AIP Journal of Applied Physics

Nonlinear modulation of TM waves in a circular waveguide

Omar R. Asfar and Ali H. Nayfeh

Citation: Journal of Applied Physics **47**, 88 (1976); doi: 10.1063/1.322302
View online: http://dx.doi.org/10.1063/1.322302
View Table of Contents: http://scitation.aip.org/content/aip/journal/jap/47/1?ver=pdfcov
Published by the AIP Publishing

Articles you may be interested in

Nonlinear polymer-clad silicon slot waveguide modulator with a half wave voltage of 0.25 V
Appl. Phys. Lett. **92**, 163303 (2008); 10.1063/1.2909656

Linear theory of the electron cyclotron maser based on TM circular waveguide mode
Phys. Plasmas **13**, 073104 (2006); 10.1063/1.2221678

Control of TM waves by strong TE nonlinear guided waves
Appl. Phys. Lett. **52**, 1 (1988); 10.1063/1.99326

Wave Propagation in a Circular Waveguide with an Absorbing Wall
J. Appl. Phys. **41**, 433 (1970); 10.1063/1.1658369

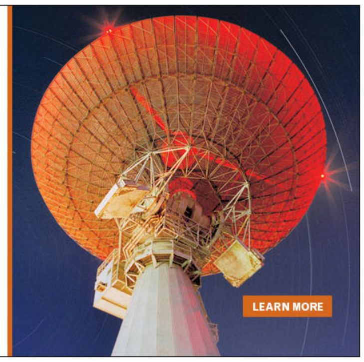
The Susceptance of a Thin Iris in Circular Wave Guide with the TM 01 Mode Incident
J. Appl. Phys. **22**, 1316 (1951); 10.1063/1.1699857

MIT LINCOLN LABORATORY CAREERS

Discover the satisfaction of innovation and service to the nation

- Space Control
- Air & Missile Defense
- Communications Systems & Cyber Security
- Intelligence, Surveillance and Reconnaissance Systems
- Advanced Electronics
- Tactical Systems
- Homeland Protection
- Air Traffic Control

LINCOLN LABORATORY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY



[This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to] IP: 128.173.126.47 On: Mon, 04 May 2015 21:20:10

Nonlinear modulation of TM waves in a circular waveguide

Omar R. Asfar* and Ali H. Nayfeh

Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

(Received 6 February 1975; in final form 23 September 1975)

The method of multiple scales is used to derive a nonlinear Schrödinger equation for the temporal and spatial amplitude and phase modulations of TM waves in a perfectly conducting guide containing a nonlinear isotropic medium. This equation is used to show that monochromatic waves are stable if the mechanism producing the nonlinearity is an electric or magnetic polarization and unstable if the nonlinearity is due to electrostriction or magnetostriction. It is also used to determine the amplitude dependence of the cutoff frequencies.

PACS numbers: 84.40.S

INTRODUCTION

Electromagnetic wave propagation in a waveguide containing a nonlinear isotropic medium was treated by Keller and Millman[1] using a generalization of the Lindstedt-Poincaré perturbation technique; they presented numerical results for rectangular waveguides only. Taniuti and Yajima[2] used the method of multiple scales in order to derive a nonlinear Schrödinger equation for nonlinear wave modulation and applied the technique to nonlinear optics.[3]

In this paper, we derive a nonlinear Schrödinger equation governing amplitude and phase modulations in a circular waveguide using the method of multiple scales.[4] We follow Nayfeh's treatment of the propagation of a wave packet in a hard-walled circular acoustic duct[5] and use Schrödinger's equation to study the stability of monochromatic wave solutions. The mechanisms producing the nonlinearity play a key role in deciding the stability of monochromatic wave solutions in much the same way that it characterizes self-focusing and defocusing of laser beams[6] in nonlinear optics.

FORMULATION

We consider a circular waveguide for long-distance transmission of microwave power filled with a homogeneous isotropic nonlinear medium whose constitutive relations are given by

$$\mathrm{D}=\epsilon(|\mathrm{E}|)\mathrm{E}, \quad \mathrm{B}=\mu(|\mathrm{H}|)\mathrm{H}. \tag{1}$$

The dielectric constant and the magnetic permeability are assumed to have the following isotropic nonlinear functions:

$$\epsilon=\epsilon_0(1+\epsilon_2|\mathrm{E}|^2+\cdots), \tag{2a}$$

$$\mu=\mu_0(1+\mu_2|\mathrm{H}|^2+\cdots), \tag{2b}$$

where ϵ_0 and μ_0 correspond to the linear problem while ϵ_2 and μ_2 are the nonlinear indices. We use the cylindrical coordinates ρ', ϕ, z' and introduce dimensionless variables ρ and z by using the radius of the cylinder R as a reference quantity so that $\rho=\rho'/R$ and $z=z'/R$. We also make time dimensionless by using R and the linear phase speed of the waves in the medium $v=(\mu_0\epsilon_0)^{-1/2}$ so that $t=vt'/R$ is the dimensionless time.

For nonlinear media, the Maxwell equations can be reduced to the following governing equation:

$$\nabla^2\mathrm{E}+\nabla\left(\frac{1}{\epsilon}(\mathrm{E}\cdot\nabla\epsilon)\right) =(\mu_0\epsilon_0)^{-1/2}\frac{\partial}{\partial t}\left((\mu_0\epsilon_0)^{-1/2}\mu\frac{\partial}{\partial t}(\epsilon\mathrm{E})+\nabla\mu\times\mathrm{H}\right). \tag{3}$$

The boundary condition to be satisfied in the case of TM waves and perfectly conducting walls is

$$E_z=0 \quad \text{at } \rho=1. \tag{4}$$

To determine an approximate solution for Eq. (3) subject to boundary condition (4), for small but finite amplitudes, we use the method of multiple scales and let

$$\mathrm{E}(\rho,\phi,z,t)=\sum_{n=1}^{3}\delta^n\mathrm{E}_n(\rho,\phi,Z_0,Z_1,Z_2,T_0,T_1,T_2)+0(\delta^4), \tag{5a}$$

$$\mathrm{H}(\rho,\phi,z,t)=\sum_{n=1}^{3}\delta^n\mathrm{H}_n(\rho,\phi,Z_0,Z_1,Z_2,T_0,T_1,T_2)+0(\delta^4), \tag{5b}$$

where δ is a small but finite dimensionless parameter characterizing the amplitude of the wave and

$$Z_m=\delta^m z, \quad T_m=\delta^m t. \tag{6}$$

Here Z_0 is a length scale of the order of a wavelength, Z_1 and Z_2 are length scales characterizing the slow spatial amplitude and phase modulations, T_0 is a time scale of the order of a period of the wave, and T_1 and T_2 are time scales characterizing the slow temporal amplitude and phase modulations. Using Eq. (6) and the chain rule, we express the temporal and axial derivatives as

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial Z_0}+\delta\frac{\partial}{\partial Z_1}+\delta^2\frac{\partial}{\partial Z_2}+\cdots, \tag{7a}$$

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial T_0}+\delta\frac{\partial}{\partial T_1}+\delta^2\frac{\partial}{\partial T_2}+\cdots. \tag{7b}$$

Substituting Eqs. (5)–(7) into Eqs. (3) and (4), using Eq. (2) and equating coefficients of like powers of δ, we obtain the following for E_z:

Order δ

$$L(E_{1z})\equiv\nabla_0^2E_{1z}-\frac{\partial^2E_{1z}}{\partial T_0^2}=0, \tag{8a}$$

$$\nabla_0^2=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial Z_0^2}, \tag{8b}$$

 Copyright © 1976 American Institute of Physics

[This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to] IP: 128.173.126.47 On: Mon, 04 May 2015 21:20:10

$$E_{1z}=0 \quad \text{at } \rho=1. \tag{8c}$$

Order δ^2

$$\mathcal{L}(E_{2z})=2\left(\frac{\partial^2 E_{1z}}{\partial T_0\,\partial T_1}-\frac{\partial^2 E_{1z}}{\partial Z_0\,\partial Z_1}\right), \tag{9a}$$

$$E_{2z}=0 \quad \text{at } \rho=1. \tag{9b}$$

Order δ^3

$$\begin{aligned}\mathcal{L}(E_{3z}) &= \frac{\partial^2 E_{1z}}{\partial T_1^2}+2\frac{\partial^2 E_{1z}}{\partial T_0\,\partial T_2}+2\frac{\partial^2 E_{2z}}{\partial T_0\,\partial T_1}-\frac{\partial^2 E_{1z}}{\partial Z_1^2}-2\frac{\partial^2 E_{1z}}{\partial Z_0\,\partial Z_2}\\ &\quad -2\frac{\partial^2 E_{2z}}{\partial Z_0\,\partial Z_1}+\epsilon_2\frac{\partial^2}{\partial T_0^2}(|\mathrm{E}_1|^2 E_{1z})+\mu_2\frac{\partial}{\partial T_0}\left(|\mathrm{H}_1|^2\frac{\partial E_{1z}}{\partial T_0}\right)\\ &\quad +\mu_2\left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}\frac{\partial}{\partial T_0}\left(\nabla_0(|\mathrm{H}_1|^2)\times \mathrm{H}_1\right)_z\\ &\quad -\epsilon_2\frac{\partial}{\partial Z_0}\left(\nabla_0(|\mathrm{E}_1|^2)\cdot \mathrm{E}_1\right),\end{aligned} \tag{10a}$$

$$\nabla_0=\hat{e}_\rho\frac{\partial}{\partial\rho}+\hat{e}_\phi\frac{\partial}{\partial\phi}+\hat{e}_z\frac{\partial}{\partial Z_0}, \tag{10b}$$

$$E_{3z}=0 \quad \text{at } \rho=1. \tag{10c}$$

Here, $\hat{e}_\rho$, $\hat{e}_\phi$, and $\hat{e}_z$ are unit vectors in the ρ, ϕ, and z directions, respectively.

SOLUTION

We take the solution of Eq. (8) that is bounded at the axis in the form of a traveling wave packet centered at frequency ω and wavenumber k_z; that is, we let

$$E_{1z}=A(Z_1,Z_2,T_1,T_2)J_m(k_\rho\rho)\exp(i\theta)+\text{cc}, \tag{11a}$$

$$\theta=k_z Z_0-\omega T_0+m\phi, \tag{11b}$$

where J_m is Bessel's function of order m, cc represents the complex conjugate of the preceding terms, and ω and k_z satisfy the dispersion relationship

$$\omega^2-k_z^2=k_\rho^2. \tag{12}$$

Substituting Eq. (11) into Eq. (8c), we have

$$J_m(k_\rho)=0. \tag{13}$$

To carry the solution to higher orders, we have to determine the remaining field components in terms of E_{1z}. Manipulation of the Faraday and Ampère laws in the set of Maxwell equations and use of the explicit Z_0 dependence [Eq. (11)] leads to the determination of the transverse fields as

$$E_{1\rho}=i(k_z/k_\rho)A\,J_m'(k_\rho\rho)\exp(i\theta)+\text{cc}, \tag{14a}$$

$$E_{1\phi}=-(m/\rho)(k_z/k_\rho^2)AJ_m(k_\rho\rho)\exp(i\theta)+\text{cc}, \tag{14b}$$

$$H_{1\rho}=(m\omega/\rho k_\rho^2)\epsilon_0/\mu_0)^{1/2}AJ_m(k_\rho\rho)\exp(i\theta)+\text{cc}, \tag{14c}$$

$$H_{1\phi}=i(\omega/k_\rho)(\epsilon_0/\mu_0)^{1/2}AJ_m'(k_\rho\rho)\exp(i\theta)+\text{cc}. \tag{14d}$$

The function A is not determined at this level of approximation; it will be determined from the solvability conditions of the second- and third-order problems.

Substituting Eq. (11) into Eq. (9), we obtain

$$\mathcal{L}(E_{2z})=-2i\left(\omega\frac{\partial A}{\partial T_1}+k_z\frac{\partial A}{\partial Z_1}\right)J_m(k_\rho\rho)\exp(i\theta)+\text{cc}. \tag{15}$$

Secular terms will be eliminated if we let

$$\omega\frac{\partial A}{\partial T_1}+k_z\frac{\partial A}{\partial Z_1}=0. \tag{16}$$

Since the homogeneous solution of Eq. (15) is proportional to E_{1z}, one can set it equal zero, without loss of generality. If it were included, the results will not change provided that one redefines the amplitude and the phase, though the algebra becomes more involved. Thus, in this paper, we choose

$$\mathrm{E}_2=0. \tag{17}$$

Substituting Eqs. (11), (14), and (17) into Eq. (10a) and carrying out the indicated algebraic manipulations, we obtain

$$\begin{aligned}\mathcal{L}(E_{3z}) &= \left[-2i\left(\omega\frac{\partial A}{\partial T_2}+k_z\frac{\partial A}{\partial Z_2}\right)+\frac{\partial^2 A}{\partial T_2^2}-\frac{\partial^2 A}{\partial Z_1^2}\right]\\ &\quad \times J_m(k_\rho\rho)\exp(i\theta)+A^2\bar{A}F(\rho)\exp(i\theta)+\text{NST}+\text{cc},\end{aligned} \tag{18}$$

where $F(\rho)$ is given in the Appendix and NST stands for terms that do not produce secular terms, i.e., terms involving higher harmonics in θ. Since the homogeneous third-order problem consisting of Eqs. (10c) and (18) has a nontrivial solution, the corresponding inhomogeneous problem has a solution if and only if a solvability condition is satisfied.

To determine this solvability condition, we seek a particular solution of the form

$$E_{3z}=\psi(\rho,Z_1,Z_2,T_1,T_2)\exp(i\theta)+\text{cc}. \tag{19}$$

Substituting this solution into Eqs. (10c) and (18) and equating the coefficients of $\exp(i\theta)$ on both sides, we obtain

$$\begin{aligned}&\frac{\partial^2\psi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}+\left(k_\rho^2-\frac{m^2}{\rho^2}\right)\psi\\ &\quad =-\left[2i\left(\omega\frac{\partial A}{\partial T_2}+k_z\frac{\partial A}{\partial Z_2}\right)-\frac{\partial^2 A}{\partial T_1^2}+\frac{\partial^2 A}{\partial Z_1^2}\right]J_m(k_\rho\rho)\\ &\qquad +A^2\bar{A}F(\rho),\end{aligned} \tag{20}$$

$$\psi=0 \quad \text{at } \rho=1. \tag{21}$$

Multiplying Eq. (20) by $\rho J_m(k_\rho\rho)$, integrating by parts from $\rho=0$ to $\rho=1$, and using Eq. (21), we obtain the solvability condition

$$2i\left(\omega\frac{\partial A}{\partial T_2}+k_z\frac{\partial A}{\partial Z_2}\right)-\frac{\partial^2 A}{\partial T_1^2}+\frac{\partial^2 A}{\partial Z_1^2}=\Lambda A^2\bar{A}, \tag{22}$$

where

$$\Lambda=\left[\int_0^1\rho F(\rho)J_m(k_\rho\rho)\,d\rho\right]\left[\int_0^1\rho J_m^2(k_\rho\rho)\,d\rho\right]^{-1}. \tag{23}$$

Eliminating $\partial^2 A/\partial Z_1^2$ from Eq. (22) by using Eq. (16), we obtain

$$2i\left(\omega\frac{\partial A}{\partial T_2}+k_z\frac{\partial A}{\partial Z_2}\right)+\left(\frac{\omega^2}{k_z^2}-1\right)\frac{\partial^2 A}{\partial T_1^2}=\Lambda A^2\bar{A}. \tag{24}$$

To simplify Eq. (24), we differentiate the dispersion relationship [Eq. (12)] with respect to ω and obtain

$$k_z k_z'=\omega, \tag{25}$$

[This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to] IP: 128.173.126.47 On: Mon, 04 May 2015 21:20:10

where $k_z' = dk_z/d\omega$, the inverse of the group velocity. Differentiating Eq. (25) with respect to ω, we obtain

$$k_z k_z'' = 1 - k_z'^2 = 1 - \omega^2/k_z^2. \quad (26)$$

Using Eqs. (25) and (26), letting $T_n = \delta^n t$ and $Z_n = \delta^n z$, and rearranging terms, we rewrite Eq. (24) as

$$\frac{\partial A}{\partial z} + k_z' \frac{\partial A}{\partial t} + \tfrac{1}{2} i k_z'' \frac{\partial^2 A}{\partial t^2} = -\tfrac{1}{2} i \delta^2 \frac{\Lambda}{k_z} A^2 \bar{A}. \quad (27)$$

Changing the independent variables from z and t to

$$\xi = t - k_z' z, \quad \eta = z, \quad (28)$$

we express Eq. (27) in the form

$$\frac{\partial A}{\partial \eta} + \tfrac{1}{2} i k_z'' \frac{\partial^2 A}{\partial \xi^2} = -\tfrac{1}{2} i \delta^2 \frac{\Lambda}{k_z} A^2 \bar{A}, \quad (29)$$

which is a nonlinear Schrödinger equation. Letting $A = \frac{1}{2} a \exp(i\beta)$ with real a and β in Eq. (29) and separating real and imaginary parts, we obtain

$$\frac{\partial a}{\partial \eta} - k_z'' \left(\frac{\partial a}{\partial \xi} \frac{\partial \beta}{\partial \xi} + \tfrac{1}{2} a \frac{\partial^2 \beta}{\partial \xi^2} \right) = 0, \quad (30)$$

$$\frac{\partial \beta}{\partial \eta} + \tfrac{1}{2} k_z'' \left[\frac{1}{a} \frac{\partial^2 a}{\partial \xi^2} - \left(\frac{\partial \beta}{\partial \xi} \right)^2 \right] = -\tfrac{1}{8} \delta^2 \frac{\Lambda}{k_z} a^2. \quad (31)$$

The case of monochromatic waves

For monochromatic waves, $\partial a/\partial \xi = \partial \beta/\partial \xi = 0$, and Eqs. (30) and (31) can be integrated to give

$$a = a_0 \quad \text{and} \quad \beta = -\tfrac{1}{8} \delta^2 (\Lambda/k_z) a_0^2 \eta + \beta_0, \quad (32)$$

where a_0 and β_0 are constants. Substituting for a and β from Eq. (32) into Eq. (11), we rewrite E_z in the form

$$E_z = \delta a_0 J_m(k_\rho \rho) \cos(kz - \omega t + m\phi + \beta_0) + 0(\delta^3), \quad (33)$$

where

$$k = k_z - \tfrac{1}{8} \delta^2 \Lambda k_z^{-1} a_0^2. \quad (34)$$

Equation (34) shows that the nonlinearity of the medium results in a wavenumber shift from k_z to k. To determine whether this shift increases or decreases the wavenumber one has to examine the sign of the nonlinear factor Λ. This depends on the signs of μ_2 and ϵ_2 which, in turn, depend on the mechanism producing the nonlinearity, a point we shall consider in some detail in the following discussion on the stability of monochromatic waves.

Equations (30) and (31) can be used to analyze the stability of the monochromatic wave solution, Eqs. (33) and (34). To do this, we let

$$a = a_0 + a_1, \quad \beta = -\tfrac{1}{8} \delta^2 (\Lambda/k_z) a_0^2 \eta + \beta_0 + \beta_1, \quad (35)$$

where a_1 and β_1 are small compared with the preceding terms. Substituting Eq. (35) into Eqs. (30) and (31) and neglecting the nonlinear terms in a_1 and β_1, we obtain

$$\frac{\partial a_1}{\partial \eta} - \tfrac{1}{2} k_z'' a_0 \frac{\partial^2 \beta_1}{\partial \xi^2} = 0, \quad (36)$$

$$\frac{\partial \beta_1}{\partial \eta} + \tfrac{1}{2} k_z'' a_0^{-1} \frac{\partial^2 a_1}{\partial \xi^2} = -\tfrac{1}{4} \delta^2 \frac{\Lambda}{k_z} a_0 a_1. \quad (37)$$

Since Eqs. (36) and (37) are linear, we seek their solution in the form

$$a_1 = \tilde{a}_1 \exp[i(\tilde{k}\eta - \tilde{\omega}\xi)],$$
$$\beta_1 = \tilde{\beta}_1 \exp[i(\tilde{k}\eta - \tilde{\omega}\xi)], \quad (38)$$

where $\tilde{a}_1$ and $\tilde{\beta}_1$ are constants. Substituting this solution into Eqs. (36) and (37) and eliminating $\tilde{a}_1$ and $\tilde{\beta}_1$, we obtain

$$\tilde{k}^2 = \tfrac{1}{4} k_z''^2 \tilde{\omega}^2 (\tilde{\omega}^2 - \tfrac{1}{2} \delta^2 \Lambda a_0^2 / k_z k_z''), \quad (39)$$

which shows that if $\Lambda/k_z'' < 0$, $\tilde{k}$ is always real for all values of $\tilde{\omega}$ so that monochromatic waves are neutrally stable. On the other hand, if $\Lambda/k_z'' > 0$, $\tilde{k}^2$ is negative for all $\tilde{\omega} < \delta a_0 (\Lambda/2k_z k_z'')^{1/2}$; consequently, disturbances grow exponentially with η and monochromatic waves are unstable. Now, Eqs. (12) and (26) show that $k_z'' = -k_\rho^2/k_z^3 < 0$; hence, the stability of monochromatic waves depends on the sign of Λ.

Apart from the signs of ϵ_2 and μ_2, our unpublished numerical results show that the signs of the coefficients of ϵ_2 and μ_2 are always negative in the microwave region. As previously mentioned, the signs of ϵ_2 and μ_2 depend on the mechanism responsible for the nonlinear behavior. Considering a nonlinear dielectric medium, we have the following possibilities according to Ref. 6: (i) nonlinear polarization due to saturation of the electronic polarizability which means $\epsilon_2 < 0$, and hence $\Lambda/k_z'' < 0$ (neutral stability); (ii) electrostriction gives rise to $\epsilon_2 > 0$, and hence $\Lambda/k_z'' > 0$ (instability).

In a medium where magnetic properties are dominant the mechanisms responsible for nonlinearity are nonlinear polarization and magnetostriction. The same conclusions regarding stability in the case of dielectric media hold correspondingly in the magnetic case. We thus see how the type of material is important for the question of stability. If both strictional effects and polarization are present in a material, they will compete. Usually strictional effects dominate thereby producing instability.

Solution near cutoff frequencies

Although Eqs. (33) and (34) are valid for a wide range of frequencies, they break down as $k_z \to 0$ (i.e., near the linear cutoff frequencies) because the wavenumber shift approaches infinity. However, the basic equations [Eqs. (16) and (22)] governing the temporal and spatial modulations are valid for all frequencies. In this section, we specialize these equations to frequencies near cutoff. Eliminating $\partial^2 A/\partial T_1^2$ from Eq. (22) by using Eq. (16), we obtain

$$2i \left(\omega \frac{\partial A}{\partial T_2} + k_z \frac{\partial A}{\partial Z_2} \right) + \left(1 - \frac{k_z^2}{\omega^2} \right) \frac{\partial^2 A}{\partial Z_1^2} = \Lambda A^2 \bar{A}. \quad (40)$$

Letting $T_2 = \delta^2 t$ and $Z_n = \delta^n z$ in Eq. (40), we rewrite it as

$$2i \left(\omega \frac{\partial A}{\partial t} + k_z \frac{\partial A}{\partial z} \right) + \left(1 - \frac{k_z^2}{\omega^2} \right) \frac{\partial^2 A}{\partial z^2} = \delta^2 \Lambda A^2 \bar{A}, \quad (41)$$

which can also be written in the form

$$\frac{\partial A}{\partial t} + \omega' \frac{\partial A}{\partial z} - \tfrac{1}{2} i \omega'' \frac{\partial^2 A}{\partial z^2} = -\tfrac{1}{2} i \Lambda \omega^{-1} A^2 \bar{A}. \quad (42)$$

For monochromatic waves, $\partial A/\partial t = 0$ and Eq. (41) becomes

[This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to] IP: 128.173.126.47 On: Mon, 04 May 2015 21:20:10

$$2ik_z\frac{dA}{dz}+\left(1-\frac{k_z^2}{\omega^2}\right)\frac{d^2A}{dz^2}=\delta^2\Lambda A^2\bar{A}, \tag{43}$$

which is valid for all frequencies away from zero. We take the solution of Eq. (43) in the form

$$A=\tfrac{1}{2}a\exp(i\beta), \tag{44}$$

where a is constant and

$$\frac{d\beta}{dz}=\left\{-k_z+\left[k_z^2-\tfrac{1}{4}\delta^2\left(1-\frac{k_z^2}{\omega^2}\right)\Lambda a^2\right]^{1/2}\right\}\left(1-\frac{k_z^2}{\omega^2}\right)^{-1}. \tag{45}$$

Away from cutoff frequencies, k_z is away from zero and the radical in Eq. (45) can be expanded for small δ yielding,

$$\frac{d\beta}{dz}=-\tfrac{1}{8}\delta^2\Lambda k_z^{-1}a^2, \tag{46}$$

in agreement with the monochromatic wave solution obtained above. On the other hand, when $k_z\to 0$, Eq. (45) tends to

$$\frac{d\beta}{dz}=-k_z+(k_z^2-\tfrac{1}{4}\delta^2\Lambda a^2)^{1/2}. \tag{47}$$

Substituting for A from Eq. (44) into Eq. (11), using Eq. (47), and letting $T_0=t$ and $Z_n=\delta^n z$, we obtain Eq. (33); however, k of Eq. (34) is modified to

$$k=(k_z^2-\tfrac{1}{4}\delta^2\Lambda a^2)^{1/2}. \tag{48}$$

Therefore, the cutoff frequencies are solutions of

$$k_z^2-\tfrac{1}{4}\delta^2\Lambda a^2=0. \tag{49}$$

Since $k_z^2=\omega^2-k_\rho^2$ according to Eq. (12), then the cutoff frequencies are

$$\omega=k_\rho+\tfrac{1}{8}\delta^2a^2\Lambda(k_\rho)+\cdots, \tag{50}$$

where $\Lambda(k_\rho)$ stands for the value of Λ when $\omega=k_\rho$. Equation (41) can be used to study the stability of monochromatic solutions as before.

APPENDIX

$$\begin{aligned}F(\rho)=&J_m(k_\rho\rho)J_m'^2(k_\rho\rho)\{2\epsilon_2k_z^2[1-(k_z/k_\rho)^2]-\epsilon_2(\omega k_z/k_\rho)^2\\&+\omega^4\mu_2(\epsilon_0/\mu_0)[4(m/\rho k_\rho^2)^2-(3/k_\rho^2)]\}\\&+J_m^3(k_\rho\rho)\{2\epsilon_2k_z^2[1+(m/\rho k_\rho)^2][1+(mk_z/\rho k_\rho^2)^2]\\&-3\epsilon_2\omega^2[1+(mk_z/\rho k_\rho^2)^2]-\omega\mu_2(\epsilon_0/\mu_0)(\omega/k_\rho^2)^3\\&\times(m/\rho)^2[2(m/\rho)^2-k_\rho^2]\}+6\omega\mu_2k_\rho(\epsilon_0/\mu_0)(\omega/k_\rho)^3\\&\times J_m'^2(k_\rho\rho)J_m''(k_\rho\rho)-J_m'(k_\rho\rho)J_m^2(k_\rho\rho)\\&\times\{(2/\rho^3)(m^2/k_\rho^5)[\omega^4\mu_2(\epsilon_0/\mu_0)+\epsilon_2k_z^4]\}.\end{aligned}$$

*Present address: College of Engineering, Box 800, Riyadh University, Riyadh, Saudi Arabia.

[1]J. B. Keller and M. H. Millman, Phys. Rev. 181, 1730 (1969).
[2]T. Taniuti and N. Yajima, J. Math. Phys. 10, 1369 (1969).
[3]T. Taniuti and N. Yajima, J. Math. Phys. 14, 1389 (1973).
[4]A. H. Nayfeh, *Perturbation Methods* (Wiley-Interscience, New York, 1973), Chap. VI.
[5]A. H. Nayfeh, J. Acoust. Soc. Am. 57, 803 (1975).
[6]S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Usp. Fiz. Nauk. 93, 19 (1967).

[This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to] IP: 128.173.126.47 On: Mon, 04 May 2015 21:20:10