

$$u_{a_{ij}} a_{ij} \geq (x_i - a_j)^2 + (y_i - b_j)^2 \quad \forall(i, j) \quad (5.23l)$$

$$w \in W, 0 \leq a \leq u_a. \quad (5.23m)$$

5.1.2 Convexification /Linearization Phase

For constructing the proposed lower bounding problem, we consider the developed reformulated problem (5.23) and create new RLT variables, in addition to the original variables $\{ x_i, y_i, w_{ij}, \alpha_{ij} \}$, that represent a linearization of the following terms.

$$\begin{aligned} g_j &= [w_{ij} a_{ij}] \\ q_{ij} &= [w_{ij} x_i] \\ f_{ij} &= [w_{ij} y_i]. \end{aligned} \quad (5.24)$$

We substitute (5.24) for the appropriate nonlinear terms, and hence, we obtain the following convex programming lower bounding problem, which will be referred to as RLT(Ω) based on the hyperrectangle $\Omega = \{ w: l_{ij} \leq w_{ij} \leq u_{ij}, \forall(i, j) \}$.

$$\mathbf{RLT}(\Omega): \text{Minimize } \sum_i \sum_j c_{ij} g_{ij} \quad (5.25)$$

subject to

$$l_{ij} a_{ij} \leq g_j \leq u_{ij} a_{ij} \quad \forall(i, j) \quad (5.25a)$$

$$u_{a_{ij}} w_{ij} + a_{ij} u_{ij} - u_{a_{ij}} u_{ij} \leq g_j \leq u_{a_{ij}} w_{ij} + a_{ij} l_{ij} - u_{a_{ij}} l_{ij} \quad \forall(i, j) \quad (5.25b)$$

$$\sum_i \sum_j c_{ij} w_{ij}^* a_{ij} \geq n^* \quad (5.25c)$$

$$\sum_i \sum_j a_{ij} \geq n_1 \quad (5.25d)$$

$$\sum_i \sum_j c_{ij} \bar{w}_{ij} a_{ij} \geq \bar{n} \quad (5.25e)$$

$$q_{ij} - w_{ij} a_j + l_{ij}(a_{ij} - x_i + a_j) \leq g_j \leq q_{ij} - w_{ij} a_j + u_{ij}(a_{ij} - x_i + a_j),$$

$$w_{ij} a_j - q_{ij} + l_{ij}(a_{ij} + x_i - a_j) \leq g_j \leq w_{ij} a_j - q_{ij} + u_{ij}(a_{ij} + x_i - a_j),$$

$$\begin{aligned}
f_{ij} - w_{ij} b_j + l_{ij}(a_{ij} - y_i + b_j) &\leq g_j \leq f_{ij} - w_{ij} b_j + u_{ij}(a_{ij} - y_i + b_j), \\
w_{ij} b_j - f_{ij} + l_{ij}(a_{ij} + y_i - b_j) &\leq g_j \leq w_{ij} b_j - f_{ij} + u_{ij}(a_{ij} + y_i - b_j)
\end{aligned}
\quad \forall(i, j) \quad (5.25f)$$

$$\begin{aligned}
q_{ij} + f_{ij} - w_{ij}(a_j + b_j) + l_{ij}(\sqrt{2} a_{ij} - x_i + a_j - y_i + b_j) &\leq \sqrt{2} g_j \leq q_{ij} + f_{ij} - w_{ij}(a_j + b_j) + u_{ij}(\sqrt{2} a_{ij} - x_i + a_j - y_i + b_j) \\
q_{ij} - \phi_{ij} + w_i(-a_j + b_j) + l_{ij}(\sqrt{2} a_{ij} - x_i + a_j + y_i - b_j) &\leq \sqrt{2} g_j \leq q_{ij} - \phi_{ij} + w_{ij}(-a_j + b_j) + u_{ij}(\sqrt{2} a_{ij} - x_i + a_j + y_i - b_j) \\
f_{ij} - q_{ij} + w_{ij}(a_j - b_j) + l_{ij}(\sqrt{2} a_{ij} + x_i - a_j - y_i + b_j) &\leq \sqrt{2} g_j \leq f_{ij} - q_{ij} + w_{ij}(a_j - b_j) + u_{ij}(\sqrt{2} a_{ij} + x_i - a_j - y_i + b_j) \\
-f_{ij} - q_{ij} + w_{ij}(a_j + b_j) + l_{ij}(\sqrt{2} a_{ij} + x_i - a_j + y_i - b_j) &\leq \sqrt{2} g_j \leq -f_{ij} - q_{ij} + w_{ij}(a_j + b_j) + u_{ij}(\sqrt{2} a_{ij} + x_i - a_j + y_i - b_j)
\end{aligned}
\quad \forall(i, j) \quad (5.25g)$$

$$\sum_j q_{ij} - s_i x_i = 0 \quad \forall i \quad (5.25h)$$

$$\sum_j f_{ij} - s_i y_i = 0 \quad \forall i \quad (5.25i)$$

$$w_{ij} Y_{0k} - u_{ij}(Y_{0k} - Y_{1k} x_i - Y_{2k} y_i) \leq Y_{1k} q_{ij} + Y_{2k} f_{ij} \leq w_{ij} Y_{0k} - l_{ij}(Y_{0k} - Y_{1k} x_i - Y_{2k} y_i)
\quad \forall(i, j, k) \quad (5.25j)$$

$$u_{a_{ij}} a_{ij} \geq (x_i - a_j)^2 + (y_i - b_j)^2 \quad \forall(i, j) \quad (5.25k)$$

$$w \in W, 0 \leq a \leq u_a. \quad (5.25l)$$

Beside the constraints of the Problem RLT(Ω), we will consider surrogating some of these constraints at the initial node of the branch-and-bound algorithm and adding the resulting surrogate constraint to this problem. Hence this surrogate constraint will also be present in the computation of the lower bounding problem at every intermediate node, and its purpose is merely to aid in the Lagrangian dual ascent scheme. The construction of this constraint will be discussed in Section 5.3.

Note that except for the convex quadratic constraints (5.25k), Problem RLT(Ω) is a linear program. Also, note that just as (5.6) and (5.8) enforce the minimum feasible value of a_{ij} to satisfy (5.2b) along the rib-framework of Figure 1, constraint (5.25k) enforces this relationship along the boundary of the circle. For example, if $(a_j, b_j) = (0, 0)$, $u_{a_{ij}} = 1$, then for any $(x_i, y_i) = (\cos q, \sin q)$ for any angle q , (5.5) and (5.7) enforce that $a_{ij} \geq |\cos q|$, a_{ij}

$\geq |\sin \theta|$, and $\alpha_{ij} \geq (|\cos \theta| + |\sin \theta|)/\sqrt{2}$. Hence, if we take $\theta = 30^\circ$, these inequalities yield $a_{ij} \geq .866$, $\alpha_{ij} \geq 0.5$, and $a_{ij} \geq .966$. Hence, $a_{ij} = .966$ is feasible to these restrictions. However, in this case, (5.25k) would require $a_{ij} \geq 1$ which is a stronger restriction.

Consequently, it is beneficial to retain these nonlinear separable convex constraints (5.25k), and to exploit their structure in a Lagrangean relaxation framework. On the other hand, deleting these constraints would yield an LP that could be solved by standard LP packages.

The Problem $RLT(\Omega)$ has $(2+5m)n$ variables and at most $(3+21m+m^2)n+m+3$ constraints, since the value of K is at most m . This is a large-scale problem for even moderate values of m and n . For example, a test problem with $n = 2$ and $m = 5$ would yield Problem $RLT(\Omega)$ having up to 54 variables and 274 constraints.

5.2 Lagrangian Relaxation Scheme for Solving Problem $RLT(\Omega)$

For the sake of notational convenience, let us write the constraints in (5.25b), (5.25c), (5.25d), (5.25f), (5.25g), (5.25h), and (5.25k) jointly as

$$H_1 w + H_2 a + H_3 x + H_4 y + H_5 q + H_6 f + H_7 g \geq h.$$

Furthermore, let us assume without loss of generality that

$$\min_j \{a_j\} = 0 \text{ and } \min_j \{b_j\} = 0$$

so that Z is embedded in the nonnegative quadrant. Also, let us accordingly denote $\bar{a} = \max_j \{a_j\}$ and $\bar{b} = \max_j \{b_j\}$. Then Problem $RLT(\Omega)$ can be more compactly rewritten as follows. (Note that for algorithmic convenience, we have added implied bounds on the x , y , q , and f variables below.)

$$\text{RLT}(\Omega): \quad \text{Minimize } \sum_i \sum_j c_{ij} \gamma_{ij} \quad (5.26a)$$

subject to

$$H_1 w + H_2 a + H_3 x + H_4 y + H_5 q + H_6 f + H_7 g \geq h \quad (5.26b)$$

$$u_{a_{ij}} a_{ij} \geq (x_i - a_j)^2 + (y_i - b_j)^2 \quad \forall (i, j) \quad (5.26c)$$

$$\sum_{i=1}^n w_{ij} = d_j \quad j = 1, \dots, m \quad (5.26d)$$

$$\sum_i \sum_j c_{ij} \bar{w}_{ij} a_{ij} \geq \bar{n} \quad (5.26e)$$

$$\sum_{j=1}^m w_{ij} = s_i \quad i = 1, \dots, n, \quad l \leq w \leq u \quad (5.26f)$$

$$\sum_j q_{ij} - s_i x_i = 0 \quad \forall i, \quad l_{ij} x_i \leq q_{ij} \leq u_{ij} x_i \quad \forall (i, j) \quad (5.26g)$$

$$\sum_j f_{ij} - s_i y_i = 0 \quad \forall i, \quad l_{ij} y_i \leq f_{ij} \leq u_{ij} y_i \quad \forall (i, j) \quad (5.26h)$$

$$l_{ij} a_{ij} \leq g_j \leq u_{ij} a_{ij} \quad \forall (i, j) \quad (5.26i)$$

$$0 \leq x_i \leq \bar{a} \quad \forall i, \quad 0 \leq y_i \leq \bar{b} \quad \forall i \quad (5.26j)$$

$$0 \leq a_{ij} \leq u_{a_{ij}} \quad \forall (i, j). \quad (5.26k)$$

Denoting $l^1 \geq 0$, $l^2 \geq 0$ and l^3 (unrestricted in sign) as the Lagrange multipliers associated with the constraints (5.26b), (5.26c), (5.26d), respectively, we can formulate the following Lagrangian dual for problem $\text{RLT}(\Omega)$.

$$\text{Maximize } \{ n(l^1, l^2, l^3) : l^1 \geq 0, l^2 \geq 0 \text{ and } l^3 \text{ (unrestricted in sign)} \} \quad (5.27a)$$

where

$$\begin{aligned} n(l^1, l^2, l^3) = & l^1 h + l^3 d + \min \{ (c - l^1 H_7) g - (l^1 H_1 w + \sum_i \sum_j l^3_j w_{ij}) + \\ & \sum_i \sum_j l^2_{ij} [(x_i - a_j)^2 + (y_i - b_j)^2] - (\sum_i \sum_j l^2_{ij} u_{a_{ij}} a_{ij} + l^1 H_2 a) \\ & - l^1 (H_3 x + H_4 y + H_5 q + H_6 f) \} \end{aligned}$$