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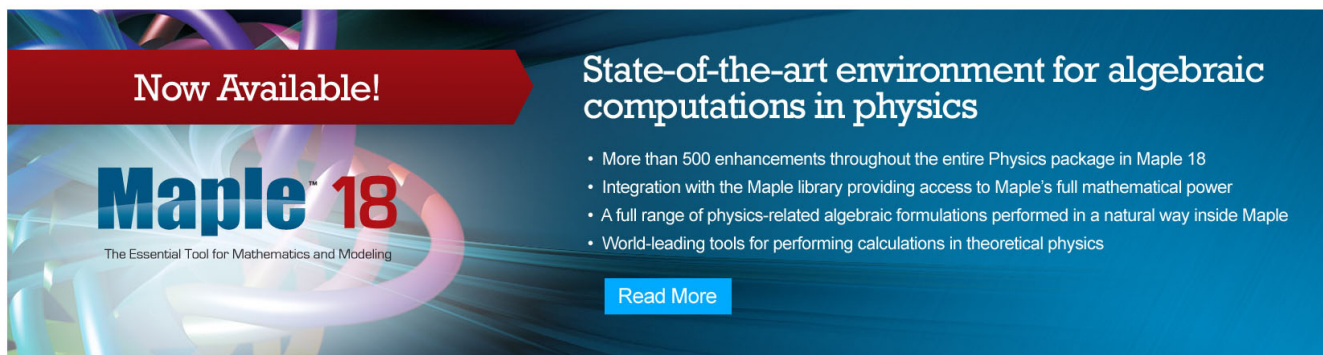
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In studying the spheroidal problem of a charged particle scattered by two charged centers, we have had to deal with a differential equation. Its solution was complicated. In this paper, we study a very similar differential equation, which appears in the Coulomb problem. The solution for this equation, however, can be put in a simple form. For completeness the spheroidal analysis of the Coulomb scattering amplitude is also discussed.

1. INTRODUCTION

In a series of papers[1] we have studied spheroidal potential scattering. The importance of such a study is two fold. First, it denotes a class of real physical problems, such as the scattering of electrons by diatomic molecules,[2] and of deformed nuclei.[3] Second, it describes the scattering between a particle and the simplest composite system which can be formed by two particles separated at a fixed distance.

The spheroidal scattering is less understood than the spherical scattering because of its complexity. In spherical scattering, after the separation of variables, only the radial equation depends on the potential. For normal cases, the equation contains two singularities, one at $r = 0$ and the other at $r = \infty$. But in spheroidal scattering, both the radial and angle equations can depend on the potential. For example, the Coulomb potential of a pair of fixed unequal charges appears in both the radial and angle equations. Thus, we have to solve two ordinary differential equations instead of just one as in the case of spherical scattering. These equations, even without the potential term, have two regular and one irregular singularities, and are in the class of Lamé differential equations.[4] In general, they are more difficult to solve.

In spheroidal scattering, the scattering amplitude is expressed in terms of spheroidal angle functions. This decomposition may be referred to as spheroidal analysis,[5] as compared with the partial wave analysis in spherical scattering. These angle functions do not obey conventional recurrence relations.[6] The so-called "recurrence relations" for these functions are actually the integral relations among them. These functions cannot be simply expressed, and their numerical construction is also troublesome. The trouble leads to a dilemma on the numerical treatment of spheroidal problems. The dilemma is on making a choice between the one-center or two-center approaches. This is a choice between spherical and spheroidal expansions of the scattering amplitude. The former expansion is an unnatural one and leads to a very slow convergent series; but each term in the series is easy to calculate. The later expansion is a natural one and leads to a very fast convergent series, but each term in the series is difficult to calculate. The difficulty is mainly due to the troublesome numerical construction of spheroidal angle functions, which makes the spheroidal analysis in the numerical approach unattractive.

The complexity of the spheroidal problem has led to a theoretical effort in a different direction.[7] Before treating the spheroidal scattering problem, one often uses some kind of approximations and tries to make problems simple and manageable. A good example is the scattering theory of electrons by polar molecules, in which the dipole is replaced by a point dipole.[7] According-

ing to its definition, a point dipole is a dipole without a spatial extent. The main idea behind the effort is to reduce the spheroidal problem into a spherical problem.

Despite its complexity, the spheroidal problem is exactly solvable.[8] It is a problem describing scattering by a composite system. Most problems in this category are not solvable without some kind of approximations. The well-founded scattering theories, whether they are potential or quantum field types, are basically theories for pointlike particles. Therefore, a fully understood scattering theory for any composite system, such as a spheroidal one, is of great value.

Our interest here is in the spheroidal aspect of the Coulomb problem. The importance of spheroidal scattering and its encountered difficulties suggest that we first deal with similar problems with fewer complications. The Coulomb problem is just one of them. Our interest in the Coulomb problem has a stronger basis. In studying the spheroidal scattering,[1] we have had to deal with a differential equation describing a charged particle scattered by two-charged centers. It was treated by constructing an integro-differential equation which, in turn, was solved by means of a substracted iteration method.[1] The final expression was messy. It was observed later that a very similar differential equation also appeared in the Coulomb problem. It is obtained through a spheroidal separation of the wave equation. This has led us to the interesting question, as to whether this equation can be solved differently and simply.

In this paper, along with the problem of main interest to us, we present a detailed spheroidal analysis of the Coulomb problem. In Sec. 2, the spheroidal symmetry of the Coulomb problem is discussed and the resulting angle function is considered. In Sec. 3, we treat the solution of the radial equation. This equation is very similar to that obtained from a charged particle scattered by two charged centers. In Sec. 4, the spheroidal analysis of the Coulomb scattering amplitude is given.

2. COULOMB EQUATION AND ANGLE FUNCTIONS

A fixed charge Q is located on the z axis with coordinates $\mathbf{r}_a = (0, 0, d/2)$, where d is the interfocal distance of the prolate spheroidal coordinates[4]:

$$x = (\tfrac{1}{2}d)[(1 - \eta^2)(\xi^2 - 1)]^{1/2} \cos\phi,$$
$$y = (\tfrac{1}{2}d)[(1 - \eta^2)(\xi^2 - 1)]^{1/2} \sin\phi, \qquad (2.1)$$
$$z = (\tfrac{1}{2}d)\eta\xi,$$

with $1 \le \xi < \infty, \quad -1 \le \eta \le 1, \quad 0 \le \phi \le 2\pi$.
The Coulomb potential V at distance $\mathbf{r}$ has the form

$$V = \frac{Q}{|\mathbf{r} - \mathbf{r}_a|} = \frac{2Q}{d} \frac{\xi + \eta}{\xi^2 - \eta^2}. \qquad (2.2)$$

This is a potential with prolate spheroidal symmetry. The Schrödinger equation for describing a charged particle scattered by the potential in Eq. (2.2) has the form

$$- (\hbar^2/2\mu) \nabla^2 \psi + V\psi = (\hbar^2 k^2/2\mu)\psi, \qquad (2.3)$$

where μ is the mass, q the charge, and k the momentum of the incident particle. The above equation may be expressed simply in the prolate spheroidal coordinates as

$$\left[\frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi} (\xi^2 - 1) \frac{\partial}{\partial \xi}\right.$$
$$+ \left(\frac{1}{1 - \eta^2} + \frac{1}{\xi^2 - 1}\right) \frac{\partial^2}{\partial \phi^2} + (\tfrac{1}{2} kd)^2(\xi^2 - \eta^2)$$
$$\left. - \frac{\mu d Q q}{\hbar^2} (\xi + \eta)\right] \psi = 0. \qquad (2.4)$$

The equation is separable and its solution can be obtained in the form of the Lamé products:

$$\psi_{mn} = R_{mn}(A, C; \xi) S_{mn}(A, C; \eta) \begin{cases} \cos m\phi \\ \sin m\phi. \end{cases} \qquad (2.5)$$

Functions $R_{mn}(A, C; \xi)$ and $S_{mn}(A, C; \eta)$ satisfy ordinary differential equations

$$\frac{d}{d\xi}\left((\xi^2 - 1) \frac{d}{d\xi} R_{mn}(A, C; \xi)\right)$$
$$- \left(\Lambda_{mn}(A, C) - C^2\xi^2 + AC\xi + \frac{m^2}{\xi^2 - 1}\right) R_{mn}(A, C, \xi) = 0, \qquad (2.6)$$

$$\frac{d}{d\eta}\left((1 - \eta^2) \frac{d}{d\eta} S_{mn}(A, C; \eta)\right)$$
$$+ \left(\Lambda_{mn}(A, C) - C^2\eta^2 - AC\eta - \frac{m^2}{1 - \eta^2}\right) S_{mn}(A, C; \eta) = 0, \qquad (2.7)$$

where

$$C = \tfrac{1}{2} kd, \qquad A = (2\mu Q q)/k\hbar^2 \qquad (2.8)$$

and constant m is an integer, which comes from the single-value requirement of the wavefunction in Eq. (2.1). The separation constant $\Lambda_{mn}(A, C)$ is chosen so as to ensure that the solution $S_{mn}(A, C; \eta)$ of Eq. (2.7) which is free of logarithmic terms is regular in the entire range $-1 \le \eta \le 1$. The functions $S_{mn}(A, C; \eta)$ also appear in scattering of a charged particle by a dipole. For convenience we will refer to $S_{mn}(A, C; \eta)$ as the Coulomb spheroidal angle function and $R_{mn}(A, C; \eta)$ as the Coulomb spheroidal radial function.

When C vanishes, the differential Eq. (2.7) becomes the one which is satisfied by the associated Legendre functions. It follows that the angle functions must reduce to the associated Legendre functions of the integral order and degree, as C goes to zero. Therefore,

$$\Lambda_{mn}(0, 0) = n(n + 1). \qquad (2.9)$$

When C is not zero, Eq. (2.7) differs from the associated Legendre equation by having an irregular singularity at infinity. This suggests for the angle functions, an infinite sum of the form

$$S_{mn}(A, C; \eta) = \sum_{r=0}^{\infty} D_r^{mn} (A, C) P_{m+r}^m (\eta). \qquad (2.10)$$

Substitution of Eq. (2.10) in Eq. (2.7), with the subsequent

use of the associated Legendre differential equation and of the recursion formulas for the associated Legendre functions, yields the following recursion formulas for the coefficients $D_r^{mn}(A, C)$:

$$\frac{(2m + r + 2)(2m + r + 1)C^2}{(2m + 2r + 3)(2m + 2r + 5)} D_{r+2}^{mn}(A, C)$$
$$+ \frac{(2m + r + 1)AC}{(2m + 2r + 3)} D_{r+1}^{mn}(A, C)$$
$$+ \left((m + r)(m + r + 1) - \Lambda_{mn}(A, C)\right.$$
$$\left. + \frac{2(m + r)(m + r + 1) - 2m^2 - 1}{(2m + 2r - 1)(2m + 2r + 3)} C^2\right) D_r^{mn}(A, C)$$
$$+ \frac{rAC}{2m + 2r - 1} D_{r-1}^{mn}(A, C)$$
$$+ \frac{r(r - 1)C^2}{(2m + 2r - 3)(2m + 2r - 1)} D_{r-2}^{mn}(A, C) = 0. \qquad (2.11)$$

The recursion formula for the expansion coefficients constitutes a linear homogeneous difference equation of the fourth order. When A vanishes, the recursion formula reduces to one which is satisfied by the expansion coefficients $d_r^{mn}(C)$ of the spheroidal angle functions in terms of the associated Legendre functions. We now substitute in Eq. (2.11) the expansions

$$D_r^{mn}(A, C) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} B_{r(j,k)}^{mn} C^{2j}(AC)^k, \qquad (2.12)$$

$$\Lambda_{mn}(A, C) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} l_{(j,k)}^{mn} C^{2j}(AC)^k \qquad (2.13)$$

and use the perturbation method to calculate expansion coefficients $B_{r(j,k)}^{mn}$ and $l_{(j,k)}^{mn}$. The lower-order coefficients are found to be

$$l_{(0,0)}^{mn} = n(n + 1),$$

$$l_{(1,0)}^{mn} = \tfrac{1}{2} \left(1 - \frac{(2m - 1)(2m + 1)}{(2n - 1)(2n + 3)}\right),$$

$$l_{(2,0)}^{mn} = \tfrac{1}{2} \left(\frac{(n - m - 1)(n - m)(n + m - 1)(n + 3)}{(2n - 3)(2n - 1)^3(2n + 1)}\right.$$
$$\left. - \frac{(n - m + 1)(n - m + 2)(n + m + 1)(n + m + 2)}{(2n + 1)(2n + 3)^3(2n + 5)}\right)$$

$$l_{(0,1)}^{mn} = 0, \qquad (2.14)$$

$$l_{(0,2)}^{mn} = \frac{(n - m)(n + m)}{2n(2n - 1)(2n + 1)} - \frac{(n + m + 1)(n - m + 1)}{(2n + 3)^2(2n + 1)},$$

$$l_{(1,1)}^{mn} = 0,$$

$$B_{r(0,0)}^{mn} = \delta_{n-m,r},$$

$$B_{r(1,0)}^{mn} = \tfrac{1}{2} \delta_{n-m,r+2} \frac{(2m + r + 2)(2m + r + 1)}{(2m + 2r + 3)^2(2m + 2r + 5)}$$
$$- \tfrac{1}{2} \delta_{n-m,r-2} \frac{r(r - 1)}{(2m + 2r - 3)(2m + 2r - 1)},$$

$$B_{r(0,1)}^{mn} = \tfrac{1}{2} \delta_{n-m,r+1} \frac{(2m + 2r + 1)}{(2m + 2r + 3)(m + r + 1)}$$
$$- \tfrac{1}{2} \delta_{n-m,r-1} \frac{r}{(2m + 2r - 1)(m + r)}.$$

We have normalized the coefficients so that each Coulomb spheroidal angle function reduces exactly to the corresponding associated Legendre function when C becomes zero. From the general theory of Sturm–Liouville differential equations it follows that the functions $S_{mn}(A, C; \eta)$ form a complete orthogonal set on the interval $(-1, 1)$. Thus

$$\int_{-1}^{1} S_{mn}(A, C; \eta) S_{mn'}(A, C; \eta) d\eta = \delta_{nn'} N_{mn}(A, C), \quad (2.15)$$

where $N_{mn}(A, C)$ is easily found with the use of the normalization factor of the associated Legendre functions to be

$$N_{mn}(A, C) = 2 \sum_{r=0}^{\infty} \frac{(r + 2m)!}{(2r + 2m + 1)r!} [D_r^{mn}(A, C)]^2. \quad (2.16)$$

Near the end points $\eta = \pm 1$, the associated Legendre functions $P_n^m(\eta)$ have the following behavior:

$$P_n^m(\eta) = O[(1 - \eta^2)^{m/2}] \quad \text{for } (n - m) \text{ even,}$$
$$P_n^m(\eta) = O[\eta(1 - \eta^2)^{m/2}] \quad \text{for } (n - m) \text{ odd.} \quad (2.17)$$

From these relations, one obtains the following behavior of the Coulomb spheroidal wavefunction near $\eta = \pm 1$ with the aid of Eq. (2.10):

$$S_{mn}(A, C; \eta) = O[(1 - \eta^2)^{m/2}]. \quad (2.18)$$

3. RADIAL FUNCTION

The differential equation (2.6) satisfied by the Coulomb spheroidal radial function is very similar to one encountered in the scattering of two charged centers.[1] The difference between these two is in the parameter dependence. The differential equation (2.6) depends on the Coulomb spheroidal eigenvalue $\Lambda_{mn}(A, C)$. The other one depends on the spheroidal eigenvalue $\lambda_{mn}(C) \equiv \Lambda_{mn}(0, C)$. We have developed a method for solving the latter equation. In the method, an integro-differential equation is used to express the solution of the differential equation. After removal of the second-order derivative, the integro-differential equation is solved through an iteration procedure. The series is convergent under a proper substraction. The same method can also be directly applied to the present equation (2.6). Due to the complexity of the method and the messiness of the expression, we shall approach the present problem through a different path.

From the general theory of integral representations of solutions of differential equations, it follows that the function defined by the integral

$$\int_{a}^{b} K_m(\xi, \eta) S_{mn}(A, C; \eta) d\eta \quad (3.1)$$

is a solution of the Coulomb radial differential equation (2.6), provided that the limits a and b are so chosen that the bilinear concomitant,

$$(1 - \eta^2)\left(\frac{\partial K_m(\xi, \eta)}{\partial \eta} S_{mn}(A, C; \eta) - K_m(\xi, \eta)\frac{\partial}{\partial \eta} S_{mn}(A, C; \eta)\right), \quad (3.2)$$

vanishes at both limits, and that the kernel $K_m(\xi, \eta)$ satisfies the differential equation

$$\left(\frac{\partial}{\partial \eta}(1 - \eta^2)\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}(\xi^2 - 1)\frac{\partial}{\partial \xi}\right)$$
$$- m^2\left(\frac{1}{1 - \eta^2} + \frac{1}{\xi^2 - 1}\right) + C^2(\xi^2 - \eta^2)$$
$$- AC(\xi + \eta)]K_m(\xi, \eta) = 0. \quad (3.3)$$

By comparing Eq. (3.3) with Eq. (2.4) we observe that the kernel $K_m(\xi, \eta)$ is actually a solution of the Coulomb scattering equation[9] with the azimuthal angle ϕ dependence factored out. If a fixed charge Q is at the origin of a coordinate system, one of the solutions for the Coulomb scattering equation has the form

$$e^{(ik/2)(\xi_0 - \eta_0)}\xi_0^{m/2}\eta_0^{m/2} \, _1F_1(-\tfrac{1}{2}Ai; m + 1; ik\eta_0), \quad (3.4)$$

where the function $_1F_1(x; y; z)$ is a confluent hypergeometric function,[10] ξ_0 and η_0 are the parabolic coordinates of the above coordinate system. In the present investigation we used a different coordinate system and placed the fixed charge Q on the z axis with coordinates $r_a = (0, 0, \tfrac{1}{2}d)$. The relations of these two coordinate systems are as follows:

$$\eta_0 = \tfrac{1}{2}d(\xi + 1)(1 - \eta), \quad \xi_0 = \tfrac{1}{2}d(\xi - 1)(1 + \eta). \quad (3.5)$$

By using the above relations the solution in Eq. (3.4) may be expressed as

$$(\tfrac{1}{2}d)^m e^{ic(\xi\eta - 1)}(\xi^2 - 1)^{m/2}(1 - \eta^2)^{m/2} {}_1F_1(-\tfrac{1}{2}Ai; m + 1; ic(\xi + 1)(1 - \eta)). \quad (3.6)$$

It may be verified directly that the function in Eq. (3.6) satisfies the differential equation in Eq. (3.3). We denote this function as kernel $K_m^{(1)}(\xi, \eta)$. The two obvious limits a and b, at which the bilinear concomitant in Eq. (3.2) vanishes, are

$$a = -1, \quad b = 1. \quad (3.7)$$

The Coulomb spheroidal radial function based on the kernel $K_m^{(1)}(\xi, \eta)$ will be called $R_{mn}^{(1)}(A, C; \xi)$:

$$R_{mn}^{(1)}(A, C; \xi) \equiv \int_{-1}^{1} K_m^{(1)}(\xi, \eta) S_{mn}(A, C; \eta) d\eta. \quad (3.8)$$

The function $R_{mn}^{(1)}(A, C; \xi)$ is regular at $\xi = 1$, at which the differential equation (2.6) has a regular singularity. In the following we will discuss its asymptotic forms, which has a special interest in the scattering theory. After the substitution of the expansion of Coulomb spheroidal angle function $S_{mn}(A, C; \eta)$ in Eq. (2.10) to Eq. (3.8), we have

$$R_{mn}^{(1)}(A, C; \xi) = \sum_{r=0}^{\infty} D_r^{mn}(A, C)\int_{-1}^{1} K_m^{(1)}(\xi, \eta) P_{m+r}^m(\eta) d\eta$$
$$= \sum_{r=0}^{\infty} D_r^{mn}(A, C)\left(\frac{d}{2}\right)^m (\xi^2 - 1)^{m/2} \int_{-1}^{1} d\eta \, P_{m+r}^m(\eta)$$
$$\times e^{ic(\xi\eta - 1)}(1 - \eta^2)^{m/2}$$
$$\times \, _1F_1(-\tfrac{1}{2}Ai; m + 1; ic(\xi + 1)(1 - \eta)). \quad (3.9)$$

The confluent hypergeometric function $_1F_1(x; y; z)$ has the integral representation

$$_1F_1(\alpha; \gamma; z) = -\frac{1}{2\pi i}\frac{\Gamma(1 - \alpha)\Gamma(\gamma)}{\Gamma(\gamma - \alpha)} \times \oint_{c'} e^{tz}(-t)^{\alpha-1}(1 - t)^{\gamma-\alpha-1} dt. \quad (3.10)$$

If γ is a positive integer and $\text{Re}(\gamma - \alpha) > 0$, contour c' can be any contour which passes around both the points $t = 0$ and $= 1$. We recall the integration

$$\int_{-1}^{1} e^{i\eta z}(1 - \eta^2)^{m/2} P_{m+r}^m(\eta) d\eta = \frac{2i^n(2m + r)!}{r!}\frac{j_{m+r}(z)}{z^m}, \quad (3.11)$$

where $j_{m+r}(z)$ is a spherical Bessel function. After

using Eqs. (3.10) and (3.11) the function $R_{mn}^{(1)}(A, C; \xi)$ in Eq. (3.9) becomes

$$R_{mn}^{(1)}(A, C; \xi) = 2\left(\frac{d}{2}\right)^m (\xi^2 - 1)^{m/2} \frac{\Gamma(1 + \frac{1}{2}Ai)\Gamma(m + 1)}{\Gamma(m + 1 + \frac{1}{2}Ai)}$$

$$\times \sum_{r=0}^{\infty} D_r^{mn}(A, C) \frac{i^r (2m + r)!}{r!} K_r^m(A, C; \xi), \quad (3.12)$$

where

$$K_r^m(A, C; \xi) \equiv -\frac{1}{2\pi i} \oint_{c'} dt \{C[\xi(1 - t) - t]\}^{-m}$$

$$\times j_{m+r}\{c[\xi(1-t)-t]\}(- t)^{-(Ai/2)-1}(1 - t)^{m+Ai/2}$$

$$\times e^{ic[t(\xi+1)-1]}. \quad (3.13)$$

The spherical Bessel function is related to the confluent hypergeometric function

$$z^{-n}j_n(z) = \frac{\sqrt{\pi}}{2^{n+1}\Gamma(n + \frac{3}{2})} e^{-iz} {}_1F_1(n + 1; 2n + 2; 2iz). \quad (3.14)$$

The confluent hypergeometric function satisfies the Kummer transformation

$${}_1F_1(\alpha; \gamma; z) = e^z {}_1F_1(\gamma - \alpha; \gamma; - z). \quad (3.15)$$

With the help of Eqs. (3.14) and (3.15), we rewrite Eq. (3.13) as

$$K_r^m(A, C; \xi) = -\frac{1}{2\pi i} \frac{\sqrt{\pi} \, 2^{-(m+r+1)}}{\Gamma(m +r + \frac{3}{2})} e^{ic(\xi-1)}$$

$$\times \oint_{c'} dt [C\{\xi(1 - t) - t\}]^r (-t)^{-(Ai/2)-1}(1 - t)^{m+(Ai/2)}$$

$$\times {}_1F_1(m + r + 1; 2m + 2r + 2; - 2ic[\xi(1 - t) - t]). \quad (3.16)$$

The confluent hypergeometric function has a series expansion

$${}_1F_1(\alpha; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + k)}{\Gamma(\gamma + k)\Gamma(k + 1)} z^k. \quad (3.17)$$

The hypergeometric function has a contour integral representation

$${}_2F_1(\alpha, \beta; \gamma; z) \equiv -\frac{1}{2\pi i} \frac{\Gamma(1 - \alpha)\Gamma(\gamma)}{\Gamma(\gamma - \alpha)}$$

$$\times \oint_{c'} dt (-t)^{\alpha-1}(1 - t)^{\gamma-\alpha-1}(1 - tz)^{-\beta}. \quad (3.18)$$

From Eqs. (3.17) and (3.18) we may express Eq. (3.16) as

$$K_r^m(A, C; \xi) = \frac{2^{m+r}\Gamma(m + 1 + Ai/2)}{\Gamma(1 + Ai/2)\Gamma(m + 1)} e^{ic(\xi-1)}$$

$$\times \sum_k \frac{\Gamma(m + r + 1 + k)}{\Gamma(2m + 2r + 2 + k)\Gamma(k + 1)} (c\xi)^{r+k}(-2i)^k$$

$$\times {}_2F_1(-Ai/2, - r - k; m + 1; (\xi + 1)/\xi). \quad (3.19)$$

In reaching Eq. (3.19) these identities are also used:

$$\Gamma(\tfrac{1}{2}) = \pi,$$

$$\Gamma(n + \tfrac{1}{2}) = \frac{1.3.5\ldots(2n - 1)}{2^n} \Gamma(\tfrac{1}{2}). \quad (3.20)$$

Equation (3.12) together with Eq. (3.19) yields a complete series solution of the Coulomb spheroidal radial function $R_{mn}^{(1)}(A, C; \xi)$. If we used the general method,[1]

which was previously suggested to solve the integro-differential equation arising from the spheroidal scattering problem we would obtain another series expansion for the function $R_{mn}^{(1)}(a, C, \xi)$. However the latter expansion has more complicated form. The asymptotic form of the function $R_{mn}^{(1)}(A, C; \xi)$ has considerable interests in the scattering theory. This form follows directly from the series expansion presented here.

The hypergeometric function satisfies a relation

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} {}_2F_1(\alpha, \beta; \alpha + \beta + 1 - \gamma;$$

$$1 - z) + \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1 - z)^{\gamma-\alpha-\beta}$$

$$\times {}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma + 1 - \alpha - \beta; 1 - z). \quad (3.21)$$

Equation (3.21) leads us to the following asymptotic form function $K_r^m(A, C; \xi)$ in Eq. (3.19):

$$K_r^m(A, C; \xi) \xrightarrow[\xi \to \infty]{} \frac{2^{m+r}\Gamma(m + r + 1 + Ai/2)}{\Gamma(1 + Ai/2)\Gamma(2m + 2r + 2)} (c\xi)^r e^{ic(\xi-1)}$$

$$\times {}_1F_1(m + r + 1 + Ai/2; 2m + 2r + 2; - 2ic\xi). \quad (3.22)$$

In arriving at Eq. (3.22), we also used Eq. (3.17) and the equation

$$1/\Gamma(-n) = 0, \quad n = 0, 1, 2, \cdots. \quad (3.23)$$

The spherical regular Coulomb wave function $F_L(\tfrac{1}{2}A, \rho)$ has the form

$$F_L(A/2, \rho) = C_L(A/2)\rho^{L+1}e^{i\rho} {}_1F_1(L + 1 + Ai/2,$$

$$2L + 2; - 2i\rho), \quad (3.24)$$

where

$$C_L\left(\frac{A}{2}\right) = \frac{2^L e^{-\pi A/4}|\Gamma(L + 1 + iA/2)|}{\Gamma(2L + 2)}. \quad (3.25)$$

From Eqs. (3.12), (3.22), and (3.24) we obtain the asymptotic form of the Coulomb spheroidal radial function

$$R_{mn}^{(1)}(A, C; \xi) \xrightarrow[\xi \to \infty]{} \frac{2\Gamma(m + 1)}{C\xi k^m \Gamma(m + 1 + Ai/2)} \exp\left(-ic + \frac{\pi A}{4}\right)$$

$$\times \sum_r D_r^{mn}(A, C) \frac{i^r (2m + r)!}{r!} e^{i\sigma_{m+r}} F_{m+r}\left(\frac{A}{2}, C\xi\right), \quad (3.26)$$

where

$$\sigma_{m+r} = \arg\Gamma(m + r + 1 + Ai/2) \quad (3.27)$$

is the spherical Coulomb phase shift. The spherical regular Coulomb wave function $F_L(A/2, \rho)$ has the asymptotic form

$$F_L(\tfrac{1}{2}A, \rho) \xrightarrow[\rho \to \infty]{} \sin(\rho - \tfrac{1}{2}A \ln 2\rho - \tfrac{1}{2}L\pi + \sigma_L). \quad (3.28)$$

The final asymptotic form of the Coulomb spheroidal radial function follows directly from Eqs. (3.26) and (3.28);

$$R_{mn}^{(1)}(A, C; \xi) \xrightarrow[c\xi \to \infty]{} \frac{2\Gamma(m + 1)}{c\xi k^m \Gamma(m + 1 + \frac{1}{2}Ai)} \exp\left(-ic + \frac{\pi A}{4}\right)$$

$$\times (H_{mn}^2 + I_{mn}^2)^{\frac{1}{2}} \sin[c\xi - \tfrac{1}{2}A \ln(2c\xi) - \tfrac{1}{2}n\pi + \sigma_n + \Sigma_n^m], \quad (3.29)$$

where

$$H_{mn} = \sum_{r=0} D_r^{mn}(A,C) \frac{i^r(2m+r)!}{r!} e^{i\sigma_{m+r}}$$

$$\times \cos[(n-m-r)\pi/2 + \sigma_{m+r} - \sigma_n],$$

$$I_{mn} = \sum_{r=0} D_r^{mn}(A,C) \frac{i^r(2m+r)!}{r!} e^{i\sigma_{m+r}} \qquad (3.30)$$

$$\times \sin[(n-m-r)\pi/2 + \sigma_{m+r} - \sigma_n],$$

$$\Sigma_n^m = \tan^{-1}[I_{mn}/H_{mn}].$$

4. SCATTERING AMPLITUDE

In this section the spheroidal expression of the Coulomb scattering amplitude is discussed. The complete Coulomb wavefunction (incident plus scattered wave) is known. This wavefunction is the starting point of the present discussion. If the fixed charge Q is at the origin of the spherical coordinate system, the complete Coulomb wavefunction has the form

$$U_c = v^{-1/2} \Gamma(1 + \tfrac{1}{2}Ai)e^{-A\pi/4}$$

$$\times e^{ikr_0 \cos\Theta} {}_1F_1(-\tfrac{1}{2}iA; 1; 2ikr_0 \sin^2\tfrac{1}{2}\Theta), \quad (4.1)$$

where v is the velocity of the incident particle, Θ is the angle between the position vector $\mathbf{r}_0$ and the incident momentum vector $\mathbf{k}$. In Eq. (4.1) the incident beam is normalized to unit flux. In the present prolate spheroidal coordinate system, the fixed charge is placed on the z axis with coordinates $\mathbf{r}_a = (0, 0, \tfrac{1}{2}d)$. The relations between these two coordinate systems are as follows:

$$r_0 = \tfrac{1}{2}d(\xi - \eta), \qquad (4.2)$$

$$z_0 = \tfrac{1}{2}d(\xi\eta - 1). \qquad (4.3)$$

The Coulomb wavefunction in Eq. (4.1) is regular at $\xi = 1$, and satisfies Eq. (2.4). This function has the following asymptotic forms:

$$U_c \xrightarrow[\xi\to\infty]{} v^{-1/2}\Bigg[\exp i\{c\xi\cos\Theta + \tfrac{1}{2}A\ln[c\xi(1-\cos\Theta)]\}$$

$$\times \left(1 + \frac{A^2}{4ic\xi(1-\cos\Theta)}\right)U(\Theta)$$

$$+ \frac{k}{c\xi} f_c(\Theta)\exp i\left(c\xi - \frac{A}{2}\ln(2c\xi)\right)\Bigg] \qquad (4.4)$$

or

$$U_c \xrightarrow[\xi\to\infty]{} v^{-1/2}\frac{1}{2ic\xi} \sum_{n=0}^{\infty} \{\exp i(c\xi - \tfrac{1}{2}A\ln(2c\xi) + 2\sigma_n)$$

$$- \exp(-i)[c\xi - n\pi - \tfrac{1}{2}A\ln(2c\xi)]\}(2n+1)P_n(\cos\Theta), \qquad (4.5)$$

where $U(\Theta)$ is a step function

$$U(\Theta) = \begin{cases} 1, & \text{for } \Theta > 0, \\ 0, & \text{for } \Theta \le 0, \end{cases} \qquad (4.6)$$

and $f_c(\Theta)$ is the Coulomb scattering amplitude

$$f_c(\Theta) = \frac{A}{4k\sin^2\tfrac{1}{2}\Theta} \exp(-i)[\tfrac{1}{2}A\ln(\sin^2\tfrac{1}{2}\Theta) - \pi - 2\sigma_0]$$

$$= \frac{1}{2ik} \sum_{n=0}^{\infty} (2n+1)e^{2i\sigma_n}P_n(\cos\Theta). \qquad (4.7)$$

The angle Θ can be expressed as

$$\cos\Theta = \cos\theta_0 \cos\theta' + \sin\theta_0 \sin\theta' \cos(\phi - \phi'), \quad (4.8)$$

where the incident momentum $\mathbf{k}$ and the position vector

$\mathbf{r}_0$ have the spherical coordinates θ', ϕ' and θ_0, $\phi_0 \equiv \phi$, respectively. In the asymptotic region there is no difference between the spheroidal coordinate η and $\cos\theta_0$:

$$\cos\Theta \xrightarrow[\xi\to\infty]{} \eta\cos\theta' + (1-\eta^2)^{1/2}\sin\theta'\cos(\phi-\phi'). \quad (4.9)$$

Now we wish to express the Coulomb scattering amplitude $f_c(\Theta)$ in terms of the spheroidal analysis. To start with the complete Coulomb wavefunction U_c in Eq. (4.1) is expanded as

$$U_c = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} B_{mn}(c, \theta')R_{mn}^{(1)}(A,C;\xi)S_{mn}(A,C,\eta)\cos[m(\phi-\phi')]. \qquad (4.10)$$

In writing down Eq. (4.10), we use the fact that the Coulomb wavefunction can be expressed in terms of the Lamé products in Eq. (2.5) and is rotational invariant along the z axis. The expansion coefficients have the form

$$B_{mn}(c, \theta')R_{mn}^{(1)}(A,C,\xi) = [2\pi N_{mn}(A,C]^{-1}(2-\delta_{0m})$$

$$\times \int_{-1}^{1} d\eta \int_0^{2\pi} d\phi U_c S_{mn}(A,C,\eta)\cos[m(\phi-\phi')]. \qquad (4.11)$$

To find function $B_{mn}(c, \theta')$, we would like to discuss the asymptotic behavior of the above equation on both sides. The asymptotic form on the left-hand side of Eq. (4.11) is easy to find and follows directly from Eqs. (3.26) and (3.28):

$$\frac{2\Gamma(m+1)}{c\xi k^m \Gamma(m+1+\tfrac{1}{2}Ai)} B_{mn}(c, \theta')\exp\left(-ic + \frac{\pi A}{4}\right)$$

$$\times \sum_{r=0}^{\infty} D_r^{mn}(A,C)\frac{i^r(2m+r)!}{r!}e^{\sigma_{m+r}}$$

$$\times \sin[c\xi - \tfrac{1}{2}A\ln(2c\xi) - \tfrac{1}{2}(m+r)\pi + \sigma_{m+r}]. \qquad (4.12)$$

The asymptotic form on the right-hand side of Eq. (4.11) deserves some careful consideration. The complete Coulomb wavefunction in Eq. (4.1) can be expanded as

$$U_c = v^{-1/2}e^{-A\pi/4} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{\Gamma(n+1+\tfrac{1}{2}iA)}{(2n)!}(2ikr_0)^n$$

$$\times e^{ikr_0} {}_1F_1(n+1+\tfrac{1}{2}iA; 2n+2; -2ikr_0)$$

$$\times (2-\delta_{0m})\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta_0)P_n^m(\cos\theta')$$

$$\cos[m(\phi'-\phi_0)]. \qquad (4.13)$$

In the asymptotic region $\xi \to \infty$, we have

$$r_0 \to \tfrac{1}{2}d\xi, \qquad \cos\theta_0 \to \eta_0. \qquad (4.14)$$

By using Eqs. (4.13) and (4.14), the asymptotic form on the right-hand side of Eq. (4.11) can be written as

$$(2\pi N_{mn}(A,C))^{-1}(2-\delta_{0m})v^{-1/2}e^{-A\pi/4}$$

$$\times \sum_{m'=n'}^{\infty} \sum_{n'=0}^{\infty} \frac{\Gamma(n'+1+\tfrac{1}{2}iA)}{(2n')!}(2ic\xi)^{n'}e^{ic\xi}(2-\delta_{0m'})$$

$$\times \frac{(n'-m')!}{(n'+m')!}P_{n'}^{m'}(\cos\theta')_1F_1(n'+1+\tfrac{1}{2}iA; 2n'+2; -2ic\xi)$$

$$\times \int_{-1}^{1} d\eta \int_0^{2\pi} d\phi S_{mn}(A,C;\eta)P_{n'}^{m'}(\eta)\cos m'(\phi'-\phi)$$

$$\times \cos m(\phi'-\phi). \qquad (4.15)$$

The associated Legendre functions are orthogonal:

$$\int_{-1}^{1} d\eta P_n^m(\eta) P_{n'}^m(\eta) = \delta_{nn'} \frac{2(n+m)!}{(2n+1)(n-m)!} . \quad (4.16)$$

We carry out the integrations in Eq. (4.15) through the help of Eqs. (2.10) and (4.16) and obtain

$$\frac{2(2-\delta_{0m})}{N_{mn}(A,C)} v^{-1/2} e^{-A\pi/4} \sum_{n'=m}^{\infty} \frac{\Gamma(n'+1+iA/2)}{(2n')!}$$

$$\times (2ic\xi)^{n'} e^{ic\xi} \frac{1}{2n'+1} D_{n'-m}^{mn}(A,C) P_{n'}^m(\cos\theta')$$

$$\times {}_1F_1(n'+1+\tfrac{1}{2}iA; 2n'+2; -2ic\xi). \quad (4.17)$$

From Eqs. (3.24) and (3.28) we obtain the final asymptotic form on the right-hand side of Eq. (4.11):

$$\frac{2(2-\delta_{0m})}{c\xi N_{mn}(A,C)} v^{-1/2} \sum_{r=0} i^{r+m} e^{i\sigma_{r+m}} D_r^{mn}(A,C)$$

$$\times P_{r+m}^m(\cos\theta') \sin[c\xi - \tfrac{1}{2} A \ln(2c\xi)$$

$$- \tfrac{1}{2}(r+m)\pi + \sigma_{m+r}]. \quad (4.18)$$

The function $B_{mn}(c,\theta')$ is then found by matching Eqs. (4.12) and (4.18):

$$B_{mn}(c,\theta') = \frac{(ik)^m \Gamma(m+1+\tfrac{1}{2}iA)}{\Gamma(m+1)} \exp\left(ic - \frac{A\pi}{4}\right) \frac{(2-\delta_{0m})}{N_{mn}(A,C)}$$

$$\times v^{-1/2} \left(\sum_{r=0}^{\infty} (-1)^r \frac{(2m+r)!}{r!} D_r^{mn}(A,C) \right)^{-1}$$

$$\times S_{mn}(A,C; -\cos\theta'). \quad (4.19)$$

The spherical expression of the Coulomb scattering amplitude in Eq. (4.7) is obtained through the comparison of the exponential dependence on the factor $\exp i[c\xi - \tfrac{1}{2}A \ln(2c\xi)]$ between the asymptotic form in Eq. (4.4) and the spherical asymptotic expansion in Eq. (4.5) of the complete Coulomb wavefunction. By the same token, we obtain the spheroidal expression of the Coulomb scattering amplitude through the comparison between the asymptotic form in Eq. (4.4) and the spheroidal asymptotic expansion, which follows from Eqs. (4.10) and (4.12):

$$f_c(\theta) = v^{1/2}(-i) \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{\Gamma(m+1)}{k^{m+1}\Gamma(m+1+\tfrac{1}{2}Ai)}$$

$$\times \exp\left(-ic + \frac{\pi A}{4}\right) \left(\sum_{r=0}^{\infty} D_r^{mn}(A,C) \frac{(-i)^m(2m+r)!}{r!} \right)$$

$$\times e^{2i\sigma_{m+r}} \right) B_{mn}(c,\theta') S_{mn}(A,C;\eta) \cos m(\phi - \phi')$$

$$= \frac{1}{ik} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{2-\delta_{0m}}{N_{mn}(A,C)} e^{2i\tilde\sigma_{mn}} S_{mn}(A,C; -\cos\theta')$$

$$\times S_{mn}(A,C;\eta) \cos m(\phi - \phi'), \quad (4.20)$$

where

$$e^{2i\tilde\sigma_{mn}} = \sum_{r=0}^{\infty} \frac{(2m+r)!}{r!} D_r^{mn}(A,C) e^{2i\sigma_{m+r}} \Big/$$

$$\sum_{r=0}^{\infty} (-)^r \frac{(2m+r)!}{r!} D_r^{mn}(A,C). \quad (4.21)$$

The quantity $\tilde\sigma_{mn}$ may be called the spheroidal Coulomb phase shift and is not real. For a real spheroidal potential we always have a real spheroidal phase shift.[1] The complexedness of the spheroidal Coulomb phase shift $\tilde\sigma_{mn}$ may be the cause of the dual spherical[11] and spheroidal nature for the Coulomb potential

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