AN APPROXIMATE SOLUTION FOR THE FLEXURAL AND IN-PLANE STRESS EFFECTS OF A LATERALLY LOADED SKEWED FOLDED PLATE STRUCTURE

by

GEORGE W. SWIFT, B.S., M.S.

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LIST OF SYMBOLS

A. LIST OF SYMBOLS APPEARING IN SECTION I THROUGH SECTION V

a	Plate dimension
A_{ij}	Constant appearing in the deflection function for the energy method
b	Plate dimension
C_{ij}	Constant appearing on the deflection function for the energy method
D	Flexural rigidity
E	Modulus of elasticity
h	Plate thickness
ℓ	Plate dimension
M_x^A	Flexural moment in plate A
M_y^A	Flexural moment in plate A
M_{xy}^A	Twisting moment in plate A
M_ξ^B	Flexural moment in plate B
M_η^B	Flexural moment in plate B
$M_{\xi\eta}^B$	Twisting moment in plate B
$M_{\bar{x}}^C$	Flexural moment in plate C
$M_{\bar{y}}^C$	Flexural moment in plate C
$M_{\bar{x}\bar{y}}^C$	Twisting moment in plate C
M_{Az}	Resultant hinge reaction moment on plate A
M_{Ax}	Resultant hinge reaction moment on plate A
$M_{C\bar{x}}$	Resultant hinge reaction moment on plate C
$M_{C\bar{z}}$	Resultant hinge reaction moment on plate C

N_x^A Normal stress resultant in plate A

N_y^A Normal stress resultant in plate A

N_{xy}^A Shear stress resultant in plate A

N_ξ^B Normal stress resultant in plate B

N_η^B Normal stress resultant in plate B

$N_{\xi\eta}^B$ Shear stress resultant in plate B

$N_{\bar{x}}^C$ Normal stress resultant in plate C

$N_{\bar{y}}^C$ Normal stress resultant in plate C

$N_{\bar{x}\bar{y}}^C$ Shear stress resultant in plate C

N_P Total unbalanced force perpendicular to the joints acting on the skewed plate due to an unsymmetric loading.

n_T Total unbalanced force perpendicular to the joints acting on the skewed plate due to a unit side sway

P Magnitude of a uniform load on the skewed plate

P_j Root of the boundary condition simultaneous equations determinant used in the energy method

R_{Ax} Resultant reaction on the hinge of plate A

R_{Az} Resultant reaction on the hinge of plate A

$R_{C\bar{x}}$ Resultant reaction of the hinge of plate C

$R_{C\bar{z}}$ Resultant reaction on the hinge of plate C

S_i Root of the boundary condition simultaneous equation determinant used in the energy method

u^A Displacement in the x-direction in plate A

u^B Displacement perpendicular to the η- direction in plate B

u^C Displacement in the $\bar{x}$-direction in plate C

v^A	Displacement in the y direction in plate A
v^B	Displacement perpendicular to the ξ -direction in plate B
v^C	Displacement in the $\bar{y}$-direction in plate C
V_x^A	Kirchhoff shear in plate A
V_y^A	Kirchhoff shear in plate A
V_ξ^B	Kirchhoff shear in plate B
V_η^B	Kirchhoff shear in plate B
$V_{\bar{x}}^{\underline{C}}$	Kirchhoff shear in plate C
$V_{\bar{y}}^{\underline{C}}$	Kirchhoff shear in plate C
w^A	Deflection of plate A
w^B	Deflection of plate B
w^C	Deflection of plate C
W_A	Flexural potential energy in plate A
W_B	Flexural potential energy in plate B
W_C	Flexural potential energy in plate C
W'_A	Extensional potential energy in plate A
W'_B	Extensional potential energy in plate B
W'_C	Extensional potential energy in plate C
x, y, z	Coordinates in plate A
$\bar{x}, \bar{y}, \bar{z}$	Coordinates in plate C
α	Angle of skew
Δ	Side sway translation of the joints
μ	Poisson's ratio
ξ, η, ζ	Coordinates in plate B

B. LIST OF SYMBOLS APPEARING IN SECTION VII THROUGH XVI

a	Arbitrary constant
d	Distance between boundary and position of a balancing load
d_i	Same as above but identified with ith boundary interval
E	Modulus of elasticity
E	One or zero depending upon which face of an infinitesimal element the extensional results are to be computed
F	One or zero depending upon which face of an infinitesimal element the flexural results are to be computed
h	Plate thickness
L	Length of plot line along which results are to be presented
M	Concentrated flexural moment load
M_j	Concentrated flexural moment load applied at the jth point of balancing load application
$\overline{M}_i$	Flexural moment calculated at point i
M_m^L	The mth concentrated applied flexural moment load
M_N^P	Moment normal to the reference line due to a load P = 1
M_N^M	Moment normal to a reference line due to a concentrated moment M
M_N^Q	Moment normal to a reference line due to a uniform load
M_{ij}^P	Moment at boundary interval i due to a load P at the jth balancing load position
M_{ij}^M	Moment at boundary interval i due to a moment M = 1 at the jth balancing load position
M_{ic}^Q	Moment at boundary interval i due to a unit uniform load
N	Number of intervals into which the boundary is divided

n	Number of points along a plot line at which results are to be calculated
P	Concentrated force
P_j	Concentrated force applied at the jth point of balancing load application
P_m^L	The mth applied concentrated force perpendicular to the boundary for extensional problems and perpendicular to the plate for flexural problems
Q	Intensity of a uniform load
Q_r	Flexural shear resultant acting on a face perpendicular to the position vector r
Q_θ	Flexural shear resultant parallel to the position vector r
r	Position vector from point of application of loads to point where the effect of the load is to be calculated.
r_{ij}	Position vector from jth balancing load position to midpoint of the ith boundary interval
S_{ij}^P	Slope of the bent plate at the ith boundary interval due to a load at the jth balancing load position
S_{ij}^M	Slope of the bent plate at the ith boundary interval due to a concentrated moment load at the jth balancing load position
S_{ic}^Q	Slope of the bent plate at the ith boundary interval due to a uniform load
T	A concentrated force applied parallel to a boundary
T_j	Concentrated balancing load applied parallel to the boundary at the jth balancing load point for extensional problems

T_m^L The mth applied concentrated force parallel to the boundary

u_i Displacement normal to the ith boundary interval

u_N Displacement normal to a reference line

u_r Displacement in the direction of the position vector r

u_θ Displacement perpendicular to the position vector r

u_{ij}^P Displacement normal to the ith boundary interval due to a balancing load P = 1 applied at the jth balancing load point

u_{ij}^T Displacement normal to the ith boundary interval due to a balancing load T = 1 applied at the jth balancing load point

v_i Displacement normal to the ith boundary interval

v_N Displacement parallel to a reference line

v_{ij}^P Displacement parallel to the ith boundary interval due to a balancing load P = 1 applied at the jth balancing load point

v_{ij}^T Displacement parallel to the ith boundary interval due to a balancing load T = 1 applied at the jth balancing load point

V_N^P Kirchhoff shear normal to a boundary due to a force P

V_N^M Kirchhoff shear normal to a boundary due to a concentrated moment M

V_N^Q Kirchhoff shear calculated at a boundary due to a uniform load Q

V_{ij}^P Kirchhoff shear calculated at the ith boundary interval due to a force P = 1 applied at the jth balancing load position

V_{ij}^M Kirchhoff shear calculated at the ith boundary interval due to a concentrated moment applied at the jth balancing load position

V_{ic}^Q Kirchhoff shear calculated at the ith boundary interval due to a unit uniform load

w^P Deflection of an infinite plate due to a concentrated force

w^M Deflection of an infinite plate due to a concentrated moment

w^Q Deflection of an infinite plate due to a uniform load

w^P_{ij} Deflection at boundary interval i due to a balancing load P = 1 applied at the jth balancing load position

w^M_{ij} Deflection at boundary interval i due to a balancing load M = 1 applied at the jth balancing load position

w^Q_{ic} Deflection at boundary interval i due to a unit uniform load

x^o_i General position coordinate referred to a reference axis on the infinite plate

x^o_I Coordinate of the initial end of a plot line

x^o_T Coordinate of the terminal end of the plot line

x^P_i Coordinate of the midpoint of a boundary interval

x^Q_i Coordinate of the ith balancing load position

y^o_i General position coordinate referred to a reference axis on the infinite plate

y^o_I Coordinate of the initial end of a plot line

y^o_T Coordinate of the terminal end of a plot line

y^P_i Coordinate of the midpoint of the ith boundary interval

y^Q_i Coordinate of the ith balancing load position

α Angle measured from the position vector, r, to the outward normal to a boundary

α_{ij} Angle measured from the position vector from the jth balancing load to the outward normal to the ith boundary interval

β_i	Azimuth measured from x^o axis to the inward normal to the ith boundary interval
β_j	Azimuth measured from the x^o axis to the inward normal to the jth boundary interval corresponding to the jth balancing load
β_A^E	Azimuth of the plot line for extensional problems
β_A^F	Azimuth of the plot line for flexural problems
$\gamma_{r\theta}$	Shearing strain
ε_r	Strain in the direction of the position vector r
ε_θ	Strain in a direction perpendicular to the position vector r
θ	Angle between an x axis through the point of application of a load and the position vector r
θ_{ij}	Angle between the jth inward normal near the jth balancing load and the position vector from the jth balancing load to the ith boundary interval midpoint
λ_{ij}	Azimuth of the position vector from the jth balancing load to the ith boundary interval midpoint
μ	Poisson's ratio
σ_r	Normal stress in the direction of the position vector
σ_θ	Normal stress perpendicular to the position vector r
σ_N	Stress normal to a boundary
σ_i	Stress normal to the ith boundary
σ_{ij}^P	Normal stress at the ith boundary interval due to a balancing load P = 1 applied at the jth balancing load position

σ_{ij}^{T} Normal stress at the ith boundary interval due to a balancing load $T = 1$ applied at the jth balancing load position

$\tau_{r\theta}$ Shear stress

τ_{N} Shear stress normal to a boundary

τ_{i} Shear stress normal to the ith boundary interval

τ_{ij}^{P} Shear stress at the ith boundary interval due to a balancing load $P = 1$ applied at the jth balancing load position

τ_{ij}^{T} Shear stress at the ith boundary interval due to a balancing load $T = 1$ applied at the jth balancing load position

I. Introduction

A folded plate consists of two or more plates joined together along their edges such that their middle surfaces form a dihedral angle. The joints are assumed to preserve continuity of bending moments, slopes, and deflections.

The particular folded plate configuration to be considered here is composed of two rectangular plates joined to opposite edges of a skewed plate in such a way that the middle surfaces form 90° dihedral angles. Put together in this way the folded plate system forms a structure similar to the one illustrated in Figure 1.

The structure is to be simply supported along the bottom edges of each rectangular plate and loaded vertically, perpendicular to the skewed plate. The remaining sides of each plate are free.

The loads will cause the plates to deflect setting up flexural moments and shears. The flexural shears in turn will load the plates parallel to their middle surfaces causing in-plane stresses and displacements to exist. It is to be assumed that these two effects can be superimposed.

The desirable results of an analysis of a structure such as this would be a description of the flexural moments and shear resultants per unit length in each of the plates and some information about how the in-plane reactions normal to the bases of the simply supported rectangular plates are distributed.

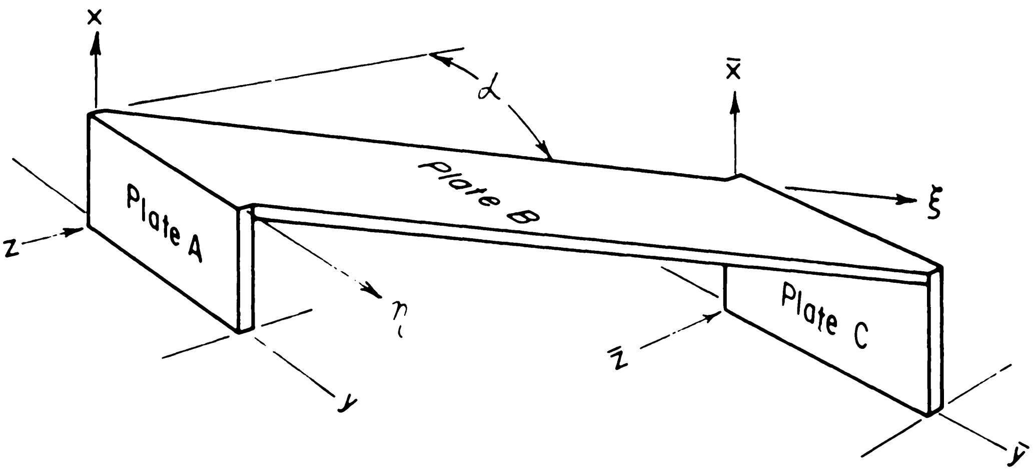
The first indication that such a structure presented a problem in analysis came from the failure of a skewed concrete railroad arch at Bendigo, Austrailia just after the turn of the century.[1] W. C. Kernot, a professor of engineering at the University of Melbourne was called upon to investigate the mishap. His conclusions were that the existing practice, due as he put it to Rankine, was incorrect. The usual approach at that time was to divide the arch into laminae parallel to the free sides and to treat each piece as an independent arch. Professor Kernot made several tests on models of the arch. The results enabled him to state that the thrusts were concentrated near the obtuse corners of the arch. Inquires to contemporary elasticians yielded no theoretical explanations for his experimental findings.

In 1923 a skewed arch designed on the basis of the laminae theory failed in Tacoma, Washington.(2) This led Professor J. C. Rathburn of the University of Washington to develop a more realistic analytical approach which was published in 1924.(3) His ideas were considered an important advancement in the art and engineers up to the present have been using procedures based upon his method to design the skewed bridges in existence today.

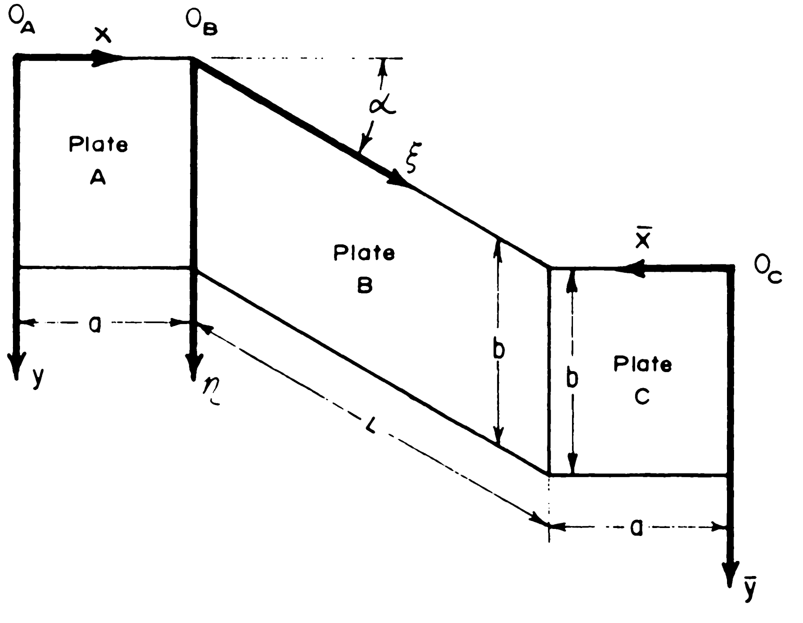
The original theory as set forth by Professor Rathburn was tedious. It contained terms which were not significant. Other engineers found ways of modifying the original theory so that bridge design could be carried out in a design office by the average engineer with a desk calculator. The major contributors to the refinements and simplifications of Rathburn's original theory were Rathburn,(4) Hayden,(5) Weiner,(6) Gifford,(7) Hodges,(8) Weiner,(9) Baron,(10) and Michalos(11).

The theory assumes that the members of the bridge are very wide beams which form a rigid frame. The frame is skewed so that it does not lie in one plane. When the structure is loaded, moments, thrusts, and shears are set up which act in three mutually perpendicular directions at any point along the structure.

One of the more recent modifications of Professor Rathburn's method was to apply the shear torsion analogy to the problem which reduces the calculations to simple arithmetic[11]. This approach still treats the structure as a set of very wide beams. It is certainly sufficient for an initial order of magnitude analysis. A more detailed accounting of stresses and displacements, however, must come from an approach based upon classical plate theory.



Perspective view of the skewed folded plate



Plan view of the skewed folded plate

Figure 1

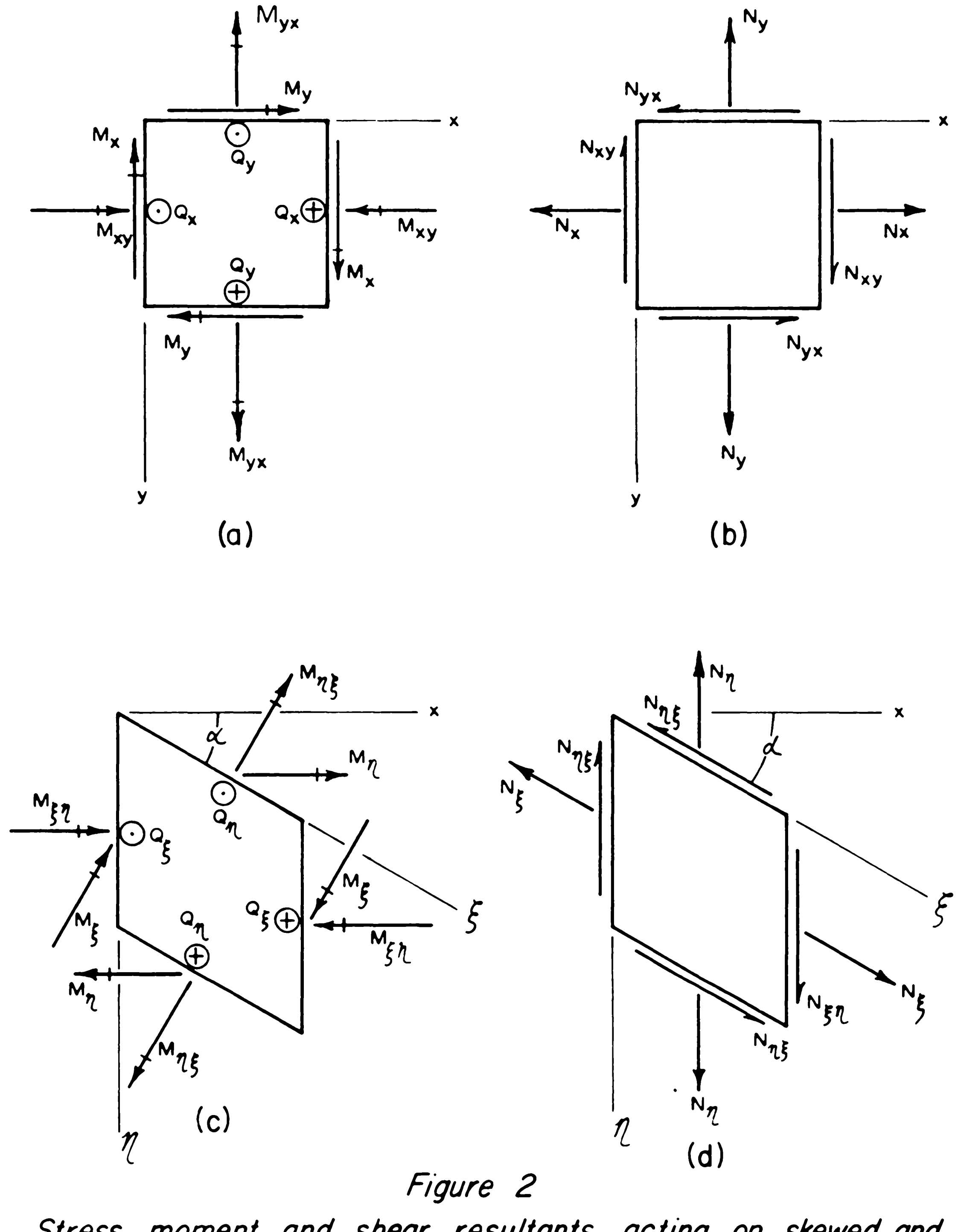


Figure 2

Stress, moment, and shear resultants acting on skewed and rectangular elements

II. Mathematical Definition of the Problem

In order to outline the various approaches to the solution of the problem, it will be convenient to write down the differential equations and boundary condition equations that must be satisfied. The equations are associated with plates A, B, and C the coordinates, dimensions, and shapes of which are shown in Figure 1.

The stress and moment resultants that are usually associated with generalized plane stress and classical plate problems are represented in their positive directions on the infinitesimal elements shown in Figure 2. Figures 2a and 2b represent an elemental piece of the rectangular plates and Figures 2c and 2d show similar elements for the skewed plate.

Since the boundary conditions which are to be listed are mixed, they are expressed in terms of transverse deflections, denoted by w, and in-plane displacements denoted by u and v. The positive directions of these terms are indicated in Figure 4. Each term in the equations will carry an A, B, or C superscript to indicate the plate with which it is associated.

The equations were all obtained by particularizing the tensor equations in Chapter 7 of Green and Zerna[(12)]. Another excellent derivation of the flexural equations listed here is to be found in a bulletin by Odman[(13)].

For deflections in the rectangular plates, the governing equation is

$$\frac{\partial^4 w^A}{\partial x^4} + \frac{\partial^4 w^A}{\partial x^2 \partial y^2} + \frac{\partial^4 w^A}{\partial y^4} = -\frac{P(x,y)}{D} \qquad (1)$$

This equation written in skewed coordinates for the deflections of plate B is

$$\frac{\partial^4 w^B}{\partial \xi^4} + \frac{\partial^4 w^B}{\partial \eta^4} + 2(1 + 2\sin^2 a)\frac{\partial^4 w^B}{\partial \xi^2 \partial \eta^2}$$

$$-4 \sin a \left[\frac{\partial^4 w^B}{\partial \xi \partial \eta^3} + \frac{\partial^4 w^B}{\partial \xi^3 \partial \eta}\right] = -\frac{\cos^4 a \, P(x,y)}{D} \qquad (2)$$

The governing equations for the in-plane displacements are the equilibrium equations for the in-plane stress resultants written in terms of the u's and v's. For the rectangular plates the governing equations are

$$\left(\frac{2}{1-\mu}\right)\frac{\partial^2 u^A}{\partial x^2} + \frac{\partial^2 u^A}{\partial y^2} + \left(\frac{1+\mu}{1-\mu}\right)\frac{\partial^2 v^A}{\partial x \partial y} = 0 \qquad (3)$$

$$\left(\frac{2}{1-\mu}\right)\frac{\partial^2 v^A}{\partial y^2} + \frac{\partial^2 v^A}{\partial x^2} + \left(\frac{1+\mu}{1-\mu}\right)\frac{\partial^2 u^A}{\partial x \partial y} = 0 \qquad (4)$$

where μ is Poisson's Ratio.

Two more equations similar to these but expressed in skewed coordinates for plate B have the form

$$\left(\frac{2}{1-\mu}\right)\left[\frac{1}{\cos^2 a}\left\{\frac{\partial^2 u^B}{\partial \xi^2} - 2\sin a\frac{\partial^2 u^B}{\partial \xi \partial \eta} + \sin^2 a\frac{\partial^2 u^B}{\partial \eta^2}\right\}\right.$$

$$\left. + \frac{\mu}{\cos^2 a}\left\{\cos a\frac{\partial^2 v^B}{\partial \eta \partial \xi} - \sin a \cos a\frac{\partial^2 v^B}{\partial \eta^2}\right\}\right]$$

$$+\left[\frac{\partial^2 u^B}{\partial \eta^2} + \frac{1}{\cos^2 a}\left\{\cos a\frac{\partial^2 v^B}{\partial \eta \partial \xi} - \sin a \cos a\frac{\partial^2 v^B}{\partial \eta^2}\right\}\right] = 0 \qquad (5)$$

$$\left(\frac{2}{1-\mu}\right)\left[\frac{\partial v^B}{\partial \eta^2} + \frac{\mu}{\cos^2 a}\left\{\cos a \frac{\partial^2 u^B}{\partial \eta \partial \xi} - \sin a \cos a \frac{\partial^2 u^B}{\partial \eta^2}\right\}\right]$$

$$+ \frac{1}{\cos^2 a}\left[\cos a \frac{\partial^2 u^B}{\partial \xi \partial \eta} - \sin a \cos a \frac{\partial^2 u^B}{\partial \eta^2}\right.$$

$$\left. + \frac{\partial^2 v^B}{\partial \xi^2} - 2 \sin a \frac{\partial^2 v^B}{\partial \xi \partial \eta} + \sin^2 a \frac{\partial^2 v^B}{\partial \eta^2}\right] = 0 \qquad (6)$$

One of the methods to be discussed is based upon a variational principle. In order to use this approach expressions for the potential energy stored in the plates due to flexure and due to extension are needed. In rectangular coordinates, the flexural potential energy equation is

$$W_A = \frac{D}{2}\int_A \left\{\left[\frac{\partial^2 w^A}{\partial x^2} + \frac{\partial^2 w^A}{\partial y^2}\right]^2 + 2(1-\mu)\left[\left(\frac{\partial^2 w^A}{\partial x \partial y}\right)^2 - \frac{\partial^2 w^A}{\partial x^2}\frac{\partial^2 w^A}{\partial y^2}\right]\right\} dA \qquad (7)$$

where D is the flexural rigidity of the plate and the integration is carried out over the area of the plate. Expressed in skewed coordinates for plate B, this equation becomes

$$W_B = \frac{D}{2\cos^4 a}\int_A \left\{\left[\frac{\partial^2 w^B}{\partial \xi^2} + \frac{\partial^2 w^B}{\partial \eta^2} - 2 \sin a \frac{\partial^2 w^B}{\partial \xi \partial \eta}\right]^2 + 2\cos^2 a(1-\mu)\left[\left(\frac{\partial^2 w^B}{\partial \xi \partial \eta}\right)^2 - \frac{\partial^2 w^B}{\partial \xi^2}\frac{\partial^2 w^B}{\partial \eta^2}\right]\right\} dA \qquad (8)$$

The potential energy equation due to extension for one of the rectangular plates has the form

$$W'_A = \frac{E}{2(1-\mu^2)} \int_A \left\{ \left[\frac{\partial u^A}{\partial x} + \frac{\partial v^A}{\partial y} \right]^2 \right.$$

$$\left. + 2(1-\mu) \left[\frac{1}{4} \left(\frac{\partial u^A}{\partial y} + \frac{\partial v^A}{\partial x} \right)^2 - \frac{\partial u^A}{\partial x} \frac{\partial v^A}{\partial y} \right] \right\} dA \qquad (9)$$

For the extensional energy of plate B, the skewed coordinate version of this equation is

$$W'_B = \frac{E}{2 \cos \alpha (1-\mu^2)} \int_A \left\{ \left[\frac{\partial u^B}{\partial \xi} + \frac{\partial v^B}{\partial \eta} - \sin \alpha \left(\frac{\partial u^B}{\partial \eta} + \frac{\partial v^B}{\partial \xi} \right) \right]^2 \right.$$

$$\left. + 2 \cos^2 \alpha (1-\mu) \left[\frac{1}{4} \left(\frac{\partial u^B}{\partial \eta} + \frac{\partial v^B}{\partial \xi} \right)^2 - \frac{\partial u^B}{\partial \xi} \frac{\partial v^B}{\partial \eta} \right] \right\} dA \qquad (10)$$

Regardless of the approach, the boundary conditions and continuity conditions remain the same. The boundary conditions imposed on u, v, and w for plate A are listed below. Along the hinged edge where x = 0

$$w^A \Big|_{x=0} = 0 \qquad (11)$$

$$M_x^A \Big|_{x=0} = -D \left[\frac{\partial^2 w^A}{\partial x^2} + \mu \frac{\partial^2 w^A}{\partial y^2} \right]_{x=0} = 0 \qquad (12)$$

$$u^A \Big|_{x=0} = 0 \qquad (13)$$

$$v^A \Big|_{x=0} = 0 \qquad (14)$$

Along the two free edged at y = 0 and y = b

$$M_y^A \Big|_{\substack{y=0 \\ y=b}} = -D \left[\frac{\partial^2 w^A}{\partial y^2} + \mu \frac{\partial^2 w^A}{\partial x^2} \right]_{\substack{y=0 \\ y=b}} = 0 \qquad (15)$$

$$V_y^A \Big|_{\substack{y=0\\y=b}} = -D \left[\frac{\partial^3 w^A}{\partial y^3} + (2-\mu) \frac{\partial^3 w^A}{\partial x^2 \partial y} \right]_{\substack{y=0\\y=b}} = 0 \quad (16)$$

$$N_y^A \Big|_{\substack{y=0\\y=b}} = \frac{E}{(1-\mu^2)} \left[\frac{\partial v^A}{\partial y} + \mu \frac{\partial u^A}{\partial x} \right]_{\substack{y=0\\y=b}} = 0 \quad (17)$$

$$N_{xy} \Big|_{\substack{y=0\\y=b}} = \frac{E}{2(1+\mu)} \left[\frac{\partial v^A}{\partial x} + \frac{\partial u^A}{\partial y} \right]_{\substack{y=0\\y=b}} = 0 \quad (18)$$

The conditions imposed along the free boundaries of plate B expressed in skewed coordinates are written in the following form.

$$M_\eta^B \Big|_{\substack{\eta=0\\\eta=b}} = \frac{-D}{\cos^2 a} \left[\frac{\partial^2 w^B}{\partial \eta^2} - 2 \sin a \frac{\partial^2 w^B}{\partial \xi \, \partial \eta} \right.$$

$$\left. + (\sin^2 a + \mu \cos^2 a) \frac{\partial^2 w^B}{\partial \xi^2} \right]_{\substack{\eta=0\\\eta=b}} = 0 \quad (19)$$

$$V_\eta^B \Big|_{\substack{\eta=0\\\eta=b}} = \frac{-D}{\cos^3 a} \left[\frac{\partial^3 w^B}{\partial \eta^3} - 3 \sin a \frac{\partial^3 w^B}{\partial \xi \, \partial \eta^2} \right.$$

$$+ (2 + \sin^2 a - \mu \cos^2 a) \frac{\partial^3 w^B}{\partial \xi^2 \partial \eta}$$

$$\left. + \sin a (\sin^2 a + \mu \cos^2 a - 2) \frac{\partial^3 w^B}{\partial \xi^2} \right]_{\substack{\eta=0\\\eta=b}} = 0 \quad (20)$$

$$N_{\eta}^{B}\Big|_{\substack{\eta=0\\ \eta=b}} = \frac{E}{2\cos^2\alpha(1-\mu^2)}\left[(\sin^2\alpha+\mu\cos^2\alpha)\frac{\partial u^B}{\partial\xi}\right.$$

$$\left.-\sin\alpha\left(\frac{\partial v^B}{\partial\xi}+\frac{\partial u^B}{\partial\eta}\right)+\frac{\partial v^B}{\partial\eta}\right]_{\substack{\eta=0\\ \eta=b}} = 0 \qquad (21)$$

$$N_{\eta\xi}^{B}\Big|_{\substack{\eta=0\\ \eta=b}} = \frac{E}{2\cos^2\alpha(1-\mu^2)}\left[-2\sin\alpha\left(\frac{\partial u^B}{\partial\xi}+\frac{\partial v^B}{\partial\eta}\right)\right.$$

$$\left.+(1+\sin^2\alpha-\mu\cos^2\alpha)\left(\frac{\partial v^B}{\partial\xi}+\frac{\partial u^B}{\partial\eta}\right)\right]_{\substack{\eta=0\\ \eta=b}} = 0 \qquad (22)$$

Boundary conditions for plate C must be written for three of its edges. These expressed in rectangular coordinates are of the following form. Along the hinged edge

$$w^C\Big|_{\bar{x}=0} = 0 \qquad (23)$$

$$M_{\bar{x}}^{C}\Big|_{\bar{x}=0} = -D\left[\frac{\partial^2 w^C}{\partial\bar{x}^2}+\mu\frac{\partial^2 w^C}{\partial\bar{y}^2}\right]_{\bar{x}=0} = 0 \qquad (24)$$

$$u^C\Big|_{\bar{x}=0} = 0 \qquad (25)$$

$$v^C\Big|_{\bar{x}=0} = 0 \qquad (26)$$

Along the two free edges of plate C

$$M_{\bar{y}}^{C}\Big|_{\substack{\bar{y}=0\\ \bar{y}=b}} = -D\left[\frac{\partial^2 w^C}{\partial\bar{y}^2}+\mu\frac{\partial^2 w^C}{\partial\bar{x}^2}\right]_{\substack{\bar{y}=0\\ \bar{y}=b}} = 0 \qquad (28)$$

$$V_{\bar{y}}^{C}\Big|_{\substack{\bar{y}=0\\ \bar{y}=b}} = -D\left[\frac{\partial^3 w^C}{\partial\bar{y}^3}+2(1-\mu)\frac{\partial^3 w^C}{\partial\bar{x}^2\partial\bar{y}}\right]_{\substack{\bar{y}=0\\ \bar{y}=b}} = 0 \qquad (29)$$

$$N^C_{\bar{y}}\Bigg|_{\substack{\bar{y}=0\\ \bar{y}=b}} = \frac{E}{(1-\mu^2)}\left[\frac{\partial v^C}{\partial \bar{y}} + \mu\frac{\partial u^C}{\partial \bar{x}}\right]_{\substack{\bar{y}=0\\ \bar{y}=b}} = 0 \qquad (30)$$

$$N^C_{\bar{x}\bar{y}}\Bigg|_{\substack{\bar{y}=0\\ \bar{y}=b}} = \frac{E}{2(1+\mu)}\left[\frac{\partial v^C}{\partial \bar{x}} + \frac{\partial u^C}{\partial \bar{y}}\right]_{\substack{\bar{y}=0\\ \bar{y}=b}} = 0 \qquad (31)$$

If continuity is to be preserved across the joints, all moments, slopes, in-plane forces, and displacements must transmit across the joints between the plates unchanged. Figure 3 shows all of the forces and moments acting at the joints. Note that N^B_ξ is broken into components along the ξ and η directions. Each force and couple is considered to act over the same distance in the y and η directions so each factor will represent a force or couple acting over the same magnitude of area when they are multiplied by this length. For instance at the joint between plates A and B, equilibrium of the forces in the x direction requires that

$$dy\, N^A_x\Big|_{x=a} = -d\eta\; V^B_\xi\Big|_{\xi=0}$$

but since at the joint $dy = d\eta$, the differentials of length along the joints will cancel from all of the force equations associated with a free body of the joint. Hence equilibrium of the joint between plates A and B requires that the following continuity conditions be satisfied.

$$\Sigma F_x = 0 = dyN^A_x\Big|_{x=0} + d\eta\; V^B_\xi\Big|_{\xi=0} = 0$$

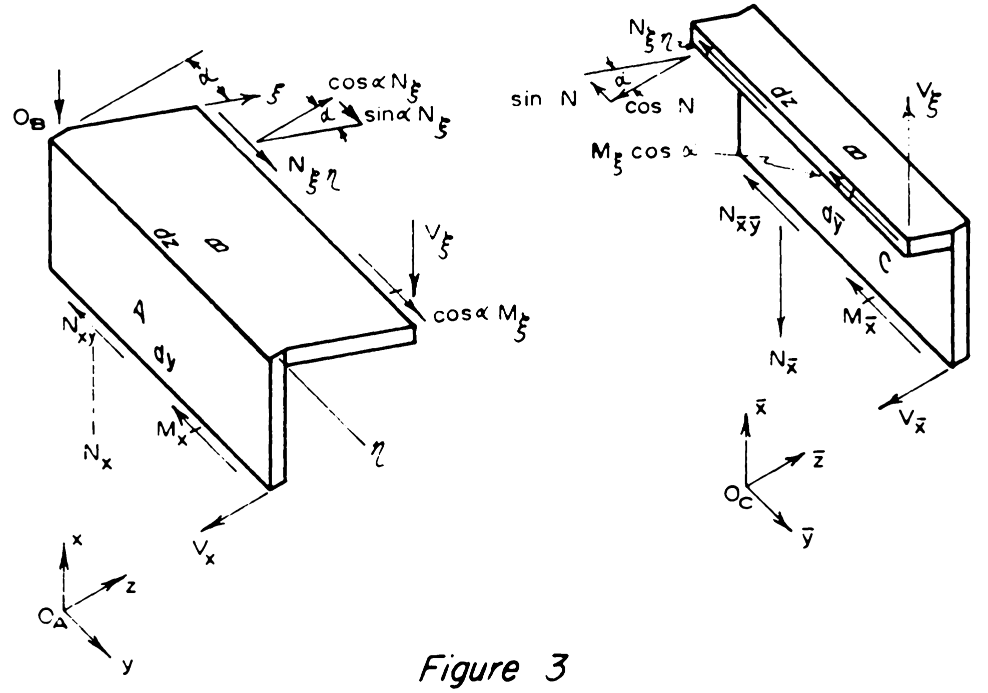


Figure 3

Free body diagram of the joints

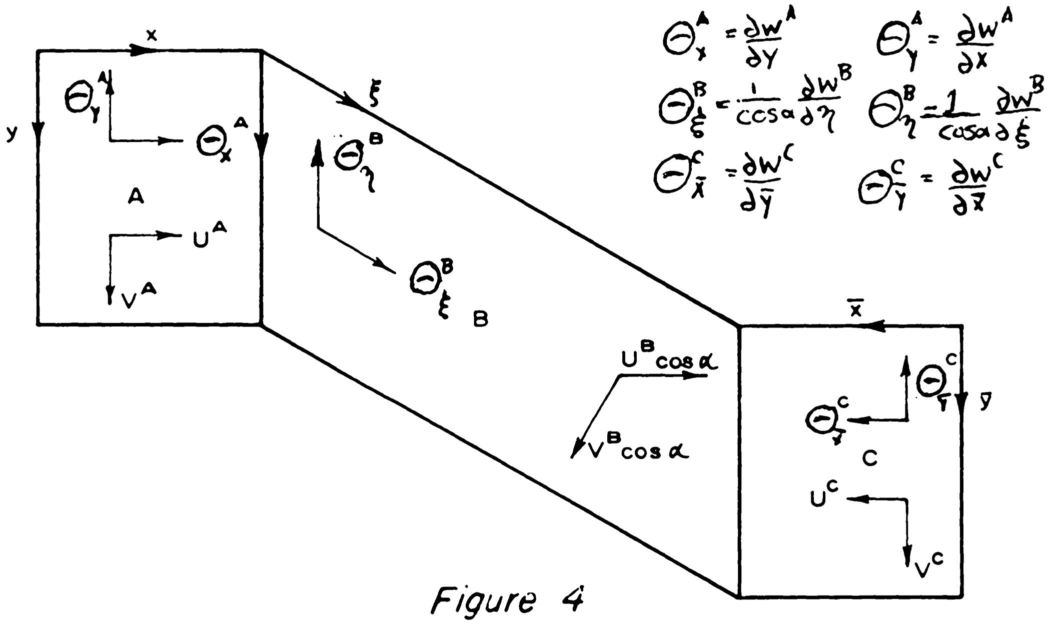


Figure 4

Positive direction of rotation and displacement vector

or

$$\frac{E}{(1-\mu^2)} \left[\frac{\partial u^A}{\partial x} + \mu \frac{\partial v^A}{\partial y} \right]_{x=a} = \frac{D}{\cos^3 a} \left[\frac{\partial^3 w^B}{\partial \xi^3} \right.$$

$$-3 \sin a \frac{\partial^3 w^B}{\partial \xi^2 \partial \eta} + (2 + \sin^2 a - \mu \cos^2 a) \frac{\partial^3 w^B}{\partial \xi \partial \eta^2}$$

$$\left. + \sin a(\sin^2 a + \mu \cos^2 a - 2) \frac{\partial^3 w^B}{\partial \eta^3} \right]_{\xi = 0} = 0 \qquad (32)$$

$$\Sigma F_y = 0 = d\eta \left[N^B_{\xi\eta} + \sin a \, N^B_{\xi} \right]_{\xi = 0} - dy N^A_{xy} \Big|_{x = a} = 0$$

or

$$\frac{E}{\cos^2 a(1-\mu^2)} \left[(1 + \sin^2 a - \mu \cos^2 a) \left(\frac{\partial v^B}{\partial \xi} + \frac{\partial u^B}{\partial \eta} \right) \right.$$

$$\left. -2 \sin a \left(\frac{\partial u^B}{\partial \xi} + \frac{\partial v^B}{\partial \eta} \right) \right]_{\xi = 0}$$

$$+ \frac{E}{\cos^2 a(1-\mu^2)} \left[\sin a \frac{\partial u^B}{\partial \xi} - \sin^2 a \left(\frac{\partial u^B}{\partial \eta} + \frac{\partial v^B}{\partial \xi} \right) \right.$$

$$\left. + \sin a (\sin^2 a + \mu \cos^2 a) \frac{\partial v^B}{\partial \eta} \right]_{\xi = 0}$$

$$- \frac{E}{2(1+\mu)} \left[\frac{\partial u^A}{\partial y} + \frac{\partial v^A}{\partial x} \right]_{x = a} = 0$$

This equation reduces to

$$\left[\frac{\partial u^B}{\partial \eta} + \frac{\partial v^B}{\partial \xi} - 2 \sin a \frac{\partial v^B}{\partial \eta} \right]_{\xi = 0} - \left[\frac{\partial u^A}{\partial x} + \frac{\partial v^A}{\partial y} \right]_{x = a} = 0 \qquad (33)$$

$$\Sigma F_z = 0 = d\eta \cos a \, N_\xi^B \Big|_{\xi = 0} - dy \, V_x^A \Big|_{x = a} = 0$$

or

$$\frac{E}{\cos a(1-\mu^2)} \left[\frac{\partial u^B}{\partial \xi} - \sin a \left(\frac{\partial u^B}{\partial \eta} + \frac{\partial v^B}{\partial \xi} \right) \right.$$

$$\left. + (\sin^2 a + \mu \cos^2 a) \frac{\partial v^B}{\partial \eta} \right] \Big|_{\xi = 0}$$

$$+ D \left[\frac{\partial^3 w^A}{\partial x^3} + (2-\mu) \frac{\partial^3 w^A}{\partial x \partial y^2} \right]_{x = a} = 0 \qquad (34)$$

$$\Sigma M_y = 0 = d\eta \cos a \, M_\xi^B \Big|_{\xi = 0} - dy \, M_x^A \Big|_{x = a} = 0$$

or

$$\frac{D}{\cos^2 a} \left[\frac{\partial^2 w^B}{\partial \xi^2} - 2 \sin a \frac{\partial^2 w^B}{\partial \xi \partial \eta} \right.$$

$$\left. + (\sin^2 a + \mu \cos^2 a) \frac{\partial^2 w^B}{\partial \eta^2} \right]_{\xi = 0}$$

$$- D \left[\frac{\partial^2 w^A}{\partial x^2} + \mu \frac{\partial^2 w^A}{\partial y^2} \right]_{x = a} = 0 \qquad (35)$$

The slopes represented by rotation vectors as shown in Figure 4 take the same positive directions as the moment vectors of Figure 2. The positive directions of u^B and v^B are also shown in Figure 4. Using these as a guide, the continuity conditions involving the geometry of the joint between plates A and B will have the following form

$$\theta_y^A \Big|_{x = a} = \left[\theta_\eta^B - \sin a \, \theta_\xi^B \right]_{\xi = 0}$$

or

$$\left.\frac{\partial w^A}{\partial x}\right|_{x=a} = \frac{1}{\cos a}\left[\frac{\partial w^B}{\partial \xi} - \sin a \frac{\partial w^B}{\partial \eta}\right]_{\xi = 0} \tag{36}$$

The deflection of plate A at $x = a$ equals the in-plane displacement of plate B at $\xi = 0$ or

$$\left. w^A \right|_{x=a} = \left[u^B - \sin a \, v^B\right]_{\xi = 0} \tag{37}$$

The deflection of plate B at $\xi = 0$ equals the in-plane displacement of plate A at $x = a$ or

$$\left. w^B \right|_{\xi = 0} = \left. u^A \right|_{x = a} \tag{38}$$

The displacement of plate A at $x = a$ equals the displacement of plate B at $\xi = 0$ both displacements being parallel to the joint or

$$\left. v^A \right|_{x=a} = \left. v^B \cos a \right|_{\xi = 0} \tag{39}$$

Statical equilibrium of the joint between plates B and C will provide the following set of continuity equations to be satisfied by the deflections and displacement functions for plates B and C.

$$\Sigma F_{\overline{x}} = 0 = d\eta \left. V_{\xi}^{B} \right|_{\xi = \ell} - \left. N_{\overline{x}}^{C} \, dy \right|_{\overline{x} = a}$$

or

$$\frac{D}{\cos^3 a}\left[\frac{\partial^3 w^B}{\partial \xi^3} - 3 \sin a \frac{\partial^3 w^B}{\partial \xi^2 \partial \eta} + (2 + \sin^2 a - \mu \cos^2 a)\frac{\partial^3 w^B}{\partial \xi \partial \eta^2} + \sin a (\sin^2 a + \mu \cos^2 a - 2)\frac{\partial^3 w^B}{\partial \xi^3}\right]_{\xi = \ell}$$

$$+ \frac{E}{(1-\mu^2)}\left[\frac{\partial u^C}{\partial \overline{x}} + \mu \frac{\partial v^C}{\partial y}\right]_{\overline{x} = \ell} = 0 \tag{40}$$

$$\Sigma F_{\bar{y}} = 0 = -d\eta \quad N^B_{\xi\eta} + \sin a \cdot N^B_{\xi} \Big|_{\xi = \ell} - d\bar{y}\, N^C_{xy} \Big|_{\bar{x} = a}$$

or

$$\left[\frac{\partial u^B}{\partial \eta} + \frac{\partial v^B}{\partial \xi} - 2 \sin a \frac{\partial v^B}{\partial \eta}\right]_{\xi = \ell} - \left[\frac{\partial u^C}{\partial \bar{y}} + \frac{\partial v^C}{\partial \bar{x}}\right]_{\bar{x} = a} = 0 \qquad (41)$$

$$\Sigma F_{\bar{z}} = 0 = -d\eta \cos a\, N^B_{\xi} \Big|_{\xi = \ell} - V^C_{\bar{x}} \Big|_{\bar{x} = a}$$

or

$$\frac{E}{(1-\mu^2)\cos a}\left[\frac{\partial u^B}{\partial \xi} - \sin a \left(\frac{\partial u^B}{\partial \eta} + \frac{\partial v^B}{\partial \xi}\right) + (\sin^2 a + \mu \cos^2 a)\frac{\partial v^B}{\partial \eta}\right]_{\xi = \ell}$$

$$- D \left[\frac{\partial^3 w^C}{\partial \bar{x}^3} + (2 - \mu)\frac{\partial^3 w^C}{\partial \bar{x}\,\partial \bar{y}^2}\right]_{\bar{x} = a} = 0 \qquad (42)$$

$$\Sigma M_{\bar{y}} = 0 = -d\eta \cos a\, M^B_{\xi} \Big|_{\xi = \ell} - d\bar{y}\, M^C_{\bar{x}} \Big|_{\bar{x} = a}$$

or

$$\frac{-D}{\cos^2 a}\left[\frac{\partial^2 w^B}{\partial \xi^2} - 2 \sin a \frac{\partial^2 w^B}{\partial \xi\,\partial \eta} + (\sin^2 a + \mu \cos^2 a)\frac{\partial^2 w^B}{\partial \eta^2}\right]_{\xi = \ell}$$

$$-D\left[\frac{\partial^2 w^C}{\partial \bar{x}^2} + \mu \frac{\partial^2 w^C}{\partial \bar{y}^2}\right]_{\bar{x} = a} = 0 \qquad (43)$$

The equations which satisfy geometrical continuity at the joint between plates B and C, again referring to the directions shown in Figure 4, have the following form. Continuity of slope requires that

$$\frac{\partial w^C}{\partial \bar{x}}\Big|_{\bar{x} = a} = \frac{1}{\cos a}\left[\frac{\partial w^B}{\partial \xi} - \sin a \frac{\partial w^B}{\partial \eta}\right]_{\xi = \ell} \qquad (44)$$

and continuity of deflections and displacements requires that

$$w^C\Big|_{\bar{x}=a} = \text{[illegible]} \left[u^B - \sin a \, v^B \right]_{\xi=\ell} \tag{45}$$

$$w^B\Big|_{\xi=\ell} = u^C\Big|_{\bar{x}=a} \tag{46}$$

and

$$v^C\Big|_{\bar{x}=a} = \cos a \, v^B\Big|_{\xi=\ell} \tag{47}$$

III. Assumptions and Mechanical Considerations

All of the basic theory governing the work done on this problem employs the assumption of small deflections and displacements so that the equations are all linear.

The plates are assumed to come under the classification of a thin plate so that stresses acting on areas parallel to the lateral surfaces of the plates can be considered negligible. Further, the thickness is assumed to be small enough compared to the other dimensions of the plates so that the extensional effects will be secondary in magnitude to the flexural effects. With this assumption, the plates can be treated as inextensible during the part of the solution dealing with flexure.

Let us make some apriori judgements at this point regarding the mechanical behavior of the structure when the skewed plate is subjected to a transverse loading. The skewed plate will tend to develop a larger bending moment at its obtuse corners than at its acute corners due to its shape. The moments at the obtuse corners of the skewed plate will cause the moments at the joints in the rectangular plates to be largest near these corners. An assumed distribution of moments along the joints is shown in Figure 5a. From the standpoint of statics, the moments at the edges of the plates should affect to some extent the magnitudes of the flexural shear resultants along the edges. Since the moments are largest near the obtuse corners, the shear resultants along the edges should be distributed roughly as shown in Figure 5b. The flexural shear resultants act as in-plane edge loads on each adjacent plate. The flexural shear resultants from the skewed plate will transmit through plates A and C to their hinged edges. Hence the vertical reactions along the hinged supports should be largest under the obtuse corners of the skewed plate as shown in Figure 5c. The resultants of these vertical reactions would be the moments M_{Az} and $M_{C\bar{z}}$ and the forces R_{Ax} and $R_{C\bar{x}}$ as shown in Figure 5d.

The flexural shear resultants on the hinged edges of plates A and C should also be largest near the obtuse corners of the skewed plate as shown in Figure 5b. At the hinged edges of plates A and C, these should produce horizontal reactions R_{Az} and $R_{C\bar{z}}$ and moments M_{Ax} and $M_{C\bar{x}}$ as shown in Figure 5d.

The flexural shears at the joints from plates A and C, distributed as shown in Figure 5b, will produce a couple in the plane of the skewed plate. This couple would be resisted by in-plane shear forces acting along the joints in the rectangular plates as shown in Figure 5c. The effect of these in-plane shear forces at the hinges would be a reduction in the moments M_{Ax} and $M_{C\bar{x}}$.

If the skewed plate is subjected to a rigid body rotation in its own plane due to this couple, the hinges would have to translate. This is not consistant with the boundary conditions. A rotation of the skewed plate could occur if the distortion in the plane of the rectangular plates were taken into account. However, this would violate the assumption that the in-plane displacements do not affect the deflections.

This point is important if side sway is to be considered. If the skewed plate can not rotate, then the joints will only translate during side sway. This motion will occur in a direction perpendicular to the hinged supports. Consequently the actual amount of side sway that would occur from an unsymmetric loading could be computed from the condition that the in-plane forces acting on the edges of the skewed plate due to translation of the joints are in equilibrium with similar forces caused by an unsymmetric loading. If we let N_P represent the resultant of all of the in-plane edge forces in the direction of the side sway, due to an unsymmetric applied load, and let n_T represent the resultant of these forces due to a unit translation, then equilibrium of the skewed plate requires that

$$\Delta n_T + N_P = 0$$

or

$$\Delta = -\frac{N_P}{n_T} \qquad (48)$$

would be the actual amount of side sway.

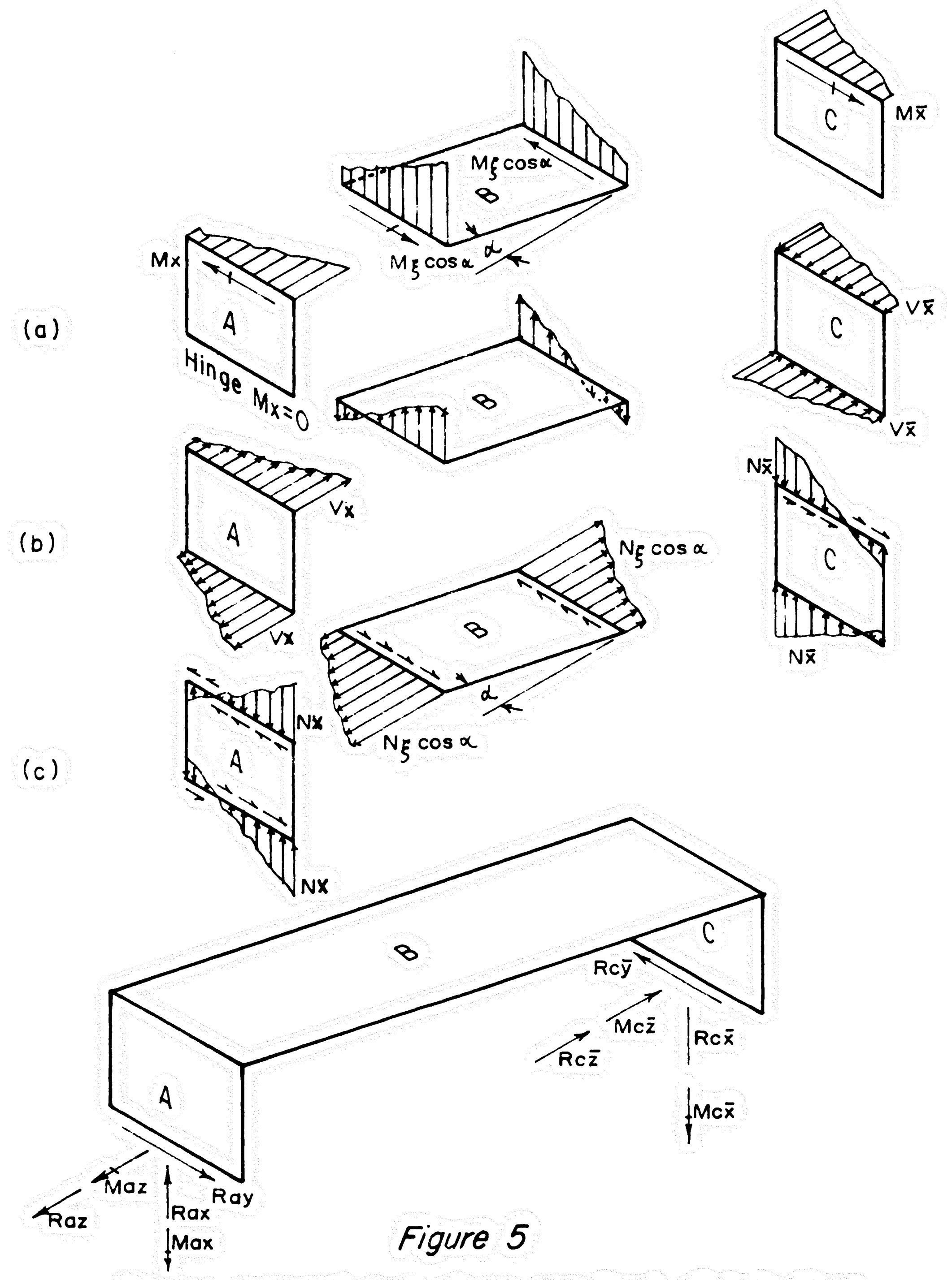


Figure 5

Stress, moment, and shear resultant distributions

IV. Formulation of the Exact Solution

Bottenhoffer,[14] Brown,[15] and Parsons[16] formulated the solution according to classical plate theory. They obtained solutions to equations (1) through (4) by a separation of variables technique. Some of the constants of integration appearing in the solution to these equations were evaluated by satisfying the boundary conditions, equations(11) through (31). The remaining constants were to be evaluated by satisfying the continuity conditions. The solution at this point contained the same number of integration constants as there were continuity conditions. In order to evaluate these remaining constants, a set of simultaneous equations independent of y formed from substitution of the solution into the continuity conditions had to be solved. Since each remaining integration constant was multiplied by a different function of y, it was necessary to express each continuity condition as a Fourier Series in y so that the functions of y would appear as a common factor in the simultaneous equations. The set of simultaneous equations thus obtained were independent of y but they were now theoretically infinite in number since each continuity condition was expressed as a Fourier Series. If these equations are solved, the exact solution to the problem will be obtained.

V. Variational Formulation of the Problem

It was thought that perhaps less manual work would be involved if minimization of the potential energy stored in the structure using the Rayleigh-Ritz procedure was programmed for a computer.

After examining the work of Bottenhoffer(14), Brown(15), and Parsons(16) it was obvious that no simple function could be found that would even partially satisfy the boundary and continuity conditions. The task of developing such a function was eased slightly by relaxing the continuity conditions prescribed by equations (37), (38), (45), and (46). Instead of these conditions, a condition of vanishing deflections at the joints was used. This did not violate the assumptions and it allowed the flexural and extensional parts of the problem to be solved independently.

The flexural part of the problem was attacked first. The usual procedure was followed. A deflection equation in series form containing a set of minimizing parameters was developed that satisfied the boundary conditions. The deflection function was to be substituted into the energy equations (7) and (8), and minimized with respect to the parameters. This would produce a set of simultaneous equations the solution of which would yield the value of the parameters. The sum of the deflection function series with the known values of the parameters in it would be the solution.

A scheme was devised by which a series deflection function could be developed that would automatically satisfy the boundary and continuity conditions. To perform such a task a deflection function was initially assumed in the following abbreviated form.

$$w = \sum\sum \Big\{(A_{1i} + \xi A_{2i}) \cos S_i \xi + (A_{3i} + \xi A_{4i}) \sin S_i \xi + (A_{5i} + \xi A_{6i}) \cosh S_i \xi + (A_{7i} + \xi A_{8i}) \sinh S_i \xi\Big\} \Big\{(C_{1j} + \eta C_{2j}) \cos P_j \eta + (C_{3j} + \eta C_{4j}) \sin P_j \eta + (C_{5j} + \eta C_{6j}) \cosh P_j \eta + (C_{7j} + \eta C_{8j}) \sinh P_j \eta\Big\} \tag{49}$$

When the indicated multiplication was performed there were 64 different AxC combinations for each term in the series. At this point these would be unknown constants that were to be determined by satisfying the boundary conditions. The arguments of the functions of ξ and η contained unknown parameters S_i and P_j. If the boundary conditions with the deflection function substituted into them were evaluated along the boundaries to which they apply, a set of homogeneous, simultaneous, transcendental equations would be generated. This set of equations would be independent of ξ and η and would contain the unknown constant AxC products and the unknown parameters S_i and P_j. There are 16 boundary and continuity equations, for instance, for the skewed plate, but 64 simultaneous equations would have been generated by these conditions.

An example of why this is true can be obtained from examination of equations (19) and (20), the free edge boundary conditions for the skewed plate. These two equations evaluated at $\eta = 0$ and $\eta = b$ must be independent of ξ. Since they involve the deflection function and its first, second, and third derivatives with respect to ξ, it is impossible to factor all of the functions of ξ out of these equations. Instead of four members of the set of simultaneous equations arising from equations (19) and (20), there would be 32 -- one for each different function of ξ. The other 32 members of the set of simultaneous equations would be generated from the continuity conditions for the skewed plate.

A deflection function similar to equation (49) was used for the two rectangular plates. The hinged edge conditions, equations (11), (12), (23), and (24), made it possible to simplify the deflection functions for the rectangular plates so that only 32 AxC products were needed for each rectangular plate.

For the entire structure then there were a total of 128 simultaneous equations with 128 unknown AxC products. The determinant of the coefficients of the unknown products was to be made to vanish by a proper selection of P_j and S_i which would insure that the equations were actually homogeneous. With these proper values of P_j and S_i substituted into the simultaneous equations, 127 of the unknown products were to be determined in terms of the 128th which would insure that all of the boundary and continuity conditions were satisfied. This procedure was to be repeated with new values of S_i and P_j until enough terms in the deflection function series could be obtained. The members of the set of the 128th constants corresponding to the various values of P_j and S_i were to be used as the minimizing parameters in the energy equations.

Since it was impractical to invert a matrix with a rank of 128 the number of times that would be required to carry out this procedure, something had to be changed so that there would be fewer constant AxC products and fewer equations. It was decided that the free edge boundary conditions would be relaxed to where only the integrals along the free edges of the moments and Kirchhoff shears would be required to vanish instead of the moments and shears themselves.

The validity of this change would depend on the way the free edge moments and shears as calculated from the final answers were distributed. If the moments and shears would change signs many times along the free edges, this last change would be a valid one.

By changing the problem in this way, the number of simultaneous equations was reduced to 28. A procedure for generating these equations and then computing their determinant was programmed for a computer. The program was so arranged that given a value of P_j, the magnitude of the determinant for many values of S_i would be computed until some of the roots of the determinant were found. For each root 27 of the constant products were determined in terms of the 28th. The 28th constant being reserved for the role of minimizing parameter in a programmed version of the Rayleigh-Ritz procedure.

Though the method seemed to hold promise and though the several programs necessary for final solution were completed and operational, the approach was abandoned because a sufficient number of roots of the determinant to insure convergence to the deflection function series could not be obtained.

The difficulty, which seems to be inherent in this method of finding roots to determinants of transcendental equations using a computer, arose from the sensitivity of the magnitude of the determinant to small changes in the trial values of the roots. For instance it is not uncommon for a change of one digit in the sixth place in the value of the trial root to cause the value of the determinant to change from a number of the order

10^{12} to 10^{-12}. For this particular set of equations this condition developed for values of trial roots greater than three. Two or three values of the roots, S_i, for each assumed value of P_j were obtained which were less than three, but these were deemed an insufficient quantity to insure convergence of the deflection function series.

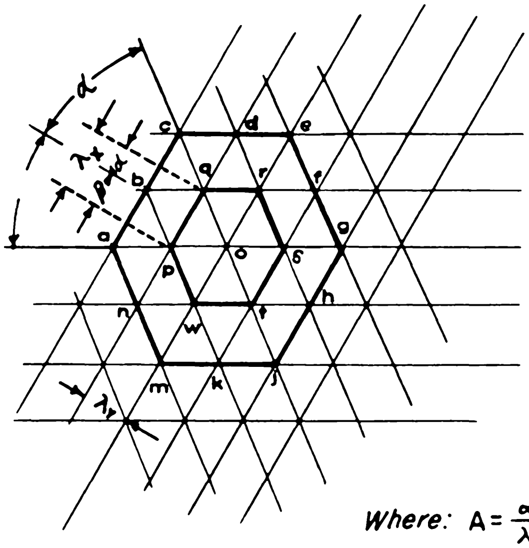
VI. Finite Difference Approach to the Folded Plate Analysis

The basic paper on the finite difference method applied to skewed plates was written by Jensen[17]. Solutions were presented in his paper for skewed plates which were simply supported on two edges and free along the other two. The ∇^4 operator used was developed by Jensen so that the finite difference net would fit the shape of the skewed plate for an arbitrary angle of skew. Modified operators were also developed for all of the various plate boundary conditions.

The deflections and moments due to a uniform load for a simply supported-free plate with a 45° angle of skew as obtained by Jensen are shown in the Table 1 along with a solution for the same plate as obtained by the method finally adopted for this thesis.

Jensen's skewed operator could not be used directly for the folded plate structure because the difference net points in the rectangular plates could not be made to match the points in the skewed plate along the joints. A modification of Jensen's operator was developed, and following the method described in his paper, boundary and continuity condition operators for the flexural part were derived for the folded plate structure. Jensen's skewed operator and the one adapted from it that would fit the skewed and the rectangular plates is shown in Figure 6. The derivation of the finite difference equations for this problem is much too long to be reasonably included in this paper. A solution to the flexural part of the problem was obtained by programming the equations for a computer and it will be published separately in the future.

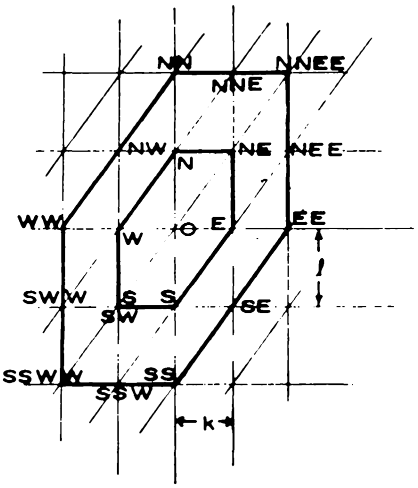
Since the boundary conditions for the extensional part of the problem



$$\frac{Q\lambda^4}{D} = 2W_o(2+A^2+B^2+4C+3C^2)$$
$$-2(W_p+W_s)(2A+2AC-BC)$$
$$-2(W_q+W_t)(2B+2BC-AC)$$
$$-2(W_r+W_w)(2C+2C^2-AB)$$
$$+(W_a+W_g)A^2+(W_c+W_j)B^2$$
$$+(W_e+W_m)C^2+2(W_b+W_h)AB$$
$$+(2)(W_d+W_r)BC+2(W_p+W_n)AC$$

Where: $A=\frac{\alpha}{\lambda_x}j$ $B=\frac{\beta}{\lambda_x}$ $C=\frac{\lambda_y^2}{\lambda_x^2}-\frac{\alpha\beta}{\lambda_x^2}$

(a) Finite difference ∇^4 operator for skewed plate as developed by Jensen(17) page 39.



$$\frac{Q}{D} = W_oA + W_NB + W_SC + W_ED$$
$$-W_NE + W_{SS}F + W_{EE}G + (W_{NE}+2W_{SE}+W_{SW})H$$
$$+W_{NNE}K + W_{SWW}L$$

Where:

$$A=\frac{5}{k^4}+\frac{8}{k^2l^2}+\frac{5}{l^4} \qquad B=-\frac{2}{l^4}-\frac{2}{k^4}-\frac{(1-\mu)}{k^2+l^2}\left(\frac{2}{l^2}-\frac{1}{k^2}\right)$$

$$C=-4\left(\frac{1}{l^4}+\frac{1}{k^2l^2}\right) \qquad D=-4\,\frac{1}{k^4}+\frac{1}{k^2l^2} \qquad H=\frac{1}{k^2l^2}$$

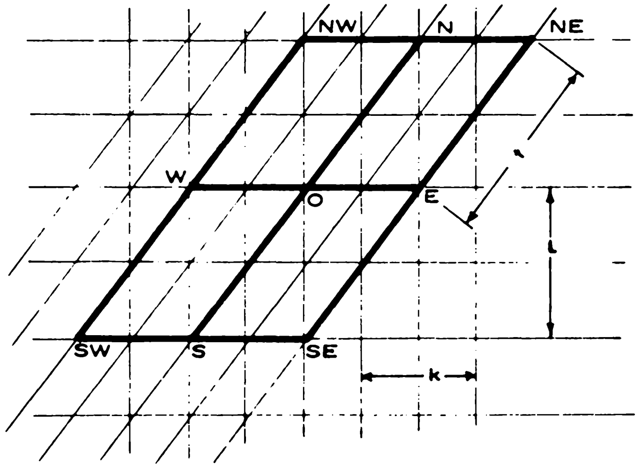
$$E=\frac{2}{k^4}+\frac{2}{k^2l^2}+\frac{(1-\mu)}{k^2+l^2}\left(\frac{2}{k^2}-\frac{1}{l^2}\right) \qquad F=\frac{1}{l^4} \qquad G=\frac{1}{k^4}$$

$$K=\frac{(1-\mu)}{l^2(k^2+l^2)} \qquad L=\frac{(1-\mu)}{k^2(l^2+k^2)}$$

(b) Rectangular-skewed finite difference ∇^4 operator

Figure 6

Flexural finite difference operators for a skewed plate



$$U_o \frac{4L}{t}\left(\frac{1}{k}+\frac{1}{L^2}\right) - (U_n+U_s)\left(\frac{2}{tL}\right) - (U_E+U_W)\left(\frac{2L}{tk^2}\right)$$

$$+(U_{NE}-U_{SE}-U_{NW}-U_{SW})\left(\frac{1}{4kL}\right)\left[1+\frac{3L^2}{t^2}+\frac{\mu k^2}{t^2}\right]$$

$$-V_o\left[\frac{4}{L^2}+\frac{2}{k^2}\left(1+\frac{L^2}{t^2}-\frac{\mu k^2}{t^2}\right)\right]+(V_N+V_S)\frac{2}{L^2}+(V_{NW}-V_{SW}-V_{NE}+V_{SE})\frac{1}{kt}$$

$$+(V_E+V_W)\frac{1}{k^2}\left(1+\frac{L^2}{t^2}-\frac{\mu k^2}{t^2}\right)=0$$

$$V_o\frac{4L}{t}\left[\frac{1}{k^2}+\frac{1}{L^2}\right]-(V_N+V_S)\left(\frac{2}{tL}\right)-(V_E+V_W)\left(\frac{2L}{tk^2}\right)$$

$$+(V_{NE}-V_{SE}-V_{NW}-V_{SW})\left(\frac{1}{4hL}\right)\left[1+\frac{3L^2}{t^2}+\frac{k^2}{t^2}\right]$$

$$-U_o\ \frac{4}{L}+\frac{2}{k}\ 1+\frac{L}{t}-\frac{k}{t}\ +(U\ +U\ )\frac{2}{L}+(U_{NW}-U_{SW}-U_{NE}+U_{SE})\frac{1}{kt}$$

$$+(U_E+U_W)\frac{1}{k^2}\left(1+\frac{L^2}{t^2}-\frac{\mu k^2}{t^2}\right)=0$$

Figure 7

Finite difference form of equation (5) and (6) were mixed, it was necessary to express the difference operators for this part of the problem in terms of the in-plane displacements. This involved writing equations (3) through (6) in finite difference form. The skewed and rectangular operators for in-plane displacements are shown in Figure 7.

The extensional boundary condition operators were not derived because a certain amount of difficulty arose in writing the finite difference form of the continuity conditions. It was found that there was no way of expressing the flexural shear directly from the finite difference equations available at net points falling on the joints. Hence equations (32), (34), (40), and (42) could not be satisfied. Since these equations are actually loading functions for the extensional part of the problem, it was feared that a further approximation of the shears along the joints by some curve fitting procedure would produce erroneous extensional results. In view of this the finite difference method was abandoned as a means of obtaining the complete flexural-extensional solution.

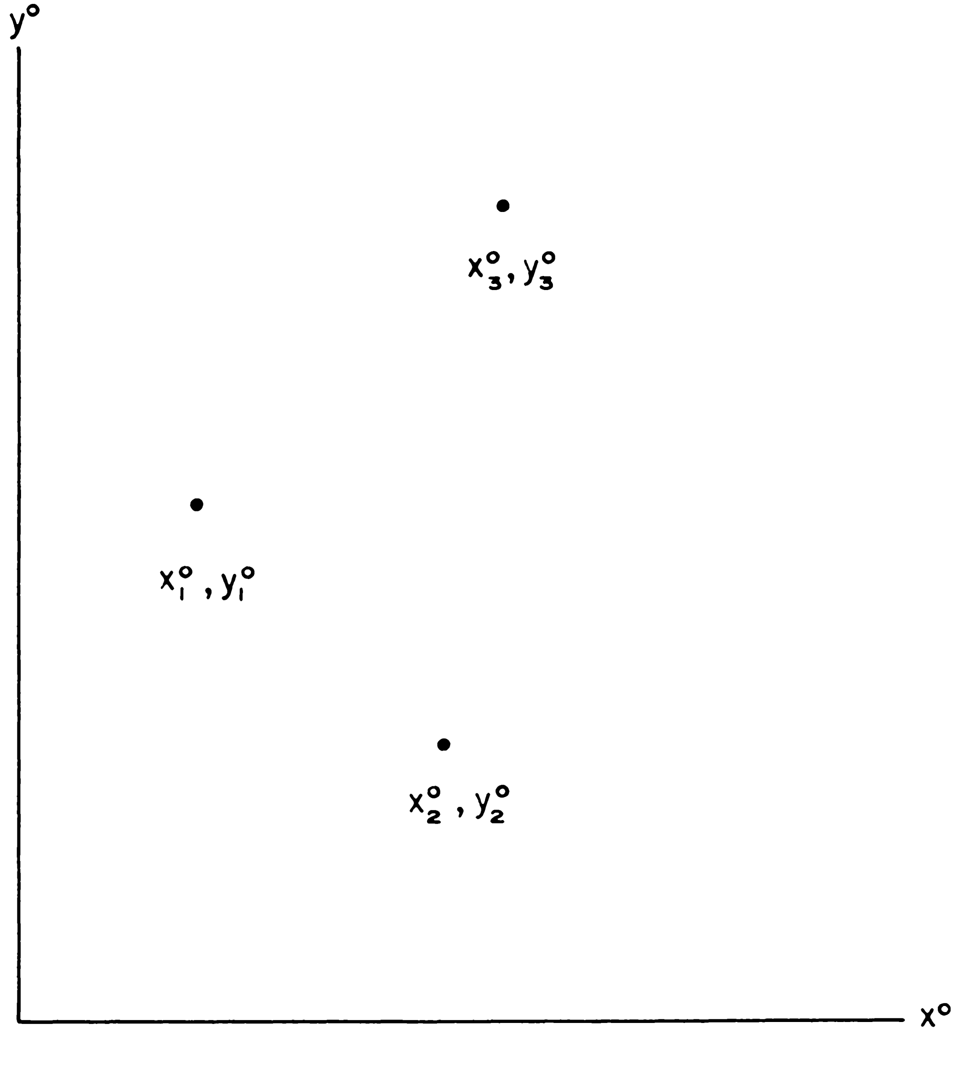


Figure 8

Location of physical points in an infinite plane

VII. General Description of the Solution of Boundary Value Problems by the Method of Superimposed Potentials

The general method will apply to any problem where the solution to its governing equations can be expressed as the potential due to a point source or point doublet in an infinite medium. The source is always considered to be applied at the origin of a set of coordinates in the infinite medium. Since the medium is infinite, the origin can be moved about without changing the form of the equations representing the effect of the potential source. Let a physical point in the infinite medium be defined as one that remains in the same place regardless of where the origin of the coordinates for the point source is placed. Using this definition the effects at the physical point of several point sources located at different places in the infinite medium can be superimposed provided the effect of one source doesn't affect the nature of the other.

Suppose for instance that in an infinite plane three physical points as shown in Figure 8 are considered. These are located with respect to a set of reference axes, x^o, y^o. Let an effect due to a point source be $P = P(x, y)$ where x and y are measured from the point of application of the point source. Then the effect at physical point 3 of point sources at physical points 1 and 2 is

$$P = P_1(x_3^o - x_1^o, y_3^o - y_1^o) + P_2(x_3^o - x_2^o, y_3^o - y_2^o) \qquad (50)$$

Consider some body of a given shape which shall be called the actual body. The body is loaded in some way with forces or heat sources or some other agency of this nature which shall be called applied loads.

Over the surface of this actual body certain boundary conditions must be satisfied. Now visualize the shape of the actual body enclosing some region of an infinite medium. Call the boundary of the region the physical boundary and it shall be defined in the infinite medium by the physical coordinates (x^o, y^o). The applied loads are applied to the physical boundary in the same way that they were applied to the actual body. Their effects will be represented, however, as the potential due to a point source at the origin of an infinite medium. The effect of the applied loads at some physical point on the physical boundary can be determined by a superposition technique similar to equation (50).

Suppose there are M boundary conditions that must be satisfied at each point on the surface of the actual body. These conditions must then also be satisfied at all the points in the physical boundary. Assume that the effects on the physical boundary of the applied loads do not satisfy these conditions. Then the boundary value problem would not be solved and there would only exist a system which would provide the effects of a group of point sources located at various points in an infinite medium. If other point sources similar in nature to the sources representing the applied loads were placed at prescribed points outside of the physical boundary, their intensities could be adjusted in such a way that their effects added to the effects of the applied loads would satisfy the boundary conditions. This would result in a region of the infinite medium enclosed by the physical boundary which would exhibit the same conditions on its boundary as those prescribed for the surface of the actual body. Hence the superimposed effects inside the physical boundary of all the point sources in the infinite medium would be the same as the required solution for the actual body.

Let the set of point sources required to satisfy the boundary conditions be called balancing loads.

In general if there were M boundary conditions associated with each point in the physical boundary, M balancing loads would be required for each boundary point. Obviously an exact solution obtained in this way would require an infinite number of point sources. An approximate solution can be obtained by reducing the number of point sources along the boundary. This could be accomplished by dividing the physical boundary into a set of intervals and requiring that the boundary conditions be satisfied only at the midpoints of the intervals. If enough intervals are used so that the boundary conditions are fairly well approximated but not enough to make the numerical work impractical, the method could provide an acceptable approximate solution.

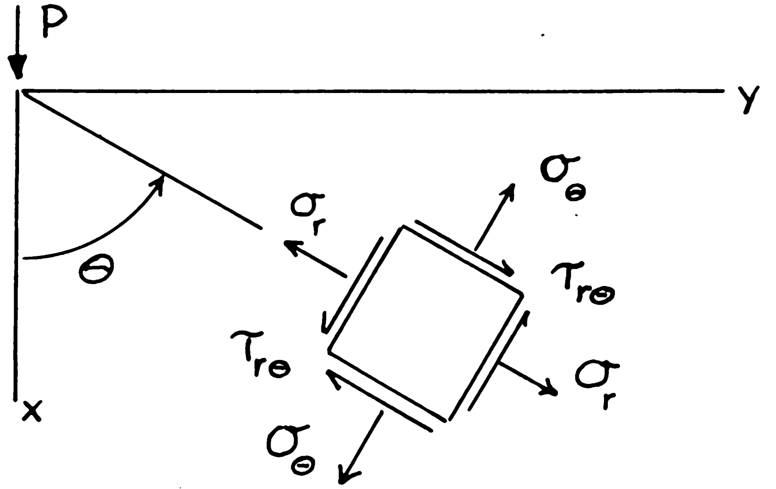


Figure 9 Coordinate System for Stresses and Displacements Due to a Concentrated Force Acting in the plane of an Infinite Plate

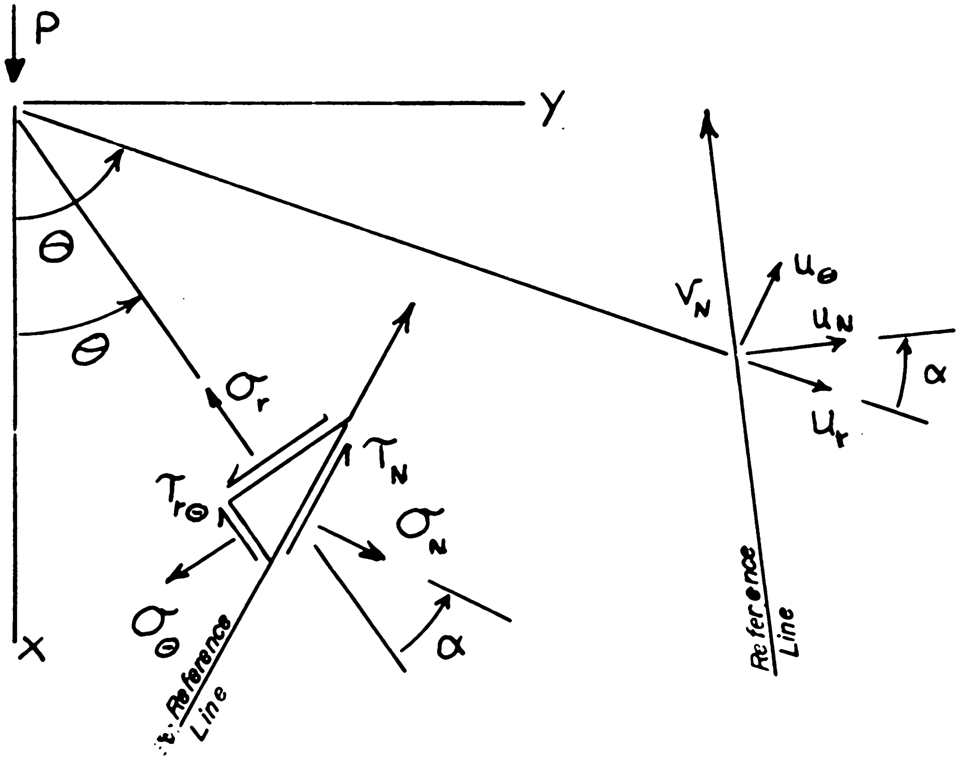


Figure 10 Stress and Displacement Transformations Across a Reference Line

VIII. Solution of the Extensional Problem by the Method of Superimposed Potentials

These ideas applied to a plane stress problem were brought to the author's attention by Dr. Ricardo Chicurel Professor of Engineering Mechanics at Virginia Polytechnic Institute in a private communication[24]. The point source in this case would be a concentrated load applied in the plane of an infinite plate. With the axes shown in Figure 9, the effects from the concentrated force are stresses of the form:

$$\sigma_r = -\frac{(3+\mu)\,P\cos\theta}{4\pi r} \tag{51}$$

$$\sigma_\theta = \frac{(1-\mu)\,P\cos\theta}{4\pi r} \tag{52}$$

$$\tau_{r\theta} = \frac{(1-\mu)\,P\sin\theta}{4\pi r} \tag{53}$$

where P is a force per unit thickness of plate and μ is Poisson's ratio. These equations are taken from equations 76 of Timoshenko and Goodier's Theory of Elasticity[18].

If the boundary conditions involving displacements are to be considered, expressions for in-plane displacements are needed. The following displacement equations for a concentrated force in an infinite plate are derived in Appendix I.

$$u_r = \frac{P\cos\theta}{4\pi E}\,(\mu^2 - 2\mu - 3)\,\mathrm{Log}\, r \tag{54}$$

$$u_\theta = \frac{P\sin\theta}{4\pi E}\left[(\mu^2 + 2\mu + 1) - (\mu^2 - 2\mu - 3)\log r\right] \tag{55}$$

Since stresses from a number of concentrated forces must be superimposed at points around the physical boundary, they must be transformed so that they will all act normal and parallel to the boundary. According to Figure 10 the transformation equations will have the form:

$$\sigma_N = \sigma_r \cos^2 a + \sigma_\theta \sin^2 a + \tau_{r\theta} \sin 2a \tag{56}$$

$$\tau_N = \frac{(\sigma_\theta - \sigma_r)}{2} \sin 2a + \tau_{r\theta} \cos 2a \tag{57}$$

$$u_N = u_r \cos a + u_\theta \sin a \tag{58}$$

$$v_N = u_\theta \cos a - u_r \sin a \tag{59}$$

The substitution of equations (51) through (55) into these equations is shown in detail in Appendix I. The final forms are:

$$\sigma_N = \frac{P}{4\pi r}\left\{ \left[(1-\mu)\sin^2 a - (3+\mu)\cos^2 a\right] \sin\theta + (1-\mu)\sin 2a \sin\theta \right\} \tag{60}$$

$$\tau_N = \frac{P}{4\pi r}\left\{ 2\cos\theta \sin 2a + (1-\mu)\cos 2a \sin\theta \right\} \tag{61}$$

$$u_N = \frac{P}{4\pi E}\left\{ (\mu^2 - 2\mu - 3)\log r \cos(\theta + a) + (\mu^2 + 2\mu + 1)\sin\theta \sin a \right\} \tag{62}$$

$$v_N = \frac{P}{4\pi E}\left\{ (\mu^2 - 2\mu - 3)\log r \sin(\theta + a) - (\mu^2 + 2\mu + 1)\sin\theta \cos a \right\} \tag{63}$$

Note that a is measured counterclockwise from the direction of the position vector, r, to the direction of the outward normal to the boundary.

Figure 11 illustrates how r, θ, and a are measured when these

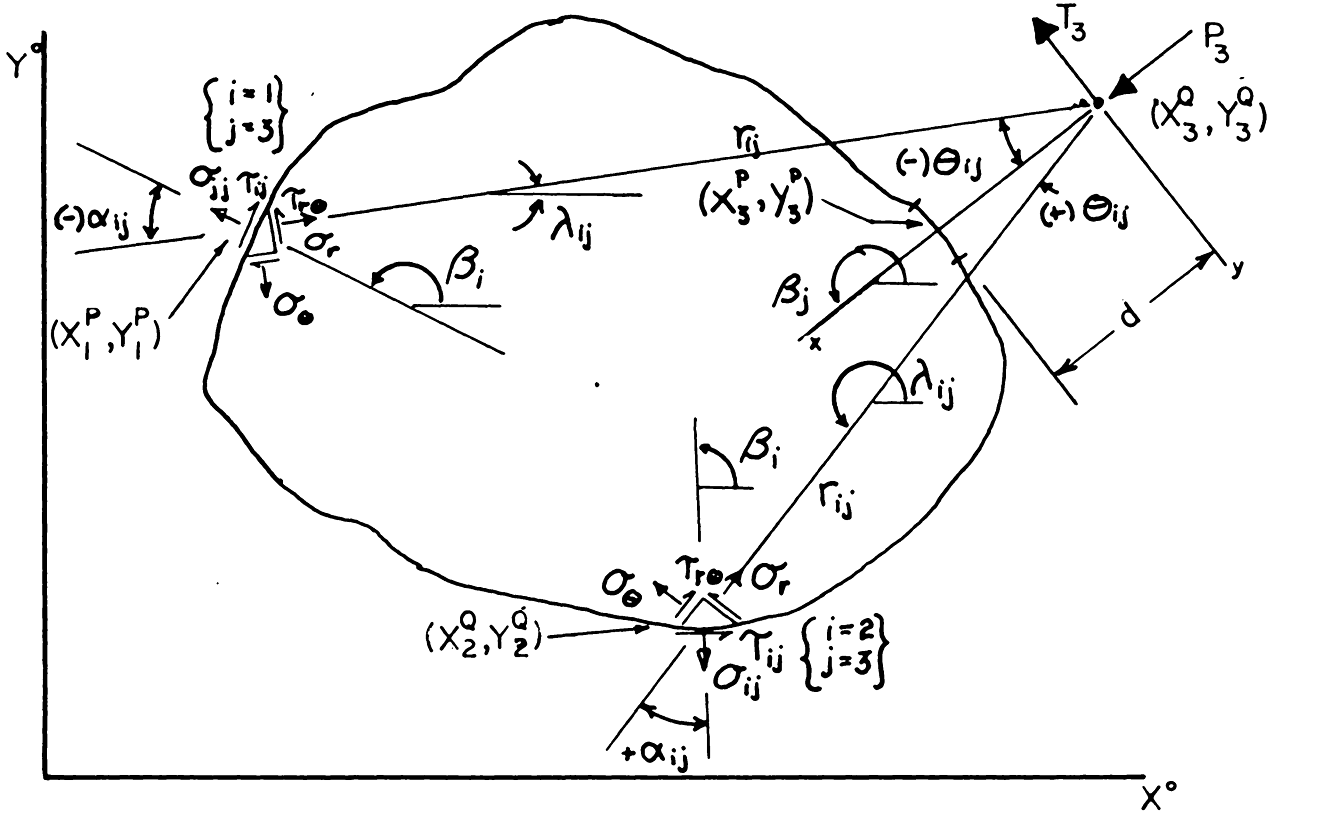


Figure 11 Stresses at Two Points on a Boundary Due to Balancing Loads Near a Third Point on the Boundary

equations are applied to a particular boundary. Applied in this way, the x axis of Figure 10 becomes the inward normal to a point on the physical boundary. This point through which the x axis passes is one of the points where the boundary conditions are to be satisfied. The position vector, r, extends from the point where P is applied to any point on the boundary where the boundary conditions are to be satisfied including the one through which the x axis passes.

Since there are in general two boundary conditions associated with each point, two balancing loads must be applied outside of the physical boundary near the points where the boundary conditions are to be satisfied. This second force, T, is applied at the same position as P and is considered positive when acting as shown in Figure 11. The same equations for stress and displacement developed for P can be used for T if $\frac{\pi}{2}$ is added to θ where ever it appears.

Suppose that it is decided that the boundary conditions will be satisfied at only N points around the boundary. The physical boundary will then be divided into N intervals. The intervals will be numbered consecutively in a counterclockwise direction. The end points of each interval can be defined by specifying their physical coordinates, (x_i^o, y_i^o) and (x_{i+1}^o, y_{i+1}^o). The midpoints of each interval where the boundary conditions are to be satisfied are defined by

$$x_i^P = \frac{x_{i+1}^o - x_i^o}{2} \qquad (64)$$

$$y_i^P = \frac{y_{i+1}^o - y_i^o}{2} \qquad (65)$$

Let the perpendicular distance from the midpoint of an interval to the position of its balancing load be denoted by d, and the azimuth of the inward normal to the boundary through the midpoint of the interval be defined by

$$\beta_i = \mathrm{Tan}^{-1}\left(\frac{x_i^o - x_{i+1}^o}{y_{i+1}^o - y_i^o}\right) \tag{66}$$

Then the positions of the points of application of the balancing loads are defined by

$$x_i^Q = -d_i \cos\beta_i + x_i^P \tag{67}$$

$$y_i^Q = -d_i \sin\beta_i + y_i^P \tag{68}$$

The length of the position vector, r, from the jth balancing load to the ith point on the boundary is defined by

$$r_{ij} = \left[(x_i^P - x_j^Q)^2 + (y_i^P - y_j^Q)^2\right]^{1/2} \tag{69}$$

and its azimuth is defined by

$$\lambda_{ij} = \mathrm{Tan}^{-1}\left(\frac{y_j^Q - y_i^P}{x_j^Q - x_i^P}\right) \tag{70}$$

With the definitions of the inward normal direction, β_i and the direction of the position vector, r_{ij} from a balancing load at j to an interval midpoint at i, we can define α_{ij} and θ_{ij} as follows:

$$\alpha_{ij} = \beta_i - \lambda_{ij} + \pi \tag{71}$$

$$\theta_{ij} = \lambda_{ij} - \beta_j \tag{72}$$

Let us now define a set of influence coefficients based upon equations (60) through (63) with P and T replaced by one pound per inch forces. The superscripts indicate whether the stress or displacement is caused by a one pound P or a one pound T and the subscripts are meant to read "the stress or displacement at some point identified with subscript i due to a load applied at a point identified with subscript j".

$$\sigma_{ij}^{P} = \frac{1}{4\pi r_{ij}}\left\{\left[(1-\mu)\sin^2\alpha_{ij} - (3+\mu)\cos^2\alpha_{ij}\right]\sin\theta_{ij} + (1-\mu)\sin 2\alpha_{ij}\cos\theta_{ij}\right\} \tag{73}$$

$$\sigma_{ij}^{T} = \frac{1}{4\pi r_{ij}}\left\{\left[(1-\mu)\sin^2\alpha_{ij} - (3+\mu)\cos^2\alpha_{ij}\right]\sin(\theta_{ij}+\frac{\pi}{2}) + (1-\mu)\sin 2\alpha_{ij}\cos(\theta_{ij}+\frac{\pi}{2})\right\} \tag{74}$$

$$\tau_{ij}^{P} = \frac{1}{4\pi r_{ij}}\left\{2\cos\theta_{ij}\sin 2\alpha_{ij} + (1-\mu)\cos 2\alpha_{ij}\sin\theta_{ij}\right\} \tag{75}$$

$$\tau_{ij}^{T} = \frac{1}{4\pi r_{ij}}\left\{2\cos(\theta_{ij}+\frac{\pi}{2})\sin 2\alpha_{ij} + (1-\mu)\cos 2\alpha_{ij}\sin(\theta_{ij}+\frac{\pi}{2})\right\} \tag{76}$$

$$u_{ij}^{P} = \frac{1}{4\pi E}\left\{(\mu^2-2\mu-3)\log r_{ij}\cos(\theta_{ij}+\alpha_{ij}) + (\mu^2+2\mu+1)\sin\theta_{ij}\sin\alpha_{ij}\right\} \tag{77}$$

$$u_{ij}^{T} = \frac{1}{4\pi E}\left\{(\mu^2-2\mu-3)\log r_{ij}\cos(\theta_{ij}+\alpha_{ij}+\frac{\pi}{2}) + (\mu^2+2\mu+1)\sin(\theta_{ij}+\frac{\pi}{2})\sin\alpha_{ij}\right\} \tag{78}$$

$$v_{ij}^{P} = \frac{-1}{4\pi E}\left\{(\mu^2-2\mu-3)\log r_{ij}\sin(\theta_{ij}+\alpha_{ij})\right.$$

$$-(\mu^2 + 2\mu + 1) \sin\theta_{ij} \cos\alpha_{ij} \Big\} \quad (79)$$

$$v_{ij}^T = \frac{-1}{4\pi E} \Big\{ (\mu^2 - 2\mu - 3) \log r_{ij} \sin(\theta_{ij} + \alpha_{ij} + \frac{\pi}{2})$$

$$-(\mu^2 + 2\mu + 1) \sin(\theta_{ij} + \frac{\pi}{2}) \cos\alpha_{ij} \Big\} \quad (80)$$

Each of the quantities are associated with a line, shown in Figure 10, passing through the point at which they are computed. Call this line the reference line. The reference line is defined as a vector by prescribing its intial and terminal x^o and y^o coordinates. The "outward normal" to the reference line means a direction obtained by rotating the vector through 90^o clockwise. The quantities σ_{ij} and u_{ij} are positive when they are directed along the outward normal to the reference line on which they are computed, and the quantities τ_{ij} and v_{ij} are positive when they have the same direction as the reference line on which they are computed.

Associated with each interval there is a P and T. If the values of all of the P's and T's are known, the stresses and displacements can be computed at some point (x_i^o, y_i^o) by evaluating one of the following summations.

$$\sigma_i = P_j \sigma_{ij}^P + T_j \sigma_{ij}^T + P_m^L \sigma_{ij}^P + T_m^L \sigma_{ij}^T \quad (81)$$

$$\tau_i = P_j \tau_{ij}^P + T_j \tau_{ij}^T + P_m^L \tau_{ij}^P + T_m^L \tau_{ij}^T \quad (82)$$

$$u_i = P_j u_{ij}^P + T_j u_{ij}^T + P_m^L u_{ij}^P + T_m^L u_{ij}^T \quad (83)$$

$$v_i = P_j v_{ij}^P + T_j v_{ij}^T + P_m^L v_{ij}^P + T_m^L v_{ij}^T \quad (84)$$

where P_m^L and T_m^L are the applied loads.

Initially the P_j and T_j are unknowns, but in this case either σ_i and τ_i or u_i and v_i are prescribed at the midpoint of each interval and equations (81) through (84) can become a set of 2N simultaneous equations with N unknown P's and N unknown T's. The left hand sides will be zero if the stresses and displacements are supposed to be zero at the ith midpoint. If nonvanishing stresses or displacements are prescribed at the ith midpoint, the left hand sides will become any two of the following:

$$\sigma_i = P_i^L \sigma_{ii}^P + T_i^L \sigma_{ii}^T \tag{85}$$

$$\tau_i = P_i^L \tau_{ii}^P + T_i^L \tau_{ii}^T \tag{86}$$

$$u_i = P_i^L u_{ii}^P + T_i^L u_{ii}^T \tag{87}$$

$$v_i = P_i^L v_{ii}^P + T_i^L v_{ii}^T \tag{88}$$

where T_i^L and P_i^L are the applied loads acting at the ith midpoint of the physical boundary.

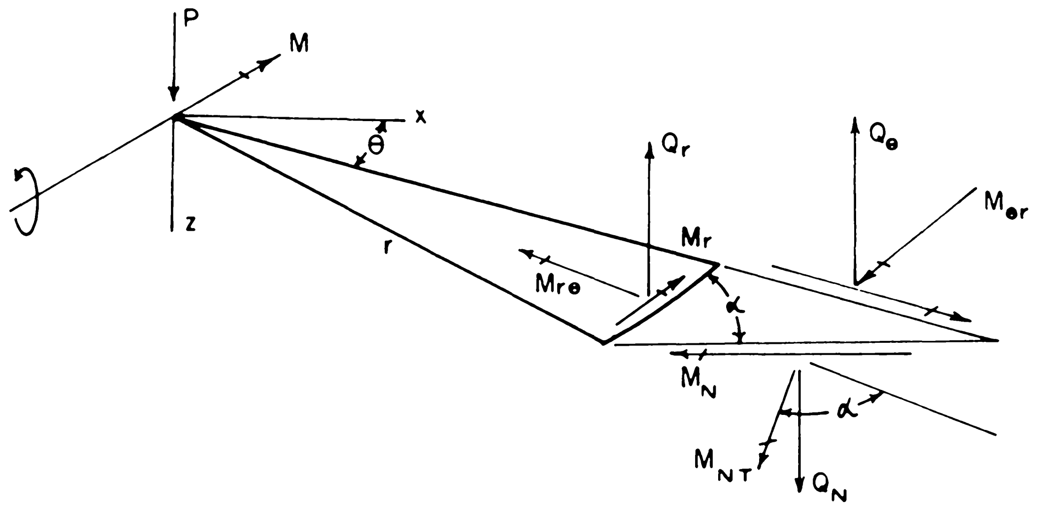


Figure 12

Moment and shear transformations across a reference line

IX. Solution of the Flexural Problem by the Method of Superimposed Potentials

The method of superimposing potentials can also be applied to flexural plate problems as suggested by Doctor Chicurel (24) if the point source is changed to that of a concentrated force applied perpendicular to an infinite plate. The potential due to a doublet is also useful in flexural problems when boundary conditions involve slopes and moments. Potentials such as these are given by equations 206 and 207 in Timoshenko and Winowsky-Krieger's Theory of Plates and Shells[19]. The potential effect for the concentrated force, P, is the deflection of an infinite plate due to a concentrated force located at its origin. It has the form

$$w^P = \frac{Pr^2 \log(r/a)}{8\pi D} \tag{89}$$

where D is the flexural rigidity and a is an arbitrary constant.

The potential effect for the doublet is the deflection due to a concentrated moment, M, acting along the y axis in an infinite plate as shown in Figure 12. It has the form

$$w^M = \frac{M\, r \log(r/a)}{4\pi D} \tag{90}$$

The application of these functions to the solution of flexural boundary value problems was developed in detail by Eskridge[20]. In this paper the equations necessary to apply these functions to the flexural problem were derived. These derivations are also summarized in this thesis in Appendix II. The results of derivations such as these are in the form of flexural deflections, slopes, moments, and shears associated with a boundary. The deflections for concentrated loads and moments are given by equations (89) and (90) and the deflection due to a uniform load, applied to the entire infinite plate as shown in Appendix II is

$$w^Q = \frac{Qr^4}{64D} \tag{91}$$

The functions used to satisfy boundary conditions involving slopes normal to a boundary have the following form

$$\frac{\partial w^P}{\partial N} = \frac{P}{8\pi D}\left[2r\log(r/a) + r\right]\cos a \tag{92}$$

$$\frac{\partial w^M}{\partial N} = \frac{M}{4\pi D}\left[\log(r/a)\cos(\theta + a) + \cos\theta\cos a\right] \tag{93}$$

$$\frac{\partial w^Q}{\partial N} = \frac{Qr^3}{16D}\cos a \tag{94}$$

The functions developed for satisfying moment and kirchhoff Shear conditions normal to a boundary are

$$M_N^P = \frac{-P}{8\pi}\left[2(1+\mu)\log(r/a) + (3+\mu)\cos^2 a + (1+3\mu)\sin^2 a\right] \tag{95}$$

$$M_N^M = \frac{-M}{4\pi r}\left[(1+\mu)\cos\theta - (1-\mu)\sin\theta\sin 2a\right] \tag{96}$$

$$M_N^Q = \frac{-Qr^2}{16}\left[(3+\mu)\cos^2 a + (1+3\mu)\sin^2 a\right] \tag{97}$$

$$V_N^P = \frac{-P\cos a}{2\pi r} \tag{98}$$

$$V_N^M = \frac{M}{4\pi r^2}\left[2 + (1-\mu)\cos 2a\right]\cos(\theta - a) \tag{99}$$

$$V_N^Q = \frac{-Qr}{8}\left[4\cos a + (1-\mu)\sin a\sin 2a\right] \tag{100}$$

Figure 12 illustrates how r, θ, and α are measured when these equations are applied to a particular boundary. The x axis of Figure 12 becomes the inward normal to a point on the physical boundary as described in Section VIII.

There are in general two boundary conditions associated with each interval into which the boundary is divided. These will involve a combination of either moment and deflection, slope and deflection, or moment and Kirchhoff shear. These conditions can be satisfied by placing balancing concentrated loads and concentrated moments outside of the physical boundary in exactly the same way the balancing loads P and T were applied in the extensional solution.

Referring to the geometrical quantities defined by equations (64) through (72), let us define a set of influence coefficients based upon equations (89) through (100) with P replaced by a one pound force and M replaced by a one inch pound moment. The influence coefficient for deflections, slopes, moments, and Kirchhoff shears are denoted respectively by the base letters w, S, M, and V. The superscripts P, M, and Q indicate the type of load with which the coefficient is associated, and the subscripts are meant to read "the coefficient evaluated at some point identified by subscript i due to a load applied at a point identified by subscript j". The only exception to this subscript rule arises when the coefficients for the one pound per square inchuniform load functions are used. For this case the subscript c will be used instead of j to indicate that the origin for the uniform load functions is fixed at some physical point at the center of the area enclosed by the physical boundary. The influence coefficients are:

$$w^P_{ij} = \frac{r^2_{ij}\log(r_{ij}/a)}{8\pi D} \tag{101}$$

$$w^M_{ij} = \frac{r_{ij}\log(r_{ij}/a)\cos\theta_{ij}}{4\pi D} \tag{102}$$

$$w^Q_{ic} = \frac{r^4_{ic}}{64D} \tag{103}$$

$$S^P_{ij} = \frac{1}{8\pi D}\left[2r_{ij}\log(r_{ij}/a) + r_{ij}\right]\cos a_{ij} \tag{104}$$

$$S^M_{ij} = \frac{1}{4\pi D}\left[\log(r_{ij}/a)\cos(\theta_{ij} + a_{ij}) + \cos\theta_{ij}\cos a_{ij}\right] \tag{105}$$

$$S^Q_{ic} = \frac{r^3_{ic}\cos a_{ic}}{16\,D} \tag{106}$$

$$M^P_{ij} = \frac{-1}{8\pi}\Big[2(1+\mu)\log(r_{ij}/a) + (3+\mu)\cos^2 a_{ij} + (1+3\mu)\sin^2 a_{ij}\Big] \tag{107}$$

$$M^M_{ij} = \frac{-1}{4\pi r_{ij}}\left[(1+\mu)\cos\theta_{ij} - (1-\mu)\sin\theta_{ij}\sin 2a_{ij}\right] \tag{108}$$

$$M^Q_{ic} = \frac{-r^2_{ic}}{16}\left[(3+\mu)\cos^2 a_{ic} + (1+3\mu)\sin^2 a_{ic}\right] \tag{109}$$

$$V^P_{ij} = \frac{-\cos a_{ij}}{2\pi r_{ij}} \tag{110}$$

$$V_{ij}^{M} = \frac{1}{4\pi r^2} \left[2 + (1 + \mu) \cos 2a_{ij} \right] \cos(\theta_{ij} - a_{ij}) \tag{111}$$

$$V_{ic}^{Q} = \frac{-r_{ic}}{8} \left[4 \cos a_{ic} + (1 - \mu) \sin a_{ic} \sin 2a_{ic} \right] \tag{112}$$

Each of these quantities except the deflections are associated with a reference line as defined in Section VIII. The shears act in a cross section of the plate containing the reference line, the moments act along this line, add the slopes are normal to it. A positive deflection means the plate has moved in the direction of the positive z axis as defined in Figure 12. A positive slope means that a vector representing the rotation of a cross section containing the reference line is directed in sense opposite to that of the reference line. A positive moment means that the plate is bent in such a way that the flexural stress is tensile on the side of the plate from which the positive z axis extends. A positive Kirchhoff shear will act in the direction of the positive z axis.

If values of all of the balancing forces and moments are known, the stresses and displacements can be computed with respect to a reference line through some point (x_i^o, y_i^o) by evaluating one of the following summations.

$$w_i = P_j w_{ij}^P + M_j w_{ij}^M + P_m^L w_{im}^P + M_m^L w_{im}^M + Q w_{ic}^Q \tag{1B3}$$

$$S_i = P_j S_{ij}^P + M_j S_{ij}^M + P_m^L S_{im}^P + M_m^L S_{im}^M + Q S_{ic}^Q \tag{114}$$

$$\overline{M}_i = P_j M_{ij}^P + M_j M_{ij}^m + P_m^L M_{im}^P + M_m^L M_{im}^M + Q M_{ic}^Q \tag{115}$$

$$V_i = P_j V_{ij}^P + M_j V_{ij}^M + P_m^L V_{im}^P + M_m^L V_{im}^M + Q V_{ic}^Q \qquad (116)$$

where P_m^L and M_m^L are applied loads as defined in Section VII.

Initially the P_j and M_j are unknowns, but either w_i and S_i, w_i and $\overline{M}_i$, or $\overline{M}_i$ and V_i will be prescribed at the midpoint of each interval. In this case equations (113) through (116) can become a set of 2N equations with N unknown P's and N unknown M's. The left hand sides will be the values of the deflections, slopes, moments, or shears that are prescribed by the boundary conditions at points on the actual body corresponding to the ith midpoints of intervals on the physical boundary.

X. Method of Calculating Results

Once the appropriate set of balancing loads are obtained, equations (81) through (84) and (113) through (116) can be employed to calculate the results at any point x_i^o, y_i^o within or on the physical boundary. The usual presentation of the results of extensional and flexural analyses in the form of a distribution of deflections, stresses, and so forth along some line in the subject of the analysis was used here. We shall call this line along which the results would be distributed the plot line. It is defined as a vector by prescribing its physical end point coordinates. Its initial point is prescribed by x_I^o, y_I^o and its terminal coordinates by x_T^o, y_T^o. Its length is defined by

$$L = \left[(x_T^o - x_I^o)^2 + (y_T^o - y_I^o)^2 \right]^{1/2} \tag{117}$$

and its azimuth is defined by

$$\beta_A^F = \mathrm{Tan}^{-1} \left(\frac{y_T^o - y_I^o}{x_T^o - x_I^o} \right) \tag{118}$$

for the flexural calculations and by

$$\beta_A^E = \mathrm{Tan}^{-1} \left(\frac{y_I^o - y_T^o}{x_I^o - x_T^o} \right) \tag{119}$$

for the extensional calculations.

The results are to be computed at n points along the plot line. The coordinates of these points are defined in the following manner.

$$x_i^P = x_I^o + (L/n)\,(i - 1) \cos \beta_A^F \tag{120}$$

$$y_i^P = y_I^o + (L/n)(i-1)\sin\beta_A^F \tag{121}$$

where $i = 1, 2, \text{---} n+1$ for the flexural part and

$$x_i^P = x_I^o - (L/n)(i-1)\cos\beta_A^E \tag{122}$$

$$y_i^P = y_I^o - (L/n)(i-1)\sin\beta_A^E \tag{123}$$

where $i = 1, 2, \text{---} n+1$ for the extensional calculations.

With these new definitions of x_i^P and y_i^P, r_{ij} and its azimuth, λ_{ij}, are still defined by equations (69) and (70) with the exception that now the subscript i refers to a point on the plot line. The angle θ_{ij} is still defined by equation (72) with the new definition of i, and α_{ij} is redefined in the following way.

$$\alpha_{ij} = \beta_A^F + \frac{\pi}{2}(F) - \lambda_{ij} + \pi \tag{124}$$

for the flexural computation of results and

$$\alpha_{ij} = \beta_A^E + \frac{\pi}{2}(E) - \lambda_{ij} + \pi \tag{125}$$

for the extensional computations.

If $F = 0$, moments, slopes, and shears associated with a plane perpendicular to the plot line will be computed. If $F = 1$, the quantities will be associated with a plane containing the plot line. The direction of the reference line as described in Section VIII will be the same as that of the plot line if $F = 1$ and will be rotated 90^o counterclockwise from the direction of the plot line if $F = 0$.

If $E = 0$, the terminal end of the reference line will be rotated through 90^o counterclockwise from the terminal end of the plot line and if E = 1, the reference line will be directed from the terminal to the initial points of the plot line.

Though these dual definitions introduce seemingly unnecessary complexity, they were introduced so that the signs of the final results of the extensional part of the problem would appear consistent with the directions of the applied loads. This correction concerned only the signs of the displacements. If the definitions carrying the superscript F were used to calculate the in-plane displacements, say, for an axially loaded strut, the points in the body would move toward the base but the signs of the displacements would be positive. One usually conceives of such a displacement as negative. Therefore the change was considered worth the trouble.

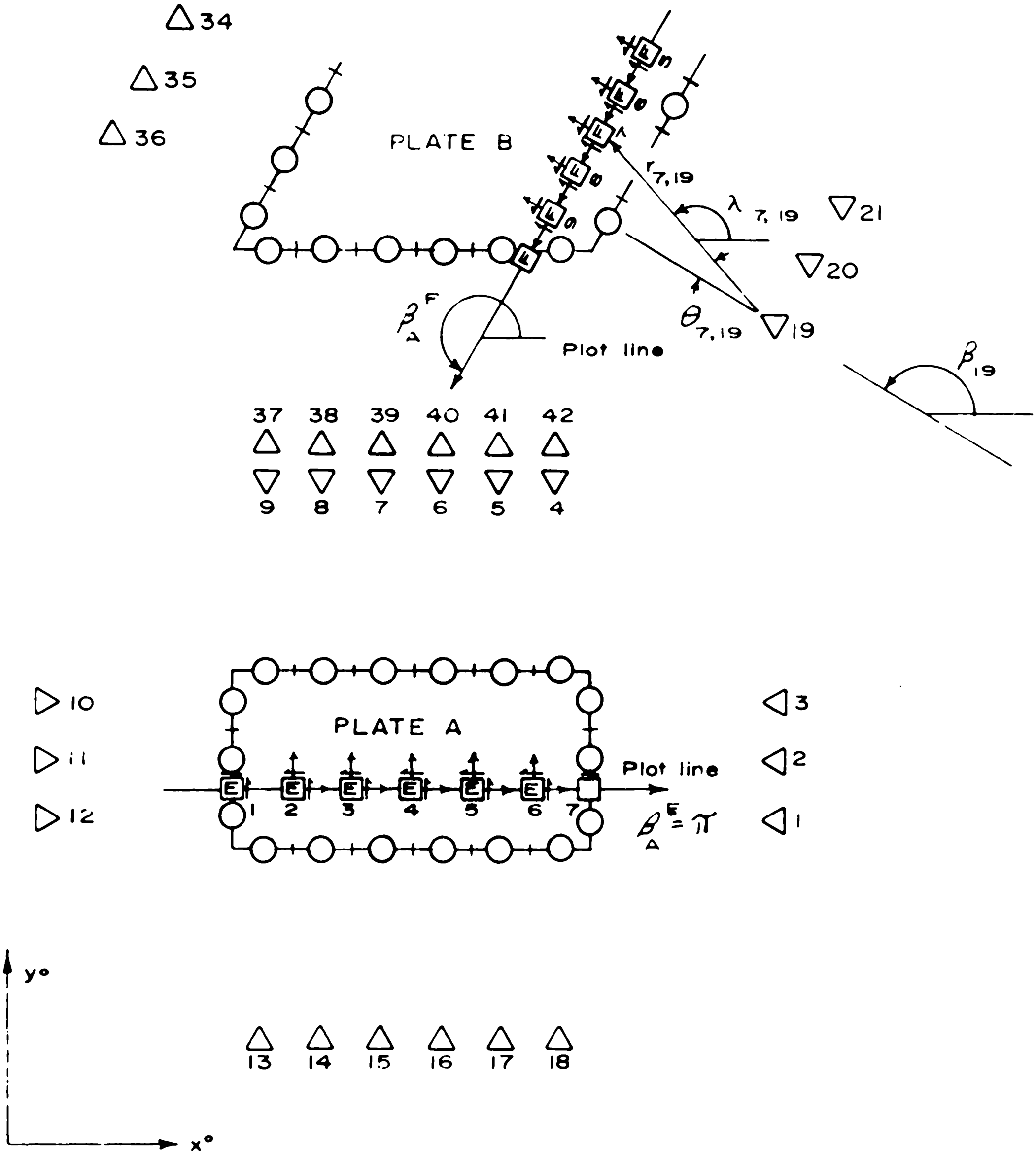
Figure 13 shows half of the physical boundary of the folded plate structure as it would appear in the infinite plate. The boundary of the two rectangular plates are divided into 18 intervals and the boundary of the skewed plate is divided into 24 intervals in the illustration. The locations of each of the midpoints of the intervals as defined by equations (64) and (65) are represented by circles. The balancing load positions as defined by equations (67) and (68) are represented by triangles in Figure 13. The numbering of the interval midpoints is the same as that of the balancing load positions.

Two plot lines are shown in Figure 13. The one through the rectangular plate indicates the positive directions of extensional quantities computed at points represented by squares along the plot line. Extensional stresses and displacements computed on the sides of the squares parallel to the plot line are obtained when $E = 0$ and the quantities on the other sides are obtained when $E = 1$.

The other plot line shown in the skewed plate indicates the positive directions of flexural results. The flexural moments or slopes on sides of the squares along this plot line are obtained when $F = 0$. If F is set equal to one, flexural quantities on the sides perpendicular to the plot line are computed.

The azimuths β_A^F and β_A^E are also shown in Figure 13 along with the various other geometrical terms required for the computation of results. The geometrical terms shown are associated with point number seven on the plot line in the skewed plate and balancing load point number 19.

The continuity conditions were satisfied by equating slopes and moments at each interval midpoint due to all of the balancing loads and applied loads to the slopes and moments at adjacent midpoints across the joints caused by the same balancing and applied loads. For instance, in the flexural problem the slope at midpoint five minus the slope at midpoint 41 due to all of the balancing and applied loads must be equal to zero. Similar relationships must be satisfied for the bending moments in the flexural problem and the in-plane shear stress and in-plane displacement parallel to the joints for the extensional part of the problem. The actual setting up of these equations was provided for in the computer program by a subscripting manipulation which caused the proper influence coefficients to be combined.



LEGEND:

- ○ Midpoints of intervals around the plate boundaries
- ▷ Positions of balancing loads
- □ Point at which results are computed along a plot line

Figure 13

Effect of E and F on the positive directions of results computed with respect to the plot line

XI. Presentation of Results

Application of the equations of the three previous sections leads immediately to such a tremendous amount of numerical work that the method is impractical unless it is programed for a computer. With such a program, however, the method becomes a powerful tool in the analysis of many kinds of steady state two dimensional problems dealing with homogeneous, isotropic media of constant thickness. The main advantage of this method is that it can be made into one program that will solve a boundary value problem dealing with a simply or multiply connected region of any shape with any type of boundary conditions acted upon by applied loads located at any point in the region. In order to initiate a solution, it is only necessary to provide the program with a few data cards specifying the shape of the region, types of boundary conditions and magnitudes and positions of the applied loads. The running time depends upon the number of intervals into which the boundary is divided. The program compiled for an IBM 1620 requires about an hour and a half for eight intervals and up to fifteen hours for sixty intervals.

The program does four jobs. It develops all of the geometrical relationships it needs, uses equations (81) through (84) or (113) through (116) to develop a set of simultaneous equations with unknown balancing loads, obtains the values of the balancing loads, and calculates the answers, again using equations (81) through (84) or (113) through (116).

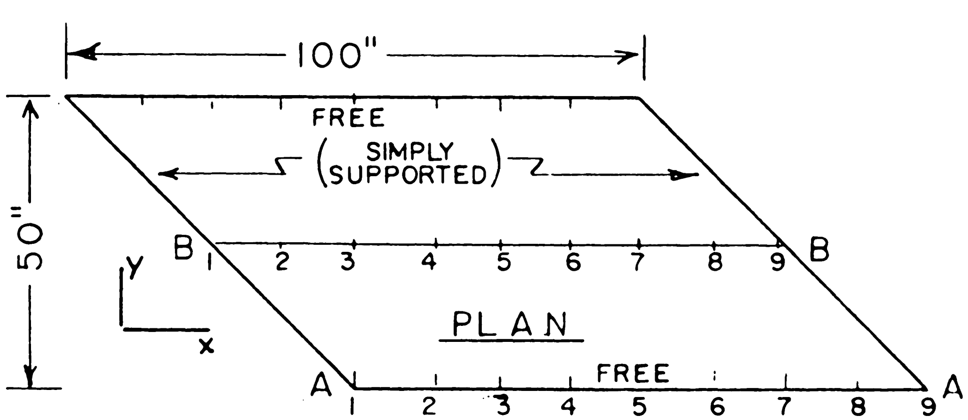
The results obtained from this program are presented in this section for a uniformly loaded skewed plate simply supported along two opposite edges and free along the other two, and for skewed folded plate structure.

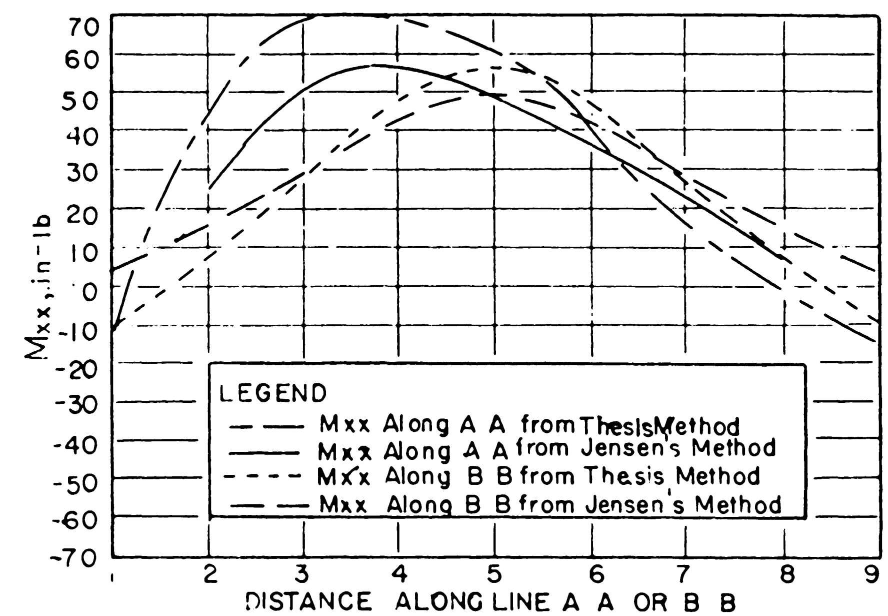
The simply support-free skewed plate was investigated by Jensen[17] who used a finite difference approach to obtain deflections and moments. These values are presented in Table I with corresponding values obtained from the method used in this thesis. Table 2 is a listing of the results of the skewed folded plate analysis. Of particular interest is the listing of the stress resultant acting along and normal to the hinged support. The stress resultant here is an indication of the foundation pressure under the abutments of a highway bridge. A comparison between the data presented from an analysis by Michalos[11] and the distribution of this stress resultant gives and idea of the accuracy of assuming that the plate can be treated as a very wide beam.

The physical properties and dimensions used as computer input for the skewed folded plate problem were taken from experimental theses by Ruff[21] and Cusano[22]. Results taken from curves plotted in these theses are also shown in Table 2 where they can be compared with corresponding results from the present analysis.

TABLE I DEFLECTIONS AND MOMENTS FOR A UNIFORMLY LOADED SKEWED PLATE WITH TWO EDGES SIMPLY SUPPORTED AND TWO EDGES FREE

	VALUES COMPUTED FROM JENSEN'S PAPER (17)		VALUES COMPUTED BY THE METHOD USED IN THIS THESIS	
POSITION	W inches	Mxx in lb	W inches	Mxx in lb
LINE A A				
1	.000	—	.000	-2.2
2	.069	25.4	.117	41.2
3	.126	52.2	.193	69.7
4	.157	56.3	.231	68.6
5	.161	49.7	.225	61.5
6	.141	37.1	.190	40.8
7	.102	22.1	.133	16.2
8	.053	8.0	.076	1.0
9	.000	—	.000	-15.0
LINE B B				
1	.000	3.4	.000	-10.0
2	.049	16.9	.059	8.2
3	.092	29.6	.113	28.5
4	.121	42.4	.156	48.2
5	.133	48.1	.175	57.3
6	.121	42.4	.156	48.2
7	.092	29.6	.113	28.5
8	.049	16.9	.059	8.2
9	.000	3.4	.000	-10.0





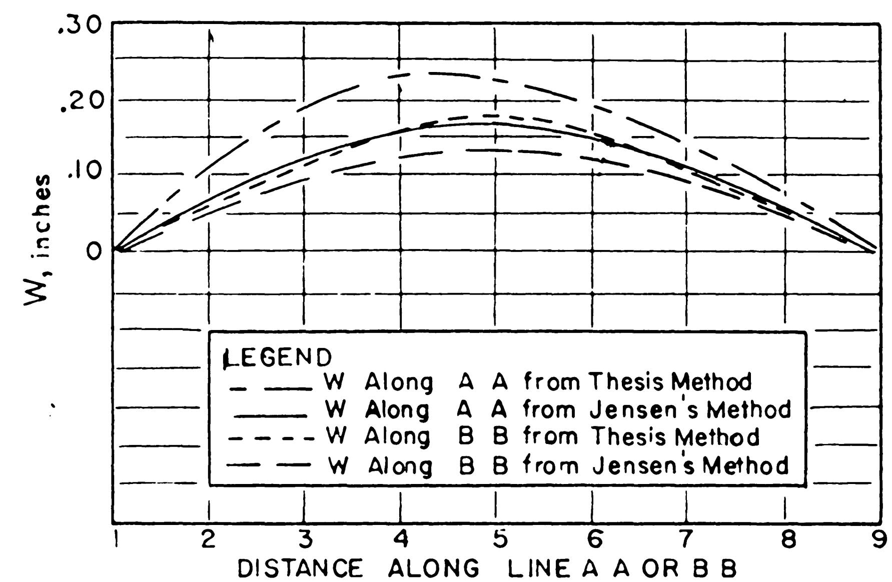
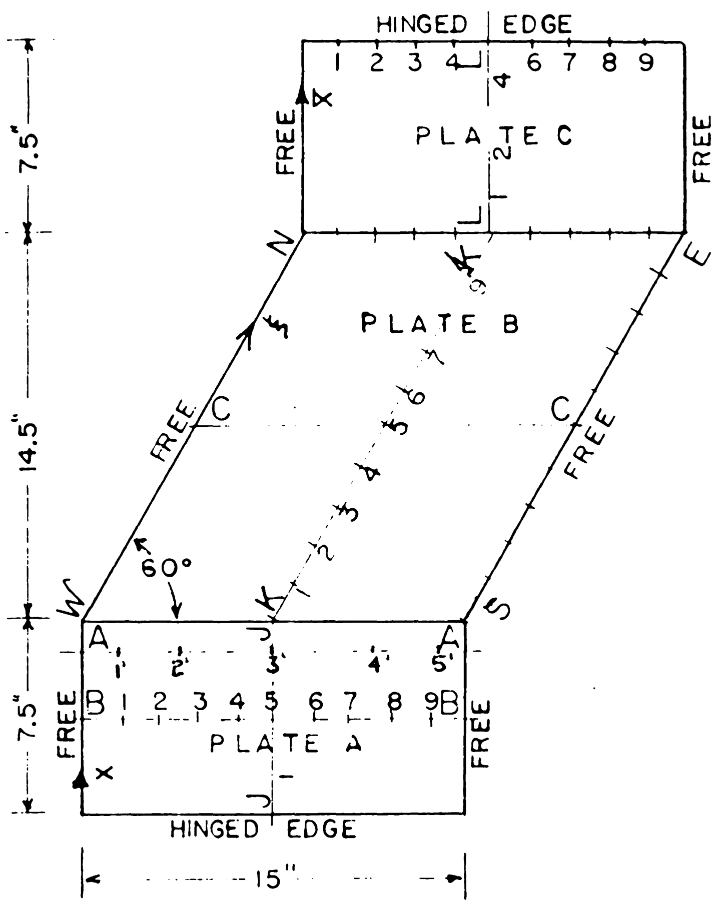


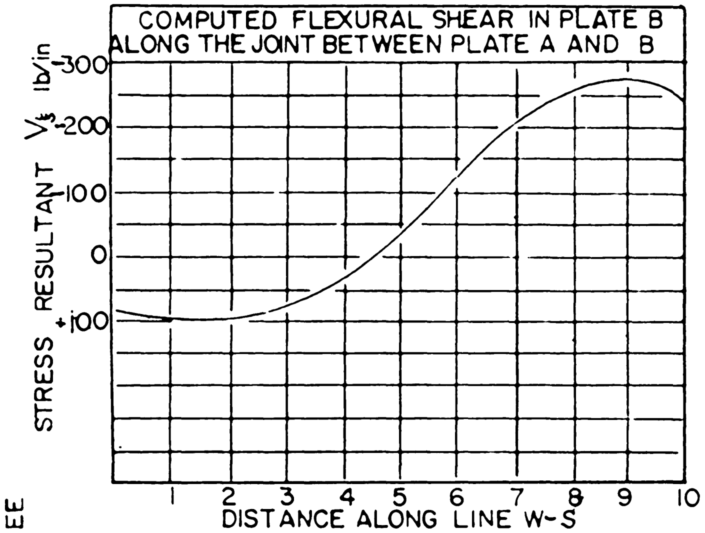
TABLE 2 EXPERIMENTAL AND THEORETICAL RESULTS FOR THE SKEWED FOLDED PLATE

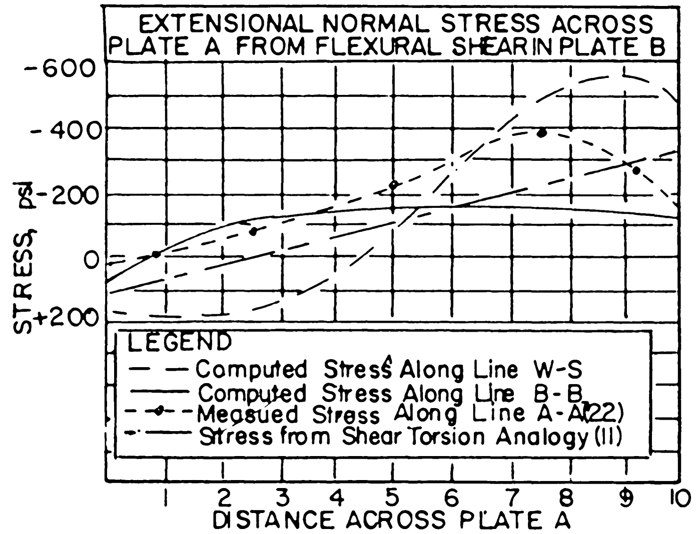
NORMAL STRESS DISTRIBUTION ALONG SECTION A A FROM EXPERIMENTS BY CUSANO (22)

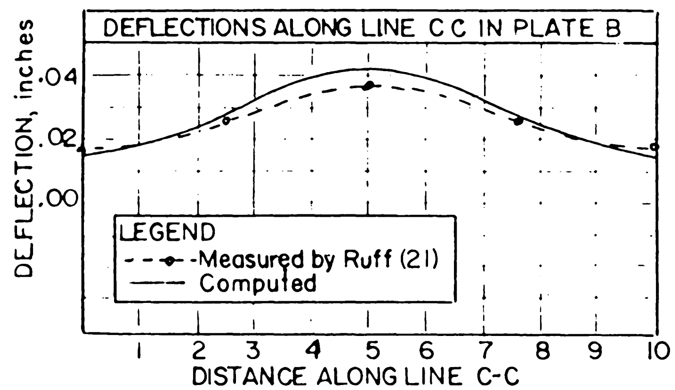
MEASURED COMBINED STRESS IN PLATE A ALONG LINE A-A			EXPERIMENTAL NORMAL STRESS ALONG A-A (AVERAGE OF COMBINED STRESS ALONG LINE A-A)
POSITION	STRESS ON INSIDE OF PLATE A psi	STRESS ON OUTSIDE OF PLATE A psi	
1'	-1050	1050	0
2'	-1550	1400	- 75
3'	-2950	2550	- 200
4'	-4750	4000	- 375
5'	-4350	3850	- 250

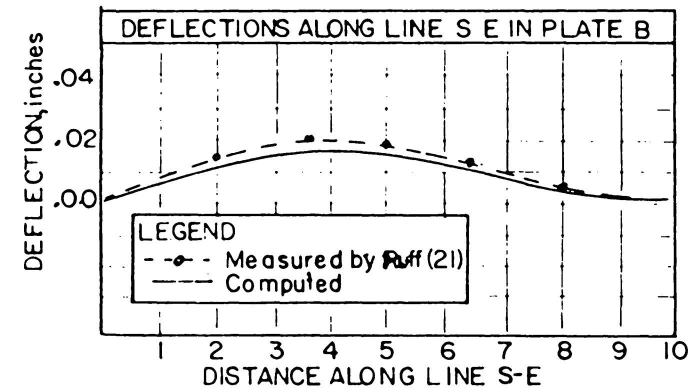


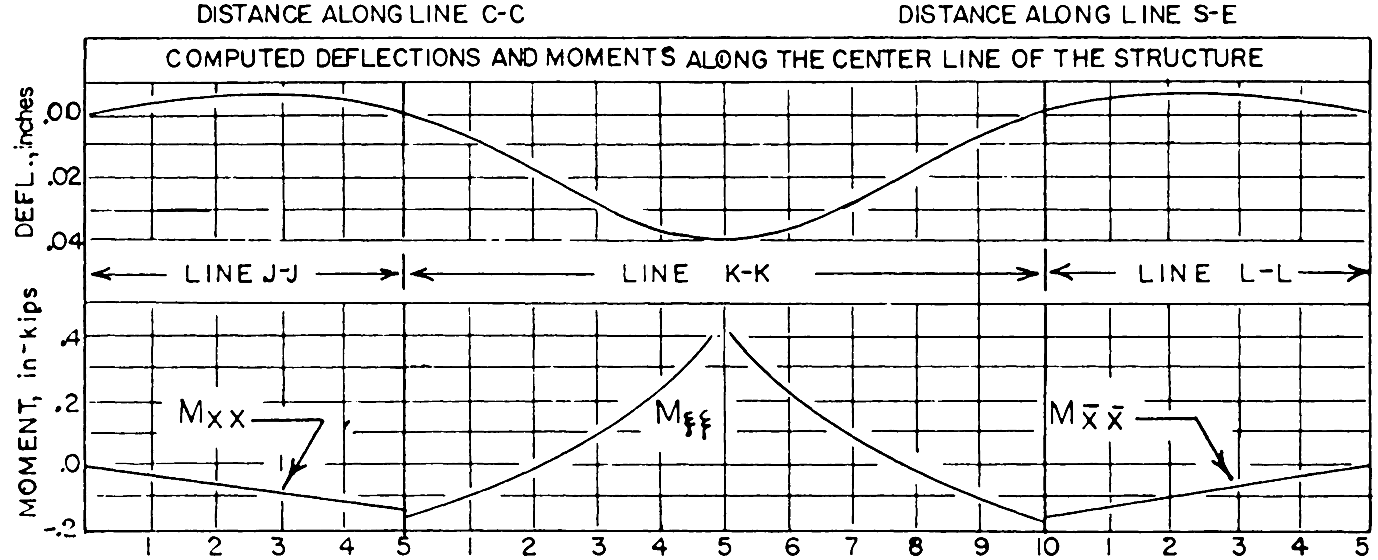
POISSON'S RATIO = 0.32
MODULUS OF ELASTICITY = 10.4×10^6 psi
PLATE THICKNESSES = 0.5"
LOAD = 1500 lb AT INTERSECTION OF C-C AND K-K











XII. Acknowledgements

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XV. APPENDIX I

Extensional Stress and Displacement Functions

The derivation of in-plane displacements due to a concentrated load on the plane of an infinite plate is developed as follows. Given the stresses due to a concentrated load in the plane of the plate

$$\sigma_r = \frac{-(3+\mu)P\cos\theta}{4\pi r} \tag{126}$$

$$\sigma_\theta = \frac{(1-\mu)P\cos\theta}{4\pi r} \tag{127}$$

$$\tau_{r\theta} = \frac{(1-\mu)P\sin\theta}{4\pi r} \tag{128}$$

the strains are

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \mu\sigma_\theta) = \frac{-P\cos\theta}{4\pi r E}\Big[3+\mu+\mu(1-\mu)\Big] \tag{129}$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \mu\sigma_r) = \frac{P\cos\theta}{4\pi r E}\Big[1-\mu+\mu(3+\mu)\Big] \tag{130}$$

$$\gamma_{r\theta} = \frac{2(1+\mu)}{E}\,\tau_{r\theta} = \frac{2(1-\mu^2)P\sin\theta}{4\pi r E} \tag{131}$$

In terms of displacements these equations become

$$\varepsilon_r = \frac{\partial u_r}{\partial r} = \frac{P\cos\theta}{4\pi r E}\Big[\mu^2 - 2\mu - 3\Big] \tag{132}$$

$$\varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta} = \frac{P\cos\theta}{4\pi r E}\Big[\mu^2 + 2\mu + 1\Big] \tag{133}$$

$$\gamma_{r\theta} = \frac{1}{r}\,\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = \frac{P\sin\theta}{4\pi r E}\Big[2(1-\mu^2)\Big] \tag{134}$$

Integrating equation (132) we have

$$u_r = \frac{P\cos\theta}{4\pi E}\left(\mu^2 - 2\mu - 3\right)\log r + f(\theta) \tag{135}$$

If equation (135) is substituted into equation (133) and the resulting equation is multiplied by r,

$$\frac{\partial u_\theta}{\partial \theta} = \frac{P\cos\theta}{4\pi E}\left(\mu^2 + 2\mu + 1\right) - \frac{P\cos\theta}{4\pi E}\left(\mu^2 - 2\mu - 3\right)\log r - f(\theta)$$

and

$$u_\theta = \frac{P\sin\theta}{4\pi E}\left[\left(\mu^2 + 2\mu + 1\right) - (\mu^2 - 2\mu - 3)\log r\right] - \int f(\theta)d\theta + F(r). \qquad (136)$$

Substituting equation (135) and (136) into (134) we have:

$$\frac{-P\sin\theta}{4\pi E}\left(\mu^2 - 2\mu - 3\right)\frac{\log r}{r} + \frac{1}{r}\frac{df(\theta)}{d\theta}$$

$$- \frac{P\sin\theta}{4\pi E r}(\mu^2 - 2\mu - 3) + \frac{dF(r)}{dr}$$

$$- \frac{P\sin\theta}{4\pi E r}\left(\mu^2 + 2\mu + 1\right) - (\mu^2 - 2\mu - 3)\log r$$

$$+ \frac{1}{r}\int f(\theta)d\theta - \frac{F(r)}{r} = \frac{P\sin\theta}{4\pi E r}\,2(1 - \mu^2)$$

which after multiplying by r and performing algebraic cancellations becomes

$$\frac{df(\theta)}{d\theta} + \int f(\theta)d\theta + r\frac{dF(r)}{dr} - F(r) = 0$$

Hence if $F(r) = f(\theta) = 0$, the strain-displacement relations will be satisfied and the resulting displacement equations will be free of terms which would associate them with a particular boundary condition.

From Figure 10 the transformation equations are

$$\sigma_n = \sigma_r\cos^2\alpha + \sigma_\theta\sin^2\alpha + \tau_{r\theta}\sin 2\alpha$$

or

$$\sigma_n = \frac{P}{4\pi r}\left\{ \left[(1-\mu)\sin^2 a - (3+\mu)\cos^2 a\right] \cos\theta + (1-\mu)\sin 2a \sin\theta \right\} \quad (137)$$

$$\tau_N = \left(\frac{\sigma_\theta - \sigma_r}{2}\right) \sin 2a + \tau_{r\theta}\cos 2a$$

or

$$\tau_N = \frac{P}{4\pi r}\left\{2\cos\theta\sin 2a + (1-\mu)\cos 2a \sin\theta\right\} \quad (138)$$

$$u_N = u_r \cos a + u_\theta \sin a ,$$

or

$$u_N = \frac{P}{4\pi E}\left\{ (\mu^2 - 2\mu - 3) \log r \cos\theta \cos a + \left[\mu^2 + 2\mu + 1 - (\mu^2 - 2\mu - 3)\log r\right] \sin\theta \sin a \right\},$$

or

$$u_N = \frac{P}{4\pi E}\left\{ (\mu^2 - 2\mu - 3)\log r \cos(\theta + a) + (\mu^2 + 2\mu + 1) \sin\theta \sin a \right\} \quad (139)$$

$$v_N = u_\theta \cos a - u_r \sin a ,$$

or

$$v_N = \frac{P}{4\pi E}\left\{ \left[\mu^2 + 2\mu + 1 - (\mu^2 - 2\mu - 3)\log r\right] \sin\theta\cos a - (\mu^2 - 2\mu - 3) \log r \cos\theta \sin a \right\} ,$$

or

$$v_N = \frac{-P}{4\pi E}\left\{ (\mu^2 - 2\mu - 3) \log r \sin(\theta + a) - (\mu^2 + 2\mu + 1)\sin\theta\cos a \right\} \quad (140)$$

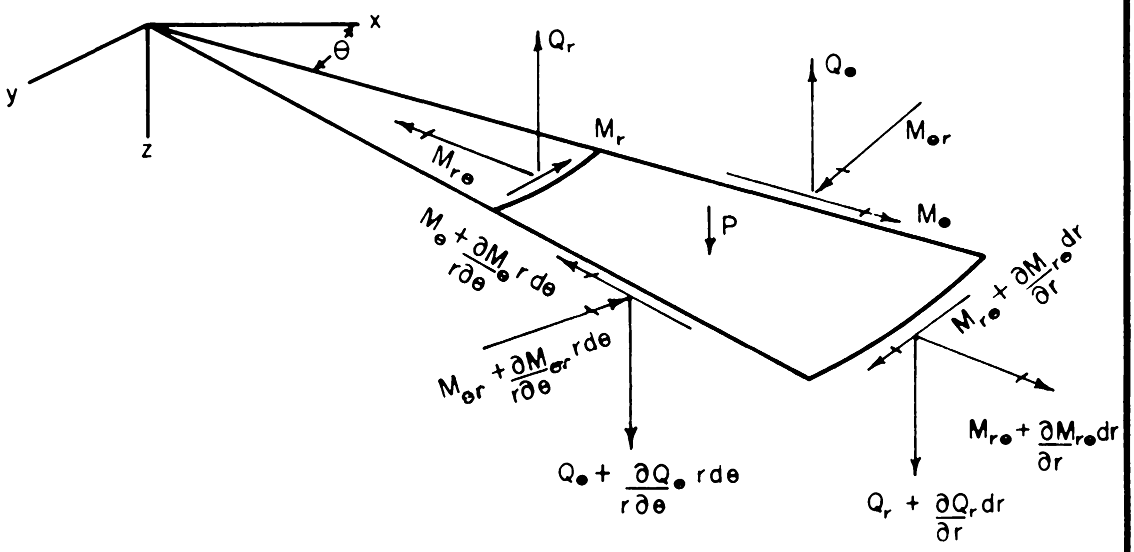

Figure 14

Shear and moment resultants acting on an infinitesimal element

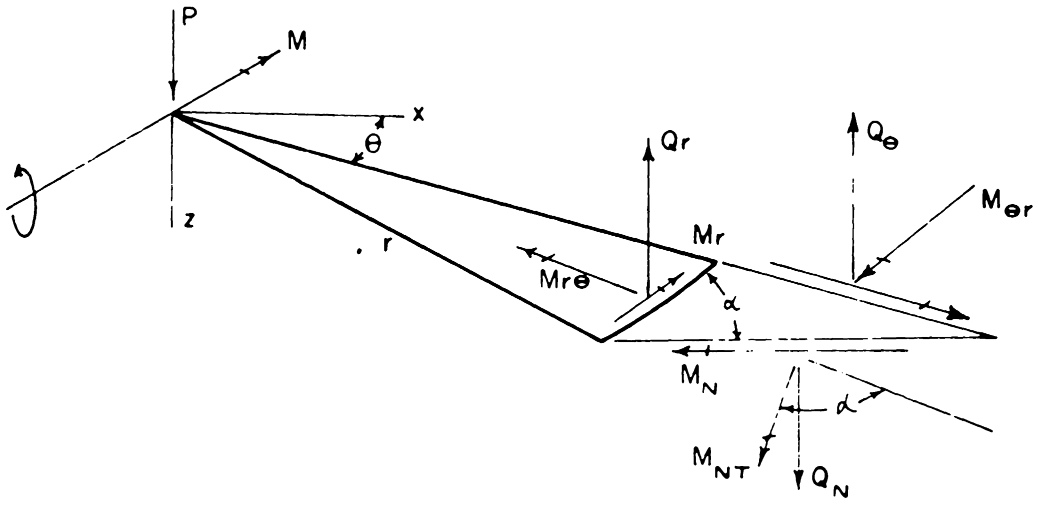

Figure 16

Moment and shear transformations across a reference line

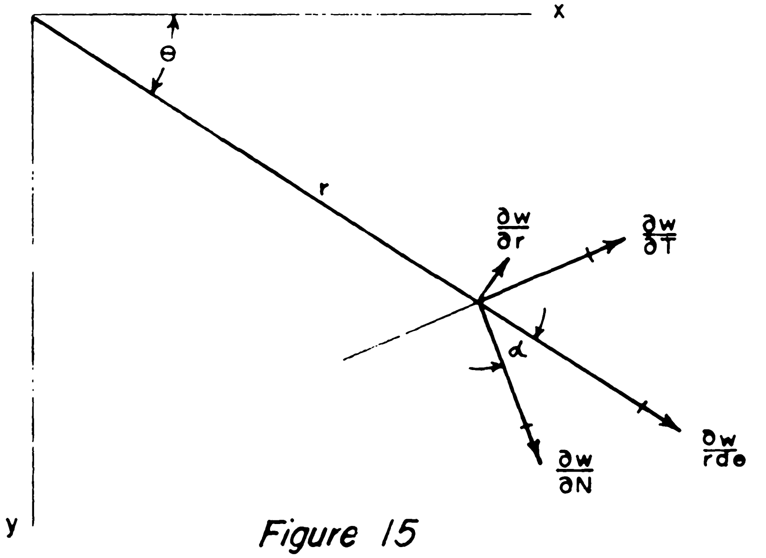

Figure 15

Transformation of rotation vectors across a reference line

XVI. APPENDIX II

Summary of the Derivation of the Flexural Plate Equations

An infinitesimal element of a plate is shown in Figure 14. This figure and the following equations for shear and moment resultants are taken from a derivation given in Chapter 11 of Wang[(23)].

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \mu\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\right] \tag{141}$$

$$M_\theta = -D\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{\mu\partial^2 w}{\partial r^2}\right] \tag{142}$$

$$M_{r\theta} = (1-\mu)\,D\,\frac{1}{r}\left[\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial\theta}\right] \tag{143}$$

$$Q_r = -D\frac{\partial}{\partial r}(\Delta^2 w) = -D\left[\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^3 w}{\partial r\partial\theta^2} - \frac{2}{r}\frac{\partial^2 w}{\partial\theta^2}\right] \tag{144}$$

$$Q_\theta = -D\frac{\partial}{\partial\theta}(\Delta^2 w) = -D\left[\frac{1}{r}\frac{\partial^3 w}{\partial r^2\partial\theta} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r\partial\theta} + \frac{1}{r^3}\frac{\partial^3 w}{\partial\theta^3}\right] \tag{145}$$

The slope normal to a boundary referring to Figure 15 has the following form.

$$\frac{\partial w}{\partial N} = \frac{\partial w}{\partial r}\frac{dr}{dN} + \frac{1}{r}\frac{\partial w}{\partial\theta}\frac{rd\theta}{dN} = \frac{\partial w}{\partial r}\cos\alpha + \frac{\partial w}{\partial\theta}\frac{\sin\alpha}{r} \tag{146}$$

Referring to Figure 16, the moment and shear resultants normal to a boundary are obtained from the following transformation equations

$$M_N = M_r \cos^2 a + M_\theta \sin^2 a - M_{r\theta} \sin 2a \tag{147}$$

$$M_{NT} = \frac{1}{2}(M_r - M_\theta) \sin 2a + M_{r\theta} \cos 2a \tag{148}$$

$$M_T = M_r \sin^2 a + M_\theta \cos^2 a + M_{r\theta} \sin 2a \tag{149}$$

$$Q_N = Q_r \cos a + Q_\theta \sin a \tag{150}$$

$$Q_T = Q_r \sin a - Q_\theta \cos a \tag{151}$$

$$\begin{aligned} V_N &= Q_N - \frac{\partial M_{NT}}{\partial T} \\ &= Q_r \cos a + Q_\theta \sin a - \frac{\partial M_{NT}}{\partial r} \frac{dr}{dT} - \frac{\partial M_{NT}}{r \partial \theta} \frac{r d\theta}{dT} \\ &= Q_r \cos a + Q_\theta \sin a + \frac{\partial M_{NT}}{\partial r} \sin a - \frac{\partial M_{NT}}{\partial \theta} \frac{\cos a}{r} \end{aligned} \tag{152}$$

From equation (89) the deflection due to a concentrated load acting at the origin of an infinite plate is

$$w = \frac{Pr^2}{8\pi D} \log \frac{r}{a} \tag{89}$$

where r is shown in Figure 16. The flexural rigidity, D, is given by

$$D = \frac{Eh^3}{12(1-\mu)^2} \tag{153}$$

In this equation E is the modulus of elasticity, μ is Poisson's ratio, and h is the thickness of the plate. The letter a in equation (89) is an arbitrary constant.

The representation of the moment and shear resultants normal to a boundary in terms of the deflection, w, requires the following expressions.

$$\frac{\partial w}{\partial r} = \frac{P}{8\pi D} \left(2r \log \frac{r}{a} + r\right) \tag{154}$$

$$\frac{\partial^2 w}{\partial r} = \frac{P}{8\pi D}\left(2 \log \frac{r}{a} + 3\right) \tag{155}$$

$$\frac{\partial^3 w}{\partial r^3} = \frac{P}{4\pi D r} \tag{156}$$

Substitution of these equations into equations (141) through (145) yields the following expressions for the moment and shear resultants in polar coordinates.

$$M_r = \frac{-P}{8\pi}\left[2 \log \frac{r}{a}\ (1 + \mu)\ \text{\&}\ 3 + \mu\right] \tag{157}$$

$$M_\theta = \frac{-P}{8\pi}\left[2 \log \frac{r}{a}\,(1 + \mu) + 1 + 3\mu\right] \tag{158}$$

$$M_{r\theta} = 0 \tag{159}$$

$$Q_r = \frac{-P}{2\pi r} \tag{160}$$

$$Q_\theta = 0 \tag{161}$$

With these expressions substituted into equations (146) through (152) the following boundary condition equations are obtained.

$$M_N = \frac{-P}{8\pi}\left[2(1 + \mu)\log \frac{r}{a} + (3 + \mu)\cos^2 a + (1 + 3\mu)\sin^2 a\right] \tag{162}$$

$$M_{NT} = \frac{-P}{8\pi}\ (1 - \mu)\sin 2a \tag{163}$$

$$M_T = \frac{-P}{8\pi}\left[2(1 + \mu)\log \frac{r}{a} + (3 + \mu)\sin^2 a + (1 + 3\mu)\cos^2 a\right] \tag{164}$$

$$\frac{\partial w}{\partial N} = \frac{P}{8\pi D}\left[2r \log \frac{r}{a} + r\right] \cos a \tag{165}$$

$$Q_N = \frac{-P}{2\pi r} \cos a \tag{166}$$

$$Q_T = \frac{-P}{2\pi r} \text{ Sin } a \tag{167}$$

$$V_N = \frac{-P}{2\pi r} \cos a \tag{168}$$

In order to obtain boundary condition expressions for a concentrated moment action along the y axis as shown in Figure 16, the following derivatives of equation (90) are needed.

$$\frac{\partial w}{\partial r} = \frac{M}{4\pi D}\left[\log \frac{r}{a} + 1\right] \cos\theta \tag{169}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{M \cos\theta}{4\pi D} \tag{170}$$

$$\frac{\partial^3 w}{\partial r^3} = -\frac{M \cos\theta}{4\pi D r^2} \tag{171}$$

$$\frac{\partial w}{\partial \theta} = -\frac{M r \log \frac{r}{a} \sin\theta}{4\pi D} \tag{172}$$

$$\frac{\partial^2 w}{\partial \theta^2} = -\frac{M r \log \frac{r}{a} \cos\theta}{4\pi D} \tag{173}$$

$$\frac{\partial^3 w}{\partial \theta^3} = \frac{M r \log \frac{r}{a} \sin\theta}{4\pi D} \tag{174}$$

$$\frac{\partial^2 w}{\partial r \partial \theta} = -\frac{M}{4\pi D} \log \frac{r}{a} \sin\theta \tag{175}$$

$$\frac{\partial^3 w}{\partial r^2 \partial \theta} = -\frac{M \sin\theta}{4\pi D r} \tag{176}$$

$$\frac{\partial^2 w}{\partial r \partial \theta^2} = -\frac{M}{4\pi D}\left[\log \frac{r}{a} + 1\right] \cos\theta \tag{177}$$

With these expressions substituted into equations (141) through (145), the following expressions for the moment and shear resultants in polar coordinates for a concentrated moment are obtained.

$$M_r = -\frac{M(1+\mu)\cos\theta}{4\pi r} \qquad (178)$$

$$M_\theta = -\frac{M(1+\mu)\cos\theta}{4\pi r} \qquad (179)$$

$$M_{r\theta} = -\frac{M(1-\mu)\sin\theta}{4\pi r} \qquad (180)$$

$$Q_r = \frac{M\cos\theta}{2\pi r^2} \qquad (181)$$

$$Q_\theta = \frac{M\sin\theta}{2\pi r^2} \qquad (182)$$

If this last set of equations is used in equations (146) through (152) the boundary condition equations for a concentrated moment are the following.

$$\frac{\partial w}{\partial N} = \frac{M}{4\pi D}\left[\log\frac{r}{a}\cos(\theta+a) + \cos\theta\cos a\right] \qquad (183)$$

$$M_N = \frac{-M}{4\pi r}\left[(1+\mu)\cos\theta - (1-\mu)\sin\theta\sin 2a\right] \qquad (184)$$

$$M_{NT} = \frac{-M}{4\pi r}(1-\mu)\sin\theta\cos 2a \qquad (185)$$

$$M_T = \frac{-M}{4\pi r}\left[(1+\mu)\cos\theta + (1-\mu)\sin\theta\sin 2a\right] \qquad (186)$$

$$Q_N = \frac{M}{2\pi r^2}\cos(\theta - a) \qquad (187)$$

$$Q_T = \frac{M}{2\pi r^2}\sin(\theta - a) \qquad (188)$$

$$V_N = \frac{M}{4\pi r^2}\left[2 + (1-\mu)\cos 2a\right]\cos(\theta - a) \qquad (189)$$

The deflection equation for a uniform load, Q, distributed over the entirity of the infinite plate can be obtained from the particular solution of the equation

$$ {}^4(w) = \frac{1}{r}\frac{\partial}{\partial r}\left\{ r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right]\right\} = \frac{Q}{D} \tag{190} $$

The solution is

$$ w = \frac{Qr^4}{16D} \tag{191} $$

The derivatives of this equation with respect to r when substituted into equations (141) through (145) yield

$$ M_r = \frac{-Qr^2(3+\mu)}{16} \tag{192} $$

$$ M_\theta = \frac{-Qr^2(1+3\mu)}{16} \tag{193} $$

$$ M_{r\theta} = 0 \tag{194} $$

$$ Q_r = \frac{-Qr^2}{2} \tag{195} $$

$$ Q_\theta = 0 \tag{196} $$

and these in turn when substituted into equations (146) through (152) provide the boundary condition equations for a uniform load in the following form.

$$ \frac{\partial w}{\partial N} = \frac{Qr^3}{16D}\cos a \tag{197} $$

$$ M_N = \frac{-Qr^3}{16}\left[(3+\mu)\cos^2 a + (1+3\mu)\sin^2 a\right] \tag{198} $$

$$ M_{NT} = \frac{-Qr^2}{16}(1-\mu)\sin 2a \tag{199} $$

$$M_r = \frac{-Qr^2}{16}\left[(3+\mu)\sin^2 a + (1+3\mu)\cos^2 a\right] \tag{200}$$

$$Q_N = \frac{-Qr}{2}\cos a \tag{201}$$

$$Q_T = \frac{-Qr}{2}\sin a \tag{202}$$

$$V_N = \frac{-Qr}{8}\left[4\cos a + (1-\mu)\sin a \sin 2a\right] \tag{203}$$

ABSTRACT

G. W. Swift. "An Approximate Solution for the Flexural and In-Plane Effects of a Laterally Loaded Skewed Folded Plate Structure.", Doctoral Thesis, Engineering Mechanics Department. Virginia Polytechnic Institute, Blacksburg Virginia, 90 pp. 19[illegible]4.

The particular folded plate configuration considered here is composed of two rectangular plates joined to opposite edges of a skewed plate in such a way that the middle surfaces form 90° dihedral angles. The structure is simply supported along the bottom edges of each rectangular plate and loaded vertically, perpendicular to the skewed plate. The remaining sides are free.

The results of this analysis are distributions of flexural deflections, moments, and shears and extensional stress resultants along several lines in each of the plates.

Abrief description of the application of several methods of analysis that were found unsuitable is followed by a detailed account of the method that was finally adopted. The methods that could not be made to provide a solution were the application of the solution to the bharmonic plate equations, an energy approach, and a finite difference approach. In the descriptions of these methods, the factors which caused them to be unsuitable are discussed.

The method that provided the solution is called the reflection method. It involves the positioning of the shape of the elastic body to be

investigated on an infinite body. Loads are applied to a region marked off in the infinite body at points corresponding to load positions in the body to be investigated. The effects of the applied loads are calculated at a finite number of points around the boundary scribed in the infinite body. Corrective loads are then applied to the infinite body in such a way that the effects of the applied loads are cancelled and boundary conditions associated with the elastic body under consideration are satisfied.

The effects of the applied loads and the corrective loads are computed at each point on the boundary at which the boundary conditions are to be satisfied. The method is approximate in that the boundary conditions are to be satisfied at only N points around the boundary. Hence 2N simultaneous equations involving 2N unknown corrective load magnitudes must be set equal to the required boundary effects if two boundary conditions are associated with each of the N boundary points. Once the corrective loads are obtained, the solution can be obtained by computing the effects inside the boundary of all of the loads.