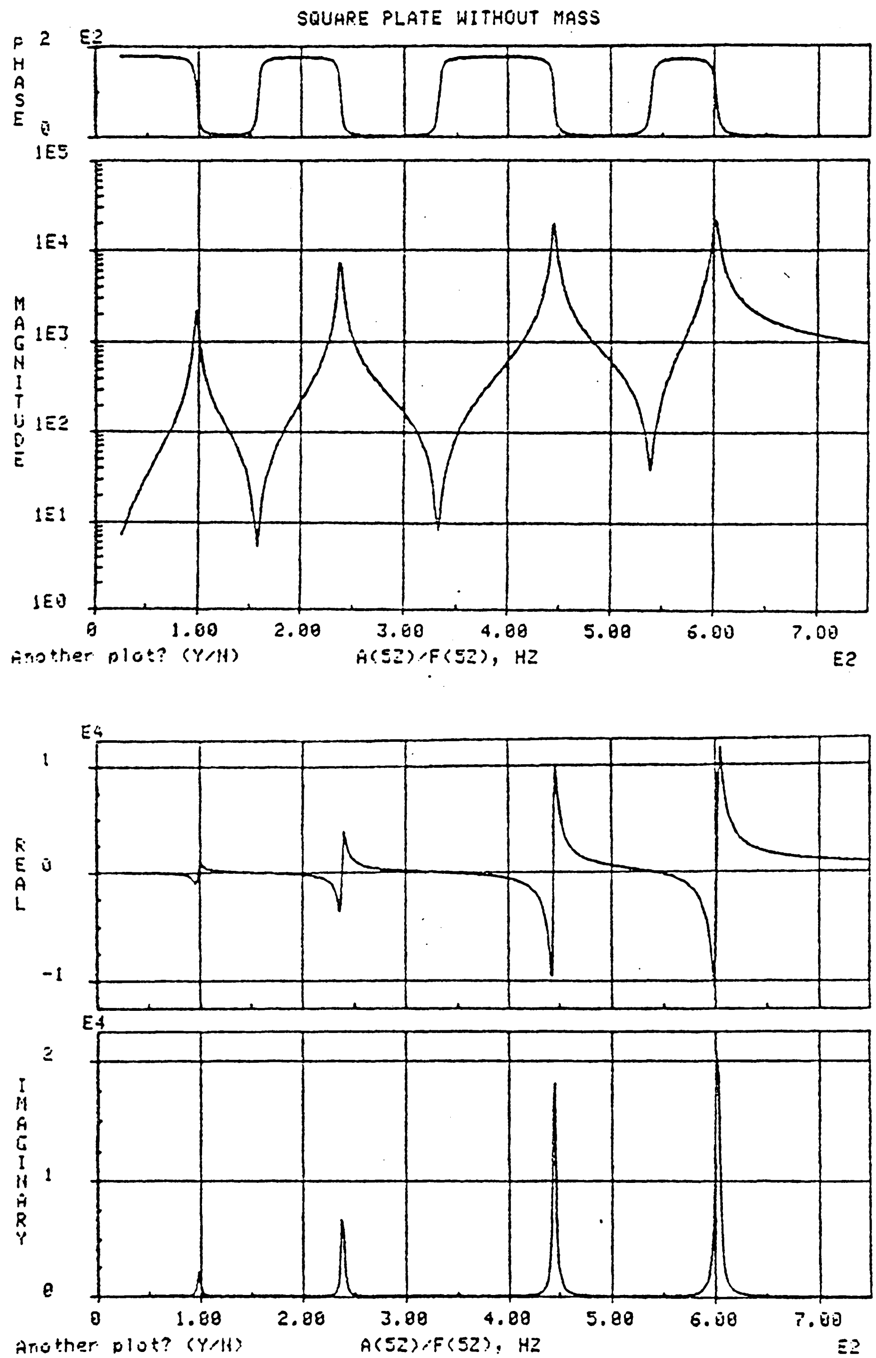


Fig. 32 Frequency response function at driving point synthesized by modal parameters of square plate without mass.

ment point number 1 on the square plate. The predicted modal parameters of the modified system is listed in Fig. 33. The frequency response function at the driving point of the modified system is synthesized using the predicted modal parameters, as shown in Fig. 34. Magnitude and phase plots of both original and modified systems are shown together in Fig. 35. The predicted deflected plots of the modified system are given in Fig. 36.

In order to verify the analytical result, one can actually attach a point mass of 0.072 kg. by a screw to point number 1 on the square plate. Then the response is measured using the FFT analyzer. The frequency response function generated are shown in Fig. 37. If one compares these plots to those obtained analytically, one can see that the experimental plots have some small peaks on both sides of the four modes, which probably are due to noise in the system. The phase plots have substantial jumpings between 0 to -360°, which should be the same points. The analytical frequency response function shown in Fig. 34 does not have these effects. Except for these effects, both the experimental and analytical frequency response function match quite closely to each other. A peak search is performed to find where the four significant peaks are located. The modified system experimental circle fit of mode #4 frequency-damping table with effective parameters, modal parameters, and modal matrix, and the deflected plots are shown in Figs. 38, 39, and 40. These experimentally obtained frequency response function and deflected plots of the modified system can be used for comparison to

Title: SQUARE PLATE WITH 0.072 KG OF MASS AT PT. #1
MODIFIED PARAMETERS:

Mode and Dir.	Natural Frequency (Hz)	Modal Mass	Modal Stiffness	Modal Damping	% Critical Damping (%)
1:Z	96.29	2.629E-003	9.621E+002	5.048E-002	1.59
2:Z	228.90	7.037E-004	1.456E+003	1.270E-002	0.63
3:Z	443.22	9.753E-004	7.564E+003	2.109E-002	0.39
4:Z	583.79	8.873E-004	1.194E+004	2.603E-002	0.40

Title: SQUARE PLATE WITH 0.072 KG OF MASS AT PT. #1
Deflection Data:

Dir:Z	Z	Z	Z	Z	Z
Mode #1					
0.47172	0.05695	0.27417	0.27820	-0.42590	0.26582
0.41705	0.40848	0.21663			
Mode #2					
0.34876	-0.29797	0.34494	-0.35282	0.27426	-0.32497
0.38841	0.38972	0.24069			
Mode #3					
0.12318	0.44151	0.23564	0.43538	0.38593	0.38106
0.28943	0.29168	0.27656			
Mode #4					
0.33478	0.23355	0.24034	-0.12642	-0.42243	-0.13871
0.37328	0.37511	0.33178			

Fig. 33 Predicted modal parameters and modal matrix for square plate with 0.072 kg. of mass added on point #1.

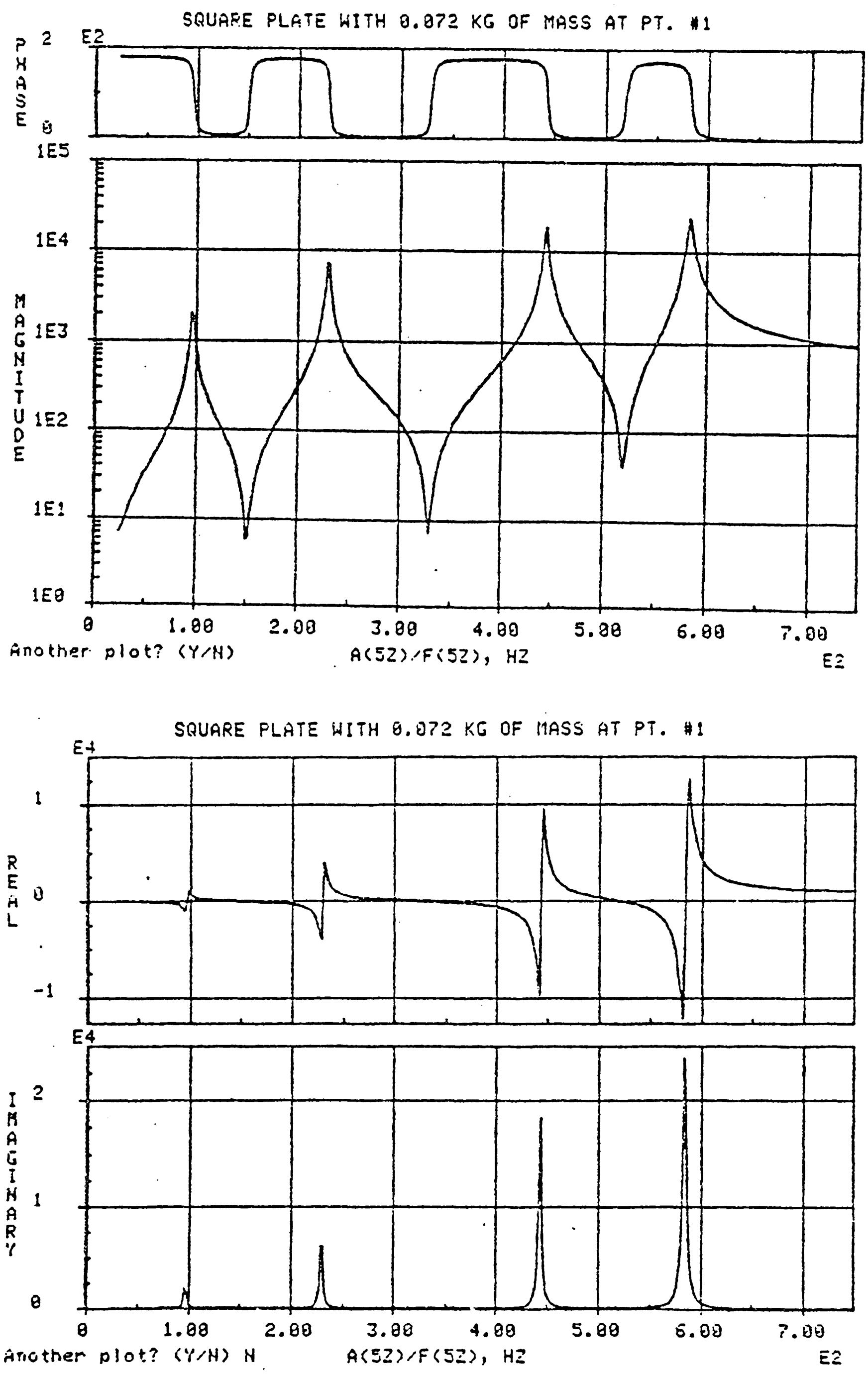


Fig. 34 Frequency response function at driving point synthesized by predicted modal parameters for the square plate with 0.072 kg. of mass added on point #1.

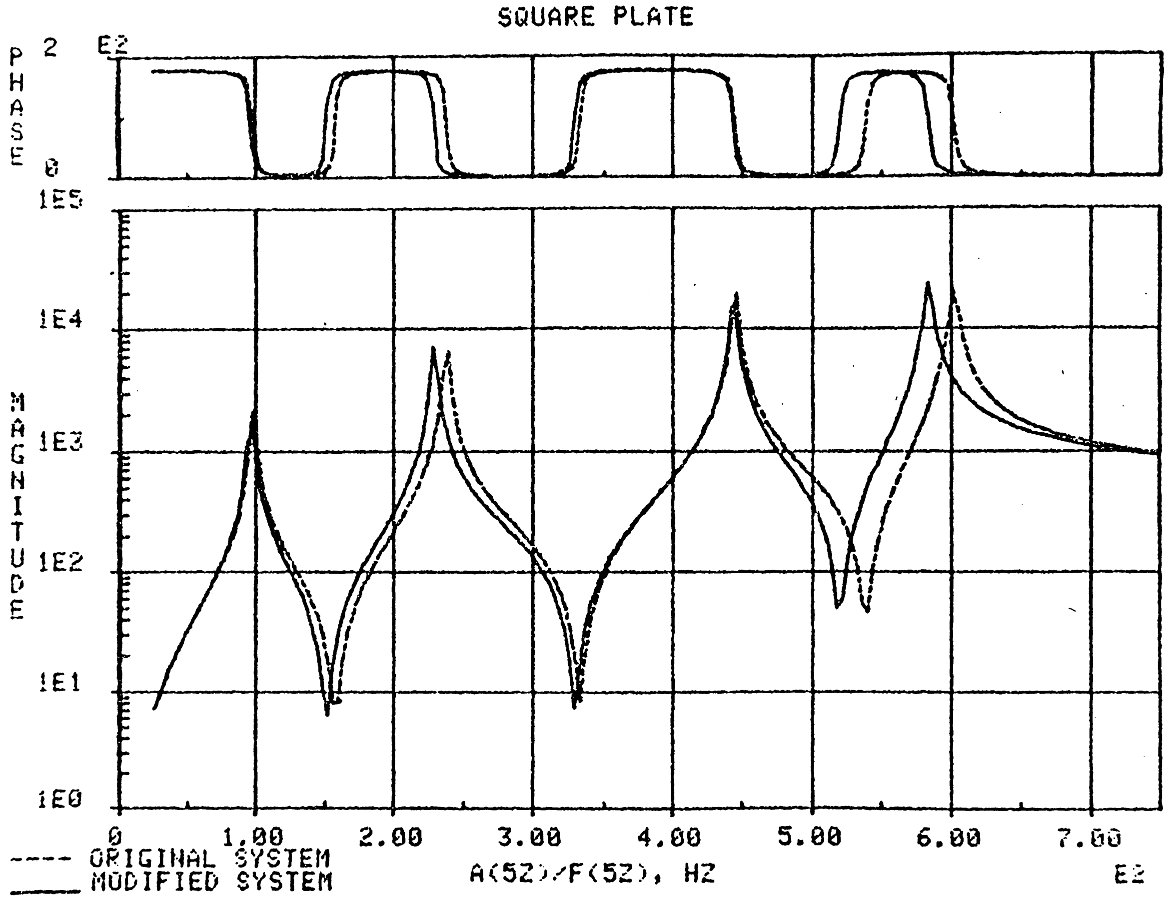


Fig. 35 Magnitude and phase of frequency response function at driving point for the original and modified systems.

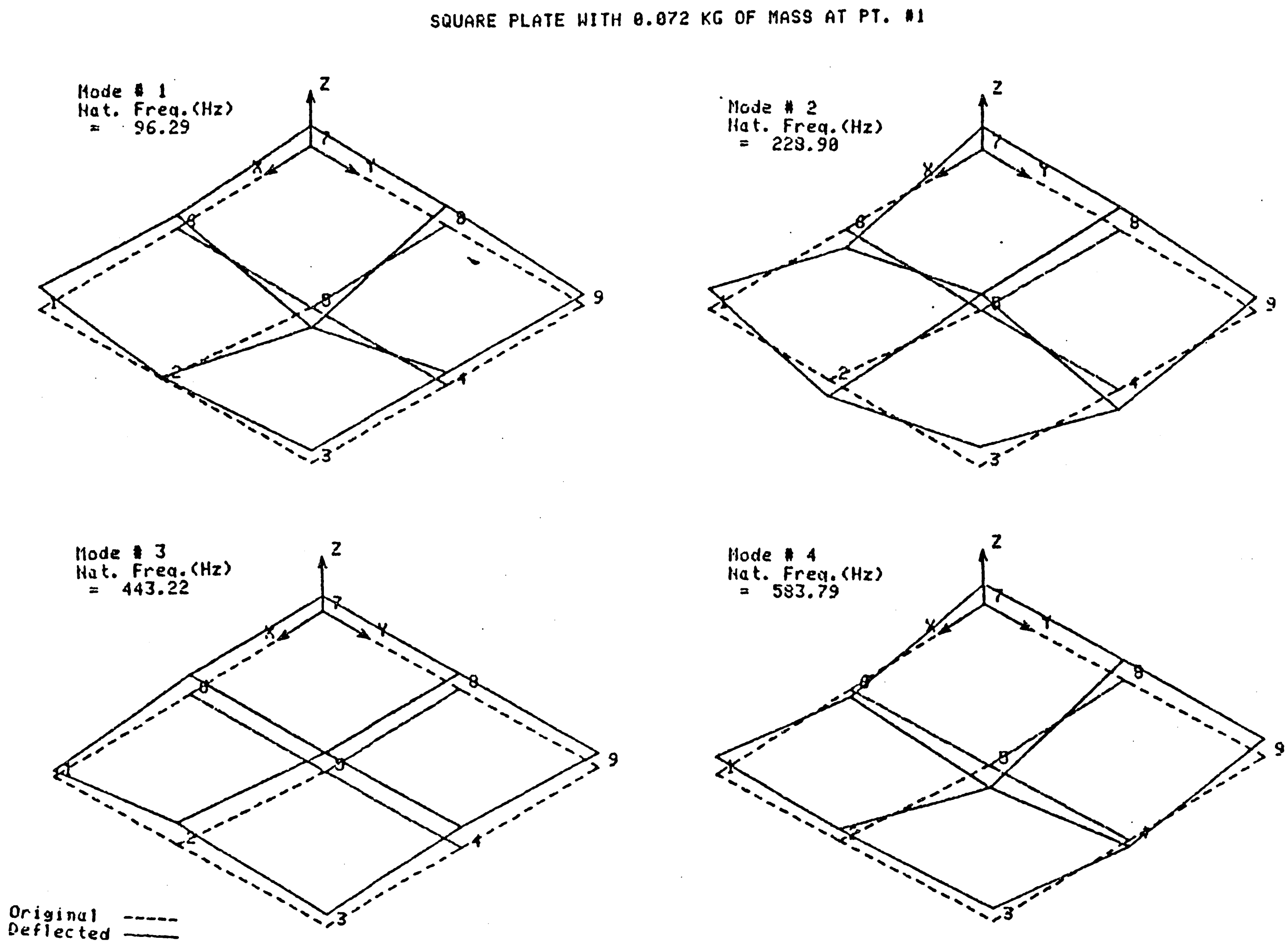


Fig. 36 Predicted deflected plots for mode #1, 2, 3, and 4 of square plate with 0.072 kg. mass.

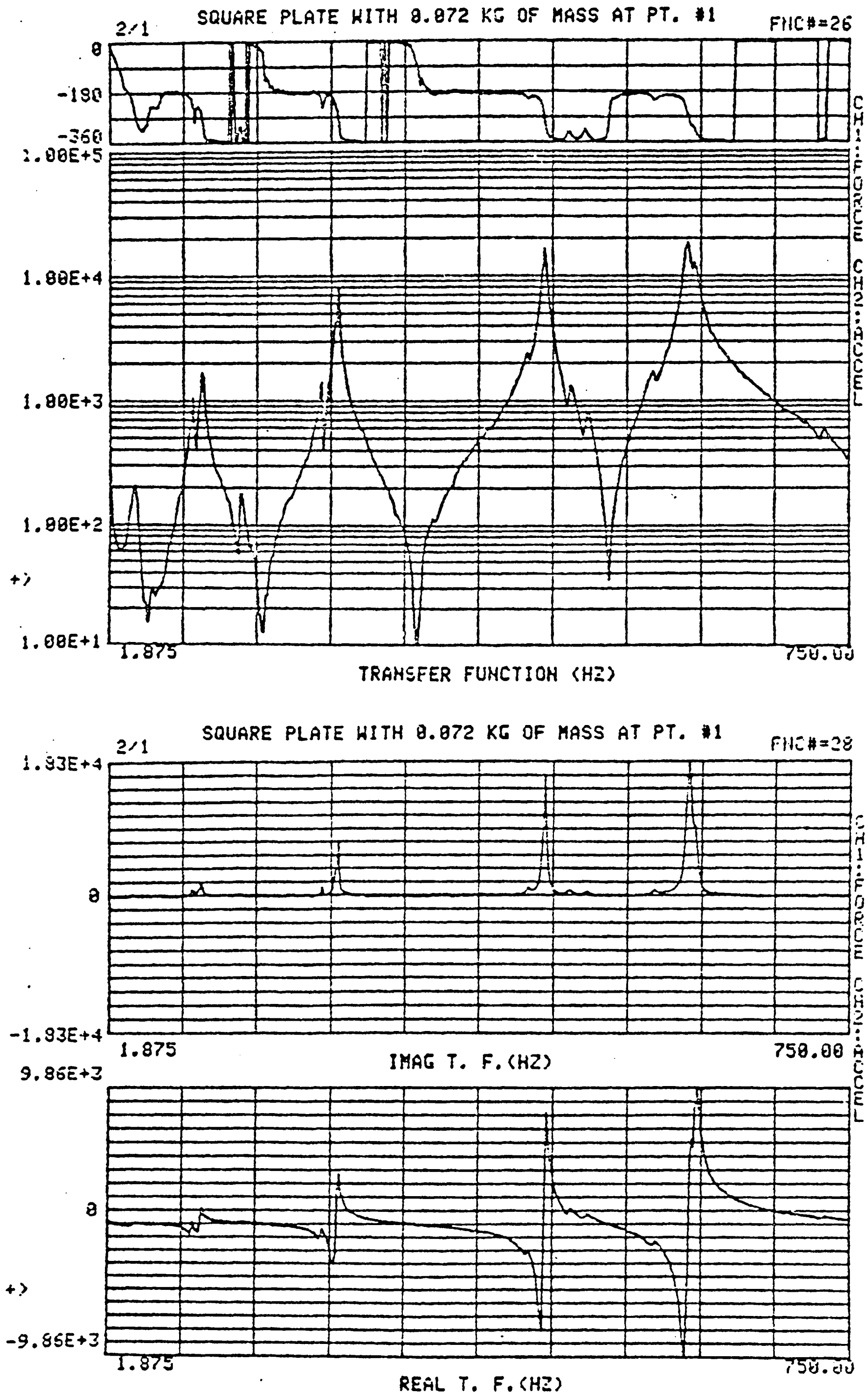


Fig. 37 Experimental frequency response function at driving point for square plate with 0.072 kg. mass.

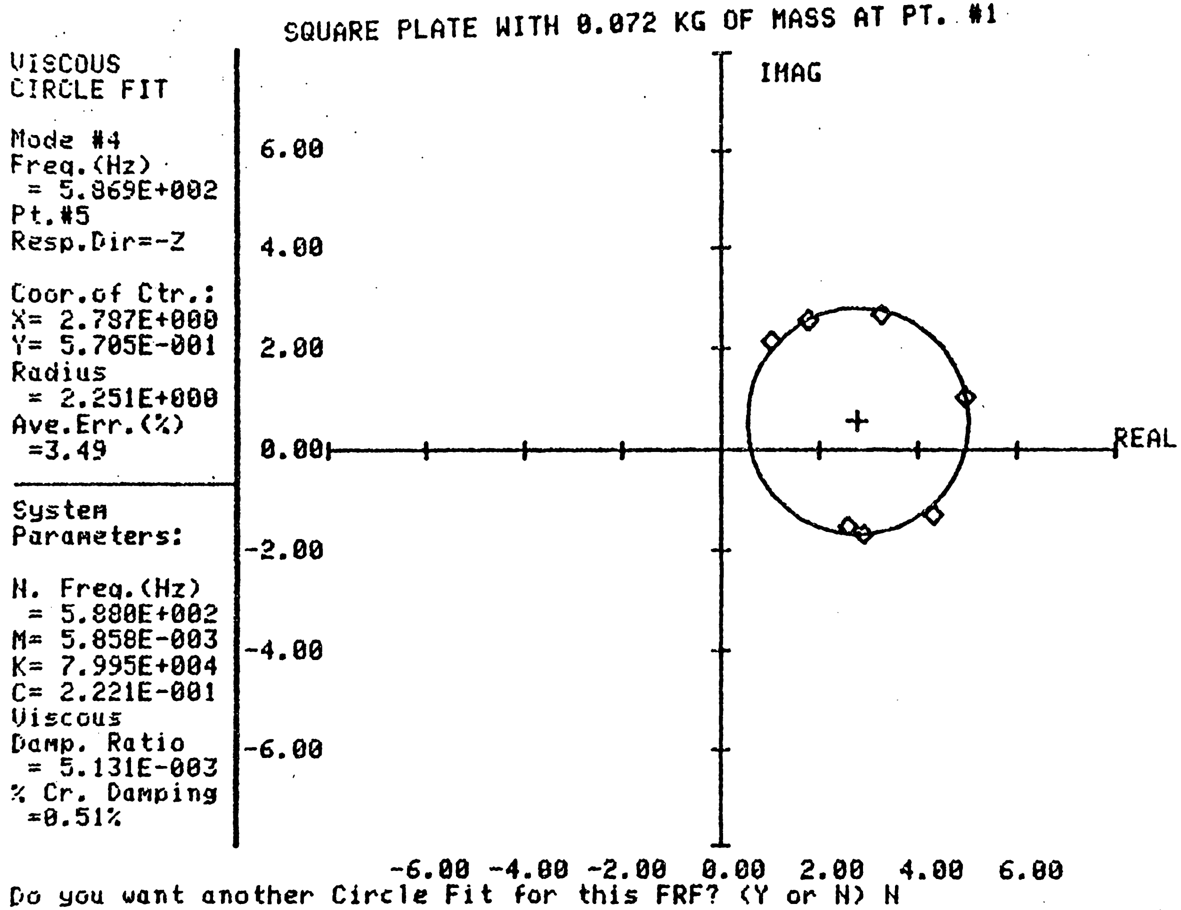


Fig. 38 Circle fitting in Nyquist plane for mode #4 of square plate with 0.072 kg. mass.

VISCOUS DAMPING MODELING

COEFFICIENTS DERIVED FROM POINT#5

FREQUENCY, DAMPING, MASS AND STIFFNESS TABLE

Mode and Dir.	Peak Frequency (Hz)	Natural Frequency (Hz)	Effective Mass	Effective Stiffness	Effective Viscous Damping,C	% Critical Damping (%)
1:Z	95.63	94.51	2.106E-002	7.428E+003	3.602E-001	1.44
2:Z	234.38	232.62	1.282E-002	2.739E+004	1.958E-001	0.52
3:Z	442.50	442.37	6.874E-003	5.311E+004	1.734E-001	0.45
4:Z	586.88	587.99	5.858E-003	7.995E+004	2.221E-001	0.51

Mode and Dir.	Peak Frequency (Hz)	Natural Frequency (Hz)	Modal Mass	Modal Stiffness	Modal Damping	% Critical Damping (%)
1:Z	95.63	94.51	2.672E-003	9.424E+002	4.569E-002	1.44
2:Z	234.38	232.62	1.070E-003	2.287E+003	1.635E-002	0.52
3:Z	442.50	442.37	1.172E-003	9.056E+003	2.956E-002	0.45
4:Z	586.88	587.99	1.000E-003	1.365E+004	3.792E-002	0.51

Title: SQUARE PLATE WITH 0.072 KG OF MASS AT PT. #1
Deflection data: (Read points in ascending order from left to right)

Dir:Z	Z	Z	Z	Z	Z
Mode #1					
0.31112	-0.12209	0.50853	0.35833	-0.35618	0.30898
0.39695	-0.05944	0.34331			
Mode #2					
-0.14072	0.26305	-0.48406	0.25479	-0.28894	0.42403
-0.12346	0.34673	-0.46155			
Mode #3					
0.07833	-0.53056	-0.33796	-0.36789	-0.41294	-0.00428
-0.30032	-0.32534	-0.31033			
Mode #4					
0.06749	-0.23186	0.49127	0.27318	-0.41322	0.27089
0.43158	-0.22393	0.38108			

Fig. 39 Effective and modal parameters of square plate with 0.072 kg. mass.

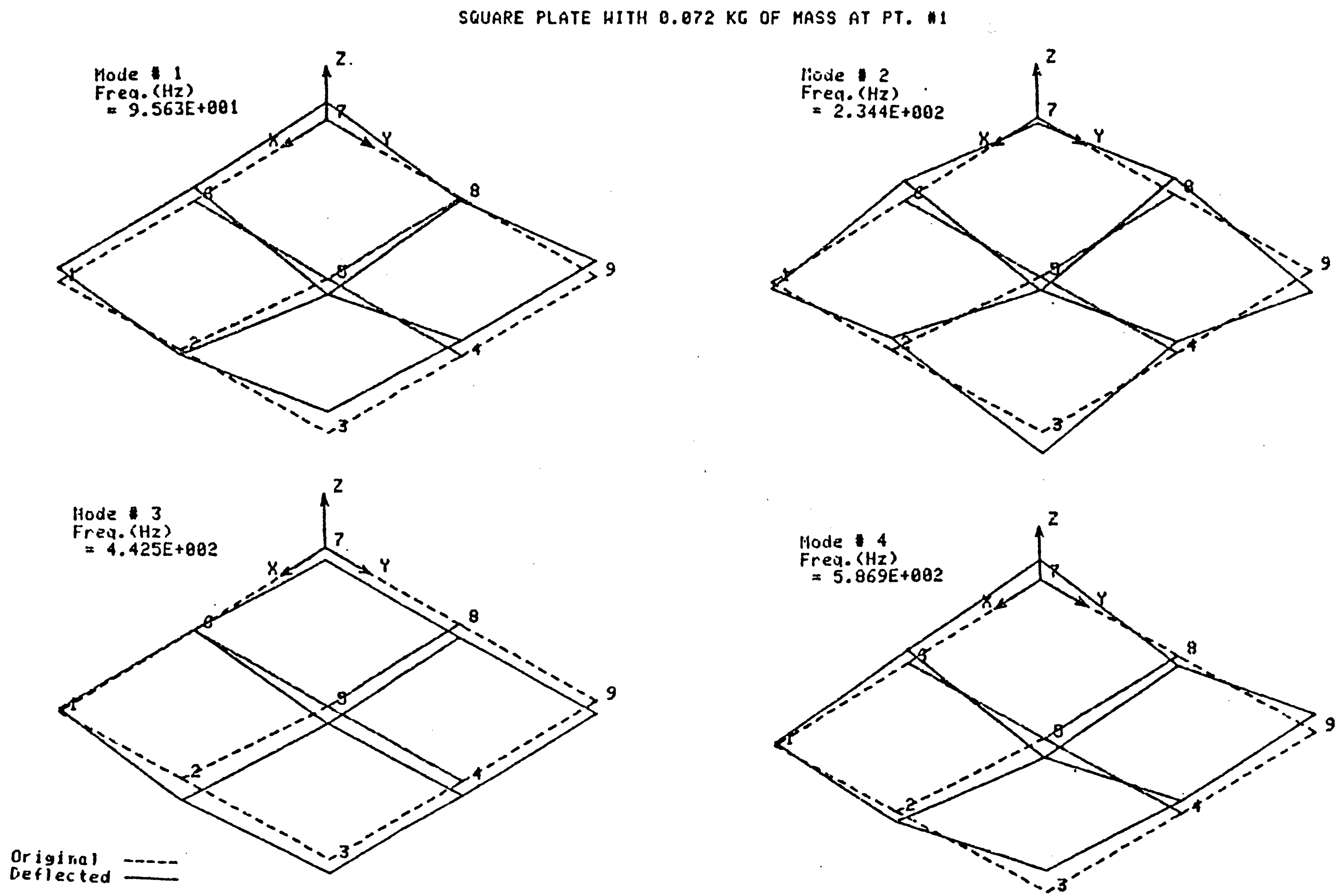


Fig. 40 Deflected plots for mode #1, 2, 3, and 4 of square plate with 0.072 kg. mass, obtained experimentally.

the theoretically predicted ones shown in Figs. 34 and 40. With experimental error in consideration, they agree quite well. The discrepancies occur because the model does not account for the rotatory inertial effect of the mass.

Next, the resonance specification routine is verified. Using the original system, one asks the routine to determine the amount of mass modification needed to change the peaks to those of a modified system. This procedure is repeated for all four modes. The results are correct to the third decimal places, as can be seen in Fig. 41.

Next, the same modification is done in the physical coordinates. Since this is a truncated system, the pseudo-inverse method has to be used. The modal parameters used are those in Fig. 30, which is for the original system. The pseudo-inverse of the modal matrix is generated and shown in Fig. 42. The physical mass, damping, and stiffness, obtained by using Eq. 3.57, are listed in Fig. 43. An eigenanalysis is performed, and the natural frequencies and modal matrix are shown in Fig. 44. Notice that only the first four natural frequencies and first four columns of the modal matrix are valid; the rest should be discarded because they represent the non-existing part of the system.

Since the physical parameters of the original system have been determined, the mass modification can be added. The physical parameters of the modified system are listed in Fig. 45. Notice that the physical damping and stiffness remain the same as the original

```
*** RESONANCE SPECIFICATION:
Title: SQUARE PLATE WITHOUT MASS

Enter unit desired (1=in-lb-sec, 2=ft-lb-sec, 3=m-N-sec) = 1


Enter Desired Natural Frequency (Hz) = 96.29
Type of Modification allowed (1=Mass, 2=Stiff.)= 1
Enter location # = 1

*** SOLUTION:
Weight change for location #   1 = 0.1581 lb    (.0718 Kg)

Want another Resonance to be specified? (Y/N)Y


Enter Desired Natural Frequency (Hz) = 228.90
Type of Modification allowed (1=Mass, 2=Stiff.)= 1
Enter location # = 1

*** SOLUTION:
Weight change for location #   1 = 0.1586 lb    (.0720 Kg)


Enter Desired Natural Frequency (Hz) = 443.22
Type of Modification allowed (1=Mass, 2=Stiff.)= 1
Enter location # = 1

*** SOLUTION:
Weight change for location #   1 = 0.1590 lb    (.0723 Kg)

Want another Resonance to be specified? (Y/N)Y


Enter Desired Natural Frequency (Hz) = 583.79
Type of Modification allowed (1=Mass, 2=Stiff.)= 1
Enter location # = 1

*** SOLUTION:
Weight change for location #   1 = 0.1586 lb    (.0720 Kg)
```

Fig. 41 Interactive session of resonance specification of square plate.

Modal Matrix:

0.4632	-0.3586	-0.1312	0.3872
0.0622	0.3017	-0.4246	0.2279
0.2659	-0.3487	-0.2441	0.2912
0.2852	0.3454	-0.4288	-0.1321
-0.4311	-0.2622	-0.4033	-0.3940
0.2724	0.3175	-0.3757	-0.1446
0.4075	-0.3948	-0.2972	0.4352
0.3989	-0.3948	-0.2993	0.4369
0.2104	-0.2415	-0.2781	0.3735

Pseudo-inverse of Modal Matrix:

0.6930	-1.1958	0.1285	0.7624	-0.5531	0.8327	0.1890	0.1488	-0.4749
-0.3602	0.9296	-0.3127	0.2046	-0.6786	0.1369	-0.2661	-0.2553	0.0683
0.1673	-0.6857	-0.1476	-0.2975	-0.7697	-0.2263	-0.1375	-0.1500	-0.3092
-0.3248	1.5153	-0.0428	-0.7089	-0.5989	-0.7952	0.0839	0.1200	0.6945

Fig. 42 Modal matrix and its pseudo-inverse of square plate without mass.

Physical Mass:

0.0014	-0.0028	0.0003	0.0014	-0.0008	0.0016	0.0003	0.0003	-0.0011
-0.0028	0.0066	-0.0005	-0.0029	0.0010	-0.0033	-0.0005	-0.0004	0.0026
0.0003	-0.0005	0.0001	0.0003	0.0001	0.0003	0.0001	0.0001	-0.0001
0.0014	-0.0029	0.0003	0.0020	-0.0006	0.0022	0.0003	0.0002	-0.0012
-0.0008	0.0010	0.0001	-0.0006	0.0020	-0.0007	-0.0001	0.0000	0.0005
0.0016	-0.0033	0.0003	0.0022	-0.0007	0.0024	0.0003	0.0002	-0.0014
0.0003	-0.0005	0.0001	0.0003	-0.0001	0.0003	0.0002	0.0001	-0.0001
0.0003	-0.0004	0.0001	0.0002	0.0000	0.0002	0.0001	0.0001	-0.0001
-0.0011	0.0026	-0.0001	-0.0012	0.0005	-0.0014	-0.0001	-0.0001	0.0011

Physical Damping:

0.0292	-0.0614	0.0058	0.0307	-0.0139	0.0345	0.0066	0.0048	-0.0239
-0.0614	0.1531	-0.0110	-0.0674	0.0128	-0.0769	-0.0092	-0.0051	0.0615
0.0058	-0.0110	0.0026	0.0059	0.0022	0.0065	0.0026	0.0023	-0.0032
0.0307	-0.0674	0.0059	0.0449	-0.0071	0.0486	0.0059	0.0038	-0.0290
-0.0139	0.0128	0.0022	-0.0071	0.0431	-0.0083	-0.0021	-0.0014	0.0063
0.0345	-0.0769	0.0065	0.0486	-0.0083	0.0529	0.0064	0.0040	-0.0328
0.0066	-0.0092	0.0026	0.0059	-0.0021	0.0064	0.0033	0.0030	-0.0023
0.0048	-0.0051	0.0023	0.0038	-0.0014	0.0040	0.0030	0.0028	-0.0006
-0.0239	0.0615	-0.0032	-0.0290	0.0068	-0.0328	-0.0023	-0.0006	0.0261

Physical Stiffness:

Row #1

2.1325E+003	-8.0794E+003	2.2985E+002	2.7963E+003	1.3539E+003
3.3064E+003	-2.3697E+002	-4.2648E+002	-3.4599E+003	

Row #2

-8.0794E+003	3.3841E+004	-5.8506E+002	-1.1990E+004	-7.2127E+003
-1.4108E+004	1.6691E+003	2.4527E+003	1.4916E+004	

Row #3

2.2985E+002	-5.8506E+002	3.4522E+002	6.9899E+002	1.4093E+003
7.0339E+002	2.5480E+002	2.4038E+002	-1.0208E+002	

Row #4

2.7963E+003	-1.1990E+004	6.9899E+002	7.3406E+003	6.2379E+003
7.9485E+003	-3.4748E+002	-6.5354E+002	-5.5596E+003	

Row #5

1.3539E+003	-7.2127E+003	1.4093E+003	6.2379E+003	9.7680E+003
6.4749E+003	3.5777E+002	1.8149E+002	-3.0212E+003	

Row #6

3.3064E+003	-1.4108E+004	7.0339E+002	7.9485E+003	6.4749E+003
8.6961E+003	-4.6984E+002	-8.2398E+002	-6.4871E+003	

Row #7

-2.3697E+002	1.6681E+003	2.5480E+002	-3.4748E+002	3.5777E+002
-4.6984E+002	3.6527E+002	4.0336E+002	9.1096E+002	

Row #8

-4.2648E+002	2.4527E+003	2.4038E+002	-6.5354E+002	1.8149E+002
-8.2398E+002	4.0336E+002	4.6006E+002	1.2617E+003	

Row #9

-3.4599E+003	1.4916E+004	-1.0208E+002	-5.5596E+003	-3.0212E+003
-6.4871E+003	9.1096E+002	1.2617E+003	6.7554E+003	

Fig. 43 Physical parameters of square plate without mass.

Natural Frequencies:

1	9.79733E+001 Hz
2	2.37756E+002 Hz
3	4.44740E+002 Hz
4	6.02077E+002 Hz
5	1.87418E+009 Hz
6	3.38849E+009 Hz
7	4.55478E+009 Hz
8	1.48643E+010 Hz
9	7.37614E+009 Hz

Modal Matrix:

0.4632	-0.3586	-0.1312	0.3872	-0.3170	0.2852	-0.3734	0.1234	-0.1527
0.0622	0.3017	-0.4246	0.2279	-0.0473	0.0369	-0.1159	0.0839	0.1067
0.2659	-0.3487	-0.2441	0.2912	0.3287	-0.3247	-0.1426	0.1848	0.8557
0.2852	0.3454	-0.4288	-0.1321	-0.3684	0.3766	-0.1766	0.1122	0.0644
-0.4311	-0.2622	-0.4033	-0.3940	-0.0817	0.0754	-0.0514	0.0080	-0.0919
0.2724	0.3175	-0.3757	-0.1446	0.4243	-0.4279	0.2369	-0.1566	-0.1079
0.4075	-0.3948	-0.2972	0.4352	0.4881	-0.4835	0.7673	-0.7220	-0.2895
0.3989	-0.3948	-0.2993	0.4369	-0.4767	0.4962	-0.3747	0.5918	0.0008
0.2104	-0.2415	-0.2781	0.3735	0.0379	-0.0350	0.0883	-0.1868	-0.3536

Fig. 44 Natural frequencies and modal matrix obtained by physical parameters for a square plate without mass.

Physical Mass of Modified System:

0.0015	-0.0029	0.0003	0.0014	-0.0008	0.0016	0.0004	0.0003	-0.0011
-0.0029	0.0066	-0.0006	-0.0029	0.0010	-0.0033	-0.0006	-0.0004	0.0026
0.0003	-0.0006	0.0001	0.0003	0.0001	0.0003	0.0001	0.0001	-0.0001
0.0014	-0.0029	0.0003	0.0020	-0.0006	0.0022	0.0003	0.0002	-0.0012
-0.0008	0.0010	0.0001	-0.0006	0.0020	-0.0007	-0.0001	-0.0001	0.0005
0.0016	-0.0033	0.0003	0.0022	-0.0007	0.0024	0.0004	0.0002	-0.0014
0.0004	-0.0006	0.0001	0.0003	-0.0001	0.0004	0.0002	0.0002	-0.0001
0.0003	-0.0004	0.0001	0.0002	-0.0001	0.0002	0.0002	0.0002	-0.0001
-0.0011	0.0026	-0.0001	-0.0012	0.0005	-0.0014	-0.0001	-0.0001	0.0011

Physical Damping of Modified System:

0.0292	-0.0614	0.0059	0.0307	-0.0139	0.0345	0.0066	0.0048	-0.0239
-0.0614	0.1531	-0.0110	-0.0674	0.0128	-0.0769	-0.0092	-0.0051	0.0615
0.0059	-0.0110	0.0026	0.0059	0.0022	0.0065	0.0026	0.0023	-0.0032
0.0307	-0.0674	0.0059	0.0449	-0.0071	0.0486	0.0059	0.0038	-0.0290
-0.0139	0.0128	0.0022	-0.0071	0.0431	-0.0083	-0.0021	-0.0014	0.0068
0.0345	-0.0769	0.0065	0.0486	-0.0083	0.0529	0.0064	0.0040	-0.0328
0.0066	-0.0092	0.0026	0.0059	-0.0021	0.0064	0.0033	0.0030	-0.0023
0.0048	-0.0051	0.0023	0.0038	-0.0014	0.0040	0.0030	0.0028	-0.0006
-0.0239	0.0615	-0.0032	-0.0290	0.0068	-0.0328	-0.0023	-0.0006	0.0261

Physical Stiffness of Modified System:

Row #1
2.1325E+003 -8.0784E+003 2.2985E+002 2.7963E+003 1.3539E+003
3.3064E+003 -2.3697E+002 -4.2648E+002 -3.4599E+003

Row #2
-8.0784E+003 3.3841E+004 -5.8506E+002 -1.1990E+004 -7.2127E+003
-1.4108E+004 1.6681E+003 2.4527E+003 1.4916E+004

Row #3
2.2985E+002 -5.8506E+002 3.4522E+002 6.9899E+002 1.4093E+003
7.0339E+002 2.5480E+002 2.4038E+002 -1.0208E+002

Row #4
2.7963E+003 -1.1990E+004 6.9899E+002 7.3406E+003 6.2379E+003
7.9485E+003 -3.4748E+002 -6.5354E+002 -5.5596E+003

Row #5
1.3539E+003 -7.2127E+003 1.4093E+003 6.2379E+003 9.7680E+003
6.4749E+003 3.5777E+002 1.8149E+002 -3.0212E+003

Row #6
3.3064E+003 -1.4108E+004 7.0339E+002 7.9485E+003 6.4749E+003
8.6961E+003 -4.6984E+002 -8.2398E+002 -6.4871E+003

Row #7
-2.3697E+002 1.6681E+003 2.5480E+002 -3.4748E+002 3.5777E+002
-4.6984E+002 3.6527E+002 4.0336E+002 9.1096E+002

Row #8
-4.2648E+002 2.4527E+003 2.4038E+002 -6.5354E+002 1.8149E+002
-8.2398E+002 4.0336E+002 4.6006E+002 1.2617E+003

Row #9
-3.4599E+003 1.4916E+004 -1.0208E+002 -5.5596E+003 -3.0212E+003
-6.4871E+003 9.1096E+002 1.2617E+003 6.7554E+003

Fig. 45 Physical parameters of a square plate with 0.072 kg. mass.

system because they were not modified. An eigenanalysis is again performed, and the natural frequencies and modal matrix resulted are shown in Fig. 46. Again, only those corresponding to the first four modes are valid. They compare quite well with those obtained in modal coordinates. Therefore, all the theories have been verified.

Natural Frequencies:

1	9.62851E+001	Hz
2	2.28901E+002	Hz
3	4.43223E+002	Hz
4	5.83791E+002	Hz
5	1.17739E+009	Hz
6	2.57054E+009	Hz
7	2.83069E+009	Hz
8	3.19011E+009	Hz
9	4.16776E+009	Hz

Modal Matrix:

0.4675	-0.3497	-0.1235	0.3679	-0.7289	0.1524	-0.1469	0.1524	-0.2452
0.0564	0.2988	-0.4427	0.2566	-0.3205	-0.1806	0.0217	-0.0104	0.1226
0.2717	-0.3459	-0.2363	0.2641	-0.1870	0.2615	-0.4870	-0.3802	0.7847
0.2757	0.3538	-0.4365	-0.1389	0.1724	0.1869	0.1967	-0.0012	-0.2312
-0.4221	-0.2750	-0.3870	-0.4642	-0.1276	-0.0321	0.0343	0.0550	-0.0950
0.2634	0.3259	-0.3821	-0.1524	-0.0277	-0.0709	-0.2567	0.0034	0.1946
0.4133	-0.3895	-0.2902	0.4102	0.3717	-0.2571	0.7403	-0.5839	-0.1679
0.4048	-0.3898	-0.2925	0.4122	0.2442	-0.5434	-0.0384	0.6951	0.0653
0.2147	-0.2413	-0.2773	0.3646	0.2939	0.6881	-0.2924	0.0711	-0.4199

Fig. 46 Natural frequencies and modal matrix of a square plate with 0.072 kg. mass.

6.2.2 C-CLAMP WITH STIFFNESS MODIFICATION

Since there is no convenient way to make appropriate stiffness modification on the square plate, a large standard 0.305 meter C-clamp is used for this verification. The C-clamp was hung vertically in the air by a small pulley and a piece of shock cord as shown in Fig. 47. To prevent the screw from rocking when it is open, a soft helical spring of 270 N/m stiffness is clamped inside the openings. The stiffness of the helical spring is computed by using the formula presented in Shigley [25]. The total stiffness is given by the helical spring in series with the screw.

Three-dimensional data are acquired by using a triaxial accelerometer, an impact hammer, and the four channels Zonic 6080 Fast Fourier Transform Analyzer. First the energy level at all frequencies of interest is checked for the input and output signal so that the maximum dynamic range of the analyzer can be utilized. A peak search is performed on a frequency response function at the driving point, from which the three most significant peaks are chosen for this analysis. Data are acquired using the BASIC MODAL®. A typical set of frequency response function of x direction obtained at point number one is shown in Fig. 48.

The deflected plots of the three modes produced by BASIC MODAL® are shown in Fig. 49. The effective parameters, modal parameters, and modal matrix are given in Fig. 50. These data are used as the base model for modification.

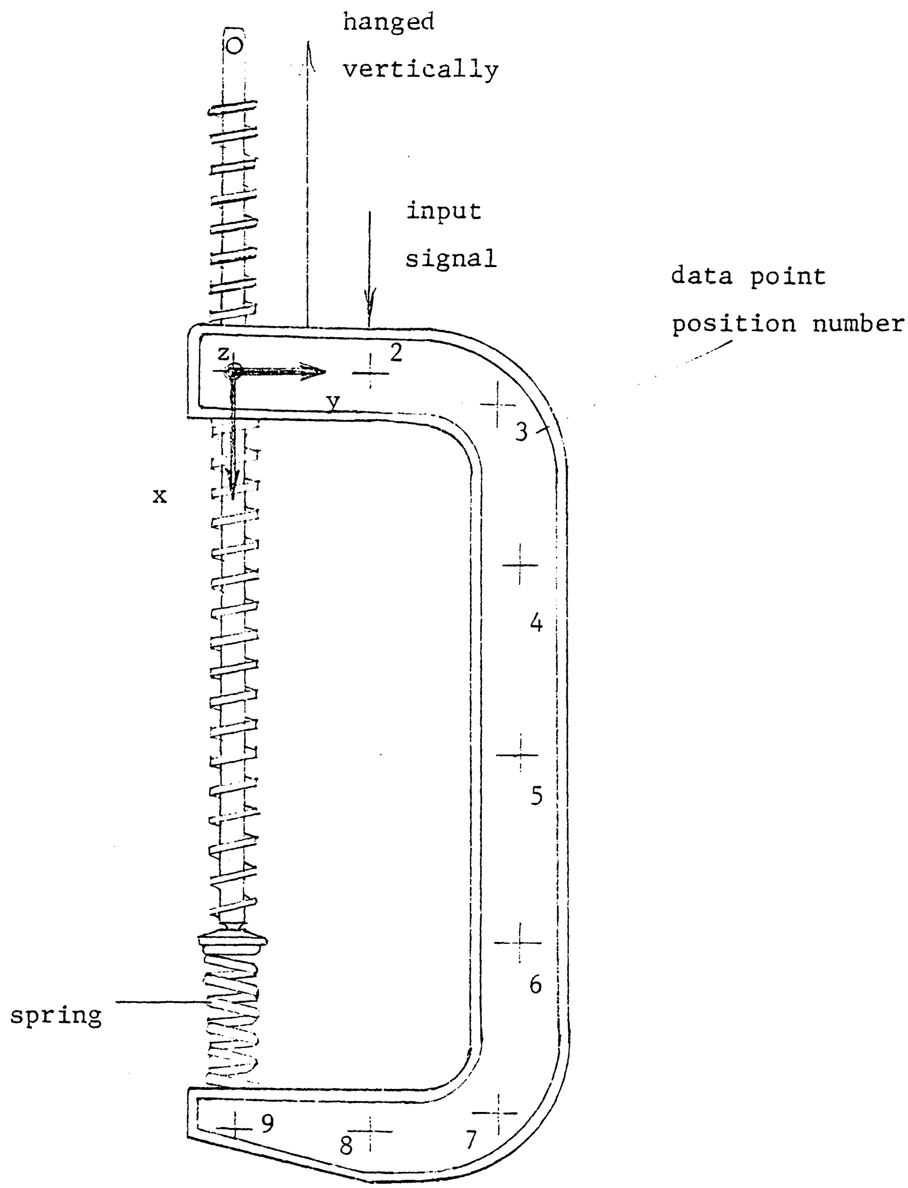


Fig. 47 C-clamp.

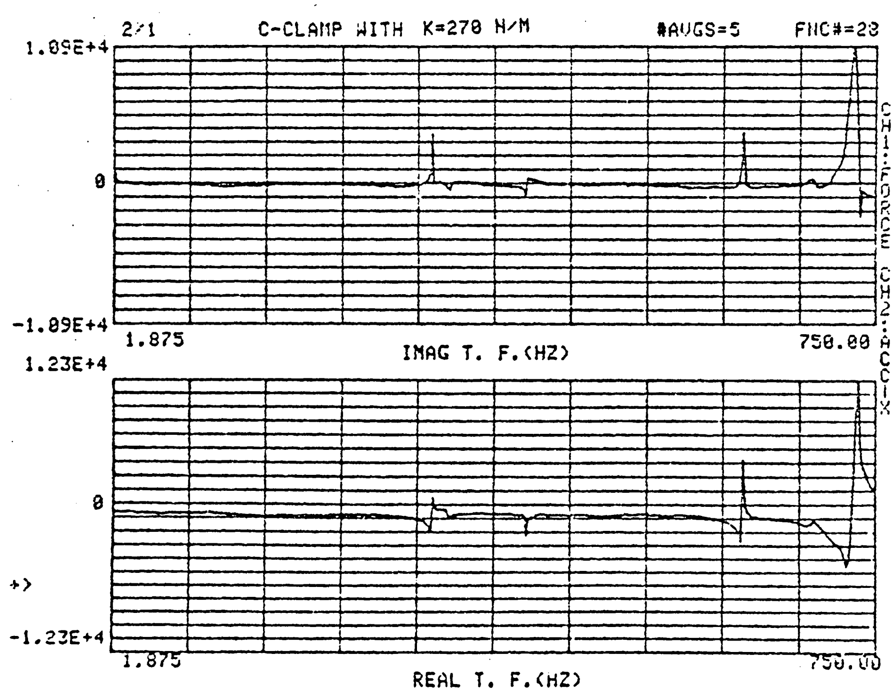


Fig. 48 Frequency response function of x direction at point #1 of C-clamp with 270 N/m stiffness.

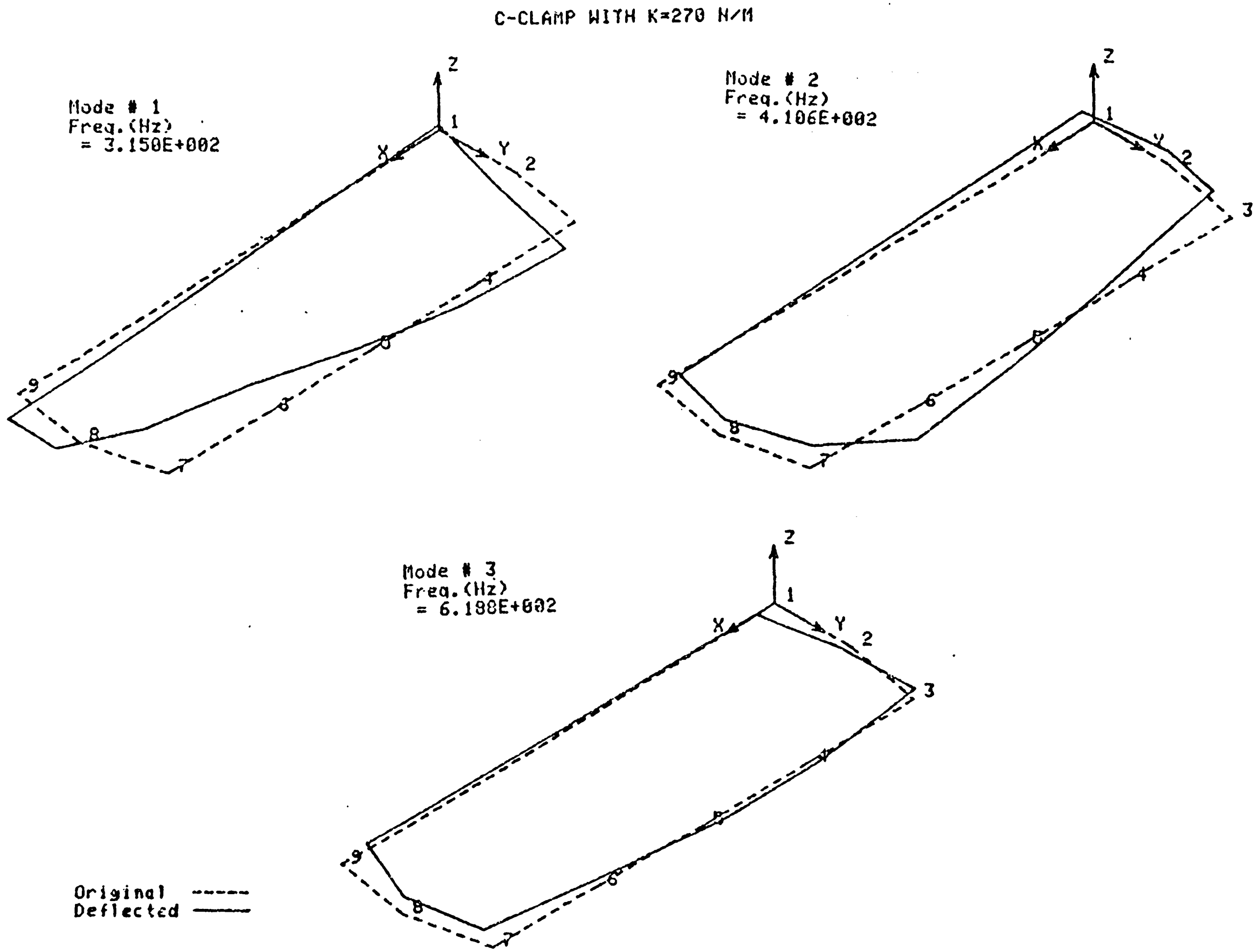


Fig. 49 Deflected plots of C-clamp with 270 N/m stiffness for mode #1, 2, and 3.

VISCOUS DAMPING MODELING

COEFFICIENTS DERIVED FROM POINT#2 X

FREQUENCY, DAMPING, MASS AND STIFFNESS TABLE

Mode and Dir.	Peak Frequency (Hz)	Natural Frequency (Hz)	Effective Mass	Effective Stiffness	Effective Viscous Damping, C	% Critical Damping (%)
1:X	315.00	313.87	9.760E-002	3.796E+005	6.367E-001	0.17
2:X	410.63	411.90	3.989E+000	2.672E+007	1.858E+000	0.01
3:X	618.75	606.25	1.915E-002	2.778E+005	1.693E+000	1.16
1:Y	315.00	315.08	2.642E+000	1.036E+007	5.015E+002	4.79
2:Y	410.63	417.94	1.042E+000	7.187E+006	2.653E+002	4.35
3:Y	618.75	600.17	1.981E+000	2.818E+007	1.777E+002	1.19
1:Z	315.00	313.55	2.771E-001	1.076E+006	1.595E+000	0.15
2:Z	410.63	412.08	1.826E+000	1.224E+007	7.384E+000	0.08
3:Z	618.75	605.00	4.833E-002	6.984E+005	3.843E+000	1.05

Mode and Dir.	Peak Frequency (Hz)	Natural Frequency (Hz)	Modal Mass	Modal Stiffness	Modal Damping	% Critical Damping (%)
1:X	315.00	313.87	5.325E-003	2.071E+004	3.474E-002	0.17
2:X	410.63	411.90	7.520E-002	5.037E+005	3.503E-002	0.01
3:X	618.75	606.25	1.596E-004	2.316E+003	1.411E-002	1.16

Title: C-CLAMP WITH K=270 N/M
Deflection data: (Read points in ascending order from left to right)

Dir:X	Y	Z	X	Y	Z
Mode #1					
0.02914	0.02914	0.09361	0.23357	0.04691	-0.09997
0.15996	0.00191	-0.24721	0.11724	-0.07534	-0.22449
0.06798	-0.15814	-0.02518	0.02914	-0.24266	0.26629
0.02914	-0.29810	0.46533	0.02914	-0.31083	-0.21267
0.02914	-0.11542	-0.37717			
Mode #2					
-0.18166	-0.36715	-0.13673	-0.13738	-0.16025	0.04398
0.12162	-0.16044	0.34994	0.06865	-0.22565	-0.11837
-0.03442	-0.05450	-0.22947	-0.05125	-0.03346	-0.48571
-0.05813	-0.01738	0.26198	-0.12544	-0.04054	0.10976
-0.36141	-0.03232	-0.06731			
Mode #3					
0.30888	-0.05158	-0.05356	0.09130	-0.14455	-0.02511
-0.12857	-0.09890	0.06048	-0.12325	0.10499	0.00951
-0.21911	0.08140	-0.01202	-0.17270	-0.07608	0.04238
-0.14683	-0.33095	0.05424	-0.31497	-0.28530	-0.00619
-0.62233	-0.13618	-0.01529			

Fig. 50 Effective and modal parameters of C-clamp with 270 N/m stiffness.

The modified system is obtained by using another helical spring whose stiffness is computed as 2546 N/m. This information is entered into BASIC MODIFICATION®, and the predicted modal parameters of the modified system are listed in Fig. 51. Also, the predicted deflected plots of all three modes are shown in Fig. 52.

These analytical results are then compared to the actual experimental results. The 2546 N/m spring was clamped inside the C-clamp, and the testing was repeated using a frequency range of 1000 Hz because the predicted results show that the third mode should have moved to 790.53 Hz. The FFT analyzer was set up as before with only the frequency range changed. A peak search is performed on a driving point measurement. With the predicted frequencies in mind, one chooses the three modes with values close to the predicted frequencies. However, these frequencies must have their mode shapes check against the predicted mode shape to verify correct selection. The typical frequency response function plots at point number one for x direction is in Fig. 53. The deflected plots, effective parameters, and modal parameters of this experimentally obtained model are in Figs. 54 and 55.

The natural frequencies predicted are very close, within 3.5%, to those obtained experimentally. The mode shape predictions are fair. The predicted mode shape of mode number 2 is 180° phase shifted from the experimental one, which, of course, is no problem. The magnitude of the mode shape are different at some measured points, but on the whole, they look alike. They have about the same bending, twisting, and compression effect in the structure.

Title: C-CLAMP WITH K=2546 N/M, PREDICT BY BASIC MODIFICATION
MODIFIED PARAMETERS:

Mode and Dir.	Natural Frequency (Hz)	Modal Mass	Modal Stiffness	Modal Damping	% Critical Damping (%)
1:X	313.87	5.325E-003	2.071E+004	3.474E-002	0.17
2:X	411.91	7.432E-002	4.978E+005	3.479E-002	0.01
3:X	790.53	1.596E-004	3.937E+003	1.411E-002	0.89

Title: C-CLAMP WITH K=2546 N/M, PREDICT BY BASIC MODIFICATION
Deflection Data:

Dir:X	Y	Z	X	Y	Z
Mode #1					
0.02914	0.02914	0.09361	0.23357	0.04691	-0.09997
0.15996	0.00191	-0.24721	0.11724	-0.07534	-0.22445
0.06798	-0.15814	-0.02518	0.02914	-0.24266	0.26629
0.02914	-0.29810	0.46533	0.02914	-0.31083	-0.21267
0.02914	-0.11542	-0.37717			
Mode #2					
0.21410	0.35939	0.13011	0.14639	0.14362	-0.04645
-0.13485	0.14876	-0.34131	-0.08162	0.23570	0.11870
0.01045	0.06301	0.22691	0.03221	0.02501	0.49744
0.04186	-0.01862	-0.25455	0.09053	0.00935	-0.10979
0.29177	0.01735	0.06525			
Mode #3					
0.30884	-0.05167	-0.05359	0.09126	-0.14459	-0.02510
-0.12955	-0.09894	0.06056	-0.12323	0.10494	0.00948
-0.21912	0.08139	-0.01207	-0.17271	-0.07609	0.04226
-0.14685	-0.33095	0.05431	-0.31500	-0.28531	-0.00617
-0.62241	-0.13619	-0.01531			

Fig. 51 Predicted modal parameters of C-clamp with 2546 N/m stiffness.

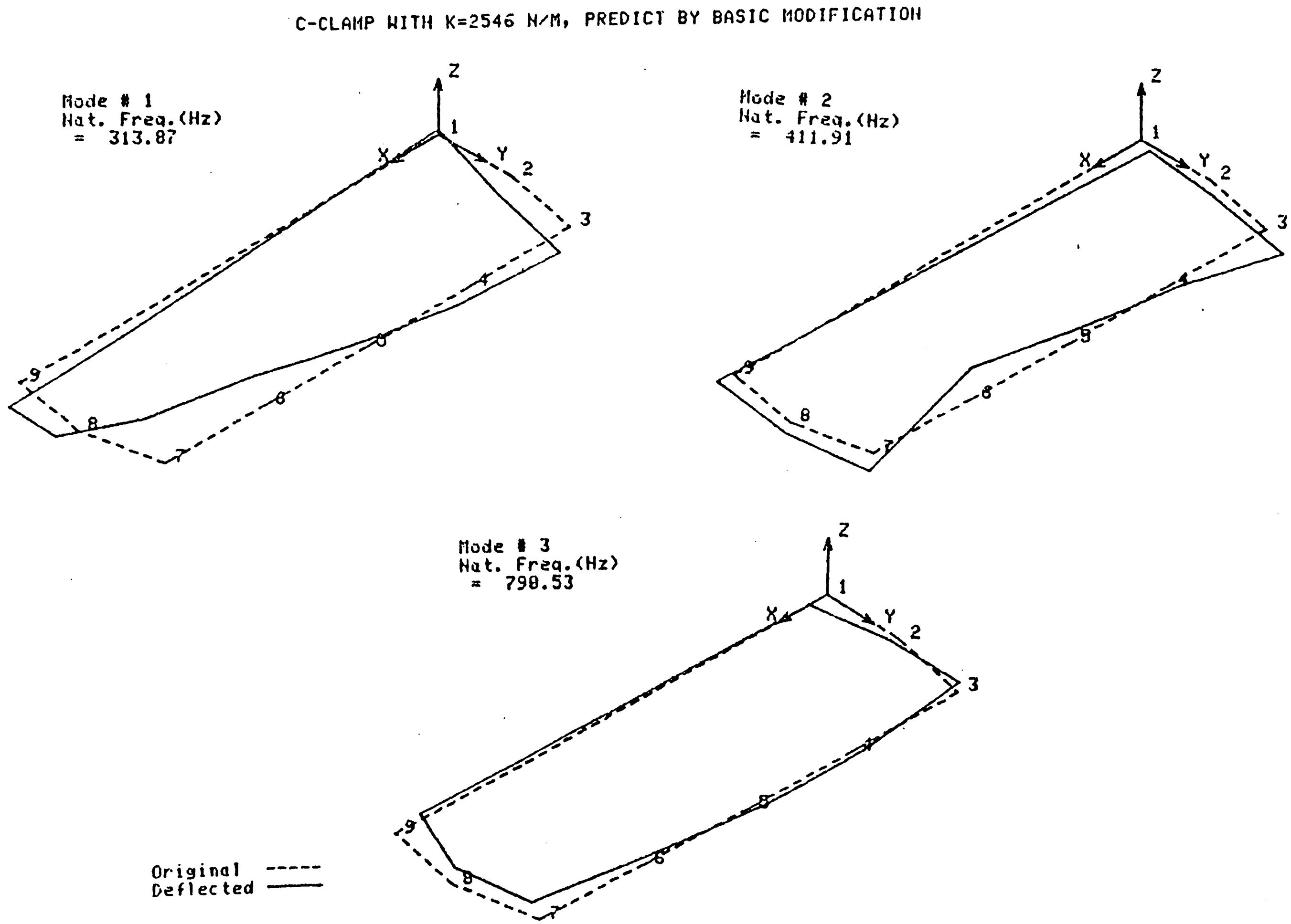
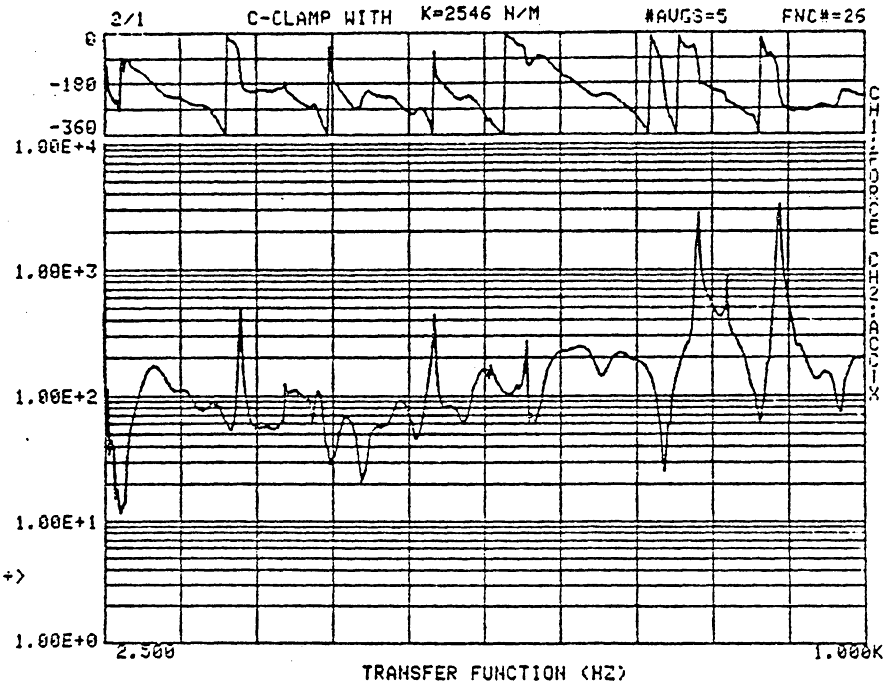


Fig. 52 Predicted deflected plots of C-clamp with 2546 N/m stiffness.



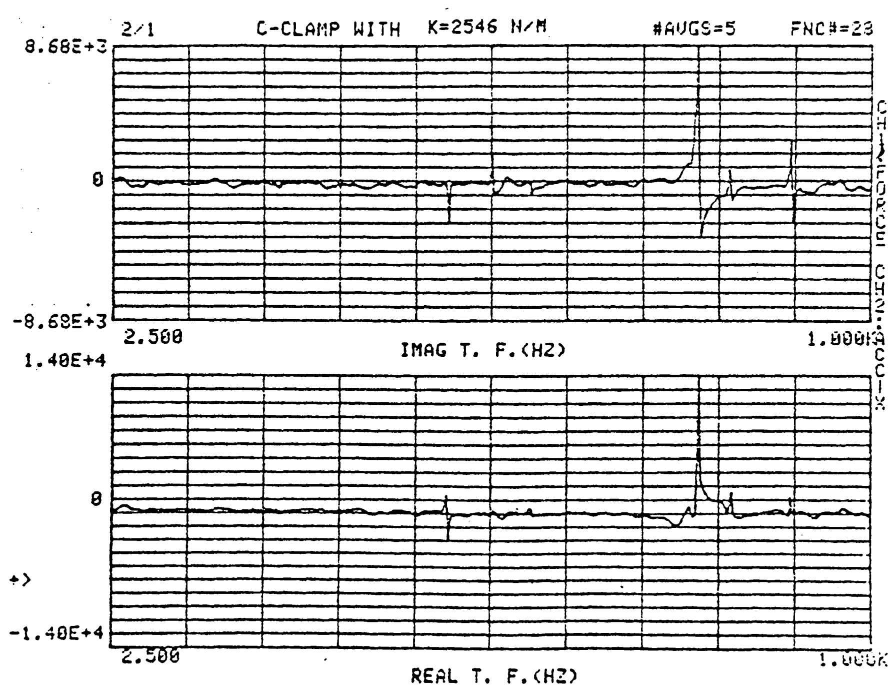


Fig. 53 Experimental frequency response function (x direction) at point #1 of C-clamp with 2546 N/m stiffness.

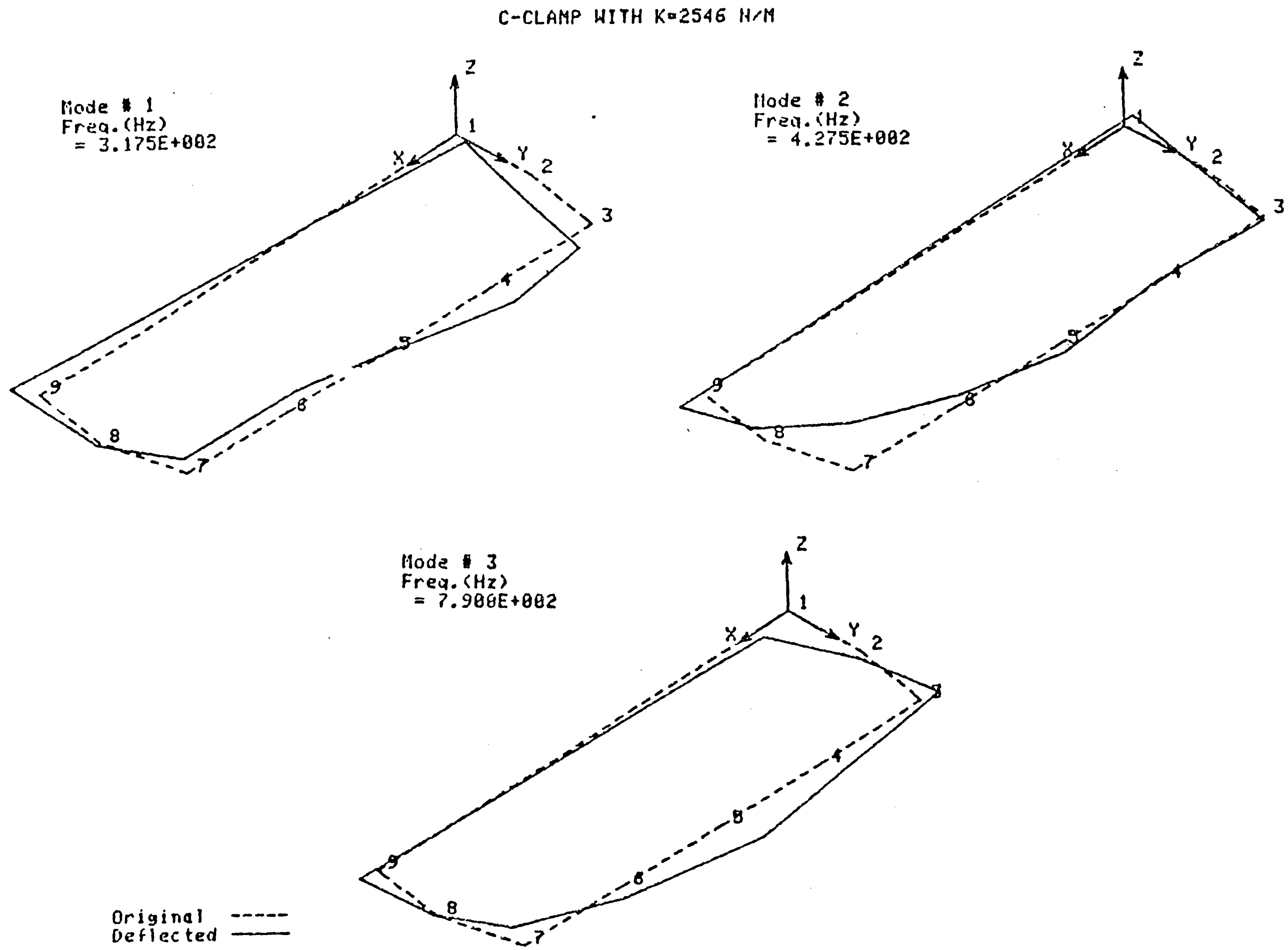


Fig. 54. Deflected plots of C-clamp with 2546 N/m stiffness, obtained experimentally.

VISCOUS DAMPING MODELING

COEFFICIENTS DERIVED FROM POINT#2 X

FREQUENCY, DAMPING, MASS AND STIFFNESS TABLE

Mode and Dir.	Peak Frequency (Hz)	Natural Frequency (Hz)	Effective Mass	Effective Stiffness	Effective Viscous Damping,C	% Critical Damping (%)
1:X	317.50	312.75	2.088E-001	8.064E+005	3.003E+001	3.66
2:X	427.50	426.80	2.535E-001	1.823E+006	3.535E+000	0.26
3:X	790.00	789.91	1.062E-002	2.615E+005	2.555E-001	0.24

Mode and Dir.	Peak Frequency (Hz)	Natural Frequency (Hz)	Modal Mass	Modal Stiffness	Modal Damping	% Critical Damping (%)
1:X	317.50	312.75	1.305E-002	5.037E+004	1.876E+000	3.66
2:X	427.50	426.80	1.451E-003	1.044E+004	2.024E-002	0.26
3:X	790.00	789.91	1.075E-004	2.649E+003	2.583E-003	0.24

Title: C-CLAMP WITH K=2546 N/M
Deflection data: (Read points in ascending order from left to right)

Dir:X	Y	Z	X	Y	Z
Mode #1					
0.14228	0.23841	0.11536	0.24994	-0.07491	0.06399
0.24994	0.10075	-0.04660	-0.04491	0.23072	-0.04660
0.24994	0.20765	0.20765	-0.19150	-0.10305	0.05430
0.22687	0.19226	0.34915	0.20840	0.24994	0.24225
0.24994	-0.06352	0.15843			
Mode #2					
-0.00508	0.00138	0.10160	0.07567	0.04456	0.01384
-0.03851	-0.03968	-0.10879	-0.03985	-0.15285	-0.10879
-0.06205	0.04708	-0.14662	-0.30771	-0.20009	-0.02976
-0.19169	-0.22195	0.62551	-0.09660	-0.25222	0.04288
0.20178	-0.25726	-0.27072			
Mode #3					
0.50267	0.00437	-0.06377	0.10064	0.04362	-0.03311
-0.20234	0.03621	-0.01517	-0.15944	0.26019	-0.01517
-0.21037	0.46859	-0.06953	0.22033	0.21480	0.01528
-0.12401	-0.31223	0.05702	0.01594	-0.04296	0.00276
0.27126	-0.02104	-0.00159			

Fig. 55 Effective and modal parameters of C-clamp with 2546 N/m stiffness.

The reason for the fair mode shape prediction is due to the fact that a real mode approach is used. The C-clamp does not necessarily behave linearly. The fact that the screw is not tightly fixed at the body contributes to an increase of damping, probably dry friction or coulomb damping which is non-linear in nature. Looseness also complicates the testing, and the quality of frequency response function suffers. Besides, the nine measured points used in the analysis may not define the model sufficiently. For example, no data is taken on the screw thread, a major element in the structural system.

Chapter 7

SUMMARY AND CONCLUSION

Two types of damping models, structural and viscous damping, are provided because for actual structures, the damping may result from a combination of effects which may fit neither of these models perfectly. In the case where damping is small, the damping ratio is less than 1%, then the difference between these two models is slight. One way to determine which is a better model is to look at the circle fit error which is the average error between each data point and the least square fit circle. As long as the half-power bandwidth of the two adjacent modes do not overlap, this single degree-of-freedom model method will give reasonably good results.

This dissertation shows that if a single degree-of-freedom model is used on each mode of vibration, the resulting mass, stiffness, and damping are effective parameters. If the first element of each column of the modal matrix which corresponds to the driving point is normalized to unity, and if the effective parameters are derived from the driving point measurement, then the effective and modal parameters will be the same. This can be seen quite easily by looking at Eq. 3.14, which is the relation between effective and modal parameters. If p_i^j and p_k^j are unity, then the effective and modal parameters will equal to each other. This can be done by using measurement at the driving point, and normalizing the entry of the modal matrix that corresponds to the driving point to unity for every mode. But this is not possible for using transfer point measurement. Therefore, the use of effective parameters will be applied to any frequency response function, and the global modal paramters can be derived when desired.

It is also found that to completely define a system in physical coordinates, the physical mass, stiffness, and damping are sufficient. But in modal coordinates, the modal mass, stiffness, and damping, as well as the modal matrix, are required to describe a system completely because the modal parameters are all arbitrary and depend on the modal matrix. Therefore, each modal matrix, normalized in some arbitrary way, will have different modal parameters. To define a system completely, the modal parameters with the corresponding modal matrix are needed.

The difference between the complete and truncated system is discussed. The method utilizing pseudo-inverses can deal with general physical systems, even if singular or non-square modal matrix are used. These result from truncated modal descriptions where the number of degree-of-freedom is far more than the number of modes identified.

The system modification allows the engineer or designer to predict the effect of physical changes such as mass, stiffness, and damping on a structure's dynamic properties before the expense of an actual change is incurred. This will eliminate the uncertainty and the time-consuming and costly process of a trial-and-error approach. An optimized solution, which is physically possible and economically practical, can be obtained easily. Three ways of making modifications are presented and compared to find the most efficient way. This comparison is based on the assumption that a mathematical model in modal space is already obtained. Table 5 gives the required matrix operations of all three ways for both viscous and structural damping models. The most efficient way is to make modifications in modal coordinates. The number of matrix operations are fewer, and only one ordinary matrix inverse is required. Moreover, the matrix size is $\ell \times \ell$, which is usually much smaller than n x n. Even the resonance specification and the frequency response function synthesis, when done in modal coordinates, will save a large number of matrix operations, as shown in Table 5. Therefore, it saves a great deal of computational time and memory space if the modifications, resonance specification, and frequency response function synthesis are all done in modal coordinates. A new approach using modal space I and modal space II has to be used when making modifications solely in modal coordinates, however.

The technique of developing a general stiffness or damping modification matrix makes the system modification easier, especially when the system has many degrees-of-freedom in all three dimensions. The technique developed requires only to specify the point number of the two measured points on the structure. Using the geometry data (the x, y, z coordinates specifying the location of a measured point), the direction cosines are computed and the required entries of the physical stiffness or damping matrices are determined. In this way, the

Table 5 Comparison of Matrix Operation required for the three Modification Methods.

Matrix Operation Required

	Physical Coordinates Model	Physical and Modal Coordinates Model	Modal Coordinates Model
Modification	28x, 4+, 2T 3PIV, 1EA	22x, 4+, 2T 3PIV, 1EA	16x, 4+, 2T 1IV, 1EA
Matrix Size	n x n	n x n	$\ell \times \ell$
Resonance Specification	13x, 3+, 1T 3PIV per step of Half-Interval Search	13x, 3+, 1T 3PIV per step of Half-Interval Search	5x, 3+, 1T 1IV per step of Half-Interval Search
Matrix Size	n x n	n x n	$\ell \times \ell$
Frequency Response Synthesis	8x, 2+, 1T 3PIV per point plotted	8x, 2+, 1T 3PIV per point plotted	No matrix operation, done by summation of ℓ scalar terms
Matrix Size	2n x 2n	2n x 2n	Not applicable

Keys:

x = matrix multiplication
+ = matrix addition
T = transpose of a matrix
PIV = pseudo-inverse of a matrix
IV = ordinary inverse of a matrix
EA = Eigenanalysis
n = no. of degree-of-freedom
ℓ = no. of modes

direction of a point need not to be told in order that corrected entries in the physical matrix can be determined.

Both the system modeling and modification, when in modal coordinates, are very computationally efficient so that they can be implemented in a small microprocessor having only 54K of memory. Therefore, mini- or main-framed computers which are still extremely expensive need not be used.

Chapter 8

RECOMMENDATION

For further work, it is recommended that a similar approach, but in complex mode, be developed to give a better mathematical model of the system. Although most real-world structures have real mode behavior, some structures which have non-proportional or high damping, such as those which contain elastomeric mounts, shock absorbers, or other localized areas of high damping, will exhibit complex modes.

Another way to improve this method is to use a multi-mode approach, i.e., to analyze simultaneously all the contiguous structure's modes using multi-mode curve fitting instead of analyzing one mode at a time. This should handle the case where two adjacent modes are very close.

Some structures, such as helicopters, have non-linear behavior which can contribute large errors to the modal analysis process. If such large errors cannot be eliminated by testing the structure in its linear range, then testing via modal analysis should be terminated. It is, therefore, necessary to develop methods of analyzing measured data that can identify and quantify non-linearities so that the engineer may make decisions concerning data validity.

For the system modification, it is recommended that methods be developed which will account for the effects of the rotatory inertia of the modification masses and the bending stiffness of the modification springs. This improvement will give a better model of a real-world structural modification and thus a better predicition of the modified modal frequencies and mode shapes.

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APPENDIX

I DETERMINATION OF THE DIAMETER OF A MODAL CIRCLE IN NYQUIST PLANE

i) VISCOUS DAMPING MODEL

Since only mobility-type of frequency response function for the case of viscous damping is a perfect circle in a Nyquist plane, the modal circle of this type is discussed. The effective equations of motion will be used, but for simplicity, all superscripts and subscripts are deleted.

The equation of motion for one mode in a mobility-type frequency response function form is

$$i\omega mv + cv - \frac{ik}{\omega} v = F \tag{A.1}$$

The real and imaginary part of the frequency response function for steady-state response are

$$\left.\begin{aligned} Re\left(\frac{v}{F}\right) &= \frac{c}{c^2 + (m\omega - \frac{k}{\omega})^2} \\ Im\left(\frac{v}{F}\right) &= \frac{-(m\omega - \frac{k}{\omega})}{c^2 + (m\omega - \frac{k}{\omega})^2} \end{aligned}\right\} \tag{A.2}$$

The equation of a circle is

$$(x - R)^2 + (y - R)^2 = R^2 \tag{A.3}$$

where R is the radius of the circle. One knows that the circle of v/F locates at the positive real axis, with the real part as the x axis, and the imaginary part as the y axis. Eq. A.3 can be written as

$$(Re - R)^2 + Im^2 = R^2 \tag{A.4}$$

Expanding and rearranging, one arrives at

$$R = \frac{Re^2 + Im^2}{2Re} \tag{A.5}$$

Substituting Eq. A.2 into Eq. A.5, and after simplifying, one obtains

$$R = \frac{1}{2c} \tag{A.6}$$

or the diameter of the modal circle is $\frac{1}{c}$.

If a similar analysis is repeated for x/F and for A/F type of frequency response function, the diameter of the modal circle can be found to be $\frac{1}{c\omega}$ and $\frac{\omega}{c}$, respectively.

ii) STRUCTURAL DAMPING MODEL

The effective equation of motion for one mode of the dynamic compliance type frequency response function is

$$m\ddot{x} + k(1+ig)x = F \tag{A.7}$$

The real and imaginary part of the frequency response function for steady-state response are

$$\left.\begin{aligned} Re\left(\frac{x}{F}\right) &= \frac{(k - m\omega^2)}{(k-m\omega^2) + ikg} \\ Im\left(\frac{x}{F}\right) &= \frac{-kg}{(k-m\omega^2) + (kg)^2} \end{aligned}\right\} \quad (A.8)$$

Knowing that the circle lies in the negative imaginary axis, the equation of the circle for this case is

$$Re^2 + (Im + R)^2 = A^2 \quad (A.9)$$

Expanding and rearranging, one arrives at

$$R = -\left(\frac{Re^2 + Im^2}{2Im}\right) \quad (A.10)$$

Substituting Eq. A.8 into Eq. A.10, and simplifying, one obtains

$$R = \frac{-1}{2kg} \quad (A.11)$$

or the diameter of the circle is $\frac{1}{kg}$.

For v/F and A/F type of frequency response function, the diameter of the circle is $\frac{\omega}{kg}$ and $\frac{\omega^2}{kg}$ respectively.

II. LEAST SQUARES ERROR FIT OF A CIRCLE [26]

The general equation of a circle can also be written as

$$x^2 + y^2 + ax + by + c = 0 \quad (A.12)$$

Setting this equal to an error function E, the least squares error term is formed by a summation over the r discrete frequencies in the area of the resonance peak.

$$\sum_{i=1}^{r} E^2 = \sum_{i=1}^{r} (x_i^2 + y_i^2 + ax_i + by_i + c)^2 \tag{A.13}$$

The partial derivatives of the least-squares error term with respect to the constants a, b, and c should be zero; we arrive

$$\frac{\partial \sum_{i=1}^{r} E^2}{\partial a} = 2\sum_{i=1}^{r} (x_i^2 + y_i^2 + ax_i + by_i + c)x = 0 \tag{A.14}$$

$$\frac{\partial \sum_{i=1}^{r} E^2}{\partial b} = 2\sum_{i=1}^{r} (x_i^2 + y_i^2 + ax_i + by_i + c)y = 0 \tag{A.15}$$

$$\frac{\partial \sum_{i=1}^{r} E^2}{\partial c} = 2\sum_{i=1}^{r} (x_i^2 + y_i^2 + ax_i + by_i + c) = 0 \tag{A.16}$$

Rearranging them in matrix form,

$$\begin{bmatrix} \sum x^2 & \sum xy & \sum x \\ \sum xy & \sum y^2 & \sum y \\ \sum x & \sum y & r \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} -\sum(x^3 + xy^2) \\ -\sum(x^2 + y^3) \\ -\sum(x^2 + y^2) \end{Bmatrix} \tag{A.17}$$

The constants a, b, and c can be found by

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{bmatrix} \sum x^2 & \sum xy & \sum x \\ \sum xy & \sum y^2 & \sum y \\ \sum x & \sum y & r \end{bmatrix}^{-1} \begin{Bmatrix} -\sum(x^3 + y^2) \\ -\sum(x^2 y + y^3) \\ -\sum(x^2 + y^2) \end{Bmatrix} \quad (A.18)$$

Therefore, the location of the center of the circle can be found and the diameter can be calculated in terms of a, b, and c as

$$x_{center} = -\frac{a}{2}$$

$$y_{center} = -\frac{b}{2} \quad (A.19)$$

$$\text{Diameter} = 2\left[\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 - c\right]^{\frac{1}{2}}$$

III. DEFINITION OF PSEUDO-INVERSE, A^+, OF A MATRIX A

i) $AA^+A = A$

ii) $A^+AA^+ = A^+$

iii) $(A^+A)^T = A^+A$

iv) $(AA^+)^T = AA^+$

where A can be a rectangular matrix, or a square but singular matrix.

IV. PROPERTIES OF PSEUDO-INVERSE

i) the pseudo-inverse is unique,

ii) for a non-singular matrix, it reduces to the ordinary inverse

iii) $(A^{+})^{+} = A$

V. A RECURSIVE ALGORITHM FOR COMPUTING THE PSEUDO-INVERSE OF A RECTANGULAR MATRIX.

Greville [27] presented the following recursive algorithm for computing the pseudo-inverse of a rectangular matrix. Let a_k denote the k^{th} column of a given matrix A, and let A_k denote the sub-matrix consisting of the first k columns. Consider A_k in the partitioned form (A_{k-1}, a_k). To initiate the process, one needs to compute A_1^+ . If a_1 is a zero vector, take $A_1^+ = 0$; otherwise, compute A_1^+ by

$$A_1^+ = (a_1^T a_1)^{-1} a_1^T \qquad (A.20)$$

Then compute

$$d_k = A_{k-1}^+ a_k \qquad (A.21)$$

and

$$C_k = a_k - A_{k-1} d_k \tag{A.22}$$

If $C_k \neq 0$, then set $b_k = C_k^+$ (A.23)

If $C_k = 0$, then find

$$b_k = (1 + d_k^T d_k)^{-1} d_k^T A_{k-1}^+ \tag{A.24}$$

And

$$A_k^+ = \begin{bmatrix} A_{k-1}^+ & -d_k b_k \\ \\ \\ & b_k \end{bmatrix} \tag{A.25}$$

This algorithm can be simply implemented in a computer program. It requires no conventional inverse of a matrix.

VI. TO SHOW: THE EQ. $[P]^T [M] [P] = \lceil m \rfloor$ IS VALID IN BOTH COMPLETE AND TRUNCATED SYSTEMS

n = no. of degree-of-freedom, m = no. of modes.

Let [P] be partitioned such that

$$[P] = \begin{bmatrix} P_1 & | & P_2 \\ \hline P_3 & | & P_4 \end{bmatrix}$$

[P] is of order n x n

P_1 is a square matrix of order m x m

P_2 is of order m x (n - m)

P_3 is of order (n - m) x m

P_4 is of order (n - m) x (n - m)

and [M] is also partitioned in the following way:

$$[M] = \begin{bmatrix} M_1 & | & M_2 \\ \hline M_3 & | & M_4 \end{bmatrix}$$

[M] is of order n x n

M_1 is of order m x m

M_2 is of order m x (n - m)

M_3 is of order (n - m) x m

M_4 is of order (n - m) x (n - m)

For a complete system

$[P]^T [M] [P]$

$$= \begin{bmatrix} P_1 & | & P_3 \\ \hline P_2 & | & P_4 \end{bmatrix} \begin{bmatrix} M_1 & | & M_2 \\ \hline M_3 & | & M_4 \end{bmatrix} \begin{bmatrix} P_1 & | & P_2 \\ \hline P_3 & | & P_4 \end{bmatrix}$$

$$= \begin{bmatrix} P_1M_1 + P_3M_3 & P_1M_2 + P_3M_4 \\ P_2M_1 + P_4M_3 & P_2M_2 + P_4M_4 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} P_1M_1P_1 + P_3M_3P_1 + P_1M_2P_3 + P_3M_4P_3 & P_1M_1P_2 + P_3M_3P_2 + P_1M_2P_4 + P_3M_4P_4 \\ \hline P_2M_1P_1 + P_4M_3P_1 + P_2M_1P_3 + P_4M_3P_3 & P_2M_1P_2 + P_4M_3P_2 + P_2M_2P_4 + P_4M_4P_4 \end{array}\right]$$

For a truncated system, $P_2 = P_4 = 0$

$[P]^T [M] [P]$

$$= \left[\begin{array}{c|c} P_1M_1P_1 + P_3M_3P_1 + P_1M_2P_3 + P_3M_4P_3 & 0 \\ \hline 0 & 0 \end{array}\right]$$

For a truncated system,

$$[P] = \left[\begin{array}{c} P_1 \\ \hline P_3 \end{array}\right]$$

$[P]$ is of order n x m

P_1 is of order m x m

P_3 is of order (n - m) x m

$[M]$ is of order n x n

$$[M] = \left[\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array}\right]$$

M_1 is of order m x m

M_2 is of order m x (n - m)

M_3 is of order (n - m) x m

M_4 is of order (n - m) x (n - m)

$[P]^T [M] [P]$

$$= [P_1 \mid P_3] \left[\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array}\right] \left[\begin{array}{c} P_1 \\ \hline P_3 \end{array}\right]$$

$$= [P_1M_1 + P_3M_3 \mid P_1M_2 + P_3M_4] \left[\begin{array}{c} P_1 \\ \hline P_3 \end{array}\right]$$

$$= [P_1M_1P_1 + P_3M_3P_1 + P_1M_2P_3 + P_3M_4P_3] \quad m \times m$$

which is the same result as before.

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SYSTEM MODELING AND MODIFICATION

VIA MODAL ANALYSIS

by

Yiu Wah Luk

(ABSTRACT)

A new method is developed for experimentally determining the system parameters of a structure that is suitable for implementation in microprocessor-based systems. It uses single degree-of-freedom models to describe a multi-degree-of-freedom system. The system is assumed to be describable by a linear, proportionally and lightly damped, lumped parameter model. Two types of damping models, viscous and structural damping, are provided.

The effective mass, stiffness, and damping are obtained by fitting the experimental data in the inverse Nyquist plane. The effective mass, stiffness, and damping are convertible to global modal mass, stiffness, and damping through normal mode corrections. Then a physical space mathematical model may be assembled from the modal properties for complete and truncated modal vector system descriptions. Therefore, this method will deal with the general case where the number of degree-of-freedom exceeds the number of identified modes.

After a mathematical model is developed, different ways of modifying the structure analytically are investigated. This modified

model is used to predict the new dynamic characteristics of the modified structure due to changes in its mass, stiffness, or damping properties. There are three ways that modifications can be made. They are: 1) modifications made in the physical coordinates model; 2) modifications made in both the physical and modal coordinates models; and 3) modifications made in the modal coordinates model. The last way is found to be the most efficient way; therefore, model modifications should be done totally in modal spaces, modal space I and II. The derivation of mass, stiffness, and damping modification matrices for general structure is also presented.

The resonance specification and frequency response function synthesis are two useful techniques that aid in system modification and are, therefore, included. A resonant peak can be shifted to another frequency by making certain modifications to the structure, thus avoiding undesired vibration. The resonance specification will determine the amount of physical change needed. It is not practical to store all the frequency response function measurements of a structure during testing. Therefore, a frequency response function synthesis is needed, such that any one can be synthesized from the model developed.

A theoretical three degree-of-freedom system and two experimental systems--a square plate and a C-clamp--were used to verify the techniques developed.