Numerical Approximation and Identification Problems for Singular Neutral Equations

by

Graciela M. Cerezo

Thesis submitted to the faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematics

APPROVED:

Dr. Terry Herdman, Committee Chairman

Dr. Max Gunzburger

Dr. John Burns

Dr. Janet Peterson

Dr. Eugene Cliff

April 25, 1994
Blacksburg, Virginia

Numerical Approximation and Identification Problems for Singular Neutral Equations

by

Graciela M. Cerezo

Commitee Chairman: Dr. Terry L. Herdman

Mathematics

(ABSTRACT)

A collocation technique in non-polynomial spline space is presented to approximate solutions of singular neutral functional differential equations (SNFDEs). Using solution representations and general well-posedness results for SNFDEs convergence of the method is shown for a large class of initial data including the case of discontinuous initial function. Using this technique, an identification problem is solved for a particular SNFDE. The technique is also applied to other different examples. Even for the special case in which the initial data is a discontinuous function the identification problem is successfully solved.

Acknowledgments

I am especially grateful to Dr. John A. Burns and Dr. Terry L. Herdman for encouraging me to come to this country to get a higher degree of education. I am also very grateful to them for their constant and sincere advice without which I would have mever been able to get this degree.

I am also very grateful to my parents who support me in the difficult desicion of moving to a foreign country and to my friends here who stand behind me at every difficult situation.

I want to thank Dr. E. Cliff for carefully reading this thesis and pointing out many usefull corrections.

Finally, I want to thank Dr. Gunzburger, Dr. Peterson, Dr. Cliff and Dr. Burns for serving on my committee.

Contents

List of Figures

List of Tables

Chapter 1

Projection Methods: A short summary of results

1.1 Introduction

1.1.1 Classification of Volterra Integral Equations

An Integral Equation is a functional equation in which the unknown function appears under one or several integral signs. An integral equation, for the unknown function y, of the form

$$y(t) = g(t) + \int_0^t K(t, s, y(s))ds, \quad t \in I = [a, b] \tag{1.1.1}$$

is called a Nonlinear Volterra Integral Equation of the Second Kind. Note that $g(t)$ and $K(t, s, y(s))$, the kernel of the integral equation, are given real valued functions. The function $g(t)$ is known as the initial function.

If the unknown function occurs only under the integral sign, we will have a Nonlinear Volterra Integral Equation of the first kind, ie.

$$\int_0^t k(t, s, y(s))ds = g(t) \quad t \in I. \tag{1.1.2}$$

A Volterra integral equation is said to be linear if its kernel has the following form

$$k(t, s, y) = k(t, s)y \tag{1.1.3}$$

When the kernel function is the product of a smooth function and a weakly singular one,

$$k(t, s, y(s)) = (t - s)^{-\alpha}\gamma(t, s, y(s)), \quad 0 < \alpha < 1 \tag{1.1.4}$$

with γ smooth, the equation is called an integral equation of Abel type.

1.2 Existence and Uniqueness of Solutions

1.2.1 Volterra Linear Integral Equations of the Second Kind

Consider equation (1.1.1) with a linear kernel K and a real valued continuous initial function g.

Definition 1.2.1 *Let $K_1(t, s) = K(t, s)$ and set*

$$K_n(t, s) = \int_s^t K_1(t, \psi)K_{n-1}(\psi, s)d\psi, \quad n \geq 1 \tag{1.2.1}$$

the functions K_n $n \geq 1$ are called the iterated kernels associated with the given kernel K in (1.1.1).

The introduction of the iterated kernels of K is motivated by the Picard method for constructing successive approximations $\{y_n(t), \quad n \geq 1\}$ to the exact solution of (1.1.1) by means of

$$(1.2.2) \qquad y_n(t) = g(t) + \int_0^t K(t,s)y_{n-1}(s)ds, \quad y_0(t) = g(t), \quad t \in I$$

the iterate y_n can be expressed in terms of the given function g and the iterated kernels $K_1 \cdots K_n$, namely

$$(1.2.3) \qquad y_n(t) = g(t) + \int_0^t \sum_{m=1}^{n} K_m(t,s)g(s)ds, \quad n \geq 1.$$

For a given kernel that satisfies $\|K(t,s)\| \leq M$ for all $(t,s) \in S = IxI$ it follows that for all $n \geq 1$

$$(1.2.4) \qquad \|K_n(t,s)\| \leq M^n \frac{(t-s)^{n-1}}{(n-1)!} \leq M^n \frac{T^{n-1}}{(n-1!)}.$$

Thus the series

$$(1.2.5) \qquad \sum_{n=1}^{\infty} K_n(t,s) = \lim_{n \to \infty} \sum_{m=1}^{n} K_m(t,s)$$

converges absolutely and uniformly on S to a continuous function $R(t,s) = \sum_{m=1}^{\infty} K_m(t,s)$ called the resolvent kernel of the given kernel K. The function y given by

$$(1.2.6) \qquad y(t) = g(t) + \int_0^t R(t,s)g(s)ds, \quad t \in I$$

is a continuous solution of (1.1.1) on I.

Theorem 1.2.2 *Let the functions g and K characterizing the integral equation* (1.1.1) *be continuous on I and S respectively. Then this equation has a unique solution $y \in C(I)$ given by*

$$(1.2.7) \qquad y(t) = g(t) + \int_0^t R(t,s)g(s)ds, \quad t \in I$$

where $R \in C(S)$ is the resolvent kernel associated with the given kernel K . The resolvent kernel satisfies the identity
$R(t,s) = K(t,s) + \int_s^t K(t,\psi)R(\psi,s)d\psi$ for all $(t,s) \in S$.

For a proof of this theorem see [4].

1.2.2 Volterra Linear Integral Equations of the First Kind

The existence of a unique solution $y \in C(I)$ of the first kind integral equation (1.1.1) is assured under certain conditions on K and g.

Theorem 1.2.3 *Let K and g satisfy*

a) $g \in C^1(I)$ with $g(0) = 0$, this assures the continuity of the solution at $t = 0$;

b) $K \in C(S)$, $\frac{\partial K}{\partial t} \in C(S)$;

c) $K(t,t) \neq 0$ for all $t \in I$.

Under these conditions, the first kind equation (1.1.1) *has a unique solution $y \in C(I)$.*

Under these conditions the first kind equation (1.1.1) can be transform into an integral equation of the second kind for which Theorem 1.2.2 can be applied. For a proof of this theorem see [4].

1.3 Projection Methods

Projection Methods include such techniques as collocation, Galerkin's method and Least Squares methods. We will describe briefly this methods and analyze some interesting convergence results.

Let X be a Banach space and consider the operator $D : X \longrightarrow X$ defined by

$$Du = \int_a^b K(t,s)u(s)ds \tag{1.3.1}$$

for $K(t,s)$ continuous or with certain types of integrable singularities, D introduces a compact operator on X, see [1] [10]. Equation (1.1.1) can be written in operator form as $y = g + Dy$.

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of finite-dimensional subspaces of X such that $\bigcup_{n=1}^{\infty} X_n$ is dense in X. Let Y_n be another sequence of finite dimensional subspaces of X and let $P_n : X \rightarrow Y_n$ define a sequence of projection operators. Then to solve the original equation, we intend to approximate y by a sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in X_n$.

Let $R(y_n) = y_n - Dy_n - g$ denote the residual. If $y_n = y$, then $R(y_n) = 0$. As this is in general not the case, one tries to select y_n so that $R(y_n)$ is small. In projections methods this is accomplished by requiring the projection of $R(y_n)$ onto Y_n to be equal to zero. That is, picking y_n such that $P_n R(y_n) = 0$ or equivalently $P_n y_n - P_n D y_n = P_n g$.

In the most common implementation of this technique we have $X_n = Y_n, n \geq 1$, so that $P_n y_n = y_n$ and y_n solves the equation of the second kind,

$$y_n - P_n D y_n = P_n g. \tag{1.3.2}$$

To do numerical calculations we take $\{\varphi_k\}_{k=1}^n$ as a basis for X_n and write

$$y_n = \sum_{k=1}^{n} a_k \varphi_k \tag{1.3.3}$$

the coefficients $\{a_k\}_{k=1}^n$ will be obtained by solving the projected equation

$$\sum_{k=1}^{n} l_j(\varphi_k) a_k - \sum_{k=1}^{n} l_j(P_n K \varphi_k) a_k = l_j(g_n) \quad j = 1, \cdots, n. \tag{1.3.4}$$

where $\{l_j\}_{j=1}^n$ are polynomials such that $l_j(\varphi_k) = 1$, for $j = k$ and vanish otherwise.

Table 1.2: Polynomial Collocation

Space X	Basis for X_n	Projected Equation
$C[a,b], \|\cdot\|_\infty$	$\{l_k\}_{k=1}^n$,Lagrange Polynomials	$y_n(t) - \int_a^b K_n(t,s)y_n(s)ds = g_n(t)$

Table 1.3: Galerkin's Method

Space X	Basis for X_n	Projected Equation
$L_2[a,b], \|\cdot\|_2$	$\{\psi_k\}_{k=1}^n$, orthonormal Polynomials	$y_n(t) - \int_a^b K_n(t,s)y_n(s)ds = g_n(t)$

In the following tables it is shown how the three most standard projection methods are selected.

Letting $T = I - D$ the following table holds for Least Squares method.

Table 1.4: Least Square Method

Space X	Basis for X_n	Projected Equation
$L_2[a,b], \|\cdot\|_2$	$\{T\psi_k\}_{k=1}^n$, orthonormal Polynomials	$\sum_{j=1}^n a_j \int_a^b T^*T\psi_j\psi_k dt = \int_a^b T^* g\psi_k$

So the approximate solution for an integral equation will be the solution y_n to the corresponding project equation.

1.4 Convergence Analysis

1.4.1 General Convergence Result for Projection Methods

Let us write (1.2.2) as

$$y_n - D_n y_n = g_n \tag{1.4.1}$$

where

$$D_n = P_n D \tag{1.4.2}$$
$$g_n = P_n g \tag{1.4.3}$$

Then the following convergence result holds:

Theorem 1.4.1 *Let the operator D and the projection P_n be such that $||D - D_n|| \to 0$ and $P_n g \to g$. Then for all $n \geq n_0$ $(I - D_n)^{-1} \in X$, $||(I - D_n)^{-1}||$ is uniformly bounded, $||u - u_n|| \to 0$, and the error estimate*

$$||y - y_n|| \leq ||(I - D_n)^{-1}|| \, ||y - P_n y|| \tag{1.4.4}$$

holds.

For the proof of this theorem see [10], [1].

It is interesting to note that convergence results could be drastically improved by changing adequately the projections spaces and the partition of the interval used. We will show this kind of behavoir of Projection Methods for the particular case of Collocation Methods, similar analysis could be done for the other methods mention above.

1.4.2 Analysis of a particular case: The Collocation Method

The most common formulation of Polynomial Collocation is to take the projection space as $X = (C[a,b], ||.||_\infty)$ where $C[a,b]$ is the space of continuous functions on $[a,b]$ and $||.||$ is the supremum norm. X_n is the subspace of polynomials of degree $n-1$. In this case we have $Y_n = X_n$, and P_n will be the operator which maps a function y in X onto the polynomial which interpolates it on the set of points $\{t_k\}_{k=1}^{n}$,

$$P_n(y(t)) = \sum_{k=1}^{n} y(t_k) l_k(t) \tag{1.4.5}$$

where $\{l_k(t)\}_{k=1}^n$ are the fundamental polynomials of Lagrange interpolation . Then applying the projection P_n to D we will have

$$(P_n Dy)(t) = \sum_{k=1}^{n} l_k(t) \int_a^b K(t_k, s)y(s)ds. \tag{1.4.6}$$

Let $p_k(t)_{k=1}^n$ be a basis for X_n, then the projection equation (1.4.1) becomes

$$y_n(t) - \sum_{k=1}^{n} l_k(t) \int_a^b K(t_k, s)y_n(s)ds = \sum_{k=1}^{n} l_k(t)g(t_k). \tag{1.4.7}$$

Evaluating both sides of (1.4.7) at $t = t_j \;\; j = 1, \cdots, n$ and using the fact that $l_k(t_j) = \delta_{kj}$, we obtain

$$y_n(t_j) - \int_a^b K(t_j, s)y_n(s)ds = g(t_j), \quad j = 1, \cdots, N. \tag{1.4.8}$$

Writting

$$y_n(t) = \sum_{k=1}^{n} a_k p_k(t) \tag{1.4.9}$$

and substituing into (1.4.8) we obtain the usual collocation equations, (see [1])

$$\sum_{k=1}^{n} a_k p_k(t_j) - \sum_{k=1}^{n} a_k \int_a^b K(t_j, s)p_k(s) = g(t_j), \quad j = 1, \cdots, N. \tag{1.4.10}$$

For numerically solving the projected equation, a partition of the interval is need. For simplicity, we will work from now on for the interval $I = [0, 1]$.

Let $h_j = t_{j+1} - t_j$ and $h = max h_i, \quad i \geq 1$ and assume that $Nh_j \leq Nh \leq T$ (quasi uniform partition). Let the finite subset $X(N)$ of $[0, 1]$ be given by $X(N) = \bigcup_{j=0}^{N-1} X_j$ with

$$X_j = \{t_{j,n} = t_j + c_n h_j \;\; 0 \leq c_1 \leq c_2 \leq \cdots \leq c_n \leq 1\}. \tag{1.4.11}$$

The set $X(N)$ is called the set of collocation points and the points $\{c_n\} \;\; n = 1, \cdots, N$ are called the collocation parameters, the l_j is the $j - th$ Lagrange fundamental polynomial for the collocation parameters $\{c_j\}$,

$$l_j(s) = \prod_{r=1, r\neq j}^{m} \frac{(s - c_r)}{(c_j - c_r)}. \tag{1.4.12}$$

Definition 1.4.2 *Let m and d denote integers satisfying $-1 \le d \le m-1$. Then $S_m^d(z_N) = \{u : u|_{(t_n,t_{n+1})} = u_n\}$; $u_n^j(t_n) = u_{n-1}^j(t_n)$ for $j = 0,\cdots,d$ and $t_n \in Z_N$ is the linear space of (real) piecewise polynomials (or polynamials splines) of degree m which are in the continuity class $C^d([0,1])$ and which have the nodes Z_N.*

Note that: If $d = -1$, then an element $u \in S_m^{-1}(Z_N)$ (with $m \ge 0$) has jump discontinuities on the set Z_N. The step functions with breakpoints Z_N correspond to $S_0^{-1}(Z_N)$.

The dimension of $S_m^d(Z_N)$ is given by

$$dim(S_m^d(Z_N)) = N(m-d) + (d+1) \quad (-1 \le d \le m-1) \tag{1.4.13}$$

For more details about the properties of this spaces see [6], [11].

Theorem 1.4.3 *Let g and K be continuous functions. If the collocation parameters$\{c_n\}$ are such that $a = c_1 \le c_2 \le \cdots \le c_m = 1$, then the collocation approximation u defined above is an element of $S_{m-1}^0(Z_N)$.*

For a proof of this theorem see [4].

1.4.3 Improvement of Convergence

Importance of the choice of the collocation parameters, Global Convergence Results

We will analyze convergence results assuming that the integrals needed for the approximation scheme are evaluated analitically, we will not take into account the effect of the error due to numerical integration in the accurancy of the polynomial collocation.

Let $e = y - y_n$ and denote its restriction to the subinterval $(t_j, t_{j+1}]$ by e^j. Note that this error function will in general have finite jumps on Z_N.We will use the notation $\|e\|_\infty = sup\{|e^j(t)|,\ t \in (t_j, t_{j+1}],\ j = 0,\cdots,N-1\}$ with $h = max(t_{j+1} - t_j),\ \ j \ge 1$, the maximun width of the given mesh.

For integral equations of the first kind the following global convergence result holds.

Theorem 1.4.4 *Let g and K in (1.1.2) possess continuous derivatives of order $m+1$ on their respectives domains, and suppose that $g(0) = 0$, $\|K(t,t)\| \ge \varepsilon > 0$ for all $t \in [0,1]$. Then the collocation equation (1.4.10), defines for each quasi-uniform mesh sequence with sufficiently small $h = maxh_n$, $n \ge 1$ a unique approximation $y \in S_{m-1}^{-1}(Z_N)$. This collocation approximation yields an error with $\|e\|_\infty = O(h^m)$, as $h \to 0$ and $Nh \le \gamma T$ for all collocation parameters c_j with $0 < c_1, \cdots < c_m = 1$. If $0 < c_1 < \cdots < c_m < 1$ then the same result holds for all $m \ge 1$ if and only if*

$$\prod_{j=1}^{m} \frac{(1-c_j)}{c_j} < 1 \tag{1.4.14}$$

For the proof of this theorem see [4].

For integral equations of the second kind we have the following global convergence result:

Theorem 1.4.5 *Consider the equation* (1.1.1) *with* $g \in C^m(I)$, $K \in C^m(S)$. *Then there exists* $h^* > 0$ *such that the collocation equation defines for each* $h \in (0, h^*)$ *a unique element* $y \in S_{m-1}^{-1}(Z_N)$. *The error induced by this approximation to the exact solution* y *of* (1.1.1) *satisfies for every choice of the collocation parameters* $\{c_j\}$ *with* $0 \leq c_1 < \cdots < c_m \leq 1$ *and for all quasi -uniform mesh sequences* $\|e\|_\infty \leq Ch^m$. *Where* C *denotes some finite constant independent of* h, *but depending on the* $\{c_j\}$.

For the proof of this theorem see [4].

It is interesting to note that for particular choices of the collocation parameters and the projection spaces, the highest order of convergence can be locally achieved. This is analysed in the following sections.

Local superconvergence results for the collocation method: Analysis of the superconvergence for integral equations of the first kind

Theorem 1.4.6 *Let* $g \in C^{m+2}$ *and* $K \in C^2$, *and let* $y \in S_m^{-1}(Z_N)$ *is uniquely determined by collocation on* $X^*(N) = \bigcup_{k=0}^{N-1} X_k^*$ *where* $X_k^* = \{t_{k,j} = x_k + c_j h : 0 < c_0 < c_1 < \cdots, c_m \leq 1\}$ *and* c_j *are the zeros of* $P_{m+2}(s) - P_m(s) \in (0,1]$, *where* $P_i(s)$ *are the Legendre Polynimials for* $[0,1]$ *of degree* i, *with the normalization* $P_i(1) = 1$. *The collocation parameters mention here are known as the Lobatto points which lie in the left open interval* $(0,1]$, *with* $c_m^* = 1$.
Then

$$\|e_k(x)\| \leq ch^{m+2}, \quad forall x \in Q_k^*, k = 0, 1, \cdots, N-1 \tag{1.4.15}$$

Where $Q_k = \{q_{k,j} = x_k + u_j^* h : 0 \leq u_0 < u_1 < \cdots < u_m \leq 1\}$, *and* u_j^*, $j = 0, 1, \cdots, m$ *are the zeros of* $P_{m+1}(s)$ *and as above* $P_i(s)$ *denots the Legendre polynomial for* $[0,1]$ *of degree* i *such that* $P_i(1) = 1$

A proof of this result can be found in [7].

In this paper it is also shown that the order of superconvergence $p = m + 2$ is the best posible, for the corresponding sets $X(N)$ and $Q(N)$, it cannot be replaced by $m + 3$.

Analysis of the superconvergence for integral equations of the second kind

As for the case of the equation of the first kind, for second kind equations we can (locally) improve the convergence by choosing the best collocation parameters. The following local superconvergence theorem on $\{Z_N\}$ holds

Theorem 1.4.7 *Let $u \in S_{m-1}^{-1}(Z_N)$ denote the collocation approximation to the solution of the integral equation* (1.2.2) *and assume that g and K satisfy $g \in C^{2m-v}(I)$, , $K \in C^{2m-v}(S)$ for some $v \in \{0,1,2\}$ and with $m \geq [v/2]+1$.*

a) If the collocation parameters $\{c_j\}$ are the zeros of $P_{m-1}(2s-1) - P_m(2s-1)$, ie. the Radou II points for $(0,1]$, then for $v = 1$ we have,

$$\max_{t_n \in Z_N} \|e(t_n)\| = O(h^{2m-1}), \quad ash \to 0Nh \leq T \tag{1.4.16}$$

b) If the collocation parameters $\{c_j\}$ are the zeros of $s(1-s)P'_{m-1}(2s-1)$, ie. the Lobatto points for $[0,1]$, and if $v = 2$,then it follows that

$$\max_{t_n \in Z_N} \|e(t_n)\| = O(h^{2m-2}), \quad ash \to 0Nh \leq T \tag{1.4.17}$$

c) If the collocation parameters $\{c_j\}$ are the zeros of $P_m(2s-1)$, ie. the Gauss points for $(0,1)$, and if $v = 0$ then we obtain

$$\max_{t_n \in Z_N} \|e(t_n)\| = O(h^{m}), \quad ash \to 0Nh \leq T \tag{1.4.18}$$

then the collocation at the Gauss abscissas does not lead to local superconvergence on Z_N.

d) If the first $m-1$ of the collocation parameters $\{c_j\}$ are the zeros of $P_{m-1}(2s-1)$ and if $c_m = 1$ then with $v = 2$ the same result that in b holds, ie. we have the same order of local superconvergence as for Lobatto points.

For a proof of this theorem see [4], also see [6].

Note that all these superconvergence results hold for the case in which the kernel K depend linearly on the function $y(t)$, for the more general case of nonlinear integral equations these results do not hold.

Importance of the choice of the projection spaces

Making use of the special structure of each problem we can get higher order convergence results. In [5] it can be seen that more imformation about the form of the solution of a particular problem allows us to build up special non-polynomial spline projection spaces for which the order of convergence of the method is drastically increased. It is interesting to note that in this case the convergence results does not depend on the choice of the collocation parameters.

The imformation about the form of the solution plays an essential role in the choosing of the adequated projection spaces. This special choice of the projection spaces turns out to be the key for numerical approximations that leads us to succesfully solve identification problems for the parameter of IVP with weak singularities.

For the particular IVP we studied, we found very usefull to apply this technique. An introduction to this problem can be found in Chapter II, a complete analysis of the numerical scheem is given in Chapter III and finally in Chapter IV the identification problem is solved.

Chapter 2

The Aerodynamic Problem and Its Semigroup Setting

2.1 Introduction

During the last ten years the study of feasibility and advantages of active control surfaces to reduce maneuver, gust and fatigue loads and dampen vibration that contributes to flutter had been an important field of research. Some of the most important problems studied in this field are related to well-posed mathematical models of aeroelastic systems. To carefully develop a procedure for control design one needs a realistic mathematical model that predicts the dynamic behavior of the physical system, in [12], a complete dynamical model was formulated in terms of functional differential equation of neutral type for the elastic motions of a three-degree-of-freedom typical airfoil section, with flap, in a two dimensional, incompressible flow. For the sake of completness, we'll briefly present the model here. In this work we've studied a simpler case suggested by the singularity presented in the general model.

2.2 The Aerodynamic Model

As can be seen in [12] a mathematical model that gives as an input- output relation between the motion of an airfoil and the resulting forces on the airfoil is given by

$$(2.2.1)\quad \frac{d}{dt}[Ax(t) + \int_{-\infty}^{0} A(s)x(t+s)ds] = Bx(t) + \int_{-\infty}^{0} B(s)x(t+s)ds + Gu(t)$$

where the matrix $A \in R^{8x8}$

$$A = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$$

has $A_{12} = 0$, $A_{21} = 0$, A_{11} is nonsingular

$$A_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

and the matrix $B \in R^{8x8}$

$$B = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix}$$

has

$$B_{22} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

and $B_{11} = B_{12} = B_{21} = 0$

The state vector $X(t)$ is given by $x(t) = col(h(t), \theta(t), \beta(t), \dot{h}(t), \dot{\theta}(t), \dot{\beta}(t), \gamma_q(t), \dot{\gamma_q}(t))$ with $h(t)$ the plunge coordinate, $\theta(t)$ the pitch coordinate, $\beta(t)$ the flap angle and $\gamma(t)$ the extended total airfoil circulation.

Note that although the matrix A_{11} is nonsingular, the matrix A is singular (since the eighth column is identically zero). The matrix functions $A(s)$ and $B(s)$ have the first seventh columns all zeros and $B_{8i}(s) = 0$ except for $i = 5, 6$. The matrix $A(s)$ has $A_{8i}(s) = 0$ except for $i = 4, 5, 6$ and the function $A_{88}(s) = \sqrt{(\frac{Us-2}{s})}$. Notice that $A_{88}(s)$ is weakly singular and that $\int_{-\infty}^{0} A(s)x(t+s)ds$ is impossible to integrated by parts since $\dot{A}(s)$ is not L_1 on neighborhoods of zero.

For each $x : (-\infty, +\infty) \to R^8$ let $x_t(s) = x(t+s)$. For $\psi : (-\infty, 0] \to R^8$ satisfying $\psi \in L_1 \cap L_\infty$ the followings operators are defined

(2.2.2) $$D\varphi = A\varphi(0) + \int_{-\infty}^{0} A(s)\varphi(s)ds$$

and

(2.2.3) $$L\varphi = B\varphi(0) + \int_{-\infty}^{0} B(s)\varphi(s)ds$$

so that the above equation can be written in the form

(2.2.4) $$\frac{d}{dt}Dx_t = Lx_t + Gu(t).$$

Note that the D operator is nonatomic at $s = 0$, since the matrix A is singular. However this particular problem has a very interesting structure since the singularity in the matrix A is at precisely the rigth position to be compensated by the weak singularity of the kernel function $A_{88}(s)$ at $s = 0$. This is, to have any imformation on what's going on with $\dot{x}_8(t)$ we must solve an integral equation with a weakly singular kernel at $t = 0$.

In this work we isolated the eighth coordinate and studyed a much more simpler problem in which the weak singularity of the kernel at $t = 0$ is still present.

2.2.1 Statement of the Problem

The problem we investigate is the following:

$$\frac{d}{dt}Dx_t = Lx_t \tag{2.2.5}$$

with

$$x_0(s) = \varphi(s), \qquad \varphi(s) \in C[-1,0] \tag{2.2.6}$$

where $x_t(s) = x(t+s) \qquad s \in [-1,0] \quad t \geq 0.$

The operators D and L are given by

$$D\varphi(s) = \int_{-1}^{0} \varphi(\sigma)(-\sigma)^{-\alpha} d\sigma, \qquad 0 < \alpha < 1 \tag{2.2.7}$$

and

$$\frac{d}{dt}\int_{-1}^{0} x(t+s)(-s)^{-\alpha} = Lx_t(s). \tag{2.2.8}$$

2.3 Well-Posedness of Singular Neutral Equations in a Semigroup Setting

Concerning the well-posedness of (2.2.5)-(2.2.6) we have the following result.

Theorem 2.3.1 *(i) For each $\varphi \in C$ the initial value problem $Dx_t = D\varphi, \quad t > 0, \quad x_o = \varphi$ has a unique continuous solution $x(\cdot, \varphi)$ on $[0, \infty)$. The family of operators $S(t)\varphi = x_t(\cdot, \varphi)$ $t \geq 0$ defines a C_o semigroup on C.*

(ii) If $p < 1/(1-\alpha)$ then for each $(\eta, \varphi) \in R \times L_p$ the initial value problem $Dx_t = \eta, \quad t > 0, \quad x_o = \varphi$ has a unique solution $x(\cdot, \varphi)$ defined a.e. on $[0, \infty)$. Moreover, the family of operators $S(t)(\eta, \varphi) = (Dx_t, x_t), \quad t \geq 0$ defines a C_o semigroup on $R \times L_p$.

(iii) If $p \geq 1/(1-\alpha)$ then (2.3.1) has a unique solution for (η, φ) in a dense subset of $R \times L_p$. However, the family of operators $S(t)$ defined as above fails to define a C_o semigroup on $R \times L_p$.

The operator D defined by (2.2.5) is bounded on $C[-1,0]$ for all $\alpha \in (0,1)$, and is bounded on $L_p[-1,0]$ if $p > 1/(1-\alpha)$. However the operator D is unbounded and densely defined on $L_p[-1,0]$ if $p \leq 1/(1-\alpha)$. A proof of this existence and uniqueness results for the initial value problem can be found in [13].

For the numerical approximation (Chapter III), we will exploit the form of the solutions of equation (2.2.5). The result needed is given in the following Theorem.

Theorem 2.3.2 *Let* $\varphi \in C[-1,0]$. *Then the IVP for* (2.2.5) *has the unique integrable solution*

$$\begin{aligned} x(t) &= \frac{\sin \alpha\pi}{\pi} \int_{-1}^{0} \frac{1}{t-s} \left| \frac{t}{s} \right|^{\alpha} \varphi(s)\, ds \\ &\quad + \frac{\sin \alpha\pi}{\pi} \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{(t-s)+1} \varphi(s-1)\, ds, \quad 0 < t \leq s. \end{aligned} \tag{2.3.9}$$

For a proof of this theorem see [13]. Note that the solution $x(t)$ in Theorem 2.3.2 can be

written in the form

$$x(t) = t^{\alpha} v_0(t) + v_1(t) \tag{2.3.10}$$

where $v_0 \in C^{\infty}[0,1]$ and $v_1 \in C^{m}[0,1]$. So we can approximate both $v_0(t)$ and $v_1(t)$ using Taylor's expantion arround any $t_0 \in [0,1]$. This idea will be used in the next Chapter to choose the best projection spaces in each interval of the numerical approximation.

Chapter 3

A Numerical Solution For The Initial Value Problem

3.1 Introduction

We now direct our attention to the numerical approximation of solutions to

$$\frac{dDx_t}{dt} = Lx_t, \quad 0 < t < T. \tag{3.1.1}$$

where $L = 0$, with initial data

$$x_0(s) = \varphi(s), \quad s \in [-1, 0] \tag{3.1.2}$$

where $x_t(s) = x(t+s)$ for $s \in [-1, 0]$, $t \geq 0$ and the linear operator D has the following representation for $\varphi \in C[-1, 0]$

$$D\varphi = \int_{-1}^{0} \varphi(s)(-s)^{-\alpha} ds, \quad 0 < \alpha < 1. \tag{3.1.3}$$

We use the non-polynomial spline collocation technique which will be described in this Chapter, also see [14]. Note that equation (3.1.1) is a neutral functional differential equation, which is singular due to the nonatomicity of the difference operator, D, in (3.1.3). By reformulating (3.1.1)-(3.1.2) as a well-posed Abel-Volterra equation of first kind on the state spaces $L_{2,g}[-1, 0]$ (g is a weight function) and $C[-1, 0]$, we can apply a collocation technique recursively on approximation spaces generated by functions (non-polynomials) whose form is suggested by the weak singularity appearing in the equation, see [5], [14]. By using the well-posedness of (3.1.1)-(3.1.2) on $L_{2,g}$ convergence of the collocation technique can be shown even in the case of discontinuous initial data.

3.2 Approximation Scheme

Collocation techniques can be formulated as projection methods in spaces of continuous functions. Here we use collocation in a recursive way on approximation spaces generated by non-polynomial basis functions. (Note that the same idea is applied by Brunner in [5] to approximate Volterra equations of the second kind with weakly singular kernels.)

Using the density of $C(-1, 0]$ in $L_{2,g}$ estimate solutions to IVP (3.1.1)-(3.1.2) can be approximated in two steps:

i) Approximate $L_{2,g}$ initial function by continuous initial functions,
and
ii) Use collocation technique on $C[-1, 0]$ to approximate solutions to IVP (3.1.1)-(3.1.2) in the continuous data case.

In the space $X = C[0, 1]$, with $\|x\| = \max_{t \in [0,1]} |x(t)|$, $t \in [0, 1]$ we consider the finite dimensional subspaces $X^N = \text{span}\{e_0^N, e_1^N, \ldots, e_N^N\}$, with projections $P^N : X \rightarrow$

X^N, $P^N x(t) = \sum_{j=0}^{N} x(\tau_j) e_j^N(t)$, $x \in C[0,1]$; where $0 \le \tau_0 \le \tau_1 \le \ldots \le \tau_N \le 1$, are the collocation points. We define the family of operators $\bar{D}^N$, $N \ge 1$ by $\bar{D}^N : X^N \to X^N$, $\bar{D}^N = P^N \bar{D}$ and the family of functions f^N, $N \ge 1$ as $f^N : X \to X^N$, $f^N = P^N f$. The sequence of approximating functions $x^N(t) = \sum_{j=0}^{N} \gamma_j e_j^N(t)$ is the solution of the system $\bar{D}^N x^N = f^N$ or equivalently, $\sum_{j=0}^{N} \gamma_j \bar{D}^N e_j^N(\tau_j) = f(\tau_i)$, $i = 1, \ldots, N$. This is an algebraic system in the unknowns $\gamma_1, \gamma_2, \ldots, \gamma_N$. In matrix notation we have: $Q^N = A^N F^N$, where $Q_{ik}^N = \bar{D} e_i^N(\tau_k)$, $A^N = (\gamma_1, \gamma_2, \ldots, \gamma_N)^T$, and $F^N = (f^N(\tau_1), \ldots, f^N(\tau_N))^T$.

Integrating (3.1.1) and splitting the integral in the operator D we get

$$\int_{-1}^{-t} x(t+s)(-s)^{-\alpha} ds + \int_{-t}^{0} x(t+s)(-s)^{-\alpha} ds = c$$

or equivalently,

$$\int_{-t}^{0} x(t+s)(-s)^{-\alpha} ds = c - \int_{-1}^{-t} \varphi(t+s)(-s)^{-\alpha} ds.$$

Setting

$$\overline{D}x(t) = \int_0^t x(t-s) s^{-\alpha} ds \tag{3.2.1}$$

and

$$f(t) = c - \int_{-1}^{-t} \varphi(t+s)(-s)^{-\alpha} ds \tag{3.2.2}$$

the integrated form of IVP (3.1.1)-(3.1.2) can be written as

$$\overline{D}x(t) = f(t). \tag{3.2.3}$$

Note that the left hand side of this last equation contains the unknown function x and in the right hand side we have already known "past" data. Accordingly, collocation is used in a recursive way. Let N be an integer, $h = 1/N$ and partition the interval $[0, T]$, $T > 0$ by $t_n = nh$. Denote by σ_0 the interval $[0, t_1]$, and by σ_n the interval $(t_n, t_{n+1}]$, $n = 1, ..., N-1$. The recursion is motivated by the fact that for $t \in \sigma_n$, that is $t = t_n + s$, $s \in [0, h)$, so equation (3.2.3) can be written as:

$$\int_0^s x(t-u) u^{-\alpha}\, du = f(t) - \int_s^{t_n+s} x(t-u) u^{-\alpha}\, du, \qquad t \in \sigma_n. \tag{3.2.4}$$

Note that in the integral on the left hand side the argument of the function x lies in the interval σ_n, while in the right hand side its argument belongs to the interval of "past values" $[t_0, t_{n-1})$. This will set the recursion.

Our scheme consists on using collocation in each interval σ_n on equation (3.2.3). The basis functions used to build the projection spaces are suggested by the form of the solution given by the following lemma.

Lemma 3.2.1 *Consider the problem* (3.1.1)-(3.1.2) *with* $\varphi \in C^m[-1,0]$ *and assume* $\varphi(0) = \varphi'(0) = \ldots = \varphi^{(m-1)}(0) = 0$. *Then the unique solution of the IPV has the form*

$$x(t) = t^\alpha v_0(t) + v_1(t) \tag{3.2.5}$$

where v_0 *belongs to* $C^m[0,1]$.

Using Taylor's expansion for the functions v_o and v_1 we may write

$$x(t) = t^\alpha \sum_{k=0}^{m-1} A_k t^k + \sum_{k=0}^{m-1} B_k t^k + t^\alpha R_o(t) + R_1(t), \quad t \in [0,h]. \tag{3.2.6}$$

Let

$$x_o^N(t) = \sum_{k=0}^{m-1} \Xi_k^o t^{k+\alpha} + \sum_{k=0}^{m-1} \Phi_k^o t^k, \quad t \in [0,h] \tag{3.2.7}$$

be the approximate solution of (3.2.3) obtained using collocation on σ_0.

Denote by X_n the space $C[t_n, t_{n+1}]$ with $\|x\| = \max_{t \in \sigma_n} |x(t)|$. Let m be as in the lemma, then the above expantion suggest the following projection spaces X_n^N, $n = 0, 1, \ldots, N-1$ as $X_0^N = \text{span}\{1, s, \ldots s^{m-1}, s^\alpha, s^{\alpha+1}, \ldots, s^{\alpha+m-1}\}$, and, $X_n^N = \text{span}\{1, s, \ldots, s^{m-1}, (1+s/n)^\alpha, (1+s/n)^\alpha s, \ldots, (1+s/n)^\alpha s^{m-1}\}$, $n = 1, \ldots, N-1$.

The space X_n^N is a subspace of X_n. Denote the projection from X_n into X_n^N by P_n^N. The sequence of approximating functions $\{x_n^N\}_{N \geq 1}$, each of them defined on the interval σ_n, is the solution of the system

$$(P_n^N \overline{D} x_n^N)(t) = (P_n^N f)(t), \quad t \in \sigma_n \tag{3.2.8}$$

or equivalently

$$P_n^N \int_0^{t_{n+1}-t} x_n^N(t-s) s^{-\alpha} ds =$$
$$P_n^N \left(f(t) - \int_{t_{n+1}-t}^t x_n^N(t-s) s^{-\alpha} ds \right), \quad t \in \sigma_n.$$

Let $t = t_n + \tau_{n_i}^N$, where τ_{n_i} $i = 0, 1, \ldots, m-1$ are the collocation points on the interval σ_n. Equation (3.2.3) yields the set of $2m$ equations

$$\int_0^{\tau_{n_i}^N} x_n^N(t_n+\tau_{n_i}^N-s) s^{-\alpha} ds = f(t_n+\tau_{n_i}^N) - \int_{\tau_{n_i}^N}^{t_n+\tau_{n_i}^N} x_n^N(t_n+\tau_{n_i}^N-s) s^{-\alpha} ds, \quad n = 0, 1, \ldots, 2m-1.$$
(3.2.9)

Now define the function $x^N(t)$, $t \in [0,T]$ as $x^N(t) = x_n^N(t)$, $t \in \sigma_n$, with $\|x\| = \max_{t \in [0,T]} |x(t)|$ for $x \in X = C[0,1]$.

The following theorem establishes the convergence of the scheme.

Theorem 3.2.2 *Let $x(t)$ be the solution of* (3.2.3) *with initial condition* $\varphi \in C^m[-1,0]$, $\varphi(0) = \varphi'(0) = \ldots = \varphi^{(m-1)}(0) = 0$. *Then*

$$\|x^N - x\| \leq Kh^{m+1} \tag{3.2.10}$$

with K a constant which depends on the collocation points and on the initial function.

For a proof of this Theorem see Chapter V.

3.3 Numerical Examples

All computations were carried out on a IBM 3090. The numerical integrations were performed using a Gauss Quadrature Method with 5 nodes. An ordinary Gauss Method solves the system of $(k_0 + k_1 + 2)$ equations with $(k_0 + k_1 + 2)$ unknowns.

Example 3.3.1 Smooth Case We consider the problem with initial function $\varphi(s) = -s$. The solution $x(t,\alpha)$ for $\alpha = 0.5$ is $x(t, 0.5) = -t + \frac{4}{\pi}t^{1/2}$

Figure 1 shows the approximated solutions for $N = 4$, $N = 10$ and $N = 50$. For this choice of the basis we obtained a relative error smaller than 0.006. Since $\varphi \in C^\infty[-1,0]$ we could have used spaces X_n^N of higher dimension, to obtain a smaller error.

Example 3.3.2 Nonsmooth Case Consider formula (3.2.3), where the RHS is given by $f(t) = \frac{3}{8}\pi t^2$. The true solution, for the case $\alpha = 1/2$ is $x(t) = t^{3/2}$.

Figure 2 shows approximates solutions for this problem for $N = 4$ and $N = 10$.

Example 3.3.3 Discontinuous Case

We consider the scalar equation

$$\int_{-1}^{0} (-s)^{-\alpha} x(t+s)\, ds = 1, \qquad t > 0 \tag{3.3.1}$$

with initial data

$$\varphi(s) = \begin{cases} 0 & \text{if } s \in (-1,0) \\ 1/2 & \text{if } s = 0. \end{cases} \tag{3.3.2}$$

This example is a generalization of Example 4.1 in [2].

The solution of equation (3.3.1)-(3.3.2) is given by

$$x(t,\alpha) = \frac{1}{\pi} t^{\alpha-1} \sin \alpha\pi, \quad t \in [0,1). \tag{3.3.3}$$

Note that in this case the initial data φ, is not a continuous function therefore our lemma about the form of the solution does not apply so the scheme described in Section 3 can not be applied directly. To estimate the solution we approximate, in a suitable way, the initial function φ by a sequence of differentiable functions.

In particular, for $\varepsilon \in (-1, 0)$ consider the family of functions φ_ε given by

$$\varphi_\varepsilon(s) = \begin{cases} 0 & \text{if } s < \varepsilon \\ a(\varepsilon)s^2 + b(\varepsilon)s + 1/2 & \text{if } s \in (\varepsilon, 0) \\ 1/2 & \text{otherwise.} \end{cases}$$

Let $\{\varepsilon_n\}$ be a sequence with $\varepsilon_n \to 0$. The sequence φ_{ε_n} is not a Cauchy sequence in $(C^1[-1,0], \|\cdot\|_\infty)$, but converges to φ in $L_{2,g}$, therefore from Theorem 3.2.2, $\lim_{n\to\infty}(x_n)_t = x_t$ in $L_{2,g}$.

Figure 3 shows the approximations to the solution of (3.3.1)-(3.3.2) for $t \in [0, 3]$ and $\alpha = 0.5$. The approximations are plotted for $\varepsilon = 10^{-3}$ and values $N = 4$, $N = 10$ and $N = 50$.

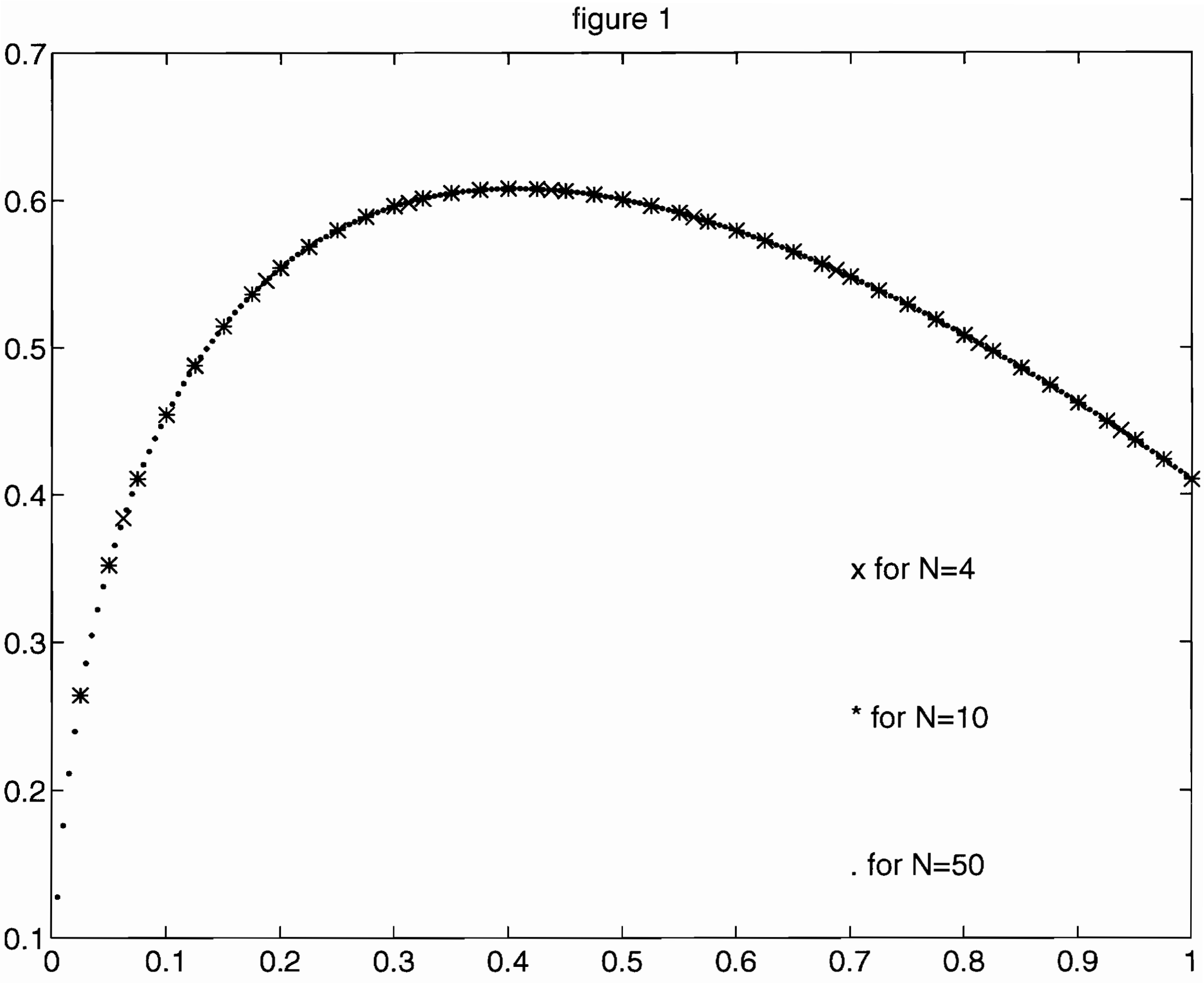

Figure 3.1: Real and Numeric Solution for a Smooth Case

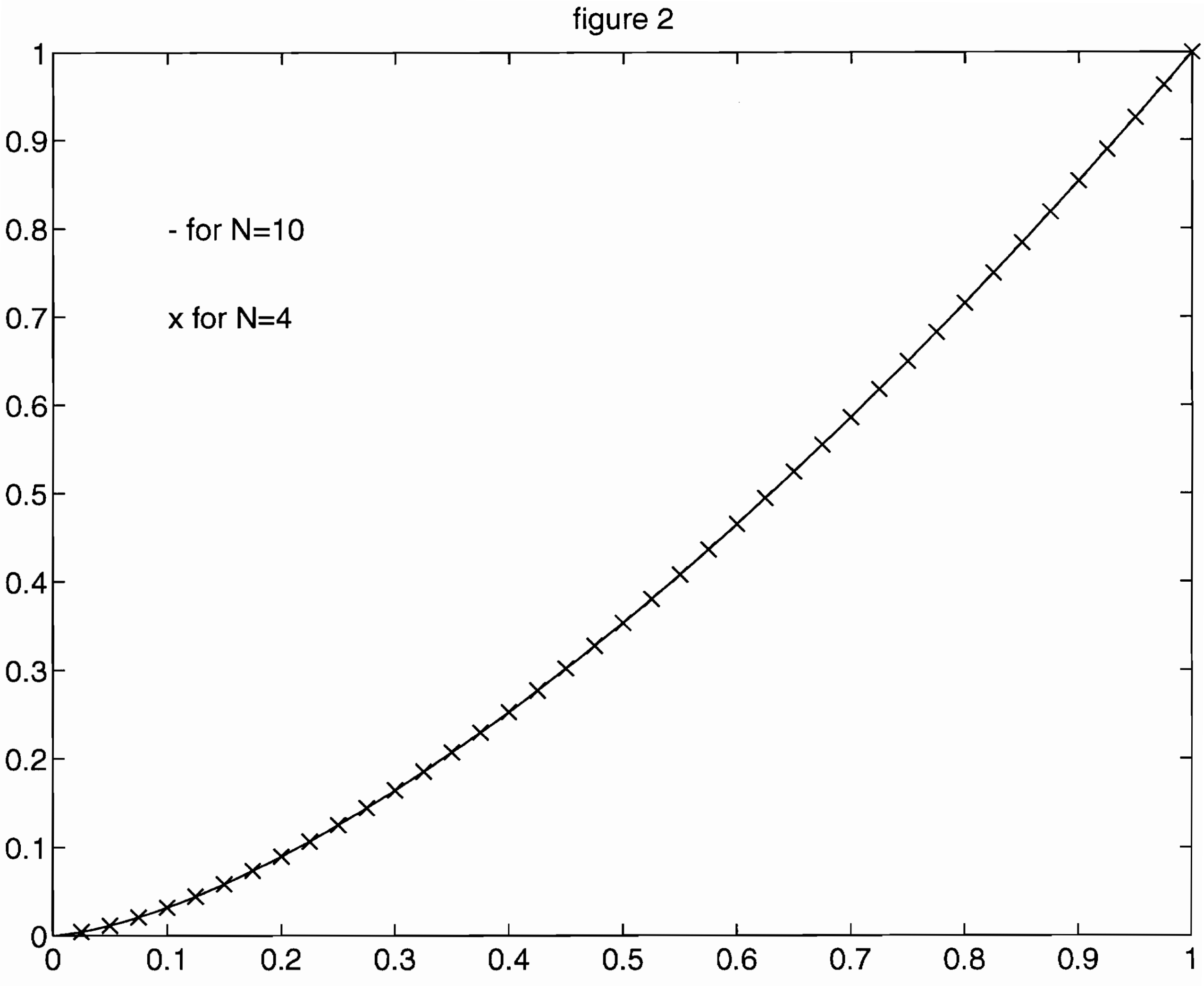

Figure 3.2: Real and Numeric Solution for a Non-Smooth Case

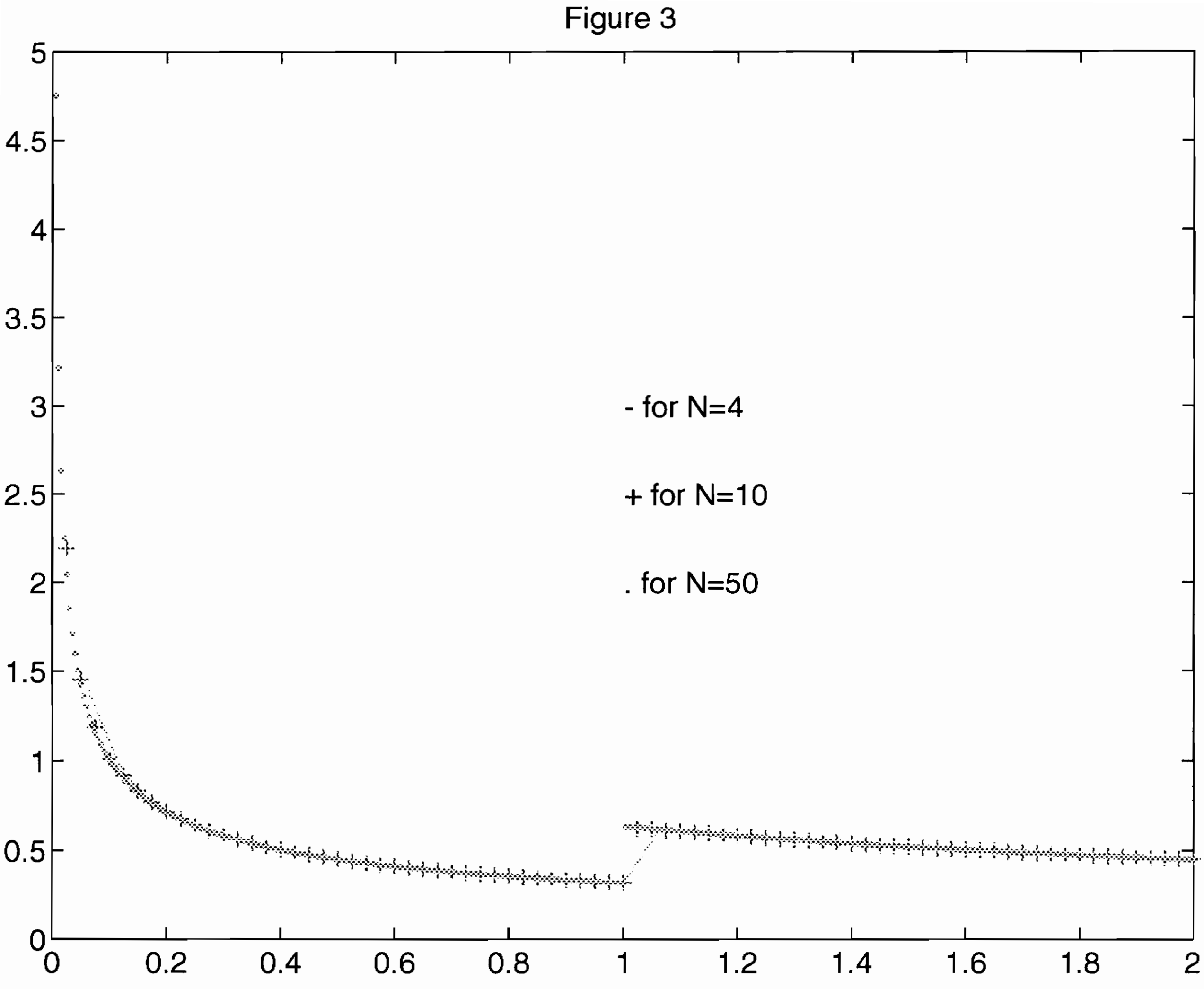

Figure 3.3: Real and Numeric Solution for a Discontinuous Case

Chapter 4

Solving The Identification Problem

4.1 Introduction

As it was discuss in Chapter II, a realistic mathematical model is need for the aerodynamic problem. For a model to be reliable it is important that all the parameters involved are correct. The identification of parameters involved in the L operator, the right hand side of the IVP we are studying , is studied by Herdman et all in [3]. In this Chapter we present the identification problem for the parameter in the D operator of our IVP. It is important to note that a differentiability result for this case is given.

4.2 The Identification Problem

Consider the problem:

Given the measurements $y_i, i = 1, \cdots, n$ of $x(t)$, solution of (3.1.1)-(3.1.2), at discrete points $t_i, i = 1, \cdots, n$ we want to identify the parameter α. Thus, we consider the quadratic cost function

(4.2.1)
$$J(\alpha) = \sum_{i=1}^{n} \|x(t_i, \alpha) - y_i\|^2$$

and the minimization problem

(4.2.2)
$$\min_{\alpha \in (0,1)} J(\alpha).$$

Associated with problem (4.2.2) we have the approximated problems,

(4.2.3)
$$\min_{\alpha \in (0,1)} J^N(\alpha)$$

where

(4.2.4)
$$J^N(\alpha) = \sum_{i=1}^{n} \|x^N(t_i, \alpha) - y_i\|^2$$

and $x^N(t, \alpha)$ are the functions obtained by the scheme in Chapter III.

In order to study the convergence of the identification procedure we establish the following differentiability results.

Lemma 4.2.1 *For every $t \in (0,1), x(t,\alpha)$ given by Lemma 1 is differenciable with respect to α and the function $x_\alpha(t,\alpha)$ is continuous for all $t \in (0,1), \alpha \in (0,1)$.*

For a proof of this Lemma see Chapter V.

4.3 Numerical Examples

In the two examples below we identify the parameter values $\alpha = 0.5$ and $\alpha = 0.75$ using the approximation scheme discused previously. In figures I and II we plotted some of the curves for the α's furnished by the minimization subroutines as they converge to $\alpha = 0.75$ and $\alpha = 0.5$, respectively. The optimal values obtained are $\alpha_{opt} = 0.7499$ and $\alpha_{opt} = 0.4999$.

4.3.1 The Special Case of Discontinuous Initial Data

Below we show the numerical results of identifying the parametrer α on the IVP for (2.2.5). Note that this is example 3 from Chapter III.

We considered measurements on $[0,1]$ for the solutions with $\alpha = 0.5$. In figure III we show the curves furnished by the minimization subroutine. The initial guess is 0.4. The optimun value obtained is $\alpha_{opt} = 0.518$.

In figure IV we identify $\alpha = 0.5$ but the measurements x_i are taken for t_i in the interval $[1,2]$. The initial guess is 0.4. The optimum value obtained is $\alpha_{opt} = 0.512$.

Figure V, shows the results for the identification problem for $\alpha = 0.5$, for $t \in (0,2)$.

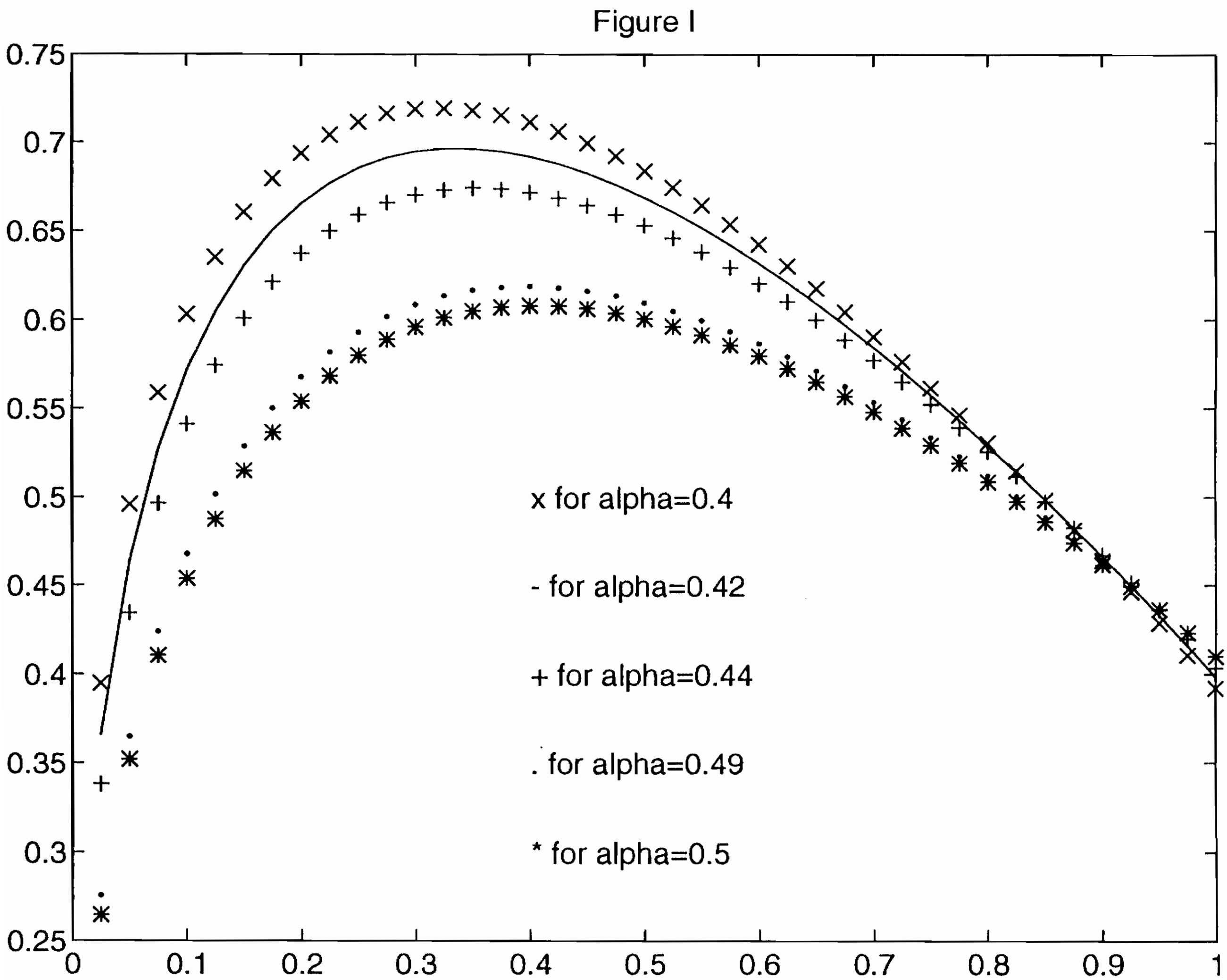

Figure 4.1: Identification for $\alpha = 0.5$, Smooth Case

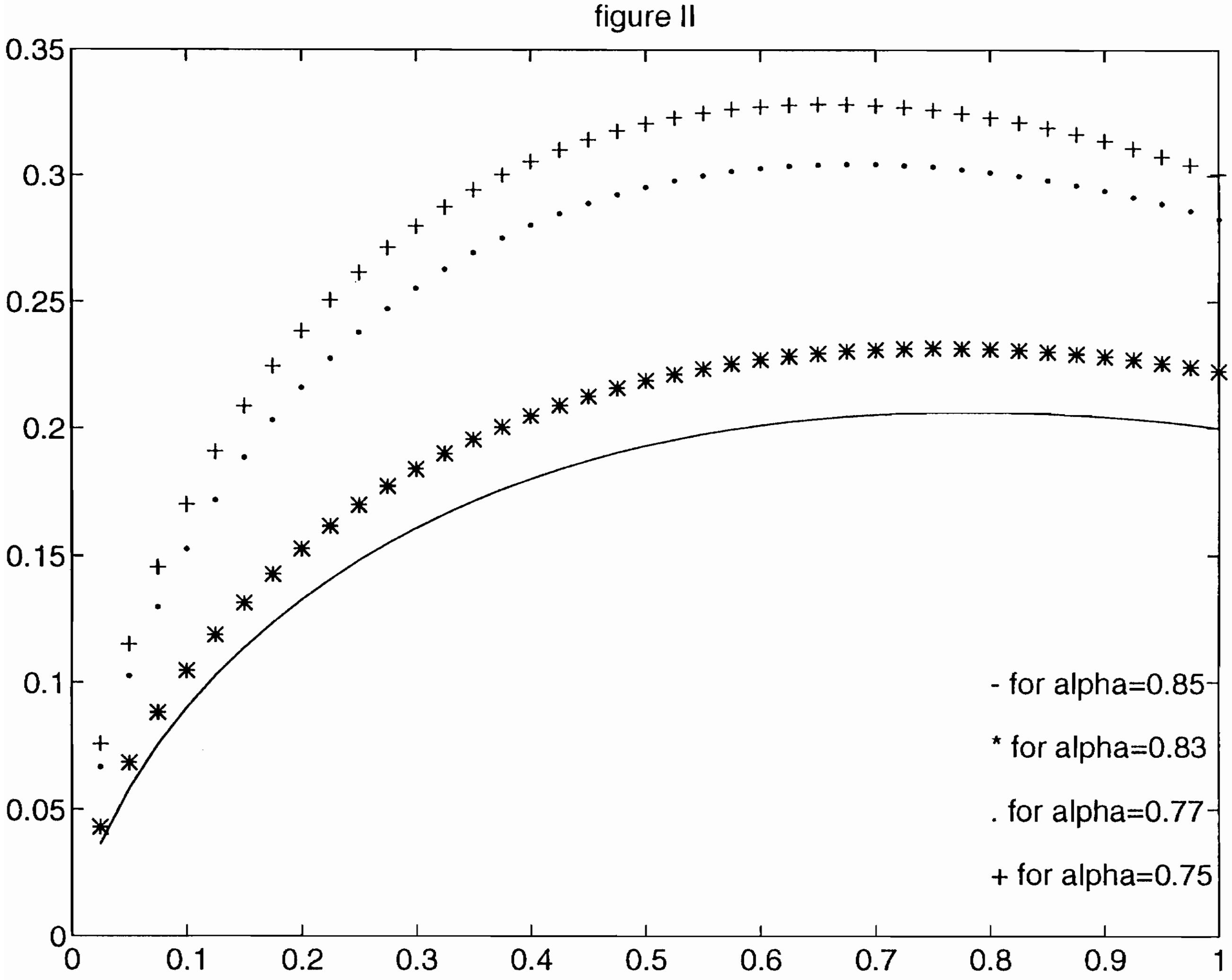

Figure 4.2: Identification for $\alpha = 0.75$, Smooth Case

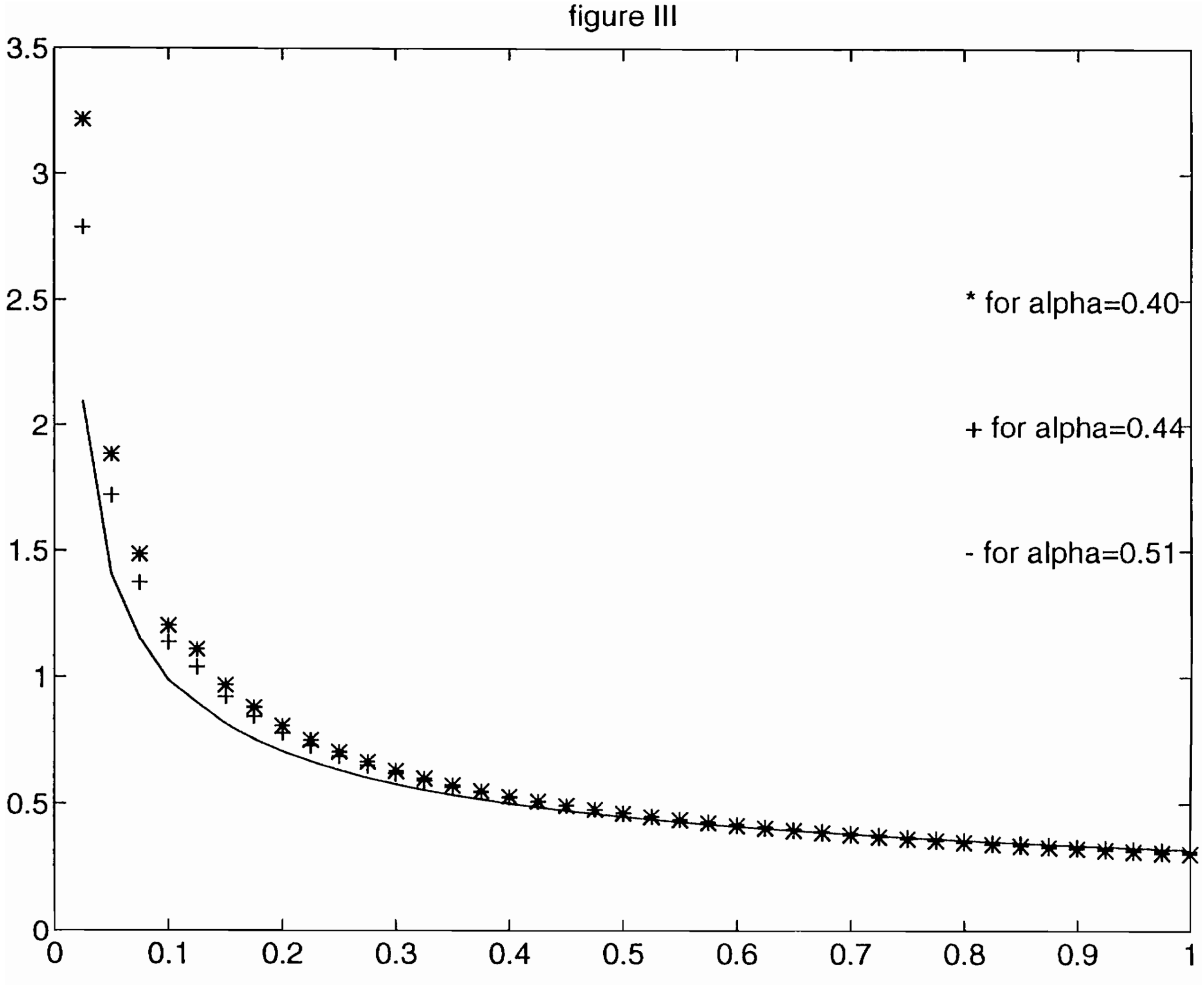

Figure 4.3: Identification of $\alpha = 0.5$, Discontinuouns Case, $t \in [0, 1]$

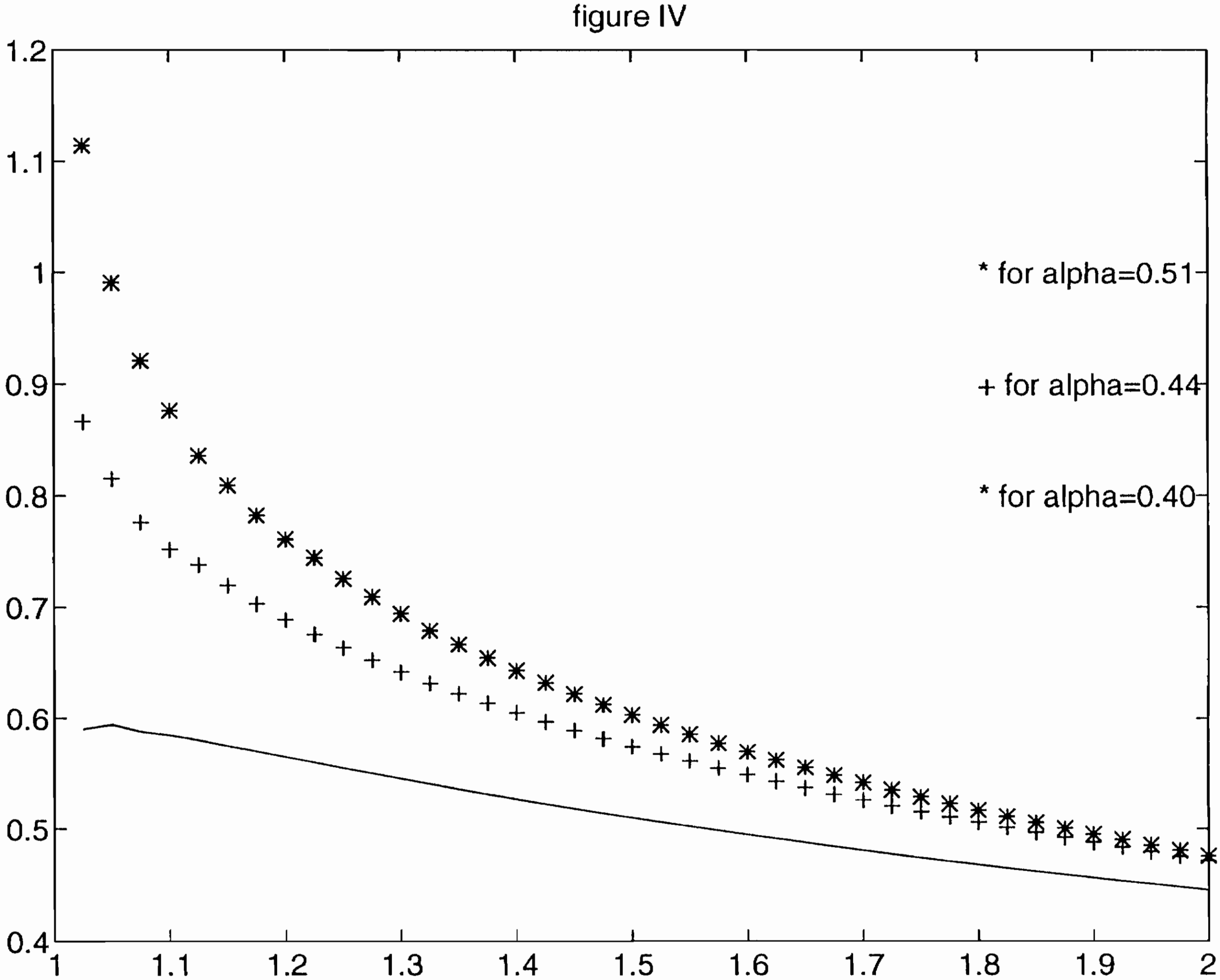

Figure 4.4: Identification for $\alpha = 0.5$, Discontinuous Case, $t \in [1, 2]$

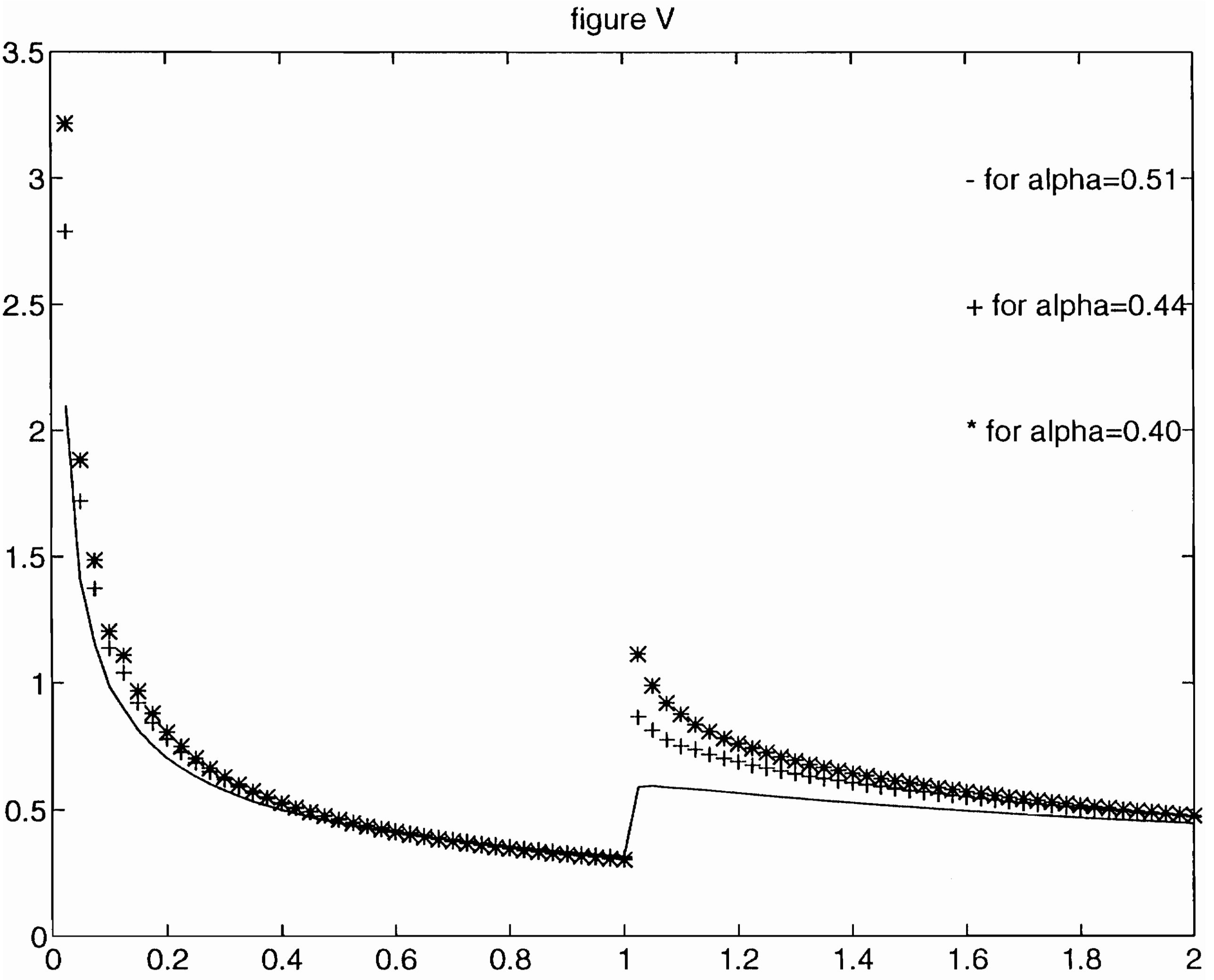

Figure 4.5: Identification for $\alpha = 0.5$, Discontinuous Case, $t \in [0, 2]$

Chapter 5

Conclusions and Proofs

5.1 Conclusions

The technique studied is a very effective tool to built approximations for the solutions to our problem. Since it is both accurate and easy to compute, it allows us to succesfully solve the identification problem for the parameter that appears in the D operator. The accurancy of this tecnique, however, is not much better than others that can be found in the literature, see [2], but the greatest advantages of it seens to be that it doesn't requires a small partition of the interval to get a "good approximation" of the solution. By "good approximation" we mean an approximation that is accurate enough to succesfully run the identification routine. Recall that the identification scheme builds an approximation to the solution at each of its steps, so it needs an approximation method for which the error is small with the less computational work possible. The results shown for the identification problem with $t \in [1,2]$ gives a nice example of the possibilities of this technique.

Using all the information we have about the form of the solution gives us the possibility to build this technique to be very accurate, (since we are choosing the projection spaces). In solving the identification problem this is a good advantage, however, if we only want to get a numerical approximation for the solution of an IVP, using so much imformation about the analytic solution might be a problem. In solving the identification problem, this not only gives us a great advantage over the usual approximation techniques (including the classical collocation that is descrived at the beginig of this work), but also allows us to prove the differentiability of the solution with respect to the parameter, which is a nessesary condition for the identification scheme to work.

5.2 Proofs

5.2.1 Proofs of results in Chapter III

Proof for Theorem 3.2.2: From Lemma (3.2.1) we have that the function x can be

expressed in the form:

$$x(t) = t^{\alpha} v_o(t) + v_1(t) \tag{5.2.1}$$

where $v_o, v_1 \in C^m[0,1]$.

Using Taylor's expansion for the functions v_o and v_1 we may write

$$x(t) = t^{\alpha} \sum_{k=0}^{m-1} A_k t^k + \sum_{k=0}^{m-1} B_k t^k + t^{\alpha} R_o(t) + R_1(t), \quad t \in [0,h]. \tag{5.2.2}$$

Let

$$x_o^N(t) = \sum_{k=0}^{m-1} \Xi_k^o t^{k+\alpha} + \sum_{k=0}^{m-1} \Phi_k^o t^k, \quad t \in [0,h] \tag{5.2.3}$$

be the approximate solution of (3.2.3) obtained using collocation on σ_0. Recall that we can write (1.2.7) in terms of the basis of X_o^N $x_o^N(t) = \sum_{k=0}^{m-1} \tilde{\Xi}_k^o s^{k+\alpha} + \sum_{k=0}^{m-1} \tilde{\Phi}_k^o s^k, \quad s \in [0,1]$ where

$$\begin{aligned} \tilde{\Xi}_k^o &= \Xi_k^o h^{-(k+\alpha)} \\ \tilde{\Phi}_k^o &= \Phi_k^o h^{-(k+\alpha)}. \end{aligned}$$

Let $\zeta_{n_i}^N = c_i h, \quad c_i \in [0,1], i = 0,1,\ldots,2m-1$ be the collocation points on σ_0. Then for each $\tau_{n_i}^N$ we have

$$\overline{D}x^N(\zeta_{n_i}^N) = f(\tau_{n_i}^N), \qquad i = 1,\ldots,2m \tag{5.2.4}$$

Also we have that

$$\overline{D}x(\tau_{n_i}^N) = f(\tau_{n_i}^N), \qquad i = 1,\ldots,2m \tag{5.2.5}$$

Formulas (5.2.4) and (5.2.5) yield

$$\overline{D}(x^N - x)(\tau_{n_i}^N) = 0, \qquad i = 1,\ldots,2m \tag{5.2.6}$$

Simple manipulations and the form of $\overline{D}$ yield

$$\begin{aligned} &\sum_{k=0}^{m-1}(\Xi_k^o - A_k)\int_0^{\tau_{n_i}^N}(\tau_{n_i}^N - s)^{k+\alpha}s^{-\alpha}\,ds + \sum_{k=0}^{m-1}(\Phi_k^o - B_k)\int_0^{\tau_{n_i}^N}(\tau_{n_i}^N - s)^k s^{-\alpha}\,ds \\ &\quad - \int_0^{\tau_{n_i}^N} R_o(\tau_{n_i}^N - s)s^{-\alpha}\,ds - \int_0^{\tau_{n_i}^N} R_1(\tau_{n_i}^N - s)s^{-\alpha}\,ds \\ &\quad = \sum_{k=0}^{m-1}(\Xi_k^o - A_k){\tau_{n_i}^N}^{k+1} + \sum_{k=0}^{m-1}(\Phi_k^o - B_k){\tau_{n_i}^N}^{k-\alpha+1} \\ &\quad - \int_0^{\tau_{n_i}^N} R_o(\tau_{n_i}^N - s)s^{-\alpha}\,ds - \int_0^{\tau_{n_i}^N} R_1(\tau_{n_i}^N - s)s^{-\alpha}\,ds = 0, \quad i = 0,1,\ldots,2m-1 \end{aligned}$$

This yields the following system of algebraic equations on $(\Xi_k^o - A_k)$ and $(\Phi_k^o - B_k)$

$$\begin{aligned} &(\Xi_o^o - A_o)\tau_{n_i}^N + (\Xi_1^o - A_1){\tau_{n_i}^N}^2 + \ldots + (\Xi_{m-1}^o - A_{m-1}){\tau_{n_i}^N}^m \\ &\quad + (\Phi_o^o - B_o)\tau_{n_i}^N + (\Phi_1^o - B_1){\tau_{n_i}^N}^2 + \ldots + (\Phi_{m-1}^o - B_{m-1}){\tau_{n_i}^N}^m \\ &= \frac{v_o^{(m+1)}(\zeta){\tau_{n_i}^N}^{m+1}}{(m+1)!} + \frac{v_1^{(m+1)}(\xi){\tau_{n_i}^N}^{m+1-\alpha}}{(m+1)!}, \qquad i = 0,1,\ldots,2m-1 \end{aligned}$$

Using Cramer's rule we can write the solution as $\Xi_k^o - A_k = \Delta_{A_k}/\Delta, \;\; \Phi_k^o - B_k = \Delta_{B_k}/\Delta$ $k = 0,1,\ldots,m-1$ where Δ, Δ_{A_k} and Δ_{B_k} are

$$\begin{aligned} \Delta &= M(c_1,\ldots,c_{2m})h^{(m+1)m}, \\ \Delta_{A_k} &= M_k(c_1,\ldots,c_{2m})h^{m+1-k+(m+1)m} \end{aligned}$$

and

$$\Delta_{B_k} = \tilde{M}_k(c_1, \ldots, c_{2m})h^{m+1-k+(m+1)m} \qquad k = 0, 1, \ldots, m-1.$$

In the equation above M, M_k and $\tilde{M}_k$ are independent of h. Simple calculations using the form of the coefficients of $x^N - x$ given by the formulas obtained above yield $\max\{|x(t) - x^N(t)|, t \in \sigma_o\} \leq K_o h^{m+1}$. For the interval σ_n, $n > 0$ the proof follows the same lines as the one for σ_o.

If $t \in \sigma_n$ it has the form $t = t_n + \zeta$, with $t_n = nh$. Writing $\zeta = sh$ for $s \in (0, 1]$, and $t = t_n(1 + \frac{\zeta}{nh}) = t_n(1 + \frac{s}{n})$ Using Taylor's expansion on $v_o(t)$ and $v_1(t)$ about t_n for each subinterval σ_n we have $v_o(t) = t_n^\alpha(1 + \frac{s}{n})^\alpha \sum_{k=0}^m \Xi_k^n s^k$, and $v_1(t) = \sum_{k=0}^m \Phi_k^n s^k$ In this way we get a representations analogous to (3.2.6) for the intervals σ_n. This proves the theorem.

5.2.2 Proofs of results in Chapter IV

Proof of Lemma 4.2.1:

Using Lemma 3.2.1 we can compute $\frac{dx}{d\alpha}$ in the following way:

$$\frac{dx}{d\alpha} = t^\alpha ln(t) v_0(t) + \frac{dv_0}{d\alpha} t^\alpha + \frac{dv_1}{d\alpha}$$

where

$$\begin{aligned} \frac{dv_0}{d\alpha} &= \int_{-1}^0 t^\alpha ln(t)\varphi(s)/((t-s)\|s\|^\alpha)ds - \int_{-1}^0 \|t\|^\alpha ln(\|s\|)\varphi(s)/((t-s)\|s\|^\alpha)ds \\ &= I - \int_{-1}^0 \|t\|^\alpha ln(\|s\|)\varphi(s)/((t-s)\|s\|^\alpha)ds \end{aligned}$$

Note that the second term is similar to $v_0(t)$ but for a problem with $ln(s)\varphi(s)$ as initial condition instead of only $\varphi(s)$.

$$\begin{aligned} \|I\| &= \|t^\alpha ln(t) \int_{-1}^0 \varphi(s)/((t-s)\|s\|^\alpha)ds \\ &= \|ln(t) \int_0^{1/t} \varphi(-t\sigma)t/((t+t\sigma)t^\alpha\sigma^\alpha)d\sigma\| \\ &= \|ln(t) \int_0^{1/t} \varphi(-t\sigma)/((1+\sigma)\sigma^\alpha)d\sigma\|. \end{aligned}$$

If $t, \sigma < 1$ then

$$\|ln(t)\| \leq \|ln(t)\| + \|ln(\sigma)\| = \|ln(t) + ln(\sigma)\| = \|ln(t\sigma)\|$$

$$\begin{aligned}
||I|| &\leq ||ln(t)|| || \int_0^1 \varphi(-t\sigma)/((1+\sigma)\sigma^\alpha)d\sigma|| + || \int_1^{1/t} \varphi(-\sigma t)/((1+\sigma)\sigma^\alpha)d\sigma|| \\
&\leq || \int_0^1 ln(\sigma t)\varphi(-t\sigma)/((1+\sigma)\sigma^\alpha)d\sigma|| + ||ln(t)|| || \int_1^{1/t} \varphi(-\sigma t)/((1+\sigma)\sigma^\alpha)d\sigma|| \\
&= J_1 + J_2
\end{aligned}$$

Note that since $ln(\sigma t)\varphi(\sigma t)$ is continuous at 0 we can write

$$||J_1|| \leq ||ln(\sigma t)\varphi(\sigma t)||_\infty \beta(1,\alpha)$$

where $\beta(.,.)$ is the usual Beta function, and $ln(t) = ln(\sigma t) - ln(\sigma)$, then

$$\begin{aligned}
J_2 &= || \int_1^{1/t} ln(\sigma t)\varphi(-\sigma t)/((1+\sigma)\sigma^\alpha)d\sigma - \int_1^{1/t} ln(\sigma)\varphi(-\sigma t)/((1+\sigma)\sigma^\alpha)d\sigma|| \\
&\leq J_3 + J_4
\end{aligned}$$

where

$$||J_4|| \leq ||\varphi||_\infty \int_1^{1/t} ln(\sigma)/((1+\sigma)\sigma^\alpha)d\sigma.$$

We have $\frac{1}{1+\sigma} \leq \frac{1}{\sigma}$, thus

$$\begin{aligned}
||J_4|| &\leq ||\varphi||_\infty \int_1^{1/t} ln(\sigma)\sigma^{-\alpha-1}d\sigma = ||\varphi||_\infty (ln(\sigma)\sigma^{-\alpha}/(-\alpha)|_1^{1/t} + \frac{1}{\alpha}\int_1^{1/t} \sigma^{-\alpha-1}d\sigma) \\
&= ||\varphi||_\infty (ln(t)t^\alpha/\alpha - \frac{1}{\alpha^2}[(\frac{1}{t})^{-\alpha} - 1]) = ||\varphi||_\infty (ln(t)t^\alpha/\alpha - \frac{1}{\alpha^2}(t^\alpha - 1)).
\end{aligned}$$

For $\frac{dv_1}{d\alpha}$ we have,

$$\frac{dv_1}{d\alpha} = \int_0^t ln(t-s)(t-s)^{\alpha-1}\varphi(s-1)/((t-s+1)d\sigma$$

and changing variables $s = t\sigma$ we have

$$\begin{aligned}
\frac{dv_1}{d\alpha} &= \int_0^1 ln(t-t\sigma)(t-t\sigma)^{\alpha-1}\varphi(t\sigma-1)t/(t-t\sigma+1)d\sigma \\
&= t^\alpha \int_0^1 ln(t(1-\sigma))(1-\sigma)^{\alpha-1}\varphi(t\sigma-1)/(t(1-\sigma)+1)d\sigma \\
&= t^\alpha(\int_0^1 ln((1-\sigma)(1-\sigma)^{\alpha-1}\varphi(t\sigma-1)/(t(1-\sigma)+1)d\sigma \\
&\quad + ln(t)\int_0^1 (1-\sigma)^{\alpha-1}\varphi(t\sigma-1)/(t(1-\sigma)+1)d\sigma) \\
&= t^\alpha I_1 + t^\alpha ln(t) I_2.
\end{aligned}$$

If $\varphi \in L^\infty$ we have

$$\|J_1\| \leq \|\varphi\|_\infty \int_0^1 ln(1-\sigma)(1-\sigma)/(t(1-\sigma)+1)d\sigma.$$

Recall that

$$\lim_{t\to 0}(t^\alpha ln(t)) = \lim_{t\to 0}(ln(t)/t^{-\alpha}) = \lim_{t\to 0}(t^{-1}/(t^{-\alpha-1})) = \lim_{t\to 0}(\frac{1}{t^{-\alpha}}) = 0$$

so the lemma is proved.

Bibliography

[1] K. E. Atkinson, *A Survey of numerical methods for the solution of Fredholm Integral Equations of the second kind, Society for Industrial and Applied Mathematics, Philadelphia, 1976*

[2] K. Ito and J. Turi, *Numerical Methods for a Class of Singular Integro-Differential Equations Based on Semigroup Approximation, Society for Industrial and Applied Mathematics, Vol 28, No 6,pp.1698-1722, Dec. 1991*

[3] D. W. Brewer and T. L. Herdman, *A parameter dependence problem in neutral functional differential equations, Pitman Resaerch Notes in Mathematics Series, Vol 273*

[4] H. Brunner. *The numerical solution of Volterra Equations, North-Holland, 1986*

[5] H. Brunner, *Nonpolynomial Spline Collocation for Volterra Equations with Weakly singular Kernels, Siam J. Num Anal, Vol 20, No 6, Dec 1983*

[6] H. Brunner, *On the discretization of Volterra integral equations, Nieuw Archief Voor Wisskunde (4), Vol. 2, 1984*

[7] H. Brunner, *Superconvergence of Collocation Methods for Volterra Integral Equations of the First Kind, Computing 21, 151-157, 1979.*

[8] H. Brunner, *A Note on Collocation Methods for Volterra Integral Equations of the First Kind. Computing 23, 179-187, 1979.*

[9] P.P. Eggermont, *Collocation for Volterra Integral Equations of the first kind with iterated kernel. SIAM J. Numer Amal. Vol 20, No5, Oct. 1983*

[10] M. A. Golberg, *A Survey of Numerical Methods for Integral Equations, Solution Methods for Integral Equations. Theory and Applications. Plenum Press N.Y.*

[11] L. Schumaker, *Spline Functions: Basic theory, Wiley, N. York, 1981.*

[12] J. Burns, E.Cliff and T. Herdmann,*A State-Space Model for an aeroelastic system*, Proc. 22nd IEEE Conf. on Decision and Control, San Antonio, Tx, Dec 14-16,1983, pp. 1074-1077.

[13] J. Burns, T. Herdman and H. Stech, *Linear Functional Differential Equations as Semigroups on Product Spaces*, SIAM J. Math. Anal., 14 (1983), pp. 98-116.

[14] G. Cerezo, E. Fernandez Berdaguer, T. Herdman, J. Turi, *Colocation Techniques for the Approximation of Singular Neutral Equations*, preprint.

Vita

Graciela was born in 1962 in Buenos Aires Argentina. She finished her studies in Applied Mathematics at the University of Buenos Aires in 1990 with the degree of Licenciada. At the same year she start working as a research assinstant at the National Comision of Atomic Energy, at Buenos Aires. There she new about the possibility of comming here to continue her education. Since she always thought that a complete education involves much more than a university degree,the possibility of travelling, meeting new people and different cultures immidiatly fasinated her. In 1992 she came here as a graduate student and she is planning to stay here for her Ph. D.