

PERTURBATION ANALYSIS OF SOME AXISYMMETRIC PROBLEMS

IN PLATES AND SHELLS

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NOMENCLATURE

ϵ	a small perturbation parameter
ζ	independent outer variable in Cochran's method
η	independent inner variable
t	thickness of shell
h	thickness of plate
a	measure of radius (a constant)
u	radial displacement in a circular plate
w	displacement normal to a surface
ω	nondimensionalized displacement (w/a)
ρ, θ	polar coordinates
r	dimensionless radial coordinate (ρ/a)
M_x, N_θ	stress resultants in a circular cylinder
M_s, N_θ, Q_s	stress resultants in a conical shell
$M_\rho, M_\theta, N_\rho, N_\theta$	stress resultants in a circular plate
ψ	stress function
G	nondimensionalized stress function (ψ/B_1)
B_1	a constant having units of pounds and depending upon the physical parameters in the problem for its magnitude
L_i, λ_i, δ_i	perturbation parameters
β	the rotation of an element on the mid-surface of a circular plate
K, \bar{K}	dimensionless constants
$O()$	the order symbol
Plus miscellaneous symbols defined in the text.	

CHAPTER I

INTRODUCTION

This investigation demonstrates the applicability of perturbation methods to the solutions of some linear and nonlinear problems in the theory of plates and shells. Solutions are obtained for conical and circular cylindrical shells of variable thickness undergoing axisymmetric linear deformation. At the present, solutions exist [4, 5]* for these problems but only for specific types of thickness variation. A more general class of thickness variation is treated here by using a new perturbation method. In addition to the linear shell problems a study is made of the nonlinear post-buckling behavior of a radially loaded circular plate and of the nonlinear bending of a circular plate with edge moments. The latter problems demonstrate how a perturbation method can be used to evaluate and compare classical linear and nonlinear (von Karman) theory with finite deflection (Reissner) theory. Using Reissner's finite deflection theory for circular plates [17], one can show that in some cases the von Karman equations give acceptable accuracy for rotations greater than order one, even though from an order of magnitude viewpoint this is not to be expected.

To apply the method of perturbation, one must first nondimensionalize the governing differential equations. Upon combining the various constants in the equations nondimensional parameters may appear which for practical applications may be small. In an attempt to solve the differential equations, the dependent variable is expanded in a power series in terms of these small parameters. The various

* Numbers in square brackets refer to the bibliography.

approximations to the differential equations are obtained by substituting this series into the governing differential equations and equating to zero the coefficients of like powers of the small parameters [1, 23].

Occasionally the straightforward perturbation expansion fails due to a loss of higher order derivatives and corresponding loss of boundary conditions in the first approximate equations. A nonuniformity may also arise due to higher order terms being more singular than the first term in the series. When no irregularities exist the problem is called a regular or ordinary perturbation problem. Regions in plates and shells where nonuniformities exist are frequently called boundary-layers after Prandtl who first discovered them in fluid mechanics.

Various techniques exist for treating singular perturbation problems depending upon the type of nonuniformity. The techniques most appropriate for solving singular perturbation problems in plates and shells are the method of matched asymptotic expansions [23], the method of composite expansions [13], Langer's method [12] and Cochran's method [3]*. To date the last technique has not been used in the solution of either plate or shell problems.

The shell problems studied herein are solved by Cochran's method. Other investigators [4, 5, 20] have solved similar problems. However, they have used different methods of analysis and their solutions are restricted to particular thickness variations. The solutions presented here are for a more general class of thickness variation.

By a systematic order of magnitude study, Reissner's

* This method has been extended by Nayfeh [15].

equations for the finite deflections of circular plates are analyzed for two specific problems. The nonuniformities (or boundary-layers) are handled by the method of inner and outer expansions. This method leads directly to equations which facilitate the evaluation of the von Karman assumptions for large rotations. The von Karman equations are based on the assumption of rotations being less than order one. However, it is found that the equations are surprisingly good for the problems considered here even when the rotations are of order one.

CHAPTER II

REVIEW OF LITERATURE

A. Perturbation Methods

The method of matched asymptotic expansions is one of the most commonly used techniques for dealing with singular perturbation problems. For a brief history of the development of this method the reader is referred to Van Dyke [23, pp. 77]. This method has been used in a less formal manner in the area of plates and shells by Reissner [18] and by Johnson and Reissner [10]. A formal application of the method to a nonlinear plate problem has been made by Tolefson [22].

One of the early methods developed for solving singular perturbation problems is due to Langer [12]. However, his method is limited to second order differential equations of a particular type and is only good for obtaining a first approximation. Clark [2] used this method to solve the toroidal shell equations.

In 1951 Latta [13] presented a formal discussion of what is referred to as the method of composite expansions [23]. In this method a uniformly valid expansion is constructed from knowledge of the form of the inner solution. The basic idea in obtaining the form for the inner expansion is found in the works of Hildebrand [9] who investigated axisymmetric deformations of shells of constant thickness. Other work in this area can be found in Flugge [5] who reviews Geckeler's and Blumenthal's asymptotic solutions of similar problems.

In 1962 Cochran introduced a method which is sometimes called the method of multiple scales. He demonstrated, by example, how the method may be equal to or superior to the methods mentioned above in particular cases. Nayfeh [15, 16] presented some additional work on Cochran's method in 1964 and 1965. No trace can be found of the use of Cochran's method in the solution of either plate or shell problems.

B. Shells of Variable Thickness

DeSilva and Naghdi [4] have solved some problems on the axisymmetric linear deformation of shells with an allowance for a variation in thickness. They use the Langer method. However, as noted previously, the Langer method is only applicable to equations of a particular type. Therefore, it is necessary to put a restriction on the shell thickness variation to obtain this form. The restriction is in terms of a differential equation for the thickness function. This restriction can be removed and higher order approximations obtained through the use of Cochran's method.

Other solutions for shells of variable thickness exist in the literature, but they all deal with more specific types of variation. Spotts [20] solves some spherical shell problems with axisymmetric deformation for the case where the thickness is a quadratic function of the coordinate of latitude. References to the work of other authors can be found in Flugge's book [5].

C. Large Deflection of Circular Plates

Reissner [17] has developed differential equations for the finite axisymmetric deflection of plates. In his work he has shown that the von Karman equations can be obtained by expanding the sine and cosine terms containing the angle of rotation in a Taylor series. Terms are kept up to and including the second degree. This approximation to the full Reissner equations should break down when rotations approach an order of magnitude of one. The solutions given here determine the limitations of this assumption for large finite rotations in particular problems.

Hart and Evans [8] and Tolefson [22] have compared the solution of the Reissner equations to the solution of the von Karman equations for an annular plate. Hart and Evans solved the full Reissner equations by a numerical method. However, they did not carry out their solutions for large enough deflections to establish the point at which the von Karman approximation of the angle of rotation breaks down in their problem. They give solutions of the equations for small finite rotations and find that for such rotations the von Karman theory agrees quite well with the Reissner theory. However, this is to be expected, since then the sine and cosine of the angle of rotation are approximated quite well by the first terms of a series expansion. Tolefson obtained the boundary-layer equations for large finite rotations but did not give their solution. Like Hart and Evans he did not determine the extent to which it is practical to use the von Karman equations for large rotations in the annular plate problem.

Way [24] has solved the von Karman equations for the bending of a circular plate loaded only by a uniformly distributed bending moment at its edge. He considered small finite rotations and used the method of Frobenius to obtain his solution.

Friedrichs and Stoker [6, 7] solved the von Karman equations for the post-buckling of a circular plate loaded only by a uniformly distributed radial force. They used both a perturbation method and the method of Frobenius. Keller and Reiss [11] also solved the von Karman equations for the same problem. They used a numerical integration method.

CHAPTER III

A CIRCULAR CYLINDRICAL SHELL OF VARIABLE THICKNESS

The axisymmetric linear deformation of a circular cylindrical shell of variable thickness will now be considered. The surface loading is normal to the shell surface and, like the thickness, is assumed to vary in an arbitrary, but smooth, manner along the axis of the shell. This problem demonstrates the utilization of perturbation methods in solving differential equations with variable coefficients.

A. Governing Differential Equation

Consider the non-dimensionalized differential equation,

$$\epsilon^2 [f^3(\bar{x}) \omega''(\bar{x})]'' + b f(\bar{x}) \omega(\bar{x}) = P(\bar{x}) \quad (3.1)$$

where,

$$\epsilon^2 \ll 1. \quad (3.2)$$

All other quantities in (3.1) are assumed to be of order one. Differentiation with respect to the argument of a function will always be represented by a prime. Equation (3.1) has the same form as the differential equation governing the axisymmetric linear deformation of a circular cylinder having a pressure loading and thickness that vary in an arbitrary, but smooth, fashion along its axis [5]. For such a shell the quantities in (3.1) are defined as:

$$f = \frac{t}{t_0}, \quad (3.3a)$$

$$\bar{x} = \frac{x}{a}, \quad (3.3b)$$

$$\omega = \frac{w}{a}, \quad (3.3c)$$

$$P = \frac{P_r a}{E t_0}, \quad (3.3d)$$

$$b = 1 \quad (3.3e)$$

and

$$\epsilon^2 = \frac{1}{12(1-\nu^2)} \frac{t_0^2}{a^2} \quad (3.3f)$$

where t is the thickness at a particular cross-section of the shell, a is the radial distance from the axis of revolution to the middle surface of the shell, x is the distance measured from the edge of the shell along its axis, t_0 is some reference thickness of the shell, P_r is the normal pressure, E is Young's modulus, ν is Poisson's ratio, and w is the radial displacement of a point on the middle surface of the shell. Although b is equal to one for the circular cylindrical shell it will be carried along as an unspecified constant of order one for the sake of generality.

In what follows it will be assumed that the boundary layer is at \bar{x} equal to zero only. This corresponds to a circular cylinder supported in a prescribed manner at \bar{x} equal to zero but free of external constraint at all other points. This is only done for convenience and is not a limitation of the method. Other problems could be handled by a simple extension of the approach.

To aid in the analysis let,

$$f_1 = f^3, \quad (3.4a)$$

$$f_2 = 6f^2 f', \quad (3.4b)$$

$$f_3 = 6(f')^2 f + 3f^2 f'', \quad (3.4c)$$

and,

$$f_4 = bf. \quad (3.4d)$$

Then (3.1) can be written as,

$$\epsilon^2 [f_1 \omega'''' + f_2 \omega'''' + f_3 \omega'''] + f_4 \omega = P. \quad (3.5)$$

A straightforward perturbation solution of this equation is obviously singular due to the loss of higher order derivatives in the first approximation.

The Langer method has been applied to a differential equation of a form similar to (3.5) by DeSilva and Naghdi [4]. They had to specify the form of the coefficients f_i , which depend upon the thickness variation, to use complex variables and reduce the problem to the solution of a second order differential equation. The method of inner and outer expansions could be used to solve (3.5), but Cochran's method reduces the amount of work.

The exact form of the f_i in (3.5) does not have to be specified when using Cochran's method. It is only necessary

to assume that they are smooth well behaved functions of order one in the region of interest. However, this is not an added restriction since this would also be assumed in applying the other methods.

B. Application of the Perturbation Method

Following Cochran [3] the singularity in (3.5) is taken care of by replacing \bar{x} by,

$$\eta = \frac{g(\zeta)}{\epsilon^{1/2}}, \quad (3.6a)$$

and,

$$\zeta = \bar{x}. \quad (3.6b)$$

The governing differential equation then becomes,

$$\begin{aligned} f_1 \{ & (g')^4 \omega_{\eta\eta\eta\eta} + \epsilon^{1/2} [6(g')^2 g'' \omega_{\eta\eta\eta} + 4(g')^3 \omega_{\zeta\eta\eta\eta}] \\ & + \epsilon [4g'g'''\omega_{\eta\eta} + 12g'g''\omega_{\zeta\eta\eta} + 3(g'')^2 \omega_{\eta\eta} \\ & + 6(g')^2 \omega_{\zeta\zeta\eta\eta}] \} + f_2 \{ \epsilon^{1/2} (g')^3 \omega_{\eta\eta\eta} + \epsilon [3g'g'''\omega_{\eta\eta} \\ & + 3(g')^2 \omega_{\zeta\eta\eta}] \} + f_3 \{ \epsilon (g')^2 \omega_{\eta\eta} \} + f_4 \omega + O(\epsilon^{3/2}) = P. \end{aligned} \quad (3.7)$$

Let,

$$\omega(\bar{x}, \epsilon) = \omega(\zeta, \eta, \epsilon) = \sum_{m=0}^{\infty} \omega_m(\zeta, \eta) \epsilon^{m/2}, \quad (3.8)$$

* This is the order symbol defined by Van Dyke [23], $f(\epsilon) = O[\delta(\epsilon)]$ as $\epsilon \rightarrow 0$ if $\lim_{\epsilon \rightarrow 0} \frac{f(\epsilon)}{\delta(\epsilon)} < \infty$.

and then substitute into (3.7) and equate the coefficients of like powers of ϵ to get:

First order terms (ϵ^0),

$$f_1(g')^4 \omega_{0\eta\eta\eta\eta} + f_4 \omega_0 = P, \quad (3.9)$$

Second order terms ($\epsilon^{1/2}$),

$$f_1(g')^4 \omega_{1\eta\eta\eta\eta} + f_4 \omega_1 = -[6f_1(g')^2 g'' + f_2(g')^3] \omega_{0\eta\eta\eta} - 4f_1(g')^3 \omega_{0\zeta\eta\eta\eta}, \quad (3.10)$$

Third order terms (ϵ),

$$f_1(g')^4 \omega_{2\eta\eta\eta} + f_4 \omega_2 = -[6f_1(g')^2 g'' + f_2(g')^3] \omega_{1\eta\eta\eta} - 4f_1(g')^3 \omega_{1\zeta\eta\eta\eta} - [4f_1 g' g'' + 3f_1(g'')^2 + 3f_2 g' g'' + f_3(g')^2] \omega_{0\eta\eta} - [12f_1 g' g'' + 3f_2(g')^2] \omega_{0\zeta\eta\eta} - 6f_1(g')^2 \omega_{0\zeta\zeta\eta\eta}, \quad (3.11)$$

and so on for higher order approximations. However, higher order approximations are of no immediate use in the practical application of the results since it will be found that for

most problems the number of terms taken here will be sufficient.

1. First Approximation

The first approximation is obtained from (3.9) and (3.10). Using the relations given in (3.4) equation (3.9) becomes:

$$\omega_{onnnn} + 4\beta^4 \omega_o = \frac{P(\zeta)}{(g')^4 f_1} , \quad (3.12)$$

where,

$$\beta^4 = \frac{b}{(g')^4 4f^2} . \quad (3.13)$$

Satisfying the requirement of boundedness [3],

$$\omega_o = \omega_{oP} + [C_o(\zeta) \cos \beta\eta + D_o(\zeta) \sin \beta\eta] e^{-\beta\eta} , \quad (3.14)$$

where,

$$\omega_{oP} = \frac{P(\zeta)}{f_4(\zeta)} . \quad (3.15)$$

The quantities $g(\zeta)$, $C_o(\zeta)$ and $D_o(\zeta)$ are determined on the basis of what will be referred to as Lighthill's principle. Lighthill's principle requires that, "each approximation shall be no more singular than its predecessor-or vanish no more slowly-as $\epsilon \rightarrow 0$ for arbitrary values of the independent variables" [23].

Applying Lighthill's principle to the second order problem (3.10) gives:

$$0 = \omega_{1\eta\eta\eta\eta} + 4\beta^4 \omega_1, \quad (3.16)$$

and

$$0 = [6(g')^2 g'' f_1 + (g')^3 f_2] \omega_{0\eta\eta\eta\eta} + 4f_1 (g')^3 \omega_{0\zeta\eta\eta\eta}. \quad (3.17)$$

Substituting (3.14) into (3.17) and setting the coefficients of $e^{-\beta\eta} \cos \beta\eta$, $\eta e^{-\beta\eta} \cos \beta\eta$, $e^{-\beta\eta} \sin \beta\eta$, $\eta e^{-\beta\eta} \sin \beta\eta$ equal to zero gives:

$$0 = \beta', \quad (3.18a)$$

$$0 = [F_{02} \beta^3 + 3F_{01} \beta^2 \beta'] D_0 + F_{01} \beta^3 D'_0, \quad (3.18b)$$

$$0 = [F_{02} \beta^3 + 3F_{01} \beta^2 \beta'] C_0 + F_{01} \beta^3 C'_0, \quad (3.18c)$$

where,

$$F_{01} = 4f_1 (g')^3, \quad (3.19a)$$

and,

$$F_{02} = 6f_1 (g')^2 g'' + f_2 (g')^3. \quad (3.19b)$$

Integration of equations (3.18) gives:

$$\eta = \frac{1}{\varepsilon^{1/2}} \left(\frac{b}{4}\right)^{1/4} \int_0^{\bar{x}} \frac{d\zeta}{f^{1/2}}, \quad (3.20)$$

$$C_0 = \frac{C}{f^{3/4}}, \quad (3.21a)$$

and,

$$D_0 = \frac{D}{f^{3/4}}, \quad (3.21b)$$

where C and D are constants of integration to be determined from the boundary conditions at $\bar{x} = 0$. Since β is an arbitrary constant it has been set equal to one in the above without any loss in generality.

2. Second Approximation

The solution to the second order problem is obtained by following the same procedure as was used for the first approximation. The general form of the solution to the second order problem satisfying (3.16) is:

$$\omega_1 = [C_1(\zeta) \cos \eta + D_1(\zeta) \sin \eta] e^{-\eta}, \quad (3.22)$$

where η is given by (3.20), and C_1 and D_1 are found by applying Lighthill's principle to (3.11). This results in the following differential equations:

$$0 = \omega_{2\eta\eta\eta\eta} + 4\omega_2, \quad (3.23)$$

and,

$$\begin{aligned} 0 = & [6f_1(g')^2 g'' + f_2(g')^3] \omega_{1\eta\eta\eta} + 4f_1(g')^3 \omega_{1\zeta\eta\eta\eta} \\ & + [4f_1 g' g'' + 3f_1(g'')^2 + 3f_2 g' g'' + f_3(g')^2] \omega_{0\eta\eta} \\ & + [12f_1 g' g'' + 3f_2(g')^2] \omega_{0\zeta\eta\eta} + 6f_1(g')^2 \omega_{0\zeta\zeta\eta\eta}. \end{aligned} \quad (3.24)$$

Substituting ω_0 and ω_1 , from (3.14) and (3.22) respectively, into (3.24) it is found that,

$$C_1 = \frac{C_{01}}{f^{3/4}} - \frac{(C + D)}{8} \frac{F}{f^{3/4}}, \quad (3.25a)$$

and,

$$D_1 = \frac{D_{01}}{f^{3/4}} + \frac{(D - C)}{8} \frac{F}{f^{3/4}}, \quad (3.25b)$$

in which,

$$F = \int \left[\frac{7}{2} \frac{f''}{f^{1/2}} + \frac{3}{8} \frac{(f')^2}{f^{3/2}} \right] d\zeta. \quad (3.26)$$

C_{01} and D_{01} are constants of integration to be determined from the boundary conditions at \bar{x} equal to zero. The constant of integration in (3.26) is dropped since the second term in (3.25a) and (3.25b) corresponds to the particular solution of the differential equations governing C_1 and D_1 , respectively.

3. Summary of Results

From the preceding it is seen that a two term asymptotic approximation of ω to the differential equation (3.1) is given by,

$$\omega = \omega_0 + \omega_1 \epsilon^{1/2} + o(\epsilon^{3/2}), \quad (3.27)$$

where,

$$\omega_0 = \frac{P}{bf} + [C_0(\zeta) \cos \eta + D_0(\zeta) \sin \eta] e^{-\eta}, \quad (3.28a)$$

$$\omega_1 = [C_1(\zeta) \cos \eta + D_1(\zeta) \sin \eta] e^{-\eta}, \quad (3.28b)$$

$$C_0 = \frac{C}{f^{3/4}}, \quad (3.28c)$$

$$D_1 = \frac{D}{f^{3/4}}, \quad (3.28d)$$

$$C_1 = \frac{C_{01}}{f^{3/4}} - \frac{(C + D)}{8} \frac{F}{f^{3/4}}, \quad (3.28e)$$

$$D_1 = \frac{D_{01}}{f^{3/4}} + \frac{(D - C)}{8} \frac{F}{f^{3/4}}, \quad (3.28f)$$

$$\eta = \frac{1}{\varepsilon^{1/2}} \left(\frac{b}{4}\right)^{1/4} \int_0^{\bar{x}} \frac{d\zeta}{f^{1/2}}, \quad (3.28g)$$

and,

$$F = \int \left[\frac{7}{2} \frac{f''}{f^{1/2}} + \frac{3}{8} \frac{(f')^2}{f^{3/2}} \right] d\zeta. \quad (3.28h)$$

C , D , C_{01} , and D_{01} are constants of integration to be determined from the boundary conditions at \bar{x} equal to zero. The constant of integration in the expression for F is set equal to zero.

C. Discussion

When f is constant and P is a linear function of \bar{x} the perturbation solution reduces to the exact closed form solution,

$$\omega = \frac{P}{bf} + [C \cos \eta + D \sin \eta] e^{-\eta},$$

and,

$$\eta = \frac{1}{\varepsilon^{1/2}} \left(\frac{b}{4}\right)^{1/4} \bar{x},$$

where C and D are constants.

The exact solution also exists [5] for a circular cylindrical tank having a linearly varying thickness as shown in Figure 1. The exact solution and the perturbation solution of this problem are plotted in Figure 2 and 3 for two typical cases. The solution for a cylinder of constant thickness t_0 is also given for comparative purposes. Without a doubt the perturbation solution is satisfyingly accurate in all cases.

From Tables 1 and 2 of Appendix A it is seen that the second order effect is negligible and could have been dropped.

The exact solution for the circular cylindrical tank with linearly varying thickness is in terms of Kelvin functions. When the argument of a Kelvin function is larger than about ten it can be approximated by an asymptotic series [14]. This approximation was used in determining the values for the exact solution in Figure 3 and Table 2. In Figure 2 and Table 1 the exact series forms of the Kelvin functions were used.

The extension of the results to a circular cylindrical shell with other thickness variations involves only simple integrations and some elementary algebra to obtain the various terms in (3.2).

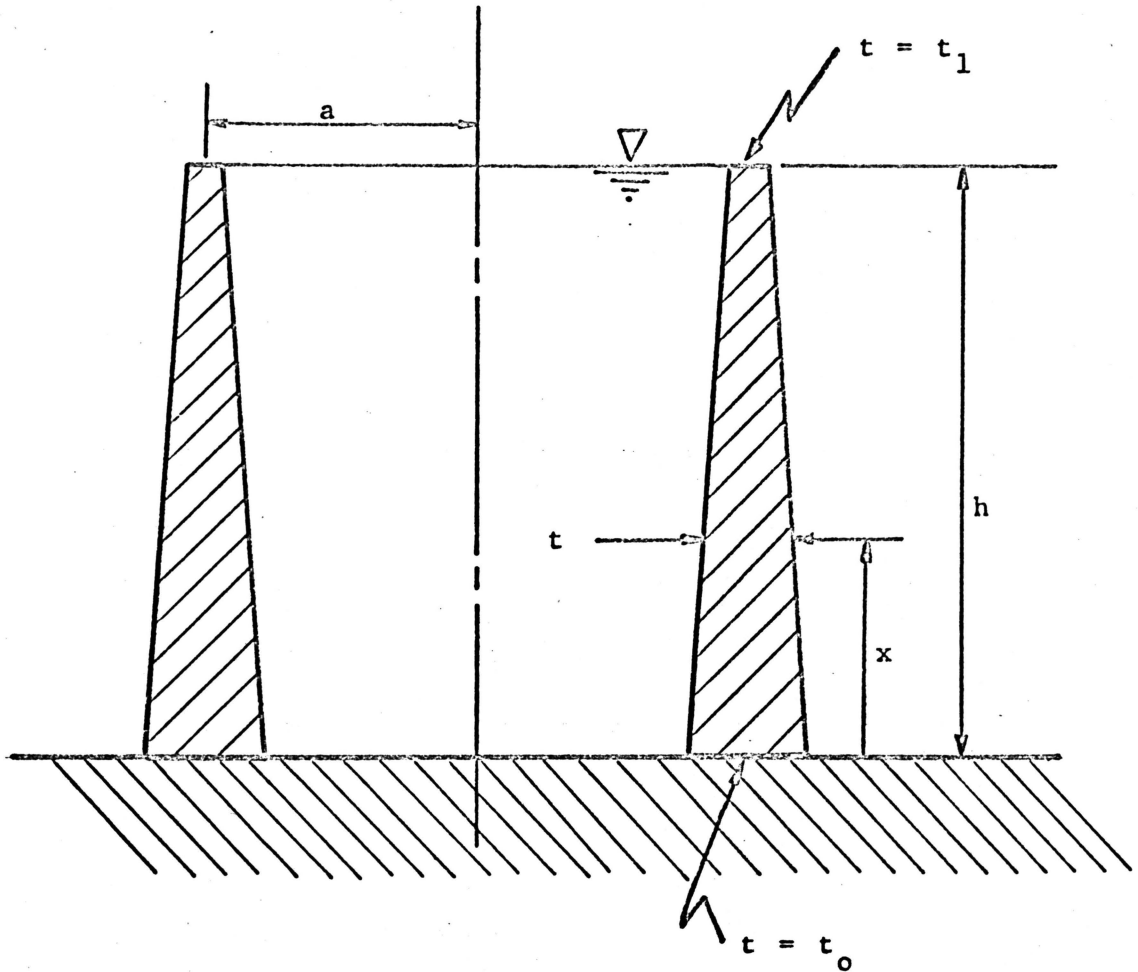


Figure 1. A CIRCULAR CYLINDRICAL TANK OF LINEARLY VARYING THICKNESS

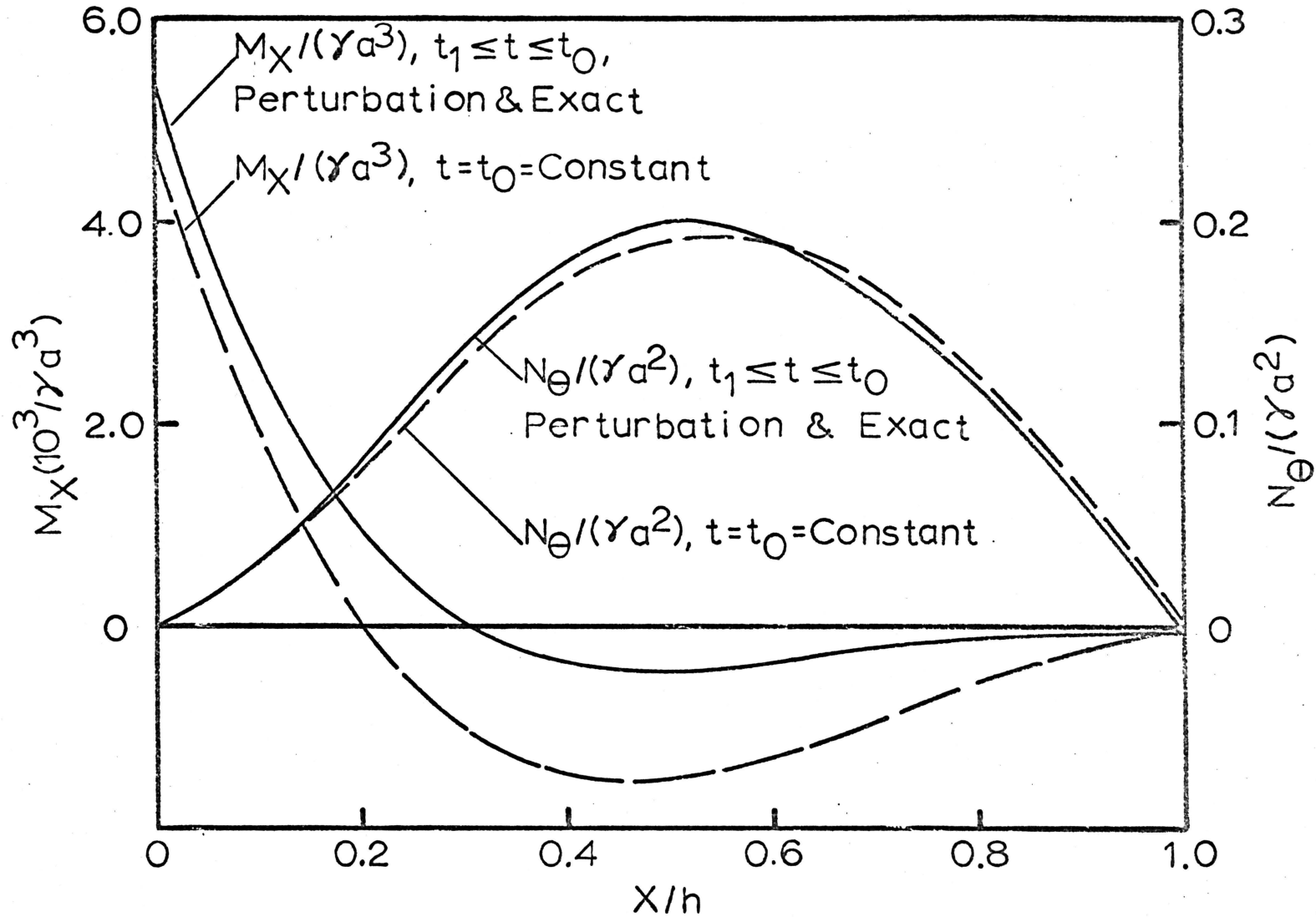


Figure 2. THE STRESS RESULTANTS IN A CIRCULAR CYLINDRICAL TANK WITH $t_0 = 14$ in., $t_1 = 2$ in., $r = 24$ ft., $h = 12$ ft., $\nu = 0.25$ AND $\epsilon = 0.014$

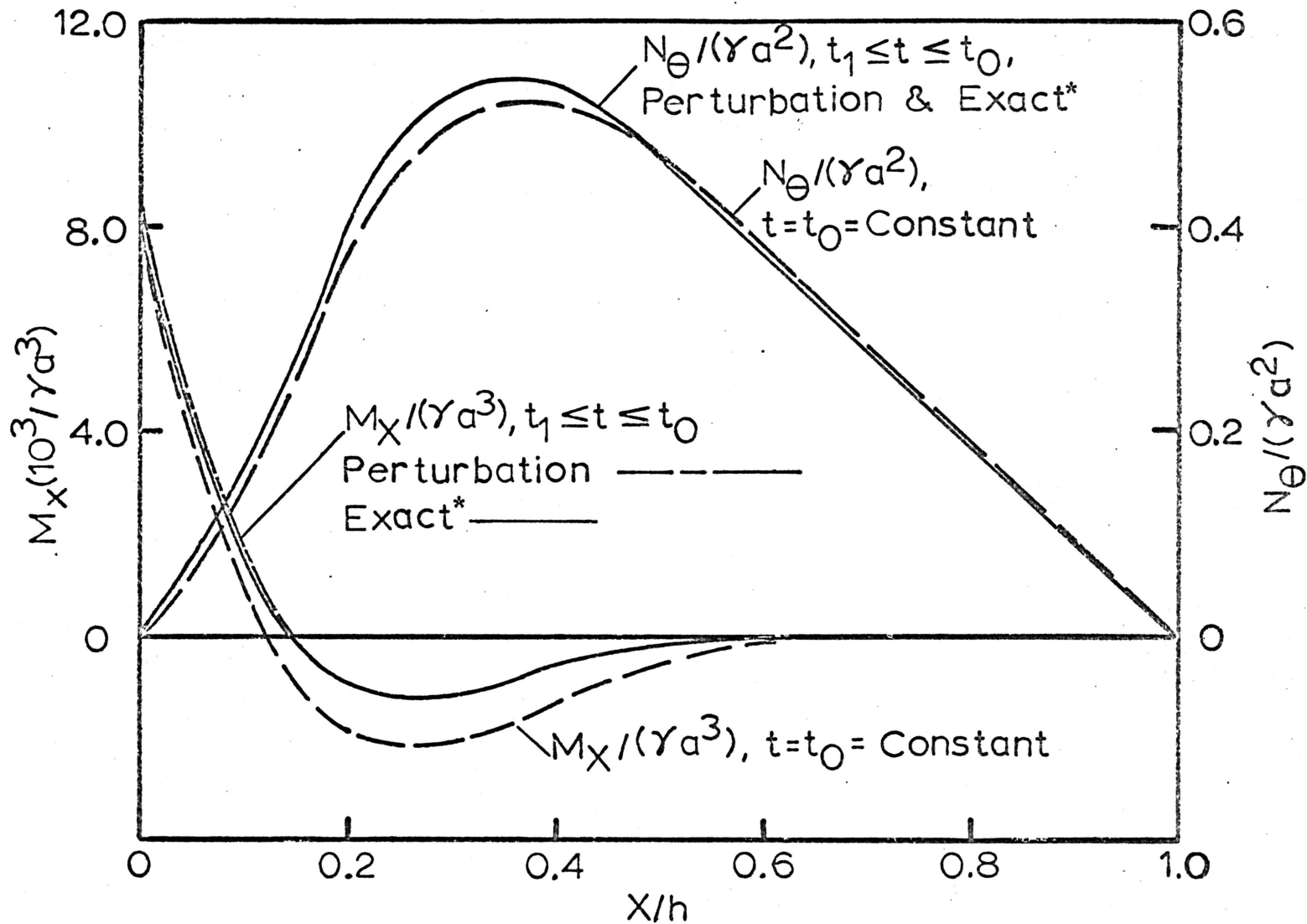


Figure 3. THE STRESS RESULTANTS IN A CIRCULAR CYLINDRICAL TANK WITH $t_0 = 14$ in., $t_1 = 3.5$ in., $a = 30$ ft., $h = 26$ ft., $\nu = 0.25$ AND $\epsilon = 0.012$

CHAPTER IV

A CONICAL SHELL OF VARIABLE THICKNESS

The applicability of Cochran's method to another problem having a differential equation with variable coefficients will be demonstrated in this chapter. The problem is that of the axisymmetric linear deformation of a conical shell. In this case the thickness will be assumed to vary in an arbitrary, but smooth, manner along the shell's generatrix.

A. Governing Differential Equation

Adopting the coordinate system shown in Figure 4 the homogeneous differential equation governing the axisymmetric linear deformation of a conical shell of variable thickness is [5],

$$\epsilon^2 [L_2 L_2(U) - 2fL_2(U)] + 4U = 0, \quad (4.1)$$

where,

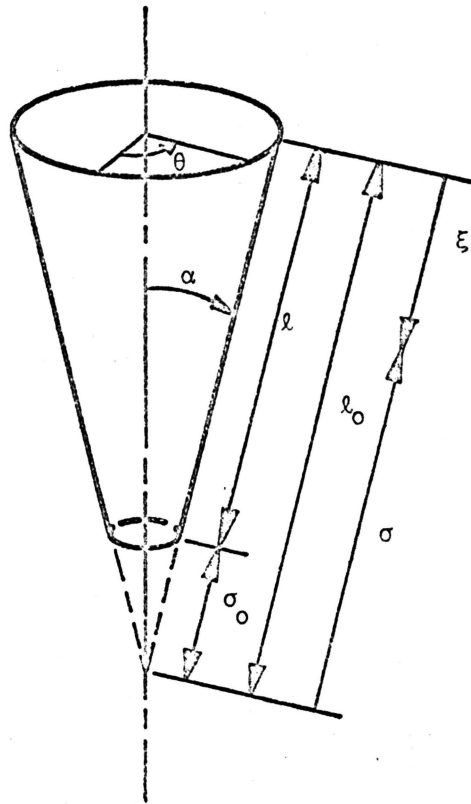
$$\epsilon^2 = \frac{1}{3(1-\nu^2)} \left(\frac{t_0}{l} \cot \alpha \right)^2 \ll 1, \quad (4.2)$$

$$L_2(\) = \sigma\tau(\)'' + (\tau - \sigma\tau')(\)' + (\nu\tau' - \frac{\tau}{\sigma})(\), \quad (4.3)$$

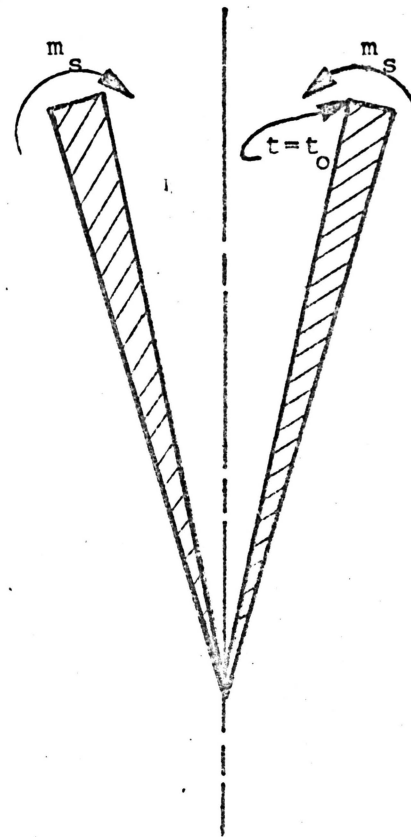
$$f = \sigma\tau'' + (1-\nu)\tau', \quad (4.4)$$

$$\tau = \frac{t}{t_0}, \quad (4.5)$$

$$\sigma = \frac{s}{l} \quad (4.6)$$



(a) MIDDLE SURFACE



(b) MOMENT LOADING OF CONICAL SHELL WITH LINEARLY VARYING THICKNESS

Figure 4. A CONICAL SHELL

s is the distance measured from the apex along the generatrix of the shell, ν is Poisson's ratio, l is the length of the shell measured along its generatrix, t_0 is some reference thickness, and

$$U = l\sigma \cot \alpha Q_s, \quad (4.7)$$

where Q_s is the transverse shearing stress resultant. It is assumed that f and U are of order one in the region of interest.

B. Application of the Perturbation Method

Assuming that the boundary layer is at the end of the shell nearest the apex ($\sigma = \sigma_0$) equation (4.1) is transformed by letting,

$$\eta = \frac{g(\zeta)}{\epsilon^{1/2}}, \quad \zeta = \sigma, \quad (4.8)$$

$$U \approx \sum_{m=0}^{\infty} U_m \epsilon^{m/2}. \quad (4.9)$$

Proceeding as in the problem for the circular cylindrical shell one obtains the following for:

First order terms (ϵ^0)

$$T_1(g')^4 U_{onnnn} + 4U_0 = 0, \quad (4.10)$$

Second order terms ($\epsilon^{1/2}$)

$$T_1(g')^4 U_{1nnnn} + 4U_1 = - \{6(g')^2 g'' T_1 + (g')^3 T_2\} U_{onnn} \\ - 4(g')^3 T_1 U_{o\zeta nnn}, \quad (4.11)$$

Third order terms (ϵ)

$$\begin{aligned}
T_1 (g')^4 U_{2nnnn} + 4U_2 = & - \{6(g')^2 g'' T_1 + (g')^3 T_2\} U_{1nnn} \\
& - 4\{(g')^3 T_1\} U_{1\zeta nnn} \\
& - \{[4g'g'' + 3(g'')^2] T_1 \\
& + 3g'g'' T_2 + (g')^2 T\} U_{onn} \\
& - \{12g'g'' T_1 + 3(g')^2 T_2\} U_{o\zeta nn} \\
& - 6(g')^2 T_1 U_{o\zeta\zeta nn} , \quad (4.12)
\end{aligned}$$

where,

$$T = T_3 - 2\{\sigma\tau'' + (1-\nu)\tau'\} q_1 , \quad (4.13a)$$

$$T_1 = (q_1)^2 , \quad (4.13b)$$

$$T_2 = 2q_1 q_1' + 2q_1 q_2 , \quad (4.13c)$$

$$T_3 = q_1 q_1'' + 2q_1 q_2' + 2q_1 q_3 + q_2 q_1' + q_2^2 , \quad (4.13d)$$

and,

$$q_1 = \tau\sigma , \quad (4.14a)$$

$$q_2 = \tau - \sigma\tau' , \quad (4.14b)$$

$$q_3 = \nu\tau' - \frac{\tau}{\sigma} . \quad (4.14c)$$

The remaining steps are somewhat lengthy and repetitious of what was done in Chapter III. Therefore, they will be skipped and the results given. They are:

$$U_0 = [C_0 \cos \eta + D_0 \sin \eta] e^{-\eta}, \quad (4.15a)$$

$$U_1 = [C_1 \cos \eta + D_1 \sin \eta] e^{-\eta}, \quad (4.15b)$$

$$\eta = \frac{1}{\epsilon^{1/2}} \int_{\sigma_0}^{\sigma} \frac{d\zeta}{(\tau\zeta)^{1/2}}, \quad (4.16)$$

$$C_0 = C \frac{\tau^{3/4}}{\sigma^{1/4}}, \quad (4.17a)$$

$$D_0 = D \frac{\tau^{3/4}}{\sigma^{1/4}}, \quad (4.17b)$$

$$C_1 = F_1 \left(C^* + \frac{D+C}{2} I \right), \quad (4.18a)$$

$$D_1 = F_1 \left(D^* + \frac{D-C}{2} I \right), \quad (4.18b)$$

$$I = \int \left\{ \frac{1}{F_1 F_2} (F_1 F_3 + F_1' F_4 + F_1'' F_5) \right\} d\sigma, \quad (4.19)$$

$$F_1 = \frac{\tau^{3/4}}{\sigma^{1/4}}, \quad (4.20a)$$

$$F_2 = 4(g')^3 T_1, \quad (4.20b)$$

$$F_3 = \{4g'g'' + 3(g'')^2\} T_1 + 3g'g''T_2 + (g')^2 T, \quad (4.20c)$$

$$F_4 = 12g'g''T_1 + 3(g'')^2 T_2 \quad (4.20d)$$

and,

$$F_5 = 6(g')^2 T_1 . \quad (4.20e)$$

C, D, C* and D* are constants of integration to be determined from the boundary conditions. When evaluating the integral I in (4.19) the constant of integration is dropped, since this corresponds to the particular solution of the differential equations governing C_1 and D_1 in the second order problem.

If the boundary layer had been at the edge $\xi = 0$ (see Figure 4) the solution would be:

$$U_0 = (\bar{C}_0 \cos \bar{\eta} + \bar{D}_0 \sin \bar{\eta}) e^{-\bar{\eta}} , \quad (4.21a)$$

$$U_1 = (\bar{C}_1 \cos \bar{\eta} + \bar{D}_1 \sin \bar{\eta}) e^{-\bar{\eta}} , \quad (4.21b)$$

and,

$$\bar{\eta} = \frac{1}{\epsilon^{1/2}} \int_0^\xi \frac{d\zeta}{[\tau(\frac{\ell_0}{\ell} - \zeta)]^{1/2}} \quad (4.22)$$

The other quantities are obtained from (4.17) through (4.20) merely by making the substitution $\sigma = (\ell_0/\ell - \xi)$ and by taking the new expressions for C_0 , D_0 , C_1 and D_1 as the expressions for \bar{C}_0 , \bar{D}_0 , \bar{C}_1 and \bar{D}_1 respectively.

Since (4.1) is linear the two solutions (4.15) and (4.21) can be superimposed to study the problem of a frustrum of a cone having a boundary layer at each end.

C. Discussion

The solution for a conical shell whose thickness varies proportionately to the distance from the apex exists in the literature [5]. This is a closed form solution and presents an opportunity to check the perturbation solution. Figures 5 and 6 show a comparison of the two solutions for such a cone loaded only by a uniformly distributed bending moment at the edge where ξ is equal to zero. (See Figure 4(b)).

As ϵ goes to zero the perturbation solution approaches the exact solution and only the first approximation is necessary for good results. However, as ϵ approaches one the error is considerable when using only the first approximation. Then the second approximation is needed to bring the answers within a practical range of the correct results.

The numerical data corresponding to Figures 5 and 6 can be found in Tables 3 through 5 of Appendix A.

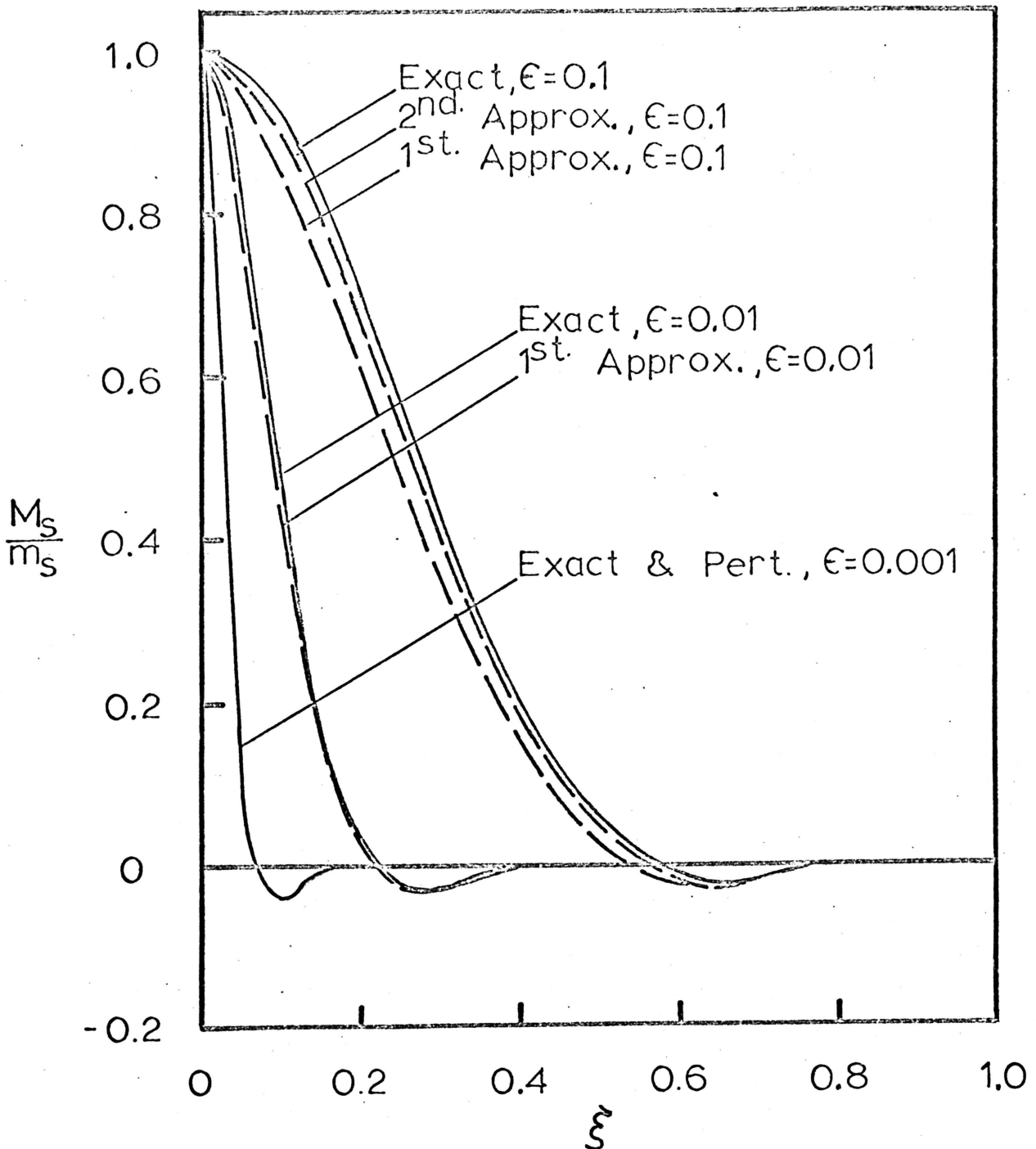


Figure 5. THE STRESS RESULTANT M_s IN A CONICAL SHELL OF LINEARLY VARYING THICKNESS WITH A UNIFORMLY DISTRIBUTED BENDING MOMENT, m_s , AT $\xi = 0$.

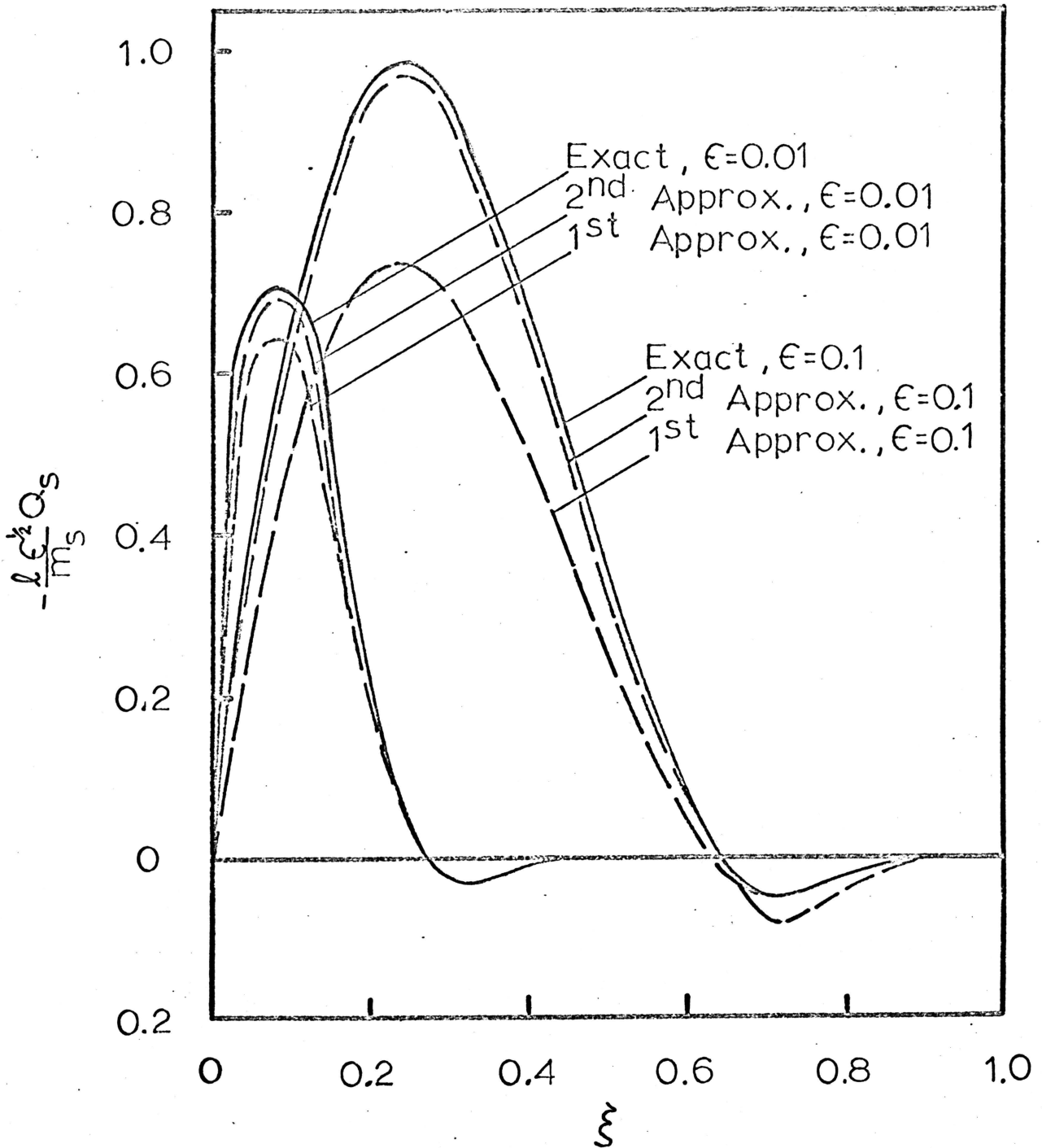


Figure 6. THE TRANSVERSE SHEAR STRESS RESULTANT IN A CONICAL SHELL OF LINEARLY VARYING THICKNESS WITH A UNIFORMLY DISTRIBUTED BENDING MOMENT, m_s , AT $\xi = 0$.

CHAPTER V

TWO CIRCULAR PLATE PROBLEMS ON FINITE SYMMETRICAL DEFORMATION

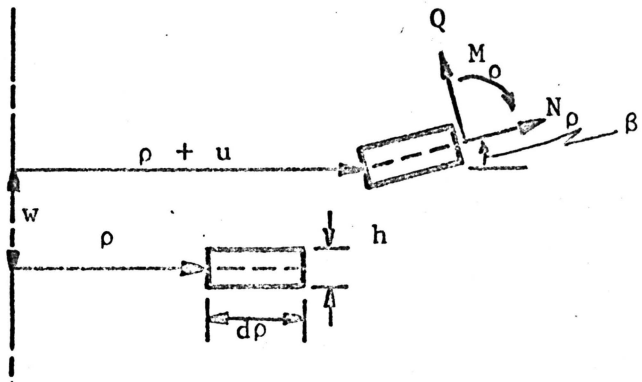
Two problems concerning the finite symmetrical deformation of circular plates are considered in this chapter. A circular plate loaded by a uniformly distributed bending moment along its edge is analyzed in section B. The post-buckling behavior of a circular plate loaded by a uniformly distributed radial compressive force is discussed in section D. The plates, which are unrestrained in both cases, are shown in Figures 7(a) and 7(b) for the bending and buckling problems, respectively. The perturbation method is used to compare classical linear and nonlinear (von Karman) theory with finite deflection (Reissner) theory for these problems.

A. Governing Differential Equations

The finite deformation of the circular plates under consideration can be described by Reissner's differential equations [17] for the finite axisymmetric bending of a circular plate. These equations are based on small strains. Transverse shear deformation is also neglected. Hooke's law and the Love-Kirchhoff assumption regarding normals to the midsurface are both assumed to be valid. For the problems to be discussed Reissner's differential equations are:

$$D(\beta'' + \frac{1}{\rho}\beta' - \frac{1}{2}\cos\beta\sin\beta) - \frac{1}{\rho}\psi\sin\beta = 0, \quad (5.1a)$$

and,



(a) AN ELEMENT OF THE PLATE BEFORE AND AFTER DEFORMATION



(b) CIRCULAR PLATE ACTED UPON BY EDGE MOMENTS



(c) CIRCULAR PLATE ACTED UPON BY A UNIFORM COMPRESSIVE FORCE

Figure 7. FINITE DEFLECTIONS OF A CIRCULAR PLATE

$$A(\psi'' + \frac{1}{\rho} \psi' - \frac{1}{\rho^2} \{\cos^2 \beta - \nu \rho \beta' \sin \beta\} \psi) + \frac{1 - \cos \beta}{\rho} = 0, \quad (5.1b)$$

where ψ is a stress function having units of pounds, ρ is the radial distance from the center of the plate to a point on the middle surface before deformation, β is the amount of rotation experienced by an element of the midsurface upon deformation and ν is Poisson's ratio. (See Figure 7 for additional information regarding the notation used.) The constants A and D are defined as follows:

$$A = \frac{1}{Eh}, \quad (5.2a)$$

and,

$$D = \frac{Eh^3}{12(1 - \nu^2)}, \quad (5.2b)$$

where E is Young's modulus, and h is the thickness of the plate. Knowing the form of ψ and β the stress resultants, stress couples and displacements can be obtained from,

$$N_\rho = \frac{\psi}{\rho} \cos \beta, \quad N_\theta = \psi', \quad (5.3a)$$

$$M_\rho = -D(\beta' + \frac{\nu}{\rho} \sin \beta), \quad (5.3b)$$

$$M_\theta = -D(\frac{1}{\rho} \sin \beta + \nu \beta'), \quad (5.3c)$$

$$u = \frac{\rho}{Eh} (N_\theta - \nu N_\rho), \quad (5.3d)$$

and,

$$w' = [1 + A(N_\rho - \nu N_\theta)] \sin \beta . \quad (5.3e)$$

Equations (5.1) are written in nondimensional form by introducing the dimensionless variables,

$$r = \frac{\rho}{a} , \quad (5.4a)$$

and,

$$G = \frac{\psi}{B_i} , \quad (5.4b)$$

where a is the radial distance to the edge of the plate, G is of order one and B_i is a constant having units of pounds. The relation for B_i is to be determined in the course of analysis. The subscript i is used to facilitate in the discussion of the various cases that might arise. Upon adopting the notation,

$$\lambda_i^2 = \frac{B_i a}{D} , \quad (5.5a)$$

and,

$$L_i^2 = 12(1 - \nu^2) \frac{a^2}{h^2} \frac{1}{\lambda_i^2} , \quad (5.5b)$$

equations (5.1) can be written as,

$$\beta'' + \frac{1}{r} \beta' - \frac{1}{r^2} \cos \beta \sin \beta - \frac{\lambda_i^2}{r} G \sin \beta = 0, \quad (5.6a)$$

and,

$$G'' + \frac{1}{r} G' - \frac{1}{r^2} (\cos^2 \beta - \nu r \beta' \sin \beta) G + \frac{L_i^2}{r} (1 - \cos \beta) = 0. \quad (5.6b)$$

These equations will be referred to as the full Reissner equations.

If β is not of order one it can be written as,

$$\beta = \delta_i \alpha, \quad (5.7)$$

where δ_i is a new perturbation parameter and α , which is a function of r , is of order one. Then by a Taylor series expansion of the sine and cosine terms which contain β , equations (5.6) become:

$$0 = \alpha'' + \frac{1}{r} \alpha' - \frac{1}{r^2} \alpha + \frac{2}{3} \frac{\delta_i^2}{r^2} \alpha^3 + \dots - \frac{\lambda_i^2}{r} G(\alpha - \frac{\delta_i^2}{6} \alpha^3 + \dots) \quad (5.8a)$$

and,

$$0 = G'' + \frac{1}{r} G' - \frac{1}{r^2} G(1 - \delta_i^2 \alpha^2 + \dots - \nu r \delta_i^2 \alpha' \alpha + \dots) + \frac{1}{r} L_i^2 \delta_i^2 (\frac{1}{2} \alpha^2 - \frac{\delta_i^2}{24} \alpha^4 + \dots). \quad (5.8b)$$

B. A Circular Plate Loaded by a Uniformly Distributed Bending Moment at the Edge

When the only loading is a uniformly distributed bending moment $-m_0$ at the edge of the plate, the boundary conditions are,

$$\beta' + \nu \sin \beta = \frac{m_0 a}{D}, \quad (5.9a)$$

and,

$$G = 0, \quad (5.9b)$$

at r equal to one and finiteness at r equal to zero. For the case corresponding to (5.7) these become,

$$\alpha' + \nu \left(\alpha - \frac{1}{6} \delta_f^2 \alpha^3 + \dots \right) = \frac{m_0 a}{D \delta_f}, \quad (5.10a)$$

and,

$$G = 0, \quad (5.10b)$$

at r equal to one and finiteness at r equal to zero. In the following, case 1 refers to (5.8) and (5.10), which are the equations resulting from taking β in the form given in (5.7), and case 2 refers to (5.6) and (5.9), which are the equations for β of order one.

Case 1

The analysis of the problem begins with the consideration of the differential equations (5.8) and (5.10). From (5.5)

it is seen that λ_i can take on various values depending upon the magnitude of the loading. Thus, $\lambda_i^2 \ll 1$, $\lambda_i^2 = 0(1)$, and $\lambda_i^2 \gg 1$ will be considered.

$$a. \quad \underline{\lambda_i^2 = \lambda_1^2 \ll 1}$$

If $\lambda_i^2 \ll 1$ the first approximation for α is of the form,

$$\alpha'' + \frac{1}{r} \alpha' - \frac{1}{r^2} \alpha = 0. \quad (5.11)$$

However, it is not known what the magnitude of $L_i^2 \delta_i^2$ is and various possibilities must be considered.

$$(i) \quad \underline{L_i^2 \delta_i^2 \ll 1}$$

For this case the differential equation for G is of the form,

$$G'' + \frac{1}{r} G' - \frac{1}{r^2} G = 0,$$

which, with the boundary conditions (5.10), leads to the trivial solution of G equal to zero.

$$(ii) \quad \underline{L_i^2 \delta_i^2 = L_1^2 \delta_1^2 = 0(1)}$$

For this case the differential equation for G is of the form,

$$G'' + \frac{1}{r} G' - \frac{1}{r^2} G + \alpha^2 = 0, \quad (5.12)$$

with,

$$L_1^2 = \frac{1}{\delta_1^2} . \quad (5.13)$$

Equations (5.11) and (5.12) are recognized as the equations of linear bending theory and their solution gives meaningful results.

To proceed further set,

$$\alpha = \alpha_0 + \delta_1^2 \alpha_1 + \dots , \quad (5.14a)$$

and,

$$G = G_0 + \delta_1^2 G_1 + \dots . \quad (5.14b)$$

Then substitute into (5.8) and equate coefficients of like powers of δ_1 obtaining,

$$\alpha_0'' + \frac{1}{r} \alpha_0' - \frac{1}{r^2} \alpha_0 - \frac{\lambda_1^2}{r} G_0 \alpha_0 = 0 , \quad (5.15a)$$

$$G_0'' + \frac{1}{r} G_0' - \frac{1}{r^2} G_0 + \frac{1}{2r} \alpha_0^2 = 0 , \quad (5.15b)$$

$$\begin{aligned} \alpha_1'' + \frac{1}{r} \alpha_1' - \frac{1}{r^2} \alpha_1 - \lambda_1^2 (G_0 \alpha_1 + G_1 \alpha_0) = \\ - \frac{2}{3} \frac{1}{r^2} \alpha_0^2 - \frac{\lambda_1^2}{6r} \alpha_0^3 G_0 , \end{aligned} \quad (5.16a)$$

and,

$$\begin{aligned} G_1'' + \frac{1}{r} G_1' - \frac{1}{r^2} G_1 + \frac{1}{r} \alpha_0 \alpha_1 = - \frac{1}{r^2} \alpha_0^2 G_0 - \frac{\nu}{r} \alpha_0 \alpha_0' G_0 \\ + \frac{1}{24r} \alpha_0^4 , \end{aligned} \quad (5.16b)$$

where,

$$\alpha'_0 + v\alpha_0 = \frac{m_0 a}{\delta_1 D}, \quad (5.17a)$$

and,

$$G_0 = 0, \quad (5.17b)$$

$$\alpha'_1 + v\alpha_1 = \frac{v}{6} \alpha_0^3, \quad (5.18a)$$

$$G_1 = 0, \quad (5.18b)$$

at r equal to one and all functions are finite at r equal to zero. Then let,

$$\alpha_0 = \alpha_{00} + \lambda_1^2 \alpha_{01} + \lambda_1^4 \alpha_{02}, \quad (5.19a)$$

$$G_0 = G_{00} + \lambda_1^2 G_{01} + \lambda_1^4 G_{02}, \quad (5.19b)$$

and substitute into (5.15) and follow the same procedure as above to obtain:

$$\alpha''_{00} + \frac{1}{r} \alpha'_{00} - \frac{1}{r^2} \alpha_{00} = 0, \quad (5.20a)$$

$$G''_{00} + \frac{1}{r} G'_{00} - \frac{1}{r^2} G_{00} + \frac{1}{2r} \alpha_{00}^2 = 0, \quad (5.20b)$$

$$\alpha''_{01} + \frac{1}{r} \alpha'_{01} - \frac{1}{r^2} \alpha_{01} = \frac{1}{r} G_{00} \alpha_{00}, \quad (5.21a)$$

$$G''_{01} + \frac{1}{r} G'_{01} - \frac{1}{r^2} G_{01} + \frac{1}{r} \alpha_{00} \alpha_{01} = 0, \quad (5.21b)$$

and etc.. Substituting (5.19) into the boundary conditions (5.17) and applying the least degeneracy principle gives,

$$\alpha'_{00} + v\alpha_{00} = 1, \quad (5.22a)$$

$$G_{00} = 0, \quad (5.22b)$$

$$\alpha'_{01} + v\alpha_{01} = 0, \quad (5.23a)$$

$$G_{01} = 0, \quad (5.23b)$$

at r equal to one and,

$$\delta_1 = \frac{m_o a}{D}. \quad (5.24)$$

Then from (5.5),

$$\lambda_1^2 = 12(1 - v^2) \frac{a^4 m_o^2}{h^2 D^2}, \quad (5.25a)$$

and,

$$B_1 = 12(1 - v^2) \frac{a^3 m_o^2}{h^2 D}. \quad (5.25b)$$

Noting that the homogeneous portions of equations (5.20) and (5.21) are of the Euler-Cauchy type their solutions may be obtained quite readily. They are,

$$\alpha_{00} = \frac{1}{1+v} r, \quad (5.26a)$$

$$G_{00} = \frac{1}{16(1+v)^2} (r - r^3), \quad (5.26b)$$

$$\alpha_{01} = \frac{1}{1680(1+\nu)^2} [7r^4 - 3r^6 - (10 + 4\nu)r] , \quad (5.27a)$$

and,

$$G_{01} = \frac{a_1}{63} r^8 + \frac{a_2}{35} r^6 + \frac{a_0 r^3}{8} - \left(\frac{a_0}{8} + \frac{a_2}{35} + \frac{a_1}{63} \right) r, \quad (5.27b)$$

where,

$$a_0 = \frac{1}{840} \frac{5 + 2\nu}{(1 + \nu)^4} , \quad (5.28a)$$

$$a_1 = \frac{1}{560} \frac{1}{(1+\nu)^3} , \quad (5.28b)$$

and,

$$a_2 = - \frac{1}{240} \frac{1}{(1+\nu)^3} . \quad (5.28c)$$

Higher approximations can be obtained in the same manner by assuming series forms similar to (5.19) for α_1 , G_1 and etc..

$$(iii) \underline{L_i^2 \delta_i^2} \gg 1$$

This case leads to a contradiction, since,

$$\alpha^2 = 0 ,$$

and equation (5.11) cannot both govern the first approximation. Thus, for this problem, $L_i^2 \delta_i^2$ equal to one is the only meaningful case when $\lambda_i^2 \ll 1$.

$$\text{b. } \underline{\lambda_i^2 = 0} \quad (1)$$

$$\text{(i) } \underline{L_i^2 \delta_i^2 \ll 1}$$

If λ_i^2 is of order one and $L_i^2 \delta_i^2 \ll 1$ the differential equations for the first order problem are,

$$\alpha''' + \frac{1}{r} \alpha' - \frac{1}{r^2} \alpha - \frac{1}{r} G \alpha \lambda_i^2 = 0, \quad (5.29a)$$

and,

$$G''' + \frac{1}{r} G' - \frac{1}{r^2} G = 0, \quad (5.29b)$$

which lead to the ridiculous result that G is indentially equal to zero. Therefore, this case cannot exist for the bending problem.

$$\text{(ii) } \underline{L_i^2 \delta_i^2 = 0} \quad (1)$$

When $L_i^2 \delta_i^2$ is of order one,

$$\alpha''' + \frac{1}{r} \alpha' - \frac{1}{r^2} G - G \alpha \lambda_i^2 = 0, \quad (5.30a)$$

and,

$$G''' + \frac{1}{r} G' - \frac{1}{r^2} G + \frac{1}{2} \alpha^2 = 0, \quad (5.30b)$$

are the differential equations for the first order problem where,

$$\delta_i^2 = \frac{1}{L_i^2} = \frac{1}{12(1-\nu^2)} \frac{h^2}{a^2}, \quad (5.31)$$

without any loss in generality. This is the condition for small finite deflections. These equations are the von Karman equations discussed by Way [24] and Timoshenko [21].

$$(iii) \quad \underline{L_i^2 \delta_i^2} \gg 1$$

For $L_i^2 \delta_i^2 \gg 1$ the following inconsistent equations result for the first approximation:

$$\alpha'' + \frac{1}{r} \alpha' - \frac{1}{r^2} \alpha = 0,$$

and,

$$\alpha^2 = 0.$$

$$c. \quad \underline{\lambda_i^2 = \lambda_2^2} \gg 1$$

If $\lambda_i^2 \gg 1$ the differential equation for the first order problem resulting from (5.8a) is of the form,

$$\alpha G = 0,$$

and since G is not equal to zero the only way this can be satisfied is to have,

$$\alpha = 0. \quad (5.32)$$

Then the equation for G is of the form,

$$G'' + \frac{1}{r} G' - \frac{1}{r^2} G = 0, \quad (5.33)$$

which gives,

$$G = c_1 r, \quad (5.34)$$

if we apply the condition of finiteness at r equal to zero. This is a singular perturbation problem since the boundary conditions at r equal to one are lost.

The application of Cochran's method to equations (5.8) results in a set of differential equations for the first order problem that are,

$$[g'(\zeta)]^2 \frac{\partial^2 \alpha}{\partial \eta^2} - \alpha G = 0,$$

and,

$$[g'(\zeta)]^2 \frac{\partial^2 G}{\partial \eta^2} + \frac{1}{2} \alpha^2 = 0.$$

These equations do not possess closed form solutions. Therefore, since there is no simple composite expansion there is no need to consider the problem by the generalized method and further analysis will be based upon the method of inner and outer expansions.

Define the inner variable,

$$\eta = \frac{1}{\tau_2} (1 - r). \quad (5.35)$$

Then substitute into the differential equations (5.8) and boundary conditions (5.10), using the series expansions (5.14) and (5.19) with $\lambda_1 = 1/\lambda_2$ and $\delta_1 = \delta_2$, to obtain the following:

$$0 = \alpha''_{oo} - \lambda_2^2 \tau_2^2 G_{oo} \alpha_{oo} + \dots, \quad (5.36a)$$

and,

$$0 = G''_{oo} + \frac{\tau_2^2 L_2^2 \delta_2^2}{2} \alpha_{oo}^2 + \dots, \quad (5.36b)$$

with,

$$\frac{\tau_2^{m_o} a}{D\delta_2} = -\alpha'_{oo} + \dots, \quad (5.37a)$$

and,

$$0 = G_o, \quad (5.37b)$$

at $r = 0$. Applying the principle of least degeneracy gives,

$$1 = \lambda_2^2 \tau_2^2, \quad (5.38a)$$

$$1 = \tau_2^2 L_2^2 \delta_2^2, \quad (5.38b)$$

and,

$$1 = \frac{\tau_2^{m_o} a}{D\delta_2}. \quad (5.38c)$$

Then from (5.5),

$$\lambda_2 = K^{1/3} \frac{m_o a}{D}, \quad (5.39a)$$

$$\tau_2 = \frac{1}{K^{1/3}} \frac{D}{m_o a}, \quad (5.39b)$$

$$\delta_2 = \frac{1}{K^{1/3}}, \quad (5.39c)$$

$$B_2 = K^{2/3} \frac{m_o^2 a}{D}, \quad (5.39d)$$

and,

$$K^2 = 12(1 - v^2) \frac{a^2}{h^2} \left(\frac{D}{m_o a}\right)^4. \quad (5.39e)$$

Thus, the system of equations governing the first order problem in the inner region (boundary layer) is,

$$0 = \alpha_{oo}''' - G_{oo} \alpha_{oo}, \quad (5.40a)$$

and,

$$0 = G_{oo}'' + \frac{1}{2} \alpha_{oo}^2. \quad (5.40b)$$

One set of boundary conditions for (5.40) is,

$$-1 = \alpha_{oo}', \quad (5.41a)$$

$$0 = G_{oo}, \quad (5.41b)$$

at η equal to zero. The remaining boundary conditions are found by matching which proceeds as follows for the rotation β :

1-term inner:	$\beta \approx \delta_2 \alpha_{00}(\eta)$
rewritten in outer variables:	$= \delta_2 \alpha_{00}(\lambda_2 [1-r])$
expand for large $\lambda_{(2)}$:	$= \delta_2 \alpha_{00}(\infty)$
1-term outer of 1-term inner:	$= \delta_2 \alpha_{00}(\infty)$
1-term outer:	$\beta \approx 0$
1-term inner of 1-term outer:	$= 0,$

the stress function G :

1-term inner:	$G \approx G_{00}(\eta)$
rewritten in outer variables:	$= G_{00}(\lambda_2 [1-r])$
expand for large $\lambda_{(2)}$:	$= G_{00}(\infty)$
1-term outer of 1-term inner:	$= G_{00}(\infty)$
1-term outer:	$G \approx c_1 r$
rewritten in inner variables:	$= c_1 (1 - \frac{\eta}{\lambda_2})$

expand for large λ_2 : $=c_1$

1-term outer of 1-term inner: $=c_1$.

Then since the 1-term inner of the 1-term outer must equal the 1-term outer of the 1-term inner,

$$\alpha_{oo} \approx 0, \quad (5.42a)$$

$$G_{oo} \approx c_1, \quad (5.42b)$$

as η goes to infinity or,

$$\alpha_{oo} \approx 0, \quad (5.43a)$$

$$G'_{oo} \approx 0, \quad (5.43b)$$

as η goes to infinity.

Case 2

When β is of order one it is not possible to have $\lambda_1^2 \ll 1$. This should be obvious since λ_1 is proportional to the loading and therefore for small λ_1 small rotations should result. However, it can also be shown to be true by considering the differential equations that result from this assumption.

For $\lambda_1^2 = \lambda_3^2 \gg 1$ the differential equation governing β in the first approximation is of the form,

$$G \sin \beta = 0. \quad (5.44)$$

Therefore, β or G must be zero and, obviously, the choice is,

$$\beta = 0 . \quad (5.45)$$

The corresponding differential equation for G becomes,

$$G'' + \frac{1}{r} G' - \frac{1}{r^2} G = 0 , \quad (5.46)$$

and from the finiteness condition at r equal to zero the solution is,

$$G = cr . \quad (5.47)$$

Due to the loss of the higher order derivatives in the first approximation this is a singular perturbation problem.

To obtain the governing equations for the inner solution let,

$$\eta = \frac{1}{\tau_3} (1 - r) , \quad (5.48)$$

$$\beta = \beta_0 + \frac{1}{\lambda_3} \beta_1 + \dots , \quad (5.49a)$$

and,

$$G = G_0 + \frac{1}{\lambda_3} G_1 + \dots . \quad (5.49b)$$

Then from (5.6) this results in the first approximation,

$$0 = \beta_0'' - \tau_3^2 \lambda_3^2 G_0 \sin \beta_0 , \quad (5.50a)$$

and,

$$0 = G'' + L_3^2 \tau_3^2 (1 - \cos \beta_0) . \quad (5.50b)$$

Applying the least degeneracy principle to these equations, it is found that,

$$0 = \beta_0'' - G_0 \sin \beta_0 , \quad (5.51a)$$

$$0 = G_0'' + K^2 (1 - \cos \beta_0) \quad (5.51b)$$

$$\lambda_3 = \frac{m_0 a}{D} , \quad (5.52a)$$

$$\tau_3 = \frac{D}{m_0 a} , \quad (5.52b)$$

and,

$$B_3 = \frac{m_0^2 a}{D} , \quad (5.52c)$$

where,

$$-1 = \beta_0' , \quad (5.53a)$$

and,

$$0 = G_0 , \quad (5.53b)$$

at η equal to zero. The constant K is as defined in (5.39e)

and is assumed to be of order one. The remaining boundary conditions are determined by applying the matching technique and are,

$$\beta_0 = 0 , \quad (5.54a)$$

and,

$$G'_0 = 0 , \quad (5.54b)$$

as η goes to infinity.

C. Discussion

Equations (5.20) are obviously the differential equations for linear bending theory. The second approximation (5.21) reflects the effect of stretching of the middle surface which is seen to be negligible as δ_1 goes to zero. This is to be expected for small deflections and has been found to be true by other investigators [21, 22].

Equations (5.40) are the boundary layer equations resulting from the von Karman approximation. These equations apply for small finite rotations. As the rotations increase a point is reached at which these equations do not give acceptable accuracy and then the boundary layer equations (5.51) govern. To determine the error in using the approximate equations (5.40) instead of the more accurate equations (5.51) for large finite rotations, solutions were compared by integrating both sets of equations numerically.

Equations (5.40) and (5.51) were solved by using the Runge-Kutta numerical integration method. This is a step-by-step technique which is only applicable to initial value problems. However, two point boundary value problems can be solved by guessing the missing conditions at η equal to zero and using Newton's method [19] to correct the guesses. The results of the calculations are shown in Figures 8 through 12. The independent variable η in Figures 10 through 12 is the one defined by equations (5.35) and (5.39b).

The solutions of (5.51) should approach that of (5.40) as K increases and should diverge from that of (5.40) as K decreases. This effect is apparent in the plots. It is seen that the amount of error in determining either the deflection or the stress function by (5.40), the boundary layer equations of the von Karman approximation, instead of (5.51), the boundary layer equations of the full Reissner equations, is less than 3% for rotations up to about 60° . The rotations are predicted to the same degree of accuracy for rotations up to about 70° .

The data used in plotting Figures 8 through 12 can be found in Appendix A, Tables 6 and 7. The computations from which the data were taken were carried out to five significant figures.

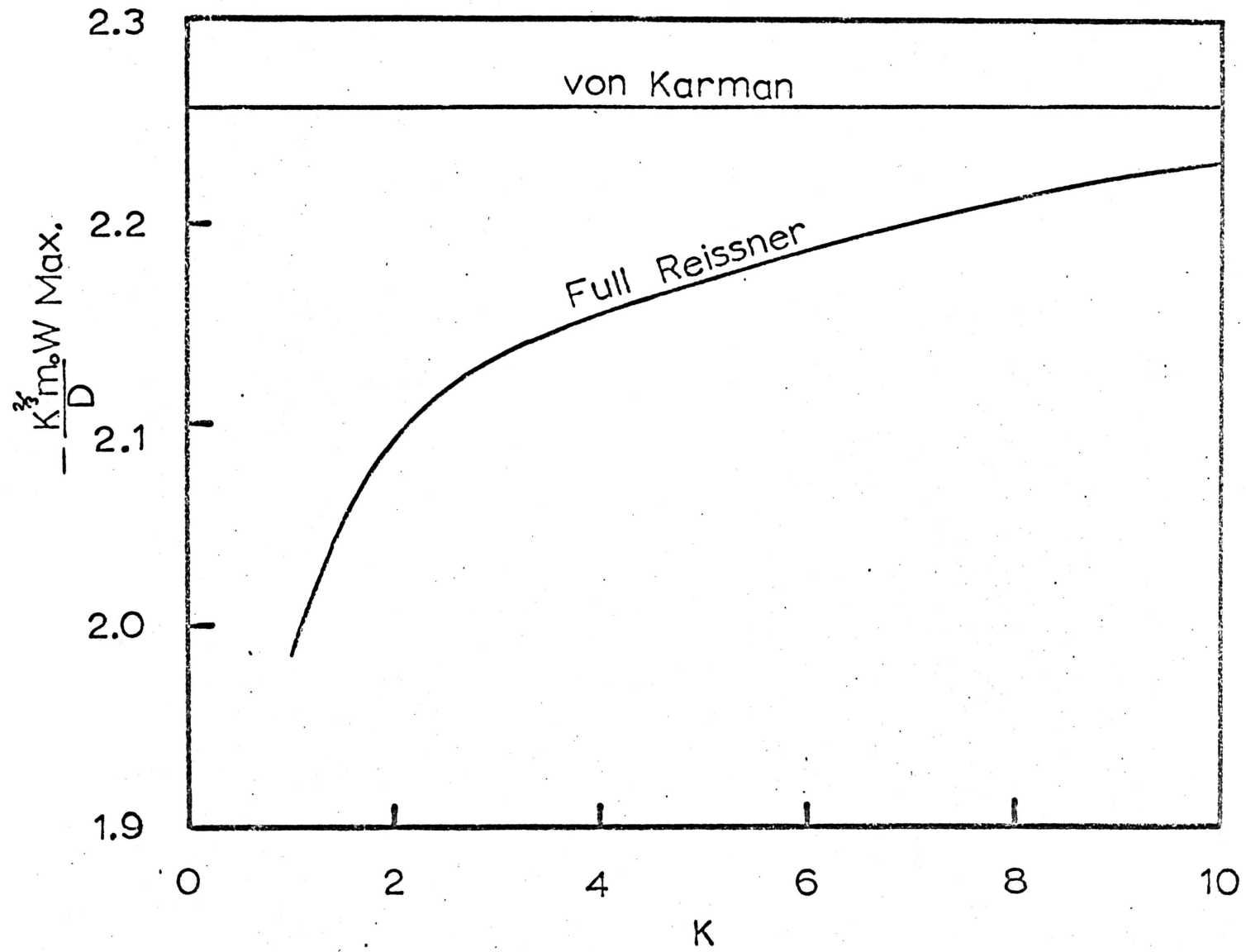


Figure 8. MAXIMUM DEFLECTION AT THE CENTER OF A CIRCULAR PLATE LOADED BY A UNIFORMLY DISTRIBUTED BENDING MOMENT, $- m_0$, AT ITS EDGE

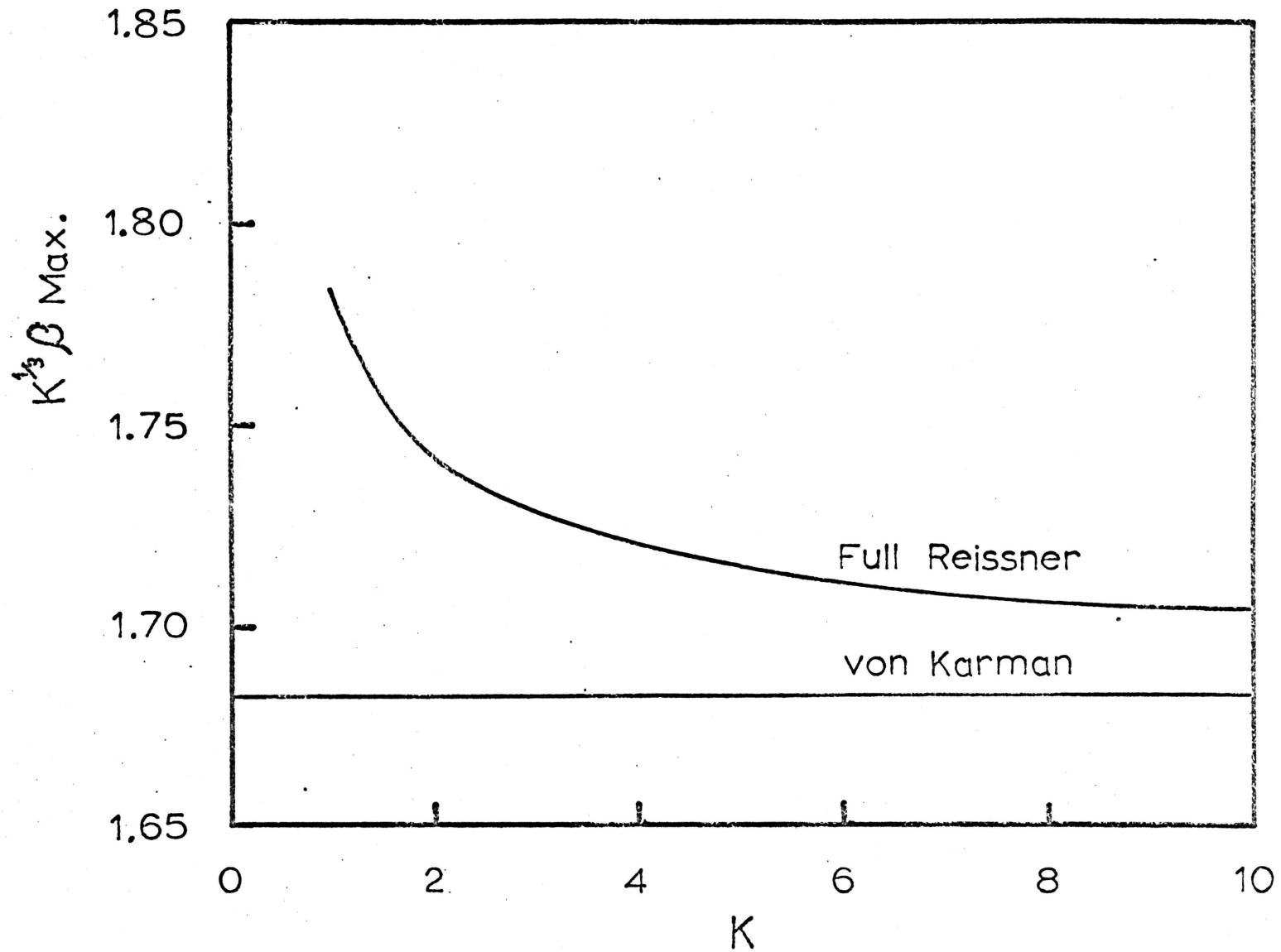


Figure 9. MAXIMUM ROTATION AT THE EDGE OF A CIRCULAR PLATE LOADED BY A UNIFORMLY DISTRIBUTED BENDING MOMENT, $-m_o$, AT ITS EDGE

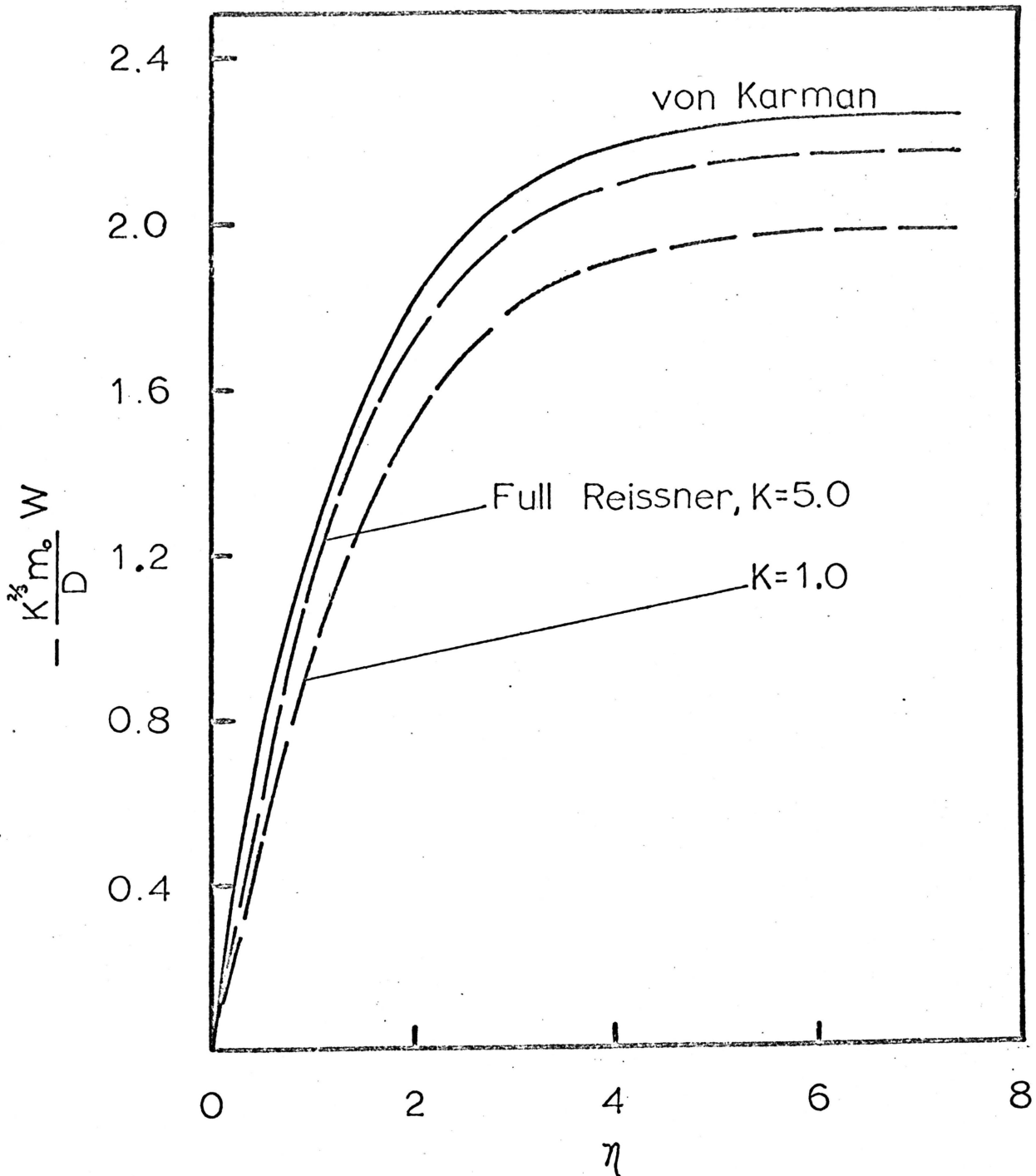


Figure 10. DEFLECTION OF A CIRCULAR PLATE LOADED BY A UNIFORMLY DISTRIBUTED BENDING MOMENT, $-m_0$, AT ITS EDGE

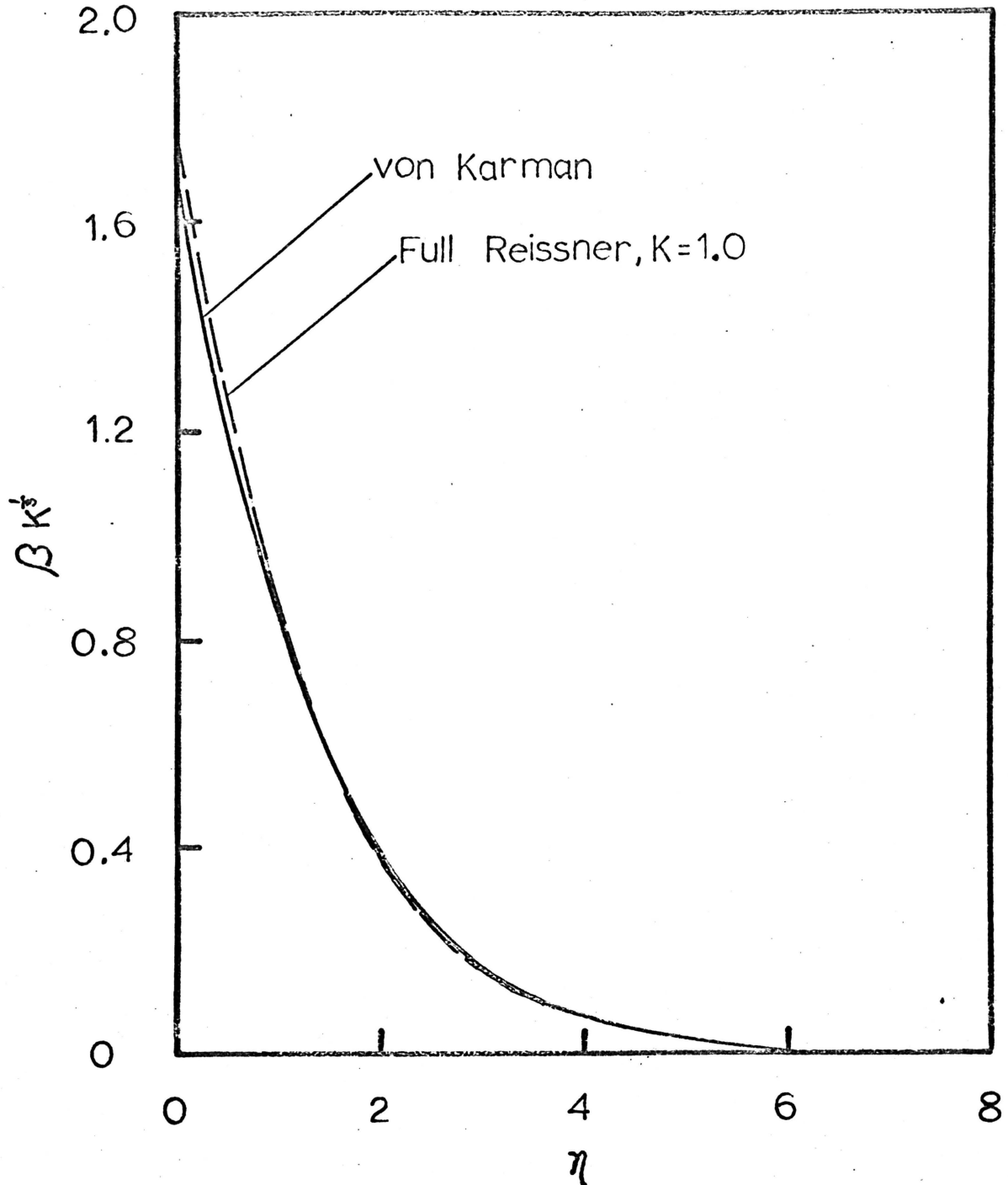


Figure 11. ROTATION OF A CIRCULAR PLATE LOADED BY A UNIFORMLY DISTRIBUTED BENDING MOMENT, $-m_0$, AT ITS EDGE.

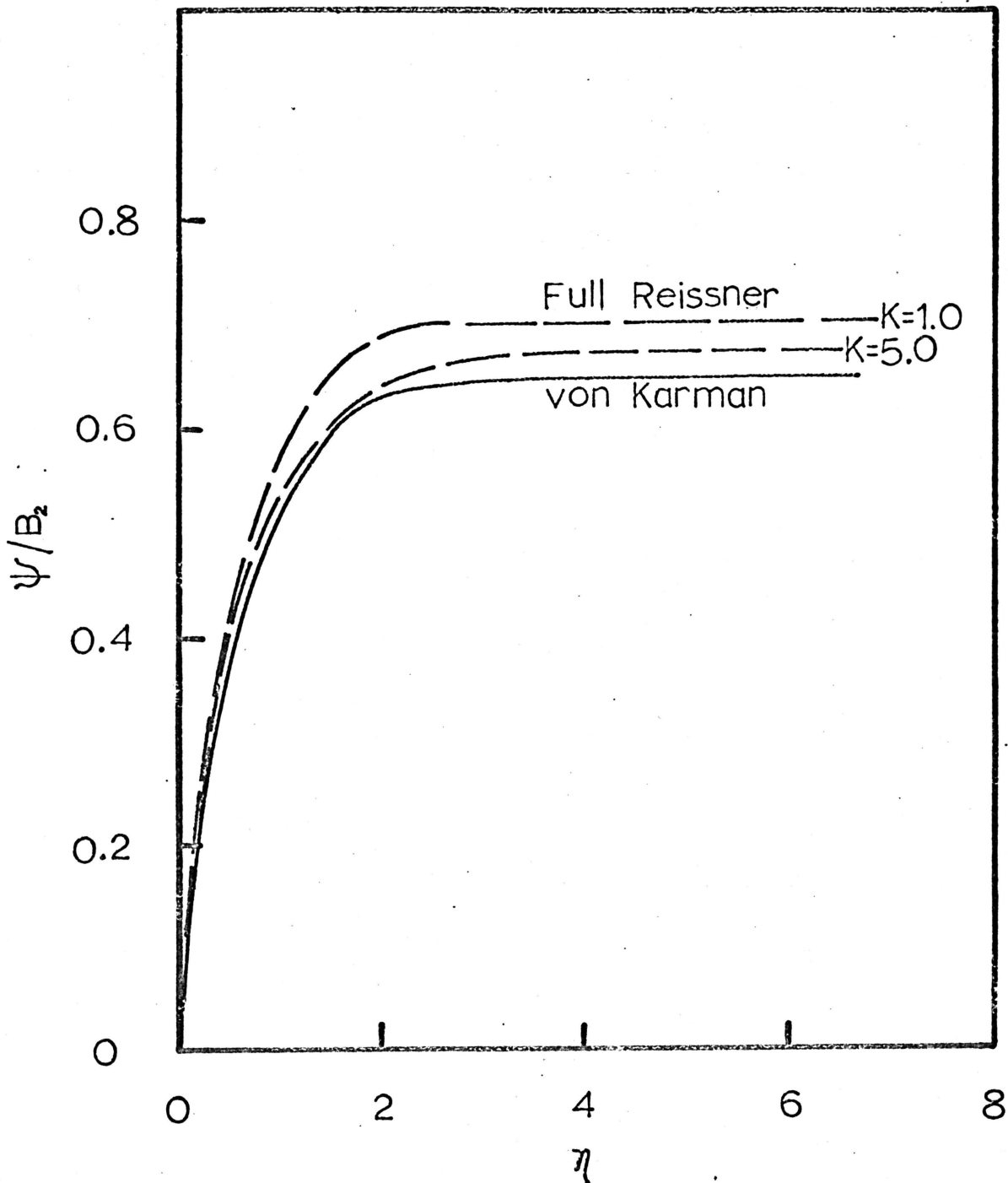


Figure 12. STRESS FUNCTION FOR A CIRCULAR PLATE LOADED BY A UNIFORMLY DISTRIBUTED BENDING MOMENT, $-m_0$, AT ITS EDGE

D. Post-Buckling Behavior of a Circular Plate Under the Action of a Radial Force

The post-buckling behavior of a circular plate loaded by a uniformly distributed radial compressive force P along its edge can also be studied by using the differential equations (5.6) and (5.8). The boundary conditions for such a problem are,

$$\beta' + \nu \sin \beta = 0 , \quad (5.55a)$$

and,

$$G = - \frac{Pa}{B_i} , \quad (5.55b)$$

at r equal to one. When β is of the form given in (5.7) these boundary conditions become,

$$\alpha' + \nu \left(\alpha - \frac{1}{6} \alpha^3 \delta_i^2 + \dots \right) = 0 , \quad (5.56a)$$

and,

$$G = - \frac{Pa}{B_i} . \quad (5.56b)$$

Also, finiteness at r equal to zero is to be satisfied in both cases.

The condition $\lambda_i^2 \ll 1$ corresponds to no load or no deflection and is, therefore, a trivial case.

Case 1

When β is assumed to be of the form given in (5.7) and λ_1^2 is of order one there are two conditions on $L_1^2 \delta_1^2$ that make sense. These are $L_1^2 \delta_1^2 = L_4^2 \delta_4^2 \ll 1$ and $L_1^2 \delta_1^2$ of order one, the first approximations of which appear in (5.29) and (5.30), respectively.

Applying the finiteness condition to the solutions for (5.29) yields,

$$G = cr , \quad (5.57a)$$

and,

$$\alpha = c_1 J_1 (\lambda_4 r) . \quad (5.57b)$$

Then from the least degeneracy principle, as applied to the boundary conditions, it is found that,

$$B_1 = B_4 = Pa , \quad (5.58)$$

$$G = -r , \quad (5.59)$$

and,

$$\lambda_4 J_0 (\lambda_4) = (1 - \nu) J_1 (\lambda_4) . \quad (5.60)$$

The last relation is an eigenvalue problem resulting from (5.56a). The lowest value for which equation (5.60) is satisfied is the critical value of λ_1 . The critical value of the radial force P can be found by substituting this value

into (5.5) and using (5.58). For example, if $\nu = 0.3$ the smallest root of (5.60) is $\lambda_4 = 2.049$. Then $4.198 \frac{D}{a^2}$ is the critical value of P .

When $\beta = \delta_1 \alpha$ and $\lambda_1^2 = \lambda_5^2 \gg 1$ the governing differential equations and boundary conditions are found by the same process of reasoning as was used in the bending problem. The results for the first term inner equations are,

$$0 = \alpha''''_{00} - G_{00} \alpha_{00}, \quad (5.61a)$$

$$0 = G''''_{00} + \frac{1}{2} \alpha_{00}^2, \quad (5.61b)$$

$$\lambda_5^2 = \frac{Pa^2}{D}, \quad (5.62a)$$

$$\tau_5^2 = \frac{D}{Pa^2}, \quad (5.62b)$$

$$\delta_5 = \frac{1}{K}, \quad (5.62c)$$

$$B_5 = Pa, \quad (5.62d)$$

and,

$$\bar{K}^2 = 12(1 - \nu^2) \frac{a^2}{h^2} \left(\frac{D}{Pa^2}\right)^2, \quad (5.62e)$$

where the independent variable is given by,

$$\eta = \frac{1}{\tau_5} (1 - r). \quad (5.63)$$

The boundary conditions are,

$$0 = \alpha'_{00} , \quad (5.64a)$$

and,

$$-1 = G_{00} , \quad (5.64b)$$

at η equal to zero and,

$$\alpha_{00} = 0 , \quad (5.65a)$$

and,

$$G'_{00} = 0 , \quad (5.65b)$$

as η goes to infinity.

Case 2

If β is of order one the first term inner equations are,

$$0 = \beta'_0 - G_0 \sin \beta_0 , \quad (5.66a)$$

and,

$$0 = G'_0 + \bar{K}^2 (1 - \cos \beta_0) , \quad (5.66b)$$

where \bar{K}^2 is defined in (5.62e) and the independent variable is,

$$\eta = \frac{1}{\tau_6} (1 - r) . \quad (5.67)$$

The perturbation parameters for (5.66) and (5.67) are,

$$\lambda_6^2 = \frac{Pa^2}{D} \gg 1, \quad (5.68a)$$

and,

$$\tau_6^2 = \frac{D}{Pa^2}. \quad (5.68b)$$

where,

$$B_6 = Pa. \quad (5.68c)$$

The boundary conditions are,

$$0 = \beta'_0, \quad (5.69a)$$

and,

$$-1 = G_0, \quad (5.69b)$$

at η equal to zero and,

$$\beta_0 \approx 0, \quad (5.70a)$$

and,

$$G'_0 \approx 0, \quad (5.70b)$$

as η goes to infinity.

E. Discussion

The boundary layer equations (5.61), which result from the von Karman approximation, and the boundary layer equations (5.66), which are for large finite rotations, were solved by using the modified Runge-Kutta numerical integration method. The results are shown in Figures 13 through 17. See also Appendix A, Tables 8 and 9, for a tabulation of the solutions.

In this problem the rotation, deflection and stress function are predicted by the boundary layer equations of the von Karman approximation to within 3% of the values resulting from the boundary layer equations of the full Reissner theory for rotations up to about 45° .

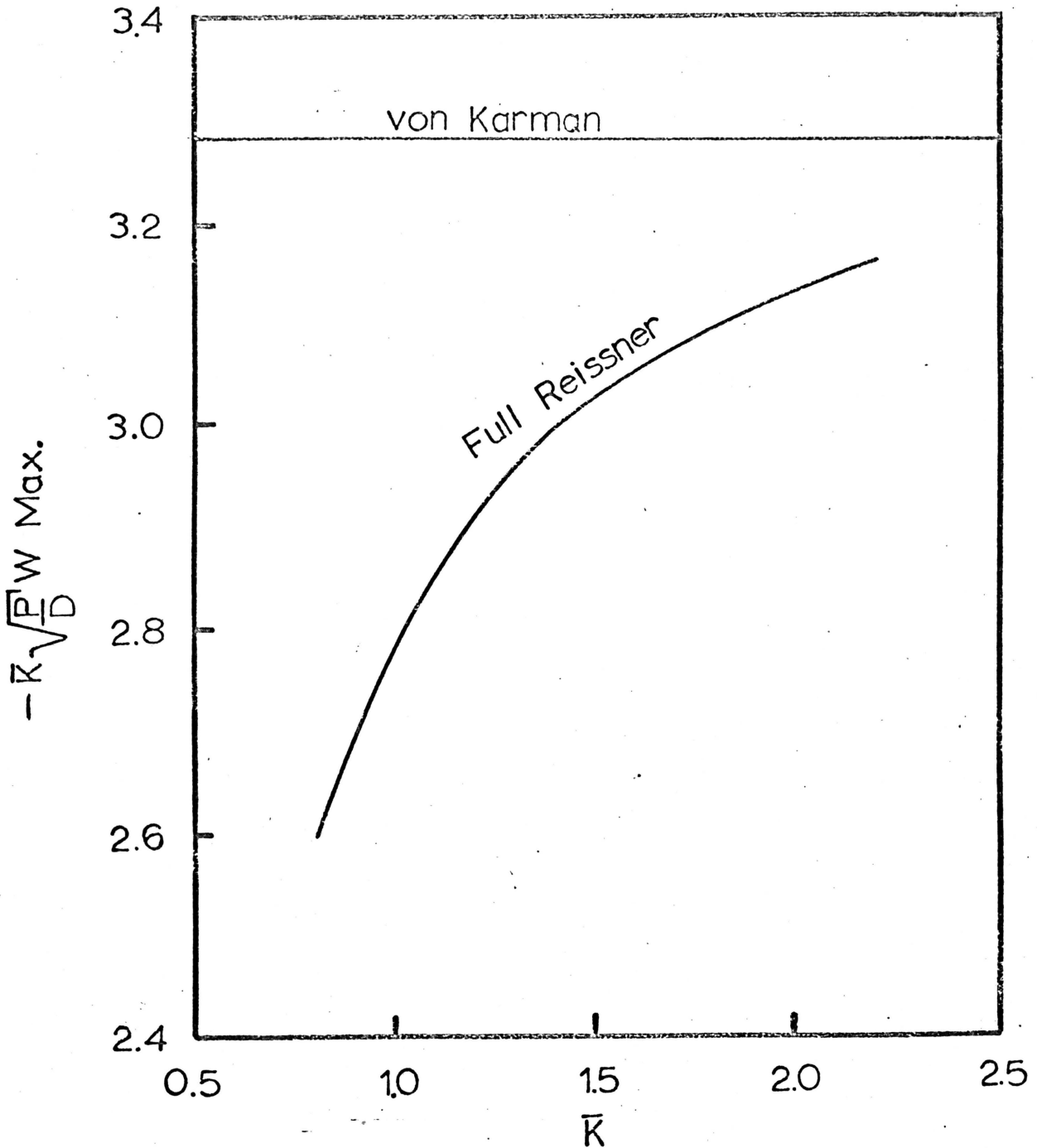


Figure 13. MAXIMUM DEFLECTION AT THE CENTER OF A BUCKLED CIRCULAR PLATE LOADED BY A UNIFORMLY DISTRIBUTED RADIAL COMPRESSIVE FORCE, P, AT ITS EDGE

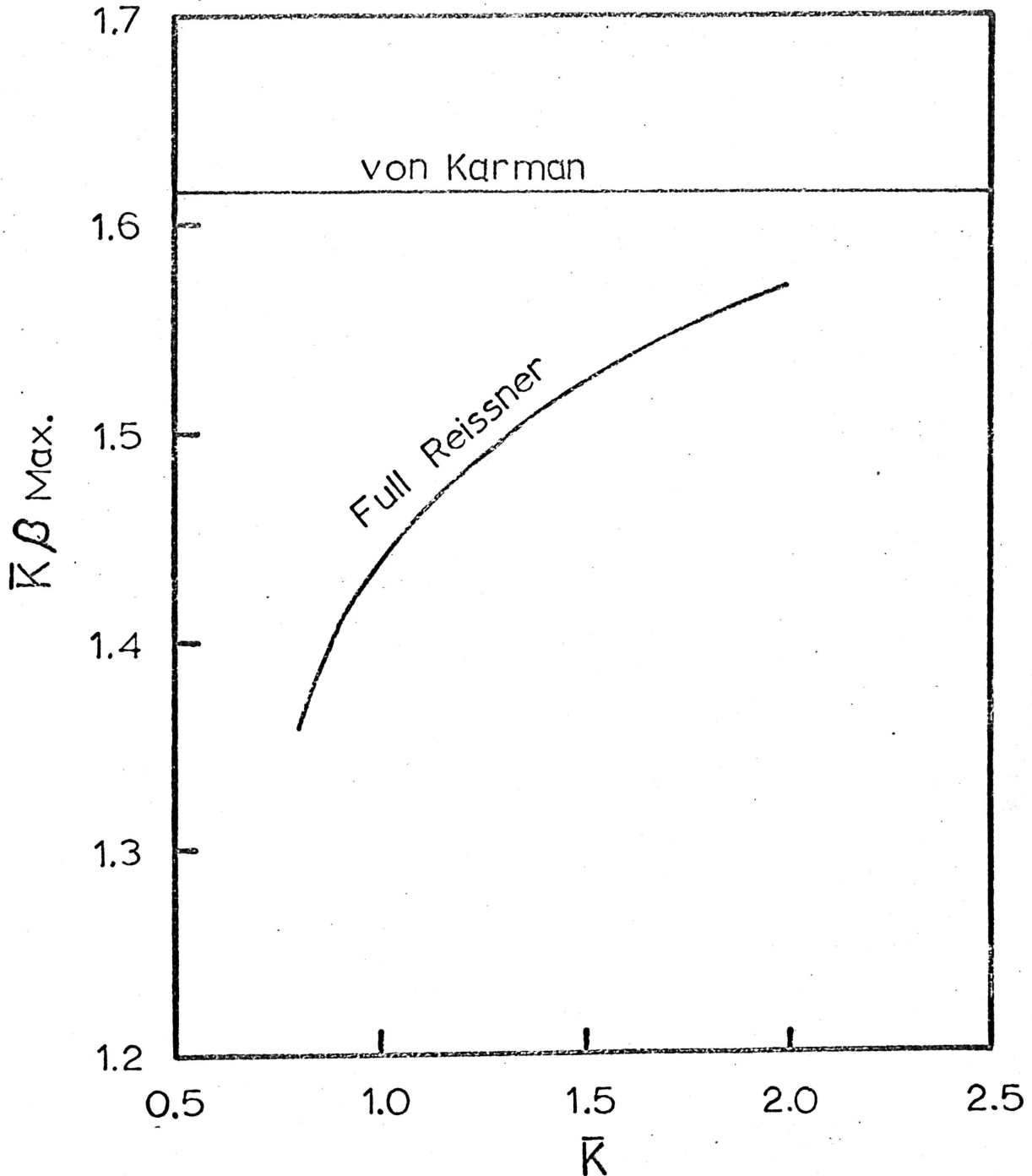


Figure 14. MAXIMUM ROTATION AT THE EDGE OF A BUCKLED CIRCULAR PLATE LOADED BY A UNIFORMLY DISTRIBUTED RADIAL COMPRESSIVE FORCE, P , AT ITS EDGE

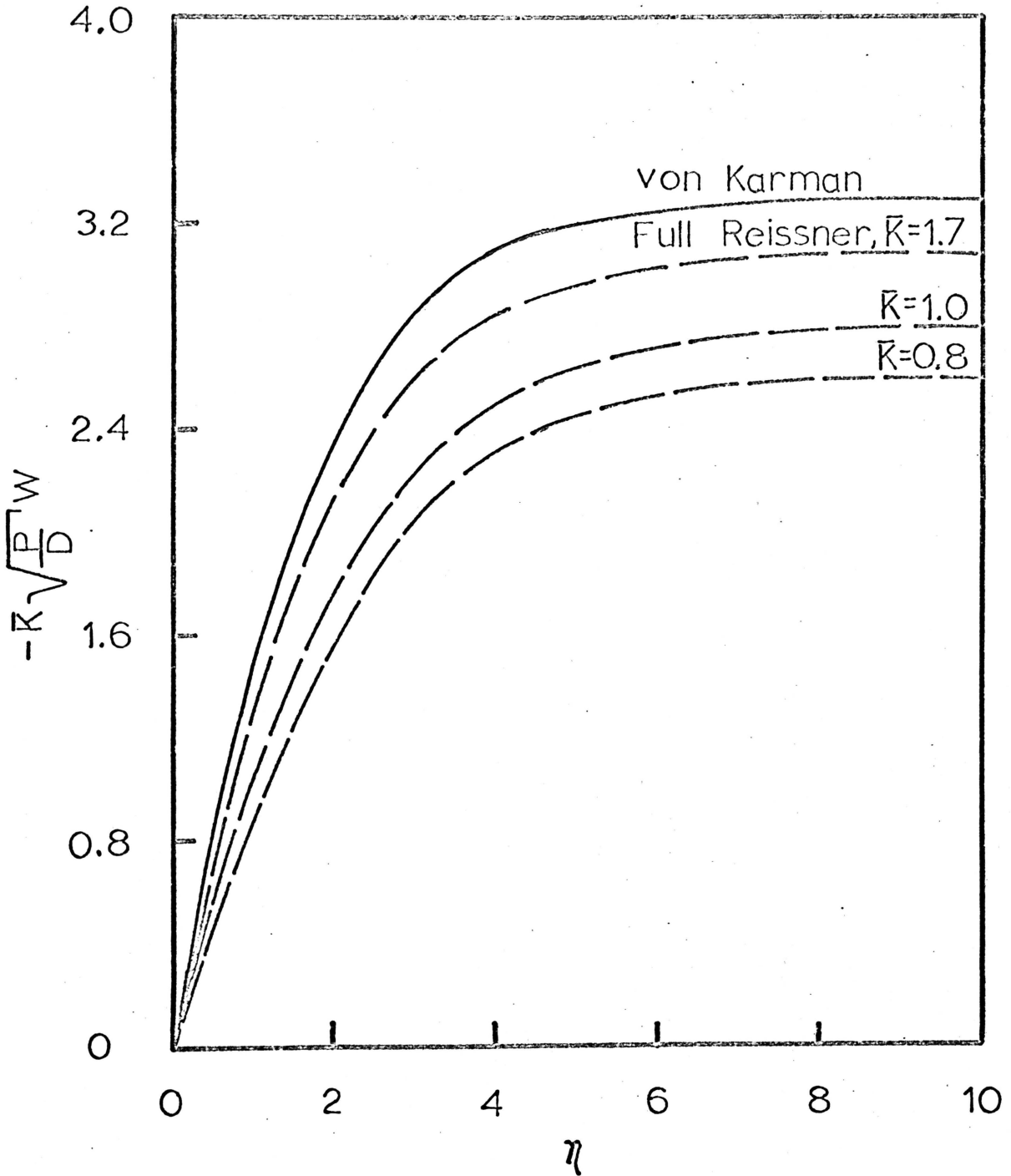


Figure 15. DEFLECTION OF A BUCKLED CIRCULAR PLATE LOADED BY A UNIFORMLY DISTRIBUTED RADIAL COMPRESSIVE FORCE, P, AT ITS EDGE

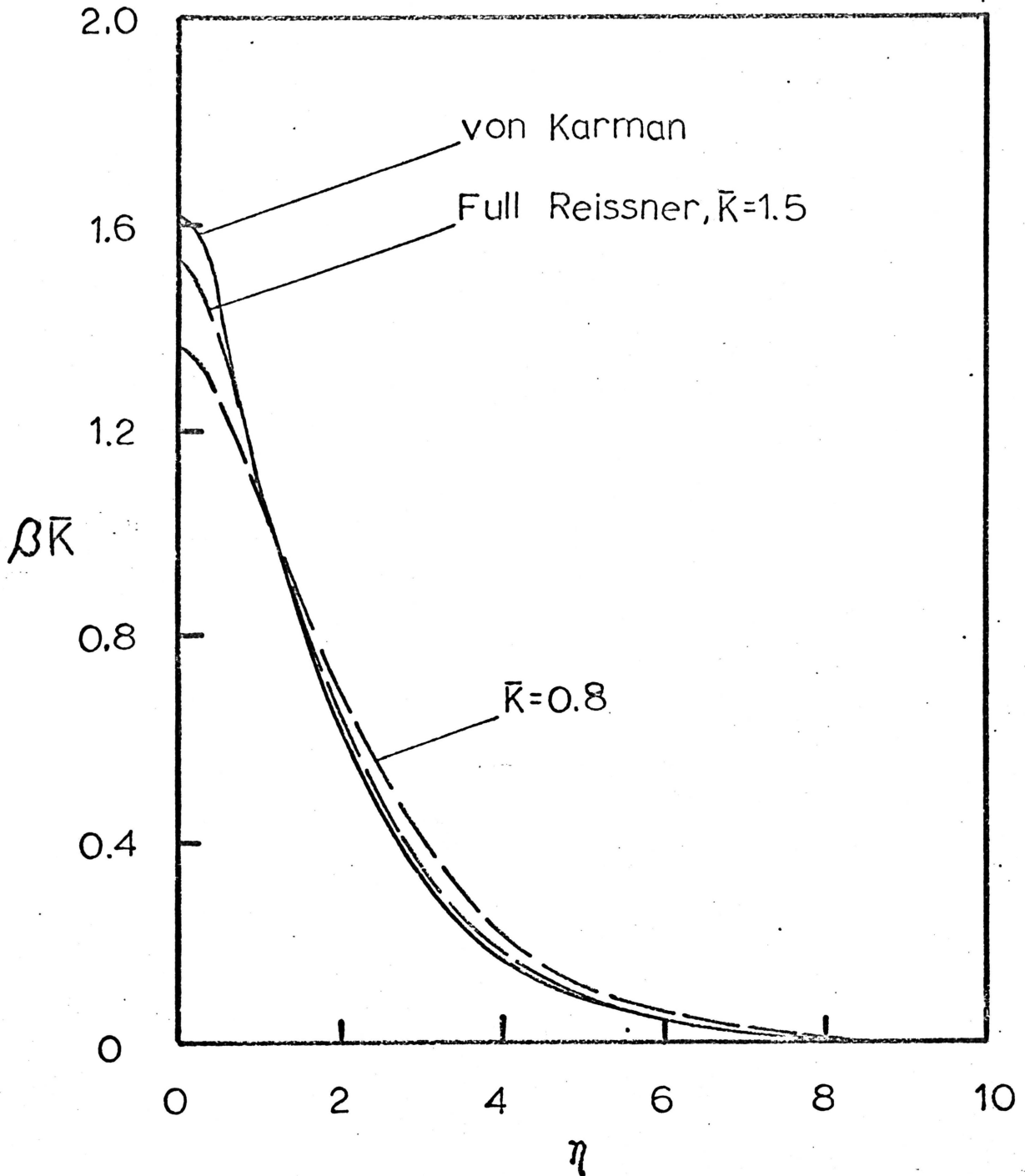


Figure 16. ROTATION OF A BUCKLED CIRCULAR PLATE LOADED BY A UNIFORMLY DISTRIBUTED RADIAL COMPRESSIVE FORCE, P, AT ITS EDGE

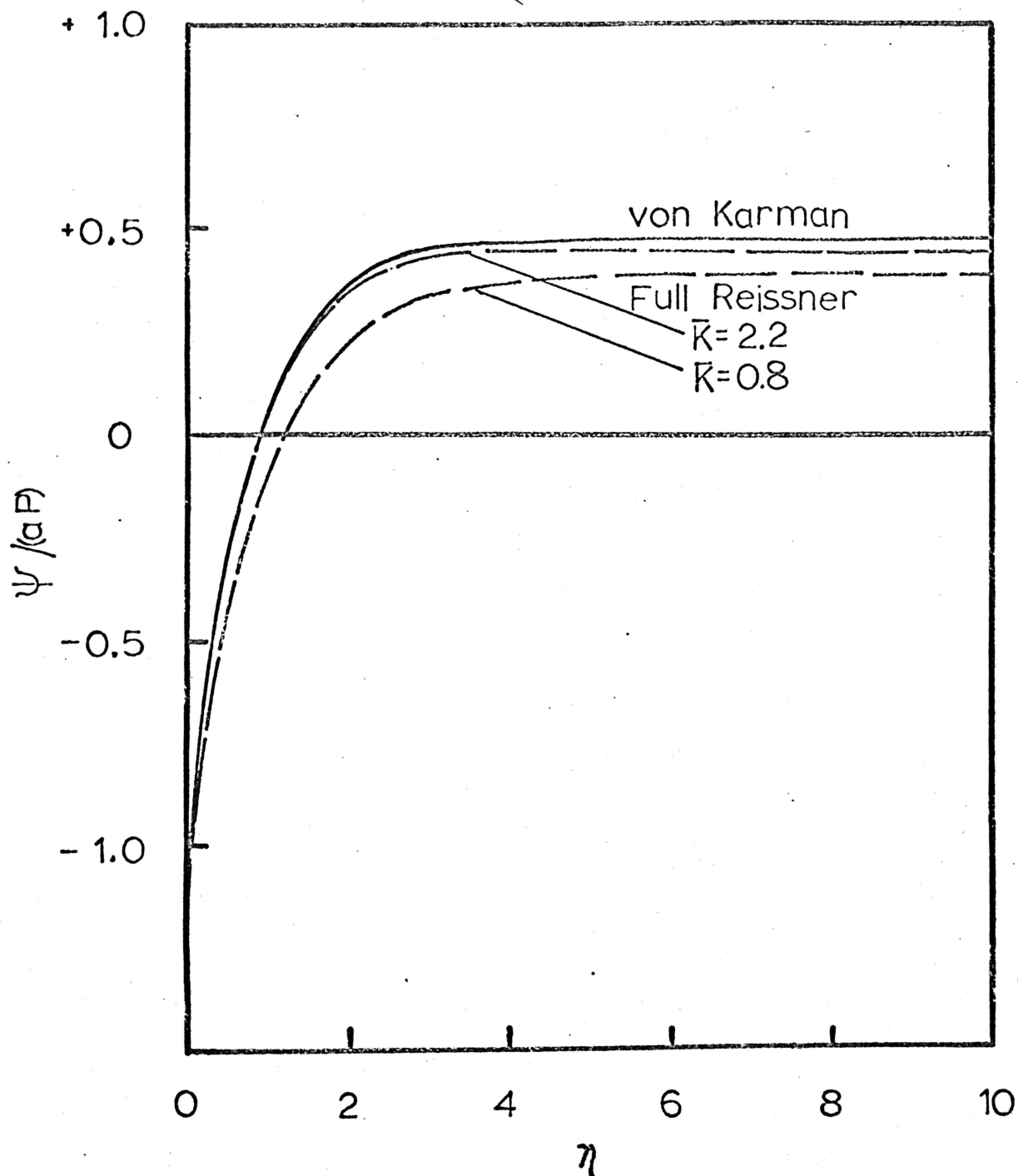


Figure 17. STRESS FUNCTION FOR A BUCKLED CIRCULAR PLATE LOADED BY A UNIFORMLY DISTRIBUTED AXIAL COMPRESSIVE FORCE, P, AT ITS EDGE

CHAPTER VI

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CHAPTER VII

BIBLIOGRAPHY

1. Bellman, R., Perturbation Techniques in Mathematics and Engineering, Holt, Rinehart and Winston, New York, 1964.
2. Clark, R. A., On the Theory of Thin Toroidal Shells, *Journal of Mathematics and Physics*, 29, 1950, pp. 146-178.
3. Cochran, J. A., Problems in Singular Perturbation Theory, Doctoral Dissertation, Stanford University, 1962.
4. DeSilva, C. N. and Naghdi, P. M., Asymptotic Solutions of Elastic Shells of Revolution with Variable Thickness, *Quarterly of Applied Mathematics*, Vol. 15, 1957, pp. 169-182.
5. Flugge, W., Stresses in Shells, Springer, Berlin, 1960.
6. Friedrichs, K. O. and Stoker, J. J., The Nonlinear Boundary Value Problem of the Buckled Plate, *American Journal of Mathematics*, 63, 1941, pp. 839-888.
7. Friedrichs, K. O. and Stoker, J. J., Buckling of the Circular Plate Beyond the Critical Thrust, *Journal of Applied Mechanics*, 9, 1942, A7-A14.
8. Hart, V. G. and Evans, D. J., Nonlinear Bending of an Annular Plate by Transverse Edge Forces, *Journal of Mathematics and Physics*, Vol. 43, 1964, pp. 275-303.
9. Hildebrand, F. B., On Asymptotic Integration in Shell Theory, *Proceedings of Symposia in Applied Mathematics*, Vol. 3, 1949, pp. 53-66.

10. Johnson, M. W. and Reissner, E., Parametric Expansions For a Class of Boundary-Value Problems of Partial Differential Equations, Journal of the Society of Industrial and Applied Mathematics, Vol. 8, June 1960, pp. 389-396.
11. Keller, H. B. and Reiss, E. L., Non-linear Bending and Buckling of Circular Plates, Proceedings of the Third National Congress of Applied Mechanics, 1958, pp. 375-385.
12. Langer, R. E., The Asymptotic Solutions of Certain Linear Ordinary Differential Equations of Second Order, Transactions of the American Mathematical Society, 36, 1934, pp. 90-106.
13. Latta, G. E., Singular Perturbation Problems, Doctoral Dissertation, California Institute of Technology, 1951.
14. McLachlan, N. W., Bessel Functions for Engineers, Oxford Press, London, Second Edition, 1955.
15. Nayfeh, A. H., A Generalized Method for Treating Singular Perturbation Problems, Department of Aeronautics and Astronautics Report, SUDAER No. 193, Stanford University, 1964.
16. Nayfeh, A. H., A Comparison of Three Perturbation Methods for Earth-Moon Spaceship Problems, AIAA Journal, Vol. 3, No. 9, September 1965, pp. 1682-1687.
17. Reissner, E., Rotationally Symmetric Problems in the Theory of Thin Elastic Shells, Proceedings of the Third U. S. National Congress of Applied Mechanics, 1958, pp. 51-69.

18. Reissner, E., On Asymptotic Expansions for Circular Cylindrical Shells, *Journal of Applied Mechanics*, June 1964, pp. 245-252.
19. Reshotko, E. and Beckwith, I., Compressible Laminar Boundary Layer Over a Yawed Infinite Cylinder With Heat Transfer and Arbitrary Prandtl Number, National Advisory Committee for Aeronautics, Report 1379, Washington.
20. Spotts, M. F., Analysis of Spherical Shells of Variable Wall Thickness, *Journal of Applied Mechanics*, 6, 1940, pp. 97-102.
21. Timoshenko, S. and Woinowsky-Krieger, S., Theory of Plates and Shells, Second Edition, McGraw-Hill, New York, 1951.
22. Tolefson, D. C., Asymptotic Solutions of a Circular Plate Problem, Doctoral Dissertation, Virginia Polytechnic Institute, 1967.
23. Van Dyke, M., Perturbation Methods in Fluid Dynamics, Academic Press, New York, 1964.
24. Way, S., Bending of Circular Plates With Large Deflection, *Transactions of A.S.M.E.*, Vol. 56, 1934, pp. 627-636.

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APPENDIX A

TABULATION OF NUMERICAL RESULTS

TABLE 1.

THE STRESS RESULTANT M_x AND N_θ FOR A CIRCULAR CYLINDRICAL TANK WITH

$t_o = 14$ in., $t_1 = 2$ in., $a = 24$ ft., $h = 12$ ft., $\nu = 0.25$, AND $\epsilon = 0.014$

x/h (in./in.)	<u>LINEAR VARYING THICKNESS ($t_1 \leq t \leq t_o$)</u>						<u>CONSTANT THICKNESS ($t = t_o$)</u>	
	<u>EXACT SOLUTION</u>			<u>PERTURBATION SOLUTION</u>			<u>EXACT & PERT. SOLN.</u>	
	$M_x / (\gamma a^3)$	$N_\theta / (\gamma a^2)$		$M_x / (\gamma a^3)$	$N_\theta / (\gamma a^2)$		$M_x / (\gamma a^3)$	$N_\theta / (\gamma a^2)$
	(10^{-3})		(10^{-3})		(10^{-3})		(10^{-3})	
0.0	5.36	0.00	5.37	0.00	5.37	0.00	4.79	0.00
0.1	2.61	0.03	2.62	0.03	2.61	0.03	1.86	0.03
0.2	0.90	0.08	0.91	0.08	0.90	0.08	0.00	0.08
0.3	0.00	0.14	0.00	0.14	0.00	0.14	-1.03	0.13
0.4	-0.36	0.18	-0.36	0.18	-0.36	0.18	-1.46	0.17
0.5	-0.42	0.20	-0.42	0.20	-0.42	0.20	-1.50	0.19
0.6	-0.35	0.19	-0.35	0.19	-0.35	0.19	-1.32	0.19
0.7	-0.24	0.16	-0.24	0.16	-0.24	0.16	-0.98	0.17
0.8	-0.16	0.12	-0.16	0.12	-0.16	0.12	-0.55	0.12
0.9	-0.12	0.06	-0.12	0.06	-0.12	0.06	-0.21	0.07
1.0	-0.10	0.00	-0.10	0.00	-0.10	0.00	-0.01	0.01

TABLE 2.

THE STRESS RESULTANTS M_x AND N_θ FOR A CIRCULAR CYLINDRICAL TANK WITH
 $t_o = 14$ in., $t_1 = 3.5$ in., $a = 30$ ft., $h = 26$ ft., $\nu = 0.25$, AND $\epsilon = 0.012$

x/h	<u>LINEAR VARYING THICKNESS ($t_1 \leq t \leq t_o$)</u>						<u>CONSTANT THICKNESS ($t = t_o$)</u>	
	<u>EXACT SOLUTION</u>			<u>PERTURBATION SOLUTION</u>			<u>EXACT & PERT. SOLN.</u>	
	$M_x / (\gamma a^3)$	$N_\theta / (\gamma a^2)$		$M_x / (\gamma a^3)$	$N_\theta / (\gamma a^2)$		$M_x / (\gamma a^3)$	$N_\theta / (\gamma a^2)$
(in./in.)	(10^{-3})		(10^{-3})		(10^{-3})	(10^{-3})		
0.0	8.28	0.00	8.63	0.00	8.65	0.00	8.28	0.00
0.1	1.34	0.17	1.52	0.16	1.51	0.16	0.88	0.15
0.2	-1.03	0.40	-0.98	0.40	-0.98	0.40	-1.81	0.37
0.3	-1.16	0.53	-1.15	0.53	-1.15	0.53	-2.01	0.50
0.4	-0.65	0.54	-0.67	0.54	-0.66	0.54	-1.34	0.52
0.5	-0.24	0.47	-0.25	0.47	-0.25	0.47	-0.63	0.47
0.6	-0.07	0.37	-0.07	0.37	-0.07	0.37	-0.17	0.38
0.7	-0.06	0.26	-0.06	0.26	-0.06	0.27	0.00	0.28
0.8	-0.06	0.17	-0.06	0.17	-0.06	0.17	0.00	0.18
0.9	-0.06	0.09	-0.06	0.09	-0.06	0.09	0.00	0.09
1.0	-0.06	0.00	-0.06	0.00	-0.06	0.00	0.00	0.00

TABLE 3.

THE STRESS RESULTANTS M_s AND Q_s FOR A CONICAL SHELL WITH $\epsilon = 0.1$

ξ	<u>EXACT SOLUTION</u>		<u>PERTURBATION SOLUTION</u>			
	M_s/m_s	$Q_s \left(\frac{\ell \epsilon^{1/2}}{m_s} \right)$	<u>FIRST APPROXIMATION:</u>		<u>SECOND APPROXIMATION:</u>	
			M_s/m_s	$Q_s \left(\frac{\ell \epsilon^{1/2}}{m_s} \right)$	M_s/m_s	$Q_s \left(\frac{\ell \epsilon^{1/2}}{m_s} \right)$
0.0	1.00	0.00	1.00	0.00	1.00	0.00
0.1	0.92	-0.67	0.86	-0.49	0.90	-0.65
0.2	0.70	-0.96	0.62	-0.72	0.68	-0.94
0.3	0.43	-0.94	0.36	-0.70	0.41	-0.92
0.4	0.20	-0.69	0.15	-0.51	0.18	-0.68
0.5	0.05	-0.36	0.02	-0.26	0.04	-0.33
0.6	-0.01	-0.08	-0.03	-0.04	-0.02	-0.04
0.7	-0.02	+0.04	-0.02	+0.05	-0.02	0.08
0.8	0.00	+0.03	0.00	0.03	0.00	0.04
0.9	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 4.

THE STRESS RESULTANTS M_s AND Q_s FOR A CONICAL SHELL WITH $\epsilon = 0.01$

ξ	<u>EXACT SOLUTION</u>		<u>PERTURBATION SOLUTION</u>			
	M_s/m_s	$Q_s \left(\frac{l\epsilon^{1/2}}{m_s} \right)$	<u>FIRST APPROXIMATION:</u>		<u>SECOND APPROXIMATION:</u>	
			M_s/m_s	$Q_s \left(\frac{l\epsilon^{1/2}}{m_s} \right)$	M_s/m_s	$Q_s \left(\frac{l\epsilon^{1/2}}{m_s} \right)$
0.0	1.00	0.00	1.00	0.00	1.00	0.00
0.05	0.83	-0.68	0.79	-0.60	0.82	-0.65
0.10	0.48	-0.71	0.45	-0.64	0.47	-0.69
0.15	0.19	-0.48	0.11	-0.43	0.19	-0.46
0.20	0.03	-0.21	0.02	-0.19	0.02	-0.21
0.25	-0.03	-0.04	-0.03	-0.03	-0.03	-0.04
0.30	-0.03	0.03	-0.03	0.03	-0.03	0.03
0.35	-0.02	0.03	-0.01	0.03	-0.02	0.03
0.40	0.00	0.02	0.00	0.01	0.00	0.02
0.45	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 5.

THE STRESS RESULTANTS M_s AND Q_s FOR A CONICAL SHELL WITH $\epsilon = 0.001$

ξ	<u>EXACT SOLUTION</u>		<u>PERTURBATION SOLUTION</u>			
	M_s/m_s	$Q_s \left(\frac{l\epsilon^{1/2}}{m_s} \right)$	<u>FIRST APPROXIMATION:</u>		<u>SECOND APPROXIMATION:</u>	
	M_s/m_s	$Q_s \left(\frac{l\epsilon^{1/2}}{m_s} \right)$	M_s/m_s	$Q_s \left(\frac{l\epsilon^{1/2}}{m_s} \right)$	M_s/m_s	$Q_s \left(\frac{l\epsilon^{1/2}}{m_s} \right)$
0.000	1.00	0.00	1.00	0.00	1.00	0.00
0.025	0.64	-0.68	0.63	-0.65	0.64	-0.67
0.050	0.19	-0.42	0.18	-0.40	0.19	-0.42
0.075	-0.01	-0.11	-0.01	-0.11	-0.01	-0.11
0.100	-0.04	0.02	-0.04	0.01	-0.04	0.01
0.125	-0.02	0.03	-0.02	0.03	-0.02	0.03
0.150	0.00	0.01	0.00	0.01	0.00	0.01
0.175	0.00	0.00	0.00	0.00	0.00	0.00
0.200	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 6.

THE STRESS FUNCTION ψ , ROTATION β AND DEFLECTION w FOR A CIRCULAR PLATE LOADED BY A UNIFORMLY DISTRIBUTED BENDING MOMENT, $-m_o$, AT ITS EDGE

<u>VON KARMAN BOUNDARY</u>				<u>FULL REISSNER BOUNDARY LAYER EQUATIONS</u>					
<u>LAYER EQUATIONS</u>				<u>K = 1.0</u>			<u>K = 2.0</u>		
η	$K^{1/3}\beta$	ψ/B_2	$-\frac{K^{2/3}m_o}{D}w$	$K^{1/3}\beta$	ψ/B_2	$-\frac{K^{2/3}m_o}{D}w$	$K^{1/3}\beta$	ψ/B_2	$-\frac{K^{2/3}m_o}{D}w$
0.000	1.683	0.000	0.000	1.783	0.000	0.000	1.742	0.000	0.000
0.550	1.165	0.378	0.779	1.257	0.394	0.543	1.219	0.388	0.633
1.050	0.801	0.532	1.266	0.868	0.564	0.974	0.840	0.551	1.088
1.550	0.541	0.601	1.597	0.584	0.642	1.302	0.566	0.625	1.418
2.050	0.363	0.633	1.820	0.388	0.677	1.532	0.377	0.659	1.645
2.550	0.242	0.647	1.970	0.256	0.692	1.688	0.250	0.673	1.798
3.050	0.162	0.653	2.069	0.169	0.698	1.792	0.166	0.680	1.900
3.550	0.108	0.656	2.136	0.111	0.701	1.861	0.110	0.683	1.968
4.050	0.077	0.657	2.180	0.073	0.702	1.906	0.072	0.684	2.013
4.550	0.048	0.657	2.210	0.048	0.703	1.936	0.048	0.684	2.042
5.050	0.031	0.658	2.229	0.031	0.703	1.955	0.031	0.685	2.062
5.550	0.020	0.658	2.242	0.020	0.703	1.968	0.020	0.685	2.075
6.050	0.013	0.658	2.250	0.013	0.703	1.976	0.013	0.685	2.083
6.550	0.008	0.658	2.255	0.007	0.703	1.981	0.007	0.685	2.088
7.050	0.004	0.658	2.258	0.003	0.703	1.983	0.003	0.685	2.090
7.550	0.000	0.658	2.259	0.000	0.703	1.984	0.000	0.685	2.091

TABLE 7.

THE STRESS FUNCTION ψ , ROTATION β AND DEFLECTION w FOR A CIRCULAR PLATE LOADED BY A UNIFORMLY DISTRIBUTED BENDING MOMENT, $-m_o$, AT ITS EDGE

FULL REISSNER BOUNDARY LAYER EQUATIONS

η	<u>K = 3.0</u>			<u>K = 5.0</u>			<u>K = 10.0</u>		
	$K^{1/3}\beta$	ψ/B_2	$-\frac{K^{2/3}m_o}{D}w$	$K^{1/3}\beta$	ψ/B_2	$-\frac{K^{2/3}m_o}{D}w$	$K^{1/3}\beta$	ψ/B_2	$-\frac{K^{2/3}m_o}{D}w$
0.000	1.727	0.000	0.000	1.714	0.000	0.000	1.702	0.000	0.000
0.550	1.205	0.385	0.669	1.193	0.383	0.701	1.183	0.381	0.730
1.050	0.830	0.546	1.132	0.821	0.542	1.171	0.813	0.538	1.207
1.550	0.560	0.619	1.462	0.554	0.614	1.502	0.549	0.609	1.538
2.050	0.374	0.652	1.688	0.370	0.646	1.727	0.367	0.641	1.762
2.550	0.248	0.667	1.840	0.247	0.661	1.879	0.245	0.655	1.913
3.050	0.165	0.673	1.942	0.164	0.667	1.980	0.163	0.661	2.013
3.550	0.109	0.676	2.009	0.109	0.670	2.047	0.108	0.664	2.080
4.050	0.072	0.677	2.054	0.072	0.671	2.091	0.072	0.665	2.125
4.550	0.048	0.677	2.084	0.048	0.671	2.121	0.048	0.666	2.154
5.050	0.031	0.677	2.103	0.031	0.672	2.140	0.031	0.666	2.174
5.550	0.020	0.678	2.116	0.020	0.672	2.153	0.020	0.666	2.186
6.050	0.013	0.678	2.124	0.013	0.672	2.161	0.013	0.666	2.195
6.550	0.008	0.678	2.129	0.007	0.672	2.166	0.008	0.666	2.200
7.050	0.004	0.678	2.132	0.003	0.672	2.169	0.004	0.666	2.202
7.550	0.000	0.678	2.133	0.000	0.672	2.170	0.000	0.666	2.203

TABLE 8.

THE STRESS FUNCTION ψ , ROTATION β AND DEFLECTION w FOR A BUCKLED CIRCULAR PLATE
LOADED BY A UNIFORMLY DISTRIBUTED RADIAL COMPRESSIVE FORCE, P , AT ITS EDGE

<u>VON KARMAN BOUNDARY</u>				<u>FULL REISSNER BOUNDARY LAYER EQUATIONS</u>					
<u>LAYER EQUATIONS</u>				<u>$\bar{K} = 0.8$</u>			<u>$\bar{K} = 1.0$</u>		
<u>n</u>	<u>$\beta\bar{K}$</u>	<u>ψ/aP</u>	<u>$-\bar{K}(\frac{P}{D})^{1/2}w$</u>	<u>$\beta\bar{K}$</u>	<u>ψ/aP</u>	<u>$-\bar{K}(\frac{P}{D})^{1/2}w$</u>	<u>$\beta\bar{K}$</u>	<u>ψ/aP</u>	<u>$-\bar{K}(\frac{P}{D})^{1/2}w$</u>
0.000	1.614	-1.000	0.000	1.359	-1.000	0.000	1.435	-1.000	0.000
1.050	1.142	0.062	1.499	1.076	-0.112	0.835	1.101	-0.591	1.009
2.050	0.626	0.365	2.365	0.670	0.224	1.535	0.658	0.268	1.767
3.050	0.322	0.445	2.823	0.378	0.336	2.010	0.360	0.370	2.242
4.050	0.163	0.466	3.057	0.206	0.369	2.288	0.192	0.399	2.506
5.050	0.082	0.471	3.174	0.111	0.379	2.441	0.101	0.407	2.647
6.050	0.041	0.472	3.233	0.060	0.382	2.524	0.053	0.409	2.722
7.050	0.020	0.472	3.263	0.032	0.383	2.568	0.028	0.409	2.761
8.050	0.010	0.472	3.277	0.016	0.383	2.590	0.014	0.410	2.781
9.050	0.004	0.472	3.284	0.006	0.383	2.600	0.006	0.410	2.791
10.050	0.000	0.472	3.290	0.000	0.383	2.600	0.000	0.410	2.793

TABLE 9.

THE STRESS FUNCTION ψ , ROTATION β AND DEFLECTION w FOR A BUCKLED CIRCULAR PLATE
 LOADED BY A UNIFORMLY DISTRIBUTED RADIAL COMPRESSIVE FORCE, P , AT ITS EDGE

FULL REISSNER BOUNDARY LAYER EQUATIONS

n	$\bar{K} = 1.5$			$\bar{K} = 1.7$			$\bar{K} = 2.2$		
	$\beta\bar{K}$	ψ/aP	$-\bar{K}(\frac{P}{D})^{1/2}w$	$\beta\bar{K}$	ψ/aP	$-\bar{K}(\frac{P}{D})^{1/2}w$	$\beta\bar{K}$	ψ/aP	$-\bar{K}(\frac{P}{D})^{1/2}w$
0.000	1.526	-1.000	0.000	1.544	-1.000	0.000	1.571	-1.000	0.000
1.050	1.125	0.003	1.243	1.129	0.015	1.293	1.134	0.033	1.370
2.050	0.643	0.318	2.061	0.639	0.328	2.122	0.634	0.342	2.215
3.050	0.340	0.408	2.530	0.336	0.416	2.589	0.331	0.427	2.678
4.050	0.176	0.433	2.778	0.173	0.440	2.834	0.169	0.450	2.919
5.050	0.061	0.439	2.907	0.089	0.445	2.960	0.086	0.455	3.042
6.050	0.031	0.441	2.973	0.045	0.447	3.025	0.044	0.457	3.104
7.050	0.024	0.441	3.007	0.023	0.448	3.058	0.020	0.457	3.136
8.050	0.011	0.441	3.024	0.011	0.448	3.075	0.011	0.457	3.151
9.050	0.005	0.441	3.032	0.004	0.448	3.082	0.004	0.457	3.156
10.050	0.000	0.441	3.034	0.000	0.448	3.084	0.000	0.457	3.160

PERTURBATION ANALYSIS OF SOME AXISYMMETRIC PROBLEMS
IN PLATES AND SHELLS

by

James R. Stafford, Jr.

A perturbation analysis is made of the following problems:

1. The axisymmetric linear deformation of a circular cylindrical shell having a thickness that varies in an arbitrary, but smooth, manner along its axis;
2. The axisymmetric linear deformation of a conical shell having a thickness that varies in an arbitrary, but smooth, manner along its generatrix;
3. The axisymmetric finite deformation of a circular plate loaded by a uniformly distributed bending moment at its edge;
4. The axisymmetric finite deformation of a buckled plate loaded by a uniformly distributed radial compressive force at its edge.

The solutions to the shell problems are in a closed form for each of the first two approximations resulting from an expansion of the dependent variable in a power series in terms of a small perturbation parameter. In the plate problems, a systematic perturbation analysis is performed on Reissner's differential equations for finite deflections, and a numerical comparison is made between the von Karman boundary layer equations and the full Reissner boundary layer equations.

Cochran's method of treating singular perturbation problems is used in the solution of the governing differential equations

in problems 1 and 2. The results for these two problems are given in a simple form requiring only elementary mathematical manipulations for specific thickness variations. The solutions are compared with known results by considering a conical shell of linearly varying thickness loaded by edge moments and a circular cylindrical tank of linearly varying thickness. In both of these problems the perturbation solutions are in excellent agreement with the exact solutions.

Reissner's equations for finite axisymmetric deformations of plates are used as the basic set of differential equations in the plate problems. A systematic study is made of the effects that the order of magnitude of various perturbation parameters have on these differential equations. These perturbation parameters reflect the changes due to the magnitude of the loading and of the physical and geometric properties of the plates.

For each of the plate problems two sets of boundary layer equations are developed for finite deformations by using the method of matched asymptotic expansions. One set of boundary layer equations represents the first approximation to the Reissner formulation for rotations of order one. The other set corresponds to the first approximation to the boundary layer equations of the von Karman theory. At some point the latter equations should become invalid and the former equations should be the ones giving a truer prediction of the stress and deformation. A comparison of the solutions to these two sets of boundary layer equations is made. It is found that the approximation is good for

rotations up to about 45° for problem 4 and about 70° for problem 3.

The boundary layer equations for the plate problems do not possess closed form solutions and are solved numerically.