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PERTURBATION THEORY FOR THE TOPOLOGICAL PRESSURE IN ANALYTIC DYNAMICAL SYSTEMS

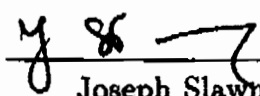
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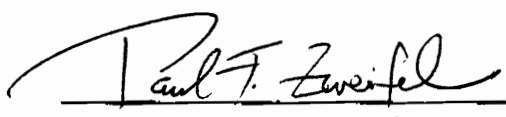
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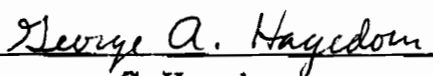
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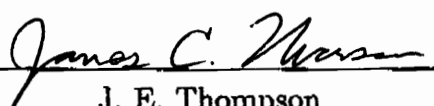
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PERTURBATION THEORY
FOR THE TOPOLOGICAL PRESSURE
IN ANALYTIC DYNAMICAL SYSTEMS

by

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(ABSTRACT)

We develop a systematic approach to the problem of finding the perturbative expansion for the topological pressure for an analytic expanding dynamics (f, M) on a Riemannian manifold M . The method is based on the spectral analysis of the transfer operator \mathcal{L} . We show that in typical cases, when f depends real-analytically on a set of perturbing parameters $\underline{\varepsilon}$, the related operators $\mathcal{L}_{\underline{\varepsilon}}$ form an analytic family. This gives rise to the rigorous construction of the power series expansion for the pressure via the analytic perturbation theory for eigenvalues, [Kato]. Consequently, the pressure and related dynamical indices, such as dimension spectra, Lyapunov

exponents, escape rates and Renyi entropies inherit the real-analyticity in ε from (f, M) .

Chapter 1 contains a review of the thermodynamic formalism for expanding maps and detailed discussion of the properties of topological pressure. Chapter 2 presents the proof of a theorem on analyticity of transfer operator families and general perturbation theory for the topological pressure including step-by-step development of the perturbation series. Chapter 3 is devoted to the detailed study of the map $f_c(z) = z^p - c$ on the Riemann sphere for small $|c|$. We use the perturbation theory for the corresponding transfer operator to construct rigorously an expansion for the pressure through the 5th order in $(\operatorname{Re} c, \operatorname{Im} c)$. This expansion is later used to obtain expressions for several dynamical indices, such as the Hausdorff dimension of the Julia set J_c and the escape rate from J_c . The earlier perturbative computations of the Hausdorff dimension of hyperbolic Julia sets [Rue3], [Wido] were not rigorous and consisted of formal manipulations on a defining expression for the topological pressure. Unlike the method presented here, they were not systematic enough to yield higher order terms in the perturbation series. All calculations in the 3rd chapter were performed with the aid of the symbolic manipulator SMP on a VAX/VMS. Our derivation of the expansion for

topological pressure shows that the transfer operator formalism has not only theoretical importance but also a practical computational power.

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Chapter 1

Introduction

1.1 Notation and general remarks

The following symbols are used in the text:

\mathbf{Z} the set of integers $\{0, \pm 1, \pm 2, \dots\}$

\mathbf{Z}_+ the set of natural numbers $\{0, 1, 2, \dots\}$

\mathbf{C} the complex plane

\mathbf{C}_∞ the compactified complex plane, the Riemann sphere

\mathbf{R} the set of real numbers

\mathbf{R}_+ the set of nonnegative real numbers

$C(X)$ the Banach space of functions $f : X \rightarrow \mathbf{C}$ continuous on the compact (metric) space X with the supremum norm

$C^\alpha(X)$ the Banach space of Hölder-continuous functions (with respect to the metric d on X) having Hölder exponent $\alpha > 0$, i.e. such $f \in C(X)$ that there exists a constant C such that

$$|f(x) - f(y)| \leq Cd(x, y)^\alpha$$

for all $x, y \in X$; the norm on $C^\alpha(X)$ is

$$\|f\| = \sup_{x \in X} |f(x)| + \sup_{x, y \in X} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}$$

$C^\omega(M)$ the class of real-analytic maps on an analytic Riemannian manifold M

$\text{cl } D$ closure of the set D

∂D boundary of the set D

$\text{int } D$ interior of the set D

Given a metric space (M, d) and a continuous map $f : M \rightarrow M$, the (topological) dynamical system (f, M) is thought of as the action of the continuous semigroup $\{f^n\}_{n \in \mathbf{Z}_+}$ on M . We will be interested in the case when M is a smooth finite-dimensional Riemannian manifold with the Riemann metric d .

Typically, one is concerned with the restriction of the dynamics (f, M) to some compact subset X of M , which is invariant with respect to f ; we write (f, X) for this restriction. Two assumptions on the dynamics (f, X) simplify the analysis significantly: first, it is required that f be expanding on an open neighbourhood U of X and, second, f is assumed to be topologically mixing on X . We define these terms below. The two assumptions allow us to use the so-called thermodynamic formalism, which provides powerful techniques and tools for the study of ergodic properties of the dynamics (f, M) .

1.2 Review of symbolic dynamics for expanding maps

In the present section we shall review more important results on the dynamics generated by expanding maps. We formulate the definitions and theorems in their original setting, more general than the analytic and conformal cases, for which our analysis is later developed. As we have mentioned earlier, the dynamics governed by the iteration of an expanding map is relatively easy to analyze; hence, the strongest results have been established in this case. The existence of Markov partitions gives rise to a symbolic description of the

dynamics, which in turn allows one to use the thermodynamic formalism — a collection of techniques developed in statistical mechanics of lattice systems, applied in the study of ergodic properties of dynamical systems in question.

For the proofs and more detailed presentation the reader is referred to [Rue1], [Rue2] and [Bow1].

Let M be a Riemannian manifold with the metric d . Let f be a continuous map $f : M \rightarrow M$.

Definition 1.1 *A compact subset $X \subseteq M$ is invariant with respect to f iff $fX=X$.*

Next, we define two properties of the dynamics (f, X) : the expanding property and topological mixing, the importance of which has been mentioned above.

Definition 1.2 *Let U be an open set such that $X \subseteq U \subseteq M$, where X is an invariant set for f . Then f is called expanding on U iff there exist $\varepsilon > 0$ and $\theta > 1$ such that for any pair of points $x, y \in U$ one has*

$$d(x, y) < \varepsilon \Rightarrow d(f(x), f(y)) \geq \theta d(x, y). \quad (1.1)$$

Moreover, f is topologically mixing on X iff for any open set $W \subseteq M$ one

has

$$W \cap X \neq \emptyset \Rightarrow \exists N > 0 \text{ such that } f^N(W \cap X) = X. \quad (1.2)$$

1.2.1 Example

Consider a mapping $f : \mathbf{C}_\infty \rightarrow \mathbf{C}_\infty$, $f_c(z) = z^p - c$, where \mathbf{C}_∞ denotes the Riemann sphere with the chordal metric ρ , $p > 1$ is an integer and $c \in \mathbf{C}$.

Suppose first that $c = 0$. The sphere \mathbf{C}_∞ itself is completely invariant, but f is not expanding on the whole \mathbf{C}_∞ ; it is a contraction near $z = 0$.

There is also a nontrivial completely invariant set for this map, namely the unit circle $J_0 = \{z : |z| = 1\}$. Moreover, there is a neighbourhood U of J_0 on which f is expanding, because $|f'_0(z)| = |pz^{p-1}| > 1$ on U . We also have the topological mixing property of f on J_0 : given an arbitrary open arc I of length η on J_0 , $f_0^k I$ is an arc of length $\min\{p^k \eta, 2\pi\}$; thus for some $N > 0$ $f_0^N I$ covers the whole J_0 . This situation persists when c varies over a neighbourhood of 0; the corresponding invariant sets J_c are Julia sets of f_c . Their geometric properties are generally much more complicated than those of J_0 , the unit circle, [Blan], [Brol]. The dynamics generated by f_c on J_c , however, exhibits many essential similarities with the case $c = 0$. In particular, the phenomenon of sensitive dependence on the initial conditions

occurs.

The dynamical systems, as characterized above, will be linked in the following sections to one-dimensional lattice models studied by statistical mechanics. This is achieved by introducing a symbolic representation of the dynamics in question, i.e. coding each point $x \in X$ as an infinite sequence $(x_i)_{i=0}^{\infty}$ of elements (symbols) from a finite set. The essential feature of this coding is the representation of the action of f on X as a shift operation on symbolic sequences, i.e. the mapping

$$(x_0, x_1, x_2, \dots) \xrightarrow{\sigma} (x_1, x_2, x_3, \dots). \quad (1.3)$$

Below we reproduce the two most important results on the existence of so-called Markov partitions and symbolic dynamics for expanding maps, [Sina], [Bow1], [Rue1], [Rue2].

Theorem 1.1 *Suppose that a continuous map $f : M \rightarrow M$ is expanding and locally invertible on an open neighbourhood U of an invariant set $X \subseteq U \subseteq M$. Then, given $\delta > 0$, there exists a finite family of closed nonempty sets, $\{X_i\}_{i=0}^r$, such that*

$$(i) \text{ diam} X_i < \delta \quad i = 0, \dots, r$$

$$(ii) \quad X = \bigcup_{i=0}^r X_i$$

$$(iii) \quad cl(intX_i) = X_i \quad \text{with the relative topology on } X$$

$$(iv) \quad X_i \cap X_j = \partial X_i \cap \partial X_j \quad \text{for } i \neq j$$

$$(v) \quad \forall i \quad fX_i = X_{j_1} \cup \dots \cup X_{j_p} \quad \text{for some } j_1, \dots, j_p, p = p(i).$$

Moreover, if f is topologically mixing on X , then the $(r+1) \times (r+1)$ matrix

$T = [t_{ij}]_{i,j=0}^r$ defined by

$$t_{ij} = \begin{cases} 1 & fX_i \supseteq X_j \\ 0 & \text{otherwise} \end{cases} \quad (1.4)$$

is irreducible, i.e. there exists $N > 0$ such that T^N has strictly positive entries.

The cover $\{X_i\}_{i=0}^r$ of X is referred to as a Markov partition of X . The matrix T is called the transition matrix. The idea behind Markov partitions is that when the cells X_i have been chosen sufficiently fine, then roughly every point $x \in X$ can be assigned a unique infinite sequence of digits from $\{0, \dots, r\}$, say (k_0, k_1, \dots) , interpreted as “dynamical history” of the point x under the action of f relative to $\{X_i\}_{i=0}^r$, or precisely, such that

$$f^j(x) \in X_{k_j} \quad j = 0, 1, \dots \quad (1.5)$$

This way the symbolic dynamics for (f, X) is constructed. In what follows, we characterize this construction in detail.

Suppose that a transition matrix $T = [t_{ij}]$ is given. Let

$$\Omega^+ = \{0, \dots, r\}^{\mathbb{Z}^+} = \{(c_i)_{i=0}^{\infty} : \forall i \ c_i \in \{0, \dots, r\}\}. \quad (1.6)$$

When the discrete topology is chosen on $\{0, \dots, r\}$, the corresponding product topology on Ω^+ makes it into a compact space. Let

$$\Omega_T^+ = \{(c_i)_{i=0}^{\infty} \in \Omega^+ : t_{c_i c_{i+1}} = 1 \ \forall i \geq 0\}. \quad (1.7)$$

One can easily verify that Ω_T^+ is compact in Ω^+ . Also, it is an elementary fact that the shift operation (1.3)

$$\sigma : \Omega^+ \rightarrow \Omega^+$$

is continuous.

Definition 1.3 *The dynamical system (σ, Ω_T^+) , where T is an irreducible transition matrix, is called a subshift of finite type.*

In what follows, we shall write $\underline{c} = (c_i)_{i=0}^{\infty}$. We have the following characterization of symbolic dynamics for (f, X) .

Theorem 1.2 *If $\{X_i\}_{i=0}^r$ is a Markov partition for (f, X) , where $\text{diam} X_i$ is sufficiently small and T is the corresponding transition matrix, then there exists a continuous surjection $\pi : \Omega_T^+ \rightarrow X$ with the following properties:*

$$(i) \pi(\underline{c}) = \bigcap_{k=0}^{\infty} f^{-k} X_{c_k}$$

$$(ii) \pi \circ \sigma = f \circ \pi$$

(iii) σ is topologically mixing on Ω_T^+ .¹

Let us remark only that the property of the map π not being injective, in general, poses some technical difficulties, which we shall not discuss here, cf. [Rue2].

1.2.2 Example

We shall construct a Markov partition and symbolic dynamics for the map of Example 1.2.1. Here we have $f(z) = z^p$ with J_0 being completely invariant.

Let

$$X_k = \left\{ e^{2\pi i\alpha} \in J_0 : \frac{k}{p} \leq \alpha \leq \frac{k+1}{p} \right\} \quad k = 0, \dots, p-1. \quad (1.8)$$

Then $\{X_k\}_{k=0}^{p-1}$ is an example of a Markov partition for (f, J_0) . One easily verifies (i) – (v) of Theorem 1.1. In particular, $fX_k = J_0$ for all k , and hence the transition matrix T has all entries equal to 1. One could find a finer partition, if necessary, by taking $m > 1$ and

$$\frac{k}{p^m} \leq \alpha \leq \frac{k+1}{p^m}$$

¹Note, that the cardinality of the set on the right side in (i) is precisely 1.

in (1.8) for $k = 0, 1, \dots, p^m - 1$. Because of the particular structure of the matrix T , in the present case we have

$$\Omega_T^+ = \Omega^+ = \{0, \dots, p-1\}^{\mathbb{Z}_+}.$$

There is an easy interpretation of the symbolic dynamics (σ, Ω^+) . Since $f(e^{2\pi i\alpha}) = e^{2\pi ip\alpha}$, we can interpret the action of f on J_0 in terms of multiplication by p modulo 1 on the unit interval,

$$\alpha \mapsto p\alpha \pmod{1}.$$

Thus, if α is represented in the form of an infinite expansion with base p ,

$$\alpha = \alpha_1 p^{-1} + \alpha_2 p^{-2} + \alpha_3 p^{-3} + \dots,$$

where $\alpha_i \in \{0, \dots, p-1\}$, then the action of f is essentially that of a shift on the sequence $(\alpha_1, \alpha_2, \alpha_3, \dots)$.

1.3 Review of statistical mechanics of one-dimensional lattice systems

We shall concentrate in turn on a few more relevant ideas and results of statistical mechanics of one-dimensional lattice models. In the present section we mainly focus our attention on the part of the theory that provides a

foundation for the development of perturbative techniques in the later chapters, specifically on the spectral theory of Ruelle–Araki transfer operators. However, we also quote a number of more general facts and results to place our analysis in a broader context of the interplay between ergodic theory of nonlinear dynamical systems and theoretical statistical mechanics, [Bow1], [Rue1], [Walt].

Given a finite set $\Delta_r = \{0, 1, \dots, r\}$ with the discrete topology, we construct the configuration space Ω as the product space

$$\Omega = \Delta_r^{\mathbf{Z}},$$

or, when given a transition matrix T ,

$$\Omega_T = \left\{ (x_i)_{i=-\infty}^{\infty} \in \Omega : t_{x_i x_{i+1}} = 1 \text{ for } i \in \mathbf{Z} \right\} \quad (1.9)$$

It will always be assumed that T is irreducible. It should be noted that the lattice presently used is \mathbf{Z} rather than \mathbf{Z}_+ . This is the conventional setting for the formulation of statistical mechanics in one dimension, [Bow1], [Rue1]. The passage to the \mathbf{Z}_+ lattice system, which is the natural setting for the study of dynamical systems and for transfer operators, is accomplished by restricting attention to the observables $\varphi \in \mathcal{C}(\Omega_T)$ depending only on coordinates x_i for $i \geq 0$.

As mentioned before, both Ω and Ω_T are compact with the product topology or, respectively, the relative product topology. Moreover, the shift operator $\sigma : \Omega_T \rightarrow \Omega_T$, given by $\sigma((x_i)_{i=-\infty}^{\infty}) = (x_{i+1})_{i=-\infty}^{\infty}$, is continuous in this topology. If $\underline{x} \in \Omega_T$, we write

$$U_n(\underline{x}) = \{ \underline{y} \in \Omega_T : x_i = y_i \text{ for } |i| \leq n \} \quad (1.10)$$

and we call the set $U_n(\underline{x})$ the cylinder of size n centered at \underline{x} . We also write $[x_{-n}, \dots, x_0, \dots, x_n]$ for the cylinder $U_n(\underline{x})$.

Definition 1.4 *Let $g \in \mathcal{C}(\Omega_T)$. We define*

$$\text{var}_n g = \max_{x_{-n}, \dots, x_n \in \Delta_r} \sup_{\underline{y}, \underline{z} \in [x_{-n}, \dots, x_n]} |g(\underline{y}) - g(\underline{z})|. \quad (1.11)$$

We say that g is of class \mathcal{F}_θ iff there exist $c > 0$ and $0 < \theta < 1$ such that

$$\forall n \geq 0 \quad \text{var}_n g \leq c\theta^n. \quad (1.12)$$

Ω_T is metrizable. One can choose the metric on Ω_T to be

$$d(\underline{x}, \underline{y}) = \eta^k \text{ iff } x_i = y_i \text{ for } |i| \leq k,$$

where $0 < \eta < 1$. Then \mathcal{F}_θ can be identified with the space of Hölder-continuous functions, having positive Hölder exponent α , i.e. those g for which there exists a constant $c > 0$ such that

$$\forall \underline{x}, \underline{y} \in \Omega_T \quad |g(\underline{x}) - g(\underline{y})| \leq c[d(\underline{x}, \underline{y})]^\alpha. \quad (1.13)$$

Hölder-continuity will play an important role in the spectral analysis of transfer operators.

Given $\varphi \in \mathcal{F}_\theta$, we define

$$\|\varphi\|_\theta = \sup_n \frac{\text{var}_n \varphi}{\theta^n} \quad (1.14)$$

and subsequently

$$\|\|\varphi\|\| = \max\{\|\varphi\|, \|\varphi\|_\theta\}, \quad (1.15)$$

where $\|\cdot\|$ denotes the sup norm in the space $\mathcal{C}(\Omega_T)$. Then $(\mathcal{F}_\theta, \|\|\cdot\|\|)$ is a Banach space. The importance of the \mathcal{F}_θ space comes from the fact that thermodynamic quantities, to be defined below, have analytic properties on \mathcal{F}_θ . The elements φ of \mathcal{F}_θ are often called interactions, which comes from the physical motivation behind lattice models. The interactions in \mathcal{F}_θ are assumed to be real-valued; when more general complex-valued interactions are to be considered, we shall write $\mathcal{F}_\theta^{\mathbb{C}}$ to indicate it explicitly.

We define the local Hamiltonian

$$H_n \varphi(\mathbf{x}) = \sum_{k=0}^{n-1} \varphi(\sigma^k \mathbf{x}), \quad (1.16)$$

where σ is the shift operator. We have the following important theorem.

Theorem 1.3 *Let $\varphi \in \mathcal{F}_\theta$ and suppose that W_n is a finite subset of Ω_T such*

that

$$\Omega_T = \bigcup_{\underline{x} \in W_n} U_n(\underline{x}),$$

where the union is disjoint. Then the following limit

$$P(\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\underline{x} \in W_n} e^{H_n \varphi(\underline{x})} \quad (1.17)$$

exists, independently of the choice of W_n , and is real-analytic on \mathcal{F}_θ .

$P(\varphi)$ is called the pressure for φ , [Bow1], [Rue1]. The limit (1.17) exists also for $\varphi \in \mathcal{C}(\Omega_T)$, but $P(\varphi)$ may not be analytic in this case.

One can also describe the pressure function in terms of a variational principle, i.e. relating $P(\varphi)$ to probability measures on Ω_T maximizing a certain quantity. We discuss this and related notions in detail below.

If $\mathcal{B} = \{B_1, \dots, B_n\}$ is a partition of Ω_T and μ is a probability measure on Ω_T , $\mu \in \mathcal{C}^*(\Omega_T)$, we define the entropy of \mathcal{B} by

$$\mathcal{H}_\mu(\mathcal{B}) = - \sum_{i=1}^n \mu(B_i) \log \mu(B_i). \quad (1.18)$$

Given two finite partitions of Ω_T , say \mathcal{B} and \mathcal{D} , one can construct a new partition in the following way:

$$\mathcal{B} \vee \mathcal{D} = \{B_i \cap D_j : B_i \in \mathcal{B}, D_j \in \mathcal{D}\}.$$

One then has

$$\mathcal{H}_\mu(\mathcal{B} \vee \mathcal{D}) \leq \mathcal{H}_\mu(\mathcal{B}) + \mathcal{H}_\mu(\mathcal{D}), \quad (1.19)$$

which follows from the concavity of the function $l(x) = -x \log x$ on $[0,1]$. It is also easy to see that if \mathcal{B} is a partition of Ω_T , then so is $\sigma^{-1}\mathcal{B} = \{\sigma^{-1}B_1, \dots, \sigma^{-1}B_n\}$.

We are ready now to define the entropy of a σ -invariant probability measure μ , i.e. $\mu \in \mathcal{C}^*(\Omega_T)$ such that for any $\varphi \in \mathcal{C}(\Omega_T)$ one has

$$\mu(\varphi) = \mu(\varphi \circ \sigma). \quad (1.20)$$

The class of σ -invariant probability measures on Ω_T is denoted by $\mathcal{M}_\sigma(\Omega_T)$.

Definition 1.5 *If $\mu \in \mathcal{M}_\sigma(\Omega_T)$, $\mathcal{B} = \{[0], [1], \dots, [r]\}$,² then the entropy of μ , denoted $S(\mu)$ is defined as the limit*

$$S(\mu) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathcal{H}_\mu(\mathcal{B} \vee \sigma^{-1}\mathcal{B} \vee \dots \vee \sigma^{-n+1}\mathcal{B}). \quad (1.21)$$

Subadditivity of \mathcal{H}_μ guarantees the existence of the limit (1.21), [Bow1].

Theorem 1.4 (The variational principle) *Let $\varphi \in \mathcal{C}(\Omega_T)$. Then*

$$P(\varphi) = \sup_{\mu \in \mathcal{M}_\sigma(\Omega_T)} S(\mu) + \int \varphi d\mu. \quad (1.22)$$

Moreover, if $\varphi \in \mathcal{F}_\theta$, then there is a unique $\mu = \mu_\varphi$, such that the supremum above is attained.

²Recall that $[k] = \{\underline{x} \in \Omega_T : x_0 = k\}$.

Every μ realizing the supremum in (1.22) is called an equilibrium state. The unique equilibrium state is also identical to the Gibbs state for φ , [Rue1], [Walt]. We should note that for any $\varphi \in \mathcal{F}_\theta$ one can construct $\psi \in \mathcal{F}_\theta$ depending only on x_i for $i \geq 0$ and such that $P(\varphi) = P(\psi)$, as well as $\mu_\varphi = \mu_\psi$, [Bow1].

The importance of the last statement arises from the fact that it is very difficult, in general, to find $P(\varphi)$ from the definitions above. There is an alternative and often simpler way, namely the transfer operator method, by which one can find $P(\psi)$ for $\psi \in \mathcal{C}(\Omega_T^+) \cap \mathcal{F}_\theta$, i.e. ψ of \mathcal{F}_θ class depending only on positive coordinates x_i , $i \geq 0$. In light of the statement above, the transfer operator method can be used for all $\varphi \in \mathcal{F}_\theta$. This fact also provides the basis for statistical mechanical interpretation of the pressure, invariant measures, etc. obtained in the context of symbolic dynamics, where one deals with functions on Ω_T^+ .

Definition 1.6 *Let $\varphi \in \mathcal{C}(\Omega_T^+)$. The transfer (Ruelle–Araki) operator $\mathcal{L}_\varphi : \mathcal{C}(\Omega_T^+) \rightarrow \mathcal{C}(\Omega_T^+)$ is defined as*

$$[\mathcal{L}_\varphi \psi](\underline{x}) = \sum_{\underline{y} \in \sigma^{-1}\{\underline{x}\}} e^{\varphi(\underline{y})} \psi(\underline{y}). \quad (1.23)$$

The following theorem, as well as its corresponding version formulated in the language of the dynamics (f, X) , has a fundamental importance for the

development of our results in later chapters.

Theorem 1.5 (Ruelle–Perron–Frobenius) *Given a real-valued interaction $\varphi \in \mathcal{C}(\Omega_T^+) \cap \mathcal{F}_\theta$, the operator \mathcal{L}_φ acting on the same space has $e^{P(\varphi)}$ as a simple eigenvalue, with the corresponding eigenvector h being strictly positive. Moreover, all isolated eigenvalues λ satisfy*

$$\theta e^{P(\varphi)} < |\lambda| < e^{P(\varphi)}. \quad (1.24)$$

The rest of the spectrum consists of the disc

$$\{z : |z| \leq \theta e^{P(\varphi)}\}.$$

The proof of a somewhat more general version of this theorem, when φ is complex valued and \mathcal{L}_φ acts on the complex Banach space $\mathcal{C}(\Omega_T^+) \cap \mathcal{F}_\theta^{\mathbb{C}}$, belongs to M. Pollicott, [Pol1], [Pol2]. We also have the following result, [Bow1].

Theorem 1.6 *With the assumptions and notation as in Theorem 1.5, there exists a measure $\nu \in \mathcal{C}^*(\Omega_T^+)$ such that*

$$(i) \quad \mathcal{L}_\varphi^* \nu = e^{P(\varphi)} \nu$$

$$(ii) \quad \nu(h) = 1$$

$$(iii) \quad \forall g \in \mathcal{C}(\Omega_T^+) \quad \|e^{-mP(\varphi)} \mathcal{L}_\varphi^m g - \nu(g) h\| \longrightarrow 0 \quad \text{as } m \rightarrow \infty.$$

In addition, the measure μ defined by

$$d\mu = h d\nu \tag{1.25}$$

is a σ -invariant probability measure on Ω_T^+ , i.e. $\mu \in \mathcal{M}_\sigma(\Omega_T^+)$.

For the measure μ in the above theorem, there is a natural way to make it into a measure on Ω_T , [Bow1]. The resulting measure $\tilde{\mu}$ is the unique Gibbs state for φ, μ_φ from Theorem 1.4.

1.4 Thermodynamic formalism

We turn to the discussion of the interplay between statistical mechanics and ergodic theory for expanding maps reviewed in the preceding sections. We use the same notation as in Section 1.2.

Let $\mathcal{B} = \{B_i\}_{i=1}^k$ and $\mathcal{B}' = \{B'_i\}_{i=1}^l$ be two partitions of a compact f -invariant set $X \subseteq M$. Then

$$\mathcal{B} \vee \mathcal{B}' = \{B_i \cap B'_j : i = 1, \dots, k, j = 1, \dots, l\}$$

is also a partition of X . Moreover, due to the invariance of X with respect to f , we have $f^{-1}X \supseteq X$ which, together with the continuity of f , implies that $\mathcal{B} \vee f^{-m}\mathcal{B} = \{B_i \cap f^{-m}B_j\}_{i,j=1}^k$ is a partition of X for any $m > 0$. We write $\text{diam}\mathcal{B} = \max_i \text{diam}B_i$.

Definition 1.7 Let \mathcal{B} be a partition of X . Define

$$\mathcal{B}^{(m)} = \mathcal{B} \vee f^{-1}\mathcal{B} \vee \dots \vee f^{-m+1}\mathcal{B}. \quad (1.26)$$

\mathcal{B} is called a *filtering partition* for (f, X) , if $\text{diam} \mathcal{B}^{(m)} \rightarrow 0$ when $m \rightarrow \infty$.

Let f be expanding on a neighbourhood U of X . Let ε and θ be the same as in Definition 1.2. If $\{X_i\}_{i=0}^r$ is a sufficiently fine Markov partition, i.e. $\max_i \text{diam} X_i < \varepsilon$, then we have

$$\text{diam} (X_{i_1} \cap f^{-1} X_{i_2} \cap \dots \cap f^{-m} X_{i_m}) < \theta^{-m} \varepsilon,$$

and hence the partition is a filtering one. When the symbolic dynamics $(\sigma, \Omega_{\mathcal{T}}^{\pm}) \xrightarrow{\pi} (f, X)$ has been constructed, one can see that for a cylinder $U_n(\underline{c})$ of size n centered at $\underline{c} = (c_i)_{i=0}^{\infty}$ the following holds:

$$\text{diam} \pi[U_n(\underline{c})] = \text{diam} \left[\bigcap_{k=0}^n f^{-k} X_{c_k} \right] \leq \varepsilon \theta^{-n}. \quad (1.27)$$

Thus, for a Hölder-continuous function $\varphi \in \mathcal{C}^{\alpha}(X)$, $\alpha > 0$, the corresponding $\tilde{\varphi} = \varphi \circ \pi$ is of class \mathcal{F}_{η} , where $\eta = \theta^{-\alpha} < 1$. With this in mind, one can then imitate (1.17), where for the set W_n one takes the collection of all admissible periodic sequences $\underline{x} \in \Omega_{\mathcal{T}}^{\pm}$ of period n . We write

$$P_f(\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in \text{Fix} f^n} e^{\varphi(x) + \varphi(f(x)) + \dots + \varphi(f^{n-1}(x))}, \quad (1.28)$$

where $\text{Fix } f^n$ is the set of all fixed points of f^n . Indeed, this limit exists and equals the corresponding $P(\tilde{\varphi})$, [Rue1]; therefore in what follows we omit the subscript f in $P_f(\varphi)$. We postpone the interpretation and discussion of the significance of $P(\varphi)$ for a specific choice of φ until the next section.

The pressure $P(\varphi)$ can also be characterized by the corresponding version of the variational principle. We start with the definition of Kolmogorov–Sinai (metric) entropy for an f -invariant probability measure μ on X , i.e. μ such that

$$\mu(W) = \mu(f^{-1}W) \tag{1.29}$$

for any measurable subset W of X . We write $\mathcal{M}_f(X)$ for the collection of f -invariant probability measures.

For a finite partition \mathcal{B} of X we define $\mathcal{H}_\mu(\mathcal{B})$ the same way as in (1.18). It is proved in [Bow1] that the following limit exists:

$$h_f(\mu, \mathcal{B}) = \lim_{n \rightarrow \infty} \frac{1}{n} \mathcal{H}_\mu(\mathcal{B}^{(n)}). \tag{1.30}$$

Definition 1.8 *The Kolmogorov–Sinai entropy (or metric entropy) of $\mu \in \mathcal{M}_f(X)$, denoted $h_f(\mu)$, is defined as*

$$h_f(\mu) = \sup_{\mathcal{B}} h_f(\mu, \mathcal{B}), \tag{1.31}$$

where the supremum is taken over all finite partitions of X .

Actually, a much stronger result holds for an expanding map f :

$$h_f(\mu) = h_f(\mu, \mathcal{B}) \quad (1.32)$$

whenever $\text{diam } \mathcal{B}$ is sufficiently small, [Bow1].

The following theorem is a version of Variational Principle 1.4, stated in the language of (f, X) , [Rue2].

Theorem 1.7 *If $\varphi \in \mathcal{C}(X)$ is real valued, then*

$$P(\varphi) = \sup_{\mu \in \mathcal{M}_f(X)} \left[h_f(\mu) + \int \varphi d\mu \right]. \quad (1.33)$$

If $\varphi \in \mathcal{C}^\alpha(X)$, then the supremum above is attained for a unique f -ergodic measure μ_φ — the Gibbs state for φ .

In the same spirit, for $\varphi \in \mathcal{C}^\alpha(X)$ one can define the transfer operator $\mathcal{L}_\varphi : \mathcal{C}^\alpha(X) \rightarrow \mathcal{C}^\alpha(X)$ by

$$(\mathcal{L}_\varphi h)(x) = \sum_{y \in f^{-1}\{x\}} e^{\varphi(y)} h(y) \quad (1.34)$$

and deduce its spectral characteristics, particularly the existence of the isolated maximal eigenvalue $e^{P(\varphi)}$. However, the spectral analysis for \mathcal{L}_φ that we need is of a different type: the domain of \mathcal{L}_φ will be assumed to be a suitable space of functions on a neighbourhood U of X , rather than X itself. This way, when a perturbation is applied to f , the resulting modification

of X need not be taken into account; the perturbation affects the operator \mathcal{L}_φ itself and not its domain. The corresponding version of Ruelle–Perron–Frobenius Theorem is quoted below.

Presently, we introduce more restrictive assumptions about the dynamics (f, M) . Thus we require that M be an analytic Riemannian manifold (i.e. of class \mathcal{C}^ω), as well as f to be analytic on M . As before, X denotes an invariant set such that f is topologically mixing on X . U is an open neighbourhood of X having compact closure in M with f is expanding on $\text{cl}U$. We note that for an expanding f , one can always choose U in such a way that $\text{cl}f^{-1}U \subseteq U$, [Rue2]. Let $\mathcal{B}(U)$ denote the Banach space of complex-valued functions analytic in U and continuous on $\text{cl}U$ — the closure of U ; $\mathcal{B}(U)$ is thus a subspace of $\mathcal{C}(\text{cl}U)$ with the supremum norm.

For $\varphi \in \mathcal{B}(U)$ we define the transfer operator $\mathcal{L}_\varphi : \mathcal{B}(U) \rightarrow \mathcal{B}(U)$ in the very same way as in (1.34). The above-mentioned property $\text{cl}f^{-1}U \subseteq U$ is essential for the correctness of the definition of \mathcal{L}_φ . We have the following theorem, [Rue2].

Theorem 1.8 *If φ is real-valued, then $e^{P(\varphi)}$ is a simple isolated eigenvalue of \mathcal{L}_φ that is maximal in modulus. The corresponding eigenvector h is strictly positive. For the dual action $\mathcal{L}_\varphi^* : \mathcal{B}^*(U) \rightarrow \mathcal{B}^*(U)$, the corresponding eigen-*

vector ν is a positive measure on U , for which one assumes the normalization $\nu(h) = 1$. Moreover, the product $\mu = h\nu$, i.e. the measure μ such that $\mu(g) = \nu(gh)$, is an f -ergodic probability measure — the unique Gibbs state for φ .

Once again the assumption $\text{cl}f^{-1}U \subseteq U$ has significant consequences. The local inverse branches of f are analytic on some sufficiently small open sets W_i forming a finite cover of $\text{cl}U$. Therefore, the operator \mathcal{L}_φ acting on a function $g \in \mathcal{B}(U)$, (1.34), improves its analytic properties: by the composition with the branches of f^{-1} it extends the domain of analyticity of $\mathcal{L}_\varphi g$ beyond $\text{cl}U$. The image of \mathcal{L}_φ is thus contained in the collection of functions $g \in \mathcal{B}(U)$ having analytic extension to a neighbourhood of $\text{cl}U$. In particular, the eigenvector h corresponding to $\exp P(\varphi)$ is of this class.

1.4.1 Example

We shall study in detail a simple example of dynamics generated by a piecewise linear expanding map on $[0,1]$. The pressure $P(\varphi)$ for a specific choice of φ will be computed in two different ways: directly from (1.28) and by using the transfer operator approach. Finally, we look closely at the role played by the assumption of φ being Hölder-continuous.

Consider a partial mapping $f : [0, 1] \rightarrow [0, 1]$ defined by

$$f_0(x) = \lambda_0 x, \quad x \in I_0 = [0, a] \quad (1.35)$$

$$f_1(x) = \lambda_1(1 - x), \quad x \in I_1 = [b, 1], \quad (1.36)$$

where $\lambda_0, \lambda_1 > 1$ and $\lambda_0^{-1} = a < b = \lambda_1^{-1}(\lambda_1 - 1)$. One can easily verify that the completely invariant set for f is

$$J = \bigcap_{n=1}^{\infty} f^{-n}[0, 1]. \quad (1.37)$$

If one uses $\{I_0, I_1\}$ as a Markov partition (or, more precisely, $\{J \cap I_0, J \cap I_1\}$), the resulting transition matrix is

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus, the symbolic dynamics is particularly simple in this case,

$$\Omega_T^+ = \Omega^+ = \{0, 1\}^{\mathbb{Z}_+}$$

and for an arbitrary binary sequence $\underline{x} = (x_i)_{i=0}^{\infty} \in \Omega^+$, for all n we have

$$\begin{aligned} U_n(\underline{x}) &= [x_0, x_1, \dots, x_n] = \\ &= \pi^{-1} \left\{ f_{x_0}^{-1} \circ f_{x_1}^{-1} \circ \dots \circ f_{x_n}^{-1}[0, 1] \cap J \right\}. \end{aligned} \quad (1.38)$$

Let $\varphi(x) = -\beta \log |f'(x)|$, where β is a parameter. The importance of this function will be discussed in the next section. We have for $j = 0, 1$

$$\varphi(x) = -\beta \log \lambda_j = \text{const} \quad \text{for } x \in I_j.$$

Thus, for $\tilde{\varphi} = \varphi \circ \pi$ we obtain

$$\tilde{\varphi}(\underline{x}) = -\beta \log \lambda_{x_0}. \quad (1.39)$$

Hence, $\tilde{\varphi}$ is constant on cylinders U_n of size $n \geq 1$ and, consequently, is of class \mathcal{F}_θ for any $0 < \theta < 1$.

The present case is so simple that one can compute $P(\tilde{\varphi})$ directly from the defining expression (1.17). We write $\text{Fix}\sigma^n$ for the set of all periodic sequences $\underline{x} \in \Omega^+$ of period n . We have

$$\begin{aligned} P(\tilde{\varphi}) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\underline{x} \in \text{Fix}\sigma^n} e^{H_n \tilde{\varphi}(\underline{x})} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x_0, \dots, x_{n-1}} e^{-\beta(\log \lambda_{x_0} + \dots + \log \lambda_{x_{n-1}})} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x_0, \dots, x_{n-1}} \prod_{k=0}^{n-1} \lambda_{x_k}^{-\beta} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\lambda_0^{-\beta} + \lambda_1^{-\beta} \right)^n \\ &= \log \left(\lambda_0^{-\beta} + \lambda_1^{-\beta} \right). \end{aligned} \quad (1.40)$$

The simplicity of this calculation comes from the fact that the potential $\tilde{\varphi}$ depends only on the first coordinate of its argument \underline{x} . In general, if $\tilde{\varphi}$ is of finite range, i.e. it is constant on cylinders of size $n \geq N$ for some N , $P(\tilde{\varphi})$ can be found explicitly with little difficulty using the transfer operator method. The spectral analysis of the operator $\mathcal{L}_{\tilde{\varphi}}$, which becomes the

standard tool in the computation of $P(\tilde{\varphi})$, is generally much more involved when the interaction $\tilde{\varphi}$, although of class \mathcal{F}_θ , has infinite range. The latter case occurs, for example, when the f_j 's are quadratic rather than linear on the I_j 's and such that $|f_j'| > 1$ on I_j . Then taking

$$\varphi(x) = -\beta \log |f_j'(x)| = -\beta \log |A_j x + B_j|$$

on I_j gives rise to $\tilde{\varphi}$ of infinite range.

Our next step is to analyze a modification of the map (1.35), (1.36).

Presently we consider

$$f_0(x) = \begin{cases} \lambda_1 x & x \in [0, c] \\ \lambda_2(x - c) + \lambda_1 c & x \in (c, a] \end{cases} \quad (1.41)$$

$$f_1(x) = \lambda_3(1 - x) \quad x \in [b, 1], \quad (1.42)$$

where $\lambda_1, \lambda_2, \lambda_3 > 1$,

$$a = \frac{1 + c(\lambda_2 - \lambda_1)}{\lambda_2}$$

$$b = \frac{\lambda_3 - 1}{\lambda_3}$$

and $0 < c < a < b < 1$.

Suppose first that $c \notin J$, i.e. there is a smallest $k > 0$ such that

$$c \notin f^{-k}[0, 1]. \quad (1.43)$$

Then one uses 2^k closed disjoint intervals

$$I_{\underline{j}} = I_{j_1 j_2 \dots j_k} = f_{j_1}^{-1} \circ f_{j_2}^{-1} \circ \dots \circ f_{j_k}^{-1}[0, 1], \quad (1.44)$$

where $j_i \in \{0, 1\}$, as a Markov partition for the system. We note that $|f'(x)|$ is constant on the sets $I_{\underline{j}}$ (taking the values λ_1, λ_2 or λ_3 in our case) and hence, in the language of statistical mechanics of the underlying one-dimensional lattice system, the function $\varphi(x) = -\beta \log |f'(x)|$ represents a finite-range interaction among k neighbouring particles. We perform the explicit evaluation of $P(\varphi)$ assuming that $k = 2$, i.e. that $a < \lambda_1 c < b$.

The Markov partition in our case is $\{I_j\}_{j=0}^3$, or specifically

$$\{[0, \lambda_1^{-1}a], [c + \lambda_2^{-1}(b - \lambda_1 c), a], [b, 1 - \lambda_3^{-1}b], [1 - \lambda_3^{-1}a, 1]\}.$$

One verifies that

$$T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Therefore, Ω_T^\pm , unlike in the previously discussed case, is a proper subset of

$\{0, 1, 2, 3\}^{\mathbb{Z}^+}$. Taking again $\varphi(x) = -\beta \log |f'(x)|$ we obtain for $\tilde{\varphi} = \varphi \circ \pi$

$$\tilde{\varphi}(\underline{x}) = \begin{cases} -\beta \log \lambda_1 & \text{if } x_0 = 0 \\ -\beta \log \lambda_2 & \text{if } x_0 = 1 \\ -\beta \log \lambda_3 & \text{if } x_0 = 2, 3. \end{cases} \quad (1.45)$$

As $\tilde{\varphi}$ is constant on cylinders of size $n \geq 1$, it is of class \mathcal{F}_θ . We apply the transfer operator analysis in order to find $P(\tilde{\varphi})$. We have for $h \in \mathcal{F}_\theta(\Omega_T^+)$

$$(\mathcal{L}_{\tilde{\varphi}} h)(\underline{x}) = \sum_{\underline{y} \in \sigma^{-1}\{\underline{x}\}} e^{\tilde{\varphi}(\underline{y})} h(\underline{y}). \quad (1.46)$$

Now, since $\tilde{\varphi}$ depends only on the first coordinate of its argument \underline{y} , it is obvious that the subspace $\mathcal{F}_\theta^{(1)}$ of \mathcal{F}_θ that consists of functions constant on cylinders of size $n \geq 1$ is invariant under the action of $\mathcal{L}_{\tilde{\varphi}}$. In the language of the system $(f, [0, 1])$, $\mathcal{F}_\theta^{(1)}$ corresponds to the collection of those functions $g \in C^\alpha([0, 1])$, for which one has

$$g|_{I_j} = \text{const} \quad \text{for } j = 0, \dots, 3.$$

Each of the functions $h \in \mathcal{F}_\theta^{(1)}$ can be represented as a constant vector

$$h = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix},$$

so that $h(\underline{x}) = h_{x_0}$. By taking into account that

$$\sigma^{-1}\{\underline{x}\} = \{\underline{y} = (y_0, x_0, x_1, \dots) : t_{y_0 x_0} = 1\},$$

it is immediate to see that $\mathcal{L}_{\tilde{\varphi}}|_{\mathcal{F}_{\theta}^{(1)}}$ actually takes the form of a matrix

$$\mathcal{L}_{\tilde{\varphi}} h = \begin{bmatrix} \lambda_1^{-\beta} & 0 & 0 & \lambda_3^{-\beta} \\ \lambda_1^{-\beta} & 0 & 0 & \lambda_3^{-\beta} \\ 0 & \lambda_2^{-\beta} & \lambda_3^{-\beta} & 0 \\ 0 & \lambda_2^{-\beta} & \lambda_3^{-\beta} & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}. \quad (1.47)$$

The largest eigenvalue of this matrix coincides with the largest eigenvalue of $\mathcal{L}_{\tilde{\varphi}}$ on \mathcal{F}_{θ} and is of the form

$$\Lambda_{\max} = \frac{1}{2} \left[\lambda_1^{-\beta} + \lambda_3^{-\beta} + \sqrt{(\lambda_1^{-\beta} - \lambda_3^{-\beta})^2 + 4\lambda_2^{-\beta}\lambda_3^{-\beta}} \right], \quad (1.48)$$

so that $P(\tilde{\varphi}) = P(\varphi) = \log \Lambda_{\max}$.

When $k > 2$ in (1.43), a similar analysis yields the transfer operator $\mathcal{L}_{\tilde{\varphi}}$ which, when restricted appropriately to the space of functions constant on the sets I_j , reduces to a $2^k \times 2^k$ matrix with entries $\lambda_1^{-\beta}$, $\lambda_2^{-\beta}$ and $\lambda_3^{-\beta}$. Consequently, the pressure $P(\varphi)$ of the system is given by

$$P(\varphi) = P(\beta) = \log \Lambda_{\max},$$

where Λ_{\max} is the largest eigenvalue of the transfer matrix \mathcal{L}_{φ} .

Similarly, if $c \in J$ but $f^k(c) = 0$ for some k (chosen to be the smallest as before), the situation is essentially the same as the one discussed above, because the discontinuity of $\varphi(x) = -\beta \log |f'(x)|$ occurs at one of the endpoints of the intervals $I_{j_1 \dots j_k}$; $\varphi(x)$ can then be continued to this one point with no influence on the final evaluation of $P(\beta)$.

Finally, suppose that $c \in J$ and $f^k(c) \neq 0$ for all k . The natural choice of Markov partition for f is $\{I_0, I_1\}$, where I_k is the domain of f_k . However, with this choice we inevitably have

$$f_0 \notin \mathcal{C}^{1+\alpha}(I_0),$$

because f'_0 is a “step” function, jumping by $\delta = \lambda_2 - \lambda_1$ at $x = c$. Consequently

$$\varphi(x) = -\beta \log |f'(x)| \notin \mathcal{C}^\alpha. \quad (1.49)$$

Going to the symbolic dynamics for (f, J) we get $\Omega_T^+ = \{0, 1\}^{\mathbb{Z}_+}$; suppose that $\underline{c} = \pi^{-1}(c)$. Then, as in (1.49), we observe that the variation of $\tilde{\varphi} = \varphi \circ \pi$ on $U_n(\underline{c})$ for all n is $|\log \lambda_1 / \lambda_2|$ and therefore

$$\text{var}_n \tilde{\varphi} = \text{const} \not\rightarrow 0.$$

Consequently, $\tilde{\varphi}$ is not of class \mathcal{F}_θ for all $0 < \theta < 1$; one cannot apply the transfer operator method in this setting (cf. [Bow1], [Maye] or [Rue1]).

The proper way to tackle the above-mentioned difficulty becomes apparent when we compare the latter two cases discussed: a Markov partition $\{I_j\}_{j=0}^r$ is to be chosen so that c is a common endpoint of two I_j 's, say

$$\{c\} = I_p \cap I_{p+1}. \quad (1.50)$$

The purpose is to have $\varphi(x)$ behave like a \mathcal{C}^α class function on each of the sets I_j (in our case, $\varphi(x)$ will be constant on I_j 's). However, for a Markov partition it is also required that each I_j be mapped by f onto a union of I_k 's: the property that makes the symbolic dynamics (σ, Ω_T^+) into a subshift of finite type. Therefore, $f(c)$ must be an endpoint of some I_k and so must be $f^2(c), f^3(c), \dots$ etc. From the finiteness of Markov partitions it follows now that the sequence

$$c, f(c), f^2(c), \dots$$

has to be eventually periodic (this, of course, includes the previously discussed case of $c \in J$, $f^k(c) = 0$).

To sum up, the partition to be chosen must be precisely the subdivision of $[0, a]$ and $[b, 1]$ by the points $c, f(c), \dots, f^m(c)$, where $m \geq 0$ is the minimal integer such that $f^{m+1}(c) \in \{0, a, b, 1, c, f(c), \dots, f^m(c)\}$.

As for the evaluation of $P(\beta)$, we observe that $\varphi(x)$ is constant on the intervals I_k (except for I_p or I_{p+1} , where the discontinuity occurs exactly at c ,

i.e. at one of the endpoints, and hence it can be removed as before). Again, the operator $\mathcal{L} = \mathcal{L}_\varphi$ reduces to an $(r+1) \times (r+1)$ matrix when restricted to the subspace of functions constant on the sets I_k . Let $\alpha_j = f'(x)$ for $x \in I_j$.

We have

$$\mathcal{L}_{ij} = \begin{cases} |\alpha_j|^{-\beta} & \text{if } I_i \subseteq fI_j \\ 0 & \text{otherwise,} \end{cases} \quad (1.51)$$

and as before

$$P(\beta) = \log \Lambda_{\max}(\mathcal{L}).$$

The above method fails however if c is not eventually periodic: one cannot construct a Markov partition with $\{c\} = I_p \cap I_{p+1}$, because the orbit of c visits infinitely many points, and consequently (σ, Ω) will not be a subshift of finite type.

The use of the thermodynamic formalism depends critically on the assumption that $\varphi(x) \in \mathcal{C}^\alpha$ on the elements of the cover of X . Thus, in the example discussed here, (1.41)–(1.42), the domain $[0, a]$ has to be subdivided into $[0, c] \cup [c, a]$ as a prerequisite for the application of the transfer operator method. One might hope that even if the orbit of c does not define a finite cover of X , namely its Markov partition, a different finite, but not necessarily Markov, partition could be used, making the \mathcal{L} operator method work anyway. We argue that it is impossible.

Thus let $\{I_j\}_{j=0}^r$ be such a partition. The operator $\mathcal{L} = \mathcal{L}_\varphi$ acting on the space of functions $h = [h_0, \dots, h_r]^\dagger$, $h_j \in C^\alpha(I_j)$, has the general form

$$(\mathcal{L}h)(x) = \sum_{y \in f^{-1}\{x\}} e^{\varphi(y)} h_{i(y)}(y), \quad (1.52)$$

where $i(y)$ is the index j such that $y \in I_j$. Let $m \geq 0$ be the largest integer such that $\{f^m(c)\} = I_p \cap I_{p+1}$ for some p ; moreover, suppose that $f^{m+1}(c) \in \text{int}I_q$.

First, consider the case $m = 0$. We evaluate $\mathcal{L}\mathbf{1}$, where $\mathbf{1}$ denotes the constant function $h_j = 1$, $j = 0, \dots, r$. We immediately obtain

$$[\mathcal{L}\mathbf{1}]_q \notin C^\alpha(I_q)$$

since, as it can be easily computed from (1.52), for $x \in I_q$

$$[\mathcal{L}\mathbf{1}]_q(x) = \begin{cases} \lambda_1^{-\beta} + \lambda_3^{-\beta} & x \leq f(c) \\ \lambda_2^{-\beta} + \lambda_3^{-\beta} & x > f(c). \end{cases} \quad (1.53)$$

On the other hand, if $m > 0$, then by taking $h = [h_0, \dots, h_r]^\dagger$ with

$$h_j = \begin{cases} \eta & j = p \\ \rho & j = p + 1 \\ 1 & \text{otherwise,} \end{cases} \quad (1.54)$$

we obtain (up to the exchange of the inequalities on the right)

$$[\mathcal{L}h]_q(x) = \begin{cases} \lambda_i^{-\beta} + \lambda_j^{-\beta} \eta & x \leq f^{m+1}(c) \\ \lambda_i^{-\beta} + \lambda_j^{-\beta} \rho & x > f^{m+1}(c), \end{cases} \quad (1.55)$$

where $i, j \in \{1, 2, 3\}$, $i \neq j$. Consequently,

$$[\mathcal{L}h]_q \in \mathcal{C}^\alpha(I_q)$$

only if $\eta = \rho$ or, more generally, if $h_p(x)$ and $h_{p+1}(x)$ can be “glued” in \mathcal{C}^α fashion over $I_p \cup I_{p+1}$; therefore, we can disregard the point $f^m(c)$ and work with a coarser partition, where $I_{p'}$ replaces $I_p \cup I_{p+1}$. Repeating this argument, we reduce the analysis to the case $m = 0$. From this we see that the operator \mathcal{L} , as defined above, does not preserve continuity; it differs substantially from the classical Ruelle–Araki transfer operator and Theorem 1.8 cannot be applied here.

One can expect, however, that small changes in c do not affect the pressure $P(\beta)$ too dramatically. It is thus conceivable that by choosing an eventually periodic point $c^{(n)}$ close to c (we observe that eventually periodic points are dense in J), one could approximate $P(\beta)$ with $P_n(\beta)$ — the pressure function obtained as above, when one uses the Markov partition generated by the orbit of $c^{(n)}$. It can be anticipated that when choosing $c^{(n)} \rightarrow c$ as $n \rightarrow \infty$, the corresponding sequence $P_n(\beta)$ should have some reasonable convergence properties.

1.5 The pressure function

In the previous sections we reviewed general facts and results on the dynamics generated by expanding maps and we signaled the importance of thermodynamic formalism in the study of such dynamics. Presently, we shall focus attention on the function $P(\varphi)$, which for a specific choice of φ , can be considered a “generating function” for numerous important indices characterizing the system (f, X) from both geometric and dynamical perspectives. These indices include in particular Lyapunov exponents, Renyi entropies and spectra of fractal dimensions for specific ergodic measures, [Bess], [Coll], [Hasl], [Rand], [Tél].

Assuming M to be a smooth Riemannian manifold and f to be at least of class $\mathcal{C}^{1+\alpha}$, we fix φ to be

$$\varphi(x) = -\beta \log \|T_x f\|, \quad (1.56)$$

where $T_x f$ is the tangent map of f at x and β is a parameter. Due to the continuity of the operator norm $\|T_x f\|$ with respect to x (M is Riemannian) and the fact that f is expanding, $\varphi(x)$ is uniformly bounded on X . Also, since $f \in \mathcal{C}^{1+\alpha}$, φ is Hölder-continuous.

This particular choice of φ is motivated in the following way. From Vari-

ational Principle 1.7 one can see that

$$P(\beta) = P(\varphi) = h_f(\mu_\beta) - \beta \int_X \log \|T_x f\| d\mu_\beta(x), \quad (1.57)$$

where μ_β is the unique Gibbs state for φ — an f -ergodic measure supported on X . Applying Birkhoff's Ergodic Theorem, one can thus write for the integral above

$$\begin{aligned} \int_X \log \|T_x f\| d\mu_\beta &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \log \|T_{f^k(x)} f\| \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \log \left(\|T_{f^{N-1}(x)} f\| \cdot \dots \cdot \|T_{f(x)} f\| \|T_x f\| \right) \end{aligned} \quad (1.58)$$

for μ_β -almost all x . If f is a conformal map, i.e. $T_x f$ equals a scalar times an isometry with respect to the Riemann metric, the last expression in (1.58) can be written as

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \|T_x f^N\| \quad (1.59)$$

by using conformality and the chain rule. In turn, (1.59) is the unique Lyapunov exponent of μ_β , denoted $\lambda(\mu_\beta)$. Thus we have

$$P(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in \text{Fix} f^n} \|T_x f^n\|^{-\beta} = h_f(\mu_\beta) - \beta \lambda(\mu_\beta). \quad (1.60)$$

The conformal case received extensive attention from many authors and we shall review some important results here. On the other hand, the non-conformal case has not been studied very deeply; the pioneering paper of Falconer, [Falc], is an exception.

One of the most elegant results originating from the thermodynamic formalism is the celebrated Bowen's formula for the Hausdorff dimension of conformal repellers, [Bow1], [Rue3].

Theorem 1.9 *If $f \in \mathcal{C}^{1+\alpha}(X)$ is a conformal expanding map, then the Hausdorff dimension of X , β_H , is determined by*

$$P(\beta_H) = 0. \tag{1.61}$$

Moreover, μ_{β_H} is equivalent to the corresponding Hausdorff measure on X .

μ_{β_H} is often called the uniform measure; the origin of this terminology will be explained later. The above theorem shows that one can extract purely geometric information about the dynamics (f, X) from $P(\beta)$, i.e. the Hausdorff dimension of X .

It is an easy consequence of (1.61) and the implicit function theorem that, when analytic dependence of f on a parameter (or parameters) ε induces analytic dependence of P on ε , then the Hausdorff dimension of the corresponding repeller X_ε is also analytic in ε . This has been established by Ruelle, [Rue3], where the author used a zeta function techniques to derive the required analyticity of P in ε . In particular, when one considers the

dynamics generated by a rational map

$$f(z) = \frac{R(z)}{Q(z)}$$

on the Riemann sphere \mathbb{C}_∞ , where R, Q are polynomials such that

$$\max\{\deg R, \deg Q\} > 1,$$

then under the condition that the Julia set J_f is repelling³, i.e.

$$\exists c > 0 \quad \exists \theta > 1 \quad \forall x \in J_f \quad \|T_x f^n\| \geq c \theta^n \quad (1.62)$$

for all $n > 0$, the Hausdorff dimension of J_f depends analytically on the coefficients of R and Q .

Two other important dynamical indices can be immediately retrieved from $P(\beta)$. Thus setting $\beta = 0$, one obtains from the variational principle

$$P(0) = \sup_{\varrho} h_f(\varrho). \quad (1.63)$$

This supremum is attained for $\varrho = \mu_0$ — the invariant measure with maximal entropy. It is easy to picture this measure in the context of symbolic dynamics: $\mu_0(U) = 1/N$ for all N cylinders U of the same length. $P(0) = h_f(\mu_0)$ is called the topological entropy, [Rue1], and it is the upper bound on the

³This, in particular, means that J_f is isolated from the critical points of f .

metric entropies of all $\mu \in \mathcal{M}_f(X)$. It characterizes, roughly speaking, the speed of mixing of points in X by the action of f . The maximal entropy measure is also called the balanced measure. It should be stressed that even if μ_0 is easily constructed in terms of symbolic dynamics, in general it has a very complicated singular and multifractal character when looked upon as a measure on $X \subseteq M$, [Bess].

When $\beta = d$, the dimension of M , $P(d)$ characterizes the so-called escape rate from X , another important dynamical index, [Kada], [Vai1]. In the conformal case, the escape rate

$$\log r = -P(d) \tag{1.64}$$

has the following intuitive interpretation.

Suppose a small neighbourhood U of X and a number $N \gg 1$ of points distributed uniformly in U are given. Let N_k denote the number of those points that remain in U after k iterations of f . Then, for large k , one has the following asymptotic scaling behaviour

$$\frac{N_k}{N} \sim r^{-k}.$$

Due to the analogy with Axiom-A attractors, the corresponding Gibbs state μ_d is called the Sinai–Ruelle–Bowen (or simply SRB) measure, [Bohr].

An important class of quantities, the so-called generalized dimension spectrum of an invariant measure, was introduced by Hentschel and Procaccia in [Hent], (cf. also [Coll], [Gra1], [Gra2], [Hasl], [Rand], [Vai2]). Given an f -invariant probability measure μ on X and a measurable partition $\mathcal{B} = \{B_i\}_{i=1}^r$ of X , one studies the asymptotic behaviour of

$$\sum_{\mathcal{B}} [\mu(B_i)]^q \quad q \in \mathbf{R}, \quad (1.65)$$

when $\text{diam } \mathcal{B} = \delta \rightarrow 0$. One is interested in characterizing the quantities $D_\mu(q)$ defined by the asymptotic relation

$$\sum_{\mathcal{B}} [\mu(B_i)]^q \sim \delta^{(q-1)D_\mu(q)}. \quad (1.66)$$

For $q = 0$, one verifies that $D_\mu(0)$ equals the Hausdorff dimension of the support of μ . Letting $q \rightarrow 1$, the so-called information dimension of μ , denoted $D_\mu(1)$, is recovered. Taking log of both sides of (1.66) and taking the limit as $q \rightarrow 1$ yields the following relation:

$$-\sum_{\mathcal{B}} \mu(B_i) \log \mu(B_i) \sim D_\mu(1) \log \frac{1}{\delta}, \quad (1.67)$$

that is, the scaling law for the amount of information necessary to specify the position of a point in X with accuracy δ as $\delta \rightarrow 0$, all with respect to the measure μ .

Substituting $q = 2$ in (1.66), we obtain $D_\mu(2)$, the correlation dimension of μ , introduced in [Gra2], [Gra3] as an important indicator of underlying determinism in the analysis of chaotic time series. In this case, (1.66) gives rise to

$$\sum_{\mathcal{B}} [\mu(B_i)]^2 \sim \delta^{D_\mu(2)}; \quad (1.68)$$

thus $D_\mu(2)$ is the scaling exponent for spatial correlations, i.e. it says how the probability of two μ -random points on X falling into the same partition box scales with the diameter of the partition, (cf. also [Bess]).

It is an important result of Vaienti, [Vai1], to relate the spectrum $D_\mu(q)$ for a Gibbs state $\mu = \mu_\beta$ to the pressure $P(\beta)$ in the case of conformal dynamics. The relation in question reads

$$P(\beta q - (q - 1)D_{\mu_\beta}(q)) = qP(\beta). \quad (1.69)$$

We note that for $q = 0$ we recover Bowen's formula

$$P(D_{\mu_\beta}(0)) = 0. \quad (1.70)$$

Also, setting $\beta = \beta_H$ turns (1.69) into

$$P(\beta_H q - (q - 1)D_{\mu_{\beta_H}}(q)) = 0$$

and hence

$$\beta_H q - (q - 1)D_{\mu_{\beta_H}}(q) = \beta_H$$

or, more explicitly

$$D_{\mu_{\beta_H}}(q) = \beta_H. \quad (1.71)$$

Consequently, all the generalized dimensions of μ_{β_H} coincide; for this reason μ_{β_H} has been called the uniform measure.

For the balanced measure μ_0 we obtain from (1.69) by substituting $\beta = 0$

$$P((1-q)D_{\mu_0}(q)) = qP(0) = qh_{\text{top}}.$$

As we shall see later when studying the example of the map $f(z) = z^p - c$, unlike $D_{\mu_{\beta_H}}(q)$, the spectrum $D_{\mu_0}(q)$ is in general a nontrivial one. The measure μ_0 has a multifractal nature, [Tél].

Renyi entropies of an invariant measure μ are defined as thermodynamic limits

$$(1-q)h_\mu(q) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\mathcal{B}^{(n)}} [\mu(B_i^{(n)})]^q, \quad (1.72)$$

where the partitions $\mathcal{B}^{(n)}$, (1.26), arise from a filtering partition \mathcal{B} . Taking the limit

$$\lim_{q \rightarrow 1} h_\mu(q),$$

one obtains the Kolmogorov–Sinai entropy of μ and the topological entropy is recovered when $q = 0$. Renyi entropy of the second order, $h_\mu(2)$, attracted the attention of some authors, [Gra3], for practical reasons. First, it is much

harder to estimate the Kolmogorov entropy than its lower bound $h_\mu(2)$ from an experimental signal; secondly, $h_\mu(2)$ is infinite for random systems and hence, like $D_\mu(2)$, it provides a useful practical test of randomness vs. determinism in an experimental time series.

Renyi entropies of the equilibrium measures μ_β can be expressed in terms of $P(\beta)$, [Bess]:

$$(1 - q) h_{\mu_\beta}(q) = P(q\beta) - qP(\beta). \quad (1.73)$$

For conformal maps f , the property of $P(\beta)$

$$-\frac{dP}{d\beta} = \lambda(\mu_\beta) \quad (1.74)$$

motivates the definition of the entropy function for Lyapunov exponents, [Bohr]. (1.73) specifies the function $\beta \mapsto \lambda(\beta)$, assigning to each β the unique Lyapunov exponent of the corresponding Gibbs state μ_β . $P(\beta)$ is either linear or strictly convex in β , [Rue1]: in the latter case $\lambda(\beta)$ is invertible and one can find $\lambda \mapsto \beta(\lambda)$. The entropy function $S(\lambda)$ is now defined as

$$S(\lambda) = P(\beta(\lambda)) + \lambda\beta(\lambda), \quad (1.75)$$

i.e. the Legendre transform of $P(\beta)$. We note that, if $P(\beta)$ is linear in β , then $\lambda(\beta) = \text{const}$ and there is no variation in the Lyapunov exponents with respect to β ; consequently, $S(\lambda)$ does not exist.

We define $U_\lambda \subseteq X$ as the set of all those $x \in X$ for which

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \log \|T_{f^k(x)} f\| = \lambda. \quad (1.76)$$

It has been proven, [Bohr], that $S(\lambda)$ is the noncompact topological entropy of the set U_λ , [Bow3]. Moreover, the Hausdorff dimension of U_λ is given by

$$D(\lambda) = \frac{S(\lambda)}{\lambda}. \quad (1.77)$$

In Chapter 3 we study the example of the map $f(z) = z^p - c$ on \mathbf{C}_∞ . We evaluate the pressure function $P(\beta)$ for this map in the form of a perturbation series in powers of the real and imaginary parts of c . From the series we later obtain expansions for dimension spectra, Renyi entropies and the entropy function for Lyapunov exponents.

Chapter 2

Perturbation theory for $P(\beta)$

2.1 Analyticity of transfer operator families

Suppose that the Riemannian manifold M as well as the expanding map f on M are of class C^ω . If f depends real-analytically on a parameter ε , i.e. $f = f_\varepsilon$, where ε varies over some open interval, then one expects that analyticity in ε can be carried over to the corresponding family of transfer operators \mathcal{L}_ε . Then, by virtue of Theorem 1.8 and regular perturbation theory for eigenvalues, [Kato],

$$P_\varepsilon(\beta) = P(-\beta \log \|T_x f_\varepsilon\|) \tag{2.1}$$

would be analytic in ε and, more importantly, one would be able to use standard perturbative techniques to derive a power series expansion for $P_\varepsilon(\beta)$.

A transfer operator is always composed of multiplication and composition parts. Below we demonstrate that under suitable regularity assumptions, operators of this type form analytic families.

Definition 2.1 *Let \mathcal{K}_ε be a family of bounded operators acting between two fixed Banach spaces, where $\varepsilon \in V \subseteq \mathbf{R}$. We say that \mathcal{K}_ε is an analytic family (with respect to ε) iff \mathcal{K}_ε admits a norm-convergent Taylor series expansion around each $\varepsilon_0 \in V$, i.e. iff the series*

$$\sum_{n=0}^{\infty} \frac{(\varepsilon - \varepsilon_0)^n}{n!} \cdot \left. \frac{d^n}{d\varepsilon^n} \mathcal{K}_\varepsilon \right|_{\varepsilon=\varepsilon_0} \quad (2.2)$$

converges in operator norm.

If ε ranges over an open subset V of the complex plane rather than the real line, then analyticity of \mathcal{K}_ε is equivalent to the requirement that $\ell(\mathcal{K}_\varepsilon h)$ be holomorphic in V for all h in the domain of \mathcal{K}_ε and ℓ in the dual of the target space, [Kato].

Theorem 2.1 *Let W and U be two open subsets of an analytic manifold M having compact closures in M . Let $\phi_\varepsilon : W \rightarrow \mathbf{C}$ be an analytic map that is also real-analytic¹ in ε for $|\varepsilon| < \eta_0$, such that $\phi_\varepsilon \in \mathcal{B}(W)$ and $\|\phi_\varepsilon\|$ is bounded*

¹i.e. admitting Taylor series expansion in ε when represented in local coordinates.

uniformly in ε . Also, let $\psi_\varepsilon : W \rightarrow U$ be an analytic map, real-analytic in ε for $|\varepsilon| < \eta_0$, such that $cl\psi_\varepsilon W \subseteq U$ for all ε .

Then the family of operators $\mathcal{K}_\varepsilon : \mathcal{B}(U) \rightarrow \mathcal{B}(W)$ defined by

$$(\mathcal{K}_\varepsilon h)(x) = \phi_\varepsilon(x)h(\psi_\varepsilon(x)) \quad (2.3)$$

is an analytic family for ε ranging over a sufficiently small neighbourhood of 0.

PROOF: The map ϕ_ε can be analytically continued in ε to a complex neighbourhood V of the interval $(-\eta_0, \eta_0)$. Let

$$\eta_1 = \sup\{\rho \leq \eta_0 : |z| < \rho \Rightarrow z \in V\}.$$

Expanding ϕ_ε in powers of ε around $\varepsilon = 0$, one realizes that the norm of ϕ_ε is still bounded uniformly in ε (now complex) for $|\varepsilon| < \eta_1$. Let

$$\sup_{|\varepsilon| < \eta_1} \|\phi_\varepsilon\| = C < \infty. \quad (2.4)$$

For the function ψ_ε we want to ensure that analytic continuation in ε to complex values leaves the image of ψ_ε inside U for all ε . Only then is composition with ψ_ε a legitimate bounded operation on $\mathcal{B}(U)$. However, complexifying ψ_ε in ε may require the complexification of the manifold M itself. This is achieved as follows.

Let $\{(D_\alpha, \zeta_\alpha)\}_{\alpha \in A}$ be a system of local charts on M , i.e.

$$\zeta_\alpha : D_\alpha \longrightarrow E_\alpha \subseteq \mathbf{R}^n.$$

Since W and U are assumed to have compact closures in M , there is a finite system of charts

$$\{(D_i, \zeta_i)\}_{i=1}^p \tag{2.5}$$

covering $\text{cl}(W \cup U)$.

Let $U_i = U \cap D_i$, $i = 1, \dots, p$. Because of the finiteness of the cover (2.5), one can find $\delta > 0$ such that the collection of sets

$$\hat{U}_i = \{x \in U_i : \text{dist}(x, M \setminus D_i) > \delta\}, \tag{2.6}$$

$i = 1, \dots, p$, covers U as $\{U_i\}_{i=1}^p$ does. Let $W_i = W \cap \psi_0^{-1} \hat{U}_i$. Without loss of generality we can assume that

$$\text{cl } \psi_\varepsilon W_i \subseteq U_i \quad \forall \varepsilon \in (-\eta_0, \eta_0) \tag{2.7}$$

because if (2.7) does not hold, we can restrict ε to sufficiently small values.

Suppose that W_i is covered by the charts D_{i_1}, \dots, D_{i_q} , $q = q(i)$. Let $A_{i_j} = \zeta_{i_j} W_i \subseteq E_{i_j}$ and $B_i = \zeta_i U_i \subseteq E_i$. For each of the maps

$$\zeta_i \circ \psi_\varepsilon \circ \zeta_{i_k}^{-1} : A_{i_k} \longrightarrow B_i, \tag{2.8}$$

which we shall denote $\psi_\varepsilon^{(i_k, i)}$, (2.7) implies that for $\varepsilon \in (-\eta_0, \eta_0)$

$$\text{cl } \psi_\varepsilon^{(i_k, i)} A_{i_k} \subseteq B_i \subseteq E_i. \quad (2.9)$$

Consider an arbitrary $h \in \mathcal{B}(U)$ and let

$$h_i = h \circ \zeta_i^{-1} \Big|_{B_i}.$$

Each of the functions h_i , $i = 1, \dots, p$ can be continued analytically to a complex neighbourhood $B_i^{\mathbb{C}}$ of B_i . Let \tilde{h}_i denote the continuation. One can find a positive δ' such that for all $i = 1, \dots, p$

$$\sup_{B_i^{\mathbb{C}}} |\tilde{h}_i| \leq \|h\| + \delta',$$

where $\|h\|$ denotes the norm of h in $\mathcal{B}(U)$.

When the maps $\psi_\varepsilon^{(i_k, i)}$ are continued analytically in ε to a complex neighbourhood of $(-\eta_0, \eta_0)$, one can find $\eta_2^{(i)} > 0$ such that for all complex ε satisfying $|\varepsilon| < \eta_2^{(i)}$, using (2.9) one obtains

$$\text{cl } \tilde{\psi}_\varepsilon^{(i_k, i)} A_{i_k} \subseteq B_i^{\mathbb{C}}, \quad (2.10)$$

where we have written $\tilde{\psi}_\varepsilon^{(i_k, i)}$ for the analytic continuation of $\psi_\varepsilon^{(i_k, i)}$.

For $x \in W_i \cap D_{i_k}$ and real ε one has

$$h(\psi_\varepsilon(x)) = \tilde{h}_i \left(\tilde{\psi}_\varepsilon^{(i_k, i)}(\underline{x}) \right),$$

where $\underline{x} = \zeta_{i_k}(x) \in A_{i_k}$. Due to (2.10), when ε ranges over the disc $|\varepsilon| < \eta_2^{(i)}$, we have

$$\left| \tilde{h}_i \left(\tilde{\psi}_\varepsilon^{(i_k, i)}(\underline{x}) \right) \right| \leq \|h\| + \delta'. \quad (2.11)$$

Then choosing η such that

$$0 < \eta < \min \left\{ \eta_0, \eta_1, \eta_2^{(1)}, \dots, \eta_2^{(p)} \right\},$$

we obtain using Cauchy's formula

$$\begin{aligned} \mathcal{K}_n h &= \frac{1}{n!} \frac{d^n}{d\varepsilon^n} (\mathcal{K}_\varepsilon h) \Big|_{\varepsilon=0} \\ &= \frac{1}{n!} \frac{d^n}{d\varepsilon^n} [\phi_\varepsilon(x) h(\psi_\varepsilon(x))] \Big|_{\varepsilon=0} \\ &= \frac{1}{2\pi i} \int_{|\xi|=\eta} \frac{\phi_\xi(x) \tilde{h}_i \left(\tilde{\psi}_\xi^{(i_k, i)}(\underline{x}) \right)}{\xi^{n+1}} d\xi. \end{aligned} \quad (2.12)$$

Therefore

$$\|\mathcal{K}_n h\| \leq \eta^{-n} C (\|h\| + \delta').$$

Taking the supremum over h with $\|h\| = 1$ we get

$$\|\mathcal{K}_n\| \leq \eta^{-n} C'.$$

Therefore, for $|\varepsilon| < \eta$ the Taylor series

$$\mathcal{K}_\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \mathcal{K}_n. \quad (2.13)$$

converges in operator norm. The proof is complete. \square

We can now specialize the above result to the case of a transfer operator \mathcal{L} for the dynamical system (f, X) . If f depends real-analytically on a parameter, $f = f_\varepsilon$, its invariant set X will also vary, in general, with ε , i.e. $X = X^{(\varepsilon)}$. Let ε be restricted to an open set $V \subseteq \mathbf{R}$, so that $X^{(\varepsilon)} \subseteq U$ for all $\varepsilon \in V$; as before, U is a neighbourhood of X with compact closure in M . As we have mentioned in Section 1.4, the dependence of the set X on ε can make the spectral analysis of $\mathcal{L}_\varepsilon : \mathcal{C}^\alpha(X) \rightarrow \mathcal{C}^\alpha(X)$ very complicated, or even impossible. This is why one wants to define \mathcal{L}_ε acting on the space $\mathcal{B}(U)$, rather than mimicking the definition of the classical transfer operator on $\mathcal{F}_\theta(\Omega_T^\pm)$, here acting on $\mathcal{C}^\alpha(X)$. Due to the fact that f_ε is expanding, U itself can be chosen so that $\text{cl } f_\varepsilon^{-1}U \subseteq U$, [Rue2], for all ε sufficiently small.

Let $\{X_i^{(\varepsilon)}\}_{i=0}^r$ be a Markov partition for $(f_\varepsilon, X^{(\varepsilon)})$, which is sufficiently fine for f_ε to be invertible in a neighbourhood of $X_i^{(\varepsilon)}$ for all i . Suppose also that the transition matrix $T = [t_{ij}]$ does not depend on ε . Choosing open sets $U_i \supseteq X_i^{(\varepsilon)}$ for $\varepsilon \in V$ so that

$$\bigcup_{i=0}^r U_i = U,$$

we have

$$\psi_{ij}^{(\varepsilon)} = f_\varepsilon^{-1} : U_j \longrightarrow U_i \quad (2.14)$$

and $\text{cl } \psi_{ij}^{(\varepsilon)} U_j \subseteq U_i$ whenever $f_\varepsilon X_i^{(\varepsilon)} \supseteq X_j^{(\varepsilon)}$.

Let $\varphi_\varepsilon \in \mathcal{B}(U)$ depend analytically on ε . Given a pair of indices i and j such that $t_{ij} = 1$, we identify the sets U and W in Theorem 2.1 with U_i and U_j respectively and we put

$$\begin{aligned} \phi_\varepsilon &= \varphi_\varepsilon \circ \psi_{ij}^{(\varepsilon)}, \\ \psi_\varepsilon &= \psi_{ij}^{(\varepsilon)}. \end{aligned}$$

Theorem 2.1 now yields the analyticity of the operator family $\mathcal{L}_{ij}^{(\varepsilon)} : \mathcal{B}(U_i) \rightarrow \mathcal{B}(U_j)$ defined by

$$\left(\mathcal{L}_{ij}^{(\varepsilon)} h\right)(x) = e^{\varphi_\varepsilon(\psi_{ij}^{(\varepsilon)}(x))} h(\psi_{ij}^{(\varepsilon)}(x)) \quad \text{for } x \in U_j. \quad (2.15)$$

Therefore, we can state the following.

Corollary 2.1 *The transfer operator family $\mathcal{L}_\varepsilon : \mathcal{B}(U) \rightarrow \mathcal{B}(U)$, defined in terms of $\mathcal{L}_{ij}^{(\varepsilon)}$ by*

$$(\mathcal{L}_\varepsilon h)(x) = \sum_{i: t_{ij}=1} \left(\mathcal{L}_{ij}^{(\varepsilon)} h\right)(x) \quad \text{for } x \in U_j, \quad (2.16)$$

is an analytic family.

2.2 Power series expansion for $P_\varepsilon(\beta)$

Suppose that an analytic family of transfer operators

$$\mathcal{L}_\varepsilon = \mathcal{L}_{\varphi_\varepsilon},$$

on $\mathcal{B}(U)$ is given with $\varphi_\varepsilon \in \mathcal{B}(U)$. If φ_ε is real valued, then by Theorem 1.8 $\exp P(\varphi_\varepsilon)$ is a simple isolated eigenvalue of \mathcal{L}_ε . By standard theorems of analytic perturbation theory for eigenvalues, [Kato], one can now obtain the analyticity of the eigenvalue and, subsequently, the analyticity of the pressure $P(\varphi_\varepsilon)$ with respect to ε . Similarly, studying the adjoint family $\mathcal{L}_\varepsilon^*$, which, like \mathcal{L}_ε , is analytic [Kato], one immediately concludes that the Gibbs state $\mu_\beta^{(\varepsilon)}$ is also analytic in ε in the sense of weak-* topology on $\mathcal{B}(U)^*$, (cf. [Rue3]). In particular, for (f_ε, M) of \mathcal{C}^ω class with f_ε expanding on a neighbourhood U of X for all admissible values of ε , the operator family

$$(\mathcal{L}_\varepsilon h)(x) = \sum_{y \in f_\varepsilon^{-1}\{x\}} \|T_y f_\varepsilon\|^{-\beta} h(y) \quad (2.17)$$

(i.e. $\varphi_\varepsilon(x) = -\beta \log \|T_x f_\varepsilon\|$) is analytic in ε and so is

$$P_\varepsilon(\beta) = P(-\beta \log \|T_x f_\varepsilon\|). \quad (2.18)$$

Now our goal is to find a power series expansion for $P_\varepsilon(\beta)$ around $\varepsilon = 0$,

$$P_\varepsilon(\beta) = P^{(0)}(\beta) + \varepsilon P^{(1)}(\beta) + \varepsilon^2 P^{(2)}(\beta) + \dots, \quad (2.19)$$

where $P^{(0)}(\beta) = P_0(\beta)$ arises as the logarithm of the eigenvalue of the unperturbed operator \mathcal{L}_0 .

The present section is devoted to the discussion of general aspects of the development of the perturbation series for $P_\varepsilon(\beta)$.

Suppose that the characteristic equation

$$\mathcal{L}_\varepsilon h_\varepsilon = \Lambda_\varepsilon h_\varepsilon \quad (2.20)$$

with $h_\varepsilon \in \mathcal{B}(U)$ is satisfied for all $|\varepsilon| < \eta$. Since \mathcal{L}_ε is an analytic family and Λ_0 is the isolated eigenvalue $\exp P(\beta)$ for the unperturbed system, we conclude analyticity of both Λ_ε and h_ε with respect to ε , [Kato]. Thus, there exist Taylor series expansions

$$\mathcal{L}_\varepsilon = \mathcal{L}_0 + \varepsilon \mathcal{L}_1 + \frac{\varepsilon^2}{2} \mathcal{L}_2 + \cdots \quad (2.21)$$

$$h_\varepsilon = h_0 + \varepsilon h_1 + \frac{\varepsilon^2}{2} h_2 + \cdots \quad (2.22)$$

$$\Lambda_\varepsilon = \Lambda_0 + \varepsilon \Lambda_1 + \frac{\varepsilon^2}{2} \Lambda_2 + \cdots \quad (2.23)$$

around $\varepsilon = 0$. Substituting the above in (2.20) and equating terms of the same order in ε , we obtain the following system of equations:

$$\mathcal{L}_0 h_0 = \Lambda_0 h_0 \quad (2.24)$$

$$\mathcal{L}_1 h_0 + \mathcal{L}_0 h_1 = \Lambda_1 h_0 + \Lambda_0 h_1 \quad (2.25)$$

$$\frac{1}{2} \mathcal{L}_2 h_0 + \mathcal{L}_1 h_1 + \frac{1}{2} \mathcal{L}_0 h_2 = \frac{1}{2} \Lambda_2 h_0 + \Lambda_1 h_1 + \frac{1}{2} \Lambda_0 h_2 \quad (2.26)$$

...

Let μ be the eigenvector of \mathcal{L}_0^* corresponding to the eigenvalue Λ_0 , i.e.

$$\mathcal{L}_0^* \mu = e^{P_0(\beta)} \mu, \quad (2.27)$$

or in other words,

$$\mu(\mathcal{L}_0 g) = \Lambda_0 \mu(g) \quad \forall g \in \mathcal{B}(U).$$

Moreover, we assume that the following normalization condition holds:

$$\mu(h_0) = 1.$$

Applying μ on both sides of the equations (2.24), (2.25), ... and combining like terms, one obtains

$$\Lambda_1 = \mu(\mathcal{L}_1 h_0) \quad (2.28)$$

$$\Lambda_2 = \mu(\mathcal{L}_2 h_0) + 2\mu(\mathcal{L}_1 h_1) - 2\Lambda_1 \mu(h_1) \quad (2.29)$$

$$\Lambda_3 = \mu(\mathcal{L}_3 h_0) + 3\mu(\mathcal{L}_2 h_1) + 3\mu(\mathcal{L}_1 h_2) \quad (2.30)$$

$$-3\Lambda_2 \mu(h_1) - 3\Lambda_1 \mu(h_2)$$

...

where the vectors h_1 and h_2 are determined by the equations (2.25), (2.26),

or more explicitly

$$[\Lambda_0 - \mathcal{L}_0] h_1 = [\mathcal{L}_1 - \Lambda_1] h_0 \quad (2.31)$$

$$[\Lambda_0 - \mathcal{L}_0]h_2 = [\mathcal{L}_2 - \Lambda_2]h_0 + 2[\mathcal{L}_1 - \Lambda_1]h_1 \quad (2.32)$$

...

Consequently, when Λ_1 is determined from (2.28), it fixes the right-hand side of (2.31), which can be solved for h_1 . In turn, once h_1 is known, Λ_2 can be evaluated from (2.29) and substituted along with h_1 in (2.32) to determine h_2 and so on.

Denoting $\mathcal{K} = \Lambda_0^{-1}\mathcal{L}_0$ we shall study the general equation

$$[1 - \mathcal{K}]h = g \quad (2.33)$$

on $\mathcal{B}(U)$. We obviously have $\|\mathcal{K}\| = 1$, since Λ_0 equals the spectral radius of \mathcal{L}_0 in our case.

Because $\text{Span}\{h_0\}$ is the kernel of the operator $1 - \mathcal{K}$, we can assume that h in (2.33) satisfies

$$\mu(h) = 0,$$

i.e. that h has no component in $\text{Span}\{h_0\}$. Moreover, we necessarily have $\mu(g) = 0$. Thus, it is sufficient to consider the action of \mathcal{K} restricted to the invariant subspace $\{\mu\}^\perp$, the complement of $\text{Span}\{h_0\}$ in $\mathcal{B}(U)$. In what follows, we use the same symbol \mathcal{K} for this restriction.

The original version of the Ruelle–Perron–Frobenius Theorem (Theorem 1.5), [Pol1], [Pol2], stresses the fact that except for the eigenvalue Λ_0 , the rest

of the spectrum of the transfer operator lies strictly inside a disc of radius $\rho = \Lambda_0 - \delta$ for some $\delta > 0$. This is because all the eigenvalues in question are isolated. In fact, \mathcal{L} acting on $\mathcal{B}(U)$ admits much stronger statements about its spectrum, for it is a nuclear operator, [Maye]. It follows that the operator \mathcal{K} on $\{\mu\}^\perp$ satisfies $\|\mathcal{K}\| < 1$. Therefore, $1 - \mathcal{K}$ in (2.33) has a bounded inverse and

$$[1 - \mathcal{K}]^{-1} = 1 + \mathcal{K} + \mathcal{K}^2 + \dots \quad (2.34)$$

Thus, we have the general solution to equations of (2.33) form:

$$h = \sum_{n=0}^{\infty} \mathcal{K}^n g. \quad (2.35)$$

With the above comments in mind, we can rewrite equations (2.29) through (2.32) in a simplified form:

$$\Lambda_2 = \mu(\mathcal{L}_2 h_0) + 2\mu(\mathcal{L}_1 h_1) \quad (2.36)$$

$$\Lambda_3 = \mu(\mathcal{L}_3 h_0) + 3\mu(\mathcal{L}_2 h_1) + 3\mu(\mathcal{L}_1 h_2) \quad (2.37)$$

...

as well as

$$h_1 = \Lambda_0^{-1} [1 - \mathcal{K}]^{-1} [\mathcal{L}_1 - \Lambda_1] h_0 \quad (2.38)$$

$$h_2 = \Lambda_0^{-1} [1 - \mathcal{K}]^{-1} ([\mathcal{L}_2 - \Lambda_2] h_0 + 2[\mathcal{L}_1 - \Lambda_1] h_1) \quad (2.39)$$

...

One can obtain corrections to Λ_ε of higher orders in ε systematically, in a similar way as above. However, the computations become more involved and, in practice, they can be carried with the use of a symbolic manipulator.

Chapter 3

The pressure function for the map $z \mapsto z^p - c$

3.1 Development of the perturbation series

In the present section we consider an analytic map $f : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ given by

$$f(z) = f_c(z) = z^p - c, \quad (3.1)$$

where $p > 1$ is an integer and $c \in \mathbb{C}$. Assuming $|c|$ to be small, we will solve the perturbation problem for the eigenvalue $\exp P_0(\beta)$ of the transfer operator \mathcal{L}_0 corresponding to the unperturbed map $f_0(z) = z^p$. Working in

polar coordinates

$$z \mapsto (R, \varphi) \quad (3.2)$$

$$c \mapsto (\varepsilon, \alpha), \quad (3.3)$$

we will compute $\exp P(\beta)$ through 5th order in ε .

It should be noted that the eigenvalue $\exp P_c(\beta)$ (and hence $P_c(\beta)$ itself) is real-analytic in 2 variables: the real and imaginary parts of c . Sines and cosines of α occur as factors in the terms of the perturbation series in ε , which we construct below. Thus, each term of the series is a binomial in $\operatorname{Re} c$ and $\operatorname{Im} c$ but, to state it explicitly, $\exp P_c(\beta)$ is not holomorphic in c .

We begin the construction of the transfer operator \mathcal{L}_ε by finding the analytic branches of f_c^{-1} . These are the maps

$$z \mapsto \sqrt[p]{z + c}$$

defined in the annular sectors

$$D_k = (R_1, R_2) \times \left(\frac{2k\pi}{p} - \eta, \frac{2(k+1)\pi}{p} + \eta \right), \quad (3.4)$$

where $k = 0, \dots, p-1$. We assume that

$$p^{\frac{1}{1-p}} < R_1 < 1 < R_2 < \infty, \quad (3.5)$$

and that η is a small positive number. Having introduced the notation

$$\tilde{R} = |z + c| = \sqrt{R^2 + \varepsilon^2 + 2R\varepsilon \cos(\varphi - \alpha)} \quad (3.6)$$

$$\tilde{\varphi} = \text{Arg}(z + c) = \varphi - \arcsin \frac{\varepsilon \sin(\varphi - \alpha)}{\tilde{R}}, \quad (3.7)$$

we express the inverse branches of f in the following form:

$$\begin{aligned} \Psi_{kl} : D_l &\longrightarrow D_k, \\ \Psi_{kl} : \begin{bmatrix} R \\ \varphi \end{bmatrix} &\longmapsto \begin{bmatrix} \tilde{R}^{1/p} \\ \frac{\tilde{\varphi} + 2k\pi}{p} \end{bmatrix}, \end{aligned} \quad (3.8)$$

where $k, l = 0, \dots, p-1$. Moreover, writing $D = \bigcup D_i$, we define $\Phi : D \rightarrow \mathbf{R}$ by

$$\begin{aligned} \Phi(R) &= -\beta \log |f'_c(R, \varphi)| \\ &= -\beta \log(pR^{p-1}) \end{aligned} \quad (3.9)$$

We notice that both Ψ_{kl} and Φ are real-analytic in the corresponding sets D_l and continuous on $\text{cl } D_l$. One can easily see that with a careful choice of the constants R_1 , R_2 and η one has $\text{cl}(\Psi_{kl}D_l) \subseteq D_k$. In addition, Ψ_{kl} are analytic in $\varepsilon \sin \alpha$ and $\varepsilon \cos \alpha$ for ε sufficiently small.

The transfer operator $\mathcal{L}_\varepsilon : \mathcal{B}(D) \rightarrow \mathcal{B}(D)$ is specified by

$$(\mathcal{L}_\varepsilon h)(R, \varphi) = \sum_{k=0}^{p-1} \exp \Phi(\Psi_{kl}(R, \varphi)) h(\Psi_{kl}(R, \varphi))$$

$$\begin{aligned}
&= \exp \Phi(\tilde{R}^{1/p}) \cdot \sum_{k=0}^{p-1} h(\Psi_{kl}(R, \varphi)) \\
&= \left[p \tilde{R}^{\frac{1-p}{p}} \right]^{-\beta} \cdot \sum_{k=0}^{p-1} h(\tilde{R}^{1/p}, (\tilde{\varphi} + 2k\pi)/p)
\end{aligned} \tag{3.10}$$

for $(R, \varphi) \in D_l$, $l = 0, \dots, p-1$. As it has been shown in Section 2.1, \mathcal{L}_ε is an analytic family.

For $\varepsilon = 0$ the unperturbed operator \mathcal{L}_0 acting on the space $\mathcal{C}^{1+\delta}(J_0)$, where J_0 is the unit circle, has a simple eigenvalue $\Lambda_0 = p^{1-\beta}$ dominating the rest of the spectrum with the corresponding eigenvector $h_0 \equiv 1$. The corresponding eigenvector of the dual action \mathcal{L}_0^* is μ , the normalized Lebesgue measure on J_0 . The eigenvector h_0 lies in the invariant subspace of \mathcal{L}_0 consisting of isotropic functions, i.e. those dependent on R only. This suggests the general form of the eigenvector h_0 of the operator \mathcal{L}_0 acting on $\mathcal{B}(D)$, namely $h_0(R, \varphi) = h_0(R)$. Indeed, assuming that

$$h_0(R) = R^\gamma,$$

we find

$$\begin{aligned}
(\mathcal{L}_0 h_0)(R) &= \left[p R^{\frac{p-1}{p}} \right]^{-\beta} \sum_{k=0}^{p-1} h_0(R^{1/p}) \\
&= p^{-\beta} R^{\beta \frac{1-p}{p}} \cdot p R^{\gamma/p} \\
&= p^{1-\beta} R^{\frac{\beta - \beta p + \gamma}{p}}.
\end{aligned} \tag{3.11}$$

By equating γ and $\frac{\beta - \beta p + \gamma}{p}$, we find $\gamma = -\beta$ and therefore

$$h_0(R) = R^{-\beta}. \quad (3.12)$$

Obviously, for the above mentioned eigenvector μ , we also have $\mu \in \mathcal{B}^*(D)$, $\mathcal{L}_0^* \mu = p^{1-\beta} \mu$ on $\mathcal{B}^*(D)$. Therefore, Theorem 1.8 ensures that $h_0(R) = R^{-\beta}$ and μ — the normalized Lebesgue measure on the unit circle are the eigenvectors we have been looking for.

The formulas (2.28), (2.36), ..., express the corrections $\Lambda_1, \Lambda_2, \dots$ to the eigenvalue $\Lambda_\varepsilon = \exp P_c(\beta)$ in terms of various expectation values of the general form $\mu(\mathcal{L}_k h_j)$. For $h \in \mathcal{B}(D)$ $\mathcal{L}_k h$ is obtained by the appropriate differentiation of $\mathcal{L}_\varepsilon h$, i.e.

$$\mathcal{L}_k h = \left. \frac{d^k}{d\varepsilon^k} \mathcal{L}_\varepsilon h \right|_{\varepsilon=0}. \quad (3.13)$$

As h will be superposed with Ψ_{kl} 's dependent on ε (3.10), the chain rule must be applied in (3.13), so that the last expression will involve partial derivatives of h . In spite of the fact that differentiation on $\mathcal{B}(D)$ is not a bounded operation (loss of continuity on the boundary ∂D may easily result), the derivative in (3.13) is well defined, precisely because of the composition of h with Ψ_{kl} 's. Since Ψ_{kl} 's are contractions on D , they map strictly into the interior of D , away from the boundary, where all the derivatives of h exist.

We are now prepared to compute the subsequent corrections $\Lambda_1, \Lambda_2, \dots$ to the eigenvalue Λ_ε . As a preparatory step, we evaluate

$$\left. \frac{d\tilde{R}}{d\varepsilon} \right|_{\varepsilon=0} = \cos(\varphi - \alpha) \quad (3.14)$$

$$\left. \frac{d\tilde{\varphi}}{d\varepsilon} \right|_{\varepsilon=0} = -\frac{\sin(\varphi - \alpha)}{R}. \quad (3.15)$$

For Λ_1 we use the formula (2.28), first finding

$$\begin{aligned} \mathcal{L}_1 h_0 &= \left. \frac{d}{d\varepsilon} (\mathcal{L}_\varepsilon h_0) \right|_{\varepsilon=0} \\ &= \left. \frac{d}{d\varepsilon} (p^{1-\beta} \tilde{R}^{-\beta}) \right|_{\varepsilon=0} \\ &= -p^{1-\beta} \beta \tilde{R}^{-\beta-1} \tilde{R}' \Big|_{\varepsilon=0} \\ &= -p^{1-\beta} \beta R^{-\beta-1} \cos(\varphi - \alpha). \end{aligned} \quad (3.16)$$

Consequently

$$\Lambda_1 = \mu(\mathcal{L}_1 h_0) = -p^{1-\beta} \beta \frac{1}{2\pi} \int_0^{2\pi} \cos(\varphi - \alpha) d\varphi = 0, \quad (3.17)$$

since for $h \in \mathcal{B}(D)$, $h = h(R, \varphi)$ we have

$$\mu(h) = \frac{1}{2\pi} \int_0^{2\pi} h(1, \varphi) d\varphi.$$

We turn to (2.36) next to calculate Λ_2 . First, however, we have to find the correction h_1 to the eigenvector h_ε ; this is done with the aid of (2.38).

Making use of (3.16) and (3.17), we obtain

$$\begin{aligned} h_1 &= \Lambda_0^{-1}[1 - \mathcal{K}]^{-1}\mathcal{L}_1 h_0 \\ &= -\beta[1 - \mathcal{K}]^{-1} \left(R^{-\beta-1} \cos(\varphi - \alpha) \right). \end{aligned} \quad (3.18)$$

It can be seen that

$$\mathcal{K}(R^{-\beta-1} \cos(\varphi - \alpha)) = R^{-\beta-1/p} \cdot \frac{1}{p} \sum_{k=0}^{p-1} \cos\left(\frac{\varphi}{p} - \alpha + \frac{2k\pi}{p}\right) = 0, \quad (3.19)$$

which follows from the fact that the sum of the p -th roots of unity vanishes for all p . Therefore, using (2.34) we finally get

$$h_1(R, \varphi) = -\beta R^{-\beta-1} \cos(\varphi - \alpha). \quad (3.20)$$

Exactly the same argument as above shows that $\mathcal{L}_\varepsilon h_1 \equiv 0$. Thus, in particular $\mathcal{L}_1 h_1 = 0$ and consequently (2.36) reduces to $\Lambda_2 = \mu(\mathcal{L}_2 h_0)$. Presently we have

$$\begin{aligned} \mathcal{L}_2 h_0 &= \frac{d^2}{d\varepsilon^2} (\mathcal{L}_\varepsilon h_0)|_{\varepsilon=0} \\ &= p^{1-\beta} \frac{d^2}{d\varepsilon^2} (\tilde{R}^{-\beta})|_{\varepsilon=0} \\ &= p^{1-\beta} \left[\beta(\beta+1)\tilde{R}^{-\beta-2}(\tilde{R}')^2 - \beta\tilde{R}^{-\beta-1}\tilde{R}'' \right]|_{\varepsilon=0} \\ &= p^{1-\beta} R^{-\beta-2} \beta [(\beta+1)\cos^2(\varphi - \alpha) - \sin^2(\varphi - \alpha)] \\ &= p^{1-\beta} R^{-\beta-2} \frac{\beta(\beta+2)\cos 2(\varphi - \alpha) + \beta^2}{2}, \end{aligned} \quad (3.21)$$

and consequently

$$\begin{aligned}
\Lambda_2 &= \mu(\mathcal{L}_2 h_0) \\
&= p^{1-\beta} \frac{\beta^2}{2} + p^{1-\beta} \frac{\beta(\beta+2)}{2} \cdot \frac{1}{2\pi} \int_0^{2\pi} \cos 2(\varphi - \alpha) d\varphi \quad (3.22) \\
&= p^{1-\beta} \frac{\beta^2}{2}.
\end{aligned}$$

Similarly, using $\mathcal{L}_2 h_1 \equiv 0$ we find from (2.37)

$$\Lambda_3 = \mu(\mathcal{L}_3 h_0) + 3\mu(\mathcal{L}_1 h_2), \quad (3.23)$$

where h_2 is determined by (2.39), i.e.

$$\begin{aligned}
h_2 &= \Lambda_0^{-1} [1 - \mathcal{K}]^{-1} (\mathcal{L}_2 - \Lambda_2) h_0 \quad (3.24) \\
&= [1 - \mathcal{K}]^{-1} \left(\frac{\beta^2}{2} R^{-\beta} (R^2 - 1) + \frac{\beta(\beta+2)}{2} R^{-\beta-2} \cos 2(\varphi - \alpha) \right).
\end{aligned}$$

$[1 - \mathcal{K}]^{-1}$ applied to the first term yields

$$h_2^{(1)}(R) = \sum_{n=0}^{\infty} R^{-\beta} (R^{-2/p^n} - 1), \quad (3.25)$$

which converges uniformly on $\text{cl}D$.

In order to evaluate $[1 - \mathcal{K}]^{-1}$ on the second term, we first prove the following.

Lemma 3.1 *For any $m > 0$*

$$\mathcal{K} \cos m\varphi = \delta_{p,m} R^{\beta \frac{1-p}{p}} \cos \varphi \quad (3.26)$$

where $\delta_{p,m}$ is the Kronecker symbol. The same is true for the sine function.

PROOF: We have

$$\mathcal{K} \cos m\varphi = R^{\beta \frac{1-p}{p}} \cdot \frac{1}{p} \sum_{k=0}^{p-1} \cos \frac{m}{p}(\varphi + 2k\pi). \quad (3.27)$$

If $m = p$, the formula is obvious. Without loss of generality assume that m and p are relatively prime. We want to show that the sum of m -th powers of the p -th roots of unity vanishes in this case or, equivalently, that $m\mathbf{Z}_p = \mathbf{Z}_p$ iff $(m, p) = 1$, where \mathbf{Z}_p denotes the ring of integers modulo p .

Suppose that $(m, p) = 1$ and that $mr = 0$ for some $r \in \mathbf{Z}_p$. Then necessarily $p|r$, i.e. $r = 0$ and hence $m\mathbf{Z}_p = \mathbf{Z}_p$.

If on the contrary $(m, p) \neq 1$, e.g. $m = kn$, $p = ln$, then while $l \neq 0$ and $l \in \mathbf{Z}_p$, we have $ml = knl = kp = 0$ and hence $m\mathbf{Z}_p \neq \mathbf{Z}_p$. \square

Finally, we find, using the above lemma

$$\begin{aligned} h_2^{(2)}(R, \varphi) &= [1 - \mathcal{K}]^{-1}(R^{-\beta-2} \cos 2(\varphi - \alpha)) \\ &= R^{-\beta-2} \cos 2(\varphi - \alpha) + \delta_{p,2} R^{-\beta-1} \cos(\varphi - 2\alpha) \end{aligned} \quad (3.28)$$

and consequently

$$\begin{aligned} h_2(R, \varphi) &= \frac{\beta^2}{2} h_2^{(1)} + \frac{\beta(\beta+2)}{2} h_2^{(2)} \\ &= \frac{\beta^2}{2} \sum_{n=0}^{\infty} R^{-\beta} (R^{-2/p^n} - 1) \\ &\quad + \frac{\beta(\beta+2)}{2} [R^{-\beta-2} \cos 2(\varphi - \alpha) + \delta_{p,2} R^{-\beta-1} \cos(\varphi - 2\alpha)]. \end{aligned} \quad (3.29)$$

Evaluating $\mathcal{L}_1 h_2 = \frac{d}{d\varepsilon} (\mathcal{L}_\varepsilon h_2)|_{\varepsilon=0}$ in turn yields

$$\begin{aligned} \mathcal{L}_1 h_2 = & -p^{1-\beta} \left[\frac{\beta^2}{2} R^{-\beta-1} \cos(\varphi - \alpha) \times \right. \\ & \times \sum_{n=1}^{\infty} [\beta(R^{-2p^{-n}} - 1) + 2p^{-n} R^{-2p^{-n}}] \\ & \left. + \delta_{p,2} \frac{\beta(\beta+2)}{4} R^{-\beta-2} [(\beta+2) \cos(2\varphi - 3\alpha) + \beta \cos \alpha] \right], \end{aligned} \quad (3.30)$$

where we have made use of the uniform convergence of $\mathcal{L}_\varepsilon h_2^{(1)}$ for termwise differentiation as well as the fact that

$$\mathcal{L}_\varepsilon \cos m\varphi = \delta_{p,m} \tilde{R}^{\beta \frac{1-p}{p}} \cos \tilde{\varphi}. \quad (3.31)$$

Thus we obtain

$$\mu(\mathcal{L}_1 h_2) = -p^{1-\beta} \delta_{p,2} \frac{\beta^2(\beta+2)}{4} \cos \alpha. \quad (3.32)$$

It can be easily verified that $\mu(\mathcal{L}_3 h_0) = 0$, and therefore

$$\Lambda_3 = -p^{1-\beta} \delta_{p,2} \frac{3\beta^2(\beta+2)}{4} \cos \alpha. \quad (3.33)$$

Before we attempt to explain the peculiar dependence on p in Λ_3 , let us examine the evaluation of Λ_4 and Λ_5 . Here we have

$$\Lambda_4 = \mu(\mathcal{L}_4 h_0) + 6\mu(\mathcal{L}_2 h_2) + 4\mu(\mathcal{L}_1 h_3) \quad (3.34)$$

$$\Lambda_5 = \mu(\mathcal{L}_5 h_0) + 10\mu(\mathcal{L}_3 h_2) + 10\mu(\mathcal{L}_2 h_3) + 5\mu(\mathcal{L}_1 h_4), \quad (3.35)$$

where

$$h_3 = \Lambda_0^{-1}[1 - \mathcal{K}]^{-1}([\mathcal{L}_3 - \Lambda_3]h_0 + 3\mathcal{L}_1h_2 - 3\Lambda_2h_1) \quad (3.36)$$

$$h_4 = \Lambda_0^{-1}[1 - \mathcal{K}]^{-1}([\mathcal{L}_4 - \Lambda_4]h_0 + 6[\mathcal{L}_2 - \Lambda_2]h_2 + 4\mathcal{L}_1h_3 - 4\Lambda_3h_1). \quad (3.37)$$

After a substantial amount of algebra one obtains

$$\begin{aligned} \Lambda_4 &= p^{1-\beta} \beta^2 \left[\frac{3(\beta+2)^2}{8} + 6 \frac{(p+1)\beta+1}{p^2-1} \right. \\ &\quad \left. + \frac{\beta+2}{2} [\delta_{p,2}3(\beta+2) + \delta_{p,3}(\beta+4)] \cos 2\alpha \right] \\ \Lambda_5 &= p^{1-\beta} \frac{5\beta^2}{4} \left[\delta_{p,2} \frac{7\beta^3 - 518\beta^2 - 1204\beta - 472}{14} \cos \alpha \right. \\ &\quad \left. - \delta_{p,2}3(\beta+2)^3 \cos 3\alpha \right. \\ &\quad \left. - \delta_{p,4} \frac{(\beta+2)(\beta+4)(\beta+6)}{4} \cos 3\alpha \right]. \end{aligned} \quad (3.39)$$

The intriguing dependence of Λ_3 , Λ_4 and Λ_5 on p reflects the invariance of the pressure $P_c(\beta)$ with respect to the action of the Möbius group on \mathbf{C}_∞ .

Recall that

$$P(\beta) = P_f(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{z \in \text{Fix} f^n} |(f^n)'(z)|^{-\beta}. \quad (3.40)$$

Given a Möbius transformation $\phi : \mathbf{C}_\infty \rightarrow \mathbf{C}_\infty$ let

$$\tilde{f} = \phi \circ f \circ \phi^{-1}.$$

Then we have

$$(i) \ z \in \text{Fix} f^n \iff \phi(z) \in \text{Fix} \tilde{f}^n$$

$$(ii) \ \text{since } \phi'(z) \neq 0 \text{ for all } z, \quad \left. \frac{d}{dz} f^n \right|_z = \left. \frac{d}{dz} \tilde{f}^n \right|_{\phi(z)}.$$

Therefore

$$P_f(\beta) = P_{\tilde{f}}(\beta).$$

Consider in particular the group of rotations $\{S_\theta\}$ commuting with f_0 ,
i.e.

$$z^p = f_0 = S_{-\theta} \circ f_0 \circ S_\theta = e^{i(p-1)\theta} z^p. \quad (3.41)$$

Therefore, one must have $\theta = \frac{2k\pi}{p-1}$, $k = 0, \dots, p-2$. Then for $c = \epsilon e^{i\alpha} \neq 0$
one gets

$$S_{-\theta} \circ f_c \circ S_\theta = z^p - \epsilon e^{i(\alpha-\theta)} \quad (3.42)$$

Taking into account the above-mentioned invariance of $P(\beta)$, we conclude
that the terms in the perturbation series for $e^{P(\beta)} = \Lambda_\epsilon$ must not change
when we replace α with $\alpha - \frac{2k\pi}{p-1}$, $k = 0, \dots, p-2$. However, α appears in
the factors $\cos m\alpha$; therefore the following must hold:

$$\cos m\alpha = \cos m \left(\alpha - \frac{2k\pi}{p-1} \right) \quad (3.43)$$

for $k = 0, \dots, p-2$, i.e. $(p-1)|m$, or otherwise the corresponding term must
vanish. In particular, the 3rd order correction, Λ_3 , containing the factor $\cos \alpha$
appears only for $p = 2$; subterms in the expression for Λ_4 containing $\cos 2\alpha$

will show up only for $p = 2$ or 3 , while those of Λ_5 containing $\cos 3\alpha$, only for $p = 2$ or 4 .

Similar observations about the symmetries of the $P_c(\beta)$ expansion have been reported in [Wido].

To sum up, we have found the following perturbative expansion:

$$\begin{aligned}
 e^{P_c(\beta)} = p^{1-\beta} & \left[1 + \varepsilon^2 \frac{\beta^2}{4} - \varepsilon^3 \delta_{p,2} \frac{\beta^2(\beta+2)}{8} \cos \alpha \right. \\
 & + \varepsilon^4 \frac{\beta^2}{16} \left(\frac{(\beta+2)^2}{4} + 4 \frac{(p+1)\beta+1}{p^2-1} \right. \\
 & \quad + \delta_{p,2} (\beta+2)^2 \cos 2\alpha \\
 & \quad \left. + \delta_{p,3} \frac{(\beta+2)(\beta+4)}{3} \cos 2\alpha \right) \\
 & + \varepsilon^5 \frac{\beta^2}{32} \left(\delta_{p,2} \frac{7\beta^3 - 518\beta^2 - 1204\beta - 472}{42} \cos \alpha \right. \\
 & \quad - \delta_{p,2} (\beta+2)^3 \cos 3\alpha \\
 & \quad \left. + \delta_{p,4} \frac{(\beta+2)(\beta+4)(\beta+6)}{12} \cos 3\alpha \right) \\
 & \left. + O(\varepsilon^6) \right]. \tag{3.44}
 \end{aligned}$$

Solving the above for $P_c(\beta)$ gives

$$\begin{aligned}
 P_c(\beta) = (1-\beta) \log p & + \varepsilon^2 \frac{\beta^2}{4} - \varepsilon^3 \delta_{p,2} \frac{\beta^2(\beta+2)}{8} \cos \alpha \\
 & + \varepsilon^4 \frac{\beta^2}{16} \left[\frac{4+4\beta-\beta^2}{4} + 4 \frac{1+(p+1)\beta}{p^2-1} \right. \\
 & \quad \left. + \delta_{p,2} (\beta+2)^2 \cos 2\alpha \right]
 \end{aligned}$$

$$\begin{aligned}
& + \delta_{p,3} \frac{(\beta+2)(\beta+4)}{3} \cos 2\alpha \Big] \tag{3.45} \\
& + \varepsilon^5 \frac{\beta^2}{32} \left[\delta_{p,2} \frac{49\beta^3 - 434\beta^2 - 1204\beta - 472}{42} \cos \alpha \right. \\
& \quad - \delta_{p,2} (\beta+2)^3 \cos 3\alpha \\
& \quad \left. - \delta_{p,4} \frac{(\beta+2)(\beta+4)(\beta+6)}{12} \cos 3\alpha \right] \\
& + O(\varepsilon^6).
\end{aligned}$$

3.2 Calculating dimensions, entropies and Lyapunov exponents from the pressure $P(\beta)$

The perturbation series (3.45) for $P_c(\beta)$ derived in the previous section will be used now to obtain dynamical indices for (f_c, J_c) , such as the Hausdorff dimension of J_c , generalized dimensions for Gibbs states μ_β and their Renyi entropies. We also determine the entropy function $S(\lambda)$ for characteristic exponents for $c \neq 0$, [Bohr].

We recall that the Julia set for the map $f_0(z) = z^p$ is the unit circle J_0 (cf. the discussion in the Examples 1.2.1 and 1.2.2), with Hausdorff dimension 1. When c varies over a small neighbourhood of 0, the corresponding invariant sets J_c are Jordan curves — homeomorphic images of J_0 . However, there is a change in the Hausdorff dimension of J_c : it is no longer 1 and therefore

the sets J_c are fractal objects. This change in the Hausdorff dimension is smooth with respect to the real and imaginary parts of c . Below we derive the corresponding perturbation series for the dimension.

By Theorem 1.9 we have

$$P_c(\beta_H) = 0.$$

Due to the analytic dependence of $P_c(\beta)$ on both $(\text{Re}c, \text{Im}c)$ and β , the Implicit Function Theorem allows us to seek an expression for β_H in the form

$$\beta_H = \beta_0 + \varepsilon\beta_1 + \cdots + \varepsilon^5\beta_5 + O(\varepsilon^6). \quad (3.46)$$

Substituting (3.46) into the expansion for P_c , (3.45), and equating the resulting expression with 0, we require that all the terms of distinct order in ε vanish separately. This yields

$$\begin{aligned} \beta_H = & 1 + \frac{\varepsilon^2}{4 \log p} - \varepsilon^3 \delta_{p,2} \frac{3 \cos \alpha}{8 \log p} \\ & + \frac{\varepsilon^4}{16 \log p} \left[\frac{7}{4} + \frac{2}{\log p} + 4 \frac{p+2}{p^2-1} + (9 \delta_{p,2} + 5 \delta_{p,3}) \cos 2\alpha \right] \\ & - \frac{\varepsilon^5}{32 \log p} \left[\delta_{p,2} \left(\frac{687 \log p + 182}{14 \log p} \cos \alpha + 27 \cos 3\alpha \right) \right. \\ & \quad \left. + \delta_{p,4} \frac{35 \cos 3\alpha}{4} \right] \\ & + O(\varepsilon^6). \end{aligned} \quad (3.47)$$

This expansion was obtained earlier by Ruelle, [Rue3], through the 2nd order

and by Widom, [Wido], through the 3rd order. In both cases the authors presented formal calculations based on the study of meromorphy properties of the zeta function. Topological pressure is related to the location of a pole of the zeta function. An analysis of this singularity for vanishing $P(\beta)$ together with an approximation of periodic points in the perturbed system yielded an asymptotic relation from which the authors formally obtained the first three terms in (3.47). Unlike our derivation of (3.45) and (3.47), this approach is not systematic and does not have a rigorous background. It could not yield the terms of 4th and 5th order in (3.45) and (3.47) because of the increasing complexity of intermediate expressions in the derivation. On the other hand, the methodical perturbation-theoretic approach admits the use of a symbolic manipulator, an inevitable necessity in dealing with hundred-term expressions. Moreover, the derivation of the power series expansion becomes as rigorous as the regular perturbation theory itself.

The appearance of Kronecker δ 's in (3.47) is once again related to the invariance of $P_c(\beta)$ with respect to the group of rotations $\{S_\theta\}$ commuting with f_0 . As we have discussed in the previous section, the pressure expansion does not change under conjugation of f_c by S_θ . Consequently, the Hausdorff dimension β_H inherits this invariance from P_c .

Two dynamical indices discussed in Section 1.5, the topological entropy and the escape rate, are easily obtained from $P_c(\beta)$ by the substitutions $\beta = 0$ and $\beta = 2$, respectively. We obtain

$$h_{\text{top}} = P_c(0) = \log p, \quad (3.48)$$

which is also the upper bound on the Lyapunov exponents of the measures μ_β for $\beta > 0$. On the other hand, for the escape rate we find

$$\begin{aligned} \log r &= -P_c(2) \\ &= \log p - \varepsilon^2 + \varepsilon^3 2\delta_{p,2} \cos \alpha \\ &\quad - \varepsilon^4 \left[\frac{1}{2} + \frac{2p+3}{p^2-1} + (4\delta_{p,2} + 2\delta_{p,3}) \cos 2\alpha \right] \\ &\quad + \varepsilon^5 \left[\frac{23}{2} \delta_{p,2} \cos \alpha + (8\delta_{p,2} + 2\delta_{p,4}) \cos 3\alpha \right] \\ &\quad + O(\varepsilon^6). \end{aligned} \quad (3.49)$$

Thus, to the second order in ε we can observe a decrease in the escape rate when the perturbing parameter $|c|$ gets larger. This phenomenon is related to the similar variation of Lyapunov exponents $\lambda(\mu_\beta)$ with c for $\beta > 0$. As we have mentioned above, these exponents never exceed the topological entropy, $\log p$, which is also equal to $\lambda(\mu_\beta)$ for all β for the unperturbed system with $c = 0$. Thus, when c departs slightly from 0, smaller Lyapunov exponents appear among $\lambda(\mu_\beta)$ for $\beta > 0$, which, as we shall discuss later in this section,

accounts for the decay in the escape rate.

The dimension spectrum $D_\mu(q)$ for a Gibbs state $\mu = \mu_\beta$ is retrieved from $P_c(\beta)$ by means of the formula (1.69). We assume that $D_\mu(q)$ takes the form of a power series

$$D_{\mu_\beta}(q) = D_0(q) + \varepsilon D_1(q) + \varepsilon^2 D_2(q) + \dots, \quad (3.50)$$

which is again justified by the analytic properties $P_c(\beta)$ and the application of the Implicit Function Theorem. Performing the appropriate substitutions in (1.69), we obtain by equating terms of the same order in ε

$$\begin{aligned} D_{\mu_\beta}(q) = & 1 + \varepsilon^2 \frac{1 - q(\beta - 1)^2}{4 \log p} \\ & - \varepsilon^3 \delta_{p,2} \frac{\cos \alpha}{8 \log p} \left[(q - 1)(q - 3) \right. \\ & \quad \left. - q\beta[q(\beta^2 - 3\beta + 3) + \beta(\beta + 2) - 7] \right] \\ & + O(\varepsilon^4). \end{aligned} \quad (3.51)$$

We have omitted the terms of 4th and 5th order, which, in their complicated general form, are not particularly enlightening. Instead, three values of the parameter q deserve special attention: $q = 0, 1, 2$. Substituting $q = 0$ in (3.51) we recover the expansion for the Hausdorff dimension, which is in agreement with (1.70). When $q = 1$, the expansion for the information dimension of μ_β

is produced. Here we find

$$\begin{aligned}
D_{\mu_\beta}(1) = & 1 + \varepsilon^2 \frac{\beta(2-\beta)}{4 \log p} - \varepsilon^3 \delta_{p,2} \frac{\beta(4+\beta-2\beta^2)}{8 \log p} \cos \alpha \\
& + \varepsilon^4 \frac{\beta}{4 \log p} \left[\frac{(\beta-2)(3\beta^2-6\beta+4)}{16} \right. \\
& \quad + \frac{2-\beta}{p^2-1} + \frac{\beta(3-2\beta)}{p-1} + \frac{\beta(2-\beta)}{2 \log p} \\
& \quad - \delta_{p,2} \frac{(\beta+2)(3\beta^2-2\beta-4)}{4} \cos 2\alpha \\
& \quad \left. - \delta_{p,3} \frac{3\beta^3+8\beta^2-10\beta-16}{12} \cos 2\alpha \right] \quad (3.52) \\
& + \varepsilon^5 \frac{\beta}{24 \log p} \left[\delta_{p,2} \cos \alpha \left(\frac{3\beta}{4 \log p} (7\beta^2-4\beta-16) \right. \right. \\
& \quad \left. \left. - \frac{196\beta^4-1547\beta^3-672\beta^2+3140\beta+944}{56} \right) \right. \\
& \quad + \delta_{p,2} \frac{3 \cos 3\alpha}{4} (\beta+2)^2 (4\beta^2-3\beta-4) \\
& \quad \left. + \delta_{p,4} \frac{\cos 3\alpha}{16} (4\beta^4+31\beta^3+40\beta^2-84\beta-96) \right] \\
& + O(\varepsilon^6).
\end{aligned}$$

Finally, setting $q = 2$, we obtain the following expression for the correlation dimension

$$\begin{aligned}
D_{\mu_\beta}(2) = & 1 + \varepsilon^2 \frac{1+4\beta-2\beta^2}{4 \log p} + \varepsilon^3 \delta_{p,2} \frac{6\beta^3-8\beta^2-2\beta+1}{8 \log p} \cos \alpha \\
& + \frac{\varepsilon^4}{4 \log p} \left[\frac{14\beta^4-56\beta^3+64\beta^2-16\beta+1}{16} \right. \\
& \quad - \frac{4\beta^3-10\beta^2+6\beta-1}{2 \log p} \\
& \quad \left. - \frac{2\beta^2-4\beta+1}{p^2-1} - \frac{6\beta^3-12\beta^2+6\beta-1}{p-1} \right]
\end{aligned}$$

$$\begin{aligned}
& -\delta_{p,2} \frac{14\beta^4 - 8\beta^3 - 16\beta^2 + 1}{4} \cos 2\alpha \\
& -\delta_{p,3} \frac{14\beta^4 + 4\beta^3 - 32\beta^2 - 4\beta + 3}{12} \cos 2\alpha \Big] \quad (3.53) \\
& + \frac{\varepsilon^5}{24 \log p} \left[\delta_{p,2} \frac{3 \cos \alpha}{4} \left(\frac{(2\beta - 1)(24\beta^3 - 38\beta^2 - 2\beta + 3)}{\log p} \right. \right. \\
& \quad \left. \left. - \frac{490\beta^5 - 1372\beta^4 + 3528\beta^3 + 376\beta^2 - 458\beta + 83}{14} \right) \right. \\
& \quad + \delta_{p,2} \frac{3 \cos 3\alpha}{4} (30\beta^5 + 4\beta^4 - 40\beta^3 - 24\beta^2 + 2\beta + 1) \\
& \quad \left. + \delta_{p,4} \frac{\cos 3\alpha}{16} (30\beta^5 + 88\beta^4 - 40\beta^3 - 184\beta^2 - 14\beta + 15) \right] \\
& + O(\varepsilon^6).
\end{aligned}$$

The above quantity characterizes the scaling behaviour for the μ_β -probability of two trajectories staying within a distance of $\delta > 0$ from each other, when $\delta \rightarrow 0$.

We will take a closer look at the dimension spectrum of the balanced measure μ_0 . By setting $\beta = 0$ in the formulas (3.51)¹, (3.52) and (3.53) we can see that

$$D_{\mu_0}(0) = \beta_H \quad (3.54)$$

$$D_{\mu_0}(1) = 1 \quad (3.55)$$

$$D_{\mu_0}(2) = 1 + \frac{\varepsilon^2}{4 \log p} + \frac{\varepsilon^3}{8 \log p} \delta_{p,2} \cos \alpha + \dots \quad (3.56)$$

¹See also formula (1.70).

The spectrum of the generalized dimensions for the balanced measure μ_0 is thus nontrivial, unlike that for the uniform measure μ_{β_H} characterized in (1.71).

Presently, we use formulas (1.72) and (3.45) for the determination of Renyi entropies of the equilibrium states μ_β . One finds

$$\begin{aligned}
h_{\mu_\beta}(q) = & \log p - \varepsilon^2 \frac{q\beta^2}{4} + \varepsilon^3 \delta_{p,2} \frac{q\beta^2(\beta q + \beta + 2)}{8} \cos \alpha \\
& + \varepsilon^4 \frac{q\beta^2}{16} \left[\frac{\beta^2(q^2 + q + 1) - 4\beta(q + 1) - 4}{4} \right. \\
& \quad - 4 \frac{\beta(q + 1)(p + 1) + 1}{p^2 - 1} \\
& \quad - \delta_{p,2}(\beta^2(q^2 + q + 1) + 4\beta(q + 1) + 4) \cos 2\alpha \\
& \quad \left. - \delta_{p,3} \frac{\beta^2(q^2 + q + 1) + 6\beta(q + 1) + 8}{3} \cos 2\alpha \right] \\
& + \varepsilon^5 \frac{q\beta^2}{32} \left[\delta_{p,2} \frac{\cos \alpha}{42} [472 + 1204\beta(q + 1) \right. \\
& \quad \left. + 434\beta^2(q^2 + q + 1) - 49\beta^3(q + 1)(q^2 + 1)] \right. \\
& \quad + \delta_{p,2} \cos 3\alpha [8 + 12\beta(q + 1) \\
& \quad \left. + 6\beta^2(q^2 + q + 1) + \beta^3(q + 1)(q^2 + 1)] \right. \\
& \quad + \delta_{p,4} \cos 3\alpha \frac{\beta(q + 1) + 6}{48} \times \\
& \quad \left. \times [8 + 6\beta(q + 1) + \beta^2(q^2 + 1)] \right] \\
& + O(\varepsilon^6).
\end{aligned} \tag{3.57}$$

Taking $\beta = 0$, one finds that for μ_0 , the maximal entropy measure, all the

Renyi entropies equal h_{top} . Also $h_{\mu_\beta}(0) = h_{\text{top}}$ for all the measures μ_β . Moreover, the Kolmogorov–Sinai entropy of the state μ_β takes the following form

$$\begin{aligned}
 h_{\mu_\beta}(1) = & \log p - \varepsilon^2 \frac{\beta^2}{4} + \varepsilon^3 \delta_{p,2} \frac{\beta^2(\beta+1)}{4} \cos \alpha \\
 & + \varepsilon^4 \frac{\beta^2}{16} \left[\frac{3\beta^2 - 8\beta - 4}{4} - 8 \frac{\beta(p+1) + 1}{p^2 - 1} \right. \\
 & \quad \left. - \delta_{p,2}(\beta+2)(3\beta+2) \cos 2\alpha \right. \\
 & \quad \left. - \delta_{p,3} \frac{3\beta^2 + 12\beta + 8}{3} \cos 2\alpha \right] \\
 & + O(\varepsilon^5).
 \end{aligned} \tag{3.58}$$

The Renyi entropy of the second order $h_\mu(2)$ satisfies

$$h_\mu(2) \leq h_\mu(1), \tag{3.59}$$

the corresponding power series being

$$\begin{aligned}
 h_{\mu_\beta}(2) = & \log p - \varepsilon^2 \frac{\beta^2}{2} + \varepsilon^3 \delta_{p,2} \frac{\beta^2(3\beta+2)}{4} \cos \alpha \\
 & + \varepsilon^4 \frac{\beta^2}{8} \left[\frac{(\beta-2)(7\beta+2)}{28} - 4 \frac{3\beta(p+1) + 1}{p^2 - 1} \right. \\
 & \quad \left. - \delta_{p,2}(7\beta^2 + 12\beta + 4) \cos 2\alpha \right. \\
 & \quad \left. - \delta_{p,3} \frac{(\beta+2)(7\beta+4)}{7} \cos 2\alpha \right] \\
 & + O(\varepsilon^6).
 \end{aligned} \tag{3.60}$$

A result of Ruelle, [Rue4], binds $h_\mu(1)$ with the Lyapunov exponents of μ for

general hyperbolic dynamical systems:

$$h_\mu(1) \leq \sum_i^+ \lambda_i(\mu), \quad (3.61)$$

where \sum^+ extends over all positive Lyapunov exponents λ_i of μ . Thus, whenever $h_\mu(1) > 0$ (or whenever $h_\mu(2) > 0$, a stronger condition), positive Lyapunov exponents appear and the dynamics has the property of sensitive dependence on the initial conditions. This is, obviously, the case of the dynamics considered here. We shall notice also that $h_{\mu_\beta}(1)$ is very well approximated by $h_{\mu_\beta}(2)$, as it has been suggested in [Gra2], [Gra3].

The knowledge of $P(\beta)$ expansion makes it possible to obtain the entropy function for characteristic exponents, [Bohr], of the dynamical system under consideration. The entropy function $S(\sigma)$, being the Legendre transform of $P(\beta)$,

$$S(\sigma) = P[\beta(\sigma)] + \sigma \beta(\sigma), \quad (3.62)$$

is interpreted in the following way. Let Λ_σ be a collection of all these points $\xi \in J_c$ which give rise to a Lyapunov exponent equal to σ , i.e.

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |f'_c(f^k(\xi))| = \lambda(\xi) = \sigma. \quad (3.63)$$

Then $S(\sigma)$ equals the noncompact topological entropy of Λ_σ in the sense of Bowen, [Bow3], and $D(\sigma) = \sigma^{-1}S(\sigma)$ is the Hausdorff dimension of Λ_σ .

Finding the function $S(\sigma)$ in the form of a perturbation series in ε poses some technical problems in our case, due to the fact that for $\varepsilon = 0$, $P_0(\beta)$ is linear in β , and hence not strictly convex; the Legendre transform does not exist. The spectrum of characteristic exponents then collapses to one point $\sigma = \log p$; $\Lambda_\sigma = J_0$ is the unit circle. On the other hand, once $\varepsilon > 0$, $P_c(\beta)$ becomes strictly convex so that $S(\sigma)$ exists around the point $\sigma = \log p$.

In dealing with this singular behaviour of $S(\sigma)$, we consider ε to be different from 0 and we introduce a “separated scale” variable, $\omega = \log p - \sigma$, assumed to be smaller than ε^2 . Denoting $\zeta = \omega\varepsilon^{-2} < 1$ and using the extremality condition for $S(\sigma)$,

$$\sigma = -\frac{dP(\beta)}{d\beta} = \log p - \varepsilon^2 \frac{\beta}{2} + \dots, \quad (3.64)$$

we obtain

$$\begin{aligned} \zeta(\beta) = \frac{\log p - \sigma}{\varepsilon^2} &= \frac{\beta}{2} + \varepsilon \delta_{p,2} \frac{\beta(3\beta + 4)}{8} \cos \alpha \\ &- \varepsilon^2 \frac{\beta}{2} \left[\frac{2 + 3\beta - \beta^2}{8} + \frac{1}{p^2 - 1} + \frac{3\beta}{2(p-1)} \right. \\ &+ \delta_{p,2} \frac{(\beta + 1)(\beta + 2)}{2} \cos 2\alpha \\ &\left. + \delta_{p,3} \frac{8 + 9\beta + 2\beta^2}{12} \cos 2\alpha \right] + \dots \end{aligned} \quad (3.65)$$

In order to invert this expression, i.e. to find $\beta = \beta(\zeta)$, we assume that $\beta(\zeta)$

takes the form

$$\beta(\zeta) = \beta_0(\zeta) + \varepsilon\beta_1(\zeta) + \varepsilon^2\beta_2(\zeta) + \dots \quad (3.66)$$

Substituting the above for β in (3.65) and using the fact that $\zeta(\beta(\zeta)) = \zeta$,

we obtain

$$\begin{aligned} \beta(\zeta) &= 2\zeta + \varepsilon\delta_{p,2}\zeta(2 + 3\zeta)\cos\alpha \\ &+ \varepsilon^2\frac{\zeta}{2}\left[2\zeta^2 - 3\zeta - 1 - \frac{4}{p-1}\left(\frac{1}{p+1} + 3\zeta\right)\right. \\ &\quad \left.+ \delta_{p,2}[(3\zeta+1)(3\zeta+2) + (\zeta^2 - 3\zeta - 2)\cos 2\alpha]\right. \\ &\quad \left.- \delta_{p,3}\frac{2\cos 2\alpha}{3}(4\zeta^2 + 9\zeta + 4)\right] \\ &+ \varepsilon^3\frac{\zeta}{6}\left[\delta_{p,2}\cos\alpha\left(\frac{4025\zeta^3 + 7868\zeta^2 + 4956\zeta + 440}{56}\right.\right. \\ &\quad \left.- 6\frac{9\zeta+4}{p^2-1} - 108\zeta\frac{2\zeta+1}{p-1}\right) \\ &\quad \left.+ \delta_{p,2}\frac{3\cos 3\alpha}{8}(8 - 44\zeta^2 - 25\zeta^3)\right. \\ &\quad \left.+ \delta_{p,4}\frac{\cos 3\alpha}{2}(12 + 33\zeta + 24\zeta^2 + 5\zeta^3)\right] + \dots \end{aligned} \quad (3.67)$$

Substituting in (3.62), we subsequently obtain

$$\begin{aligned} S(\zeta) &= P[\beta(\zeta)] + (\log p - \zeta\varepsilon^2)\beta(\zeta) = \\ &= \log p - \varepsilon^2\zeta^2 - \varepsilon^3\delta_{p,2}\zeta^2(\zeta+1)\cos\alpha \\ &\quad + \varepsilon^4\frac{\zeta^2}{4}\left[1 + 2\zeta - \zeta^2 + \frac{1}{p-1}\left(\frac{1}{p+1} + 2\zeta\right)\right. \\ &\quad \left.- \delta_{p,2}\frac{(3\zeta+2)^2 + (\zeta^2 - 4\zeta - 4)\cos 2\alpha}{2}\right] \end{aligned}$$

$$\begin{aligned}
& + \delta_{p,3} \frac{4(\zeta+1)(\zeta+2)}{3} \cos 2\alpha \Big] \tag{3.68} \\
& - \varepsilon^5 \frac{\zeta^2}{16} \left[\delta_{p,2} \cos \alpha \left(\frac{805\zeta^3 + 1967\zeta^2 + 1652\zeta + 220}{21} \right. \right. \\
& \quad \left. \left. - \frac{3\zeta+2}{p^2-1} - 3\zeta \frac{3\zeta+2}{p-1} \right) \right. \\
& \quad - \delta_{p,2}(\zeta+2)(5\zeta^2 + \zeta - 2) \cos 3\alpha \\
& \quad \left. - \delta_{p,4} \frac{4(\zeta+1)(\zeta+2)(\zeta+3)}{3} \cos 3\alpha \right] \\
& + \dots
\end{aligned}$$

Moreover, for the dimension spectrum, $D(\zeta)$, we find

$$\begin{aligned}
D(\zeta) &= \frac{S(\zeta)}{\log p - \varepsilon^2 \zeta} = S(\zeta) \frac{1}{\log p} \sum_{n=0}^{\infty} \left[\frac{\varepsilon^2 \zeta}{\log p} \right]^n = \\
&= 1 + \varepsilon^2 \frac{\zeta(1-\zeta)}{\log p} - \varepsilon^3 \delta_{p,2} \frac{\zeta^2(\zeta+1)}{\log p} \cos \alpha \\
&+ \varepsilon^4 \frac{\zeta^2}{4 \log p} \left[1 + 2\zeta - \zeta^2 + 4 \frac{1-\zeta}{\log p} \right. \\
& \quad \left. + \frac{1}{p^2-1} + \frac{2\zeta}{p-1} \right. \tag{3.69} \\
& \quad \left. - \delta_{p,2} \frac{(3\zeta+2)^2 + (\zeta^2 - 4\zeta - 4) \cos 2\alpha}{2} \right. \\
& \quad \left. + \delta_{p,3} \frac{4(\zeta+1)(\zeta+2)}{3} \cos 2\alpha \right] \\
& - \varepsilon^5 \frac{\zeta^2}{16 \log p} \left[\delta_{p,2} \cos \alpha \left(\frac{805\zeta^3 + 1967\zeta^2 + 1652\zeta + 220}{21} \right. \right. \\
& \quad \left. \left. - \frac{3\zeta+2}{p^2-1} - 3\zeta \frac{3\zeta+2}{p-1} - 16 \frac{\zeta(\zeta+1)}{\log p} \right) \right. \\
& \quad \left. - \delta_{p,2}(\zeta+2)(5\zeta^2 + \zeta - 2) \cos 3\alpha \right]
\end{aligned}$$

$$- \delta_{p,4} \left. \frac{4(\zeta + 1)(\zeta + 2)(\zeta + 3)}{3} \cos 3\alpha \right] \\ + \dots$$

The condition $\varepsilon = 0$ requires ζ to be 0 as well; the only Lyapunov exponent in this case ($c = 0$) is $\sigma_0 = \log p$ and the Hausdorff dimension of $\Lambda_{\sigma_0} = J_0$ is 1. When $\varepsilon \neq 0$, variation in σ and hence in ζ is allowed. Looking at the leading order behaviour in the formulas (3.64) and (3.69) we can see now that the Hausdorff dimension of the subset $\Lambda_\sigma \subseteq J_c$ (i.e. $D(\zeta)$) increases to more than 1 when ζ assumes small positive values and decreases below 1 for negative ζ . The former case ($\zeta > 0$) is thus connected with the appearance of Lyapunov exponents σ smaller than σ_0 in the perturbed system. These are the exponents of the equilibrium states μ_β for $\beta > 0$. Similarly, the characteristic exponents of μ_β for $\beta < 0$ (which corresponds to $\zeta < 0$) are greater than $\log p$ and they are “rare” in the system, i.e. $D(\zeta) < 1$.

As it can be expected, the “geometrically dominating” subset of J_c is Λ_{σ_H} , where σ_H is the Lyapunov exponent of μ_{β_H} , the uniform measure (β_H is the Hausdorff dimension of J_c , (3.47)). Indeed, a straightforward calculation, [Bohr], shows that

$$\max_{\zeta} D(\zeta) = \beta_H \tag{3.70}$$

so that the Hausdorff dimensions of Λ_{σ_H} and J_c are equal. One can easily

compute the critical point $\zeta = \zeta_H$ of $D(\zeta)$:

$$\begin{aligned}\zeta_H &= \frac{1}{2} - \varepsilon \frac{7}{8} \delta_{p,2} \cos \alpha + \dots \\ &= (\log p - \lambda(\mu_{\beta_H})) \varepsilon^{-2},\end{aligned}\tag{3.71}$$

where

$$\begin{aligned}\lambda(\mu_{\beta_H}) &= -\left. \frac{dP}{d\beta} \right|_{\beta=\beta_H} \\ &= \log p - \frac{\varepsilon^2}{2} + \varepsilon^3 \frac{7}{8} \delta_{p,2} \cos \alpha \\ &\quad - \frac{\varepsilon^4}{4} \left[1 + \frac{1}{2 \log p} + \frac{3p+5}{p^2-1} \right. \\ &\quad \quad \left. + \left(6\delta_{p,2} + \frac{19}{6} \delta_{p,3} \right) \cos 2\alpha \right] \\ &\quad + \frac{\varepsilon^5}{16} \left[5\delta_{p,2} \left(\log p + \frac{6047}{84} \right) \cos \alpha \right. \\ &\quad \quad \left. + \left(8\delta_{p,2} + \frac{281}{24} \delta_{p,4} \right) \cos 3\alpha \right] \\ &\quad + O(\varepsilon^6).\end{aligned}\tag{3.72}$$

The facts mentioned above are in agreement with the observation that the escape rate from the Julia set decreases with ε to the leading order, (3.49). Subsets $\Lambda_\sigma \subseteq J_c$ for the Lyapunov exponents smaller than $\log p$ “fill” the neighbouring space around J_0 , as their Hausdorff dimensions are greater than 1. This domination of smaller Lyapunov exponents causes the decay in the escape rate.

To sum up, the geometry of J_c is intimately related to the variation in the Lyapunov exponents, specifically to the appearance of a continuum of those smaller than the unique exponent $\log p$ in the unperturbed system. This produces a change in the escape rate from J_c as c departs from 0.

3.3 The use of symbolic manipulation software

All the terms in the expansion for $P_c(\beta)$, (3.45), and in formulae derived from it in Section 3.2 were generated with the aid of the interactive SMP package on a VAX/VMS computer. Using standard features of the SMP such as substitution operations and macrogeneration, one begins the computations by defining the basic expressions (3.6), (3.7) and, in parametric form, the action of \mathcal{L}_ε on a function h , (3.10). Expressions of (3.13) type are then explicitly generated by the built-in symbolic differentiation operation. SMP provides the framework for defining certain simplification patterns, such as particular trigonometric identities in our case, which are used then in automatic reduction of the generated expressions. In our computations we used multiple-angle and sum-of-two-angles identities for sines and cosines, which

resulted in the automatic elimination of powers higher than 1 of trigonometric functions. This would simplify symbolic integration of the terms in (3.34) and (3.35) in the next steps. In handling expressions involving infinite sums, e.g. (3.25), one operates on a single term in its general form, which, by the way is justified here by the uniform convergence of the power series in question.

Expressions in Section 3.2 were obtained by operating on the truncated expansion for $P_c(\beta)$ — the 5th order polynomial in ε . For example, the construction of the Hausdorff dimension expansion (3.47) proceeds in the following way. First, the expression (3.46) (up to the ε^5 term) is defined and substituted for β in (3.45). Next, after expanding the resulting expression, terms of the same order in ε are grouped and extracted. β_0 is obtained by instructing the SMP to solve the equation “0th order term = 0” for β_0 . The computed expression for β_0 is automatically substituted for all the occurrences of β_0 in higher order terms. Next, the equation “1st order term = 0” is solved for β_1 and the resulting expression substituted in all higher order terms, etc.

It is worth noting that most of the programming steps described above consist in general of a single SMP instruction. The computations are very

efficient, although memory requirements are in general very high. It should also be stressed that terms of the 6th and higher orders in (3.45) can be computed systematically, however the dramatic increase in the complexity of calculations, especially in the number of subterms, has to be expected. Computation of the 5th order term in (3.45) and in other formulae involved operations on intermediate expressions with over a hundred terms in the expanded form. The use of a symbolic manipulator becomes a necessity rather than a convenience in this context.

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Vita

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